University of BRISTOL

Machine Learning

Distributions

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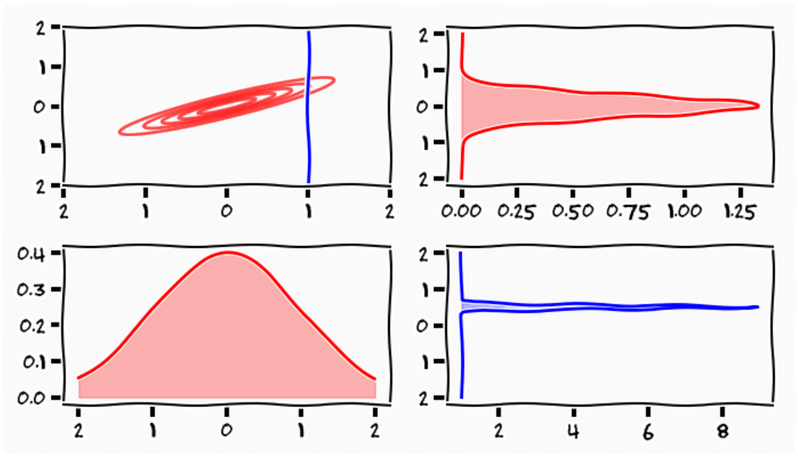
Introduction

Basic Probabilities



The Rules of Probability

Sum Rule

$$p(X) = \sum_{Y} p(X, Y)$$

Product Rule

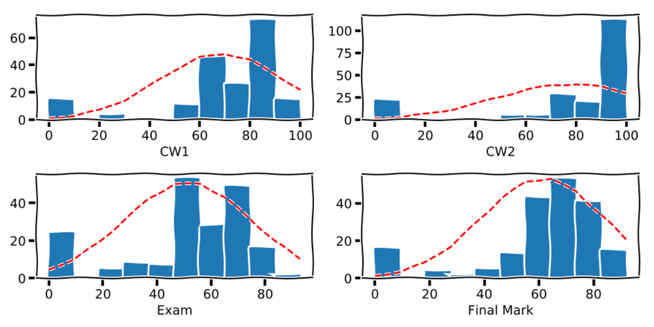
$$p(X, Y) = p(Y|X)p(X)$$

$\Rightarrow$ **Bayes Rule**

$$p(X|Y) = \frac{P(Y|X)p(X)}{p(Y)}$$

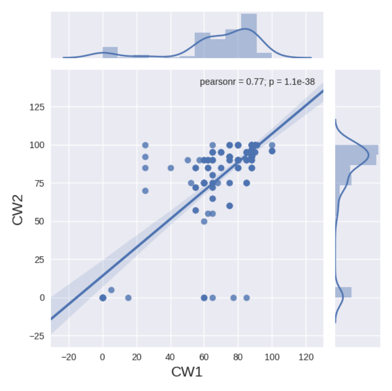
$$p(\text{CW1}, \text{CW2}, \text{Exam})$$

Marginals



60
40
20
0
0
20
40
60
80
100
CW1
100
75
50
25
0
0
20
40
60
80
100
CW2
40
20
0
0
20
40
60
80
Exam
40
20
0
0
20
40
60
80
Final Mark

Marginal



$$p(\mathrm{CW1}, \mathrm{CW2}) = \sum_{x=1}^{100} p(\mathrm{CW1}, \mathrm{CW2}, \mathrm{Exam} = x) = \sum_{x=1}^{100} p(\mathrm{CW1}, \mathrm{CW2} | \mathrm{Exam} = x) p(\mathrm{Exam} = x)$$

Questions

Exam

$$p(\text{Exam} = 100 | \text{CW1} = 20, \text{CW2} = 30)$$

- What is the probability of me getting Exam=100 if CW1=20 and CW2=30
- As you will get a result on the exam the probability for all results sums to 1

$$\sum_{x=0}^{x=100} p(\text{Exam} = x | \text{CW1} = 20, \text{CW2} = 30) = 1.0$$

Questions

Coursework

$$p(\text{Exam} = 70|\text{CW1} = 70) = \sum_{x=0}^{x=100} p(\text{Exam} = 70, \text{CW2} = x|\text{CW1} = 70)$$

- What is the probability that I will get Exam=70 if I got 70 on the coursework CW1

Questions

- Remember that each conditional is a probability
- However rare it is that you get 100% on both courseworks the conditional probability over all possibe exam results will sum to one

$$\sum_{x=1}^{x=100} p(\text{Exam} = x | \text{CW1} = 100, \text{CW2} = 100) = 1.0$$

Questions

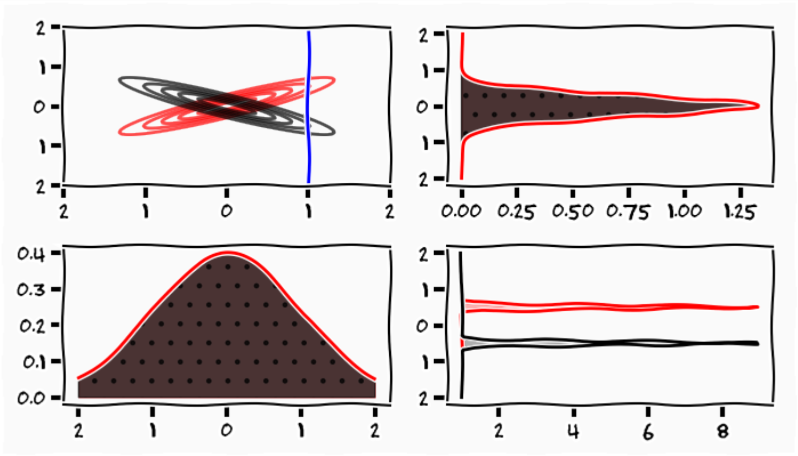
- Remember that each conditional is a probability
- However rare it is that you get 100% on both courseworks the conditional probability over all possibe exam results will sum to one

$$\sum_{x=1}^{x=100} p(\text{Exam} = x | \text{CW1} = 100, \text{CW2} = 100) = 1.0$$

- What shows that it is rare is that the probability for getting

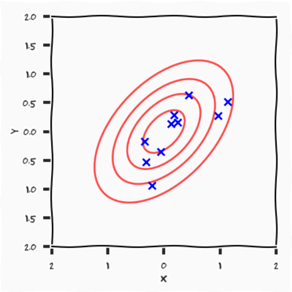
$$\sum_{x=1}^{x=100} p(\text{Exam} = x, \text{CW1} = 100, \text{CW2} = 100)$$
$$= p(\text{CW1} = 100, \text{CW2} = 100) \leq 1.0$$

Dangers of Marginals



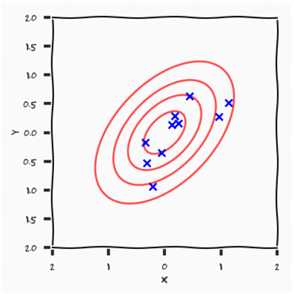
Learning with Distributions

Machine Learning



- Our goal is to understand realisations of a system

[1] https://en.wikipedia.org/wiki/All_models_are_wrong



- Our goal is to understand realisations of a system
- If we can, then we can "equate" our model with the system

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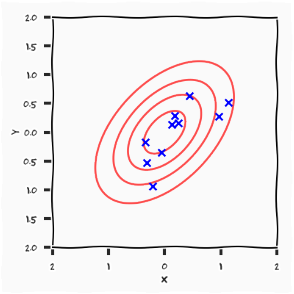
Machine Learning



- Our goal is to understand realisations of a system
- If we can, then we can "equate" our model with the system
- Importantly not as truth, but as a useful hypothesis related to our assumptions[1]

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Bayes Rule

$$p(\theta|Y) = \frac{p(Y|\theta)p(\theta)}{p(Y)}$$

Bayes Rule

$$\underbrace{p(\theta|Y)}_{\text{posterior}} = \underbrace{P(Y|\theta)}_{\text{likelihood}} \cdot \underbrace{p(\theta)}_{\text{prior}} \cdot \frac{1}{\underbrace{p(Y)}_{\text{evidence}}}$$

Likelihood how likely is the data to come from the model specific model indexed by θ

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Prior what do I believe the specific model to be, i.e. how likely do I believe different θ to be

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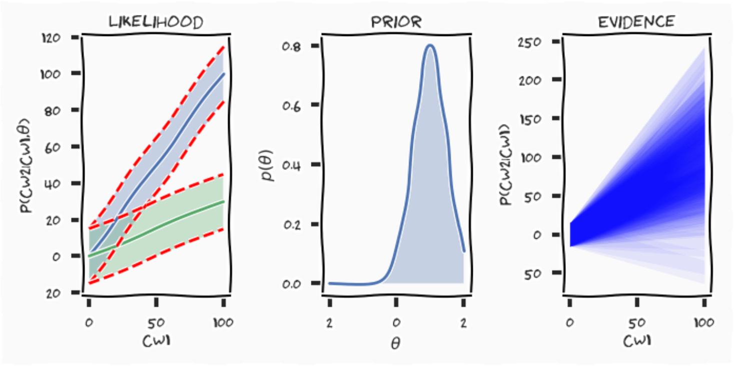
Posterior which distributions of models do I believe have generated this data



$$\mathrm{CW2} = \theta \cdot \mathrm{CW1} \pm 15\%$$

$$\theta \sim \mathcal{N}(1.0, 0.5)$$

Discrete Distributions

Bernoulli Distribution

- Distribution over binary random variable $x \in \{0, 1\}$

$$p(x = 1|\mu) = \mu$$

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- Due to binary outcome

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- Due to binary outcome

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- Distribution

$$\mathrm{Bern}(x|\mu) = \mu^x (1 - \mu)^{1-x}$$

Data

- We want to figure out what μ is for a specific coin
- Toss the coin N times, $\mathcal{D} = \{x_1, x_2, \ldots, x_N\}$

$$p(\mathcal{D}|\mu) = \prod_{n=1}^{N} p(x_n|\mu) = \prod_{n=1}^{N} \mu^{x_n}(1-\mu)^{1-x_n}$$

- What happens if we blindly trust this one experiment?

Maximum Likelihood

$$\mu_{ML} = \text{argmax}_{\mu} p(\mathcal{D}|\mu) = \frac{1}{N} \sum_{n=1}^{N} x_n$$

- if we get 3 heads in a row, we believe it will always be heads
- we need to include an assumption as a prior over μ

$$p(\mu|\mathcal{D}) = \frac{p(\mathcal{D}|\mu)p(\mu)}{p(\mathcal{D})}$$

- Also gives us an uncertainty related to our knowledge

Posterior

$$p(\mu|\mathcal{D}) = \frac{p(\mathcal{D}|\mu)p(\mu)}{p(\mathcal{D})}.$$

- if we can specify a prior $p(\mu)$ we can reach the posterior belief

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- what do we know about coins?

Posterior

$$p(\mu|\mathcal{D}) = \frac{p(\mathcal{D}|\mu)p(\mu)}{p(\mathcal{D})}.$$

- if we can specify a prior $p(\mu)$ we can reach the posterior belief
- what do we know about coins?
- how do I make that knowledge mathematicall explicit?

Conjugate Prior

- If we have a prior belief μ we want the posterior belief to have the same functional form

$$\text{posterior} \propto \text{likelihood} \times \text{prior}$$

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- Likelihood

$$p(\mathcal{D}|\mu) = \prod_{n=1}^{N} p(x_n|\mu) = \prod_{n=1}^{N} \mu^{x_n}(1-\mu)^{1-x_n}$$

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- Conjugate prior

$$p(\mu|\theta) = f_1(\theta)\mu^{f_2(\theta)}(1-\mu)^{f_3(\theta)}$$

$$\int_0^1 p(\mu|\theta)\mathrm{d}\mu = 1$$

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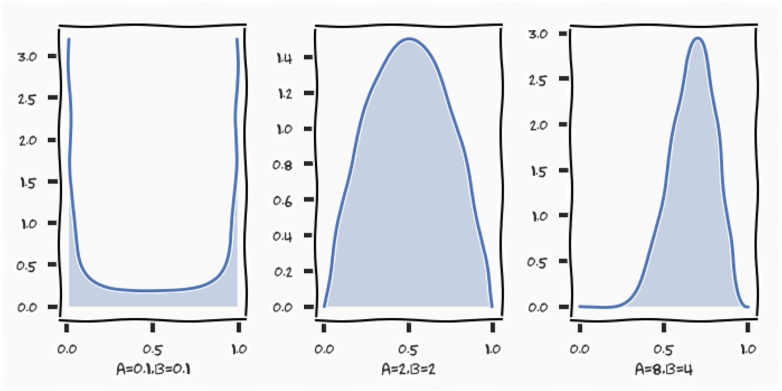
- *Does this make philosophical sense?*

Beta Distribution



$$\text{Beta}(\mu|a, b) = \frac{\Gamma(a+b)}{\Gamma(a)+\Gamma(b)}\mu^{a-1}(1-\mu)^{b-1}$$

Posterior

$$p(\mu|\mathcal{D}) \propto p(\mathcal{D}|\mu)p(\mu)$$

Posterior

$$
\begin{aligned}
p(\mu|\mathcal{D}) &\propto p(\mathcal{D}|\mu)p(\mu) \\
&= \prod_{i=1}^{N} \mathrm{Bern}(x|\mu)\mathrm{Beta}(\mu|a, b)
\end{aligned}
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&= \mu^{\sum_i x_i}(1-\mu)^{\sum_i (1-x_i)}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\mu^{a-1}(1-\mu)^{b-1}
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&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\mu^{\sum_i x_i + a - 1}(1-\mu)^{\sum_i (1-x_i) + b - 1}.
\end{aligned}
$$

Posterior

Churn the handle

$$p(\mu|\mathcal{D}) = \frac{p(\mathcal{D}|\mu)p(\mu)}{p(\mathcal{D})} = \frac{p(\mathcal{D}|\mu)p(\mu)}{\underbrace{\int p(\mathcal{D}|\mu)p(\mu)\mathrm{d}\mu}_{\text{This is hard}}}$$

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Conjugacy

- We know the functional form of the posterior

Posterior

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Conjugacy

- We know the functional form of the posterior
- We know that the posterior is proportional to the likelihood times the prior

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Conjugacy

- We know the functional form of the posterior
- We know that the posterior is proportional to the likelihood times the prior
- *Use these facts to avoid the integral*

Posterior

- Because we know the form of the posterior, we can *identify* its parameters

$$\text{Beta}(\mu|a_n, b_n) \propto \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \mu^{\overbrace{\sum_i x_i + a}^{a_n} - 1} (1-\mu)^{\overbrace{\sum_i (1-x_i) + b}^{b_n} - 1}$$

$$\text{Beta}(\mu|a, b) = \frac{\Gamma(a+b)}{\Gamma(a)+\Gamma(b)} \mu^{a-1}(1-\mu)^{b-1}$$

Posterior

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$$\text{Beta}(\mu|a_n, b_n) \propto \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\mu^{\underbrace{\sum_i x_i + a - 1}_{a_n}}(1-\mu)^{\underbrace{\sum_i (1-x_i) + b - 1}_{b_n}}$$

$$\text{Beta}(\mu|a, b) = \frac{\Gamma(a+b)}{\Gamma(a)+\Gamma(b)}\mu^{a-1}(1-\mu)^{b-1}$$

- This leads to the following posterior

$$\text{Beta}(\mu|a_n, b_n) = \frac{\Gamma\left(\sum_i x_i + a + \sum_i (1-x_i) + b\right)}{\Gamma\left(\sum_i x_i + a\right)\Gamma\left(\sum_i (1-x_i) + b\right)}\mu^{\sum_i x_i + a - 1}(1-\mu)^{\sum_i (1-x_i) + b - 1}$$

Lab 2

Multinomial

- If we have a variable that can take K different states

$$\mathbf{x} = [0, 0, 1, 0, 0, 0]^{\mathrm{T}}$$

Multinomial

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$$\mathbf{x} = [0, 0, 1, 0, 0, 0]^{\mathrm{T}}$$

- Multinomial

$$p(\mathbf{x}|\boldsymbol{\mu}) = \prod_{k=1}^{K} \mu_k^{x_k}$$

$$\boldsymbol{\mu} = [\mu_1, \ldots, \mu_k]^{\mathrm{T}}, \sum_k \mu_k = 1$$

Multinomial

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$$\boldsymbol{\mu} = [\mu_1, \ldots, \mu_k]^{\mathrm{T}}, \sum_k \mu_k = 1$$

- Likelihood

$$p(\mathbf{D}|\boldsymbol{\mu}) = \prod_{n=1}^{N} \prod_{k=1}^{K} \mu_k^{x_{nk}}$$

Dirichlet

- Multinomial

$$p(\mathbf{x}|\boldsymbol{\mu}) = \prod_{k=1}^{K} \mu_k^{x_k}$$

Dirichlet

- Multinomial

$$p(\mathsf{x}|\boldsymbol{\mu}) = \prod_{k=1}^{K} \mu_k^{x_k}$$

- Conjugate prior

$$p(\boldsymbol{\mu}|\boldsymbol{\alpha}) \propto \prod_{k=1}^{K} \mu_k^{\alpha_k - 1}$$

Dirichlet

- Multinomial

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- Dirichlet Distribution

$$\mathrm{Dir}(\boldsymbol{\mu}|\boldsymbol{\alpha}) = \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1) \cdot \ldots \cdot \Gamma(\alpha_K)} \prod_{k=1}^{K} \mu_k^{\alpha_k - 1}$$

Dirichlet

- Posterior

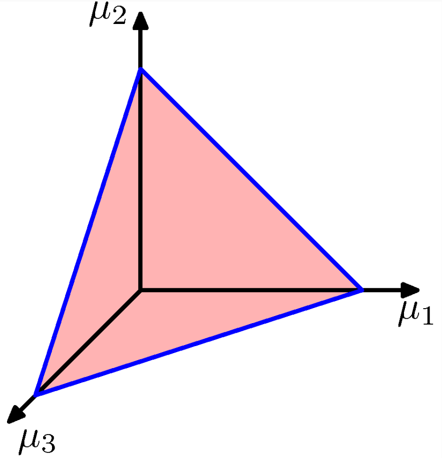
$$p(\boldsymbol{\mu}|\mathcal{D}, \boldsymbol{\alpha}) \propto p(\mathcal{D}|\boldsymbol{\mu})p(\boldsymbol{\mu}|\boldsymbol{\alpha}) \propto \prod_{k=1}^{K} \mu_k^{\alpha_k + m_k + 1}$$

$$m_k = \sum_n x_{nk}$$

- Normalised Form

$$p(|\mathcal{D}, \boldsymbol{\alpha}) = \frac{\Gamma(\alpha_0 + N)}{\Gamma(\alpha_1 + m_1) \cdot \ldots \cdot \Gamma(\alpha_K + m_K)} \prod_{k=1}^{K} \mu_k^{\alpha_k + m_k + 1}$$

Dirichlet Prior



Spans the plane $\mu_1 + \mu_2 + \mu_3 = 1$

Hyperparameters

$$p(\mu|\mathcal{D}, \boldsymbol{\alpha}) = \frac{p(\mathcal{D}|\boldsymbol{\mu})p(\boldsymbol{\mu}|\boldsymbol{\alpha})}{p(\mathcal{D}|\boldsymbol{\alpha})}$$

- all these priors have parameters, where do they come from?

Hyperparameters

$$p(\mu|\mathcal{D}, \boldsymbol{\alpha}) = \frac{p(\mathcal{D}|\boldsymbol{\mu})p(\boldsymbol{\mu}|\boldsymbol{\alpha})}{p(\mathcal{D}|\boldsymbol{\alpha})}$$

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Hyperparameters

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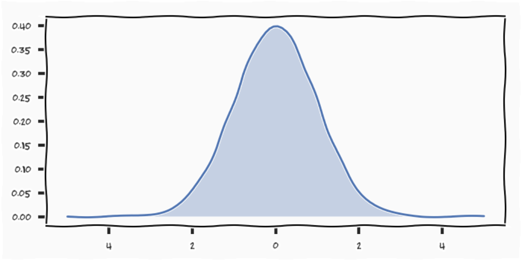
- all these priors have parameters, where do they come from?
- either we know them
- if we don't then place a prior over the priors parameters and go again
- the idea is to build up a hierarchy until you can input your knowledge/assumptions

Continous Distributions

Gaussian Distribution



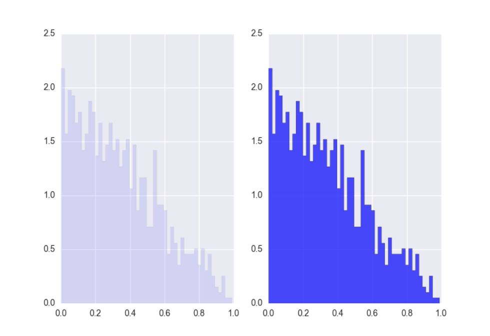
$$p(x|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{\frac{D}{2}} |\boldsymbol{\Sigma}|^{\frac{1}{2}}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{x}-\boldsymbol{\mu})}$$

Central Limit Theorem[2]

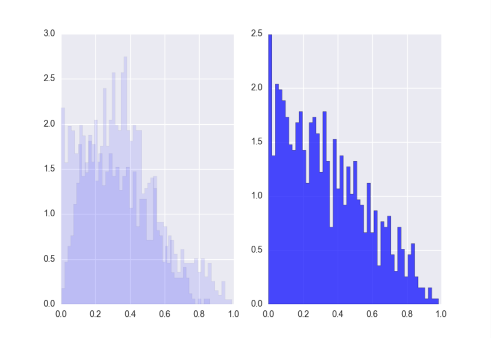
The central limit theorem states that the distribution of the sum (or average) of a large number of independent, identically distributed variables will be approximately normal, regardless of the underlying distribution.

[2] https://www.youtube.com/watch?v=wadzsURQFT4

Central Limit Theorem



Central Limit Theorem



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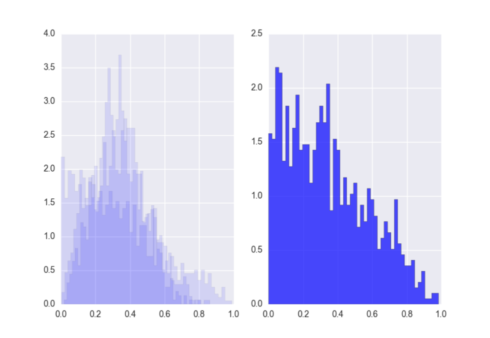

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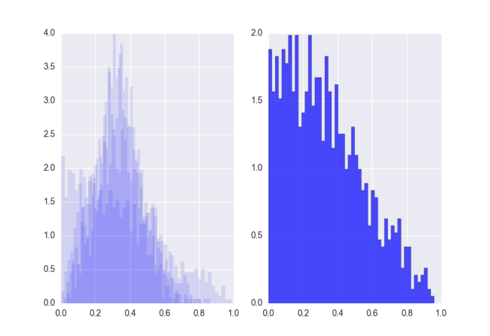

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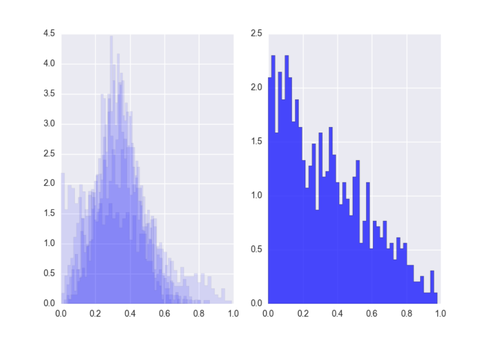

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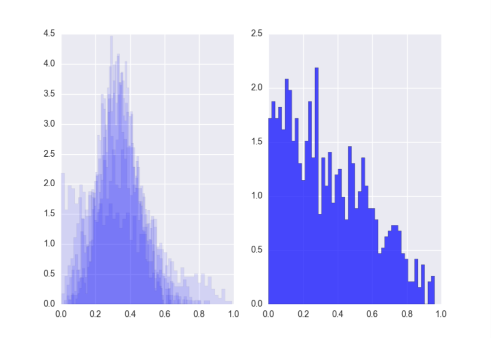

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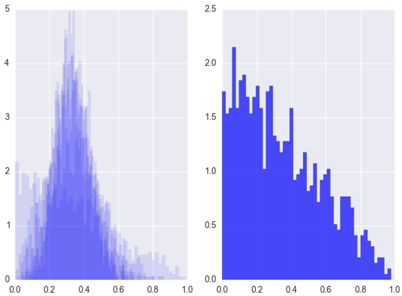

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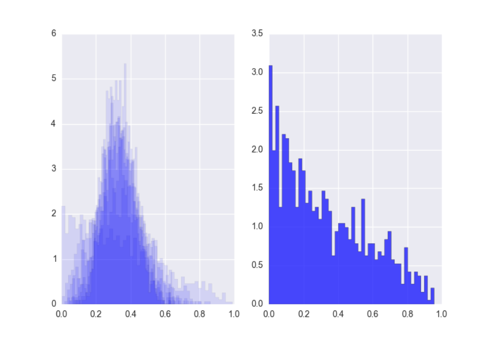

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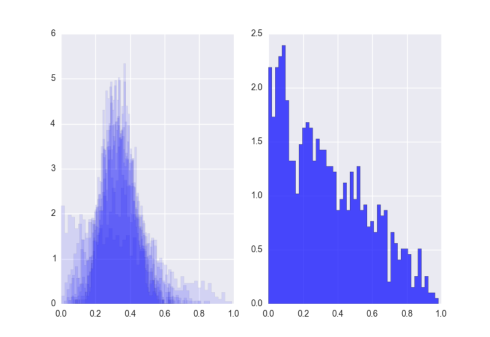

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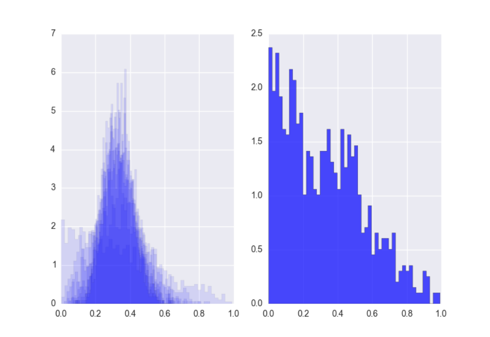

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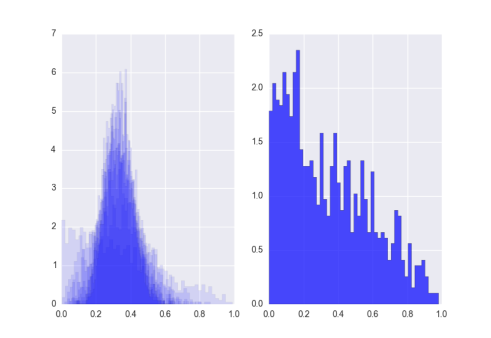
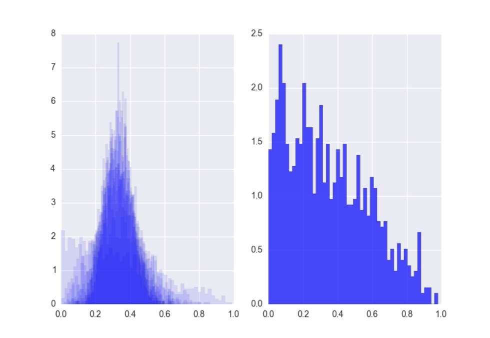
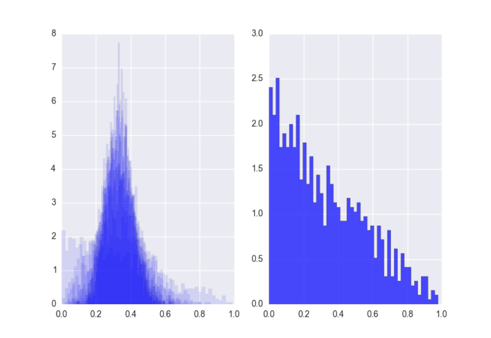


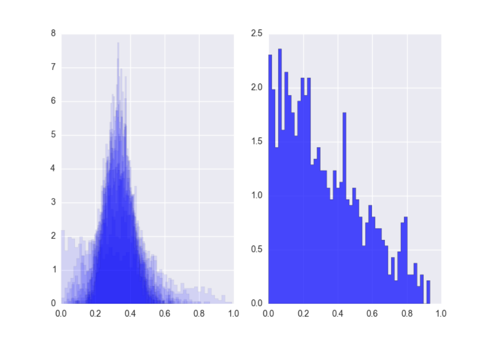
Central Limit Theorem



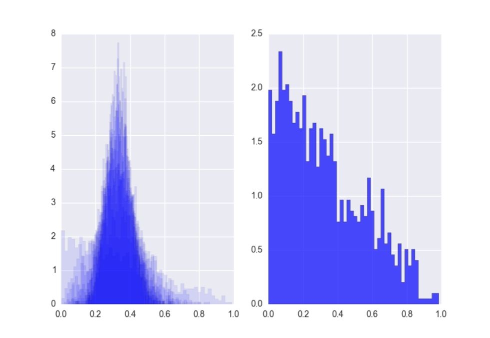
Central Limit Theorem



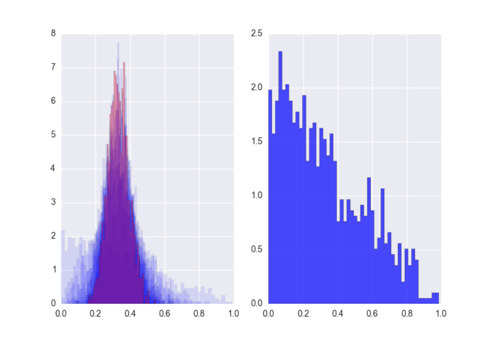
Central Limit Theorem

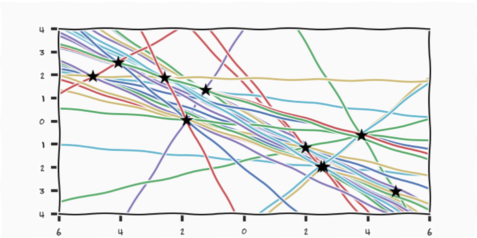


Central Limit Theorem



Central Limit Theorem





The search for Cerces

Gauss made the assumption that Piazzi's measurment errors where *independent* draws from a *unknown* distribution that was *fixed*.

This we often know as `i.i.d` *Independent and Identically Distributed*

Conjugate Prior[3]

- Gaussians are self-conjugate
  - Gaussian likelihood + Gaussian Prior $\rightarrow$ Gaussian Posterior
- Gaussian distribution
  - Conjugate prior for $\boldsymbol{\mu}$ is Gaussian
  - Conjugate prior for $\boldsymbol{\Sigma}$ is Inverse-Wishard

[3] https://en.wikipedia.org/wiki/Conjugate_prior

Gaussian Distribution

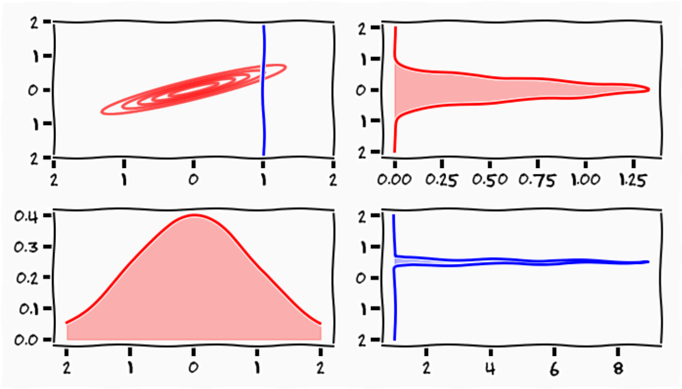
$$p(x_1, x_2) = \mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right)$$

Posterior $p(x_1|x_2) \propto p(x_2|x_1)p(x_1)$

Marginal $p(x_1) = \int p(x_1, x_2)\mathrm{d}x_2$

Conditional $p(x_1|x_2) = \frac{p(x_1,x_2)}{p(x_2)}$

Gaussian Identities



Tuesday Lecture

Exponential Family [1] Ch. 2.4

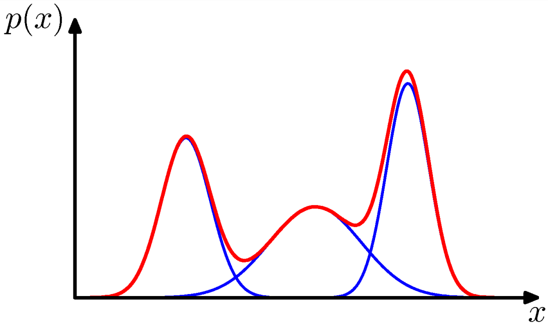
- Most distributions are parametetrised using exponentials
- Exponential family natural parametrisation

$$p(\mathbf{x}|\boldsymbol{\eta}) = h(\mathbf{x})g(\boldsymbol{\eta})e^{\boldsymbol{\eta}^{\mathrm{T}}\mathbf{u}(\mathbf{x})}$$

- Conjugate prior

$$p(\boldsymbol{\eta}|\boldsymbol{\chi}, \nu) = f(\boldsymbol{\chi}, \nu)g(\boldsymbol{\chi})^{\nu}e^{\nu\boldsymbol{\eta}^{\mathrm{T}}\boldsymbol{\chi}}$$

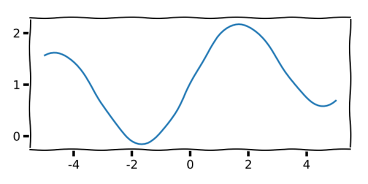
Mixtures of Gaussians [1] Ch 2.3.9

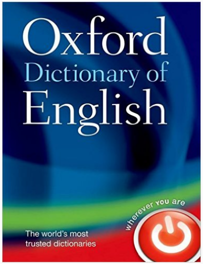


$$p(\mathbf{x}) = \sum_{k=1}^{K} p(k) \underbrace{p(\mathbf{x}|k)}_{\mathcal{N}(\mu_k, \Sigma_k)}$$

Stochastic Process

$$\infty$$





Kologrovs Existence Theorem

Defines what a distribution needs to forfill in order for a process to exist. Each finite instantiaton of the process is this distribution.

Example

Decisions

[4]

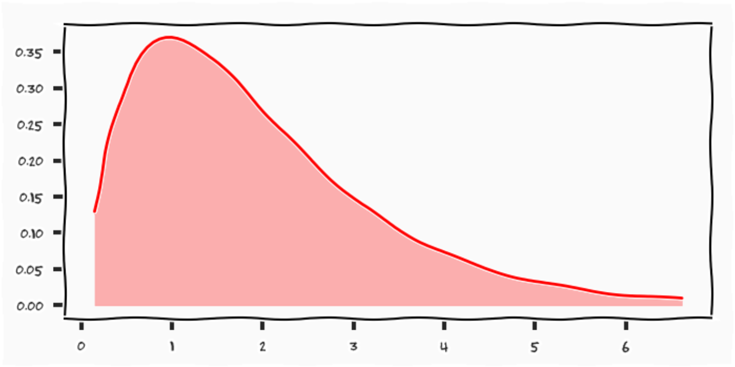
[4] Reservoir Dogs Tipping Scene YouTube

Tip

$$p(y)$$

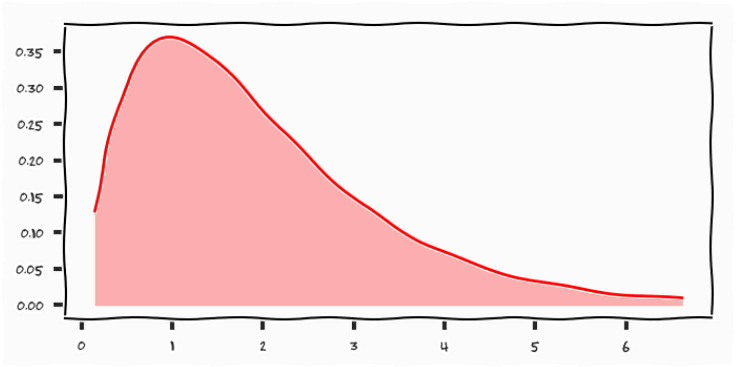
- what do I believe about tip before I see data?
- what is a sensible tip?

Tipping



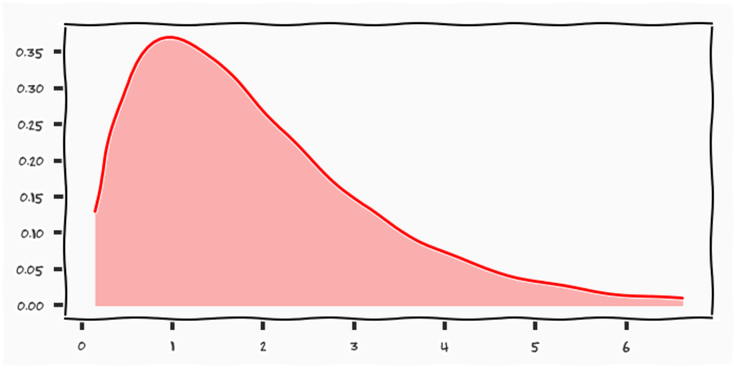
- I believe that 1£ is a sensible tip
- You cannot tip negative
- There is potentially an upper bound

Tipping



- I believe that 1£ is a sensible tip
- You cannot tip negative
- There is potentially an upper bound
- This is not a model, its just a belief in a variable

Tipping



- I believe that 1£ is a sensible tip
- You cannot tip negative
- There is potentially an upper bound
- This is not a model, its just a belief in a variable
- *a model relates new phenomenon to knowledge*

Tipping

- it is quite hard to say something about tip without any other knowledge
- Assumption the value of tip is related to the quality of the food

Likelihoods

$$p(y|x)$$

- how likely do I think the observed data y is to come from this specific x.

Tipping
: if I know the quality of the food what do I believe the tip should be

What is the tip that I should expect to get?

$$\mathbb{E}_{p(x)}[p(y|x)] = \int p(y|x)p(x)\mathrm{d}x = p(y)$$

- What should I expect to get in tip
- I have an idea of the general distribution of quality of food
- *Understanding is when we can relate knowledge to new phenomenon*

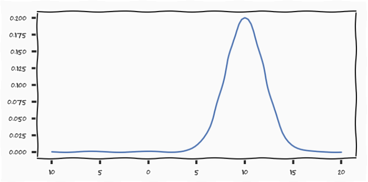
Tipping

$$p(x|c)$$

Hierarchical distribution

- Its quite hard to think of a prior over quality of food
- Can we parametrise the quality?

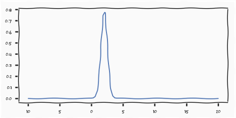
$$p(x|c) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu_c)(x-\mu_c)}{2\sigma^2}}$$



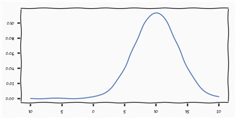
Hierarchical distribution

- Its quite hard to think of a prior over quality of food
- Can we parametrise the quality?
- Lets assume that if we know the cusine we have an idea

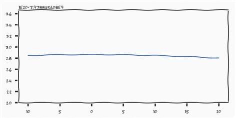
Tipping



Swedish



Italian



Uzbeki

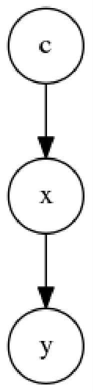
Hierarchical distribution

- Its quite hard to think of a prior over quality of food
- Can we parametrise the quality?
- Lets assume that if we know the cusine we have an idea
- *Relating to knowledge!*

Tipping model

$$p(y, x, c) = p(y|x)p(x|c)p(c)$$

- Graphical Model shows dependency structure
- Shows "minimal" factorisation of joint distribution (model)



Decisions

Knowing the tip

- Which cusine did they eat if?
  - $p(c|y)$
- What was the quality of the food?
  - $p(x|y)$

Summary

Summary

- Distributions allows us to make our assumptions explicit
- Conjugacy implies that the posterior and the prior is in the same family
- Now we have the tools that allows us to do Machine Learning

eof

Appendix

Posterior

$$p(x_1|x_2) \propto p(x_2|x_1)p(x_1)$$

1. Multiply right-hand side
2. Look at the exponents
3. Find the three terms, constant, mixed and quadratic
4. Complete the square to find the parameters

Marginal

$$p(x_1) = \int p(x_1, x_2)\mathrm{d}x_2 = \mathcal{N}(\mu_1, \Sigma_{11})$$

1. Write out the exponent of the joint distribution
2. Complete Square and collect terms with $x_1 - \mu_1$ (as we know the result)
3. Compute integral by knowing that densities always integrates to one

Conditional

$$p(x_1|x_2) = \frac{p(x_1, x_2)}{p(x_2)}$$

1. Factorise the problem as $p(x_1, x_2) = p(x_1|x_2)p(x_2)$
2. We know the marginal and the joint
3. Use Schur complement to re-write the covariance matrix on block form

References

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