Machine Learning

Dirichlet Processes

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Introduction

Bayes Rule

$$p(\theta|\mathcal{Y}) = p(\mathcal{Y}|\theta)p(\theta)\frac{1}{p(\mathcal{Y})}$$

Evidence

$$p(\mathcal{Y}) = \int p(\mathcal{Y}|\theta)p(\theta)\mathrm{d}\theta$$

Regression Models

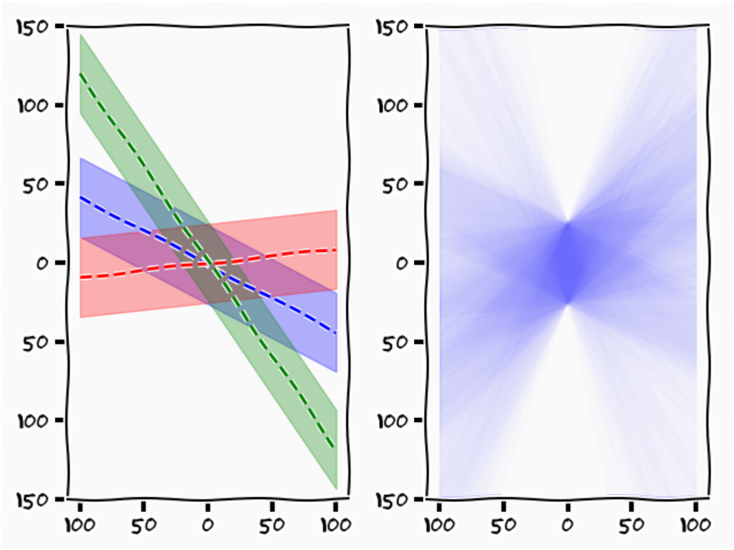
Linear Linear Model

$$p(y_i|x_i, \mathbf{w}) = \mathcal{N}(w_0 + w_1 \cdot x_i, \beta^{-1})$$

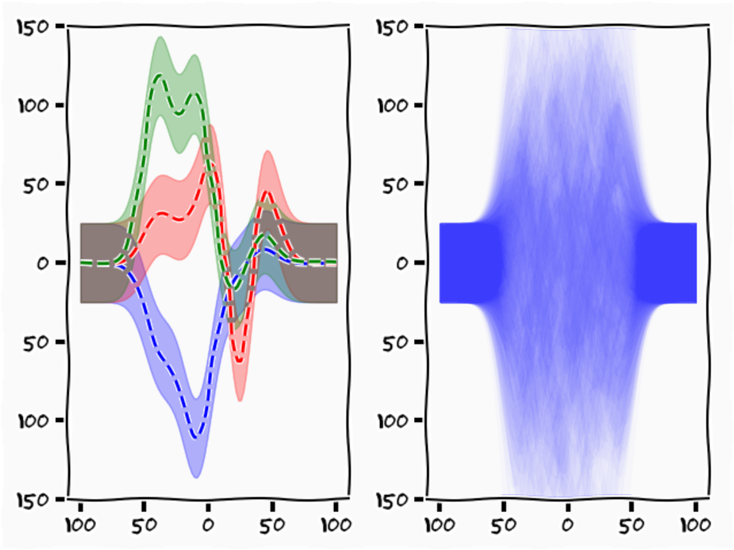
Basis function

$$p(y_i|x_i, \mathbf{w}) = \mathcal{N}(\sum_{i=1}^{6} w_i \phi(x_i), \beta^{-1})$$

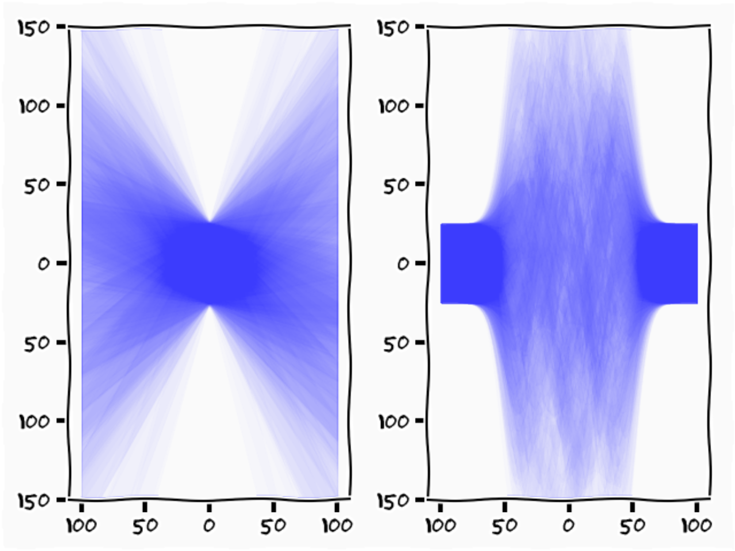
Model Linear Linear



Model Linear Basis



Evidence



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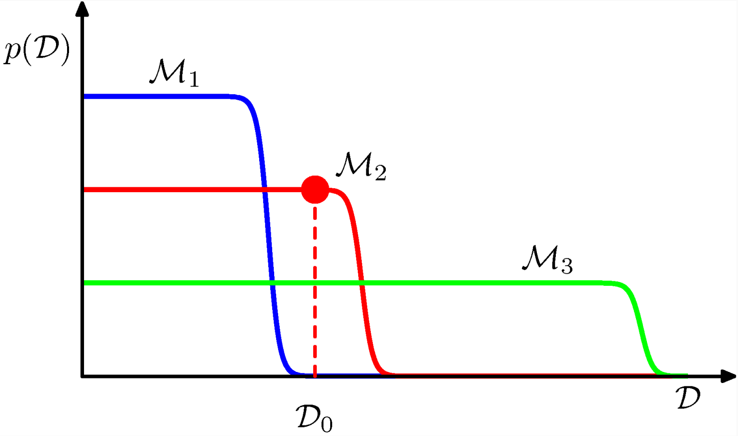
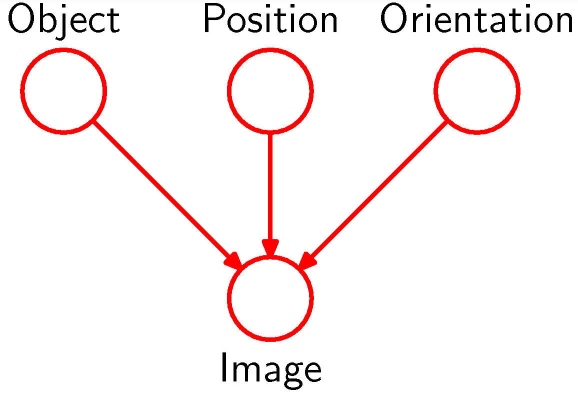
Model Selection



Explaining Away



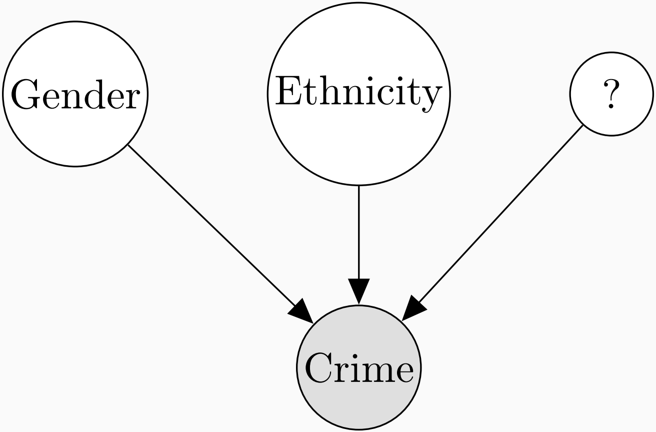
- The Object variable *explains away* variance associated with objects from the image
  - → position won't contain object variations
  - → orientation won't contain object variations

Explaining Away

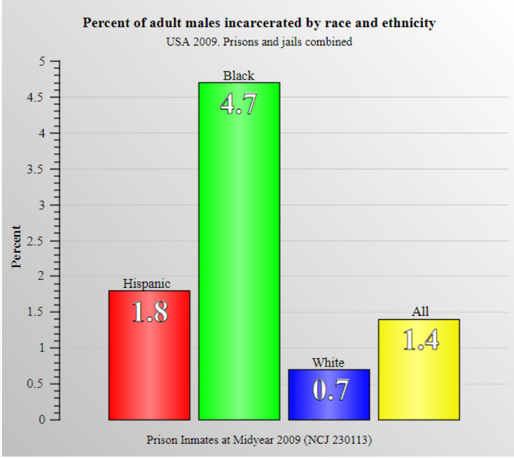
$$p(\text{Image}) = \int p(\text{Image}|\text{Object}, \text{Position}, \text{Orientation})p(\text{Object})p(\text{Position})$$

- $p(\text{Object}|\text{Image})$ what is the object
- $p(\text{Object}|\text{Image}, \text{Orientation})$ what is the object given that I know the orientation

Explaining Away

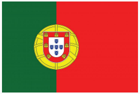


Explaining Away[1]



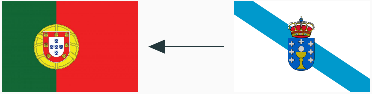
[1] https://artificialintelligence-news.com/2019/01/22/ai-sentencing-people-risk-assessment/

Hierarchical Knowledge



$p(\text{Portugal})$

Hierarchical Knowledge

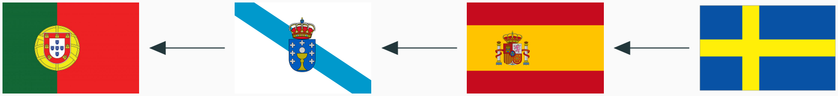


$$p(\text{Portugal}) = \int p(\text{Portugal}|\text{Galicia})p(\text{Galicia})\mathrm{d}\text{Galicia}$$

Hierarchical Knowledge

$$p(\text{Portugal}) = \int p(\text{Portugal}|\text{Galicia})p(\text{Galicia}|\text{Spain})p(\text{Spain})$$

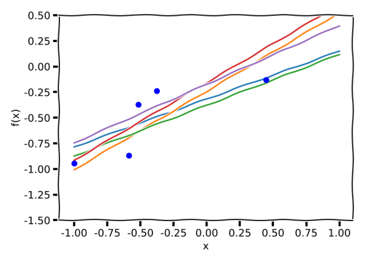
Hierarchical Knowledge

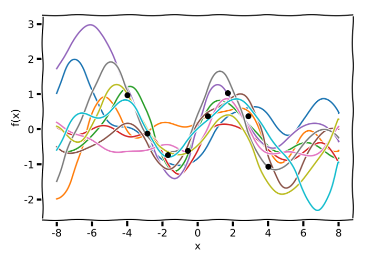


$$p(\text{Portugal}) = \int p(\text{Portugal}|\text{Galicia})p(\text{Galicia}|\text{Spain})p(\text{Spain}|\text{Sweden})p(\text{Sweden})$$

Non-parametrics

Non-parametrics





Parametric model

- Number of parameters fixed with respect to sample size
- fixed parameter space

Non-parametric model

- Number of parameters grows with sample size
- ∞ -dimensional parameter space

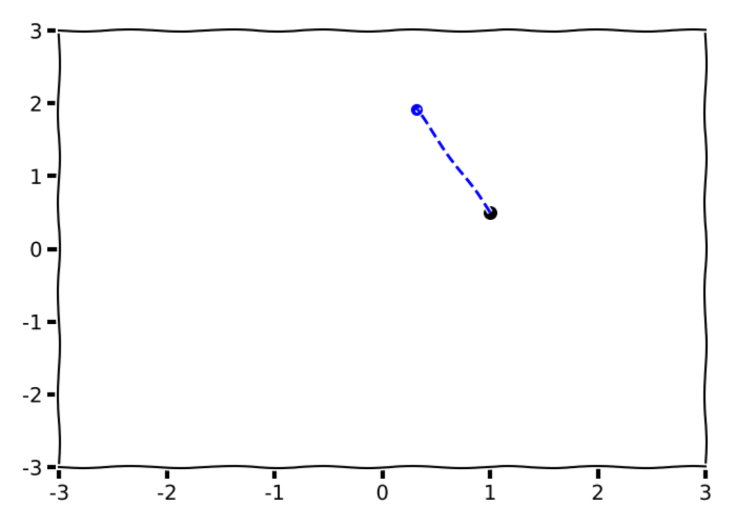
Nearest Neighbour

- Training data: $\{\mathsf{x}_i, y_i\}_{i=1}^{N}$
- Test data: $\{\mathsf{x}_i\}_{i=1}^{M}$
- Inference

$$\hat{i} = \text{argmin}_i D(\mathsf{x}_*, \mathsf{x}_i)$$

- Complexity grows with number of training data
- Does not generalise at all

Nearest Neighbour



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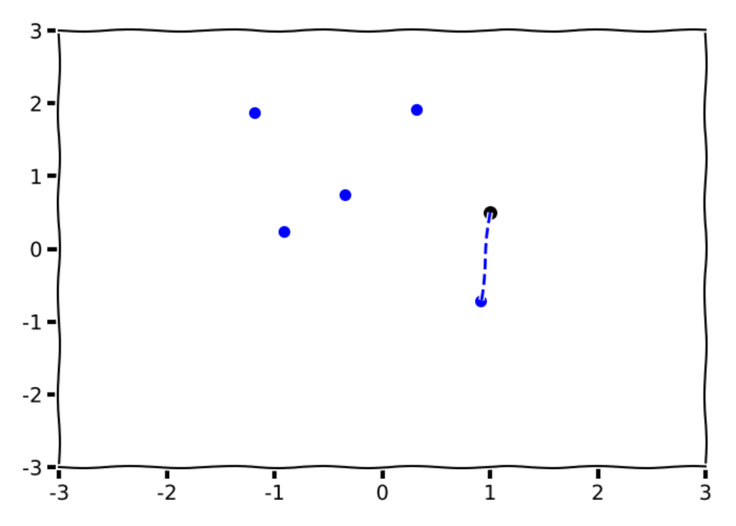

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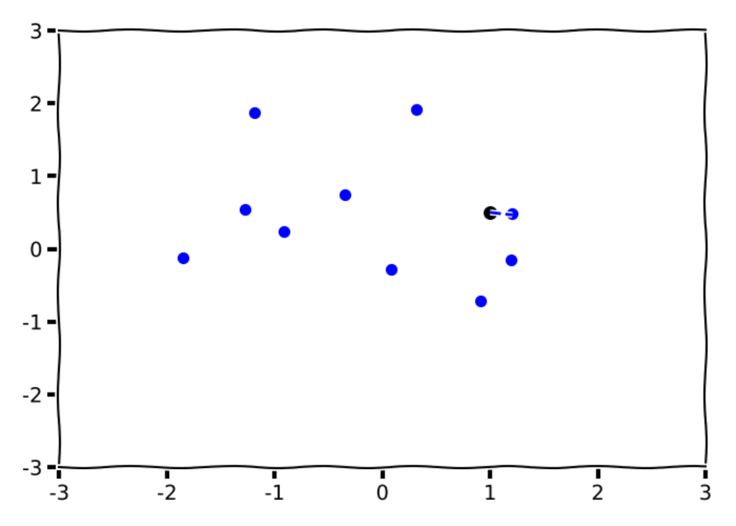
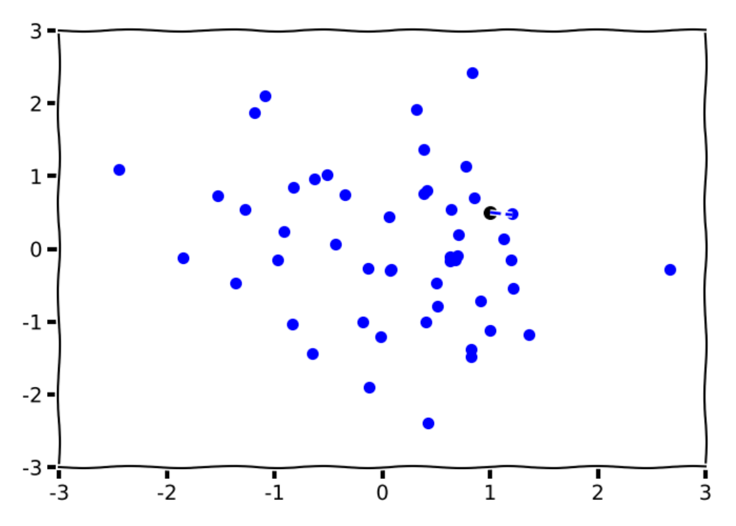
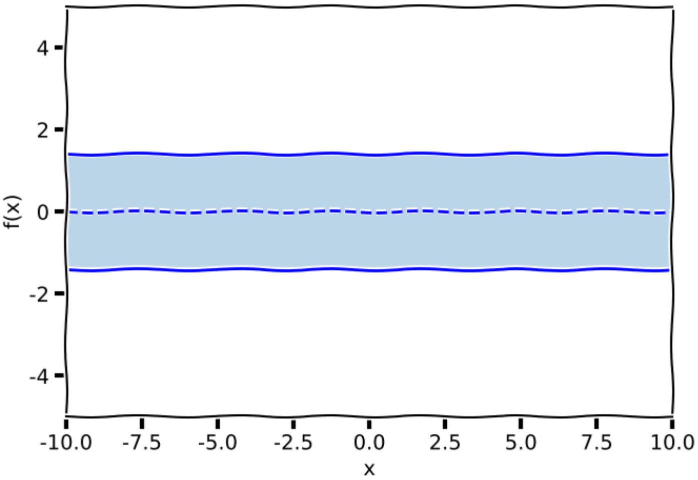


Nearest Neighbour



Gaussian Processes



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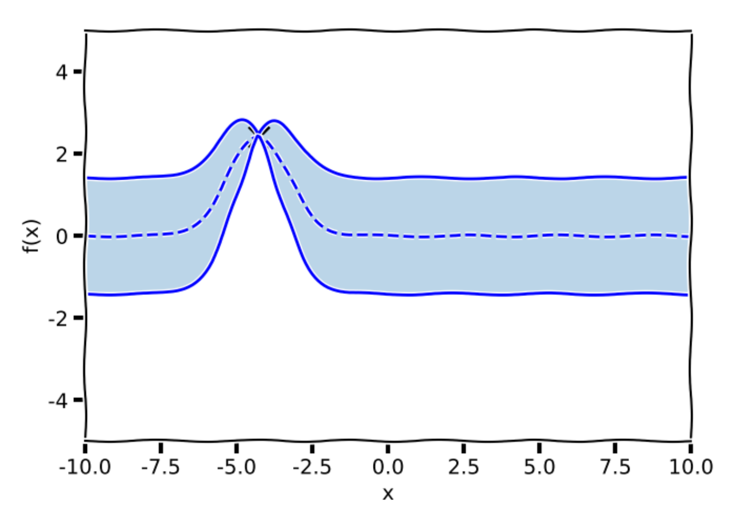

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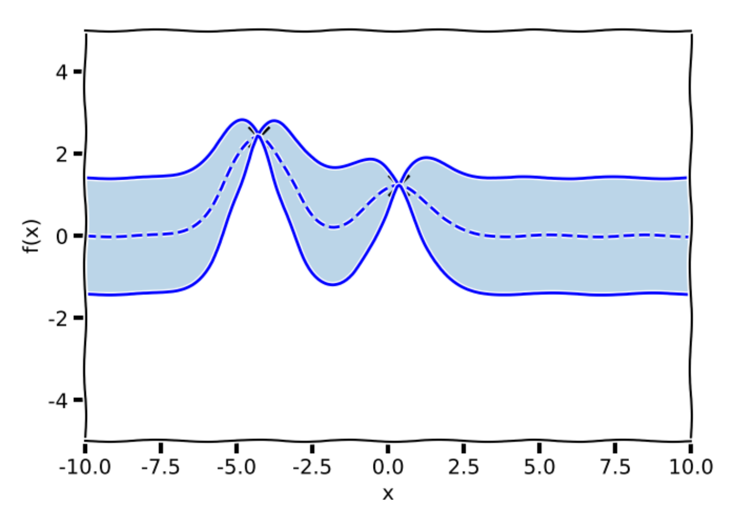

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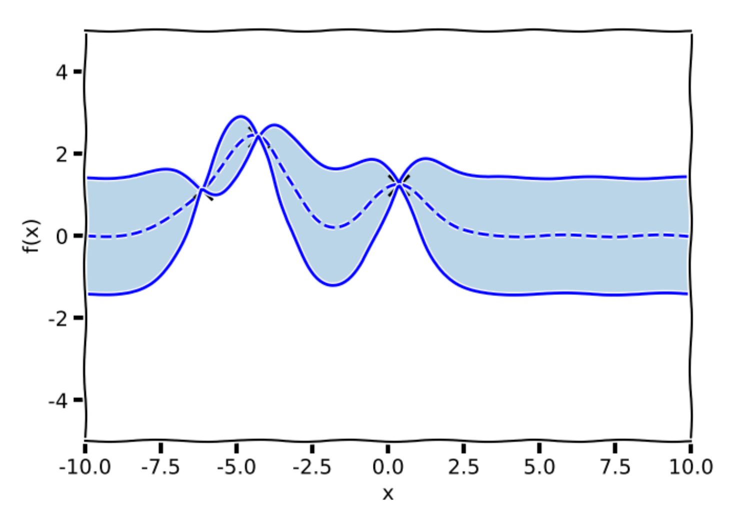

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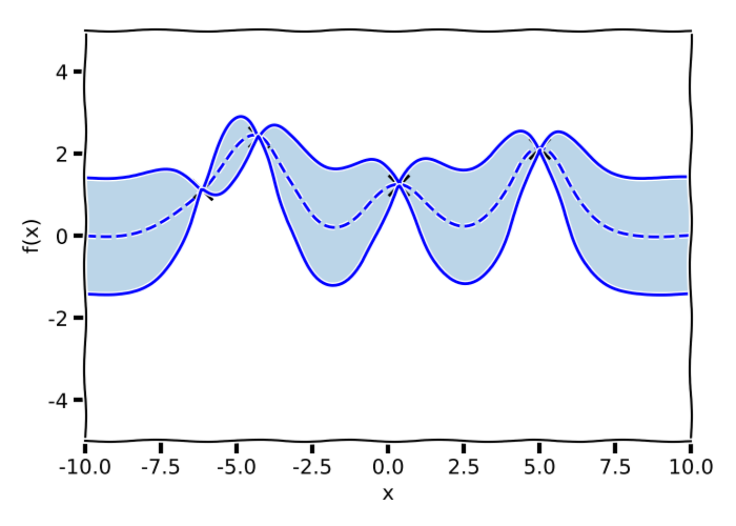

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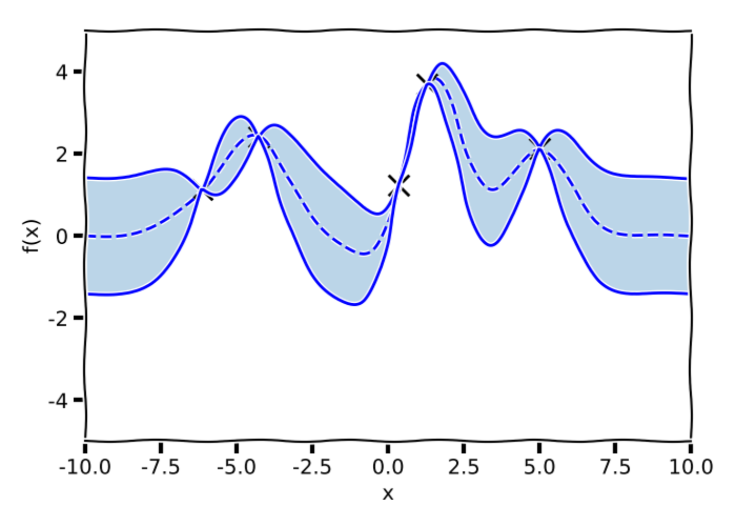

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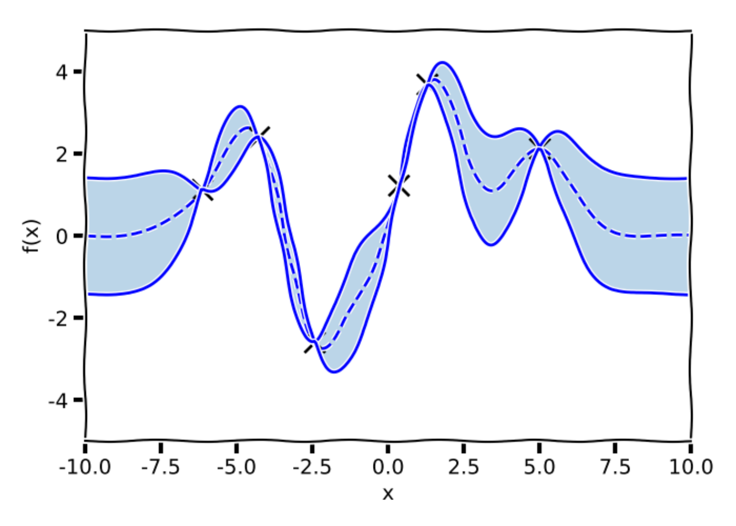
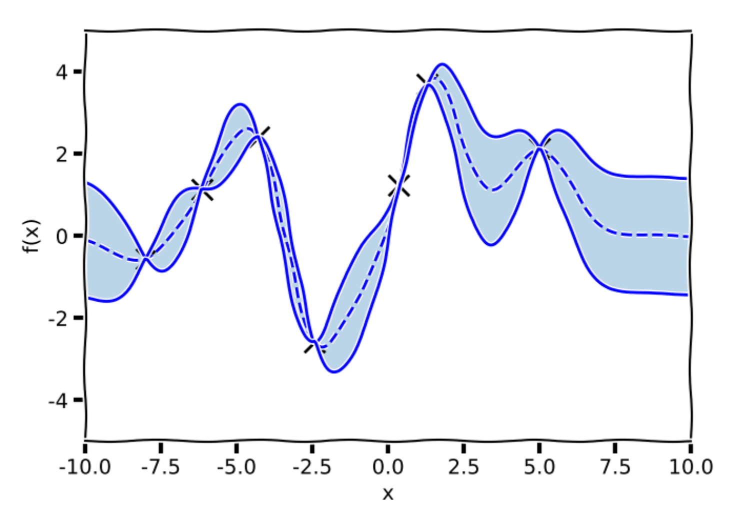


Gaussian Processes



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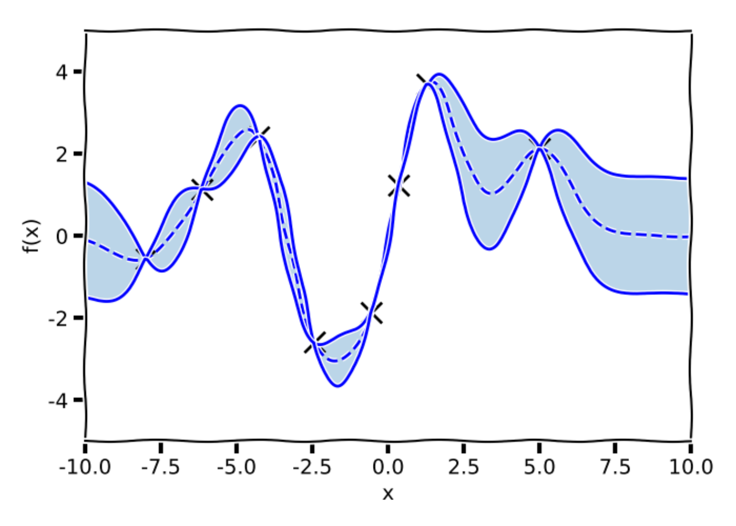

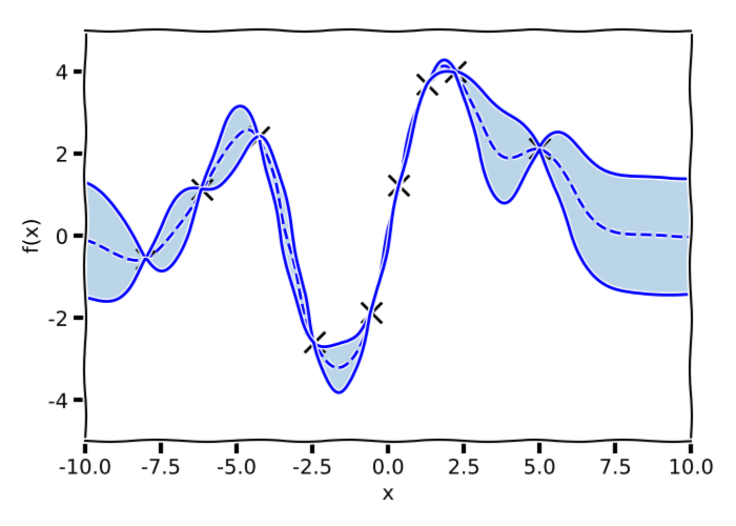
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-2.5
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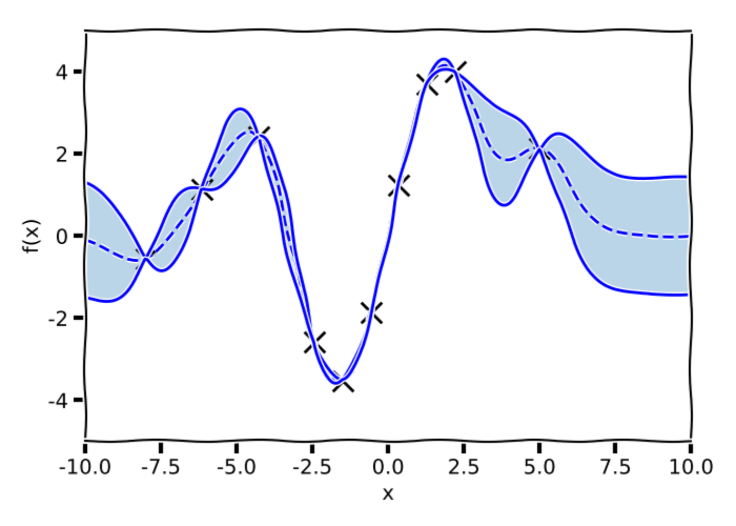
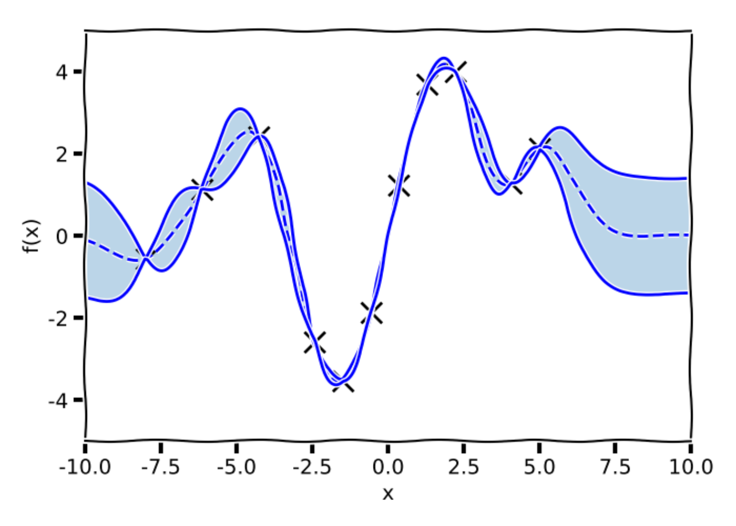
2.5
5.0
7.5
10.0
x

Gaussian Processes



Gaussian Processes



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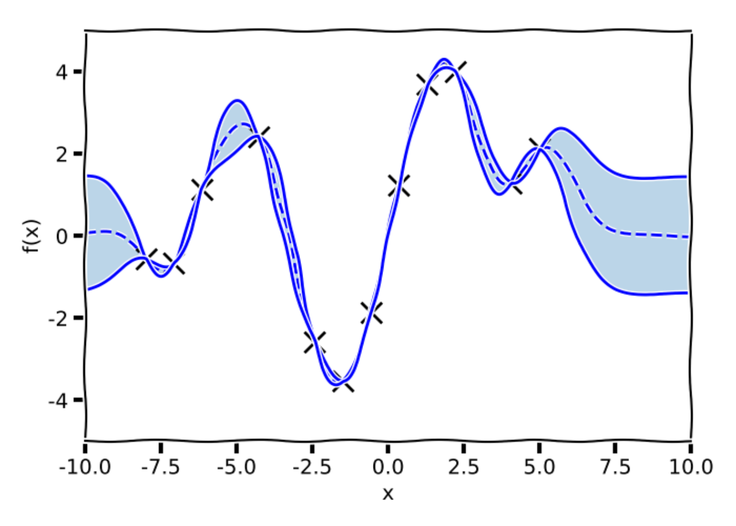

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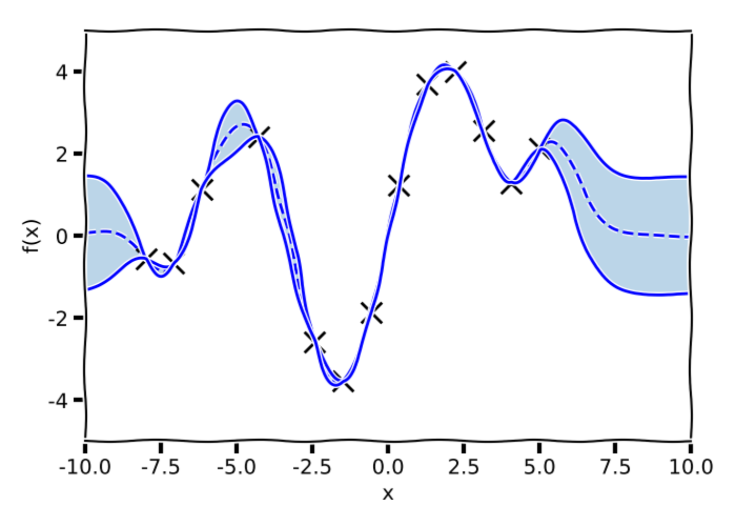

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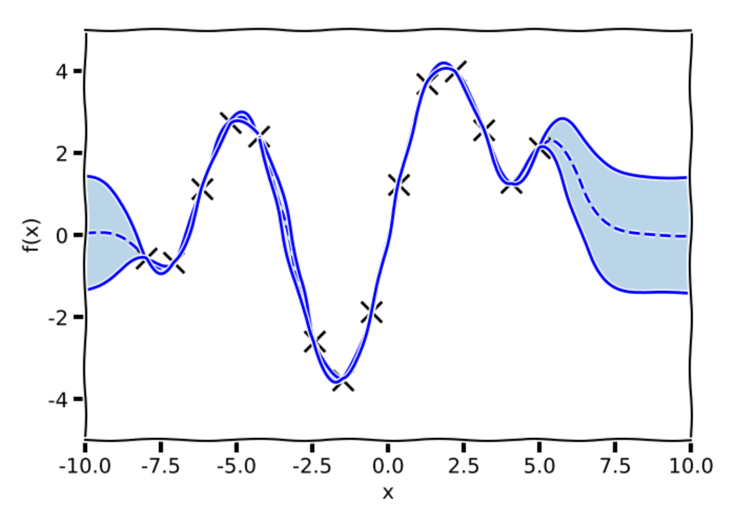

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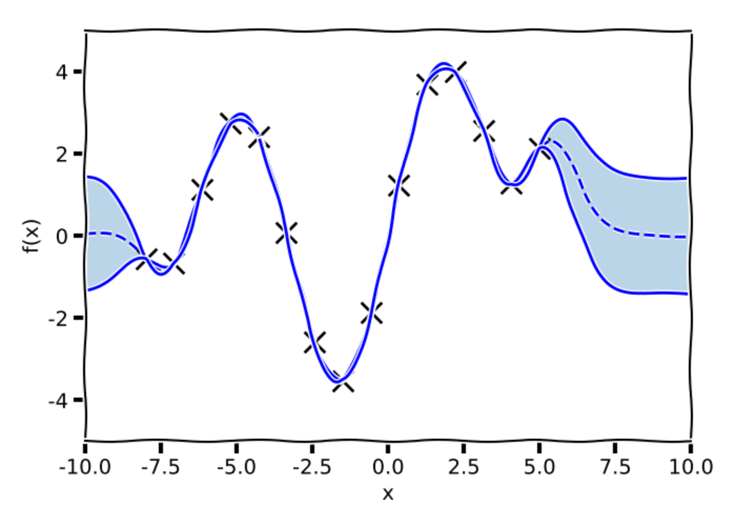

Process $\rightarrow$ Distribution $\rightarrow$ value

- Each evaluation of a process is a distribution

$$\mathcal{N}(0, \Sigma) \sim \mathcal{N}(0, k(\mathbf{X}, \mathbf{X}))$$

Process → Distribution → value

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- Each evaluation of a distribution is a value

$$y \sim \mathcal{N}(y|0, \Sigma)$$

- Parametric models are definedy by distributions and non-parametric by processes

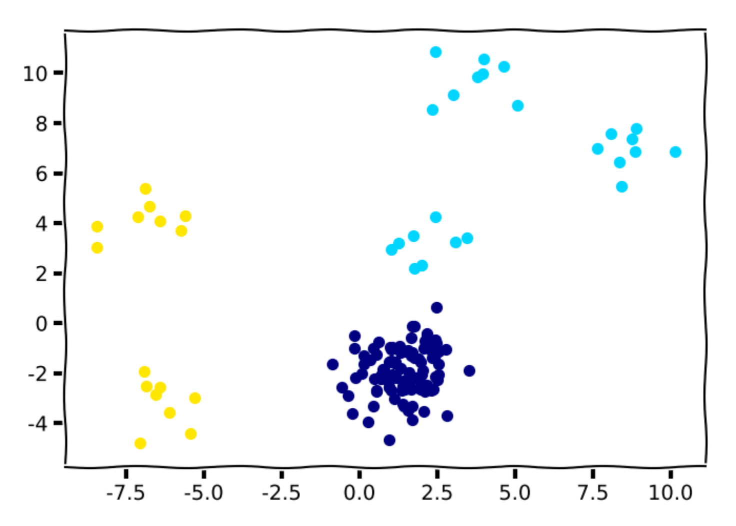
Gaussian Process

- Formulate process
- Evaluate process at specific location $x \rightarrow$ distribution
- Evaluate distribution at any location y
- GP is defined over uncountable infinite space

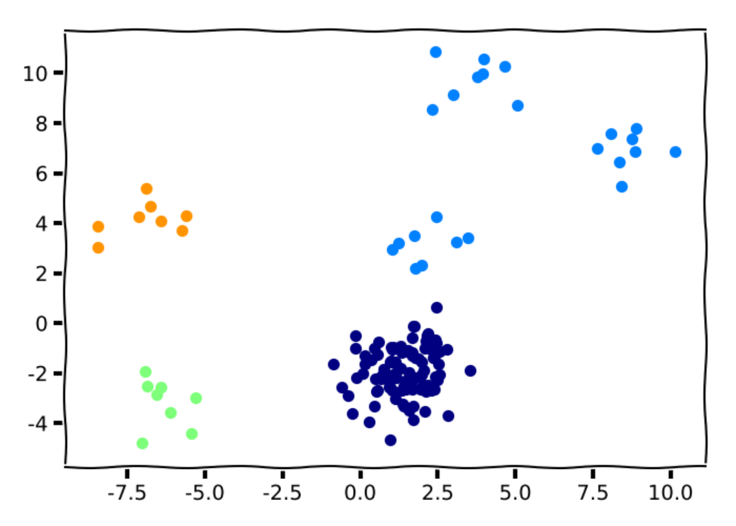
Gaussian Process

- Formulate process
- Evaluate process at specific location $x \rightarrow$ distribution
- Evaluate distribution at any location y
- GP is defined over uncountable infinite space
- *What about countable objects?*

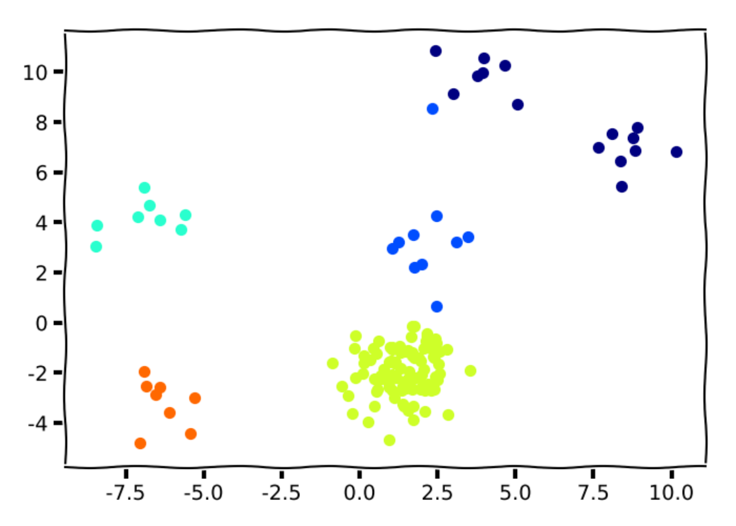
Gaussian Mixture Model



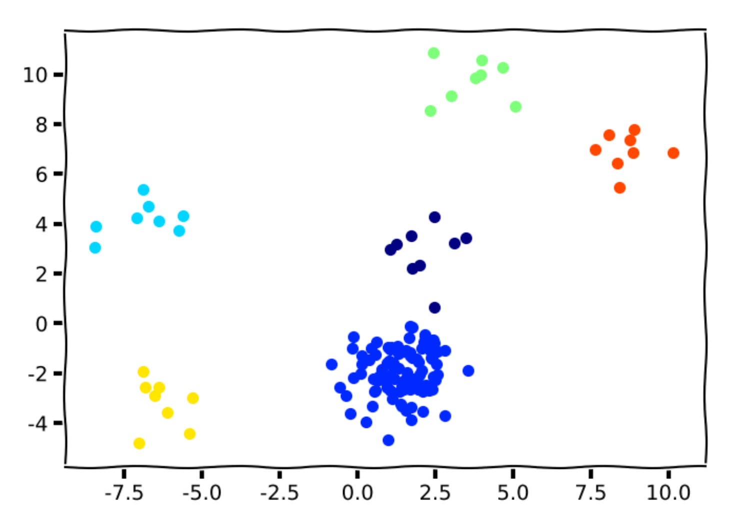
Gaussian Mixture Model



Gaussian Mixture Model



Gaussian Mixture Model



Gaussian Mixture Models

$$p(\mathbf{X}) = \sum_{k=1}^{K} p(\mathbf{X}|k)p(k) = \sum_{k=1}^{K} \mathcal{N}(\mathbf{X}|\boldsymbol{\mu}_k, \Sigma_k)p(k)$$

- Represent the probability of $\mathbf{X}$ as a combination or *mixture* of distributions
- What should K be?
- Can we make K infinite?

Non-parametrics

Gaussian Process

$$p(\mathbf{y}|\mathbf{X}, \theta) = \int p(\mathbf{y}|f)p(f|\mathbf{X}, \theta)\mathrm{d}f$$

Infinite Mixture Model

$$p(\mathbf{X}) = \sum_{k=1}^{\infty} p(\mathbf{X}|k)p(k) = \sum_{k=1}^{\infty} \mathcal{N}(\mathbf{X}|\boldsymbol{\mu}_k, \Sigma_k)p(k)$$

- When we build models we describe how the data has been generated

Unsupervised Linear Regression

1. Sample (pick) a weight

Clustering

Generative Models

- When we build models we describe how the data has been generated

Unsupervised Linear Regression

1. Sample (pick) a weight
2. Sample (pick) a input location

Clustering

Generative Models

- When we build models we describe how the data has been generated

Unsupervised Linear Regression

1. Sample (pick) a weight
2. Sample (pick) a input location
3. Generate observation

Clustering

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Unsupervised Linear Regression

1. Sample (pick) a weight
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Clustering

1. Sample cluster identity

Generative Models

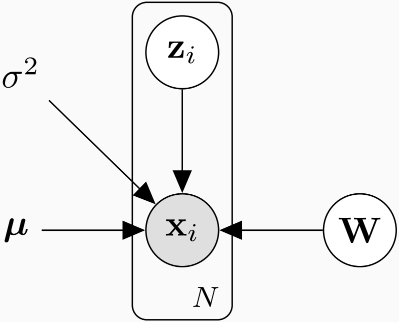
- When we build models we describe how the data has been generated

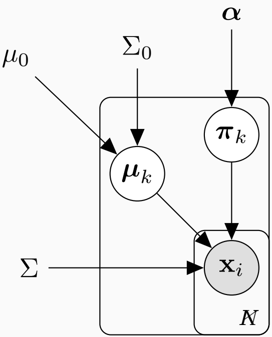
Unsupervised Linear Regression

1. Sample (pick) a weight
2. Sample (pick) a input location
3. Generate observation

Clustering

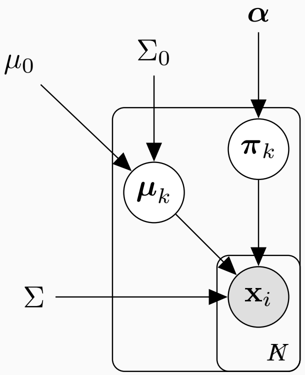
1. Sample cluster identity
2. Sample point from cluster





- Graphical model clearly shows generative proceedure

Gaussian Mixture Model



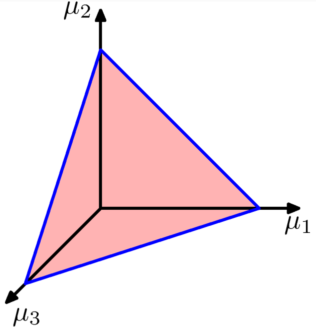
1. Sample proportions
2. Sample cluster id given proportions
3. Sample cluster mean
4. Sample data

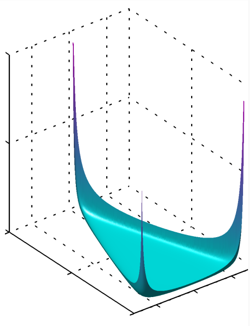
Distributions over partitionings [1] Ch 2.2.0

$$\text{Mult}(m_1, m_2, \ldots, m_k | \boldsymbol{\mu}, N) = \binom{N}{m_1, m_2, \ldots, m_K} \prod_{k=1}^{K} \mu_k^{m_k}$$

- Joint distribution over $m_1, m_2, \ldots, m_K$
- The parameter $\boldsymbol{\mu}$ says how likely each component is

Dirichlet Distribution

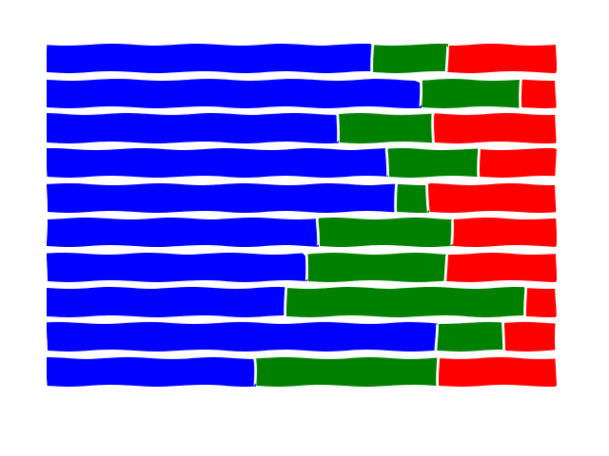




- Conjugate prior to multinomial

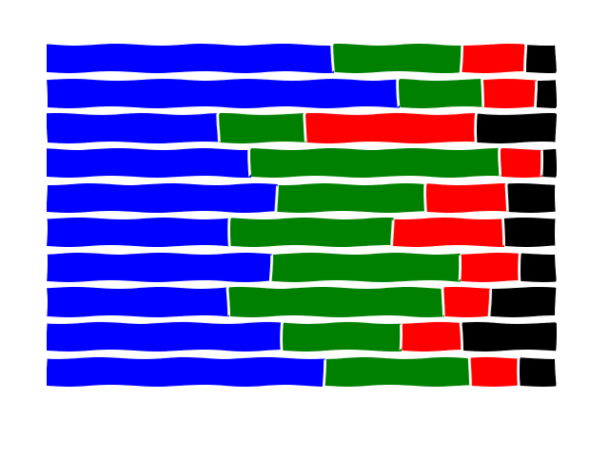
$$\mathrm{Dir}(\boldsymbol{\mu}|\boldsymbol{\alpha}) = \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\cdot \ldots \cdot \Gamma(\alpha_K)} \prod_{k=1}^{K} \mu_k^{\alpha_k - 1}$$

Dirichlet Distribution



$\mathrm{Dir}(10, 5, 3)$

Dirichlet Distribution



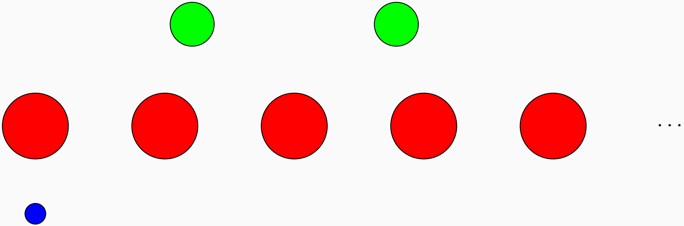
$\mathrm{Dir}(7, 5, 3, 2)$

Dirichlet Processes

Dirichlet Process

- Is the infinite dimensional generalisation of a Dirichlet distribution
    - just as Gaussian process is of Gaussian distribution
- Generates a partitioning of (possibly) infinite number of elements
- Not as intuitive to write down
- Best written constructively

Chinese Resturant Process



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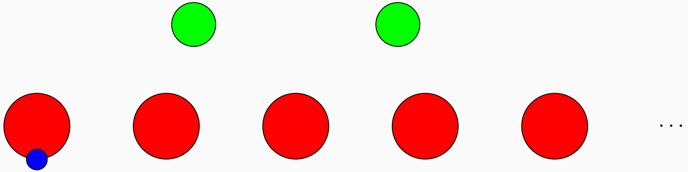

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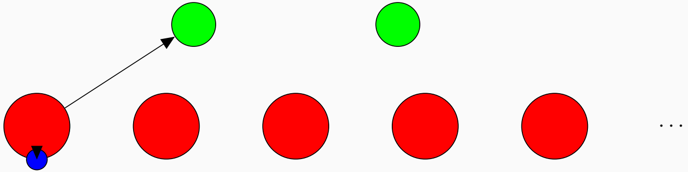

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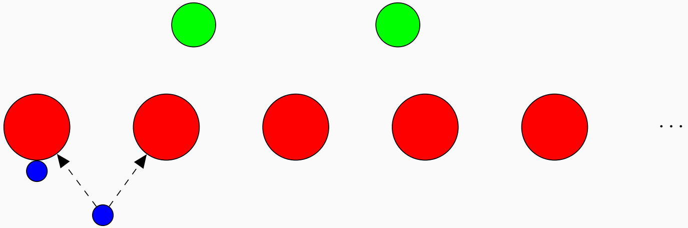

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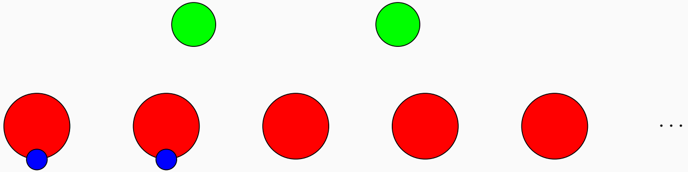

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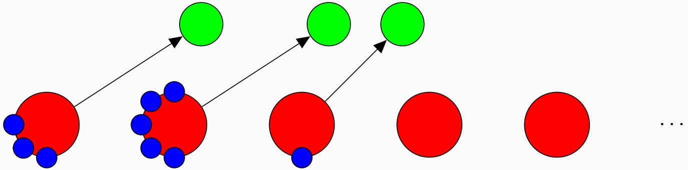
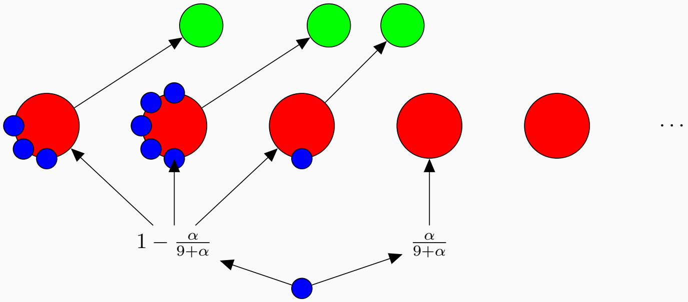


- Go to new table $\frac{\alpha}{N-1+\alpha}$
- If not choose table as $\frac{n_i}{N}$ where n_i number of diners at table *in*

Chinese Resturant Process



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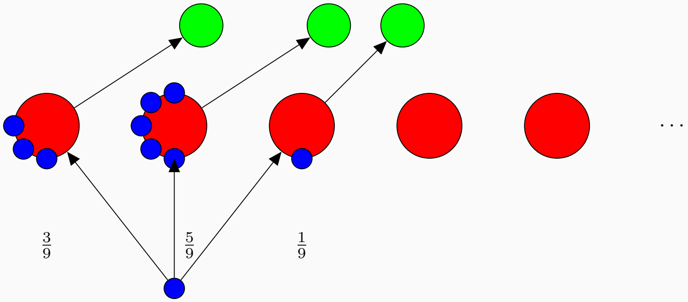
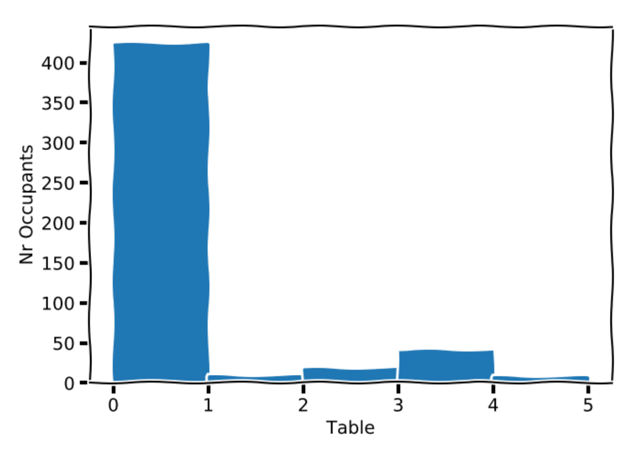


Chinese Resturant Process Code

Code

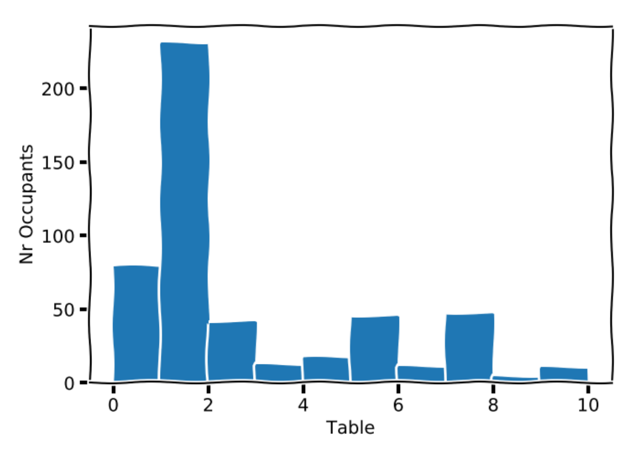
```
def chrp(N, alpha):
    table_assignments = np.zeros(N)
    next_open_table = 0
    for i in range(0,N):
        r = np.random.random()
        if r < (alpha/(i+alpha)):
            table_assignments[i] = next_open_table
            next_open_table += 1
        else:
            index = int(np.round((i-1)*np.random.random()
            table_assignments[i] = table_assignments[inde

    return table_assignments
```

Chinese Resturant Process



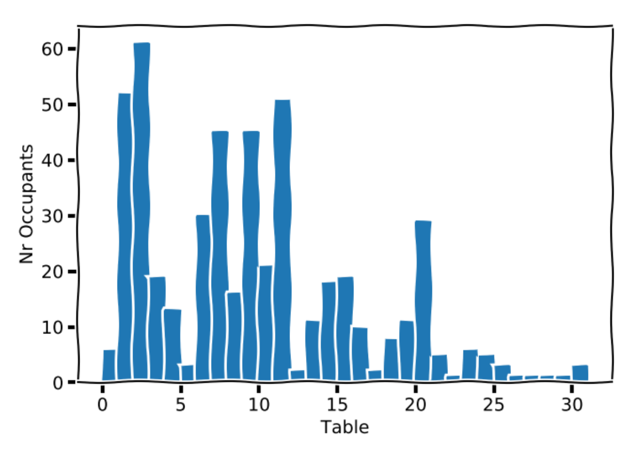
$$N = 500 \quad \alpha = 1.0$$

Chinese Resturant Process



$$N = 500 \quad \alpha = 2.0$$

Chinese Resturant Process



$$N = 500 \quad \alpha = 10.0$$

Chinese Resturant Process

N	α	μ_K	σ_K
500	3	14.3	8.81
500	5	21.6	15.64
500	10	39.1	27.29
500	20	64.3	63.8
500	100	180	69.69

Infinite Mixture Model

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Infinite Mixture Model

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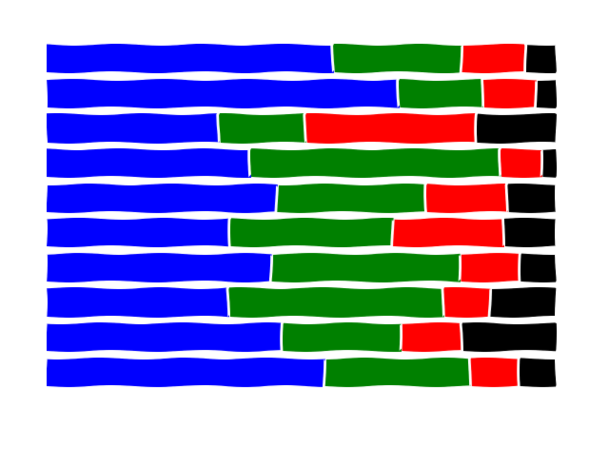
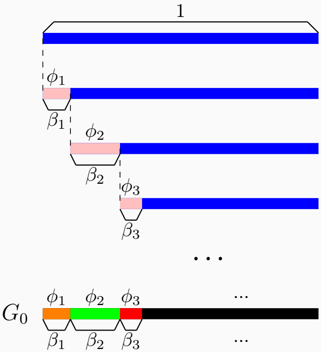
4. Next data-point
5. Associate with a new mixture or create new
6. If new, pick new parameters
7. Repeat

Dirichlet Distribution



$\mathrm{Dir}(7, 5, 3, 2)$

Stick-Breaking



$$G \sim \mathrm{DP}(\mathcal{H}, \alpha)$$

$$\hat{\beta}_k \sim \mathrm{Beta}(1, \alpha)$$

$$\beta_k = \hat{\beta}_k \prod_{l=1}^{k-1} (1 - \hat{\beta}_l)$$

$$\Phi \sim \mathcal{H}$$

$$G_0 = \sum_{k=1}^{\infty} \beta_k \Phi_k$$

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- *We are so far only looking at how to formulate models*

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- Take home message: generative models
- Tomorrow: Latent Dirichlet Allocation

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References

Christopher M. Bishop.
Pattern Recognition and Machine Learning (Information Science and Statistics).
Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2006.