

Machine Learning

Basic Probabilities

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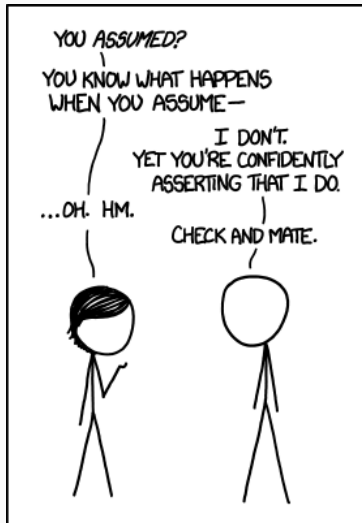
<http://www.carlhenrik.com>

Introduction

Assumptions

- Observations cannot be argued with
- Interpretations of observations are relative to assumptions
- Good assumptions structures the world in a useful manner
- Wrong assumptions can be very scary

Assumptions



Laplace Demon [1]

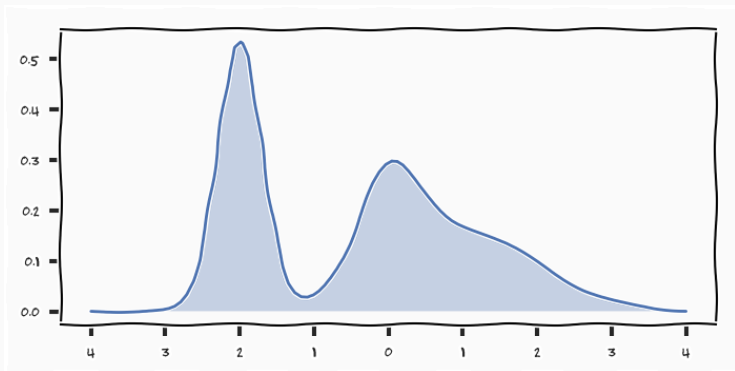


Laplace's Demon [1]

An intelligence which at a given instant knew all the forces acting in nature and the position of every object in the universe - if endowed with a brain sufficiently vast to make all necessary calculations - could describe with a single formula the motions of the largest astronomical bodies and those of the smallest atoms. To such an intelligence, nothing would be uncertain; the future, like the past, would be an open book.

All these efforts in the search for truth tend to lead the mind continuously towards the intelligence we have just mentioned, although it will always remain infinitely distant from this intelligence.

Uncertainty



- Uncertainty is a "realisation" of an assumption
- Probabilities are a quantification of uncertainty

Deterministic Variable

```
int x = 3
```

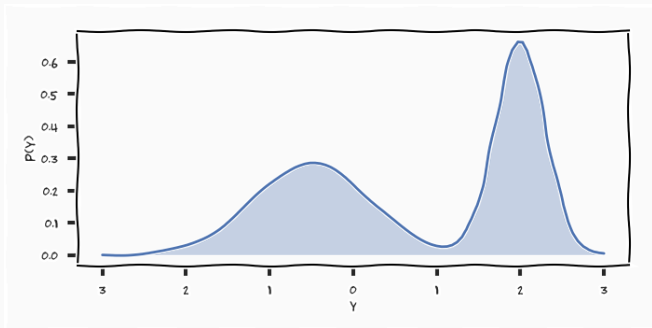
```
float y = 3.14
```

Stochastic Variable

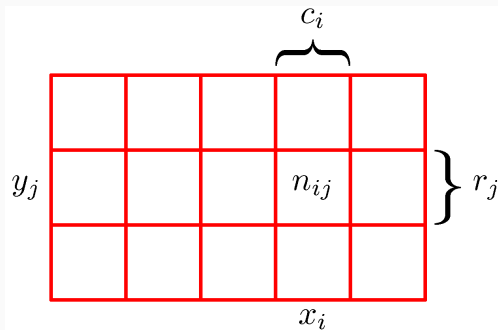
$$x \sim p(x) \quad y \sim \mathcal{N}(0, I)$$

Probabilities

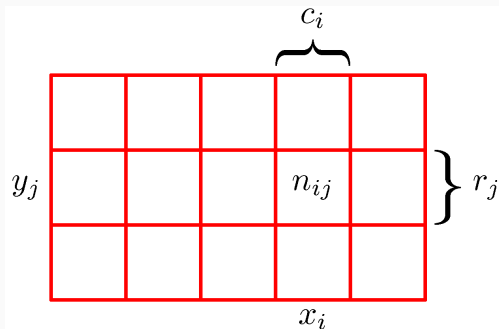
Probability Theory



- Probability theory is a framework for manipulating uncertainty
- Random variable, is a stochastic variable that follows a distribution
- Random does **not** mean max entropy

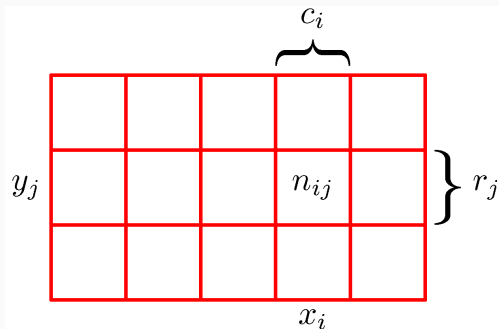


$$\begin{aligned} \{X = x_i, Y = y_j\} &= n_{ij} & \{X = x_i\} &= c_i & \{Y = y_j\} &= r_j \\ X &= \{x_i\}_{i=1}^M & Y &= \{y_j\}_{j=1}^N \end{aligned}$$



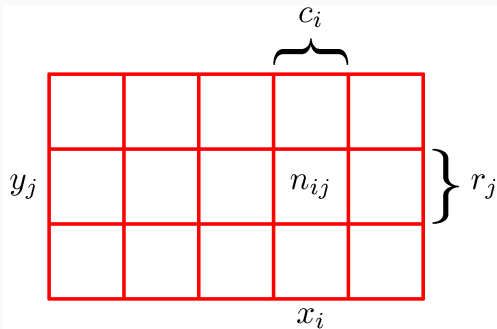
Joint Probability

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{\sum_{kl} n_{kl}} = \frac{n_{ij}}{N}$$



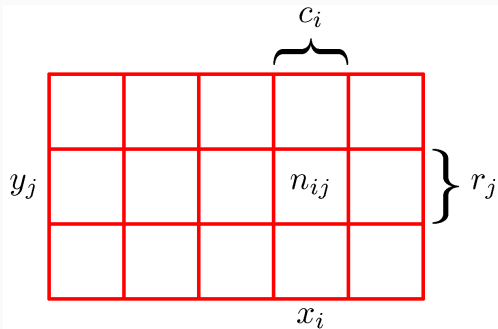
Marginal Probability

$$p(X = x_i) = \frac{\sum_j n_{ij}}{N} = \frac{c_i}{N}$$



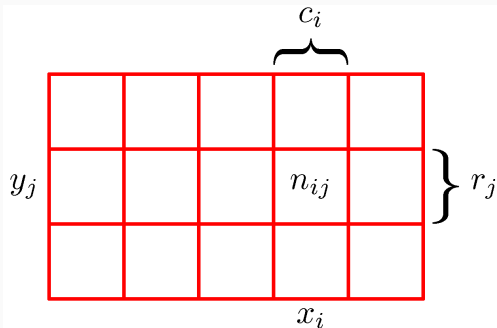
Sum rule

$$p(X = x_i) = \frac{\sum_j n_{ij}}{N}$$



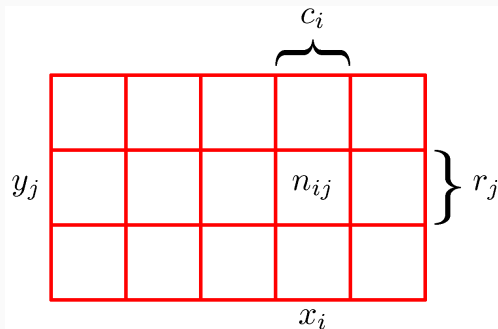
Sum rule

$$p(X = x_i) = \frac{\sum_j n_{ij}}{N} = \sum_j \frac{n_{ij}}{N}$$



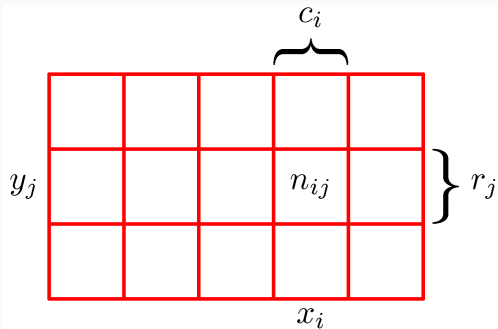
Sum rule

$$p(X = x_i) = \frac{\sum_j n_{ij}}{N} = \sum_j \frac{n_{ij}}{N} = \sum_j p(X = x_i, Y = y_j)$$



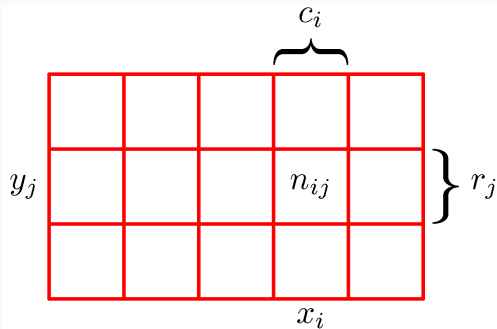
Conditional

$$p(Y = y_j | X = x_i) = \frac{n_{ij}}{c_i}$$



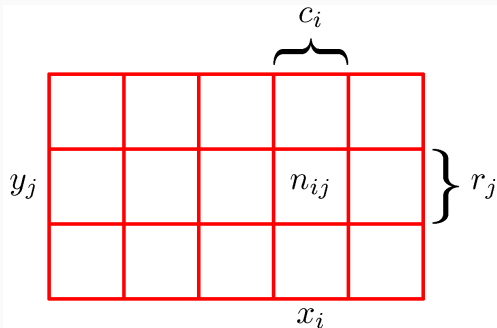
Product rule

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N}$$



Product rule

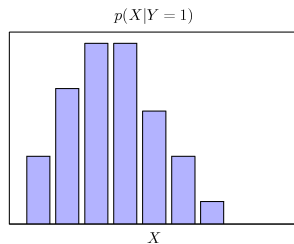
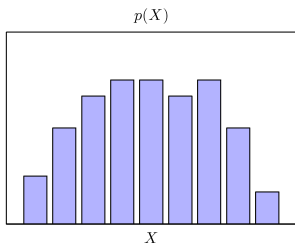
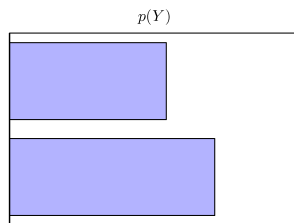
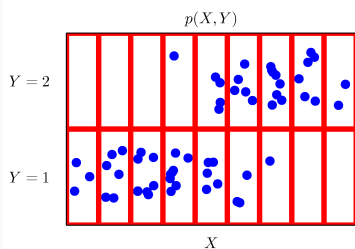
$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N} = \frac{n_{ij}}{c_i} \cdot \frac{c_i}{N}$$



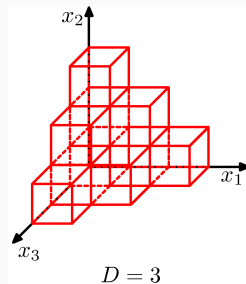
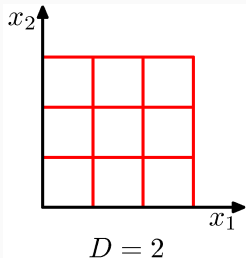
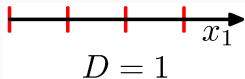
Product rule

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N} = \frac{n_{ij}}{c_i} \cdot \frac{c_i}{N} = p(Y = y_j | X = x_i) p(X = x_i)$$

Probability Theory [2] ch 1.2



Curse of Dimensionality [2] ch 1.4



Curse of Dimensionality

```
for i in range(0,N1):  
    for j in range(0,N2):  
        for k in range(0,N3):  
            ...  
            ...  
            ...
```


Notation

- The probability distribution over the random variable X

$$p(X) = p(X = x_i)$$

Notation

- The probability distribution over the random variable X

$$p(X) = p(X = x_i)$$

- The probability distribution over X evaluated at x_i

$$p(x_i)$$

The Rules of Probability

Sum Rule

$$p(X) = \sum_Y p(X, Y)$$

Product Rule

$$p(X, Y) = p(Y|X)p(X)$$

$$p(X, Y) = p(Y|X)p(X)$$

$$p(X, Y) = p(Y|X)p(X)$$

$$p(X, Y) = p(X|Y)p(Y)$$

$$p(X, Y) = p(Y|X)p(X)$$

$$p(X, Y) = p(X|Y)p(Y)$$

$$p(X|Y)p(Y) = p(Y|X)p(X)$$

Baye's Rule

$$p(X, Y) = p(Y|X)p(X)$$

$$p(X, Y) = p(X|Y)p(Y)$$

$$p(X|Y)p(Y) = p(Y|X)p(X)$$

$$p(X|Y) = \frac{p(Y|X)p(X)}{p(Y)}$$

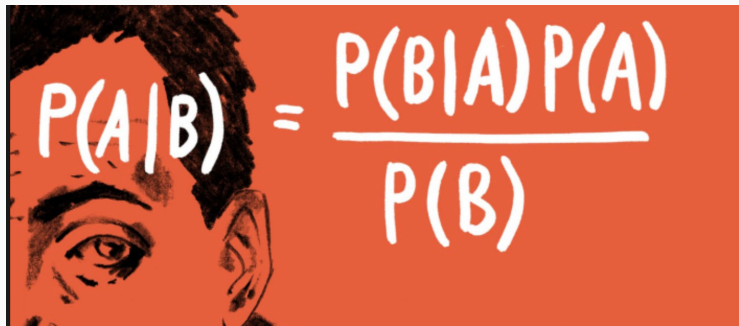
$$p(X, Y) = p(Y|X)p(X)$$

$$p(X, Y) = p(X|Y)p(Y)$$

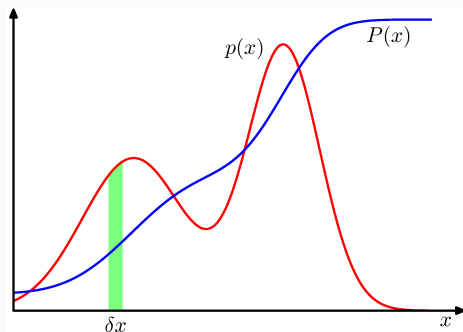
$$p(X|Y)p(Y) = p(Y|X)p(X)$$

$$\begin{aligned} p(X|Y) &= \frac{p(Y|X)p(X)}{p(Y)} \\ &= \frac{p(Y|X)p(X)}{\sum_X p(Y|X)p(X)} \end{aligned}$$

Baye's Rule


$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

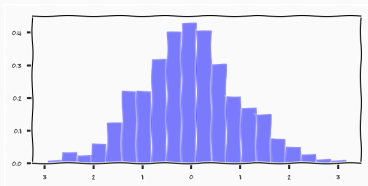
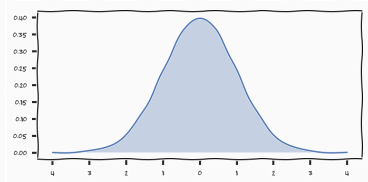
Probability Densities [2] ch 1.2.1



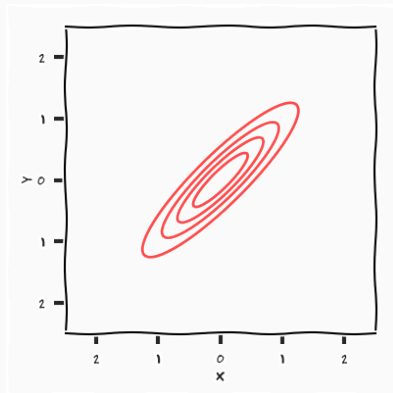
$$\lim_{\delta x \rightarrow 0} p(x \in (x, x + \delta x)) = \lim_{\delta x \rightarrow 0} \int_x^{x+\delta x} p(x) dx = p(x) \cdot \delta x$$

$$p(x) \geq 0, \quad \int_{-\infty}^{\infty} p(x) dx = 1$$

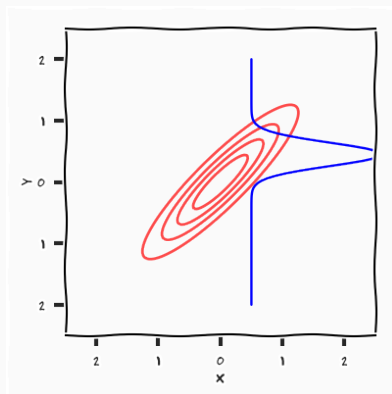
Discrete vs. Continuous


$$\Sigma$$

$$\int$$

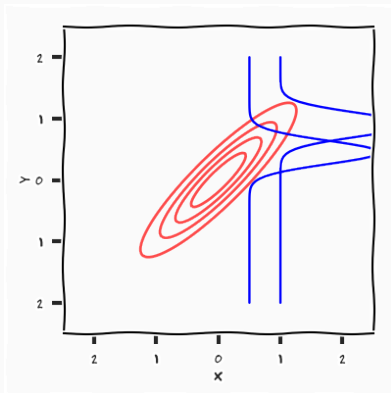
Continuous: Conditional



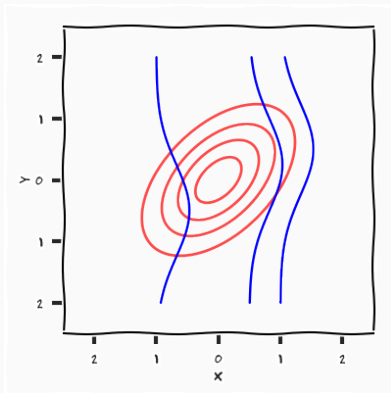
Continuous: Conditional



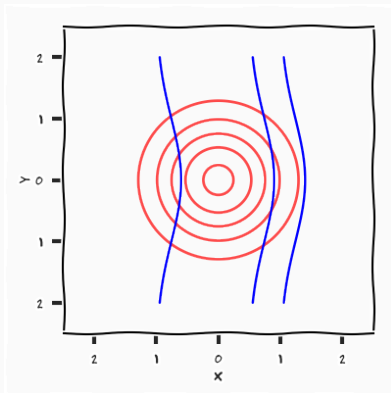
Continuous: Conditional



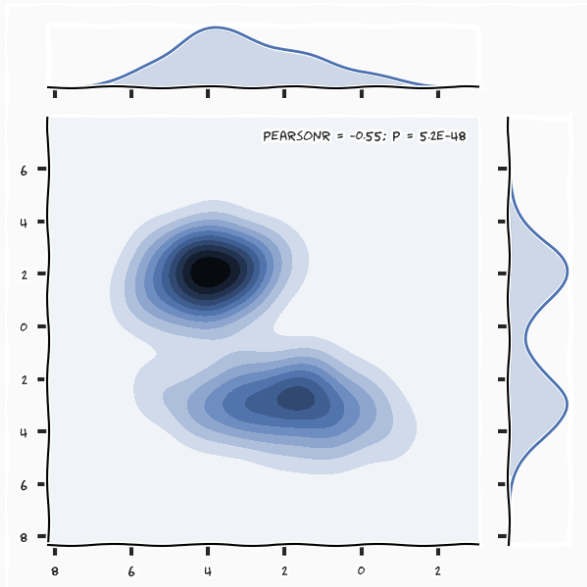
Continuous: Conditional



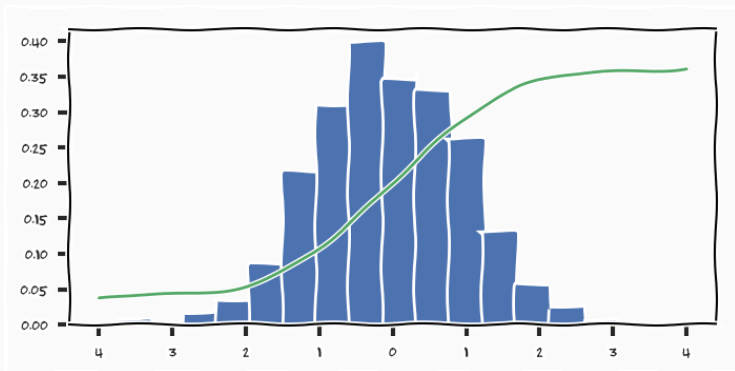
Continuous: Conditional



Continuous: Marginal



Expectations



$$\mathbb{E}[f] = \sum_x p(x)f(x)$$

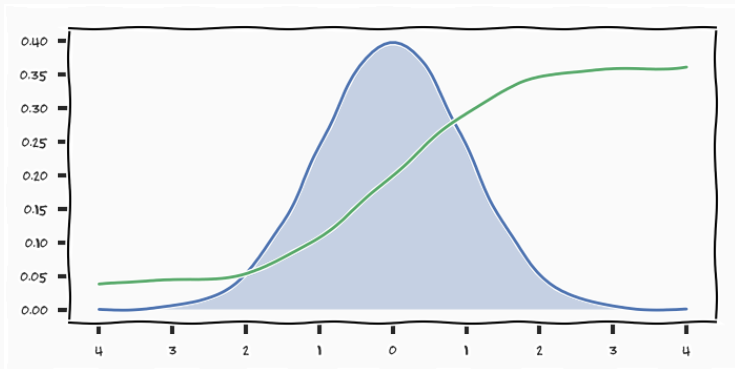
Expectations

```
e = 0.0
for x in range(Xmin,Xmax):
    e += f(x)*p(x)

return e
```

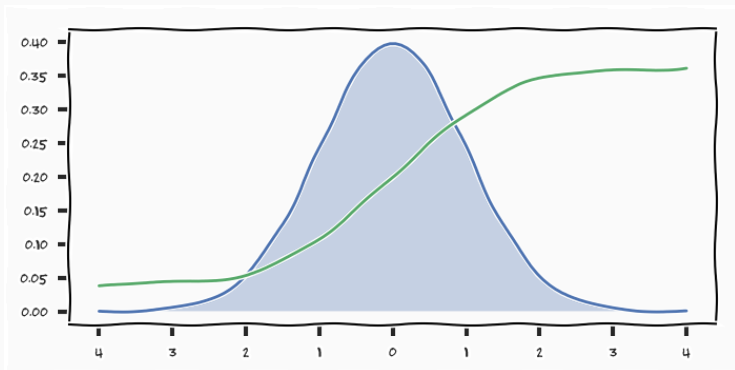
- simple to write
- can be infeasible to compute when domain is high dimensional

Expectations



$$\mathbb{E}[f] = \int p(x)f(x)dx$$

Expectations



$$\mathbb{E}[f] = \int p(x)f(x)dx \approx \frac{1}{N} \sum_i^N f(x_i)$$

$$x_i \sim p(x)$$

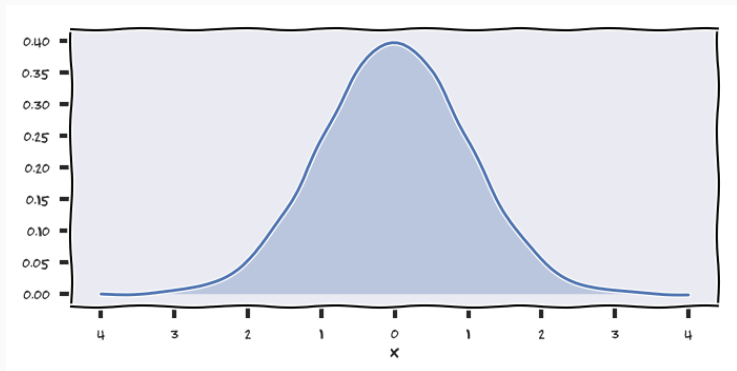
Expectations

```
e = 0.0
for i in range(0,N):
    x = 0.0 + 1.0*np.random.randn(1)
    e += f(x)

return e/N
```

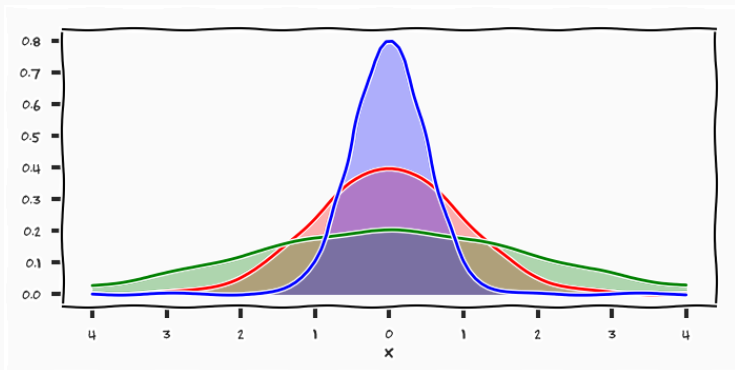
- drawing samples might be tricky
- can be infeasible when entropy of $p(x)$ is large, i.e. many samples

Expectations



$$\mathbb{E}[x] = \int xp(x)dx$$

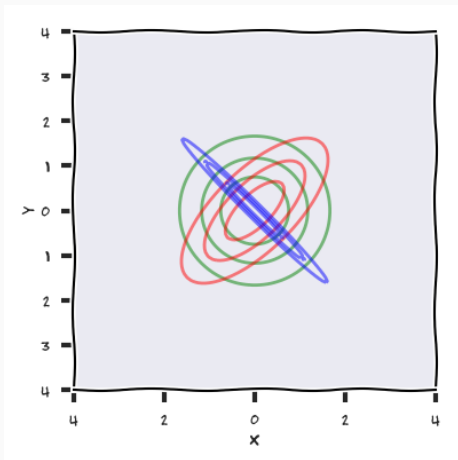
Variance



$$\text{var}[f] = \mathbb{E} \left[(f(x) - \mathbb{E}[f(x)])^2 \right]$$

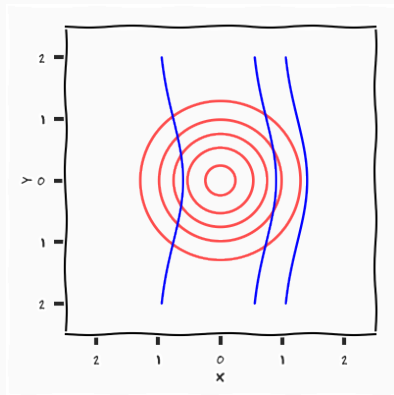
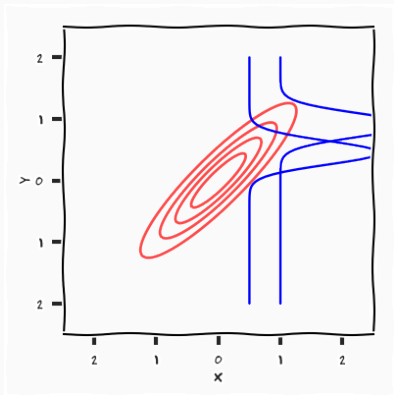
$$\text{var}[x] = \mathbb{E} \left[(x - \mathbb{E}[x])^2 \right]$$

Covariance



$$\text{cov}[x, y] = \mathbb{E}[(x - \mathbb{E}[x])(y - \mathbb{E}[y])]$$

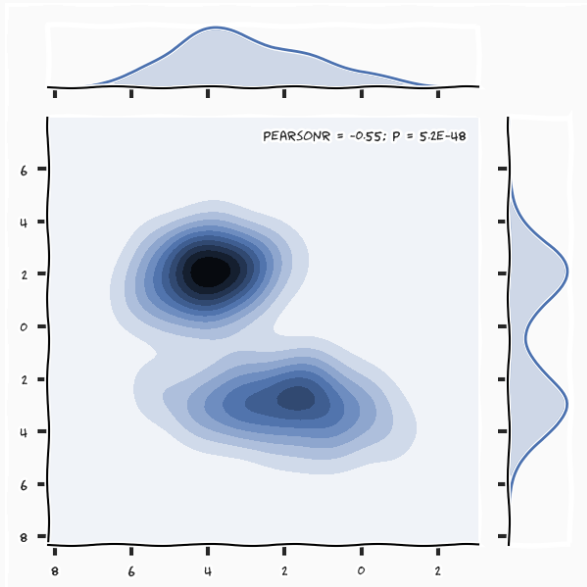
Covariance



$$p(y) = \int p(x, y) dx = \int p(y|x)p(x) dx$$

- Marginalisation accounts for all your belief in a variable
- Importantly it does not "remove" the effect of the variable
- Marginalisation is an **expectation** over a conditional distribution

Continuous: Marginal



What does Expectations Mean?



What does Expectations Mean?



1000 GBP

$$p(x = \text{dad}) = 0.2$$



5000 GBP

$$p(x = \text{mum}) = 0.5$$



500 GBP

$$p(x = \text{sister}) = 0.3$$

What does Expectations Mean?



1000 GBP

$$p(x = \text{dad}) = 0.2$$



5000 GBP

$$p(x = \text{mum}) = 0.5$$



500 GBP

$$p(x = \text{sister}) = 0.3$$

$$1000 \cdot 0.2 + 5000 \cdot 0.5 + 500 \cdot 0.3 = 2850$$

What does Expectations Mean?

*Next time you want to give your friends a compliment, tell them that you have completely **marginalised** them from your life*

Summary

Distributions

Joint $p(x, y)$

Rules

Summary

Distributions

Joint $p(x, y)$

Marginal $p(x), p(y)$

Rules

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Distributions

Joint $p(x, y)$

Marginal $p(x), p(y)$

Conditional $p(y|x), p(x|y)$

Rules

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Rules

Sum $p(x) = \sum_y p(y, x)$

Summary

Distributions

Joint $p(x, y)$

Marginal $p(x), p(y)$

Conditional $p(y|x), p(x|y)$

Rules

Sum $p(x) = \sum_y p(y, x)$

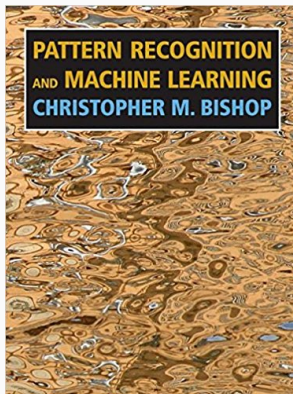
Product $p(x, y) = p(y|x)p(x)$

Probability Mass/Density Functions

it is important to note, these are just like any function, and you can deal with them in the same way, the difference is just that they have the additional constraints

$$p(x) \geq 0, \forall x, \quad \int_{-\infty}^{\infty} p(x) dx = 1$$

When we call them $p(y|x)$ is just a semantic added to the function



Ch 1.0, 1.2.1-1.2.2

Bayesian Probabilities

Frequentist

- a probability is a frequency of a repeatable random event

Bayesian

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- a probability is a frequency of a repeatable random event
- have no beliefs

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Frequentist

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- can be seen as an extension to Boolean logic for uncertain events

Frequentist

- a probability is a frequency of a repeatable **random** event
- have no beliefs

Bayesian

- a probability is a quantification of a belief
- probabilities are usually attributed to random/stochastic variables
- can be seen as an extension to Boolean logic for uncertain events
- requires beliefs

DID THE SUN JUST EXPLODE?
(IT'S NIGHT, SO WE'RE NOT SURE.)

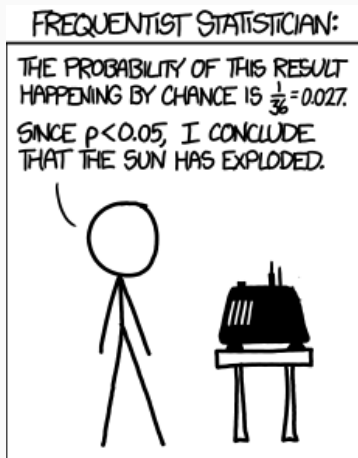
THIS NEUTRINO DETECTOR MEASURES
WHETHER THE SUN HAS GONE NOVA.

THEN, IT ROLLS TWO DICE. IF THEY
BOTH COME UP SIX, IT LIES TO US.
OTHERWISE, IT TELLS THE TRUTH.

LET'S TRY.
DETECTOR! HAS THE
SUN GONE NOVA?



¹XKCD

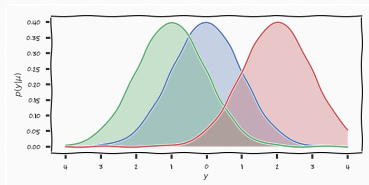


¹XKCD



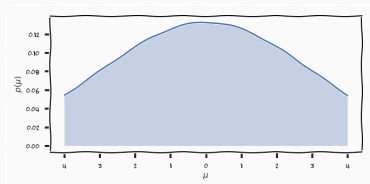
¹XKCD

Interpretations



$$p(y|\mu) = \mathcal{N}(\mu, 1.0)$$

Likelihood

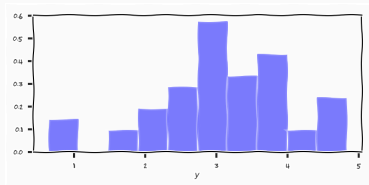


$$p(\mu)$$

Prior

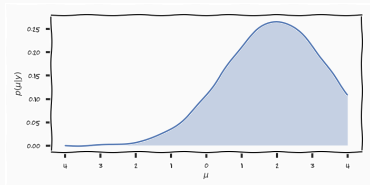
$$p(\mu|y) = \frac{p(y|\mu)p(\mu)}{\int p(y|\mu)p(\mu)}$$

Bayes Rule



y

Data



$p(\mu|y)$

Posterior

Are beliefs objective?

NO of course not and therefore we all learn different things from the same data

Are beliefs objective?

NO of course not and therefore we all learn different things from the same data

YES if two exact copies of the same "person" have different beliefs they cannot be the same person, therefore beliefs are objective, its only different amount of data/knowledge that generates differences

"Scientific modelling is a scientific activity, the aim of which is to make a particular part or feature of the world easier to understand, define, quantify, visualize, or simulate by referencing it to existing and usually commonly accepted knowledge." ²

²Wikipedia

- Davey, S., Gordon, N., Holland, I., Rutten, M., & Williams, J., Bayesian methods in the search for mh370 (2016), : Springer Singapore. [3]
- Stone, L. D., Keller, C. M., Kratzke, T. M., & Strumpfer, J. P., Search for the wreckage of air france flight af 447, Statistical Science, 29(1), 69–80 (2014). [4]

Summary

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Summary

- Learning can only be made by making assumptions
- You don't learn something that is true, you accept it
- Uncertainty is a "realisation" of an assumption
- Probabilities are a quantification of uncertainty
- Probabilities does not need to be frequencies of events
- More assumptions means less data (if I'm right)

eof

References



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A philosophical essay on probabilities, 1814.



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Pattern Recognition and Machine Learning (Information Science and Statistics).

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Sam Davey, Neil Gordon, Ian Holland, Mark Rutten, and Jason Williams.

Bayesian Methods in the Search for MH370.

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Lawrence D. Stone, Colleen M. Keller, Thomas M. Kratzke, and Johan P. Strumpfer.

Search for the wreckage of air france flight af 447.

Statistical Science, 29(1):69–80, 2014.

Appendix

Decisions



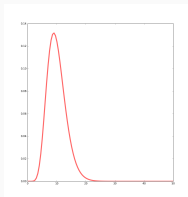
3

³Reservoir Dogs Tipping Scene YouTube

$$p(y)$$

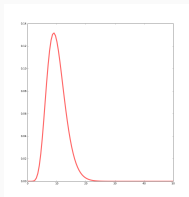
- what do I believe about tip **before** I see data?
- what is a sensible tip?

Tipping



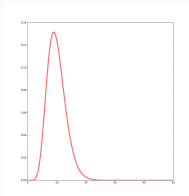
- I believe that 10£ is a sensible tip
- You cannot tip negative
- There is potentially an upper bound

Tipping



- I believe that 10£ is a sensible tip
- You cannot tip negative
- There is potentially an upper bound
- This is not a model, its just a belief in a variable

Tipping



- I believe that 10£ is a sensible tip
- You cannot tip negative
- There is potentially an upper bound
- This is not a model, its just a belief in a variable
- *a model relates new phenomenon to knowledge*

Tipping



- it is quite hard to say something about tip without any other knowledge
- **Assumption** the value of tip is related to the quality of the food

$$p(y|x)$$

- how likely do I think the observed data y is to come from this specific x .

Tipping if I know the quality of the food what do I believe the tip should be

What is the tip that I should expect to get?

$$\mathbb{E}_{p(x)}[p(y|x)] = \int p(y|x)p(x)dx = p(y)$$

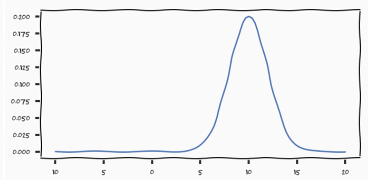
- What should I expect to get in tip
- I have an idea of the general distribution of quality of food
- *Understanding is when we can relate knowledge to new phenomenon*

$$p(x|c)$$

Hierarchical distribution

- Its quite hard to think of a prior over quality of food
- Can we parametrise the quality?

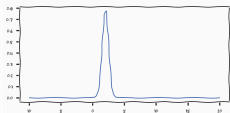
$$p(x|c) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu_c)(x-\mu_c)}{2\sigma^2}}$$



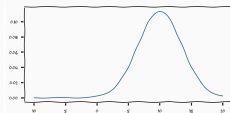
Hierarchical distribution

- Its quite hard to think of a prior over quality of food
- Can we parametrise the quality?
- Lets assume that if we know the cuisine we have an idea

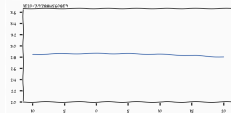
Tipping



Swedish



Italian



Uzbeki

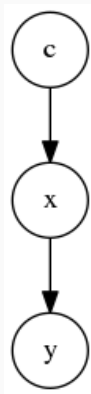
Hierarchical distribution

- Its quite hard to think of a prior over quality of food
- Can we parametrise the quality?
- Lets assume that if we know the cuisine we have an idea
- *Relating to knowledge!*

Tipping model

$$p(y, x, c) = p(y|x)p(x|c)p(c)$$

- Graphical Model shows dependency structure
- Shows "minimal" factorisation of joint distribution (model)



Tipping model

$p(y|x)$ Lets assume that tip is linearly related to quality of food

- $y = Wx + m$

$p(x|c)$ We saw them before

$p(c)$ What do I **believe** the proportions of restaurants to be?

- $p(c = \text{"swedish"}) = 0.04$
- $p(c = \text{"italian"}) = 0.90$
- $p(c = \text{"uzbeki"}) = 0.06$

Knowing the tip

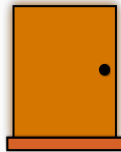
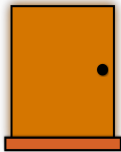
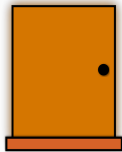
- Which cuisine did they eat if?
 - $p(c|y)$
- What was the quality of the food?
 - $p(x|y)$



Monty Hall



Monty Hall



Monty Hall

- The doors are exchangeable
- You choose door 1 and Monty chooses door 3
- *should you switch door?*
- Lets call C the door with the car behind and D the door Monty chooses

Monty Hall

- What is the probability that the car is behind any specific door?

Monty Hall

- What is the probability that the car is behind any specific door?
- $p(C = \{1, 2, 3\}) = \frac{1}{3}$

Monty Hall

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Monty Hall

- What is the probability that the car is behind any specific door?
- $p(C = \{1, 2, 3\}) = \frac{1}{3}$
- What is the probability that Monty opens door 3 if the car is behind door 1?
- $p(D = 3|C = 1) = \frac{1}{2}$

Monty Hall

- What is the probability that the car is behind any specific door?
- $p(C = \{1, 2, 3\}) = \frac{1}{3}$
- What is the probability that Monty opens door 3 if the car is behind door 1?
- $p(D = 3|C = 1) = \frac{1}{2}$
- What is the probability that Monty opens door 3 if the car is behind door 2?

Monty Hall

- What is the probability that the car is behind any specific door?
- $p(C = \{1, 2, 3\}) = \frac{1}{3}$
- What is the probability that Monty opens door 3 if the car is behind door 1?
- $p(D = 3|C = 1) = \frac{1}{2}$
- What is the probability that Monty opens door 3 if the car is behind door 2?
- $p(D = 3|C = 2) = 1$

Monty Hall

- What is the probability that the car is behind any specific door?
- $p(C = \{1, 2, 3\}) = \frac{1}{3}$
- What is the probability that Monty opens door 3 if the car is behind door 1?
- $p(D = 3|C = 1) = \frac{1}{2}$
- What is the probability that Monty opens door 3 if the car is behind door 2?
- $p(D = 3|C = 2) = 1$
- Marginal of $D = 3$

Monty Hall

- What is the probability that the car is behind any specific door?
- $p(C = \{1, 2, 3\}) = \frac{1}{3}$
- What is the probability that Monty opens door 3 if the car is behind door 1?
- $p(D = 3|C = 1) = \frac{1}{2}$
- What is the probability that Monty opens door 3 if the car is behind door 2?
- $p(D = 3|C = 2) = 1$
- Marginal of $D = 3$
- $p(D = 3) = p(D = 3|C = 1)p(C = 1) + p(D = 3|C = 2)p(C = 2) + p(D = 3|C = 3)p(C = 3) = \frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = \frac{1}{2}$

- Should I change from door 1 to 2?

$$p(C = 1|D = 3) = \frac{p(D = 3|C = 1)p(C = 1)}{p(D = 3)} = \frac{1/2 \cdot 1/3}{1/2} = 1/3$$

$$p(C = 2|D = 3) = \frac{p(D = 3|C = 2)p(C = 2)}{p(D = 3)} = \frac{1 \cdot 1/3}{1/2} = 2/3$$