Machine Learning

Distributions

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October 2, 2017

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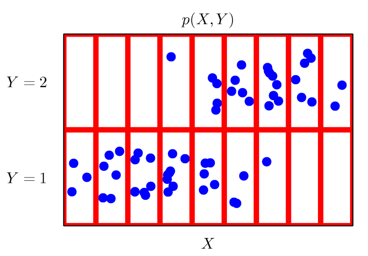
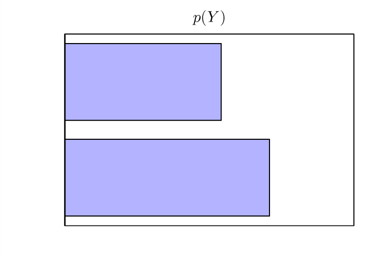
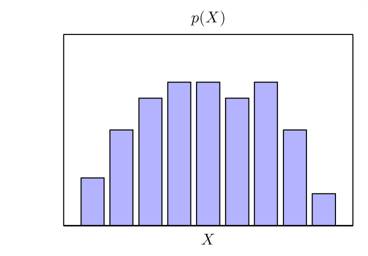
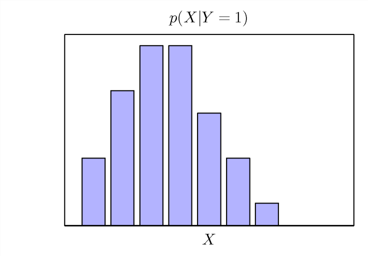
Introduction

Basic Probabilities

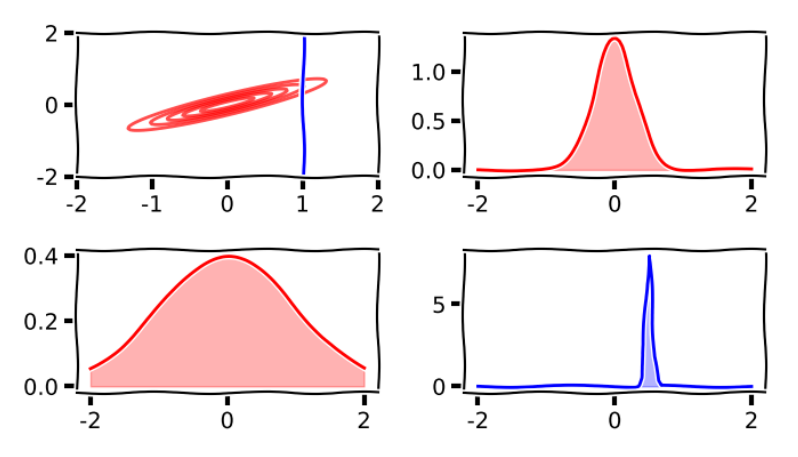








Basic Probabilities



The Rules of Probability

Sum Rule

$$p(X) = \sum_{Y} p(X, Y)$$

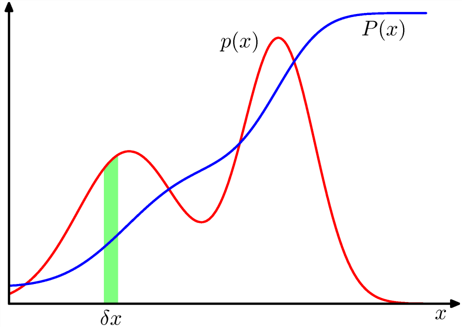
Product Rule

$$p(X, Y) = p(Y|X)p(X)$$

$\Rightarrow$ **Bayes Rule**

$$p(X|Y) = \frac{P(Y|X)p(X)}{p(Y)}$$

Probability Densities [1] ch 1.2.1



$$\lim_{\delta x \to 0} p(x \in (x, x + \delta x)) = \lim_{\delta x \to 0} \int_x^{x+\delta x} p(x)\mathrm{d}x = p(x) \cdot \delta x$$

$$p(x) \geq 0, \quad \int_{-\infty}^{\infty} p(x)\mathrm{d}x = 1$$

Machine Learning

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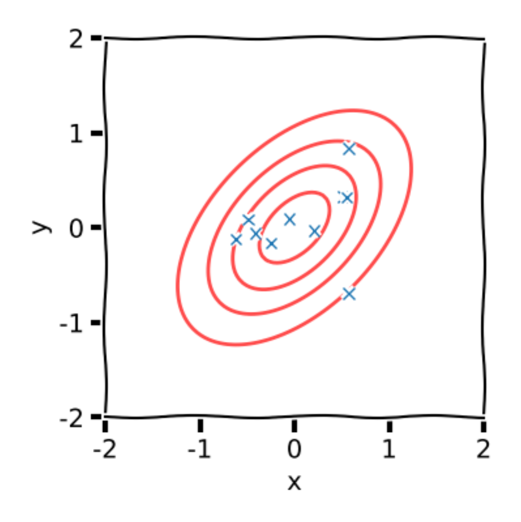
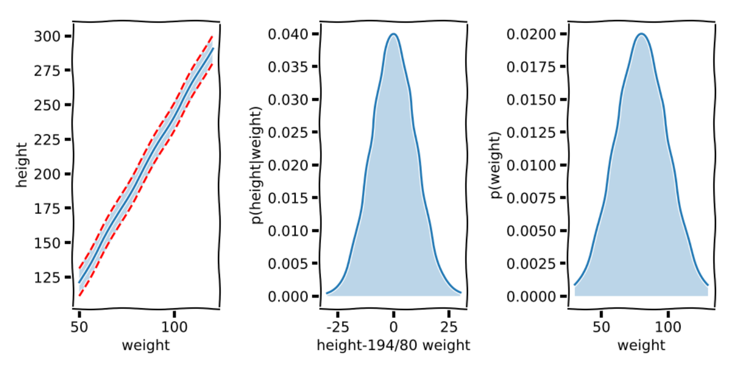
- If we can, then we can "equate" our model with the system
- if you are observing a system and never get surprised, would you say that you understand the system?
- *if you think of the probability as a measure of "suprisedness", if you have probability 0 and you see data you will be very surprised."*


2
1
0
-1
-2
y
-2
-1
0
1
2
x

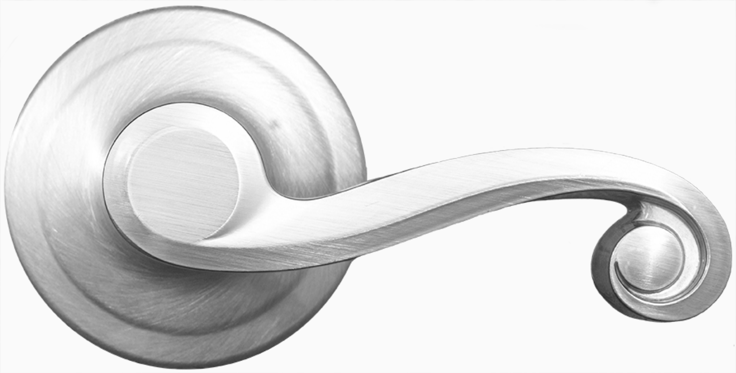
Bayes Rule

$$\underbrace{p(X|Y)}_{\text{posterior}} = \underbrace{P(Y|X)}_{\text{likelihood}} \cdot \underbrace{p(X)}_{\text{prior}} \cdot \frac{1}{\underbrace{p(Y)}_{\text{evidence}}}$$

$$\text{posterior} \propto \text{likelihood} \times \text{prior}$$



$$p(h|w) = \mathcal{N}(w \cdot \frac{194}{80}, 10^2)$$
$$p(w) = \mathcal{N}(80, 20^2)$$



Discrete Distributions

Bernoulli Distribution

- Distribution over binary random variable $x \in \{0, 1\}$

$$p(x = 1|\mu) = \mu$$

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$$\mathrm{Bern}(x|\mu) = \mu^x (1 - \mu)^{1-x}$$

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$$\mathrm{Bern}(x|\mu) = \mu^x (1 - \mu)^{1-x}$$

- Binomial

$$\mathrm{Bin}(m|N, \mu) = \binom{N}{m} \mu^m (1 - \mu)^{N-m}$$

Data

- We want to figure out what μ is for a specific coin
- Toss the coin N times, $\mathcal{D} = \{x_1, x_2, \ldots, x_N\}$

$$p(\mathcal{D}|\mu) = \prod_{n=1}^{N} p(x_n|\mu) = \prod_{n=1}^{N} \mu^{x_n}(1-\mu)^{1-x_n}$$

- What happens if we blindly trust this one experiment?

Maximum Likelihood

$$\mu_{ML} = \text{argmax}_{\mu} p(\mathcal{D}|\mu) = \frac{1}{N}\sum_{n=1}^{N} x_n$$

- if we get 3 heads in a row, we believe it will always be heads
- we need to include an assumption as a prior over μ

$$p(\mu|\mathcal{D}) = \frac{p(\mathcal{D}|\mu)p(\mu)}{p(\mathcal{D})}$$

- Also gives us an uncertainty related to our knowledge

Conjugate Prior

- If we have a prior belief μ we want the posterior belief to have the same functional form

$$\text{posterior} \propto \text{likelihood} \times \text{prior}$$

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- Conjugate prior

$$p(\mu|\theta) = f_1(\theta)\mu^{f_2(\theta)}(1-\mu)^{f_3(\theta)}$$

$$\int_0^1 p(\mu|\theta)\mathrm{d}\mu = 1$$

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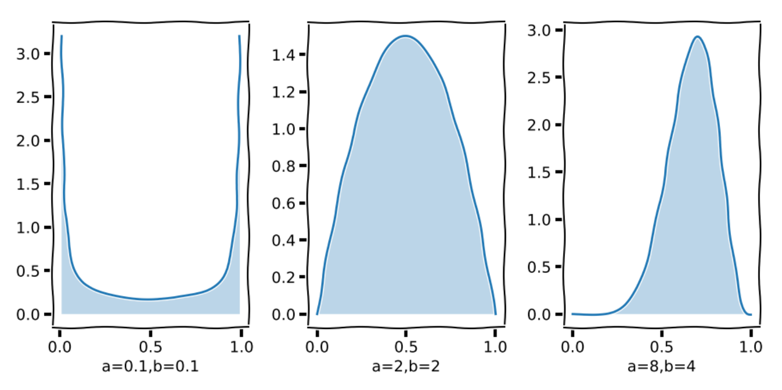
- Does this make philosophical sense?

Beta Distribution



$$\mathrm{Beta}(\mu|a,b) = \frac{\Gamma(a+b)}{\Gamma(a)+\Gamma(b)}\mu^{a-1}(1-\mu)^{b-1}$$

Binomial Posterior

$$
\begin{aligned}
p(\mu|m,a,b) &\propto p(\mathcal{D}|\mu)p(\mu) \\
&= \binom{N}{m}\mu^m(1-\mu)^{N-m}\cdot\frac{\Gamma(a+b)}{\Gamma(a)+\Gamma(b)}\mu^{a-1}(1-\mu)^{b-1} \\
&\propto \mu^{m+a-1}(1-\mu)^{N-m+b-1}
\end{aligned}
$$

- the parameters of the prior have a clear interpretation

  a number of extra observations of $x = 0$

  b number of extra observations of $x = 1$

Multinomial

- If we have a variable that can take K different states

$$\mathbf{x} = [0, 0, 1, 0, 0, 0]^{\mathrm{T}}$$

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- Multinomial

$$p(\mathbf{x}|\boldsymbol{\mu}) = \prod_{k=1}^{K} \mu_k^{x_k}$$

$$\boldsymbol{\mu} = [\mu_1, \ldots, \mu_k]^{\mathrm{T}}, \sum_k \mu_k = 1$$

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- Likelihood

$$p(\mathbf{D}|\boldsymbol{\mu}) = \prod_{n=1}^{N} \prod_{k=1}^{K} \mu_k^{x_{nk}}$$

Dirichlet

- Multinomial

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Dirichlet

- Multinomial

$$p(\mathsf{x}|\boldsymbol{\mu}) = \prod_{k=1}^{K} \mu_k^{x_k}$$

- Conjugate prior

$$p(\boldsymbol{\mu}|\boldsymbol{\alpha}) \propto \prod_{k=1}^{K} \mu_k^{\alpha_k - 1}$$

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- Dirichlet Distribution

$$\mathrm{Dir}(\boldsymbol{\mu}|\boldsymbol{\alpha}) = \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1) \cdot \ldots \cdot \Gamma(\alpha_K)} \prod_{k=1}^{K} \mu_k^{\alpha_k - 1}$$

Dirichlet

- Posterior

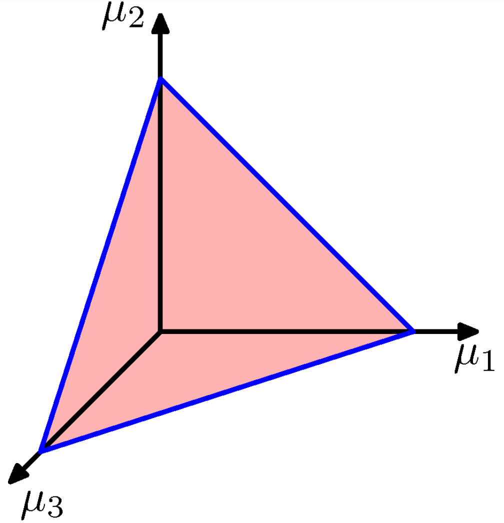
$$p(|\mathcal{D}, \boldsymbol{\alpha}) \propto p(\mathcal{D}|\boldsymbol{\mu})p(\boldsymbol{\mu}|\boldsymbol{\alpha}) \propto \prod_{k=1}^{K} \mu_k^{\alpha_k + m_k + 1}$$

$$m_k = \sum_n x_{nk}$$

- Normalised Form

$$p(|\mathcal{D}, \boldsymbol{\alpha}) = \frac{\Gamma(\alpha_0 + N)}{\Gamma(\alpha_1 + m_1) \cdot \ldots \cdot \Gamma(\alpha_K + m_K)} \prod_{k=1}^{K} \mu_k^{\alpha_k + m_k = 1}$$

Dirichlet Prior



Hyperparameters

$$p(\mu|\mathcal{D}, \boldsymbol{\alpha}) = \frac{p(\mathcal{D}|\boldsymbol{\mu})p(\boldsymbol{\mu}|\boldsymbol{\alpha})}{p(\mathcal{D}|\boldsymbol{\alpha})}$$

- all these priors have parameters, where do they come from?

Hyperparameters

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- either we know them

Hyperparameters

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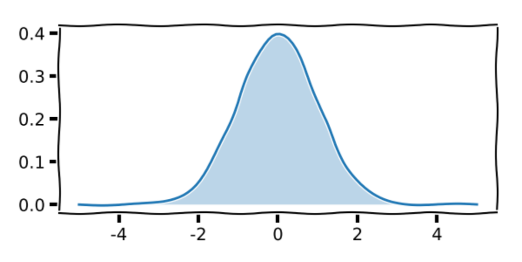
- all these priors have parameters, where do they come from?
- either we know them
- if we don't then place a prior over the priors parameters and go again

Continous Distributions

Gaussian Distribution



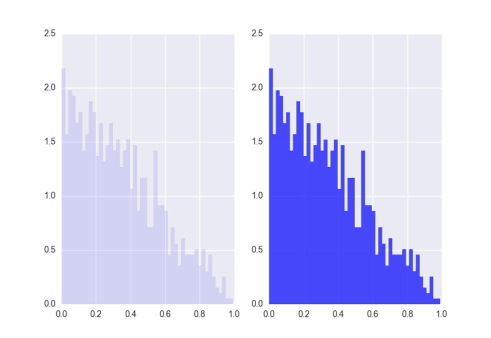
$$p(x|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{\frac{D}{2}} |\boldsymbol{\Sigma}|^{\frac{1}{2}}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{x}-\boldsymbol{\mu})}$$

Central Limit Theorem[1]

The central limit theorem states that the distribution of the sum (or average) of a large number of independent, identically distributed variables will be approximately normal, regardless of the underlying distribution.

[1] `https://www.youtube.com/watch?v=wadzsURQFT4`

Central Limit Theorem



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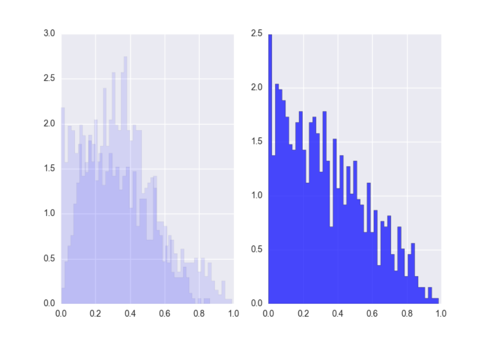

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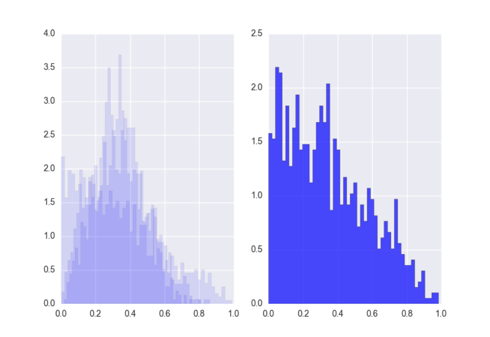

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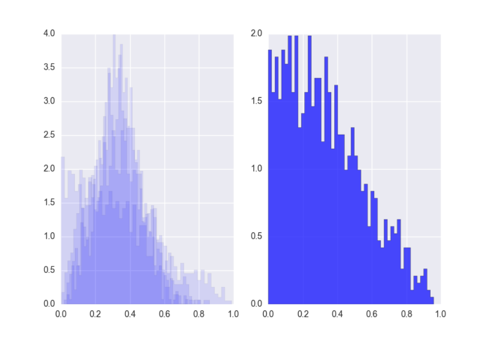

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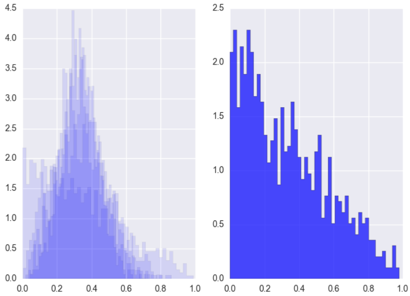

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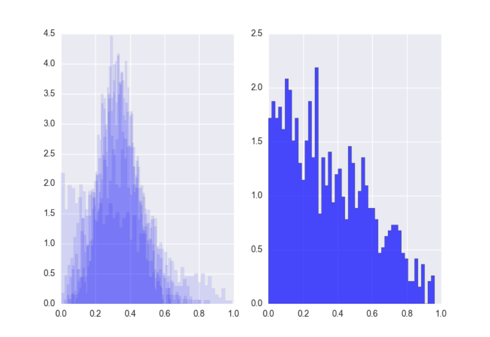

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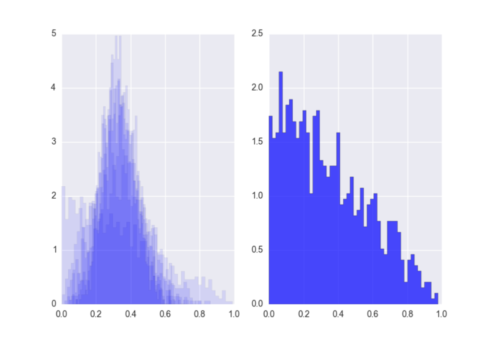

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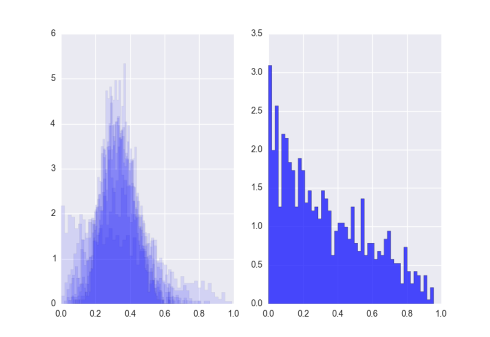

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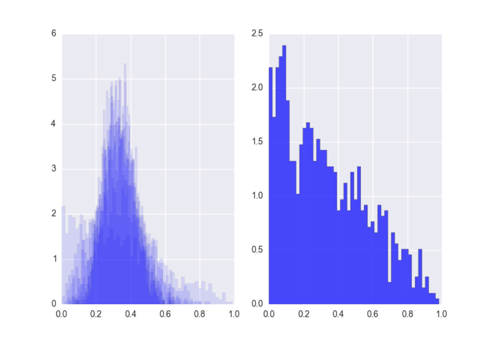

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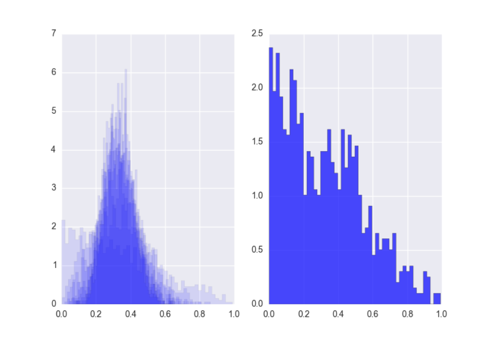

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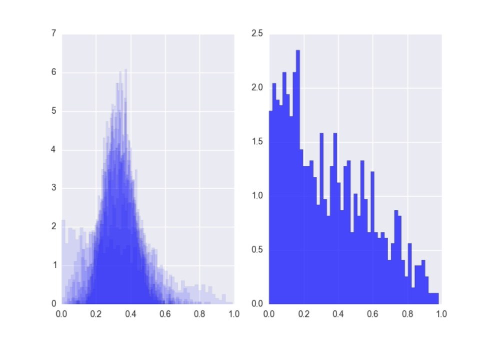

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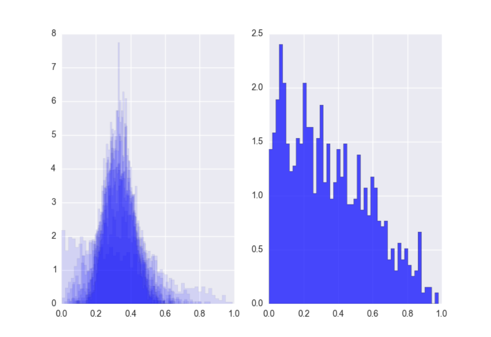

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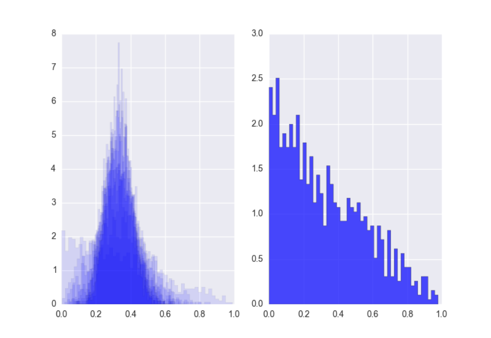

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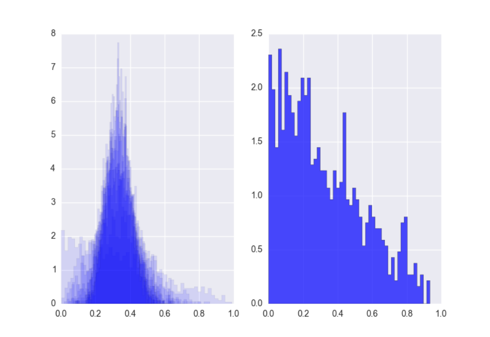

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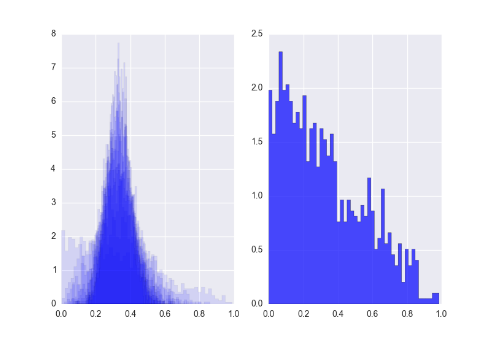

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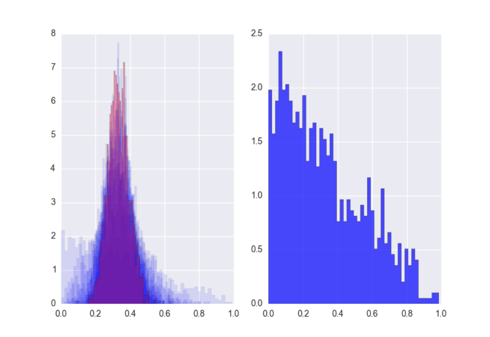


Central Limit Theorem Carl

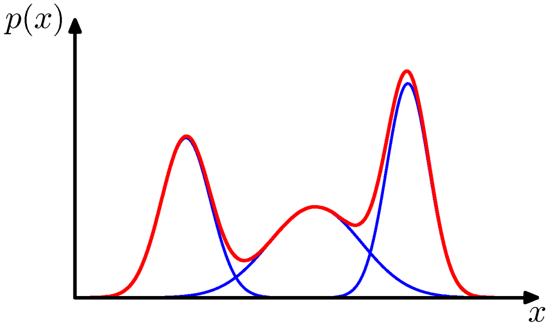
- If I do not know anything at all except that I think that there is some distribution something comes from
- A sensible idea (potentially) would be to think of an average
- As it turns out, the average of anything that I do not know (but are identically distributed) is Gaussian
- Therefore I think it kinda makes sense

Conjugate Prior[2]

- Gaussians are self-conjugate
  - Gaussian likelihood + Gaussian Prior $\rightarrow$ Gaussian Posterior
- Gaussian distribution
  - Conjugate prior for $\boldsymbol{\mu}$ is Gaussian
  - Conjugate prior for $\boldsymbol{\Sigma}$ is Inverse-Wishard

[2] https://en.wikipedia.org/wiki/Conjugate_prior

Mixtures of Gaussians [1] Ch 2.3.9



$$p(\mathbf{x}) = \sum_{k=1}^{K} p(k) \underbrace{p(\mathbf{x}|k)}_{\mathcal{N}(\mu_k, \Sigma_k)}$$

- Exponential family natural parametrisation

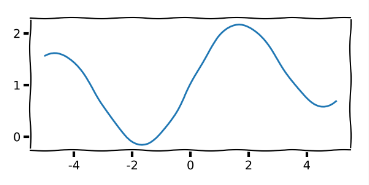
$$p(\mathsf{x}|\boldsymbol{\eta}) = h(\mathsf{x})g(\boldsymbol{\eta})e^{\boldsymbol{\eta}^{\mathrm{T}}\mathbf{u}(\mathsf{x})}$$

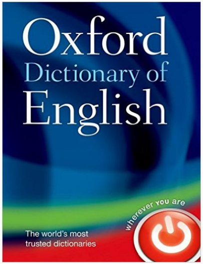
- Conjugate prior

$$p(\boldsymbol{\eta}|\boldsymbol{\chi}, \nu) = f(\boldsymbol{\chi}, \nu)g(\boldsymbol{\chi})^{\nu} e^{\nu\boldsymbol{\eta}^{\mathrm{T}}\boldsymbol{\chi}}$$

Stochastic Process

$$\infty$$





Kologrovs Existence Theorem

Defines what a distribution needs to forfill in order for a process to exist. Each finite instantiaton of the process is this distribution.

Gaussian Identities

Gaussian Distribution

$$p(x_1, x_2) = \mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right)$$

Posterior $p(x_1|x_2) \propto p(x_2|x_1)p(x_1)$

Marginal $p(x_1) = \int p(x_1, x_2)\mathrm{d}x_2$

Conditional $p(x_1|x_2) = \frac{p(x_1,x_2)}{p(x_2)}$

Posterior

$$p(x_1|x_2) \propto p(x_2|x_1)p(x_1)$$

1. Multiply right-hand side
2. Look at the exponents
3. Find the three terms, constant, mixed and quadratic
4. Called completing the square and very very useful to have done

Marginal

$$p(x_1) = \int p(x_1, x_2)\mathrm{d}x_2 = \mathcal{N}(\mu_1, \Sigma_{11})$$

1. Write out the exponent of the joint distribution
2. Complete Square and collect terms with $x_1 - \mu_1$ (as we know the result)
3. Compute integral by knowing that densities always integrates to one

Conditional

$$p(x_1|x_2) = \frac{p(x_1, x_2)}{p(x_2)}$$

1. Factorise the problem as $p(x_1, x_2) = p(x_1|x_2)p(x_2)$
2. We know the marginal and the joint
3. Use Schur complement to re-write the covariance matrix on block form

Gaussian Identities

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1

$$N(x|\mu,\Sigma) = \frac{1}{(2\pi)^{\frac{D}{2}}|\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}\underbrace{(x-\mu)^T\Sigma^{-1}(x-\mu)}}$$

$$(x-\mu)^T\Sigma^{-1}(x-\mu)$$

1 Σ = diagonal matrix

$$\Sigma = \begin{bmatrix} \Sigma_1 & & \emptyset \\ & \Sigma_2 & \\ \emptyset & \ddots & \Sigma_d \end{bmatrix} \Rightarrow \Sigma^{-1} = \begin{bmatrix} \frac{1}{\Sigma_1} & & \emptyset \\ & \frac{1}{\Sigma_2} & \\ \emptyset & \ddots & \frac{1}{\Sigma_d} \end{bmatrix}$$

$$(x-\mu)^T\Sigma^{-1}(x-\mu) = [(x_1-\mu_1), \dots, (x_D-\mu_D)] \begin{bmatrix} \frac{1}{\Sigma_1} & & \\ & \ddots & \\ & & \frac{1}{\Sigma_D} \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ \vdots \\ x_D-\mu_D \end{bmatrix} =$$

$$= (x_1-\mu_1)\cdot\frac{1}{\Sigma_1}\cdot(x_1-\mu_1) + \dots + (x_D-\mu_D)\cdot\frac{1}{\Sigma_D}(x_D-\mu_D) =$$

$$= \sum_i^D \underbrace{\frac{1}{\Sigma_i}(x_i-\mu_i)^2}_{A}$$

A - alway positive
- small value $(x_i-\mu_i) \rightarrow$ close to mean
- Σ_i - scales this value

$\Rightarrow$ Σ_i - big $\Rightarrow$ uncertain about this dimenson $\Rightarrow$ large deviation from mean doesn't matter

Σ_i - small $\Rightarrow$ certain about this dimension $\Rightarrow$ large deviation from mean matters a lot

Summary

Summary

- Distributions allows us to make our assumptions explicit
- Conjugacy implies that the posterior and the prior is in the same family
- Exponential family defines a *natural* parametrisation of distributions that we can work with
- Gaussian Identities!

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