Sanya, China

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Lectures on AdS/CFT Quantum Spectral Curve and Fishnet CFT

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Plan: many faces of N=4 SYM

Lecture I

- Many faces of N=4 SYM: AdS/CFT, quantum and classical planar integrability, Fishnet CFT, BPS physics and localization, black holes, etc
- Field content and symmetries of N=4 SYM
- Some results from integrability and localization

Lecture II

- 1-loop integrability. T- and Q-systems of gl(N) Heisenberg (super)spin chains
- (super)Hasse diagram, QQ relations and nested Bethe ansatz equations

Lecture III

- All-loop integrability for spectrum: AdS/CFT quantum spectral curve (QSC).
- Some applications: one loop from P-mu, …
- Review of applications (QSC numerics and PT, BFKL, cusp) and perspectives (structure constants, correlators, BH)

Lecture IV

- From N=4 SYM to Fishnet CFT
- Integrability. More general FCFTs (Loom for FCFTs)
- Applications: multi-loop graphs from QSC, BFKL…

Lecture I

N=4 Super-Yang Mills theory and AdS/CFT correspondence

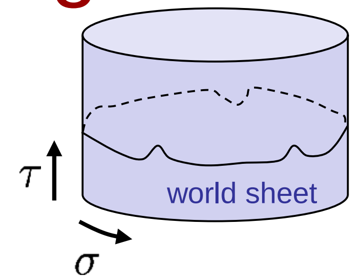
N=4 SYM dual to superstring on $AdS_5 \times S^5$

$$\mathcal{S}_{SYM} = \frac{1}{\lambda}\int d^4x \, \mathrm{Tr}\left(F^2 + (\mathcal{D}\Phi)^2 + \bar{\Psi}\mathcal{D}\Psi + \bar{\Psi}\Phi\Psi + [\Phi,\Phi]^2\right)$$

super-conformal theory:
β-function=0, no massive particles

$$\lambda = N g_{YM}^2 \equiv 16\pi^2 g^2$$

Exact equivalence

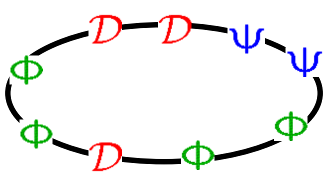


$$\mathcal{S}_{sigma} = \sqrt{\lambda}\int d\tau \int_0^L d\sigma \left[\left(\partial\vec{X}\right)^2 + \left(\partial\vec{Y}\right)^2 + \text{fermions}\right]$$

Maldacena
Gubser,Klebanov,Polyakov
Witten

Metsaev, Tseytlin

$$\mathcal{O}(x) = \mathrm{Tr}\left[\mathcal{D}\mathcal{D}\Psi\Psi\Phi\Phi\mathcal{D}\Psi\ldots\right](x)$$

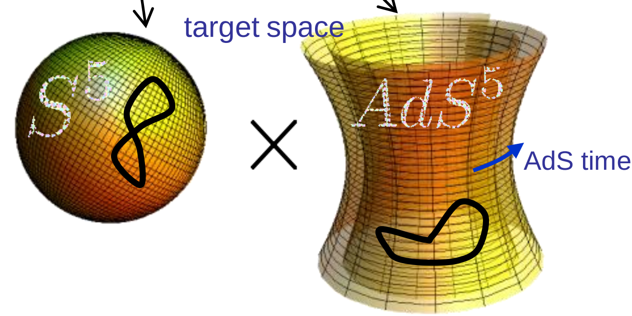


CFT/AdS duality

$\longleftrightarrow$

weak / strong

$$\lambda = g_{YM}^2 N = \frac{R^4}{\alpha'^2} = g_s N$$



$$\mathcal{O}_A(\xi x) \;\to\; \xi^{\Delta_A(\lambda)}\mathcal{O}_A(x)$$

$$\mathrm{AdS}_5: \quad -X_{-1}^2 - X_0^2 + X_7^2 + X_8^2 + X_9^2 + X_{10}^2 = -R^2$$

$$S^5: \quad X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 = R^2$$

$$\hat{D}\,\mathcal{O} = \Delta\,\mathcal{O}$$

Anomalous dimension $\Delta_{\mathcal{O}}$ = Energy of the dual string state

Super-conformal symmetry PSU(2,2|4) isometry of string target space
($AdS_5 \times S^5$ is its bosonic part).

Maldacena'97
Gubser,Klebanov,Polyakov'98
Witten'98

AdS/CFT correspondence

$\mathcal{N}=4$ SYM	Strings on $AdS_5 \times S^5$
't Hooft coupling: $g^2 = g_{YM}^2 N$	String tension: $T = \frac{1}{2\pi\alpha'} = \frac{g}{2\pi}$
Number of colors: N	String coupling: $g_s = \frac{g^2}{N}$
Large-N limit	Free strings
Strong coupling	Classical strings
Local operators Scaling dimension: Δ	String states Energy: E

Conformal transformations SO(2,4)~SU(2,2)

- Scale invariance: stress tensor is traceless, which implies a larger symmetry
- Conformal invariance: local scale invariance

$$x^\mu \to x'^\mu, \qquad dx'^2 = \rho^2(x) dx^2$$

Infinitesimal: $\delta x^\mu = v^\mu(x), \qquad \partial_\mu v_\nu + \partial_\nu v_\mu = \frac{2}{d}\eta_{\mu\nu}\,\partial\cdot v\,,$

$$v^\mu = \left\{a^\mu, \omega^\mu{}_\nu x^\nu, x^\mu, x^2 a^\mu - 2(a\cdot x)x^\mu\right\}$$

(translation P, Lorentz L, dilatation D, special conf. K)

$$\rho(x) = 1 + \frac{\partial\cdot v}{d}$$

Isometry: $\rho(x) = 1$

- K := Spec.conf. = inversion × shift × inversion

- Full conformal group:

$$\begin{aligned} D &= -ix_\mu \partial_{x_\mu} - i\Delta \\ L_{\mu\nu} &= ix_\mu\partial_{x_\nu} - ix_\nu\partial_{x_\mu} \\ P_\mu &= -i\partial_{x_\mu} \\ K_\mu &= 2\,x^\nu L_{\nu\mu} - ix^2\partial_{x_\mu} - 2i\Delta x_\mu \end{aligned}$$

$$\begin{aligned} [L_{\mu\nu}, L_{\lambda\rho}] &= \eta_{\mu\lambda}L_{\nu\rho} + \eta_{\nu\rho}L_{\mu\lambda} - \eta_{\mu\rho}L_{\nu\lambda} - \eta_{\nu\lambda}L_{\mu\rho} \\ [L_{\mu\nu}, P_\lambda] &= \eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu \\ [L_{\mu\nu}, K_\lambda] &= \eta_{\mu\lambda}K_\nu - \eta_{\nu\lambda}K_\mu \\ [D, P_\mu] &= P_\mu \\ [D, K_\mu] &= -K_\mu \\ [P_\mu, K_\nu] &= 2L_{\mu\nu} - 2\eta_{\mu\nu}D. \end{aligned}$$

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Conformal 2- and 3-point function

- We consider only planar limit. Operators from local fields

$$\mathcal{O}(x) = \text{linear combinations of Tr}\,[\mathcal{D}\mathcal{D}\lambda\lambda\phi\phi\mathcal{D}\phi\lambda\ldots]\,(x)$$

- 2- and 3-point correlators (structure functions):

$$G_{II}(x_1,x_2) = \langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle \qquad G_{III}(x_1,x_2,x_3) = \langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}(x_3)\rangle$$

- Under special conformal: $\delta \log(x_1 - x_2)^2 = -a\cdot(x_1+x_2)\,, \qquad \rho(x) = 1 - 2a\cdot x + O(a^2)$

$$\delta G_{II}(x_1,x_2) = -\frac{\partial G_{II}}{\partial \log r_{12}} a\cdot(x_1+x_2) = 2(\Delta_1 a\cdot x_1 + \Delta_2 a\cdot x_2) G_{II}$$

- Possible only for equal dimensions, then $\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle = \dfrac{\delta_{ij}}{|x|^{2\Delta_j(\lambda)}}$

- Similarly, for 3-point corr.:

$$\delta G_{III}(x_1,x_2,x_2) = -\sum_{i>j} \frac{\partial G_{III}}{\partial \log r_{ij}} a\cdot(x_i+x_j) = 2\sum_i \Delta_i a\cdot x_i G_{III}$$

- Solution: $\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \dfrac{C_{ijk}(\lambda)}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k}|x_{23}|^{\Delta_j+\Delta_k-\Delta_i}|x_{31}|^{\Delta_i+\Delta_k-\Delta_j}}$

They describe the whole conformal theory via operator product expansion

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k \frac{C_{ijk}(\lambda)}{|x|^{\Delta_i+\Delta_j-\Delta_k}} \mathcal{O}_k(0) + \text{decendants}$$

*

Zero-magnon 4-point correlator and exact OPE data

- Exact expression of a 4-point correlator (only from conformal symmetry!)

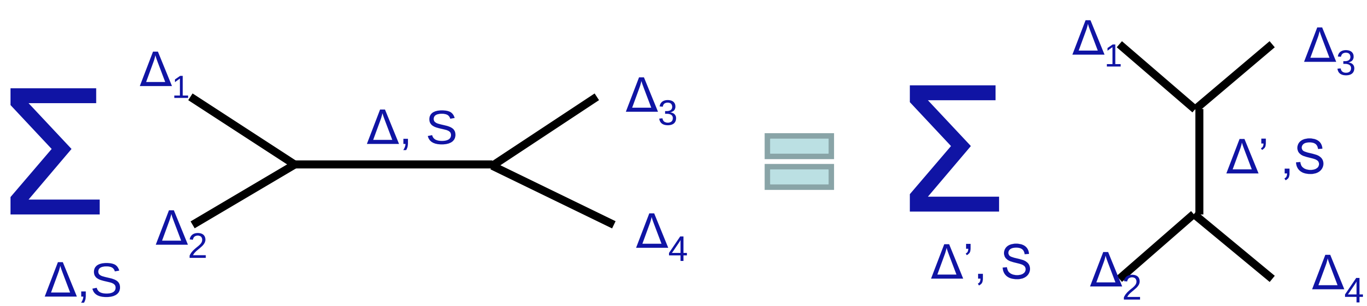
$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \mathcal{G}(u,v)\frac{(x_{24}/x_{14})^{\Delta_1-\Delta_2}(x_{14}/x_{13})^{\Delta_3-\Delta_4}}{x_{12}^{\Delta_1+\Delta_2}x_{34}^{\Delta_3+\Delta_4}}$$

$$\mathcal{G}(u,v) = \sum_{\Delta,S} C_{12}^{S,\Delta}\, C_{34}^{S,\Delta}\, u^{\frac{\Delta-S}{2}}\, g_{\Delta,S}\left(\frac{\Delta-\Delta_1+\Delta_2-S}{2}, \frac{\Delta-\Delta_4+\Delta_3-S}{2}; u,v\right)$$

cross-ratios

$$\begin{aligned} u &= \quad z\bar{z} &&= x_{12}^2 x_{34}^2/(x_{13}^2 x_{24}^2) \\ v &= (1-z)(1-\bar{z}) &&= x_{14}^2 x_{23}^2/(x_{13}^2 x_{24}^2) \end{aligned}$$

- Conformal block g(…) explicitly expressed through hypergeometric function ${}_2F_1(a,b,c;z)$, where a,b,c are linear functions of spin S and dim Δ
- OPE for 4-point correlator in two different channels:



crossing

Gliozzi,Scherk,Olive'77

N=4 SYM as a superconformal 4d QFT

- Can be realized as reduction to 4D of 10D N=1 SYM (so(6) setting)

$$\mathcal{L}_{\rm YM}[\mathcal{W}] = \tfrac{1}{4}\,\mathrm{Tr}\,\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} + \tfrac{1}{2}\,\mathrm{Tr}\,\mathcal{D}^{\mu}\Phi^{n}\mathcal{D}_{\mu}\Phi_{n} - \tfrac{1}{4}g^{2}\,\mathrm{Tr}\,[\Phi^{m},\Phi^{n}][\Phi_{m},\Phi_{n}] \qquad ($$
$$+ \mathrm{Tr}\,\dot{\Psi}^{a}_{\dot\alpha}\sigma^{\dot\alpha\beta}_{\mu}\mathcal{D}^{\mu}\Psi_{\beta a} - \tfrac{1}{2}ig\,\mathrm{Tr}\,\Psi_{\alpha a}\sigma^{ab}_{m}\varepsilon^{\alpha\beta}[\Phi^{m},\Psi_{\beta b}] - \tfrac{1}{2}ig\,\mathrm{Tr}\,\dot{\Psi}^{a}_{\dot\alpha}\sigma^{m}_{ab}\varepsilon^{\dot\alpha\dot\beta}[\Phi_{m},\dot{\Psi}^{b}_{\dot\beta}]$$

- Fields in adjoint irrep of gauge group su(N_c) - $N_c \times N_c$ matrices
- Tensors with global superconformal $psu(2,2|4)$

Gauge field
Lorentz symmetry SO(4)

$$\mathcal{D}_\mu = \partial_\mu + iA_\mu$$

Scalar fields
R-symmetry SO(6)~SU(4)

$$\Phi_m\,, \qquad m = 1,2,\ldots,6$$

$$X = \Phi_1 + i\Phi_4$$
$$Y = \Phi_2 + i\Phi_5$$
$$Z = \Phi_3 + i\Phi_6$$

16 Weyl spinors: $\Psi_{\alpha a},\ \Psi^{\dot\alpha}_{a}, \quad a = 1,\ldots,4 : su(4), \quad \alpha,\dot\alpha = 1,2 : so(4) \simeq su(2)_L \times su(2)_R$

Gamma matrices: 4D spinors and 6D spinors

$$\sigma^{\{\mu}\sigma^{\nu\}} = \eta^{\mu\nu}, \qquad \sigma^{\{m}\sigma^{n\}} = \eta^{mn}$$
$$\sigma^{m,ab} = \tfrac{1}{2}\varepsilon^{abcd}\sigma^{m}_{cd}, \qquad \sigma_{m,ab} = \tfrac{1}{2}\varepsilon_{abcd}\sigma^{cd}_{m}$$

Fierz id. 4D: $\sigma^{\dot\alpha\beta}_{\mu}\sigma^{\mu}_{\dot\gamma\delta} = 2\delta^{\dot\alpha}_{\dot\gamma}\delta^{\beta}_{\delta}, \qquad \sigma^{\dot\alpha\beta}_{\mu}\sigma^{\mu,\dot\gamma\delta} = 2\varepsilon^{\dot\alpha\dot\gamma}\varepsilon^{\beta\delta}, \qquad \sigma_{\mu,\dot\alpha\beta}\sigma^{\mu}_{\dot\gamma\delta} = 2\varepsilon_{\dot\alpha\dot\gamma}\varepsilon_{\beta\delta}$

Fierz id. 6D: $\sigma^{ab}_{m}\sigma^{m}_{cd} = 2\delta^{a}_{d}\delta^{b}_{c} - 2\delta^{a}_{c}\delta^{b}_{d}, \qquad \sigma^{ab}_{m}\sigma^{m,cd} = -2\varepsilon^{abcd}, \qquad \sigma_{m,ab}\sigma^{m}_{cd} = -2\varepsilon_{abcd}$

SU(4) notations and topological term

- Spinorial notations for fields:

$$\mathcal{F}_{\pm}^{\mu\nu} = \pm\frac{1}{2}\epsilon^{\mu\nu\sigma\rho}\mathcal{F}_{\pm\sigma\rho}$$

$$\sigma^{\mu\nu}_{\alpha\beta} = \sigma^{[\mu}_{\alpha\dot\alpha}\sigma^{\nu]}_{\beta\dot\beta}\,\epsilon^{\dot\alpha\dot\beta}$$

$$\mathcal{F}_{+\alpha\beta} \equiv \frac{1}{2}\sigma^{\mu\nu}_{\alpha\beta}\mathcal{F}_{+\mu\nu} = \mathcal{F}_{+\beta\alpha}, \quad \mathcal{F}_{-\dot\alpha\dot\beta} \equiv \frac{1}{2}\sigma^{\mu\nu}_{\dot\alpha\dot\beta}\mathcal{F}_{-\mu\nu} = \mathcal{F}_{-\dot\beta\dot\alpha} \qquad \mathcal{D}_{\alpha\beta} \equiv \sigma^{\mu}_{\alpha\beta}\mathcal{D}_{\mu}$$

$$\Phi_{ab} = \Phi_m\,\sigma^m_{ab}. \qquad \Phi_m, \quad m = 1,\ldots,6 \in SO(6); \quad \Phi^{ab} \in SU(4) \sim SO(6)$$

$$\Phi_{ab} = \begin{pmatrix} 0 & \Phi_3 + i\Phi_6 & -\Phi_2 - i\Phi_5 & \Phi_1 - i\Phi_4 \\ -\Phi_3 - i\Phi_6 & 0 & \Phi_1 + i\Phi_4 & \Phi_2 - i\Phi_5 \\ \Phi_2 + i\Phi_5 & -\Phi_1 + i\Phi_4 & 0 & \Phi_3 - i\Phi_6 \\ -\Phi_1 + i\Phi_4 & -\Phi_2 + i\Phi_5 & -\Phi_3 + i\Phi_6 & 0 \end{pmatrix}$$

- Lagrangian in these notations:

$$\mathcal{L} = N_c \mathrm{tr}[-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D^{\mu}\Phi^{\dagger}_{AB}D_{\mu}\Phi^{BA} + \frac{g^2}{8}[\Phi^{\dagger}_{AB},\Phi^{\dagger}_{CD}][\Phi^{BA},\Phi^{DC}] +$$

$$+2i\bar\psi^{\dot\alpha}_{A}D^{\alpha}_{\dot\alpha}\psi^{A}_{\alpha} - \sqrt{2}g\psi^{\alpha\,A}[\Phi^{\dagger}_{AB},\psi^{B}_{\alpha}] + \sqrt{2}g\bar\psi_{\dot\alpha\,A}[\Phi^{AB},\bar\psi^{\dot\alpha}_{B}]\ ]$$

- Topological term:

$$\theta\int d^4x \,\mathrm{Tr} F\wedge F$$

complexified coupling

$$\tau_{YM} = \frac{1}{g^2_{YM}} + \frac{\theta}{2\pi} \Leftarrow \tau_{str} = \frac{i}{g_s} + \chi \qquad g^2 = N_c g^2_{YM}$$

SL(2,Z) S -duality, related to the string coupling

$$\tau = -\frac{1}{\tau}$$

Operators, planar graphs and 1/N expansion

- Local operators: single trace of products of matrix fields (only planar limit!)

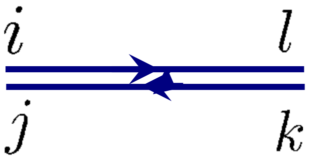
$$\mathcal{O}(x) = \text{tr}\,[\chi_1(x), \chi_2(x)\ldots\chi_L(x)] \;+\; \text{perm.}$$

$$\chi \in \{\mathcal{D}_{\dot\alpha\beta}, \Phi_{ab}, \Psi_{\alpha b}, \Psi^{b}_{\dot\alpha}, \mathcal{F}_{\alpha\beta}, \dot{\mathcal{F}}_{\dot\alpha\dot\beta}\}$$

- Dimensions of operators can be read of from 2-point correlators
- They can be computed via perturbation theory

- Propagators in double-line notations (only color indices exposed):

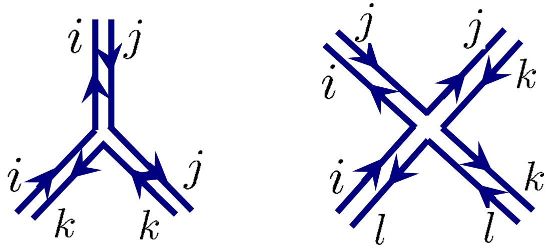
$$\left\langle \chi^{ij}(y)\chi^{kl}(x)\right\rangle_0 = \delta^{il}\delta^{jk}\, D_\chi(x-y)\delta_{mn}$$



$$i, j, k, l = 1\ldots N_c$$

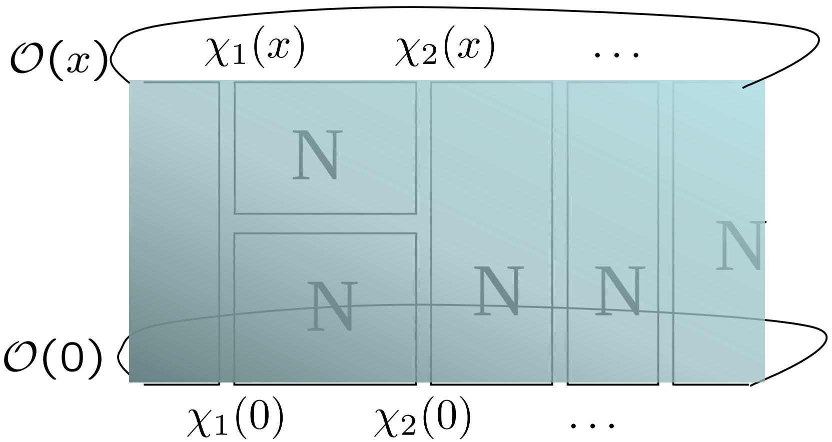
Index is conserved along each line

- Vertices

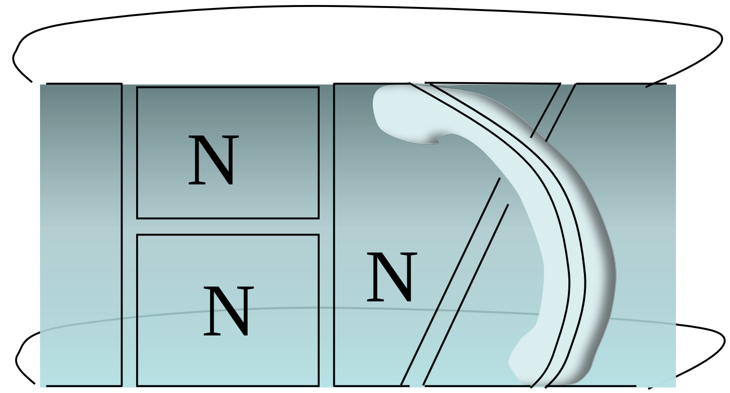


Planar graphs and large-N expansion

- Compare two cylindric graphs for two-point correlators



$(g_{YM}^2 N)^5$



$(g_{YM}^2 N)^5 \times \frac{1}{N^2}$

- The one with bigger number of faces (smaller genus) dominates

Double expansion:

$$\langle\langle O(x)O(0)\rangle\rangle_{N,g} = \sum_{h=0}^{\infty} \frac{1}{N^{2h}} \sum_{m=0}^{\infty} g^{2m} K_m^{(h)}(|x|)$$

Perturbative, in 't Hooft coupling:

$$g^2 = g_{YM}^2 N$$

another frequent notation

$\lambda \equiv 16\pi^2 g^2$

Topological, in string coupling:

$$g_s = \frac{1}{N}$$

- We will always deal with planar quantities (in 't Hooft limit), like $K_m^{(0)}$

Superconformal Symmetry of N=4 SYM

Action obeys superconformal psu(2,2|4) symmetry.
It persists on QM level

Generators of global superconformal psu(2,2|4) symmetry:

$$
\begin{array}{cc}
 & \begin{array}{ccc} \text{Lorentz} & \text{susy} & \text{translations} \end{array} \\
\text{superconf.} & \left(\begin{array}{c|c|c}
L^{\alpha}_{\beta} & \dot{Q}^{a}_{\dot{\alpha}} & P_{\dot{\alpha}\beta} \\ \hline
\dot{S}_{\dot{\alpha}a} & \overset{\text{su}(4)_R}{R^{a}_{b}} & \overset{\text{susy}}{Q_{\alpha a}} \\ \hline
K_{\alpha\dot{\beta}} & S^{a}_{\alpha} & \dot{L}^{\dot{\alpha}}_{\dot{\beta}}
\end{array}\right) \\
 & \begin{array}{ccc} \text{spec. conf.} & \text{superconf.} & \text{Lorentz} \end{array}
\end{array}
$$

$$a, b = 1, \ldots, 4$$
$$\alpha, \dot{\alpha}, \beta, \dot{\beta} = 1, 2$$

Some algebra relations: $\{Q, Q\} = \{S, S\} = \{Q, \dot{S}\} = \{\dot{Q}, S\} = 0$

$$\{Q, \dot{Q}\} = P, \qquad \{S, \dot{S}\} = K$$

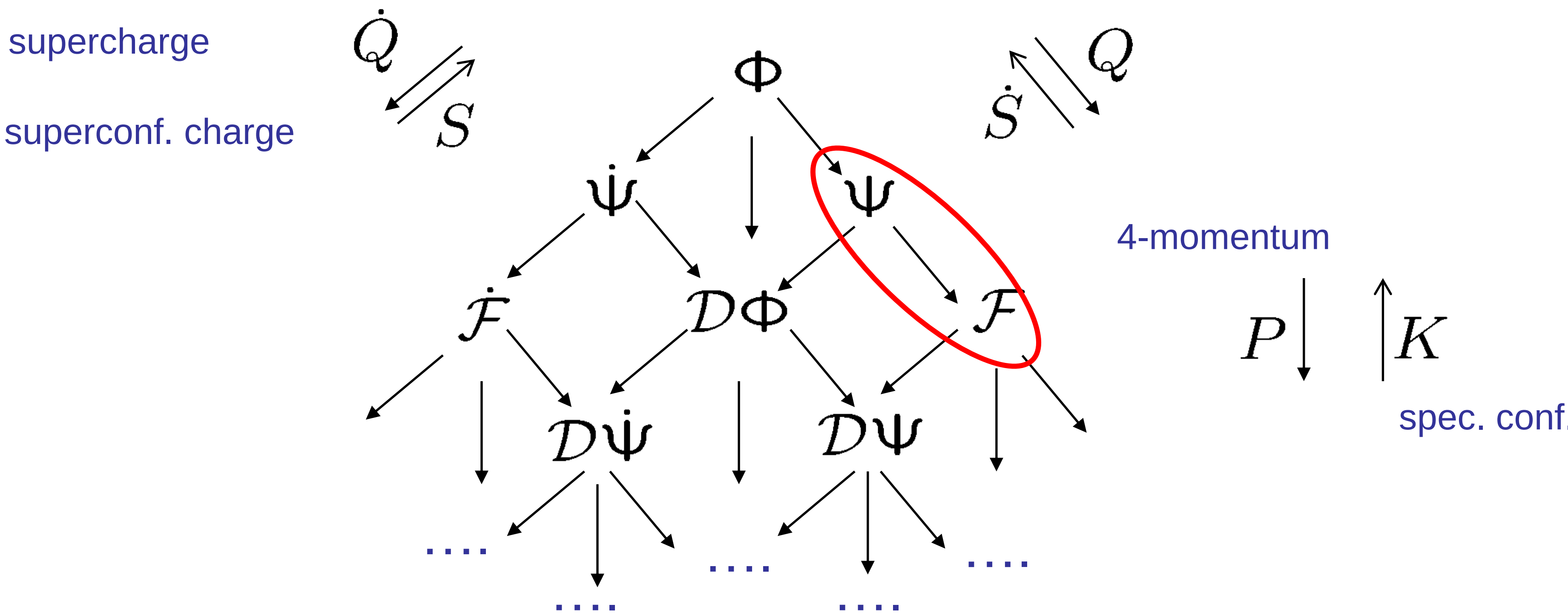
Dilatation operator is a part of algebra! $\{Q, S\} = D + R + L$

$$\{Q, K\} = S, \qquad \{D, Q\} = Q, \text{ etc}$$

Cartan charges: $\{\underset{\text{e.v. of D}}{\Delta}, \underbrace{S_1, S_2}_{\text{Lorentz}} | \underbrace{J_1, J_2, J_3}_{\text{su}(4)_R}\}$

Transformation of fields in supermultiplet

SYM symmetric under global superconformal symmetry: PSU(2,2|4)



Example:

$$Q\,\Psi_{\alpha a} \;=\; -\epsilon_a^{\gamma}\mathcal{F}_{\alpha\gamma} + \frac{i}{2}\sqrt{\lambda}\varepsilon_{\alpha\beta}\epsilon_c^{\beta}\left[\Phi_{ab},\Phi^{bc}\right]$$

Operators and superalgebra at zero loop order

psu(2,2|4) superalgebra

$$\{E_{mn}, E_{kl}\} = \delta_{nk} E_{ml} - (-1)^{(p_m+p_n)(p_k+p_l)} \delta_{ml} E_{kn}$$

$$m, n \in \{\underbrace{1, 2}_{\text{Lorentz}}, \underbrace{\dot{1}, \dot{2}}, \underbrace{\hat{1}, \hat{2}, \hat{3}, \hat{4}}_{\text{su(4)}_R}\}$$

$$E^\dagger_{mn} = (-1)^{c_m + c_n} E_{nm}$$

	α	$\dot{\alpha}$	$\hat{\alpha}$
p	0	0	1
c	0	1	0

Projectivity:

$$C = \sum_n E_{nn} = 0 \implies C = -2 - n_b + n_f + n_a = 0$$

Super-unimodularity:

$$\operatorname{Str} E = \sum_n (-1)^F E_{nn} = 0$$

Oscillator formalism for psu(2,2|4) superalgebra

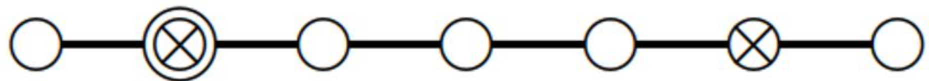
Oscillators:

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}, \quad \alpha, \beta \in \{1, 2\},$$

$$[b_{\dot\alpha}, b_{\dot\beta}^\dagger] = \delta_{\dot\alpha\dot\beta}, \quad \dot\alpha, \dot\beta \in \{\dot 1, \dot 2\},$$

$$\{f_a, f_b^\dagger\} = \delta_{ab}, \quad \hat a, \hat b \in \{\hat 1, \hat 2, \hat 3, \hat 4\}$$

Dynkin diagram



Generators (in « beauty » grading):

$$E_{mn} = \left(\begin{array}{c|c|c} -b_{\dot\alpha} b_{\dot\beta}^\dagger & -b_{\dot\alpha} f_b & -b_{\dot\alpha} a_\beta \\ \hline f_a^\dagger b_{\dot\beta}^\dagger & f_a^\dagger f_b & f_a^\dagger a_\beta \\ \hline a_\alpha^\dagger b_{\dot\beta}^\dagger & a_\alpha^\dagger f_b & a_\alpha^\dagger a_\beta \end{array}\right)$$

Bosonic generators: $E_{\alpha\beta} = a_\alpha a_\beta^\dagger, \quad E_{\dot\alpha\dot\beta} = -b_{\dot\alpha} b_{\dot\beta}^\dagger, \quad E_{\hat a \hat b} = f_{\hat a} f_{\hat b}^\dagger$

Supercharges: $E_{\dot\alpha b} = -b_{\dot\alpha} f_b, \quad \text{etc}$

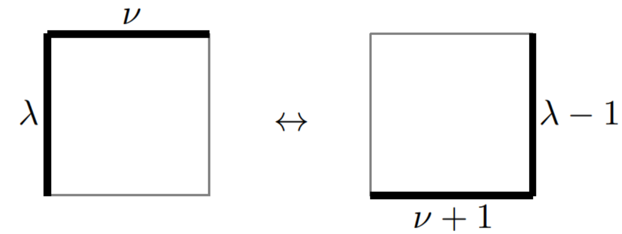
Number of fermions of type « i »: $n_f \equiv \sum_{i=1}^{4} n_{fi}$, of type « i »: $n_{fi} \equiv f_i^\dagger f_i$

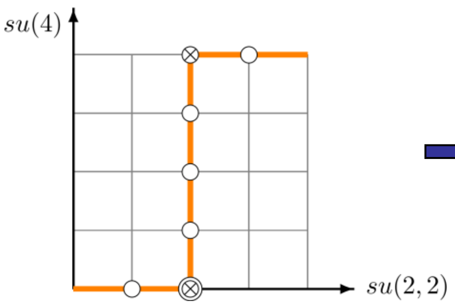
Fermionic duality and Dynkin labels

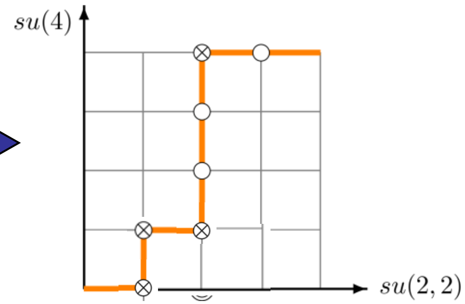
- Various gradings: Duality interchanges odd generators:

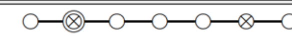
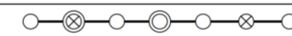
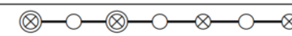
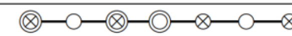
$$\begin{pmatrix} E_{mm} & E_{mn} \\ E_{nm} & E_{nn} \end{pmatrix} \leftrightarrow \begin{pmatrix} E_{nn} & E_{nm} \\ E_{mn} & E_{mm} \end{pmatrix} \qquad p_m + p_n = 1$$

e.g. $\quad E_{mn} = -b_\alpha f_a \;\rightarrow\; E_{nm} = b^\dagger_\alpha f^\dagger_a$







Name	grading	short-hand notation	Dynkin diagram
compact beauty	$12\hat{1}\hat{2}\hat{3}\hat{4}34$	2222	
non-compact beauty	$\hat{1}\hat{2}1234\hat{3}\hat{4}$	0044	
compact ABA	$\hat{1}12\hat{2}\hat{3}34\hat{4}$	0224	
non-compact ABA	$1\hat{1}\hat{2}23\hat{3}\hat{4}4$	1133	

- Dynkin labels:

su(4) $\qquad \lambda_a = f^\dagger_a f_a = n_{f_a}\,, \quad a = 1, \ldots, 4$

su(2,2) $\qquad \nu_i = \left\{ -b_{\dot\alpha} b^\dagger_{\dot\alpha}, a^\dagger_\alpha a_\alpha \right\}_i = \{-L - n_{b_{\dot\alpha}}, n_{a_\alpha}\}_i\,, \quad i = 1, \ldots, 4$

Fields and operators from oscillators

Bare vacuum

$$a_\alpha|0\rangle = b_{\dot\alpha}|0\rangle = f_a|0\rangle = 0$$

Hignest weight state

$$E_{mn}|\mathrm{HWS}\rangle = 0 \quad \text{for } m < n$$

Operator fixed by oscillator numbers

$$[n_{b_1}, n_{b_2} \mid n_{f_1}, n_{f_2}, n_{f_3}, n_{f_4} \mid n_{a_1}, n_{a_2}]$$

Length and bare dimension of operator (conserved only for g=0)

$$2L = n_a - n_b + n_f, \qquad \Delta_0 = \frac{n_f}{2} + n_a$$

Field interpretation		Content	Δ_0	Components
scalar	Φ_{ab}	$\mathbf{f}_a^\dagger \mathbf{f}_b^\dagger \|0\rangle$	1	6
fermion	$\Psi_{a\alpha}$	$\mathbf{f}_a^\dagger \mathbf{a}_\alpha^\dagger \|0\rangle$	$\frac{3}{2}$	8
	$\bar\Psi_{a\dot\alpha}$	$\epsilon_{abcd}\mathbf{f}_b^\dagger \mathbf{f}_c^\dagger \mathbf{f}_d^\dagger \mathbf{b}_{\dot\alpha}^\dagger \|0\rangle$	$\frac{3}{2}$	8
field strength	$\mathcal{F}_{\alpha\beta}$	$\mathbf{a}_\alpha^\dagger \mathbf{a}_\beta^\dagger \|0\rangle$	2	3
	$\bar{\mathcal{F}}_{\dot\alpha\dot\beta}$	$\mathbf{f}_1^\dagger \mathbf{f}_2^\dagger \mathbf{f}_3^\dagger \mathbf{f}_4^\dagger \mathbf{b}_{\dot\alpha}^\dagger \mathbf{b}_{\dot\beta}^\dagger \|0\rangle$	2	3
covariant derivative	$\mathcal{D}_{\alpha\dot\alpha}$	$\mathbf{a}_\alpha^\dagger \mathbf{b}_{\dot\alpha}^\dagger$	1	4

Example of operator

$$\mathrm{Tr}\,[\mathcal{Z}\mathcal{D}_{12}\Psi_{11}\mathcal{F}_{12}] = f_1^\dagger f_2^\dagger|0\rangle \otimes (a_1^\dagger)^2 b_2^\dagger f_1^\dagger|0\rangle \otimes a_1^\dagger a_2^\dagger|0\rangle$$

Super-Multiplets

Superconformal multiplet – a set of operators (irreps) related by global psu(2,2|4) symmetry, all having the same anomalous dimension. Contains a few conformal primaries: operators annihilated by

$$K_{\alpha,\dot{\alpha}} \sim a_\alpha \, a_{\dot{\alpha}}$$

One of them is the HWS. Which one – depends on the grading. Standart choice is the chiral primary having the lowest bare (classical) dimension

Examples

$$\begin{tikzcd} \mathcal{Z}\mathcal{Z}\mathcal{Y}\mathcal{Y} \arrow[r, shift left] & \mathcal{Z}\mathcal{Z}\mathcal{X}\mathcal{Y} \arrow[l, shift left] \arrow[r, shift left, "\mathbf{f}^\dagger\mathbf{f}"] & \mathcal{Z}\mathcal{Z}\mathcal{X}\mathcal{X} \arrow[l, shift left] \arrow[r, shift left, "\mathbf{a}^\dagger\mathbf{b}^\dagger"] \arrow[d, shift left, "\mathbf{a}^\dagger\mathbf{f}"] & \mathcal{D}\mathcal{Z}\mathcal{Z}\mathcal{X}\mathcal{X} \arrow[l, shift left] \arrow[r, shift left] & \mathcal{D}^2\mathcal{Z}\mathcal{Z}\mathcal{X}\mathcal{X} \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \\ & & \mathcal{Z}\mathcal{Z}\mathcal{X}\Psi \arrow[u, shift left] & & & \end{tikzcd}$$

1/2BPS multiplet (beauty grading): $|\mathrm{HWS}\rangle_{\mathrm{BPS}} = \mathrm{Tr} Z^L \, , \quad n = [0,0|L,L,0,0|0,0]$

More general expression: $|\mathrm{HWS}\rangle_{1/2\mathrm{BPS}} = \mathrm{Tr}(n_m \Phi^m)^L \, , \quad n_m n^m = 0$

psu(2,2|4) Cartan charges: $(\Delta, S_1, S_2 | J_1, J_2, J_3)$

$$\mathrm{Tr} Z^L \, , \quad (L,0,0|L,0,0)$$
$$\mathrm{Tr} X^L \, , \quad (L,0,0|0,L,0)$$
$$\mathrm{Tr} Y^L \, , \quad (L,0,0|0,0,L)$$

Conserves half of the susy. Conformal dimension at any coupling Δ=L

Dynkin labels at finite coupling

Anomalous dimension $\gamma = \Delta - \Delta_0$

Shift of Dynkin labels for conformal group

$$\nu_i = \nu_i\big|_{g=0} + \frac{\gamma}{2}\{-1,-1,1,1\}_i$$

Operators with different lengths can mix
– length is not a conserved charge at finite coupling!

$$\left.\begin{array}{l} \{L, n_{b_\alpha}\} \leftrightarrow \{L-1, n_{b_\alpha}+1\} \\ \{L, n_{f_\alpha}, n_{a_\alpha}\} \leftrightarrow \{L-1, n_{f_\alpha}-1, n_{a_\alpha}+1\} \end{array}\right\} \quad \lambda_a \leftrightarrow \lambda_a - 1 \text{ and } \nu_i \leftrightarrow \nu_i + 1$$

Examples

n^{2222}	L	Field content example	λ_a	ν_j
$[1,1\|2,2,2,2\|1,1]$	4	$\Psi_{11}\Psi_{12}\bar{\Psi}_{11}\bar{\Psi}_{12}$	$\{2,2,2,2\}$	$\{-5,-5,1,1\}$
$[0,0\|2,2,2,2\|1,1]$	5	$\Psi_{11}\Psi_{12}\bar{\mathcal{Z}}\bar{\mathcal{X}}\bar{\mathcal{Y}}$	$\{2,2,2,2\}$	$\{-5,-5,1,1\}$
$[1,1\|3,3,3,3\|0,0]$	5	$\bar{\Psi}_{11}\bar{\Psi}_{12}\mathcal{Z}\mathcal{X}\mathcal{Y}$	$\{3,3,3,3\}$	$\{-6,-6,0,0\}$

Some conclusions to lecture I

- N=4 SYM obeys a superconformal symmetry PSU(2,2|4), uniting the Poincare and conformal bosonic subgroups
- Local operators form conformal supermultiplets, with the same anomalous dimension.
- In large N limit N=4 SYM single trace operators form a complete set
- Spectrum of conformal dimensions of these operators is encoded in the dilatation operator D(g). D is a part of superconformal algebra.
- At g=0 the generators of PSU(2,2|4) can be efficiently described via the oscillator algebra. Length L and bare (classical) Δ_0 dimension conserve
- At $g\neq 0$ the susy generators and D (and thus Δ) depend non-trivially on g
- In the next lecture we will compute D perturbatively and diagonalize it using quantum integrability

References:

- J. Minahan, https://arxiv.org/pdf/1012.3983.pdf
- N. Beisert, https://arxiv.org/pdf/hep-th/0407277(thesis, chapt.1,2)
- C. Marboe & D. Volin, https://arxiv.org/pdf/1701.03704