Project1: Constrained motion on surfaces

The bob mass in the simple pendulum is attached to a light string of fixed length that is fixed at the other end to a certain support point. Therefore, the motion of the bob mass resembles a circular arc of radius equal to the length of the string.

One can alternatively reproduce the motion of the bob mass by allowing it to slide on a circular vertical hoop of radius equal to the string length. The normal force from the hoop plays the role of the tension in the string.

Although the motion appears to take place in a two dimensional plane, the Cartesian coordinates of the bob mass are related to each other since the path is restricted to be circular. Therefore, one can describe the position of the mass by a single parameter θ:

$$x = \ell \sin\theta$$
$$z = -\ell \cos\theta$$

where we have assumed that the centre of the circular hoop (or the support point of the string) is at the origin.

The equation of motion of the parameter θ is

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

where friction is neglected.

While the system above has been well studied in the class, one can investigate the motion of the mass along other curves (which you will do in the first part of the project). One can also generalize the motion to be along two dimensional surfaces (which you will do in the second part of the project). Details are given below.

Part I

In this part of the project, you will consider two different curves. The first is a parabolic path lying in the xz plane (gravity is along the $-z$ direction) which can be described by

$$z = \frac{1}{2}\left(\frac{x^2}{\ell} - \ell\right) \tag{1}$$

where ℓ is a parameter characterizing the shape and the size of the parabola. Similarly, since the motion is constrained along the parabolic curve, a single parameter θ is enough to describe the location of the particle along the path. One can parametrize the parabolic curve in Eq. 1 as

$$x = \ell\frac{\sin\theta}{1+\cos\theta} \tag{2}$$
$$z = -\ell\frac{\cos\theta}{1+\cos\theta} \tag{3}$$

where $-\pi < \theta < \pi$.

The corresponding equation of motion of the parameter θ can be shown to read

$$\ddot{\theta} = -\frac{g}{2\ell} \sin\theta \,(1 + \cos\theta) - \frac{3\dot{\theta}^2 \sin\theta}{2\,(1 + \cos\theta)} \tag{4}$$

The second curve that you will investigate is, again, parametrized by θ, whose equation of motion is described by

$$\ddot{\theta} = \frac{\sin\theta \left(\dot{\theta}^2 - \frac{g}{\ell}\right)}{2\,(1 + \cos\theta)} \tag{5}$$

and $-\pi < \theta < \pi$.

In this part, you should perform a comprehensive analysis to the motion of the particle described by equations 3 and 5. The corresponding curves are referred to as $C_1(\ell)$ and $C_2(\ell)$, respectively. The investigations involve the following:

1. Confirm that the particle undergoes an oscillatory motion on both curves. Determine the equilibrium point(s) and find the properties of the motion.
2. For fixed value of ℓ, investigate the dependence of the period of the oscillation on the oscillations amplitude.
3. For fixed value of amplitude, investigate the dependence of the period of the oscillation on the parameter ℓ.
4. **(Optional):** Combine the analysis of points 2 and 3 above and provide a full analysis of how the period changes with ℓ or with the amplitude. Your goal is to come up with a formula that takes in the values of amplitude and ℓ and gives back the corresponding period.
5. Identify the shape of C_2.
6. Update equations 3 and 5 by adding a damping term (of strength q) and a sinusoidal driving force of strength F_d and frequency ω_d. Investigate the motion and see if they can show a chaotic behaviour.
7. Produce beautiful and informative animations for the motion of the mass along the mentioned curves at different conditions.

Part II

In the second part of the project, you will investigate the motion of the mass along a spherical surface. The motion mimics the one of the so-called spherical pendulum. Unlike the first part, the particle motion must be described using two degrees of freedom (θ, φ). These can be chosen by noting the relation between the particle Cartesian coordinates x, y and z:

$$
\begin{aligned}
x &= \sin\theta\cos\varphi \\
y &= \sin\theta\sin\varphi \\
z &= -\cos\theta
\end{aligned}
$$

where $0 \leq \theta < \pi$ and $0 \leq \varphi < 2\pi$.

Again, taking the gravity to be pointing along the $-z$ direction, the equations of motion of θ and φ can be expressed as

$$\ddot{\theta} = \dot{\varphi}^2 \sin\theta\cos\theta - \frac{g}{l}\sin\theta \tag{6}$$

$$\ddot{\varphi} = -\frac{2\dot{\theta}\dot{\varphi}\cos\theta}{\sin\theta} \tag{7}$$

assuming that the surface is frictionless. In this part, you should investigate the motion of the particle on the given surface taking into account the points below:

1. Describe the motion on the sphere by starting from different initial conditions.
2. Confirm that you can reproduce the motion on the circular path in the first part of the project by selecting proper initial conditions.
3. Come up with initial conditions that give rise to a closed-path curve of motion along the surface.
4. Obtain different Lissajous-like figures for the motion on the spherical surfaces.
5. Check if the motion can show a chaotic behaviour.
6. Consider only one of the following ideas to investigate. The rest are optional.
   (a) Study the motion of two point-like masses on the spherical surface interacting by hard-sphere collisions.
   (b) Update the equations of motion to properly add damping effects and study the consequences on the motion.
   (c) Investigate the motion on a different surface (or surfaces). Note that this requires you to carefully choose the degrees of freedom and derive the corresponding equations of motion.
   (d) Ask your own question: Think about a certain aspect of this problem or a similar problem and investigate it.
7. In all cases, produce selected beautiful and informative animations for the motion of the mass along the surface.

General Notes

- In all relevant cases given above, you need to confirm that the energy is conserved. You should also check the conservation of angular momentum about the z axis for the motion on the spherical surface.
- Use the 4^{th} order Runge-Kutta algorithm (RK4) to integrate all equations of motion in this project. You can also use Euler or Euler-Cromer method to compare with and give your conclusions. A separate document will be provided that briefly explains the RK4 method.
- All results should be supported by proper discussion and analysis. These should be included in your written report.

Enjoy!