**Theory Behind the Stroh Class**

The atomman.defect.Stroh class solves the anisotropic elasticity theory for a continuum defect that is infinitely long and periodic along the z-axis such that the stress, strain and displacements are invariant in that direction. The solution was first developed by Eshelby, et al. [1]. Shortly after, Stroh [2, 3] explored the solution in-depth and showed how the problem can be rewritten in an easier to solve form.

1. **Theory**

Starting with the fundamental equations of elasticity

Compatibility:

 1.1

Strain definition:

 1.2

Force equilibrium:

 1.3

Combining these equations and using the symmetries of Cijkl reveals the partial differential equation (PDE)

. 1.4

The infinite periodicity of the defect along the z-axis makes the stress, strain, and displacements invariant along the z direction. Accordingly, . With this, a solution for Eq. 1.4 is obtained

. 1.5

Substituting the solution back into 1.4 gives

, 1.6

or more simply

, 1.7

or

. 1.8

In order for *Ak* to be non-zero, the determinate of *aik* must be zero. The determinate expression is a sixth-order polynomial expression, giving it six complex roots, *pn* with *n* = 1,2,3,4,5,6. Since the coefficients of the determinate polynomial are real, the roots *pn* come in complex conjugate pairs.

With this knowledge, the expression for *uk* can be written in one of two forms:

Eshelby: For, then

 1.9a

Stroh: For having positive imaginary terms and, then

 1.9b

Stroh also defines a stress function vector, *ϕ*, such that

 1.10

Now, we can find

 1.11

where

. 1.12

Additionally,

 1.13

Going back to the displacement expression,

 1.14

consider the functions *fα*. The requirements for *fα* are that it allow for *uk* to be multivalued such that a complete circuit around the defect results in a Δ*uk* = *bk*, while the stresses remain single valued and continuous except at the origin. The most general solution for this is

 1.15

The ln term gives *f* a change of ±D for every revolution (where the ‘cut’ is oriented in the +x direction), while the other terms give single valued displacements, which do not apply/matter for an infinitely straight periodic dislocation. So,

 1.16

and

. 1.17

 1.18

The *Dα* terms are imaginary can be solved by considering the net force on the dislocation.

From Stroh, if the singularity is subsonic, then it can be shown that

. 1.19

Considering only a stationary dislocation, *Fi* = 0, so

 1.20

Defining

 1.21

. 1.22

With this, the displacements can be calculated by knowing *pα*, *Akα*, and *Lkα*.

1. **Finding solutions**

Obtaining values for *pα*, *Akα*, and *Lkα* is not a simple task due to the sixth order complex roots. Stroh introduced an idea to create a 6x6 matrix such that the problem can be treated as an eigenvalue expression instead of as a polynomial. Various forms for the eigenvalue expression exist, and atomman.defect.Stroh uses the form listed in Hirth and Lothe [4].

Start by defining three orthogonal vectors . In general, these three vectors can be any orientation, but our analysis need only consider , , and . This definition also makes direct comparisons with the previous section possible.

Given two arbitrary vectors  and , use the notation  to indicate a matrix given by:

 2.1

Now, combining this with the previous analysis

 2.2

Equation 1.7 becomes

 2.3

And equation 1.12 becomes

 2.4

From this, the problem is rewritten as a six dimensional eigenequation:

 for *r*, *s* = 1,2,3,4,5,6 2.5

Where *N* is a 6x6 matrix composed of joining three 3x3 component blocks, *NA, NB, NC,* and *ND*, and  is a six dimensional vector with components of *A* and *L.* Written out:

 2.6

The four component blocks that satisfy Eqs. 2.3 and 2.4 are:

 2.7a

 2.7b

 2.7c

 2.7d

The eigenvalues and eigenvectors of 2.6 directly give values for *pα*, *Akα*, and *Lkα*.

1. **Additional Useful Expressions**

Stress:



Elastic energy per unit length in circular ring with inner and outer radii *r*0 and *R*:



Where is the pre-ln energy factor



and *Gij* is the energy coefficient tensor



A number of consistency checks exist that be used to verify that the eigenvalues and vectors have been properly calculated:















[1] J.D. Eshelby, W.T. Read, W. Shockley, Acta Metall Mater, 1 (1953) 251-259.

[2] A.N. Stroh, Philosophical Magazine, 3 (1958) 625-&.

[3] A.N. Stroh, J Math Phys Camb, 41 (1962) 77-&.

[4] J.P. Hirth, J. Lothe, Theory of Dislocations, 2nd ed., Wiley, New York, 1982.