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New Identities for F_{2n} and F_{2n+1}

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Abstract

In this study, we obtain a new identities for Fibonacci numbers F_{2n} and F_{2n+1}, where $n \geq 1$.

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1 Introduction

It is well-known that the Fibonacci numbers are given the recurrence relation

$$F_{n+1} = F_n + F_{n-1}, \qquad \textit{where} \qquad n \geq 1 \tag{1}$$

with the initial conditions $F_0 = 0$, $F_1 = 1$. There are a lot of identities about the Fibonacci numbers. We obtain a new identities for even and odd Fibonacci numbers.

2 A New Identities for F_{2n} and F_{2n+1}

Theorem 2.1 $F_{2n} = F_n^2 + 2F_nF_{n-1}$ *, where* $n \geq 1$.
Proof. *We write the Fibonacci recurrence relation*

$$F_{n+1} = F_n + F_{n-1}. \tag{2}$$

By squaring of the each side of equation (2), we get

$$F_{n+1}^2 = F_n^2 + 2F_nF_{n-1} + F_{n-1}^2, \tag{3}$$

that is

$$F_{n+1}^2 - F_{n-1}^2 = F_n^2 + 2F_nF_{n-1}. \tag{4}$$

Obtained by setting m=n in equation $F_{m+n} = F_{m+1}F_{n+1} - F_{m-1}F_{n-1}$ *(Mana, 1969) is*

$$F_{2n} = F_{n+1}^2 + F_{n-1}^2. \tag{5}$$

From equalities (4) and (5), it is clearly seen that

$$F_{2n} = F_n^2 + 2F_nF_{n-1}. \tag{6}$$

So the theorem is proved.

Theorem 2.2

$$F_{2n+1} = F_{n-1}^2 + 2F_nF_{n+1}, \quad where \quad n \geq 1. \tag{7}$$

Proof. *Proving the identity is to apply the principle of mathematical induction (PMI). Since*

$$F_3 = 2F_1F_2 + F_0^2 = 2.1.1 + 0 = 2, \tag{8}$$

the given identity is true when $n = 1$. *Now we assuma that it is true for arbitrary positive integer k:*

$$F_{2k+1} = F_{k-1}^2 + 2F_kF_{k+1} \tag{9}$$

Then, we write

$$F_k^2 + 2F_{k+1}F_{k+2} = (F_{k+1} - F_{k-1})^2 + 2F_{k+1}F_{k+2} \tag{10}$$

$$= F_{k+1}^2 - 2F_{k+1}F_{k-1} + F_{k-1}^2 + 2F_{k+1}(F_{k+1} + F_k \tag{11}$$

$$= 3F_{k+1}^2 - 2F_{k+1}F_{k-1} + F_{2k+1} \tag{12}$$

$$= F_{k+1}^2 + 2F_{k+1}(F_{k+1} - F_{k-1}) + F_{2k+1} \tag{13}$$

$$= F_{k+1}^2 + 2F_{k+1}F_k + F_{2k+1}. \tag{14}$$

Using the Theorem 1, we obtain

$$F_k^2 + 2F_{k+1}F_{k+2} = F_{2k+1} + F_{2k+2} = F_{2k+3}. \tag{15}$$

Thus the formula is true for $n = k + 1$. *So, by PMI, the identity is true for every integer* $n \geq 1$. *Since by using Fibonacci numbers, Fibonacci numbers with negative indices can be written as*

$$F_{-n} = (-1)^{n+1}F_n, n \geq 1. \tag{16}$$

we conclude the following result.

Theorem 2.3 *For* $n \geq 1$,

$$F_{-2n} = F_n^2 - 2F_nF_{n+1}F_{-2n+1} = F_{n+1}^2 - 2F_nF_{n-1}. \tag{17}$$

References

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