# The Fast Fourier and Computational Analysis

Asis A Sotelo
*Physics Department, University of California, Santa Barbara, CA 93106*
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Leonard Euler invented integral transforms in order to solve second-order differential equations. The creation of Euler's worke eventually led to the transforms we know today. Especifically the Fourier Transform is of the most powerful and widely utilized mathematical concepts in numerical analysis. Named after Jean-Baptiste Joseph Fourier (1768-1830), the Fourier transform is the extension of his Fourier Series which he developed in order to solve the heat equations [5]. The Fourier Transform is now applied in a broad range of fields including electrical engineering, acoustics, and physics. Its use data analysis is what we will explore in this paper. To highlight its functionality we will consider an example in the field of mathematical finance. In particular, the analysis of stock price fluctuations over time.

## I. INTRODUCTION

Implementation of the Fourier Transform in modern data analysis has been around since the creation of the modern Fast Fourier Transform (FFT) in 1965. This algorithm created by James Cooley and John Tukey created a new tool for data scientists implement. Its use in physics is widespread as it allows one to take a difficult problem in one space, say t-space, transform it to k-space, where it might be much easier to solve and then invert the transformation to get a solution in the original space [4]. Specifically, the importance of the Fourier Transform is remarked upon by Shitoshna Nepa in his article "Signal Processing" [3]. Nepa explains how the composition of a "noisy" signal, where the signal can have various frequencies along its spectrum, can be decomposed by the application of the Fourier transform.

This identification of distinct frequencies in a signal allows for a better description of the signal's information. From this decomposition we can distill the signal to its most important data contributions while removing the "noise".

In this paper we will begin by defining some financial definitions that will be used throughout the paper. Although this information is not needed to understand the overall method, it provides a background to the abstract mathematics. Next we demonstrate the standard method to find the characteristic behavior or best curve fit. Then, we take the line of best fit and take the fluctuations between the values that are larger than the fit and lower than the fit. The higher the polynomial fit the more oscillations across the fit line occur. We then take these oscillations under a certain value to find a very accurate representation of the value without noise. [8].

### A. Definition of Terms

A stock is simply a percentage of ownership of a company that a member of the market may buy and is traded in a stock exchange. What is a market? The market is simply a place where two parties can buy and sell this percentages of ownership. The stock exchange is an intuition created to facilitate the legalities associated with these trades. The price of these stock fluctuates as the company performs well, its price will go up, or performs poorly, its price will go down. This fluctuation in price is tracked by the milliseconds and contains many underlying complex factors that are important to anyone within the company or who owns the stock.

What interests us is that this is a time series that contains within it rapid cyclical fluctuations. Fluctuations that can be analyzed utilizing the Fourier Transform.

To define the Fourier Transform we first begin with the Fourier Series. According to Boas [1] the Fourier Transform is an expansion in a series of sines and cosines:

$$F(x) = a_0 + a_1 cosx + a_2 cos2x... \tag{1}$$

By using the orthogonality of sine and cosine we can then find coefficients $a_n$ and $b_n$. We can then expand this to the complex form of the Fourier Series which looks like this:

$$f(x) = c_o + c_1 e^{ix} + c_{-1} e^{-ix} ... = \sum_{n=-\infty}^{n=+\infty} c_n e^{inx} \tag{2}$$

From this we can use the average of this function over its period $2\pi$ to find the coefficients n:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \tag{3}$$

We can then make this work for any arbitrary period $[-l, l]$ and thus changing the x in the original equation too $\frac{\pi x}{l}$. Allowing this length $l$ to from $-\infty$ to $\infty$ we have the general representation of the Fourier Expansion of a function:

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{n\pi x}{l}} \tag{4}$$

Now that we have defined the discrete Fourier Series we can define this formula for a continuous range of functions [1].

$$f(x) = \int_{-\infty}^{\infty} g(k) e^{ikx} dk \tag{5}$$

$$g(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \tag{6}$$

Having these two functions we can return to the signal. Any given signal in financial analytic may be thought as containing two aspects: the observed price signal and the noise around the signal. Our goal is to separate the noise from the observed signal and extract the actual signal of the data so that we may use this data to make more accurate predictions of future time.

### B. Linear Regression and Polynomial Curve Fitting

Linear regression and curve fitting is used in data analysis to identify the relationship between a dependent and independent variable. In our example the stock price is our dependent and our time is the independent. As shown in Fig I B after plotting the scattered data point one can see the pattern in the data. Hence linear regression takes all of the points and finds a line through the data that minimizes error by making a line that is as closest to as many points as possible. We do this by minimizing what is known as the error function.

Least Square Method

$$\sum_{i=1}^{n} r_i^2 \tag{7}$$

This presents an equation of the line that is of the form $y = cx + d$ FIG. 2, with the c acting as a weight and d is the shift in the y-directions. If we take this to more powers of x we can find what is known as the polynomial regression which is of the form $y = ax^3+bx^2+cx+d$ as shown in Fig II A

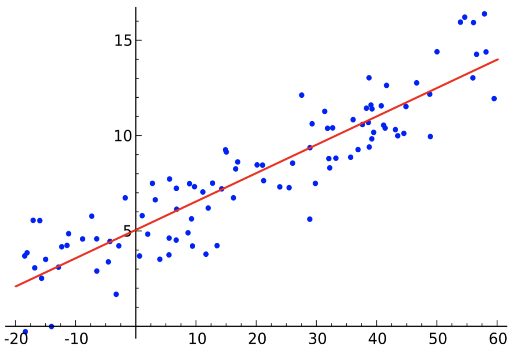

FIG. 1: A linear fit a scatter of plot [7]

### C. The FFT Algorithm

### D. The FFT Algorithm

The following is the form of the FFT algorithm which takes the the method of the Fourier Transform and separates it into discrete values of the frequency N [9].

This form of the Fourier transform will be used and manipulated and implemented using numpy.fft.fft as described in [14], [12], and [3] .

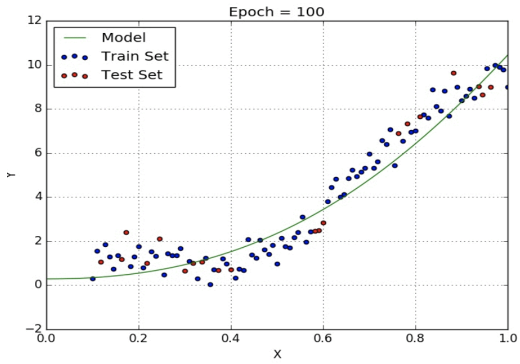


FIG. 2: Polynomial Regression of a scatter plot [7]

*FFT Algorithm*

$$\sum_{n=0}^{\frac{N}{2}-1} a_{2n} e^{\frac{-2\pi(2n)ik}{N}} + \sum_{n=1}^{\frac{N}{2}-1} a_{2n+1} e^{\frac{-2\pi(2n)ik}{N}} \tag{8}$$

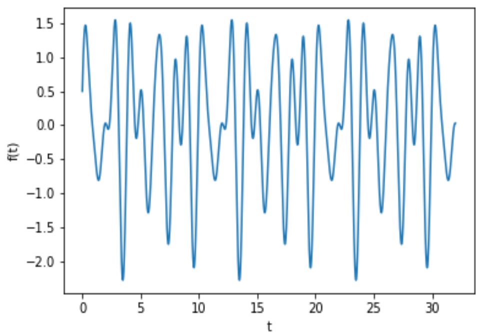


FIG. 3: Plot of A Sinusoidal Function Comprised of 1Hz, .8 Hz, and .5 Hz Signals

## II. MATERIALS AND METHODS

### A. Methods

In order to analyze the data of our example quickly and efficiently we will be using Python implemented with Jupyter Notebook (PDF included at the end of this pa-

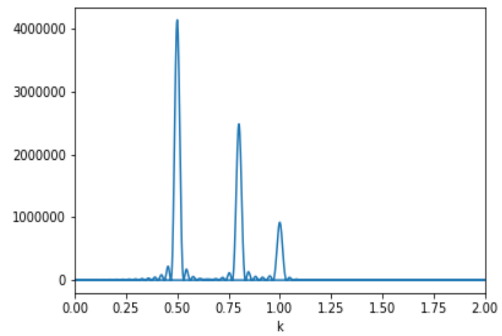


FIG. 4: Fourier Transform of the Sinusoidal Function, one can identify the frequencies that make up the signal.

per).The modules consist of numpy, pandas-datareader, matplotlib.py, datetime, and warnings. We ran all of the code on Python 3.5.3. The implementation of the algorithm and the algorithm in pseudocode can be seen Carr's [3], *Option Valuation using the Fourier Transform*. We used Apple stock (AAPL) from October 1st, 2006 to January 1st, 2012. We then take the polynomial fit and find the difference between the actual price and the fit of the curve. We take this set of data which should resemble an oscillating signal. We then take the FFT of this data and plots its absolute value to identify which frequencies affect the signal the most. We take this data and zero out all frequencies below this threshold. We then take the inverse of that data and and subtract it from the price which generates a much closer fit to the data .

## III. RESULTS AND IMPLICATIONS

The data period that was sampled was taken over 1,323 days and had price mean of \$28.89 throughout the period. The dat set had a minimum of \$10.37 and a maximum value of \$60.95 US dollars. We took in the

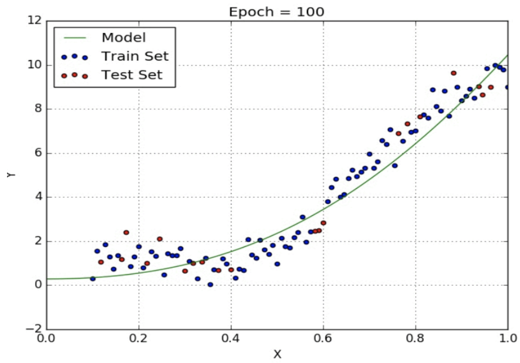


FIG. 5: Polynomial Regression of a scatter plot [7]

data as shown in Fig 6 and could quickly notice a couple of things. the data is trending upwards with a slight dip in 2008 due to the Great Recession and but then continuing its upward trend. There is quite a bit of noise in the signal.


FIG. 6: Stock Price of Apple over 1,323 Days

The polynomial fit of 20 degrees created a very good approximation of the data as shown in Fig. 7. After performing Fourier transform, zeroing out values above the threshold frequency and taking the inverse we got a fit that dampened the noise of the single quite significantly creating a polynomial fit on these values created a polynomial with a much less significant error. The polynomial fit of the graph before transforming was a 91% data fit while after the transformation at the same amount of degrees was within 98.12%.

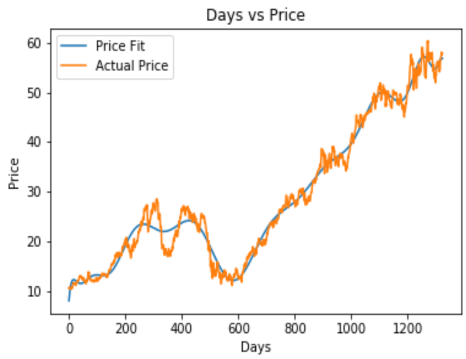


FIG. 7: Polynomial fit of Degree 20 of Price

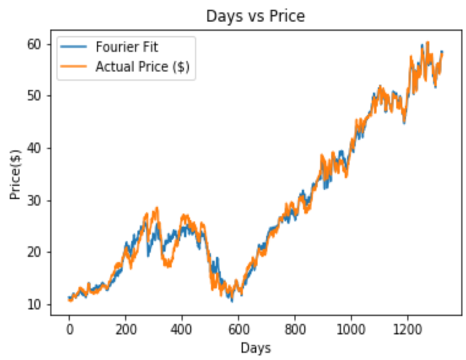


FIG. 8: Plot of the Fourier Fit and Actual Price after Eliminating Frequencies

## IV. DISCUSSION

This small demonstration of the implementation of the Fourier Transform is just one of the many ways it can be used in data analysis to minimize error in our results and in our experiments. This method of refining polynomial curve fits can be carried over to a myriad of scientific fields. Moving forward with this example we can attempt to introduce more underlying variables instead of just time. The

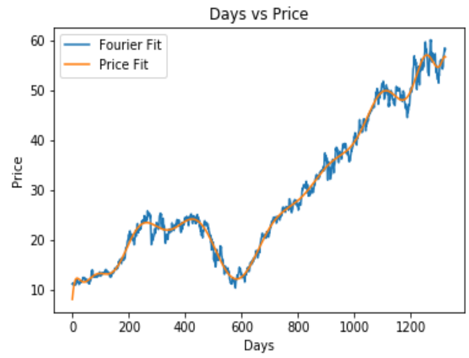


FIG. 9: Plot of the Fourier Fit and the Polynomial Fit after Eliminating Frequencies

price of a stock is much more complicated and factors in prices of other commodities as well as world events. To create a model that factors this in to our price analysis would be the next logical step.

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