PH504M Lab 6: Linear Regression

Vikram Khaire

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Q1. Justifying the Minimum Condition in Least Squares Approximation

In class, we derived the conditions for least squares approximation by setting the partial derivatives.

$$\frac{\partial f}{\partial m} = 0 \quad \text{and} \quad \frac{\partial f}{\partial c} = 0$$

where

$$f = \sum (y_i - (mx_i + c))^2,$$

which ensures an extremum. However, these conditions do not guarantee that we obtain a minimum.

To justify that these conditions lead to a minimum, consider a dataset generated from the linear equation

$$y = 4x + 5$$

where x varies from 0 to 10.

A) Verifying the Minimum for Slope m

- Compute the function f for different values of m ranging from 1 to 7 with an interval of 0.1, while keeping $c = 5$.
- Plot f as a function of m and verify that the minimum occurs at $m = 4$.
- Numerically find the minimum of f.

B) Verifying the Minimum for Intercept c

- Compute f for different values of c ranging from 2 to 8 with an interval of 0.1, while keeping $m = 4$.
- Plot f as a function of c and verify that the minimum occurs at $c = 5$.
- Numerically find the minimum of f.

Write a Python script to perform these computations, generate the plots keeping in mind the dependent and independent variables, and confirm that the least squares conditions indeed give a minimum.

Q2. Simulating the Pale Blue Dot Image Brightness

On **February 14, 1990** (**Valentine's Day, just like today!**), NASA's *Voyager 1* spacecraft took the famous *Pale Blue Dot* image of Earth from a distance of 6×10^9 km. The image was taken at a phase angle of approximately **32 degrees** (the angle between the Sun, Earth, and *Voyager 1*).

Figure 1: NASA's *Voyager 1* *"Pale Blue Dot"* image, taken from 6×10^9 km away.

Computational Task

Write a Python program to simulate how Earth would appear from Voyager 1's position by modeling the brightness of Earth as a function of distance and phase angle using the **Lambertian reflection model**:

$$I = I_0 \cdot \frac{R^2}{d^2} \cdot \cos(\theta)$$

where:

- I is the observed intensity,
- I_0 is the intrinsic brightness of Earth (assume $I_0 = 1$ in arbitrary units),
- R is Earth's radius ($R = 6.37 \times 10^3$ km),
- d is the distance to Voyager 1,
- θ is the phase angle (**32° for Voyager 1**).

Tasks:

1. Write a function to calculate the observed intensity I.
2. Compare how Earth's brightness changes for phase angles **0° (full illumination)**, **32° (Voyager 1's view)**, and **60° (partial illumination)**.
3. Calculate Earth's brightness1's current distance) with an interval of 1 AU). Note that **1 AU** is 1.5×10^8 km (whereas 10 AU is the distance to Saturn).
4. Plot the observed intensity (on a log scale) as a function of distance (in AU) for the three phase angles mentioned above. Highlight the brightness seen by the Voyager 1 in the plot.

This calculation will help you understand how the brightness of a planet changes with distance and viewing angle, just like *Voyager 1*'s view of the *Pale Blue Dot*.

Hint: To improve visualization in your plot, use the following functions:

- `plt.axhline(y, linestyle="--", color="gray")` to draw a horizontal reference line at y.
- `plt.axvline(x, linestyle="--", color="red")` to draw a vertical reference line at x.
- `plt.yscale("log")` to set the y-axis to a logarithmic scale for better readability of intensity values over large distances.