PH504M Lab 3: Series Summation and Root Finding Methods

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Problem 1: Series Problems

A) Approximation of $\cos(x)$

The cosine function can be represented as an infinite series:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

1. Write a function that would sum the series to find $\cos(x)$.
2. Approximate $\cos(\pi/4)$ using the first four terms and compare your result to the actual value of $\cos(\pi/4)$ (use `numpy.cos` function).
3. Discuss how the accuracy improves as more terms are included.

B) Convergence of the Harmonic Series

The harmonic series is defined as:

$$S = \sum_{n=1}^{\infty} \frac{1}{n}.$$

1. Write a function that would sum the series given above.
2. Compute the partial sums S_5 (where you sum first 5 terms), S_{10}, and S_{20} .
3. Verify whether the harmonic series converges or diverges by explaining the behavior of S as $n \to \infty$

Problem 2: Using the False Position Method to Find the Time of Free Fall

Background

The motion of a falling object under gravity can be described by the equation:

$$y(t) = y_0 + v_0 t - \frac{1}{2} g t^2,$$

where:

- $y(t)$ is the height of the object at time t (in meters),
- y_0 is the initial height (in meters),
- v_0 is the initial velocity (in meters per second),
- g is the acceleration due to gravity ($9.8\,\mathrm{m/s^2}$),
- t is the time (in seconds).

When the object hits the ground, $y(t) = 0$. Solving for t gives us the time of free fall, but for complex scenarios or when an analytical solution is cumbersome, numerical methods such as the **False Position Method** are used.

The False Position Method

The False Position (Regula Falsi) method is a bracketing numerical technique to find the root of a function $f(x)$. It starts with an interval $[a, b]$ where the function changes sign ($f(a) \cdot f(b) < 0$). The method then approximates the root by connecting the points $(a, f(a))$ and $(b, f(b))$ with a straight line and finding where this line crosses the x-axis. The root is iteratively refined until it meets a desired tolerance.

The formula for the root estimate is:

$$x_r = b - \frac{f(b) \cdot (b - a)}{f(b) - f(a)}.$$

Steps of the False Position Method

Step 1: Initial Setup

1. Define the function $f(x)$ for which the root is to be found.
2. Choose an initial interval $[a, b]$ such that $f(a) \cdot f(b) < 0$. This ensures that a root lies within the interval (based on the Intermediate Value Theorem).
3. If $f(a) \cdot f(b) \geq 0$, the method cannot proceed. Choose a different interval.

Step 2: Compute the Root Estimate

1. Approximate the root x_r between $(a, f(a))$ and $(b, f(b))$:

$$x_r = b - \frac{f(b) \cdot (b - a)}{f(b) - f(a)}.$$

2. This formula finds the point where the straight line connecting $(a, f(a))$ and $(b, f(b))$ crosses the x-axis.

Step 3: Check the Root

1. Evaluate $f(x_r)$:
    - If $f(x_r) = 0$, x_r is the root, and the algorithm stops.
    - Otherwise, proceed to the next step.

Step 4: Update the Interval

1. Determine which subinterval contains the root:
    - If $f(a) \cdot f(x_r) < 0$, the root lies in $[a, x_r]$. Update $b = x_r$.
    - If $f(b) \cdot f(x_r) < 0$, the root lies in $[x_r, b]$. Update $a = x_r$.
    - If $f(a) \cdot f(b) > 0$, there is no guarantee of a root. Stop and recheck your initial guesses.

Problem

An object is dropped from a height of $y_0 = 80\,\mathrm{m}$ with an initial downward velocity of $v_0 = 5\,\mathrm{m/s}$. Assuming no air resistance, use the False Position Method to find the time t it takes for the object to hit the ground.

1. Write a function that uses false position methods to find the root of the equation given above to find time t for the object to hit the ground. (Hint: Choose initial guesses for t (e.g., $a = 0\,\mathrm{s}$ and $b = 5\,\mathrm{s}$) where $f(a) \cdot f(b) < 0$.)

2. Stop if the error is within $0.01\,\mathrm{s}$.

3. Use the function that you wrote for the bisection method in class to find the solution. Feel free to copy that function from the class notes.

Hints

- Substitute $y_0 = 80$, $v_0 = 5$, and $g = 9.8$ into the equation to get $f(t)$.
- Verify $f(a) \cdot f(b) < 0$ before proceeding.
- For error, use:

$$\text{Error} = \left| \frac{t_{\text{new}} - t_{\text{old}}}{t_{\text{new}}} \right| \times 100\%.$$