PH504M Lab 9 (Assignment): System of linear equations & Integration

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Problem 1: Kirchhoff's Circuit

Consider the Kirchhoff's circuit below, which consists of three resistors and two voltage sources. The currents I_1, I_2, and I_3 flow through different branches of the circuit.

$$\begin{aligned} -10I_2 + 20I_3 &= 5 \\ 10I_1 + 5I_2 &= 20 \\ 5I_1 + 15I_2 - 10I_3 &= 10 \end{aligned}$$

Solve for the current vector $\mathbf{I}$ (i.e, three components I_1, I_2 and I_3) by performing the Gaussian elimination.

Problem 2: Time Evolution Operator in Quantum Mechanics

In quantum mechanics, the time evolution of a quantum state $|\psi(t)\rangle$ is governed by the Schrödinger equation:

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

where H is the Hamiltonian matrix of the system. For a given system, the evolution operator is given by:

$$U = e^{-iHt/\hbar}$$

For small time intervals, we approximate this as:

$$U \approx I - \frac{iHt}{\hbar}$$

To analyze the system, we need to compute the inverse of U, which allows us to determine past quantum states.

Consider the Hamiltonian matrix:

$$H = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

with $\hbar = 1$ and a small time step $t = 0.1$.

1. Construct the evolution matrix $U = I - iHt$.
2. Compute U^{-1} using Gauss-Jordan elimination (use the function from class notes).
3. Verify that $U^{-1}U = I$ by performing matrix multiplication.

Problem 3: Quantum States using Eigenvectors

In quantum mechanics, the *Hamiltonian* of a system determines its energy levels. The eigenvalues of the Hamiltonian correspond to possible energy levels, and the associated eigenvectors represent quantum states.

Consider the Hamiltonian matrix:

$$H = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix}$$

The eigenvalues of H are given as:

$$\lambda_1 = 2, \quad \lambda_2 = 4, \quad \lambda_3 = 6$$

For each eigenvalue λ_i, solve for the corresponding eigenvector $\mathbf{v}_i$ by using either the Gaussian elimination or Gauss-Jordan elimination method. Also, Normalize the eigenvectors so that their magnitude is 1.

Problem 4: Evaluating the Work Done by a Force Field

The work done by a force $F(x)$ over a displacement $x \in [a, b]$ is given by the integral:

$$W = \int_a^b F(x)\, dx$$

Consider a variable force acting on a particle given by:

$$F(x) = x^2 e^{-x}$$

where x is in meters and $F(x)$ is in Newtons.

1. **Write functions** to compute the integral numerically using:
   - Midpoint rule
   - Trapezoidal rule
   - Simpson's 1/3rd rule
   - Simpson's 3/8th rule (formula given below)
2. **Compute the work done** by the force over the interval $x \in [0, 2]$ using each method with the same step size h.
3. **Compare the numerical results**:
   - Use the same step size h for all four methods.
   - Compute the percentage change in results when comparing different methods.
   - Discuss which method gives the most accurate result and why Simpson's rules typically perform better than the others.

Simpson's 3/8th Rule Formula

For an interval $[a, b]$ divided into n segments (where n is a multiple of 3), the integral is approximated as:

$$\int_a^b f(x)dx \approx \frac{3h}{8}\left[f(x_0) + 3\sum_{\text{odd } i} f(x_i) + 3\sum_{\text{even } i} f(x_i) + 2\sum_{\text{multiples of 3 } i} f(x_i) + f(x_n)\right]$$

where $h = \frac{b-a}{n}$ is the step size.

Problem 5: Radiation Energy Flux

In statistical physics, the energy density of blackbody radiation inside a cavity at temperature T is given by,

$$u = \frac{8\pi}{c^3h^3}\int_0^\infty \frac{h\nu^3}{e^{h\nu/k_BT} - 1}d\nu$$

where $h = 6.626 \times 10^{-34}$ J·s, $k_B = 1.381 \times 10^{-23}$ J/K and $c = 3.0 \times 10^8$ m/s. To analyze the total energy emitted in a given direction, we express the energy flux as:

$$I = \int_0^{\pi/2}\int_0^{2\pi}\int_0^\infty B_\nu(\nu, T)\cos\theta \, d\nu \, d\phi \, d\theta$$

where the *Planck function* is

$$B_\nu(\nu, T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/k_BT} - 1}$$

Compute I for temperature of $T = 6000$ K using trapezoidal integration method and compare with the expected value 4.7×10^6 W/m^2. Consider frequency range of $\nu \in [10^{12}, 10^{15}]$ Hz and verify if these limiting values are appropriate. Note that angular range follows $\theta \in [0, \pi/2]$ and $\phi \in [0, 2\pi]$.

Hint: Handling Complex Numbers in Python

In Python, complex numbers are represented using the imaginary unit j (instead of i, as in mathematics). To ensure that calculations handle complex numbers correctly, use the following:

- Define a complex number:

$$z = 3 + 4j$$

- Create a NumPy array with complex numbers:

$$A = \texttt{np.array}([\,[1 + 2j, 2 - 1j],\ [3 + 4j, 3]\,], \texttt{dtype=complex})$$

- The argument `dtype=complex` ensures that NumPy correctly treats all elements as complex numbers.

- Try multiplying `1j*1j to verify` $i^2 = -1$.

This is useful when working with quantum mechanics problems or any calculations involving imaginary numbers.