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ALMOST EVERYWHERE CONVERGENCE OF THE GRADIENTS OF SOLUTIONS TO ELLIPTIC AND PARABOLIC EQUATIONS

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1. INTRODUCTION

IN THIS paper, we consider the nonlinear equations

$$-\operatorname{div} a(x, u_n, Du_n) = f_n + g_n \qquad \text{in } \Omega \tag{1.1}$$

where Ω is a bounded subset of $\mathbb{R}^N$ and where $A(u) = -\operatorname{div} a(x, u, Du)$ is a Leray–Lions operator defined on $W^{1,p}(\Omega)$.

Assuming that

$$u_n \rightharpoonup u \qquad \text{weakly in } W^{1,p}(\Omega) \tag{1.2}$$

$$f_n \to f \qquad \text{strongly in } W^{1,p'}(\Omega) \tag{1.3}$$

$$g_n \rightharpoonup g \qquad \text{weakly} * \text{ in } \mathcal{M}(\Omega) \tag{1.4}$$

(where $\mathcal{M}(\Omega)$ denotes the space of Radon measures on Ω), we prove (theorem 2.1) that

$$Du_n \to Du \qquad \text{strongly in } (L^q(\Omega))^N \ \forall\, q < p. \tag{1.5}$$

This result allows one to pass to the limit in equation (1.1) as well as in the corresponding boundary value problems (see remark 2.1).

If hypothesis (1.4) is replaced by the stronger hypothesis

$$g_n \rightharpoonup g \qquad \text{weakly in } L^1(\Omega) \tag{1.6}$$

we improve (1.5) by proving (theorem 3.1) that for any fixed k the truncation of u_n at height k satisfies

$$DT_k(u_n) \to DT_k(u) \qquad \text{strongly in } (L^p_{\text{loc}}(\Omega))^N. \tag{1.7}$$

These two results are shown to be sharp (see counterexamples in remarks 2.3 and 3.2 below).

In Section 4 we prove the corresponding results for the parabolic case, i.e. when equation (1.1) is replaced by

$$\frac{\partial u_n}{\partial t} + A(u_n) = f_n + g_n \qquad \text{in } \Omega \times (0, T). \tag{1.8}$$

Our method of proof is elementary and is based on the use of test functions $T_\eta(u_n - u)$ (see [7]) or $T_k(u_n) - T_k(u)$, which however confines the proofs to the case where A is a second-order elliptic operator.

Theorem 2.1 (and its parabolic analogue) extends previous results obtained in [1, Section 3] (this paper deals with the quasilinear elliptic case; the proof presented there is easy but is based on Meyers' regularity theorem [2] and on Murat's lemma [3]), in [4, theorem 2] (this paper deals with the p-Laplacian for systems and uses capacity methods), in [5, Section 4] (where nonlinear systems are also considered) and in [6, theorem 4.1] (this paper deals with the quasilinear parabolic case and uses the $C^{0,\alpha}$ regularity of the solutions of parabolic equations). Theorem 2.1 is also similar to [7, Subsection 5] (where the almost everywhere convergence of Du_n is proved without assuming u_n to be bounded in $W^{1,p}(\Omega)$, but assuming $f_n = 0$ and a mild hypothesis of "strong monotonicity" on A).

In contrast, theorem 3.1 (and its parabolic analogue) seems to be new.

Besides their own interest, the results presented here have interesting applications. In particular, they allow one to prove that solutions of approximating nonlinear elliptic or parabolic equations converge to the solution of the original equation, a method which provides existence results: see, e.g. [1, 5, 6, 8–10]. They were also recently used in [11] as a step to prove the compactness of the intervals of $W^{-1,p'}(\Omega)$.

Finally, note that the results presented here apply to unilateral variational inequalities. Indeed, if u_n is a solution of the obstacle problem

$$\begin{cases} u_n \in K(\psi) \\ \langle A(u_n), v - u_n \rangle \geq \langle f_n, v - u_n \rangle \qquad \forall\, v \in K(\psi) \end{cases} \tag{1.9}$$

where $K(\psi) = \{v \in W_0^{1,p}(\Omega) : v \geq \psi \text{ a.e. in } \Omega\}$ and f_n strongly converges to f in $W^{-1,p'}(\Omega)$, one has

$$u_n \text{ bounded in } W_0^{1,p}(\Omega),\ A(u_n) = f_n + g_n \text{ where } g_n \text{ is a positive measure.}$$

Remark 2.2 then allows one to apply theorem 2.1 and to obtain the almost everywhere convergence of Du_n. Theorem 3.2 can also be used in this context (see remark 3.5).

Notation. In the whole of this paper, $\langle f, u \rangle$ will denote the duality pairing between an element f of $W^{-1,p'}(\Omega)$ and an element u of $W_0^{1,p}(\Omega)$. For $k > 0$, $T_k : \mathbb{R} \to \mathbb{R}$ will denote the truncation at height k, defined by

$$T_k(s) = s \quad \text{if } |s| \leq k, \qquad T_k(s) = ks/|s| \quad \text{if } |s| \geq k. \tag{1.10}$$

Finally, $|E|$ will denote the Lebesgue measure of a measurable set E.

2. ALMOST EVERYWHERE CONVERGENCE OF THE GRADIENTS IN THE ELLIPTIC CASE

Let Ω be a bounded open subset of $\mathbb{R}^N$ (no smoothness is assumed on $\partial\Omega$) and p and p' be such that

$$1 < p,\ p' < +\infty, \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

Consider the operator A defined on $W^{1,p}(\Omega)$ by

$$A(u) = -\operatorname{div} a(x, u, Du) \tag{2.1}$$

where $a: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the classical Leray–Lions hypotheses (see [12, 13])

$$|a(x, s, \zeta)| \le c(x) + k_1|s|^{p-1} + k_2|\zeta|^{p-1} \tag{2.2}$$

$$[a(x, s, \zeta) - a(x, s, \zeta^*)][\zeta - \zeta^*] > 0 \tag{2.3}$$

$$\frac{a(x, s, \zeta)\zeta}{|\zeta| + |\zeta|^{p-1}} \to +\infty \qquad \text{if } |\zeta| \to +\infty \tag{2.4}$$

$$\text{a.e. } x \in \Omega,\ \forall s \in \mathbb{R},\ \forall \zeta, \zeta^* \in \mathbb{R}^N,\ \zeta \neq \zeta^*$$

where $c(x)$ belongs to $L^{p'}(\Omega)$, $c \ge 0$, and k_1, k_2 to $\mathbb{R}^+$.

Consider the nonlinear elliptic equations

$$-\operatorname{div} a(x, u_n, Du_n) = f_n + g_n \qquad \text{in } \mathcal{D}'(\Omega) \tag{2.5}$$

and assume that

$$u_n \rightharpoonup u \qquad \text{weakly in } W^{1,p}(\Omega), \text{ strongly in } L^p_{\text{loc}}(\Omega) \text{ and a.e. in } \Omega \tag{2.6}$$

$$f_n \to f \qquad \text{strongly in } W^{-1,p'}(\Omega). \tag{2.7}$$

In view of (2.2) and (2.5)–(2.7), g_n belongs to (and is bounded in) $W^{-1,p'}(\Omega)$. Assume, moreover, that g_n is bounded in the space $\mathfrak{M}(\Omega)$ of Radon measures, i.e. that

$$|\langle g_n, \varphi\rangle| \le C_K\|\varphi\|_{L^\infty(\Omega)} \qquad \text{for any } \varphi \in \mathcal{D}(\Omega) \text{ with } \operatorname{supp}(\varphi) \subset K \tag{2.8}$$

where C_K is a constant which depends on the compact set K.

THEOREM 2.1. Assume that (2.2)–(2.8) hold true. Then

$$Du_n \to Du \qquad \text{strongly in } (L^q(\Omega))^N \text{ for any } q < p. \tag{2.9}$$

Remark 2.1. From (2.9) one deduces that, extracting a subsequence n', one has

$$Du_{n'} \to Du \qquad \text{a.e. in } \Omega \tag{2.10}$$

which justifies the title of this section.

Since $a(x, u_n, Du_n)$ is bounded in $(L^{p'}(\Omega))^N$ (see (2.2), (2.6)), convergence (2.10) implies that

$$\begin{cases} a(x, u_{n'}, Du_{n'}) \to a(x, u, Du) & \text{a.e. in } \Omega \\ a(x, u_n, Du_n) \rightharpoonup a(x, u, Du) & \text{weakly in } (L^{p'}(\Omega))^N \end{cases}$$

which allows one to pass to the limit in (2.5), obtaining that u satisfies

$$-\operatorname{div} a(x, u, Du) = f + g \qquad \text{in } \mathcal{D}'(\Omega). \tag{2.11}$$

Remark 2.2. Hypothesis (2.8) is of course satisfied when g_n is a sequence of $L^1_{\text{loc}}(\Omega)$ which is bounded in this space.

Note also that hypothesis (2.8) is satisfied whenever g_n satisfies for some s in $\mathbb{R}$ and some r, $1 \le r \le +\infty$

$$\begin{cases} g_n & \text{bounded in } W^{-s,r}_{\text{loc}}(\Omega) \text{ in the sense of } \mathcal{D}'(\Omega) \\ g_n \ge 0 & \end{cases} \tag{2.12}$$

(the last assertion implies that g_n is a positive Radon measure). Indeed, for a given compact set K with $K \subset \Omega$, consider a function $\phi_K \in \mathfrak{D}(\Omega)$ with $0 \le \phi_K \le 1$ in Ω and $\phi_K = 1$ on K; whenever (2.12) holds true one has for any $\varphi \in \mathfrak{D}(\Omega)$ with $\operatorname{supp}(\varphi) \subset K$

$$
\begin{aligned}
|\langle g_n, \varphi\rangle| = |\langle g_n, \varphi\phi_K\rangle| &= \left|\int_\Omega \varphi\phi_K \, \mathrm{d}g_n\right| \\
&\le \|\varphi\|_{L^\infty(\Omega)} \int_\Omega \phi_K \, \mathrm{d}g_n = \|\varphi\|_{L^\infty(\Omega)}\langle g_n, \phi_K\rangle \le \|\varphi\|_{L^\infty(\Omega)} C_K
\end{aligned}
$$

which implies (2.8).

Remark 2.3. The strong convergence (2.9) does not in general hold true with $q = p$, as shown by the following counterexample.

Consider the case where $N \ge 2$, $\Omega = B_R = \{x \in \mathbb{R}^N : |x| < R\}$, $p = 2$, $A(u) = -\Delta u$ and let

$$
\begin{cases}
u_n(x) = \dfrac{1}{\sqrt{n}} \inf\left(n, \dfrac{1}{|x|^{N-2}}\right) & \text{in } B_R \text{ if } N \ge 3 \\[2ex]
u_n(x) = \dfrac{1}{\sqrt{n}} \inf(n, -\log|x|) & \text{in } B_R \text{ if } N = 2 \\[2ex]
f_n = 0, \, g_n = -\Delta u_n & \text{in } B_R.
\end{cases}
\tag{2.13}
$$

Easy computations yield

$$
\begin{cases}
\displaystyle\int_{B_{R'}} |Du_n|^2 \, \mathrm{d}x = C(R') & \forall\, R' \le R, \; B_{R'} = \{x : |x| < R'\} \\
u_n \rightharpoonup 0 & \text{weakly in } H^1(B_R) \\
g_n \ge 0, \; \|g_n\|_{H^{-1}(B_R)} = C, & g_n \text{ satisfies (2.8)}
\end{cases}
\tag{2.14}
$$

which shows that in this example the hypotheses of theorem 2.1 are fulfilled while (2.9) does not hold true with $q = p = 2$.

Another (more complicated) example where (2.9) does not hold true with $p = q$ is provided by the example studied in remark 3.2 below.

Remark 2.4. It follows from equation (2.5) that g_n automatically satisfies

$$
g_n \text{ is bounded in } W^{-1,p'}(\Omega) \tag{2.15}
$$

whenever hypotheses (2.2), (2.6) and (2.7) hold true.

Hypothesis (2.8) thus appears as a further condition on g_n, which plays a prominent role in order to obtain the strong convergence (2.9): if only (2.2), (2.5)–(2.7) and (2.15) are satisfied, (2.9) does not in general hold true for any q.

Consider indeed the case where $p = 2$, $A(u) = -\Delta u$ and let

$$
u_n(x) = \frac{1}{n}\psi(nx), \qquad f_n = 0, \qquad g_n = -\Delta u_n \qquad \text{in } \mathbb{R}^N \tag{2.16}
$$

where $\psi(y)$ is a function in $C^\infty(\mathbb{R}^N)$ which is periodic with respect to $y_1, y_2, \ldots, y_N$. It is then well known that Du_n converges weakly to 0 in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for any $1 \le q < +\infty$, but does not converge strongly in any of these spaces. Therefore convergence (2.9) cannot be obtained assuming only (2.15).

Remark 2.5. Equation (2.5) is a "local" problem in the sense that no boundary condition is assumed on u_n.

If one complements (2.5) by the homogeneous Dirichlet boundary condition and if one assumes the further coerciveness hypothesis

$$\frac{1}{\|u\|_{W_0^{1,p}(\Omega)}}\int_\Omega a(x,u,Du)Du\,\mathrm{d}x \to +\infty \qquad \text{if } \|u\|_{W_0^{1,p}(\Omega)} \to +\infty \tag{2.17}$$

it is easy to deduce from (2.7) and (2.15) that u_n is bounded in $W_0^{1,p}(\Omega)$, and thus that (2.6) holds true after extracting a subsequence. In this context (2.15) appears as a hypothesis necessary to obtain (2.6) and then to apply theorem 2.1, except if some structure assumptions are made on g_n, which automatically guarantee that the solution u_n of the boundary value problem associated with (2.5) is bounded in $W_0^{1,p}(\Omega)$: this is for example the case if one a priori knows that

$$\langle g_n, u_n\rangle \geq 0 \tag{2.18}$$

a condition which is an immediate consequence of the so-called "sign condition" in many examples (see, e.g. [1, 5, 6, 8–10]). In such a case, hypothesis (2.15) has no longer to be assumed in order to obtain (2.6).

Remark 2.6. The proof of theorem 2.1 presented below strongly uses the truncation operators, which act in $W^{1,p}(\Omega)$ but not in the higher order spaces $W^{m,p}(\Omega)$ with $m \geq 2$. This is the reason why the framework of theorem 2.1 is limited to second-order operators.

Proof of theorem 2.1. *Step* 1. Fix a compact set K with $K \subset \Omega$ and a function ϕ_K in $\mathfrak{D}(\Omega)$ with $0 \leq \phi_K \leq 1$ in Ω and $\phi_K = 1$ on K. Using in (2.5) the test function

$$v_n = \phi_K T_\eta(u_n - u) \in W_0^{1,p}(\Omega)$$

where T_η is the truncation at height η defined by (1.10), we obtain

$$\begin{aligned}\int_\Omega \phi_K[a(x,u_n,Du_n) - a(x,u_n,Du)]DT_\eta(u_n - u)\,\mathrm{d}x \\ = -\int_\Omega T_\eta(u_n - u)a(x,u_n,Du_n)D\phi_K\,\mathrm{d}x \\ -\int_\Omega \phi_K a(x,u_n,Du)DT_\eta(u_n - u)\,\mathrm{d}x \\ + \langle f_n, \phi_K T_\eta(u_n - u)\rangle + \langle g_n, \phi_K T_\eta(u_n - u)\rangle.\end{aligned} \tag{2.19}$$

From (2.6) one deduces that

$$T_\eta(u_n - u) \rightharpoonup 0 \qquad \text{weakly in } W^{1,p}(\Omega) \text{ and strongly in } L^p_{\mathrm{loc}}(\Omega)$$

which implies that for η fixed, the first three terms of the right-hand side of (2.19) tend to 0 when n tends to infinity. On the other hand, for any smooth function ψ, we have in view of (2.8)

$$|\langle g_n, \phi_K\psi\rangle| = \left|\int_\Omega \phi_K\psi\,\mathrm{d}g_n\right| \leq C_K\|\psi\|_{L^\infty(\Omega)}.$$

Mollifying $T_\eta(u_n - u)$ thus implies that

$$|\langle g_n, \phi_K T_\eta(u_n - n)\rangle| \leq C_K \eta. \tag{2.20}$$

We have proved that, for η fixed

$$\limsup_{n\to+\infty} \int_K [a(x, u_n, Du_n) - a(x, u_n, Du)]DT_\eta(u_n - u)\,\mathrm{d}x \leq C_K\eta. \tag{2.21}$$

Step 2. Define now the (nonnegative) function e_n by

$$e_n(x) = [a(x, u_n, Du_n) - a(x, u_n, Du)][Du_n - Du] \tag{2.22}$$

and fix θ with $0 < \theta < 1$. Splitting the set K into

$$S_n^\eta = \{x \in K : |u_n(x) - u(x)| \leq \eta\}, \qquad G_n^\eta = \{x \in K : |u_n(x) - u(x)| > \eta\}$$

and using Hölder's inequality one has

$$\int_K e_n^\theta\,\mathrm{d}x = \int_{S_n^\eta} e_n^\theta\,\mathrm{d}x + \int_{G_n^\eta} e_n^\theta\,\mathrm{d}x \leq \left(\int_{S_n^\eta} e_n\,\mathrm{d}x\right)^\theta |S_n^\eta|^{1-\theta} + \left(\int_{G_n^\eta} e_n\,\mathrm{d}x\right)^\theta |G_n^\eta|^{1-\theta}. \tag{2.23}$$

Since, for η fixed, $|G_n^\eta|$ tends to 0 if n tends to infinity, and since e_n is bounded in $L^1(\Omega)$, one deduces from (2.21) and (2.23) that

$$\limsup_{n\to+\infty} \int_K e_n^\theta\,\mathrm{d}x \leq (C_K\eta)^\theta|\Omega|^{1-\theta}. \tag{2.24}$$

Letting η tend to 0 implies that e_n^θ tends strongly to 0 in $L^1(K)$ and thus, using a sequence of compact sets K, there exists a subsequence n' such that

$$e_{n'}(x) \to 0 \qquad \text{a.e. } x \in \Omega. \tag{2.25}$$

From a lemma due to Leray and Lions (see, e.g. [13, lemma 2.2, pp. 184–185]; see also remark 2.7 below) one deduces from (2.25) that

$$Du_{n'}(x) \to Du(x) \qquad \text{a.e. } x \in \Omega. \tag{2.26}$$

Since Du_n is bounded in $(L^p(\Omega))^N$, Vitali's theorem implies that

$$Du_n \to Du \qquad \text{strongly in } (L^q(\Omega))^N \text{ for any } q < p \tag{2.27}$$

which is the desired result. Note that in (2.27) the whole sequence u_n (and not only a subsequence) converges because the limit Du is independent of the subsequence n'. ■

Remark 2.7. The end of the above proof becomes simpler when a "strong monotonicity" assumption is made on the function a. In this case it is not necessary to use the lemma due to Leray and Lions which is quoted above and the proof is elementary. Indeed if one assumes that the strict monotonicity assumption (2.3) is strengthened in

$$[a(x, s, \zeta) - a(x, s, \zeta^*)][\zeta - \zeta^*] \geq \begin{cases} \alpha|\zeta - \zeta^*|^p & \text{if } p \geq 2 \\ \alpha\dfrac{|\zeta - \zeta^*|^2}{(d(x) + |\zeta| + |\zeta|^*)^{2-p}} & \text{if } p \leq 2 \end{cases} \tag{2.28}$$

$$\text{a.e. } x \in \Omega,\ \forall s \in \mathbb{R},\ \forall \zeta, \zeta^* \in \mathbb{R}^N$$

for some $\alpha > 0$ and d in $L^p(\Omega)$, $d \geq 0$ (such an assumption is satisfied if $a(x, s, \zeta) = |\zeta|^{p-2}\zeta$) one immediately deduces (2.26) from (2.25).

3. STRONG CONVERGENCE OF THE TRUNCATIONS IN THE ELLIPTIC CASE

In this section, we will replace the coerciveness hypothesis (2.4) on the function a and hypothesis (2.8) on g_n by the stronger hypotheses

$$\begin{cases} a(x, s, \zeta)\zeta \geq \alpha|\zeta|^p \qquad \text{for some } \alpha > 0 \\ \text{a.e. } x \in \Omega, \ \forall s \in \mathbb{R}, \ \forall \zeta \in \mathbb{R}^N \end{cases} \tag{3.1}$$

$$g_n \in L^1(\Omega), \qquad g \in L^1(\Omega), \qquad g_n \rightharpoonup g \qquad \text{weakly in } L^1(\Omega). \tag{3.2}$$

THEOREM 3.1. Assume that (2.2), (2.3), (2.5)–(2.7), (3.1) and (3.2) hold true. Then, for any $k > 0$ fixed, the truncation of u_n at height k satisfies

$$DT_k(u_n) \to DT_k(u) \qquad \text{strongly in } (L^p_{\text{loc}}(\Omega))^N. \tag{3.3}$$

Remark 3.1. Compared with the result (2.9) of theorem 2.1, convergence (3.3) is only concerned with the areas where $|u_n(x)| \leq k$. It is, however, stronger than (2.9), since it takes place in $(L^p_{\text{loc}}(\Omega))^N$ and not only in $(L^q(\Omega))^N$ with $q < p$.

Note, also, that convergence (3.3) is a local result. It becomes global (i.e. the convergence takes place in $(L^p(\Omega))^N$) if u_n is assumed to belong to $W_0^{1,p}(\Omega)$.

Remark 3.2. If hypothesis (3.2) is replaced by

$$\begin{cases} g_n \in L^1(\Omega), \ g \in L^1(\Omega), \qquad g_n \text{ bounded in } L^1(\Omega) \\ g_n \rightharpoonup g \qquad \text{weakly } * \text{ in } \mathcal{M}(\Omega) \text{ (or equivalently in } \mathcal{D}'(\Omega)) \end{cases} \tag{3.4}$$

[which is very similar to hypothesis (2.8)], the result of theorem 3.1 does not continue to hold, as shown by the following counterexample inspired by [14].

Consider a covering of $\mathbb{R}^N$ ($N \geq 2$) by disjoint cubes Q_n^i with sides parallel to the coordinates axes, of size $2/n$ and of centres $x_n^i = (2i_1/n, 2i_2/n, \ldots, 2i_N/n)$ where the multi-index i belongs to $\mathbb{Z}^N$. Define B_n^i and T_n^i as the balls of centre x_n^i and of radius $1/n$ and $(1/n)^{N/(N-2)}$ respectively (in the case $N = 2$ the radius of T_n^i has to be taken equal to e^{-n^2}). Define the function $u_n \in H^1_{\text{loc}}(\mathbb{R}^N)$ by

$$\begin{cases} u_n = 1 & \text{in } T_n^i \\ \Delta u_n = 0 & \text{in } B_n^i \backslash T_n^i \\ u_n = 0 & \text{in } Q_n^i \backslash B_n^i \end{cases} \tag{3.5}$$

(with continuity of u_n on the spheres ∂T_n^i and ∂B_n^i). Set also

$$f_n = 0, \qquad g_n = -\Delta u_n \qquad \text{in } \mathcal{D}'(\mathbb{R}^N). \tag{3.6}$$

It is easily seen that g_n is a measure supported by the union of all the spheres ∂T_n^i and ∂B_n^i, which is constant on each of these spheres, and that

$$0 \leq u_n \leq 1. \tag{3.7}$$

It can also be proved (see [14, Section 2]) that for any smooth open bounded subset Ω of $\mathbb{R}^N$

$$\begin{cases} u_n \rightharpoonup 0 \qquad \text{weakly in } H^1(\Omega) \\ \|u_n\|_{H^1(\Omega)} \to C_1 \\ \|g_n\|_{\mathcal{M}(\Omega)} = \displaystyle\int_\Omega d|g_n| \to C_2. \end{cases} \tag{3.8}$$

As in the example presented in remark 2.3, the hypotheses of theorem 2.1 are satisfied by u_n, $f_n = 0$ and g_n (with $p = 2$ and $A(u) = -\Delta u$) but u_n does not converge strongly to 0 in $H^1(\Omega)$.

For a fixed function ρ which belongs to $\mathfrak{D}(\mathbb{R}^N)$ with $\rho \geq 0$ and $\int \rho(x)\,dx = 1$, define now

$$\bar{u}_n = u_n * \rho_n, \qquad \bar{g}_n = g_n * \rho_n \tag{3.9}$$

where $\rho_n(x) = c_n^N \rho(c_n x)$, and choose c_n sufficiently large in order to have

$$\|\bar{u}_n - u_n\|_{H^1(\Omega)} \leq 1/n \tag{3.10}$$

and in order for the support of $\bar{g}_n$ to be made of small disjoint coronas and to have a measure smaller than $1/n$; note that $\bar{g}_n$ is bounded in $L^1(\Omega)$. The main difference between $(\bar{u}_n, \bar{g}_n)$ and (u_n, g_n) is that $\bar{g}_n$ is now an $L^1(\Omega)$ function while all the fundamental features of u_n and g_n are retained.

This provides the desired counterexample in the case where hypothesis (3.2) of theorem 3.1 is replaced by (3.4). Indeed $\bar{g}_n$ and $\bar{g} = 0$ now satisfy (3.4) (but not (3.2): since g_n is a measure which is concentrated on the spheres ∂T_n^i and ∂B_n^i and does not belong to $L^1(\Omega)$, g_n cannot be equi-integrable (a property which is known to be equivalent to the weak compactness in $L^1(\Omega)$); regularization of g_n by ρ_n produces $\bar{g}_n$ which now belongs to $L^1(\Omega)$, but the support of $\bar{g}_n$ is so small that $\bar{g}_n$ again is not equi-integrable). On the other hand, since $0 \leq \bar{u}_n \leq 1$ one has

$$T_k(\bar{u}_n) = \bar{u}_n \qquad \text{for any } k \geq 1$$

which proves that $T_k(\bar{u}_n)$ does not converge strongly to 0 in $H_0^1(\Omega)$ when $k \geq 1$.

Remark 3.3. Similarly to the proof of theorem 2.1, the proof of theorem 3.1 that we will give below strongly uses the truncation and does not extend to higher order equations.

Proof of theorem 3.1. *Step* 1. Fix a compact set K with $K \subset \Omega$ and a function Φ_K in $\mathfrak{D}(\Omega)$ with $0 \leq \phi_K \leq 1$ in Ω and $\phi_K = 1$ on K. When u_n belongs to $W_0^{1,p}(\Omega)$, one can take $\phi_K = 1$ on Ω and the result (3.3) becomes global (see remark 3.1). Using in (2.5) the test function

$$v_n = (T_k(u_n) - T_k(u))\phi_K \in W_0^{1,p}(\Omega)$$

yields

$$\begin{aligned}\int_\Omega \phi_K a(x, u_n, Du_n)[DT_k(u_n) - DT_k(u)]\,dx \\ = -\int_\Omega (T_k(u_n) - T_k(u))a(x, u_n, Du_n)D\phi_K\,dx \\ + \langle f_n, (T_k(u_n) - T_k(u))\phi_K\rangle + \langle g_n, (T_k(u_n) - T_k(u))\phi_K\rangle.\end{aligned} \tag{3.11}$$

Since

$$T_k(u_n) - T_k(u) \rightharpoonup 0 \qquad \text{weakly in } W^{1,p}(\Omega), \text{ strongly in } L^p_{\text{loc}}(\Omega) \text{ and a.e. in } \Omega \tag{3.12}$$

the first two terms of the right-hand side of (3.11) tend to 0.

On the other hand, since g_n belongs to $W^{-1,p'}(\Omega) \cap L^1(\Omega)$ while $(T_k(u_n) - T_k(u))\phi_K$ belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ we have

$$\langle g_n, (T_k(u_n) - T_k(u))\phi_K\rangle = \int_\Omega g_n(T_k(u_n) - T_k(u))\phi_K\,dx.$$

Using Egorov's theorem on $(T_k(u_n) - T_k(u))$ and hypothesis (3.2), it is easy to prove (see [15, lemma 3.4] if necessary) that the last expression tends to 0.

We thus have proved that

$$\int_\Omega \phi_K a(x, u_n, Du_n)[DT_k(u_n) - DT_k(u)]\,dx \to 0 \qquad \text{when } n \to +\infty. \tag{3.13}$$

Step 2. Splitting Ω into

$$S_n = \{x \in \Omega : |u_n(x)| \le k\}, \qquad G_n = \{x \in \Omega : |u_n(x)| > k\}$$

and denoting by χ_{G_n} the characteristic function of G_n one has

$$\begin{aligned} E_n &= \int_\Omega \phi_K[a(x, u_n, DT_k(u_n)) - a(x, u_n, DT_k(u))][DT_k(u_n) - DT_k(u)]\,dx \\ &= \int_\Omega \phi_K a(x, u_n, Du_n)[DT_k(u_n) - DT_k(u)]\,dx \\ &\quad - \int_\Omega \phi_K[a(x, u_n, Du_n) - a(x, u_n, 0)]\chi_{G_n}[DT_k(u_n) - DT_k(u)]\,dx \\ &\quad - \int_\Omega \phi_K a(x, u_n, DT_k(u))[DT_k(u_n) - DT_k(u)]\,dx. \end{aligned} \tag{3.14}$$

In view of (3.13) and (3.12) the first and third terms of the right-hand side of (3.14) tend to 0. Since $\chi_{G_n} DT_k(u_n) = 0$, the second term reads as

$$\int_\Omega \phi_K[a(x, u_n, Du_n) - a(x, u_n, 0)]\chi_{G_n} DT_k(u)\,dx \tag{3.15}$$

which tends to 0 since $[a(x, u_n, Du_n) - a(x, u_n, 0)]$ is bounded in $(L^{p'}(\Omega))^N$ while $\chi_{G_n} DT_k(u)$ tends strongly to 0 in $(L^p(\Omega))^N$ by Lebesgue's theorem.

We have proved that E_n tends to 0, which yields

$$\int_K [a(x, u_n, DT_k(u_n)) - a(x, u_n, DT_k(u))][DT_k(u_n) - DT_k(u)]\,dx \to 0. \tag{3.16}$$

By a refinement of Leray and Lions' original proof (see, e.g. [16, pp. 13 and 27] or [17, lemma 5]) this implies the convergence of $DT_k(u_n)$ to $DT_k(u)$ in the strong topology of $(L^p(K))^N$ whenever the coerciveness hypothesis (3.1) is assumed to hold. This proves the desired result (3.3).

Remark 3.4. The end of the above proof becomes simpler when hypothesis (3.1) is replaced by the "strong monotonicity" assumption (2.28). In this case, it is not necessary to use the refinement of Leray and Lions' proof which is quoted above and the proof is elementary. Indeed when (2.28) holds true and when $p \ge 2$, convergence (3.16) immediately implies that

$$\alpha \int_K |DT_k(u_n) - DT_k(u)|^p\,dx \to 0$$

i.e. (3.3); when $p \leq 2$, Hölder's inequality yields

$$\int_K |DT_k(u_n) - DT_k(u)|^p \, dx \leq \left(\int_K \frac{|DT_k(u_n) - DT_k(u)|^2}{(d(x) + |DT_k(u_n)| + |DT_k(u)|)^{2-p}} \, dx \right)^{p/2}$$

$$\left(\int_K (d(x) + |DT_k(u_n)| + |DT_k(u)|)^p \, dx \right)^{(2-p)/2}$$

which combined to (3.16) and to the boundedness of u_n in $W^{1,p}(\Omega)$ again implies (3.3).

Remark 3.5. Consider two sequences $\bar{f}_n$ and $\bar{g}_n$ such that

$$\bar{f}_n \to \bar{f} \quad \text{strongly in } W^{-1,p'}(\Omega), \qquad \bar{g}_n \to \bar{g} \quad \text{weakly in } L^1(\Omega) \tag{3.17}$$

and a measurable function ψ with values in $\bar{\mathbb{R}}$, and define

$$K(\psi) = \{v \in W_0^{1,p}(\Omega) : v \geq \psi \text{ a.e. in } \Omega\}. \tag{3.18}$$

Assume that there exists a solution u_n of the unilateral variational inequality

$$\begin{cases} u_n \in K(\psi), \qquad u_n \bar{g}_n \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x, u_n, Du_n)[Dv - Du_n] \, dx \geq \langle \bar{f}_n, v - u_n \rangle + \int_\Omega \bar{g}_n (v - u_n) \, dx \\ \hfill \forall\, v \in K(\psi) \text{ with } v - u_n \in L^\infty(\Omega) \end{cases} \tag{3.19}$$

and that

$$u_n \rightharpoonup u \qquad \text{weakly in } W_0^{1,p}(\Omega) \text{ and a.e. in } \Omega. \tag{3.20}$$

It is not clear in general that there exists a solution of (3.19) and even that a possible solution of (3.19) is bounded in $W_0^{1,p}(\Omega)$, which will imply (3.20) for a subsequence. This is nevertheless the case in many examples when $-\bar{g}_n$ satisfies a "sign condition" (see, e.g. [8, 18]).

The function $v_n = u_n - [T_k(u_n) - T_k(u)]$ clearly satisfies $v_n - u_n \in L^\infty(\Omega)$ and can be proved (using in particular the fact that u belongs to $K(\psi)$) to belong to $K(\psi)$. Using v_n as a test function in (3.19) one obtains

$$\int_\Omega a(x, u_n, Du_n)[DT_k(u_n) - DT_k(u)] \, dx \leq \langle \bar{f}_n, T_k(u_n) - T_k(u) \rangle + \int_\Omega \bar{g}_n (T_k(u_n) - T_k(u)) \, dx$$

which is nothing but the analogue of (3.11) with $\phi_K = 1$.

The above proof then applies and implies that (3.3) again holds true in this context.

4. PARABOLIC CASE

In this section, we extend theorems 2.1 and 3.1 to the parabolic case.

Let Ω be a bounded open subset of $\mathbb{R}^N$ and let $T > 0$ be fixed. Set $Q = \Omega \times (0, T)$. Consider the nonlinear parabolic equations

$$\frac{\partial u_n}{\partial t} - \operatorname{div} a(x, t, u_n, Du_n) = f_n + g_n \qquad \text{in } \mathcal{D}'(Q) \tag{4.1}$$

where $A(u) = -\operatorname{div} a(x, t, u, Du)$ is a Leray–Lions operator defined on $L^p(0, T; W^{1,p}(\Omega))$ from a Carathéodory function $a: \Omega \times (0, T) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying

$$|a(x, t, s, \zeta)| \leq c(x, t) + k_1|s|^{p-1} + k_2|\zeta|^{p-1} \tag{4.2}$$

$$[a(x, t, s, \zeta) - a(x, t, s, \zeta^*)][\zeta - \zeta^*] > 0 \tag{4.3}$$

$$a(x, s, \zeta)\zeta \geq \alpha|\zeta|^p \qquad \text{for some } \alpha > 0 \tag{4.4}$$

$$\text{a.e. } (x, t) \in Q, \ \forall s \in \mathbb{R}, \ \forall \zeta, \zeta^* \in \mathbb{R}^N, \ \zeta \neq \zeta^*$$

where $c(x, t)$ belongs to $L^{p'}(Q)$, $c \geq 0$, and k_1, k_2 to $\mathbb{R}^+$.

Assume that

$$u_n \rightharpoonup u \qquad \text{weakly in } L^p(0, T; W^{1,p}(\Omega)) \tag{4.5}$$

$$f_n \to f \qquad \text{strongly in } L^{p'}(0, T; W^{-1,p'}(\Omega)) \tag{4.6}$$

$$g_n \in L^1(Q), \qquad g \in L^1(Q), \qquad g_n \rightharpoonup g \qquad \text{weakly in } L^1(Q). \tag{4.7}$$

Then one has the analogue of theorem 3.1.

THEOREM 4.1. Assume that (4.1)–(4.7) hold true. Then for any fixed $k > 0$

$$DT_k(u_n) \to DT_k(u) \qquad \text{strongly in } (L^p_{\text{loc}}(Q))^N. \tag{4.8}$$

One of the steps of the proofs of theorems 4.1 and 4.3 below consists of replacing Rellich's theorem (which asserts that $W^{1,p}(\Omega)$ is compactly embedded in $L^p_{\text{loc}}(\Omega)$) by the following result.

LEMMA 4.2. Assume that

$$\frac{\partial u_n}{\partial t} = h_n + k_n \qquad \text{in } \mathcal{D}'(\Omega) \tag{4.9}$$

where h_n and k_n satisfy

$$h_n \text{ bounded in } L^{p'}(0, T; W^{1,p'}(\Omega)), \qquad k_n \text{ bounded in } \mathfrak{M}(Q) \tag{4.10}$$

while u_n satisfies (4.5). Then

$$u_n \to u \qquad \text{strongly in } L^p_{\text{loc}}(Q). \tag{4.11}$$

Proof. Consider a function $\phi(x, t)$, $\phi(x, t) = \psi(x)\eta(t)$ with ψ in $\mathcal{D}(\Omega)$ and h in $\mathcal{D}(0, T)$ and set

$$v_n = \phi u_n, \qquad \alpha_n = \phi h_n + \frac{\partial \phi}{\partial t} u_n, \qquad \beta_n = \phi k_n.$$

Then for any bounded open subset ω with $\operatorname{supp} \psi \subset \omega \subset \Omega$, we have

$$\begin{cases} \dfrac{\partial v_n}{\partial t} = \alpha_n + \beta_n \quad \text{in } \mathcal{D}'(\omega \times (0, T)), & v_n \text{ bounded in } L^p(0, T; W_0^{1,p}(\omega)) \\ \alpha_n \text{ bounded in } L^{p'}(0, T; W^{-1,p'}(\omega)), & \beta_n \text{ bounded in } \mathfrak{M}(\omega \times (0, T)) \end{cases} \tag{4.12}$$

and all these functions have their support included in the same compact subset of $\omega \times (0, T)$.

For a fixed function ρ which belongs to $\mathfrak{D}(\mathbb{R}^{N+1})$ with $\rho \geq 0$ and $\int \rho(x, t)\,dx\,dt = 1$ define now

$$\bar{v}_n = v_n * \rho_n, \qquad \bar{\alpha}_n = \alpha_n * \rho_n, \qquad \bar{\beta}_n = \beta_n * \rho_n$$

where $\rho_n(x, t) = C_n^{N+1}\rho(c_n x, c_n t)$, and choose c_n sufficiently large in order to have

$$\|\bar{v}_n - v_n\|_{L^p(Q)} \leq 1/n. \tag{4.13}$$

The function $\bar{\beta}_n$ now belongs to $L^1(0, T; L^1(\omega))$ and is bounded in this space while $\bar{\alpha}_n$ is bounded in $L^{p'}(0, T; W^{-1,p'}(\omega))$. Since $L^1(\omega) \subset W^{-1,r'}(\omega)$ for any r' with $1 \leq r' < N/(N-1)$ and since $\partial\bar{v}_n/\partial t = \bar{\alpha}_n + \bar{\beta}_n$ in $\mathfrak{D}'(\omega \times (0, T))$ one has

$$\begin{cases} \bar{v}_n \text{ bounded in } L^p(0, T; W_0^{1,p}(\omega)) \\ \dfrac{\partial \bar{v}_n}{\partial t} \text{ bounded in } L^1(0, T; W^{-1,q}(\omega)) \qquad \text{with } q < \inf\left(p', \dfrac{N}{N-1}\right). \end{cases} \tag{4.14}$$

Since the embedding of $W_0^{1,p}(\omega)$ in $L^p(\omega)$ is compact, a refinement of Aubin's lemma (see, e.g. [19, Section 8, corollary 4] ensures that $\bar{v}_n$ is relatively compact in $L^p(0, T; L^p(\omega))$. In view of (4.13) this implies (4.11).

Proof of theorem 4.1. *Step* 1. Recall first that if $T_k: \mathbb{R} \to \mathbb{R}$ is the truncation at height k defined by (1.10) and if S_k is defined by $S_k(s) = \int_0^s T_k(r)\,dr$ one has

$$\begin{cases} \text{for any } \phi \in \mathfrak{D}(Q) \text{ and any } v \text{ in } L^p(0, T; W^{1,p}(\Omega)) \\ \text{with } \dfrac{\partial v}{\partial t} \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q) \\ \left\langle\!\!\left\langle \dfrac{\partial v}{\partial t}, \varphi T_k(v) \right\rangle\!\!\right\rangle = -\displaystyle\int_Q \dfrac{\partial \phi}{\partial t} S_k(v)\,dx\,dt \end{cases} \tag{4.15}$$

where $\langle\langle\ ,\ \rangle\rangle$ denotes the duality pairing between $L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)$ and $L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$; formula (4.15) is easily proved by approximating v by convolution.

Fix now a compact set K with $K \subset Q$ and a function ϕ_K in $\mathfrak{D}(Q)$ with $0 \leq \phi_K \leq 1$ in Q and $\phi_K = 1$ on K. Using in (4.1) the test function

$$v_n = (T_k(u_n) - T_k(u))\phi_K$$

is licit because g_n belongs to $L^1(Q)$; this yields

$$\begin{aligned} -\int_Q \frac{\partial \phi_K}{\partial t} S_k(u_n)\,dx\,dt &- \left\langle\!\!\left\langle \frac{\partial u_n}{\partial t}, \phi_K T_k(u) \right\rangle\!\!\right\rangle \\ &+ \int_Q \phi_K a(x, t, u_n, Du_n)[DT_k(u_n) - DT_k(u)]\,dx\,dt \\ &+ \int_Q (T_k(u_n) - T_k(u))a(x, t, u_n, Du_n)D\phi_K\,dx\,dt \\ = \int_0^T \langle f_n, (T_k(u_n) - T_k(u))\phi_K\rangle\,dt &+ \int_Q g_n(T_k(u_n) - T_k(u))\phi_K\,dx\,dt. \end{aligned} \tag{4.16}$$

Because of lemma 4.2, one has for a subsequence n'

$$T_k(u_{n'}) - T_k(u) \rightharpoonup 0 \text{ weakly in } L^p(0, T; W^{1,p}(\Omega)), \text{ strongly in } L^p_{\mathrm{loc}}(Q) \text{ and a.e. in } Q.$$

Therefore, the last three terms of (4.16) tend to 0 [for the latest, use Egorov's theorem on $(T_k(u_{n'}) - T_k(u))$ and hypothesis (4.7)] (see [15, lemma 3.4] if necessary).

Because of lemma 4.2, $S_k(u_n)$ tends to $S_k(u)$ strongly in $L^p_{\mathrm{loc}}(Q)$, which yields

$$-\int_Q \frac{\partial \phi_K}{\partial t} S_k(u_n)\,\mathrm{d}x\,\mathrm{d}t \to -\int_Q \frac{\partial \phi_K}{\partial t} S_k(u)\,\mathrm{d}x\,\mathrm{d}t.$$

Concerning the second term of (4.16), since $\phi_K T_k(u)$ belongs to $L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ while $\partial u_n/\partial t$ is the sum of a bounded term in $L^{p'}(0, T; W^{-1,p'}(\Omega))$ and of g_n which weakly converges in $L^1(Q)$ one has

$$-\left\langle\!\!\left\langle \frac{\partial u_n}{\partial t}, \phi_K T_k(u) \right\rangle\!\!\right\rangle \to -\left\langle\!\!\left\langle \frac{\partial u}{\partial t}, \phi_K T_k(u) \right\rangle\!\!\right\rangle = \int_Q \frac{\partial \phi_K}{\partial t} S_k(u)\,\mathrm{d}x\,\mathrm{d}t$$

where (4.15) has been used to obtain the last equality [note that g belongs to $L^1(Q)$].

We thus have proved that

$$\int_Q \phi_K a(x, t, u_n, Du_n)[DT_k(u_n) - DT_k(u)]\,\mathrm{d}x\,\mathrm{d}t \to 0 \qquad \text{when } n \to +\infty. \tag{4.17}$$

Step 2. The proof is now exactly the same as in step 2 of the proof of theorem 3.1. One first deduces from (4.17) that

$$\int_K [a(x, t, u_n, DT_k(u_n)) - a(x, t, u_n, DT_k(u))][DT_k(u_n) - DT_k(u)]\,\mathrm{d}x\,\mathrm{d}t \to 0. \tag{4.18}$$

This immediately implies (4.8) when the "strong monotonicity" hypothesis (4.19) is assumed (see remark 3.4). If only (4.4) is assumed, the refinement of Leray and Lions' original argument already used in the elliptic case again proves (4.8).

We now turn to the analogue of theorem 2.1. We strengthen the hypotheses on a, replacing (4.3) and (4.4) by

$$[a(x, t, s, \zeta) - a(x, t, s, \zeta^*)][\zeta - \zeta^*] \geq \begin{cases} \alpha|\zeta - \zeta^*|^p & \text{if } p > 2 \\ \alpha \dfrac{|\zeta - \zeta^*|^2}{(d(x, t) + |\zeta| + |\zeta^*|)^{2-p}} & \text{if } p \leq 2 \end{cases}$$

$$\text{a.e. } (x, t) \in Q,\ \forall s \in \mathbb{R},\ \forall \zeta, \zeta^* \in \mathbb{R}^N \tag{4.19}$$

for some $\alpha > 0$, where $d(x, t)$ belongs to $L^p(Q)$, $d \geq 0$. Such a hypothesis is satisfied by $a(x, t, s, \zeta) = |\zeta|^{p-2}\zeta$.

We also replace hypothesis (4.7) on g by the following ones

$$g_n \in L^{p'}(0, T; W^{-1,p'}(\Omega)) \tag{4.20}$$

$$g_n \text{ bounded in } \mathcal{M}(Q). \tag{4.21}$$

THEOREM 4.3. Assume that (4.1), (4.2), (4.5), (4.6) and (4.19)–(4.21) hold true. Then

$$Du_n \to Du \quad \text{strongly in } (L^q(\Omega))^N \text{ for any } q < p. \tag{4.22}$$

Remark 4.1. Theorem 4.3 has been proved in [6, theorem 4.1] in the quasilinear case (i.e. when $p = 2$ and $a(x, t, s, \xi) = b(x, t, s)\xi$ for some coercive matrix b) by a completely different method based on the $C^{0,\alpha}$ regularity of the solution to the adjoint linear problem with some special right-hand side.

In the same paper, we applied the almost everywhere convergence of the gradients to prove [6, theorem 2.1] that the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div} a(x, t, u, Du) + g(x, t, u, Du) = f & \text{in } \mathcal{D}'(Q) \\ u(0) = 0 \end{cases} \tag{4.23}$$

has at least one solution u in $L^p(0, T; W_0^{1,p}(\Omega)) \cap C^0([0, T]; W^{-1,r'}(\Omega))$ with $1 \le r' < \inf(p', N/(N-1))$, satisfying further $g(x, t, u, Du)$ and $ug(x, t, u, Du)$ in $L^1(Q)$. This result is proved when f belongs to $L^{p'}(0, T; W^{-1,p'}(\Omega))$, g is a Carathéodory function which satisfies

$$sg(x, t, s, \zeta) \ge 0, \qquad |g(x, t, s, \zeta)| \le b(|s|)(h(x, t) + |\zeta|^q) \tag{4.24}$$

(with $1 \le q < p$, h in $L^1(Q)$ and $b\colon \mathbb{R} \to \mathbb{R}$ a continuous increasing function) and the function a satisfies (4.2)–(4.4). Although the present theorem 4.3 is not proved under so weak hypotheses on a, let us point out that the proof of theorem 2.1 of [6] is completely correct; indeed in that case the approximation $g_n(x, t, u_n, Du_n)$ is weakly compact in $L^1(Q)$ and we can use the present theorem 4.1 as well as theorem 4.3.

Proof of theorem 4.3. *Step* 1. Fix a compact set K with $K \subset Q$ and a function ϕ_K in $\mathcal{D}(Q)$ with $0 \le \phi_K \le 1$ in Q and $\phi_K = 1$ on K. Since g_n is assumed to belong to $L^{p'}(0, T; W^{-1,p'}(\Omega))$ [see (4.20)], $\partial u_n/\partial t$ belongs to the same space, which allows one to perform the usual integrations by parts. However $\partial u/\partial t$ only belongs to $L^{p'}(0, T; W^{-1,p'}(\Omega)) + \mathcal{M}(Q)$ in general and this prevents one using the test function $\phi_K T_\eta(u_n - u)$. We thus use, as in [20], u_m as an approximation of u, and consider the test function

$$v_{n,m} = \varphi_K T_\eta(u_n - u_m).$$

Defining $S_\eta\colon \mathbb{R} \to \mathbb{R}$ by $S_\eta(s) = \int_0^s T_\eta(r)\,\mathrm{d}r$, we obtain from equations (4.1) relative to both u_n and u_m

$$\begin{aligned} &-\int_Q \frac{\partial \phi_K}{\partial t} S_\eta(u_n - u_m)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_Q \phi_K[a(x, t, u_n, Du_n) - a(x, t, u_m, Du_m)]DT_\eta(u_n - u_m)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_Q T_\eta(u_n - u_m)[a(x, t, u_n, Du_n) - a(x, t, u_m, Du_m)]D\phi_K\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \langle f_n - f_m, \phi_K T_\eta(u_n - u_m)\rangle\,\mathrm{d}t + \int_0^T \langle g_n - g_m, \phi_K T_\eta(u_n - u_m)\rangle\,\mathrm{d}t. \end{aligned} \tag{4.25}$$

Because of lemma 4.2, when n and m tend to infinity one has

$$\begin{cases} T_\eta(u_n - u_m) \rightharpoonup 0 & \text{weakly in } L^p(0, T; W^{1,p}(\Omega)) \text{ and strongly in } L^p_{\text{loc}}(Q) \\ S_\eta(u_n - u_m) \to 0 & \text{strongly in } L^p_{\text{loc}}(Q). \end{cases} \tag{4.26}$$

From (4.2), (4.5), (4.6) and (4.26) it follows that the first, third and fourth terms of (4.25) tend to 0. On the other hand, similarly to (2.20), the last term is bounded by $C_K\eta$. We have proved that for η fixed

$$\limsup_{n,m\to+\infty} \int_K [a(x, t, u_n, Du_n) - a(x, t, u_m, Du_m)]DT_\eta(u_n - u_m)\,\mathrm{d}x\,\mathrm{d}t \le C_K\eta. \tag{4.27}$$

Step 2. Define the functions $e_{n,m}$, $\bar{e}_{n,m}$ and $r_{n,m}$ by

$$\begin{cases} e_{n,m}(x, t) = [a(x, t, u, Du_n) - a(x, t, u, Du_m)][Du_n - Du_m] \\ \bar{e}_{n,m}(x, t) = [a(x, t, u_n, Du_n) - a(x, t, u_m, Du_m)][Du_n - Du_m] \\ r_{n,m}(x, t) = [a(x, t, u_n, Du_n) - a(x, t, u, Du_n)][Du_n - Du_m] \end{cases} \tag{4.28}$$

and note that

$$e_{n,m} = \bar{e}_{n,m} - r_{n,m} - r_{m,n}. \tag{4.29}$$

Fixing θ with $0 < \theta < 1$, we obtain from (4.27), by a proof similar to the proof of (2.24), that

$$\limsup_{n,m\to+\infty} \int_K \bar{e}^\theta_{n,m}\,\mathrm{d}x\,\mathrm{d}t \le (C_K\eta)^\theta |Q|^{1-\theta}. \tag{4.30}$$

We now claim that for any $\lambda > 0$ fixed one has

$$\limsup_{n\to+\infty} \int_K |r_{n,m}|^\theta\,\mathrm{d}x\,\mathrm{d}t \le C_0/|\lambda|^{p(1-\theta)} \tag{4.31}$$

where the constant C_0 does not depend on λ. Indeed, define

$$\bar{S}^\lambda_{n,m} = \{(x, t) \in K : |Du_n(x, t)| + |Du_m(x, t)| \le \lambda\}$$

$$\bar{G}^\lambda_{n,m} = \{(x, t) \in K : |Du_n(x, t)| + |Du_m(x, t)| > \lambda\}.$$

Hölder's inequality yields

$$\begin{aligned} \int_K |r_{n,m}|^\theta\,\mathrm{d}x\,\mathrm{d}t &= \int_{\bar{S}^\lambda_{n,m}} |r_{n,m}|^\theta\,\mathrm{d}x\,\mathrm{d}t + \int_{\bar{G}^\lambda_{n,m}} |r_{n,m}|^\theta\,\mathrm{d}x\,\mathrm{d}t \\ &\le \left(\int_{\bar{S}^\lambda_{n,m}} |r_{n,m}|\,\mathrm{d}x\,\mathrm{d}t\right)^\theta |\bar{S}^\lambda_{n,m}|^{1-\theta} + \left(\int_{\bar{G}^\lambda_{n,m}} |r_{n,m}|\,\mathrm{d}x\,\mathrm{d}t\right)^\theta |\bar{G}^\lambda_{n,m}|^{1-\theta}. \end{aligned} \tag{4.32}$$

Since Du_n and Du_m are bounded in $(L^p(Q))^N$, $|\lambda|^p|\bar{G}^\lambda_{n,m}|$ is bounded, which, combined with the fact that $r_{n,m}$ is bounded in $L^1(Q)$, implies that the second term of the right-hand side of (4.32) is bounded by $C_0/|\lambda|^{p(1-\theta)}$. Concerning the first term, set

$$\bar{S}^\lambda_n = \{(x, t) \in K : |Du_n(x, t)| \le \lambda\}, \qquad R_n(x, t) = |a(x, t, u_n, Du_n) - a(x, t, u, Du_n)|.$$

Denoting by $\chi_{\bar{S}_n^\lambda}$ the characteristic function of $\bar{S}_n^\lambda$, one has for any m and λ

$$\int_{\bar{S}_{n,m}^\lambda} |r_{n,m}| \, dx \, dt \le \lambda \int_K \chi_{\bar{S}_n^\lambda} R_n \, dx \, dt. \tag{4.33}$$

Because of Vitali's theorem, the right-hand side of (4.33) tends to 0 when n tends to infinity: indeed because of (4.2) $\chi_{\bar{S}_n^\lambda} R_n$ is dominated by $2c + k_1(|u_n|^{p-1} + |u|^{p-1}) + 2k_2|\lambda|^{p-1}$ which converges strongly in $L^{p'}(K)$, while extracting a subsequence n' such that (see lemma 4.2) $u_{n'}$ tends to u almost everywhere in K, $\chi_{\bar{S}_{n'}^\lambda} R_{n'}$ can be proved to converge almost everywhere to 0. This completes the proof of (4.31).

Choosing η small and λ large and using (4.29)-(4.31) and

$$|e_{n,m}|^\theta \le 3^\theta[|\bar{e}_{n,m}|^\theta + |r_{n,m}|^\theta + |r_{m,n}|^\theta]$$

proves that

$$\limsup_{n,m \to +\infty} \int_K |e_{n,m}|^\theta \, dx \, dt = 0. \tag{4.34}$$

Using hypothesis (4.19), this immediately implies that Du_n is a Cauchy sequence in $(L^{\theta p}(K))^N$ when $p \ge 2$; when $p \le 2$, Hölder's inequality yields

$$\int_K |Du_n - Du_m|^{\theta p} \, dx \, dt \le \left(\int_K \frac{|Du_n - Du_m|^{2\theta}}{(d(x,t) + |Du_n| + |Du_m|)^{(2-p)\theta}} \, dx \, dt \right)^{p/2} \cdot \left(\int_K (d(x,t) + |Du_n| + |Du_m|)^{p\theta} \, dx \, dt \right)^{(2-p)/2} \tag{4.35}$$

which combined to (4.19) and (4.34) gives the same result. This proves that

$$Du_n \to Du \qquad \text{strongly in } (L^q(K))^N \tag{4.36}$$

for any $q < p$ and any compact set K with $K \subset Q$. Since Du_n is bounded in $(L^p(Q))^N$, Hölder's inequality proves that (4.36) holds true for $K = Q$, which completes the proof of theorem 4.3. ■

Note added in proof—In a slightly different context, a result similar to the result of theorem 3.1 above is one of the main tools in the proof of the existence of "renormalized solutions for some nonlinear elliptic equations": see the paper to appear by Lions and Murat "Sur les solutions renormalisées d'equations elliptiques non linéaires". A generalization of theorem 2.1 above is lemma 1 of the paper "Nonlinear elliptic equations with right hand side measures" by Boccardo and Gallouet, to appear in *Communs partial diff. Eqns.*

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