Economic Scenario Generation

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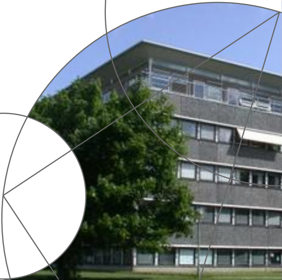


What is Economic Scenario Generation?

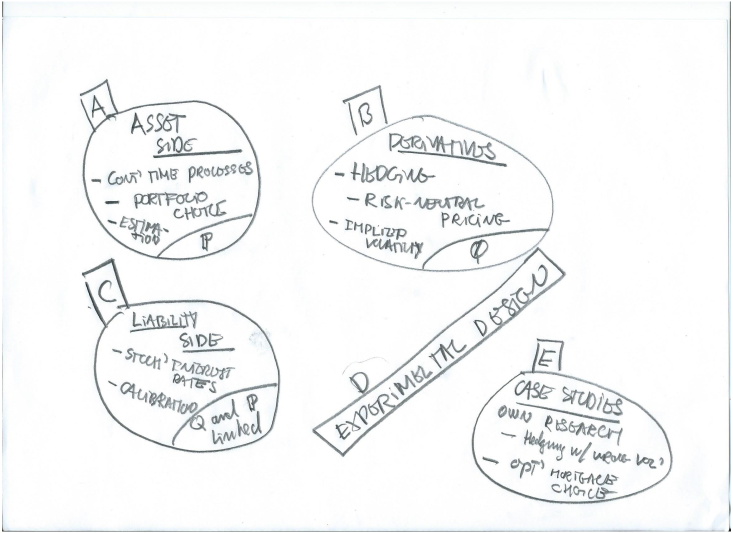
The Society of Actuaries in their guidelines :

1. WHAT IS AN ECONOMIC SCENARIO GENERATOR?

An economic scenario generator (ESG) is a computer-based model of an economic environment that is used to produce simulations of the joint behavior of financial market values and economic variables. Two common applications are driving the increased utilization of ESGs:

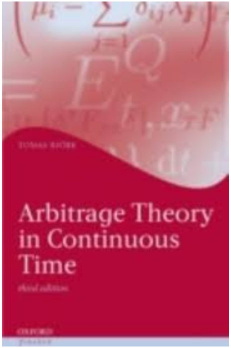
1. **Market-consistent (risk-neutral) valuation work for pricing complex financial derivatives and insurance contracts with embedded options.** These applications are mostly concerned with mathematical relationships within and among financial instruments and less concerned with forward–looking expectations of economic variables.
2. **Risk management work for calculating business risk, regulatory capital and rating agency requirements.** These applications apply real-world models that are concerned with forward-looking potential paths of economic variables and their potential influence on capital and solvency.

Mind-map: Many moving parts



Generic references

Textbook:



Not textbook: Poulsen (2017)

Part A: Stock market modelling, asset side, $\mathbb{P}$

Geometric Brownian motion as a continuous-time workhorse.

Estimation:

- $\widehat{\mu} \sim$ last - first, higher obs' frequency does not help, imprecisely estimated
- $\widehat{\sigma} \to \sigma$ as obs' frequency increases

Portfolio choice concepts: Utility functions, risk-aversion, Sharpe ratio. Sensible/consensus/prudent choices.

~~Multiple stocks: Single index CAPM, Black-Litterman~~

Model failings: Heavy tails; stochastic volatility.

Part B: Derivatives, risk-neutral pricing, $\mathbb{Q}$

Dynamic Δ-hedging in the Black-Scholes model: PDE, call-price formula. Drift doesn't matter. Volatility does.

Representing arbitrage-free prices: Martingale measures and risk-neutral pricing.

A discrete hedge experiment.

Implied volatility.

Incomplete or misspecified models. The fundamental theorem of derivative trading: With a decent idea of volatility, we're good.

Part C: Bond markets, liability side, $\mathbb{P}$ vs. $\mathbb{Q}$

As a catch-phrase: Risk-management is also important on the liability side. (Cases abound)

Stochastic interest rate: A specific, general case. The term structure PDE.

Vasicek as a worked example, including $\mathbb{P}$-$\mathbb{Q}$ specifics and yield curve calibration.

~~Multi-factor models, Heath-Jarrow-Morton formalism~~.

Joint stock-bond modelling; correlations.

~~Build on same ideas: FX, inflation~~.

Part D: Experiments; simulation and empirical

Commonly seen errors:

- How not to simulate Brownian motion
- Discretization biases; stop/loss paradoxes
- Variance and scenario reduction

$\mathbb{Q}$-simulations: Sanity (martingale) checks, a prudent approach.

$\mathbb{P}$-simulations: Risk/return trade-offs.

Bootstrapping and back-testing: In-sample/out-of-sample considerations.

Part E: Case studies, aka my own papers

(Realistically: No time.)

Option hedging:
The fundamental theorem of derivative trading.

Mortgage choice and asset-liability management:

- A Gaussian 3-factor model, a mortgage bond pricing approximation, and a stock price model.
- Mortgagors, criteria and benchmarks.

Continuous-time stochastic processes

The work-horse: Geometric Brownian motion, which solves the stochastic differential equation

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),$$

where W is a Brownian motion. We call μ the drift (rate) and σ the volatility. Here, they are constants.

Intuition/interpretation:

$$\begin{aligned} \mathrm{E}_t\left(\frac{S(t+dt)-S(t)}{S(t)}\right) &\approx \mu dt, \\ \mathrm{var}_t\left(\frac{S(t+dt)-S(t)}{S(t)}\right) &\approx \sigma^2 dt. \end{aligned}$$

Intuition/approximation in simulation: The Euler scheme

$$S(t+dt) = S(t) + \mu S(t)dt + \sigma S(t)\sqrt{dt}\,\epsilon_{t+dt},$$

where all the ϵ's are independent $N(0,1)$.

The previous results also hold with non-constant drift and volatility.

An explicit representation of geometric Brownian motion is

$$S(u) = S(t)e^{(\mu-\sigma^2/2)(u-t)+\sigma(W(u)-W(t))} \text{ for } u \geq t \geq 0$$

Splits into measurable and independent. Highlights continuous nature. Derived w/ Ito on log. (Or the other way round.)

Parameter estimation

A useful exercise in the properties of (geometric) Brownian motion.

Suppose we have split the time interval $[0, T]$ into n pieces. Put $\Delta t = T/n$, $t_i = i\Delta t$ for $i = 1, \ldots, n$ and look at the logarithmic rates of return,

$$\xi_i = \ln\left(\frac{S(t_i)}{S(t_{i-1})}\right).$$

Under geometric Brownian motion, the ξ_i's are independent and $N(\alpha\Delta t, \sigma^2\Delta t)$, where $\alpha = \mu - \sigma^2/2$.

We can use the estimator

$$\widehat{\alpha} = \frac{1}{\Delta t n} \sum_{i=1}^{n} \xi_i = \frac{\ln(S(T)/S(0))}{T} \sim N(\alpha, \sigma^2/T)$$

So we see that increasing the sampling frequency (a higher n) does not help estimate the drift rate; only a longer observation period does.

For (squared) volatility we can use the estimator

$$\widehat{\sigma}^2 = \frac{1}{T} \sum_{i=1}^{n} \xi_i^2.$$

(A commonly used pragmatic alternative is the exponentially weighted moving average method [analyzed here .)

We immediately see that $\mathrm{E}(\widehat{\sigma}^2) = \sigma^2 + O(1/n)$.

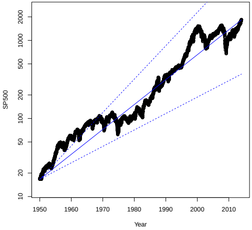
Moreover, using independence and moment properties of the normal distribution (in particular that $\mathrm{E}(N(0,\sigma^2)) = 3\sigma^4$) we have

$$\begin{aligned}
\mathrm{var}(\widehat{\sigma}^2) &= \frac{1}{T}\sum_{i=1}^{n}(\mathrm{E}(\xi_i^4) - (\mathrm{E}(\xi_i))^2) \\
&= \frac{n}{T}(3\sigma^4\Delta t^2 - \sigma^4\Delta t^2 + O(\Delta t^3)) \\
&= \frac{2\sigma^4 T}{n}(1 + O(1/n)) \to 0
\end{aligned}$$

So under perfect conditions increasing the sampling frequency (over an arbitrarily short time interval) tells us exactly what the volatility is.

Estimation applied to 60+ years of SP500-data.

- Drift rate est' ~7.5%, but it could as well be 5% or 10%
- Vol' est' ~15%; stable across observation frequency – doesn't happen by construction



α: est' $\pm$ std' err' $= 0.073 \pm 0.025$

Frequency	$\widehat{\sigma}$
Yearly	0.152
Monthly	0.146
Weekly	0.153
Daily	0.155

More favorite data sources:

- $\boxed{\text{Kenneth French}}$
- $\boxed{\text{Robert Shiller}}$

Good when you need interest rates and dividends.

Risk-Aversion

Utility functions are used to capture quantitatively the trade-off between risk and expected reward.

We imagine (in the simplest case) an agent who wants to act in order to maximize $\mathrm{E}(u(W(T)))$, where $W(T)$ is (random, but controlled) wealth at time T. Concave $u \sim$ risk-aversion.

A common utility function is

$$u(x) = \frac{x^{1-\gamma} - 1}{1-\gamma},$$

for which relative risk-aversion is constant, $-\frac{xu''}{u'} = \gamma$.

Utility functions are invariant to affine transformations; improvements must be measured through $\boxed{\text{wealth equivalents}}$ (or something of that nature).

In the most basic dynamic portfolio choice set-up, it is optimal for such an agent to invest the fraction

$$w^* = \frac{\mu - r}{\gamma \sigma^2}$$

of wealth in the stock.

Note: (a) Constant fraction $\rightsquigarrow$ dynamic trading, (b) $\boxed{\text{horizon independence}}$.

Risk-aversion parameter levels:

- Experiments and survey data: 2–5 (maybe 10)
- Mathematical convenience: 1 ($\sim$ log-utility)
- Numerical convenience: $\frac{1}{2}$ ($\rightsquigarrow$ utility bounded at 0)
- Consumption-based CAPM backwards: $\sim$ 50 (unrealistically high – see e.g. page 27 here – the equity premium puzzle)

Caveat: Parametrization in some way other than through relative risk-aversion.

Equity premium $= \mu_t - r_t$; $\boxed{\text{data puzzling}}$, $\boxed{\text{theory ongoing}}$

The Sharpe ratio is a measure of risk-reward tradeoff,

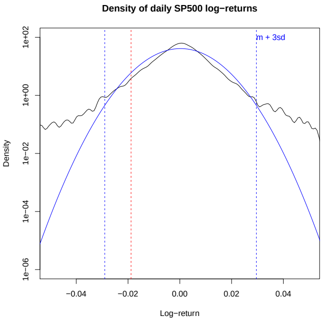
$$SR_t = \frac{\mu_t - r_t}{\sigma_t}.$$

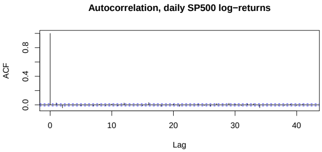
(Careful about time-scaling; σ, not σ^2.)

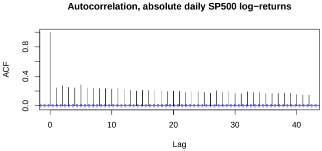
For something like S&P500, $SR \sim$ 0.25-0.5.

Can/should be calculated for any investment strategy;
$SR > 1$ raises suspicion. (Bernie Madoff: 2.5–4)

Empirical failings of geom. Brownian motion







Tails (in particular the left) are heavier than normal. But 99% of the time things look fine.

Returns are uncorrelated over time; absolute returns absolutely are not.

Derivatives, risk-neutral pricing, $\mathbb{Q}$

Important concept: Arbitrage (opportunity).

Intuitively: A free lunch; free money – or at least free lottery tickets.

More precisely: A portfolio strategy that does not cost us anything, and/but always pays off positively (or: non-negatively).

A way to spot: A risk-free portfolio that doesn't earn the risk-free rate.

Note:

- Assuming no arbitrage (there is no such thing as a free lunch) is not wishful thinking (an efficient, invisible benevolent hand), but rather a prudent, conservative view.
- A favorable, a good deal is not an arbitrage, Having $\mu > r$ is not arbitrage.

Look at a specific call option and assume its underlying follow a geometric Brownian motion.

Suppose (an *ansatz*) $\text{Call}(t) = F(t, S(t))$ where F is some as of yet unknown function.

At time t form a portfolio that is short 1 call-option and long $a(t)$ units of the stock. The value of the portfolio is

$$V(t) = a(t)S(t) - F(t, S(t)).$$

By Ito's formula and w/ shorthand notation we have

$$dV = adS - \sigma F_S S dW - (F_t + \mu S F_S + \frac{1}{2}\sigma^2 S^2 F_{SS})dt$$

If we chose $a = F_S =: \Delta$, the dW-terms cancel.

So, in fact, do the terms with μ.

Multiplying and dividing by $V = F_S S - F$ we have

$$dV(t) = V(t)\left(\frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{SS}}{F - SF_S}\right) dt.$$

Now, since V is locally risk-free, to preclude arbitrage, its drift rate (the term in the big parenthesis above)must equal the risk-free rate, r.

Rearranging that leads to the Black-Scholes partial differential equation for the arbitrage-free price function F,

$$F_t + rSF_S + \frac{1}{2}\sigma^2 S^2 F_{SS} = rF,$$

where for the call-option $F(T, S) = (S - K)^+$.

The price can be represented (by Girsanov's theorem, which is what tells us that $dW^{\mathbb{Q}} = dW^{\mathbb{P}} + \lambda dt$ defines a new Bronwian motion; Björk's Theorem 11.3),

$$\mathrm{Call}(t) = e^{-r(T-t)} \mathrm{E}_t^{\mathbb{Q}} \left((S(T) - K)^+ \right),$$

where $\mathbb{Q}$ is a probability measure such that

$$dS(u) = rS(u)dt + \sigma S(u) dW^{\mathbb{Q}}(t),$$

with $W^{\mathbb{Q}}$ being a $\mathbb{Q}$-Brownian motion.

Similar results holds with stochastic interest rate, dividends, general volatility, and jumps in the stock price.

This is called risk-neutral valuation/pricing and it is a deceptively simple result/principle:

- The arbitrage-free price can be represented as a conditional expected value. Arguably, it is a computational trick, inarguably not an assumption about agents.
- Prices can be backed up constructively by hedging or replicating trading strategies: Hold $\Delta := F_S$ units of the underlying.
- As catch-phrases: Risk-neutral pricing does not assume risk-neutrality; prices are not literally expected values.
- Abstract nonsense formulations: Fundamental theorems of asset pricing. 1st: No arbitrage $\Leftrightarrow \exists\ \mathbb{Q}$; (2nd) Completeness $\Leftrightarrow \mathbb{Q}$ unique

Other words and terms that mean sort of the same thing:

- Pricing kernel (prices as $\mathbb{P}$-expectations)
- Stochastic discount factor (ditto)
- State price deflator (when you just like integral transforms)
- Numeraires (when the (locally) risk-free assets is not given a special role)

Why does the option price not depend on μ?

The abstract nonsense subtleties on the previous slide, but the μ-independence of option prices is a concrete and quite counter-intuitive conclusion.

Correct, if technical argument:

- The replication argument gave us a unique price based solely on no arbitrage-considerations.
- Arbitrages are invariant to equivalent measure changes.
- Another way to formulate Girsanov's theorem is that via equivalent measure changes, we can make the drift anything.
- Hence, the option price can depend on what the drift is.

One-period binomial: If you change p, the strange thing isn't that the option price doesn't change, but that $S(0)$ doesn't.

Tests for faulty thinking:

- Option prices *do* depend on volatility.
- The put-call-parity. Left hand side does not depend on $\mathrm{E}^{\mathbb{P}}(S(T))$, so at least μ must affect calls and puts in the same way.
- Beware incompleteness.

The Black-Scholes Formula

The call-option PDE is solved by the Black-Scholes formula, i.e.

$$F(t,x) = x\Phi(d_+(x,t)) - Ke^{-r(T-t)}\Phi(d_-(x,t)),$$

where Φ is the standard normal distribution function and

$$d_\pm(x,t) = \frac{\ln(x/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

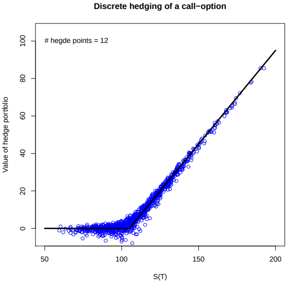
We have $\Delta := F_x = \Phi(d_+)$.

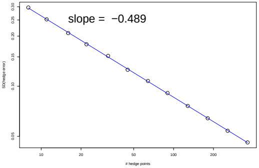
Proof Direct calculations (page 130 here) — that I really don't like when people leave out of textbooks. (They are lengthier than the final expressions suggest.)

A discrete hedge experiment

The core of $\boxed{\text{the code}}$:

```
St=St*exp(α*dt+σ*sqrt(dt)*rnorm(0,1))
Vpf=a*St+b*exp(dt*r)
a=BlackScholesDelta(St,T-t,K,r,σ_H)
b=Vpf-a*St
⋮
hedgeerror<-(Vpf-(St-strike,0)^+)
```





Hedging works along each path – not just *on average*.

The standard deviation of the hedge error (for a call option) goes to 0 like $1/\sqrt{n}$.

Implied volatility

We can view the Black-Scholes call price formula as a function of the volatility σ; $\text{call}^{BS}(\sigma; \ldots)$.

Result Call- and put option prices are increasing in volatility. Their Vegas (derivatives wrt. σ) are positive; $\text{Vega}(t) = S_t \phi(d_+(S(t), t))\sqrt{T - t}$.

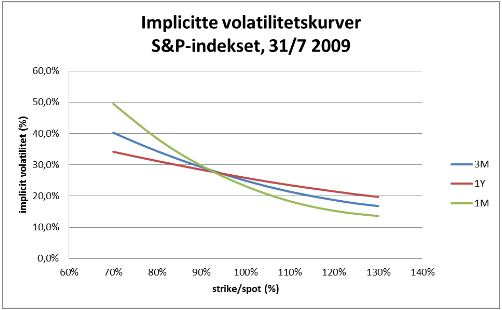
So for a certain observed call price in the market, call^{obs}, we can define and calculate its implied volatility, σ^{imp} as the solution to

$$\text{call}^{BS}(\sigma^{imp}; \ldots) = \text{call}^{obs}.$$

Often market participants quote (or parameterize) call option prices in terms of implied volatilities.

This does not mean that they believe the Black-Scholes model holds. *Au contraire*: If it did, all implied volatilities would be the same.

A typical picture with $\boxed{\text{real data}}$



Implied volatilities display skews (stocks, interest rates) or smiles (FX) in strikes, particularly at short expiries.
Implied volatility is a forward looking forecast of future realized volatility.

Empirical evidence (a) implied volatilities are typically above realized volatilities (volatility risk premia), (b) implied volatilities contain some information not easily captured by historical, backward-looking estimation estimation methods.

Volatility benchmark: $\boxed{\text{VIX}}$

The Fundamental Thm. of Derivative Trading

Suppose our bank has sold the option and we are now told to hedge the position.

Imagine we do that as if we were in a Black-Scholes model with volatility σ_{hedge}.

However, the real world is more complicated:

$$dS(u) = \sigma_{actual}(u)S(u)dW(u),$$

Theorem Our total portfolio value, our profit-and-loss, our P&L at option expiry breaks down as

$$
\begin{aligned}
\text{P\&L}_T &= V_{implied}(0) - V_{hedge}(0) \\
&\quad + \frac{1}{2}\int_0^T S^2(u)\Gamma(S(u),u)(\sigma^2_{hedge} - \sigma^2_{actual}(u))du,
\end{aligned}
$$

where Γ is the Black/Scholes-Gamma of the option being hedged.

Proof: Not as difficult as you'd think. Careful application of Ito's formula, the Black-Scholes pricing PDE and the self-financing condition.

1st term on the right-hand side is what we get when we sell the option less what it costs to set up the hedge ($= 0$ if we hedge with implied volatility).

2nd term is hedge error of the dynamic strategy. Note that it is "only dt, not dW"; *misspecification causes bleeding, not blow-up*.

The theorem says — in quite a quantitative way — that if we Δ-hedge dynamically, then successful derivative trading comes down to estimating/predicting/guessing volatility.

Corollary Option trading is not a 0-sum game between buyer and seller.

$\mathbb{P}$-expected option returns

$$dCall(t) = \left(F_t + \mu S(t) F_S + \frac{1}{2}\sigma^2 S^2(t) F_{SS} \right) dt + \sigma S(t) F_S dW^{\mathbb{P}}.$$

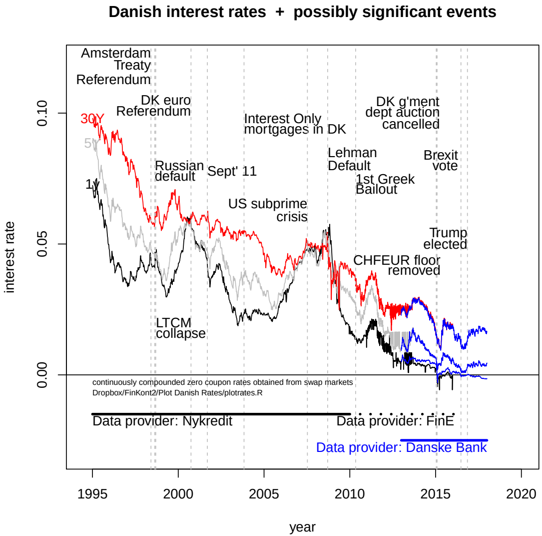
This we may rewrite this as

$$dCall(t) = Call(t) \left(\underbrace{\left(\frac{(\mu - r) S(t) F_S}{F} + r \right)}_{:= \mu^{Call}} dt + \frac{\sigma S(t) F_S}{F} dW^{\mathbb{P}} \right),$$

meaning that the instantaneous $\mathbb{P}$-expected excess rate of return satisfies the CAPM-like equation

$$\mu^{Call} - r = \frac{S(t) F_S}{F} (\mu - r).$$

Interest rates; Stories in a $\boxed{\text{picture}}$

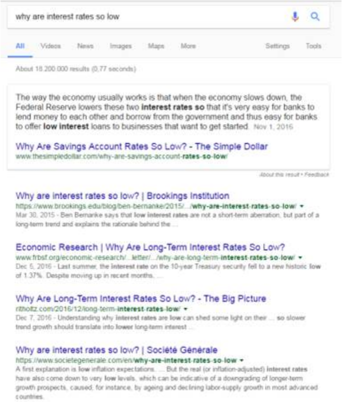


The elephant in the room: Why have interest rates become so low?

What does Google say?

Be critical of sources – none here are academic and/or peer reviewed.

I think I like SocGen's piece best.



What determines interest rates? What moves them around?

Loads of suspects:

- Agents' preferences for consuming now vs. saving for later.
- The borrower's creditworthiness.
- Supply and demand; fear and greed. (Speculation sounds less nice.)
- Macroeconomics: Expected growth rates and inflation, fiscal policy, monetary policy (central banks but subject to political pressure).
- Institutional structure (labor, housing and mortgage markets, legal and political systems.
- Life and pensions angle: Solvency II, financial authorities, and Smith-Wilson curves. (There! I said it.)

Overload! Too much to model in detail

Borrowing/lending is the the biggest thing in finance, so we must attack it.

Let's just accept that randomness is a very large component and build empirically plausible stochastic models. Worked well for stocks.

Nice: The martingale formalism, our fundamental theorems of asset pricing (absence of arbitrage, completeness, . . .) carry over.

We will be looking only at non-trivial special cases. (And the more special, the more non-trivial.)

A quote that has aged well

Wilmott and Rasmussen (2003)

1990s Credit risk

If we were to award marks out of 10 for the scientific accuracy of financial models, we would probably give about 7 for the lognormal random walk model for equities. There are one or two problems to do with stability of parameters, serial autocorrelation, the distribution of returns and continuity of asset paths but, what the hell, it's still pretty good. If we were to give a mark for interest rate models, we would have to give them 3, or maybe 4. It doesn't take a statistical genius to show how poor these models actually are. But then again, so what? If they are popular, does it matter? Now, there's a question. . . . but not one we'll be answering here.

Moving on to credit risk models, we'd be hard pressed to give them even 1 out of 10. Mathematical models for credit risk have taken the same well-worn path trodden successfully by equity product models and, less successfully, by interest rate product models. But credit risk is an entirely different beast. For many different reasons, not least the inability to hedge and the extreme nature of returns, credit risk modelling is a whole new ball game. To spare their blushes, we have not named the guilty parties.

A model where only real-world short rate dynamics are specified is not complete. To guarantee (replicate) 1 at time T, we'd have to invest $\exp(-\int_0^T r(u)du)$ in the bank-account at time 0 — but we don't yet know that number.

We get consistency relation between ZCBs of different maturities. So nice we may forget we have a problem at all.

We extend the usual PDE derivation to stochastic interest rates. A line of reasoning first done in Vasicek (1977).

Look at the case where

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW^{\mathbb{P}}(t)$$

where μ and σ are functions and $W^{\mathbb{P}}$ is a 1-dimensional Brownian motion under the real-world probability measure $\mathbb{P}$.

The formal equation

$$P(t;T) = \mathsf{E}_t^{\mathbb{Q}}\left(\exp\left(-\int_t^T r(u)du\right)\right)$$

and the Markov property of r makes us conjecture that

$$P(t;T) = F(t, r(t); T)$$

for some smooth F (of 3 variables.)

We now go through similar reasoning as when we derived the Black-Scholes PDE with two zero coupon bonds with different maturities (say, S and T) in the roles of the traded assets.

A key observation is that to prevent arbitrage we must have

$$\frac{\mu^S - r(t)}{\sigma^S} = \frac{\mu^T - r(t)}{\sigma^T}$$

LHS doesn't depend on T, RHS doesn't depend on $S \Rightarrow$ the ratio is independent of maturity,

$$\frac{\alpha^S - r(t)}{\sigma^S} := \lambda(r(t); t)$$

λ is called the market price of risk; interpretation as excess expected return relative to volatility; a Sharpe ratio. Has to be exogenously specified.
Usually: Postulate form that gives same structure under $\mathbb{P}$ and $\mathbb{Q}$ — a subtlety that people may be obtuse about.

Substitute back and get the term structure PDE:

$$F_t + (\mu - \lambda\sigma)F_r + \frac{1}{2}\sigma^2 F_{rr} = rF \quad \text{and } F(T, r; T) = 1$$

This may be Feynman-Kac represented (see Björk's Exercise 5.12) and we may change measure:

$$F(t, r(t); T) = \mathsf{E}_t^{\mathbb{Q}}\left(\exp\left(-\int_t^T r(s)ds\right)\right)$$

where

$$dr(s) = (\mu - \lambda\sigma)ds + \sigma dW^{\mathbb{Q}}(s)$$

Note: Clearly $P(t; T)/\beta(t)$ is a $\mathbb{Q}$-martingale, where $\beta(t) = e^{\int_0^t r(u)du}$ is the bank-account.

The Vasicek model

$\mathbb{Q}$-dynamics

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW(t)$$

AKA Ornstein-Uhlenbeck process and various other aliases.

Mean-reversion: The drift pulls the process back towards the long-term (stationary) level θ.

The force of the mean-reversion governed by κ. Quantitative intuition: $\frac{\ln 2}{\kappa}$ is the half-life of expected deviances from θ.

We call σ volatility, despite it being on absolute (not proportional) form.

The Ito formula on $e^{\kappa t} r(t) \rightsquigarrow$

$$r(t) = r(0)e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \underbrace{\int_0^t \sigma e^{-\kappa u} dW(u)}_{\sim N(0, \frac{\sigma^2(1-e^{-2\kappa t})}{2\kappa})}.$$

Zero coupon bond prices have the form

$$P(t, T) = e^{A(t,T) - B(t,T)r(t)},$$

where $B(t, T) = \frac{1-e^{-\kappa(T-t)}}{\kappa}$ and

$$A(t, T) = \left(\theta - \frac{\sigma^2}{2\kappa^2}\right)((T - t) - B(t, T)) - \frac{\sigma^2 B^2(t, T)}{4\kappa}$$

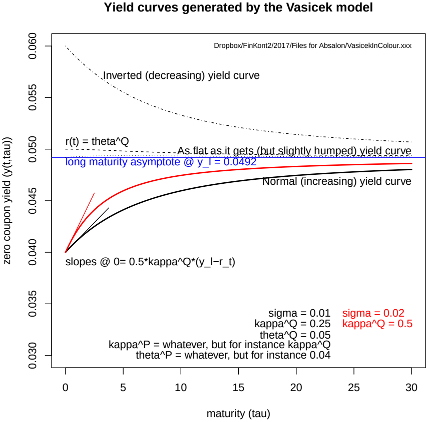
Can be derived by a direct calculation or by an affine trick $\rightsquigarrow$ ODEs.

What the Vasicek model does – in a picture

Current short rate r_t and $\mathbb{Q}$-parameters $(= (\mathbb{P}, \lambda))$ $\rightsquigarrow$ model yield curve.

The model yield curve may look nothing like the market's.

As r_t move around over time, the model yield curve moves. But like a dog and its tail.



Mind your $\mathbb{P}$s and $\mathbb{Q}$s

When $r_t = \theta$, which is the typical value, the yield curve is almost flat (slightly inverted to be precise).

Empirically violated. So have we created a useless model?

No. Assuming that λ is an honest-to-god constant

$$dr(t) = \kappa \left(\underbrace{\theta + \frac{\lambda\sigma}{\kappa}}_{=:\theta^{\mathbb{P}}} - r(t) \right) dt + \sigma dW^{\mathbb{P}}(t)$$

The sign artefact.

Vasicek calibration

If we use the model

$$dr(t) = \kappa(\theta(t) - r(t)) + \sigma dW^{\mathbb{Q}}(t)$$

where the deterministic function θ is

$$\theta(t) = (f_T^{obs}(0,t) + g'(t))/\kappa + (f^{obs}(0,t) + g(t)),$$

with $g(t) = \sigma^2 B(0,t)/2$ and $f^{obs}(0,T)$ being the initially observed forward rate curve, **then** theoretical and observed zero coupon bond prices match perfectly.

Try Björk's exercise 24.7 for fun.

Arguably, calibration is vital: How can you trust a model that does not price the most basic things – whose prices we can observe – correctly?

Formula involves derivative of forward rate curve; not that easy to find. But often we may not actually need to calculate it, we just need the prices of calibration instruments.

Recalibration: Conceptually, it makes you head hurt. Universal practice. Are we refining our model or just digging a deeper hole?

The Cox-Ingersoll-Ross model

CIR dynamics (AKA a $\sqrt{}$-process or a Feller-process)

$$dr(t) = \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t)$$

Affine model: Drift and volatility squared are affine functions of r (i.e. $a + br$).

This means that — like in the Vasicek model — zero-coupon bond prices are of the form

$$P(t, T) = e^{A^{CIR}(t,T) - B^{CIR}(t,T)r(t)},$$

where the functions A^{CIR} and B^{CIR} satisfy ODEs that can solved explicitly.

Analytical tractability combined with several appealing features: Mean-reversion/stationarity, level dependent volatility, positivity.

Suggested other uses in the literature: Volatility itself (the Heston model), default intensity in credit risk modelling.

My warning, though: Stay away from it. It looks good on paper, but does not do well in practice/production. Simulation tricky, Heston's formula tricky to implement (see Roger Lord's work), very rarely empirically superior to both Ornstein-Uhlenbeck and log-O-U.

Ilmanen (2003): "The causality from bond prices to stock prices is positive [...], while the causality from stock to bond prices is negative"

Kraft & Munk (2011)

Kraft and Munk: *Optimal Housing, Consumption, and Investment Decisions over the Life Cycle*
Management Science 57(6), pp. 1025–1041, ©2011 INFORMS

Table 1 Benchmark Parameter Values

	Interest rate, bond	
κ	Mean reversion speed	0.2
$\bar{r}$	Long-term short rate	0.02
σ_r	Short rate volatility	0.015
λ_B	Sharpe ratio of bond	0.1
T_{bond}	Bond maturity	20
σ_B	Bond volatility	0.0736
ξ_B	In speculative part of π_B	−0.1679
ζ_B	In hedging part of π_B	−1.3835
	Stock	
σ_S	Stock volatility	0.2
λ_S	Sharpe ratio of stock	0.25
ξ_S	In speculative part of π_S	0.0439
ζ_S	In hedging part of π_S	−0.8335
	Preferences	
δ	Time preference rate	0.03
γ	Relative risk aversion	4
β	Weight, nonhousing cons.	0.8
$\tilde{T}$	Years to retirement	30
T	Years to live	50
ε	Bequest weight	0

	Housing	
σ_H	House price volatility	0.12
λ_H	Sharpe ratio of house	0.325
r^{imp}	Imputed rent	0.05
ν	Rental rate	0.05
λ'_H	Adj. Sharpe ratio of house	0.325
ξ_l	In speculative part of φ_l	0.4122
ζ_l	In hedging part of φ_l	1.6669
	Labor income	
b	Slope of growth	0.5
$\bar{\mu}_Y$	Fixed part of growth	0.01
$\overline{\sigma}_Y$	Income volatility	0.075
Υ	Replacement ratio	0.6
λ_Y	Income "Sharpe ratio"	0.1950
	Correlations	
ρ_{SB}	Stock-bond correlation	0
ρ_{HB}	House-bond correlation	0.65
ρ_{HS}	House-stock correlation	0.5
ρ_{YB}	Income-bond correlation	−0.3
ρ_{YS}	Income-stock correlation	0
ρ_{HY}	House-income correlation	0.3509

Simulation errors

Use explicit expressions if possible.

The Euler scheme works, but forces you to use small time-steps. Or at least think about (program) step sizes.

However, using Euler is usually not what causes your strange results. Nor is a bad random number generator.

But these mistakes may:

- `eps=rnorm(0,0,1); for (i in 1:n) W[i] = sqrt(i*dt)*eps`
- `for (i in 1:n) W[i] = rnorm(1,0,sqrt(i*dt))`
- `dW[i,1] = rnorm(1,0,sqrt(dt))`

  `dW[i,2] = rho* rnorm(1,0,sqrt(dt)) + sqrt(1-rho`^{2}`) rnorm(1,0,sqrt(dt))`

$\mathbb{Q}$-simulations

Traded assets: Drift rates $= r_t -$ dividend rate. (Or a jump condition with discrete dividends.)

Non-traded assets (interest rate, volatility): Use $\mathbb{Q}$-parameters — which may (possible, if not plausible) be equal to $\mathbb{P}$-parameters.

Should I use implied or actual/historical volatility?
Hmmm . . . that depends; not one right answer. But if I can't sit on the fence: Implied (if you've got it/them).

Value process $V^{\phi}(t) = \phi(t) \cdot \pi(t)$; β bank-account.

A trading strategy ϕ is self-financing if

$$\phi(t + dt) \cdot \pi(t + 1) = \phi(t) \cdot (\pi(t + 1) + \delta(t + 1)),$$

where π denotes generic prices.

In discrete time: Simple bookkeeping.

Sanity check: For any self-financing trading strategy, its discounted value process V^{ϕ}/β is a $\mathbb{Q}$-martingale. This holds irrespective of trading frequency and what the strategy might or might not replicate.