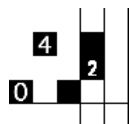


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Stochastic Processes and Advanced Mathematical Finance

Properties of Geometric Brownian Motion



Rating

Mathematically Mature: may contain mathematics beyond calculus with proofs.



Section Starter Question

What is the relative rate of change of a function?

For the function defined by the ordinary differential equation

$$\frac{dx}{dt} = rx \quad x(0) = x_0$$

what is the relative rate of growth? What is the function?



Key Concepts

1. Geometric Brownian Motion is the continuous time stochastic process $z_0 \exp(\mu t + \sigma W(t))$ where $W(t)$ is standard Brownian Motion.
2. The mean of Geometric Brownian Motion is

$$z_0 \exp(\mu t + (1/2)\sigma^2 t).$$

3. The variance of Geometric Brownian Motion is

$$z_0^2 \exp(2\mu t + \sigma^2 t)(\exp(\sigma^2 t) - 1).$$



Vocabulary

1. **Geometric Brownian Motion** is the continuous time stochastic process $z_0 \exp(\mu t + \sigma W(t))$ where $W(t)$ is standard Brownian Motion.

2. A random variable X is said to have the **lognormal** distribution (with parameters μ and σ) if $\log(X)$ is normally distributed ($\log(X) \sim N(\mu, \sigma^2)$). The p.d.f. for X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp((-1/2)[(\ln(x) - \mu)/\sigma]^2).$$



Mathematical Ideas

Geometric Brownian Motion

Geometric Brownian Motion is the continuous time stochastic process $X(t) = z_0 \exp(\mu t + \sigma W(t))$ where $W(t)$ is standard Brownian Motion. Most economists prefer Geometric Brownian Motion as a simple model for market prices because it is everywhere positive (with probability 1), in contrast to Brownian Motion, even Brownian Motion with drift. Furthermore, as we have seen from the stochastic differential equation for Geometric Brownian Motion, the relative change is a combination of a deterministic proportional growth term similar to inflation or interest rate growth plus a normally distributed random change

$$\frac{dX}{X} = r dt + \sigma dW .$$

(See Itô's Formula and Stochastic Calculus.) On a short time scale this is a sensible economic model.

A random variable X is said to have the **lognormal** distribution (with parameters μ and σ) if $\log(X)$ is normally distributed ($\log(X) \sim N(\mu, \sigma^2)$). The p.d.f. for X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp((-1/2)[(\ln(x) - \mu)/\sigma]^2).$$

Theorem 1. *At fixed time t , Geometric Brownian Motion $z_0 \exp(\mu t + \sigma W(t))$ has a lognormal distribution with parameters $(\ln(z_0) + \mu t)$ and $\sigma\sqrt{t}$.*

Proof.

$$\begin{aligned}
F_X(x) &= \mathbb{P}[X \leq x] \\
&= \mathbb{P}[z_0 \exp(\mu t + \sigma W(t)) \leq x] \\
&= \mathbb{P}[\mu t + \sigma W(t) \leq \ln(x/z_0)] \\
&= \mathbb{P}[W(t) \leq (\ln(x/z_0) - \mu t)/\sigma] \\
&= \mathbb{P}\left[W(t)/\sqrt{t} \leq (\ln(x/z_0) - \mu t)/(\sigma\sqrt{t})\right] \\
&= \int_{-\infty}^{(\ln(x/z_0) - \mu t)/(\sigma\sqrt{t})} \frac{1}{\sqrt{2\pi}} \exp(-y^2/2) dy
\end{aligned}$$

Now differentiating with respect to x , we obtain that

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma x\sqrt{t}} \exp((-1/2)[(\ln(x) - \ln(z_0) - \mu t)/(\sigma\sqrt{t})]^2).$$

□

Calculation of the Mean

We can calculate the mean of Geometric Brownian Motion by using the m.g.f. for the normal distribution.

Theorem 2. $\mathbb{E}[z_0 \exp(\mu t + \sigma W(t))] = z_0 \exp(\mu t + (1/2)\sigma^2 t)$

Proof.

$$\begin{aligned}
\mathbb{E}[X(t)] &= \mathbb{E}[z_0 \exp(\mu t + \sigma W(t))] \\
&= z_0 \exp(\mu t) \mathbb{E}[\exp(\sigma W(t))] \\
&= z_0 \exp(\mu t) \mathbb{E}[\exp(\sigma W(t)u)]|_{u=1} \\
&= z_0 \exp(\mu t) \exp(\sigma^2 t u^2/2)|_{u=1} \\
&= z_0 \exp(\mu t + (1/2)\sigma^2 t)
\end{aligned}$$

since $\sigma W(t) \sim N(0, \sigma^2 t)$ and $\mathbb{E}[\exp(Yu)] = \exp(\sigma^2 t u^2/2)$ when $Y \sim N(0, \sigma^2 t)$. See Moment Generating Functions, Theorem 4. □

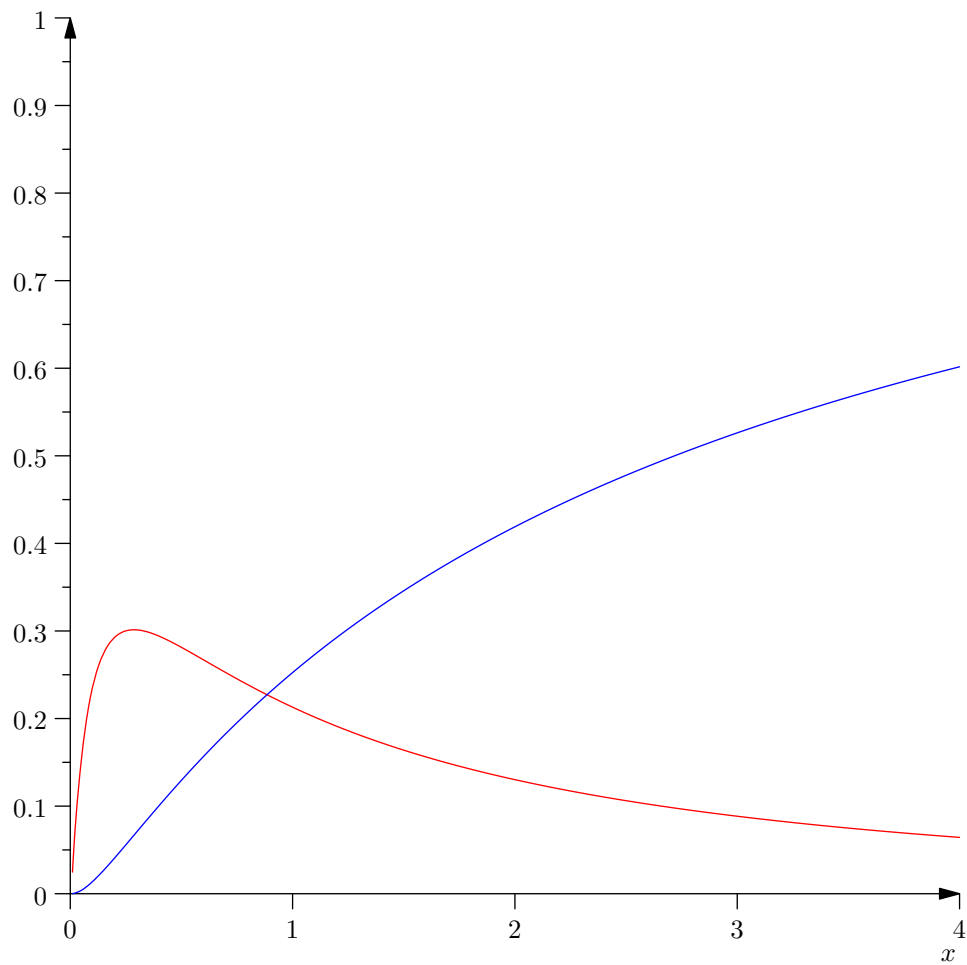


Figure 1: The p.d.f. and c.d.f. for a lognormal random variable with $m = 1$, $s = 1.5$.

Calculation of the Variance

We can calculate the variance of Geometric Brownian Motion by using the m.g.f. for the normal distribution, together with the common formula

$$\text{Var} [X] = \mathbb{E} [(X - \mathbb{E} [X])^2] = \mathbb{E} [X^2] - (\mathbb{E} [X])^2$$

and the previously obtained formula for $\mathbb{E} [X]$.

Theorem 3. $\text{Var} [z_0 \exp(\mu t + \sigma W(t))] = z_0^2 \exp(2\mu t + \sigma^2 t) [\exp(\sigma^2 t) - 1]$

Proof. First compute:

$$\begin{aligned} \mathbb{E} [X(t)^2] &= \mathbb{E} [z_0^2 \exp(\mu t + \sigma W(t))^2] \\ &= z_0^2 \mathbb{E} [\exp(2\mu t + 2\sigma W(t))] \\ &= z_0^2 \exp(2\mu t) \mathbb{E} [\exp(2\sigma W(t))] \\ &= z_0^2 \exp(2\mu t) \mathbb{E} [\exp(2\sigma W(t)u)]|_{u=1} \\ &= z_0^2 \exp(2\mu t) \exp(4\sigma^2 t u^2 / 2)|_{u=1} \\ &= z_0^2 \exp(2\mu t + 2\sigma^2 t) \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Var} [z_0 \exp(\mu t + \sigma W(t))] &= z_0^2 \exp(2\mu t + 2\sigma^2 t) - z_0^2 \exp(2\mu t + \sigma^2 t) \\ &= z_0^2 \exp(2\mu t + \sigma^2 t) [\exp(\sigma^2 t) - 1]. \end{aligned}$$

□

Note that this has the consequence that the variance starts at 0 and then grows exponentially. The variation of Geometric Brownian Motion starts small, and then increases, so that the motion generally makes larger and larger swings as time increases.

Stochastic Differential Equation and Parameter Summary

If a Geometric Brownian Motion is defined by the stochastic differential equation

$$dX = rX \, dt + \sigma X \, dW \quad X(0) = z_0$$

then the Geometric Brownian Motion is

$$X(t) = z_0 \exp((r - (1/2)\sigma^2)t + \sigma W(t)).$$

At each time the Geometric Brownian Motion has lognormal distribution with parameters $(\ln(z_0) + rt - (1/2)\sigma^2 t)$ and $\sigma\sqrt{t}$. The mean of the Geometric Brownian Motion is $\mathbb{E}[X(t)] = z_0 \exp(rt)$. The variance of the Geometric Brownian Motion is

$$\text{Var}[X(t)] = z_0^2 \exp(2rt)[\exp(\sigma^2 t) - 1]$$

If the primary object is the Geometric Brownian Motion

$$X(t) = z_0 \exp(\mu t + \sigma W(t)).$$

then by Itô's formula the SDE satisfied by this stochastic process is

$$dX = (\mu + (1/2)\sigma^2)X(t) dt + \sigma X(t) dW \quad X(0) = z_0.$$

At each time the Geometric Brownian Motion has lognormal distribution with parameters $(\ln(z_0) + \mu t)$ and $\sigma\sqrt{t}$. The mean of the Geometric Brownian Motion is $\mathbb{E}[X(t)] = z_0 \exp(\mu t + (1/2)\sigma^2 t)$. The variance of the Geometric Brownian Motion is

$$z_0^2 \exp(2\mu t + \sigma^2 t)[\exp(\sigma^2 t) - 1].$$

Ruin and Victory Probabilities for Geometric Brownian Motion

Because of the exponential-logarithmic connection between Geometric Brownian Motion and Brownian Motion, many results for Brownian Motion can be immediately translated into results for Geometric Brownian Motion. Here is a result on the probability of victory, now interpreted as the condition of reaching a certain multiple of the initial value. For $A < 1 < B$ define the “duration to ruin or victory”, or the “hitting time” as

$$T_{A,B} = \min\{t \geq 0 : \frac{z_0 \exp(\mu t + \sigma W(t))}{z_0} = A, \frac{z_0 \exp(\mu t + \sigma W(t))}{z_0} = B\}$$

Theorem 4. *For a Geometric Brownian Motion with parameters μ and σ , and $A < 1 < B$,*

$$\mathbb{P}\left[\frac{z_0 \exp(\mu T_{A,B} + \sigma W(T_{A,B}))}{z_0} = B\right] = \frac{1 - A^{1-(2\mu-\sigma^2)/\sigma^2}}{B^{1-(2\mu-\sigma^2)/\sigma^2} - A^{1-(2\mu-\sigma^2)/\sigma^2}}$$

Quadratic Variation of Geometric Brownian Motion

The quadratic variation of Geometric Brownian Motion may be deduced from Itô's formula:

$$dX = (\mu - \sigma^2/2)X \, dt + \sigma X \, dW$$

so that

$$(dX)^2 = (\mu - \sigma^2/2)^2 X^2 \, dt^2 + (\mu - \sigma^2/2)\sigma X^2 \, dt \, dW + \sigma^2 X^2 (dW)^2.$$

Guided by the heuristic principle that terms of order $(dt)^2$ and $dt \cdot dW \approx (dt)^{3/2}$ are small and may be ignored, and that $(dW)^2 = dt$, we obtain:

$$(dX)^2 = \sigma^2 X^2 \, dt.$$

Continuing heuristically, the expected quadratic variation is

$$\begin{aligned} \mathbb{E} \left[\int_0^T (dX)^2 \right] &= \mathbb{E} \left[\int_0^T \sigma^2 X^2 \, dt \right] \\ &= \sigma^2 \int_0^T [\mathbb{E} [X^2]] \, dt \\ &= \sigma^2 \left[\int_0^T z_0^2 \exp(2\mu t + 2\sigma^2 t) \, dt \right] \\ &= \frac{\sigma^2 z_0^2}{2\mu + 2\sigma^2} (\exp((2\mu + 2\sigma^2)T) - 1). \end{aligned}$$

Note the assumption that the order of the integration and the expectation can be interchanged.

Sources

This section is adapted from: *A First Course in Stochastic Processes, Second Edition*, by S. Karlin and H. Taylor, Academic Press, 1975, page 357; *An Introduction to Stochastic Modeling* 3rd Edition, by H. Taylor, and S. Karlin, Academic Press, 1998, pages 514–516; and *Introduction to Probability Models* 9th Edition, S. Ross, Academic Press, 2006.



Algorithms, Scripts, Simulations

Algorithm

Given values for μ , σ and an interval $[0, T]$, the script creates `trials` sample paths of Geometric Brownian Motion, sampled at N equally-spaced values on $[0, T]$. The scripts do this by creating `trials` Brownian Motion sample paths sampled at N equally-spaced values on $[0, T]$ using the definition of Brownian Motion having normally distributed increments. Adding the drift term and then exponentiating the sample paths creates `trials` Geometric Brownian Motion sample paths sampled at N equally-spaced values on $[0, T]$. Then the scripts use semi-logarithmic least-squares statistical fitting to calculate the relative growth rate of the mean of the sample paths. The scripts also compute the predicted relative growth rate to compare it to the calculated relative growth rate. The problems at the end of the section explore plotting the sample paths, comparing the sample paths to the predicted mean with standard deviation bounds, and comparing the mean quadratic variation of the sample paths to the theoretical quadratic variation of Geometric Brownian Motion.

Scripts

Geogebra GeoGebra applet for Geometric Brownian Motion

R R script for Geomtric Brownian Motion.

```
1 mu <- 1
2 sigma <- 0.5
3 T <- 1
4 # length of the interval [0, T] in time units
5
6 trials <- 200
7 N <- 200
8 # number of end-points of the grid including T
9 Delta <- T/N
10 # time increment
11
12 t <- t( seq(0,T, length=N+1) * t(matrix(1, trials, N+1))
13 )
14 # Note use of the R matrix recycling rules, by columns,
15   so transposes
```

```

14 W <- cbind(0, t( apply(sqrt(Delta) * matrix(rnorm(trials
      * N), trials, N), 1, cumsum)))
15 # Wiener process, Note the transpose after the apply, (
      side effect of
16 # apply is the result matches the length of individual
      calls to FUN,
17 # then the MARGIN dimension/s come next. So it's not so
      much
18 # "transposed" as that being a consequence of apply in 2D
      .) Note
19 # use of recycling with cbind to start at 0
20
21 GBM <- exp( mu*t + sigma*W)
22
23 meanGBM <- colMeans(GBM)
24
25 meanGBM_rate <- lm(log(meanGBM) ~ seq(0,T, length=N+1))
26 predicted_mean_rate = mu + (1/2)*sigma^2
27
28 cat(sprintf("Observed meanGBM relative rate: %f \n",
      coefficients(meanGBM_rate)[2] ))
29 cat(sprintf("Predicted mean relative rate: %f \n",
      predicted_mean_rate))
30

```

Octave Octave script for Geometric Brownian Motion

```

1 mu = 1;
2 sigma = 0.5;
3 T = 1;
4 # length of the interval [0, T] in time units
5
6 trials = 200;
7 N = 100;
8 # number of end-points of the grid including T
9 Delta = T/N;
10 # time increment
11
12 W = zeros(trials, N+1);
13 # initialization of the vector W approximating
14 # Wiener process
15 t = ones(trials, N+1) .* linspace(0, T, N+1);
16 # Note the use of broadcasting (Octave name for R
      recycling)

```

```

17 W(:, 2:N+1) = cumsum( sqrt(Delta) * stdnormal_rnd(trials
    ,N), 2);
18
19 GBM = exp(mu*t + sigma*W);
20
21 meanGBM = mean(GBM);
22
23 A = [transpose((0:Delta:T)) ones(N+1, 1)];
24 meanGBM_rate = A\transpose( log(meanGBM) )
25 predicted_mean_rate = mu + (1/2)*sigma^2
26

```

Perl Perl PDL script for Geometric Brownian Motion

```

1 $mu = 1;
2 $sigma = 0.5;
3 $T = 1.;
4 # length of the interval [0, T] in time units
5
6 $trials = 200;
7 $N = 100;
8 # number of end-points of the grid including T
9 $Delta = $T / $N;
10 # time increment
11
12 $W = zeros( $N + 1, $trials );
13 # initialization of the vector W approximating
14 # Wiener process
15 $t = ones( $N+1, $trials ) * zeros( $N + 1 )->xlinvals(
    0, $T );
16 # Note the use of PDL dim 1 threading rule (PDL name for
    R recycling)
17 $W ( 1 : $N, : ) .= cumsumover( sqrt($Delta) * grandom(
    $N, $trials) );
18
19 $GBM = exp( $mu * $t + $sigma * $W );
20
21 $meanGBM = sumover($GBM->xchg(0,1))/ $trials;
22
23 use PDL::Fit::Linfit;
24
25 $fitFuncs = cat ones($N + 1), zeros( $N + 1 )->xlinvals(
    0, $T );

```

```

26 ( $linfitfunc, $coeffs ) = linfit1d log($meanGBM),
    $fitFuncs;
27 print "Observed Mean GBM Rate: ", $coeffs(1), "\n";
28 print "Predicted Mean GBM Rate: ", $mu + (1./2.)*$sigma
    **2, "\n";
29

```

SciPy Scientific Python script for Geometric Brownian Motion

```

1 import scipy
2
3 mu = 1.;
4 sigma = 0.5;
5 T = 1.
6 # length of the interval [0, T] in time units
7
8 trials = 200
9 N = 100
10 # number of end-points of the grid including T
11 Delta = T/N;
12 # time increment
13
14 W = scipy.zeros((trials, N+1), dtype = float)
15 # initialization of the vector W approximating
16 # Wiener process
17 t = scipy.ones( (trials, N+1), dtype = float) * scipy.
    linspace(0, T, N+1)
18 # Note the use of recycling
19 W[:, 1:N+1] = scipy.cumsum(scipy.sqrt(Delta)*scipy.random
    .standard_normal( (trials, N)), axis = 1,)
20
21 GBM = scipy.exp( mu*t + sigma*W)
22 meanGBM = scipy.mean(GBM, axis=0)
23 predicted_mean_GBM_rate = mu + (1./2.)*sigma**2
24
25 meanGBM_rate = scipy.polyfit( scipy.linspace(0, T, N+1),
    scipy.log(meanGBM), 1)
26
27 print "Observed Mean GBM Relative Rate:", meanGBM_rate
    [0];
28 print "Predicted Mean GBM Relative Rate:",
    predicted_mean_GBM_rate;

```



Problems to Work for Understanding

1. Differentiate

$$\int_{-\infty}^{(\ln(x/z_0) - \mu t)/(\sigma\sqrt{t})} \frac{1}{\sqrt{2\pi}} \exp(-y^2/2) \, dy$$

to obtain the p.d.f. of Geometric Brownian Motion.

2. What is the probability that Geometric Brownian Motion with parameters $\mu = -\sigma^2/2$ and σ (so that the mean is constant) ever rises to more than twice its original value? In economic terms, if you buy a stock or index fund whose fluctuations are described by this Geometric Brownian Motion, what are your chances to double your money?
3. What is the probability that Geometric Brownian Motion with parameters $\mu = 0$ and σ ever rises to more than twice its original value? In economic terms, if you buy a stock or index fund whose fluctuations are described by this Geometric Brownian Motion, what are your chances to double your money?
4. Derive the probability of ruin (the probability of Geometric Brownian Motion hitting $A < 1$ before hitting $B > 1$).
5. Modify the scripts to plot several sample paths of Geometric Brownian Motion all on the same set of axes.
6. Modify the scripts to plot several sample paths of Geometric Brownian Motion and the mean function of Geometric Brownian Motion and the mean function plus and minus one standard deviation function, all on the same set of axes.
7. Modify the scripts to measure the quadratic variation of each of many sample paths of Geometric Brownian Motion, find the mean quadratic

variation and compare to the theoretical quadratic variation of Geometric Brownian Motion.



Reading Suggestion:

References

- [1] S. Karlin and H. Taylor. *A Second Course in Stochastic Processes*. Academic Press, 1981.
- [2] Sheldon M. Ross. *Introduction to Probability Models*. Academic Press, 9th edition, 2006.
- [3] H. M. Taylor and Samuel Karlin. *An Introduction to Stochastic Modeling*. Academic Press, third edition, 1998.



Outside Readings and Links:

- 1. Graphical Representations of Brownian Motion and Geometric Brownian Motion
- 2. Wikipedia Geometric Brownian Motion

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