IV Leaks

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The IV bit of the title means implied volatility; the connotations of the full title I leave to the reader.

We can view the Black-Scholes call price formula as a function of the volatility σ; $\text{call}^{BS}(\sigma; \ldots)$. A straightforward calculation shows that call and put option prices are increasing in volatility. Specifically, their Vegas (derivatives wrt. σ) are

$$\text{Vega}(t) = e^{-\delta(T-t)} S(t) \phi(d_+(S(t), t)) \sqrt{T-t}.$$

Intuitively, higher volatility pushes mass into the tails, and with the convexity of the payoff functions, we gain more from the good outcomes than we lose on the bad ones. If you claim that this sounds more obvious than it is because the lognormal distribution is not symmetric, I won't disagree strongly. But fortunately, we can just do the differentiation. For a call price observed in the market, call^{obs}, that does not violate mild arbitrage bounds (Merton's Tunnel) we can safely define its implied volatility, σ^{imp}, as the solution to

$$\text{call}^{BS}(\sigma^{imp}; \ldots) = \text{call}^{obs}.$$

How not to calculate implied volatilities. There is no explicit expression for the implied volatility,[1] the defining equation must be solved numerically. That is not difficult to do by bisection as the left-hand side is a smooth, monotone function and a root is typically easy to bracket. Newton-Raphson can be used to speed up convergence, but divergence can occur, so safety checks are needed. There can also be programming challenges in handling efficiently and safely this situation where we view *something* as a function of *this* but not *that*. Lambda expressions in C++11+ are a tool. Unless you want to pick a fight with the put-call parity, European call and put options with the same strike and expiry have the same implied volatility. However, if you get the interest rate and dividend payments wrong, you may get something like the kinky Figure 1.

What do implied volatilities look like? Often market participants quote call and put prices in terms of implied volatilities. This does not mean that they believe the Black-Scholes model holds. *Au contraire*: If it did, all implied volatilities would be the same. But in this way we look at numbers on the same order of magnitude when comparing different options. Some say that implied volatilities measure relative values; I'm not sure I like that formulation. Implied volatilities display skews (stocks, interest rates) or smiles (FX; and necessarily far-away-from-the-money) in strike, particularly at short expiries. And expiry dependence. And they definitely move around over time. But does a non-flat implied

[1] Although Stefanica & Radoicic (2017) may beg to differ.

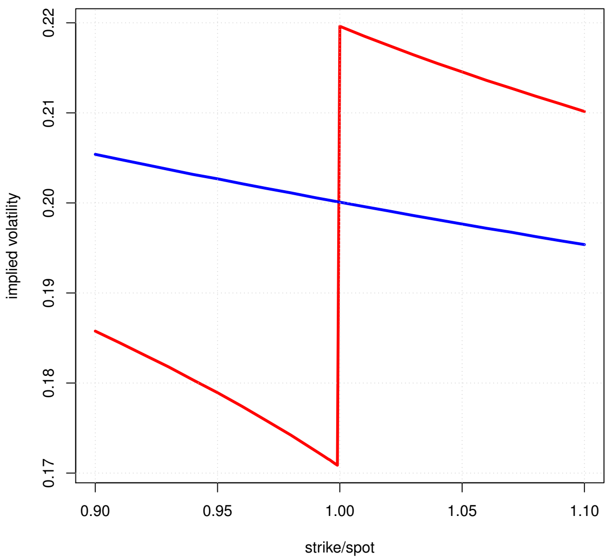

Figure 1: Blue curve: Implied volatilities for a market that uses 4% interest rate and 2% dividend rate when pricing options. Red curve: Implied volatilities for a modeler who assumes zero-rates and uses only out-of-the-money options as data.

volatility surface not imply that there are arbitrage opportunities? After all, that was how we derived the Black-Scholes formula. It is possible but unlikely. It is much more plausible that the Black-Scholes model's geometric Brownian motion assumption is violated. An often-seen way to make the abstract concrete: Calculate (arbitrage-free) option-prices in some non-Black-Scholes model — Bachelier, constant elasticity of variance (CEV), Heston, jump-diffusion, ... — and convert these into (non-flat, but arbitrage-free) implied volatility surfaces.

What do implied volatilities tell us? Intuitively, implied volatility is a forward looking measure of volatility. What can we expect it to tell us? To this end assume that

$$dS(t) = (r - \delta)S(t)dt + \sigma(t)S(t)dW^Q(t),$$

where $\sigma = \{\sigma(t)\}_t$ is some random process. It is now convenient to work with realized variance,

$$RV(t,T) = \int_t^T \sigma^2(u)du,$$

and accordingly to view the Black-Scholes call formula as a function of $\sigma^2(T-t)$ rather than σ; $\widetilde{\text{call}}^{BS}(x;\ldots) = \text{call}^{BS}(\sqrt{x/(T-t)};\ldots)$. A mixing result going back to Hull & White (1987) then says: *If σ is independent of W, then*

$$\text{call}(t) = \text{E}_t^Q(\widetilde{\text{call}}^{BS}(RV(t,T);\ldots)).$$

Thus under independence: If $\widetilde{\text{call}}^{BS}$ were linear, then implied volatility (squared and time-to-expiry-scaled) would be a (Q-)unbiased forecast of the integrated variance over the life of the option. But alas:

- Linearity does not exactly hold.[2]
- In stock markets, S and σ are quite strongly negatively correlated (< -0.5, typically).
- What about P vs. Q?

The last point manifests itself empirically in a volatility risk premium; implied volatilities are typically above realized volatilities (suitably defined). No form of $\Delta-$hedging will work perfectly; the option seller needs compensation for that. There are ways to investigate and remedy the first two points. A noteworthy result[3] is that it is possible — in

[2] If anything, at-the-money prices are $\sim$ linear in σ, not σ^2.

[3] This was derived independently in "quant dungeons" in the 90'ies; Bruno Dupire, Emanuel Derman and Peter Carr with their respective groups can all lay claim to it. Possibly a future column, but for now I'll just mention Avellaneda (2000) as my favorite short derivation.

a largely model-free and explicit way — to construct a portfolio of puts and calls whose value is equal to $\mathrm{E}_t^Q(RV(t,T))$. This is the idea behind the CBOE's volatility benchmark VIX — although you wouldn't know it from the official technical document.[4] Finally, let me summarize quite a large empirical literature[5] in one sentence: Implied volatilities contain some information not easily captured by backward-looking estimation methods.

Modelling. It is surprisingly difficult (in my view, nobody has really succeeded yet; many have tried, so this is a contentious point) to write arbitrage-free model dynamics directly in terms of implied volatilities – which would seem the natural approach given everything above. To get an indication why, consider the following example from Savine (2017): Suppose all options trade at the same implied volatility. We sell an out-of-the-money option and buy $\text{Vega}^{OTM}/\text{Vega}^{ATM}$ at-the-money options. This makes the portfolio Vega-neutral. But because ratios of Vegas are the same as ratios of Gammas,[6] it is also Gamma-neutral. Now add an amount of the underlying to make the portfolio Delta-neutral. So from the Black-Scholes PDE (and the definition of implied volatility), its theta is 0. Finally, by choosing a sufficiently out-of-the-money option, we we can make the portfolio's Volga (= second σ-derivative) positive, since for each option Volga = Vega $\cdot\, d_+ \cdot d_-/\sigma$. So if, over time, only flat shifts in flat volatilities are possible, there is arbitrage. This is akin to the Barbell-example from interest rate modelling,[7] but here no Heath-Jarrow-Morton result saves us. It seems we have to go via something else; Dupire local volatility, an affine factor model, . . .

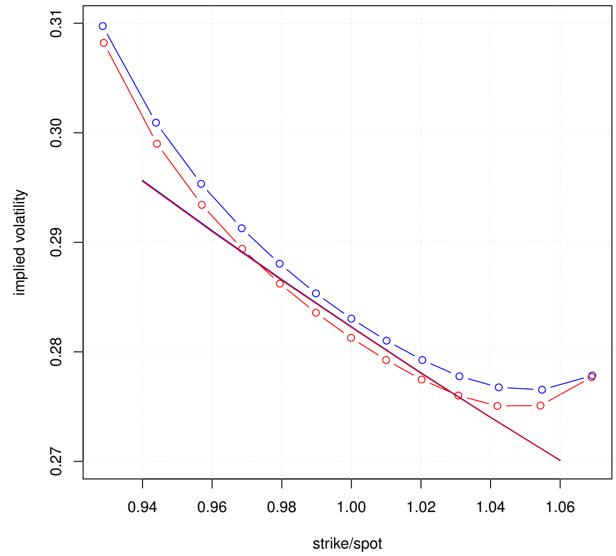

Figure 2: The o-o-o curves are observed implied volatilities of one-month calls (blue) and puts (red) on Apple's stock in early September 2012 ($r = 0.005$, $\delta = 0.015$). Average absolute difference $\sim$ 20 basis points. The fully drawn curves – of which there are three – are theoretical American (based on finite difference PDE solution) and European (black) implied volatilities of the best-fitting CEV model. Average absolute difference $\sim$ 0.5 basis points. Ready-to-run R-code at https://tinyurl.com/y8lgslnz

Exotic implied volatility. For exotic options, prices may not be monotone in volatility; arguably, that could be the definition of exotic. For instance barrier options often have multiple sensible implied volatilities, which may cause havoc as hinted at in Jessen & Poulsen (2013, Footnote 6). For American calls and puts, implied Black-Scholes volatility is defined similarly to European options, with the complication that for an observed American option price, different guesses of volatility

[4] See `https://tinyurl.com/yb5rges2`

[5] To get one strand of said literature click on 'Cited by' for Christensen & Prabhala (1998) in Google Scholar.

[6] A neat fact also used in Poulsen (2007).

[7] See for instance section 3.6.1 at https://tinyurl.com/yb6n2q5u

have to be run through a numerical solution method (finite difference for instance) to produce a match; double numerics so to say. With non-zero rates implied volatilities for American calls and puts with the same strike and expiry need *not* be the same. Such differences are often seen in data. In my experience the theoretical models have problems generating effects of the empirical magnitude, as Figure 2 shows. Finance has many puzzles; is this another one?

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