STT 465

I. Multivariate Normal Distribution
II. Bayesian Multiple Linear Regression

Multivariate Normal Distribution

$$x \sim MVN\left[\mu, \Sigma\right]$$

$$x = \left(x_1, \ldots, x_q\right)' \qquad \mu = \left(\mu_1, \ldots, \mu_q\right)'$$

$$\Sigma = Cov\left(x, x'\right) = \begin{bmatrix} Cov\left(x_1, x_1\right) & \cdots & Cov\left(x_1, x_q\right) \\ \vdots & \ddots & \vdots \\ Cov\left(x_q, x_1\right) & \cdots & Cov\left(x_q, x_q\right) \end{bmatrix}$$

Multivariate Normal Distribution

$$p(x) = (2\pi)^{-n/2} \|\Sigma\|^{-1/2} \, Exp\left\{ -\frac{1}{2}(x-\mu)' \Sigma^{-1} (x-\mu) \right\}$$

Multivariate Normal Distribution

$$x = (x_1', x_2')' \quad \mu = (\mu_1', \mu_2')' \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

Important Results

- All marginal are normal
- All conditional distributions are also normal
- The normal distribution is closed under linear transformations (i.e., linear transformations of MVN random variables are also MVN).

Multivariate Normal Distribution

$$x \sim MVN\left[\mu, \Sigma\right] \quad x = \left(x_1', x_2'\right)' \quad \mu = \left(\mu_1', \mu_2'\right)' \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

Marginal distributions are normal: $x_j \sim MVN\left[\mu_j, \Sigma_{jj}\right]$

Conditional distributions are normal: $x_2 \mid x_1 \sim MVN\left[\mu_{2|1}, \Sigma_{2|1}\right]$

where: $\mu_{2|1} = E[x_2 \mid x_1] = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}\left(x_1 - \mu_1\right)$

and $\Sigma_{2|1} = Cov\left(x_2 \mid x_1\right) = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$

II. Bayesian Multiple Linear Regression

- Known Variance parameters
- Uknown variance parameters

Bayesian Multiple Linear Regression

- Gaussian Linear Regression Model

$$y_i = \sum_{j=1}^{p} x_{ij}\beta_j + \varepsilon_i \qquad \varepsilon \sim MVN(0, R)$$

- Matrix representation

Let $x_i' = (1, x_{i1}, x_{i2}, \ldots, x_{ip})$ $\beta = (\mu, \beta_1, \beta_2, \ldots, \beta_p)'$

Then $y_i = x_i'\beta + \varepsilon_i$

Stack equations 1-n to get $y = X\beta + \varepsilon$

Probability assumptions $\varepsilon \sim MVN(0, R)$

- Likelihood

$$[y \mid \beta] \sim MVN(X\beta, R)$$

Likelihood (cont.)

MVN Density

$$p(x) = (2\pi)^{-n/2} \left\|\Sigma\right\|^{-1/2} Exp\left\{-\frac{1}{2}(x-\mu)' \Sigma^{-1} (x-\mu)\right\}$$

Set $\mu = X\beta$ and $\Sigma = R$

$$p(y \mid X, \beta) = (2\pi)^{-n/2} \left\|R\right\|^{-1/2} Exp\left\{-\frac{1}{2\sigma_\varepsilon^2}(y-X\beta)' R^{-1} (y-X\beta)\right\}$$

Prior

Normal prior for reg. coefficients

$$\beta \sim N\left(\beta_0, \Sigma_0\right) = \left(2\pi\right)^{-n/2} \left\|\Sigma_0\right\|^{-1/2} Exp\left\{-\left(\beta-\beta_0\right)' \Sigma_0^{-1}\left(\beta-\beta_0\right)\right\}$$

Posterior Density

Likelihood

$$p(y \mid X,\beta) = (2\pi)^{-n/2} \|R\|^{-1/2} Exp\left\{-\frac{1}{2}(y - X\beta)' R^{-1}(y - X\beta)\right\}$$

Prior

$$\beta \sim N(\beta_0,\Sigma_0) = (2\pi)^{-n/2} \|\Sigma_0\|^{-1/2} Exp\left\{-(\beta - \beta_0)' \Sigma_0^{-1}(\beta - \beta_0)\right\}$$

Posterior (derivation presented in class)

$$p(\beta \mid y) \sim N\left(C^{-1}rhs, C^{-1}\right)$$

$$C = \left[X'R^{-1}X + \Sigma_0^{-1}\right] \qquad \left[X'R^{-1}X + \Sigma_0^{-1}\right]\hat{\beta} = X'R^{-1}y + \Sigma_0^{-1}\beta_0$$

$$rhs = X'R^{-1}y + \Sigma_0^{-1}\beta_0$$

Special (most commonly used) case

Likelihood [iid residuals] $R = I\sigma_{\varepsilon}^{2}$

Prior [iid mean-zero effects] $R = I\sigma_{\varepsilon}^{2}$

$$\beta \sim N\left(0, I\sigma_{0}^{2}\right)$$

Posterior

$$p\left(\beta \mid y\right) \sim N\left(C^{-1}rhs, C^{-1}\right)$$

$$C = \left[X'X\sigma_{\varepsilon}^{-2} + I\sigma_{0}^{2}\right]$$

$$rhs = X'y\sigma_{\varepsilon}^{-2}$$