STT 465

Lecture 2:
Belief, probability, independence, conditional independence, exchangeability

(G. de los Campos)

Beliefs & Probability

$\Longrightarrow$ Beliefs and probability (read the chapter in the book)

$\Longrightarrow$ Review of probability

(1) Events $\{A_i\}$ & Sample space $S=\{A_1,\ldots\}$

(2) Probability (map from events to numbers in the [0,1] interval that follows a few rules).

(3) $P(A_i) \geq 0$

(4) $P(S)=1$

(5) Probability of the union: $P(A_i \cup A_j)=P(A_i)+P(A_j)-P(A_i \cap A_j)$ [U=OR, ∩=&]

$\Longrightarrow$ Marginal, conditional and joint probabilities

(6) Marginal probability: $P(A)$

(7) Joint probability: $P(A\&B)=P(A,B)$ [we will commonly use ',' for joint]

(8) Conditional probability: $p(A|B)$

(9) Factorization: $P(A,B)=P(A)\,P(B|A)=P(B)P(A|B)$

(10) Rule of marginal probability: $P(A)=P(A|B)\,P(B)+P(A|\text{Not } B)P(\text{Not } B)$

$\Longrightarrow$ Bayes rule: $p(A|B)=P(B|A)P(A)/P(B)$

Examples

$\Rightarrow$ Review income/education example (p 16)

$\Rightarrow$ Example: genetics of disease

Genotype	Status
AA	H
AB | BA	H
BB	D

p(A)=0.8

$\Rightarrow$ Determine

(1) The (marginal) probability of each genotype under random mating, p(G)
(2) The conditional probability of disease given genotypes, p(D|G)
(3) The probability of disease of a randomly sampled individual, P(D)
Hint: use the law of marginal probabilities.
(4) The joint probability of disease and genotypes, p(D,G). Hint: use p(A,B)=P(A|B)p(B)
(5) The conditional probability of genotype given disease, p(G|D). Hint: use Bayes rule.

Conditional Independence

$\Longrightarrow$ X,Y are independent if $p(X,Y)=p(X)p(Y)$

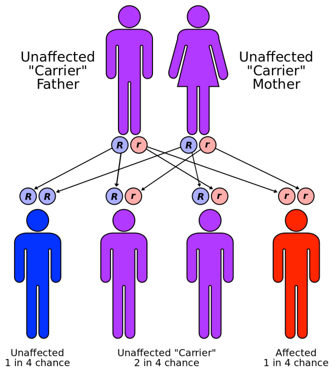
$\Longrightarrow$ X,Y are said <u>conditionally independent</u> if, given a third variable (Z), X and Y are independe
that is:

<u>Conditional Independence:</u> $p(X,Y|Z)=P(X|Z)*P(Y|Z)$

Let's see an example

Bayesian Learning

⟹ Goal: to make statements about the genotype of one of the parents given the phenotype of one or more offspring.



Source: Wikipedia

⟹ Compute the probability that a parent is a carrier given that the first offspring is healthy p(G_0=AB|O_1=H)

⟹ Ho would that probability be updated if you know that the second offspring is also healthy? Hint: you can assume conditional independence; therefore,

$$\text{(always)} \qquad p\left(G_0 = AB \mid O_1 = H, O_2 = H\right) = \frac{p\left(O_1 = H, O_2 = H \mid G_0 = AB\right) p\left(G_0 = AB\right)}{p\left(O_1 = H, O_2 = H\right)}$$

$$\text{(due to cond. independence)} \qquad = \frac{p\left(O_1 = H \mid G_0 = AB\right) p\left(O_2 = H \mid G_0 = AB\right) p\left(G_0 = AB\right)}{p\left(O_1 = H, O_2 = H\right)}$$

REVIEW HW 1

Questions?

Remarks

$\Longrightarrow$ The joint probability contains all the information needed to arrive at the marginal s and conditionals

$\Longrightarrow$ We cannot arrive at the joint distribution from the marginal distributions (under independence this is trivial, but in other cases we need to know the conditional distributions and one of the marginal to get to the joint distribution).

$\Longrightarrow$ Distinction between independence and conditional independence (see HW1)

Exchangeability

$\Rightarrow$ Define Exchangeable Sequence of Random Variables

- Sequence of Random Variables $Y_1, Y_2, \ldots.\ Y_N$
- Permutation: $\pi=\{\pi_1, \pi_2, \ldots, , \pi_N\}$

$\Rightarrow$ A sequence of RV is said to be exchangeable if

$p(Y_{\pi 1}, Y_{\pi 2}, \ldots., Y_{\pi N})$ is the same for any permutation π

$\Rightarrow$ Note: sequence of IID RVs are exchangeable (discuss), but the converse is not TRUE

$\Rightarrow$ Importantly, independence is not required for exchangeability (discuss MVN case)

de Finetti's Theorem

⇒ Bruno de Finetti established an important relationship between exchangeable sequences of RVs and Conditional independence (de Finetti's theorem)

⇒ If Y_1, Y_2 ,...., Y_n is an exchangeable sequence of random variables then, for some parameter (θ), some prior density, p(θ) and some conditional density $p(Y_i|\theta)$, the joint distribution of the sequence can be expressed as:

$$p(Y_1, Y_2, \ldots, Y_n) = \int \left\{ \prod_{i=1}^{n} p(Y_i \mid \theta) \right\} p(\theta) d\theta$$

⇒ Note:

$$\text{[Allways]: } p(Y_1, Y_2, \ldots, Y_n) = \int p(Y_1, Y_2, \ldots, Y_n, \theta) d\theta = \int p(Y_1, Y_2, \ldots, Y_n \mid \theta) p(\theta) d\theta$$

⇒ De Finetti's theorem tells us that if the sequence is exchangeable, we can assume conditional IID

$$p(Y_1, Y_2, \ldots, Y_n \mid \theta) = \prod_{i=1}^{n} p(Y_i \mid \theta)$$