

STT 465

Bayesian Multiple Linear Regression:

- Summary of model specification
- Conditional distribution of effects
- Model with unknown variances
- Gibbs sampler I
- Fully conditional distribution of single effects
- Gibbs sampler II

Bayesian Multiple Linear Regression

- Gaussian Linear Regression Model

$$y_i = \sum_{j=1}^p x_{ij} \beta_j + \varepsilon_i$$

- Matrix representation

Stack equations 1-n to get $y = X\beta + \varepsilon$

- Likelihood (assuming iid normal errors) $\varepsilon \sim MVN(0, I\sigma_\varepsilon^2)$

$$[y | \beta] \sim MVN(X\beta, R)$$

$$p(y | X, \beta, \sigma_\varepsilon^2) = (2\pi)^{-n/2} \|I\sigma_\varepsilon^2\|^{-1/2} \text{Exp} \left\{ -\frac{1}{2\sigma_\varepsilon^2} (y - X\beta)' (y - X\beta) \right\}$$

Prior with known variance components

Normal prior for reg. coefficients

$$\beta \sim N(\beta_0, \Sigma_0) = (2\pi)^{-n/2} \|\Sigma_0\|^{-1/2} \text{Exp} \left\{ -\frac{1}{2} (\beta - \beta_0)' \Sigma_0^{-1} (\beta - \beta_0) \right\}$$

If we assume IID, zero-mean prior we have

$$\beta \sim N(0, I\sigma_\beta^2) = (2\pi)^{-n/2} \|I\sigma_\beta^2\|^{-1/2} \text{Exp} \left\{ -\frac{\beta'\beta}{2\sigma_\beta^2} \right\}$$

Posterior Density

Likelihood

$$p(y | X, \beta) = (2\pi)^{-n/2} \|I\sigma_\varepsilon^2\|^{-1/2} \text{Exp} \left\{ -\frac{1}{2} (y - X\beta)' [I\sigma_\varepsilon^2]^{-1} (y - X\beta) \right\}$$

Prior

$$\beta \sim N(0, I\sigma_\beta^2) = (2\pi)^{-n/2} \|I\sigma_\beta^2\|^{-1/2} \text{Exp} \left\{ -\frac{1}{2} \beta' [I\sigma_\beta^2]^{-1} \beta \right\}$$

General for of the posterior density (derivation presented in class)

$$p(\beta | y) \sim N(C^{-1}rhs, C^{-1})$$

$$C = [X'R^{-1}X + \Sigma_0^{-1}] \quad [X'R^{-1}X + \Sigma_0^{-1}] \hat{\beta} = X'R^{-1}y + \Sigma_0^{-1}\beta_0$$

$$rhs = X'R^{-1}y + \Sigma_0^{-1}\beta_0$$

Special (most commonly used) case

Likelihood

[iid residuals] $R = I\sigma_\varepsilon^2$

Prior

[iid mean-zero effects] $R = I\sigma_\varepsilon^2$

$$\beta \sim N(0, I\sigma_0^2)$$

Posterior

$$p(\beta | y) \sim N(C^{-1}rhs, C^{-1})$$

$$C = [X'X\sigma_\varepsilon^{-2} + I\sigma_0^2]$$

$$rhs = X'y\sigma_\varepsilon^{-2}$$

Discuss shrinkage and
connection to Ridge
Regression.

Model with unknown variances

Joint prior

$$\begin{aligned} p(\beta, \sigma_\varepsilon^2, \sigma_\beta^2) &= p(\beta | \sigma_\beta^2) p(\sigma_\beta^2) p(\sigma_\varepsilon^2) \\ &= N(\beta | 0, I \sigma_\beta^2) \chi^{-2}(\sigma_\beta^2 | df_\beta, S_\beta) \chi^{-2}(\sigma_\varepsilon^2 | df_\varepsilon, S_\varepsilon) \\ &\propto \left[\|I \sigma_\beta^2\|^{-1/2} e^{-\frac{\beta' \beta}{2 \sigma_\beta^2}} \right] \times \left[(\sigma_\beta^2)^{-(1+df_\beta/2)} e^{-\frac{S_\beta}{2 \sigma_\beta^2}} \right] \times \left[(\sigma_\varepsilon^2)^{-(1+df_\varepsilon/2)} e^{-\frac{S_\varepsilon}{2 \sigma_\varepsilon^2}} \right] \end{aligned}$$

Model with unknown variances

Joint prior

$$\begin{aligned} p(\beta, \sigma_\varepsilon^2, \sigma_\beta^2) &= p(\beta | \sigma_\beta^2) p(\sigma_\beta^2) p(\sigma_\varepsilon^2) \\ &= N(\beta | 0, I \sigma_\beta^2) \chi^{-2}(\sigma_\beta^2 | df_\beta, S_\beta) \chi^{-2}(\sigma_\varepsilon^2 | df_\varepsilon, S_\varepsilon) \\ &\propto \left[(\sigma_\beta^2)^{-p/2} e^{-\frac{\beta' \beta}{2 \sigma_\beta^2}} \right] \times \left[(\sigma_\beta^2)^{-(1+df_\beta/2)} e^{-\frac{S_\beta}{2 \sigma_\beta^2}} \right] \times \left[(\sigma_\varepsilon^2)^{-(1+df_\varepsilon/2)} e^{-\frac{S_\varepsilon}{2 \sigma_\varepsilon^2}} \right] \end{aligned}$$

Model with unknown variances

Joint posterior

$$p(\beta, \sigma_\varepsilon^2, \sigma_\beta^2 | y) = p(y | \beta, \sigma_\varepsilon^2) p(\beta | \sigma_\beta^2) p(\sigma_\beta^2) p(\sigma_\varepsilon^2)$$

$$p(\beta, \sigma_\varepsilon^2, \sigma_\beta^2 | y) \propto (\sigma_\varepsilon^2)^{-n/2} \text{Exp} \left\{ -\frac{1}{2\sigma_\varepsilon^2} (y - X\beta)' (y - X\beta) \right\} \\ \times \left[(\sigma_\beta^2)^{-p/2} e^{-\frac{\beta'\beta}{2\sigma_\beta^2}} \right] \times \left[(\sigma_\beta^2)^{-(1+df_\beta/2)} e^{-\frac{S_\beta}{2\sigma_\beta^2}} \right] \times \left[(\sigma_\varepsilon^2)^{-(1+df_\varepsilon/2)} e^{-\frac{S_\varepsilon}{2\sigma_\varepsilon^2}} \right]$$

Gibbs Sampler

1. Initialize parameters

- Any initial values with non-null prior prob. are valid.

2. Sample effects given variances

$$p(\beta | ELSE) \propto \text{Exp} \left\{ -\frac{1}{2\sigma_\varepsilon^2} (y - X\beta)' (y - X\beta) \right\} \left[e^{-\frac{\beta'\beta}{2\sigma_\beta^2}} \right]$$

$$[\text{from previous results}] = N(C^{-1}rhs, C^{-1}) \quad [1]$$

$$C = [X'X\sigma_\varepsilon^{-2} + I\sigma_0^2]$$

$$rhs = X'y\sigma_\varepsilon^{-2}$$

Gibbs Sampler

3. Sample error variance given effects

$$p(\sigma_\varepsilon^2 \mid ELSE) = (\sigma_\varepsilon^2)^{-n/2} \text{Exp} \left\{ -\frac{1}{2\sigma_\varepsilon^2} (y - X\beta)' (y - X\beta) \right\} \left[(\sigma_\varepsilon^2)^{-(1+df_\varepsilon/2)} e^{-\frac{S_\varepsilon}{2\sigma_\varepsilon^2}} \right]$$

- Given effects the errors are known $\varepsilon = y - X\beta$
- Therefore

$$p(\sigma_\varepsilon^2 \mid ELSE) \propto (\sigma_\varepsilon^2)^{-n/2} \text{Exp} \left\{ -\frac{\varepsilon' \varepsilon}{2\sigma_\varepsilon^2} \right\} \left[(\sigma_\varepsilon^2)^{-(1+df_\varepsilon/2)} e^{-\frac{S_\varepsilon}{2\sigma_\varepsilon^2}} \right]$$

$$\propto (\sigma_\varepsilon^2)^{-[1+(n+df_\varepsilon)/2]} \text{Exp} \left\{ -\frac{\varepsilon' \varepsilon + S_\varepsilon}{2\sigma_\varepsilon^2} \right\}$$

$$= \chi^{-2} \left(\sigma_\varepsilon^2 \mid S = \varepsilon' \varepsilon + S_\varepsilon, df = n + df_\varepsilon \right) \quad [2]$$

Gibbs Sampler

4. Sample the variance of effects

$$\begin{aligned} p(\sigma_\beta^2 | ELSE) &\propto \left[(\sigma_\beta^2)^{-p/2} e^{-\frac{\beta'\beta}{2\sigma_\beta^2}} \right] \times \left[(\sigma_\beta^2)^{-(1+df_\beta/2)} e^{-\frac{S_\beta}{2\sigma_\beta^2}} \right] \\ &\propto \left[(\sigma_\beta^2)^{-[1+(p+df_\beta)/2]} e^{-\frac{\beta'\beta+S_\beta}{2\sigma_\beta^2}} \right] \quad \varepsilon = y - X\beta \\ &== \chi^{-2}(\sigma_\beta^2 | S = \beta'\beta + S_\beta, df = p + df_\beta) \quad [3] \end{aligned}$$