STT 465
Single-parameter models: Beta-Binomial & Poisson

(G. de los Campos)

Beta-Binomial

OUTLINE:

$\Longrightarrow$ Elements of the models:

(1) Sampling model $p(Y|\theta)$

(2) Prior distribution $p(\theta)$

(3) From (1) and (2) and using Bayes Rule, we derive the posterior distribution of the parameter given the data, $p(\theta|Y)$

(4) In this case the posterior distribution has a recognizable form.

$\Longrightarrow$ Inference

- Posterior mean and posterior variance
- Is the Bayesian estimator unbiased?
- What happens as sample size increases? (consider both the effects on bias and variance)
- Posterior credibility regions (interpretation, types,…)

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Beta-Binomial

⟹ Sampling model

$$p(y_i \mid \theta) = \theta^{y_i} (1-\theta)^{1-y_i}$$

[Assuming IID]

$$p(y_1,\ldots,y_n \mid \theta) = \prod_{i=1}^{n} \theta^{y_i} (1-\theta)^{1-y_i} = \theta^{n_1} (1-\theta)^{n_0} \qquad n_1 = \sum_{i=1}^{n} y_i \ ; \ n_0 = n - n_1$$

⟹ Prior: we will consider a Beta distribution (the uniform is a special case)

$$p(\theta \mid \alpha,\beta) = \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha,\beta)} \quad ; \ B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$p(\theta \mid \alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

⟹ Discuss: Kernel vs Integrating constant.

Posterior Distribution

$\Rightarrow$ According to Bayes' rule

$$p(\theta \mid y_1,\ldots,y_n) = \frac{p(y_1,\ldots,y_n \mid \theta)\, p(\theta)}{p(y_1,\ldots,y_n)} \propto p(y_1,\ldots,y_n \mid \theta)\, p(\theta)$$

$$p(\theta \mid y_1,\ldots,y_n) \propto \theta^{n_1}(1-\theta)^{n_0}\,\frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha,\beta)} \propto \theta^{\alpha+n_1-1}(1-\theta)^{\beta+n_0-1}$$

$\Rightarrow$ Finding the integrating constant

$$\int_0^1 \frac{\theta^{\alpha+n_1-1}(1-\theta)^{\beta+n_0-1}}{c(y)}\,d\theta = 1$$

$$\frac{1}{c(y)}\int_0^1 \theta^{\alpha+n_1-1}(1-\theta)^{\beta+n_0-1}\,d\theta = 1$$

$$\int_0^1 \theta^{\alpha+n_1-1}(1-\theta)^{\beta+n_0-1}\,d\theta = c(y) = B(\alpha+n_1,\beta+n_0,)$$

Posterior Distribution

$\Rightarrow$ Therefore

$$p(\theta \mid y_1,\ldots,y_n) = \frac{p(y_1,\ldots,y_n \mid \theta)\, p(\theta)}{p(y_1,\ldots,y_n)} \propto p(y_1,\ldots,y_n \mid \theta)\, p(\theta)$$

$$p(\theta \mid y_1,\ldots,y_n) = \frac{\theta^{\alpha+n_1-1}(1-\theta)^{\beta+n_0-1}}{B(\alpha+n_1,\beta+n_0)} = Beta(\theta \mid \alpha+n_1, \beta+n_0)$$

$\Rightarrow$ Because the posterior distribution has the same form as that of the prior, we say that the Beta prior is Conjugate to the Binomial likelihood.

$\Rightarrow$ Discuss:

- Posterior Vs. Prior Mean
- Posterior variance Vs sample size
- Posterior credibility regions (definition & interpretation).