Quantum Max Flow Analysis

Moreno Giussani

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Abstract

This document will describe the analysis I have performed in the last months about the quantum implementation of the Max Flow algorithm. The max flow problem involves finding a feasible flow through a single-source, single-sink flow network that is maximum. I have been working for finding a practical solution to this problem using a quantum algorithm which could be at least as efficient as a classical one, without succeding in it.

Part I

Introduction

Before considering the algorithm, I have tried to understand most of the concepts which lies behind a quantum algorithm.

For what regards quantum computing, the standard model of computation is the quantum circuit. A quantum circuit is a scheme composed of some elementary blocks, which are qubits and quantum logic gates. Rows of this scheme represents qubits, while in columns are inserted quantum logic gates.

1 Qubit

Qubits are the quantum equivalent of bits. A single qubit $|Q_0\rangle$is usually described by a 2-dimensional column vector which is a particular linear combination of its orthonormal bases $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. When the qubit is measured (an equivalent operation of reading a bit value in classical computing), its value collapses to either $|0\rangle$ or $|1\rangle$ (their orthonormal bases).

Suppose $|Q_0\rangle$is defined as

$$|Q_0\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \;\; \alpha \in \mathbb{C}, \beta \in \mathbb{C}$$

Then α and β must respect the rule $|\alpha|^2 + |\beta|^2 = 1$, because $|\alpha|^2$ represents the probability that a measurement outputs $|0\rangle$ and $|\beta|^2$ represents the probability that a measurement outputs $|1\rangle$. During the computation the qubit can assume an "overlapped" state (both state 0 and state 1), but when measured, its expressivity power reduces to a classical bit. When both α and β are different from 0 ,Q_0 is said to be in superposition.

Qubits have also another interesting property: they cannot be copied. There is no way to create an identical copy of an arbitrary unknown quantum state (*no cloning theorem*).

Now, things get a bit tricky when considering a N-qubit quantum computer. If there are two or more qubits, their representation is made as the tensor product of all of the qbits. Suppose to have a 3-qubit quantum computer which uses qubits Q_a, Q_b, Q_c. The representation of the state of the quantum system becomes:

$$|Q_x\rangle = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad x \in \{a, b, c\} \tag{1}$$

$$|Q_{ab}\rangle = |Q_a\rangle \otimes |Q_b\rangle = \begin{bmatrix} a_1 \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\ a_2 \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_1b_1 \\ a_1b_2 \\ a_2b_1 \\ a_2b_2 \end{bmatrix} \tag{2}$$

$$|Q_{abc}\rangle = |Q_a\rangle \otimes |Q_b\rangle \otimes |Q_c\rangle = |Q_{ab}\rangle \otimes |Q_c\rangle = \begin{bmatrix} a_1b_1 \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ a_1b_2 \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ a_2b_1 \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ a_2b_2 \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_1b_1c_1 \\ a_1b_1c_2 \\ a_1b_2c_1 \\ a_1b_2c_2 \\ a_2b_1c_1 \\ a_2b_1c_2 \\ a_2b_2c_1 \\ a_2b_2c_2 \end{bmatrix} \tag{3}$$

In many cases it is impossible to consider $Q_a, Q_b$and $Q_c$separately, because in a quantum system some quantum logic gates may cause to obtain a "mixed" state from which is not possible to find some suitable $Q_a, Q_b$and Q_c which satisfies (3). This concept, which is called entanglement, will be described in detail later. Of course, the quadratic sum of all elements of (3) must be 1.

2 Quantum logic gates

Quantum logic gates are represented by means of unitary square matrices. A matrix U is said unitary if

$$UU^\dagger = U^\dagger U = I$$

where $U^\dagger$ is the Hermitian conjugate of U. The Hermitian conjugate could be described as a conjugate traspose of U.

The most known unitary logic gates are:

Name	Symbol	Matrix
Hadamard	H	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$