Quantum Max Flow Analysis

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Abstract

This document will describe the analysis I have performed in the last months about the quantum implementation of the Max Flow algorithm. The max flow problem involves finding a feasible flow through a single-source, single-sink flow network that is maximum. I have been working for finding a practical solution to this problem using a quantum algorithm which could be at least as efficient as a classical one, without succeding in it.

Part I

Introduction

Before considering the algorithm, I have tried to understand most of the concepts which lies behind a quantum algorithm.

For what regards quantum computing, the standard model of computation is the quantum circuit. A quantum circuit is a scheme composed of some elementary blocks, which are qubits and quantum logic gates. Rows of this scheme represents qubits, while in columns are inserted quantum logic gates.

1 Qubit

Qubits are the quantum equivalent of bits. They are usually represented using the bra-ket notation. A single qubit $|Q_0\rangle$is usually described by a 2-dimensional column vector (Ket nontation) which is a specific linear combination of its orthonormal bases $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. When the qubit is measured (an equivalent operation of reading a bit value in classical computing), its value collapses to either $|0\rangle$ or $|1\rangle$ (their orthonormal bases).

Suppose $|Q_0\rangle$is defined as

$$|Q_0\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \;\; \alpha \in \mathbb{C}, \beta \in \mathbb{C}$$

Then α and β must respect the rule $|\alpha|^2 + |\beta|^2 = 1$, because $|\alpha|^2$ represents the probability that a measurement outputs $|0\rangle$ and $|\beta|^2$ represents the probability that a measurement outputs $|1\rangle$. During the computation the qubit can assume an “overlapped” state (both state 0 and state 1), but when measured, its expressivity power reduces to a classical bit. When both α and β are different from 0 ,Q_0 is said to be in superposition.

Qubits have also another interesting property: they cannot be copied. There is no way to create an identical copy of an arbitrary unknown quantum state (*no cloning theorem*).

Now, things get a bit tricky when considering a N-qubit quantum computer. If there are two or more qubits, their representation is made as the tensor product of all of the qbits, so they often cannot be considered as separated qubits. Suppose to have a 3-qubit quantum computer which uses qubits Q_a, Q_b, Q_c. The representation of the state of the quantum system becomes:

$$|Q_x\rangle = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad x \in \{a, b, c\} \tag{1}$$

$$|Q_{ab}\rangle = |Q_a\rangle \otimes |Q_b\rangle = \begin{bmatrix} a_1 \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\ a_2 \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{bmatrix} \tag{2}$$

$$|Q_{ab}\rangle = \begin{bmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{bmatrix} = a_1 b_1 (|0\rangle \otimes |0\rangle) + a_1 b_2 (|0\rangle \otimes |1\rangle) + a_2 b_1 (|1\rangle \otimes |0\rangle) + a_2 b_2 (|1\rangle \otimes |1\rangle) \tag{3}$$

$$|Q_{abc}\rangle = |Q_a\rangle \otimes |Q_b\rangle \otimes |Q_c\rangle = |Q_{ab}\rangle \otimes |Q_c\rangle = \begin{bmatrix} a_1 b_1 \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ a_1 b_2 \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ a_2 b_1 \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ a_2 b_2 \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_1 b_1 c_1 \\ a_1 b_1 c_2 \\ a_1 b_2 c_1 \\ a_1 b_2 c_2 \\ a_2 b_1 c_1 \\ a_2 b_1 c_2 \\ a_2 b_2 c_1 \\ a_2 b_2 c_2 \end{bmatrix} \tag{4}$$

In many cases it is impossible to consider $Q_a, Q_b$and $Q_c$separately, because in a quantum system some quantum logic gates may cause to obtain a “mixed” state from which is not possible to find some suitable $Q_a, Q_b$and Q_c which satisfies (4). This concept, which is called entanglement, will be described in detail later. The quadratic sum of all elements of a Ket must be 1, like said before for a single qubit.

2 Quantum logic gates

In a quantum circuit model, quantum logic gates are transformation matrices which describes the behaviour of the physical quantum logic gates. Quantum logic gates are represented by means of unitary square matrices of size 2^n,where n is the number of qubits to which a gate can be applied. A matrix U is said unitary if

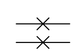
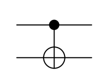
$$UU^\dagger = U^\dagger U = I$$

where $U^\dagger$ is the Hermitian transpose of U. The Hermitian transpose can be obtained by transposing U and then calculating the complex conjugate of all elements in U^T matrix.

The description of the state obtained from the application of a generic quantum gate $G$from quantum state$\left|S^0\right\rangle$ can be calculated as:

$$\left|S^1\right\rangle = G\left|S^0\right\rangle$$

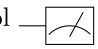
Some of the most known unitary logic gates are:

Name	Symbol	Matrix	Circuit
Hadamard	H	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$	$-\boxed{H}-$
Pauli-X (Not)	X(or NOT)	$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$	$-\boxed{X}-$
Pauli-Y	Y	$\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$	$-\boxed{Y}-$
Pauli-Z	Z or R_π	$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$	$-\boxed{Z}-$
Swap	$SWAP$	$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$	
Controlled Not	$CNOT$	$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}$	
Identity	I	$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$	— or $-\boxed{I}-$

When applied to a single qubit in one of its bases ($\|0\rangle$ or $\|1\rangle$), an H gate will put the qubit in a superstate.

There are many more controlled gates which are represented using a C prefix, like for the CNOT gate. Their structure is

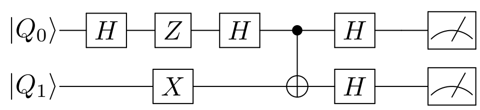
$$U = \begin{bmatrix} u_{11} & u_{12} & ... & u_{1m} \\ u_{21} & u_{22} & ... & u_{2m} \\ ... & ... & ... & ... \\ u_{m1} & u_{m2} & ... & u_{mm} \end{bmatrix} \; CU = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & u_{11} & ... & u_{1m} \\ ... & ... & ... & ... & ... \\ 0 & 0 & u_{m1} & ... & u_{mm} \end{bmatrix}$$

The “measurement” operator has symbol 

3 Quantum circuits

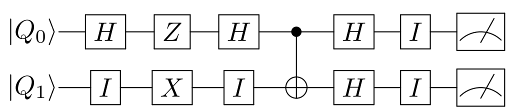
Quantum circuits can be represented via a ladder-like scheme. Each row (horizontal lines) represents a distinct qubit, and the gates which have to be applied to the given qubit are inserted on that line, from left to right. Gates on the same column has to be applied at the same time.

Here’s an example with a 2-qubit circuit:



(5)

which is equivalent to the given circuit:



(6)

Identity gates columns will not change the state, they can be ignored, because

$$I_{n\,x\,n} \otimes I_{m\,x\,m} = I_{nm\,x\,nm}$$

As said before, in a multi qubit computer, considering $Q_0$and Q_1as independent qubits would often lead to mistakes, because the application of a gate to a qubit would cause some side effects on other qubits.

If not specified, like in this case, every qubit is conventionally initialized to state $|0\rangle$.

Knowing that the above circuit is a 2-qubit circuit, the initial state $\left|S^0\right\rangle$is described as $|Q_0\rangle \otimes |Q_1\rangle = |00\rangle$.

Then, the applied gate is

$$H \otimes I = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

.

So, the state of the circuit after the application of $H \otimes I$ is:

$$\left|S^1\right\rangle = (H\otimes I)\left|S^0\right\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\left|00\right\rangle + \frac{1}{\sqrt{2}}\left|10\right\rangle$$

Now, simulating the whole execution:

$$\left|S^2\right\rangle = (Z\otimes X)\left|S^1\right\rangle = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\left|S^3\right\rangle = (H\otimes I)\left|S^2\right\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\left|S^4\right\rangle = CNOT\left|S^3\right\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\left|S^5\right\rangle = (H\otimes H)\left|S^4\right\rangle = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\left|S^6\right\rangle = (I\otimes I)\left|S^5\right\rangle = \left|S^5\right\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Then, the measurement would show $\left|01\right\rangle$as final result ($Q_0 = 0, Q_1 = 1$).

By definition, all the quantum gates that have been applied to this circuit are unitary. For this, by replacing recursively S^i, $0 < i < 6$ with S^{i-1} in the example circuit we obtain:

$$\left|S^6\right\rangle = (I\otimes I)(H\otimes H)CNOT(H\otimes I)(Z\otimes X)(H\otimes I)\left|S^0\right\rangle$$

So, if the circuit is unitary, then we could reverse the execution from the final state to the initial one. This property is called reversibility. In the example:

$$[(I \otimes I)(H \otimes H)CNOT(H \otimes I)(Z \otimes X)(H \otimes I)]^{-1} \left|S^6\right\rangle = \left|S^0\right\rangle$$

$$[(I \otimes I)(H \otimes H)CNOT(H \otimes I)(Z \otimes X)(H \otimes I)]^{\dagger} \left|S^6\right\rangle = \left|S^0\right\rangle$$

$$\left|S^0\right\rangle = (H \otimes I)^{\dagger}(Z \otimes X)^{\dagger}(H \otimes I)^{\dagger}CNOT^{\dagger}(H \otimes H)^{\dagger}(I \otimes I)^{\dagger} \left|S^6\right\rangle$$

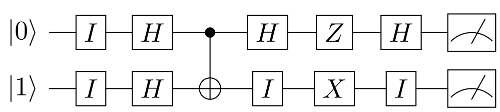
In our example, there are no complex gates, so

$$\left|S^0\right\rangle = (H \otimes I)'(Z \otimes X)'(H \otimes I)'CNOT'(H \otimes H)'(I \otimes I)' \left|S^6\right\rangle$$

All the applied matrices are symmetrical, so

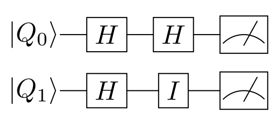
$$\left|S^0\right\rangle = (H \otimes I)(Z \otimes X)(H \otimes I)CNOT(H \otimes H)(I \otimes I) \left|S^6\right\rangle$$

This means that from every state of the system, we could rewind the execution flow. This property is called reversible computing and it is a property of every quantum computing algorithm. In this case, we can also build a circuit that from the final state S^6could reobtain the initial state S^0. The circuit that allows to do that is the original circuit flipped horizontally, that is:



(7)

Sometimes, considering qubits as independent from others would cause mistakes. Consider the following simple circuit:



(8)

In a naïve approach, we could say that $|Q_0\rangle = HH\,|0\rangle = I\,|0\rangle = |0\rangle$, which would have been true if $|Q_1\rangle$ would have not existed in the same circuit. But in this case, $|Q_1\rangle$ and $|Q_0\rangle$ influence eachother.

$$\left|S^0\right\rangle = |00\rangle$$