PRIME NUMBER THEOREM

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Abstract. Prime numbers have always been seen as the building blocks of all integers, but their behavior and distribution are often puzzling. The prime number theorem gives an estimate for how many prime numbers there are under any given positive number. By using complex analysis, we are able to find a function $\pi(x)$ that for any input will give us approximately the number of prime numbers less than the input.

Contents

1. The Prime Number Theorem

A **prime number** is an interger $\geqq 2$ which is divisible only by itself and 1. Thus the prime numbers start with the sequence 2,3,5,7,11,13,17,19, ... Since these numbers are indivisible but anything other than itself and 1, we can see them as the building blocks of all other numbers. In fact, the Fundamental Theorem of Arithmetic tells us that every natural number can be factored into a unique set of primes. It would therefore be beneficial for us to find out these prime numbers, or how they are distributed amongst the natural numbers.

By looking at the first few prime numbers, it is hard to believe that there could be any pattern among the prime numbers, but using complex analysis, we are able to come out with a very simple representation of the distribution of prime numbers.

In order to do so, we must first define a function $\pi(x)$ which will output the number of primes $\leqq x$. The prime number theorem then describes how $\pi(x)$ behaves for real numbers x.

Theorem 1.1. (Prime Number Theorem) *We have*

$$\pi(x) \sim \frac{x}{\log x}.$$

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Somehow, we are able to relate the distribution of prime numbers to a simple fraction, $\frac{x}{logx}$. To understand how this makes sense, we will need to start from a more basic function, the zeta function.

2. The Zeta Function

Let s be a complex variable. For $\mathrm{Re}(s) > 1$ the series

$$\sum_{n=1}^{\infty} \frac{1}{n^s},$$

converges absolutely, and uniformly for $\mathrm{Re}(s) \geqq 1+\delta$, with any $\delta > 0$. This series is called the **zeta function**. One sees this convergence by estimating

$$\left|\frac{1}{n^s}\right| \leqq \frac{1}{n^{1+\delta}},$$

and by using the integral test on the real series $\sum 1/n^{1+\delta}$, which has positive terms.

The Reimann Zeta Function has been used in algebraic number theory because of its usefulness in dealing with prime numbers. We will see its power in the next proposition.

Proposition 2.1. *The product*

$$\prod_p \left(1 - \frac{1}{p^s}\right),$$

converges absolutely for Re$(s) > 1$*, and uniformly for Re*$(s) \geqq 1+\delta$ *with* $\delta > 0$*, and we have*

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Proof. The product converges since each term is less than 1 and thus
In the same region $Re(s) \geqq 1+\delta$, we can use the geometric series estimate to conclude that

$$\left(1 - \frac{1}{p^s}\right)^{-1} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots = E_p(s).$$

Using a basic fact from elementary number theory that every positive integer has unique factorization into primes, up to the order of the factors, we conclude that in the product of the terms $E_p(s)$ for all primes p, the expression $1/(n^s)$ will occur exactly once, thus giving the series for the zeta functino in the region $\mathrm{Re}(s) > 1$. This concludes (Euler's) proof. $\square$

The product

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

is called the **Euler product**. The representation of the zeta function as such a product shows that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$. Since this product relates all primes to all natural numbers, if we are able to study this function, we can conclude the distribution of primes amongst the natural numbers. In order to make this function more usable, we want to extend it to all $\mathrm{Re}(s) > 0$, however we also have to make sure that the resulting function behaves in a way we want it to so we can analyze it further.

Theorem 2.2. *The function*

$$\zeta(s) - \frac{1}{s-1},$$

extends to a holomorphic function on the region $Re(s) > 0$

Proof. For $\mathrm{Re}(s) > 1$, we have

$$\begin{aligned}\zeta(s) - \frac{1}{s-1} &= \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^{\infty} \frac{1}{x^s} dx \\ &= \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx.\end{aligned}$$

We estimate each term in the sum by using the relations

$$f(b) - f(a) = \int_a^b f'(t)dt \quad \text{and so} \quad |f(b) - f(a)| \leqq \max_{a \leqq t \leqq b} |f'(t)||b-a|,$$

therefore each term is estimated as follows:

$$\begin{aligned}\left| \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx \right| \leqq \max_{n \leqq x \leqq n+1} \left| \frac{1}{n^s} - \frac{1}{x^s} \right| &\leqq \max \left| \frac{s}{x^{s+1}} \right| \\ &\leqq \frac{|s|}{n^{Re(s)+1}}.\end{aligned}$$

Thus the sum of the terms coverge absolutely and uniformly for $\mathrm{Re}(s) > \delta$. This concludes the proof of the theorem

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We note that the zeta function has a certain symmetry about the x-axis, namely

$$\zeta(\bar{s}) = \overline{\zeta(s)}.$$

This is immediate from the Euler product and the series expansion of Theorem 2.2. It follows that if s_0 is a complex number where ζ has a zero of order m (which may be a pole, in which case m is negative), then the complex conjugate $\bar{s_0}$ is a complex number where ζ has a zero of the same order m.

We now define

$$\varphi(x) = \sum_{p \leqq x} \log p \qquad \text{and} \qquad \Phi(s) = \sum_{p} \frac{\log p}{p^s} \qquad \text{for } Re(s) > 1.$$

We will now try to use log because it allows us to turn the Euler product into sums which may be more manageable.

The sum defining $\Phi(s)$ converges uniformly and absolutely for

$$\mathrm{Re}(s) \geqq 1 + \delta,$$

by the same argument as for the sum defining the zeta function. We merely use the fact that given $\epsilon > 0$

$$\log n \leqq n^{\epsilon} \qquad \text{for all} \quad n \geqq n_0(\epsilon)$$

Theorem 2.3. *The function Φ is meromorphic for $Re(s) > \frac{1}{2}$. Furthermore, for $Re(s) \geqq 1$, we have $\zeta(s) \neq 0$ and*

$$\Phi(s) - \frac{1}{s-1},$$

has no poles for $Re(s) \geqq 1$

Proof. For $\mathrm{Re}(s) > 1$, The Euler product shows that $\zeta(s) \neq 0$. By taking the derivative of $\log \zeta(s)$ we get

$$-\frac{\zeta'}{\zeta(s)} = \sum_p \frac{\log p}{p^s - 1}.$$

Using the geometric series we get the expansion

$$\begin{aligned} \frac{1}{p^s-1} = \frac{1}{p^s}\frac{1}{1-1/p^s} &= \frac{1}{p^s}\left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) \\ &= \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots, \end{aligned}$$

so

$$-\frac{\zeta'}{\zeta(s)} = \Phi(s) + \sum_p h_p(s) \quad \text{where} \quad |h_p(s)| \leqq C\frac{\log p}{|p^{2s}|}, \tag{2.4}$$

for some constant C. But the series $\sum(\log n)/n^{2s}$ converges absolutely and uniformly for $\mathrm{Re}(s) \geqq \frac{1}{2} + \delta$ with $\delta > 0$, so Theorem 2.2 and Equation 2.4 imply that Φ is meromorphic for $\mathrm{Re}(s) > \frac{1}{2}$, and has a pole at $s = 1$ and at the zeros of ζ, but no other poles in this region.

There remains only to be proved that ζ has no zero on the line $\mathrm{Re}(s) = 1$. Suppose ζ has a zero of order m at $s = 1 + ib$ with $b \neq 0$. Let n be the order of the zero at $s = 1 + 2ib$. Then $m, n \geqq 0$ by Theorem 2.2.

From Equation 2.4 we get

$$\begin{aligned} \lim_{\epsilon\to 0} \epsilon\Phi(1+i\epsilon) = 1, \quad &\lim_{\epsilon\to 0} \epsilon\Phi(1+\epsilon \pm ib) = -m, \\ &\lim_{\epsilon\to 0} \epsilon\Phi(1+\epsilon \pm 2ib) = -n. \end{aligned}$$

Indeed, if a meromorphic function $f(z)$ has the factorization

$$f(z) = (z - z_0)^k h(z) \quad \textit{with} \quad h(z_0) \neq 0, \infty$$

then

$$\frac{f'}{f(z)} = \frac{k}{z - z_0} + \text{holomorphic terms at } z_0,$$

whence our three limits follow immediatley from the definitions and the pole of the zeta function at 1, with residue 1.

Now the following identity is trivially verified:

$$\begin{aligned}&\Phi(s+2ib)+\Phi(s-2ib)+4\Phi(s+ib)+4\Phi(s-ib)+6\Phi(s)\\&\quad=\sum_p \frac{\log p}{p^{2s}}(p^{ib/2}+p^{-ib/2})^4\end{aligned}$$

For $s = 1+\epsilon > 1$ we see that the above expression is $\geqq 0$. using our limits, we conclude that

$$-2n-8m+6 \geqq 0,$$

whence $m = 0$. This concludes the proof of Theorem 1.4. □

Proposition 2.5. *For Re(s) > 1 we have*

$$\Phi(s) = s\int_1^\infty \frac{\varphi(x)}{x^{s+1}}dx$$

Proof. To prove this, compute the integral on the right between successive prime numbers, where φ is constant. Then sum by parts. □

3. The Main Lemma and its Application

Theorem 3.1. (Chebyshev) *We have*

$$\varphi(x) = O(x).$$

Proof. Let n be a positive integer. Then

$$2^{2n} = (1+1)^{2n} = \sum_j \binom{2n}{j} \geqq \binom{2n}{n} \geqq \prod_{n<p\leqq 2n} p = e^{\varphi(2n)-\varphi(n)},$$

hence we get the equality

$$\varphi(2n) - \varphi(n) \leqq 2n\log 2.$$

But if x increases by 1, then $\varphi(x)$ increases by at most $\log(x+1)$, which is $O(\log x)$. Hence there is a constant $C > \log 2$ such that for all $x \geqq x_0(C)$ we have

$$\varphi(x) - \varphi(x/2) \leqq Cx.$$

We apply this inequality in succession to $x, x/2, x/2^2, \cdots, x/2^r$ and sum.
This yields

$$\varphi(x) \leqq 2Cx + O(1),$$

which proves the theorem. □

We shall now state the main lemma, which constitutes the delicate part of the proof. Let f be a function defined on the real numbers $\geqq 0$, and assume for simplicity that f is bounded, and piecewise continuous. What we prove will hold under much less restrictive condition: instead of piecewise continuous, it would suffice to assume that the integral

$$\int_a^b |f(t)|dt,$$

exists for every pair of numbers $a, b \geqq 0$. We shall associate to f the Laplace transform g defined by

$$g(z) = \int_0^\infty f(t)e^{-zt}dt \text{ for } \mathrm{Re}(z) > 0.$$

Lemma 3.2. (Main Lemma) *Let f be bounded, piecewise continuous on the reals $\geqq 0$. Let $g(z)$ be defined by*

$$g(z) = \int_0^\infty f(t)e^{-zt}dt \textit{ for } Re(z) > 0$$

If g extends to an analytic function for $Re(z) \geqq 0$, then

$$\int_0^\infty f(t)\,dt$$

exists and is equal to $g(0)$

The main lemma will be proved in the next section, for now we will apply the main lemma to prove Lemma 3.3

Lemma 3.3. *The integral*

$$\int_1^\infty \frac{\varphi(x) - x)}{x^2}\,dx,$$

converges

Proof. Let

$$f(t) = \varphi(e^t)e^{-t} - 1 = \frac{\varphi(e^t) - e^t}{e^t}.$$

f is certainly piecewise continuous, and is bounded by Theorem 2.1. Making the substitution $x = e^t$ in the desired integral, $dx = e^t dt$, we see that

$$\int_1^\infty \frac{\varphi(x) - x}{x^2}\,dx = \int_0^\infty f(t)\,dt.$$

It suffices to prove that the integral on the right converges. By the main lemma, it suffices to prove that the Laplace transform of f is analytic for $\mathrm{Re}(z) \geqq 0$, so we have to compute this Laplace transform. We claim that in this case,

$$g(z) = \frac{\Phi(z+1)}{z+1} - \frac{1}{z}.$$

Once we have proved this formula, we can apply Theorem 1.3 to conclude that g is analytic for $\mathrm{Re}(z) \geqq 0$, thus concluding the proof of Lemma 2.3. Now to compute $g(z)$, we use the integral formula of Proposition 1.4. By this formula, we obtain

$$\frac{\Phi(s)}{s} - \frac{1}{s-1} = \int_1^\infty \frac{\varphi(x) - x}{x^{s+1}}\,dx.$$

By substituting $z+1$ for s we get

$$\begin{aligned}\frac{\Phi(z+q)}{z+1} - \frac{1}{z} &= \int_1^\infty \frac{\varphi(x)-x}{x^{s+1}}\,dx \\ &= \int_0^\infty \frac{\varphi(e^t)-e^t}{e^{2t}} e^{-zt} e^t\,dt \\ &= \int_0^\infty f(t) e^{-zt}\,dt.\end{aligned}$$

This gives us the Laplace transform of f, and concludes the proof of Lemma 2.3.

$\square$

Let f_1 and f_2 be functions defined for all $x \geqq x_0$, for some x_0. We say that f_1 is asymptotic to f_2, and write

$$f_1 \sim f_2 \quad \text{if and only if} \quad \lim_{x\to\infty} f_1(x)/f_2(x) = 1$$

Theorem 3.4. *We have $\varphi(x) \sim x$*

Proof. The assertion of the theorem is logically equivalent to the comibnation of teh following two assertions:

Given $\lambda > 1$, the set of x such that $\varphi(x)/x \geqq \lambda\, x$ is bounded;
Given $0 < \lambda < 1$, the set of x such that $\varphi(x) \leqq \lambda\, x$ is bounded.

Let us prove the first. Suppose the first assertion is false. Then there is some $\lambda > 1$ such that for arbitrarily large x we have $\varphi(x)/x \geqq \lambda$. Since φ is monotone and increasing, we get for such x:

$$\int_x^{\lambda x} \frac{\varphi(t)-t}{t^2}\,dt \geqq \int_x^{\lambda x} \frac{\lambda x - t}{t^2}\,dt = \int_1^{\lambda} \frac{\lambda - t}{t^2}\,dt > 0.$$

The number on the far right is independent of x. Since there are arbitrarily large x satisfying the above inequality, it follows that the integral of Lemma 2.3 does not converge, a contradiction. So the first assertion is proved. The second assertion is proved in the same way. $\square$

Theorem 3.5. (Prime Number Theorem) *We have*

$$\pi(x) \sim \frac{x}{\log x}.$$

Proof. We have

$$\varphi(x) = \sum_{p \leqq x} \log p \leqq \sum_{p \leqq x} \log x = \pi(x)\log x;$$

and given $\epsilon > 0$,

$$\begin{aligned}\varphi(x) \geqq \sum_{x^{1-\epsilon} \leqq p \leqq x} \log p &\geqq \sum_{x^{1-\epsilon} \leqq p \leqq x} (1-\epsilon)\log x \\ &= (1-\epsilon)\log x\, \pi(x) + O(x^{1-\epsilon}).\end{aligned}$$

Using Theorem 2.4 that $\varphi(x) \sim x$ concludes the proof of the prime number theorem

$\square$

4. Proof of the Main Lemma

We recall the main lemma

Let f be bounded, piecewise continuous on the reals $\geqq 0$. Let

$$g(z) = \int_0^\infty f(t)e^{-zt}\, dt \quad for\ \mathrm{Re}(z) > 0$$

If g extends to an analytic function for $\mathrm{Re}(z) \geqq 0$, *then*

$$\int_0^\infty f(t)\, dt \text{ exists and is equal to } g(0)$$

Proof. For $T > 0$ define

$$g_T(z) = \int_0^T f(t)e^{-zt} dt.$$

g_T is an entire function, as follows at once by differentiating under the integral sign. We have to show that

$$\lim_{T\to\infty} g_T(0) = g(0).$$

Let $\delta > 0$ and let C be the path consisting of the line segment $\mathrm{Re}(z) = -\delta$ and the arc of circle $|z| = R$ and $\mathrm{Re}(z) \geqq -\delta$,

By our assumption that g extends to an analytic function for $\mathrm{Re}(z) \geqq 0$, we can take δ small enough so that g is analytic on the region bounded by C, and on its boundary. Then

$$g(0) - g_T(0) = \frac{1}{2\pi i}\int_C (g(z) - g_T(z))e^{Tz}(1 + \frac{z^2}{R^2})\frac{dz}{z} = \frac{1}{2\pi i}\int_C H_T(z)\, dz,$$

where $H_T(z)$ abbreviates the expression under the integral sign. Let B be a bound for f, that is $|f(t)| \leqq B$ for all $t \geqq 0$.

In order to prove $|g(0) - g_T(0)| < \epsilon$, we will split up the path C into C^- and C^+

Let C^+ be the semicircle $|z| = R$ and $\mathrm{Re}(z) \geqq 0$. Then we can prove

$$\left|\frac{1}{2\pi i}\int_{C^+} H_T(z)\, dz\right| \leqq \frac{2B}{R} \tag{4.1}$$

by first noting that for $\mathrm{Re}(z) > 0$ we have

$$\begin{aligned}|g(z) - g_T(z)| = \left|\int_T^\infty f(t)e^{-zt}\, dt\right| &\leqq B\int_T^\infty |e^{-zT}|\, dt \\ &= \frac{B}{\mathrm{Re}(z)}e^{-\mathrm{Re}(z)T}\end{aligned}$$

and for $|z| = R$,

$$\left|e^{Tz}\left(1 + \frac{z^2}{R^2}\right)\frac{1}{z}\right| = e^{\mathrm{Re}(z)T}\left|\frac{R}{z} + \frac{z}{R}\right|\frac{1}{R} = e^{\mathrm{Re}(z)T}\frac{2|\mathrm{Re}(z)|}{R^2}.$$

Taking the product of the last two estimates and multiplying by the length of the semicircles gives the bound for the integral over the semicircle, and proves the claim

Now let C^- be the part of the path C with $\mathrm{Re}(z) < 0$. We wish to estimate

$$\frac{1}{2\pi i}\int_C^- (g(z) - g_T(z))\, e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}$$

Now we estimate separately the expression under the integral with g and g_T. We want to show that

$$\left|\frac{1}{2\pi i}\int_{C^-} g_T(z)e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}\right| \leqq \frac{B}{R}. \tag{4.2}$$

we do this by letting S^- be the semicircle with $|z| = R$ and $\mathrm{Re}(z) < 0$. Since g_T is entire, we can replace C^- by S^- in the integral without changing the value of the integral, because the integrand has no pole to the left of the y-axis. Now we estimate the expression under the integral sign on S^-. We have

$$\begin{aligned} |g_T(z)| = \left|\int_0^T f(t)e^{-zT}\,dt\right| &\leqq B\int_0^T e^{-\mathrm{Re}(z)T}\,dt \\ &\leqq \frac{B\,e^{-\mathrm{Re}(z)T}}{-\mathrm{Re}(z)} \end{aligned}$$

For the other factor we use the same estimate as previously. We take the product of the two estimates, and multiply by the length of the semicircle to give the desired bound in (3.2).

Third, we claim that

$$\int_{C^-} g(z)e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z} \to 0 \quad \text{as } T\to\infty \tag{4.3}$$

We can write the expression under the integral sign as

$$g(z)e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z} = h(z)e^{Tz} \quad \text{where } h(z) \text{ is independent of } T.$$

Given any compact subset K of the region defined by $\mathrm{Re}(z) < 0$, we note that

$$e^{Tz} \to 0 \text{ rapidly uniformly for } z \in K, \text{as } T\to\infty$$

The word "rapidly" means that the expression divided y any power T^N also tends to 0 uniformly for z in K, as $T \to \infty$. From this our claim (3.3) follows easily
We may now prove the main lemma. We have

$$\int_0^\infty f(t)\,dt = \lim_{T\to\infty} g_T(0) \text{ if this limit exists.}$$

But given ϵ, pick R so large that $2B/R < \epsilon$. Then by (3.3), pick T so large that

$$\left| \int_{C^-} g(z) e^{Tz} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} \right| < \epsilon$$

Then by (3.1), (3.2), and (3.3) we get $|g(0) - g_T(0)| < 3\epsilon$. This means that the equation converges, proving the main lemma. This concludes the proof of the prime number theorem.

□

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6. References

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