Algebraic Topology with different coefficients

Günthner

WS24

This paper will be an exercise in Algebraic Topology by developing singular homology with arbitrary coefficients.

For this paper let $\mathbf{R}$ be a commutative ring and T as topological space.

1 The homology of the point

1.1 An Introduction of Homology

First let us define the k-Simplex:

$$\Delta^k := \{ (t_i) \in \mathbb{R}^{k+1} : \sum_{i=1}^{k+1} t_i = 1 \text{ and } 0 \leq t_i \leq 1 \} \tag{1}$$

Then we define $C_k(T)$ as the free module over $\mathbf{R}$ with basis:

$$\{ \Delta^k \to T \} \tag{2}$$

Now we define $d_k : C_k(T) \to C_{k-1}(T)$ to make the $C_k T$ a chain complex, meaning that $d^2 = 0$:

$$d_k(\varphi) = \sum_{i=0}^{k} (-1)^i d_k^i(\varphi) \tag{3}$$

The d_k^i is defined as follows:

$$d_k^i(\varphi)(t_j) = \varphi(t_1, t_2, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_{k-2}, t_{k-1}) \tag{4}$$

Homology can now be defined as:

$$H_k(T) := {}^{\ker(d_k)}/_{\mathrm{img}(d_{k+1})} \tag{5}$$

1.2 The calculation

To calculate $H_k(*)$, we will need to figure out when d_k is the identity and when it is the zero-map. First it will be good to understand the underlying chaincomplex:

$$C_k(*) = \{\, r \cdot (\Delta^k \to * \text{ constant}) : r \in \mathbf{R} \,\} \tag{6}$$

In other words $\dim(C_k(*)) = 1$.

Now if k is odd there is an even number of boundries and all the terms in the alternating sum of d_k cancel each other out and $d_k = 0$.

If—however—k is even it is not the null-map and (as $\dim(C_k(*)) = 1$) d_k is an isomorphism.

This can be used to compute the kernel and the image of d_k:

$$\ker(d_k) = \begin{cases} \mathbf{R} & \text{if } k \text{ odd} \\ 0 & \text{if } k \text{ even} \end{cases} \tag{7}$$

$$\operatorname{img}(d_k) = \begin{cases} 0 & \text{if } k \text{ odd} \\ \mathbf{R} & \text{if } k \text{ even} \end{cases} \tag{8}$$

This yields for $k > 0$:

$$H_k(*) = 0 \tag{9}$$

For $k = 0$:

$$H_0(*) = {}^{\mathbf{R}}\!/_{0} = \mathbf{R} \tag{10}$$