

Notes about Prime Constellations

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1 Introduction

Definition 1. *A Constellation is a function: $\chi : \mathbb{R} \rightarrow \mathbb{R}^d$
We call d the degree of the constellation.*

We are now looking for n such that all $\chi(n)$ are prime! Or rather we are looking for their count in the range $[1, n]$. To do this we will first “generalize” prime numbers because we have to closed form expression for $\pi(n)$, the number of primes in the range $[1, n]$. Instead we have an upper bound.

The general idea is to step over all “prime” numbers and eliminate all t such that not all $\chi(t)$ are prime. Then we use the upper bound to establish a lower bound for the resulting count.

1.1 Example Constellations

1.1.1 Basic Prime Numbers

The constellation $\chi(t) = t$ yields the primes.

1.1.2 Twin Primes

The constellation $\chi(t) = (t - 2, t)$ yields twin primes.

1.1.3 Goldbach-n-primes

Definition 2. *Goldbach-n-prime*

For n even, a number $p < n$ is a Goldbach-n-prime iff both p and $n - p$ are prime

Conjecture 1 (The Goldbach-Conjecture). *For every $n \geq 4$ there exists at least one Goldbach-n-prime*

The constellation $\chi(t) = (t, n - t)$ yields Goldbach-n-primes

1.1.4 Landau’s Fourth Problem

The constellation $\chi(t) = t^2 + 1$ yields primes for Landau’s fourth problem.

1.1.5 Mersenne Primes

The constellation $\chi(t) = 2^t - 1$ yields Mersenne Primes.

1.1.6 Sophie Germain Primes

The constellation $\chi(t) = (t, 2t + 1)$ yields Sophie Germain Primes.

2 Basic Definitions

Let's define a function that generalizes prime numbers:

$$\psi : \mathbb{Z} \rightarrow \mathbb{Z} \quad (1)$$

We will also need an inverse function for χ :

$$\begin{aligned} \chi^{-1} : \mathbb{R} &\rightarrow \mathbb{R}^d \\ \chi^{-1}(t) &= (\chi_k^{-1}(t))_{k \in [d]} \end{aligned} \quad (2)$$

Now to define the numbers “coprime” to the first k “prime” numbers: $M_\chi^\psi(k)$

$$\begin{aligned} M_\chi^\psi(0) &:= M_0 := \mathbb{Z} \setminus \{t \in \mathbb{Z} : \exists l \in [d] : \chi_l(t) = 0 \vee \chi_l(t) \in \mathbb{Z}^\times\} \\ &= \mathbb{Z} \setminus \chi^{-1}(0) \setminus \chi^{-1}(\mathbb{Z}^\times) \end{aligned} \quad (3)$$

$$\begin{aligned} M_\chi^\psi(k) &:= M_\chi^\psi(k-1) \setminus \{t \in \mathbb{Z} : \exists l \in \chi(t) : \psi(k) \mid l\} \\ &= M_\chi^\psi(k-1) \setminus \{t \in \mathbb{Z} : \exists l \in [d], m \in M_0 : m\psi(k) = \chi_l(t)\} \\ &= M_\chi^\psi(k-1) \setminus \{t \in \mathbb{Z} : \exists l \in [d], m \in M_0 : \chi_l^{-1}(m\psi(k)) = t\} \\ &= M_\chi^\psi(k-1) \setminus \chi^{-1}(\psi(k) \cdot M_0) \\ &= M_\chi^\psi(k-1) \setminus \left(M_\chi^\psi(k-1) \cap \chi^{-1}(\psi(k) \cdot M_0) \right) \end{aligned} \quad (4)$$

Definition 3.

$$\Psi_\chi^\psi = \lim_{k \rightarrow \infty} M_\chi^\psi(k) \quad (5)$$

Definition 4.

$$\pi_\chi^\psi(t) = \#(\Psi_\chi^\psi \cap [t]) \quad (6)$$

3 Derivation

3.1 Primes

For $\chi = \text{id}$:

$$\begin{aligned} M_\chi^\psi(k-1) \cap \chi^{-1}(\psi(k) \cdot M_0) &= M_\chi^\psi(k-1) \cap (\psi(k) \cdot M_0) \\ &= \psi(k) \cdot M_\chi^\psi(k-1) \end{aligned} \quad (7)$$

$$\pi_{\text{id}}^\psi(t) = \gamma_{\psi^{-1}(t)}^\psi(t) \quad (8)$$

$$\gamma_k^\psi(t) := M_\chi^\psi(t) \cap [t] \quad (9)$$

$$\gamma_0^\psi(t) = t \quad (10)$$

$$\gamma_k^\psi(t) = \gamma_{k-1}^\psi(t) - \gamma_{k-1}^\psi\left(\left\lfloor \frac{t}{\psi(t)} \right\rfloor\right) \quad (11)$$

3.2 Twin Primes

For $\chi(t) = \{t-2, t\}$:

$$\begin{aligned} & M_\chi^\psi(k-1) \cap \chi^{-1}(\psi(k) \cdot M_0) \\ &= M_\chi^\psi(k-1) \cap (\psi(k) \cdot M_0 \cup (\psi(k) \cdot M_0 + 2)) \\ &= (M_\chi^\psi(k-1) \cap \psi(k)M_0) \cup (M_\chi^\psi(k-1) \cap \psi(k)M_0 + 2) \end{aligned} \quad (12)$$

3.3 Goldbach-n-primes

For $n \in \mathbb{N}$ and $\chi(t) = \{t, n-t\}$:

$$\begin{aligned} & M_\chi^\psi(k-1) \cap \chi^{-1}(\psi(k) \cdot M_0) \\ &= (M_\chi^\psi(k-1) \cap \psi(k)M_0) \cup (M_\chi^\psi(k-1) \cap (n - \psi(k)M_0)) \end{aligned} \quad (13)$$

3.4 Landau's Fourth Problem

$$M_\chi^\psi(k-1) \cap \chi^{-1}(\psi(K)M_0) = M_\chi^\psi(k-1) \cap \sqrt{\psi(k)M_0 - 1} \quad (14)$$

3.5 Mersenne Primes

$$M_\chi^\psi(k-1) \cap \chi^{-1}(\psi(K)M_0) = M_\chi^\psi(k-1) \cap \log_2(\psi(k)M_0 + 1) \quad (15)$$

3.6 Sophie Germain Primes

$$\begin{aligned} & M_\chi^\psi(k-1) \cap \chi^{-1}(\psi(K)M_0) \\ &= (M_\chi^\psi(k-1) \cap \psi(k)M_0) \cup (M_\chi^\psi(k-1) \cap \frac{\psi(k)M_0 - 1}{2}) \end{aligned} \quad (16)$$

4 Results

Lemma 1.

$$\mathbb{P} = \Psi_{\text{id}}^{\pi^{-1}} \quad (17)$$

Lemma 2.

$$\pi(t) = \pi_{\text{id}}^{\pi^{-1}}(t) \quad (18)$$