Notes about Prime Constellations

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1 Introduction

Definition 1. *A Constellation is a function:* $\chi : \mathbb{R} \to \mathbb{R}^d$
We call d *the degree of the constellation.*

We are now looking for n such that all $\chi(n)$ are prime! Or rather we are looking for their count in the range $[1, n]$. To do this we will first "generalize" prime numbers because we have to closed form expression for $\pi(n)$, the number of primes in the range $[1, n]$. Instead we have an upper bound.

The general idea is to step over all "prime" numbers and eliminate all t such that not all $\chi(t)$ are prime. Then we use the upper bound to establish a lower bound for the resulting count.

1.1 Example Constellations

1.1.1 Basic Prime Numbers

The constellation $\chi(t) = t$ yields the primes.

1.1.2 Twin Primes

The constellation $\chi(t) = (t - 2, t)$ yields twin primes.

1.1.3 Goldbach-n-primes

Definition 2. *Goldbach-n-prime*
For n *even, a number* $p < n$ *is a Goldbach-n-prime iff both* p *and* $n - p$ *are prime*

Conjecture 1 (The Goldbach-Conjecture)**.** *For every* $n \geq 4$ *there exists at least one Goldbach-n-prime*

The constellation $\chi(t) = (t, n - t)$ yields Goldbach-n-primes

1.1.4 Landau's Fourth Problem

The constellation $\chi(t) = t^2 + 1$ yields primes for Landau's fourth problem.

1.1.5 Mersenne Primes

The constellation $\chi(t) = 2^t - 1$ yields Mersenne Primes.

1.1.6 Sophie Germain Primes

The constellation $\chi(t) = (t, 2t + 1)$ yields Sophie Germain Primes.

2 Basic Definitions

Let's define a function that generalizes prime numbers:

$$\psi : \mathbb{Z} \to \mathbb{Z} \tag{1}$$

We will also need an inverse function for χ:

$$\begin{aligned} \chi^{-1} &: \mathbb{R} \to \mathbb{R}^d \\ \chi^{-1}(t) &= (\chi_k^{-1}(t))_{k\in[d]} \end{aligned} \tag{2}$$

Now to define the numbers "coprime" to the first k "prime" numbers: $M_\chi^\psi(k)$

$$\begin{aligned} M_\chi^\psi(0) \coloneqq M_0 \coloneqq \mathbb{Z} \setminus \{\, t \in \mathbb{Z} \;:\; \exists l \in [d] : \chi_l(t) = 0 \vee \chi_l(t) \in \mathbb{Z}^\times \} \\ = \mathbb{Z} \setminus \chi^{-1}(0) \setminus \chi^{-1}(\mathbb{Z}^\times) \end{aligned} \tag{3}$$

$$\begin{aligned} M_\chi^\psi(k) \coloneqq &M_\chi^\psi(k-1) \setminus \{\, t \in \mathbb{Z} \;:\; \exists l \in \chi(t) : \psi(k) \mid l \,\} \\ = &M_\chi^\psi(k-1) \setminus \{\, t \in \mathbb{Z} \;:\; \exists l \in [d], m \in M_0 : m\psi(k) = \chi_l(t) \,\} \\ = &M_\chi^\psi(k-1) \setminus \{\, t \in \mathbb{Z} \;:\; \exists l \in [d], m \in M_0 : \chi_l^{-1}(m\psi(k)) = t \,\} \\ = &M_\chi^\psi(k-1) \setminus \chi^{-1}\left(\psi(k) \cdot M_0\right) \\ = &M_\chi^\psi(k-1) \setminus \left(M_\chi^\psi(k-1) \cap \chi^{-1}\left(\psi(k) \cdot M_0\right)\right) \end{aligned} \tag{4}$$

3 Derivation

For $\chi = \mathrm{id}$:

$$\begin{aligned} M_\chi^\psi(k-1) \cap \chi^{-1}\left(\psi(k) \cdot M_0\right) &= M_\chi^\psi(k-1) \cap \left(\psi(k) \cdot M_0\right) \\ &= \psi(k) \cdot M_\chi^\psi(k-1) \end{aligned} \tag{5}$$

For $\chi(t) = \{\, t-2, t \,\}$:

$$\begin{aligned} &M_\chi^\psi(k-1) \cap \chi^{-1}\left(\psi(k) \cdot M_0\right) \\ &\quad = M_\chi^\psi(k-1) \cap \left(\psi(k) \cdot M_0 \cup (\psi(k) \cdot M_0 + 2)\right) \\ &\quad = \left(M_\chi^\psi(k-1) \cap \psi(k)M_0\right) \;\cup\; \left(M_\chi^\psi(k-1) \cap \psi(k)M_0 + 2\right) \end{aligned} \tag{6}$$

For $n \in \mathbb{N}$ and $\chi(t) = \{\, t, n-t \,\}$:

$$\begin{aligned} &M_\chi^\psi(k-1) \cap \chi^{-1}\left(\psi(k) \cdot M_0\right) \\ &\quad = \left(M_\chi^\psi(k-1) \cap \psi(k)M_0\right) \cup \left(M_\chi^\psi(k-1) \cap (n - \psi(k)M_0)\right) \end{aligned} \tag{7}$$

4 Results

Definition 3.

$$\Psi_{\chi}^{\psi} = \lim_{k \to \infty} M_{\chi}^{\psi}(k) \tag{8}$$

Lemma 1.

$$\mathbb{P} = \Psi_{\mathrm{id}}^{\pi^{-1}} \tag{9}$$

5 Notes

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