MATH 442: Homework 3

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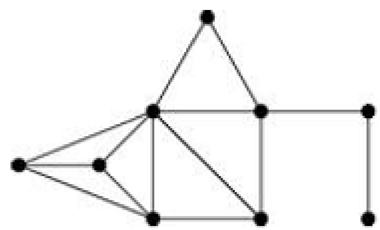
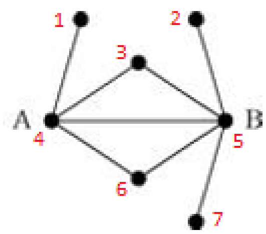
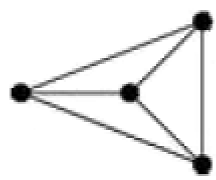
1. Consider the following two networks:





a) Find a 3-core in the first network.



b) Write down the modularity matrix for the second network.

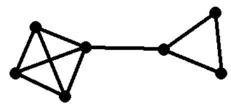
$$B_{ij} = A_{ij} - \frac{k_i k_j}{2M}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad M = 7 \qquad \mathbf{k} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \\ 5 \\ 2 \\ 1 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} -0.063 & -0.063 & -0.125 & 0.750 & -0.313 & -0.125 & -0.063 \\ -0.063 & -0.063 & -0.125 & -0.250 & 0.688 & -0.125 & -0.063 \\ -0.125 & -0.125 & -0.250 & 0.500 & 0.375 & -0.250 & -0.125 \\ 0.750 & -0.250 & 0.500 & -1.000 & -0.250 & 0.500 & -0.250 \\ -0.313 & 0.688 & 0.375 & -0.250 & -1.563 & 0.375 & 0.688 \\ -0.125 & -0.125 & -0.250 & 0.500 & 0.375 & -0.250 & -0.125 \\ -0.063 & -0.063 & -0.125 & -0.250 & 0.688 & -0.125 & -0.063 \end{pmatrix}$$

2. What is the difference between a 2-component and a 2-core? Draw a small network which has one 2-core but two 2-components.

A 2-component is a maximal subset of vertices such that each is reachable from each of the others by at least 2 vertex-independent paths. A 2-core is a maximal subset of vertices such that each is connected to at least 2 others in the subset. The network below has one 2-core 2-component and one 3-core 2-component.



3. Consider a "line graph" consisting of n vertices in a line.

a) Show that if we divide the network into two parts by cutting any single edge, such that one part has r vertices and the other has $n-r$, the modularity takes the value

$$Q = \frac{3-4n+4rn-4r^2}{2(n-1)^2}$$

It is easier to use the simplified equation for modularity $Q = \sum_r (e_{rr} - a_r^2)$ where e_r is the fraction of edges among vertices of type r and a_r is the fraction of ends of edges which are attached to vertices in r.

The number of edges between vertices in part A of the divided network can be expressed as $M_r^A = r-1$, where r is the number of vertices in part A, and the number of edges between vertices in part B of the divided network can be expressed as $M_r^B = n-r-1$. The number of ends of edges attached to r can be written as $2(r-1)+1$ since there are two ends for each edge between vertices in r plus one extra end for the edge that was cut. Then,

$$e_r^A = \frac{r-1}{n-1}, \quad e_r^B = \frac{n-r-1}{n-1}$$
$$a_r^A = \frac{2(r-1)+1}{2(n-1)}, \quad a_r^B = \frac{2(n-r-1)+1}{2(n-1)}$$

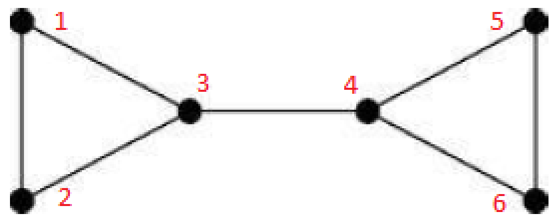
So to find modularity,

$$\begin{aligned}
Q &= (e_r^A - (a_r^A)^2) + (e_r^B - (a_r^B)^2) \\
&= \left(\frac{r-1}{n-1} - \left(\frac{2(r-1)+1}{2(n-1)}\right)^2\right) + \left(\frac{n-r-1}{n-1} - \left(\frac{2(n-r-1)+1}{2(n-1)}\right)^2\right) \\
&= \left(\frac{r-1}{n-1} + \frac{n-r-1}{n-1}\right) - \left(\left(\frac{2(r-1)+1}{2(n-1)}\right)^2 + \left(\frac{2(n-r-1)+1}{2(n-1)}\right)^2\right) \\
&= \frac{n-2}{n-1} - \frac{(2r-1)^2 + (2n-2r-1)^2}{4(n-1)(n-1)} \\
&= \frac{4(n-1)(n-2) - \left((2r-1)^2 + (2n-2r-1)^2\right)}{4(n-1)^2} \\
&= \frac{(2n^2-6n+4) - (2n^2-2n-4rn+4r^2+1)}{2(n-1)^2} \\
&= \frac{3-4n+4rn-4r^2}{2(n-1)^2}
\end{aligned}$$

b) If we assume n is even, what is the division that maximizes modularity?

We can apply the ratio cut partitioning approach, which is to minimize the ratio $\frac{R}{n_1 n_2}$, where n_1 and n_2 are the sizes of the two groups. The ratio is reduced by the largest amount when the denominator $n_1 n_2$ has its largest value, which is when $n_1 = n_2 = \frac{1}{2}n$. So the division that maximizes modularity is in the middle of the line graph.

4. Using your favorite numerical software for finding eigenvectors of matrices, construct the Laplacian and the modularity matrix for this small network:



$$\mathbf{L} = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} -0.286 & 0.714 & 0.571 & -0.429 & -0.286 & -0.286 \\ 0.714 & -0.286 & 0.571 & -0.429 & -0.286 & -0.286 \\ 0.571 & 0.571 & -0.643 & 0.357 & -0.429 & -0.429 \\ -0.429 & -0.429 & 0.357 & -0.643 & 0.571 & 0.571 \\ -0.286 & -0.286 & -0.429 & 0.571 & -0.286 & 0.714 \\ -0.286 & -0.286 & -0.429 & 0.571 & 0.714 & -0.286 \end{pmatrix}$$

a) Find the eigenvector of the Laplacian corresponding to the second smallest eigenvalue and hence perform a spectral bisection of the network into two equally sized parts.

```r
# Laplacian matrix
x <- matrix(c(2,-1,-1,0,0,0,
              -1,2,-1,0,0,0,
              -1,-1,3,-1,0,0,
              0,0,-1,3,-1,-1,
              0,0,0,-1,2,-1,
              0,0,0,-1,-1,2),nrow=6,byrow=T)
eig <- eigen(x)
lambda2 <- sort(eigen(x)$values)[2]    # This is the second smallest eigenvalue
eigvec2 <- eig$vectors[,eig$values==lambda2]    # Eigenvector corresponding to lambda2
cat("Second smallest eigenvalue: ",lambda2,"\n",sep = "")
cat("Corresponding eigenvector:","[",eigvec2,"]")
```

```
Second smallest eigenvalue: 0.4384472
Corresponding eigenvector: [ 0.4647051 0.4647051 0.2609565 -0.2609565 -0.4647051 -0.4647051 ]
```

Since there are three elements that are clearly more positive and three elements that are clearly more negative, we find:

$$\mathbf{s} = (1, 1, 1, -1, -1, -1)$$

So, $n_1 = n_2 = 3$ and $R = \frac{n_1 n_2}{n}\lambda = \frac{9}{6} \cdot 0.4384 = 0.6577$.

b) Find the eigenvector of the modularity matrix corresponding to the largest eigenvalue and hence divide the network into two communities.

```r
# Modularity matrix
b <- matrix(c(-0.286,0.714,0.571,-0.429,-0.286,-0.286,
              0.714,-0.286,0.571,-0.429,-0.286,-0.286,
              0.571,0.571,-0.643,0.357,-0.429,-0.429,
              -0.429,-0.429,0.357,-0.643,0.571,0.571,
              -0.286,-0.286,-0.429,0.571,-0.286,0.714,
              -0.286,-0.286,-0.429,0.571,0.714,-0.286),nrow=6,byrow=T)
eig <- eigen(b)
lambda <- sort(eigen(b)$values,decreasing = T)[1]   # This is the largest eigenvalue
eigvec <- eig$vectors[,eig$values==lambda]   # Eigenvector corresponding to lambda
cat("Largest eigenvalue: ",lambda,"\n",sep = "")
cat("Corresponding eigenvector:","[",eigvec,"]")
```

```
Largest eigenvalue: 1.732051
Corresponding eigenvector: [ -0.4440369 -0.4440369 -0.3250576 0.3250576 0.4440369 0.4440369 ]
```

$$\mathbf{s} = (-1, -1, -1, 1, 1, 1)$$

We can divide the communities so that $n_1 = n_2 = 3$. Group 1 has vertices (1,2,3), and group 2 has vertices (4,5,6).