

Sheet 3

1) - Problem 1

1. Define the operations

$$\text{Initial State: } |\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\text{Free Evolution: } H = \frac{\hbar \omega_0}{2} \sigma_z$$

$$U(\tau) = e^{-iH\tau/\hbar}$$

$$U(\tau) = e^{-i(\frac{\hbar \omega_0}{2} \sigma_z) \tau / \hbar} = e^{-i(\omega_0 \tau / 2) \sigma_z}$$

$$U(\tau) = e^{-i(\omega_0 \tau / 2) \sigma_z} \quad (\cancel{= e^{-i(\frac{\hbar \omega_0}{2} \sigma_z) \tau / \hbar}} = e^{-i(\omega_0 \tau / 2) \sigma_z})$$

$$\sigma_z |0\rangle = |0\rangle, \sigma_z |1\rangle = -|1\rangle$$

$$U(\tau) |0\rangle = e^{-i\omega_0 \tau / 2} |0\rangle, U(\tau) =$$

$$U(\tau) |1\rangle = e^{i\omega_0 \tau / 2} |1\rangle$$

$$\text{Gate Operation: } X = \sigma_x, X |0\rangle = |1\rangle, X |1\rangle = |0\rangle$$

$$X |0\rangle = |1\rangle$$

$$X |1\rangle = |0\rangle$$

$$U_{\text{cycle}} = X U(\tau), N_{\text{cycle}}, |\psi_N\rangle = (U_{\text{cycle}})^N |\psi(0)\rangle$$

$$N = k \ln n$$

$$4) |\psi_N\rangle = (\text{Vcycle})^N |\psi(0)\rangle, N-\text{even}$$

2. Calculate the state after one cycle ($N=1$)

- Step 1: Evolve for τ

$$|\psi(\tau^-)\rangle = U(\tau^-) |\psi(0)\rangle = \frac{1}{\sqrt{2}} (U(\tau^-)|0\rangle + iU(\tau^-)|1\rangle)$$

$$|\psi(\tau^-)\rangle = U(\tau^-) |\psi(0)\rangle = \frac{1}{\sqrt{2}} (U(\tau^-)|0\rangle + iU(\tau^-)|1\rangle)$$

$$|\psi(\tau^-)\rangle = \frac{1}{\sqrt{2}} (e^{-i\omega_0 \tau/2} |0\rangle + ie^{i\omega_0 \tau/2} |1\rangle)$$

- Step 2: Apply X gate

$$|\psi_1\rangle = X |\psi(\tau^-)\rangle = \frac{1}{\sqrt{2}} (e^{-i\omega_0 \tau/2} X |0\rangle + ie^{i\omega_0 \tau/2} X |1\rangle)$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} (e^{-i\omega_0 \tau/2} |1\rangle + ie^{i\omega_0 \tau/2} |0\rangle)$$

3. Calculate the state after cycles ($N=2$)

- Step 3: Evolve for another τ

$$|\psi(2\tau^-)\rangle = U(\tau^-) |\psi_1\rangle = \frac{1}{\sqrt{2}} (ie^{i\omega_0 \tau/2} U(\tau^-)|0\rangle + e^{-i\omega_0 \tau/2} U(\tau^-)|1\rangle)$$

1) 3. Calculate the state after cycles ($N=2$)

-Step 3: Evaluate for another τ

$$|\psi(2\tau^-)\rangle = \frac{1}{\sqrt{2}} (ie^{i\omega_0\tau/2} (e^{-i\omega_0\tau/2}|0\rangle) + e^{-i\omega_0\tau/2} (e^{i\omega_0\tau/2}|1\rangle))$$

$$\Rightarrow |\psi(2\tau^-)\rangle = \frac{1}{\sqrt{2}} (ie^0|0\rangle + e^0|1\rangle) = \frac{1}{\sqrt{2}} (i|0\rangle + |1\rangle)$$

-Step 4: Apply second X gate

$$|\psi_2\rangle = X|\psi(2\tau^-)\rangle = \frac{1}{\sqrt{2}} (iX|0\rangle + X|1\rangle)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} (i|1\rangle + |0\rangle)$$

$$\boxed{|\psi_2\rangle = \frac{1}{\sqrt{2}} (|0\rangle + i|1\rangle)}.$$

4. Explanation and Final State

$$|\psi_2\rangle = |\psi(0)\rangle$$

$$N-\text{even}, N=2m: |\psi_N\rangle = (\text{cycle})^N |\psi(0)\rangle =$$

$$= (\text{cycle})^{2m} |\psi(0)\rangle = ((\text{cycle})^2)^m |\psi(0)\rangle$$

$$\Rightarrow |\psi_N\rangle = ((\text{cycle})^2)^m |\psi(0)\rangle$$

$$\cancel{\text{If } N=1} \quad \text{If } m=1: |\psi_N\rangle = (\text{cycle})^2 |\psi(0)\rangle = |\psi(0)\rangle$$

$$|\psi_N\rangle = ((\text{cycle})^2)^{m-1} ((\text{cycle})^2 |\psi(0)\rangle) = \dots = |\psi(0)\rangle$$

$$\boxed{\text{Final answer: } N-\text{even}, |\psi_N\rangle = |\psi(0)\rangle = \frac{1}{\sqrt{2}} (|0\rangle + i|1\rangle)}$$

2) - Problem 2

1. Analyse the problem and contradiction

- Hamiltonian: $H(t) = \frac{\hbar \omega}{2} \cos(\omega t) \hat{O}_Z$

- Gate Rule: $\cos(\omega t) = 0, \omega t = \frac{\pi}{2} + k\pi, k=0, 1, 2, \dots$
 (for NOT(X) gate)

- Gate Times: $t_k = \frac{\pi}{\omega} \left(k + \frac{1}{2} \right)$

$$t_0 = \frac{\pi}{2\omega}, t_1 = \frac{3\pi}{2\omega}, \dots, t_{N-1} = \frac{\pi}{\omega} \left(N - \frac{1}{2} \right)$$

- Final Time: $t_f = \frac{2\pi N}{\omega}$

- Contradiction: $t_k \leq t_f \Rightarrow$

$$\Rightarrow \frac{\pi}{\omega} \left(k + \frac{1}{2} \right) \leq \frac{2\pi N}{\omega} \Rightarrow k + \frac{1}{2} \leq 2N \Rightarrow$$

$$\Rightarrow k \leq 2N - \frac{1}{2}$$

The largest integer k is $2N-1$, from $k=0$ to
 to $k=2N-1 \Rightarrow (2N-1)-0+1=2N$ gates

- Assumption: $t_f = \frac{2\pi N}{\omega}$ correct (assume all
 correct). $2N$ gate in total. (Assume the rule
 of "apply gates at zero") and t_f)

2. Define the Evolution Operator

$$\{H(t_1), H(t_2)\} = 0$$

2) Define the Evolution Operator

$$[H_1(t_1), H(t_2)] = 0$$

$$U(t_b, t_a) = \exp\left(-\frac{i}{\hbar} \int_{t_a}^{t_b} H(t') dt'\right) =$$

$$= \exp\left(-\frac{i}{\hbar} \int_{t_a}^{t_b} \frac{\hbar \omega_0}{2} \cos(\omega t') dt' \cdot \hat{\sigma}_z\right) =$$

$$\Rightarrow U(t_a, t_b) = \exp\left(-\frac{i \omega_0}{2} \left[\frac{\sin(\omega t_b)}{\omega}\right]_{t_a}^{t_b} \cdot \hat{\sigma}_z\right)$$

$$\phi(t_a, t_b) = \frac{\omega_0}{2\omega} (\sin(\omega t_b) - \sin(\omega t_a))$$

$$\Rightarrow \phi(t_a, t_b) = \frac{\omega_0}{2\omega} (\sin(\omega t_b) - \sin(\omega t_a))$$

$$\Rightarrow [H(t_a), U(t_b, t_a)] = e^{-i\phi(t_b, t_a)\hat{\sigma}_z} \quad \boxed{\phi_0 = \frac{\omega_0}{2\omega}}$$

3. Trace the states

- Initial state: $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

- Evolve $0 \rightarrow 0$ (Gate 1):

$$\phi_1 = \phi(t_0, 0) = \phi_0 (\sin(\pi/2) - \sin(0)) = \phi_0$$

$$U_1 = e^{-i\phi_0 \hat{\sigma}_z}$$

$$|\psi(t_0^-)\rangle = U_1 |\psi_0\rangle = \frac{1}{\sqrt{2}}(e^{-i\phi_0}|0\rangle + ie^{i\phi_0}|1\rangle)$$

$$|\psi(t_0^+)\rangle = X |\psi(t_0^-)\rangle = \frac{1}{\sqrt{2}}(ie^{i\phi_0}|0\rangle + e^{-i\phi_0}|1\rangle)$$

2) 3. Trace the state

- Evolve to t_1 (Gate 2):

$$\phi_2 = \phi(t_1, t_0) = \phi_0 (\sin(3\pi/2) - \sin(\pi/2)) =$$

$$= \phi_0(-1-1) = -2\phi_0$$

$$U_2 = e^{-i\phi_2 \hat{\sigma}_z} = e^{i2\phi_0 \hat{\sigma}_z}$$

$$|\psi(t_1^-)\rangle = U_2 |\psi(t_0^+)\rangle = \frac{1}{\sqrt{2}} (ie^{i\phi_0}(e^{-i2\phi_0})|0\rangle +$$

$$+ e^{-i\phi_0}(e^{-i2\phi_0})|1\rangle) = \frac{1}{\sqrt{2}} (ie^{i3\phi_0}|0\rangle + e^{-i3\phi_0}|1\rangle).$$

$$|\psi(t_1^+)\rangle = X |\psi(t_1^-)\rangle = \frac{1}{\sqrt{2}} (e^{-i3\phi_0}|0\rangle +$$

$$+ ie^{i3\phi_0}|1\rangle)$$

- Evolve $t_1 \rightarrow t_2$ (Gate 3):

$$\phi_3 = \phi(t_2, t_1) = \phi_0 (\sin(5\pi/2) - \sin(3\pi/2)) =$$

$$= \phi_0(1 - (-1)) = 2\phi_0$$

$$U_3 = e^{-i2\phi_0 \hat{\sigma}_z}$$

$$|\psi(t_2^-)\rangle = U_3 |\psi(t_1^+)\rangle = \frac{1}{\sqrt{2}} (e^{-i3\phi_0}(e^{-i2\phi_0})|0\rangle +$$

$$+ ie^{i3\phi_0}(e^{-i2\phi_0})|1\rangle) = \frac{1}{\sqrt{2}} (e^{-i5\phi_0}|0\rangle + ie^{i5\phi_0}|1\rangle)$$

$$|\psi(t_2^-)\rangle = \frac{1}{\sqrt{2}} (e^{-i5\phi_0}|0\rangle + ie^{i5\phi_0}|1\rangle)$$

$$|\psi(t_2^+)\rangle = X |\psi(t_2^-)\rangle = \frac{1}{\sqrt{2}} (ie^{i5\phi_0}|0\rangle +$$

$$+ e^{-i5\phi_0}|1\rangle)$$

$$= \boxed{|\psi(t_2^+)\rangle = \frac{1}{\sqrt{2}} (ie^{i5\phi_0}|0\rangle + e^{-i5\phi_0}|1\rangle)}$$

2)

4. Find the state after $2N$ gates

$$|\psi_k\rangle = |\psi(t_{k-1}^+)\rangle$$

$$|\psi_1\rangle_{\text{odd}}: \frac{1}{\sqrt{2}}(ie^{i\phi_0}|0\rangle + e^{-i\phi_0}|1\rangle) \Rightarrow \text{Phase is } (2-1-1)\phi_0$$

$$|\psi_2\rangle_{\text{even}}: \frac{1}{\sqrt{2}}(e^{-i3\phi_0}|0\rangle + ie^{i3\phi_0}|1\rangle) \Rightarrow \text{Phase is } (2-2-1)\phi_0$$

$$|\psi_3\rangle_{\text{odd}}: \frac{1}{\sqrt{2}}(ie^{i5\phi_0}|0\rangle + e^{-i5\phi_0}|1\rangle) \Rightarrow \text{Phase is } (2-3-1)\phi_0$$

$$k-\text{odd}: |\psi_k\rangle = \frac{1}{\sqrt{2}}(ie^{i(2k-1)\phi_0}|0\rangle + e^{-i(2k-1)\phi_0}|1\rangle)$$

$$k-\text{even}: |\psi_k\rangle = \frac{1}{\sqrt{2}}(e^{-i(2k-1)\phi_0}|0\rangle + ie^{i(2k-1)\phi_0}|1\rangle)$$

$k=2N \Rightarrow |\psi_{2N}\rangle$ at time t_{2N-1}

$$\boxed{|\psi_{2N}\rangle = \frac{1}{\sqrt{2}}(e^{-i(2(2N)-1)\phi_0}|0\rangle + ie^{i(2(2N)-1)\phi_0}|1\rangle)}$$

$$\boxed{|\psi_{2N}\rangle = \frac{1}{\sqrt{2}}(e^{-i(4N-1)\phi_0}|0\rangle + ie^{i(4N-1)\phi_0}|1\rangle)}$$

5. Final Evaluation

$$t_i = t_{2N-1} \text{ to } t_f = \frac{2\pi N}{\omega}$$

$$t_i = t_{2N-1} = \frac{\pi}{\omega}(2N-1+\frac{1}{2}) = \frac{\pi(2N-\frac{1}{2})}{\omega}$$

2) \bar{A}
5. Final Evolution

$$t_i = t_{2N-1} = \frac{\pi}{\omega} \left(2N-1 + \frac{1}{2}\right) = \frac{\pi(2N-\frac{1}{2})}{\omega}$$

$$\sin(\omega t_i) = \sin\left(\omega \frac{\pi(2N-\frac{1}{2})}{\omega}\right) = \sin(2N\pi - \frac{\pi}{2}) = \\ = \sin(-\frac{\pi}{2}) = -1$$

$$\sin(\omega t_f) = \sin\left(\omega \frac{2\pi N}{\omega}\right) = \sin(2\pi N) = 0$$

Final phase:

$$\phi_{\text{final}} = \phi(t_f, t_i) = \phi_0 (\sin(\omega t_f) - \sin(\omega t_i)) = \\ = \phi_0 (0 - (-1)) = \phi_0 = \boxed{\phi_{\text{final}} = \phi_0}$$

$$U_{\text{final}} = e^{-i\phi_{\text{final}} \hat{\sigma}_z} = e^{-i\phi_0 \hat{\sigma}_z} = \\ \Rightarrow \boxed{U_{\text{final}} = e^{-i\phi_0 \hat{\sigma}_z}}$$

6. Final State Calculation

$$|\psi(t_f)\rangle = U_{\text{final}} |\psi_{2N}\rangle = e^{-i\phi_0 \hat{\sigma}_z} \sqrt{\frac{1}{2}} \left[e^{-i(4N-1)\phi_0} |0\rangle + \right. \\ \left. + ie^{i(4N-1)\phi_0} |1\rangle \right] = \frac{1}{\sqrt{2}} \left[e^{-i4N\phi_0} |0\rangle + ie^{+i4N\phi_0} |1\rangle \right]$$

$$\phi_0 = \frac{\omega_0}{2\omega} : 4N\phi_0 = 4N \left(\frac{\omega_0}{2\omega} \right) = \frac{2N\omega_0}{\omega}$$

Final answer: $t = \frac{2\pi N}{\omega}$

$$\Rightarrow \boxed{|\psi(t_f)\rangle = \frac{1}{\sqrt{2}} \left(e^{-i2N\omega_0/\omega} |0\rangle + ie^{i2N\omega_0/\omega} |1\rangle \right)}$$

$$\boxed{|\psi(t_f)\rangle = \frac{1}{\sqrt{2}} \left(e^{-i2N\omega_0/\omega} |0\rangle + ie^{i2N\omega_0/\omega} |1\rangle \right)} \quad \cancel{\left(e^{\frac{i2N\omega_0}{\omega}} \right)}$$

Sheet 3

1. Suppose that you want to measure a parameter ω_0 . You choose a two-level system as your quantum probe. This parameter shows up in the Hamiltonian of your probe: $H = \frac{\hbar\omega_0}{2}\sigma_z$. You initialize the probe in the state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$. Here $\sigma_z|0\rangle = |0\rangle$ and $\sigma_z|1\rangle = -|1\rangle$. Now, however, after every time interval τ you apply a quantum NOT gate (or a X gate). What is the state just after applying N such gates (here N is even)? Explain.

2. Now consider the same situation as in the previous problem, except that we now have an oscillating field so that the Hamiltonian is now $H = \frac{\hbar\omega_0}{2}\cos(\omega t)\sigma_z$. You again apply the quantum NOT gates, but these are now applied whenever the field becomes zero. What is then the state at time $t = 2\pi N/\omega$? Here N is the number of gates applied.