Sheet 1

Problem 1:

1)

a) $U = e^{i\omega t \sigma_z/2}$

$$H = UHU^+ + i\hbar(\partial_t U)U^+$$

$$UHU^+ : H = \frac{\hbar\omega_0}{2}\sigma_z + \hbar\Omega(\sigma_+ + \sigma_-)\cos(\omega t)$$

$$\cancel{U} \; U\left(\frac{\hbar\omega_0}{2}\sigma_z\right)U^+ = \frac{\hbar\omega_0}{2}\sigma_z$$

$$\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$H_{int} \simeq \frac{\hbar\Omega}{2}\left(\sigma_+ e^{-\omega t} + \sigma_- e^{i\omega t}\right)$$

$$UH_{int}U^+ = \frac{\hbar\Omega}{2}\left(U\sigma_+ U^+ e^{-i\omega t} + U\sigma_- U^+ e^{i\omega t}\right)$$

$$e^{i\theta\sigma_z}\sigma_\pm e^{-i\theta\sigma_z} = \sigma_\pm e^{\pm i2\theta}, \quad \theta = \omega t/2$$

$$U\sigma_+ U^+ = \sigma_+ e^{i\omega t}, \quad U\sigma_- U^+ = \sigma_- e^{-i\omega t}$$

$$UH_{int}U^+ = \frac{\hbar\Omega}{2}\left(\sigma_+ e^{i\omega t} e^{-i\omega t} + \sigma_- e^{-i\omega t} e^{i\omega t}\right) =$$

$$= \cancel{\hbar\Omega} = \frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-) = \frac{\hbar\Omega}{2}\sigma_x$$

$$\cancel{UH_{int}U^+} \quad \boxed{UH_{int}U^+ = \frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-) = \frac{\hbar\Omega}{2}\sigma_x}$$

1)

b)

1) a)

b) $i\hbar(\partial_t U)U^\dagger$, $\partial_t U = \partial_t\left(e^{i\omega t\sigma_z/2}\right) = \frac{i\omega\sigma_z}{2} e^{i\omega t\sigma_z/2}$

$= \cancel{i\omega\sigma_z}\ \partial_t U = \frac{i\omega\sigma_z}{2} e^{i\omega t\sigma_z/2} = \frac{i\omega\sigma_z}{2} U$

$\partial_t U = \frac{i\omega\sigma_z}{2} U$

$\cancel{i\hbar(\partial_t U}\ i\hbar(\partial_t U)U^\dagger = i\hbar\left(\frac{i\omega\sigma_z}{2}U\right)U^\dagger = -\frac{\hbar\omega}{2}\sigma_z$

$\tilde{H} = \frac{\hbar\omega_0}{2}\sigma_z + \frac{\hbar\Omega}{2}\sigma_x - \frac{\hbar\omega}{2}\sigma_z$

$\tilde{H} = -\frac{\hbar(\omega-\omega_0)}{2}\sigma_z + \frac{\hbar\Omega}{2}\sigma_x$

$\Delta = \omega - \omega_0$, $\boxed{\tilde{H} = \frac{\hbar}{2}(\Omega\sigma_x - \Delta\sigma_z)}$

$\boxed{\tilde{H} = \frac{\hbar}{2}(\Omega\sigma_x - \Delta\sigma_z)}$ $\boxed{\tilde{H} = \frac{\hbar}{2}(\Omega\sigma_x - \Delta\sigma_z)}$

1) b) $|\psi(0)\rangle = |1\rangle$, $|\tilde{\psi}(t)\rangle = e^{-i\tilde{H}t/\hbar}|\tilde{\psi}(0)\rangle$, $U(0) = I$

$|\tilde{\psi}(0)\rangle = |1\rangle$, $e^{-i\tilde{H}t/\hbar} = e^{-i\frac{t}{2}(\Omega\sigma_x - \Delta\sigma_z)}$

$\cancel{e^{-i\theta\hat{n}}}\ e^{-i\theta(\hat{n}\cdot\sigma)}$, $\theta = \frac{\Omega_R t}{2}$

$\hat{n} = \frac{1}{\Omega_R}(\Omega, 0, -\Delta)$, $\Omega_R = \sqrt{\Omega^2+\Delta^2}$

$e^{-i\theta(\hat{n}\cdot\sigma)} = \cos(\theta)I - i\sin(\theta)(\hat{n}\cdot\sigma)$

$\cancel{|\tilde{\psi}(t)\rangle = \left[\cos\left(\frac{\Omega_R t}{2}\right)I - i\sin(\theta\right.}$

$|\tilde{\psi}(t)\rangle = \left[\cos\left(\frac{\Omega_R t}{2}\right)I - i\sin\left(\frac{\Omega_R t}{2}\right)\frac{\Omega\sigma_x - \Delta\sigma_z}{\Omega_R}\right]|1\rangle$

$\sigma_x|1\rangle = |0\rangle$, $\sigma_z|1\rangle = -|1\rangle$:

1) b) $\sigma_x|1\rangle = |0\rangle$, $\sigma_z|1\rangle = -|1\rangle$

$$|\tilde{\psi}(t)\rangle = \cos\left(\frac{\Omega_R t}{2}\right)|1\rangle - i\frac{\sin\left(\frac{\Omega_R t}{2}\right)}{\Omega_R}(\Omega|0\rangle + \Delta|1\rangle)$$

$$|\tilde{\psi}(t)\rangle = i\frac{\Omega}{\Omega_R}\sin\left(\frac{\Omega_R t}{2}\right)|0\rangle + \left[\cos\left(\frac{\Omega_R t}{2}\right) - i\frac{\Delta}{\Omega_R}\sin\left(\frac{\Omega_R t}{2}\right)\right]|1\rangle$$

$U^{+} = e^{-i\omega t\sigma_z/2}$:

$$|\tilde{\psi}(t)\rangle = U^{+}(t)\,|\tilde{\psi}(t)\rangle = e^{-i\omega t/2}|0\rangle\langle 0|\tilde{\psi}(t)\rangle + e^{i\omega t/2}|1\rangle\langle 1|\tilde{\psi}(t)\rangle$$

$$|\psi(t)\rangle = -\frac{i\Omega}{\Omega_R}\sin\left(\frac{\Omega_R t}{2}\right)e^{-i\omega t/2}|0\rangle + [\cos($$

~~$|\psi(t)\rangle = -\frac{i\Omega}{\Omega_R}\sin\left(\frac{\Omega_R t}{2}\right)e^{-i\omega t/2}|0\rangle + [\cos\left(\frac{\Omega_R t}{2}\right) - i\frac{\Delta}{\Delta_R}$~~

$$\boxed{|\psi(t)\rangle = -i\frac{\Omega}{\Omega_R}\sin\left(\frac{\Omega_R t}{2}\right)e^{-i\omega t/2}|0\rangle + \left[\cos\left(\frac{\Omega_R t}{2}\right) - i\frac{\Delta}{\Omega_R}\sin\left(\frac{\Omega_R t}{2}\right)\right]e^{-i\omega t/2}|1\rangle}$$

1) c) $P_0(t) = |\langle 0|\psi(t)\rangle|^2$, $-i\frac{\Omega}{\Omega_R}\sin\left(\frac{\Omega_R t}{2}\right)e^{-i\omega t/2}$

$$P_0(t) = \left|-i\frac{\Omega}{\Omega_R}\sin\left(\frac{\Omega_R t}{2}\right)e^{-i\omega t/2}\right|^2$$

$$\boxed{P_0(t) = \frac{\Omega^2}{\Omega^2+\Delta^2}\sin^2\left(\frac{\sqrt{\Omega^2+\Delta^2}}{2}t\right)}$$

$P_0(t)$: $A = \frac{\Omega^2}{\Omega^2+\Delta^2}$, $\Omega_R = \sqrt{\Omega^2+\Delta^2}$

1) d) w_0 , $\Delta = w - w_0$, P_0

$\Delta t \approx t$

$\Delta E = \hbar \Delta w$

$w = w_0 \Rightarrow \Delta = 0$

$A = \frac{\Omega^2}{\Omega^2+\Delta^2} \Rightarrow \Delta = 0 \Rightarrow w = w_0$

$\Delta t \approx t \Rightarrow \Delta E = \hbar \Delta w$

2)

a) $|0\rangle, |1\rangle \Rightarrow P_0 = P_1 = \frac{1}{2}$

$P_0(t) = \frac{1}{2}$, $P_0(t) = \frac{\Omega^2}{\Omega^2+\Delta^2} \sin^2\left(\frac{\Omega_R t}{2}\right) = \frac{1}{2}$

$\Delta = 0$, $\Omega_R = \Omega \Rightarrow P_0(t) = \sin^2\left(\frac{\Omega t}{2}\right) = \frac{1}{2}$

$\sin\left(\frac{\Omega t}{2}\right) = \pm\frac{1}{\sqrt{2}} \Rightarrow \frac{\Omega t}{2} = \frac{\pi}{4} \Rightarrow \boxed{t = \frac{\pi}{2\Omega}}$

~~$w = w_0 \Rightarrow$~~ $\boxed{w = w_0 , t = \frac{\pi}{2\Omega}}$

2)

b) 1. $\frac{\pi}{2}$: $|\tilde{\psi}_1\rangle = \cos\left(\frac{\Omega t}{2}\right)|1\rangle - i\sin\left(\frac{\Omega t}{2}\right)|0\rangle$

$|\tilde{\psi}_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle - i|0\rangle)$

2. ~~Free~~ Free Evolution: T, $\Omega = 0$, $\tilde{H}_{Free} = -\frac{\hbar\Delta\sigma_z}{2}$

$|\tilde{\psi}_2\rangle = e^{-i\tilde{H}_{Free}T/\hbar}|\tilde{\psi}_1\rangle = e^{i\Delta T\sigma_z/2}|\tilde{\psi}_1\rangle$

$|\tilde{\psi}_2\rangle = \frac{1}{\sqrt{2}}\left(e^{i\Delta T\sigma_z/2}|1\rangle - ie^{i\Delta T\sigma_z/2}|0\rangle\right)$

$|\tilde{\psi}_2\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\Delta T/2}|1\rangle - ie^{-i\Delta T/2}|0\rangle\right)$

2)

b) 3. $\frac{\pi}{2}$: $|\tilde{\psi}_f\rangle = \left(\frac{1}{\sqrt{2}}(I - i\sigma_x)\right)|\hat{\psi}_2\rangle$

$$|\tilde{\psi}_f\rangle = \frac{1}{2}(I - i\sigma_x)\left(e^{-i\Delta T/2}|1\rangle - ie^{i\Delta T/2}|0\rangle\right)$$

$$~~|\tilde{\psi}_f\rangle = \frac{1}{2}\left[e^{-i\Delta T/2}|1\rangle - ie^{i\Delta T/2}|0\rangle\, i(e^{-i\Delta T/2}\right. =~~$$

$$|\tilde{\psi}_f\rangle = \frac{1}{2}\left[e^{-i\Delta T/2}|1\rangle - ie^{i\Delta T/2}|0\rangle - i\left(e^{-i\Delta T/2}|0\rangle - ie^{i\Delta T/2}|1\rangle\right)\right]$$

$$|\tilde{\psi}_f\rangle = \frac{1}{2}\left[-i\left(e^{i\Delta T/2} + e^{-i\Delta T/2}\right)|0\rangle + \left(e^{-i\Delta T/2} - e^{i\Delta T/2}\right)|1\rangle\right]$$

$$2\cos(x) = e^{ix} + e^{-ix}, \quad -2i\sin(x) = e^{-ix} - e^{ix}$$

$$\Longrightarrow \boxed{|\tilde{\psi}_f\rangle = -i\cos(\Delta T/2)|0\rangle - i\sin(\Delta T/2)|1\rangle}$$

$$\boxed{P_0 = \cos^2\left(\frac{\Delta T}{2}\right) = \frac{1}{2}(1 + \cos(\Delta T))}$$

2) c) $P_0, \Delta = \omega_0 - \omega, T \Rightarrow P_0(\Delta) = \cos^2\left(\frac{\Delta T}{2}\right) = \frac{1}{2}(1 + \cos(\Delta T))$

$$\Delta = \frac{2}{T}\arccos\left(\pm\sqrt{P_0}\right)$$

$$P_0(t) = \frac{\Omega^2}{\Omega_R^2}\sin^2\left(\frac{\Omega_R t}{2}\right), \quad \Omega_R = \sqrt{\Delta^2 + \Omega^2}$$

Rabi Oscillations (scheme 1): $t_{max} \sim \frac{1}{\Omega_R}$

Ramsey Interferometry (Scheme 2):

$$\frac{\partial P_0}{\partial \Delta} = -T\sin(\Delta T), \quad \Delta T \approx \frac{\pi}{2}, \quad \delta\Delta = \frac{\delta P_0}{|\partial P_0/\partial\Delta|}$$

$$|\partial P_0/\partial\Delta| \propto T, \quad \delta\Delta \Rightarrow \boxed{\delta\Delta \propto \frac{1}{T}}$$

Sheet 1

1. Consider a two-level atom interacting with a classical electromagnetic field. The excited state is $|0\rangle$ while the ground state is $|1\rangle$. Treat the atom as an electric dipole. The Hamiltonian is then $H = \frac{\hbar\omega_0}{2}\sigma_z + \hbar\Omega(\sigma_+ + \sigma_-)\cos(\omega t)$. Here, the operator σ_z is defined as $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$. Also, $\sigma_+ = |0\rangle\langle 1|$ and $\sigma_- = |1\rangle\langle 0|$.

   (a) A common trick to solve for the dynamics is to use a unitary transformation. You saw one example in the lecture. Here is another unitary transformation you can use: $U = e^{i\omega\sigma_z t/2}$. This corresponds to transforming to 'the frame of the fields'. With this unitary transformation, show that the effective Hamiltonian can be written as $\widetilde{H} = UHU^\dagger + i\hbar(\partial_t U)U^\dagger$ (see equation 5.33 on the reading uploaded on two-level atoms). Use this to work out what $\widetilde{H}$ is.

   (b) Use this effective Hamiltonian to work out what the state of the atom is at time t (assuming that it starts from the ground state).

   (c) What is the probability that, at time t, the atom is in the excited state? Plot this probability as a function of time.

   (d) How can we measure the parameter ω_0 using your answer to the previous question? In other words, how can you measure the parameter $\Delta \equiv \omega - \omega_0$? What happens as you increase the time for which the atom interacts with the electromagnetic field?

2. To continue the previous problem, let's suppose again that our atom is initially in the ground state.

   (a) How can you use the interaction of this atom with an electromagnetic field to prepare an equal superposition state of $|0\rangle$ and $|1\rangle$?

   (b) After interacting with this field, you turn off the field for time T - this means that Ω becomes zero in your effective Hamiltonian. Then you again apply the same field as before. After doing this, you perform a measurement. What is the probability of finding the atom to be in the excited state?

   (c) How can you now measure Δ? Is this scheme better than the scheme in the first question?