Sheet 2

1)

a) $|\psi(t)\rangle = U(t)|\psi(0)\rangle$, $U(t) = e^{-iHt/\hbar}$

$$H = \frac{\hbar\omega_0}{2}\sigma_z$$

$$U(t) = e^{-i\left(\frac{\hbar\omega_0}{2}\sigma_z\right)t/\hbar} = e^{-i\frac{\omega_0 t}{2}\sigma_z}$$

$$U(t)|0\rangle = e^{-i\frac{\omega_0 t}{2}(+1)}|0\rangle = e^{-i\omega_0 t/2}|0\rangle$$

$$U(t)|1\rangle = e^{-i\frac{\omega_0 t}{2}(-1)}|1\rangle = e^{+i\omega_0 t/2}|1\rangle$$

$$\cancel{|\psi_0\rangle}\ \cancel{|\psi(t)\rangle = U(t)\frac{1}{\sqrt{2}}}\ |\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|\psi(t)\rangle = U(t)\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = \frac{1}{\sqrt{2}}(U(t)|0\rangle + iU(t)|1\rangle)$$

$$\Longrightarrow \boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t/2}|0\rangle + ie^{i\omega_0 t/2}|1\rangle\right)}$$

1) b) $X = |0\rangle\langle 1| + |1\rangle\langle 0|$

The Eigenvalues of Pauli matrix, σ_x, are -1 ~~an~~ and 1 (1 and -1).

$$+1: |+\rangle_x = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$-1: |-\rangle_x = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle_x$$

$$P(+1) = |\langle +_x|\psi(t)\rangle|^2 \cdot \langle +_x|\psi(t)\rangle =$$

$$\langle +_x|\psi(t)\rangle = \frac{1}{2}\left(e^{-i\omega_0 t/2} + ie^{i\omega_0 t/2}\right)$$

$$P(+1) = \left|\frac{1}{2}\left(e^{-i\omega_0 t/2} + ie^{i\omega_0 t/2}\right)\right|^2$$

1)
b) $P(+1) = \frac{1}{4}\left(e^{-i\omega_0 t/2} + ie^{i\omega_0 t/2}\right)\left(e^{i\omega_0 t/2} - ie^{-i\omega_0 t/2}\right)$

$P(+1) = \frac{1}{4}\left(2 + 2\sin(\omega_0 t)\right) = \frac{1}{2}\left(1 + \sin(\omega_0 t)\right)$

$-1: P(-1) = 1 - P(+1)$

$$\boxed{\begin{aligned} P(+1) &= \tfrac{1}{2}\left(1 + \sin(\omega_0 t)\right) \\ P(-1) &= \tfrac{1}{2}\left(1 - \sin(\omega_0 t)\right) \end{aligned}}$$

1)
c) $F(\omega_0) = \frac{1}{P(+1)}\left(\frac{\partial P(+1)}{\partial \omega_0}\right)^2 + \frac{1}{P(-1)}\left(\frac{\partial P(-1)}{\partial \omega_0}\right)^2$

$\frac{\partial P(+1)}{\partial \omega_0} = \frac{\partial}{\partial \omega_0}\left[\frac{1}{2}\left(1 + \sin(\omega_0 t)\right)\right] = \frac{t}{2}\cos(\omega_0 t)$

$\frac{\partial P(-1)}{\partial \omega_0} = \frac{\partial}{\partial \omega_0}\left[\frac{1}{2}\left(1 - \sin(\omega_0 t)\right)\right] = -\frac{t}{2}\cos(\omega_0 t)$

$F(\omega_0) = \frac{\left(\frac{t}{2}\cos(\omega_0 t)\right)^2}{\frac{1}{2}\left(1 + \sin(\omega_0 t)\right)} + \frac{\left(-\frac{t}{2}\cos(\omega_0 t)\right)^2}{\frac{1}{2}\left(1 - \sin(\omega_0 t)\right)}$

$F(\omega_0) = \frac{t^2\cos^2(\omega_0 t)}{2}\left[\frac{2}{1 - \sin^2(\omega_0 t)}\right] = \frac{t^2\cos^2(\omega_0 t)}{2} \cdot$

$\cdot\left[\frac{2}{\cos^2(\omega_0 t)}\right] \Longrightarrow \boxed{F(\omega_0) = t^2}$

1) d) $H(t) = \frac{\hbar\omega_0}{2}\cos(\omega t)\sigma_z$

$U(t) = e^{-\frac{i}{\hbar}\int_0^t H(t')\,dt'}$

$\int_0^t H(t')\,dt' = \int_0^t \frac{\hbar\omega_0}{2}\cos(\omega t')\sigma_z\,dt'$

1)

d) $\int_0^t H(t')\,dt' = \int_0^t \frac{\hbar \omega_0}{2}\cos(\omega t')\sigma_z\,dt' =$

$\int_0^t H(t')\,dt' = \frac{\hbar\omega_0}{2}\sigma_z\left[\frac{\sin(\omega t')}{\omega}\right]_0^t =$

$\int_0^t H(t')\,dt' = \frac{\hbar\omega_0}{2\omega}\sin(\omega t)\sigma_z$

$U(t) = e^{-\frac{i}{\hbar}\left(\frac{\hbar\omega_0}{2\omega}\sin(\omega t)\sigma_z\right)}$

$U(t) = e^{-i\frac{\omega_0}{2\omega}\sin(\omega t)\sigma_z}$

$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

$|\psi(t)\rangle = U(t)|\psi(0)\rangle$

$e^{-i\phi\sigma_z}|0\rangle = e^{-i\phi}|0\rangle, \; e^{-i\phi\sigma_z}|1\rangle = e^{+i\phi}|1\rangle$

$\phi = \frac{\omega_0}{2\omega}\sin(\omega t) \Longrightarrow$

$\Longrightarrow \boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\frac{\omega_0}{2\omega}\sin(\omega t)}|0\rangle + i e^{i\frac{\omega_0}{2\omega}\sin(\omega t)}|1\rangle\right)}$

1).

e) $F'(\omega_0) = \left(\frac{d}{d\omega_0}\left[\omega_0\frac{\sin(\omega t)}{\omega}\right]\right)^2 = \left(\frac{\sin(\omega t)}{\omega}\right)^2$

$\boxed{F'(\omega_0) = \frac{\sin^2(\omega t)}{\omega^2}}$

Scenario 1 – Constant Field: $\boxed{F(\omega_0) = t^2}$ – ω_0 grow

Scenario 2 – Oscillating Field: $\boxed{F'(\omega_0) = \frac{\sin^2(\omega t)}{\omega^2}}$

Scenario 1 – Constant Field: $F(w0) = t^2$ – w0: grows without bound

Scenario 2 – Oscillating Field: $F(w0) = \frac{\sin^2(wt)}{w^2}$ – Oscillates and bounded by a maximum value of $\frac{1}{w^2}$

Sheet 2

1. Suppose that you want to measure a parameter ω_0. You choose a two-level system as your quantum probe. This parameter shows up in the Hamiltonian of your probe: $H = \frac{\hbar\omega_0}{2}\sigma_z$. You initialize the probe in the state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$. Here $\sigma_z\,|0\rangle = |0\rangle$ and $\sigma_z\,|1\rangle = -\,|1\rangle$.

   (a) What is the state of the probe at time t?

   (b) Suppose that at time t, I measure the observable $X = |0\rangle\langle 1| + |1\rangle\langle 0|$. What are the possible measurement results? What are the corresponding probabilities?

   (c) Given these probabilities, what is the Fisher information for the estimation of ω_0? Sketch this versus time.

   (d) Suppose now that the Hamiltonian is $H = \frac{\hbar\omega_0}{2}\cos(\omega t)$. This happens when we have to measure an oscillating field. What is the time at time t now?

   (e) We again perform a measurement of the observable X. What is the Fisher information now? Compare with the previous scenario and comment.