

Sheet 3

1) - Problem 1

1. Define the operations

$$\text{Initial State: } |\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$\text{Free Evolution: } H = \frac{\hbar\omega_0}{2}\sigma_z$$

$$U(\tau) = e^{-iH\tau/\hbar}$$

$$U(\tau) = \exp\left(-i\left(\frac{\hbar\omega_0}{2}\sigma_z\right)\frac{\tau}{\hbar}\right) = e^{-i(\omega_0\tau/2)\sigma_z}$$

$$U(\tau) = e^{-i(\omega_0\tau/2)\sigma_z} (= e^{-iH\tau/\hbar} = e^{-i(\frac{\hbar\omega_0}{2}\sigma_z)\tau/\hbar})$$

$$\sigma_z|0\rangle = |0\rangle, \sigma_z|1\rangle = -|1\rangle$$

$$U(\tau)|0\rangle = e^{-i\omega_0\tau/2}|0\rangle, U(\tau)|1\rangle =$$

$$U(\tau)|1\rangle = e^{i\omega_0\tau/2}|1\rangle$$

$$\text{Gate Operation: } X = \sigma_x, X|0\rangle = |1\rangle, X|1\rangle = |0\rangle$$

$$X|0\rangle = |1\rangle$$

$$X|1\rangle = |0\rangle$$

$$U_{\text{cycle}} = X U(\tau), N_{\text{cycle}}, |\psi_N\rangle = (U_{\text{cycle}})^N |\psi(0)\rangle$$

$N = 1000$

$$4) |\psi_N\rangle = (U_{\text{cycle}})^N |\psi(0)\rangle, N - \text{even}$$

2. Calculate the state after one cycle ($N=1$)

- Step 1: Evaluate for τ

$$|\psi(\tau^-)\rangle = U(\tau) |\psi(0)\rangle = \frac{1}{\sqrt{2}} (U(\tau) |0\rangle + i U(\tau) |1\rangle)$$

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$$\boxed{|\psi(\tau^-)\rangle = \frac{1}{\sqrt{2}} (e^{-i\omega_0 \tau/2} |0\rangle + i e^{i\omega_0 \tau/2} |1\rangle)}$$

- Step 2: Apply X gate

$$|\psi_1\rangle = X |\psi(\tau^-)\rangle = \frac{1}{\sqrt{2}} (e^{-i\omega_0 \tau/2} X |0\rangle + i e^{i\omega_0 \tau/2} X |1\rangle)$$

$$\boxed{|\psi_1\rangle = \frac{1}{\sqrt{2}} (e^{-i\omega_0 \tau/2} |1\rangle + i e^{i\omega_0 \tau/2} |0\rangle)}$$

3. Calculate the state after cycles ($N=2$)

- Step 3: Evaluate for another τ

$$|\psi(2\tau^-)\rangle = U(\tau) |\psi_1\rangle = \frac{1}{\sqrt{2}} (i e^{i\omega_0 \tau/2} U(\tau) |0\rangle + e^{-i\omega_0 \tau/2} U(\tau) |1\rangle)$$

1) 3. Calculate the state after cycles ($N=2$)

- Step 3: Evaluate for another τ

$$|\psi(2\tau^-)\rangle = \frac{1}{\sqrt{2}} \left(i e^{i\omega_0\tau/2} (e^{-i\omega_0\tau/2} |0\rangle) + e^{-i\omega_0\tau/2} (e^{i\omega_0\tau/2} |1\rangle) \right)$$

$$\Rightarrow |\psi(2\tau^-)\rangle = \frac{1}{\sqrt{2}} (i e^0 |0\rangle + e^0 |1\rangle) = \frac{1}{\sqrt{2}} (i|0\rangle + |1\rangle)$$

- Step 4: Apply second X gate

$$|\psi_2\rangle = X |\psi(2\tau^-)\rangle = \frac{1}{\sqrt{2}} (iX|0\rangle + X|1\rangle)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} (i|1\rangle + |0\rangle)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} (|0\rangle + i|1\rangle)$$

4. Explanation and Final State

$$|\psi_2\rangle = |\psi(0)\rangle$$

$$N - \text{even}, N = 2m: |\psi_N\rangle = (U_{\text{cycle}})^N |\psi(0)\rangle = (U_{\text{cycle}})^{2m} |\psi(0)\rangle = ((U_{\text{cycle}})^2)^m |\psi(0)\rangle$$

$$\Rightarrow |\psi_N\rangle = ((U_{\text{cycle}})^2)^m |\psi(0)\rangle$$

$$\text{If } N=1 \text{ If } m=1: |\psi_N\rangle = (U_{\text{cycle}})^2 |\psi(0)\rangle = |\psi(0)\rangle$$

$$|\psi_N\rangle = ((U_{\text{cycle}})^2)^{m-1} ((U_{\text{cycle}})^2 |\psi(0)\rangle) = \dots = |\psi(0)\rangle$$

$$\text{Final answer: } N - \text{even}, |\psi_N\rangle = |\psi(0)\rangle = \frac{1}{\sqrt{2}} (|0\rangle + i|1\rangle)$$

2) - Problem 2

1. Analyse the problem and Contradiction

- Hamiltonian: $H(t) = \frac{\hbar \omega_0}{2} \cos(\omega t) \sigma_z$

- Gate Rule: $\cos(\omega t) = 0$, $\omega t = \frac{\pi}{2} + k\pi$, $k=0,1,2,\dots$
(for NOT(X) gate)

- Gate Times: $t_k = \frac{\pi}{\omega} \left(k + \frac{1}{2} \right)$

$$t_0 = \frac{\pi}{2\omega}, t_1 = \frac{3\pi}{2\omega}, \dots, t_{N-1} = \frac{\pi}{\omega} \left(N - \frac{1}{2} \right)$$

- Final Time: $t_f = \frac{2\pi N}{\omega}$

- Contradiction: $t_k \leq t_f \Rightarrow$

$$\Rightarrow \frac{\pi}{\omega} \left(k + \frac{1}{2} \right) \leq \frac{2\pi N}{\omega} \Rightarrow k + \frac{1}{2} \leq 2N \Rightarrow$$

$$\Rightarrow k \leq 2N - \frac{1}{2}$$

The largest integer k is $2N-1$, from $k=0$ to $k=2N-1 \Rightarrow (2N-1) - 0 + 1 = 2N$ gates

- Assumption: $t_f = \frac{2\pi N}{\omega}$ correct (assume are correct). $2N$ gate in total. (Assume the rule of "apply gates at zero" and t_f)

2. Define the Evolution Operator

$$[H(t_1), H(t_2)] = 0$$

2) Define the Evolution Operator

$$[H_1(t_1), H(t_2)] = 0$$

$$U(t_b, t_a) = \exp\left(-\frac{i}{\hbar} \int_{t_a}^{t_b} H(t') dt'\right) =$$

$$= \exp\left(-\frac{i}{\hbar} \int_{t_a}^{t_b} \frac{\hbar \omega_0}{2} \cos(\omega t') dt' \cdot \sigma_z\right) \Rightarrow$$

$$\Rightarrow \boxed{U(t_a, t_b) = \exp\left(-i \frac{\omega_0}{2} \left[\frac{\sin(\omega t')}{\omega}\right]_{t_a}^{t_b} \cdot \sigma_z\right)}$$

$$\phi(t_a, t_b) = \frac{\omega_0}{2\omega} (\sin(\omega t_b) - \sin(\omega t_a))$$

$$\Rightarrow \boxed{\phi(t_a, t_b) = \frac{\omega_0}{2\omega} (\sin(\omega t_b) - \sin(\omega t_a))}$$

$$\Rightarrow \boxed{U(t_a, t_b) = e^{-i\phi(t_b, t_a)\sigma_z}} \quad \boxed{\phi_0 = \frac{\omega_0}{2\omega}}$$

3. Trace the States

- Initial State: $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

- Evolve $0 \rightarrow t_0$ (Gate 1):

$$\phi_1 = \phi(t_0, 0) = \phi_0 (\sin(\pi/2) - \sin(0)) = \phi_0$$

$$U_1 = e^{-i\phi_0\sigma_z}$$

$$|\psi(t_0^-)\rangle = U_1 |\psi_0\rangle = \frac{1}{\sqrt{2}} (e^{-i\phi_0}|0\rangle + i e^{i\phi_0}|1\rangle)$$

$$|\psi(t_0^+)\rangle = X |\psi(t_0^-)\rangle = \frac{1}{\sqrt{2}} (i e^{i\phi_0}|0\rangle + e^{-i\phi_0}|1\rangle)$$

2) 3. trace the state

- Evolve $t_0 \rightarrow t_1$ (Gate 2):

$$\phi_2 = \phi(t_1, t_0) = \phi_0 (\sin(3\pi/2) - \sin(\pi/2)) =$$

$$= \phi_0(-1-1) = -2\phi_0$$

$$U_2 = e^{-i\phi_2 \sigma_z} = e^{i2\phi_0 \sigma_z}$$

$$|\psi(t_1^-)\rangle = U_2 |\psi(t_0^+)\rangle = \frac{1}{\sqrt{2}} (ie^{i\phi_0}(e^{i2\phi_0}|0\rangle +$$

$$+ e^{-i\phi_0}(e^{-i2\phi_0})|1\rangle) = \frac{1}{\sqrt{2}} (ie^{i3\phi_0}|0\rangle + e^{-i3\phi_0}|1\rangle).$$

$$|\psi(t_1^+)\rangle = X |\psi(t_1^-)\rangle = \frac{1}{\sqrt{2}} (e^{-i3\phi_0}|0\rangle +$$

$$+ ie^{i3\phi_0}|1\rangle)$$

- Evolve $t_1 \rightarrow t_2$ (Gate 3):

$$\phi_3 = \phi(t_2, t_1) = \phi_0 (\sin(5\pi/2) - \sin(3\pi/2)) =$$

$$= \phi_0 (1 - (-1)) = 2\phi_0$$

$$U_3 = e^{-i2\phi_0 \sigma_z}$$

$$|\psi(t_2^-)\rangle = U_3 |\psi(t_1^+)\rangle = \frac{1}{\sqrt{2}} (e^{-i3\phi_0}(e^{-i2\phi_0})|0\rangle +$$

$$+ ie^{i3\phi_0}(e^{i2\phi_0})|1\rangle) = \frac{1}{\sqrt{2}} (e^{-i5\phi_0}|0\rangle + ie^{i5\phi_0}|1\rangle)$$

$$|\psi(t_2^-)\rangle = \frac{1}{\sqrt{2}} (e^{-i5\phi_0}|0\rangle + ie^{i5\phi_0}|1\rangle)$$

$$|\psi(t_2^+)\rangle = X |\psi(t_2^-)\rangle = \frac{1}{\sqrt{2}} (ie^{i5\phi_0}|0\rangle +$$

$$+ e^{-i5\phi_0}|1\rangle)$$

$$\Rightarrow |\psi(t_2^+)\rangle = \frac{1}{\sqrt{2}} (ie^{i5\phi_0}|0\rangle + e^{-i5\phi_0}|1\rangle)$$

2)

4. Find the state after $2N$ gates

$$|\psi_k\rangle = |\psi(t_{k-1}^+)\rangle$$

$$|\psi_1\rangle (\text{odd}): \frac{1}{\sqrt{2}} (ie^{i\phi_0}|0\rangle + e^{-i\phi_0}|1\rangle) \Rightarrow \text{Phase is } (2 \cdot 1 - 1)\phi_0$$

$$|\psi_2\rangle (\text{even}): \frac{1}{\sqrt{2}} (e^{-i3\phi_0}|0\rangle + ie^{i3\phi_0}|1\rangle) \Rightarrow \text{Phase is } (2 \cdot 2 - 1)\phi_0$$

$$|\psi_3\rangle (\text{odd}): \frac{1}{\sqrt{2}} (ie^{i5\phi_0}|0\rangle + e^{-i5\phi_0}|1\rangle) \Rightarrow \text{Phase is } (2 \cdot 3 - 1)\phi_0$$

$$k - \text{odd}: |\psi_k\rangle = \frac{1}{\sqrt{2}} (ie^{-i(2k-1)\phi_0}|0\rangle + e^{-i(2k-1)\phi_0}|1\rangle)$$

$$k - \text{even}: |\psi_k\rangle = \frac{1}{\sqrt{2}} (e^{-i(2k-1)\phi_0}|0\rangle + ie^{i(2k-1)\phi_0}|1\rangle)$$

$$k = 2N \Rightarrow |\psi_{2N}\rangle \text{ at time } t_{2N-1}$$

$$|\psi_{2N}\rangle = \frac{1}{\sqrt{2}} (e^{-i(2(2N)-1)\phi_0}|0\rangle + ie^{i(2(2N)-1)\phi_0}|1\rangle)$$

$$|\psi_{2N}\rangle = \frac{1}{\sqrt{2}} (e^{-i(4N-1)\phi_0}|0\rangle + ie^{i(4N-1)\phi_0}|1\rangle)$$

5. Final Evolution

$$t_i = t_{2N-1} \text{ to } t_f = \frac{2\pi N}{\omega}$$

$$t_i = t_{2N-1} = \frac{\pi}{\omega} (2N - 1 + \frac{1}{2}) = \frac{\pi(2N - \frac{1}{2})}{\omega}$$

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2) Final Evolution

$$t_i = t_{2N-1} = \frac{U}{\omega} \left(2N-1 + \frac{1}{2} \right) = \frac{U(2N-\frac{1}{2})}{\omega}$$

$$\sin(\omega t_i) = \sin\left(\omega \frac{U(2N-\frac{1}{2})}{\omega}\right) = \sin(2N\pi - \frac{\pi}{2}) = -1$$

$$\sin(\omega t_f) = \sin\left(\omega \frac{2UN}{\omega}\right) = \sin(2UN) = 0$$

Final phase:

$$\begin{aligned} \phi_{\text{final}} = \phi(t_f, t_i) &= \phi_0 (\sin(\omega t_f) - \sin(\omega t_i)) \\ &= \phi_0 (0 - (-1)) = \phi_0 \Rightarrow \boxed{\phi_{\text{final}} = \phi_0} \end{aligned}$$

$$U_{\text{final}} = e^{-i\phi_{\text{final}} \sigma_z} = e^{-i\phi_0 \sigma_z} \Rightarrow$$

$$\Rightarrow \boxed{U_{\text{final}} = e^{-i\phi_0 \sigma_z}}$$

6. Final State Calculation

$$\begin{aligned} |\psi(t_f)\rangle &= U_{\text{final}} |\psi_{2N}\rangle = e^{-i\phi_0 \sigma_z} \left[\frac{1}{\sqrt{2}} (e^{-i(4N-1)\phi_0} |0\rangle + e^{i(4N-1)\phi_0} |1\rangle) \right] \\ &= \frac{1}{\sqrt{2}} [e^{-i4N\phi_0} |0\rangle + e^{i4N\phi_0} |1\rangle] \end{aligned}$$

$$\phi_0 = \frac{\omega_0}{2\omega} : 4N\phi_0 = 4N \left(\frac{\omega_0}{2\omega} \right) = \frac{2N\omega_0}{\omega}$$

Final answer: $t = \frac{2UN}{\omega}$

$$\Rightarrow \boxed{|\psi(t_f)\rangle = \frac{1}{\sqrt{2}} (e^{-i2N\omega_0/\omega} |0\rangle + e^{i2N\omega_0/\omega} |1\rangle)}$$

$$\boxed{|\psi(t_f)\rangle = \frac{1}{\sqrt{2}} (e^{-i2N\omega_0/\omega} |0\rangle + e^{i2N\omega_0/\omega} |1\rangle)} \quad \left(\frac{i2N\omega_0}{\omega} \right)$$

Sheet 3

1. Suppose that you want to measure a parameter ω_0 . You choose a two-level system as your quantum probe. This parameter shows up in the Hamiltonian of your probe: $H = \frac{\hbar\omega_0}{2}\sigma_z$. You initialize the probe in the state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$. Here $\sigma_z|0\rangle = |0\rangle$ and $\sigma_z|1\rangle = -|1\rangle$. Now, however, after every time interval τ you apply a quantum NOT gate (or a X gate). What is the state just after applying N such gates (here N is even)? Explain.
2. Now consider the same situation as in the previous problem, except that we now have an oscillating field so that the Hamiltonian is now $H = \frac{\hbar\omega_0}{2}\cos(\omega t)\sigma_z$. You again apply the quantum NOT gates, but these are now applied whenever the field becomes zero. What is then the state at time $t = 2\pi N/\omega$? Here N is the number of gates applied.