8.ex.7

EE24BTECH11008 - ASLIN GARVASIS

Question : Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$

(a) Theoretical Solution :

1) **Find Points of Intersection:**
For $x^2 = \frac{4y}{3}$:

$$V_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad u_1 = \begin{pmatrix} 0 \\ \frac{-2}{3} \end{pmatrix}, \quad f_1 = 0$$

The quadratic form is:

$$\begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} 0 & \frac{-2}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 0 = 0$$

For $2y = 3x + 12$:

$$V_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1.5 \\ 1 \end{pmatrix}, \quad f_2 = -12$$

The quadratic form is:

$$\begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} -1.5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + -12 = 0$$

2) **Write the two equations in matrix form:**
- From $Q_1(x, y)$:

$$\begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} 0 \\ \frac{-2}{3} \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

Simplifies to:

$$3x^2 - 4y = 0$$

- From $Q_2(x, y)$:

$$2 \begin{pmatrix} -1.5 \\ 1 \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} - 12 = 0$$

Simplifies to:

$$-3x + 2y - 12 = 0 \quad \text{or equivalently, } 2y = 3x + 12$$

3) **Substitute $2y = 3x + 12$ into $4y = 3x^2$:**

$$2(3x + 12) = 3x^2 \tag{1}$$
$$3x^2 - 6x - 24 = 0 \tag{2}$$
$$x^2 - 2x - 8 = 0 \tag{3}$$
$$(x - 4)(x + 2) = 0 \tag{4}$$

4) **Solve for x:**

$$x = 4 \quad \text{or} \quad x = -2$$

5) **Find y for each x:**
   - For $x = 4$, $y = \frac{3}{4}(4)^2 = 12$
   - For $x = -2$, $y = \frac{3}{4}(-2)^2 = 3$

6) **The intersection points are:**

$$(4, 12) \quad \text{and} \quad (-2, 3)$$

7) **Set Up the Integral:**
The parabola is $4y = 3x^2 \implies y = \frac{3x^2}{4}$, and the line is $2y = 3x + 12 \implies y = \frac{3x+12}{2}$. To calculate the area, we integrate the difference between the parabola and the line in terms of x, from $x = -2$ to $x = 4$:

$$\text{Area} = \int_{x=-2}^{x=4} \left(\frac{3x + 12}{2} - \frac{3x^2}{4} \right) dx. \tag{5}$$

8) **Evaluate the Integral:**
Expand the integral:

$$\text{Area} = \int_{-2}^{4} \frac{3x + 12}{2}\, dx - \int_{-2}^{4} \frac{3x^2}{4}\, dx. \tag{6}$$

First term:

$$\frac{1}{2} \int_{-2}^{4} (3x + 12) = \frac{1}{2} \left(\frac{3x^2}{2} + 12x \right)_{-2}^{4} = 45 \tag{7}$$

Second term:

$$\int_{-2}^{4} \frac{3x^2}{4}\, dx = \frac{1}{4} \int_{-2}^{4} 3x^2\, dx = \frac{1}{4} \left(x^3 \right)_{-2}^{4} = \frac{1}{4} \cdot 72 = 18. \tag{8}$$

Now subtract:

$$\text{Area} = 45 - 18 = 27 \tag{9}$$

(b) Numerical Solution / Simulation :

We aim to compute the integral:

$$I = \int_{x=-2}^{x=4} \left(\frac{3x + 12}{2} - \frac{3x^2}{4} \right) dx.$$

using the trapzoidal trule approach.

9) **Discretize the Interval:** Divide the interval $[-2, 4]$ into $N = 100$ equal subintervals. The step size is:

$$h = \frac{4 - -2}{N} = \frac{6}{100} = 0.06. \tag{10}$$

The discrete points are:

$$x_i = -2 + i \cdot h, \quad \text{for } i = 0, 1, 2, \ldots, 100. \tag{11}$$

For example:

$$x_0 = -2, \quad x_1 = -1.94, \quad x_2 = -1.88, \ldots, x_{100} = 4. \tag{12}$$

10) **Define the Function:** The function to integrate is:

$$f(x) = \frac{3x + 12}{2} - \frac{3x^2}{4}. \tag{13}$$

11) **Establish the difference equation:**
Using the trapazoidal rule ,

$$I_n = I_{n-1} + \frac{h}{2} \left(f(x_n) + f(x_{n-1}) \right) \tag{14}$$

where:

a) I_n: The approximate integral value up to the n-th point,
b) $h = 0.06$: The step size,
c) $f(x_n) = \frac{3x_n+12}{2} - \frac{3x_n^2}{4}$: The function evaluated at x_n.

So, the integral can be approximated as,

$$I_n = I_{n-1} + \frac{h}{2} \left(\left(\frac{3x_n + 12}{2} - \frac{3x_n^2}{4} \right) + \left(\frac{3x_{n-1} + 12}{2} - \frac{3x_{n-1}^2}{4} \right) \right) \tag{15}$$

$$x_n = x_{n-1} + h \tag{16}$$

12) **Iterative Computation:** The recurrence relation is applied iteratively starting with the initial condition:

$$I_0 = 0. \tag{17}$$

Each step updates I_n using the values of $f(x_n)$ and $f(x_{n-1})$.

13) **Final Value:** After iterating up to $n = 100$, the value of the integral at the upper bound $x = 4$ is:

$$I[100] = 27 \tag{18}$$

Using the difference equation (15) **we can code to simulate the area pretty easily . Choosing n = 100 we get area as 27 which verifies with the theoretical solution.**

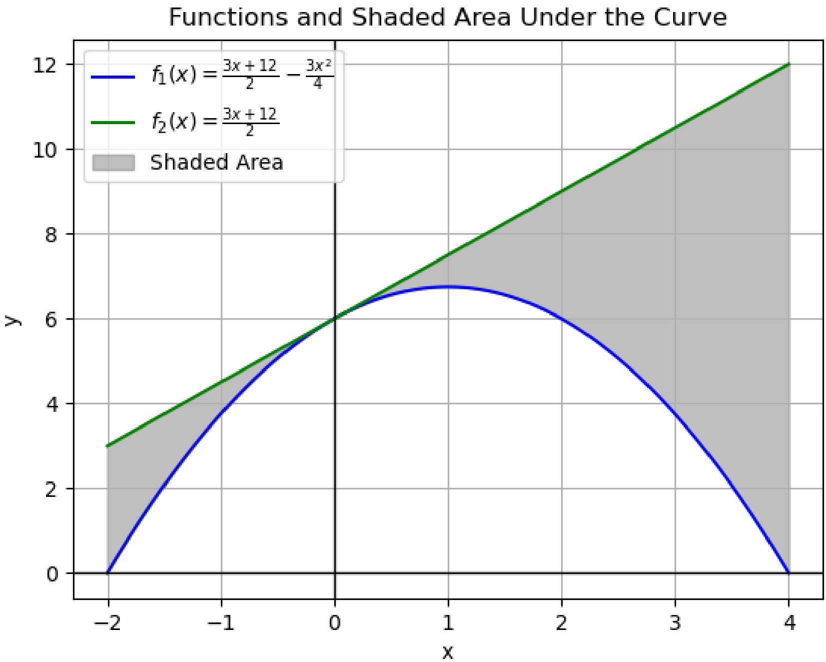

Fig. 13: Graph