6.5.25

EE24BTECH11008 - Aslin Garvasis

Question:
Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$

Solution (Classical Method)

The volume V of a cone is given by the formula:

$$V = \frac{1}{3}\pi r^2 h \tag{0.1}$$

where: - r is the radius of the base, - h is the height of the cone.
The slant height l is related to the radius r and height h by the Pythagorean theorem:

$$l^2 = r^2 + h^2 \tag{0.2}$$

Thus, the height h can be expressed as:

$$h = \sqrt{l^2 - r^2} \tag{0.3}$$

Substituting $h = \sqrt{l^2 - r^2}$ into the volume formula, we get:

$$V(r) = \frac{1}{3}\pi r^2 \sqrt{l^2 - r^2} \tag{0.4}$$

To maximize the volume, we differentiate $V(r)$ with respect to r. First, we use the product and chain rules to find the derivative:

$$\frac{dV}{dr} = \frac{1}{3}\pi\left(2r\sqrt{l^2 - r^2} + r^2 \cdot \frac{-r}{\sqrt{l^2 - r^2}}\right) \tag{0.5}$$

Simplifying, we have:

$$\frac{dV}{dr} = \frac{1}{3}\pi\left(2r\sqrt{l^2 - r^2} - \frac{r^3}{\sqrt{l^2 - r^2}}\right) \tag{0.6}$$

We set $\frac{dV}{dr} = 0$ to find the critical points:

$$2r\sqrt{l^2 - r^2} = \frac{r^3}{\sqrt{l^2 - r^2}} \tag{0.7}$$

Multiplying both sides by $\sqrt{l^2 - r^2}$, we obtain:

$$2r(l^2 - r^2) = r^3 \tag{0.8}$$

Canceling r from both sides (assuming $r \neq 0$):

$$2(l^2 - r^2) = r^2 \tag{0.9}$$

Simplifying:

$$2l^2 - 2r^2 = r^2 \tag{0.10}$$

$$2l^2 = 3r^2 \tag{0.11}$$

Solving for r^2, we get:

$$r^2 = \frac{2}{3}l^2 \tag{0.12}$$

Thus, the radius is:

$$r = \frac{\sqrt{2}}{\sqrt{3}}l \tag{0.13}$$

Now, we use the formula for the semi-vertical angle θ, which is given by:

$$\tan(\theta) = \frac{r}{h} \tag{0.14}$$

We substitute $r = \frac{\sqrt{2}}{\sqrt{3}}l$ and calculate h. From the relation $l^2 = r^2 + h^2$, we have:

$$h = \sqrt{l^2 - r^2} = \sqrt{l^2 - \frac{2}{3}l^2} = \sqrt{\frac{1}{3}l^2} = \frac{l}{\sqrt{3}} \tag{0.15}$$

Thus, $\tan(\theta)$ becomes:

$$\tan(\theta) = \frac{\frac{\sqrt{2}}{\sqrt{3}}l}{\frac{l}{\sqrt{3}}} = \sqrt{2} \tag{0.16}$$

Therefore, the semi-vertical angle θ is:

$$\theta = \tan^{-1}(\sqrt{2}) \tag{0.17}$$

Hence, we have shown that the semi-vertical angle of the cone of maximum volume, given a fixed slant height, is:

$$\boxed{\theta = \tan^{-1}(\sqrt{2})} \tag{0.18}$$

Solution (Gradient Descent Method)

We will now solve the problem using the gradient descent method. We want to maximize the volume $V(r)$ of the cone, which is given by:

$$V(r) = \frac{1}{3}\pi r^2 \sqrt{l^2 - r^2} \tag{0.19}$$

To do so, we use the gradient descent algorithm to iteratively find the value of r that maximizes the volume.

Step 1: Gradient of the Volume Function

The gradient (derivative) of the volume with respect to r is:

$$\frac{dV}{dr} = \frac{1}{3}\pi \left(2r\sqrt{l^2 - r^2} - \frac{r^3}{\sqrt{l^2 - r^2}} \right) \tag{0.20}$$

This is the function we will use for gradient descent.

Step 2: Iterative Update Rule

In gradient descent, we update the variable r iteratively using the following rule:

$$r_{n+1} = r_n - \alpha \frac{dV}{dr}(r_n) \tag{0.21}$$

where α is the learning rate, and r_n is the current value of r.

Step 3: Choosing an Initial Guess

We start with an initial guess for r, say $r_0 = \frac{l}{2}$, and choose a small learning rate α, for example $\alpha = 0.01$.

Step 4: Convergence Criterion

We iterate the update rule until the change in volume is sufficiently small, i.e., when $|r_{n+1} - r_n| < \epsilon$, where ϵ is a small threshold (e.g., $\epsilon = 10^{-6}$).

Step 5: Conclusion

After running the gradient descent algorithm for a sufficient number of iterations, the algorithm converges to the value of r, which maximizes the volume. Using the previously derived result, we know that this value of r corresponds to the semi-vertical angle θ where:

$$\tan(\theta) = \sqrt{2} \tag{0.22}$$

Thus, the semi-vertical angle is:

$$\boxed{\theta = \tan^{-1}(\sqrt{2})} \tag{0.23}$$

This confirms that the gradient descent method yields the same result as the classical approach.

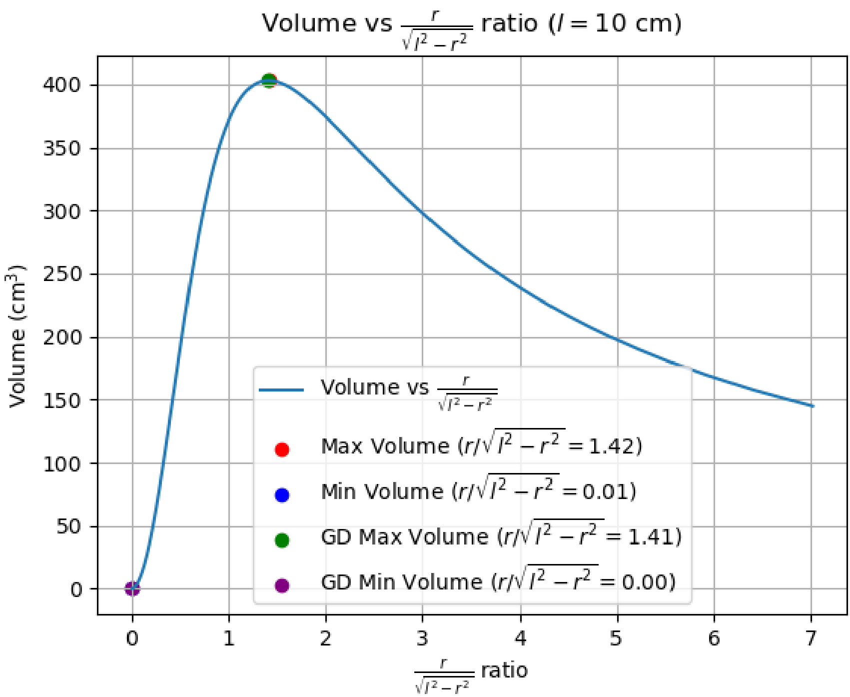

Fig. 0.1: Plot of volume versus r/h where $h = \sqrt{l^2 - r^2}$