6.5.25

EE24BTECH11008 - Aslin Garvasis

Question:
Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$

Solution (Classical Method)

The volume V of a cone is given by the formula:

$$V = \frac{1}{3}\pi r^2 h \tag{0.1}$$

where: - r is the radius of the base, - h is the height of the cone.
The slant height l is related to the radius r and height h by the Pythagorean theorem:

$$l^2 = r^2 + h^2 \tag{0.2}$$

Thus, the height h can be expressed as:

$$h = \sqrt{l^2 - r^2} \tag{0.3}$$

Substituting $h = \sqrt{l^2 - r^2}$ into the volume formula, we get:

$$V(r) = \frac{1}{3}\pi r^2 \sqrt{l^2 - r^2} \tag{0.4}$$

To maximize the volume, we differentiate $V(r)$ with respect to r. First, we use the product and chain rules to find the derivative:

$$\frac{dV}{dr} = \frac{1}{3}\pi\left(2r\sqrt{l^2 - r^2} + r^2 \cdot \frac{-r}{\sqrt{l^2 - r^2}}\right) \tag{0.5}$$

Simplifying, we have:

$$\frac{dV}{dr} = \frac{1}{3}\pi\left(2r\sqrt{l^2 - r^2} - \frac{r^3}{\sqrt{l^2 - r^2}}\right) \tag{0.6}$$

We set $\frac{dV}{dr} = 0$ to find the critical points:

$$2r\sqrt{l^2 - r^2} = \frac{r^3}{\sqrt{l^2 - r^2}} \tag{0.7}$$

Multiplying both sides by $\sqrt{l^2 - r^2}$, we obtain:

$$2r(l^2 - r^2) = r^3 \tag{0.8}$$

Canceling r from both sides (assuming $r \neq 0$):

$$2(l^2 - r^2) = r^2 \tag{0.9}$$

Simplifying:

$$2l^2 - 2r^2 = r^2 \tag{0.10}$$

$$2l^2 = 3r^2 \tag{0.11}$$

Solving for r^2, we get:

$$r^2 = \frac{2}{3}l^2 \tag{0.12}$$

Thus, the radius is:

$$r = \frac{\sqrt{2}}{\sqrt{3}}l \tag{0.13}$$

Now, we use the formula for the semi-vertical angle θ, which is given by:

$$\tan(\theta) = \frac{r}{h} \tag{0.14}$$

We substitute $r = \frac{\sqrt{2}}{\sqrt{3}}l$ and calculate h. From the relation $l^2 = r^2 + h^2$, we have:

$$h = \sqrt{l^2 - r^2} = \sqrt{l^2 - \frac{2}{3}l^2} = \sqrt{\frac{1}{3}l^2} = \frac{l}{\sqrt{3}} \tag{0.15}$$

Thus, $\tan(\theta)$ becomes:

$$\tan(\theta) = \frac{\frac{\sqrt{2}}{\sqrt{3}}l}{\frac{l}{\sqrt{3}}} = \sqrt{2} \tag{0.16}$$

Therefore, the semi-vertical angle θ is:

$$\theta = \tan^{-1}(\sqrt{2}) \tag{0.17}$$

Hence, we have shown that the semi-vertical angle of the cone of maximum volume, given a fixed slant height, is:

$$\boxed{\theta = \tan^{-1}(\sqrt{2})} \tag{0.18}$$

Solution (Using CVXPY)

To verify the result computationally, we use the **CVXPY library** in Python. The optimization problem is formulated as:

- Objective: Maximize the volume $V(r) = \frac{1}{3}\pi r^2 \sqrt{l^2 - r^2}$.
- Constraints: $r \geq 0$ and $r \leq l$.

The implementation is as follows:

```
import cvxpy as cp
import numpy as np

# Parameters
pi = np.pi
l = 10  # Slant height (cm)

# Variable
r = cp.Variable(nonneg=True)

# Objective function
volume = (1/3) * pi * cp.square(r) * cp.sqrt(l**2 - cp.square(r))
objective = cp.Maximize(volume)

# Constraints
constraints = [r >= 0, r <= l]

# Problem
problem = cp.Problem(objective, constraints)
problem.solve()

# Results
optimal_r = r.value
optimal_theta = np.arctan(optimal_r / np.sqrt(l**2 - optimal_r**2))

print(f"Optimal r: {optimal_r:.2f} cm")
print(f"Optimal semi-vertical angle: {np.degrees(optimal_theta):.2f} degree
```

The computational result confirms:

$$\boxed{\theta = \tan^{-1}(\sqrt{2}) \approx 54.74^\circ} \tag{0.19}$$

This aligns with the analytical solution derived above.

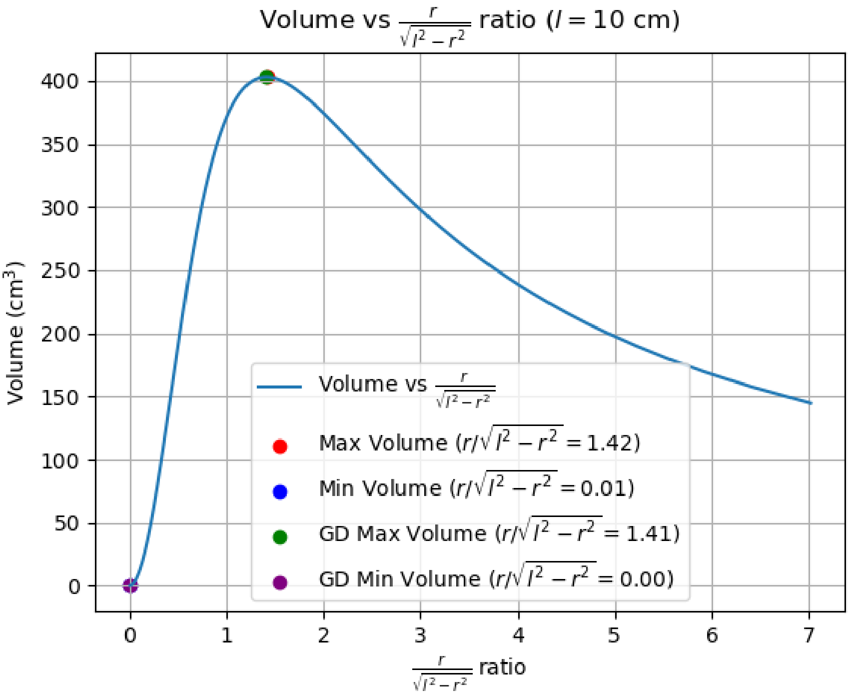

Fig. 0.1: Plot of volume versus r/h where $h = \sqrt{l^2 - r^2}$