# 10.3.4.1.4

EE24BTECH11008 - Aslin Garvasis

**Question**:
Solve the following pair of of linear equation by the elimination method and substitution method:

$$\frac{x}{2}+\frac{2y}{3}=-1 \tag{0.1}$$

$$x-\frac{y}{3}=3 \tag{0.2}$$

**Represent the system in matrix form**
The system of equations can be written as:

$$A\mathbf{x}=\mathbf{b}, \tag{0.3}$$

where

$$A=\begin{pmatrix} \frac{1}{2} & \frac{2}{3} \\ 1 & \frac{-1}{3} \end{pmatrix}, \mathbf{x}=\begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{b}=\begin{pmatrix} -1 \\ 3 \end{pmatrix}. \tag{0.4}$$

**Perform LU Decomposition using Doolittle's Algorithm**
We decompose the matrix $A$ into the product of a lower triangular matrix $L$ and an upper triangular matrix $U$, i.e.,

$$A=LU, \tag{0.5}$$

where

$$L=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{pmatrix}, \quad U=\begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & u_{n-1,n} \\ 0 & 0 & 0 & \cdots & u_{nn} \end{pmatrix}. \tag{0.6}$$

The elements of these matrices are calculated as follows:
Elements of the $U$ Matrix:
For each column $j$:

$$U_{ij}=A_{ij} \text{ if } i=0, \tag{0.7}$$

$$U_{ij}=A_{ij}-\sum_{k=0}^{i-1} L_{ik}U_{kj} \text{ if } i>0. \tag{0.8}$$

Elements of the $L$ Matrix:

For each row $i$:

$$L_{ij} = \frac{A_{ij}}{U_{jj}} \text{ if } j = 0, \tag{0.9}$$

$$L_{ij} = \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} \text{ if } j > 0. \tag{0.10}$$

$$L = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix}, U = \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}. \tag{0.11}$$

Now, let's compute the LU decomposition step by step.

First, we find the elements of $U$ and $L$:

$$u_{11} = a_{11} = \frac{1}{2}, u_{12} = a_{12} = \frac{2}{3}. \tag{0.12}$$

Next, we compute $l_{21}$ and $u_{22}$:

$$l_{21} = \frac{a_{21}}{u_{11}} = \frac{1}{\frac{1}{2}} = 2, \tag{0.13}$$

$$u_{22} = a_{22} - l_{21} u_{12} = \frac{-1}{3} - 2 \times (\frac{2}{3}) = \frac{-5}{3}. \tag{0.14}$$

So the LU decomposition is:

$$L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, U = \begin{pmatrix} \frac{1}{2} & \frac{2}{3} \\ 0 & \frac{-5}{3} \end{pmatrix}. \tag{0.15}$$

**Solve for x using $LU$ decomposition**

Now we solve the system in two steps using forward substitution and backward substitution.

First, solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$:

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}. \tag{0.16}$$

This gives:

$$y_1 = -1 \tag{0.17}$$

$$2y_1 + y_2 = 3 \Rightarrow 2(-1) + y_2 = 3 \Rightarrow y_2 = 5. \tag{0.18}$$

Thus, $\mathbf{y} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$.

Next, solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$:

$$\begin{pmatrix} \frac{1}{2} & \frac{2}{3} \\ 0 & \frac{-5}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}. \tag{0.19}$$

This gives:

$$\frac{-5}{3}y = 5 \Rightarrow y = -3, \tag{0.20}$$

$$\frac{1}{2}x + \frac{2}{3}y = -1 \Rightarrow \frac{1}{2}x + \frac{2}{3} \times -3 = -1 \Rightarrow x = 2. \tag{0.21}$$

Thus, the solution is $x = 2$ and $y = -3$.

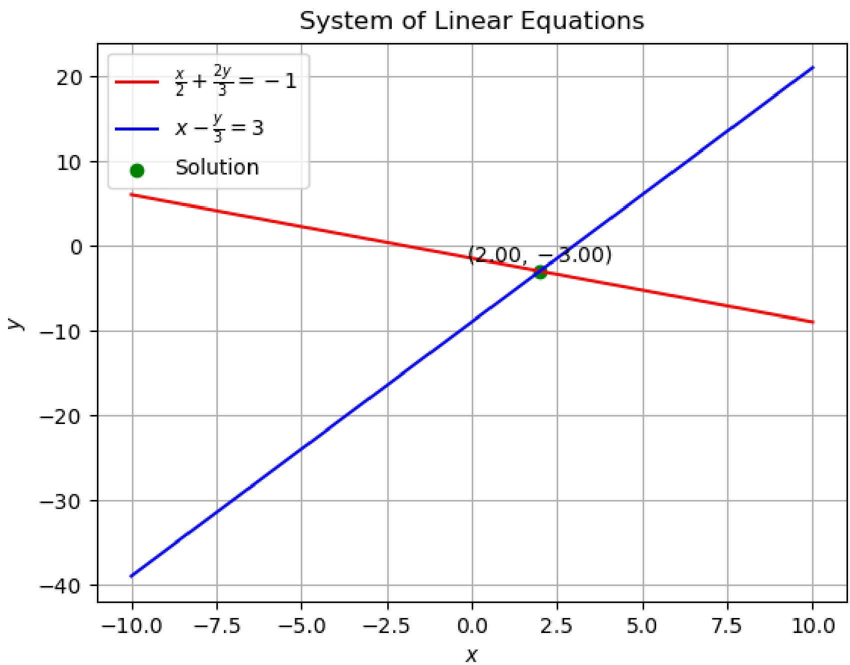


Fig. 0.1: Solving the system of equations