1 Mixed-Integer Programming Formulation

Quick and dirty definitions of the variables and indices.

$s \in S$	the set of all scenarios,
$j \in J$	the set of all generators we control,
$j \in J^c$	the set of all generators of the competitors,
$i \in I$	the set of all possible bid prices.

The model parameters:

p_s	the probability that the scenario s will be realised,
d_s	the total demand in the scenario s,
$\lambda^c_{s,j}$	the price bid by the competitors in the scenario s for their generator j,
$\bar{g}^c_{s,j}$	the quantity bid by the competitors in the scenario s for the generator j,
g^{c*}_j	the maximum quantity which can be produced by the competitors generator j,
g^*_j	the maximum quantity which can be produced by our generator j,
c_j	the operating cost to produce one unit of energy by our generator j.

The model variables:

R_s	our profit during scenario s,
R	our average profit for all scenarios,
π_s	the spot price during scenario s,
λ_j	the price bid by us for our generator j,
$\bar{g}_j$	the quantity bid by us for our generator j,
$g^c_{s,j}$	the actual quantity produced by the competitors generator j during scenario s,
$g_{s,j}$	the actual quantity produced by our generator j during scenario s.

The goal is to maximize profit, which can be modeled as such:

$$\max R, \tag{1}$$

$$
\begin{aligned}
\text{s.t.} \qquad R &= \sum_{s \in S} p_s R_s, && && (2)\\
R_s &= \sum_{j \in J} (\pi_s - c_j)\, g_{s,j} && \forall s \in S, && (3)\\
\sum_{j \in J} g_{s,j} + \sum_{j \in J^C} g^c_{s,j} &= d_s && \forall s \in S, && (4)\\
g^c_{s,j} &\geq 0 && \forall s \in S, j \in J^c, && (5)\\
g^c_{s,j} &\leq \bar{g}^c_j && \forall s \in S, j \in J^c, && (6)\\
\lambda^c_{s,j} < \pi_s &\rightarrow g^c_{s,j} = \bar{g}^c_j && \forall s \in S, j \in J^c, && (7)\\
\lambda^c_{s,j} > \pi_s &\rightarrow g^c_{s,j} = 0 && \forall s \in S, j \in J^c, && (8)\\
\bar{g}_{s,j} &\geq 0 && \forall s \in S, j \in J, && (9)\\
\bar{g}_{s,j} &\leq g^*_j && \forall s \in S, j \in J, && (10)\\
g_{s,j} &\geq 0 && \forall s \in S, j \in J, && (11)\\
g_{s,j} &\leq \bar{g}_j && \forall s \in S, j \in J, && (12)\\
\lambda_j < \pi_s &\rightarrow g_{s,j} = \bar{g}_j && \forall s \in S, j \in J, && (13)\\
\lambda_j > \pi_s &\rightarrow g_{s,j} = 0 && \forall s \in S, j \in J, && (14)\\
\pi_s &\in \left\{\lambda^c_{s,j}, \forall j \in J^c\right\} \cup \{\lambda_j, \forall j \in J\} && \forall s \in S. && (15)
\end{aligned}
$$

It should be noted that in this formulation, the spot price π_s is not set to the lowest price for which all the bids of lower value satisfy the demand, but instead is set to the highest price for which all the bids of lower value does not surpass the demand. While this is different from the real definition of the spot price, we could reduce our own bid quantities by some infinitesimal amount and get the same effect, with only an infinitesimal reduction in profit.

Since that formulation is highly inconvenient, we would like to switch to a linear or bilinear formulation. In order to do so, we need to simplify the constraints on the spot prices π_s. If the bid prices λ_j were constrained to be from a finite subset of values, then π_s could be easily defined using those same values.

In this bidding problem, we can prove that the optimal value for each bid price λ_j is always equal to one of the possible competitor bid prices $\lambda^c_{s,j}$, assuming that ties in prices are resolved in our advantage.

Proof: Let assume that the one of our bids λ_j is strictly between two other bids (either from ourself or from competitors) λ^c_{s,j_1} and λ^c_{s,j_2} (with no other bids in between):

$$\lambda^c_{s,j_1} < \lambda_j < \lambda^c_{s,j_2}, \tag{16}$$

then the expected profit R would either increase or remain constant if λ_j was to be increased to λ^c_{s,j_2}, for all possible scenarios. The possibilities are:

- $\lambda^c_{s,j_2} \leq \pi_s$: The bid price is above λ^c_{s,j_2}, so any value of λ_j lesser or equal than λ^c_{s,j_2} would not change the bid price. And since it implies that $\lambda_j \leq \pi_s$, then all of the production for generator j will be sold, for any bid value. So R_s would stay constant from this change in bid price.

- $\lambda^c_{s,j_1} \geq \pi_s$: The bid price is below λ_j, so none of the production from generator j is sold. Increasing the bid price does not change that fact, so R_s would remain constant from this change in bid price.

- $\lambda_j = \pi_s$: The bid price is equal to λ_j. This means that the production from generator j is necessary to satisfy the scenario demand. Therefore, increasing the bid price will also increase the spot price by the same amount, as long as the bid price is not increased past another bid price (λ^c_{s,j_2} in this case). So, increasing λ_j up to λ^c_{s,j_2} would increase R_s.

Since increasing λ_j to the next higher bid λ^c_{s,j_2} always lead to an increasing or a constant R_s, the same can be said of the average profits R. It is thus always optimal to only select bid prices amongst the set of all possible competitors bid prices.

We will denote using λ^c_i a competitor bid price from any scenario, using the i index instead of s, j. Those prices are assumed to be sorted in ascending order ($i' > i \to \lambda^c_{i'} \geq \lambda^c_i$). This adds extra variables to our model:

$y_{s,i}$ binary variable that indicates if the spot price in scenario s is set to λ^c_i,
x_{ij} binary variable that indicates if λ_j is set to λ^c_i.

Rewriting our model, we now have:

$$\max R, \tag{17}$$

$$
\begin{aligned}
\text{s.t.} \qquad & R = \sum_{s \in S} p_s R_s, && && (18)\\
& R_s = \sum_{j \in J} \left(\sum_{i \in I} y_{s,i} \lambda_i^c - c_j \right) g_{s,j} && \forall s \in S, && (19)\\
& \bar{g}_j \geq 0 && \forall s \in S, j \in J, && (20)\\
& \bar{g}_j \leq g_j^* && \forall s \in S, j \in J, && (21)\\
& x_{ij} \in \{0, 1\} && \forall i \in I, j \in J, && (22)\\
& \sum_{i \in I} x_{ij} = 1 && \forall j \in J, && (23)\\
& y_{s,i} \in \{0, 1\} && \forall s \in S, i \in I, && (24)\\
& \sum_{i \in I} y_{s,i} = 1 && \forall s \in S, && (25)\\
& g_{s,j} \geq 0 && \forall s \in S, j \in J, && (26)\\
& g_{s,j} \geq \bar{g}_j x_{ij} y_{s,i'} && \forall s \in S, j \in J, i, i' \in I, i < i' && (27)\\
& g_{s,j} \leq \bar{g}_j && \forall s \in S, j \in J, && (28)\\
& g_{s,j} \leq \bar{g}_j \left(1 - x_{ij} y_{s,i'}\right) && \forall s \in S, j \in J, i, i' \in I, i > i', && (29)\\
& \sum_{j \in J} g_{s,j} + \sum_{\substack{j \in J^c \\ \lambda_{s,j}^c < \lambda_i}} \bar{g}_{s,j}^c \leq d^s y_{s,i} + \left(\max_{s' \in S} d^{s'} \right) (1 - y_{s,i}) && \forall s \in S, i \in I, && (30)\\
& \sum_{j \in J} g_{s,j} + \sum_{\substack{j \in J^c \\ \lambda_{s,j}^c \leq \lambda_i}} \bar{g}_{s,j}^c \geq d^s y_{s,i} && \forall s \in S, i \in I. && (31)
\end{aligned}
$$

Since some of those constraints have non-linear terms, we need to linearize them. Constraint (27) can be replaced by:

$$
g_{s,j} \geq \bar{g}_j - g_j^* (2 - x_{ij} - y_{s,i'}) \qquad \forall s \in S, j \in J, i, i' \in I, i < i', \tag{32}
$$

constraint (29) can be replaced by:

$$
g_{s,j} \leq g_j^* (2 - x_{ij} - y_{s,i'}) \qquad \forall s \in S, j \in J, i, i' \in I, i > i', \tag{33}
$$

while each $y_{s,i} g_{s,j}$ term in equation (19) must be replaced by a new variable

$yg_{s,ij}$ which follows the following constraints:

$$yg_{s,ij} \geq 0, \tag{34}$$
$$yg_{s,ij} \leq g_{s,j}, \tag{35}$$
$$yg_{s,ij} \leq g_j^* y_{s,i}, \tag{36}$$
$$yg_{s,ij} \geq g_{s,j} - g_j^*(1 - y_{s,i}). \tag{37}$$

With this formulation, we have a total of $O(|S||J||I|)$ continuous variables, $O((|S| + |J|)|I|)$ binary variables, and $O(|S||J||I|^2)$ constraints.

2 Shortest-Path Algorithm

We now relax the strategic-pricing problem to allow us to split the capacity of each generator into an arbitrary number of unique bids. The generators that will be used to fulfill the bids will then simply be the n-cheapest generators, where n is chosen such that their capacity is at least equal to the amount we end up having to produce. Accordingly, it is no longer useful to associate each bid to a specific generator. Instead, each bid will only be associated to a specific price λ_i. The quantity bid on price λ_i will be denoted using g_i.

Before introducing the shortest-path algorithm to solve the relaxed strategic-pricing problem, we have to prove a few properties. We define the cumulative bid quantities $G_i = \sum_{\substack{i' \in I \\ i' \leq i}} g_{i'}$, the residual demands $r_{s,i} = d_s - \sum_{\substack{j \in J^c \\ \lambda_{s,j}^c < \lambda_i}} \bar{g}_{s,j}^c$, and the effective capacities $c_i^e = \sum_{\substack{j \in J \\ c_j < \lambda_i}} g_j^*$. We also note that we can alternatively define the bids using the cumulative bid quantities G_i instead of the bid quantities g_i, with the restrictions that $G_{i'} \geq G_i$ if $i' > i$ (do not allow negative bid quantities) and $G_i \leq c_i^e$ (do not offer production at a loss).

The first property we need is that impact on profit of modifying a consecutive list of cumulative bid quantities $\{G_i, \ldots, G_{i'}\}$ only depends on the modified quantities and their two neighbouring quantities G_{i-1} and $G_{i'+1}$.

Proof: Let consider a specific scenario s. When calculating the profits R_s, there are three possibilities:

- $G_{i-1} > r_{s,i}$: the bids at values lower than λ_i are sufficient to satisfy the whole demand. Therefore, the spot price $\pi_s < \lambda_i$ and will not vary based on $\{G_i, \ldots, G_{i'}\}$. The production sold will also not be depending on the modified bids. Therefore, R_s will be constant.

- $G_{i'+1} \leq r_{s,i'+1}$: the bids at values lower than $\lambda_{i'+1}$ are not sufficient to satisfy the whole demand. Therefore, the spot price $\pi_s \geq \lambda_{i'+1}$ and will not vary based on $\{G_i, \ldots, G_{i'}\}$. The production sold will also not be depending on the modified bids, since all bids up to $G_{i'+1}$ will be fully sold. Therefore, R_s will be constant.

- $G_{i-1} \leq r_{s,i} \wedge G_{i'+1} > r_{s,i'+1}$: the bids at lower values than λ_i does not suffice to satisfy the whole demand, and the bids at lower values than $\lambda_{i'}$ will be enough. In this case, the precise values of the modified bid quantities $\{G_i, \ldots, G_{i'}\}$ will have an impact on the scenario profit. Because of the values of G_{i-1} and $G_{i'+1}$, the spot price π_s must be in $\{\lambda_i, \ldots, \lambda_{i'}\}$. Which one will be the spot price will depend on the precise values of the cumulative bid quantities. Since the profits for a single scenario only depend on the spot price and the cumulative bid quantity at that value, then they will only depend on $\{G_i, \ldots, G_{i'}\}$.

Since for each scenario the profit is only a function of the modified cumulative bid quantities or their two neighbouring quantities, the same will be true for the full profit function R, thus proving the property.

A consequence of this property is that the profit function R can be written as a sum of incremental profits for individual bids:

$$R\left((\lambda_1, g_1), \ldots, (\lambda_n, g_n)\right) = R\left((\lambda_1, g_1)\right) + \sum_{k=2}^{n} \left[R\left((\lambda_{k-1}, G_{k-1}), (\lambda_k, g_k)\right) - R\left((\lambda_{k-1}, G_{k-1})\right)\right]. \tag{38}$$

The second property is that at least one of the set of optimal bids has all of the cumulative bid quantities G_i equal to one of the following thresholds:

- The effective capacity at that price, c_i^e.

- The residual demand for some scenario s and for the next higher bid price $r_{s,i+1}$, if lower than the effective capacity.

- The bid quantity is equal to the following bid quantity G_{i+1}, which itself must be at a threshold. Recursively, we can find that these thresholds are all of the $r_{s,i'}$ for $i' > i + 1$.

Proof: This property can be proven by showing that if we have a solution, then the solution can always be improved by moving one of the G_i, as long as all the G_i are not at thresholds. There are two cases to consider: $G_i > c_i^e$ and $G_i < c_i^e$.

In the case where $G_i > c_i^e$, then it means that we are offering to sell power at a value lower than our marginal cost. Therefore, it is trivial to see that reducing G_i to c_i^e will always increase our profit (if that power was to be sold at that value), or leave it unchanged (if the bid price happened to be different).

So only the $G_i < c_i^e$ case remains. Let again consider a specific scenario s, and a set of bids such that for some i, G_i is not at any of the preceding thresholds, and $G_i < G_{i+1}$. It will always be possible to find such a i if at least single $G_{i'}$ is not at a threshold, since if G_i is not at a threshold, then either $G_i < G_{i+1}$, or G_{i+1} is also not at a threshold, since by construction of the thresholds. In that case, increasing G_i to its next threshold value will depend on the value of the spot price π_s:

- $\pi_s < \lambda_i$: the demand is fully covered by previous bids, so how is bid at λ_i does not matter. So increasing G_i leaves R_s unchanged.

- $\pi_s > \lambda_{i+1}$: the demand requires at least the bids up to values λ_{i+2} to reach the demand. This indicates that bidding up to G_{i+1} will not be sufficient to reach the demand. Since G_i cannot be increased past G_{i+1} and G_{i+1} is not modified, then the spot price will also not be modified. So the R_s will again remains unchanged.

- $\pi_s = \lambda_i$: the demand is only satisfied using the bids at value λ_i. If $G_i < r_{s,i}$, increasing G_i increases how much power we sell at that bid price, therefore also increasing profit (since λ_i is higher than the marginal production cost). If $G_i \geq r_{s,i}$, then we are already selling as much as the demand allows, and increasing G_i does not change the amount of power sold, thus leaving R_s unchanged. In both cases the spot price stays at λ_i.

- $\pi_s = \lambda_{i+1}$: This is the only situation where increasing the value of G_i could lead to a decrease in the spot price value. For the spot price to decrease from $\pi_s = \lambda_{i+1}$ to $\pi_s = \lambda_i$, then the cumulative quantity bid by us at this price level must be higher than the residual demand at the next next level: $G_i > r_{s,i+1}$. But since $r_{s,i+1}$ is one of the thresholds, we are forbidden from increasing G_i past its value, so the spot price will remain constant. Therefore, increasing G_i will not have any impact on the profit, since the profit only depends on the quantity bid at the spot price level G_{i+1}, which is not modified here.

Since increasing a G_i that is lower than c_i^e and G_{i+1} and not at any of its threshold values always either increases R_s or leaves it constant, the same can be said for the full profit function R.

This second property indicates that an optimal set of bids can always be found by only looking at bids where the G_i, and thus also the g_i, take specific values (the c_i^e or $r_{s,i'}$ thresholds). Since there is only a finite number of those values, an algorithm that would checked all combinations would find the optimal solution in finite time. But such an algorithm would be very slow, taking a time exponential in the size of the problem to find the solution. This is where the first property can be used: since the profit function R can be written as an incremental sum over individual bids, we can optimise each bids iteratively. Defining $R_i^{max}(G_i)$ as the maximum profit when bidding for a total quantity G_i at price up to λ_i, we have:

$$R_i^{max}(G_i) = \max_{\substack{G_{i-1}\in\left\{c_{i-1}^e, G_i\right\}\cup\{r_{s,i} \forall s\in S\}\\ 0\leq G_{i-1}\leq G_i}} R_{i-1}^{max}(G_{i-1}) + [R\left((\lambda_{i-1}, G_{i-1}), (\lambda_i, G_i - G_{i-1})\right) - R\left((\lambda_{i-1}, G_{i-} \right. \tag{39}$$

The optimal profit is then readily obtained by taking the value of $R_i^{max}(c_i^e)$ associated with the highest bid value λ_i of the problem.

Since there are $O(|S||J^c|)$ (where $|J^c|$ is used as a shorthand for the average number of competitor bids per scenario) different bid prices, and also $O(|S||J^c|)$ different values for the bid quantities G_i, this mean that finding the optimal set of bids requires the computation of $O(|S|^3|J^c|^2)$ profit increments.

The single-scenario single-bid profit function $R_s((\lambda_i, G_i))$ has three regimes: none of the bid is sold, a partial amount of is sold at the value given, or all of it is sold at a potentially higher spot price.

$$R_s((\lambda_i, G_i)) = \begin{cases} 0 & \text{if } r_{s,i} < 0, \\ \lambda_i r_{s,i} - c(r_{s,i}) & \text{if } 0 \leq r_{s,i} < G_i, \\ G_i \left(\min_{i'\geq i} \left\{\lambda_{i'} : r_{s,i'+1} < G_i\right\}\right) - c(G_i) & \text{if } G_i \leq r_{s,i}. \end{cases} \tag{40}$$

Where $c(G)$ is the total cost to produce G units of energy with the cheapest available generator. This function can be pre-calculated for all possible thresholds in advance in $O(|S||J^c||J| + |J| \lg |J|)$ time. These profit functions can be pre-calculated in $O(|S|^2|J^c|^2)$ time, and used to calculate the profit increments.

$$R_s\left((\lambda_{i-1}, G_{i-1}), (\lambda_i, G_i - G_{i-1})\right) - R_s((\lambda_i, G_i)) = \begin{cases} 0 & \text{if } r_{s,i} < G_{i-} \\ R_s((\lambda_i, G_i)) - R_s((\lambda_{i-1}, G_{i-1})) & \text{if } G_{i-1} \leq r \end{cases} \tag{41}$$

This last formulation has a simple interpretation: there are two possibilities when adding an additional bid to a previous bid. Either there is no left-over demand at the new bid price ($r_{s,i} < G_{i-1}$) and the system is unchanged by the additional bid; or there is some left-over demand at the new bid price ($G_{i-1} \leq r_{s,i}$), and then the spot price will at least be λ_i, and the profits will as if there was only a single bid at that value. Since the full profit function is the sum over all scenarios of the single-scenario profit functions, the time to compute a single profit increment is $O(|S|)$, which lead to a total running time of the algorithm of $O(|S|^4|J^c|^2)$.