ECS 130: HW 3 - Eigendecompositions & SVD:

Problem 1:

Problem 1 (10 pts). Construct by hand an SVD for the following 2×2 matrices:

a) $\begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Do not do this in a brute force way (*e.g.*, by computing an eigendecomposition of A^TA or AA^T)! Instead, think about how the special structure of each matrix relates to the definition and geometric interpretation of the SVD. *Hint: multiplying by* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *swaps the entries of a vector in* $\mathbb{R}^2$.

Goal: Find SVD for a, b, c, d w/ geometric interpretation

↳ we have to use $A = U\Sigma V^T$

$\underbrace{[u_1 | u_2]}_{\hookrightarrow U} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\hookrightarrow \Sigma} = A \underbrace{[v_1 | v_2]}_{V}$ } both are equivalent.

a) $\begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}$ → take abs value so $\sigma_1 = 7$ & $\sigma_2 = 4$ by $A^TA = \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 49 & 0 \\ 0 & +16 \end{bmatrix}$

↳ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So, $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 7 & 0 \\ 0 & +4 \end{bmatrix}$, $V_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = I$

This means SVD is $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ → so

Now, let's apply $A = U\Sigma V^T$ again here.

$U = I$, $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

↳ must rotate greatest to least so V_1 → $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

↳ so, we know $\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$ and this does NOT have a diagonal.

we know $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ maps $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so $\sqrt{1+1} = \sqrt{2} = \sigma^1$ in Σ. & since vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ also map $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ then $\sigma^2 = 0$

So, $\Sigma = \begin{vmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{vmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$

Then, $A = U\Sigma V^T$. So, we need to find U & V^T.

We know $\sigma_1 = \sqrt{2}$ so to find orthogonality. $1/\sqrt{2} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which maps to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

So, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$= A = U\Sigma V^T$

d) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Let us use $A = U\Sigma V^T$

We know we A^TA so $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ → $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ → $\lambda = 4$

so $\sigma_1 = \sqrt{4} = 2$

so this means $\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

This time eigenvectors are $\begin{bmatrix}1\\1\end{bmatrix}$ so $\|V\| = \sqrt{1^2+1^2} = \sqrt{2}$

so, have to normalize $1/\sqrt{2}$

$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$ is updated eigenvector

Also $1/\sqrt{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$

so,

$$A = \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$$

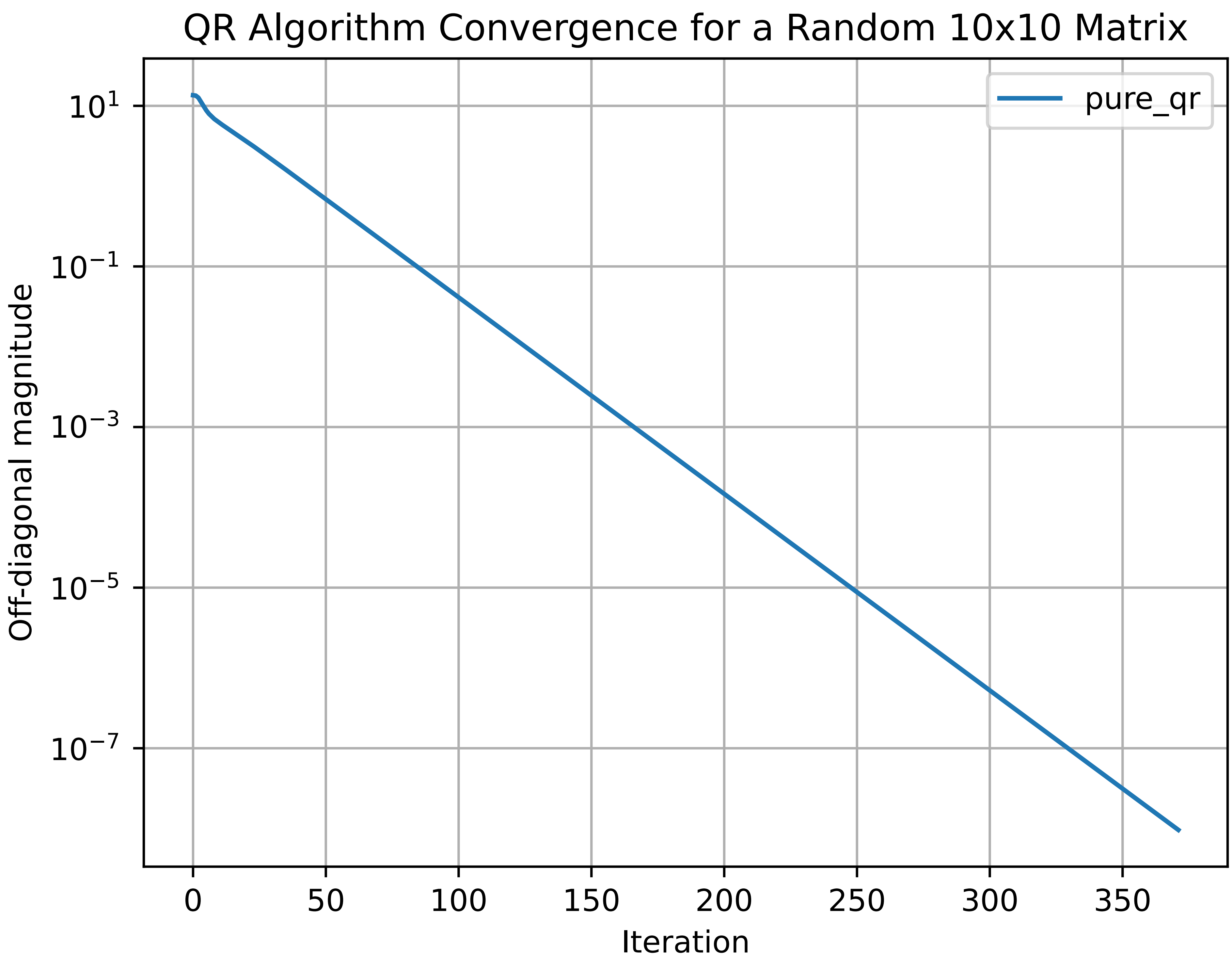
Problem 2: Comment on whether it reflects linear convergence we expect

My plot is below this and so we know v_k = normalize $(A^k v)$ = norm $\left(a_1 q_1 + a_2\left(\frac{\lambda_2}{\lambda_1}\right)^k q_2 + \ldots\right.$ ...

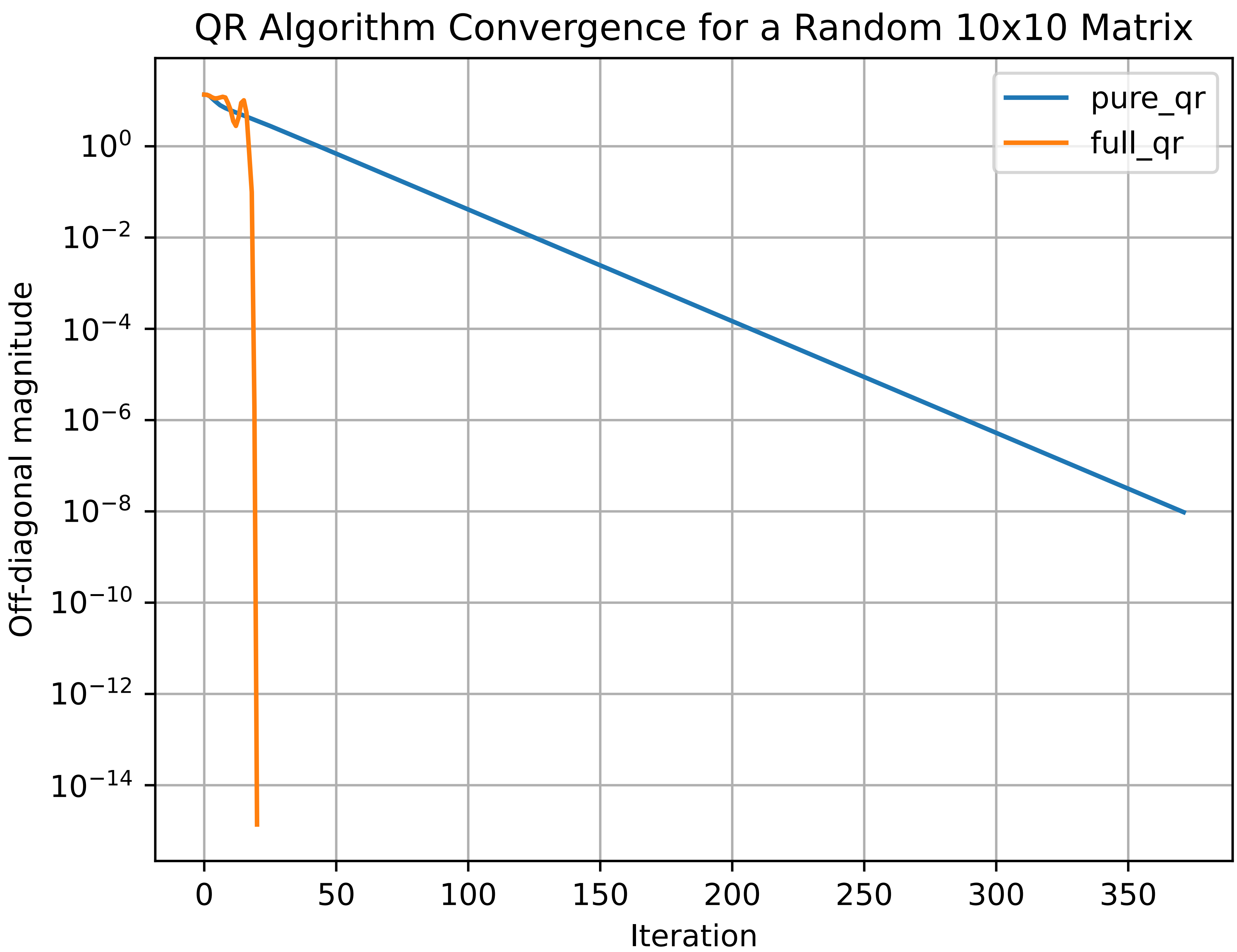
We also know ordinary power iteration converges SLOWLY for matrices. $\left.\left(\frac{\lambda_m}{\lambda_1}\right)^k q_m\right)$.

But, $|v_k - (\pm q_1)| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)$ as $k \to \infty$

So, this is a linear convergence because it ensures exponential decay in a predictable manner. I believe this is also the rate we expect b/c power iteration is known to converge slowly for matrices.



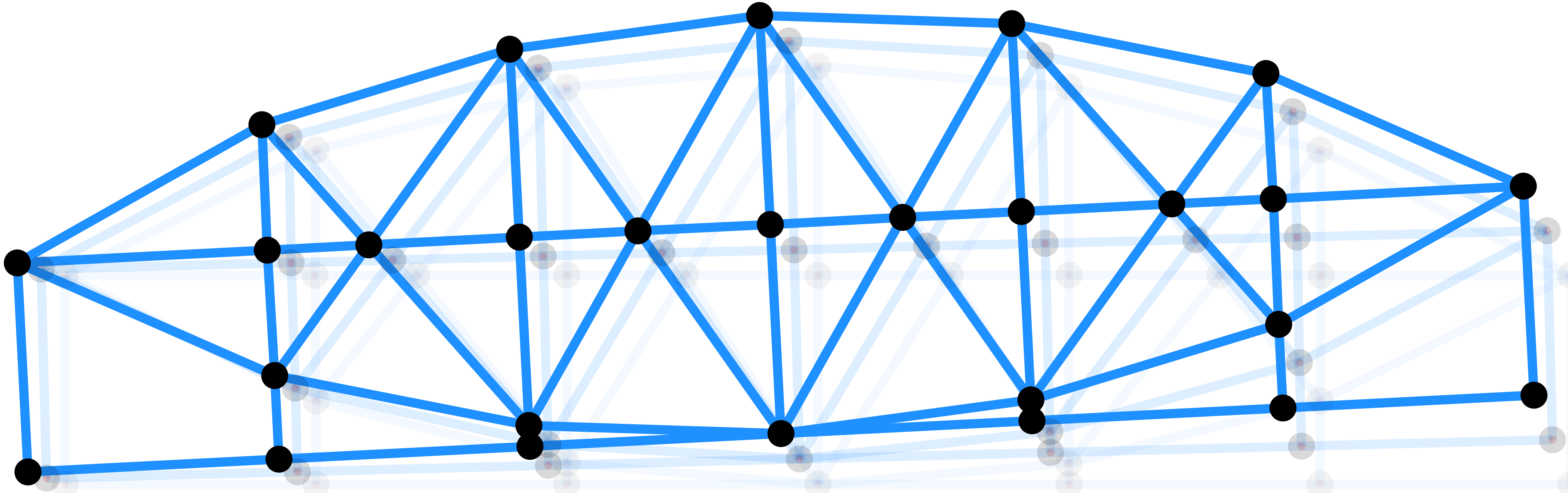
QR Algorithm Convergence for a Random 10x10 Matrix
pure_qr
Off-diagonal magnitude
10^{1}
10^{-1}
10^{-3}
10^{-5}
10^{-7}
0
50
100
150
200
250
300
350
Iteration


QR Algorithm Convergence for a Random 10x10 Matrix
pure_qr
full_qr
Off-diagonal magnitude
Iteration
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
0
50
100
150
200
250
300
350

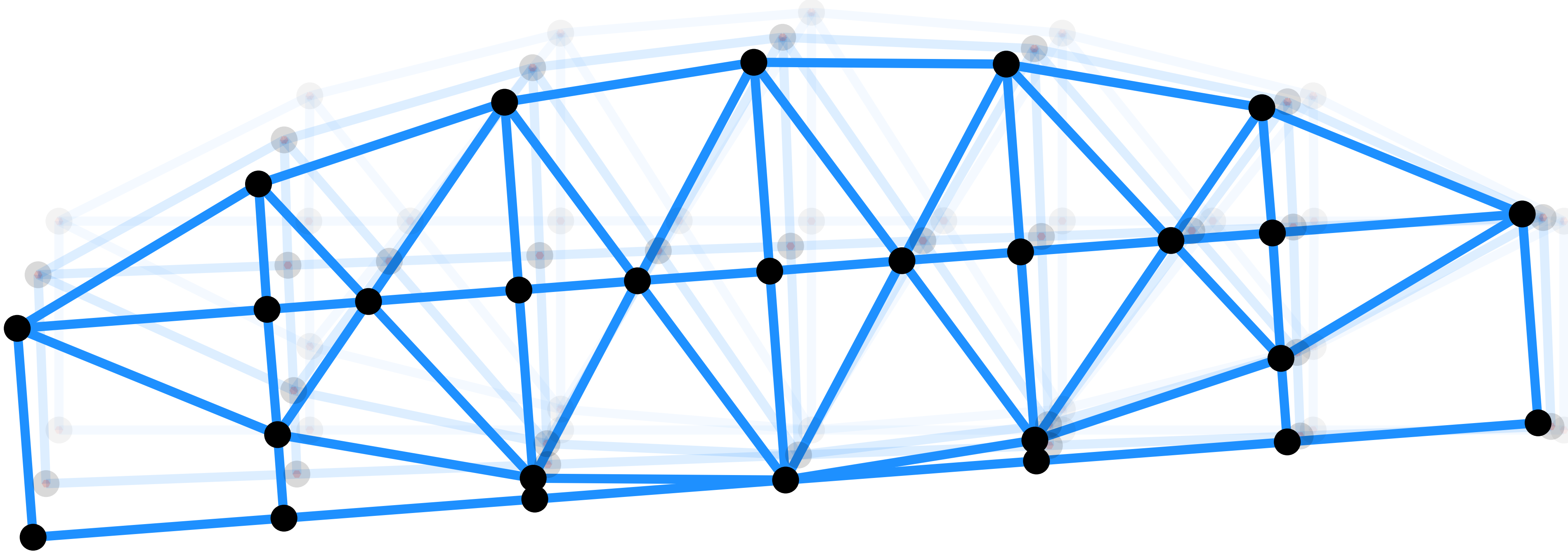
Problem 3: comment on convergence from full-qr and pure-qr.

↳ pure-qr seems to have linear convergence as it grows in a predictable manner while full-qr grows much faster & detects an error more suddenly as it converges cubically.

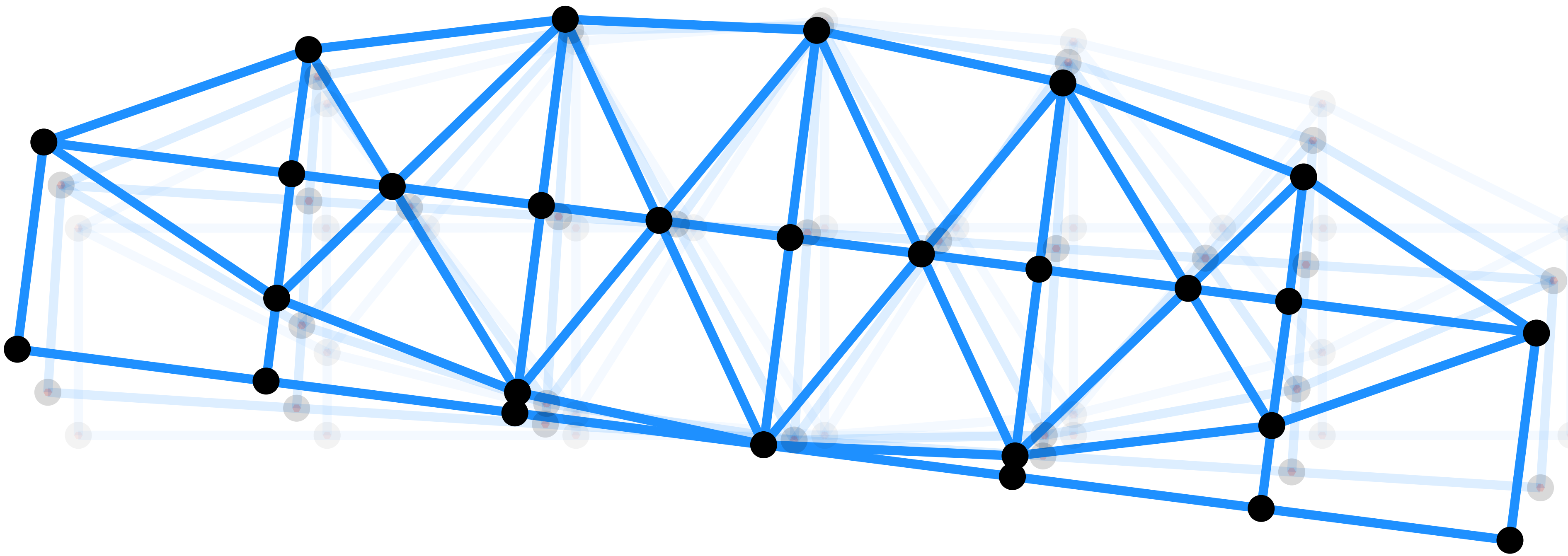
Mode 0, energy = 0.000e+00, frequency = 0.000



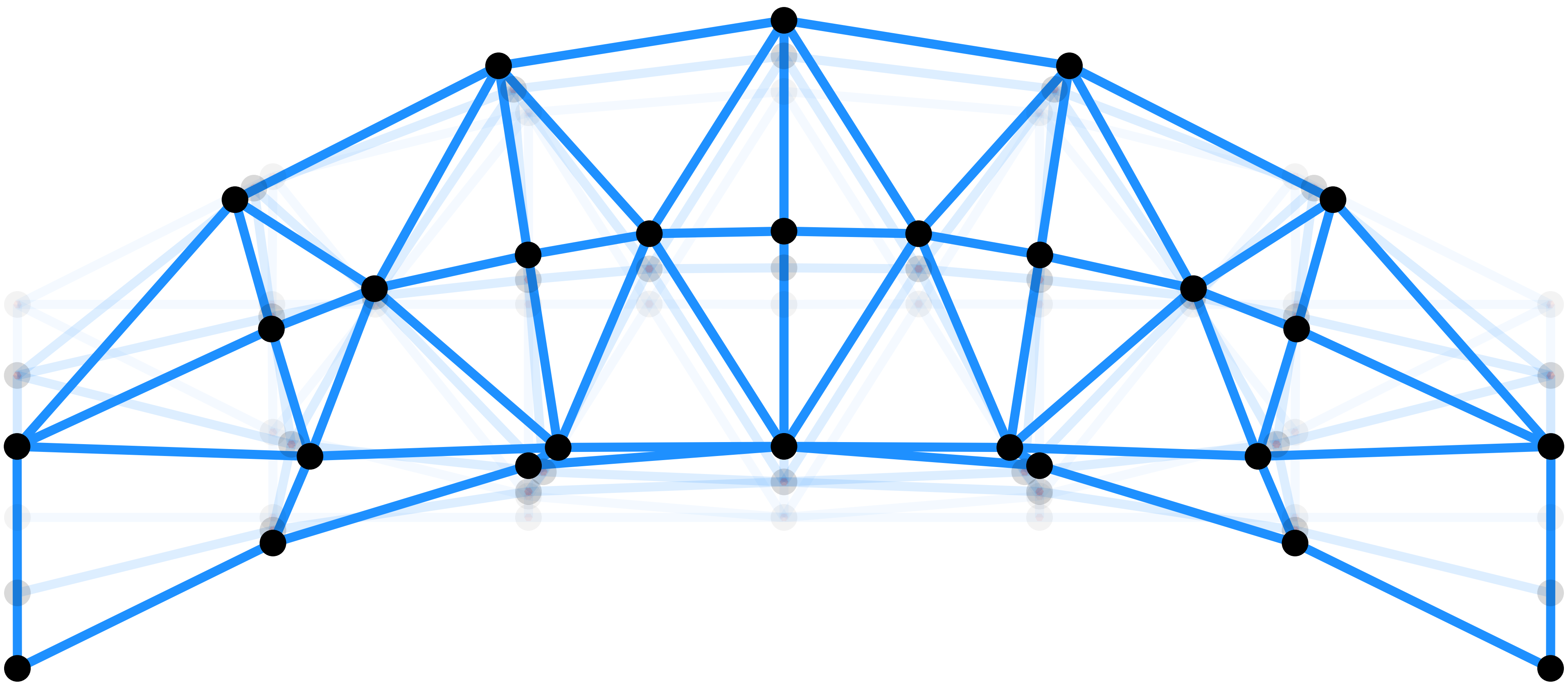
Mode 1, energy = 0.000e+00, frequency = 0.000



Mode 2, energy = 0.000e+00, frequency = 0.000



Mode 3, energy = 9.479e-01, frequency = 0.974



Problem 4: comment on what type of deformation each of these 4 modes corresponds to and what it means abt nullspace dimension of stiffness matrix K.

Mode 0 - Notice how the eigenvalues are smaller b/c there is no energy needed to apply a deformation. I believe this means this means there is no needed energy for the slight rotation + vibration. I think the nullspace dimension of stiffness matrix K is quite large as there is little to no change in Mode 0.

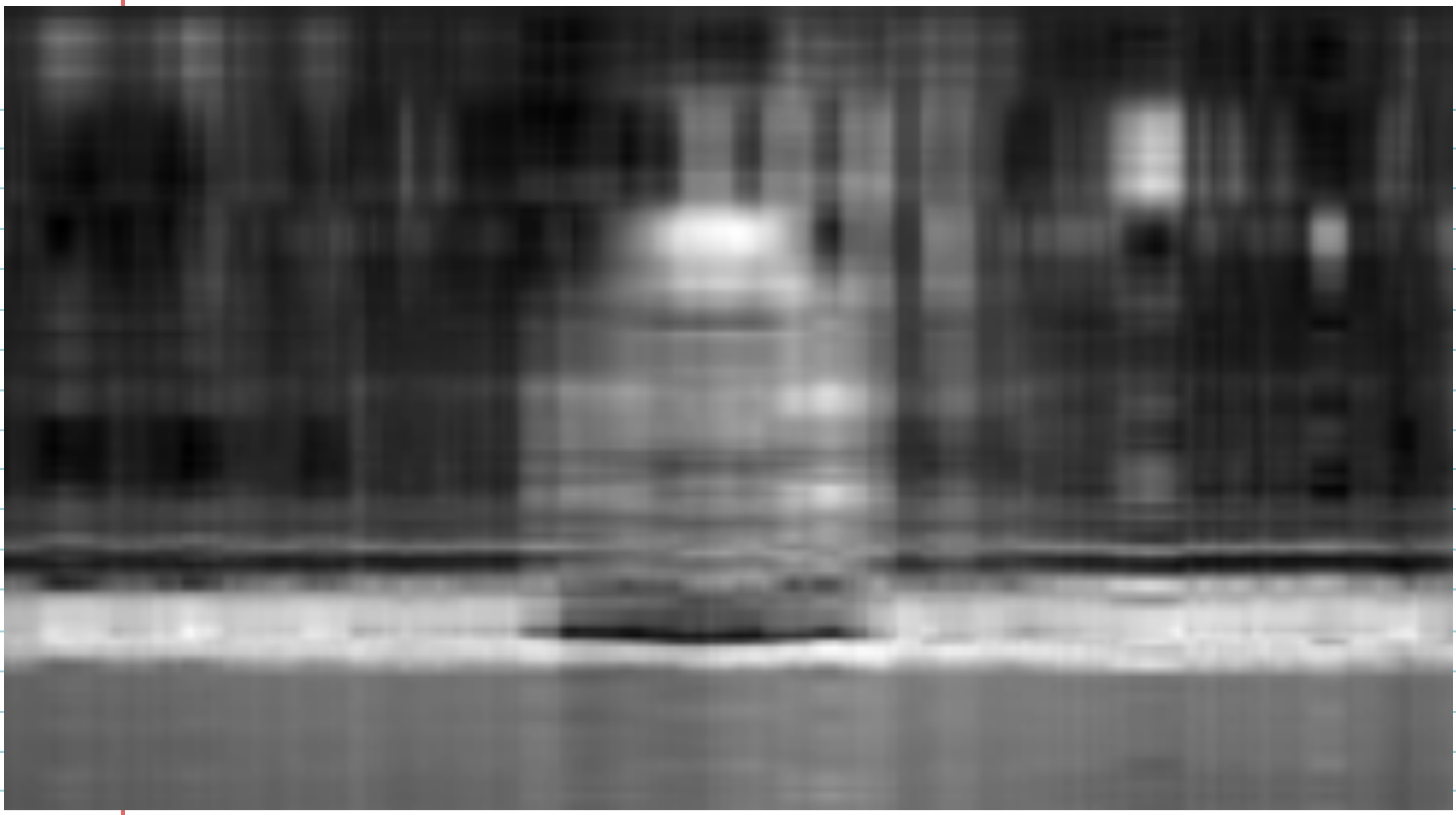
Mode 1 - The eigenvalues here are also smaller & closer to 0 as energy is not needed to apply the deformation. This also means the deformation is close to 0, and this means no energy is NEEDED to rotate. Also, notice how this image rotates much more compared to Mode 0.

Mode 2 - Lastly, there is also a deformation close to 0 as the energy is 0 here too but notice this time the bridge rotates in the opposite direction to that of Mode 1. I think since energy is still 0 it remains in a nullspace.

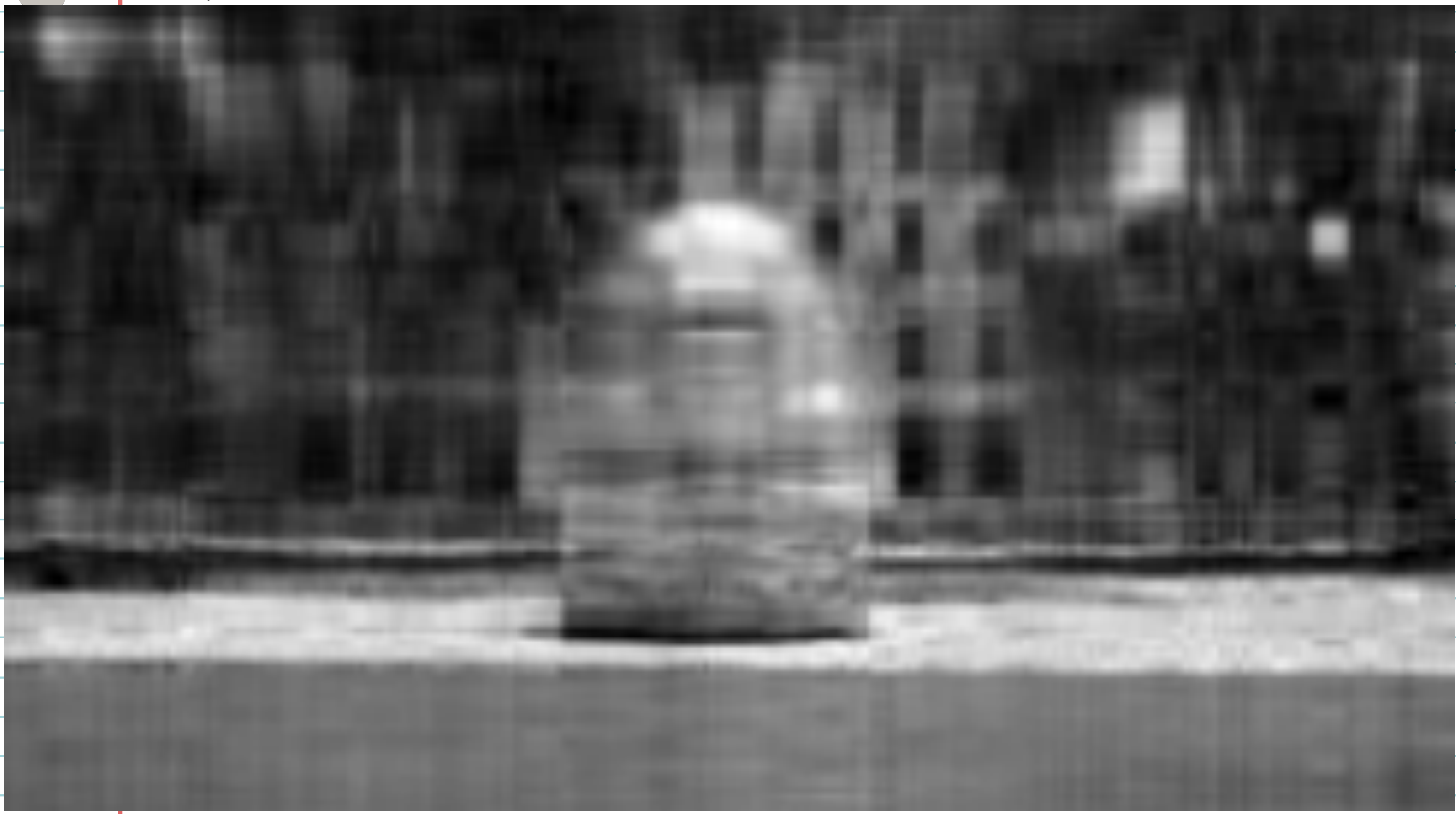
Mode 3 - Here, the eigenvalues are larger in my opinion → that's why the energy is 9.479×10^{-1} so the energy is greater, and from the Professor's hint in his discussion post this means the energy is larger and that results in the stretch. I think since energy is larger here the nullspace is small/nonexistent & does not impact the energy.

Note: I used Professor Dannetta's hint from disc post abt eigenvalues & energy

Rank 5:



Rank 10:



Rank 25:

Rank 50