ECS 130: HW 2:

Problem 1 (10 pts). Consider a Cholesky factorization of the matrix appearing in the normal equations for a least-squares problem involving matrix $A \in \mathbb{R}^{m \times n}$:

$$A^\top A = LL^\top$$

when A is full rank and $m \geq n$ (so that the least-squares solution is unique). 1) Show that the columns of the matrix $AL^{-\top}$ are orthonormal. In other words, this matrix is orthogonal apart from the technicality that orthogonal matrices are square by definition. ② Next, based on this fact, describe how to construct a reduced QR factorization of A using the Cholesky factorization of $A^\top A$.

Problem 1:

The problem asks us to consider a Cholesky factorization, involving matrix $A \in \mathbb{R}^{m \times n}$, with $A^TA = LL^T$. Given A is a full rank & $m \geq n$. & A^T is transpose (A). & L is lower triangular from Cholesky

① Show that columns of AL^{-T} are orthonormal

② Describe how to construct a reduced QR factorization of A using Cholesky factorization.

Step 1: Show columns of AL^{-T} are orthonormal => (orthogonal + normalized)

WTS: ① Each column has unit norm

② Columns are orthogonal to each other

We are given matrix $A \in \mathbb{R}^{m \times n}$ and $A^TA = LL^T$.

So, let's call $P = AL^{-T}$, where P has cols $\{p_1, \ldots, p_n\}$.

WTS: ① columns are orthogonal

② columns are normalized

So, WTS AL^{-T} are orthonormal:

$$(A^T L^{\cancel{-T}\,-1})^{\cancel{T}}(AL^{-T}) = L^{-1}A^T \cdot AL^{-T}$$

$$(AL^{-T})^T(AL^{-T}) = A^TL^{-1} \cdot AL^{-T} = L^{-1}LL^TL^{-T}$$ → we know $A^TA = LL^T$ so that's why we replace w/ LL^T

$$(AL^{-T})^T(AL^{-T}) = (AL^{-T})^T(AL^{-T}) = I$$

Step 2: construct a QR factorization using Cholesky of A^TA

We know $A^TA = LLT$ & L is lower triangular matrix in $\mathbb{R}^{m \times n}$

We can say $A = PR$

b/c P = orthnormal in $\mathbb{R}^{m \times n}$ & R = upper triangular matrix in $\mathbb{R}^{m \times n}$

$P = AL^{-T}$ by orthnormal columns

Because $A = PR$ and $P = AL^{-T}$ and $A^TA = LL^T$

$$A^TA = R^TP^TPR = R^TR$$

So, $A^TA = LL^T$ so $R^TR = L^T \Rightarrow R = L^T$.

This means $Q = AL^{-T}$ is orthonormal & $R = L^T$ is upper triangular.

Problem 2 (10 pts). Show that the stiffness matrix K defined below in Section 3 is positive *semi*definite provided that the spring constants k_e are all nonnegative. In other words, show that $\mathbf{u}^\top K\mathbf{u} \geq 0$ for all $\mathbf{u} \neq 0$. Finally, find a nonzero vector $\mathbf{u} \neq 0$ so that $\mathbf{u}^\top K\mathbf{u} = 0$. This means that K is singular (and not quite positive definite).

Problem 2:

WTS matrix K is positive semidefinite

we know:

① $u^T K u \geq 0$ for $u \neq 0$

Want to find a nonzero vector $u \neq 0$ so $u^T K u = 0$

We know:

$E_{elastic}(u) = \frac{1}{2} u^T K u$ & $E_{elastic}(u) = \sum_{e=(i,j)} \frac{k_e}{2} (\hat{a}_e \cdot (u_i - u_j))^2 = \frac{1}{2} u^T B^T D B u$

s.t $u \in \mathbb{R}^{2n}$

WTS: $u^T K u \geq 0$

We know $K = B^T D B$ so,

$$E_{elastic}(u) = \sum_{e=(i,j)} \frac{k_e}{2} (\hat{a}_e \cdot (u_i - u_j))^2 = \frac{1}{2} u^T \underbrace{B^T D}_{K} u$$

if $x = Bu$

$$u^T K u = x^T D v = \sum_{e=(i,j)}^{m} k_e(\hat{a}_e \cdot (x_i - x_j))$$

we know $k_e (x_i - x_j)^2$

Next, we need to find $v \neq 0$ but $v^T K u = 0$

$$\text{So,} \quad v^T D v = \sum_{e=1}^{m} k_e x_e^2 = 0$$

But this means $Bu = 0$, so K is singular b/c $u \neq 0$ still holds $u^T K u = 0$

Next, we find nonzero vector:

From handout:

$$u^T K u = x^T D v = \sum_{e=(i,j)}^{m} k_e(\hat{a}_e \cdot \underbrace{(v_i - v_j)})$$

$v_i - v_j$ is NOT always $1 - 1 = 0$
and b/c a factor is 0 then is nonzero v.

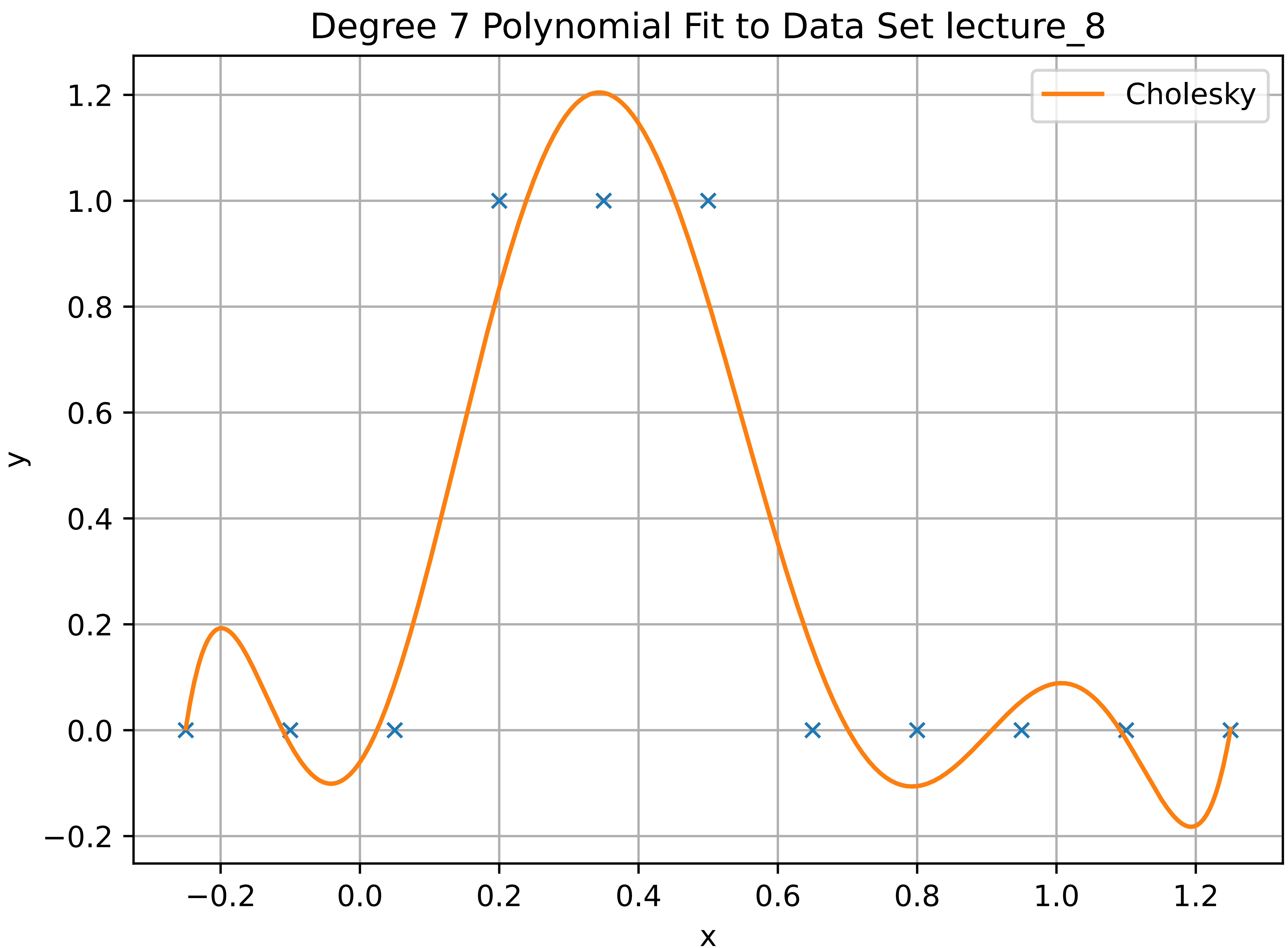
$[u_i - v_j]$ = to nonzero vector

PROBLEM 8:
(sc_env) asmaaslam@Asmas-MacBook-Air-2 ~ % python3 /users/asmaaslam/Desktop/hw2/curvefit.py 12 --example quadratic --method Cholesky
Data set quadratic:
  Cholesky: [0.05941186 0.04459575 0.07052678 0.06996184 0.06278471 0.05428775 0.04579023 0.03775874 0.03036452 0.02365081 0.0176015 0.01217364 0.00731376]; error = 0.2027869827169851

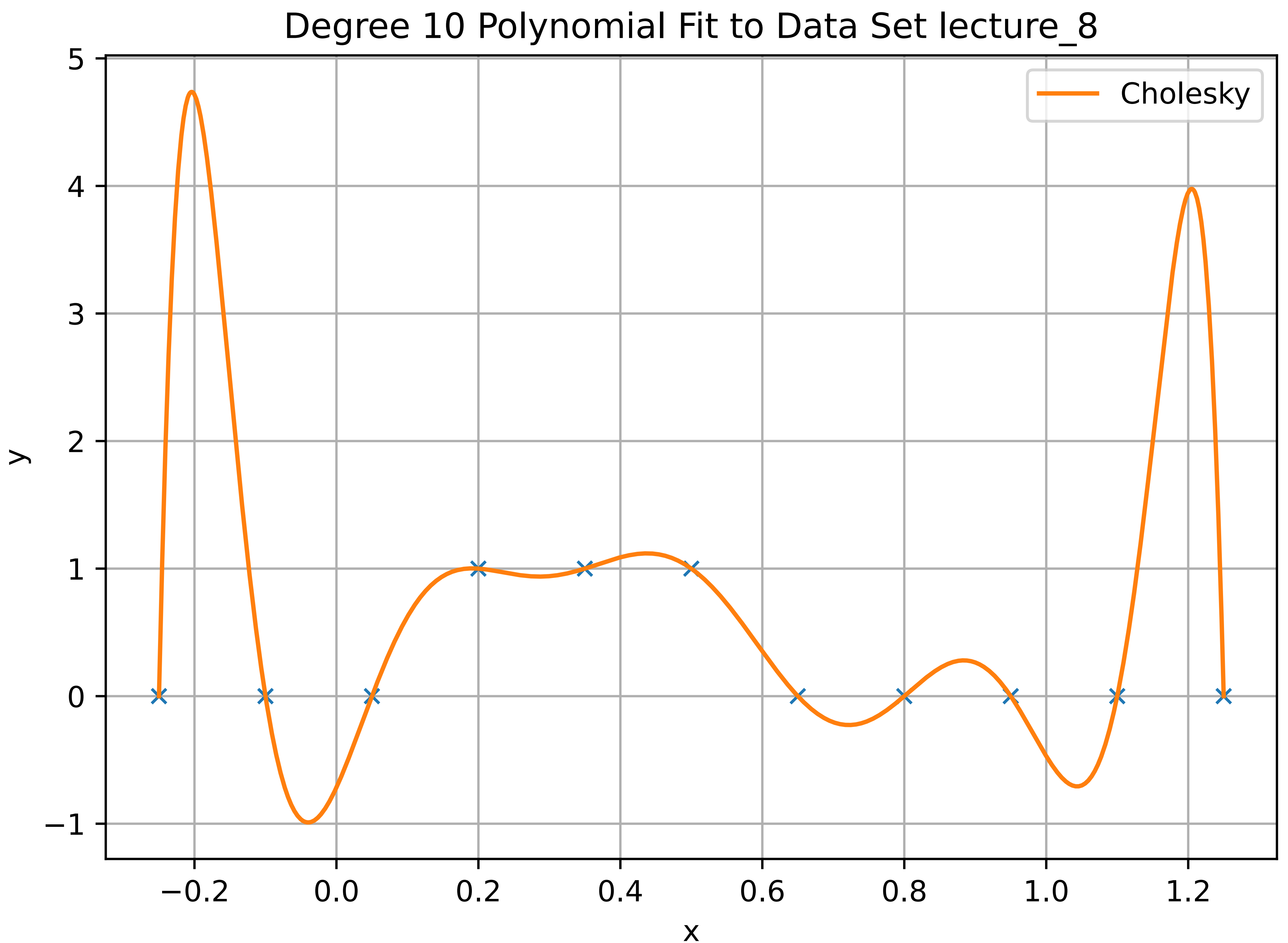
PROBLEM 10:
(sc_env) asmaaslam@Asmas-MacBook-Air-2 ~ % python3 /users/asmaaslam/Desktop/hw2/curvefit.py 12 --example quadratic --method ModifiedGramSchmidt
Data set quadratic:
  ModifiedGramSchmidt: [2.31004549e-11 -4.17581969e-08 5.00001844e-01 -3.12408907e-05 2.78214435e-04 -1.48705932e-03 5.10794503e-03 -1.16719234e-02 1.79367350e-02 -1.83212217e-02 1.19265247e-02 -4.47751413e-03 7.37738312e-04]; error = 2.3310502307746605e-10
(sc_env) asmaaslam@Asmas-MacBook-Air-2 ~ %

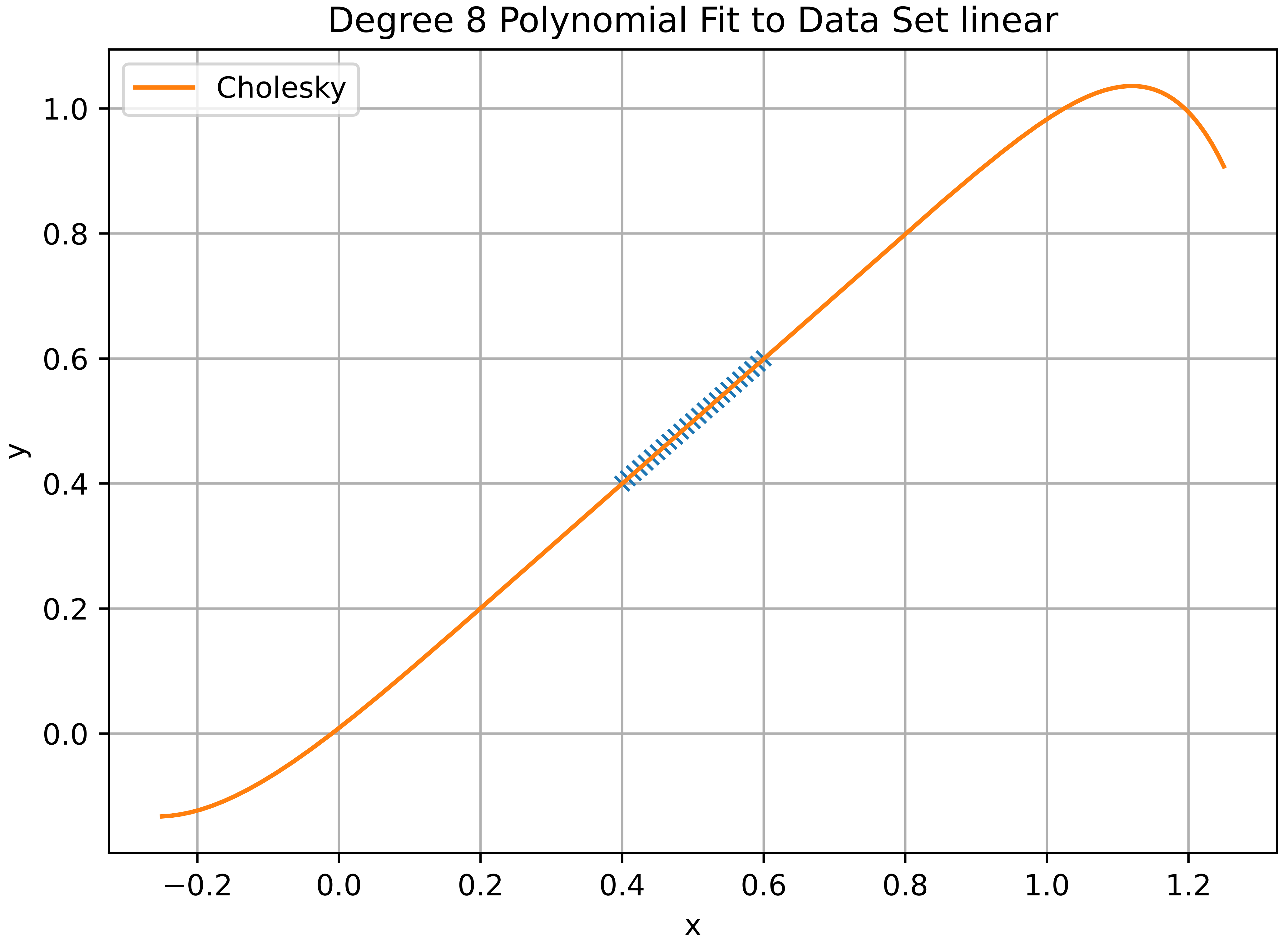
PROBLEM 13:
(sc_env) asmaaslam@Asmas-MacBook-Air-2 ~ % python3 /users/asmaaslam/Desktop/hw2/curvefit.py 12 --example quadratic --method Householder
Data set quadratic:
  Householder: [4.44089210e-17 -1.47801823e-16 5.00000000e-01 -3.90104281e-14 4.31276370e-13 -2.58955936e-12 9.41204195e-12 -2.17609186e-11 3.25792296e-11 -3.12972079e-11 1.84789471e-11 -6.03664917e-12 8.20427598e-13]; error = 2.6372244617052776e-16
(sc_env) asmaaslam@Asmas-MacBook-Air-2 ~ %

PROBLEM 16:
(sc_env) asmaaslam@Asmas-MacBook-Air-2 ~ % python3 /users/asmaaslam/Desktop/hw2/truss_simulation.py /users/asmaaslam/Desktop/hw2/data/bridge.pkl
number of nonzero entries 45.18518518518518

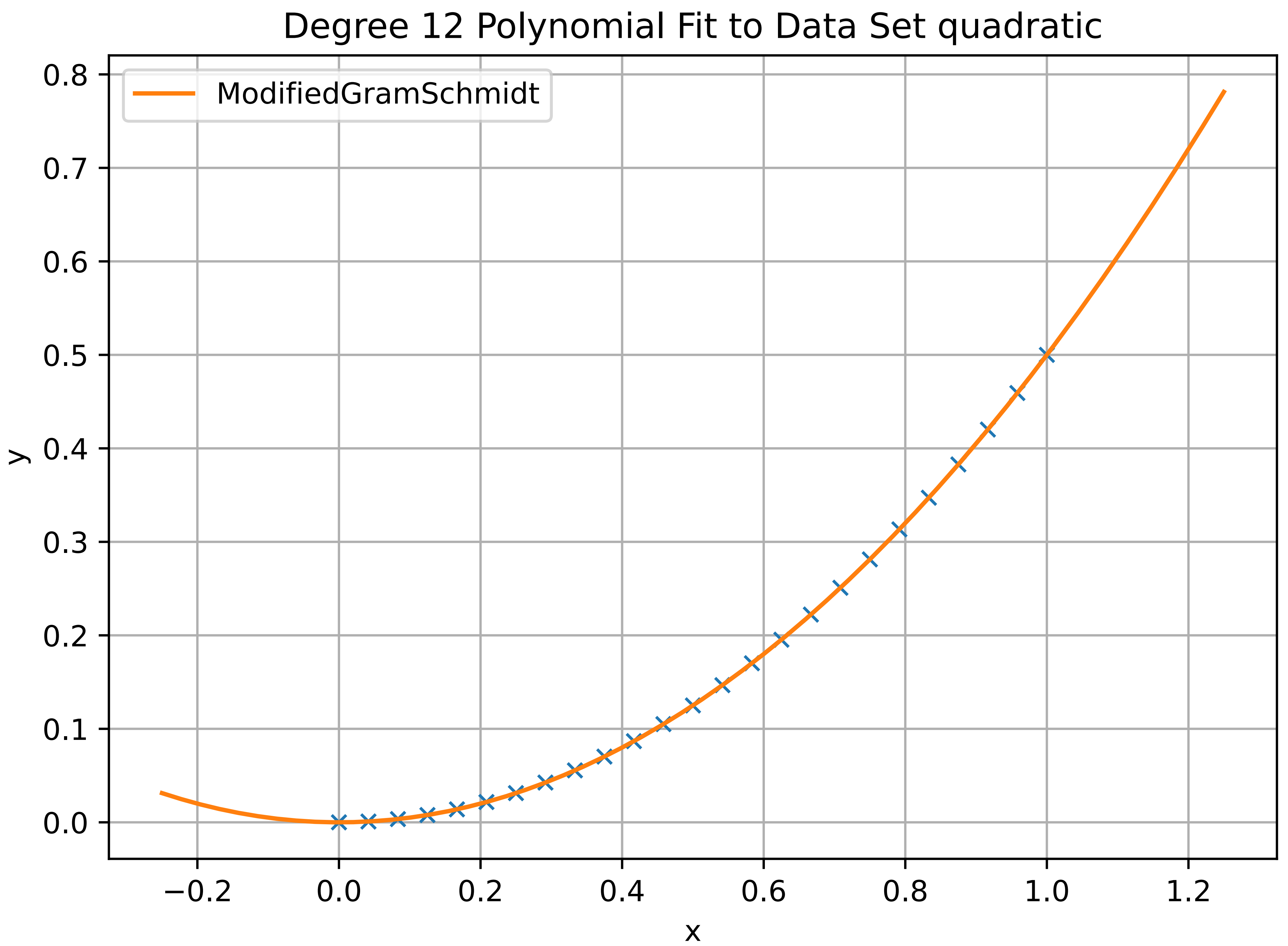


Degree 7 Polynomial Fit to Data Set lecture_8
Cholesky
y
x
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
1.2

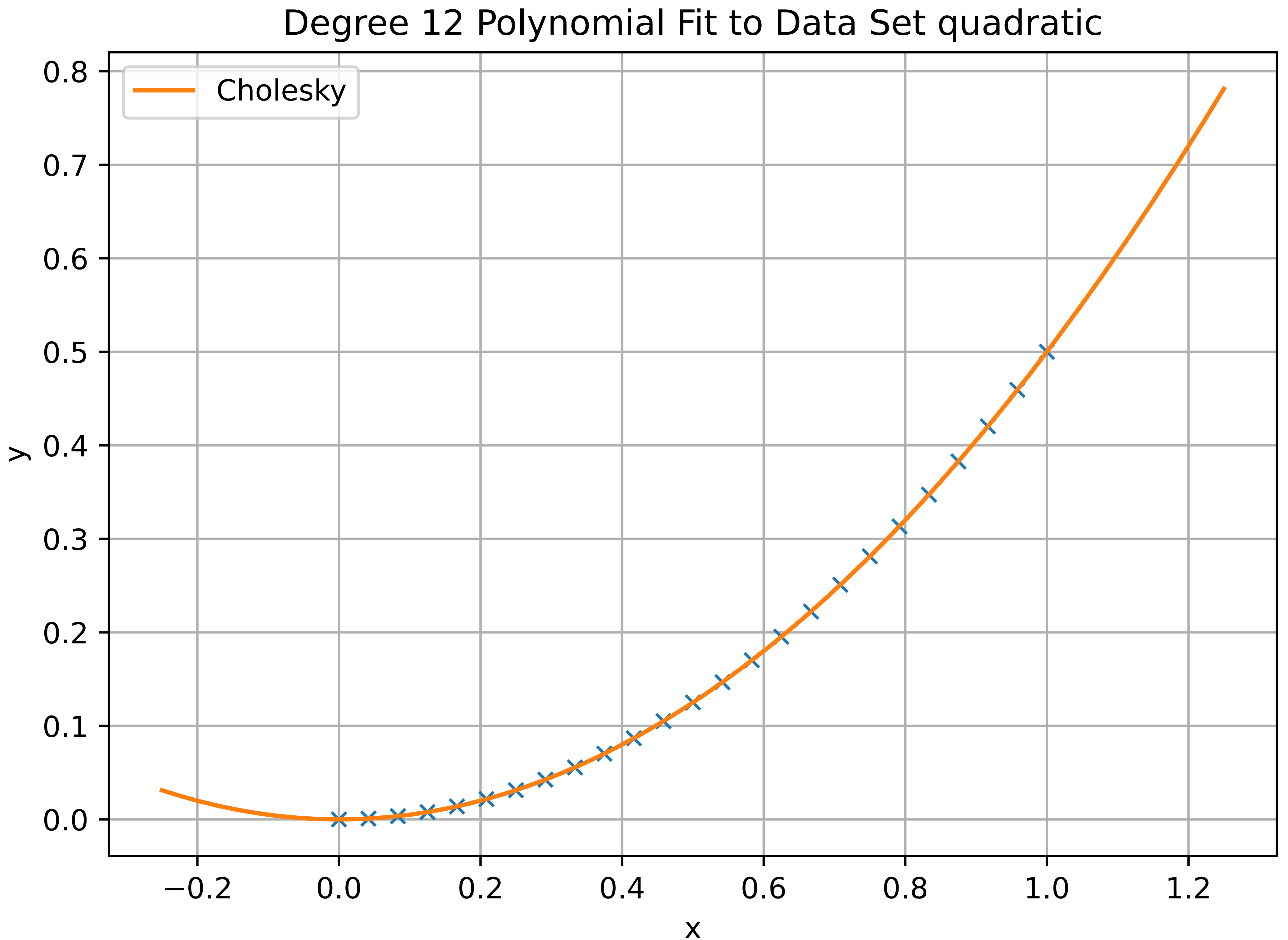

Degree 10 Polynomial Fit to Data Set lecture_8
Cholesky
y
x
5
4
3
2
1
0
−1
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
1.2



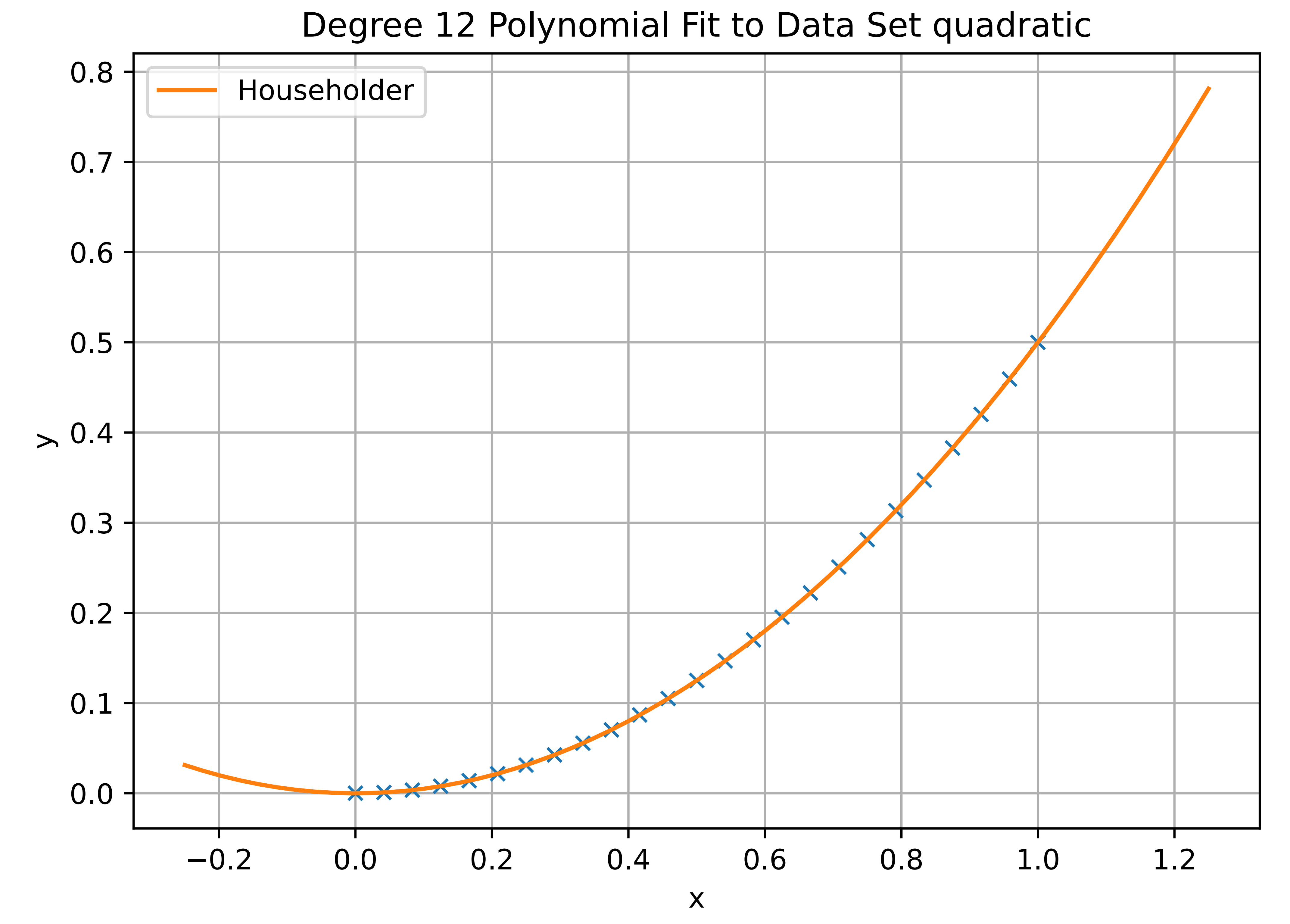
Degree 8 Polynomial Fit to Data Set linear
Cholesky
y
x
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
1.2

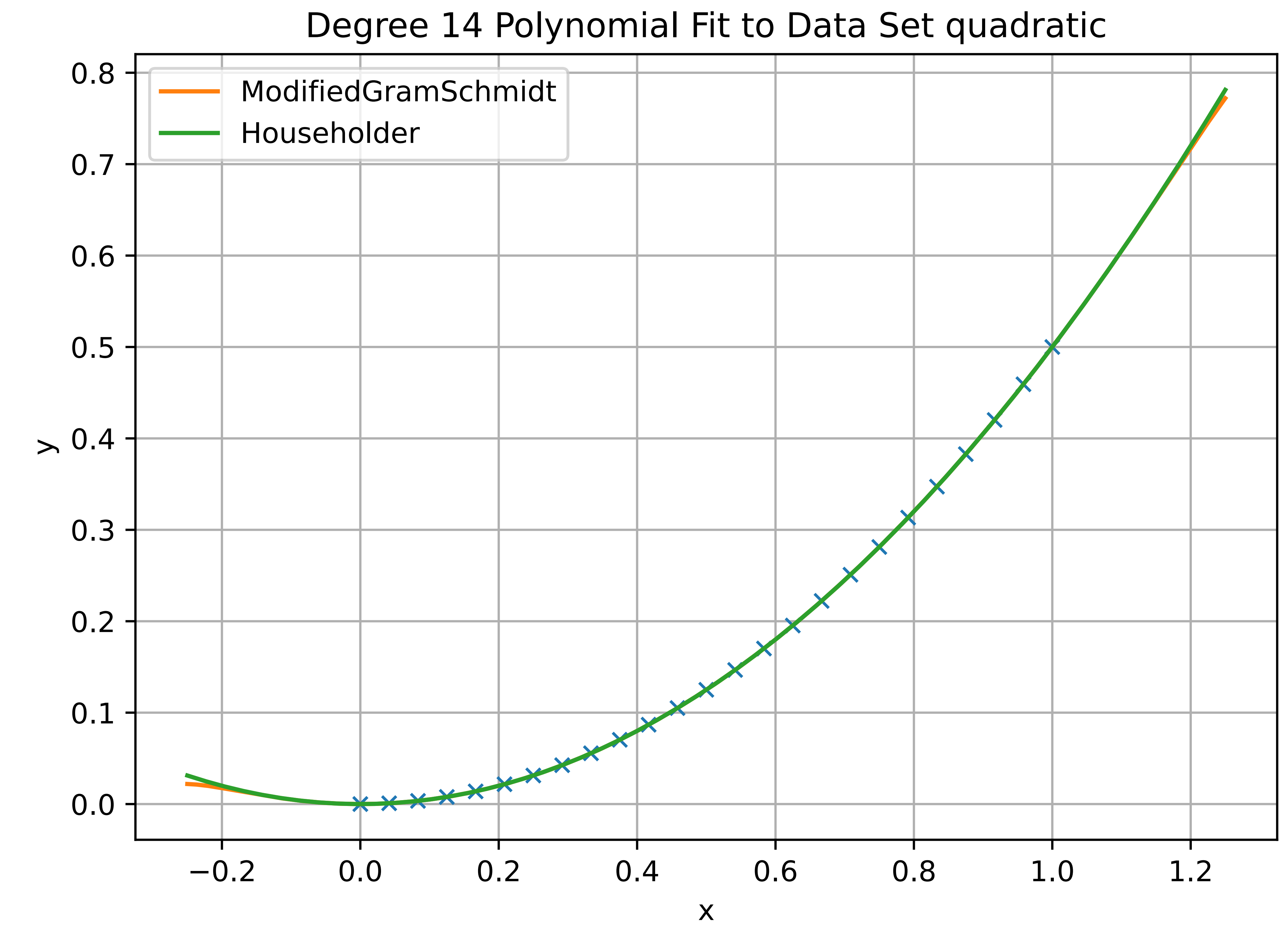


Degree 12 Polynomial Fit to Data Set quadratic
ModifiedGramSchmidt
y
x
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
1.2



Degree 12 Polynomial Fit to Data Set quadratic
Cholesky
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
y
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
1.2
x


Degree 12 Polynomial Fit to Data Set quadratic
Householder
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
y
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
1.2
x


Degree 14 Polynomial Fit to Data Set quadratic
ModifiedGramSchmidt
Householder
y
x
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
1.2