Multivariate Calculus

**Function of several variables:** suppose D is set of n-tuples of real number $(x_1, x_2, x_3, ---, x_n)$, a real valued function f on D is a rule that assigns a unique (single) real number $w = f(x_1, x_2, x_3, ---, x_n)$ to each element in D. The set D is the function's domain and w values taken on f is function's range. w is dependent and f is said to be function's of n independent variable.

14.1 Q#1 $f(x,y) = x^2 + xy^3$

1-4 i) $f(0,0)$. ii) $f(-1,1)$

ii) $f(-1,1) = (-1)^2 + (-1)(1)^3$
$= 1 - 1$
$= 0$

i) $f(0,0) = (0)^2 + (0)(0)^3$
$= 0$

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## Domain and Range

① Example: $f = \sqrt{y - x^2}$ $\qquad y \geq x^2$

Domain: $y - x^2 \geq 0$

Range: $(0, \infty)$

② $f = \frac{1}{xy}$

Domain: $xy \neq 0$

Range: $(-\infty, 0) \cup (0, \infty)$

③ $w = \sqrt{\ \ \ }$

④ $f = \sin xy$

Domain = entire space

Range = $(-1, 1)$

⑤ $f = \frac{1}{\ \ }$

# equations of basic graph

14.1 parabolla, circle, hyperbola, straight line, oval, x-mode.

13-16

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## Limits and continuity

**Limit**

A function $f(x,y)$ approaches to Limit "L" as $(x,y)$ tends to $(x_0,y_0)$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

where $(x,y) \to (x_0,y_0)$

Then the function $f(x,y) \to f(x_0,y_0)$

$\frac{\infty}{\infty}, \frac{0}{0}$

$\infty$, complex

## Ex # 14.2

In single variable when undefined form comes we use Lapital rule but in two variable we can't be one remains same, so, we use other methods

Q # 3

$$\lim_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1}$$

$$= \sqrt{(3)^2+(4)^2-1}$$

$$= \sqrt{9+16-1}$$

$$= \sqrt{25-1}$$

$$= \sqrt{24}$$

Q # 7

$$\lim_{(x,y)\to(0,\ln 2)} e^{x-y}$$

$$\lim = e^{0-\ln 2}$$

$$= e^{-\ln 2}$$

$$= e^{\ln 2^{-1}}$$

$$= 2^{-1}$$

$$= 1$$

## Q# Continous Function

A function $f(x,y)$ is continous at the point $(x_0, y_0)$ if

(i) $f$ is defined at $(x_0, y_0)$

(ii) $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists [L.H.S limit equal to R.H.S]

(iii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$ (if $x,y$ approach to that function then you point/ans also approach that function)

A function is continous if and only if it continous at every point of its domain.

## Q# 41 to 48

→ In limit we see function defined or not, the limits' value when put in function then answer should be same (continuity) $(L_1 = L_2 = L_3 = L_4)$

Q#41 (only for origin), check all 4).

. To check at origin, we have to follow four paths.

(i) along $x$-axis, $y=0 \longrightarrow L_1$

(ii) along $y$-axis, $x=0 \longrightarrow L_2$

(iii) along $y = mx \longrightarrow L_3$

(iv) along $y = mx^2$ or $y = kx^2 \Rightarrow L_4$

Q#41 $f(x,y) = \dfrac{-x}{\sqrt{x^2+y^2}} \Rightarrow \dfrac{0}{0} \quad (x,y)\to(0,0)$

(i) Along $x$-axis $y=0$

$$\lim_{(x,y)\to(0,0)} f(x,0) = \lim_{(x,y)\to(0,0)} \frac{-x}{\sqrt{x^2+0}}$$

$$\lim_{x\to 0} \frac{-x}{\sqrt{x^2}} = \lim_{x\to 0} \frac{-x}{x} = -1$$

$L_1 = -1$ —— (i)

(32) (a) $f(x,y) = \dfrac{x+y}{x-y}$

when $x=y$ Then function will be undefined otherwise defined.

The given function is undefined at $x=y$ and defined for all other point of $(x,y)$ and its limiting value also exists. So, $f(x,y) = \dfrac{x+y}{x-y}$ is continous for all values of $(x,y)$ except $x=y$.

(b) $f(x,y) = \dfrac{y}{x^2+1}$

The given function is defined for all value of $x$ so ~~sin~~ $\dfrac{y}{x^2+1}$ is defined for all real number and its limiting value also exists. So, $f(x,y) = \dfrac{y}{x^2+1}$ is continous for all values of $x$.

## Partial Derivatives

single variable

The partial derivatives of $f(x,y)$ with respect to "$x$" at point $(x_0, y_0)$ is.

$$f_x = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

→ by definition then go for these equation otherwise below.

The partial derivatives of $f(x,y)$ w.r.t. "$y$" at point $(x_0, y_0)$ is:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{k \to 0} \frac{f(x_0, y_0+k) - f(x_0, y_0)}{k}$$

Second ~~Derivat~~ order partial derivation:

double variable dependent and independent

The notation are:

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} \quad , \quad \frac{\partial^2 f}{\partial y^2} = f_{yy} \quad , \quad \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

and $\dfrac{\partial^2 f}{\partial y \partial x} = f_{yx}$

$f_{yx}$ first do "$y$" derivative then $x$

# Chain Rule

## chain Rule of two variables:

one variable chain rule parametric $(x+y)$

The chain rule formula for a differentiable function $w = f(x, y)$ when $x = x(t)$ and $y = y(t)$ are both differentiable functions of $t$.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

For three variable

$w = (x, y, z)$ $\quad x = x(t),\ y = y(t),\ z = z(t)$

$$\frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dx}$$

Example #1

$w = xy$ w.r.t $x = \cos t$, $y = \sin t$ and $t = \frac{\pi}{2}$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \quad \text{—} \ (i)$$

$\therefore w = xy$

$$\frac{\partial w}{\partial x} = (1)y = y$$

$$\frac{\partial w}{\partial y} = x(1) = x$$

$$\frac{dx}{dt} = -\sin t \quad , \quad \frac{dy}{dt} = \cos t$$

eq (i) $\Rightarrow \frac{\partial w}{\partial t} = y(-\sin t) + x(\cos t)$

$$\frac{\partial w}{\partial t} = -y\sin t + x\cos t \qquad t = \frac{\pi}{2}$$

$= \cancel{-y\sin\frac{\pi}{2} + x\cos\frac{\pi}{2}}$

$= \cancel{\#}$

# Gradient vector and Directional Derivatives:

## Gradiant vector:

The gradiant vector of $f(x,y)$ at a point $P_o(x_o, y_o)$ is the $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$ obtained by evaluating the partical ~~deritives~~ derivatives of $f$, $P_o$.

Rules of Gradiants: $\boxed{\nabla = \text{Gradiant}}$

i) $\nabla(f+g) = \nabla f + \nabla g$ (sum rule)

ii) $\nabla(f-g) = \nabla f - \nabla g$ (difference rule)

iii) $\nabla(kf) = k\nabla f$ $k$=constant

iv) $\nabla(f \cdot g) = f\nabla g + g\nabla f$ (Product Rule)

v) $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$ (Quotient Rule)

## Direct Derivatives:

If $f(x,y)$ is differentiable in an open region, containing $P_o(x_o, y_o)$ Then

$$\left.\frac{df}{ds}\right|_{u, P_o} = (\nabla f)_{P_o} \cdot \hat{u}$$ → dot product $\hat{u}$ = unit vector.

The Dot product of the gradiant $\nabla$ at $P_o$ and $\hat{u}$.

Example

$$\int_1^3\int_2^4 (40 - 2xy)\, dy\, dx$$

$$\int_2^4 40\, dy - \int_2^4 2xy\, dy$$

$$= \left|40y\right|_2^4 - 2x \left|\frac{y^2}{2}\right|_2^4$$

$$= 40(4) - 40(2) - 2x\left[\left(\frac{4^2}{2}\right) - \left(\frac{2^2}{2}\right)\right]$$

$$= 160 - 80 - 2x\left(\frac{16}{2} - \frac{4}{2}\right)$$

$$= 80 - x(12)$$

$$= 80 - 12x$$

$$\int_1^3 (80 - 12x)\, dx$$

$$= \left|80x\right|_1^3 - \left|\frac{\overset{6}{\cancel{12}}x^2}{\cancel{2}}\right|_1^3$$

$$= [80(3) - 80(1)] - [6(3)^2 - 6(1)^2]$$

$$= (240 - 80) - (54 - 6)$$

$$= 160 - 48$$

$$= 112$$

pollar coordinates
3-D → $r, \theta, z$
2-D → $r, \theta$

cartesian coordinate
2D → $x, y$
3-D → $x, y, z$

## Multiple Integrations

Let $R$ be the rectangle defined by the inequilities $c \le y \le d$ ~~If~~ and $a \le x \le b$. If $f(x,y)$ is continous on this rectangle. Then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Example # 3 $\quad z = 4 - x - y$ $\qquad \frac{1}{2}xy \quad (x^2+y^2+1)^{-1/2}$

Ex # 14.1 $\quad$ 14, $\int_0^1 \int_0^2 (x+3)\,dy\,dx$ $\qquad 2y$

Newbook $\qquad (2y^2 \cdot 1)$

Example # 3 $\quad z = 4 - x - y$ $\qquad \left(\frac{u^3}{3}\right)$

$$R = [\underset{a}{0}, \underset{b}{1}] \times [\underset{c}{0}, \underset{d}{2}]$$

$$= \int_0^2 \int_0^1 (4 - x - y)\,dx\,dy$$

$$= \int_0^2 \left| 4x - \frac{x^2}{2} - yx \right|_0^1 dy$$

$$= \int_0^2 \left(4[1-0] - \left(\frac{1}{2} - \frac{0}{2}\right) - y(1-0)\right) dy$$

$$= \int_0^2 \left(4 - \frac{1}{2} - y\right) dy$$

$$= \left| 4y - \frac{1}{2}y - \frac{y^2}{2} \right|_0^2$$

$$= 4(2-0) - \frac{1}{2}(2-0) - \left(\frac{2^2}{2} - \frac{0}{2}\right)$$

$$= 4(2) - \frac{1}{2}(2) - \left(\frac{4^2}{2} - 0\right)$$

$$= 8 - 1 - 2$$

$$= 5$$

Q#1 $\displaystyle\int_0^1\int_0^2 (x+3)\,dy\,dx$

$$= \int_0^1 \int_0^2 (xy+3y)\Big|_0^2 dx$$

$$= \int_0^1 (x(2-0)+3(2-0))\,dx = \int_0^1 (2x+6)\,dx$$

$$= \left| \cancel{x}\,\frac{2x^2}{2} + 6x \right|_0^1 = x^2+6x\Big|_0^1 = (1^2-0^2)+6(1-0) = 1+6 = 7$$

Ex #14.2 1-12 and 13-16 #(8-12+16) Assignment

Double Integrals in Polar coordinates

If $R$ is a simple region whose boundaries are the ray $\theta=\alpha$ and $\theta=\beta$ and the curves $r=r(\theta_1)$ and $r=r_2(\theta_2)$ and if $f(r,\theta)$ is continous on $R$. Then

$$\iint_R f(r,\theta)\,dA = \int_\alpha^\beta \int_{r_1(\theta_1)}^{r_2(\theta_2)} f(r,\theta)\, r\,dr\,d\theta$$

$dx\,dy = r\,dr\,d\theta$ $\quad x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$

New book

Ex #14.3

$$\int_0^{\pi/2}\int_0^{\sin\theta} r\cos\theta\,dr\,d\theta$$

$$\int_0^{\pi/2} \left| \frac{r^2}{2}\cos\theta \right|_0^{\sin\theta} d\theta$$

$$\int_0^{\pi/2} \frac{1}{2}\left| \sin^2\theta\cos\theta \right| d\theta$$

$$\frac{1}{2}\left| \frac{\sin^3\theta}{3} \right|_0^{\pi/2}$$

$$= \frac{1}{2}\cdot\frac{\sin^3\pi/3}{3}$$

$$= \frac{1}{6}\left(\sin\pi/2\right)^3 - (0)$$

$$= \frac{1}{6}(1)$$

$$= \frac{1}{6}$$

$\frac{90}{\cancel{180}}$
$\cancel{2}$