The research problem:

Consider a community of N interacting species. If each species has a population-dependent growth rate then we can describe the population dynamics using a system of differential equations of the form:

$$\frac{dN_i}{dt} = F_i(N_1, N_2, ..., N_N). \qquad i \; = 1, 2, ..., N \tag{1}$$

The aim of this research is to show that the analysis of this system using an approximate Jacobian matrix has the same result as analysis using an interaction or encounter matrix. We will do this for the specific example of the Lotka-Volterra model with two interacting species, described by the dynamical system:

$$\frac{dN_1}{dt} = r_1 N_1 (1 - \sum_{j=1}^{2} \alpha_{1,j} N_j), \tag{2}$$

$$\frac{dN_2}{dt} = r_2 N_2 (1 - \sum_{j=1}^{2} \alpha_{2,j} N_j). \tag{3}$$

To find the fixed points of the system we compute $\frac{dN_1}{dt} = 0$ and $\frac{dN_2}{dt} = 0$ using Eq(2) and Eq(3), from which we get:

$$r_1 N_1 (1 - (N_1 + \alpha_{1,2} N_2) = 0,$$
$$r_2 N_2 (1 - (\alpha_{2,1} N_1 + N_2) = 0.$$

The solution of this system gives four fixed points, at $(0,0)$, $(0,1)$, $(1,0)$ and $(\frac{\alpha_{1,2}-1}{\alpha_{2,1}\alpha_{1,2}-1}, \frac{\alpha_{2,1}-1}{\alpha_{1,2}\alpha_{2,1}-1})$ (where $\alpha_{i,i} = 1$). We will use various matrix formulations in order to analyse the dynamics of the system. We will compute eigenvalues and use the leading eigenvalue of each matrix to determine the dynamics of the system.

Jacobian matrix:

A Jacobian matrix is a matrix of first-order partial derivatives of the rate of change of each species in the system with respect to each other species.

$$\frac{\partial(\frac{dN_i}{dt})}{\partial N_j} = \frac{\partial F_i(N_k)}{\partial N_j} = J, \tag{4}$$

and is given by:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial N_1} & \frac{\partial f_1}{\partial N_2} \\ \frac{\partial f_2}{\partial N_1} & \frac{\partial f_2}{\partial N_2} \end{bmatrix}.$$

Interaction strength matrix:

Assuming that the species are completely interacting with themselves (i.e $\alpha_{i,i} = 1$), we have that:

$$\frac{\partial(\frac{1}{N_i}\frac{dN_i}{dt})}{\partial N_j} = [\alpha_{i,j}]. \tag{5}$$

The interaction matrix has the form :

$$\alpha = \begin{bmatrix} 1 & \alpha_{1,2} \\ \alpha_{2,1} & 1 \end{bmatrix}.$$

Encounter matrix:

The encounter matrix form is:

$$E = [N_i N_j]. \tag{6}$$

The matrix form for two species competing is given by:

$$E = \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_2 N_1 & N_2^2 \end{bmatrix}.$$

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \hdotsfor{5} \\ x_{d1} & x_{d2} & x_{d3} & \dots & x_{dn} \end{bmatrix}$$