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# Numerical Analysis I

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# Chapter 2

# Nonlinear Equations

## 2.1 INTRODUCTION

We have seen that expression of the form

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$$

where $a'$s are constant $(a_0 \neq 0)$ and $n$ is a positive integer, is called a polynomial in $x$ of degree $n$, and the equation $f(x) = 0$ is called a linear(or algebraic) equation of degree $n$. If $f(x)$ contains some other functions like exponential, trigonometric, logarithmic etc., then $f(x) = 0$ is called a non-linear(or transcendental) equation. For example,

$$x^3 - 3x + 6 = 0,\ x^5 - 7x^4 + 3x^2 + 36x - 7 = 0$$

are algebraic equations of third and fifth degree, whereas $xe^x - 2 = 0, x\log_{10}^x = 1.2, x^2 - 3\cos x + 1 = 0$ etc., are transcendental equations. In both the cases, if the coefficients are pure numbers, they are called numerical equations.

There are two types of methods available to find the roots of algebraic and transcendental equations of the form $f(x) = 0$.

1. **Direct Methods:** Direct methods give the exact value of the roots in a finite number of steps. We assume here that there are no round off errors. Direct methods determine all the roots at the same time.
   For example, the roots of the quadratic equation $ax^2 + bx + c = 0, a \neq 0$, can be obtained using the method
   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
2. **Indirect or Iterative Methods:** Indirect or iterative methods are based on the concept of successive approximations. The general procedure is to start with one or more initial approximation to the root and obtain a sequence of iterates $(x_k)$ which in the limit converges to the actual or true solution to the root. Indirect or iterative methods determine one or two roots at a time.

The indirect or iterative methods are further divided into two categories: bracketing and open methods. The bracketing methods require the limits between which the root lies, whereas the open methods require the initial estimation of the solution. Bisection and False position methods are two known examples of the bracketing methods. Among the open methods, the Newton-Raphson and the method of successive approximation are most commonly used. The most popular method for solving a non-linear equation is the Newton-Raphson method and this method has a high rate of convergence to a solution.

## 2.2 ORDER (OR RATE) OF CONVERGENCE OF ITERATIVE METHODS

Convergence of an iterative method is judged by the order at which the error between successive approximations to the root decreases.

The order of convergence of an iterative method is said to be $k$th order convergent if $k$ is the largest positive real number such that

$$\lim_{i\to\infty}\left|\frac{e_{i+1}}{e_i^k}\right| \le A$$

Where $A$, is a non-zero finite number called asymptotic error constant and it depends on derivative of $f(x)$ at an approximate root $x$. $e_i$ and $e_{i+1}$ are the errors in successive approximation.

In other words, the error in any step is proportional to the $k$th power of the error in the previous step. Physically, the $k$th order convergence means that in each iteration, the number of significant digits in each approximation increases $k$ times.

## 2.3 Locating Roots

**Theorem 2.1.** *If $f(x)$ is continuous on some interval $[a, b]$ and $f(a)f(b) < 0$, then the equation $f(x) = 0$ has at least one real root in the interval $(a, b)$.*

## 2.4 BISECTION METHOD

This is one of the simplest iterative method and is strongly based on the property of intervals. To find a root using this method, let the function $f(x)$ be continuous between $a$ and $b$. For definiteness, let $f(a)$ be negative and $f(b)$ be positive. Then there is a root of $f(x) = 0$, lying between $a$ and $b$. Let the first approximation to be $x_1 = \frac{1}{2}(a + b)$.

Now of $f(x_1) = 0$ then $x_1$ is a root of $f(x) = 0$. Otherwise, the root will lie between $a$ and $x_1$ or $x_1$ and $b$ depending upon whether $f(x_1)$ is positive or negative.

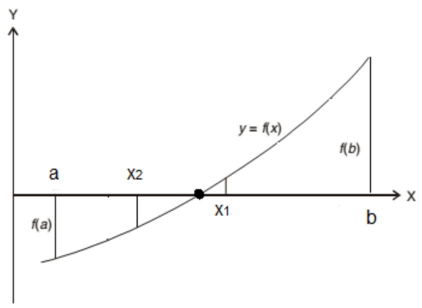


**Figure 2.1:** Solution of $f(x) = 0$ between $x = a$ and $x = b$

Then, we bisect the interval as before and continue the process until the root is found to the desired accuracy. In the above figure, $f(x_1)$ is positive; so that the root lies in between $a$ and $x_1$. The second approximation to the root is $x_2 = \frac{1}{2}(a + x_1)$. If $f(x_2)$ is negative as shown in the figure then the root lies in between $x_2$ and $x_1$. The third approximation to the root is $x_3 = \frac{1}{2}(x_2 + x_1)$, and so on.

### 2.4.1 Procedure for the Bisection Method

**Step 1:** Choose two real numbers $a$ and $b$ such that $f(a)f(b) < 0$.

**Step 2:** Evaluate $x_i = \frac{1}{2}(a+b)$ and also evaluate $f(x_i)$.

**Step 3:** **i)** If $f(a)f(x_i) < 0$, the root lies in the interval $(a, x_i)$. Therefore, set $b = x_i$ and return to step 2.

**ii)** If $f(x_i)f(b) < 0$ the root lies in the interval $(x_i, b)$. Therefore, set $a = x_i$ and return to step 2.

**iii)** If $f(a)f(x_i) = 0$, the root equals to $x_i$; terminate the computation.

**Step 4:** Stop evaluation when the difference of two successive values of $x_i$ obtained from step 2, is numerically less than the prescribed accuracy.

### 2.4.2 Order of Convergence of Bisection Method

In Bisection Method, the original interval is divided into half interval in each iteration. If we take mid points of successive intervals to be the approximations of the root, one half of the current interval is the upper bound to the error.

In Bisection Method,

$$e_{i+1} = 0.5e_i \text{ or } \frac{e_{i+1}}{e_i} = 0.5 \tag{2.1}$$

where $e_i$ and $e_{i+1}$ are the errors in $i^{th}$ and $(i+1)^{th}$ iterations respectively.

Comparing (2.1) with

$$\lim_{i\to\infty} \left|\frac{e_{i+1}}{e_i^k}\right| \leq A$$

We get $k = 1$ and $A = 0.5$. Thus the Bisection Method is first order convergent or linearly convergent.

**Example 2.1.** *Find a root of the equation $x^3 - 3x + 1 = 0$ using Bisection method correct to three decimal places.*

**Solution:** Let $f(x) = x^3 - 3x + 1 = 0$

Since $f(0) = 0^3 - 3(0) + 1 = 1$, which is positive

and $f(1) = 1^3 - 3(1) + 1 = -1$, which is negative

Therefore, $f(0)$ is positive and $f(1)$ is negative, so at least one real root lie between 0 and 1.

**First iteration:** Now using Bisection Method, we can take first approximation

$$x_1 = \frac{0+1}{2} = 0.5$$

Now,

$$\begin{aligned} f(0.5) &= (0.5)^3 - 3(0.5) + 1 \\ &= -0.375 < 0 \end{aligned}$$

Therefore, $f(0)$ is positive and $f(0.5)$ is negative, so root lies between 0 and 0.5.

**Second iteration:** The second approximation to the root is

$$x_2 = \frac{0+0.5}{2} = 0.25$$

Now, $$f(0.25) = (0.25)^3 - 3(0.25) + 1$$
$$= 0.2656 > 0$$

Therefore, $f(0.25)$ is positive and $f(0.5)$ is negative, so root lies between $0.25$ and $0.5$.

**Third iteration:** The third approximation to the root is

$$x_3 = \frac{0.25+0.5}{2} = 0.375$$

Now, $$f(0.375) = (0.375)^3 - 3(0.375) + 1$$
$$= -0.0723 < 0$$

Therefore, $f(0.25)$ is positive and $f(0.375)$ is negative, so root lies between $0.25$ and $0.375$.

**Fourth iteration:** The fourth approximation to the root is

$$x_4 = \frac{0.25+0.375}{2} = 0.3125$$

Now, $$f(0.3125) = (0.3125)^3 - 3(0.3125) + 1$$
$$= 0.093 > 0$$

Therefore, $f(0.3125)$ is positive and $f(0.375)$ is negative, so root lies between $0.3125$ and $0.375$.
Repeating the process, the successive approximations are

$$x_5 = 0.3438, \quad x_7 = 0.3516, \quad x_9 = 0.3458, \quad x_{11} = 0.3471,$$
$$x_6 = 0.3594, \quad x_8 = 0.3477, \quad x_{10} = 0.3468, \quad x_{12} = 0.3474.$$

From the last two approximations, that is

$$|x_{12} - x_{11}| = |0.3474 - 0.3471| \approx 0.0003 < 0.0005$$

Therefore, the root correct to 3 decimal places is $0.347$.

**Example 2.2.** *Find the real root of equation $x\log_{10}^{x} = 1.2$ by Bisection Method.*

**Solution:** Let $f(x) = x\log_{10}^{x} - 1.2 = 0$

So that $f(2) = 2\log_{10}^{2} - 1.2 = -0.598 < 0$

and $f(3) = 3\log_{10}^{3} - 1.2 = 0.2313 > 0$

Thus $f(2)$ is negative and $f(3)$ is positive, therefore, the root will lie between $2$ and $3$.

**First Approximation:** The first approximation to the root is

$$x_1 = \frac{2+3}{2} = 2.5$$

Now, $$f(2.5) = 2.5\log_{10}^{2.5} - 1.2$$
$$= 2.5(0.3979) - 1.2 = -0.2052 < 0$$

Thus, $f(2.5)$ is negative and $f(3)$ is positive, therefore, the root lies between $2.5$ and $3$.

**Second Approximation:** The second approximation to the root is

$$x_2 = \frac{2.5+3}{2} = 2.75$$

Now, $$f(2.75) = 2.75\log_{10}^{2.75} - 1.2$$
$$= 2.75(0.4393) - 1.2 = 0.0081 > 0$$

Thus, $f(2.75)$ is positive and $f(2.5)$ is negative, therefore, the root lies between $2.5$ and $2.75$.

**Third Approximation:** The third approximation to the root is

$$x_3 = \frac{2.5+2.75}{2} = 2.625$$

Now, $$f(2.625) = 2.625\log_{10}^{2.625} - 1.2$$
$$= 2.625(0.4191) - 1.2 = -0.0999 < 0$$

Thus, $f(2.625)$ is negative and $f(2.75)$ is positive, therefore, the root lies between $2.625$ and $2.75$.

**Fourth Approximation:** The fourth approximation to the root is

$$x_4 = \frac{2.625+2.75}{2} = 2.6875$$

Now, $$f(2.6875) = 2.6875\log_{10}^{2.6875} - 1.2$$
$$= 2.6875(0.4293) - 1.2 = -0.0463 < 0$$

Thus, $f(2.6875)$ is negative and $f(2.75)$ is positive, therefore, the root lies between $2.6875$ and $2.75$.

Proceed in similar way to obtain the iterations as follows

$$x_5 = 2.7188, \qquad x_7 = 2.742,$$
$$x_6 = 2.7344, \qquad x_8 = 2.738.$$

Hence, from the approximate value of the roots $x_7$ and $x_8$, we observed that, up to two places of decimal, the root is 2.74 approximately.

**Example 2.3.** *Using Bisection Method, Find the real root of the equation* $3x - \sqrt{1+\sin x} = 0$ *up to two decimal places.*

**Solution:** The given equation $f(x) = 3x - \sqrt{1+\sin x} = 0$ is a transcendental equation.

Then $$f(0) = 0 - \sqrt{1+\sin 0} = -1$$

and $$f(1) = 3 - \sqrt{1+\sin 1} = 1.643$$

Thus $f(0)$ is negative and $f(1)$ is positive, therefore, a root lies between $0$ and $1$.

**First iteration:** The first approximation to the root is given by

$$x_1 = \frac{0+1}{2} = 0.5$$

Now, $$f(0.5) = 3(0.5) - \sqrt{1+\sin(0.5)}$$
$$= 1.5 - \sqrt{1.4794} = 0.2837 > 0$$

Thus, $f(0.5)$ is positive and $f(0)$ is negative, therefore, a root lies between $0$ and $0.5$.
**Second iteration:** The second approximation to the root is

$$x_2 = \frac{0+0.5}{2} = 0.25$$

Now, $$f(0.25) = 3(0.25) - \sqrt{1+\sin(0.25)} = 0.75 - \sqrt{1.2474} = -0.3669 < 0.$$

Thus, $f(0.25)$ is negative and $f(0.5)$ is positive, therefore,a root lies between $0.25$ and $0.5.$
**Third iteration:** The third approximation to the root is

$$x_3 = \frac{0.25+0.5}{2} = 0.375$$

Now, $$f(0.375) = 3(0.375) - \sqrt{1+\sin(0.375)} = 1.125 - \sqrt{1.3663} = -0.0439 < 0.$$

Thus, $f(0.375)$ is negative and $f(0.5)$ is positive, therefore,a root lies between $0.375$ and $0.5.$
Repeating the process, the successive approximations are

$$x_4 = 0.4375, \qquad x_6 = 0.3907,$$
$$x_5 = 0.4063, \qquad x_7 = 0.3985.$$

From the last two observations, that is, $x_6 = 0.3907$ and $x_7 = 0.3985$, the approximate value of the root up to two places of decimal is given by $0.39$. Hence the root is $0.39$ approximately.

**Exercise 2.1.**

1. Find the smallest root of $x^3 - 9x + 1 = 0$, using Bisection Method correct to three decimal places. [**Ans.** 0.111]
2. Find the real root of $e^x = 3x$ by Bisection Method. [**Ans.** 1.5121375]
3. Find the positive real root of $x - \cos x = 0$ by Bisection Method, correct to four decimal places between 0 and 1. [**Ans.** 0.7393]
4. Find the positive root of the equation $xe^x = 1$ which lies between 0 and 1. [**Ans.** 0.5671433]
5. Compute the root of $\log x = \cos x$ correct to 2 decimal places using Bisection Method. [**Ans.** 1.5121375]
6. Find the root of $\tan x + x = 0$ up to two decimal places which lies between 2 and 2.1 using Bisection Method. [**Ans.** 2.02875625]
7. Use the Bisection Method to find out the positive square root of 30 correct to 4 decimal places. [**Ans.** 5.4771]

## 2.5 METHOD OF FALSE POSITION

The method is also called *linear interpolation method* or *regula-falsi method.*

In this method, we choose two points $x_0$ and $x_1$ such that $f(x_0)$ and $f(x_1)$ are of opposite signs. Since the graph of $y = f(x)$ crosses the X-axis between these two points, a root must lie in between these points.

Consequently, $f(x_0)f(x_1) < 0$. Equation of the line joining points $P\{x_0, f(x_0)\}$ and $Q\{x_1, f(x_1)\}$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

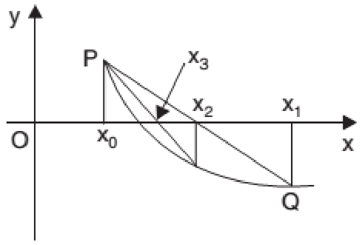

**Figure 2.2:** Method of false position

The method consists in replacing the curve PQ by means of the line PQ and taking the point of intersection of the line with X-axis as an approximation to the root. So the abscissa of the point where line cuts $y = 0$ is given by

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

The value of $x_2$ can also be put in the following form:

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

In general, the $(i + 1)$th approximation to the root is given by

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

### 2.5.1 Procedure for the False Position Method

**Step 1:** Choose two initial approximations $x_0$ and $x_1$ (where $x_1 > x_0$) such that $f(x_0)f(x_1) < 0$.

**Step 2:** Find the next approximation $x_2$ using the formula

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

and also evaluate $f(x_2)$.

**Step 3:** If $f(x_2)f(x_1) < 0$, then go to the next step. If not, rename $x_0$ as $x_1$ and then go to the next step.

**Step 4:** Evaluate successive approximations using the formula

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}, \text{ where } i = 2, 3, 4, \ldots$$

But before applying the formula for $x_{i+1}$, ensure whether $f(x_{i-1})f(x_i) < 0$; if not, rename

$x_{i-2}$ as $x_{i-1}$ and proceed.

**Step 5:** Stop the evaluation when $|x_i - x_{i-1}| < \varepsilon$ , where $\varepsilon$ is the prescribed accuracy.

### 2.5.2 Order (or Rate) of Convergence of False Position Method

The general iterative formula for False Position Method is given by

$$x_{i+1} = \frac{x_{i-1}f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})} \tag{2.2}$$

where $x_{i-1}, x_i$ and $x_{i+1}$ are successive approximations to the required root of $f(x) = 0$.

The formula given in (2.2), can also be written as:

$$x_{i+1} = x_i - \frac{(x_i - x_{i-1})f(x_i)}{f(x_i) - f(x_{i-1})} \tag{2.3}$$

Let $\alpha$ be the actual (true) root of $f(x) = 0, i.e., f(\alpha) = 0$. If $e_{i-1}, e_i$ and $e_{i+1}$ are the successive errors in $(i-1)$th, $i$th and $(i+1)$th iterations respectively, then

$$e_{i-1} = x_{i-1} - \alpha,\ e_i = x_i - \alpha,\ e_{i+1} = x_{i+1} - \alpha$$

or $$x_{i-1} = \alpha + e_{i-1},\ x_i = \alpha + e_i,\ x_{i+1} = \alpha + e_{i+1}$$

Using these in (2.3), we obtain

$$\alpha + e_{i+1} = \alpha + e_i - \frac{(e_i - e_{i-1})f(\alpha + e_i)}{f(\alpha + e_i) - f(\alpha + e_{i-1})}$$

$$e_{i+1} = e_i - \frac{(e_i - e_{i-1})f(\alpha + e_i)}{f(\alpha + e_i) - f(\alpha + e_{i-1})}$$

Expanding $f(\alpha + e_i)$ and $f(\alpha + e_{i-1})$ in Taylor's series around $\alpha$, we have

$$e_{i+1} = e_i - \frac{(e_i - e_{i-1})\left[f(\alpha) + e_i f'(\alpha) + \frac{e_i^2}{2} f''(\alpha) + \dots\right]}{\left[f(\alpha) + e_i f'(\alpha) + \frac{e_i^2}{2} f''(\alpha) + \dots\right] - \left[f(\alpha) + e_{i-1} f'(\alpha) + \frac{e_{i-1}^2}{2} f''(\alpha) + \dots\right]}$$

$$e_{i+1} = e_i - \frac{(e_i - e_{i-1})\left[f(\alpha) + e_i f'(\alpha) + \frac{e_i^2}{2} f''(\alpha)\right]}{\left[(e_i - e_{i-1})f'(\alpha) + \left(\frac{e_i^2 - e_{i-1}^2}{2}\right) f''(\alpha)\right]}, \text{[on ignoring the higher order terms]}$$

$$e_{i+1} = e_i - \frac{\left[f(\alpha) + e_i f'(\alpha) + \frac{e_i^2}{2} f''(\alpha)\right]}{\left[f'(\alpha) + \left(\frac{e_i + e_{i-1}}{2}\right) f''(\alpha)\right]}$$

$$e_{i+1} = e_i - \frac{\left[e_i f'(\alpha) + \frac{e_i^2}{2} f''(\alpha)\right]}{\left[f'(\alpha) + \left(\frac{e_i + e_{i-1}}{2}\right) f''(\alpha)\right]} \qquad [\text{since} f(\alpha) = 0]$$

$$e_{i+1} = e_i - \frac{\left[e_i + \frac{e_i^2}{2}\frac{f''(\alpha)}{f'(\alpha)}\right]}{\left[1 + \left(\frac{e_i + e_{i-1}}{2}\right)\frac{f''(\alpha)}{f'(\alpha)}\right]}, \text{[on dividing numerator and denominator by} f'(\alpha)\text{)}$$

$$e_{i+1} = e_i - \left[e_i + \frac{e_i^2}{2}\frac{f''(\alpha)}{f'(\alpha)}\right]\left[1 + \left(\frac{e_i + e_{i-1}}{2}\right)\frac{f''(\alpha)}{f'(\alpha)}\right]^{-1}$$

$$e_{i+1} = e_i - \left[e_i + \frac{e_i^2}{2}\frac{f''(\alpha)}{f'(\alpha)}\right]\left[1 - \left(\frac{e_i + e_{i-1}}{2}\right)\frac{f''(\alpha)}{f'(\alpha)}\right]$$
$$e_{i+1} = e_i - \left[e_i \frac{e_i(e_i + e_{i-1})}{2}\frac{f''(\alpha)}{f'(\alpha)} + \frac{e_i^2}{2}\frac{f''(\alpha)}{f'(\alpha)} - \frac{e_i^2(e_i + e_{i-1})}{4}\left(\frac{f''(\alpha)}{f'(\alpha)}\right)^2\right]$$
$$e_{i+1} = e_i e_{i-1}\frac{f''(\alpha)}{2f'(\alpha)} + O(e_i^2)$$

If $e_{i-1}$ and $e_i$ are very small, then ignoring $O(e_i^2)$, we get

$$e_{i+1} = e_i e_{i-1}\frac{f''(\alpha)}{2f'(\alpha)}$$

which can be written as

$$e_{i+1} = e_i e_{i-1} M, \text{ where } M = \frac{f''(\alpha)}{2f'(\alpha)} \text{ and would be a constant} \tag{2.4}$$

In order to find the order of convergence, it is necessary to find a formula of the type

$$e_{i+1} = A e_i^k \text{ with an appropriate value of } k \tag{2.5}$$

With the help of (2.5), we can write

$$e_i = A e_{i-1}^k \text{ or } e_{i-1} = (e_i/A)^{1/k}$$

Now, substituting the value of $e_{i+1}$ and $e_{i-1}$ in (2.4), we get

$$A e_i^k = e_i \left(\frac{e_i}{A}\right)^{1/k} M$$
$$e_i^k = M A^{-(1+1/k)} e^{(1+1/k)} \tag{2.6}$$

Comparing the powers of $e_i$ on both sides of (2.6), we get

$$k = 1 + (1/k)$$
$$k^2 - k - 1 = 0 \tag{2.7}$$

From (2.7), taking only the positive root, we get $k = 1.618$
By putting this value of $k$ in (2.5), we have

$$e_{i+1} = A e_i^{1.618} \text{ or } \frac{e_{i+1}}{e_i^{1.618}} = A$$

Comparing this with $\lim_{i\to\infty}\left(\frac{e_{i+1}}{e_i^k}\right) \leq A$, we see that order (or rate) of convergence of false position method is 1.618.

**Example 2.4.** *Find a real root of the equation $x^3 - 3x + 1 = 0$ by the method of false position up to three places of decimal.*

**Solution:** Let $f(x) = x^3 - 3x + 1 = 0$
Then, $f(0) = 0^3 - 3(0) + 1 = 1$, and $f(1) = 1^3 - 3(1) + 1 = -1$
Therefore, $f(0)$ is positive and $f(1)$ is negative, so a root lies between $0$ and $1$.

**First approximation:** Taking $x_0 = 0, x_1 = 1, f(x_0) = 1$ and $f(x_1) = -1$. Then by Regula-falsi method, we get

$$\begin{aligned} x_2 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\ &= 0 - \frac{1-0}{-1-1}(1) = 0.5 \\ \text{Now,} \qquad f(x_2) &= f(0.5) \\ &= (0.5)^3 - 3(0.5) + 1 = -0.375 < 0 \end{aligned}$$

Thus, the root lies between 0 and 0.5.

**Second approximation:** Taking $x_0 = 0, x_1 = 0.5, f(x_0) = 1$ and $f(x_1) = -0.375$. Then the next approximation to the root is given by

$$\begin{aligned} x_3 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\ &= 0 - \frac{0.5-0}{-0.375-1}(1) = 0.36364 \\ \text{Now,} \qquad f(x_3) &= f(0.36364) \\ &= (0.36364)^3 - 3(0.36364) + 1 = -0.04283 < 0 \end{aligned}$$

Thus, the root lies between 0 and 0.0.36364.

**Third approximation:** Taking $x_0 = 0, x_1 = 0.36364, f(x_0) = 1$ and $f(x_1) = -0.04283$. Then the next approximation to the root is given by

$$\begin{aligned} x_4 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\ &= 0 - \frac{0.36364-0}{-0.04283-1}(1) = 0.34870 \\ \text{Now,} \qquad f(x_4) &= f(0.34870) \\ &= (0.34870)^3 - 3(0.34870) + 1 = -0.00370 < 0 \end{aligned}$$

Thus, the root lies between 0 and 0.0.34870.

**Fourth approximation:** Taking $x_0 = 0, x_1 = 0.34870, f(x_0) = 1$ and $f(x_1) = -0.00370$. Then the next approximation to the root is given by

$$\begin{aligned} x_5 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\ &= 0 - \frac{0.34870-0}{-0.00370-1}(1) = 0.34741 \\ \text{Now,} \qquad f(x_5) &= f(0.34741) \\ &= (0.34741)^3 - 3(0.34741) + 1 = -0.00030 < 0 \end{aligned}$$

Thus, the root lies between 0 and 0.0.34741.

**Fifth approximation:** Taking $x_0 = 0, x_1 = 0.34741, f(x_0) = 1$ and $f(x_1) = -0.00030$. Then the next approximation to the root is given by

$$\begin{aligned} x_6 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\ &= 0 - \frac{0.34741 - 0}{-0.00030 - 1}(1) = 0.34730 \end{aligned}$$

Now, $|x_6 - x_5| = |0.34730 - 0.34741| \approx 0.0001 < 0.0005$
Thus, the root is $0.347$ correct to three decimal places.

**Example 2.5.** *Find a real root of the equation* $x^2 \log_e^x - 12 = 0$ *using Regula-Falsi method correct to three places of decimals.*

**Solution:** Let $f(x) = x^2 \log_e^x - 12 = 0$
So that, $f(3) = 3^2 \log_e^3 - 12 = -4.0986$, and $f(4) = 4^2 \log_e^4 - 12 = 2.6137$.
Therefore, $f(3)$ and $f(4)$ are of opposite signs. Therefore, a real root lies between $3$ and $4$. For the first approximation to the root, taking $x_0 = 3, x_1 = 4, f(x_0) = -4.0986$ and $f(x_1) = 2.6137$.
**First approximation:** By Regula-Falsi method, the root is

$$\begin{aligned} x_2 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\ &= 3 - \frac{4 - 3}{2.6137 + 4.0986}(-4.0986) = 3.6106 \end{aligned}$$

$$\text{Now,} \quad \begin{aligned} f(x_2) &= f(3.6106) \\ &= (3.6106)^2 - \log_e^{(3.6106)} - 12 = -0.2475. \end{aligned}$$

**Second approximation:** The root will lies between 3.6106 and 4. Therefore for next approximation, taking $x_0 = 3.6106, x_1 = 4, f(x_0) = -0.2475$ and $f(x_1) = 2.6137$. Then the root is

$$\begin{aligned} x_3 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\ &= 3.6106 - \frac{4 - 3.6106}{2.6137 + 0.2475}(-0.2475) = 3.6443 \end{aligned}$$

$$\text{Now,} \quad \begin{aligned} f(x_3) &= f(3.6443) \\ &= (3.6443)^2 - \log_e^{(3.6443)} - 12 = -0.0123. \end{aligned}$$

**Third approximation:** The root will lies between 3.6443 and 4. Therefore, taking $x_0 = 3.6443, x_1 = 4, f(x_0) = -0.0123$ and $f(x_1) = 2.6137$. Then the root is given by

$$\begin{aligned} x_4 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\ &= 3.6443 - \frac{4 - 3.6443}{2.6137 + 0.0123}(-0.0123) = 3.6459 \end{aligned}$$

$$\text{Now,} \quad \begin{aligned} f(x_4) &= f(3.6459) \\ &= (3.6459)^2 - \log_e^{(3.6459)} - 12 = -0.001. \end{aligned}$$

**Fourth approximation:** The root will lies between 3.6459 and 4. Therefore, taking $x_0 = 3.6459, x_1 = 4, f(x_0) = -0.001$ and $f(x_1) = 2.6137$. Then the root is

$$\begin{aligned}
x_5 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\
&= 3.6459 - \frac{4 - 3.6459}{2.6137 + 0.001}(-0.001) = 3.6460 \\
\text{Now,} \quad f(x_5) &= f(3.6460) \\
&= (3.6460)^2 - \log_e^{(3.6460)} - 12 = -0.0003.
\end{aligned}$$

**Fifth approximation:** The root will lies between 3.6460 and 4. Therefore, taking $x_0 = 3.6460, x_1 = 4, f(x_0) = -0.0003$ and $f(x_1) = 2.6137$. Then the root is

$$\begin{aligned}
x_6 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\
&= 3.6460 - \frac{4 - 3.6460}{2.6137 + 0.0003}(-0.0003) = 3.6461.
\end{aligned}$$

Hence the root is approximated by 3.646 correct to three decimal places.

**Example 2.6.** *Find the root correct to two decimal places of the equation $xe^x = \cos x$, using the method of false position.*

**Solution:** We have $f(x) = \cos x - xe^x = 0$

Then, $f(0) = 1$, and $f(1) = \cos 1 - e = -2.17798.$

$\therefore$ The root lies between $0$ and $1$. Therefore, taking $x_0 = 0, x_1 = 1, f(x_0) = 1$ and $f(x_1) = -2.17798$. The first approximation to the root is

$$\begin{aligned}
x_2 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\
&= 0 - \frac{1 - 0}{-2.17798 - 1}(1) = 0.31467 \\
\text{Now,} \quad f(x_2) &= f(0.31467) = 0.51986 > 0
\end{aligned}$$

Thus, the root lies between 0.31467 and 1. Then taking $x_0 = 0.31467, x_1 = 1, f(x_0) = 0.51986$ and $f(x_1) = -2.17798$.

**Second approximation:** The next approximation to the root is

$$\begin{aligned}
x_3 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\
&= 0.31467 - \frac{1 - 0.31467}{-2.17798 - 0.51986}(0.51986) = 0.44673 \\
\text{Now,} \quad f(x_3) &= f(0.44673) = 0.20354.
\end{aligned}$$

Thus, the root lies between 0.44673 and 1. Then taking $x_0 = 0.44673, x_1 = 1, f(x_0) = 0.20354$ and $f(x_1) = -2.17798$.

**Third approximation:** The next approximation to the root is

$$
\begin{aligned}
x_4 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\
&= 0.44673 - \frac{1 - 0.44673}{-2.17798 - 0.20354}(0.20354) = 0.49402
\end{aligned}
$$

Now, $\quad f(x_4) = f(0.49402) = 0.07079.$

Thus, the root lies between 0.49402 and 1.

**Fourth approximation:** Taking $x_0 = 0.49402, x_1 = 1, f(x_0) = 0.07079$ and $f(x_1) = -2.17798.$ Then the root becomes

$$
\begin{aligned}
x_5 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\
&= 0.49402 - \frac{1 - 0.49402}{-2.17798 - 0.07079}(0.07079) = 0.50995
\end{aligned}
$$

Now, $\quad f(x_5) = f(0.50995) = 0.02360.$

Thus, the root lies between 0.50995 and 1.

**Fifth approximation:** Taking $x_0 = 0.50995, x_1 = 1, f(x_0) = 0.02360$ and $f(x_1) = -2.17798.$ Then the next approximation to the root is given by

$$
\begin{aligned}
x_6 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\
&= 0.50995 - \frac{1 - 0.50995}{-2.17798 - 0.02360}(0.07079) = 0.51520
\end{aligned}
$$

Now, $\quad f(x_6) = f(0.51520) = 0.00776.$

Thus, the root lies between 0.5152 and 1.

**Sixth approximation:** Taking $x_0 = 0.51520, x_1 = 1, f(x_0) = 0.020776$ and $f(x_1) = -2.17798.$ Then the next approximation to the root is given by

$$
\begin{aligned}
x_7 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\
&= 0.51520 - \frac{1 - 0.51520}{-2.17798 - 0.020776}(0.020776) = 0.51692
\end{aligned}
$$

Now, $\quad |x_7 - x_6| = |0.51692 - 0.51520| \approx 0.00172 < 0.005$

Hence, the root is $0.51$ correct to two decimal places.

**Exercise 2.2.**

1. Find the real root of the equation $x^3 - 2x - 5 = 0$ by the method of Falsi Position correct to three decimal places. [**Ans.** 2.094]

2. Find the real root of the equation $x \log_{10}^{x} = 1.2$ by Regula-Falsi method correct to four decimal places. [**Ans.** 2.7406]

3. Find the positive root of $xe^x = 2$ by the method of Falsi Position. [**Ans.** 0.852605]

4. Apply Falsi Position method to find smallest positive root of the equation $x - e^{-x} = 0$ correct to three decimal places. [**Ans.** 0.567]

5. Find the real root of the equations:

   **(a)** $x = \tan x$ [**Ans.** 4.4934]

   **(b)** $x^2 - \log_e^x - 12 = 0$ [**Ans.** 3.5425]

   **(c)** $3x = \cos x + 1$ [**Ans.** 0.6071]

6. Find real cube root of 18 by Regula-Falsi method. [**Ans.** 2.62074]

## 2.6 ITERATION METHOD(METHOD OF SUCCESSIVE APPROXIMATION)

This method is also known as the *direct substitution method* or *method of fixed iterations.*

To find the root of the equation $f(x) = 0$ by successive approximations, we rewrite the given equation in the form

$$x = \phi(x)$$

There are many ways of rewriting $f(x) = 0$ in this form.
For example, $f(x) = x^3 - 5x + 1 = 0$, can be rewritten in the following forms.

$$x = \frac{x^3 + 1}{5}, \qquad x = (5x - 1)^{\frac{1}{3}}, \qquad x = \sqrt{\frac{5x - 1}{x}}, \text{ etc.} \tag{2.8}$$

Now, first we assume the approximate value of root (let $x_0$), then substitute it in $\phi(x)$ to have a first approximation $x_1$ given by

$$x_1 = \phi(x_0)$$

Similarly, the second approximation $x_2$ is given by

$$x_2 = \phi(x_1)$$

In general, $x_{n+1} = \phi(x_n)$

### 2.6.1 Procedure for Iteration Method To Find The Root of The Equation $f(x) = 0$

**Step 1:** Take an initial approximation as $x_0$.

**Step 2:** Find the next (first) approximation $x_1$ by using $x_1 = \phi(x_0)$

**Step 3:** Follow the above procedure to find the successive approximations $x_{n+1}$ by using $x_{n+1} = \phi(x_n), n = 1, 2, 3, \ldots$

**Step 4:** Stop the evaluation where relative error $\leq \varepsilon$ , where $\varepsilon$ is the prescribed accuracy.

**Note 2.1.** *The iteration method $x = \phi(x)$ is convergent if $|\phi'(x)| < 1$, for all $x$ in the interval.*

We can test this condition using $x_0$, the initial approximation, before the computations are done.

Let us now check whether the methods given in (2.8) converge to a root in $(0,1)$ of the equation $f(x) = x^3 - 5x + 1 = 0$.

i) We have $\phi_1(x) = \frac{x^3+1}{5}, \phi'(x) = \frac{3x^2}{5}$, and $|\phi'(x)| = \frac{3x^2}{5} \leq 1$ for all $x$ in $0 < x < 1$. Hence, the method converges to a root in $(0,1)$.

ii) We have $\phi_2(x) = (5x-1)^{\frac{1}{3}}, \phi'(x) = \frac{5}{3(5x-1)^{\frac{2}{3}}}$. Now $|\phi'(x)| < 1$, when $x$ is close to $1$ and $|\phi'(x)| > 1$ in the other part of the interval. Convergence is not guaranteed.

iii) We have $\phi_3(x) = \sqrt{\frac{5x-1}{x}}, \phi'(x) = \frac{1}{2x^{\frac{3}{2}}(5x-1)^{\frac{1}{2}}}$. Again, $|\phi'(x)| < 1$, when $x$ is close to $1$ and $|\phi'(x)| > 1$ in the other part of the interval. Convergence is not guaranteed.

### 2.6.2 Rate of Convergence of Iteration Method

Let $f(x) = 0$ be the equation which is being expressed as $x = \phi(x)$. The iterative formula for solving the equation is

$$x_{i+1} = \phi(x_i)$$

If $\alpha$ is the root of the equation $x = \phi(x)$ lying in the interval [a,b], $\alpha = \phi(\alpha)$.
Then by mean value theorem

$$x_{i+1} = \phi(\alpha) + (x_i - \alpha)\phi'(c_i) \quad \text{Where } a < c_i < b$$

$$\text{But} \qquad \phi(x) = \alpha$$

$$\Rightarrow \qquad x_{i+1} = \alpha + (x_i - \alpha)\phi'(c_i)$$

$$\Rightarrow \qquad x_{i+1} - \alpha = (x_i - \alpha)\phi'(c_i) \tag{2.9}$$

Now, if $e_{i+1}, e_i$ are the error for the approximation $x_{i+1}$ and $x_i$
Therefore, $e_{i+1} = x_{i+1} - \alpha, e_i = x_i - \alpha$
Using this in (2.9), we get

$$e_{i+1} = e_i\phi'(c_i)$$

Here $\phi(x)$ is a continuous function, therefore, it is bounded
$\therefore$ $|\phi'(c_i)| \leq k$, where $k\epsilon[a,b]$ is a constant.
$\therefore$ $e_{i+1} \leq e_i k$
or $\frac{e_{i+1}}{e_i} \leq k$
Hence, by definition, the rate of convergence of iteration method is 1. In other words, iteration method converges linearly.

**Example 2.7.** *Find a real root of the equation $x^3 - x - 10 = 0$ correct to three decimal places using iteration method.*

**Solution:** We have $f(x) = x^3 - x - 10, f(0) = -10, f(1) = -10,$
Now, we find that $f(2) = 8 - 2 - 10 = -4$ and $f(3) = 27 - 3 - 10 = 14$
Since, $f(2)f(3) < 0$, the root lies in the interval $(2,3)$.
Now, the given equation can be re-written as

$$x = (x+10)^{\frac{1}{3}} = \phi(x)$$

We obtain $\phi'(x) = \frac{1}{3(x+10)^{\frac{2}{3}}}$

We find $|\phi'(x)| < 1$ for all $x$ in the interval $(2, 3)$. Hence, the iteration converges.
Let $x_0 = 2.5$. We obtain the following results.

$$
\begin{aligned}
x_1 &= \phi(x_0) = (2.5 + 10)^{\frac{1}{3}} = 2.3208 \\
x_2 &= \phi(x_1) = (2.3208 + 10)^{\frac{1}{3}} = 2.3096 \\
x_3 &= \phi(x_2) = 2.3090 \\
x_4 &= \phi(x_3) = 2.3089.
\end{aligned}
$$

Since, $|x_4 - x_3| = |2.3089 - 2.3090| = 0.0001$, we take the required root as $x \approx 2.3089$.
**Example 2.8.** *Find a real root of* $2x - \log_{10}^{x} = 7$ *correct to four decimal places using the iteration method.*

**Solution:** We have $f(x) = 2x - \log_{10}^{x} - 7 = 0$
Now, we find that $f(3) = 6 - \log_{10}^{3} - 7 = -1.4471$ and $f(4) = 8 - \log_{10}^{4} - 7 = 0.398$
Therefore, a root lies between $3$ and $4$.
Rewriting the given equation as

$$x = \tfrac{1}{2}[\log_{10}^{x} + 7] = \phi(x),$$

we have $\phi'(x) = \frac{1}{2}\left(\frac{1}{x}\log_{10}^{e}\right)$
$\therefore$ $|\phi'(x)| < 1$ when $3 < x < 4$ $(\because \log_{10}^{e} = 0.4343)$
Hence the iteration method can be applied and we start with $x_0 = 3.6$. Then the successive approximations are

$$
\begin{aligned}
x_1 &= \phi(x_0) = \frac{1}{2}[\log_{10}^{3.6} + 7] = 3.77815 \\
x_2 &= \phi(x_1) = \frac{1}{2}[\log_{10}^{3.77815} + 7] = 3.78863 \\
x_3 &= \phi(x_2) = 3.78924 \\
x_4 &= \phi(x_3) = 3.78927.
\end{aligned}
$$

Hence, the root of the equation correct to the four places of decimal is $3.7892$.
**Exercise 2.3.**

1. Use the method of Iteration to find a positive root between 0 and 1 of the equation $xe^x = 1$. [**Ans.** 0.5671477]
2. Find the Iterative method, the real root of the equation $3x - \log_{10}^{x} = 6$ correct to four significant figures. [**Ans.** 2.108]
3. Solve by Iteration method:
   - **(a)** $x^3 + x + 1 = 0$ [**Ans.** -0.682327803]
   - **(b)** $\sin x = \frac{x+1}{x-1}$ [**Ans.** -0.420365]
   - **(c)** $x^3 - 2x^2 - 4 = 0$ [**Ans.** 2.5943]
4. By Iteration method, find $\sqrt{30}$. [**Ans.** 5.477225575]

## 2.7 NEWTON-RAPHSON METHOD

This is a very powerful method for finding the real root of an equation in the form, $f(x) = 0$. Let $x_0$ be an approximate root of $f(x) = 0$ and let $x_1 = x_0 + h$ be the exact root so that $f(x_1) = 0$. Expanding $f(x_0 + h)$ by Taylor's series, we get

$$f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \ldots = 0$$

Since $h$ is small, neglecting $h^2$ and higher powers of $h$, we get

$$f(x_0) + hf'(x_0) = 0 \text{ or } h = -\frac{f(x_0)}{f'(x_0)}, \quad f'(x_0) \neq 0.$$

Hence, if $x_0$ be the initial approximation, then next (or first) approximation $x_1$ is given by

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The next and second approximation $x_2$ is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general, $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

This formula is well known as Newton-Raphson formula.

### 2.7.1 Procedure for Newton Raphson Method to Find the Root of the Equation $f(x) = 0$

**Step 1:** Take a trial solution (initial approximation) as $x_0$. Find $f(x_0)$ and $f'(x_0)$.

**Step 2:** Find next (first) approximation $x_1$ by using the formula $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

**Step 3:** Follow the above procedure to find the successive approximations $x_{n+1}$ using the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, where $n = 1, 2, 3, \ldots$

**Step 4:** Stop the process when $|x_{n+1} - x_n| < \varepsilon$, where $\varepsilon$ is the prescribed accuracy.

### 2.7.2 Convergence of Newton-Raphson Method

To examine the convergence of Newton-Raphson formula, that is,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

we compare it with the general iteration formula

$$x_{n+1} = \phi(x_n)$$

we get

$$\phi(x_n) = x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In general, we write it as
$$\phi(x) = x - \frac{f(x)}{f'(x)}$$
which gives
$$\phi'(x) = 1 - \left[\frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2}\right] = \frac{f(x)f''(x)}{[f'(x)]^2}$$
Since the iteration method converges if $|\phi'(x)| < 1$
Therefore, Newton-Raphson method converges, provided
$$\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| < 1$$
or $\qquad |f(x)f''(x)| < [f'(x)]^2$
in the interval considered. Newton-Raphson formula therefore converges, provided the initial approximation $x_0$ is chosen sufficiently close to the root and $f(x), f'(x)$ and $f''(x)$ are continuous and bounded in any small interval containing the root.

### 2.7.3 Order of Convergence of Newton-Raphson Method

Let $\alpha$ denotes the exact value of the root of $f(x) = 0$, and let $x_n, x_{n+1}$, be two successive approximations to the actual root $\alpha$. If $e_n$ and $e_{n+1}$ are the corresponding errors, we have

$x_n = \alpha + e_n$ and $x_{n+1} = \alpha + e_{n+1}$

by Newton-Raphson's iterative formula
$$\begin{aligned}
\alpha + e_{n+1} &= \alpha + e_n - \frac{f(\alpha + e_n)}{f'(\alpha + e_n)} \\
e_{n+1} &= e_n - \frac{f(\alpha + e_n)}{f'(\alpha + e_n)} \\
e_{n+1} &= e_n - \frac{f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \dots}{f'(\alpha) + e_n f''(\alpha) + \frac{e_n^2}{2!} f'''(\alpha) + \dots} \text{(By Taylor's expansion)} \\
e_{n+1} &= e_n - \frac{e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \dots}{f'(\alpha) + e_n f''(\alpha) + \frac{e_n^2}{2!} f'''(\alpha) + \dots} \qquad [\because f(\alpha) = 0] \\
&= e_n - \frac{e_n \left[f'(\alpha) + \frac{e_n}{2} f''(\alpha) + \dots\right]}{f'(\alpha) + e_n f''(\alpha) + \dots} \\
&= \frac{1}{2}\left[\frac{e_n^2 f''(\alpha)}{f'(\alpha) + e_n f''(\alpha) + \dots}\right] \\
&= \frac{e_n^2}{2}\left[\frac{f''(\alpha)}{f'(\alpha)\left(1 + e_n \frac{f''(\alpha)}{f'(\alpha)} + \dots\right)}\right] \\
&= \frac{e_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)}\left[1 + e_n \frac{f''(\alpha)}{f'(\alpha)} + \dots\right]^{-1} \\
&= \frac{e_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)}\left[1 - e_n \frac{f''(\alpha)}{f'(\alpha)} + \dots\right] \\
&= \frac{e_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)} - \frac{e_n^3}{2}\left(\frac{f''(\alpha)}{f'(\alpha)}\right)^2 + \dots
\end{aligned}$$

Neglecting the terms containing $e_n^3$ and higher powers of $e_n$, we get

$$e_{n+1} = Ce_n^2, \text{ where } C = \frac{f''(\alpha)}{2f'(\alpha)}$$

and $$|e_{n+1}| = |C||e_n^2|.$$

Therefore, Newton's method is of order 2 or has quadratic rate of convergence.

**Example 2.9.** *Find the real root of the equation $x^3 - 2x - 5 = 0$ correct to three decimal places, using Newton-Raphson method.*

**Solution:** Let $f(x) = x^3 - 2x - 5 = 0$

Now, $f(2) = 2^3 - 2(2) - 5 = -1$

and $f(3) = 3^3 - 2(3) - 5 = 16$

Therefore, the root lies between $2$ and $3$.

Let us take $x_0 = 2.2$ and $f'(x) = 3x^2 - 2$

Now, by Newton-Raphson method, we have

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^3 - 2} \end{aligned}$$

or $$x_{n+1} = \frac{2x_n^3 + 5}{3x_n^2 - 2}, \quad n = 0, 1, 2, \dots \tag{E.a}$$

**First approximation:** Putting $n = 0$, in (E.a) we get first approximation

$$x_1 = \frac{2x_0^3 + 5}{3x_0^2 - 2}$$
$$x_1 = \frac{2(2.2)^3 + 5}{3(2.2)^2 - 2} = \frac{26.296}{12.52} = 2.1003$$

**Second approximation:** Putting $n = 1$, in (E.a) we get second approximation

$$x_2 = \frac{2x_1^3 + 5}{3x_1^2 - 2}$$
$$x_2 = \frac{2(2.1003)^3 + 5}{3(2.1003)^2 - 2} = 2.0946$$

**Third approximation:** Putting $n = 2$, (E.a) we get third approximation

$$x_3 = \frac{2x_2^3 + 5}{3x_2^2 - 2}$$
$$x_3 = \frac{2(2.0946)^3 + 5}{3(2.0946)^2 - 2} = 2.0946$$

Since $x_2 = x_3$, the required root is $2.094$ correct to three decimal places.

**Example 2.10.** *Evaluate* $\sqrt{29}$ *to five decimal places by Newton's iterative method.*

**Solution:** Let $x = \sqrt{29}$ then $x^2 - 29 = 0$
we consider $f(x) = x^2 - 29 = 0$ and $f'(x) = 2x$
The Newton's Iterative formula gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_{n+1} = x_n - \frac{x_n^2 - 29}{2x_n} = \frac{1}{2}(x_n + \frac{29}{x_n}) \tag{E.b}$$

Now since $f(5) = 25 - 29 = -4\ (-ve)$ and $f(6) = 36 - 29 = 7\ (+ve)$
Therefore the root lies between 5 and 6.
Take $x_0 = 5.3$, equation (E.b) gives

$$x_1 = \frac{1}{2}(5.3 + \frac{29}{5.3}) = 5.38585$$
$$x_2 = \frac{1}{2}(5.38585 + \frac{29}{5.38585}) = 5.38516$$
$$x_3 = \frac{1}{2}(5.38516 + \frac{29}{5.38516}) = 5.38516$$

Since, $x_2 = x_3$ up to five decimal places. So we have $\sqrt{29} = 5.38516.$
**Example 2.11.** *Find the real root of the equation* $3x = \cos x + 1$ *by Newton's method.*

**Solution:** Let $f(x) = 3x - \cos x - 1 = 0$
So that $\quad f(0) = -2$
$\qquad f(1) = 3 - \cos 1 - 1 = 1.4597$
So the root lies between 0 and 1.
Let us take $x_0 = 0.6$ and $f'(x) = 3 + \sin x$
Therefore the Newton's method gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_{n+1} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \tag{E.c}$$

**First approximation:** Putting $n = 0$, in (E.c) we get first approximation

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0}$$
$$= \frac{(0.6)\sin(0.6) + \cos(0.6) + 1}{3 + \sin(0.6)}$$
$$x_1 = \frac{(0.6)(0.5646) + 0.8253 + 1}{3 + 0.5646} = \frac{2.16406}{3.5646} = 0.6071$$

**Second approximation:** Putting $n = 0$, in (E.c) we get second approximation

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1}$$
$$= \frac{(0.6071)\sin(0.6071) + \cos(0.6071) + 1}{3 + \sin(0.6071)}$$
$$x_2 = \frac{(0.6071)(0.5705) + 0.8213 + 1}{3 + 0.5705} = \frac{2.1677}{3.5705} = 0.6071$$

Since $x_1 = x_2$. Therefore the root as 0.6071 correct to four decimal places.

**Exercise 2.4.**

1. Use Newton-Raphson method to find a root of the equation $x^3 - 3x - 5 = 0$. [**Ans.** 2.279]
2. Find the four places of decimal, the smallest root of the equation $e^x = \sin x$. [**Ans.** 0.5885]
3. Find the cube root of 10. [**Ans.** 2.15466]
4. Use Newton-Raphson method to obtain a root, correct to three decimal places of following equations:
   **(a)** $\sin x = \frac{x}{2}$ [**Ans.** 1.896]
   **(b)** $x + \log x = 2$ [**Ans.** 1.756]
   **(c)** $\tan x = x$ [**Ans.** 4.4934]
5. Find cube root of 3 correct to three decimal places by Newton's iterative method. [**Ans.** 1.442]
6. Apply Newton's formula to find the values of $(30)^{1/5}$. [**Ans.** 1.973]