Self-Assessment Exercise 3.1

1. Describe the drawbacks of Gaussian elimination method and the possible strategy to avoid such drawbacks.
2. A diabetic patient wishes to prepare a meal consisting of roasted chicken, mashed potatoes, and peas. A serving of roasted chicken contains 140 Cal, 27g of protein, and 64mg of sodium. A one cup serving of mashed potatoes contains 160 Cal, 4g of protein and 636mg of sodium. A one cup serving of peas contains 125 Cal, 8g of protein, and 139mg of sodium. How many servings of each should be used if the meal is to contain 415 Cal, 50.5g of protein, and 553mg of sodium?

Hint: Let x, y, and z represent the number of servings of chicken, mashed potatoes, and peas to be used, respectively. The resulting equation is:

$$140x+160y+125z=415,$$
$$27x+4y+8z=50.5,$$
$$64x+636y+139z=553$$

Now you try to solve the system.

3. Solve the following system of linear equations by Gaussian- Elimination method.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

3.2.2. Gauss-Jordan Elimination Method

Gauss-Jordan Elimination is a variant of Gaussian Elimination. Again, we are transforming the coefficient matrix into another matrix that is much easier to solve, and the system represented by the new augmented matrix has the same solution set as the original system of linear equations. In Gauss-Jordan Elimination, the goal is to transform the coefficient matrix into a diagonal matrix, and the zeros are introduced into the matrix one column at a time. We work to eliminate the elements both above and below the diagonal element of a given column in one pass through the matrix.

The general procedure for Gauss-Jordan Elimination can be summarized in the following steps:

1. Write the augmented matrix of the system.
2. Use row operations to transform the augmented matrix in the form, which is called the reduced row echelon form (RREF).
3. Stop process in step 2 if you obtain a row whose elements are all zeros except the last one on the right. In that case, the system is inconsistent and has no solutions. Otherwise, finish step 2 and read the solutions of the system from the final matrix.

Note: A matrix is said to be in its RREF if it satisfies the following:

(a) The rows (if any) consisting entirely of zeros are grouped together at the bottom of the matrix.

(b) In each row that does not consist entirely of zeros, the left most nonzero element is a 1(called a leading 1 or a pivot).

(c) Each column that contains a leading 1 has zeros in all other entries.

(d) The leading 1 in any row is to the left of any leading 1's in the rows below it.

Example 3.5:

Solve the following system of equations by using the Gauss-Jordan elimination method.

$$x + y + z = 5$$
$$2x + 3y + 5z = 8$$
$$4x + 5z = 2$$

Solution:

The augmented matrix of the system is the following.

$$\begin{bmatrix} 1 & 1 & 1 & \vdots 5 \\ 2 & 3 & 5 & \vdots 8 \\ 4 & 0 & 5 & \vdots 2 \end{bmatrix}$$

We will now perform row operations until we obtain a matrix in reduced row echelon form.

$$\begin{bmatrix} 1 & 1 & 1 & \vdots 5 \\ 2 & 3 & 5 & \vdots 8 \\ 4 & 0 & 5 & \vdots 2 \end{bmatrix} \xrightarrow[\text{-4R}_1+\text{R}_3]{\text{-2R}_1+\text{R}_2} \begin{bmatrix} 1 & 1 & 1 & \vdots 5 \\ 0 & 1 & 3 & \vdots -2 \\ 0 & -4 & 1 & \vdots -18 \end{bmatrix} \xrightarrow[\text{-4R}_1+\text{R}_3]{\text{4R}_2+\text{R}_3} \begin{bmatrix} 1 & 1 & 1 & \vdots 5 \\ 0 & 1 & 3 & \vdots -2 \\ 0 & 0 & 13 & \vdots -26 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{13}\text{R}_3} \begin{bmatrix} 1 & 1 & 1 & \vdots 5 \\ 0 & 1 & 3 & \vdots -2 \\ 0 & 0 & 1 & \vdots -2 \end{bmatrix} \xrightarrow[\text{-R}_3+\text{R}_1]{\text{-3R}_3+\text{R}_2} \begin{bmatrix} 1 & 1 & 0 & \vdots 7 \\ 0 & 1 & 0 & \vdots 4 \\ 0 & 0 & 1 & \vdots -2 \end{bmatrix} \xrightarrow{\text{-R}_2+\text{R}_1} \begin{bmatrix} 1 & 0 & 0 & \vdots 3 \\ 0 & 1 & 0 & \vdots 4 \\ 0 & 0 & 1 & \vdots -2 \end{bmatrix}$$

From this final matrix, we can read the solution of the system.

Hence the solution of the system is given by: $x = 3$, $y = 4$, $z = -2$.

Example 3.6:

Solve the following system by Gauss-Jordan elimination method.

$$2x_1 + 3x_2 + 4x_3 = 19$$
$$x_1 + 2x_2 + x_3 = 4$$
$$3x_1 - x_2 - x_3 = 9$$

Solution:

The augmented matrix of the system is: $\left[\begin{array}{ccc|c} 2 & 3 & 4 & 19 \\ 1 & 2 & 1 & 4 \\ 3 & -1 & 1 & 9 \end{array}\right]$.

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 19 \\ 1 & 2 & 1 & 4 \\ 3 & -1 & 1 & 9 \end{array}\right] \xrightarrow{R_1 \longleftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 3 & 4 & 19 \\ 3 & -1 & 1 & 9 \end{array}\right] \xrightarrow[R_3 \longrightarrow R_3-3R_1]{R_2 \longrightarrow R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -1 & 2 & 11 \\ 0 & -7 & -2 & -3 \end{array}\right]$$

$$\xrightarrow{R_2 \longrightarrow -R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -2 & -11 \\ 0 & -7 & -2 & -3 \end{array}\right] \xrightarrow{R_3 \longrightarrow R_3+7R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -2 & -11 \\ 0 & 0 & -16 & -80 \end{array}\right] \xrightarrow{R_3 \longrightarrow -\frac{1}{16}R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -2 & -11 \\ 0 & 0 & 1 & 5 \end{array}\right] \xrightarrow{R_1 \longrightarrow R_1-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 26 \\ 0 & 1 & 0 & -11 \\ 0 & 0 & 1 & 5 \end{array}\right] \xrightarrow[R_2 \longrightarrow R_2+2R_3]{R_1 \longrightarrow R_1-5R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Our solution is simply the right-hand side of the augmented matrix. Notice that the coefficient matrix is now a diagonal matrix with ones on the diagonal. This is a special matrix called the **identity matrix**.

Hence the above system is reduced to the linear system:

$$\begin{aligned} x_1 \quad\quad &= 1 \\ x_2 \quad &= -1 \\ x_3 &= 5 \end{aligned}$$

Hence the solution of the system is $(x_1, x_2, x_3) = (1, -1, 5)$.

When performing calculations by hand, many individuals choose Gauss-Jordan Elimination over Gaussian Elimination because it avoids the need for back substitution. However, that Gauss-Jordan elimination involves slightly more work than does Gaussian elimination, and thus it is not the method of choice for solving systems of linear equations on a computer.

Self-Assessment Exercise 3.2

1. What is the difference between Gauss Jordan and Gauss elimination methods?
2. Solve the following system by Gauss-Jordan elimination method:

a) $$\begin{aligned} -2x_1 + x_2 + 3x_3 &= -7 \\ x_1 - 4x_2 + 2x_3 &= 0 \\ x_1 - 3x_2 + x_3 &= 1 \end{aligned}$$

b) $$\begin{aligned} 3x_1 + x_2 - 5x_3 + x_4 &= 3 \\ x_1 - x_2 + x_3 - x_4 &= 1 \\ 2x_1 + 2x_2 - 6x_3 - x_4 &= 2 \end{aligned}$$

c) $$\begin{aligned} 2x_1 - x_2 - x_3 &= 0 \\ 4x_1 - 8x_2 + 3x_3 &= 0 \\ -2x_1 + 4x_2 + x_3 &= 0 \end{aligned}$$

3.2.3. LU Decomposition Method

Some linear equations $AX = B$ are relatively easy to solve. For example, if A is a lower triangular matrix, then, the elements of X can be computed recursively using *forward substitution.* Similarly if A is an upper triangular matrix, then the elements of X can be computed recursively using *backward substitution.*

Most linear equations encountered in practice, do not have a triangular coefficient matrix. In such cases, the linear equation is often best solved using the *L-U factorization Method or L-U decomposition Method*. The L-U factorization method is designed to decompose the *coefficient* matrix into the product of lower and upper triangular matrices, allowing the linear equation to be solved using a combination of backward and forward substitution.

The L-U factorization method involves two phases:

- In the first *factorization* phase, Gaussian elimination is used to factor the matrix A into the product $A=LU$ of a lower triangular matrix L and an upper-triangular matrix U.
- In the *solution* phase of the L-U factorization method, the factored linear equations $AX = (LU)X = L(UX) = B$ is solved by first solving $LY=B$ for Y using forward substitution, and then solving $UX = Y$ for X using backward substitution.

To decompose A as A = LU, we have:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Hence

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & (l_{11}u_{12}) & (l_{11}u_{13}) \\ l_{21} & (l_{21}u_{12} + l_{22}) & (l_{21}u_{13} + l_{22}u_{23}) \\ l_{31} & (l_{31}u_{12} + l_{32}) & (l_{31}u_{13} + l_{32}u_{23} + l_{33}) \end{bmatrix}$$

We can find, therefore, the elements of the matrices L and U by equating the above two matrices:

$$l_{11} = a_{11};$$
$$l_{21} = a_{21};$$
$$l_{31} = a_{31}$$

$$l_{11}u_{12} = a_{12}, \textit{hence } u_{12} = \frac{a_{12}}{l_{11}} = \frac{a_{12}}{a_{11}}$$

$$l_{21}u_{12} + l_{22} = a_{22}, \textit{hence } l_{22} = a_{22} - l_{21}u_{12}$$
$$l_{31}u_{12} + l_{32} = a_{32}, \textit{hence } l_{32} = a_{32} - l_{31}u_{12}$$

$$l_{11}u_{13} = a_{13}, hence\ u_{13} = \frac{a_{13}}{l_{11}} = \frac{a_{13}}{a_{11}}$$

$$l_{21}u_{13} + l_{22}u_{23} = a_{23}, hence\ u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}}$$

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = a_{33}, hence\ l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$$

Example 3.7:

Solve the following system of equations using the method of L-U factorization.

$$2x_1 + x_2 - x_3 = 5$$
$$x_1 + 3x_2 + 2x_3 = 5$$
$$3x_1 - 2x_2 - 4x_3 = 3$$

Solution:

The corresponding coefficient matrix of the system is given as: $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ 3 & -2 & -4 \end{bmatrix}$.

From the equation A = LU we can determine L and U as follows:

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ 3 & -2 & -4 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$ which is equivalent to:

$l_{11} = a_{11} = 2;$ $\quad l_{21} = a_{21} = 1;$ $\quad l_{31} = a_{31} = 3;$

$l_{11}u_{12} = a_{12}, hence$ $\quad u_{12} = \frac{a_{12}}{l_{11}} = \frac{a_{12}}{a_{11}} = \frac{1}{2};$

$l_{21}u_{12} + l_{22} = a_{22}, hence$ $\quad l_{22} = a_{22} - l_{21}u_{12} = 3 - \frac{1}{2} = \frac{5}{2};$

$l_{31}u_{12} + l_{32} = a_{32}, hence$ $\quad l_{32} = a_{32} - l_{31}u_{12} = -2 - 3 * \frac{1}{2} = \frac{-7}{2}$

$l_{11}u_{13} = a_{13}, hence$ $\quad u_{13} = \frac{a_{13}}{l_{11}} = \frac{a_{13}}{a_{11}} = \frac{-1}{2}$

$l_{21}u_{13} + l_{22}u_{23} = a_{23}, hence$ $\quad u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}} = \frac{2 + \frac{1}{2}}{\frac{5}{2}} = 1$

$l_{31}u_{13} + l_{32}u_{23} + l_{33} = a_{33}, hence$ $\quad l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} = -4 - 3 * \frac{-1}{2} - \frac{-7}{2} = 1$

Thus $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ 3 & -2 & -4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 5/2 & 0 \\ 3 & -7/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

As you can see, we got the decomposition right, as the multiplication of the L and U gives the original matrix A.

The original equation is equivalent to: AX = B. That is LUX =LY= B, where UX = Y.

(i) Now first **LY**= B implies $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 5/2 & 0 \\ 3 & -7/2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix}$

which is equivalent to the system: $\begin{aligned} 2y_1 &= 5 \\ y_1 + 5/2\, y_2 &= 5 \\ 3y_1 - 7/2\, y_2 + y_3 &= 3 \end{aligned}$

Solving this system of equations by forward substitution, we get:

$y_1 = 5/2$

$y_2 = \dfrac{5 - 5/2}{\frac{5}{2}} = 1$

$y_3 = 3 - \dfrac{15}{2} + \dfrac{7}{2} = -1$

Thus Y = (y_1, y_2, y_3) = (5/2, 1, -1).

(ii) From UX = Y, we have $\begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1 \\ -1 \end{bmatrix}$

which is equivalent to the system: $\begin{aligned} x_1 + 1/2\, x_2 - 1/2\, x_3 &= 5/2 \\ x_2 + x_3 &= 1 \\ x_3 &= -1 \end{aligned}$

Solving this system of equations by back substitution, we get:

$x_1 = \dfrac{5}{2} - \dfrac{1}{2}(2) + \dfrac{1}{2}(-1) = 1$

$x_2 = 1 - (-1) = 2$

$x_3 = -1$

Thus the solution of the original system of equations is given by: X = (x_1, x_2, x_3) = (1, 2, -1).

Example 3.8:

The currents $I_1, I_2,$ *and* I_3 occurring in a closed circuit having three mesh points is given by:

$$-I_1 + 3I_2 = 5$$
$$3I_1 + I_2 + 2I_3 = 3$$
$$-11I_2 - 7I_3 = -15$$

Use the LU decomposition method to determine the currents I_1, I_2, *and* I_3.

Solution:
The system can be written as AI = B, where

$$A = \begin{bmatrix} -1 & 3 & 0 \\ 3 & 1 & 2 \\ 0 & -11 & -7 \end{bmatrix}, \; I = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 3 \\ -15 \end{bmatrix}$$

We can decompose the coefficient matrix A as: $[A] = [L][U]$. That is:

$$\begin{bmatrix} -1 & 3 & 0 \\ 3 & 1 & 2 \\ 0 & -11 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Solving this gives:

$u_{11} = -1$; $\quad u_{12} = 3$; $\quad u_{13} = 0$

$l_{21}u_{11} = 3$, *hence* $\quad l_{21} = -3$

$l_{21}u_{12} + u_{22} = 1$, *hence* $\quad u_{22} = 10$

$u_{23} = 2$; $\quad l_{31} = 0$

$l_{32}u_{22} = -11$, *hence* $\quad l_{32} = \frac{-11}{10}$

$l_{32}u_{23} + u_{33} = -7$, *hence* $\quad u_{33} = \frac{-24}{5}$

Hence AI = B implies LUI = B. That is: $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & \frac{-11}{10} & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 0 \\ 0 & 10 & 2 \\ 0 & 0 & \frac{-24}{5} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -15 \end{bmatrix}$

Now assuming Y = UI, we have: LY = B. That is $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & \frac{-11}{10} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -15 \end{bmatrix}$.

And solving this system, we have: $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ \frac{24}{5} \end{bmatrix}$.

From UI = Y, we have $\begin{bmatrix} -1 & 3 & 0 \\ 0 & 10 & 2 \\ 0 & 0 & \frac{-24}{5} \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ \frac{24}{5} \end{bmatrix}$ which is equivalent to the system

$$-I_1 + 3I_2 = 5$$
$$10I_2 + 2I_3 = 18$$
$$\frac{-24}{5}I_3 = \frac{24}{5}$$

Solving this system gives $\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

Therefore the required answer is $I_1 = 1$, $I_2 = 2$ and $I_3 = -1$

Self-Assessment Exercise 3.3

1. Write a procedure to apply LU Decomposition Method?

2. Solve the following system of equations by LU Decomposition method.

$$x + y = 9$$
$$2x - 3y + 4z = 13$$
$$5y + 5z = 23$$

3.3. Iterative Mthods for Solving Systems of Linear Equations

The previous section considered the approximation of the solution of a linear system using *direct methods*, techniques that would produce the exact solution if all the calculations were performed using exact arithmetic. In this section we describe some popular *iterative* techniques which require an initial approximation to the solution. These methods will not be expected to return the exact solution even if all the calculations could be performed using exact arithmetic.

Because of round-off errors, direct methods become less efficient than iterative methods when they applied to large systems. In addition, the amount of storage space required for iterative solutions on a computer is far less than the one required for direct methods when the coefficient matrix of the system is sparse. Thus, especially for sparse matrices, iterative methods are more attractive than direct methods.

For the iterative solution of a system of equations, one starts with an arbitrary starting vector x^0 and computes a sequence of iterates x^m for m=1,2,3, … :

$$x^0 \mapsto x^1 \mapsto x^2 \mapsto \cdots \mapsto x^m \mapsto x^{m+1} \mapsto \cdots$$

The iterative methods are not always successful to all systems of equations. If the methods are to succeed, each equation of the system must possess one large coefficient and the large coefficients are along the leading diagonal of the coefficient matrix. That is iterative methods will be successful only when the system is diagonally dominant system.

For example let us consider a 3x3 linear system of equations.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

A diagonally dominant system is a system which satisfies the following conditions.

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

In general an nxn matrix A is diagonally dominant if $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|$

Under the category of iterative method, we shall describe the following two methods:

(i) Gauss-Jacobi's method

(ii) Gauss-Seidel method.

3.3.1. Gauss-Jacobi Method

Let us consider again a system of 3 linear equations with 3 unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \qquad (1)$$

Rearranging the system of equations in (1) and solving for x_1, x_2, x_3 from the 1st, 2nd and 3rd equations respectively, we get:

$$\begin{aligned} x_1 &= \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3) \\ x_2 &= \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3) \\ x_3 &= \frac{1}{a_{33}}(b_3 - a_{31}x_1 - a_{32}x_2) \end{aligned} \qquad (2)$$

Let $x_1^{(0)}$, $x_2^{(0)}$, $x_3^{(0)}$ be the initial approximations of the unknowns x_1, x_2, *and* x_3. Then substituting these initial guesses of the unknowns in the governing equation in (2), we get the first approximations as:

$$x_1^{(1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right)$$

$$x_2^{(1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(0)} - a_{23}x_3^{(0)}\right)$$

$$x_3^{(1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{(0)} - a_{32}x_2^{(0)}\right)$$

Similarly, the second approximations are given by:

$$x_1^{(2)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(1)} - a_{13}x_3^{(1)}\right)$$

$$x_2^{(2)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(1)}\right)$$

$$x_3^{(2)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}\right)$$

Proceeding in the same way, if $x_1^{(n)}$, $x_2^{(n)}$, $x_3^{(n)}$ are the nth iterations then:

$$x_1^{(n+1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(n)} - a_{13}x_3^{(n)}\right)$$

$$x_2^{(n+1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(n)} - a_{23}x_3^{(n)}\right)$$

$$x_3^{(n+1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{(n)} - a_{32}x_2^{(n)}\right)$$

The process is continued till convergence is secured.

Note 1: In the absence of any better estimates, the initial approximations are taken as $x_1^{(0)} = 0,\ x_2^{(0)} = 0, x_3^{(0)} = 0$.

2: We assume that the diagonal terms a_{11}, a_{22}, a_{33} are all nonzero and are the largest coefficients of x_1, x_2, x_3, respectively so that convergence is assured. If some of the diagonal elements are zeros, we may have to rearrange the equations.

Convergence Test

It is necessary to test for convergence of the iterates and the iterations will stop if the condition:

$\left\| x_i^{k+1} - x_i^{k} \right\| \le tolerance$ is satisfied, for i = 1, 2, 3, …, n.

In addition diagonally dominant system is a sufficient condition for Jacobi to converge.

Example 3.9:

Solve the following system of equations using Gauss-Jacobi method up to three decimal places accuracy:

$$\begin{aligned} 20x + y - 2z &= 17 \\ 3x + 20y - z &= -18 \\ 2x - 3y + 20z &= 25 \end{aligned} \qquad (1)$$

Solution:

The system in (1) is a diagonally dominant system. Hence the iteration will converge to the solution.

The given system of equation can be written in the form:

$$x = \frac{1}{20}(17 - y + 2z)$$
$$y = \frac{1}{20}(-18 - 3x + z) \qquad (2)$$
$$z = \frac{1}{20}(25 - 2x + 3y)$$

Let $x_0 = 0, y_0 = 0, z_0 = 0$ be the initial approximations of the unknowns $x,\ y, and\ z$.

First Approximation:

Then substituting these initial guesses of the unknowns in the above governing equation (2), we get the first approximations as:

$$x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = \frac{1}{20}(17 - 0 + 2(0)) = 0.85$$
$$y_1 = \frac{1}{20}(-18 - 3x_0 + z_0) = \frac{1}{20}(-18 - 3(0) + 0) = -0.9$$
$$z_1 = \frac{1}{20}(25 - 2x_0 + 3y_0) = \frac{1}{20}(25 - 2(0) + 3(0)) = 1.25$$

Second Approximation:

Substituting the first approximation in the governing equations (2), we get the second approximations as:

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = \frac{1}{20}(17 - (-0.9) + 2(1.25)) = 1.02$$
$$y_2 = \frac{1}{20}(-18 - 3x_1 + z_1) = \frac{1}{20}(-18 - 3(0.85) + 1.25) = -0.965$$
$$z_2 = \frac{1}{20}(25 - 2x_1 + 3y_1) = \frac{1}{20}(25 - 2(0.85) + 3(-0.9)) = 1.03$$

Third Approximation:

Substituting the second approximation in the governing equations (2), we get the third approximations as:

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = \frac{1}{20}(17 - (-0.965) + 2(1.03)) = 1.00125$$
$$y_3 = \frac{1}{20}(-18 - 3x_2 + z_2) = \frac{1}{20}(-18 - 3(1.02) + 1.03) = -1.0015$$
$$z_3 = \frac{1}{20}(25 - 2x_2 + 3y_2) = \frac{1}{20}(25 - 2(1.02) + 3(-0.965)) = 1.00325$$

Fourth Approximation:

Substituting the third approximation in the governing equations (2), we get the fourth approximations as:

$$x_4 = \frac{1}{20}(17 - y_3 + 2z_3) = \frac{1}{20}(17 - (-1.0015) + 2(1.00325)) = 1.0004$$

$$y_4 = \frac{1}{20}(-18 - 3x_3 + z_3) = \frac{1}{20}(-18 - 3(1.00125) + 1.00325) = -1.000025$$

$$z_4 = \frac{1}{20}(25 - 2x_3 + 3y_3) = \frac{1}{20}(25 - 2(1.00125) + 3(-1.0015)) = 0.99965$$

To terminate the iteration, we check the error of approximations and compare them with the given precision, that is, $\left\|x_i^{k+1} - x_i^{k}\right\| \leq TOL$. Here TOL = three decimal places accuracy = 0.0005.

Hence $|x_4 - x_3| = |1.0004 - 1.00125| = 0.00085 \geq TOL$

$$|y_4 - y_3| = |-1.000025 - (-1.0015| = 0.001475 \geq TOL$$

$$|z_4 - z_3| = |0.99965 - 1.00325| = 0.0036 \geq TOL$$

Since the termination criteria is not satisfied, we cannot stop at the fourth iteration.

Fifth Approximation:

Substituting the fourth approximation in the governing equations (2), we get the fifth approximations as:

$$x_5 = \frac{1}{20}(17 - y_4 + 2z_4) = \frac{1}{20}(17 - (-1.000025) + 2(0.99965)) = 0.999966$$

$$y_5 = \frac{1}{20}(-18 - 3x_4 + z_4) = \frac{1}{20}(-18 - 3(1.0004) + 0.99965) = -1.000075$$

$$z_5 = \frac{1}{20}(25 - 2x_4 + 3y_4) = \frac{1}{20}(25 - 2(1.0004) + 3(-1.000025)) = 0.999956$$

Now let us check the error:

$$|x_5 - x_4| = |1.0004 - 0.999966| = 0.000434 \leq TOL$$

$$|y_5 - y_4| = |-1.000025 - (-1.000075| = 0.00005 \leq TOL$$

$$|z_5 - z_4| = |0.999956 - 0.99965| = 0.0003 \leq TOL$$

Since the termination criteria is now satisfied, we can stop the approximation at this step. Hence the solution is (x, y, z) = (0.999966, -1.000075, 0.999956).

Proceeding in the above manner the values of x, y and z approaches 1, -1 and 1 respectively. Hence the solution of the above system can be given by (x, y, z) = (1, -1, 1).

Example 3.10:

Solve the following system of equations by Jacobi's iterations method:

$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$2x + 2y + 10z = 14 \qquad (1)$$

Solution:

The system in (1) is a diagonally dominant system. Hence the iteration will converge to the solution.

The given system of equation can be written in the form:

$$x = \frac{1}{10}(12 - y - z)$$
$$y = \frac{1}{10}(13 - 2x - z) \qquad (2)$$
$$z = \frac{1}{10}(14 - 2x - 2y)$$

Let $x_0 = 0, y_0 = 0,\ z_0 = 0$ be the initial approximations of the unknowns $x,\ y, and\ z$.

First Approximation:

Then substituting these initial guesses of the unknowns in the above governing equation (2), we get the first approximations as:

$$x_1 = \frac{1}{10}(12 - y_0 - z_0) = \frac{1}{10}(12 - 0 - 0) = 1.2$$
$$y_1 = \frac{1}{10}(13 - 2x_0 - z_0) = \frac{1}{10}(13 - 2(0) - 0) = 1.3$$
$$z_1 = \frac{1}{10}(14 - 2x_0 - 2y_0) = \frac{1}{10}(14 - 2(0) - 2(0)) = 1.4$$

Second Approximation:

Substituting the first approximation in the governing equations (2), we get the second approximations as:

$$x_2 = \frac{1}{10}(12 - y_1 - z_1) = \frac{1}{10}(12 - 1.3 - 1.4) = 0.93$$
$$y_2 = \frac{1}{10}(13 - 2x_1 - z_1) = \frac{1}{10}(13 - 2(1.2) - 1.4) = 0.92$$
$$z_2 = \frac{1}{10}(14 - 2x_1 - 2y_1) = \frac{1}{10}(14 - 2(1.2) - 2(1.3)) = 0.9$$

Third Approximation:

Substituting the second approximation in the governing equations (2), we get the third approximations as:

$$x_3 = \frac{1}{10}(12 - y_2 - z_2) = \frac{1}{10}(12 - 0.92 - 0.9) = 1.018$$

$$y_3 = \frac{1}{10}(13 - 2x_2 - z_2) = \frac{1}{10}(13 - 2(0.93) - 0.9) = 1.024$$

$$z_3 = \frac{1}{10}(14 - 2x_2 - 2y_2) = \frac{1}{10}(14 - 2(0.93) - 2(0.92)) = 1.03$$

Fourth Approximation:

Substituting the third approximation in the governing equations (2), we get the fourth approximations as:

$$x_4 = \frac{1}{10}(12 - y_3 - z_3) = \frac{1}{10}(12 - 1.024 - 1.03) = 0.9946$$

$$y_4 = \frac{1}{10}(13 - 2x_3 - z_3) = \frac{1}{10}(13 - 2(1.018) - 1.03) = 0.9934$$

$$z_4 = \frac{1}{10}(14 - 2x_3 - 2y_3) = \frac{1}{10}(14 - 2(1.018) - 2(1.024)) = 0.9916$$

Proceeding in this manner, the successive approximations of x, y and z approaches 1, 1 and 1 respectively.

Hence the solution is (x, y, z) = (1, 1, 1).

Self-Assessment Exercise 3.4

1. What is the convergence criterion for Jacobi Method?
2. A civil engineer builds three types of buildings; each type requires metal, cement, and wood in the following composition.

Product type	Metal (tones)	Cement (tones)	Wood (tones)
A	5	1	1
B	2	4	2
C	1	2	5

If he/she has 12 kg of metal, 15 of cement and 20 of wood in total, how many buildings of each type can be constructed at most?

3.3.2. Gauss-Seidel Method

Gauss-Seidel is almost the same as for Jacobi, except that each x-value is improved using the most recent approximations to the values of the other variables.Therefore **Gauss-Seidel** is a modification of Gauss-Jacobi method. Because the new values can be immediately stored in the location that held the old values, the storage requirement for x with the Gauss-Seidel method is half what it would be the Jacobi method and the rate of convergence is more rapid.

Let us again consider the system of simultaneous linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \qquad (1)$$

Rearranging the above system of equations and rewriting in terms of x_1, x_2, x_3, we get:

$$\begin{aligned} x_1 &= \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3) \quad \text{(i)} \\ x_2 &= \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3) \quad \text{(ii)} \\ x_3 &= \frac{1}{a_{33}}(b_3 - a_{31}x_1 - a_{32}x_2) \quad \text{(iii)} \end{aligned} \qquad (2)$$

We start with the initial approximations $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}$

Substituting $x_2^{(0)}$ *and* $x_3^{(0)}$ in (i) of the governing equation in (2), we get:

$$x_1^{(1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right)$$

Substituting $x_1^{(1)}$ *and* $x_3^{(0)}$ in (ii) of the governing equation in (2), we get:

$$x_2^{(1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}\right)$$

Again Substituting $x_1^{(1)}$ *and* $x_2^{(1)}$ in (iii) of the governing Equation in (2), we get:

$$x_3^{(1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}\right)$$

This process is continued till the value of x_1, x_2, x_3 are obtained to the desired degree of accuracy. In general, *the* $(n+1)^{th}$ iteration can be written as:

$$x_1^{(n+1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(n)} - a_{13}x_3^{(n)}\right)$$

$$x_2^{(n+1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(n-1)} - a_{23}x_3^{(n)}\right)$$

$$x_3^{(n-1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{(n+1)} - a_{32}x_2^{(n+1)}\right)$$

Example 3.11:

Solve the following system of equations using Gauss-Seidel method up to three decimal places accuracy.

$$\begin{aligned} 20x + y - 2z &= 17 \\ 3x + 20y - z &= -18 \\ 2x - 3y + 20z &= 25 \end{aligned} \qquad (1)$$

Solution:

The system in (1) is a diagonally dominant system. Hence the iteration will converge to the solution. Solving for x, y and z from the 1st, 2nd and 3rd equations respectively of the given system, we get the following governing system of equations:

$$x = \frac{1}{20}(17 - y + 2z) \quad \text{(i)}$$

$$y = \frac{1}{20}(-18 - 3x + z) \quad \text{(ii)} \qquad (2)$$

$$z = \frac{1}{20}(25 - 2x + 3y) \quad \text{(iii)}$$

Start with the initial guesses $x_0 = 0,\ y_0 = 0,\ z_0 = 0$.

First Approximation:

Substituting $y_0 = 0, and\ z_0 = 0$ in (i) of (2), we obtain:

$$x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = \frac{1}{20}(17 - 0 + 2(0)) = 0.85$$

Substituting $x_1 = 0.85,\ and\ z_0 = 0$ in (ii) of (2), we obtain:

$$y_1 = \frac{1}{20}(-18 - 3x_1 + z_0) = \frac{1}{20}(-18 - 3(0.85) + 0) = -1.0275$$

Substituting $x_1 = 0.85, and\ y_1 = -1.0275$ in (iii) of (2), we obtain:

$$z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1) = \frac{1}{20}(25 - 2(0.85) + 3(-1.0275)) = 1.0109$$

Second Approximation:

Substituting $y_1 = -1.0275, and\ z_1 = 1.0109$ in (i) of (2), we obtain:

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = \frac{1}{20}(17 - (-1.0275) + 2(1.0109)) = 1.0025$$

Substituting $x_2 = 1.0025, and\ z_1 = 1.0109$ in (ii) of (2), we obtain:

$$y_2 = \frac{1}{20}(-18 - 3x_2 + z_1) = \frac{1}{20}(-18 - 3(1.0025) + 1.0109) = -0.9998$$

Substituting $x_2 = 1.0025, and\ y_2 = -0.9998$ in (iii) of (2), we obtain:

$$z_2 = \frac{1}{20}(25 - 2x_2 + 3y_2) = \frac{1}{20}(25 - 2(1.0025) + 3(-0.9998)) = 0.9998$$

Third Approximation:

Substituting $y_2 = -0.9998, and\ z_2 = 0.9998$ in (i) of (2), we obtain:

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = \frac{1}{20}(17 - (-0.9998) + 2(0.9998)) = 0.99997$$

Substituting $x_3 = 0.99997,\ and\ z_2 = 0.9998$ in (ii) of (2), we obtain:

$$y_3 = \frac{1}{20}\left(-18 - 3x_3 + z_2\right) = \frac{1}{20}\left(-18 - 3(0.99997) + 0.9998\right) = -1.00000$$

Substituting $x_3 = 0.99997, and\ y_3 = -1.00000$ in (iii) of (2), we obtain:

$$z_3 = \frac{1}{20}\left(25 - 2x_3 + 3y_3\right) = \frac{1}{20}\left(25 - 2(0.99997) + 3(-1.00000)\right) = 1.00000$$

Proceeding in this manner the values x, y and z approaches 1, -1 and 1 respectively with error nearly zero.

Hence the solution of the above system is given by (x, y, z) = (1, -1, 1).

Example 3.12:

Solve the following system of equations using Gauss-Seidel method:

$$\begin{aligned} 10x + y + 2z &= 44 \\ 2x + 10y + z &= 51 \\ x + 2y + 10z &= 61 \end{aligned} \qquad (1)$$

Solution:

The system in (1) is a diagonally dominant system. Hence the iteration will converge to the solution. Solving for x, y and z from the first, second and third equations of the system respectively, we get the following governing equations:

$$\begin{aligned} x &= \frac{1}{10}(44 - y - 2z) \quad \text{(i)} \\ y &= \frac{1}{10}(51 - 2x - z) \quad \text{(ii)} \\ z &= \frac{1}{10}(61 - x - 2y) \quad \text{(iii)} \end{aligned} \qquad (2)$$

Start with the initial guesses $x_0 = 0, y_0 = 0, z_0 = 0$.

First Approximation:

Substituting $y_0 = 0, and\ z_0 = 0$ in (i) of (2), we obtain:

$$x_1 = \frac{1}{10}\left(44 - y_0 - 2z_0\right) = \frac{1}{10}\left(44 - 0 - 2(0)\right) = 4.4$$

Substituting $x_1 = 4.4, and\ z_0 = 0$ in (ii) of (2), we obtain:

$$y_1 = \frac{1}{10}\left(51 - 2x_1 - z_0\right) = \frac{1}{10}\left(51 - 2(4.4) - 0\right) = 4.22$$

Substituting $x_1 = 4.4, and\ y_1 = 4.22$ in (iii) of (2), we obtain:

$$z_1 = \frac{1}{10}\left(61 - x_1 - 2y_1\right) = \frac{1}{10}\left(61 - 4.4 - 2(4.22)\right) = 4.816$$

Second Approximation:

Substituting $y_1 = 4.22, and\ z_1 = 4.816$ in (i) of (2), we obtain:

$$x_2 = \frac{1}{10}(44 - y_1 - 2z_1) = \frac{1}{10}(44 - 4.22 - 2(4.816)) = 4.0154$$

Substituting $x_2 = 4.0157, and\ z_1 = 4.816$ in (ii) of (2), we obtain:

$$y_2 = \frac{1}{10}(51 - 2x_2 - z_1) = \frac{1}{10}(51 - 2(4.0154) - 4.816) = 3.0148$$

Substituting $x_2 = 4.0154, and\ y_2 = 3.0148$ in (iii) of (2), we obtain:

$$z_2 = \frac{1}{10}(61 - x_2 - 2y_2) = \frac{1}{10}(61 - 4.0154 - 2(3.0148)) = 5.0955$$

Third Approximation:

Proceeding in the same manner, the third approximation is:

$x_3 = 3.0794$, $y_3 = 3.9746$, $z_3 = 4.9971$.

Similarly, if we proceed up to eighth approximation, then, we obtain

$x_8 = 3.0000$, $y_8 = 4.0000$, $z_8 = 5.0000$

Hence the solution is given by (x, y, z) = (3, 4, 5)

Algorithm for Gauss Seidel Method

Objective: To solve A**x** = **B** given an initial approximation **XO**.

INPUT: the number of unknowns, n; the entries a_{ij} of the coefficient matrix, A; the entries b_i of the constant matrix, B; the initial guess of the solution, XO; tolerance, TOL and maximum number of iterations, N.

OUTPUT: the approximate solution $x_1, x_2, \ldots, x_n$ or message that the number of iterations was exceeded.

Step 1: Set k = 1

Step 2: While ($k \le N$) do Steps 3 – 6:

Step 3: For i = 1, …, n

Set $\frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j - \sum_{j=i+1}^{n} a_{ij}XO_j\right]$

Step 4: If $\|x - XO\| < TOL$, then OUTPUT ($x_1, x_2, \ldots, x_n$)

STOP (The procedure was successful)

Step 5: Set k = k+1

Step 6: For i = 1, …, n

set $XO_i = x_i$

Step 7: OUTPUT (“Maximum number of iterations exceeded”)

STOP (The procedure was unsuccessful)

Self-Assessment Exercise 3.5

1. Given a linear system of equations AX = B, what condition must the elements of A satisfy in order to guarantee convergence of the Gauss-Seidel Method?
2. What is the difference between Gauss Jacobi and Gauss Seidel Iterative methods?
3. Solve the following system of equations using the Gauss-Seidal iteration method. Assume (0, 0, 0) is the initial guess of solution.

$$25x + y - z = 28$$
$$x + 30y + 2z = 59$$
$$3x - 2y - 20z = 19$$

Miscellaneous Exercise

1. The currents $I_1, I_2, and\ I_3$ occurring in a closed circuit having three mesh points is given by

$$5I_1 - 2I_2 + I_3 = 9$$
$$-2I_1 + 10I_2 + 2I_3 = 8$$
$$I_1 + 2I_2 - 5I_3 = -1$$

Determine the currents $I_1, I_2, and\ I_3$ using ***Gauss Seidel*** method starting with $(I_1^{(0)}, I_2^{(0)}, I_3^{(0)}) = (0,0,0)$. [ANS: (2, 1, 1)]

2. Solve the following system of equations using Gauss-Elimination method:

(a) $x - y + z = 1$
$-3x + 2y - 3z = -6$
$2x - 5y + 4z = 5$ **[ANS: –2, 3, 6]**

(b) $x + 3y + 6z = 2$
$x - 4y + 2z = 7$
$3x - y + 4z = 9$ **[ANS: 2, –1,1/2]**

(c) $5x + y + z + u = 4$
$x + 7y + z + u = 12$
$x + y + 6z + u = -5$
$x + y + z + u = -6$ **[ANS: 1, 2, –1, –2]**

3. Solve the following system of equations by using Gauss Seidel and Gauss Jacobi methods.

a) 2x1 – 5 x2 + x3 = 12;
-x1 + 3x2 – x3 = -8;
-3x1 – 4x2 + 2x3 = 16.

b) 10x + 2y + z = 9
2x + 20y – 2z = - 44
-2x + 3y + 10z = 22 **[ANS: (x, y, z) = (1, -2,**
3)]

b) 10x1 – 2x2 – x3 – x4 = 3;
-2x1 + 10x2 – x3 – x4 = 15;
$-x_1 - x_2 + 10x_3 - 2x_4 = 27$;
$-x_1 - x_2 - 2x_3 + 10x_4 = -9$

Unit Summary

An important problem of applied mathematics is to find the solution of systems of linear equations which arises in a large number of areas. A system of equations $[A][X]=[B]$ is consistent if there is a solution, and it is inconsistent if there is no solution. However, a consistent system of equations does not mean a unique solution, that is, a consistent system of equations may have a unique solution or infinitely many solutions.

Numerical methods for solving linear algebraic systems may be divided into two types: direct and iterative.

Gaussian Elimination is one of the most popular techniques for solving simultaneous linear equations. Variables are eliminated from the equations until the coefficient matrix of the system is changed to upper triangular matrix so that variables are evaluated by back substitution.

Division by zero, round off error and ill conditioning are some pitfalls of Gaussian elimination method. So to avoid division by zero as well as reduce round-off error, Gaussian elimination with partial pivoting is the method of choice.

The L-U factorization method is designed to decompose the *coefficient* matrix into the product of lower and upper triangular matrices, allowing the linear equation to be solved using a combination of backward and forward substitution.

Because of round-off errors, direct methods become less efficient than iterative methods when they applied to large systems. In addition, the amount of storage space required for iterative solutions on a computer is far less than the one required for direct methods when the coefficient matrix of the system is sparse. That is iterative methods will be successful only when the system is diagonally dominant system.

Gauss-Seidel is almost the same as for Jacobi, except that each x-value is improved using the most recent approximations to the values of the other variables.Therefore **Gauss-Seidel** is a modification of Gauss-Jacobi method.

Further Reading for the unit

- R.L. Burden and D. Faires, Numerical Analysis-Fifth Edition PWS-Kent Publishing Co. Boston , 1985.
- Chapra, S.C. & Canal, R. P., "Numerical Methods for Engineers", 4th Ed., Tata McGraw Hill Pub. Co. Ltd., New Delhi, 2007.
- Sastry, S.S., "Introductory Methods of Numerical Analysis", Prentice Hall of India Ltd, New Delhi.
- Grewal B.S., Grewal J.S., "Numerical Methods in Engineering and Science", Khanna Publishers, New Delhi, 1999.