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Numerical Analysis I

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Chapter 4

Finite Differences

4.1 INTRODUCTION

The calculus of finite differences deals with the changes that take place in the value of the dependent variable due to finite changes in the independent variable from this we study the relations that exist between the values, which can be assumed by function, whenever the independent variable changes by finite jumps whether equal or unequal.

4.2 Differences

Consider a function $y = f(x)$ defined on (a, b). x and y are the independent and dependent variables respectively. If the points $x_0, x_1, \dots, x_n$ are taken at equidistance i.e., $x_i = x_0 + ih, i = 0, 1, 2, \dots, n$, then the value of y, when $x = x_i$, is denoted as y_i, where $y_i = f(x_i)$. Here, the values of x are called **arguments** and the values of y are known as **entries**. The interval h is called the difference interval. To determine the values of function $y = f(x)$ for given intermediate values of x, the following three types of differences are useful:

4.2.1 Forward Differences

The forward difference or simply difference operator is denoted by Δ and it is defined as

$$\Delta f(x) = f(x+h) - f(x) \tag{4.1}$$

or writing in terms of y, at $x = x_i$, Eq.(4.1) becomes

$$\Delta f(x_i) = f(x_i + h) - f(x_i)$$

or $$\Delta y_i = y_{i+1} - y_i, \qquad i = 0, 1, 2, \dots, (n-1)$$

To be explicit, we write

$$\begin{aligned} \Delta y_0 &= y_1 - y_0 \\ \Delta y_1 &= y_2 - y_1 \\ &\vdots \\ \Delta y_{n-1} &= y_n - y_{n-1} \end{aligned}$$

These differences are called the first forward differences of the function y and are denoted by the symbol Δy.

The differences of the first differences are called the second differences and they are denoted by $\Delta^2 y_0, \Delta^2 y_1, \dots, \Delta^2 y_i$. That is,

$$\begin{aligned} \Delta^2 y_0 &= \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0 \\ \Delta^2 y_1 &= \Delta y_2 - \Delta y_1 = (y_3 - y_2) - (y_2 - y_1) = y_3 - 2y_2 + y_1 \\ &\vdots \\ \Delta^2 y_{n-1} &= \Delta y_n - \Delta y_{n-1} \end{aligned}$$

Similarly, $\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0$

$$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1 = y_4 - 3y_3 + 3y_2 - y_1,$$
$$\vdots$$
$$\Delta^3 y_{n-1} = \Delta^2 y_n - \Delta^2 y_{n-1}$$

In general, nth forward difference are given by

$$\Delta^n y_i = \Delta^{n-1} y_{i+1} - \Delta^{n-1} y_i$$

or $\Delta^n f(x) = \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x), \;\; n = 1, 2, \ldots$

where $\Delta^0 \equiv$ identity operator i.e., $\Delta^0 f(x) = f(x)$ and $\Delta^1 = \Delta$.

The following table shows how the forward differences of all orders can be formed.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

Table 4.1: Forward differences

Example 4.1. *Construct a forward difference table for the following values:*

x	0	5	10	15	20
$f(x)$	7	11	14	18	24

Solution: Forward difference table for given data is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	7				
		4			
5	11		-1		
		3		2	
10	14		1		-1
		4		1	
15	18		2		
		6			
20	24				

Example 4.2. *If $y = x^3 + x^2 - 2x + 1$, calculate values of y for $x = 0, 1, 2, 3, 4, 5$ and form the difference table. Also find the value of y at $x = 6$ by extending the table and verify that the same value is obtained by substitution.*

Solution: For $x = 0, 1, 2, 3, 4, 5$, we get the values of y are $1, 1, 9, 31, 73, 141$. Therefore, difference table for these data is as:

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		0		
1	1		8	
		8		6
2	9		14	
		22		6
3	31		20	
		42		6
4	73		26	
		68		6
5	141		32	
		100		
6	$\boxed{241}$			

Because third differences are zero therefore

$$\begin{aligned}
& \Delta^3 y_3 = 6 && \Rightarrow && \Delta^2 y_4 - \Delta^2 y_3 = 6 \\
\Rightarrow\quad & \Delta^2 y_4 - 26 = 6 && \Rightarrow && \Delta^2 y_4 = 32 \\
\text{Now,}\quad & \Delta^2 y_4 = 32 && \Rightarrow && \Delta y_5 - \Delta y_4 = 32 \\
\Rightarrow\quad & \Delta y_5 - 68 = 32 && \Rightarrow && \Delta y_5 = 100 \\
\text{Further,}\quad & \Delta y_5 = 100 && \Rightarrow && y_6 - y_5 = 100 \\
\Rightarrow\quad & y_6 - 141 = 100 && \Rightarrow && y_6 = 241
\end{aligned}$$

Verification: For given function $x^3 + x^2 - 2x + 1$, at $x = 6, y(6) = (6)^3 + (6)^2 - 2(6) + 1 = 241$. Hence Verified.

Example 4.3. *Given* $f(0) = 3, f(1) = 12, f(2) = 81, f(3) = 200, f(4) = 100$ *and* $f(5) = 8$. *From the difference table and find* $\Delta^5 f(0)$.

Solution: The difference table for given data is as follows:

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	3					
		9				
1	12		60			
		69		-10		
2	81		50		-259	
		119		269		755
3	200		-219		496	
		-100		227		
4	100		8			
		-92				
5	8					

Hence, $\Delta^5 f(0) = 755$.

Example 4.4. *Find the function whose first difference is* e^x.

Solution: We know that $\Delta e^x = e^{x+h} - e^x = e^x(e^h - 1)$, where h is the interval of differencing.
Therefore, $e^x = \frac{1}{e^h-1}\Delta e^x = \Delta\left(\frac{e^x}{e^h-1}\right)$
Hence, required function is given by $\frac{e^x}{e^h-1}$.

4.2.2 Backward Differences

The backward difference operator is denoted by ∇ and it is defined as

$$\nabla f(x) = f(x) - f(x-h) \tag{4.2}$$

Equation (4.2) can be written as

$$\nabla y_i = y_i - y_{i-1}, \qquad i = n, (n-1), \ldots, 1$$

or

$$\begin{aligned} \nabla y_1 &= y_1 - y_0 \\ \nabla y_2 &= y_2 - y_1 \\ &\vdots \\ \nabla y_n &= y_n - y_{n-1} \end{aligned}$$

These differences are called first backward differences.
The second backward differences are denoted by $\nabla^2 y_2, \nabla^2 y_3, \ldots, \nabla^2 y_n$.
Hence

$$\begin{aligned} \nabla^2 y_2 &= \nabla y_2 - \nabla y_1 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0. \\ \nabla^2 y_3 &= \nabla y_3 - \nabla y_2 = y_3 - 2y_2 + y_1 \\ &\vdots \\ \nabla^2 y_n &= \Delta y_n - \Delta y_{n-1} \end{aligned}$$

Similarly,

$$\begin{aligned} \nabla^3 y_3 &= \nabla^2 y_3 - \nabla^2 y_2 \\ \nabla^3 y_4 &= \nabla^2 y_4 - \nabla^2 y_3 \\ \nabla^4 y_4 &= \nabla^3 y_4 - \nabla^3 y_3, \text{ and so on.} \end{aligned}$$

In general, kth backward difference are given by

$$\nabla^k y_i = \nabla^{k-1} y_i - \nabla^{k-1} y_{i-1}, \qquad i = n, (n-1), \ldots, k$$

or

$$\nabla^k f(x) = \nabla^{k-1} f(x) - \nabla^{k-1} f(x-h)$$

These backward differences can be systematically arranged for a table of values (x_i, y_i)

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
		∇y_4			
x_4	y_4				

Table 4.2: Backward differences

Example 4.5. *Given that:*

x	1	2	3	4	5	6
$f(x)$	1	8	27	64	125	216

Construct backward difference table and obtain $\nabla^5 f(6)$.

Solution: Backward difference table for given data is as:

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$	$\nabla^5 f(x)$
1	1					
		7				
2	8		12			
		19		6		
3	27		18		0	
		37		6		0
4	64		24		0	
		61		6		
5	125		30			
		91				
6	216					

Hence, $\nabla^5 f(6) = 0.$

Example 4.6. *Construct the backward difference table for* $y = \log x$ *given that:*

x	10	20	30	40	50
y	1	1.3010	1.4771	1.6021	1.6990

and find the values of $\nabla^3 \log 40$ *and* $\nabla^4 \log 50$.

Solution: For the given data, backward difference table as:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
10	1				
		0.3010			
20	1.3010		-0.1249		
		0.1761		0.0738	
30	1.4771		-0.0511		-0.0508
		0.1250		0.0230	
40	1.6021		-0.0281		
		0.0969			
50	1.6990				

Hence, $\nabla^3 \log 40 = 0.0738$ and $\nabla^4 \log 50 = -0.0508.$

4.2.3 Central Differences

The central difference operator is denoted by the symbol δ and is defined by

$$\delta f(x) = f(x + h/2) - f(x - h/2)$$

where h is the interval of differencing.

In terms of y, the first central difference is written as

$$\delta y_i = y_{i+1/2} - y_{i-1/2}$$

where $\quad y_{i+1/2} = f(x_i + h/2)$ and $y_{i-1/2} = f(x_i - h/2)$

Hence

$$\delta y_{1/2} = y_1 - y_0$$
$$\delta y_{3/2} = y_2 - y_1$$
$$\vdots$$
$$\delta y_{n-1/2} = y_n - y_{n-1}$$

The second central differences are given by

$$\delta^2 y_i = \delta y_{i+1/2} - \delta y_{i-1/2}$$
$$\delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2}$$
$$\delta^2 y_2 = \delta y_{5/2} - \delta y_{3/2} \text{ and so on.}$$

Generalising

$$\delta^n y_i = \delta^{n-1} y_{i+1/2} - \delta^{n-1} y_{i-1/2}$$

The central differences can be written in a tabular form as in Table 4.3.

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
x_0	y_0				
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

Table 4.3: Central differences

It is noted in Table 4.3 that all odd differences have fraction suffices and all the even differences are with integer suffices.

4.2.4 Other Difference Operators

(a) Shift operator, E:

The shift operator is defined as

$$Ef(x) = f(x + h),$$

or $\quad Ey_i = y_{i+1}$ for each $i = 0, 1, 2, \ldots$

Hence, shift operator shifts the function value y_i to the next higher value y_{i+1}.

The second shift operator gives

$$E^2 f(x) = E(Ef(x)) = E(f(x + h)) = f(x + 2h)$$

Generalising,

$$E^n f(x) = f(x + nh) \quad \text{or} \quad E^n y_i = y_{i+nh}$$

The inverse shift operator E^{-1} is defined as

$$E^{-1} f(x) = f(x - h),$$

In a similar manner, second and higher inverse operators are given by

$$E^{-2} f(x) = f(x - 2h) \quad \text{and} \quad E^{-n} f(x) = f(x - nh)$$

(b) Average operator, μ:

The operator μ is a averaging operator and is defined by,

$$\mu f(x) = \frac{1}{2}[f(x+h/2) + f(x-h/2)]$$

i.e.,
$$\mu y_i = \frac{1}{2}[y_{i+1/2} + y_{i-1/2}]$$

(c) Differential operator, D:

The differential operator for a function $y = f(x)$ is defined by

$$Df(x) = \frac{d}{dx}f(x)$$
$$D^2 f(x) = \frac{d^2}{dx^2}f(x)$$

4.2.5 Properties of Operators

1. The operators $\Delta, \nabla, E, \delta, \mu$ and D are all linear operators.

$$\begin{aligned} i.e., \quad \nabla[af(x+h) + bg(x+h)] &= [af(x+h) + bg(x+h)] - [af(x) + bg(x)] \\ &= a[f(x+h) - f(x)] + b[g(x+h) - g(x)] \\ &= a\nabla f(x+h) + b\nabla g(x+h) \end{aligned}$$

   Hence, ∇ is a linear operator.
   On substituting $a = 1, b = 1$, we get
   $$\nabla[f(x+h) + g(x+h)] = \nabla f(x+h) + \nabla g(x+h)$$
   Also on substituting $b = 0$, we get
   $$\nabla[af(x+h)] = a\nabla f(x+h)$$

2. The operator is distributive over addition.

3. All the operators follows the law of indices.
   i.e., $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x)$
   Also, $\Delta[f(x) + g(x)] = \Delta[g(x) + f(x)]$

4. If $\Delta f(x) = 0$, then it does not mean that either $\Delta = 0$ or $f(x) = 0$.

5. $\Delta[f(x)g(x)] = f(x+h)\Delta g(x) + g(x)\Delta f(x)$

6. $\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+h)}$

4.2.6 Relation between Different Operators

There are few relations defined between these operators. Some of them are:

1. $\Delta = E - 1$ or $E = 1 + \Delta$

2. $\nabla = 1 - E^{-1}$ or $E = (1 - \nabla)^{-1}$

3. $\delta = E^{1/2} - E^{-1/2}$

4. $\Delta = E\nabla = \nabla E = \delta E^{1/2}$

5. $E = e^{hD} = 1 + \Delta,$ where D is the differential operator.

6. $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

Proof:

1. Since $\quad \Delta f(x) = f(x+h) - f(x)$
   or $\quad \Delta f(x) = E[f(x)] - f(x) = (E-1)f(x)$
   Since $f(x)$ is arbitrary, so ignoring it, we have
   $\Delta = E - 1$ or $E = 1 + \Delta$

2. We have
$$\begin{aligned}\nabla f(x) &= f(x) - f(x-h)\\ &= f(x) - E^{-1}f(x)\\ &= (1-E^{-1})f(x)\end{aligned}$$
   Hence $\quad \nabla = 1 - E^{-1}$

3. We have
$$\begin{aligned}\delta[f(x)] &= f(x+h/2) - f(x-h/2)\\ &= E^{1/2}[f(x)] - E^{-1/2}[f(x)]\\ &= (E^{1/2} - E^{-1/2})f(x)\end{aligned}$$
   Hence $\quad \delta = E^{1/2} - E^{-1/2}$

4. We have
$$\begin{aligned}E\nabla[f(x)] &= E[f(x) - f(x-h)]\\ &= E[f(x)] - E[f(x-h)]\\ &= f(x+h) - f(x) = \Delta f(x)\end{aligned}$$
   Hence $\quad E\nabla = \Delta$
   Again,
$$\begin{aligned}\nabla E[f(x)] &= \nabla f(x+h)\\ &= f(x+h) - f(x) = \Delta f(x)\end{aligned}$$
   Hence $\quad \nabla E = \Delta$
   Also,
$$\begin{aligned}\delta E^{1/2}[f(x)] &= \delta[f(x+h/2)]\\ &= f(x+h) - f(x) = \Delta f(x)\end{aligned}$$
   Hence $\quad \delta E^{1/2} = \Delta$

5. We know
$$\begin{aligned}E[f(x) &= f(x+h)\\ &= f(x) + hf'(x) + \tfrac{h^2}{2!}f''(x) + \dots, \text{ by Taylor's series}\\ &= f(x) + hD(x) + \tfrac{h^2}{2!}D^2(x) + \dots\\ &= (1 + hD + \tfrac{h^2}{2!}D^2 + \dots)f(x) = e^{hD}f(x)\end{aligned}$$
   Hence $\quad E = e^{hD}$

6. Since
$$\begin{aligned}\mu[f(x)] &= \tfrac{1}{2}[f(x+h/2) - f(x-h/2)]\\ &= \tfrac{1}{2}[E^{1/2}f(x) + E^{-1/2}f(x)]\\ &= \tfrac{1}{2}[E^{1/2} + E^{-1/2}]f(x)\end{aligned}$$
   Hence $\quad \mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$

Example 4.7. *Evaluate the following:*

$(i)\ \Delta \tan^{-1} x \qquad\qquad (ii)\ \Delta^2(\cos 2x)$

where h is the interval of differencing.

Solution:

(i) We have $\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$
$$= \tan^{-1}\left[\frac{(x+h)-x}{1+(x+h)x}\right] = \tan^{-1}\left[\frac{h}{1+xh+x^2}\right]$$

(ii) We have $\Delta^2(\cos 2x) = (E-1)^2 \cos 2x$ because $\Delta = E-1$.
$$\begin{aligned}
&= (E^2 - 2E + 1)\cos 2x \\
&= E^2 \cos 2x - 2E\cos 2x + \cos 2x \\
&= \cos(2x+4h) - 2\cos(2x+2h) + \cos 2x \\
&= [\cos(2x+4h) - \cos(2x+2h)] - [\cos(2x+2h) + \cos 2x] \\
&= -2\sin(2x+3h)\sin h + 2\sin(2x+h)\sin h \\
&= -2\sin h[2\cos(2x+2h)\sin h] \\
&= -4\sin^2 h\cos(2x+2h).
\end{aligned}$$

Example 4.8. *Evaluate* $\Delta^2\left(\frac{5x+12}{x^2+5x+6}\right)$*; the interval of differencing being unity.*

Solution:

$$\begin{aligned}
\Delta^2\left(\frac{5x+12}{x^2+5x+6}\right) &= \Delta^2\left(\frac{5x+12}{(x+2)(x+3)}\right) \\
&= \Delta^2\left(\frac{2}{x+2} + \frac{3}{x+3}\right) = \Delta\left[\Delta\left(\frac{2}{x+2}\right) + \Delta\left(\frac{3}{x+3}\right)\right] \\
&= \Delta\left[2\left(\frac{1}{x+3} - \frac{1}{x+2}\right) + 3\left(\frac{1}{x+4} - \frac{1}{x+3}\right)\right] \\
&= -2\Delta\left(\frac{1}{(x+2)(x+3)}\right) - 3\Delta\left(\frac{1}{(x+3)(x+4)}\right) \\
&= -2\left[\frac{1}{(x+3)(x+4)} - \frac{1}{(x+2)(x+3)}\right] - 3\left[\frac{1}{(x+4)(x+5)} - \frac{1}{(x+3)(x+4)}\right] \\
&= \frac{4}{(x+2)(x+3)(x+4)} + \frac{6}{(x+3)(x+4)(x+5)} \\
&= \frac{2(5x+16)}{(x+2)(x+3)(x+4)(x+5)}
\end{aligned}$$

Example 4.9. *Evaluate* $\Delta^n e^{ax+b}$*; where the interval of differencing taken to be unity.*

Solution: Given $\Delta^n e^{ax+b}$; which shows that $f(x) = e^{ax+b}$.

Now $\qquad \Delta f(x) = f(x+1) - f(x)$

$\therefore \qquad \Delta(e^{ax+b}) = e^{a(x+1)+b} - e^{ax+b} = e^{ax+b}(e^a - 1)$

$\therefore$
$$\begin{aligned}
\Delta^2(e^{ax+b}) &= \Delta(\Delta e^{ax+b}) = \Delta\{e^{ax+b}(e^a-1)\} \\
&= (e^a-1)(\Delta e^{ax+b}) \\
&= (e^a-1)e^{ax+b}(e^a-1) \\
&= (e^a-1)^2 e^{ax+b}.
\end{aligned}$$

Proceeding in the same way, we get
$$\Delta^n e^{ax+b} = (e^a-1)^n e^{ax+b}$$

Example 4.10. *With usual notations, prove that,*

$$\Delta^n\left(\frac{1}{x}\right) = (-1)^n \frac{n!h^n}{x(x+h)\dots(x+nh)}$$

Solution:

$$\begin{aligned}
\Delta^n\left(\frac{1}{x}\right) &= \Delta^{n-1}\Delta\left(\frac{1}{x}\right) = \Delta^{n-1}\left[\frac{1}{x+h}-\frac{1}{x}\right]\\
&= \Delta^{n-1}\left[\frac{-h}{x(x+h)}\right]\\
&= (-h)\Delta^{n-2}\Delta\left[\frac{1}{x(x+h)}\right]\\
&= (-1)\Delta^{n-2}\left[\Delta\left(\frac{1}{x}-\frac{1}{x+h}\right)\right]\\
&= (-1)\Delta^{n-2}\left[\left(\frac{1}{x+h}-\frac{1}{x}\right)-\left(\frac{1}{x+2h}-\frac{1}{x+h}\right)\right]\\
&= (-1)\Delta^{n-2}\left[\frac{2}{x+h}-\frac{1}{x}-\frac{1}{x+2h}\right]\\
&= (-1)\Delta^{n-2}\left[\frac{-2h^2}{x(x+h)(x+2h)}\right]\\
&= (-1)^2\Delta^{n-2}\left[\frac{2!h^2}{x(x+h)(x+2h)}\right]\\
&= (-1)^3\Delta^{n-3}\left[\frac{3!h^3}{x(x+h)(x+2h)(x+3h)}\right]\\
&\vdots\\
&= (-1)^n\frac{n!h^n}{x(x+h)\dots(x+nh)}
\end{aligned}$$

Example 4.11. *Evaluate*

$(a)\ \Delta(e^{ax}\log bx)$ $\qquad$ $(b)\ \Delta\left(\frac{2^x}{(x+1)!}\right); h = 1.$

Solution: (a) Let $f(x) = e^{ax}$ and $g(x) = \log bx$

Hence $\Delta f(x) = e^{a(x+h)} - e^{ax} = e^{ax}(e^{ah})$

$\Delta g(x) = \log b(x+h) - \log bx = \log\left(1+\frac{h}{x}\right)$

Also

$$\begin{aligned}
\Delta[f(x)g(x)] &= f(x+h)\Delta g(x) + g(x)\Delta f(x)\\
&= e^{a(x+h)}\log\left(1+\tfrac{h}{x}\right) + (\log bx)e^{ax}(e^{ah})\\
&= e^{ax}\left[e^{ah}\log\left(1+\tfrac{h}{x}\right) + (e^{ah}-1)\log bx\right]
\end{aligned}$$

(b) Let $f(x) = 2^x$ and $g(x) = (x+1)!$

$\therefore$ $\Delta f(x) = 2^{x+1} - 2^x = 2^x$

and $\Delta g(x) = (x+1+1)! - (x+1)! = (x+1)(x+1)!$

We know that,

$$\begin{aligned}
\Delta\left[\frac{f(x)}{g(x)}\right] &= \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+h)}\\
&= \frac{(x+1)!\cdot 2^x - 2^x\cdot(x+1)(x+1)!}{(x+1+1)!(x+1)!} \qquad (\because\ h = 1)\\
&= \frac{2^x(x+1)!(1-x-1)}{(x+2)!(x+1)!} = -\frac{x}{(x+2)!}2^x.
\end{aligned}$$

Example 4.12. *Show that* $\Delta \log f(x) = \log \left[1 + \frac{\Delta f(x)}{f(x)}\right]$

Solution: Let h be the interval of differencing

$$\begin{aligned}
\Delta \log f(x) &= \log f(x+h) - \log f(x) \\
&= \log\left[\frac{f(x+h)}{f(x)}\right] = \log\left[\frac{Ef(x)}{f(x)}\right] \\
&= \log\left[\frac{(1+\Delta)f(x)}{f(x)}\right] \\
&= \log\left[\frac{f(x)+\Delta f(x)}{f(x)}\right] \\
&= \log\left[1+\frac{\Delta f(x)}{f(x)}\right]
\end{aligned}$$

Example 4.13. *Sum the following series* $1^3 + 2^3 + 3^3 + \ldots + n^3$ *using the calculus of finite differences.*

Solution: Let $1^3 = u_0, 2^3 = u_1, 3^3 = u_2, \ldots, n^3 = u_{n-1}$. Therefore sum is given by

$$\begin{aligned}
S &= u_0 + u_1 + u_2 + \ldots + u_{n-1} \\
&= (1 + E + E2 + E3 + \ldots + E_{n-1})u_0 \\
&= \left(\frac{E^n - 1}{E - 1}\right) u_0 = \left[\frac{(1+\Delta)^n - 1}{\Delta}\right] u_0 \\
&= \frac{1}{\Delta}\left[1 + n\Delta + \frac{n(n-1)}{2!}\Delta^2 + \frac{n(n-1)(n-2)}{3!}\Delta^3 + \ldots + \Delta^n - 1\right] u_0 \\
&= n + \frac{n(n-1)}{2!}\Delta u_0 + \frac{n(n-1)(n-2)}{3!}\Delta u_0 + \ldots
\end{aligned}$$

We know
$$\begin{aligned}
\Delta u_0 &= u_1 - u_0 = 2^3 - 1^3 = 7. \\
\Delta^2 u_0 &= u_2 - 2u_1 + u_0 = 3^3 - 2(2)^3 + 1^3 = 12.
\end{aligned}$$

Similarly we have obtained $\Delta^3 u_0 = 6$ and $\Delta^4 u_0, \Delta^5 u_0, \ldots$ are all zero as $u_r = r^3$ is a polynomial of third degree.

$$\begin{aligned}
\therefore \qquad S &= n + \frac{n(n-1)}{2!}(7) + \frac{n(n-1)(n-2)}{6}(12) + \frac{n(n-1)(n-2)(n-3)}{24}(6) \\
&= \frac{n^2}{4}(n^2 + 2n + 1) = \left[\frac{n(n+1)}{2}\right]^2
\end{aligned}$$

Example 4.14. *Given that* $u_0 = 3, u_1 = 12, u_2 = 81, u_3 = 200, u_4 = 100, u_5 = 8$. *Find the value of* $\Delta^5 u_0$.

Solution: We know $\Delta = E - 1$, therefore

$$\begin{aligned}
\Delta^5 u_0 &= (E-1)^5 u_0 \\
&= (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)u_0
\end{aligned}$$

$$= u_5 - 5u_4 + 10u_3 - 10u_2 + 5u_1 - u_0$$
$$= 8 - 50 + 2000 - 810 + 60 - 3$$
$$= 755.$$

PROBLEM SET 4.1

1. Form the forward difference table for given set of data:

x	10	20	30	40
y	1.1	2.0	4.4	7.9

2. Construct the difference table for the given data and hence evaluate $\Delta^3 f(2)$.

x	0	1	2	3	4
y	1.0	1.5	2.2	3.1	4.6

[**Ans.** 0.4]

3. Find the value of E^2x^2 when the values of x vary by a constant increment of 2.
[**Ans.** $x^2 + 8x + 16$]

4. Evaluate $\Delta^3(1-x)(1-2x)(1-3x)$; the interval of differencing being unity.
[**Ans.** $\Delta^3 = -36$]

5. Evaluate $\left(\frac{\Delta^2}{E}\right) x^3$ [**Ans.** $6x$]

6. Find the value of $\Delta^2 \left[\frac{a^{2x}+a^{4x}}{(a^2-1)^2}\right]$; $h = 1$ [**Ans.** $a^{2x} + (a^2+1)^2 a^{4x}$]

7. Evaluate:

   (a) $\Delta \cot 2^x$ [**Ans.** $-Cosec\, 2^{x+1}$]

   (b) $\Delta \sin h(a+bx)$ [**Ans.** $2 \sin h\frac{b}{2} \cos h(a+\frac{b}{2}+bx)$]

   (c) $\Delta \tan ax$ [**Ans.** $\frac{\sin a}{\cos ax \cos a(x+1)}$]

8. Prove that $\Delta \sin^{-1} x = [(x+1)\sqrt{1-x^2} - x\sqrt{1-(x+1)^2}]$.