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Numerical Analysis I

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Chapter 3

Linear System of Equations

3.1 INTRODUCTION

In this chapter we present the solution of n linear simultaneous algebraic equations in n unknowns. Linear systems of equations are associated with many problems in engineering and science, as well as with applications of mathematics to the social sciences and quantitative study of business and economic problems.
A system of algebraic equations has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_3 \end{aligned}$$

where the coefficients a_{ij} and the constants b_j are known and x_i represents the unknowns. In matrix notation, the equations are written as

$$\mathbf{Ax} = \mathbf{b} \tag{3.1}$$

where $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

The matrix **[A | b]**, obtained by appending the column **b** to the matrix **A** is called the augmented matrix. That is

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

We define the following.

(i) The system of equations (3.1) is consistent (has at least one solution), if
$\text{rank}(\mathbf{A}) = \text{rank}[\mathbf{A}|\mathbf{b}] = \mathbf{r}$
If $r = n$, then the system has unique solution.
If $r < n$, then the system has $(n - r)$ parameter family of infinite number of solutions.

(ii) The system of equations (3.1) is inconsistent (has no solution) if
$\text{rank}(\mathbf{A}) \neq \text{rank}[\mathbf{A}|\mathbf{b}]$

The methods of solution of the linear algebraic system of equations (3.1) may be classified as direct and iterative methods.

(a) *Direct methods* produce the exact solution after a finite number of steps (disregarding the round-off errors). In these methods, we can determine the total number of operations (additions, subtractions, divisions and multiplications). This number is called the operational count of the method.

(b) *Iterative methods* are based on the idea of successive approximations. We start with an initial approximation to the solution vector $x = x_0$, and obtain a sequence of approximate vectors $x_0, x_1, \ldots, x_k, \ldots$, which in the limit as $k \to \infty$, converge to the exact solution vector x. Now, we derive some direct methods.

3.2 DIRECT METHODS

Before we derive some direct methods, we define elementary row operations that can be performed on the rows of a matrix.

Elementary row transformations (operations) The following operations on the rows of a matrix A are called the *elementary row transformations (operations)*.

(i) Interchange of any two rows.

(ii) Division/multiplication of any row by a non-zero number p.

(iii) Adding/subtracting a scalar multiple of any row to any other row.

These row operations change the form of **A**, but do not change the row-rank of **A**. The matrix **B** obtained after the elementary row operations is said to be row equivalent with **A**. In the context of the solution of the system of algebraic equations, the solution of the new system is identical with the solution of the original system.

3.2.1 Gauss Elimination Method

The method is based on the idea of reducing the given system of equations **Ax = b**, to an upper triangular system of equations **Ux = z**, using elementary row operations. We know that these two systems are equivalent. That is, the solutions of both the systems are identical. This reduced system **Ux = z**, is then solved by the back substitution method to obtain the solution vector **x**.

We illustrate the method using the 3×3 system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \tag{3.2}$$

We write the augmented matrix **[A | b]** and reduce it to the following form

$$[\mathbf{A} \mid \mathbf{b}] \xrightarrow{\text{Gauss elimination}} [\mathbf{U} \mid \mathbf{z}]$$

The augmented matrix of the system (3.2) is

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \tag{3.3}$$

First stage of elimination

We assume $a_{11} \neq 0$. This element a_{11} in the 1×1 position is called the *first pivot*. We use this pivot to reduce all the elements below this pivot in the first column as zeros. Multiply the first row in (3.3) by a_{21}/a_{11} and a_{31}/a_{11} respectively and subtract from the second and third rows. That is, we are performing the elementary row operations $R_2-(a_{21}/a_{11})R_1$ and $R_3-(a_{31}/a_{11})R_1$ respectively. We obtain the new augmented matrix as

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & b_3^{(1)} \end{array}\right] \tag{3.4}$$

where $$a_{22}^{(1)} = a_{22} - \left(\frac{a_{21}}{a_{11}}\right)a_{11}, \qquad a_{23}^{(1)} = a_{23} - \left(\frac{a_{21}}{a_{11}}\right)a_{13}, \qquad b_2^{(1)} = b_2 - \left(\frac{a_{21}}{a_{11}}\right)b_1,$$
$$a_{32}^{(1)} = a_{32} - \left(\frac{a_{31}}{a_{11}}\right)a_{12}, \qquad a_{33}^{(1)} = a_{33} - \left(\frac{a_{31}}{a_{11}}\right)a_{13}, \qquad b_3^{(1)} = b_3 - \left(\frac{a_{31}}{a_{11}}\right)b_1.$$

Second stage of elimination

We assume $a_{22}^{(1)} \neq 0$. This element $a_{22}^{(1)}$ in the 2×2 position is called the *second pivot*. We use this pivot to reduce the element below this pivot in the second column as zero. Multiply the second row in (3.4) by $a_{32}^{(1)}/a_{22}^{(1)}$ and subtract from the third row. That is, we are performing the elementary row operation $R_3 - (a_{32}^{(1)}/a_{22}^{(1)})R_2$. We obtain the new augmented matrix as

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ 0 & 0 & a_{33}^{(2)} & b_3^{(2)} \end{array}\right] \tag{3.5}$$

where $$a_{33}^{(1)} = a_{33}^{(1)} - \left(\frac{a_{32}^{(1)}}{a_{22}^{(1)}}\right)a_{23}^{(1)}, \qquad b_3^{(2)} = b_3^{(1)} - \left(\frac{a_{32}^{(1)}}{a_{22}^{(1)}}\right)b_2^{(1)}.$$

The element $a_{33}^{(2)} \neq 0$ is called the *third pivot*. This system is in the required upper triangular form $[\mathbf{U}|\mathbf{z}]$. The solution vector $\mathbf{x}$ is now obtained by back substitution.

From the third row, we get $\quad x_3 = \frac{b_3^{(2)}}{a_{33}^{(2)}}$.

From the second row, we get $\quad x_2 = \frac{1}{a_{22}^{(1)}}(b_2^{(1)} - a_{23}^{(1)}x_3)$.

From the first row, we get $\quad x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3)$.

In general, using a pivot, all the elements below that pivot in that column are made zeros.

Alternately, at each stage of elimination, we may also make the pivot as 1, by dividing that particular row by the pivot.

Example 3.1. *Solve the system of equations by Gauss elimination method:*

$$\begin{aligned} x + 3y - 5z &= 2 \\ 3x + 11y - 9z &= 4 \\ -x + y + 6z &= 5 \end{aligned}$$

Solution: The augmented matrix is given by

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 2 \\ 3 & 11 & -9 & 4 \\ -1 & 1 & 6 & 5 \end{array}\right]$$

We perform the following elementary row transformations and do the eliminations.

$$\begin{array}{r} \\ R_2 - 3R_1 \to \\ R_3 + R_1 \to \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -5 & 2 \\ 0 & 2 & 6 & -2 \\ 0 & 4 & 1 & 7 \end{array}\right] \quad R_3 - 2R_2 \to \left[\begin{array}{ccc|c} 1 & 3 & -5 & 2 \\ 0 & 2 & 6 & -2 \\ 0 & 0 & -11 & 11 \end{array}\right]$$

Back substitution gives the solution.

$$z = -1, \qquad y = \frac{1}{2}(-2 - 6z) = \frac{1}{2}(-2 + 6) = 2, \qquad x = 2 - 3y + 5z = 2 - 6 - 5 = -9.$$

Example 3.2. *Use the method of Gaussian elimination to solve*

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 3 \\ 2x_1 - x_2 + 3x_3 &= 3 \\ 2x_2 + 3x_4 &= 1 \\ -x_1 + 2x_3 + x_4 &= 0 \end{aligned}$$

Solution: The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 0 & 3 \\ 0 & 2 & 0 & 3 & 1 \\ -1 & 0 & 2 & 1 & 0 \end{array}\right]$$

From the augmented matrix, we apply elementary transformations:

$$\xrightarrow[R_4 = R_4 + R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 & -3 \\ 0 & 2 & 0 & 3 & 1 \\ 0 & 1 & 3 & 2 & 3 \end{array}\right] \xrightarrow[R_4 = R_4 + \frac{1}{3}R_2]{R_3 = R_3 + \frac{2}{3}R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 & -3 \\ 0 & 0 & 2/3 & 5/3 & -1 \\ 0 & 0 & 10/3 & 4/3 & 2 \end{array}\right]$$

$$\xrightarrow{R_4 = R_4 - 5R_3} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 & -3 \\ 0 & 0 & 2/3 & 5/3 & -1 \\ 0 & 0 & 0 & -7 & 7 \end{array}\right]$$

Using back substitution, we obtain

$$x_4 = -\tfrac{1}{7}(7) = -1, \quad x_3 = \tfrac{3}{2}\left(-1 - \tfrac{5}{3}x_4\right) = 1,$$
$$x_2 = -\tfrac{1}{3}(-3 - x_3 - 2x_4) = 2, \quad x_1 = 3 - x_2 - x_3 - x_4 = 1.$$

Pivoting:

The Gaussian elimination method fails if any one of the pivot elements becomes zero. In such a situation, we rewritten the equations in a different order to avoid zero pivots. changing the order of equations is called Pivoting.

For example, we may have the system

$$\begin{aligned} 2y + 5z &= 7 \\ 7x + y - 2z &= 6 \\ 2x + 3y + 8z &= 13 \end{aligned}$$

in which the first pivot is zero.

Partial pivoting method:

Step 1: The numerically largest coefficient of x_1 is selected from all the equations are pivot and the corresponding equation becomes the first equation.

Step 2: The numerically largest coefficient of x_2 is selected from all the remaining equations as pivot and the corresponding equation becomes the second equation. This procedure is continued until the upper triangular system is obtained.

Complete pivoting method:

In this method, we select at each stage the numerically largest coefficient of the complete matrix of coefficients. This procedure leads to an interchange of the equations as well as interchange of the position of variables.

Remark 3.1. *When the system of algebraic equations is large, how do we conclude that it is consistent or not, using the Gauss elimination method?*

A way of determining the consistency is from the form of the reduced system (3.5). We know that if the system is inconsistent then $\text{rank}(\mathbf{A}) \neq \text{rank}[\mathbf{A}|\mathbf{b}]$. By checking the elements of the last rows, conclusion can be drawn about the consistency or inconsistency.

Suppose that in (3.5), $a_{33}^{(2)} \neq 0$ and $b_3^{(2)} \neq 0$. Then, $\text{rank}(\mathbf{A}) = \text{rank}[\mathbf{A}|\mathbf{b}] = \mathbf{3}$. The system is consistent and has a unique solution.

Suppose that we obtain the reduced system as

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ 0 & 0 & 0 & b_3^{(2)} \end{array}\right]$$

Then, $\text{rank}(\mathbf{A}) = \mathbf{2}, \text{rank}[\mathbf{A}|\mathbf{b}] = \mathbf{3}$ and $\text{rank}(\mathbf{A}) \neq \text{rank}[\mathbf{A}|\mathbf{b}]$. Therefore, the system is inconsistent and has no solution.

Suppose that we obtain the reduced system as

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then, $\text{rank}(\mathbf{A}) = \text{rank}[\mathbf{A}|\mathbf{b}] = \mathbf{2} < \mathbf{3}$. Therefore, the system has $3 - 2 = 1$ parameter family of infinite number of solutions.

Example 3.3. *Solve the system of equations*

$$\begin{aligned} x+y+z &= 7 \\ 3x+3y+4z &= 24 \\ 2x+y+3z &= 16 \end{aligned}$$

using the Gauss elimination with partial pivoting.

Solution: We have the augmented matrix as

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 3 & 3 & 4 & 24 \\ 2 & 1 & 3 & 16 \end{array}\right]$$

We perform the following elementary row transformations and do the eliminations.

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 3 & 3 & 4 & 24 \\ 1 & 1 & 1 & 7 \\ 2 & 1 & 3 & 16 \end{array}\right] \xrightarrow{R_1 = \frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 1 & 4/3 & 8 \\ 1 & 1 & 1 & 7 \\ 2 & 1 & 3 & 16 \end{array}\right]$$

$$\xrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 4/3 & 8 \\ 0 & 0 & -1/3 & -1 \\ 0 & -1 & 1/3 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 4/3 & 8 \\ 0 & -1 & 1/3 & 0 \\ 0 & 0 & -1/3 & -1 \end{array}\right]$$

Back substitution gives the solution.

$$z = (-1)(-3) = 3, \quad y = \tfrac{1}{3}z = 1, \quad x = 8 - y - \tfrac{4}{3}z = 3$$

Example 3.4. *Solve the system of equations*

$$\begin{aligned} 2x_1 + x_2 + x_3 - 2x_4 &= -10 \\ 4x_1 + 2x_3 + x_4 &= 8 \\ 3x_1 + x_2 + 2x_3 &= 7 \\ x_1 + 3x_2 + 2x_3 - x_4 &= -5 \end{aligned}$$

using the Gauss elimination with partial pivoting.

Solution: The augmented matrix is given by

$$\left[\begin{array}{cccc|c} 2 & 1 & 1 & -2 & -10 \\ 4 & 0 & 2 & 1 & 8 \\ 3 & 2 & 2 & 0 & 7 \\ 1 & 3 & 2 & -1 & -5 \end{array}\right]$$

We perform the following elementary row transformations and do the eliminations.

$$R_1 \leftrightarrow R_2 : \left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 2 & 1 & 1 & -2 & -10 \\ 3 & 2 & 2 & 0 & 7 \\ 1 & 3 & 2 & -1 & -5 \end{array}\right] \begin{array}{l} \\ R_2 - (1/2)R_1 \to \\ R_3 - (3/4)R_1 \to \\ R_4 - (1/4)R_1 \to \end{array} \left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 0 & 1 & 0 & -5/2 & -14 \\ 0 & 2 & 1/2 & -3/4 & 1 \\ 0 & 3 & 3/2 & -5/4 & -7 \end{array}\right]$$

$$R_2 \leftrightarrow R_4 : \left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 0 & 3 & 3/2 & -5/4 & -7 \\ 0 & 2 & 1/2 & -3/4 & 1 \\ 0 & 1 & 0 & -5/2 & -14 \end{array}\right] \begin{array}{l} \\ \\ R_3 - (2/3)R_2 \to \\ R_4 - (1/3)R_2 \to \end{array} \left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 0 & 3 & 3/2 & -5/4 & -7 \\ 0 & 0 & -1/2 & 1/12 & 17/3 \\ 0 & 0 & -1/2 & -25/12 & -35/3 \end{array}\right]$$

$$R_4 - R_3 : \left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 0 & 3 & 3/2 & -5/4 & -7 \\ 0 & 0 & -1/2 & 1/12 & 17/3 \\ 0 & 0 & 0 & -13/6 & -52/3 \end{array}\right]$$

Using back substitution, we obtain

$$x_4 = \left(-\tfrac{52}{3}\right)\left(-\tfrac{6}{13}\right) = 8, \qquad x_3 = -2\left(\tfrac{17}{3} - \tfrac{1}{12}x_4\right) = -10,$$
$$x_2 = \tfrac{1}{3}\left[-7 - \tfrac{3}{2}x_3 - \tfrac{5}{4}x_4\right] = 6, \qquad x_1 = \tfrac{1}{4}[8 - 2x_3 - x_4] = 5.$$

Example 3.5. *Test the consistency of the following system of equations*

$$\begin{aligned} x + 10y - z &= 3 \\ 2x + 3y + 20z &= 7 \\ 9x + 22y + 79z &= 45 \end{aligned}$$

using the Gauss elimination with partial pivoting.

Solution: We have the augmented matrix as

$$\left[\begin{array}{ccc|c} 1 & 10 & -1 & 3 \\ 2 & 3 & 20 & 7 \\ 9 & 22 & 79 & 45 \end{array}\right]$$

We perform the following elementary row transformations and do the eliminations.

$$\frac{R_2 = R_2 - 2R_1}{R_3 = R_3 - 9R1} > \left[\begin{array}{ccc|c} 1 & 10 & -1 & 3 \\ 0 & -17 & 22 & 1 \\ 0 & -68 & 88 & 18 \end{array}\right] \xrightarrow{R_3 - 4R_2} \left[\begin{array}{ccc|c} 1 & 01 & -1 & 3 \\ 0 & -17 & 22 & 1 \\ 0 & 0 & 0 & 14 \end{array}\right]$$

Now, rank$[\mathbf{A}] = \mathbf{2}$, and rank$[\mathbf{A}|\mathbf{b}] = \mathbf{3}$. Therefore, the system is inconsistent and has no solution.

3.2.2 Gauss-Jordan Method

The method is based on the idea of reducing the given system of equations **Ax = b**, to a diagonal system of equations **Ix = d**, where **I** is the identity matrix, using elementary row operations. We know that the solutions of both the systems are identical. This reduced system gives the solution vector **x**. This reduction is equivalent to finding the solution as $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

$$[\mathbf{A} \mid \mathbf{b}] \xrightarrow{\text{Gauss-Jordan method}} [\mathbf{I} \mid \mathbf{X}]$$

In this case, after the eliminations are completed, we obtain the augmented matrix for a 3×3 system as

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & d_1 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 1 & d_3 \end{array}\right] \tag{3.6}$$

and the solution is $\quad x_i = d_i, i = 1, 2, 3.$

Elimination procedure The first step is same as in Gauss elimination method, that is, we make the elements below the first pivot as zeros, using the elementary row transformations.

From the second step onwards, we make the elements below and above the pivots as zeros using the elementary row transformations. Lastly, we divide each row by its pivot so that the final augmented matrix is of the form (3.6). Partial pivoting can also be used in the solution. We may also make the pivots as 1 before performing the elimination.

Example 3.6. *Solve the following system of equations*

$$\begin{aligned} x+y+z &= 1\\ x+3y-z &= 6\\ 3x+5y+3z &= 4 \end{aligned}$$

using the Gauss-Jordan method (i) without partial pivoting, (ii) with partial pivoting.

Solution: We have the augmented matrix as

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 4 & 3 & -1 & 6\\ 3 & 5 & 3 & 4 \end{array}\right]$$

(i) We perform the following elementary row transformations and do the eliminations.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 4 & 3 & -1 & 6\\ 3 & 5 & 3 & 4 \end{array}\right] \xrightarrow[R_3=R_3-3R_1]{R_2=R_2-4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 0 & -1 & -5 & 2\\ 0 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3=R_3+2R_2]{R_1=R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 3\\ 0 & -1 & -5 & 2\\ 0 & 0 & -10 & 5 \end{array}\right] \xrightarrow[R_2=R_2-\frac{5}{10}R_3]{R_1=R_1-\frac{4}{10}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1\\ 0 & -1 & 0 & -1/2\\ 0 & 0 & -10 & 5 \end{array}\right]$$

Now, making the pivots as $1, ((R_2=-R_2),(R_3=-\frac{1}{10}R_3))$ we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1/2\\ 0 & 0 & 1 & -1/2 \end{array}\right]$$

Therefore, the solution of the system is $x=1, y=1/2, z=-1/2$.

(ii) We perform the following elementary row transformations and do the eliminations.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 4 & 3 & -1 & 6\\ 3 & 5 & 3 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 4 & 3 & -1 & 6\\ 1 & 1 & 1 & 1\\ 3 & 5 & 3 & 4 \end{array}\right] \xrightarrow{R_1=\frac{1}{4}R_1} \left[\begin{array}{ccc|c} 1 & 3/4 & -1/4 & 3/2\\ 1 & 1 & 1 & 1\\ 3 & 5 & 3 & 4 \end{array}\right]$$

$$\xrightarrow[R_3=R_3-3R_1]{R_2=R_2-R_1} \left[\begin{array}{ccc|c} 1 & 3/4 & -1/4 & 3/2\\ 0 & 1/4 & 5/4 & -1/2\\ 0 & 11/4 & 15/4 & -1/2 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 3/4 & -1/4 & 3/2\\ 0 & 11/4 & 15/4 & -1/2\\ 0 & 1/4 & 5/4 & -1/2 \end{array}\right]$$

$$\xrightarrow{R_2=\frac{4}{11}R_2} \left[\begin{array}{ccc|c} 1 & 3/4 & -1/4 & 3/2\\ 0 & 1 & 15/11 & -2/11\\ 0 & 1/4 & 5/4 & -1/2 \end{array}\right] \xrightarrow[R_3=R_3-\frac{1}{4}R_2]{R_1=R_1-\frac{3}{4}R_2} \left[\begin{array}{ccc|c} 1 & 0 & -14/11 & 18/11\\ 0 & 1 & 15/11 & -2/11\\ 0 & 0 & 10/11 & -5/11 \end{array}\right]$$

$$\xrightarrow{R_3=\frac{11}{10}R_3}\left[\begin{array}{ccc|c}1&0&-14/11&18/11\\0&1&15/11&-2/11\\0&0&1&-1/2\end{array}\right]\xrightarrow[R_2=R_2-\frac{15}{11}R_3]{R_1=R_1+\frac{14}{11}R_3}\left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&1/2\\0&0&1&-1/2\end{array}\right]$$

Therefore, the solution of the system is $x=1, y=1/2, z=-1/2$.

Remark 3.2. *The Gauss-Jordan method looks very elegant as the solution is obtained directly. However, it is computationally more expensive than Gauss elimination. For large n, the total number of divisions and multiplications for Gauss-Jordan method is almost 1.5 times the total number of divisions and multiplications required for Gauss elimination. Hence, we do not normally use this method for the solution of the system of equations. The most important application of this method is to find the inverse of a non-singular matrix. We present this method in the following section.*

3.2.3 Inverse of a Matrix by Gauss-Jordan Method

As given in Remark 3.2, the important application of the Gauss-Jordan method is to find the inverse of a non-singular matrix A. We start with the augmented matrix of A with the identity matrix I of the same order. When the Gauss-Jordan procedure is completed, we obtain

$$[\mathbf{A}\mid\mathbf{I}]\xrightarrow{\text{Gauss-Jordan method}}[\mathbf{I}\mid\mathbf{A}^{-1}]$$

since, $\mathbf{A}\mathbf{A}^{-1}=\mathbf{I}$.

Remark 3.3. *Partial pivoting can also be done using the augmented matrix* $[\mathbf{A}\mid\mathbf{I}]$. *However, we cannot first interchange the rows of A and then find the inverse. Then, we would be finding the inverse of a different matrix.*

Example 3.7. *Find the inverse of the matrix*

$$\begin{bmatrix}1&1&1\\4&3&-1\\3&5&3\end{bmatrix}$$

using the Gauss-Jordan method

Solution: Consider the augmented matrix.

$$\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\4&3&-1&0&1&0\\3&5&3&0&0&1\end{array}\right]$$

We perform the following elementary row transformations and do the eliminations.

$$\xrightarrow[R_3=R_3-3R_1]{R_2=R_2-4R_1}\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&-1&-5&-4&1&0\\0&2&0&-3&0&1\end{array}\right]\xrightarrow{R_2=-R_2}\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&5&4&-1&0\\0&2&0&-3&0&1\end{array}\right]$$

$$\xrightarrow[R_3=R_3-2R_2]{R_1=R_1-R_2}\left[\begin{array}{ccc|ccc}1&0&-4&-3&1&0\\0&1&5&4&-1&0\\0&0&-10&-11&2&1\end{array}\right]\xrightarrow{R_3=-\frac{1}{10}R_3}\left[\begin{array}{ccc|ccc}1&0&-4&-3&1&0\\0&1&5&4&-1&0\\0&0&1&\frac{11}{10}&-\frac{2}{10}&-\frac{1}{10}\end{array}\right]$$

3.3 ITERATIVE METHOD

All the previous methods seen in solving the system of simultaneous algebraic linear equations are direct methods. Now we will see some indirect methods or iterative methods.

This iterative methods is not always successful to all systems of equations. If this method is to succeed, each equation of the system must possess one large coefficient and the large coefficient must be attached to a different unknown in that equation. This condition will be satisfied if the large coefficients are along the leading diagonal of the coefficient matrix. When this condition is satisfied, the system will be solvable by the iterative method. The system,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}$$

will be solvable by this method if

$$\begin{aligned}
|a_{11}| &> |a_{12}| + |a_{13}|\\
|a_{22}| &> |a_{21}| + |a_{23}|\\
|a_{33}| &> |a_{31}| + |a_{32}|
\end{aligned}$$

In other words, the solution will exist (iterating will converge) if the absolute values of the leading diagonal elements of the coefficient matrix A of the system $AX = B$ are greater than the sum of absolute values of the other coefficients of that row. The condition is sufficient but not necessary.

Under the category of iterative method, we shall describe the following two methods:

(i) Gauss-Jacobi's method *(ii)* Gauss-Seidel method.

3.3.1 Gauss-Jacobi Method

Let us consider the system of simultaneous equations

$$\begin{aligned}
a_1x + b_1y + c_1z &= d_1\\
a_2x + b_2y + c_2z &= d_2\\
a_3x + b_3y + c_3z &= d_3
\end{aligned}$$

such that a_1, b_2, and c_3 are the largest coefficients of x, y, z, respectively. So that convergence is assured. Rearranging the above system of equations and rewriting in terms of x, y, z, as:

$$\begin{aligned}
x &= \frac{1}{a_1}(d_1 - b_1y - c_1z)\\
y &= \frac{1}{b_2}(d_2 - a_2x - c_2z)\\
z &= \frac{1}{c_3}(d_3 - a_3x - b_3y)
\end{aligned}$$

let x_0, y_0, z_0 be the initial approximations of the unknowns x, y and z. Then, the first approximation are given by

$$\begin{aligned} x_1 &= \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0) \\ y_1 &= \frac{1}{b_2}(d_2 - a_2 x_0 - c_2 z_0) \\ z_1 &= \frac{1}{c_3}(d_3 - a_3 x_0 - b_3 y_0) \end{aligned}$$

Similarly, the second approximations are given by

$$\begin{aligned} x_2 &= \frac{1}{a_1}(d_1 - b_1 y_1 - c_1 z_1) \\ y_2 &= \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_1) \\ z_2 &= \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1) \end{aligned}$$

Proceeding in the same way, if x_n, y_n, z_n are the nth iterates then

$$\begin{aligned} x_{n+1} &= \frac{1}{a_1}(d_1 - b_1 y_n - c_1 z_n) \\ y_{n+1} &= \frac{1}{b_2}(d_2 - a_2 x_n - c_2 z_n) \\ z_{n+1} &= \frac{1}{c_3}(d_3 - a_3 x_n - b_3 y_n) \end{aligned}$$

The process is continued till convergency is secured.
Note: In the absence of any better estimates, the initial approximations are taken as $x_0 = 0, y_0 = 0, z_0 = 0$.
Example 3.11. *Solve the following system of equations*

$$\begin{aligned} 20x + y - 2z &= 17 \\ 3x + 20y - z &= -18 \\ 2x - 3y + 20z &= 25 \end{aligned}$$

using the Jacobi iteration method. Obtain the result correct to three decimal places.

Solution: The given system of equations is diagonally dominant. Jacobi method gives the iterations as

$$\begin{aligned} x_{n+1} &= \frac{1}{20}(17 - y_n + 2z_n) \\ y_{n+1} &= \frac{1}{20}(-18 - 3x_n + z_n) \\ z_{n+1} &= \frac{1}{20}(25 - 2x_n + 3y_n), \quad n = 0, 1, \ldots \end{aligned}$$

Choose the initial approximation as $x_0 = y_0 = z_o = 0$. We obtain the following results.

First iteration:

$$\begin{aligned}
x_1 &= \frac{1}{20}(17 - y_0 + 2z_0) = \frac{1}{20}(17) = 0.85\\
y_1 &= \frac{1}{20}(-18 - 3x_0 + z_0) = \frac{1}{20}(-18) = -0.9\\
z_1 &= \frac{1}{20}(25 - 2x_0 + 3y_0) = \frac{1}{20}(25) = 1.25
\end{aligned}$$

Second iteration:

$$\begin{aligned}
x_2 &= \frac{1}{20}(17 - y_1 + 2z_1) = \frac{1}{20}(17 + 0.9 + 2(1.25)) = 1.02\\
y_2 &= \frac{1}{20}(-18 - 3x_1 + z_1) = \frac{1}{20}(-18 - 3(0.85) + 1.25) = -0.965\\
z_2 &= \frac{1}{20}(25 - 2x_1 + 3y_1) = \frac{1}{20}(25 - 2(0.85) + 3(-0.9)) = 1.03
\end{aligned}$$

Third iteration:

$$\begin{aligned}
x_3 &= \frac{1}{20}(17 - y_2 + 2z_2) = \frac{1}{20}(17 + 0.965 + 2(1.03)) = 1.00125\\
y_3 &= \frac{1}{20}(-18 - 3x_2 + z_2) = \frac{1}{20}(-18 - 3(1.02) + 1.03) = -1.0015\\
z_3 &= \frac{1}{20}(25 - 2x_2 + 3y_2) = \frac{1}{20}(25 - 2(1.02) + 3(-0.965)) = 1.00325
\end{aligned}$$

Fourth iteration:

$$\begin{aligned}
x_4 &= \frac{1}{20}(17 - y_3 + 2z_3) = \frac{1}{20}(17 + 1.0015 + 2(1.00325)) = 1.0004\\
y_4 &= \frac{1}{20}(-18 - 3x_3 + z_3) = \frac{1}{20}(-18 - 3(1.00125) + 1.0032) = -1.000025\\
z_4 &= \frac{1}{20}(25 - 2x_3 + 3y_3) = \frac{1}{20}(25 - 2(1.00125) + 3(-1.0015)) = 0.99965
\end{aligned}$$

Subsequent iterations result in the following:

$$x_5 = 0.999966 \qquad y_5 = -1.000078 \qquad z_5 = 0.999956$$

$$x_6 = 1.00000 \qquad y_6 = -0.999997 \qquad z_6 = 0.999992$$

We find $|x_6 - x_5| = |1.00000 - 0.99997| = 0.00003$

$$|y_6 - y_5| = |-0.99999 + 1.00008| = 0.00009$$

$$|z_6 - z_5| = |0.99999 - 0.99996| = 0.00003$$

Since, all the errors in magnitude are less than 0.0005, the required solution is

$$x = 1.0, \quad y = -0.9999, \quad z = 0.9999$$

3.3.2 Guass-Seidel Method

This is a modification of Gauss-Jacobi method. As before, the system of the linear equations.

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

is written as

$$x = \frac{1}{a_1}(d_1 - b_1y - c_1z) \tag{3.13}$$
$$y = \frac{1}{b_2}(d_2 - a_2x - c_2z) \tag{3.14}$$
$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y) \tag{3.15}$$

and we start with the initial approximation x_0, y_0, z_0. Substituting y_0 and z_0 in Eqn.(3.13), we get

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

Now substituting $x = x_1, z = z_0$ in Eqn.(3.14), we get

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_0)$$

Substituting $x = x_1, y = y_1$ in Eqn.(3.15), we get

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

This process is continued till the value of $x, y, z,$ are obtained to the desired degree of accuracy. In general, kth iteration can be written as

$$\begin{aligned} x_{k+1} &= \frac{1}{a_1}(d_1 - b_1y_k - c_1z_k) \\ y_{k+1} &= \frac{1}{b_2}(d_2 - a_2x_{k+1} - c_2z_k) \\ z_{k+1} &= \frac{1}{c_3}(d_3 - a_3x_{k+1} - b_3y_{k+1}) \end{aligned}$$

The rate of convergence of Gauss-Seidel method is roughly twice that of Gauss-Jacobi method.

Example 3.12. *Solve the following system of equation using Gauss-Seidel method*

$$\begin{aligned} 20x + y - 2z &= 17 \\ 3x + 20y - z &= -18 \\ 2x - 3y + 20z &= 25 \end{aligned}$$

Solution: The given system of equations is strongly diagonally dominant. Hence, we can expect fast convergence. Gauss-Seidel method gives the iteration

$$x_{n+1} = \frac{1}{20}(17 - y_n + 2z_n)$$
$$y_{n+1} = \frac{1}{20}(-18 - 3x_{n+1} + z_n)$$
$$z_{n+1} = \frac{1}{20}(25 - 2x_{n+1} + 3y_{n+1})$$

Starting with $x_0 = 0, y_0 = 0, z_0 = 0$, we get the following results.
First iteration:

$$\begin{aligned} x_1 &= \frac{1}{20}(17 - y_0 + 2z_0) = \frac{1}{20}(17 + 0 + 0) = 0.85 \\ y_1 &= \frac{1}{20}(-18 - 3x_1 + z_0) = \frac{1}{20}(-18 - 3(0.85) + 0) = -1.0275 \\ z_1 &= \frac{1}{20}(25 - 2x_1 + 3y_1) = \frac{1}{20}(25 - 2(0.85) + 3(-1.0275)) = 1.0109 \end{aligned}$$

Second iteration:

$$\begin{aligned} x_2 &= \frac{1}{20}(17 - y_1 + 2z_1) = \frac{1}{20}(17 + 1.0275 + 2(1.0109)) = 1.0025 \\ y_2 &= \frac{1}{20}(-18 - 3x_2 + z_1) = \frac{1}{20}(-18 - 3(1.0025) + 1.0109) = -0.9998 \\ z_2 &= \frac{1}{20}(25 - 2x_2 + 3y_2) = \frac{1}{20}(25 - 2(1.0025) + 3(-0.9998)) = 0.9998 \end{aligned}$$

Third iteration:

$$\begin{aligned} x_3 &= \frac{1}{20}(17 - y_2 + 2z_2) = \frac{1}{20}(17 + 0.9998 + 2(0.9998)) = 0.9999 \\ y_3 &= \frac{1}{20}(-18 - 3x_3 + z_2) = \frac{1}{20}(-18 - 3(0.9999) + 0.9998) = -0.9999 \\ z_3 &= \frac{1}{20}(25 - 2x_3 + 3y_3) = \frac{1}{20}(25 - 2(0.9999) + 3(-0.9999)) = 1.0000 \end{aligned}$$

Fourth iteration:

$$\begin{aligned} x_4 &= \frac{1}{20}(17 + 0.9999 + 2(1.0000)) = 0.9999 \\ y_4 &= \frac{1}{20}(-18 - 3(0.9999) + 1.0000) = -0.9999 \\ z_4 &= \frac{1}{20}(25 - 2(0.9999) + 3(-0.9999)) = 1.0000 \end{aligned}$$

Since, all the errors in magnitude are less than 0.0005, the required solution is
$x = 0.9999, \quad y = -0.9999, \quad z = 1.0$
Rounding to three decimal places, we get $x = 1.0, \ y = -1, \ z = 1.0$.