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Numerical Analysis I

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Chapter 5

Interpolation

5.1 INTRODUCTION

Interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable. The process of computing or finding the value of a function for any value of the independent variable outside the given range is called *extrapolation*. Here, interpolation denotes the method of computing the value of the function $y = f(x)$ for any given value of the independent variable x when a set of values of $y = f(x)$ for certain values of x are known or given.

Hence, if $(x_i, y_i), i = 0, 1, 2, \ldots, n$ are the set of $(n+1)$ given data points of the function $y = f(x)$, then the process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n, is called *interpolation*. There are several definitions available for the term interpolation. Hiral defines interpolation as the estimation of a most likely estimate in given conditions. It is the technique of estimating a past figure. Theile's definition of interpolation is "Interpolation is the art of reading between the lines of a table" while Harper's definition is "Interpolation consists in reading a value which lies between two extreme points".

If the function $f(x)$ is known explicitly, then the value of y corresponding to any value of x can easily be obtained. On the other hand, if the function $f(x)$ is not known, then it is very hard to find the exact form of $f(x)$ with the tabulated values (x_i, y_i). In such cases, the function $f(x)$ can be replaced by a simpler, function, say, $\phi(x)$, which has the same values as $f(x)$ for $x_0, x_1, x_2, \ldots, x_n$. The function $\phi(x)$ is called the *interpolating* or *smoothing function* and any other value can be computed from $\phi(x)$.

If $\phi(x)$ is a polynomial, then $\phi(x)$ is called the *interpolating polynomial* and the process of computing the intermediate values of $y = f(x)$ is called the *polynomial interpolation*. In the study of interpolation, we make the following assumptions:

(a) there are no sudden jumps in the values of the dependent variable for the period under consideration

(b) the rate of change of figures from one period to another is uniform.

In this chapter, we present the study of interpolation based on the calculus of finite differences.

5.2 INTERPOLATION WITH EQUAL INTERVALS

Here, we assume that for function $y = f(x)$, the set of $(n+1)$ functional values $y_0, y_1, \ldots, y_n$ are given corresponding to the set of $(n + 1)$ equally spaced values of the independent variable,

$x_i = x_0 + ih, i = 0, 1, \ldots, n$, where h is the spacing.

5.2.1 Missing Values

Let a function $y = f(x)$ is given for equally spaced values $x_0, x_1, x_2, \ldots, x_n$ of the argument and $y_0, y_1, y_2, \ldots, y_n$ denote the corresponding values of the function. If one or more values of $y = f(x)$ are missing, we can determine the missing values by employing the relationship between the operators E and Δ.

5.2.2 Newton's Binomial Expansion Formula

Suppose $y_0, y_1, y_2, \ldots, y_n$ denote the values of the function $y = f(x)$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh$ of x. Let one of the values of y is missing since n values of the functions are known. Therefore, we have

$$\Delta^n y_0 = 0$$

$$\text{or} \quad (E-1)^n y_0 = 0 \tag{5.1}$$

Expanding Eq. (5.1), we have

$$[E^n - {}^{\mathrm{n}}C_1 E^{n-1} + {}^{\mathrm{n}}C_2 E^{n-2} + \cdots + (-1)^n] y_0 = 0$$

$$\text{or} \quad E^n y_0 - nE^{n-1} y_0 + \frac{n(n-1)}{2!} E^{n-2} y_0 + \cdots + (-1)^n y_0 = 0$$

$$\text{or} \quad y_n - n y_{n-1} + \frac{n(n-1)}{2} y_{n-2} + \cdots + (-1)^n y_0 = 0 \tag{5.2}$$

Equation (5.2) is quite useful in determining the missing values without actually constructing the difference table.

Example 5.1. *Determine the missing entry in the following table.*

x	*0*	*1*	*2*	*3*	*4*
$y = f(x)$	*1*	*4*	*17*	-	*97*

Solution: Let $y_0 = 1, y_1 = 4, y_2 = 17$ and $y_4 = 97$. We are given four values of y. Let y be a polynomial of degree 3.

$$\text{Hence} \quad \Delta^4 y_0 = 0$$

$$\text{or} \quad (E-1)^4 y_0 = 0$$

$$(E^4 - 4E^3 + 6E^2 - 4E + 1) y_0 = 0$$

$$E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4E y_0 + y_0 = 0$$

$$\text{or} \quad y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0 \tag{E.1}$$

Substituting the given values for $y_0, y_1, \ldots, y_4$ in Eq.(E.1)

$$97 - (4y_3) + 6(17) - 4(4) + 1 = 0$$

$$\text{or} \quad y_3 = 46.$$

Example 5.2. *Find the missing entry in the following table.*

x	0	1	2	3	4	5
$y = f(x)$	1	3	11	-	189	491

Solution: Here, we are given $y_0 = 1, y_1 = 3, y_2 = 11, y_4 = 189$, and $y_5 = 491$. Since five values are given, we assume that y is a polynomial of degree 4.

$$
\begin{aligned}
\text{Hence} \quad && \Delta^5 y_0 &= 0 \\
\text{or} \quad && (E-1)^5 y_0 &= 0 \\
\text{That is} \quad && (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1) y_0 &= 0 \\
\text{or} \quad && y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 &= 0 \\
&& 491 - 5(189) + 10y_3 - 10(11) + 5(3) - 1 &= 0 \\
\text{or} \quad && 10y_3 &= 550 \\
\text{or} \quad && y_3 &= 55
\end{aligned}
$$

Example 5.3. *Find the missing entry in the following table.*

x	0	1	2	3	4	5
$y = f(x)$	1	-	11	28	-	116

Solution: Here, we are given $y_0 = 1, y_2 = 11, y_3 = 28$, and $y_5 = 116$. Since four values are known, we assume $y = f(x)$ as a polynomial of degree three.

$$
\begin{aligned}
\text{Hence} \quad && \Delta^4 y_0 &= 0 \\
\text{or} \quad && (E-1)^4 y_0 &= 0 \\
\text{That is} \quad && (E^4 - 4E^3 + 6E^2 - 4E + 1) y_0 &= 0 \\
\text{or} \quad && y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 &= 0 \\
&& y_4 - 4(28) + 6(11) - 4y_1 + 1 &= 0 \\
&& y_4 - 4y_1 &= 45 \qquad \text{(E.1)} \\
\text{and} \quad && \Delta^5 y_0 &= 0 \\
\text{or} \quad && (E-1)^5 y_0 &= 0 \\
\text{or} \quad && (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1) y_0 &= 0 \\
&& y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 &= 0 \\
&& 116 - 5y_4 + 10(28) - 10(11) + 5y_1 - 1 &= 0 \\
\text{or} \quad && -5y_4 + 5y_1 &= -285 \qquad \text{(E.2)}
\end{aligned}
$$

Solving Equations (E.1) and (E.2), we obtain

$y_1 = 4$ and $y_4 = 61$.

5.2.3 Newton's Forward Interpolation Formula

Let $y = f(x)$, which takes the values $y_0, y_1, y_2, \ldots, y_n$, that is the set of $(n+1)$ functional values $y_0, y_1, y_2, \ldots, y_n$ are given corresponding to the set of $(n+1)$ equally spaced values of the independent variable, $x_i = x_0 + ih, i = 0, 1, 2, \ldots, n$ where h is the spacing. Let $\phi(x)$ be a polynomial of the n^{th} degree in x taking the same values as y corresponding to $x = x_0, x_1, \ldots, x_n$. Then, $\phi(x)$ represents the continuous function $y = f(x)$ such that $f(x_i) = \phi(x_i)$ for $i = 0, 1, 2, \ldots, n$ and at all other points $f(x) = \phi(x) + R(x)$ where $R(x)$ is called the error term (remainder term) of the interpolation formula.

$$\text{Let} \quad \phi(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)(x-x_1) + A_3(x-x_0)(x-x_1)(x-x_2) + \cdots + A_n(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1}) \tag{5.3}$$

$$\text{and} \quad \phi(x_i) = y_i; \quad i = 0, 1, 2, \ldots, n \tag{5.4}$$

The constants $A_0, A_1, A_2, \ldots, A_n$ can be determined as follows:
Substituting $x = x_0, x_1, x_2, \ldots, x_n$ successively in Equation (5.3).

Therefore for $x = x_0$, $\phi(x_0) = A_0 = y_0$

For $x = x_1$, $\phi(x_1) = A_0 + A_1(x_1 - x_0)$

$$y_1 = y_0 + A_1(x_1 - x_0)$$

$$A_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

For $x = x_2$, $\phi(x_2) = A_0 + A_1(x_2 - x_0) + A_2(x_2 - x_0)(x_2 - x_1)$

$$y_2 = y_0 + \frac{\Delta y_0}{h}(2h) + A_2(2h)(h)$$

$$A_2(2h^2) = y_2 - 2y_1 + y_0$$

$$A_2 = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2!h^2}$$

Similarly, we obtain

$$A_3 = \frac{\Delta^3 y_0}{3!h^3}, \quad A_4 = \frac{\Delta^4 y_0}{4!h^4}, \ldots, A_n = \frac{\Delta^n y_0}{n!h^n}$$

Hence, from Eq.(5.3), we have

$$\phi(x) = y_0 + \frac{\Delta y_0}{h}(x-x_0) + \frac{\Delta^2 y_0}{2!h^2}(x-x_0)(x-x_1) + \frac{\Delta^3 y_0}{3!h^3}(x-x_0)(x-x_1)(x-x_2) + \cdots + \frac{\Delta^n y_0}{n!h^n}(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1}) \tag{5.5}$$

$$\begin{aligned} \text{Let} \quad & x = x_0 + uh \\ \text{or} \quad & x - x_0 = uh \\ \text{and} \quad & x - x_1 = (x - x_0) - (x_1 - x_0) = uh - h = (u-1)h \\ & x - x_2 = (x - x_1) - (x_2 - x_1) = (u-1)h - h = (u-2)h, \text{ etc.} \end{aligned} \tag{5.6}$$

Using the values from Eq. (5.6), Eq. (5.5) reduces to

$$\begin{aligned}\phi(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 \\ + \cdots + \frac{u(u-1)(u-2)\cdots(u-(n-1))}{n!}\Delta^n y_0\end{aligned} \tag{5.7}$$

The formula given in Eq. (5.7) is called the *Newton's forward interpolation formula.* This formula is used to interpolate the values of y near the beginning of a set of equally spaced tabular values. This formula can also be used for extrapolating the values of y a little backward of y_0.

Example 5.4. *Evaluate $f(15)$, given the following table of values:*

x	*10*	*20*	*30*	*40*	*50*
$y = f(x)$	*46*	*66*	*81*	*93*	*101*

Solution: We may note that $x = 15$ is very near to the beginning of the table. we use Newton's forward interpolation formula. Forming the forward difference table

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	46				
		20			
20	66		-5		
		15		2	
30	81		-3		-3
		12		-1	
40	93		4		
		8			
50	101				

We have Newton's forward interpolation formula as

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

From the above table, we have

$x_0 = 10,\ h = 10,\ y_0 = 46,\ \Delta y_0 = 20,\ \Delta^2 y_0 = -5,\ \Delta^3 y_0 = 2,\ \Delta^4 y_0 = -3.$

Let y_{15} be the value of y when $x = 15$, then

$$\begin{aligned}u &= \frac{x - x_0}{h} = \frac{15 - 10}{10} = 0.5 \\ f(15) = y_{15} &= 46 + (0.5)(20) + \frac{(0.5)(0.5-1)}{2}(-5) + \frac{(0.5)(0.5-1)(0.5-2)}{6}(2) \\ &+ \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24}(-3) \\ f(15) &= 56.8672\end{aligned}$$

Therefore, $f(15) = 56.8672$ correct to four decimal places.

Example 5.5. *A second degree polynomial passes through the points* $(1,-1), (2,-2), (3,-1)$

and $(4, 2)$. *Find the polynomial.*

Solution: The difference table is constructed with the given values of x and y as shown below:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	-1			
		-1		
2	-2		2	
		1		0
3	-1		2	
		3		
4	2			

Here $x_0 = 1,\ h = 1,\ y_0 = -1,\ \Delta y_0 = -1$ and $\Delta^2 y_0 = 2$
$\therefore\quad u = \frac{x-x_0}{h} = (x-1)$
From the Newton's forward interpolation formula, we have

$$\begin{aligned} f(x) &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 \\ &= -1 + (x-1)(-1) + \frac{(x-1)(x-1-1)}{2}(2) \\ f(x) &= x^2 - 4x + 2 \end{aligned}$$

Example 5.6. *Find* $y = e^{3x}$ *for* $x = 0.05$ *using the following table.*

x	*0*	*0.1*	*0.2*	*0.3*	*0.4*
e^{3x}	*1*	*1.3499*	*1.8221*	*2.4596*	*3.3201*

Solution: The difference table is shown in below:

x	$y = e^{3x}$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.00	1.0000				
		0.3499			
0.10	1.3409		0.1224		
		0.4723		0.0428	
0.20	1.8221		0.1652		0.0150
		0.6375		0.0578	
0.30	2.4596		0.2230		
		0.8605			
0.40	3.3201				

We have $x_0 = 0.00,\ x = 0.05,\ h = 0.1$
Hence $u = \frac{x-x_0}{h} = \frac{0.05-0.00}{0.1} = 0.5$

Using Newton's forward formula

$$\begin{aligned}
f(x) &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 \\
f(0.05) &= 1.0 + 0.5(0.3499) + \frac{0.5(0.5-1)}{2}(0.1224) + \frac{0.5(0.5-1)(0.5-2)}{6}(0.0428) \\
&\quad + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24}(0.0150) \\
f(0.05) &= 1.16172
\end{aligned}$$

Example 5.7. *The values of* $\sin x$ *are given below for different values of* x. *Find the value of* $\sin 42°$.

x	$40°$	$45°$	$50°$	$55°$	$60°$
$y = \sin x$	*0.6428*	*0.7071*	*0.7660*	*0.8192*	*0.8660*

Solution: $x = 42°$ is near the starting value $x_0 = 40°$. Hence, we use Newton's forward interpolation formula.

x	$y = \sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$40°$	0.6428				
		0.0643			
$45°$	0.7071		-0.0054		
		0.0589		-0.0004	
$50°$	07660		-0.0058		0
		0.0531		-0.0004	
$55°$	0.8192		-0.0062		
		0.0469			
$60°$	0.8660				

$u = \frac{x-x_0}{h} = \frac{42°-40°}{5} = 0.4$

We have $y_0 = 0.6428,\ \Delta y_0 = 0.0643,\ \Delta^2 y_0 = -0.0054,\ \Delta^3 y_0 = -0.0004$

Putting these values in Newton's forward interpolation formula we get

$$\begin{aligned}
f(x) &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 \\
f(42°) &= 0.6428 + 0.4(0.0643) + \frac{0.4(0.4-1)}{2}(-0.0054) + \frac{0.4(0.4-1)(0.4-2)}{6}(-0.0004) \\
&= 0.66913.
\end{aligned}$$

PROBLEM SET 5.1

1. The population of a town in the decimal census was as given below. Estimate the population for the year 1895.

Year (x):	1891	1901	1911	1921	1931
Population y(in thousands)	46	66	81	93	101

[**Ans.** 54.8528 thousands]

2. From the following table, find the value of $e^{0.24}$

x	0.1	0.2	0.3	0.4	0.5
e^x	1.10517	1.22140	1.34986	1.49182	1.64872

[**Ans.** $e^{0.24} = 1.271249$]

3. From the table, Estimate the number of students who obtained marks between 40 and 45.

Marks	30-40	40-50	50-60	60-70	70-80
No. of students	31	42	51	35	31

[**Ans.** 17]

4. Find the cubic polynomial which takes the following values

x	0	1	2	3
e^{3x}	1	2	1	10

[**Ans.** $2x^3 - 7x^2 + 6x + 1$]

5. The profits of a company (in thousands of rupees) are given below:

Year(x)	1990	1993	1996	1999	2002
Profit $y = f(x)$	120	100	111	108	99

Calculate the total profits between 1990-1991. [**Ans.** 104.93 thousand rupees.]

6. The following table give the marks secured by 100 students in Mathematics:

Marks	30-40	40-50	50-60	60-70	70-80
No. of students	25	35	22	11	7

Use Newton's forward difference interpolation formula to find

(a) the number of students who got more than 55 marks. [**Ans.** 28]

(b) the number of students who secured marks in the range from 36 to 45. [**Ans.** 36]

5.2.4 Newton's backward Interpolation Formula

Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to the values $x_0, x_1, x_2, \ldots, x_n$ of the independent variable x. Let the values of x be equally spaced with h as the interval of differencing.

That is $\quad x_i = x_0 + ih, i = 0, 1, 2, \ldots, n$

Let $\phi(x)$ be a polynomial of the n^{th} degree in x taking the same values of y corresponding to $x = x_0, x_1, \ldots, x_n$. That is, $\phi(x)$ represents $y = f(x)$ such that $f(x_i) = \phi(x_i), i = 0, 1, 2, \ldots$ Hence we can write $\phi(x)$ as

$$\phi(x_i) = y_i, \qquad i = n, n-1, \ldots, 1, 0$$

and

$$x_{n-i} = x_n - ih, \qquad i = 1, 2, \ldots, n$$

Let $$\phi(x) = A_0 + A_1(x - x_n) + A_2(x - x_n)(x - x_{n-1}) + \cdots + A_n(x - x_n)(x - x_{n-1}) \cdots (x - x_0) \tag{5.8}$$

Substituting $x = x_n, x_{n-1}, \ldots, x_1, x_0$ successively, we obtain

Therefore for $x = x_n$, $$A_0 = y_n \tag{5.9}$$

For $x = x_{n-1}$, $$y_{n-1} = A_0 + A_1(x_{n-1} - x_n)$$

$$A_1 = \frac{y_{n-1} - y_n}{x_{n-1} - x_n} = \frac{\nabla y_n}{h} \tag{5.10}$$

Similarly, we obtain

$$A_2 = \frac{\nabla^2 y_n}{2!h^2}, A_3 = \frac{\nabla^3 y_n}{3!h^3}, \ldots, A_n = \frac{\nabla^n y_n}{n!h^n} \tag{5.11}$$

Substituting the values from Eqs. (5.9), (5.10) and (5.11) in Eq. (5.8), we get

$$\phi(x) = y_n + \frac{\nabla y_n}{h}(x - x_n) + \frac{\nabla^2 y_n}{2!h^2}(x - x_n)(x - x_{n-1}) + \cdots + \frac{\nabla^n y_n}{n!h^n}(x - x_n)(x - x_{n-1}) \ldots (x - x_0) \tag{5.12}$$

Now, setting $x = x_n + uh$, we obtain

$$\begin{aligned} x - x_n &= uh \\ x - x_{n-1} &= (u+1)h \\ x - x_{n-2} &= (u+2)h \\ &\cdots \\ x - x_0 &= (u+n-1)h \end{aligned}$$

Hence, Eq. (5.12) reduces to

$$\phi(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots + \frac{u(u+1)\cdots(u+n-1)}{n!}\nabla^n y_n \tag{5.13}$$

The formula given in Eq. (5.13) is called the *Newton's backward interpolation formula.* This formula is used for interpolating values of y near the end of the tabulated values and also used for extrapolating values of y a little a head(to the right) of y_n.

Example 5.8. *Calculate the value of $f(84)$ for the data given in the table below:*

x	40	50	60	70	80	90
$f(x)$	204	224	246	270	296	324

Solution: The value of 84 is near the end of the table. Hence, we use the Newton's backward interpolation formula. The difference table is shown below.

x	$f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
40	204					
		20				
50	224		2			
		22		0		
60	246		2		0	
		24		0		0
70	270		2		0	
		26		0		
80	296		2			
		28				
90	324					

We have $x_n = 90,\ x = 84,\ h = 10,\ y_n = 324$, and $u = \frac{x-x_n}{h} = \frac{84-90}{10} = -0.6$
From Newton's backward formula

$$\begin{aligned} f(x) &= y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \dots \\ f(84) &= 324 - 0.6(28) + \frac{-0.6(-0.6+1)}{2}(2) = 306.96 \end{aligned}$$

Example 5.9. *From the following table estimate the number of students who obtained marks in computer programming between 75 and 80.*

Marks	*35-45*	*45-55*	*55-65*	*65-75*	*75-85*
No. of students	*20*	*40*	*60*	*60*	*20*

Solution: The cumulative frequency table is

Marks less than (x)	No. of students (y)	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
45	20				
		40			
55	60		20		
		60		-20	
65	120		0		-20
		60		-40	
75	180		-40		
		20			
85	200				

To find the number of students with marks less than 80
Let $x_n = 85,\ x = 80,\ h = 10,\ u = \frac{x-x_n}{h} = -0.5$
Then using Newton's backward interpolation formula we obtain

$$\begin{aligned} y &= y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n \\ &= 200 + (-0.5)(20) + \frac{-0.5(-0.5+1)}{2}(-40) + \frac{-0.5(-0.5+1)(-0.5+2)}{6}(-40) \\ &\quad + \frac{-0.5(-0.5+1)(-0.5+2)(-0.5+3)}{24}(-20) \\ &= 198.2813 \end{aligned}$$

So number of students getting marks in computer programming between 75 and 80
$= 198 - 180 = 18.$

Example 5.10. *The population of a town was as given. Estimate the population for the year 1925.*

Year (x):	*1891*	*1901*	*1911*	*1921*	*1931*
Population (y): (in thousands)	*46*	*66*	*81*	*93*	*101*

Solution: The difference table is:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

Here $x_n = 1931,\ h = 10,\ x = 1925$
$\therefore \quad u = \frac{1925-1931}{10} = -0.6$
Applying Newton's Backward difference formula, we get

$$\begin{aligned} y &= y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n \\ &= 101 + (-0.6)(8) + \frac{(-0.6)(-0.6+1)}{-4}(-4) + \frac{(-0.6)(-0.6+1)(-0.6+1+2)}{6}(-1) \\ &\quad + \frac{(-0.6)(-0.6+1)(-0.6+1+2)(-0.6+1+3)}{24}(-3) \\ &= 96.8368 \ \text{ thousands.} \end{aligned}$$

Hence the population for the year $1925 = 96836.8 \approx 96837.$

PROBLEM SET 5.2

1. The population of a town is as follows:

Year	1921	1931	1941	1951	1961	1971
Population(in lakhs)	20	24	29	36	46	51

Estimate the increase in population during the period 1955 to 1961 [**Ans.** 621036.8 lakhs.]

2. From the following table find the value of $\tan 17^0$.

θ	0	4	8	12	16	20	24
$\tan\theta$	0	0.0699	0.1408	0.2126	0.2867	0.3640	0.4402

[**Ans.** 0.3057]

3. From the following table, find y when $x = 1.84$ and 2.4

x	1.7	1.8	1.9	2.0	2.1	2.2	2.3
e^x	5.474	6.050	6.686	7.389	8.166	9.025	9.974

[**Ans.** $6.36,\ \ 11.02$]

4. From the following table of half yearly premium for policies maturing at different ages, estimate the premium for policy maturing at the age of 63:

Age	45	50	55	60	65
Premium	114.84	96.16	83.32	74.48	68.48

[**Ans.** 15.47996]

5.3 CENTRAL DIFFERENCE FORMULAE

As earlier we study formulae for leading terms and differences. These formulae are fundamental and are applicable to nearly all cases of interpolation, but they do not converge as rapidly as central difference formulae. The main advantage of central difference formulae is that they give more accurate result than other method of interpolation. Their disadvantages lies in complicated calculations and tedious expression, which are rather difficult to remember. These formulae are used for interpolation near the middle of a argument values. In this category we use the following formulae:

5.3.1 Gauss Forward Interpolation Formula

The Newton's forward interpolation formula is

$$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots \quad (5.14)$$

where $u = \frac{x-x_0}{h}$ and $x = x_0$ is the origin.
Now obtain the values of $\Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0, \ldots$
To get these values,

$$\begin{aligned}
\Delta^3 y_{-1} &= \Delta^2 y_0 - \Delta^2 y_{-1} \Rightarrow \Delta^2 y_0 = \Delta^3 y_{-1} + \Delta^2 y_{-1}\\
\text{Also,}\quad \Delta^4 y_{-1} &= \Delta^3 y_0 - \Delta^3 y_{-1} \Rightarrow \Delta^3 y_0 = \Delta^4 y_{-1} + \Delta^3 y_{-1}\\
\Delta^5 y_{-1} &= \Delta^4 y_0 - \Delta^4 y_{-1} \Rightarrow \Delta^4 y_0 = \Delta^5 y_{-1} + \Delta^4 y_{-1}\\
\Delta^6 y_{-1} &= \Delta^5 y_0 - \Delta^5 y_{-1} \Rightarrow \Delta^5 y_0 = \Delta^6 y_{-1} + \Delta^5 y_{-1} \ldots \text{and so on.}
\end{aligned}$$

Substituting these values in equation (5.14)

$$\begin{aligned}
y &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}[\Delta^3 y_{-1} + \Delta^2 y_{-1}] + \frac{u(u-1)(u-2)}{3!}[\Delta^4 y_{-1} + \Delta^3 y_{-1}]\\
&\quad + \frac{u(u-1)(u-2)(u-3)}{4!}[\Delta^5 y_{-1} + \Delta^4 y_{-1}]\\
&\quad + \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}[\Delta^6 y_{-1} + \Delta^5 y_{-1}] + \cdots\\
y &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)}{2!}\left(1 + \frac{(u-2)}{3}\right)\Delta^3 y_{-1}\\
&\quad + \frac{u(u-1)(u-2)}{6}\left(1 + \frac{(u-3)}{4}\right)\Delta^4 y_{-1} + \frac{u(u-1)(u-2)(u-3)}{24}\left(1 + \frac{(u-1)}{5}\right)\Delta^5 y_{-1}\\
&\quad + \frac{u(u-1)(u-2)(u-3)(u-4)}{120}\Delta^6 y_{-1} + \cdots\\
y &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-1}\\
&\quad + \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^5 y_{-1} + \cdots \quad (5.15)
\end{aligned}$$

But $\quad \Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2}$

$\Rightarrow \quad \Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$

and $\quad \Delta^6 y_{-2} = \Delta^5 y_{-1} + \Delta^5 y_{-2}$

$\Rightarrow \quad \Delta^5 y_{-1} = \Delta^5 y_{-2} + \Delta^6 y_{-2}$

The equation (5.15) becomes

$$\begin{aligned}
y &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} \\
&\quad + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^5 y_{-2} + \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^5 y_{-2} \\
&\quad + \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^6 y_{-2} \\
y &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} \\
&+ \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2} + \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^6 y_{-2} + \cdots
\end{aligned} \tag{5.16}$$

Equation (5.16) is known as the *Gauss's forward interpolation formula.* The Gauss's forward interpolation formula employs odd differences above the central line through y_0 and even differences on the central line. Gauss's forward formula is used to interpolate the values of the function for the value of u such that $0 < u < 1$.

Example 5.11. *Apply a central difference formula to obtain* $f(32)$ *given that:*

$$f(25) = 0.2707, \quad f(30) = 0.3027, \quad f(35) = 0.3386, \quad f(40) = 0.3794.$$

Solution: We have $h = 5$ and taking origin at 30

$\therefore \; u = \frac{x-x_0}{h} = \frac{32-30}{5} = 0.4$

The difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_{-1} = 25$	0.2707			
		0.032		
$x_0 = 30$	$\boxed{0.3027}$		$\boxed{0.0039}$	
		$\boxed{0.0359}$		$\boxed{0.0010}$
$x_1 = 35$	0.3386		0.0049	
		0.0408		
$x_2 = 40$	0.3794			

Applying Gauss's forward interpolation formula, we have

$$\begin{aligned}
y &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} \\
&= 0.3027 + (0.4)(0.0359) + \frac{(0.4)(0.4-1)}{2!}(0.0039) + \frac{(1+0.4)(0.4)(0.4-1)}{3!}(0.0010) \\
&= 0.3027 + 0.01436 - 0.000468 - 0.000056 \\
&= 0.316536.
\end{aligned}$$

Example 5.12. *Use Gauss's forward formula to find a polynomial of degree four which takes the following values of the function* $f(x)$:

x	1	2	3	4	5
$f(x)$	1	-1	1	-1	1

Solution: Taking origin at 3 and $h = 1$

$\therefore\ u = \frac{x-x_0}{h} = \frac{x-3}{1} = x - 3$

The difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_{-2} = 1$	1				
		-2			
$x_{-1} = 2$	-1		4		
		2		-8	
$x_0 = 3$	$\boxed{1}$		$\boxed{-4}$		$\boxed{16}$
		$\boxed{-2}$		$\boxed{8}$	
$x_1 = 4$	-1		4		
		2			
$x_2 = 5$	1				

Gauss's forward interpolation formula is

$$\begin{aligned}
f(x) &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} \\
&= 1 + (x-3)(-2) + \frac{(x-3)(x-4)}{2}(-4) + \frac{(x-2)(x-3)(x-4)}{6}(8) \\
&\qquad + \frac{(x-2)(x-3)(x-4)(x-5)}{24}(16) \\
&= 1 - 2x + 6 - 2x^2 + 14x - 24 + \frac{4}{3}(x^3 - 9x^2 + 26x - 24) \\
&\qquad + \frac{2}{3}(x^4 - 14x^3 + 71x^2 - 154x + 120)
\end{aligned}$$

$$\therefore\ f(x) = \frac{2}{3}x^4 - 8x^3 + \frac{100}{3}x^2 - 56x + 31.$$

Example 5.13. *The values of* e^{-x} *at* $x = 1.72$ *to* $x = 1.76$ *are given in the following table:*

x	1.72	1.73	1.74	1.75	1.76
$y = e^{-x}$	0.17907	0.17728	0.17552	0.17377	0.17204

Find the value of $e^{-1.7425}$ *using Gauss's forward interpolation formula.*

Solution: Here taking the origin at 1.74 and $h = 0.01$.

$\therefore u = \frac{x-x_0}{h} = \frac{1.7425-1.7400}{0.01} = 0.25$

The difference table is as follows:

x	$10^5 y$	$10^5 \Delta y$	$10^5 \Delta^2 y$	$10^5 \Delta^3 y$	$10^5 \Delta^4 y$
$x_{-2} = 1.72$	17907				
		-179			
$x_{-1} = 1.73$	17728		3		
		-176		-2	
$x_0 = 1.74$	$\boxed{17552}$		$\boxed{1}$		$\boxed{3}$
		$\boxed{-175}$		$\boxed{1}$	
$x_1 = 1.75$	17377		2		
		-173			
$x_2 = 1.76$	17204				

Gauss's forward interpolation formula is

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2}$$

$$\begin{aligned}
\therefore 10^5 f(0.25) &= 17552 + (0.25)(-175) + \frac{(0.25)(-0.75)}{2}(1) + \frac{(1.25)(0.25)(-0.75)}{6}(1) \\
&\qquad + \frac{(1.25)(0.25)(-0.75)(-1.75)}{24}(3) \\
&= 17552 - 43.75 - 0.09375 - 0.0390625 + 0.0512695 \\
&= 17508.16846 \\
\therefore \quad f(0.25) &= e^{-1.7425} = 0.1750816846.
\end{aligned}$$

Example 5.14. *Apply Gauss's forward formula to find the value of* y_9, *if* $y_0 = 14, y_4 = 24$, $y_8 = 32, y_{12} = 35, y_{16} = 40$.

Solution: The difference table is (taking origin at 8):

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_{-2} = 0$	14				
		10			
$x_{-1} = 4$	24		-2		
		8		-3	
$x_0 = 8$	$\boxed{32}$		$\boxed{-5}$		$\boxed{10}$
		$\boxed{3}$		$\boxed{7}$	
$x_1 = 12$	35		2		
		5			
$x_2 = 16$	40				

Here $x_0 = 8,\ h = 4,\ x = 9$

$\therefore \qquad u = \frac{x - x_0}{h} = \frac{9-8}{4} = 0.25$

Gauss's forward interpolation formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2}$$

$$\therefore\ y_9 = 32 + (0.25)(3) + \frac{(0.25)(-0.75)}{2}(-5) + \frac{(1.25)(0.25)(-0.75)}{6}(7) + \frac{(1.25)(0.25)(-0.75)(-1.75)}{24}(10)$$

$$= 32 + 0.75 + 0.46875 - 0.2734375 + 0.17089844$$

$$y_9 = 33.11621094.$$

PROBLEM SET 5.3

1. From the following table find y when $x = 1.45$

x	1.0	1.2	1.4	1.6	1.8	2.
y	0.0	-0.112	-0.016	0.336	0.992	2.0

[**Ans.** 0.047875]

2. Use Gauss's forward formula to find y_{30} for the following data.

y_{21}	y_{25}	y_{29}	y_{33}	y_{37}
18.4708	17.8144	17.170	16.3432	15.5154

[**Ans.** $y_{30} = 16.9216$]

3. Apply Gauss forward formula to find a polynomial of degree three which takes the values of y as given on next page:

x	2	4	6	8	10
y	-2	1	3	8	20

$\Big[$**Ans.** $3 + \frac{17}{6}x + \frac{3}{2}x^2 + \frac{2}{3}x^3\Big]$

4. Given that

x	25	30	35	40	45
$\log x$	1.39794	1.47712	1.54407	1.60206	1.65321

$\log 3.7 = ?$ [**Ans.** 0.56819272]

5.3.2 Gauss Backward Interpolation Formula

This formula is also solved by using Newton's forward difference formula.
Now, we know Newton's formula for forward interpolation is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots \quad (5.17)$$

Now $\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$
$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$
$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$
$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}$ ... and so on.
On substituting these values in (5.17), we get

$$\begin{aligned}
y &= y_0 + u[\Delta y_{-1} + \Delta^2 y_{-1}] + \frac{u(u-1)}{2!}\left[\Delta^2 y_{-1} + \Delta^3 y_{-1}\right] + \frac{u(u-1)(u-2)}{3!}\left[\Delta^3 y_{-1} + \Delta^4 y_{-1}\right] \\
&\quad + \frac{u(u-1)(u-2)(u-3)}{4!}[\Delta^4 y_{-1} + \Delta^5 y_{-1}] + \cdots \\
y &= y_0 + u\Delta y_{-1} + u\Delta^2 y_{-1}\left[1 + \frac{(u-1)}{2}\right] + \frac{u(u-1)}{2}\Delta^3 y_{-1}\left[1 + \frac{(u-2)}{3}\right] \\
&\quad + \frac{u(u-1)(u-2)}{3}\Delta^4 y_{-1}\left[1 + \frac{(u-3)}{4}\right] + \cdots \\
y &= y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} \\
&\quad + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-1} + \cdots \qquad (5.18)
\end{aligned}$$

Again $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$
$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$
$\Delta^5 y_{-1} = \Delta^5 y_{-2} + \Delta^6 y_{-2}$ ... and so on.
Therefore, equation (5.18) becomes

$$\begin{aligned}
y &= y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\left[\Delta^3 y_{-2} + \Delta^4 y_{-2}\right] \\
&\quad + \frac{(u+1)u(u-1)(u-2)}{4!}\left[\Delta^4 y_{-2} + \Delta^5 y_{-2}\right] + \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\left[\Delta^5 y_{-2} + \Delta^6 y_{-2}\right] \\
y &= y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} \\
&\quad + \frac{(u+1)u(u-1)}{3!}\left[1 + \frac{(u-2)}{4}\right]\Delta^4 y_{-2} + \frac{(u+1)u(u-1)(u-2)}{4!}\left[1 + \frac{(u-3)}{5}\right]\Delta^5 y_{-2} + \cdots \\
y &= y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} \\
&\quad + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2} + \cdots \qquad (5.19)
\end{aligned}$$

Equation (5.19) is called the *Gauss's backward interpolation formula.* Gauss's backward interpolation formula employs odd differences below the central line through y_0 and even differences on the central line as shown. Gauss's backward interpolation formula is used to interpolate line value of the function for a negative value of u which lies between -1 and $0(-1 < u < 0)$.

Example 5.15. *Given that*

x^0	*50*	*51*	*52*	*53*	*54*
$\tan x^0$	*1.1918*	*1.2349*	*1.2799*	*1.3270*	*1.3764*

Using Gauss's backward formula, find the value of $\tan 51^0 42'$

Solution: Take the origin at 52^0 and given $h = 1$

$\therefore \qquad u = \frac{x - x_0}{h} = 51^0 42' - 52^0 = 18' = -0.3^0$

Difference table for given data is:

x^0	$\tan x^0$	Δ	Δ^2	Δ^3
50	1.1918			
		0.0431		
51	1.2349		0.0019	
		$\boxed{0.045}$		$\boxed{0.0002}$
52	$\boxed{1.2799}$		$\boxed{0.0021}$	
		0.0471		0.0002
53	1.3270		0.0023	
		0.0494		
54	1.3764			

Now using Gauss backward formula

$$y = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2}$$

$$= 1.2799 + (-0.3)(0.045) + \frac{(-0.3)(-0.3+1)}{2}(0.0021)$$

$$+ \frac{(-0.3+1)(-0.3)(-0.3-1)}{6}(0.0002) + 0$$

$$= 1.2799 - 0.0135 - 0.0002205 - 0.0000119$$

$$= 1.266167(\text{Approx.})$$

Example 5.16. *Using Gauss backward formula, Estimate the no. of persons earning wages between Rs. 60 and Rs. 70 from the following data:*

Wages (Rs.)	*Below 40*	*40 - 60*	*60 - 80*	*80 - 100*	*100 - 120*
No. of Persons (in thousands)	*250*	*120*	*100*	*70*	*50*

Solution: Difference table for the given data is as:

Wages below	No. of Persons	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	250				
		120			
60	370		-20		
		$\boxed{100}$		$\boxed{-10}$	
80	$\boxed{470}$		$\boxed{-30}$		$\boxed{20}$
		70		10	
100	540		-20		
		50			
120	590				

$\therefore \qquad u = \frac{x-x_0}{h} = \frac{70-80}{20} = -0.5$

From Gauss backward formula

$$y = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2}$$
$$= 470 + (-0.5)(100) + \frac{(0.5)(-0.5)}{2}(-30) + \frac{(0.5)(-0.5)(-1.5)}{6}(-10)$$
$$+ \frac{(1.5)(0.5)(-0.5)(-1.5)}{24}(20)$$
$$= 470 - 50 + 3.75 - 0.625 + 0.46875$$
$$= 423.59375$$

Hence No. of Persons earning wages between Rs. 60 to 70 is $423.59375 - 370 = 53.59375$ or 54000. (Approx.)

Example 5.17. *If $f(x)$ is a polynomial of degree four find the value of $f(5.8)$ using Gauss's backward formula from the following data:*

$$f(4) = 270, \;\; f(5) = 648, \;\; \Delta f(5) = 682, \;\; \Delta^3 f(4) = 132.$$

Solution: Given $\Delta f(5) = 682$

$$\begin{aligned}
\Rightarrow \qquad & f(6) - f(5) = 682 \\
\Rightarrow \qquad & f(6) = 682 + 648 \\
\Rightarrow \qquad & f(6) = 1330 \\
\text{Also,} \qquad & \Delta^3 f(4) = 132 \\
\Rightarrow \qquad & (E-1)^3 f(4) = 132 \\
\Rightarrow \qquad & f(7) - 3f(6) + 3f(5) - f(4) = 132 \\
\Rightarrow \qquad & f(7) = 3\times 1330 - 3\times 648 + 270 + 132 \\
& f(7) = 2448
\end{aligned}$$

Now form difference table as:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
4	270			
		378		
5	648		304	
		$\boxed{682}$		$\boxed{132}$
6	$\boxed{1330}$		$\boxed{436}$	
		1118		
7	2448			

Take $x_0 = 6, h = 1$

$\therefore \quad u = \frac{x-x_0}{h} = \frac{5.8-6}{1} = -0.2$

From Gauss backward formula

$$\begin{aligned} f(-0.2) &= f(0) + u\Delta f(-1) + \frac{u(u+1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2) \\ &= 1330 + (-0.2)(682) + \frac{(-0.2)(0.8)}{2}(436) + \frac{(-0.2)(0.8)(-1.2)}{6}(132) \\ &= 1330 - 136.4 - 34.88 + 4.224 \\ f(5.8) &= 1162.944 \end{aligned}$$

PROBLEM SET 5.4

1. Find the value of $\cos 51^0 42'$ by Gauss's backward formula from the following data:

x	50^0	51^0	52^0	53^0	54^0
$\cos x$	0.6428	0.6293	0.6157	0.6018	0.5878

[**Ans.** 0.61981013]

2. The population of a town in the years are as follows:

Year	1931	1941	1951	1961	11971
Population (in thousands)	15	20	27	39	52

Find the population of the town in 1946 by applying Gauss's backward formula. [**Ans.** 22898]

3. Interpolate by means of Gauss's backward formula, the population of a town ADAMA for the year 1974, given that:

Year	1939	1949	1959	1969	1979	1989
Population (in thousands)	12	15	20	27	39	52

[**Ans.** 32.345 thousands approx.]

4. Use Gauss interpolation formula to find y_{41} from the following data:
$y_{30} = 3678.2, y_{35} = 2995.1, y_{40} = 2400.1, y_{45} = 1876.2, y_{50} = 1416.3$ [**Ans.** 2290.1]

5. Use Gauss's backward formula to find the value of y when $x = 3.75$, given the following table:

x	2.5	3.0	3.5	4.0	4.5	5.0
y_x	24.145	22.043	2.225	18.644	17.262	16.047

[**Ans.** 19.704]

5.3.3 Stirling's Formula

Consider the mean of the Gauss's forward and backward interpolation formula given by Equations (5.16) and (5.19), we get

$$y = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{(u+1)u^2(u-1)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2} + \cdots \tag{5.20}$$

Equation (5.20) is known as the Stirling's formula. Stirling formula gives the most accurate result for $0.25 \leq u \leq 0.25$. Hence, x_0 should be selected such that u satisfies this inequality.

Example 5.18. *Use Stirling's interpolation formula to find* y_{28}*, given that* $y_{20} = 48234$, $y_{25} = 47354, y_{30} = 46267, y_{35} = 44978$ *and* $y_{40} = 43389$.

Solution: Here $x = 30$ as origin and $h = 5$. Therefore $u = \frac{28-30}{5} = -0.4$.
The difference table is shown below:

x	$u = \frac{x-30}{5}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$
20	-2	48234				
			-880			
25	-1	47354		-207		
			$\boxed{-1087}$		$\boxed{5}$	
30	0	$\boxed{46267}$		$\boxed{-202}$		$\boxed{-103}$
			$\boxed{-1289}$		$\boxed{-98}$	
35	1	44978		-300		
			-1589			
40	2	43389				

The Stirling's interpolation formula is

$$\begin{aligned}
y_u &= y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} \\
&= 46267 + (-0.4)\left[\frac{-1289 - 1087}{2}\right] + \frac{(-0.4)^2}{2}(-202) + \frac{(-0.4)((-0.4)^2-1)}{6}\left[\frac{-98+5}{2}\right] \\
&\quad + \frac{(-0.4)^2((-0.4)^2-1)}{24}(-103) \\
&= 46267 + 475.2 - 16.16 - 2.604 + 0.5768 \\
&= 46724.0128
\end{aligned}$$

Example 5.19. *Apply Stirling's formula to find a polynomial of degree three which takes the following values of* x *and* y*:*

x	*2*	*4*	*6*	*8*	*10*
y	*-2*	*1*	*3*	*8*	*20*

Solution: Let $u = \frac{x-6}{2}$. Now, we construct the following difference table:

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$
2	-2	-2				
			3			
4	-1	1		-1		
			$\boxed{2}$		$\boxed{4}$	
6	0	$\boxed{3}$		$\boxed{3}$		$\boxed{0}$
			$\boxed{5}$		$\boxed{4}$	
8	1	8		7		
			12			
10	2	20				

The Stirling's formula is

$$\begin{aligned}
y_u &= y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} \\
&= 3 + (-0.4)\left[\frac{2+5}{2}\right] + \frac{(-0.4)^2}{2}(3) + \frac{(-0.4)((-0.4)^2-1)}{6}\left[\frac{4+4}{2}\right] + 0 \\
&= 3 + \frac{7}{2}u + \frac{3}{2}u^2 + \frac{2}{3}(u^3 - u) \\
&= \frac{2}{3}u^3 + \frac{3}{2}u^2 + \frac{17}{6}u + 3
\end{aligned}$$

Put $u = \frac{x-6}{2}$

$$\begin{aligned}
&= \frac{2}{3}\left(\frac{x-6}{2}\right)^3 + \frac{3}{2}\left(\frac{x-6}{2}\right)^2 + \frac{17}{6}\left(\frac{x-6}{2}\right) + 3 \\
y &= 0.0833x^3 - 1.125x^2 + 8.9166x - 19.
\end{aligned}$$

Example 5.20. *Apply Stirling's formula to find the value of* $f(1.22)$ *from the following table which gives the value of* $f(x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{x^2}{2}}dx$ *at intervals of* $x = 0.5$ *from* $x = 0$ *to* 2.

x	*0*	*0.5*	*1.0*	*1.5*	*2.0*
$f(x)$	*0*	*0.191*	*0.341*	*0.433*	*0.477*

Solution: Let the origin be at 1 and $h = 0.5$

$\therefore \quad u = \frac{x-x_0}{h} = \frac{1.22-1.00}{0.5} = 0.44$

Applying Stirling's formula

$$\begin{aligned}
y &= y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} \\
&= y_0 + (0.44)\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{(0.44)^2}{2!}\Delta^2 y_{-1} + \frac{(0.44)((0.44)^2-1)}{6}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] \\
&\qquad + \frac{(0.44)^2((0.44)^2-1)}{24}\Delta^4 y_{-2} \\
&= y_0 + (0.22)[\Delta y_0 + \Delta y_{-1}] + 0.0968\Delta^2 y_{-1} - 0.029568[\Delta^3 y_{-1} + \Delta^3 y_{-2}] - 0.06505\Delta^4 y_{-2}
\end{aligned}$$

The difference table is as follows:

u	x	y	$10^3\Delta y$	$10^3\Delta^2 y$	$10^3\Delta^3 y$	$10^3\Delta^4 y$
-2	0	0				
			191			
-1	0.5	191		-41		
			$\boxed{150}$		$\boxed{-17}$	
0	1	$\boxed{341}$		$\boxed{-58}$		$\boxed{27}$
			$\boxed{92}$		$\boxed{10}$	
1	1.5	433		-48		
			44			
2	2	477				

y_0 and the differences are being multiplied by 10^3

$$
\begin{aligned}
\therefore\quad 10^3 y_{0.44} &\approx 341 + 0.22 \times (150 + 92) + 0.0968 \times (-58) - 0.029568 \times (-17 + 10) - 0.006505 \times 27 \\
&\approx 341 + 0.22 \times 242 - 0.0968 \times 58 + 0.029568 \times 7 - 0.006505 \times 27 \\
&\approx 341 + 53.24 - 5.6144 + 0.206276 - 0.175635 \\
&\approx 388.66 \\
y_{0.44} &= 0.389 \text{ at } x = 1.22
\end{aligned}
$$

PROBLEM SET 5.5

1. Find $f(0.41)$ using Stirling's formula if,

x	0.30	0.35	0.40	0.45	0.50
$f(x)$	0.1179	0.1368	0.1554	0.1736	0.1915

[**Ans.** 0.15907168]

2. Use Stirling's formula to find y_{35}, data being:
$y_{20} = 512, y_{30} = 439, y_{40} = 346$, and $y_{50} = 243$. [**Ans.** 394.6875]

3. Apply Stirling's formula to find a polynomial of degree four which takes the values of y as given below:

x	1	2	3	4	5
y	1	-1	1	-1	1

[**Ans.** $\frac{2}{3}u^4 - \frac{8}{3}u^2 + 1$]

5.3.4 Bessel's Interpolation Formula

This is one of the another type of central difference formula and obtained by

1. shifting the origin by 1 in Gauss backward difference
2. replacing u by $(u-1)$,
3. take mean of this equation with Gauss forward formula.

Gauss backward difference formula is,

$$y = y_0 + u\Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2} + \cdots$$

Now shift the origin by one, we get

$$y = y_1 + u\Delta y_0 + \frac{(u+1)u}{2!}\Delta^2 y_0 + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-1} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-1} + \cdots$$

On replacing u by $(u-1)$

$$y = y_1 + (u-1)\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-1} + \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^5 y_{-1} + \cdots \tag{5.21}$$

Gauss forward difference formula is,

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2} + \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^6 y_{-2} + \cdots \tag{5.22}$$

Take the mean of equation (5.21) and (5.22)

$$y = \frac{y_0+y_1}{2} + \left(\frac{(u-1)+u}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right) + \frac{u(u-1)}{3!}\left(\frac{(u-2)+(u+1)}{2}\right)\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\left(\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right) + \frac{(u+1)u(u-1)(u-2)}{5!}\left(\frac{(u-3)\Delta^5 y_{-1} + (u+2)\Delta^5 y_{-2}}{2}\right) + \cdots$$

$$y = \frac{y_0+y_1}{2} + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right) + \frac{u(u-1)(u-1/2)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\left(\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right) + \frac{(u+1)u(u-1)(u-2)(u-1/2)}{5!}\Delta^5 y_{-2} + \cdots$$

This formula is very useful when $u = \frac{1}{2}$ and gives best result when $\frac{1}{4} < u < \frac{3}{4}$.

Example 5.21. *Apply Bessel's interpolation formula to obtain* y_{25}, *given that* $y_{20} = 2860$, $y_{24} = 3167, y_{28} = 3555$ *and* $y_{32} = 4112$.

Solution: The difference table is shown below:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_{-1} = 20$	2860			
		37		
$x_0 = 24$	$\boxed{3167}$		$\boxed{81}$	
		$\boxed{388}$		$\boxed{88}$
$x_1 = 28$	$\boxed{3555}$		$\boxed{169}$	
		557		
$x_2 = 32$	4112			

Here $x_0 = 24, h = 4$ and $u = \frac{x - x_0}{h} = \frac{25-24}{4} = 0.25$

The Bessel's formula is

$$
\begin{aligned}
y &= \frac{y_0 + y_1}{2} + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right) + \frac{u(u-1)(u-1/2)}{3!}\Delta^3 y_{-1} \\
&= \frac{3167 + 3555}{2} + (-0.25)(388) + \frac{0.25(0.25-1)}{2}\left(\frac{81+169}{2}\right) + \frac{0.25(0.25-1)(0.25-0.5)}{6}(88) \\
&= 3361 - 97 - 11.71875 + 0.6875 \\
&= 3252.96875
\end{aligned}
$$

Example 5.22. *The pressure* p *of wind corresponding to velocity* v *is given by following data. Estimate pressure when* $v = 25$.

v	*10*	*20*	*30*	*40*
P	*1.1*	*2*	*4.4*	*7.9*

Solution: The difference table for the given data is as:

v	P	Δ	Δ^2	Δ^3
10	1.1			
		0.9		
20	$\boxed{2}$		$\boxed{1.5}$	
		$\boxed{2.4}$		$\boxed{-0.4}$
30	$\boxed{4.4}$		$\boxed{1.1}$	
		3.4		
40	7.9			

Taking the origin at 20 and $h = 10$

$$u = \frac{x - x_0}{h} = \frac{25-20}{10} = 0.5$$

Bessel's formula for interpolation is:

$$
\begin{aligned}
P(u) &= \frac{P_0+P_1}{2} + \left(u-\frac{1}{2}\right)\Delta P_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 P_0 + \Delta^2 P_{-1}}{2}\right) + \frac{u(u-1)(u-1/2)}{3!}\Delta^3 P_{-1} \\
&= \frac{2+4.4}{2} + (0.5-0.5)(2.4) + \frac{0.5(0.5-1)}{2}\left(\frac{1.5+1.1}{2}\right) + \frac{0.5(0.5-1)(0.5-0.5)}{6}(-0.4) \\
&= 3.2 + 0 - 0.16250 + 0 \\
P_{25} &= 3.03750
\end{aligned}
$$

Example 5.23. *If third differences are constant, prove that*

$$y_{x+\frac{1}{2}} = \tfrac{1}{2}(y_x + y_{x+1}) - \tfrac{1}{16}(\Delta^2 y_{x-1} + \Delta^2 y_x)$$

Solution: Put $u = \frac{1}{2}$ in Bessel's formula, we get

$$y_{\frac{1}{2}} = \tfrac{1}{2}(y_0 + y_1) - \tfrac{1}{16}(\Delta^2 y_0 + \Delta^2 y_{-1})$$

Shifting the origin to x,

$$y_{x+\frac{1}{2}} = \tfrac{1}{2}(y_x + y_{x+1}) - \tfrac{1}{16}(\Delta^2 y_x + \Delta^2 y_{x-1})$$

Example 5.24. *Find a polynomial for the given data using Bessel's formula* $f(2) = 7, f(3) = 9, f(4) = 12, f(5) = 16$.

Solution: Let us take origin as 3 therefore,
$u = x - 3$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2	7			
		2		
3	$\boxed{9}$		$\boxed{1}$	
		$\boxed{3}$		$\boxed{0}$
4	$\boxed{12}$		$\boxed{1}$	
		4		
5	16			

The Bessel's formula is

$$
\begin{aligned}
y_u &= \frac{y_0+y_1}{2} + \left(u-\frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right) + \frac{u(u-1)(u-1/2)}{3!}\Delta^3 y_{-1} \\
&= \frac{9+12}{2} + \left(u-\frac{1}{2}\right)(3) + \frac{u(u-1)}{2}\left(\frac{1+1}{2}\right) + 0 \\
&= \frac{u^2}{2} + \frac{5}{2}u + 9
\end{aligned}
$$

Put $u = x - 3$

$$
\begin{aligned}
y &= \frac{(x-3)^2}{2} + \frac{5}{2}(x-3) + 9 \\
y &= \frac{1}{2}x^2 - \frac{1}{2}x + 6.
\end{aligned}
$$

PROBLEM SET 5.6

1. Find $y(0.543)$ from the following values of x and y:

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7
$y(x)$	2.631	3.328	4.097	4.944	5.875	6.896	8.013

[**Ans.** 6.303]

2. Find y_{25} by using Bessel's interpolation formula from the data:
   $y_{20} = 24,\ y_{24} = 32,\ y_{28} = 35$, and $y_{32} = 40$ [**Ans.** 32.9453125]

3. Apply Bessel's formula to obtain a polynomial of degree three:

x	7	8	9	10	11
$f(x)$	14	17	19	22	25

[**Ans.** $-\frac{x^2}{6} + 5x^2 - \frac{281}{6}x + 157$]

4. Given $y_0, y_1, y_2, y_3, y_4, y_5$ (fifth difference constant), prove that
   $$y_{2\frac{1}{2}} = \frac{1}{2}c + \frac{25(c-b)+3(a-c)}{256}$$
   where $a = y_0 + y_5, b = y_1 + y_4, c = y_2 + y_3$.

5.3.5 Laplace-Everett's Formula

Eliminating odd differences in Gauss's forward formula [Eq.(5.16)] by using the relation

$$\Delta y_0 = y_1 - y_0$$

We have
$$\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$$
$$\Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2} \ldots,$$

Therefore, using this in equation (5.16), we get

$$\begin{aligned}
y &= y_0 + u(y_1 - y_0) + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}(\Delta^2 y_0 - \Delta^2 y_{-1}) \\
&\quad + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}(\Delta^4 y_{-1} - \Delta^4 y_{-2}) + \cdots \\
&= (1-u)y_0 + uy_1 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} - \frac{(u+1)u(u-1)}{3!}\Delta^2 y_{-1} \\
&\qquad + \frac{(u+1)u(u-1)}{3!}\Delta^2 y_0 + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 y_{-1} \\
&\qquad + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} - \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 y_{-2} + \cdots \\
&= (1-u)y_0 + uy_1 + \frac{u(u-1)(2-u)}{3!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^2 y_0 \\
&\quad + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 y_{-1} + \frac{(u+1)u(u-1)(u-2)(3-u)}{5!}\Delta^4 y_{-2} + \cdots \\
y &= \left(uy_1 + \frac{(u+1)u(u-1)}{3!}\Delta^2 y_0 + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 y_{-1} + \cdots\right) \\
&\quad + \left((1-u)y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)(u-2)(3-u)}{5!}\Delta^4 y_{-2} + \cdots\right) \qquad (5.23)
\end{aligned}$$

Substitute $1-u=w$ in second part of equation (5.23)

$$y = \left(uy_1 + \frac{(u+1)u(u-1)}{3!}\Delta^2 y_0 + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 y_{-1} + \cdots\right)$$
$$+ \left(wy_0 + \frac{(w-1)w(w+1)}{3!}\Delta^2 y_{-1} + \frac{(w+2)(w+1)w(w-1)(w-2)}{5!}\Delta^4 y_{-2} + \cdots\right)$$

This is called *Laplace-Everett's formula.* It gives better estimate value when $u > \frac{1}{2}$.

Example 5.25. *Using Laplace Everett's formula, find* $f(30)$*, if* $f(20) = 2854,\ f(28) = 3162,\ f(36) = 7088,\ f(44) = 7984$.

Solution: Take origin at $28, h = 8$

$\therefore \quad x_0 + hu = 30$

$\Rightarrow \quad 28 + 8u = 30 \Rightarrow u = 0.25$

Also, $w = 1 - u = 1.2.5 = 0.75$

Difference table is:

u	$f(u)$	$\Delta f(u)$	$\Delta^2 f(u)$	$\Delta^3 f(u)$
-1	2854			
		308		
0	$\boxed{3162}$		$\boxed{3618}$	
		3926		-6648
1	$\boxed{7088}$		$\boxed{-3030}$	
		896		
2	7984			

By Everett's formula,

$$f(u) = \left[uf(1) + \frac{(u+1)u(u-1)}{3!}\Delta^2 f(0)\right] + \left[wf(0) + \frac{(w-1)w(w+1)}{3!}\Delta^2 f(-1)\right]$$
$$f(0.25) = \left[(0.25)(7088) + \frac{(1.25)(0.25)(-0.75)}{3!}(-3030)\right]$$
$$+ \left[(0.75)(3162) + \frac{(1.75)(0.75)(-0.25)}{3!}(3618)\right]$$
$$= 4064$$

Hence $f(30) = 4064$

Example 5.26. *Use Everett's interpolation formula to find the value of* y *when* $x = 1.60$ *from the following table.*

x	*1.0*	*1.25*	*1.50*	*1.75*	*2.0*	*2.25*
$y = f(x)$	*1.0543*	*1.1281*	*1.2247*	*1.3219*	*1.4243*	*1.4987*

Solution: The difference table is shown below:

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-2	1.00	1.0543				
			0.0738			
-1	1.25	1.1281		0.0228		
			0.0966		-0.0222	
0	1.50	$\boxed{1.2247}$		$\boxed{0.006}$		$\boxed{0.0268}$
			0.0972		0.0046	
1	1.75	$\boxed{1.3219}$		$\boxed{0.0052}$		$\boxed{-0.0378}$
			0.1024		-0.0332	
2	2.0	1.4243		-0.0280		
			0.0744			
3	2.25	1.4987				

Here $x_0 = 1.50$ and $h = 0.25$
Therefore $u = \frac{x-x_0}{h} = \frac{1.60-1.50}{0.25} = 0.4$
and $w = 1 - u = 1 - 0.4 = 0.6$
The Everett's interpolation formula is

$$
\begin{aligned}
y &= \left[uy_1 + \frac{(u+1)u(u-1)}{3!}\Delta^2 y_0 + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 y_{-1} + \cdots\right] \\
&\quad + \left[wy_0 + \frac{(w-1)w(w+1)}{3!}\Delta^2 y_{-1} + \frac{(w+2)(w+1)w(w-1)(w-2)}{5!}\Delta^4 y_{-2} + \cdots\right] \\
&= \left[0.4(1.3219) + \frac{(0.4)(0.4^2-1)}{6}(0.0052) + \frac{(0.4)(0.4^2-1)(0.4^2-4)}{120}(-0.03780)\right] \\
&\quad + \left[0.6(1.2247) + \frac{(0.6)(0.6^2-1)}{6}(0.0006) + \frac{(0.6)(0.6^2-1)(0.6^2-4)}{120}(0.02680)\right] \\
&= 1.26316
\end{aligned}
$$

PROBLEM SET 5.7

1. Find the value of $f(27.4)$ from the following table:

x	25	26	27	28	29	30
$f(x)$	4.000	3.846	3.704	3.571	3.448	3.333

[**Ans.** 3649.678336]

2. Given the table:

x	310	320	330	340	350	360
$\log x$	2.49136	2.50515	2.51851	2.53148	2.54407	2.55630

Find the value of $\log 337.5$ by Laplace Everett's formula. [**Ans.** -2.528273]

3. Obtain the values of y_{25}, given that
$y_{20} = 2854,\ y_{24} = 3162,\ y_{28} = 3544$ and $y_{32} = 3992$ [**Ans.** 3250.875]

5.4 INTERPOLATION WITH UNEQUAL INTERVALS

The interpolation formulae derived before for forward interpolation, Backward interpolation and central interpolation have the disadvantages of being applicable only to equally spaced argument values. So it is required to develop interpolation formulae for unequally spaced argument values of x. Therefore, when the values of the argument are not at equally spaced then we use two such formulae for interpolation.

1. Lagrange's Interpolation formula
2. Newton's Divided difference formula.

The main advantage of these formulas is, they can also be used in case of equal intervals but the formulae for equal intervals cannot be used in case of unequal intervals.

5.4.1 Lagrange's Interpolation Formula

Let $y = f(x)$ be a real valued continuous function defined in an interval [a, b]. Let $x_0, x_1, x_2, \ldots, x_n$ be $(n+1)$ distinct points which are not necessarily equally spaced and the corresponding values of the function are $y_0, y_1, \ldots, y_n$. Since $(n+1)$ values of the function are given corresponding to the $(n+1)$ values of the independent variable x, we can represent the function $y = f(x)$ is a polynomial in x of degree n.
Let the polynomial is represented by

$$\begin{aligned} f(x) &= A_0(x-x_1)(x-x_2)\cdots(x-x_n) + A_1(x-x_0)(x-x_1)\cdots(x-x_n) \\ &\quad + A_2(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n) + \cdots + A_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \end{aligned} \tag{5.24}$$

Each term in Equation (5.24) being a product of n factors in x of degree n, putting $x = x_0$ in Equation (5.24) we obtain

$$\begin{aligned} f(x_0) &= A_0(x-x_1)(x-x_2)\cdots(x-x_n) \\ \Rightarrow A_0 &= \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} \end{aligned}$$

Putting $x = x_1$ in Equation (5.24) we obtain

$$\begin{aligned} f(x_1) &= A_1(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n) \\ \Rightarrow A_1 &= \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} \end{aligned}$$

Similarly putting $x = x_2, x = x_3, \cdots x = x_n$ in Eq. (5.24) we obtain

$$\begin{aligned} A_2 &= \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)\cdots(x_2-x_n)} \\ &\vdots \qquad\qquad \vdots \\ A_n &= \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} \end{aligned}$$

Substituting the values of $A_0, A_1, \cdots, A_n$ in Equation (5.24) we get

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}f(x_0) + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}f(x_1)+ \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}f(x_n) \quad (5.25)$$

The formula given by Equation (5.25) is known as the *Lagrange's interpolation formula.*

Example 5.27. *Apply Lagrange's interpolation formula to find a polynomial which passes through the points* $(0,20), (1,12), (3,20)$ *and* $(4,24)$.

Solution: We have $x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 4, y_0 = f(x_0) = 20, y_1 = f(x_1) = 12, y_2 = f(x_2) = 20$ and $y_3 = f(x_3) = 24$.

The Lagrange's interpolation formula is

$$\begin{aligned} f(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1) \\ &+ \frac{(x-x_0)(x-x_)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3) \end{aligned}$$

$$\text{Hence} \quad \begin{aligned} f(x) &= \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)}(-20) + \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)}(-12) \\ &+ \frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)}(-20) + \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)}(-24) \end{aligned}$$

$f(x) = x^3 - 8x^2 + 15x + 20$ is the required polynomial.

Example 5.28. *Using Lagrange's interpolation formula, find the value of* y *corresponding to* $x = 10$ *from the following data.*

x	*5*	*6*	*9*	*11*
$y = f(x)$	*380*	*-2*	*196*	*508*

Solution: The Lagrange's interpolation formula is

$$\begin{aligned} f(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 \\ &+ \frac{(x-x_0)(x-x_)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3 \end{aligned} \quad (5.26)$$

Here, we have $x_0 = 5, x_1 = 6, x_2 = 9, x_3 = 11, y_0 = 380, y_1 = 2, y_2 = 196$ and $y_3 = 508$. Substituting these values in Eq. (5.26), we get

$$\begin{aligned} f(10) &= \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(380) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(-2) \\ &+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(196) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-9)(11-9)}(508) \\ &= 330. \end{aligned}$$

PROBLEM SET 5.8

1. Find Lagrange's interpolation polynomial fitting the points $f(1) = -3, f(3) = 0,$ $f(4) = 30, f(6) = 132$. Hence find $f(5)$. [**Ans.** 75]

2. If $y(1) = -3,\ y(3) = 9,\ y(4) = 30$ and $y(6) = 132$, find the four point Lagrange's interpolation polynomial which takes the same values as the function y at the given points. [**Ans.** $x^3 - 3x^2 + 5x - 6$]

3. The percentage of Criminals for different age group are given below:

Ages less than	25	30	40	50
Percentage of Criminals	52	67	84	94

Apply Lagrange's formula to find the percentage of criminals under 35 years of age. [**Ans.** 77]

4. The following table gives the normal weights of babies during the first 12 months of life:

Age in Months	0	2	5	8	10	12
Weight in lbs	7.5	10.25	15	16	18	21

The following table gives the normal weights of babies during the first 12 months of life: [**Ans.** 15.67]

5. Find the value of $\tan 33^0$ by Lagrange's formula if $\tan 30^0 = 0.5774,\ \tan 32^0 = 0.6249,$ $\tan 35^0 = 0.7002,\ \tan 38^0 = 0.7813.$ [**Ans.** 0.6494]

5.4.2 DIVIDED DIFFERENCE

Let $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be given $(n + 1)$ points. Let $y_0, y_1, y_2, \ldots, y_n$ be the values of the function corresponding to the values of argument $x_0, x_1, x_2, \ldots, x_n$ which are not equally spaced. Since the difference of the function values with respect to the difference of the arguments are called divided differences, so the first divided difference for the arguments $x_0, x_1,$ is given by

$$f(x_0, x_1) = [x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

Similarly $$f(x_1, x_2) = [x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1}$$ and so on.

The second divided difference for x_0, x_1, x_2 is given by

$$f(x_0, x_1, x_2) = [x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

The third divided difference for x_0, x_1, x_2, x_3 is given by

$$f(x_0, x_1, x_2, x_3) = [x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$$

The n^{th} divided difference using all the data values in the table, is defined as

$$f(x_0, x_1, x_2, \ldots, x_n) = [x_0, x_1, x_2, \ldots, x_n] = \frac{[x_1, x_2, \ldots, x_n] - [x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}$$

The divided differences can be written in a tabular form as in Table 5.1.

x	$f(x)$	First d.d	Second d.d	Third d.d
x_0	$f(x_0)$			
		$f(x_0, x_1)$		
x_1	$f(x_1)$		$f(x_0, x_1, x_2)$	
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$	
		$f(x_2, x_3)$		
x_3	$f(x_3)$			

Table 5.1: Divided differences (d.d)

5.4.2.1 Properties of Divided Differences

1. Divided differences are symmetric with respect to the arguments i.e., independent of the order of arguments.
   $i.e.,$ $[x_0, x_1] = [x_1, x_0]$
   Also, $[x_0, x_1, x_2] = [x_2, x_0, x_1]$ or $[x_1, x_2, x_0]$

2. The n^{th} divided differences of a polynomial of n^{th} degree are constant.

Example 5.29. *Construct a divided difference table for the following:*

x	*4*	*5*	*7*	*10*
$f(x)$	*48*	*100*	*294*	*900*

Solution: The divided difference table is given as,

x	$f(x)$	First d.d	Second d.d	Third d.d
4	48			
		$\frac{100-48}{5-4} = 52$		
5	100		$\frac{97-52}{7-4} = 15$	
		$\frac{294-100}{7-5} = 97$		$\frac{21-15}{10-4} = 1$
7	294		$\frac{202-97}{10-5} = 21$	
		$\frac{900-294}{10-7} = 202$		
10	900			

Example 5.30. *Find the third divided difference with arguments 2, 4, 9, 10 of the function* $f(x) = x^3 - 2x$.

Solution:

x	$f(x)$	First d.d	Second d.d	Third d.d
2	4			
		$\frac{56-4}{4-2} = 26$		
4	56		$\frac{131-26}{9-2} = 15$	
		$\frac{711-56}{9-4} = 131$		$\frac{23-15}{10-2} = 1$
9	711		$\frac{269-131}{10-4} = 23$	
		$\frac{980-711}{10-9} = 269$		
10	980			

Hence third divided difference is 1.

Example 5.31. *If* $f(x) = \frac{1}{x}$*, find the divided differences* $f(a, b), f(a, b, c), f(a, b, c, d)$.

Solution:

x	$f(x)$	First d.d	Second d.d	Third d.d
a	$\frac{1}{a}$			
		$\frac{\frac{1}{b}-\frac{1}{a}}{b-a} = -\frac{1}{ab}$		
b	$\frac{1}{b}$		$\frac{1}{abc}$	
		$-\frac{1}{bc}$		$-\frac{1}{abcd}$
c	$\frac{1}{c}$		$\frac{1}{bcd}$	
		$-\frac{1}{cd}$		
d	$\frac{1}{d}$			

From the above divided difference table, we observe that first divided differences,

$$f(a, b) = -\frac{1}{ab}$$
$$f(a, b, c) = \frac{1}{abc}$$

and
$$f(a, b, c, d) = -\frac{1}{abcd}$$

5.4.3 Newton's Divided Difference Interpolation Formula

Let $y_0, y_1, \ldots, y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots, x_n$ then from the definition of divided differences, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

so that, $$y = y_0 + (x - x_0)[x, x_0] \tag{5.27}$$

Again, $$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

which gives, $$[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1] \tag{5.28}$$

From (5.27) and (5.28),

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1] \tag{5.29}$$

Also, $$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$$

which gives, $$[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2] \tag{5.30}$$

From (5.29) and (5.30),
$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2]$
Proceeding in this manner, we get

$$\begin{aligned} y = f(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \\ \cdots + (x - x_0)(x - x_1) \cdots (x - x_{n-1})[x, x_0, x_1, x_2, \ldots x_n] \end{aligned}$$

This is called Newton's Divided Difference interpolation formula with divided differences, the last term being the remainder term after $(n + 1)$ terms.
Example 5.32. *Apply Newton's divided difference formula to find the value of* $f(8)$ *if* $f(1) = 3, f(3) = 31, f(6) = 223, f(10) = 1011$

Solution: The divided difference table is given by

x	$f(x)$	First d.d	Second d.d	Third d.d
1	3			
		$\frac{28}{2} = 14$		
3	31		$\frac{50}{5} = 10$	
		$\frac{192}{3} = 64$		$\frac{9}{9} = 1$
6	223		$\frac{133}{7} = 19$	
		$\frac{788}{4} = 197$		
10	1011			

On, applying Newton's divided difference formula, we have

$$\begin{aligned} f(x) &= y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] \\ &= 3 + (x - 1)(14) + (x - 1)(x - 3)(10) + (x - 1)(x - 3)(x - 6)(1) \end{aligned}$$

Now for $f(8)$, put $x = 8$ in above equation, we get

$$f(8) = 3 + (7)(14) + (7)(5)(10) + (7)(5)(2)(1)$$
$$f(8) = 521.$$

Example 5.33. *The following are the mean temperatures $(°F)$ on three days, 30 days apart during summer and winter. Estimate the approximate dates and values of maximum and minimum temperature.*

Days	*Summer*		*Winter*	
	Date	*Temp.*	*Date*	*Temp.*
0	*15 June*	*58.8*	*16 Dec.*	*40.7*
30	*15 July*	*63.4*	*15 Jan.*	*38.1*
60	*14 August*	*62.5*	*14 Feb.*	*39.3*

Solution: Divided difference table for summer is:

x	$f(x)$	First d.d	Second d.d
0	58.8		
		4.6	
1	63.4		-2.75
		-0.9	
2	62.5		

$$\therefore \quad f(x) = 58.8 + (x-0)(4.6) + (x-0)(x-1)(-2.75)$$
$$= -2.75x^2 + 7.35x + 58.8$$

For maximum and minimum of $f(x)$, we have

$$f'(x) = 0$$
$$\Rightarrow \quad -5.5x + 7.35 = 0 \Rightarrow x = 1.342$$

Again, $\quad f''(x) = -5.5 < 0$

$\therefore f(x)$ is maximum at $x = 1.342$

Since unit $1 \equiv 30$ days

$$1.342 = 30 \times 1.342 = 40.26 \text{ days}$$

$\therefore$ Maximum temperature was on 15 June + 40 days i.e., on 25 July and value of maximum temperature is

$$[f(x)]_{max} = [f(x)]_{1.342} = 63.711°F \text{ approximately.}$$

Divided difference table for winter is as follows:

x	$f(x)$	First d.d	Second d.d
0	40.7		
		-2.6	
1	38.1		1.9
		1.2	
2	39.3		

$$\therefore \quad f(x) = 40.7 + (x-0)(-2.6) + (x-0)(x-1)(1.9)$$
$$= 1.9x^2 - 4.5x + 40.7$$

For $f(x)$ to be maximum and minimum, we have $f'(x) = 0$

$\Rightarrow \quad 3.8x - 4.5 = 0 \Rightarrow x = 1.184$

Again, $\quad f''(x) = 3.8 > 0$

$\therefore f(x)$ is minimum at $x = 1.184$

Since unit $1 \equiv 30$ days

$$1.184 = 30 \times 1.184 = 35.52 \text{ days}$$

$\therefore$ Minimum temperature was on 16 Dec + 35.5 days i.e., on the mid night of 20th Jan. and its value can be obtained similarly.

$$[f(x)]_{min} = [f(x)]_{1.184} = 63.647^\circ F \text{ approximately.}$$

Example 5.34. *Find the value of* $\log_{10}^{656}$ *using Newton's divided difference formula from the data given below:*

x	*654*	*658*	*659*	*661*
$\log_{10} x$	*2.8156*	*2.8182*	*2.8189*	*2.8202*

Solution: Divided difference table for the given data is as:

x	$10^5 f(x)$	First d.d	Second d.d	Third d.d
654	281560			
		$\frac{260}{4} = 65$		
658	281820		$\frac{5}{5} = 1$	
		$\frac{70}{1} = 70$		$\frac{-2.66}{7} = -0.38$
659	281890		$\frac{-5}{3} = -1.66$	
		$\frac{130}{2} = 65$		
661	282020			

For the given argument, the divided difference formula is,

$$f(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3]$$
$$10^5 f(x) = 281560 + (x - 654)(65) + (x - 654)(x - 658)(1) + (x - 654)(x - 658)(x - 659)(-0.38)]$$

On Substituting $x = 656$, we get

$$\begin{aligned} 10^5 f(x) &= 281560 + 2 \times 65 + (2)(-2) + (2)(-2)(-3) \times (-0.38) \\ &= 281560 + 130 - 4 - 4.56 \\ 10^5 f(x) &= 281681.44 \\ \Rightarrow f(x) &= 2.8168144 \end{aligned}$$

$\Rightarrow \log_{10} 656 = 2.8168144$

PROBLEM SET 5.9

1. Using Newton's divided difference formula, determine $f(3)$ for the data

x	0	1	2	4	5
$f(x)$	1	14	15	5	6

[**Ans.** 10]

2. Using Newton's divided difference formula to find $f(7)$ if $f(3) = 24,\ f(5) = 120,$ $f(8) = 504,\ f(9) = 720,$ and $f(12) = 1716$ [**Ans.** 328]

3. If $f(x) = \frac{1}{x^2}$, find the divided difference $f[x_1, x_2, x_3, x_4]$. $\left[\textbf{Ans.}\ \frac{-[x_2x_3(x_1+x_4)+x_1x_4(x_2+x_3)]}{[x_1^2x_2^2x_3^2x_4^2]}\right]$

4. The observed values of a function are, respectively, 168, 120, 72, and 63 at the four positions 3, 7, 9, and 20 of the independent variable. What is the best estimate you can give for value of the function at the position 6 of the independent variable? [**Ans.** 77]

5. There is a data be given

x	0	1	2	5
$f(x)$	2	3	12	147

What is the form of the function? [**Ans.** $x^3 + x^2 - x + 2$]