Unit-4: Interpolation

Unit Learning Outcomes:

Upon completion of this unit, students will be able to:

- **define and manipulate** finite difference operators (shift, forward, backward, and central difference operators).
- **construct** tables of differences for tabulated function values.
- **derive and apply** Newton's forward, Newton's backward interpolation polynomials.
- **drive and apply** Lagrangean interpolation polynomials.
- **drive and apply** Newton's divided differences interpolation polynomial.

4.1. Introduction

The statement $y = f(x)$, for $x_0 \leq x \leq x_n$ means: corresponding to every value of x in the range $x_0 \leq x \leq x_n$, there exists one or more values of y. Assuming that f(x) is single-valued and continuous and that it is known explicitly, then the values of f(x) corresponding to certain given values of x, say $x_0, x_1, \ldots, x_n$ can easily be computed and tabulated. The central problem of numerical analysis is the converse one: Given the set of tabular values $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ satisfying the relation $y = f(x)$ where the explicit nature of f(x) is not known, it is required to find a simpler function, say $\emptyset(x)$, such that f(x) and $\emptyset(x)$ agree at the set of tabulated points. Such a process is called *interpolation*. If $\emptyset(x)$ is a polynomial, then the process is called polynomial interpolation and $\emptyset(x)$ is called the interpolating polynomial. Similarly, different types of interpolation arise depending on whether $\emptyset(x)$ is a trigonometric functions, exponential functions, etc.

Activity 4.1:

Why are Polynomials Chosen to Approximate Functions?

Functions are approximated using other functions deemed to be simple to manipulate numerically. Specifically, one uses polynomials to approximate other complicated functions, mainly because polynomials are

- Simple to evaluate
- Simple to differentiate, and
- Simple to integrate.

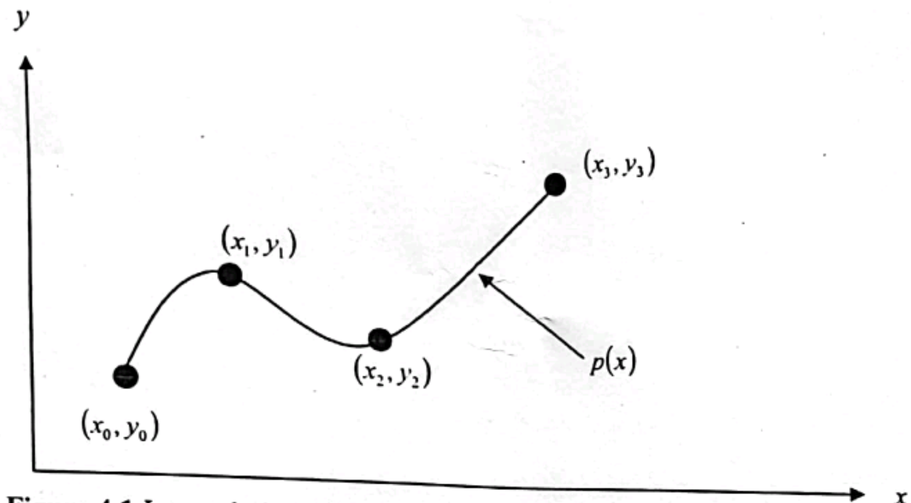

Figure 4.1 Interpolation of discrete data.

Numerical interpolation approximates functions and we approximate functions for one or several of the following reasons:

- A large number of important mathematical functions may only be known through tables of their values.
- Some functions may be known to exist but are computationally too complex to manipulate numerically.
- Some functions may be known but the solution of the problem in which they appear may not have an obvious mathematical expression to work with.

Some of the methods of interpolation that will be considered in this Unit include Newton's Forward and backward difference interpolation formulae, Newton's divided difference interpolation formula and the Lagrangian interpolation formula.

4.2. Finite Differences

Assume that we have a table of values (x_i, y_i), $i = 0, 1, 2, \ldots, n$ of any function $y = f(x)$, the values of x being equally spaced, i.e. $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots, n$. Suppose that we are required to recover the values of f(x) for some intermediate values of x, or to obtain the derivative of f(x) for some x in the range $x_0 \le x \le x_n$. The methods for the solution to these problems are based on the concept of the differences of a function which we now proceed to define.

4.2.1. Forward Differences

If $y_0, y_1, y_2, \ldots, y_n$ denotes a set of values of y, then $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the differences of y. Denoting these differences by $\Delta y_0, \Delta y_1, \Delta y_2, \ldots, \Delta y_{n-1}$ respectively, we have

$$\Delta y_0 = y_1 - y_0,$$
$$\Delta y_1 = y_2 - y_1,$$
$$\ldots$$
$$\Delta y_{n-1} = y_n - y_{n-1}.$$

Where Δ is called the forward difference operator and $\Delta y_0, \Delta y_1, \Delta y_2, \ldots,$ are called first forward differences. The differences of the first forward differences are called second forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots$. Similarly, one can define third forward differences, fourth forward differences, etc. Thus,

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$
$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0)$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$
$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = y_4 - 3y_3 + 3y_2 - y_1 - (y_3 - 3y_2 + 3y_1 - y_0)$$
$$= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

It is therefore clear that any higher-order differences can easily be expressed in terms of the ordinates, since the coefficients occurring on the right side are the binomial coefficients. Table 4.1 shows how the forward differences of all orders can be formed:

Table 4.1 Forward Difference Table

x	$y = f(x)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$.
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$.			
		Δy_4				
x_5	y_5					

Where $x_0 + h = x_1$, $x_0 + 2h = x_2$, ..., $x_0 + nh = x_n$.

Example 4.1:

Given $f(0) = 3$, $f(1) = 12$, $f(2) = 81$, $f(3) = 200$, $f(4) = 100$ and $f(5) = 8$. Construct the forward difference table and find $\Delta^5 f(0)$.

Solution:

The difference table for the given data is as follows:

x	$y = f(x)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	3					
		9				
1	12		60			
		69		-10		
2	81		50		-259	
		119		-269		755
3	200		-219		496	
		-100		227		
4	100		8			
		-92				
5	8					

Hence, $\Delta^5 f(0) = 755$.

4.2.2. Backward Differences

The differences $y_1 - y_0$, $y_2 - y_1$, ... ,$y_n - y_{n-1}$ are called first backward differences if they are denoted by ∇y_1, ∇y_2, ..., ∇y_n respectively, so that

$$\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_2, \ldots, \nabla y_n = y_n - y_{n-1},$$

where ∇ is called the backward difference operator. In a similar way, one can define backward differences of higher orders. Thus we obtain:

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2 = y_3 - 3y_2 + 3y_1 - y_0 \text{ and so on.}$$

Similarly the n[th] order backward differences can be defined as:

$$\nabla^n y_n = \nabla^{n-1} y_n - \nabla^{n-1} y_{n-1}$$

Table 4.2. Backward Difference Table

x	$y=f(x)$	∇	∇^2	∇^3	∇^4	∇^5
x_0	y_0					
		∇y_0				
x_1	y_1		$\nabla^2 y_0$			
		∇y_1		$\nabla^3 y_0$		
x_2	y_2		$\nabla^2 y_1$		$\nabla^4 y_0$	
		∇y_2		$\nabla^3 y_1$		$\nabla^5 y_0$
x_3	y_3		$\nabla^2 y_2$		$\nabla^4 y_1$	
		∇y_3		$\nabla^3 y_2$		
x_4	y_4		$\nabla^2 y_3$			
		∇y_4				
x_5	y_5					

Example 4.2:

Construct the backward difference table for y = log x given below:

x	10	20	30	40	50
y	1	1.3010	1.4771	1.6021	1.6990

Find the values of $\nabla^3 \log 40$ and $\nabla^4 \log 50$.

Solution:

The backward difference table is constructed as under:

x	$y=f(x)$	∇	∇^2	∇^3	∇^4
10	1				
		0.3010			
20	1.3010		-0.1249		
		0.1761		0.0738	
30	1.4771		-0.0511		-0.0508
		0.1250		0.0230	
40	1.6021		-0.0281		
		0.0969			
50	1.6990				

Hence, $\nabla^3 \log 40 = 0.0738$ and $\nabla^4 \log 50 = -0.0508$

4.2.3. Central Differences

The central difference operator δ is defined by the relations:

$\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$ which is equivalent to $\delta y_x = y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}$.

The first order central differences are given by:

$\delta y_{\frac{1}{2}} = y_1 - y_0$, $\delta y_{\frac{3}{2}} = y_2 - y_1$, $\delta y_{\frac{5}{2}} = y_3 - y_2$, ..., $\delta y_{n-\frac{1}{2}} = y_n - y_{n-1}$.

Similarly, higher-order central differences can be defined as:

$\delta^2 y_1 = \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}}$, $\delta^3 y_{\frac{3}{2}} = \delta^2 y_2 - \delta^2 y_1$, $\delta^4 y_2 = \delta^3 y_{\frac{5}{2}} - \delta^3 y_{\frac{3}{2}}$, and so on.

With the values of x and y as in the preceding two tables, a central difference table can be formed:

Table 4.3: Central Difference Table

x	$y = f(x)$	δ	δ^2	δ^3	δ^4	δ^5
x_0	y_0					
		$\delta y_{\frac{1}{2}}$				
x_1	y_1		$\delta^2 y_1$			
		$\delta y_{\frac{3}{2}}$		$\delta^3 y_{\frac{3}{2}}$		
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{\frac{5}{2}}$		$\delta^3 y_{\frac{5}{2}}$		$\delta^5 y_{\frac{5}{2}}$
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{\frac{7}{2}}$		$\delta^3 y_{\frac{7}{2}}$		
x_4	y_4		$\delta^2 y_4$			
		$\delta y_{\frac{9}{2}}$				
x_5	y_5					

It is clear from the three tables that in a definite numerical case, the same numbers occur in the same positions whether we use forward, backward or central differences. Thus we obtain

$\Delta y_0 = \nabla y_1 = \delta y_{\frac{1}{2}}$, $\Delta^3 y_2 = \nabla^3 y_3 = \delta^3 y_{\frac{7}{2}}$, and so on.

4.2.4. The Shift Operator E

The operator E is called shift operator or displacement or translation operator. It shows the operation of increasing the argument value x by its interval of differencing h so that:

$Ef(x) = f(x+h)$ in the case of a continuous variable x, and

$Ey_x = y_{x+1}$ in the case of a discrete variable.

Similarly, $Ef(x+h) = f(x+2h)$

Powers of the operator (positive or negative) are defined in a similar manner:

$E^n f(x) = f(x+nh)$; $E^n y_x = y_{x+nh}$

In the same manner, $E^{-1}f(x) = f(x-h)$. Also, $E^{-2}f(x) = f(x-2h)$, $E^{-n} f(x) = f(x-nh)$

4.3. Interpolation with Equally Spaced Points

4.3.1. Newton's Forward Interpolation Formula

Given the set of (n+1) values, viz., $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, of x and y, it is required to find $P_n(x)$, a polynomial of n^{th} degree such that y and $P_n(x)$ agree at the tabulated points. Let the values of x be equidistant, i.e. $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots, n$.

Since $P_n(x)$ is a polynomial of the n^{th} degree, it may be written as:

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \cdots$$
$$+ a_n(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1}) \qquad (4.1)$$

Imposing now the conditions that y and $P_n(x)$ should agree at the set of tabulated points, that is putting $x = x_0, x_1, x_2, \ldots, x_n$ successively in (4.1), we obtain:

$$a_0 = y_0; \quad a_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}; \quad a_2 = \frac{\Delta^2 y_0}{h^2 2!}; \quad a_3 = \frac{\Delta^3 y_0}{h^3 3!}; \cdots; \quad a_n = \frac{\Delta^n y_0}{h^n n!}.$$

Setting $x = x_0 + ph$ and substituting for $a_0, a_1, a_2, \ldots, a_n$, equation (3.9) gives:

$$P_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$
$$+ \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}\Delta^n y_0 \qquad (4.2)$$

where $p = \dfrac{(x-x_0)}{h}$.

This is Newton's forward difference interpolation formula and is useful for interpolation near the beginning of a set of tabular values.

4.3.2. Newton's Backward Interpolation Formula

Instead of assuming $P_n(x)$ as in (3.9) if we choose it in the form

$$P_n(x) = a_0 + a_1(x-x_n) + a_2(x-x_n)(x-x_{n-1}) + a_3(x-x_n)(x-x_{n-1})(x-x_{n-2}) + \cdots$$
$$+ a_n(x-x_n)(x-x_{n-1})(x-x_{n-2})\cdots(x-x_1) \qquad (4.3)$$

and then impose the condition that y and $P_n(x)$ should agree at the tabulated points $x_n, x_{n-1}, \ldots, x_2, x_1, x_0$, we obtain (after some simplification)

$$P_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots + \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n y_n \qquad (4.4)$$

where $p = \frac{(x-x_n)}{h}$. This is Newton's backward difference interpolation formula and it uses tabular values to the left of y_n. This formula is therefore useful for interpolation near the end of the tabular values.

Example 4.3:

The population of a town in the census was as given below.

Estimate the population for the years a) 1895 b) 1925

Year, x	1891	1901	1911	1921	1931
Population, y (in thousands)	46	66	81	93	101

Solution: First we construct the difference table:

Year, x	Population, y	Δ	Δ^2	Δ^3	Δ^4
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

a) Since interpolation is desired at the beginning of the table, we use Newton's forward difference interpolation formula.

Here $x_0 = 1891$, $y_0 = 46$, $h = 10$ and $x = 1895$, and so $p = \frac{x-x_0}{h} = \frac{1895-1891}{10} = 0.4$.

Hence Newton's forward difference interpolation formula gives:

$$P(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$P(1895) = 46 + 0.4(20) + \frac{0.4(0.4-1)}{2}(-5) + \frac{0.4(0.4-1)(0.4-2)}{6}(2) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24}(-3)$$

$= 54.8528$ thousands.

Hence the population of the town in the year 1895 is 54,853.

b) Since interpolation is desired at the end of the table, we use Newton's backward difference interpolation formula.
Here $x_n = 1931$, $y_n = 101$, $h = 10$ and $x = 1925$, and so

$$p = \frac{x - x_n}{h} = \frac{1925 - 1931}{10} = -0.6$$

Hence Newton's backward difference interpolation formula gives:

$$P(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$P(1925) = 101 + (-0.6)(8) + \frac{(-0.6)(-0.6+1)}{2}(-4) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{6}(-1) + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{24}(-3)$$

$= 96.8368$ thousands.

Hence the population of the town in the year 1925 is 96,837.

<u>**Example 4.4**</u>:

The table below gives the value of tan x for $0.10 \le x \le 0.30$

x	0.10	0.15	0.20	0.25	0.30
tan (x)	0.1003	0.1511	0.2027	0.2553	0.3093

Find (a) tan 0.12 (b) tan 0.26 (c) tan 0.50

<u>**Solution**</u>:

First we construct the difference table:

x	y = tan x	Δ	Δ^2	Δ^3	Δ^4
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3093				

a) Since interpolation is desired at the beginning of the table, we use Newton's forward difference interpolation formula.

Here $x_0 = 0.10$, $y_0 = 0.1003$, $h = 0.05$ and $x = 0.12$, and so

$$p = \frac{x - x_0}{h} = \frac{0.12 - 0.10}{0.05} = 0.4.$$

Hence Newton's forward difference interpolation formula gives:

$$P(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$P(0.12) = 0.1003 + 0.4(0.0508) + \frac{0.4(0.4-1)}{2}(0.0008) + \frac{0.4(0.4-1)(0.4-2)}{6}(0.0002) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24}(0.0002)$$

$$= 0.1205$$

Hence $\tan(0.12) \cong P(0.12) = 0.1205$.

b) Since interpolation is desired at the end of the table, we use Newton's backward difference interpolation formula.

Here $x_n = 0.30$, $y_n = 0.3093$, $h = 0.05$ and $x = 0.26$, and so

$$p = \frac{x - x_n}{h} = \frac{0.26 - 0.30}{0.05} = -0.8$$

Hence Newton's backward difference interpolation formula gives:

$$P(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$P(0.26) = 0.3093 + (-0.8)(0.0540) + \frac{(-0.8)(-0.8+1)}{2}(0.0014) + \frac{(-0.8)(-0.8+1)(-0.8+2)}{6!}(0.0004) + \frac{(-0.8)(-0.8+1)(-0.8+2)(-0.8+3)}{24}(0.0002)$$

$$= 0.2662$$

Hence $\tan(0.26) \cong P(0.26) = 0.2662$.

c) Proceeding as in the case of (a) above, we obtain $\tan(0.50) \cong P(0.50) = 0.5543$.

The actual values, correct to four decimal places, of tan(0.12), tan(0.26), and tan(0.50) are respectively 0.1206, 0.2660 and 0.5463. Comparison of the computed and actual values shows that in the first two cases (i.e. interpolation) the results obtained are fairly accurate whereas in the last case (i.e. of extrapolation) the errors are quite considerable. The example therefore demonstrates the important result that if a tabulated function is other than a polynomial, then extrapolation very far from the table limits would be dangerous-although interpolation can be carried out very accurately.

Pseudo-code for Newton Forward Interpolation Formula

```
Read n, x
For I = 1 to n by 1
   Read x[i], y[i]
End for
If ((x < x[i] or (x > x[n]))
   Print "Value lies out of boundary" Exit
End if
//Calculating p
p = (x – x [1]) / (x [2]-x [1])
// constructing Forward difference table
For j = 1 to (n-1) by 1
  For i =1 to (n - j) by 1
       If (j=1) Then d[i][j] = y [i+1] – y[i]
         Else d[i][j] = d[i+1][j-1] – d[i][j-1]
      End if
   End For
End For
// Applying Formula
Sum = y [1]
For I = 1 to (n-1) by 1
    Prod = 1; Fact = 1
    For j =0 to (i-1) by 1
      Prod = prod * (p-j)
    End for
Fact = Fact*i
Sum = sum + (d[1][i] * prod) / Fact
End For
Print "Answer is", Sum
End of the program.
```

Self-Assessment Exercise 4.1

1. In what cases do we apply Newton forward and backward Interpolation Formulas?
2. From the table, Estimate the number of students who obtained marks between 40 and 45.

Marks	30 - 40	40 - 50	50 - 60	60 - 70	70 - 80
No. of Students	35	48	70	40	22

3. Using the tabulated data for the saturation values of dissolved oxygen concentration (mg/L) as a function of temperature ($°_c$), find an estimate for the aturation values of oxygen concentration at (a) temperature , $T = 3°_c$ (b) $T = 17°_c$

4.4. Interpolation with Unequally Spaced Points

Activity 4.2:

What will happen if we apply Newton forward and backward interpolation formulas for unequally spaced data points?

The interpolation formulae derived before for forward interpolation, backward interpolation and central interpolation have the disadvantages of being applicable only to equally spaced argument values. So it is required to develop interpolation formulae for unequally spaced argument values of x. Therefore, when the values of the argument are not at equally spaced then we use two such formulae for interpolation.

1. Lagrange's interpolation formula,
2. Newton's divided difference formula.

The main advantage of these formulas is, they can also be used in case of equal intervals but the formulae for equal intervals cannot be used in case of unequal intervals.

4.4.1. Lagrang's Interpolation Formula

Let $y_0 = f(x_0)$, $y_1 = f(x_1)$, ..., $y_n = f(x_n)$ be $(n + 1)$ entries of a function $y = f(x)$. Let $P(x)$ be a polynomial of degree n corresponding to the arguments $x_0, x_1, x_2, \ldots x_n$ which can be written as:

$$P_n(x) = A_0(x-x_1)(x-x_2)\ldots(x-x_n) + A_1(x-x_0)(x-x_2)\ldots(x-x_n) + \cdots + A_n(x-x_0)(x-x_1)\ldots(x-x_{n-1}) \qquad (1)$$

Where $A_0, A_1, \ldots, A_n$, are constants to be determined.

The constants $A_0, A_1, \ldots, A_n$, will be determined by considering the tabulated function y = f(x) and the polynomial function P(x) agree at the set of tabulated points.

Putting $x = x_0, x_1, x_2, \ldots, x_n$ in (1) successively, we get the following.

For $x = x_0$, $y_0 = A_0(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)$

That is $A_0 = \dfrac{y_0}{(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)}$

For $x = x_1$, $y_1 = A_1(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)$

That is $A_1 = \dfrac{y_1}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)}$

Similarly,

For $x = x_n$, $y_n = A_n(x_n - x_0)(x_n - x_1)(x_n - x_2)\cdots(x_n - x_{n-1})$

That is $A_n = \dfrac{y_n}{(x_n - x_0)(x_n - x_1)(x_n - x_2)\cdots(x_n - x_{n-1})}$

Substituting the values of $A_0, A_1, \ldots, A_n$ in equation (1), we get

$$p_n(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)}y_2 + \ldots + \frac{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})}y_n$$

This is called Lagrange's interpolation formula and which can be written as a general form:

$$P_n(x) = L_0(x)f_0 + L_1(x)f_1 + L_2(x)f_2 + \ \ldots\ + L_n(x)f_n \quad = \quad \sum_{i=0}^{n} L_i(x)f_i \tag{2}$$

in which the terms $L_i(x)$, $\quad i = 0,1,2,\ldots,n$ given by

$L_i(x) = \dfrac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}$ are individually polynomials of degree n in x and are called the **Lagrangean interpolation coefficients.**

<u>Note:</u> $L_i(x_j) = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$

<u>Example 4.5</u>:

The percentages of criminals for different age group are given below. Determine by Lagrange's formula, the percentage number of criminals under 35 years:

Age	% no. of criminals
Under 25 year	*52*
Under 30 years	*67.3*
Under 40 years	*84.1*
Under 50 years	*94.4*

interpolation formula has the disadvantage that if any other interpolation point were added, the interpolation co-efficient will have to be recomputed. So an interpolation polynomial, which has the property that a polynomial of higher degree may be derived from it by simply adding new terms, in Newton's divided difference formula.

Let $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be given $(n + 1)$ points where $y_0, y_1, y_2, \ldots, y_n$ are the values of the function corresponding to the values of argument $x_0, x_1, x_2, \ldots, x_n$ which are not equally spaced. The difference of the function values with respect to the difference of the arguments is called divided differences.

The first order divided difference for the arguments x_0, x_1, is given by:

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}.$$

Similarly first order divided difference for the arguments x_1, x_2, is given by:

$$[x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1}.$$

In general first order divided difference for the arguments x_{n-1}, x_n, is given by:

$$[x_{n-1}, x_n] = \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.$$

The second order divided difference for the arguments x_0, x_1, x_2 is given by:

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

The third divided difference for the arguments x_0, x_1, x_2, x_3 is given by:

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$$

The n^{th} order divided differences for the arguments $x_0, x_1, x_2, \ldots, x_n$ *is given by:*

$$[x_0, x_1, x_2 \ldots, x_n] = \frac{[x_1, x_2, x_3, \ldots, x_n] - [x_0, x_1, x_2, \ldots, x_{n-1}]}{x_n - x_0}$$

Table 4.1: Table of Divided Differences

x	$y=f(x)$	First	second	Third	Fourth	Fifth
x_0	y_0					
		$[x_0,x_1]$				
			$[x_0,x_1,x_2]$			
x_1	y_1			$[x_0,x_1,x_2,x_3]$		
		$[x_1,x_2]$				
			$[x_1,x_2,x_3]$		$[x_0,x_1,x_2,x_3,x_4]$	
x_2	y_2			$[x_1,x_2,x_3,x_4]$		$[x_0,x_1,x_2,x_3,x_4,x_5]$
		$[x_2,x_3]$				
			$[x_2,x_3,x_4]$		$[x_1,x_2,x_3,x_4,x_5]$	
x_3	y_3			$[x_2,x_3,x_4,x_5]$		
		$[x_3,x_4]$				
			$[x_3,x_4,x_5]$			
x_4	y_4					
		$[x_4,x_5]$				
x_5	y_5					

Properties of Divided Differences

1. Divided differences are symmetric with respect to the arguments *i.e.*, independent of the order of arguments.

   i.e., $[x_0, x_1] = [x_1, x_0]$

   Also, $[x_0, x_1, x_2] = [x_2, x_0, x_1] = [x_1, x_2, x_0]$.

2. The n^{th} divided differences of a polynomial of n^{th} degree are constant.
3. Let $f(x) = A_0x_n + A_1x_{n-1} + \ldots + A_{n-1}x + A_n$ by a polynomial of degree n provided $A_0 \neq 0$ and arguments be equally spaced so that $x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h$.

Then first divided difference $[x_0,x_1]=\dfrac{y_1-y_0}{x_1-x_0}=\dfrac{\Delta y_0}{h}$

Second divided difference $[x_0,x_1,x_2]=\dfrac{1}{2!h^2}=\Delta^2 y_0$

Similarly $[x_0,x_1,x_2,\ldots,x_n]=\dfrac{1}{n!h^n}=\Delta^n y_0$

Since, the function is an n^{th} degree polynomial, $\Delta^n y_0$ = constant. Therefore n^{th} divided difference will also be constant.

Example 4.7:

Construct a divided difference table for the following data:

x	1	2	4	7	12
f(x)	22	30	82	106	216

Solution:

The divided difference table is given as:

x	$y=f(x)$	First	second	Third	Fourth
1	22				
		$\frac{30-22}{2-1}=8$			
2	30				
			$\frac{26-8}{4-1}=6$		
		$\frac{82-30}{4-2}=26$		$\frac{-36-6}{7-1}=-1.6$	
4	82				
			$\frac{8-26}{7-2}=-3.6$		$\frac{0.535+1.6}{12-1}=0.194$
		$\frac{106-82}{7-4}=8$		$\frac{1.75+3.6}{12-2}=0.535$	
7	106				
			$\frac{22-8}{12-4}=1.75$		
		$\frac{216-106}{5}=22$			
12	216				

Derivation of Newton's Divided difference Formula

Let $y_0, y_1, \ldots, y_n$, be the values of $y=f(x)$ corresponding to the arguments $x_0, x_1, \ldots, x_n$, then from the definition of divided differences, we have

$$[x, x_0] = \frac{y-y_0}{x-x_0}$$

So, that, $y = y_0 + (x-x_0)[x, x_0]$ (1)

Again, $[x, x_0, x_1] = \frac{[x, x_0]-[x_0, x_1]}{x-x_1}$

which gives, $[x, x_0] = [x_0, x_1] + (x-x_1)[x, x_0, x_1]$ (2)

From (1) and (2), we have: $y = y_0 + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x, x_0, x_1]$ (3)

Also $[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1]-[x_0, x_1, x_2]}{x-x_2}$

which gives, $[x, x_0, x_1] = [x_0, x_1, x_2] + (x-x_2)[x, x_0, x_1, x_2]$ (4)

From (3) and (4) we have:

$$y = y_0 + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2)[x, x_0, x_1, x_2]$$

Proceeding in this manner, we get:

$$y = y_0 + (x-x_0)[x_0,x_1] + (x-x_0)(x-x_1)[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2)[x_0,x_1,x_2,x_3] + \cdots$$
$$+ (x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})[x_0,x_1,x_2,\cdots,x_n]$$
$$+ (x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)[x,x_0,x_1,x_2,\cdots,x_n]$$

The last term being the remainder term after $(n+1)$ terms, the n^{th} degree polynomial

$$P_n(x) = y_0 + (x-x_0)[x_0,x_1] + (x-x_0)(x-x_1)[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2)[x_0,x_1,x_2,x_3] + \dots$$
$$+ (x-x_0)(x-x_1)(x-x_2)\dots(x-x_{n-1})[x_0,x_1,x_2,\dots,x_n]$$

is called Newton's general interpolation formula with divided differences.

Example 4.8:

Apply Newton's divided difference formula to find the value of $f(8)$ if $f(1)=3, f(3)=31, f(6)=223, f(10)=1011, f(11)=1343,$

Solution:

The divided difference table is given by:

x	$y=f(x)$	**First**	**second**	**Third**	**Fourth**
1	3				
		$\frac{28}{2}=14$			
3	31		$\frac{50}{5}=10$		
		$\frac{192}{3}=64$		$\frac{9}{9}=1$	
6	223		$\frac{133}{7}=19$		0
		$\frac{788}{4}=197$		$\frac{8}{8}=1$	
10	1011		$\frac{135}{5}=27$		
		$\frac{332}{1}=332$			
11	1343				

On applying Newton's divided difference formula, we have:

$$P_n(x) = y_0 + (x-x_0)[x_0,x_1] + (x-x_0)(x-x_1)[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2)[x_0,x_1,x_2,x_3] + \dots$$
$$+ (x-x_0)(x-x_1)(x-x_2)\dots(x-x_{n-1})[x_0,x_1,x_2,\dots,x_n]$$

That is $P_n(x) = 3 + (x-1)(14) + (x-1)(x-3)(10) + (x-1)(x-3)(x-6)(1)$

To find $f(8)$, we put $x = 8$ in the above equation.

$f(8) \cong P_n(8) = 3 + (7)(14) + (7)(5)(10) + (7)(5)(2)$

$f(8) = 3 + 98 + 350 + 70.$

$f(8) = 521.$

Example 4.9:

The following table gives the weight (in pounds) of one baby during the first few months of his life:

Age	0	1	3	4	7
weight	8	10	16	20	36

Estimate the **weight** of the baby when his ***age*** is***6*** months.

Solution:

The divided difference table is given by:

Age, x	Weight, y	First	second	Third	Fourth
0	8				
		$\frac{2}{1}=2$			
1	10		$\frac{1}{3}=0.3333$		
		$\frac{6}{2}=3$		0	
3	16		$\frac{1}{3}=0.3333$		0.000019
		$\frac{4}{1}=4$		$\frac{-0.0008}{6}=-0.00013$	
4	20		$\frac{1.33}{4}=0.3325$		
		$\frac{16}{3}=5.33$			
7	36				

On applying Newton's divided difference formula, we have:

$$P_n(x) = y_0 + (x-x_0)[x_0,x_1] + (x-x_0)(x-x_1)[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2)[x_0,x_1,x_2,x_3] + \dots$$
$$+ (x-x_0)(x-x_1)(x-x_2)\dots(x-x_{n-1})[x_0,x_1,x_2,\dots,x_n]$$

That is $P_n(x) = 8 + (x-0)(2) + (x-0)(x-1)(0.3333) + (x-0)(x-1)(x-3)(0)$
$+ (x-0)(x-1)(x-3)(x-4)(0.000019)$

To estimate the **weight** of the baby when his ***age*** is***6*** months, we put $x = 6$ in the above equation.

Weight (at 6) $\cong P_n(6) = 8 + (6)(2) + (6)(5)(0.3333) + (6)(5)(3)(2)(0.000019)$
$= 8 + 12 + 9.999 + 0.00342$
$= 30.$

Hence the **weight** of the baby when his ***age*** is ***6*** months is 30 pounds.

Pseudo-code for Newton's Divided Difference Interpolation Formula

read the number of data points (n), point of interpolation (x), an array of x, (ax), and an array of y, (ay)

for $i = 0$ *to* n

  $d_{i,0} = f(x_i)$

end for

for $i = 1$ *to* n

  for $j = 1$ *to* i

$$d_{i,j} = \frac{(d_{i,j-1} - d_{i-1,j-1})}{x_i - x_{i-j}}$$

  end for

end for

$sum = d_{0,0}$

$prod = 1.0$

for $i = 1$ *to* n

  $prod = prod*(x - x_{i-1})$

  $sum = sum + d_{i,i}*prod$

end for

print "approximation at", x, "is ",sum

Self-Assessment Exercise 4.3

1. Using Newton's divided difference formula, calculate the value of $f(6)$ from the following data:

x	1	2	7	8
$f(x)$	1	5	5	4

2. Find $f(3)$, using Newton's divided difference formula from the given data

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

3. In the following table, values of y are consecutive terms of a series of which 23.6 is the 6th term. Find the first and tenth terms of the series.

x	3	4	5	6	7	8	9
y	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Miscellaneous Exercise

1. From the following table of half-yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age of 46.

Age	45	50	55	60	65
Premium (in Birr)	114.84	96.16	83.32	74.48	64.48

[ANS: 110.52 Birr]

2. Suppose the volume of a liquid during a chemical reaction is given by the following table.

Time, t(in min,)	2	3	4	5	7
Volume, v(in c.c)	3	9	20	39	113

Use Find the volume v, of the liquid in the reaction at time $t = 6$ min.

3. The following table is used by marine engineers to relate the ratio of depth of water (d) to percentage increase in resistance to movement in shallow waters (r).

d	5.0	4.0	3.0	2.5
r	6.0	9.0	13.0	24.0

Use Lagrange's interpolation formula to estimate the resistance to movement (r) at water depth $(d) = 2$.

4. Find the cubic polynomial which takes the following values

x	0	1	2	3
$f(x)$	1	2	1	10

[ANS: $2x^3-7x^2+6x+1$]

5. By means of Newton's divided difference formula, find the value of $f(8)$ and $f(15)$ from the following table:

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

[ANS: 448, 3150]

Unit Summary

Given the set of tabular values (x_0, y_0), (x_1, y_1), (x_2, y_2), …, (x_n, y_n) satisfying the relation $y = f(x)$ where the explicit nature of f(x) is not known, it is required to find a simpler function, say $\emptyset(x)$, such that f(x) and $\emptyset(x)$ agree at the set of tabulated points. Such a process is called *interpolation*.

Numerical interpolation approximates functions and we approximate functions for one or several of the following reasons:

- A large number of important mathematical functions may only be known through tables of their values.
- Some functions may be known to exist but are computationally too complex to manipulate numerically.
- Some functions may be known but the solution of the problem in which they appear may not have an obvious mathematical expression to work with.