Guide for Tensor Network Techniques for Renormalization Group in Physics

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1 Intro

This is an introductory guide to the Tensor Renormalization Group (TRG) topic. Our central problem is computing the partition function Z of a many-body system. This is in general not an easy task because as we increase the number of sites on a lattice, the calculation becomes computationally too expensive and impossible to perform exactly. Therefore, in order to make computing the many-body partition function possible, the appropriate approximations need to be carried out. TRG aims to solve this problem by representing a partition function with a tensor network, which we then contract in efficient way by gradually compressing the information.

The system of interest for exploring the TRG will be the classical Ising model on a square lattice, which is convenient because it is simple and exactly solvable, so you can use the analytic solution to compare and benchmark the results. In Section 2 you can find the definition of the classical Ising model and explanation how to write its partition function as a tensor network for 1D and 2D case. The exact solution is provided at the end of the section. The TRG algorithm is explained in Section 3.

2 Ising Model partition function as a tensor network

The Ising model probably the most famous model in

2.1 1D Ising Model

As an introduction and to get some intuition to the problem, we have a first look at the classical Ising model without external field in one dimension and apply the transfer matrix method to calculate the analytical solution.

The spins of the system are arranged along a line with periodic boundary conditions, therefore each spin has two neighbours. The energy of the system is then

$$E(\sigma_1, \sigma_2, ..., \sigma_N)/J = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \cdots + \sigma_N\sigma_1 \tag{1}$$

Each σ_i can take the value 1 or -1. If not mentioned otherwise, we will assume units, where $J = 1$. The partition function of the system is then

$$Z = \sum_{\{\sigma_i\}} e^{-\beta E(\{\sigma_i\})}. \tag{2}$$

With $\{\sigma_i\}$ the set of all spin variables is denoted. The transfer matrix trick is then to rewrite the partition function as the trace of the the transfer matrix M to the power N

$$Z = \sum_{\{\sigma_i\}} e^{-\beta\sum_{i=1}^{N}\sigma_i\sigma_{i+1}} = \sum_{\{\sigma_i\}}\prod_{i=1}^{N} e^{-\beta\sigma_i\sigma_{i+1}} = \mathrm{Tr}\left(M^N\right). \tag{3}$$

Here, M is defined as

$$M_{ij} = e^{-\beta\sigma_i\sigma_j}. \tag{4}$$

The easiest way to calculate $N-$th power of the transfer matrix is to diagonalize it, since $M^N = U^{-1}D^NU$, where D is a diagonal matrix with the eigenvalues λ_1 and λ_2 of M on its diagonal. With the cyclicity property of the trace function, the partition function is then

$$Z = \mathrm{Tr}\left(M^N\right) = \mathrm{Tr}\left(D^N\right) = \lambda_1^N + \lambda_2^N. \tag{5}$$

Similarly one and two point functions can be calculated by rewriting it as product of matrices.

$$< \sigma_i >= \sum_{\{\sigma_k\}} \sigma_i P(\{\sigma_k\}) = \frac{1}{Z}\mathrm{Tr}\left(M^{j-1}\sigma_z M^{N-(j-1)}\right) = \frac{1}{Z}\mathrm{Tr}\left(\sigma_z M^N\right), \tag{6}$$

whereσ_z is the Pauli z-matrix

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

To simplify (6) we assume we know the eigenvectors$|0\rangle, |1\rangle$ with eigenvalue λ_1, λ_2 of M respectively.

$$\frac{1}{Z}\mathrm{Tr}\left(\sigma_z M^N\right) \frac{1}{Z}\sum_{i=1,2} \langle i| \sigma_z M^N |i\rangle = \frac{1}{Z}\sum_{i=1,2} \lambda_i^N \langle i| \sigma_z |i\rangle = \frac{1}{Z}\left(\lambda_0^N - \lambda_1^N\right). \tag{8}$$

Similarly, the two point correlator can be written as (assuming $i < j$)

$$\begin{aligned} < \sigma_i\sigma_j > &= \frac{1}{Z}\mathrm{Tr}\left(M^{i-1}\sigma_z M^{j-1-(i-1)}\sigma_z M^{N-(j-1)}\right) \\ &= \frac{1}{Z}\mathrm{Tr}\left(\sigma_z M^{j-i}\sigma_z M^{N-(j-i)}\right). \end{aligned} \tag{9}$$

2.2 2D Ising Model

Similar as in the 1D case, the 2D Ising Model with periodic boundary condition (pbc), the partition function Z can be written as a contraction of tensors

$$Z = \sum_{\{s_1^i, s_2^i, s_3^i, s_4^i\}_i} \prod_i A^{s_1^i s_2^i s_3^i s_4^i}. \tag{10}$$

The tensor A is defined as

$$A^{s_1 s_2 s_3 s_4} = e^{\beta(s_1 s_2 + s_2 s_3 + s_3 s_4 + s_4 s_1)}. \tag{11}$$

One way to interpret this tensor is, that it calculates the local energies between four adjacent spins, and weights the configurations, with the Boltzmann probability $\exp -E/T$ through the respective energies. Note its similarity to the one-dimensional transfer matrix M.

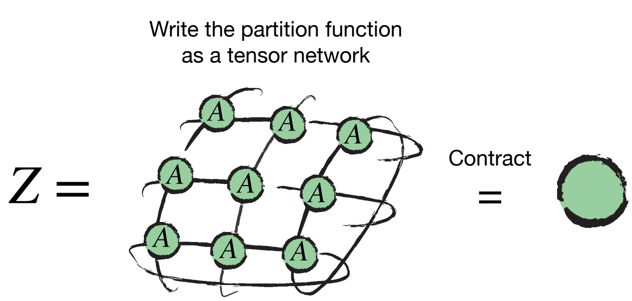

Figure 1: Partition function of Ising model on square lattice.

2.3 Exact solution 2D Ising Model

Onsager's exact solution to the 2D Ising Model [1, 2]

$$\lim_{N\to\infty} -\beta f_N = \lim_{N\to\infty} \ln(Z_N) = \ln\left(2\cosh(\beta J)\right) + \frac{1}{2\pi}\int_0^{\pi} \mathrm{d}\phi \ln\left(\frac{1}{2}\left(1+\sqrt{1-\kappa^2\sin^2(\phi)}\right)\right), \tag{12}$$

where $\kappa \equiv 2\sinh(2\beta J)/\cosh^2(2\beta J)$. Internal free energy

$$U = -J\coth(2\beta J)\left[1+\frac{2}{\pi}(2\tanh^2(2\beta J)-1)\int_0^{\pi/2}\frac{1}{\sqrt{1-4k(1+k)^{-2}\sin^2(\theta)}}d\theta\right] \tag{13}$$

Magnetization

$$M = \left[1-\sinh^{-4}(2\beta J)\right]^{1/8} \tag{14}$$

The specific heat C(T), defined by

$$C(T) = \kappa\beta^2\frac{\partial^2}{\partial\beta^2}\ln(Z) \tag{15}$$

The specific het diverges logarithmically as $T \to T_c$, therefore

$$\tanh\left(\frac{2J}{\kappa T_c}\right) = \frac{1}{\sqrt{2}} \to \frac{kT_c}{J} = 2.269185. \tag{16}$$

3 Tensor Renormalization Group Algorithm

Here we write briefly the main steps for TRG algorithm.

References

[1] Somendra M. Bhattacharjee and Avinash Khare. Fifty years of the exact solution of the two-dimensional ising model by onsager, 1995.

[2] Lars Onsager. Crystal statistics. i. a two-dimensional model with an order-disorder transition. *Phys. Rev.*, 65:117–149, Feb 1944.