Guide for Tensor Network Techniques for Renormalization Group in Physics

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1 Introduction

This is an introductory guide to the Tensor Renormalization Group (TRG) [1]. Our central problem is computing the partition function Z of a classical many-body system. This is in general not an easy task because as we increase the number of sites on a lattice, the calculation becomes computationally too expensive and impossible to perform exactly. Therefore, in order to make computing the many-body partition function possible, the appropriate approximations need to be carried out. TRG aims to solve this problem by representing a partition function with a tensor network, which we then contract in an efficient way by gradually compressing the information.

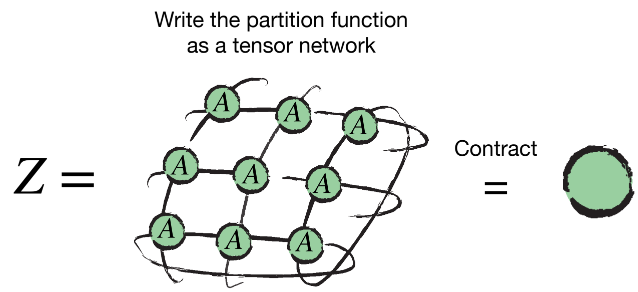

Figure 1: Main steps in TRG algorithm.

In Section 2 we go through how to recast a partition function into the tensor network, and in Section 3 we explain how to perform the contraction. The system of interest for exploring the TRG will be the classical Ising model on a square lattice, which is convenient because it is simple and exactly solvable, so you can use the analytic solution to compare and benchmark the results. In Section 2 you can find the definition of the classical Ising model and explanation how to write its partition function as a tensor network for 1D and 2D case. The exact solution is provided at the end of the section.

2 Ising Model partition function as a tensor network

The Ising model is without a doubt the most famous model in statistical physics. It was proposed in 1920 from Wilhelm Lenz to Ernst Ising to study ferromagnetism and phase transitions. While Ising himself solved the one-dimensional case which does not manifest the finite-temperature phase transition, in 1944 an exact solution for 2-dimensional lattice was presented by Onsager. Onsager showed that Ising model in 2D does exhibit a phase transition, which up to today represents one of the landmarks in theoretical physics. The importance of the model comes also from the fact that many seemingly unrelated physical systems show the same scaling behaviour at criticality and therefore belong to the same universality class as the Ising model.

The Ising model is defined on a lattice Λ, with a discrete spin variable $s_i \in \{\pm 1\}$ attached on each lattice site. A spin configuration $\{s_i\}_{i\in\Lambda}$ is then an assignment of spin values to each lattice site. The energy of a spin configuration is given by

$$H = J\sum_{\langle ij\rangle} s_i s_j - h\sum_i s_j, \tag{1}$$

where the first sum goes over nearest-neighbour spins on the lattice and h parametrizes the strength of an external magnetic field applied to the spin system. Constant J is the strength of interparticle interaction. Moreover, for this task we will consider the case without external field, i.e. $h = 0$.

2.1 1D Ising Model

As an introduction and to get some intuition to the problem, we have a first look at the classical Ising model without external field in one dimension and apply the transfer matrix method to calculate the analytic solution. The spins of the system are arranged along a line with periodic boundary conditions, therefore each spin has two neighbours. The energy of a spin configuration is

$$E(s_1, s_2, ..., s_N)/J = s_1 s_2 + s_2 s_3 + \cdots + s_N s_1. \tag{2}$$

If not mentioned otherwise, we will assume units where $J = 1$. The partition function of the system is then

$$Z = \sum_{\{s_i\}} e^{-\beta E(\{s_i\})}. \tag{3}$$

With $\{s_i\}$ the set of all spin variables is denoted. The transfer matrix trick is to rewrite the partition function as the trace of the the transfer matrix M to the power N

$$Z = \sum_{\{\sigma_i\}} e^{-\beta \sum_{i=1}^{N} \sigma_i \sigma_{i+1}} = \sum_{\{s_i\}} \prod_{i=1}^{N} e^{-\beta s_i s_{i+1}} = \mathrm{Tr}\left(A^N\right). \tag{4}$$

Here, A is defined as

$$A_{ij} = e^{-\beta\sigma_i\sigma_j}. \tag{5}$$

Notice that, when expressing the partition function via the transfer matrix, the expression corresponds to the tensor network drawn as:

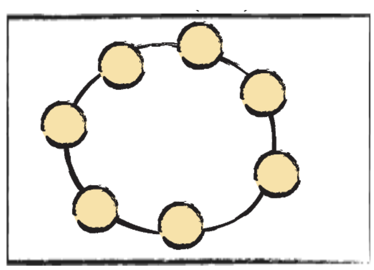

Figure 2

However, this is the easiest case scenario, and here the evaluation of the contraction can easily be found analytically. Since $A^N = U^{-1}D^N U$, where D is a diagonal matrix with the eigenvalues λ_1 and λ_2 of A on its diagonal, we can calculate the $N-$th power of the transfer matrix by diagonalizing it. With the cyclicity property of the trace function, the partition function can be rewritten as

$$Z = \mathrm{Tr}\left(A^N\right) = \mathrm{Tr}\left(D^N\right) = \lambda_1^N + \lambda_2^N. \tag{6}$$

Similarly, one-point and two-point functions can be calculated by rewriting it as product of matrices, with $P(\{\sigma_k\})$ being the probability of finding a system in certain configuration:

$$< \sigma_i >= \sum_{\{\sigma_k\}} \sigma_i P(\{\sigma_k\}) = \frac{1}{Z}\mathrm{Tr}\left(A^{j-1}\sigma_z A^{N-(j-1)}\right) = \frac{1}{Z}\mathrm{Tr}\left(\sigma_z A^N\right), \tag{7}$$

where σ_z is the Pauli z-matrix

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

Expressing Eq. (7) in terms of eigenvalues of A gives,

$$\frac{1}{Z}\mathrm{Tr}\left(\sigma_z A^N\right) \frac{1}{Z}\sum_{i=1,2} \langle i|\,\sigma_z A^N\,|i\rangle = \frac{1}{Z}\sum_{i=1,2} \lambda_i^N \langle i|\,\sigma_z\,|i\rangle = \frac{1}{Z}\left(\lambda_0^N - \lambda_1^N\right). \tag{9}$$

For the two point correlator (assuming $i < j$) we find

$$\begin{aligned} < \sigma_i \sigma_j > &= \frac{1}{Z}\mathrm{Tr}\left(A^{i-1}\sigma_z A^{j-1-(i-1)}\sigma_z A^{N-(j-1)}\right) \\ &= \frac{1}{Z}\mathrm{Tr}\left(\sigma_z A^{j-i}\sigma_z A^{N-(j-i)}\right). \end{aligned} \tag{10}$$

2.2 2D Ising Model

The transfer tensor for the two-dimensional case can be obtained similarly, by introducing

$$A^{s_1 s_2 s_3 s_4} = e^{\beta(s_1 s_2 + s_2 s_3 + s_3 s_4 + s_4 s_1)}. \tag{11}$$

Using this (transfer-) tensor, the partition function can be written as

$$Z = \sum_{\{s_1^i, s_2^i, s_3^i, s_4^i\}_i} \prod_i A^{s_1^i s_2^i s_3^i s_4^i}, \tag{12}$$

or in a tensor network form:

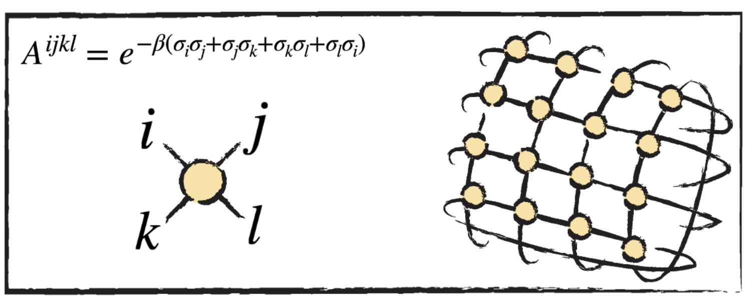

Figure 3: Transfer tensor (left) and partition function (right) of the Ising model on a square lattice.

One way to interpret this tensor is that it calculates the local energies between four adjacent spins (dual-lattice), and weights the local configurations with the Boltzmann probability $\exp(-E/T)$ through the respective energies. Note its similarity to the one-dimensional transfer matrix M.

2.3 Domain wall variables

Another way to represent the Ising model is by using domain wall variables [2, 3]. Domain walls are the boundaries between spin up or down domains in the spin system. These new set of variables α_{ij} are defined as

$$\alpha_{ij} = s_i s_j \tag{13}$$

and can take the values 1 or -1. Note that the mapping to this new set of variables is not one to one, since $\alpha_{ij} = 1$ if $s_i = s_j = 1$ or $s_i = s_j = -1$ and $\alpha_{ij} = -1$ if $s_i = -s_j$. Therefore, it is a two to one mapping. Although it seems at first that we are loosing information when we map the system to these new set of variables $\{\alpha_{ij}\}$ it turns out that its free energy has exactly the same singularities as the original free energy and is therefore a suitable representation to study the critical properties of the Ising model. In practice, one performs this mapping to domain wall variables by noting

$$e^{\beta s_a s_b} = \cosh(\beta) + s_a s_b \sinh(\beta) = \cosh(\beta) \sum_x (s_a s_b \tanh(\beta))^x. \tag{14}$$

This can then be inserted in the partition function. After performing explicitly the sum over the spin variables of the Ising model in 2D with pbc and without external magnetic field one obtains for the partition function the following contractions of tensors

$$Z = 2^N \cosh(\beta)^{2N} \sum_{\{x_1,x_2,x_3,x_4\}} \prod_i A_{x_1^i,x_2^i,x_3^i,x_4^i}, \tag{15}$$

where

$$A_{i,j,k,l} = \sqrt{\tanh(\beta)}^{i+j+k+l} \cdot \delta_{\mathrm{mod}(i+j+k+l,2),0}. \tag{16}$$

The other advantage of the domain wall variables is that you can also find a convinient expression for calculating one and two point correlation functions, by written them as contraction of tensors with one or two impurity tensors on site i and i,j for one point and two point correlation functions respectively [4].

$$< s_j > = \frac{1}{Z} \sum_{\{s\}} s_j e^{\beta H(\{s\})} = \frac{2^N \cosh(\beta)^{2N}}{Z} \sum_{\{i,j,k,l\}} B_{i^a,j^a,k^a,l^a} \prod_{b\neq a} A_{i^b,j^b,k^b,l^b}, \tag{17}$$

$$< s_i s_j > = \frac{1}{Z} \sum_{\{s\}} s_i s_j e^{\beta H(\{s\})} = \frac{2^N \cosh(\beta)^{2N}}{Z} \sum_{\{i,j,k,l\}} B_{i^a,j^a,k^a,l^a} B_{i^b,j^b,k^b,l^b} \prod_{c\neq a,b} A_{i^c,j^c,k^c,l^c}. \tag{18}$$

The impurity tensor B is defined as

$$B_{i,j,k,l} = \sqrt{\tanh(\beta)}^{i+j+k+l} \cdot \delta_{\mathrm{mod}(i+j+k+l+1,2),0}. \tag{19}$$

The rewritting in terms of domain wall variables can also be applied to the 2D Ising model with external magnetic field

$$H = -\sum_{<a,b>} s_a s_b - h \sum_a s_a. \tag{20}$$

Assuming periodic boundary conditions, the Hamiltonian can be rewritten as

$$H = -\sum_{<a,b>} s_a s_b + \frac{h}{4} (s_a + s_b). \tag{21}$$

Hence the partition function is

$$Z = \sum_{\{s\}} e^{\beta \sum_{<a,b>} s_a s_b + \frac{h}{4}(s_a+s_b)} = \sum_{\{s\}} \prod_{<a,b>} e^{\beta s_a s_b + \frac{h}{4}(s_a+s_b)}, \tag{22}$$

where the Boltzmann weights are

$$e^{\beta(s_a s_b + \frac{h}{4}(s_a+s_b))} = \sum_{x=0,1} \cosh(\beta)(s_a s_b a)^{x_1} e^{\frac{h\beta}{4} S_a} e^{\frac{h\beta}{4} S_b}. \tag{23}$$

The expression is then inserted into the partition function.

$$Z(\beta,h) = \sum_{\{s\}} \prod_{<a,b>} \cosh(\beta)(s_a s_b a)^{x_1} e^{\frac{h\beta}{4} S_a} e^{\frac{h\beta}{4} S_b}. \tag{24}$$

By summing again over the spin variables, one can rewrite the partition function as a tensor network [5]

$$T_{abcd} = \sum_i D_{ia} D_{ib} D_{ic} D_{id}, \tag{25}$$

where $\gamma = \beta h/4$ and

$$D_{ia} = \begin{pmatrix} e^{\gamma}\sqrt{\cosh\beta} & e^{\gamma}\sqrt{\sinh(\beta)} \\ e^{-\gamma}\sqrt{\cosh\beta} & e^{-\gamma}\sqrt{\sinh(\beta)} \end{pmatrix}. \tag{26}$$

2.4 Exact solution 2D Ising Model

Here we give Onsagners exact solution of the 2D Ising model without external magnetic field such that the results of the of TRG-algorithm can be compared to them. The Helmholtz free energy per site of the Ising model in the thermodynamic limit is given by [6, 7]

$$\lim_{N\to\infty} -\beta f_N = \lim_{N\to\infty} \ln(Z_N) = \ln\left(2\cosh(\beta J)\right) + \frac{1}{2\pi}\int_0^{\pi} \mathrm{d}\phi \ln\left(\frac{1}{2}\left(1+\sqrt{1-\kappa^2\sin^2(\phi)}\right)\right), \tag{27}$$

where $\kappa \equiv 2\sinh(2\beta J)/\cosh^2(2\beta J)$. Consequently, the internal free energy U and the magnetization per side are

$$U = -J\coth(2\beta J)\left[1+\frac{2}{\pi}(2\tanh^2(2\beta J)-1)\int_0^{\pi/2}\frac{1}{\sqrt{1-4k(1+k)^{-2}\sin^2(\theta)}}d\theta\right] \tag{28}$$

$$= M = \left[1-\sinh^{-4}(2\beta J)\right]^{1/8}. \tag{29}$$

From the specific heat

$$C(T) = \kappa\beta^2\frac{\partial^2}{\partial\beta^2}\ln(Z). \tag{30}$$

The critical temperature can be calculated, by looking at the divergence of the specific heat

$$\tanh\left(\frac{2J}{\kappa T_c}\right) = \frac{1}{\sqrt{2}} \to \frac{kT_c}{J} = 2.269185. \tag{31}$$

3 Tensor Renormalization Group algorithm contraction

Here we explain the main steps in TRG algorithm contraction. As explained in the previous section, a partition function of 2D Ising model can be recast in a following tensor network (for clearness of the picture we omit connecting the boundary legs, but periodic boundaries are implicitly taken into account):

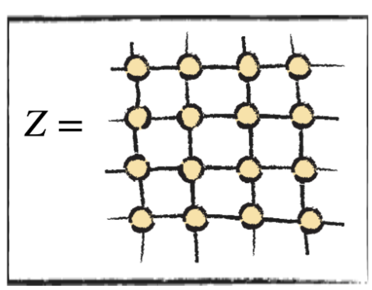

Figure 4

To obtain the value of partition function, we need to fully contract it. It is not an easy task, because contracting such tensor network exactly for large system size leads to tensors of too large dimensions, such that the contraction is impossible in terms of computational time and memory resources. For this reason, within the contraction algorithm we have to compress the information. The compression technique on which TRG (and more generally - all tensor network algorithms) rely is the singular value decomposition (SVD). SVD is decomposition of a matrix into three tensors, $A = U \cdot S \cdot V$, where U and V are unitary, and S is a diagonal matrix. The diagonal entries of the matrix S are referred to as the singular values. Supposing that the singular values are ordered from the largest to the smallest, the compression is carried out by keeping only the first m singular values. This way we keep the most important part of information, and discard the less important information. In our tensor network, we will use two possible SVDs of the transfer tensor A, shown in Fig. 5.

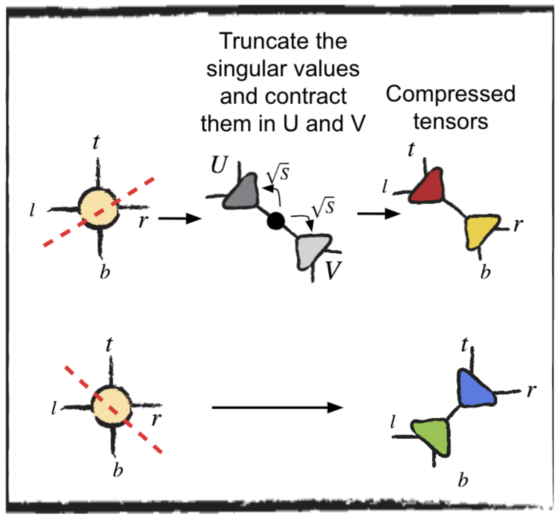

Figure 5

In these decompositions, we first need to reshape the rank-4 tensor into the matrix, i.e. rank-2 tensor. The decompositions differ in the way that we perform the reshaping, i.e. over which bipartition of legs we construct the matrix, which is marked with the red dashed line. Performing these two SVDs on the tensors of partition function, we can then expand the tensor network into the following:

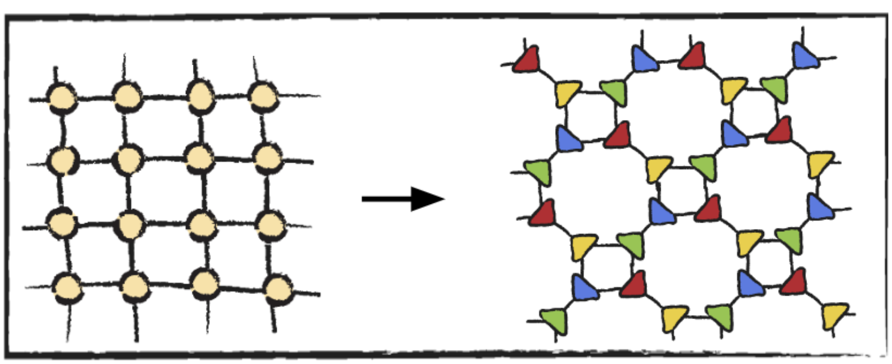

Figure 6

Now, we group the tensors in groups of four as indicated, and contract them.

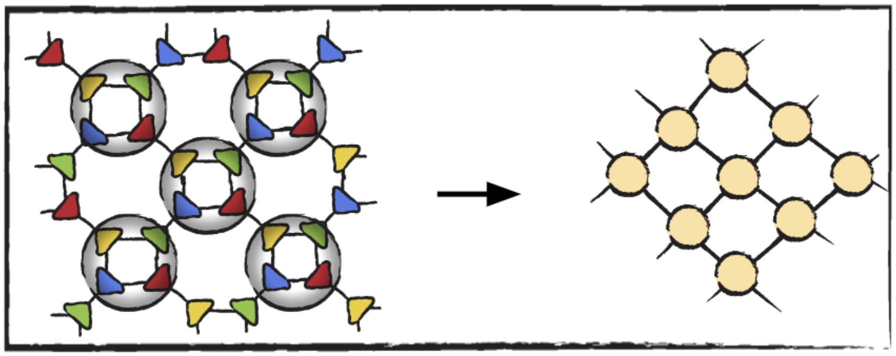

Figure 7

The resulting tensor network is now again a square lattice, but composed of half number of tensors. In other words, we compressed the transfer tensor of 1 site, A^1 into the transfer tensor of 2 sites, A^2. The described procedure represents one coarse-graining step of the TRG algorithm, which we then iteratively repeat. Notice that repeating the step n times results in a transfer tensor which represents 2^n sites. Remembering that we have periodic boundary conditions, the remaining tensor will have a shape:

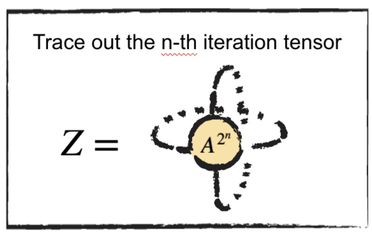

Figure 8

To obtain a partition function value, all that is left is to contract the remaining legs, i.e. perform the double trace of a tensor.

For tensor network-based renormalization group algorithms beyond TRG, see the references [8, 9, 10, 11].

References

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