Solving MaxCut with imaginary time evolution - guide through the main theoretical concepts.

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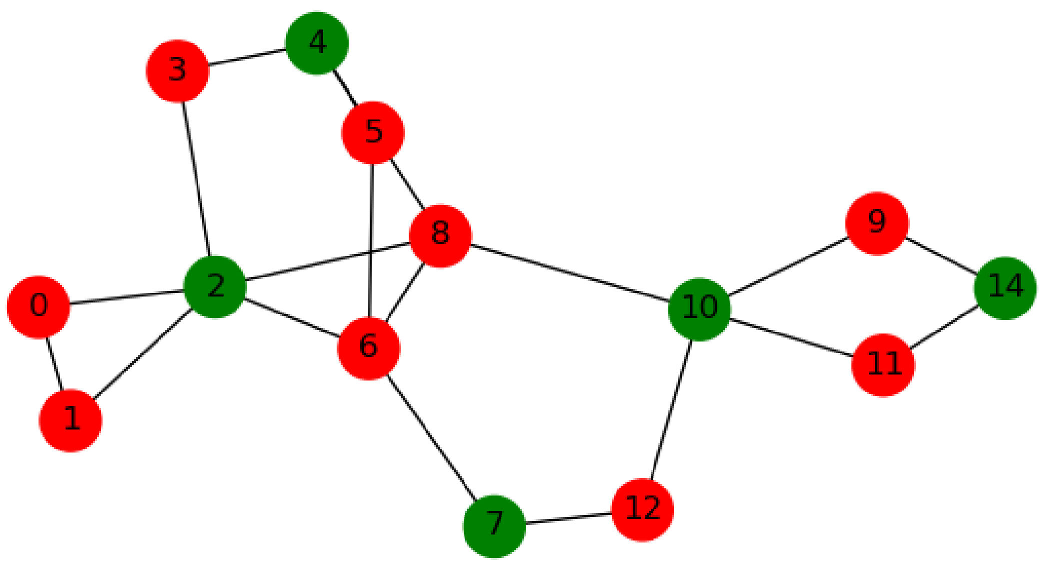
1 Ising mapping for the maximum cut problem.

The Maximum Cut (MaxCut) problem is a fundamental combinatorial optimization problem in graph theory. Given an undirected graph, the goal is to partition its vertices into two disjoint sets such that the number of edges connecting vertices from different sets is maximized. Formally, for a graph $G = (V, E)$, where V is the set of vertices and E is the set of edges, MaxCut seeks to find a partition $(S, \bar{S})$ of V that maximizes the number of edges with one endpoint in S and the other in $\bar{S}$. This problem has applications in various fields, including network designing efficient telecommunications networks, optimizing transportation routes, and maximizing the flow of goods or services in supply chain networks.



The MaxCut problem is NP-hard, meaning that no polynomial-time algorithm is currently known that can solve all instances of the problem. Therefore, finding efficient approximation algorithms for MaxCut is crucial for tackling a wide range of real-world optimization problems. The MaxCut problem can be formulated as a Quadratic Unconstrained Binary Optimization (QUBO) problem, where the goal is to minimize a quadratic function of binary variables subject to no constraints, a form particularly suitable for quantum annealing or other quantum optimization algorithms.

Indeed, the MaxCut problem can be mapped onto the search for the ground state of an Ising model. In this mapping, each vertex of the graph corresponds to a spin variable, and the interaction between spins is determined by the edges of the graph.

For further details and for an explicit construction of the Ising Hamiltonian associated to the MaxCut problem see [Max].

2 Ground state search by imaginary time evolution.

We want to find the ground state of the Ising Hamiltonian $\hat{H} = \sum_{ij} h_{ij} \sigma_1^z \otimes \sigma_j^z$ of an N spins system. Note that this Hamiltonian is diagonal in the computational basis $\{|i\rangle\}$, i.e.

$$H = \sum_{1 \le i \le 2^N} E_i |i\rangle\langle i|.$$

We start from an equal superposition of all states of the computational basis:

$$|\psi_0\rangle = \frac{1}{2^N} \sum_{1 \le i \le 2^N} |i\rangle.$$

We perform imaginary time evolution by applying the evolution operator $\hat{O}(t) = e^{-Ht}$ and then normalizing the system state. We obtain

$$|\psi(t)\rangle = \frac{\hat{O}(t)|\psi(t)\rangle}{\sqrt{\langle\psi(t)|\hat{O}(t)^\dagger \hat{O}(t)|\psi(t)\rangle}} = \frac{\sum_{1 \le i \le 2^N} e^{-E_i t}|i\rangle}{\sqrt{\sum_{1 \le j \le 2^N} e^{-2E_j t}}} \xrightarrow{t \to \infty} |\mathrm{argmin}(E)\rangle,$$

that is the Hamiltonian ground state.

References

[Max] Maximum cut. Maximum cut — Wikipedia, the free encyclopedia. [Online; accessed 19-April-2024].