Linear Algebra

- → Basic Definitions
- → Matrix Algebra
- → Invertible Matrix
- → Rank of a Matrix
- → Linearly dependent and independent vectors
- → system of linear eqns.
- → Eigen values and eigen vectors
- → Basis and dimension

Text Books: 1) Matrices by A R Vasistha

2) Linear Algebra SCHAUM SERIES MCGRAWHILL PUBLICATIONS 4^{th} edition.

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Matrix A set of mn elements (real (or) complex) are arranged in a rectangular array of m rows and n columns is called an m×n matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$A = (a_{ij})_{m\times n}$ where $1 \le i \le m$ & $1 \le j \le n$

Symmetric Matrix If A is an n×n matrix such that $A^T = A$ (or) $a_{ij} = a_{ji} \ \forall\, i,j$ then A is called a symmetric matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

arbitrary elements

arbitrary elements

Not arbitrary elements.

Skew-Symmetric Matrix If A is an $n \times n$ matrix such that $A^T = -A$ (or) $a_{ij} = -a_{ji}\ \forall i,j$ then A is called a skew-symmetric matrix.

Eg.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$$

→ arbitrary elements

→ Not arbitrary elements

Not arbitrary elements

Note 1) $tr(A) = 0$

2) Diagonal elements of skew-symmetric matrix are zeros.

3) Det. of odd order skew-symmetric matrix is zero.

4) Det. of even order skew-symmetric matrix is perfect square.

Eg. $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

$|A| = 4 = 2^2$ (perfect square)

5) If A & B are symmetric matrices then

i) $A+B$ is symmetric matrix

ii) $A-B$ is " "

iii) $AB+BA$ is symmetric matrix

iv) $AB-BA$ is skew-symmetric matrix.

Orthogonal Matrix If A is an $n \times n$ matrix such that $AA^T = A^TA = I$ (or) $A^T = A^{-1}$ then A is called an orthogonal matrix

Eg. $$A = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

Note 1) If A is an orthogonal matrix then

i) sum of squares of elements of each row is 1

ii) sum of products of corresponding elements of any two distinct rows is zero.

2) If A is an orthogonal matrix then $|A| = \pm 1$

Idempotent Matrix If A is an $n \times n$ matrix such that $A^2 = A$ then A is called an idempotent matrix

Eg. $A = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

Nilpotent Matrix For a square matrix A if there exists a natural number 'm' such that $A^m = 0$ then A is called a nilpotent matrix of index m.

Eg. $A = \begin{bmatrix} 0 & 2 & 5 & 2 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ A is a nilpotent matrix of index 4

since $A^2 = A^3 \neq 0$ and $A^4 = 0$

Involutary Matrix If A is an $n \times n$ matrix such that $A^2 = I$ then A is called an involutary matrix.

Eg. $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

Here : $A^2 = I$

$A \to$ Involutary matrix

Matrix Algebra

properties of Determinants

→ $\det(A) = \det(A^T)$

$$|A| = \begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} = 6$$

$$|A^T| = \begin{vmatrix} 5 & 1 \\ 4 & 2 \end{vmatrix} = 6$$

→ If two rows (or) columns in a determinant are interchanged then the value of its determinant is multiplied by -1.

Eg. $$\begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} = 6$$

$$R_1 \leftrightarrow R_2$$

$$\begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix} = -6$$

→ If all elements of a row (column) in a determinant are zeros then the value of its det. is zero.

Eg. $\begin{vmatrix} 1 & 2 & 3 \\ \boxed{0 \quad 0 \quad 0} \\ -4 & 5 & 7 \end{vmatrix} = 0$

→ If two rows (columns) are identical (or) proportional then the value of its det. is zero.

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -4 & 5 & 7 \end{vmatrix} = 0$$

Same rows [Identical rows]

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -4 & 5 & 7 \end{vmatrix} = 0$$

proportional rows ($R_2 = 2R_1$)

→ If sum of elements of each row (column) is zero then the value of its det. is zero

$$\begin{vmatrix} 1 & 2 & -3 \\ -10 & 2 & 8 \\ 2 & -9 & 7 \end{vmatrix} = 0$$

→ The determinant of upper triangular matrix, lower triangular matrix, diagonal matrix, scalar matrix (or) identity matrix is the product of its diagonal elements.

Eg. 1)

Det. of upper Δ^{lar} matrix

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = (1)(4)(5) = 20$$

2)

Det. of lower Δ^{lar} matrix

$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{vmatrix} = (1)(3)(6) = 18$$

3)

Det. of diagonal matrix

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = (1)(2)(3) = 6$$

4) Det. of scalar matrix

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = (2)(2)(2) = 8$$

5) Det. of identity matrix

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1)(1)(1) = 1$$

→ $|AB| = |A|\,|B|$

→ $|A+B| \neq |A| + |B|$ (In general)

→ $|A^m| = |A|^m$

→ $|KA| = K^n |A|$

problems

1) The no. of arbitrary elements in $n\times n$ symmetric matrix is

A) n B) $n(n+1)$ C) $\frac{n(n+1)}{2}$ D) $\frac{n(n-1)}{2}$

Sol. → Total no. of elements $= n\times n = n^2$

→ Total no. of diagonal elements $= n$

→ Total no. of non-diagonal elements $= n^2-n$

→ Among (n^2-n) non-diagonal elements, half of the elements $\left(\frac{n^2-n}{2}\right)$ lie above the diagonal and half of the elements $\left(\frac{n^2-n}{2}\right)$ lie below the diagonal.

→ No. of arbitrary elements $= n + \frac{n^2-n}{2}$

$$= \frac{2n+n^2-n}{2}$$

$$= \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

2) The no. of arbitrary elements in $n \times n$ skew_symmetric matrix is

A) n B) $n-1$ C) $n(n-1)$ D) $\frac{n(n-1)}{2}$

Soln. $\frac{n^2-n}{2}$ elements $= \frac{n(n-1)}{2}$

3) The different $n \times n$ symmetric matrices can be formed with each element being 0 (or) 1 is

[GATE CS- 2004]

Soln. The no. of possible 2×2 symmetric matrices with each element being 0 (or) 1

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

No. of diferent 2×2 symmetric matrices using 0 (or) 1

$$= 8$$

$$= 2^3$$

$$= 2^{\frac{2(2+1)}{2}}$$

$\frac{2(2+1)}{2}$ → No. of arbitrary elements.

2 → No. of given elements

∴ No. of diferent n×n symmetric matrices can be formed using 0 (or) 1 $= 2^{\frac{n(n+1)}{2}}$

4) the no. of different n×n symmetric matrices possible with entries chosen from the set $\{0, 1, 2, \ldots\ldots, q-1\}$ is

Ans $q^{\frac{n(n+1)}{2}}$

5) The no. of different nxn skew-symmetric matrices can be formed with each element being 0, 1 (or) -1 is

Ans $3^{\frac{n(n-1)}{2}} = 3^{\frac{n^2-n}{2}}$

6) If $A = \begin{bmatrix} 9 & 6 \\ 8 & 7 \end{bmatrix}$ then $\det(A^{99} - A^{98})$ is

[ISRO ME-2017]

A) 1 B) 48 C) 0 D) 2^{99}

soln. Given : $A = \begin{bmatrix} 9 & 6 \\ 8 & 7 \end{bmatrix}$

$|A^{99} - A^{98}| = |A^{98}(A-I)|$

$= |A^{98}| \, |A-I|$ $\{ |AB| = |A| \, |B| \}$

$= |A|^{98} |A-I|$ —(1) $\{ |A^m| = |A|^m \}$

$$|A-I| = \begin{vmatrix} 8 & 6 \\ 8 & 6 \end{vmatrix} = 0$$

$(1) \Rightarrow |A^{99} - A^{98}| = 0$

7) Matrix $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal. The values of α, β and γ respectively are

A) $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{6}}$ B) $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{2}}$

C) $\pm\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{2}}$ D) $\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{3}}$

Soln.

$4\beta^2 + \gamma^2 = 1$ ——— (1)

$\alpha^2 + \beta^2 + \gamma^2 = 1$ ——— (2)

$2\beta^2 - \gamma^2 = 0$ ——— (3)

$\alpha^2 - \beta^2 - \gamma^2 = 0$ ——— (4)

$(4) \Rightarrow \alpha^2 = \beta^2 + \gamma^2$ ——— (5)

$(2) \Rightarrow \alpha^2 + \beta^2 + \gamma^2 = 1$

$\Rightarrow \alpha^2 + \alpha^2 = 1$ {$\because$ from (5)}

$\Rightarrow 2\alpha^2 = 1$

$\Rightarrow \alpha^2 = \frac{1}{2} \Rightarrow \boxed{\alpha = \pm\frac{1}{\sqrt{2}}}$

$(2)+(3) \Rightarrow \alpha^2+3\beta^2=1$

$\Rightarrow \frac{1}{2}+3\beta^2=1$

$\Rightarrow 3\beta^2=\frac{1}{2}$

$\Rightarrow \beta^2=\frac{1}{6}$

$\Rightarrow \boxed{\beta=\pm\frac{1}{\sqrt{6}}}$

$(3) \Rightarrow \gamma^2 = 2\beta^2$

$= 2\left(\frac{1}{6}\right)$

$= \frac{1}{3}$

$\boxed{\gamma = \pm\frac{1}{\sqrt{3}}}$

8) If $\begin{bmatrix} \frac{\sqrt{5}}{3} & -\frac{2}{3} & c \\ \frac{2}{3} & \frac{\sqrt{5}}{3} & d \\ a & b & 1 \end{bmatrix}$ is a real orthogonal matrix

then the value of $a^2+b^2+c^2+d^2$ is ______

Ans 0

9) Find the determinant of the following matrices

i) $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 8 & 1 & 7 & 2 \\ 2 & 0 & 2 & 0 \\ 9 & 0 & 6 & 1 \end{bmatrix}$

GATE CE-2014

ii) $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$

GATE EC-2013

iii) $A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

iv) $A = \begin{bmatrix} 2 & 2 & 3 & 3 \\ 2 & 1 & 4 & 2 \\ 3 & 2 & 3 & 2 \\ 3 & 1 & 2 & 4 \end{bmatrix}$

Soln.

i) $|A| = \begin{array}{c} \begin{matrix} \oplus & \ominus & \oplus & \ominus \end{matrix} \\ \begin{vmatrix} 2 & 0 & 0 & 0 \\ 8 & 1 & 7 & 2 \\ 2 & 0 & 2 & 0 \\ 9 & 0 & 6 & 1 \end{vmatrix} \end{array}$

$= 2 \begin{vmatrix} 1 & 7 & 2 \\ 0 & 2 & 0 \\ 0 & 6 & 1 \end{vmatrix}$

$= 2 \{1(2-0) - 7(0-0) + 2(0-0)\} = 4$

ii) $$|A| = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}$$

$$R_3 \to R_3 - 3R_1$$

$$|A| = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 0 & -6 & -8 \\ 3 & 0 & 1 & 2 \end{vmatrix}$$

Expanding along second column

$$|A| = (-1)\begin{vmatrix} 1 & 3 & 0 \\ 2 & -6 & -8 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= (-1)\left\{(-12+8) - 3\,(4+24)\right\}$$

$$= (-1)\left\{-4-84\right\}$$

$$= 88$$

iv)

$$|A| = \begin{vmatrix} 2 & 2 & 3 & 3 \\ 2 & 1 & 4 & 2 \\ 3 & 2 & 3 & 2 \\ 3 & 1 & 2 & 4 \end{vmatrix}$$

$C_1 - 2C_2$

$C_3 - 4C_2$

$C_4 - 2C_2$

$$|A| = \begin{vmatrix} -2 & 2 & -5 & -1 \\ 0 & 1 & 0 & 0 \\ -1 & 2 & -5 & -2 \\ 1 & 1 & -2 & 2 \end{vmatrix}$$

$$|A| = (1)\begin{vmatrix} -2 & -5 & -1 \\ -1 & -5 & -2 \\ 1 & -2 & 2 \end{vmatrix}$$

$$|A| = (-2)\{-10-4\} + 5\,\cancel{(-2+2)}^{0} - 1\,(2+5)$$

$$|A| = 28 - 7$$

$$|A| = 21$$

iii)

$$|A| = \begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3 + R_4$

$$|A| = \begin{vmatrix} 5 & 5 & 5 & 5 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$$

$C_2 - C_1,\ C_3 - C_1,\ C_4 - C_1$

$$|A| = \begin{vmatrix} 5 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$$ (Det. of Lower Δ^{lar} matrix)

$|A| = (5)(1)(1)(1)$

$|A| = 5$

10) find the determinant of

i) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 0 & 2 \\ 4 & 1 & 0 & 3 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & 1 & -1 & 1 \\ 3 & 2 & 1 & -1 \\ 4 & 3 & 2 & 1 \end{bmatrix}$

Ans i) 188 ii) 27

Note

Suppose $A = [a \ b \ c]_{1\times 3}$ & $B = \begin{bmatrix} l \\ m \\ n \end{bmatrix}_{3\times 1}$

$$AB = [a \ b \ c] \begin{bmatrix} l \\ m \\ n \end{bmatrix} = [a\times l + b\times m + c\times n]_{1\times 1}$$

1) No. of multiplications $= 3 = 1\times \cancel{3}\cdot\cancel{3}\times 1 = 1\times \underline{3}\times 1$

2) No. of additions $= 2 = 1\times \cancel{3}\cdot\cancel{3}\times 1 = 1\times(3-1)\times 1 = 1\times 2\times 1$

Note Suppose A & B are $m\times n$ and $n\times p$ matrices. If A&B are multiplied then

i) no. of multiplications $= m\times \cancel{n}\cdot \cancel{n}\times p = m\times \underline{\underline{n}} \times p$

ii) no. of additions $= m\times \cancel{n}\cdot \cancel{n}\times p = m\times \underline{\underline{(n-1)}} \times p$

problem what is the minimum no. of multiplications needed in computing matrix product PQR ?

where P has 4 rows and 2 columns

Q has 2 rows and 4 columns

R has 4 rows and 1 column

[GATE CE-2013]

$$PQR \begin{cases} P(QR) = 8+8 = 16 \text{ (Min. multiplications)} \\ (PQ)R = 32+16 = 48 \end{cases}$$

Case I

$$QR = 2\times\cancel{4}\cdot\cancel{4}\times 1 = 2\times \underline{\underline{4}} \times 1 = 8$$

$$P(QR) = 4\times\cancel{2}\cdot\cancel{2}\times 1 = 4\times \underline{\underline{2}} \times 1 = 8$$

Total $= 16$

Case II

$$PQ = 4\times\cancel{2}\cdot\cancel{2}\times 4 = 4\times \underline{\underline{2}}\times 4 = 32$$

$$(PQ)R = 4\times\cancel{4}\cdot\cancel{4}\times 1 = 4\times \underline{\underline{4}} \times 1 = 16$$

Total $= 48$

problem: The min. no. of multiplications and additions needed in computing the matrix product ABC are

where A has 10 rows & 20 columns

B has 20 rows & 10 columns

C has 10 rows & 5 columns

Invertible Matrix

Defn. If A & B are two square matrices of same order such that $AB = BA = I$ then B is called inverse of A (or) A is called inverse of B

i.e. $B = A^{-1}$ (or) $A = B^{-1}$

$$\left.\begin{aligned} AB &= I \\ A^{-1}AB &= A^{-1}I \\ IB &= A^{-1}I \\ B &= A^{-1} \end{aligned}\right\} \quad \begin{aligned} BA &= I \\ B^{-1}BA &= B^{-1}I \\ IA &= B^{-1}I \\ A &= B^{-1} \end{aligned}$$

For example, consider the 3x3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

→ Minor of a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

→ cofactor of a_{11} is $A_{11} = (-1)^{1+1} M_{11}$

$\rightarrow$ Minor of a_{12} is $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$\rightarrow$ cofactor of a_{12} is $A_{12} = (-1)^{1+2} M_{12}$

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$$\text{cofactors of } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\text{adj}A = [\text{cofactors of } A]^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{\text{adj}A}{|A|}}$$ where $|A| \neq 0$

Properties of Adjoint & Inverse

1) $A(\operatorname{adj}A) = (\operatorname{adj}A)A = |A|I$

2) $(\operatorname{adj}A)^{-1} = \frac{A}{|A|}$

3) $\operatorname{adj}(\operatorname{adj}A) = |A|^{n-2}A$

4) $|\operatorname{adj}A| = |A|^{n-1}$, $|\operatorname{adj}(\operatorname{adj}A)| = |A|^{(n-1)^2}$,

$|\operatorname{adj}\operatorname{adj}(\operatorname{adj}A)| = |A|^{(n-1)^3}$ ....

5) If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$

where a, b & c are non-zero elements

6) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

problems

1) Find the inverse of the matrix $\begin{bmatrix} 3+2i & i \\ -i & 3-2i \end{bmatrix}$ [GATE CE-2010]

Soln.

$$\text{Let } A = \begin{bmatrix} 3+2i \to a & i \to b \\ -i \to c & 3-2i \to d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{(9-4i^2+i^2)}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$$

2) For a given matrix $P = \begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$, where $i = \sqrt{-1}$, the inverse of the matrix P is [GATE ME-2015]

Ans $\frac{1}{24}\begin{bmatrix} 4-3i & i \\ -i & 4+3i \end{bmatrix}$

3) The inverse of the matrix $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ is

[GATE CE-2019]

Soln.

$$\text{Let } A = \begin{array}{cl} \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} & \begin{array}{l} (3) \\ (1) \\ (2) \end{array} \end{array}$$

$$|A| = 2(12-2) - 3(16-1) + 4(8-3) = -5$$

$$\begin{array}{c} \times \\ \times \\ \\ \\ \end{array}\begin{bmatrix} 3 & 2 & 3 & 3 \\ 1 & 4 & 4 & 1 \\ 4 & 1 & 2 & 4 \\ 3 & 2 & 3 & 3 \end{bmatrix} \Rightarrow \operatorname{adj} A = \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

$$A^{-1} = \frac{\operatorname{adj} A}{|A|} = -\frac{1}{5}\begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & \frac{4}{5} & \frac{9}{5} \\ 3 & -\frac{4}{5} & -\frac{14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix} //$$

4) Find the inverse of

i) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} 3 & 2 & 0 \\ 1 & 1 & 0 \\ 4 & 1 & 2 \end{bmatrix}$

Ans i) $A^{-1} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}$

ii) $A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -4 & 0 \\ -2 & 6 & 0 \\ -3 & 5 & 1 \end{bmatrix}$

5) If $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 3 \\ 1 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ then

$|AB-BA|$ (det. of $(AB-BA)$) is ______

Soln. We know that A & B are symmetric matrices

$AB-BA$ is a skew-symmetric matrix of order 3

$\therefore |AB-BA| = 0$ [$\because$ Det. of odd order skew-symmetric matrix is zero]

$\therefore$ The inverse of $(AB-BA)$ does not exist

6) If $\operatorname{adj} A = \begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}$ then the absolute value of det(A) is ______

Sdn.

$$|\operatorname{adj} A| = |A|^{n-1}$$

$$-18(-112+20) + 11\,\cancel{(-16+16)}^{\,0} - 10(10-56) = |A|^{3-1}$$

$$-18(-92) - 10(-46) = |A|^2$$

$$\Rightarrow |A|^2 = 2116 = (46)^2$$

$$\Rightarrow |A| = \pm 46$$

$\therefore$ Absolute value of $\det(A) = 46$

7) If $\operatorname{adj} A = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ and $\det(A) = 4$ then the value of α is ______

Ans $\alpha = 11$

8) If the determinant of a 3x3 matrix A is 16 then the value of $\det[(4A)^{-1}]$ is ____

soln.

Given : $|A| = 16$

$$|(4A)^{-1}| = |4^{-1}A^{-1}|$$

$$= |\underbrace{\tfrac{1}{4}}_{K}\,\underbrace{A^{-1}}_{A}|$$

$$= (\tfrac{1}{4})^n |A^{-1}|$$

$$\boxed{\begin{array}{l} |KA| = K^n|A| \\ |A^m| = |A|^m \end{array}}$$

$$= (\tfrac{1}{4})^3 |A|^{-1}$$

$$= (\tfrac{1}{64})(16)^{-1}$$

$$= (\tfrac{1}{64})(\tfrac{1}{16})$$

$$= \frac{1}{1024}$$

9) If $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 2 & 1 & 0 \end{bmatrix}$ and $B = \operatorname{adj}A$, $C = 5A$ then the value of $\frac{|\operatorname{adj}B|}{|C|}$ is ____

soln.

$$|A| = 1(0+3) + 1(0+6) + 1(0-4) = 5$$

$$\frac{|\operatorname{adj}B|}{|C|} = \frac{|\operatorname{adj}(\operatorname{adj}A)|}{|5A|}$$

$$= \frac{|A|^{(n-1)^2}}{5^n|A|}$$

$$\boxed{\begin{array}{l} |\operatorname{adj}(\operatorname{adj}A)| = |A|^{(n-1)^2} \\ |KA| = K^n|A| \end{array}}$$

$$= \frac{|A|^{(3-1)^2}}{5^3 |A|}$$

$$= \frac{|A|^3}{5^3}$$

$$= \frac{(5)^3}{5^3}$$

$$= 1$$

10) If A is a 3×3 matrix and $\det(A) = 2$ then the value of $\det[\operatorname{adj}\operatorname{adj}(\operatorname{adj}A^{-1})]$ is

A) $\frac{1}{512}$ B) $\frac{1}{1024}$ C) $\frac{1}{128}$ ✓D) $\frac{1}{256}$

11) If the value of determinant of 3×3 matrix is 11, then the value of square of the determinant formed by its cofactors is ______

Given: $|A_{3\times3}| = 11$

$$\operatorname{adj}A = [\text{Cofactors of } A]^T$$

$$\operatorname{adj}A = P^T, \quad \text{where } P = \text{cofactors of } A$$

$$|\operatorname{adj}A| = |P^T|$$

$$|A|^{n-1} = |P| \qquad \{|P^T| = |P|\}$$

$$11^{3-1} = |P|$$

$|P| = 121$

$|\text{cofactors of } A| = 121$

$|\text{cofactors of } A|^2 = (121)^2 = 14641$

12) Given : $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ find.

i) $adj(adj A)$ ii) $(adj A)^{-1}$ iii) $|2\, adj A|$

iv) $|2\, adj(A^{-1})|$ v) $|2\, adj(adj A)|$ vi) $|2\, adj\, adj(A^{-1})|$

Ans i) $5A$

ii) $\frac{1}{5}A$

iii) 200

iv) $\frac{8}{25}$

v) 5000

vi) $\frac{8}{625}$

Linearly Dependent & Independent Vectors

Vector An ordered set of 'n' numbers is called an n-vector (or) n-tuple vector (or) n-dimensional vector (or) vector of order 'n'

Eg: $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ (or) $[x_1, x_2, \ldots\ldots x_n]$

Linear Combination of Vectors

If a vector X can be expressed as $X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$ then X is said to be linear combination of the vectors $X_1, X_2, \ldots, X_n$

Eg: $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ & $X_3 = \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix}$

$$X_3 = 3X_1 + X_2$$

$\therefore X_3$ is linear combination of the vectors X_1 & X_2

Linearly Dependent Vectors

A set of vectors $X_1, X_2, \ldots, X_n$ is said to be linearly dependent if there exists n scalars $k_1, k_2, \ldots, k_n$ (not all zeros i.e. atleast one of $k_1, k_2, \ldots, k_n$ is non-zero) such that $k_1 X_1 + k_2 X_2 + \cdots + k_n X_n = 0$

Linearly Independent Vectors

A set of vectors $X_1, X_2, \ldots, X_n$ is said to be linearly independent if the relation $k_1 X_1 + k_2 X_2 + \cdots + k_n X_n = 0$ is satisfied only when $k_1 = k_2 = \cdots = k_n = 0$

Note 1) Let $X_1, X_2, \ldots, X_n$ be n vectors

Let $A = [X_1 \; X_2 \cdots X_n]$

Find det(A) (or) rank of A

i) If $|A| \neq 0$ (or) rank of (A) = no. of given vectors then the vectors are said to be linearly independent

ii) If $|A| = 0$ (or) rank of $(A) <$ no. of given vectors then the vectors are said to be linearly dependent.

iii) If the vectors are LD then one of the vector can be expressed as linear combination of the remaining vectors.

For example $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $X_3 = \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 6 \\ 1 & 1 & 4 \end{bmatrix}$

$|A| = (12-6) - 2(4-6) + 5(1-3)$

$|A| = 6 + 4 - 10$

$|A| = 0$

$\therefore$ The vectors are LD.

$\therefore$ X_3 can be expressed as linear combination of X_1 & X_2

$X_3 = 3X_1 + X_2$

4) If one vector can be expressed as a constant multiple of other vector then the two vectors are said to be LD.

Eg. $X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $X_2 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$

$X_2 = 3X_1$ (or) $X_1 = \frac{1}{3} X_2$

$\therefore$ X_1 & X_2 are linearly dependent Vectors.

For example $X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ & $X_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

Here : X_1 & X_2 are LI Vectors

since, one Vector can not be expressed as a constant multiple of other Vector.

Theorem(1) Let S_1 and S_2 be two finite set of Vectors such that $S_1 \subset S_2$ { S_1 is a subset of S_2 }

$$S_1 = \{ (\ ), (\ ), (\ ), \cdots (\ ) \}$$

$$S_2 = \{ (\ ), (\ ), (\ ), \cdots (\ ) \}$$

If S_1 is LD then S_2 is also LD.

Theorem(2) Let S_1 & S_2 be two finite set of Vectors such that $S_1 \subset S_2$ (S_1 is a subset of S_2). If S_2 is LI then S_1 is also LI

* $\boxed{\text{Every subset of a LI set is also LI}}$ *

problem consider the following statements:

S_1 : If x_1, x_2, x_3, x_4 is LI set of Vectors then x_1, x_2, x_3 is also LI

S_2 : If x_1, x_2, x_3, x_4 is LD set of Vectors then x_1, x_2, x_3 is also LD.

which of the following is TRUE?

S_1 is true {theorem (2)}

Every subset of LI vectors is also LI

S_2 is false {theorem (1)}

subset of LD vectors need not be LD.

orthogonal vectors Two vectors x_1 & x_2 are said to be orthogonal if $x_1^T x_2 = x_2^T x_1 = 0$

Eg. $x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $x_2 = \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix}$

$$x_1^T x_2 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix} = -6+6+0=0$$

$\therefore$ x_1 & x_2 are orthogonal vectors.

Orthonormal Vectors Two vectors x_1 & x_2 are said to be orthonormal if

i) x_1 & x_2 are orthogonal vectors i.e. $x_1^T x_2 = 0$

ii) $||x_1|| = 1$ and $||x_2|| = 1$

$||x|| \rightsquigarrow$ Norm x
$\rightsquigarrow$ Length of Vector x

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$||x|| = \sqrt{x_1^2 + x_2^2}$

Eg. $x_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $x_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

Rank of a Matrix

Submatrix A matrix obtained by deleting some rows (or) columns (or) both is called a submatrix

Minor The determinant of square submatrix is called minor of a matrix.

Rank The order of largest non-zero minor is called rank of a matrix

order $\to$ Rank; largest $\to$ higest; non-zero $\to$ $\neq 0$; minor $\to$ det. of square submatrix

Example The rank of the following matrix is

GATE CE-2018

$$A = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{bmatrix}_{3\times4}$$

A) 1
B) 2
C) 3
D) 4

$$\begin{array}{l} 123 \checkmark \\ 124 \checkmark \\ 134 \checkmark \\ 234 \checkmark \end{array}$$

Let $B_1 = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 4 & 1 & 3 \end{bmatrix} \Rightarrow |B_1| = 0$

Let $B_2 = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 0 & 2 \\ 4 & 1 & 1 \end{bmatrix} \Rightarrow |B_2| = 0$

Let $B_3 = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 2 \\ 4 & 3 & 1 \end{bmatrix} \Rightarrow |B_3| = 0$

Let $B_4 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix} \Rightarrow |B_4| = 0$

$\rho(A) < 3$ — $2 \checkmark$, $1 \times$

Let $B_5 = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ be a submatrix of A.

$|B_5| = -2 \neq 0$

$\therefore \rho(A) = \text{Rank of } A = 2$ //

Rank (Another Defn.)

A number 'r' is said to be of rank of A if

i) all minors of order greater than r vanish

ii) there exists atleast one non-zero minor of order r.

Echelon Form

A matrix is said to be in echelon form if

i) all zero rows occupy last rows, if any

ii) the no. of zeros before the first non-zero element of each row is less than no. of such zeros before the first non-zero element of its succeding row.

Eg.

1) $$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = 3$$

2) $$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 6 & 7 & 8 \\ 0 & 0 & 0 & 10 & 11 \\ 0 & 0 & 0 & 0 & 13 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = 4$$

3) $$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix} \Rightarrow \rho(A) = 4$$

4) $$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix} \longrightarrow \text{Not in echelon form}$$

[Second condition not satisfied]

Note 1) Rank of a matrix in echelon form is equal to no. of non-zero rows.

2) only row operations should be performed.

i) $R_i \leftrightarrow R_j$

ii) $K R_i$ $(K \neq 0)$

iii) $K R_i + R_j$ $(K \neq 0)$

3) Row (or) column operations do not change rank of a matrix.

4) Rank of a matrix in echelon form = No. of non-zero rows

These non-zero rows are also called LI rows (or) vectors

5) Rank of a matrix is also defined as number of LI rows (or) vectors.

6) If a square matrix is in echelon form then it is an upper triangular matrix [Example (3)]

But converse need not be true.

Upper triangular matrix need not be in echelon form.

properties of Rank of a Matrix

1) $\rho(O_{m \times n})$ = Rank of zero matrix $= 0$

2) ρ(Non-zero matrix) ≥ 1

3) ρ(Non-zero vector) $= 1$

4) $\rho(I_{n \times n}) = n$

5) ρ(diagonal matrix) = no. of non-zero elements in the diagonal

E.g. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \Rightarrow \rho(A) = 3$

6) If A is a symmetric matrix then $\rho(A) = \rho(A^2)$

7) ρ(Non-zero skew-symmetric matrix) ≥ 2

8) $\rho(A) = \rho(A^T)$

9) $\rho(AA^T) = \rho(A^TA) = \rho(A)$

10) $\rho(A_{m\times n}) \leq \min(m,n)$

11) $\rho(AB) \leq \rho(A)$

$\rho(AB) \leq \rho(B)$

$\rho(AB) \leq \min\{\rho(A), \rho(B)\}$

12) $\rho(A+B) \leq \rho(A) + \rho(B)$

13) $\rho(A-B) \geq \rho(A) - \rho(B)$

14) If $\rho(A_{n\times n}) = n$ then $\rho(\text{adj}A) = n$

15) If $\rho(A_{n\times n}) = n-1$ then $\rho(\text{adj}A) = 1$

16) If $\rho(A_{n\times n}) = (n-2)$ or $(n-3)$ or $(n-4)$ ...... then $\rho(\text{adj}A) = 0$

17) If $|A_{n\times n}| \neq 0$ then $\rho(A) = n$

18) If $|A_{n\times n}| = 0$ then $\rho(A) < n$

problems

1) Let $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ where a, b, c are non-zero real numbers. Then rank of A =

A) 0 B) 1 C) 2 D) 3

soln. A is odd order (order 3) skew symmetric matrix.

$|A_{3\times3}| = 0$

$\Rightarrow \rho(A) < 3$ { property (18) }

$\Rightarrow \rho(A) \le 2$ ——— (1)

Now, $\rho(A) \geq 2$ ——— (2) { property (7) }

from (1) & (2), $\rho(A) = 2$ //

2) If V is a non-zero vector of dimension 3×1, then the matrix $A = VV^T$ has rank = ______

[GATE IN-2017]

Given : V is a non-zero vector of order 3×1

$A = VV^T$ (Given)

$\rho(A) = \rho(VV^T)$

$\rho(A) = \rho(V)$ { property (9) }

$\rho(A) = 1$ { property (3) }

Non-zero Vector

3) $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, $B = \begin{bmatrix} p^2+q^2 & pr+qs \\ pr+qs & r^2+s^2 \end{bmatrix}$. If the rank of A is N, then the rank of the matrix B is

A) $\frac{N}{2}$ B) $N-1$ C) N D) $2N$

[GATE EE-2014]

Sdn. we know that $B = AA^T$

$$\rho(B) = \rho(AA^T)$$
$$\rho(B) = \rho(A) \quad \{\text{property } (9)\}$$

$$|A_{3\times3}| = 0 \iff \rho(A) < 3 \quad \{\text{property } (18)\}$$

$$\begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$

$$\begin{vmatrix} 2x+1 & 2x+1 & 2x+1 \\ x & 1 & x \\ x & x & 1 \end{vmatrix} = 0$$

$C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_1$

Det. of lower triangular matrix

$$\begin{vmatrix} 2x+1 & 0 & 0 \\ x & 1-x & 0 \\ x & 0 & 1-x \end{vmatrix} = 0$$

$$(2x+1)(1-x)^2 = 0$$

$$\therefore x = 1, -\frac{1}{2}$$ //

28/10/2020

Note If A is an $n \times n$ upper triangular matrix with zeros in the digonal is a nilpotent matrix of index 'n'

i.e. $A^n = 0$

4) Suppose that A is an $n \times n$ upper triangular matrix such that $a_{ii} = 0$, $i = 1, 2, \ldots\ldots, n$. Then the rank of matrix A^n =

A) 0 B) 1 C) $n-1$ D) n

Soln. $A^n = 0$ ($\because$ A is an $n \times n$ upper triangular matrix with zeros in the diagonal is a nilpotent matrix of index n).

$\rho(A^n) = \rho(0)$

$\therefore \rho(A^n) = 0$ //

5) Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{bmatrix}$$ [GATE CE-2018]

Soln.

$$A = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 2^* & 0 & 2 & 2 \\ 4^* & 1 & 3 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 4R_1$

$$A = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & -2 & 2 & 6 \\ 0 & -3^* & 3 & 9 \end{bmatrix}$$

$R_3 \to 2R_3 - 3R_2$

$$A = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & -2 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ 2 LI rows (or) vectors

$\therefore \rho(A) = 2 = 2$ LI rows (or) vectors

6) find the rank of the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ -6 & 2 & 2 & 2 \\ 9 & 9 & -6 & 3 \end{bmatrix}$$

Soln.

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4^* & 4 & -3 & 1 \\ -6^* & 2 & 2 & 2 \\ 9^* & 9 & -6 & 3 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$, $R_3 \to R_3 + 6R_1$ & $R_4 \to R_4 - 9R_1$

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 8 & -4 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 8 & -4 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3^* & 3 \end{bmatrix}$$

$R_4 \to R_4 - 3R_3$

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 8 & -4 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3 LI rows (or) vectors

$\therefore\ \rho(A) = 3 = 3$ LI rows (or) vectors

7) find the rank of the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

[GATE EC-2017]

Ans 4

8) If $A = (a_{ij})$ is defined by $a_{ij} = i+j \ \forall\, i,j$ where $1 \le i \le 4$ and $1 \le j \le 5$ then rank of A is ____

Soln.

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix}$$

$$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1,\ R_4 \to R_4 - R_1$$

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$ & $R_4 \to R_4 - 3R_2$

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = 2 \;//$$

9) Let $A = (a_{ij})$, $1 \le i, j \le n$ with $n \geq 3$ and $a_{ij} = i.j$

The rank of A is

A) 0 B) 1 C) $n-1$ D) n

[GATE IN-2007]

[GATE CE-2015]

System of Linear Eqns.

Consider the following system of 'm' linear eqns. in 'n' variables

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n = b_m$$

(i) If $b_1, b_2, \ldots, b_m$ are all zeros then the system of linear eqns is said to be homogeneous.

(ii) If atleast (minimum) one of $b_1, b_2, \ldots, b_m$ is non-zero then the system of linear eqns. is said to be non-homogeneous.

(iii) If the system of eqns. has atleast one soln. then the system of eqns. is said to be consistent

(iv) If the system of eqns. has no soln. then the system of eqns. is said to be inconsistent

(v) The given system of eqns. can be written in the matrix form.

$$
\begin{array}{cccc}
x_1 & x_2 & & x_n \\
\downarrow & \downarrow & & \downarrow
\end{array}
$$

$$
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

$$A \quad X = B \rightarrow \text{constant matrix.}$$

coefficient matrix ← A

X → Variable Vector → Solution vector

vi) write the elements of matrix B in the last column of matrix A. Then the resulting matrix is called the augmented matrix and is denoted by

$$
[A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]
$$

vii) Reduce the augmented matrix $[A|B]$ to echelon form and hence find rank of A and rank of $[A|B]$

viii) If $\rho(A) < \rho(A|B)$ i.e. $\rho(A|B) \neq \rho(A)$ then the non-homogeneous system of eqns. $AX = B$ has no soln. (inconsistent)

ix) If $\rho(A|B) = \rho(A) =$ no. of Variables (no. of unknowns) then the non-homogeneous system of eqns. $AX = B$ has a unique non-trivial soln. (consistent)

x) If $\rho(A|B) = \rho(A) <$ no. of Variables then the non-homogeneous system of eqns has infinite no. of

non-trivial solns. [consistent]

Note 1) The necessary and sufficient condition for non-homogeneous system of eqns. to be consistent is $\rho(A|B) = \rho(A)$

2) If $|A| \neq 0$ then the non-homogeneous system of eqns. $AX = B$ has a unique non-trivial soln.

3) If $|A| = 0$ and $(\text{adj}A)B \neq 0$ then $AX = B$ has no soln.

4) If $|A| = 0$ and $(\text{adj}A)B = 0$ then $AX = B$ has infinitely many solns.

Homogeneous system of Linear Eqns.

$$AX = 0$$

$\|A\| \neq 0$ (or) $\rho(A) =$ no. of variables	$\|A\| = 0$ (or) $\rho(A) <$ no. of variables
$\downarrow$	$\downarrow$
Trivial Soln. $[x_1 = x_2 = \cdots\cdots = x_n = 0]$	Infinite no. of non-trivial solns.

Note 1) No. of LI solns. for homogeneous system of eqns. $AX = 0$ is $n - r$

where n = no. of variables

r = rank of coefficient matrix.

2) If the no. of eqns. less than number of variables for homogeneous system of eqns. $AX = 0$ then $AX = 0$ has infinite no. of non-trivial solns.

problems

1) consider the system of linear eqns

$$x - 2y + z = 3,$$
$$2x + \alpha z = -2,$$
$$-2x + 2y + \alpha z = 1.$$

In order to have a unique soln. to this linear system of eqns the value of α should not be equal to

A) $-\frac{2}{3}$ B) $\frac{2}{3}$ C) $\frac{4}{3}$ D) $-\frac{4}{3}$

Soln. For unique soln. $|A| \neq 0$

$$\begin{vmatrix} 1 & -2 & 1 \\ 2 & 0 & \alpha \\ -2 & 2 & \alpha \end{vmatrix} \neq 0$$

$$1(0-2\alpha)+2(2\alpha+2\alpha)+1(4+0) \neq 0$$

$$\Rightarrow 6\alpha+4 \neq 0$$

$$\Rightarrow 6\alpha \neq -4$$

$$\therefore \alpha \neq -\frac{2}{3}$$

2) The following system of eqns.

$x+y+z=3,$

$x+2y+3z=4,$

$x+4y+kz=6.$ has infinitely many solns. when

A) $k \neq 0$ B) $k=0$ C) $k=7$ D) $k \neq 7$

3) Consider the system of eqns.

$kx+y+z=1$

$x+ky+z=1$

$x+y+kz=1$

If the above system of eqns. has a unique soln. then which of the following is true?

A) $k=1$ and $k=-2$ B) $k\neq 1$ & $k\neq -2$

C) $k=1$ and $k\neq -2$ D) $k\neq 1$ & $k=-2$

Soln. For unique soln. we have $|A|\neq 0$

$$\begin{vmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix} \neq 0$$

$R_1 \to R_1 + R_2 + R_2$

$$\begin{vmatrix} k+2 & k+2 & k+2 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix} \neq 0$$

$C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_1$

Det. of lower triangular matrix

$$\begin{vmatrix} k+2 & 0 & 0 \\ 1 & k-1 & 0 \\ 1 & 0 & k-1 \end{vmatrix} \neq 0$$

$(K+2)(K-1)^2 \neq 0$

$K \neq -2$ and $K \neq 1$

3) Consider the following system of linear eqns.

$x_1 + 2x_2 = b_1$;

$2x_1 + 4x_2 = b_2$;

$3x_1 + 7x_2 = b_3$;

$3x_1 + 9x_2 = b_4$

which of the following conditions ensures that a soln. exists for the above system?

✓ A) $b_2 = 2b_1$ and $6b_1 - 3b_3 + b_4 = 0$

× B) $b_3 = 2b_1$ and $6b_1 - 3b_3 + b_4 = 0$

✓ C) $b_2 = 2b_1$ and $3b_1 - 6b_3 + b_4 = 0$

× D) $b_3 = 2b_1$ and $3b_1 - 6b_3 + b_4 = 0$

[GATE EC-2020]

Soln.

$$[A|B] = \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 2 & 4 & b_2 \\ 3 & 7 & b_3 \\ 3 & 9 & b_4 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$ and $R_4 \to R_4 - 3R_1$

$$[A|B] = \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 3b_1 \\ 0 & 3 & b_4 - 3b_1 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$[A|B] = \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 1 & b_3 - 3b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 3 & b_4 - 3b_1 \end{array}\right]$$

$R_4 \to R_4 - 3R_2$

$$[A|B] = \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 1 & b_3 - 3b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 6b_1 - 3b_3 + b_4 \end{array}\right]$$

For $b_2 = 2b_1$ & $6b_1 - 3b_3 + b_4 = 0$

$\rho(A|B) = 2 = \rho(A) = $ no. of variables

$\therefore$ The system is consistent (unique soln.)

4) Consider the matrix $A = \begin{bmatrix} K & K & K \\ 0 & K-1 & K-1 \\ 0 & 0 & K^2-1 \end{bmatrix}$

If the system $AX = 0$ has only one linearly independent soln. then $K =$

A) $0, -1$ B) $-1, 1$ C) $0, 1$ D) $0, 1, -1$

soln. No. of LI solns $= 1$

$n - r = 1$

$3 - \rho(A) = 1$

$\rho(A_{3\times3}) = 2 < 3$ {properties of Rank property (18)}

$\Rightarrow |A_{3\times3}| = 0$

$\Rightarrow K(K-1)(K^2-1) = 0$

$$\Rightarrow k = 0, 1, -1$$

$\rightarrow$ For $k=0$, $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \rho(A) = 2$ $(\checkmark)$

$\rightarrow$ For $k=1$, $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = 1$ $(\times)$

$\rightarrow$ For $k=-1$, $A = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = 2$ $(\checkmark)$

(5) consider the system of linear eqns.

$$x+y+z=0$$
$$(\lambda+1)y + (\lambda+1)z = 0$$
$$(\lambda^2-1)z = 0.$$

For what values of λ the system has 2 LI solns.

A) 0 B) 1 C) -1 D) $1, -1$

6) consider the system of eqns.

$$2x - y + 3z = 2$$
$$x + y + 2z = 2$$
$$5x - y + az = b$$

Find the values of a & b for which the system of eqns has

i) no soln.

ii) unique soln.

iii) infinite solns. [ESE-2018]

The augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 1^{*} & 1 & 2 & 2 \\ 5^{*} & -1 & a & b \end{array}\right]$$

$R_2 \to 2R_2 - R_1$ and $R_3 \to 2R_3 - 5R_1$

$$[A|B] = \left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 3 & 1 & 2 \\ 0 & 3^{*} & 2a-15 & 2b-10 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$[A|B] = \left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 2a-16 & 2b-12 \end{array}\right]$$

$\longleftarrow A \longrightarrow \quad \longleftarrow B \longrightarrow$

i) No soln. For $a = 8$ and $b \neq 6$

$\rho(A) = 2$ and $\rho(A|B) = 3$

$\Rightarrow \rho(A) < \rho(A|B)$

ii) Unique soln. For $a \neq 8$, $b =$ any value ($b \in R$)

$\rho(A) = 3 = \rho(A|B) =$ no. of variables

iii) Infinite solns For $a = 8$ and $b = 6$

$\rho(A) = 2 = \rho(A|B) <$ no. of variables

7) consider the following system of linear eqns. (where p & q are constants)

$x_1 + x_2 + x_3 = 1$
$x_1 - x_2 + 2x_3 = p$
$3x_1 - x_2 + 5x_3 = q$

This system has atleast one soln. for any p & q satisfying

A) $2p - q + 1 = 0$ B) $2q + p + 1 = 0$

C) $2p + q - 1 = 0$ D) $2q + p - 1 = 0$

8) Consider the system of eqns.

$x + y + z = 6,$
$x + 4y + 6z = 20$
$x + 4y + \lambda z = \mu$

find the values of λ & μ such that the system of eqns. has

i) no soln. [GATE EC-2011]
ii) unique soln.
iii) infinite solns.

Ans i) $\lambda = 6$ & $\mu \neq 20$

ii) $\lambda \neq 6$ & $\mu =$ any value

iii) $\lambda = 6$ & $\mu = 20$

9) In the system of eqns $AX = B$,

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ -\sqrt{2} \end{bmatrix}$$

where A is an orthogonal matrix, the sum of unknowns, $x + y + z =$ ______

Soln.

$$AX = B$$

$$A^T A X = A^T B$$

$$I X = A B \quad \left\{ \because A \text{ is symmetric and orthogonal matrix} \right\}$$

$$X = AB$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore x+y+z = -1+1+1 = 1$

(10) If the system of eqns. $x+4ay+az=0$, $x+3by+bz=0$ & $x+2cy+cz=0$ has infinite no. of non-trivial solns. then a,b,c are in

A) AP B) GP C) HP D) $a+b+c\neq 0$

soln. $|A| = 0$

$$\begin{vmatrix} 1 & 4a & a \\ 1 & 3b & b \\ 1 & 2c & c \end{vmatrix} = 0$$

$$1(3bc-2bc) - 4a(c-b) + a(2c-3b) = 0$$

$$bc - 2ac + ab = 0$$

$$\frac{bc-2ac+ab}{abc} = 0$$

$$\frac{1}{a} - \frac{2}{b} + \frac{1}{c} = 0$$

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$$

(or)

$$\frac{2}{b} = \frac{a+c}{ac}$$

(or)

$$b = \frac{2ac}{a+c}$$

Note

1) a, b, c are in AP

$$b = \frac{a+c}{2}$$

2) a, b, c are in GP

$$b = \sqrt{ac}$$

3) a, b, c are in HP

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c} \text{ (or) } b = \frac{2ac}{a+c}$$

11) Let $M = \begin{bmatrix} \alpha & 1 & 1 \\ 1 & \beta & 1 \\ 1 & 1 & \gamma \end{bmatrix}$, $\alpha\beta\gamma = 1$, where α, β, γ are real numbers and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then $MX = 0$ has infinite no. of non-trivial solns. if $tr(M) =$ ______

Ans 3

Note Let $x_1, x_2, \ldots, x_n$ be the columns of A for non-homogeneous system of eqns.

$$AX = B \quad \text{where } A = \begin{array}{c} \left[\begin{array}{cccc} \vdots & \vdots & \vdots & \cdots & \vdots \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & x_3 & \quad x_n \end{array} \end{array}$$

If matrix B is the linear combination of columns of matrix A i.e.

$B = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$ then the

non-homogeneous system of eqns. $AX = B$ is consistent i.e. $\rho(A|B) = \rho(A)$

12) Let $AX = B$ be a system of three eqns. in three Variables x, y & z. If A has 3 LI columns and B is a linear combination of the columns of matrix A, then which of the following is true?

(A) The system has unique soln.

(B) The system has infinitely many solns.

c) The system has no soln.

D) The system $Ax = 0$ has non-zero soln.

Soln.
$$AX = B$$

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}_{3\times 3}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} - \\ - \\ - \end{bmatrix}$$

$x_1 \quad x_2 \quad x_3 \longrightarrow$ 3 LI columns.

Given: B is the linear combination of columns of A

i.e. $B = c_1x_1 + c_2x_2 + c_3x_3$

$AX = B$ is consistent

$\Rightarrow \rho(A|B) = \rho(A)$ —— (1)

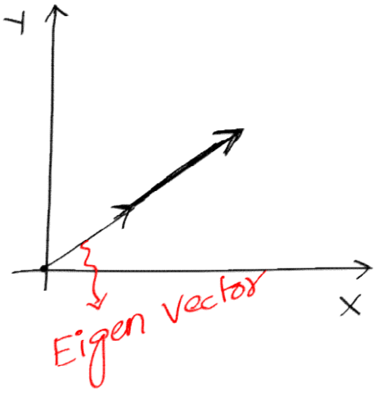
$\Rightarrow |A_{3\times3}| \neq 0$ ($\because A$ has 3 LI columns)

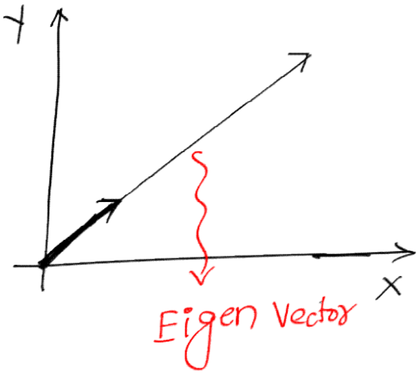
$\Rightarrow \rho(A) = 3$ {property (17), properties of rank}

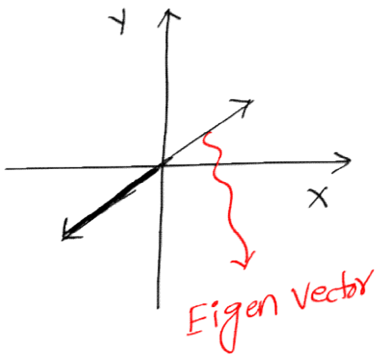
(1) $\Rightarrow \rho(A|B) = \rho(A) = 3 =$ no. of variables

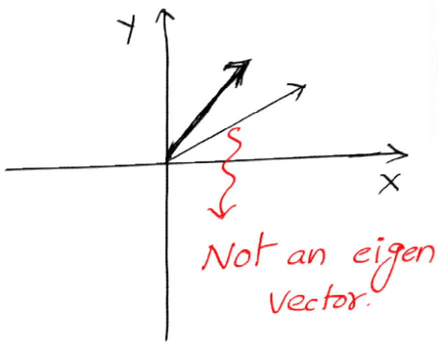
$\therefore$ $AX = B$ has unique soln. //

Eigen Values and Eigen Vectors



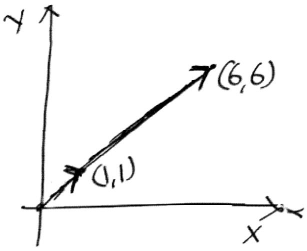






$$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4+2 \\ 2+4 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A \quad X = \lambda \quad X$$



Defn. Let A be an $n \times n$ matrix and λ is a scalar. If there exists a non-zero vector X such that $AX = \lambda X$ then λ is called an eigen value and X is called the eigen vector corresponding to the eigen value

29/10/2020

Note (1)

$$AX = \lambda X$$

$$AX - \lambda X = 0$$

$$AX - \lambda I X = 0$$

$$\boxed{(A - \lambda I)X = 0}$$ [Homogeneous system of eqns.]

$|A - \lambda I| \neq 0$ → Trivial soln. $[x_1 = x_2 = \cdots = x_n = 0]$ → Not an eigen vector

$|A - \lambda I| = 0$ → Infinite no. of non-trivial solns. → Infinite no. of eigen vectors

(2) Eigen vectors are non-zero solns. of homogeneous system of eqns. $(A - \lambda I)X = 0$

(3) The eqn. $|A - \lambda I| = 0$ is called the characteristic eqn. The roots of this characteristic eqn. are called eigen values (or) characteristic roots (or) latent roots (or) proper values.

(4) If eigen value is given then to find the corresponding eigen vector (or) vice-versa then the following formula should be used.

$$\{(A-\lambda I)X = 0\}$$

For example, Let $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

To find eigen values, $|A-\lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow (4-\lambda)^2 - 4 = 0$

$\Rightarrow \lambda^2 - 8\lambda + 16 - 4 = 0$ det(A)

$\Rightarrow \lambda^2 - 8\lambda + 12 = 0$ is the characteristic eqn. tr(A)

$\Rightarrow \lambda = 6, 2$ are eigen values of A.

Note 1) The characteristic eqn. of a 2x2 matrix is

$$\lambda^2-\lambda(tr(A))+\det(A)=0$$

2) The characteristic eqn. of a 3x3 matrix is

$$\lambda^3-\lambda^2[tr(A)]+\lambda[\underbrace{M_{11}+M_{22}+M_{33}}_{\text{sum of minors of diagonal elements}}]-\det(A)=0$$

3) The constant term in the characteristic eqn. is $\pm\det(A)$

Properties of Eigen Values

1) Sum of diagonal elements = sum of eigen values

i.e. $tr(A)=\lambda_1+\lambda_2+\cdots\cdots+\lambda_n$

2) Determinant of a matrix = product of eigen values

i.e. $\det(A)=\lambda_1\lambda_2\cdots\cdots\lambda_n$

3) Eigen values of A & A^T are Same

4) Eigen values of upper triangular matrix, lower triangular matrix, diagonal matrix, scalar matrix (or) identity matrix are its diagonal elements.

5) Eigen values of symmetric matrix are real

6) Eigen values of skew-symmetric matrix are either purely imaginary (or) zeros.

7) Eigen values of orthogonal matrix ($AA^T = A^TA = I$ (or) $A^T = A^{-1}$)

are of unit modulus.

8) Eigen values of odd order orthogonal matrix are ± 1

9) If λ is an eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is also its another eigen value.

10) The eigen values of Idempotent matrix ($A^2 = A$) are either 0 (or) 1 [ESE-2020]

11) The eigen values Involutary matrix ($A^2 = I$) are ± 1

12) All eigen values of Nilpotent matrix ($A^m = 0$) are zeros [GATE IN-2018]

13) If $a+ib$ is an eigen value of a matrix A then its complex conjugate $a-ib$ is also its another eigen value and vice-versa.

14) If λ is an eigen value of matrix A then

i) $k\lambda$ is an eigen value of kA

ii) $\frac{1}{\lambda}$ " " " A^{-1}

iii) λ^2 " " " A^2

iv) λ^m " " " A^m

v) $\frac{|A|}{\lambda}$ " " " $\text{adj}A$

vi) $(\lambda \pm k)^m$ " " $(A \pm kI)^m$

properties of Eigen Vectors

1) Eigen vectors of A & A^T are not same

2) Eigen vectors of A & A^m are same

3) Eigen vectors corresponding to distinct eigen values of a matrix are LI

$$\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \ldots\ldots \neq \lambda_m$$

$$\begin{array}{cccc} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_m \\ \downarrow & \downarrow & \downarrow & \downarrow \\ x_1 & x_2 & x_3 & x_m \end{array}$$

LI eigen vectors

4) If an eigen value is repeated 'm' times i.e. $\lambda_1 = \lambda_2 = \ldots\ldots = \lambda_m$ {algebraic multiplicity of an eigen value is 'm'} then

$$\boxed{\text{No. of LI eigen vectors} = n - \rho(A - \lambda I)}$$

Geometric multiplicity of an eigen value = {No. of variables (or) No. of columns} − {Rank of $(A-\lambda I)$}

5) The eigen vectors corresponding to distinct eigen values of a symmetric matrix are orthogonal

[GATE EC-2014]

6) If eigen values of a matrix are real then the corresponding eigen vectors may be real (or) complex.

7) If eigen values of a matrix are complex then the corresponding eigen vectors are complex.

Note 1) If A is a singular matrix i.e. $|A| = 0$ then atleast one eigen valued should be zero.

$$|A| = 0$$

$$\lambda_1 \cdot \lambda_2 \cdots \lambda_n = 0$$

$\Rightarrow$ Atleast one of $\lambda_1, \lambda_2, \ldots, \lambda_n$ is zero.

2) If a, b, c, d are integers such that $a+b = c+d$ then the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has integer

eigen values namely $\lambda_1 = a+b$ and $\lambda_2 = a-c$

For example, $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

Here: $a+b = c+d$

$\therefore \lambda_1 = 4+2 = 6$

$\lambda_2 = 4-2 = 2$

3) Given: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $b \neq 0$

If λ_1 and λ_2 are eigen values of A

then the corresponding eigen vectors are

$$X_1 = \begin{bmatrix} -b \\ a-\lambda_1 \end{bmatrix} \quad \& \quad X_2 = \begin{bmatrix} -b \\ a-\lambda_2 \end{bmatrix}$$

problems

1) The matrix $\begin{bmatrix} 2 & -4 \\ 4 & -2 \end{bmatrix}$ has [GATE CE-2018]

A) real eigen values and eigen vectors

B) real eigen values and complex eigen vectors

C) complex eigen values and real eigen vectors

D) complex eigen values and eigen vectors

Sol.

$$\lambda^2 - \lambda[\cancelto{0}{tr(A)}] + \cancelto{12}{det(A)} = 0$$

$$\lambda^2 + 12 = 0 \Rightarrow \lambda^2 = -12$$

$$\Rightarrow \lambda^2 = 12 i^2$$

$$\Rightarrow \lambda = \pm i\sqrt{12}$$

$$\lambda_1 = i\sqrt{12}, \; \lambda_2 = -i\sqrt{12}$$

The eigen vectors are

$$X_1 = \begin{bmatrix} -b \\ a-\lambda_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2-i\sqrt{12} \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -b \\ a-\lambda_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2+i\sqrt{12} \end{bmatrix}$$

2) Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

Soln.

$$A = \begin{bmatrix} 4 \xrightarrow{} a & 2 \xrightarrow{} b \\ 2 \xrightarrow{} c & 4 \xrightarrow{} d \end{bmatrix} \quad (\because a+b = c+d)$$

$$\lambda_1 = a+b = 6 \quad \& \quad \lambda_2 = a-c = 2$$

$$X_1 = \begin{bmatrix} -b \\ a-\lambda_1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} -2c \\ -2c \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -b \\ a-\lambda_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2c \\ 2c \end{bmatrix}$$

By property (5) (properties of eigen vectors), the eigen vectors are orthogonal i.e. $X_1^T X_2 = X_2^T X_1 = 0$.

$$X_1^T X_2 = [-2c \quad -2c]\begin{bmatrix} -2c \\ 2c \end{bmatrix}$$

$$= 4c^2 - 4c^2$$

$$= 0$$

$\therefore$ X_1 & X_2 are orthogonal eigen vectors.

3) find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 10 & -4 \\ 18 & -12 \end{bmatrix}$$

4) consider the matrix $M = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{bmatrix}$. one of the eigen vector of M is

A) $\begin{bmatrix}1\\-1\\1\end{bmatrix}$ B) $\begin{bmatrix}1\\1\\-1\end{bmatrix}$ C) $\begin{bmatrix}-1\\1\\1\end{bmatrix}$ ✓D) $\begin{bmatrix}1\\1\\1\end{bmatrix}$

[GATE IN-2020]

Soln. $MX = \lambda X$

option(A)

$$\begin{bmatrix}1 & -1 & 0\\1 & -2 & 1\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}1+1+0\\1+2+1\\0+1+1\end{bmatrix} = \begin{bmatrix}2\\4\\2\end{bmatrix} \neq 2\begin{bmatrix}1\\2\\1\end{bmatrix} \quad (MX \neq \lambda X)$$

option(B)

$$\begin{bmatrix}1 & -1 & 0\\1 & -2 & 1\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}1-1+0\\1-2-1\\0-1-1\end{bmatrix} = \begin{bmatrix}0\\-2\\-2\end{bmatrix} \neq -2\begin{bmatrix}0\\1\\1\end{bmatrix} \quad (MX \neq \lambda X)$$

option(C)

$$\begin{bmatrix}1 & -1 & 0\\1 & -2 & 1\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}-1-1+0\\-1-2+1\\0-1+1\end{bmatrix} = \begin{bmatrix}-2\\-2\\0\end{bmatrix} \neq -2\begin{bmatrix}1\\1\\0\end{bmatrix} \quad (MX \neq \lambda X)$$

option (D)

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-1+0 \\ 1-2+1 \\ 0-1+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (MX = \lambda X)$$

$\therefore$ The eigen vector is

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

5) Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. If -3 & -3 are two eigen values of A then the eigen vector corresponding to the third eigen value is

A) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ B) $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ C) $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ D) $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

Soln.

$$tr(A) = \lambda_1 + \lambda_2 + \lambda_3$$

$$(-2) + (1) + (0) = (-3) + (-3) + \lambda_3$$

$$\lambda_3 = 5$$

$$(A - \lambda I)X = 0$$

$$(A - 5I)X = 0 \quad (\because \lambda = 5)$$

Select any two different non-zero rows

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{cccccc} & x_1 & & x_2 & x_3 & \\ 2 & -3 & -7 & 2 \\ -4 & 6 & 2 & -4 \end{array}$$

$$\frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24} = c \text{ (say)}$$

$$x_1 = -24c, \quad x_2 = -48c, \quad x_3 = 24c$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -24c \\ -48c \\ 24c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Another Method

$$(A-\lambda I)X=0$$

$$(A-5I)X=0 \quad (\because \lambda=5)$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2^{*} & -4 & -6 \\ -1^{*} & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reducing to echelon form

$$R_2 \to 7R_2+2R_1$$

$$R_3 \to 7R_3-R_1$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -24 & -48 \\ 0 & -16^{*} & -32 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to 3R_3-2R_2$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -24 & -48 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x_1+2x_2-3x_3=0 \quad \text{—— (1)}$$

$$-24x_2-48x_3=0 \quad \text{—— (2)}$$

put $n - r = 3 - 2 = 1$ variable as arbitrary const.

put $x_3 = c$, $(2) \Rightarrow -24x_2 - 48c = 0$

$$\Rightarrow -\cancel{24}x_2 = \overset{2}{\cancel{48}}c$$

$$\Rightarrow x_2 = -2c$$

$$(1) \Rightarrow -7x_1 + 2(-2c) - 3c = 0$$

$$\Rightarrow -7x_1 - 4c - 3c = 0$$

$$\Rightarrow -7x_1 - 7c = 0$$

$$\Rightarrow -\cancel{7}x_1 = \cancel{7}c$$

$$\Rightarrow x_1 = -c$$

$\therefore$ The eigen vector is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -c \\ -2c \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} //$$

6) An eigen vector corresponding to the smallest eigen value of the matrix $\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is

A) $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ B) $\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$ C) $\begin{bmatrix}1\\0\\2\\0\end{bmatrix}$ D) $\begin{bmatrix}-1\\-1\\2\\2\end{bmatrix}$

7) For the matrix $P = \begin{bmatrix}3 & -2 & 2\\0 & -2 & 1\\0 & 0 & 1\end{bmatrix}$ one of the eigen value is -2. which of the following is an eigen vector?

A) $\begin{bmatrix}3\\-2\\1\end{bmatrix}$ B) $\begin{bmatrix}-3\\2\\-1\end{bmatrix}$ C) $\begin{bmatrix}1\\-2\\3\end{bmatrix}$ D) $\begin{bmatrix}2\\5\\0\end{bmatrix}$ [GATE EE-2005]

8) Let $A = \begin{bmatrix}1 & 0 & -1\\-1 & 2 & 0\\0 & 0 & -2\end{bmatrix}$ and $B = A^3 - A^2 - 4A + 5I$, where I is the 3×3 identity matrix. The determinant of B is ______ [GATE EE-2018]

Soln.

$$|A-\lambda I| = 0$$

$$\begin{vmatrix}1-\lambda & 0 & -1\\-1 & 2-\lambda & 0\\0 & 0 & -2-\lambda\end{vmatrix} = 0$$

$(1-\lambda)\{(2-\lambda)(-2-\lambda)\} = 0 \Rightarrow \lambda = 1, 2, -2$

If λ is an eigen value of A then

$\lambda^3 - \lambda^2 - 4\lambda + 5$ is an eigen value

of $B = A^3 - A^2 - 4A + 5I$

$$B = A^3 - A^2 - 4A + 5I$$

$$B = \lambda^3 - \lambda^2 - 4\lambda + 5$$

$$\begin{matrix} 1 \\ 2 \\ -2 \end{matrix} \rightarrow \lambda^3 - \lambda^2 - 4\lambda + 5 \rightarrow \begin{matrix} (1)^3 - (1)^2 - 4(1) + 5 = 1 \\ (2)^3 - (2)^2 - 4(2) + 5 = 1 \\ (-2)^3 - (-2)^2 - 4(-2) + 5 = 1 \end{matrix}$$

Eigen values of B are $1, 1, 1$

$tr(B) = (1) + (1) + (1) = 3$

$det(B) = (1)(1)(1) = 1$

9) If $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ then the sum and product of eigen values of the matrix $B = A^2 - 4A^{-1}$

Ans $tr(B) = 17$

$det(B) =$

10) Let $A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}$. Find i) $tr(A^{1000})$

ii) $\det(A^{1000})$

Ans i) $2^{1000}+1$ ii) 2^{1000}

11) consider the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$ whose eigen values are 1, −1 and 3. The value of $tr(A^3-3A^2)$ is ——

[GATE IN–2016] Ans −6

12) If the characteristic polynomial of a 3×3 matrix M over R (The set of real numbers) is $\lambda^3-4\lambda^2+a\lambda+30$, $a \in R$, and one of the eigen value of M is 2, then the largest among the absolute values of eigen values of M is ——

[GATE CS–2017]

Soln: The characteristic eqn. of a 3×3 matrix is

$$\lambda^3 - \lambda^2[tr(M)] + \lambda[M_{11}+M_{22}+M_{33}] - \det(A) = 0 \quad \text{—— (1)}$$

$$\lambda^3 - 4\lambda^2 + a\lambda + 30 = 0 \text{ —— } (2)$$

from (1) & (2),

$tr(M) = 4$	$-det(M) = 30$
$\lambda_1 + \lambda_2 + \lambda_3 = 4$	$det(M) = -30$
$2 + \lambda_2 + \lambda_3 = 4$	$\lambda_1\lambda_2\lambda_3 = -30$
$\lambda_2 + \lambda_3 = 2$ —(3)	$2\lambda_2\lambda_3 = -30$
	$\lambda_2\lambda_3 = -15$ — (4)

$$\lambda_2 = 5,\ \lambda_3 = -3 \text{ (or) } \lambda_2 = -3,\ \lambda_3 = 5$$

Eigen values of M are $5, -3, 2$

$\therefore$ Largest absolute eigen value $= 5$

13) one of the eigen value of a 3×3 matrix M is 3. If $tr(M) = 9$ and $det(M) = 24$, then the smallest eigen value of M^{-1} is ——

Ans $\frac{1}{4} = 0.25$

14) Let 'p' be a 2×2 complex matrix such that $tr(p) = 1$ & $det(p) = -6$. Then $tr(p^4 - p^3)$ is —— Ans 78

15) Find the no. of LI eigen vectors of the following

i) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ ii) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ [GATE ME-2016]

iii) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ iv) $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (Ans: 1)

Soln. i) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

Eigen values $\lambda = 1, 4, 6$

It has 3 LI eigen vectors ($\because$ A has 3 distinct eigen values)

ii) Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ ($\because$ A is upper Δ^{lar} matrix)

$\lambda = 2, 2, 3 \rightarrow$ 1 LI eigen vector

For $\lambda = 2$, $A - \lambda I = A - 2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 \leftrightarrow R_3$

$$A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A - \lambda I) = 2$

No. of LI eigen vectors $= n - \rho(A - \lambda I)$

$= 3 - 2$

$= 1$

$\therefore$ Total no. of LI eigen vectors $= 2$

iii) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$\lambda = 1, 1, 1$

For $\lambda = 1$, $A - \lambda I = A - 1I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\rho(A - \lambda I) = 2$

$\therefore$ No. of LI eigen vectors $= n - \rho(A - \lambda I)$

$= 3 - 2 = 1$

16) If A is a skew-symmetric matrix of order 'n', then the no. of LI eigen vectors of $(A+A^T)$ is

A) 0 B) 1 C) $n-1$ D) n

Soln. Given: A is a skew-symmetric matrix of order 'n'.

$$A^T = -A$$

$$A^T + A = 0 \rightsquigarrow (1)$$

No. of LI eigen vectors of $A = n - \rho(A - \lambda I)$

No. of LI eigen vectors of $(A+A^T) = n - \rho\{(A+A^T) - \lambda I\}$

$= n - \rho\{0 - \lambda I\}$ { from (1)}

$= n - \rho(0)$ $\{\because \lambda = 0$ i.e. λ is an eigen value of $(A+A^T)\}$

$= n$ $\{\because \rho(0) = 0\}$

17) A 4×4 matrix [P] is given below

$$[P] = \begin{bmatrix} 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

The eigen values of [P] are

A) 0, 3, 6, 6 B) 1, 2, 3, 4

C) 3, 4, 5, 7 ✓D) 1, 2, 5, 7 [GATE CE-2020]

Soln. $|P-\lambda I| = 0$

$$\begin{vmatrix} -\lambda & 1 & 3 & 0 \\ -2 & 3-\lambda & 0 & 4 \\ 0 & 0 & 6-\lambda & 1 \\ 0 & 0 & 1 & 6-\lambda \end{vmatrix} = 0$$

$C_4 \to C_4 - (6-\lambda)C_3$

$$\begin{vmatrix} -\lambda & 1 & 3 & -3(6-\lambda) \\ -2 & 3-\lambda & 0 & 4 \\ 0 & 0 & 6-\lambda & 1-(6-\lambda)^2 \\ 0 & 0 & 1 & 0 \end{vmatrix} = 0$$

$- \quad + \quad - \quad +$

x cut x cut

$$(-1)\begin{vmatrix} -\lambda & 1 & -3(6-\lambda) \\ -2 & 3-\lambda & 4 \\ 0 & 0 & 1-(6-\lambda)^2 \end{vmatrix} = 0$$

$$-\lambda\{(3-\lambda)[1-(6-\lambda)^2]\} - \{-2[1-(6-\lambda)^2]\} = 0$$

$$\{1^2-(6-\lambda)^2\}\{\lambda^2-3\lambda+2\} = 0$$

$$\{1+(6-\lambda)\}\{1-(6-\lambda)\}\{(\lambda-1)(\lambda-2)\} = 0$$

$$(7-\lambda)(-5+\lambda)(\lambda-1)(\lambda-2)$$

$$\lambda = 1, 2, 5, 7$$

30/10/2020

18) The no. of distinct eigen values of

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n\times n}$$ are

A) 0, n B) 1, n-1 C) 0, n-1 D) 1, n

Soln. (n-1) eigen values are zeros

$\lambda = 0, 0, \cdots, 0$ {(n-1) times}

$tr(A) = \lambda_1 + \lambda_n + \cdots + \lambda_{n-1} + \lambda_n$

$1+1+\cdots+1 = 0+0+\cdots+0+\lambda_n$

[n times]

$\lambda_n = n$

$\therefore$ No. of distinct eigen values = 2 {0 & n}

19) If $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ is an eigen vector of the matrix

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$ then the corresponding eigen value is ______

Soln.

$$(A-\lambda I)X = 0$$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(6-\lambda)-4+0=0$$

$$2-\lambda=0$$

$$\therefore \lambda = 2$$

20) The product of non-zero eigen values of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$ is ______

[GATE CS-2014]

$\boxed{\text{Ans: 6}}$

Cayley - Hamilton Theorem

Statement Every square matrix satisfies its own characteristic eqn.

Using Cayley-Hamilton theorem, we can find

i) inverse of a matrix

ii) higher powers of a matrix

1) If 1, 2 and 3 are eigen values of a 3×3 matrix 'A' then $6A^{-1} =$

A) $A^2+6A-11I$ B) $A^2-6A-11I$

C) $A^2-6A+11I$ D) $A^2+6A+11I$

Soln. $\lambda = 1, 2, 3$ are eigen values

$$(\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$$(\lambda-1)\{\lambda^2-5\lambda+6\} = 0$$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$ is the characteristic eqn.

By Cayley-Hamilton theorem, we have

$$A^3 - 6A^2 + 11A - 6I = 0$$

$$A^{-1}[A^3 - 6A^2 + 11A - 6I] = A^{-1}\cdot 0$$

$$A^2 - 6A + 11I - 6A^{-1} = 0$$

$$\boxed{6A^{-1} = A^2 - 6A + 11I}$$

$$A^{-1} = \frac{1}{6}(A^2 - 6A + 11I)$$

2) Let M be a 3×3 matrix and suppose that 1, 2, 3 are the eigen values of M. If $M^{-1} = \frac{M^2}{\alpha} - M + \frac{11}{\alpha} I_3$ for some scalar $\alpha \neq 0$, then $\alpha =$ ______

Ans 6

3) If $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & -1 \end{bmatrix}$ and $6P^{-1} = aI_3 + bP - P^2$ then the ordered pair (a,b) is

A) (3, 2) B) (2, 3) C) (4, 5) D) (5, 4)

4) Let A be a 3×3 singular matrix such that $AX = X$ for non-zero vector X and

$$A\begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}2/5\\0\\-2/5\end{bmatrix}. \text{ Then}$$

A) $A^3 = \frac{1}{5}(7A^2 - 2A)$ B) $A^3 = \frac{1}{4}(7A^2 - 2A)$

C) $A^3 = \frac{1}{3}(7A^2 - 2A)$ D) $A^3 = \frac{1}{2}(7A^2 - 2A)$

Soln. $\lambda = 0, 1, \frac{2}{5}$ are eigen values

$\Rightarrow \lambda(\lambda - 1)(\lambda - \frac{2}{5}) = 0$

$\Rightarrow \lambda\{\lambda^2 - \frac{2}{5}\lambda - \lambda + \frac{2}{5}\} = 0$

$\Rightarrow \lambda\{\lambda^2 - \frac{7}{5}\lambda + \frac{2}{5}\} = 0$

$\Rightarrow \lambda^3 - \frac{7}{5}\lambda^2 + \frac{2}{5}\lambda = 0$ is the characteristic eqn.

By Cayley-Hamilton theorem, we have

$$A^3 - \frac{7}{5}A^2 + \frac{2}{5}A = 0$$

$$A^3 = \frac{7}{5}A^2 - \frac{2}{5}A$$

$$A^3 = \frac{1}{5}(7A^2 - 2A)$$

5) Given that $A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

the value of A^3 is [GATE EC,EE,IN-2012]

A) $15A + 12I$ B) $19A + 30I$ ✓

C) $17A + 15I$ D) $17A + 21I$

6) For the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ the expression $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ is equivalent to

A) $A^2 + A + 5I$ B) $A + 5I$ ✓

C) $A^2 + 5I$ D) $A^2 + 2A + 6I$

[ESE-2020 prelims]

Soln.

$$\lambda^2 - \lambda(tr(A)) + det(A) = 0$$

$\lambda^2 - 4\lambda - 5 = 0$ is the characteristic eqn.

By Cayley-Hamilton theorem, we have

$$A^2 - 4A - 5I = 0 \quad \text{——} (1)$$

$$A^2 = 4A + 5I \quad \text{——} (2)$$

$$A^3 = 4A^2 + 5A$$

$$A^3 = 4(4A+5I) + 5A \quad \{\text{From eqn. } (2)\}$$

$$A^3 = 21A + 20I \quad \text{——} (3)$$

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A^3(A^2 - 4A - 7I) + 11A^2 - A - 10I$$

$$= A^3\{\underbrace{A^2 - 4A - 5I}_{0} - 2I\} + 11A^2 - A - 10I$$

$$= A^3(-2I) + 11A^2 - A - 10I \quad \{\text{From } (1)\}$$

$$= -2I\{21A + 20I\} + 11\{4A + 5I\} - A - 10I$$

$$\{\text{From } (2) \,\&\, (3)\}$$

$$= -42A - 40I + 44A + 55I - A - 10I$$

$$= A + 5I$$

7) Given that $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Evaluate $A^3 - 6A^2 + 11A - 10I$

A) Null matrix B) Identity matrix

C) $-4I$ D) None of the above

[ISRO EC-2009]

Vector space

Vector space: Let V be a non-empty set of certain objects which may be matrices, vectors, functions (or some other objects. The set V defines a vector space if for any elements a, b, c in V ($a, b, c \in V$) and any scalars α, β the following properties (axioms) are satisfied.

Axioms w.r.t. Vector addition

i) If $a, b \in V$ then $a+b \in V$

ii) $a+b = b+a$ [commulative law]

iii) $(a+b)+c = a+(b+c)$ [Associative law]

iv) $a+0 = 0+a = a$ [Existence of unique zero element in V]

v) $a+(-a) = 0$ [Existence of additive inverse]

Axioms w.r.t. Scalar Multiplication

vi) If $a \in V$ then $\alpha a \in V$

vii) $(\alpha+\beta)a = \alpha a+\beta a$ [Left distributive law]

viii) $(\alpha\beta)a = \alpha(\beta a)$

ix) $\alpha(a+b) = \alpha a+\alpha b$ [Right distributive law]

x) $1a = a$ [Existence of multiplicative inverse]

Note The above vector space V is also denoted by V(F). If F = R then the vector space is called a real vector space. If F = C then the vector space is called a complex vector space.

Subspace A non-empty subset W of a vector space V is called a subspace of V if W itself is a vector under the same operations of vector addition and scalar multiplication on V.

Note If W is a non-empty subset of vector space V, then W is a subspace of V if and only if the following axioms hold:

i) If $u, v \in W$ then $u+v \in W$

ii) If α is any scalar and $u \in W$ then $\alpha u \in W$

problems

1) Verify whether $W = \{(x, y) \mid x = 3y\}$ is a subspace of R^2

Soln. I) Let $u, v \in W$

$$\text{Let } u = \{(x_1, y_1) \mid x_1 = 3y_1\}$$

$$v = \{(x_2, y_2) \mid x_2 = 3y_2\}$$

$$u + v = (x_1, y_1) + (x_2, y_2) \quad \text{where } x_1 = 3y_1 \text{ \& } x_2 = 3y_2$$

$$u + v = (x_1 + x_2, y_1 + y_2)$$

$$x_1 + x_2 = 3y_1 + 3y_2$$

$$\therefore x_1 + x_2 = 3(y_1 + y_2)$$

$$u + v = \{(x_1 + x_2, y_1 + y_2) \mid x_1 + x_2 = 3y_1 + 3y_2\}$$

$$u + v = \{(x_1 + x_2, y_1 + y_2) \mid x_1 + x_2 = 3(y_1 + y_2)\}$$

$$u + v \in W$$

II) Let $u \in W$ and α is a scalar

$$u = \{(x_1, y_1) \mid x_1 = 3y_1\}$$

$$u = (x_1, y_1) \quad \text{where} \quad x_1 = 3y_1$$

$$\alpha u = \alpha(x_1, y_1)$$

$$= (\alpha x_1, \alpha y_1)$$

$$\therefore \alpha x_1 = \alpha(3y_1) = 3(\alpha y_1)$$

$$\alpha u = \{(\alpha x_1, \alpha y_1) \mid \alpha x_1 = 3(\alpha y_1)\}$$

$$\alpha u \in \omega$$

$\therefore$ ω is a subspace of R^2

2) Check whether the following is a subspace of R^3

$$W = \{(x, y, z) \mid y = x + z + 1\}$$

soln. Let $u, v \in \omega$

Let $u = \{(x_1, y_1, z_1) \mid y_1 = x_1 + z_1 + 1\}$

$v = \{(x_2, y_2, z_2 \mid y_2 = x_2 + z_2 + 1\}$

$u + v = (x_1, y_1, z_1) + (x_2, y_2, z_2)$ where $y_1 = x_1 + z_1 + 1$ &
$y_2 = x_2 + z_2 + 1$

$u + v = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$$y_1 + y_2 = (x_1 + z_1 + 1) + (x_2 + z_2 + 1)$$

$$y_1 + y_2 = (x_1 + x_2) + (z_1 + z_2) + 2$$

$$u + v = \{(x_1 + x_2, y_1 + y_2, z_1 + z_2) \mid y_1 + y_2 = (x_1 + x_2) + (z_1 + z_2) + 1\}$$

$$u + v \notin W$$

$\therefore W$ is not a subspace of R^3

3) Check whether the following is a subspace of R^3

$$W = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$$

Ans W is not a subspace of R^3

Span

Span The set of all the vectors that are the linear combination of the vectors in the set $S = \{V_1, V_2, \dots, V_r\}$ is called span of S and is denoted by

span S (or) span $\{V_1, V_2, \dots, V_r\}$

Vectors Spanning Vector space

The method to check if the vectors $V_1, V_2, \dots, V_r$ span the vector space V is as follows:

1) choose any arbitrary vector b
2) Express b as linear combination of $V_1, V_2, \dots, V_r$

   i.e. $b = K_1 V_1 + K_2 V_2 + \dots + K_r V_r$ —— (1)
3) If the system of eqns. in (1) is consistent for all choices of b then the vectors $V_1, V_2, \dots, V_r$ span V
4) If the system of eqns. in (1) is inconsistent for some choices of b then the vectors do not span V

Note

1) If the coeficient matrix A of eqn. (1) is non-singular i.e. $|A| \neq 0$ then the system of eqns. in (1) is consistent for all choices of b and hence the given vectors span V

2) If $|A| = 0$ then the system of eqns. in (1) is inconsistent for some choices of b and hence given vectors do not span V.

problems

1) Determine whether the following vectors span the vector space R^3

(i) $v_1 = (2,2,2)$, $v_2 = (0,0,3)$, $v_3 = (0,1,1)$

ii) $v_1 = (2,1,0)$, $v_2 = (1,-1,2)$, $v_3 = (0,3,-4)$

iii) $v_1 = (3,1,4)$, $v_2 = (2,-3,5)$, $v_3 = (5,-2,9)$

$v_4 = (1,4,-1)$

Soln. i) Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix}$

$$|A| = 2(0-3) = -6 \neq 0$$

$\therefore$ The given vectors V_1, V_2, V_3 span R^3

ii) Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 0 & 2 & -4 \end{bmatrix}$

$$|A| = 2(4-6) - 1(-4-0)$$
$$|A| = -4+4 = 0$$

$\therefore$ The given vectors do not span R^3

iii) Let $b = (b_1, b_2, b_3)$ be any arbitrary vector

$$b = k_1 V_1 + k_2 V_2 + k_3 V_3 + k_4 V_4$$

$$(b_1, b_2, b_3) = k_1(3,1,4) + k_2(2,-3,5) + k_3(5,-2,9) + k_4(1,4,-1)$$

$$(b_1, b_2, b_3) = (3k_1 + 2k_2 + 5k_3 + k_4,\ k_1 - 3k_2 - 2k_3 + 4k_4,\ 4k_1 + 5k_2 + 9k_3 - k_4)$$

$$3k_1 + 2k_2 + 5k_3 + k_4 = b_1$$
$$k_1 - 3k_2 - 2k_3 + 4k_4 = b_2$$
$$4k_1 + 5k_2 + 9k_3 - k_4 = b_3$$

$$[A|B] = \left[\begin{array}{cccc|c} 3 & 2 & 5 & 1 & b_1 \\ 1^* & -3 & -2 & 4 & b_2 \\ 4^* & 5 & 9 & -1 & b_3 \end{array}\right]$$

$33b_3 - 44\cancel{b_1}$
$-21\cancel{b_2} + \cancel{7}b_1$

$3R_2 - R_1$ and $3R_3 - 4R_1$

$$[A|B] = \left[\begin{array}{cccc|c} 3 & 2 & 5 & 1 & b_1 \\ 0 & -11 & -11 & 11 & 3b_2 - b_1 \\ 0 & 7^* & 7 & -7 & 3b_3 - 4b_1 \end{array}\right]$$

$$R_3 \to 11R_3 + 7R_2$$

$$[A|B] = \left[\begin{array}{cccc|c} 3 & 2 & 5 & 1 & b_1 \\ 0 & -11 & -11 & 11 & 3b_2 - b_1 \\ 0 & 0 & 0 & 0 & -37b_1 - 21b_2 + 33b_3 \end{array}\right]$$

$\longleftarrow A \longrightarrow \quad \longleftarrow B \longrightarrow$

$\rho(A) = 2$, $\rho(A|B) = 3$ for some choices of b

$\rho(A|B) \neq \rho(A)$

It is inconsistent

$\therefore$ The given vectors do not span R^3

2) Determine whether the following vectors span the vector space R^3 & R^4

NO i) $V_1 = (1, 2, 1, 0)$, $V_2 = (1, 1, -1, 0)$, $V_3 = (0, 0, 0, 1)$

YES ii) $V_1 = (1, 1, 0, 0)$, $V_2 = (1, 2, -1, 1)$, $V_3 = (0, 0, 1, 1)$

$V_4 = (2, 1, 2, 1)$

YES iii) $V_1 = (1, 2, 6)$, $V_2 = (3, 4, 1)$, $V_3 = (4, 3, 1)$, $V_4 = (3, 3, 1)$

NO iv) $V_1 = (1, 2, 5)$, $V_2 = (1, 3, 7)$, $V_3 = (1, -1, -1)$

Basis

Defn. The set of vectors $S = \{V_1, V_2, \ldots, V_n\}$ in a vector space V is called a basis for V if

i) S span V

ii) S is linearly independent

Note: To show that the set of vectors S to be a basis of a vector space V it is sufficient to show that the det. of coefficient matrix obtained from $b = k_1 V_1 + k_2 V_2 + \cdots + k_n V_n$ is non-zero.

→ If the det. of coefficient matrix is zero, then S does not span V and hence S is not a basis of V.

Dimension

Defn. The no. of vectors in a basis of a non-zero finite dimensional vector space V is known as the dimension of V and is denoted by dim(V)

Basis and Dimension For Soln. space of the Homogeneous system of Eqns. $AX=0$

Let $AX=0$ be the homogeneous system of eqns.

1) Solve the homogeneous system of eqns using row echelon form method. If $AX = 0$ has trivial soln. then the soln. space is $\{0\}$ which has no basis and hence the dimension of the soln. space is zero.

2) If the soln. vector x contains arbitrary constants $c_1, c_2, ---, c_p$. then express x as linear combination of $x_1, x_2, ---, x_p$. with $c_1, c_2, -----, c_p$ as coefficients.

$$\text{i.e. } x = c_1 x_1 + c_2 x_2 + \cdots\cdots + c_p x_p.$$

3) The set of vector $\{x_1, x_2, -----, x_p\}$ form a basis for the soln. space $AX = 0$ and hence dimension of the soln. space is p

problems

1) The set $\{(1,2,2), (2,1,2), (2,2,1)\}$ of vectors is

A) LI and a basis of R^3

B) LD and a basis of R^3

C) LI and not a basis of R^3

D) LD and not a basis of R^3

Soln. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$|A| = (1-4) - 2(2-4) + 2(4-2) = 5$

$|A| \neq 0$

$\therefore$ The vectors form a basis of R^3

Dimension $= 3$

2) If the set $\{(K,1,1), (0,1,1), (K,0,K)\}$ of vectors

form a basis of R^3 then k is ——

Ans $\quad k \neq 0$

3) Determine whether the following vectors form a basis for R^3

i) $(1,1,1)$, $(1,2,3)$, $(2,-1,1)$

ii) $(1,1,2)$, $(1,2,5)$, $(5,3,4)$

Ans i) Forms a basis for R^3 and dimension $= 3$

ii) Does not form a basis for R^3

4) Consider the set of [column] vectors defined by

$X = \{ x \in R^3 \mid x_1 + x_2 + x_3 = 0 \text{ where } x^T = [x_1 \ x_2 \ x_3]^T \}$

which of the following is TRUE?

A) $\{ [1,-1,0]^T, [1,0,-1]^T$ is a basis for the subspace $X \}$

B) $\{ [1,-1,0]^T, [1,0,-1]^T$ is LI set but it does not span X and therefore is not a basis of $X \}$

c) X is a not a subspace of R^3

D) None of the above [GATE CS-2007]

soln. $x_1 + x_2 + x_3 = 0$

$$[1 \quad 1 \quad 1]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (Ax = 0)$$

$\rho(A) = 1 <$ no. of variables(3)

It has infinite no. of non-trivial solns.

It forms a basis.

$x_1 + x_2 + x_3 = 0$

put $x_2 = c_1$ & $x_3 = c_2$

$x_1 + x_2 + x_3 = 0$

$x_1 = -x_2 - x_3$

$x_1 = -c_1 - c_2$

$$\therefore \text{The soln. is } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -c_1 - c_2 \\ c_1 \\ c_2 \end{bmatrix} = -c_1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - c_2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$x = -c_1 x_1 - c_2 x_2 = k_1 x_1 + k_2 x_2$$

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\} = \{x_1, x_2\}$$

Dimension $= 2$

5) Determine the dimension and basis for the subspace of R^3

$X = \{(x, y, z) \mid 3x - 2y + 5z = 0$, where $X^T = [x, y, z]\}$

Ans Forms a basis

$$\text{Basis} = \left\{ \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{5}{3} \\ 0 \\ 1 \end{bmatrix} \right\}$$

Dimension = 2

6) Determine the dimension and basis for the soln. space of the system

$x_1 + x_2 - 2x_3 = 0, \; -2x_1 - 2x_2 + 4x_3 = 0,$

$-x_1 - x_2 + 2x_3 = 0$

Ans Forms a basis

$$\text{Basis} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Dimension = 2

7) If $V_1, V_2, V_3, \ldots, V_6$ are six vectors in R^4. which one of the following statements is FALSE ?

A) If $\{V_1, V_3, V_5, V_6\}$ spans R^4 then it forms a basis for R^4

B) These vectors are not LI

C) It is necessary that these vectors span R^4

D) any four of these vectors forms a basis for R^4 ✓

[GATE EC-2020]

SYLLABUS COMPLETED