The recommended questions you should solve for each week are:

- Tutorial 1: Q. 1 to Q. 11.
- Tutorial 2: Q. 13 and Q. 14.
- Tutorial 3: Q. 15 and Q. 18.
- Tutorial 4: Q. 26 and Q. 31.
- Tutorial 5: Q. 27, Q. 28, and Q. 33.
- Tutorial 6: Q. 35 and Q. 46.
- Tutorial 9: Q. 52, Q. 53, and Q. 57.
- Tutorial 10: Q. 56, Q. 58, and Q. 60.
- Tutorial 11: Q. 65, Q. 67, and Q. 68.

All other questions are provided for additional practice and should help you to explore all aspects of the course.
Fully worked solutions are given, but you should attempt the problems without the solutions, its the only way to know what you don't know!
Where marks are given, these are indicative of the *relative* weighting each part of a question might have. Please note, the number of questions in an exam (and exam durations) have changed over the years, so the overall marks for a question may now be different to what is reported here.
All past exam questions are collected in this document.

Questions

Q.1 Question 1

Your house is 18°C inside when it is 4°C outside. If your walls are 20 cm thick and have a thermal conductivity of 0.03 W m^{-1} K^{-1}, calculate the heat lost per unit area of wall.
Notes: The heat transfer rate per unit area of wall, q (W m^{-2}), is given by:

$$q = U\,\Delta T$$

where U (W m^{-2} K^{-1}) is the heat transfer coefficient, and ΔT is the driving temperature difference. For solid, rectangular walls $U = k/L$, where k is the thermal conductivity and L is the wall thickness.
Solution:
For conduction problems, we have

$$\begin{aligned} q &= \frac{k}{L}(T_i - T_o) \\ &= \frac{0.03}{0.2}(18 - 4) = 2.1\ \mathrm{W\ m^{-2}} \end{aligned}$$

[Question end]

Q.2 Question 2
Model your house as a box 10 m×10 m×10 m and calculate the heat transfer from its side walls, is this estimate reasonable? What natural effects are missing from this model?
Notes: For simple heat transfer, the total heat transfer Q (W) is given by:

$$Q = q\,A = U\,A\,\Delta T \tag{1}$$

Solution:
There are four sides to the house, each 100 m^2; therefore, we have

$$\begin{aligned} Q &= q\,A \\ &= 400 \times 2.1 = 840\ \mathrm{W} \end{aligned}$$

This question is to get you thinking about modes of heat transfer, and remind you that you already know quite a bit about it. We are missing effects from the roof, windows/doors, and ventilation. We're also missing convection and radiation, not to mention the layered nature of each wall. This heat loss estimate is a little low, but it is sufficient to obtain an order of magnitude estimate.

[Question end]

Q.3 Question 3
What is the pressure at the bottom of the Mariana Trench (the deepest part of the ocean)?
Note: Its depth is 10.911 km and you may assume the density of water is roughly constant at ρ = 1000 kg m^{-3}.
Bernoulli's equation is

$$\frac{1}{2}\rho_1\,v_1^2 + p_1 + \rho_1\,g\,h_1 = \frac{1}{2}\rho_2\,v_2^2 + p_2 + \rho_2\,g\,h_2 \tag{2}$$

Solution:
Assuming ocean water is stationary $v_1 = v_2 = 0$, and the surface $h_1 = 0$ is at atmospheric pressure $p_1 = 1$ atm=1.013 bar, and $g = 9.81$ m s^{-2} we have:

$$\begin{aligned} \frac{1}{2}\rho_1\,\cancelto{0}{v_1^2} + p_1 + \rho_1\,g\,\cancelto{0}{h_1} &= \frac{1}{2}\rho_2\,\cancelto{0}{v_2^2} + p_2 + \rho_2\,g\,h_2 \\ 1.013 \times 10^5 + 1000 \times 9.81 \times 10.911 \times 10^3 &= p_2 \\ p_2 &\approx 1071\ \mathrm{bar} \end{aligned}$$

[Question end]

Q.4 Question 4
Assuming that blood has a density of 1060 kg m^{-3}, what is the maximum height your heart can lift your blood, given that a typical driving pressure of the heart is 100 mmHg (0.13 bar)?
Note: You can use Bernoulli's equation and as you're looking for the maximum height, you may treat the blood as stationary at both ends.

Solution:
Assuming the blood is stationary ($v_1 = v_2 = 0$) at both ends we have:

$$\frac{1}{2}\rho_1 \cancelto{0}{v_1^2} + p_1 - \rho_1\, g\, h_1 = \frac{1}{2}\rho_2 \cancelto{0}{v_2^2} + p_2 - \rho_2\, g\, h_2$$
$$\frac{p_1 - p_2}{\rho\, g} = h_1 - h_2$$
$$\frac{0.13 \times 10^5}{1060 \times 9.81} \approx 1.25m$$

Although this seems small considering we're assuming the flow is stationary (it is less than the average height in the UK), your heart only has to pump blood upwards from your chest to your head. Blood within your extremities (arms and legs) is pumped outward by the heart and is returned in part by the action of your skeletal muscles around your veins (look up skeletal-muscle pumps for more information). Giraffes have twice the blood pressure of humans.

[Question end]

Q.5 Question 5
Write the following expressions in index notation, and state whether the answer is a scalar, vector, or matrix.

	Solution
a) $\boldsymbol{a} + \boldsymbol{b}$	$a_i + b_i$ (vector)
b) $\boldsymbol{ab}$	$a_i\, b_j$ (matrix)
c) $\boldsymbol{c} \cdot \boldsymbol{ab}$	$c_i\, a_i\, b_j$ (vector)
d) $\boldsymbol{a} \cdot \boldsymbol{A}$	$a_i\, A_{ij}$ (vector)
e) $\boldsymbol{A} \cdot \boldsymbol{b}$	$A_{ij}\, b_j$ (vector)
f) $\boldsymbol{a}^2$	$a_i\, a_i$ (scalar)
g) $\boldsymbol{A}^2 \cdot \boldsymbol{b}$	$A_{ij}\, A_{jk} b_k$ (vector)
h) $\boldsymbol{abc} \cdot \boldsymbol{A} \cdot \boldsymbol{d}$	$a_i\, b_j\, c_k\, A_{kl}\, d_l$ (matrix)
i) $\nabla \cdot \boldsymbol{bc}$	$\partial(b_i\, c_j)/\partial r_i$ (vector)

[Question end]

Q.6 Question 6
Given $\boldsymbol{a} = [1, 2, 3]$ and $\boldsymbol{b} = [4, 5, 6]$, calculate the following

	Solution
a) $\boldsymbol{a} + \boldsymbol{b}$	$[5, 7, 9]$
b) $4\,\boldsymbol{a}$	$[4, 8, 12]$
c) $\boldsymbol{a} \cdot \boldsymbol{b}$	$1 \times 4 + 2 \times 5 + 3 \times 6 = 32$
d) $\boldsymbol{a}^2$	$1^2 + 2^2 + 3^2 = 14$

e) $\nabla \cdot \boldsymbol{b}$ As all elements of $\boldsymbol{b}$ are constant, its 0

f) $\nabla \boldsymbol{b}$ As all elements of $\boldsymbol{b}$ are constant, its $\begin{bmatrix} 0, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 0 \end{bmatrix}$

[Question end]

Q.7 Question 7

Write the following expressions in vector notation.

Solution

a) $a_l\, b_j$ — $\boldsymbol{ab}$

b) $a_k\, b_k$ — $\boldsymbol{a} \cdot \boldsymbol{b}$

c) $b_j\, A_{ij}\, a_i$ — $\boldsymbol{a} \cdot \boldsymbol{A} \cdot \boldsymbol{b}$

d) $a_i\, b_j\, c_k$ — $\boldsymbol{abc}$

e) $a_i\, b_j\, a_i$ — $\boldsymbol{a}^2\boldsymbol{b}$

f) $a_i\, (\partial b_j / \partial r_i)$ — $\boldsymbol{a} \cdot \nabla \boldsymbol{b}$

[Question end]

Q.8 Question 8

The del operator ($\nabla = \partial/\partial r_i$) is a "vector" version of the derivative. Like the normal derivative operation, it has a product rule. Prove the following identity:

$$\nabla \cdot \boldsymbol{ab} = \boldsymbol{b}\nabla \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \nabla \boldsymbol{b}$$

Hint: Use index notation, treat a_i and b_i as functions of x, y, z, and use the normal product rule!

Solution:

Working in Cartesian coordinates (x,y,z) we can use index notation,

$$\nabla \cdot \boldsymbol{ab} = \nabla_i\, a_i\, b_j$$

We can expand the del operator into index notation $\nabla_i = \partial/\partial r_i$. This gives

$$\nabla \cdot \boldsymbol{ab} = \frac{\partial a_i\, b_j}{\partial r_i}$$

We can use the normal product rule, as it doesn't matter what values are i or j are (they don't affect how the derivative operation will proceed). If you don't believe this, write out the full vector representation and follow it through. Applying the product rule, we have:

$$\begin{aligned} \nabla \cdot \boldsymbol{ab} &= \frac{\partial a_i\, b_j}{\partial r_i} \\ &= b_j \frac{\partial a_i}{\partial r_i} + a_i \frac{\partial b_j}{\partial r_i} \end{aligned}$$

And going back to vector notation, we have the answer!

$$\begin{aligned} \nabla \cdot \boldsymbol{ab} &= b_j \frac{\partial a_i}{\partial r_i} + a_i \frac{\partial b_j}{\partial r_i} \\ &= \boldsymbol{b}\nabla \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \nabla \boldsymbol{b} \end{aligned}$$

This tutorial question shows that it's easy and powerful to work with index notation (it works almost like normal scalar calculus). You can easily find other identities like this one

$$\nabla^2 f\, g = f\, \nabla^2 g + 2(\nabla f) \cdot (\nabla g) + g\, \nabla^2 f$$

[Question end]

Q.9 Question 9

Using index notation, prove the following vector calculus identity:

$$\nabla^2 f\, g = f\, \nabla^2 g + 2(\nabla f) \cdot (\nabla g) + g\, \nabla^2 f$$

[5 marks]

Note: You must treat f and g as functions of x, y, z.

Solution:

Converting to index notation in Cartesian coordinates (x,y,z),

$$\nabla^2 f\, g = \frac{\partial}{\partial r_i}\left(\frac{\partial}{\partial r_i} f\, g\right)$$

[1/5] $\checkmark_1$ We can't use $\partial^2/\partial r_i^2$ as there is no repeated i index. Using the product rule on the term in parenthesis

$$\frac{\partial}{\partial r_i} f\, g = f\frac{\partial g}{\partial r_i} + g\frac{\partial f}{\partial r_i}$$

[1/5] $\checkmark_1$ Using the product rule again to apply the second derivative to both of these terms gives

$$\begin{aligned}\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i} f\, g &= \frac{\partial f}{\partial r_i}\frac{\partial g}{\partial r_i} + f\frac{\partial}{\partial r_i}\frac{\partial g}{\partial r_i} + \frac{\partial g}{\partial r_i}\frac{\partial f}{\partial r_i} + g\frac{\partial}{\partial r_i}\frac{\partial f}{\partial r_i}\\ &= f\frac{\partial}{\partial r_i}\frac{\partial g}{\partial r_i} + 2\frac{\partial f}{\partial r_i}\frac{\partial g}{\partial r_i} + g\frac{\partial}{\partial r_i}\frac{\partial f}{\partial r_i}\end{aligned}$$

[2/5] $\checkmark_2$ Converting back to vector notation, yields the identity,

$$\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i} f\, g = f\, \nabla^2 g + 2(\nabla f) \cdot (\nabla g) + g\, \nabla^2 f$$

[1/5] $\checkmark_1$

[Question total: 5 marks]

Q.10 Question 10

Solve the following integration and differentiation problems:

Solution

a) $\int r\, \mathrm{d}r\, r$ — $\frac{r^3}{2} + C_1\, r$

b) $\int\int \theta\, \mathrm{d}\theta\, \mathrm{d}r$ — $r\left(\theta^2/2 + C_1\right) + C_2$

c) $\int_A^B y^{-1}\mathrm{d}y$ — $\ln\left(\frac{B}{A}\right)$

d) $\int x \sin x\, \mathrm{d}x$ (hint: by parts) — $\sin x - x\cos x + C_1$

e) $\nabla \cdot \boldsymbol{r}$ where $\boldsymbol{r} = [x, y, z]$ — 3

f) $\nabla \boldsymbol{r}$ where $\boldsymbol{r} = [x, y, z]$ — $\begin{bmatrix}1, & 0, & 0\\ 0, & 1, & 0\\ 0, & 0, & 1\end{bmatrix} = \delta_{ij}$

[Question end]

Q.11 Question 11

Solve the following integration problem by using a variable substitution of $\eta = y/H$.

$$\int_0^H \left(A\frac{y^2}{H^2} + B\frac{y}{H}\right) \mathrm{d}y$$

Solution:

As we have a variable substitution of $\eta = y/H$, this yields $\mathrm{d}y = H\mathrm{d}\eta$. Performing the substitution, we have:

$$\int_0^H \left(A\frac{y^2}{H^2} + B\frac{y}{H}\right) \mathrm{d}y = H\int_0^1 \left(A\eta^2 + B\eta\right) \mathrm{d}\eta$$
$$= H\left(A/3 + B/2\right)$$

If you've done this without the substitution, you might notice the variable substitution makes this integration simpler. You don't have to use it, but some of the problems you will face later are significantly easier if you use an appropriate variable substitution. The most obvious variable substitutions use dimensionless variables (e.g. if y is a position, then H might be a height, making η dimensionless).

[Question end]

Q.12 Question 12

2015 Exam Question Using a Cartesian control volume (as illustrated in Fig. 1):

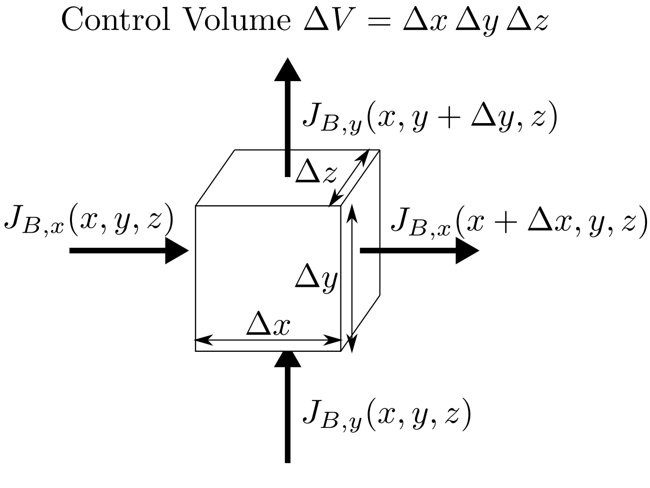


Figure 1: A differential balance of flow property B in cartesian coordinates.

a) Derive the general advection-diffusion equation for a property B, including a source term, σ_B. **[12 marks]**

Solution:

In each direction, we can perform a balance of the fluxes, J_B. Considering just the $x-$*direction*, in an interval of time Δt, we have the following fluxes

$$[\text{INPUT} - \text{OUTPUT}]_x = \Delta t\,\Delta y\,\Delta z\left(J_{B,x}(x, y, z) - J_{B,x}(x + \Delta x, y, z)\right)$$

Where the $\Delta y\,\Delta z$ term is the area of flux in the x-direction. Similar expressions can be generated for the y and z directions. We should also consider the generation of B within the control volume:

$$\text{GENERATION} = \Delta t\,\Delta x\,\Delta y\,\Delta z\,\sigma_B$$

where σ_B is the production of B per unit volume per time. The balance of fluxes, and generation terms must equal the accumulation/change in concentration over the interval:

$$\text{ACCUMULATION} = \Delta x\,\Delta y\,\Delta z\,()$$

Setting these equal, and dividing by the control volume and interval we have

$$(\Delta t\,\Delta V)^{-1}\,\text{ACCUMULATION} = (\Delta t\,\Delta V)^{-1}\,(\text{INPUT} - \text{OUTPUT} + \text{GENERATION})$$

$$\frac{C_B(t+\Delta t) - C_B(t)}{\Delta t} = \sigma_B - \frac{J_{B,x}(x+\Delta x) - J_{B,x}(x)}{\Delta x} - \frac{J_{B,y}(y+\Delta y) - J_{B,y}(y)}{\Delta y} - \ldots$$

Taking all intervals in the limit that they tend to zero, we have

$$\frac{\partial C_B}{\partial t} = \sigma_B - \frac{\partial J_{B,x}}{\partial x} - \frac{\partial J_{B,y}}{\partial y} - \frac{\partial J_{B,z}}{\partial z}$$

Writing this in vector and index form:

$$\frac{\partial C_B}{\partial t} = \sigma_B - \nabla\cdot\boldsymbol{J}_B$$

b) Set B = mass and derive the continuity equation. **[8 marks]**

Solution:

For mass, the concentration is the mass density $C_{mass} = \rho$. We typically handle systems where mass is conserved (no nuclear processes), therefore $\sigma_{mass} = 0$. For the fluxes, there is only the convective flux, which is $J_{mass,conv.} = \rho\,\boldsymbol{v}$ as mass diffusion only appears when considering a single species in a multicomponent fluid, not the overall mass. Inserting these definitions into the general balance equation we have:

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot\rho\,\boldsymbol{v}$$

or

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\,\boldsymbol{v} = 0$$

[Question total: 20 marks]

Q.13 Question 13

Using index notation:

a) Write down the continuity equation (Eq. (65)).

Solution:

Note: *The answers to these index notation questions have been expanded as much as possible for your reference! Please do not write such verbose answers yourself! With a little bit of practise you should be able to jump straight to the answer. In general, I will not expect workings out for an index notation question.*

The continuity equation fully expanded in Cartesian coordinates is

$$\frac{\partial \rho}{\partial t} = -\left(\nabla_x\,\rho\,v_x + \nabla_y\,\rho\,v_y + \nabla_z\,\rho\,v_z\right)$$

If we collect the terms on the right hand side into a sum we can write

$$\frac{\partial \rho}{\partial t} = -\sum_{i=x,y,z}\nabla_i\rho\,v_i$$

The *summation convention* (see the paragraph above) states that in index notation, whenever an index is repeated within a term a summation is implied. So in index notation the continuity equation is

$$\frac{\partial \rho}{\partial t} = -\nabla_i \rho\, v_i$$

b) Write down the Cauchy momentum equation.
Solution:
The answer is

$$\rho \frac{\partial v_i}{\partial t} = -\rho\, v_j\, \nabla_j\, v_i - \nabla_j\, \tau_{ji} - \nabla_i\, p + \rho\, g_i$$

We will now illustrate the connection between index notation and the full explicit "component" notation. This is purely an educational exercise, **do not write out the expressions in component notation**. The Cauchy momentum equation fully expanded in component notation is:

$$\begin{bmatrix} \rho\frac{\partial v_x}{\partial t} \\ \rho\frac{\partial v_y}{\partial t} \\ \rho\frac{\partial v_z}{\partial t} \end{bmatrix} = -\begin{bmatrix} \sum_{j=x,y,z} \rho\, v_j\, \nabla_j\, v_x \\ \sum_{j=x,y,z} \rho\, v_j\, \nabla_j\, v_y \\ \sum_{j=x,y,z} \rho\, v_j\, \nabla_j\, v_z \end{bmatrix} - \begin{bmatrix} \sum_{j=x,y,z} \nabla_j\, \tau_{jx} \\ \sum_{j=x,y,z} \nabla_j\, \tau_{jy} \\ \sum_{j=x,y,z} \nabla_j\, \tau_{jz} \end{bmatrix} - \begin{bmatrix} \nabla_x\, p \\ \nabla_y\, p \\ \nabla_z\, p \end{bmatrix} + \begin{bmatrix} \rho\, g_x \\ \rho\, g_y \\ \rho\, g_z \end{bmatrix}$$

Again, using the summation convention we can remove all of the sums in the above expression, as there is always a repeated index j whenever a sum is present!

$$\begin{bmatrix} \rho\frac{\partial v_x}{\partial t} \\ \rho\frac{\partial v_y}{\partial t} \\ \rho\frac{\partial v_z}{\partial t} \end{bmatrix} = -\begin{bmatrix} \rho\, v_j\, \nabla_j\, v_x \\ \rho\, v_j\, \nabla_j\, v_y \\ \rho\, v_j\, \nabla_j\, v_z \end{bmatrix} - \begin{bmatrix} \nabla_j\, \tau_{jx} \\ \nabla_j\, \tau_{jy} \\ \nabla_j\, \tau_{jz} \end{bmatrix} - \begin{bmatrix} \nabla_x\, p \\ \nabla_y\, p \\ \nabla_z\, p \end{bmatrix} + \begin{bmatrix} \rho\, g_x \\ \rho\, g_y \\ \rho\, g_z \end{bmatrix}$$

Finally, we can represent the x, y, and z components all at once by using an index which is **not** repeated within a single term. Here, the index i is not in use so we can write

$$\rho \frac{\partial v_i}{\partial t} = -\rho\, v_j\, \nabla_j\, v_i - \nabla_j\, \tau_{ji} - \nabla_i\, p + \rho\, g_i$$

[Question end]

Q.14 Question 14

In a plate heat-exchanger, water is heated by forcing it between alternating plates and heat is exchanged through the walls with a hot process stream. In order to design such an exchanger, we need to know what the relationship is between pressure drop, flow velocity, and volumetric flow-rate.
You may neglect the effect of heat transfer on the flow. Water is incompressible and Newtonian to a good approximation. For simplicity, you can also assume that the flow is laminar.

a) Simplify the continuity equation for this system:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho\, \boldsymbol{v}$$

What does your result state about the flow velocity in the x-direction? **[4 marks]**
Solution:
[1/4] If the fluid is incompressible (ρ = constant)$^{\checkmark}_{1}$, we have:

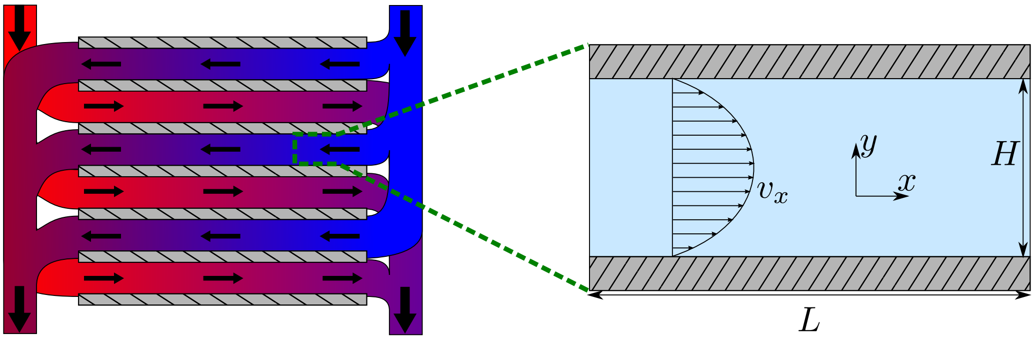


Figure 2: A plate heat exchanger (left) and the simplification to steady state, pressure driven flow between two horizontal plates (right).

$$\cancelto{0}{\frac{\partial \rho}{\partial t}} + \rho \frac{\partial v_i}{\partial r_i} = 0$$
$$\frac{\partial v_i}{\partial r_i} = 0$$

[1/4] $\checkmark_1$ As the flow is laminar (no turbulence), there will be no flow in the y-direction. We've also been told there's no flow in the z-direction, so we have $v_z = v_y = 0$ and the equation becomes:

$$\frac{\partial v_x}{\partial x} = 0 \tag{3}$$

[1/4] $\checkmark_1$ This is a statement that the steady-state velocity profile between the plates does not vary
[1/4] in the x direction. $\checkmark_1$

b) Simplify the x-component of the Cauchy momentum equation:

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} = -\rho\, \boldsymbol{v} \cdot \nabla \boldsymbol{v} - \nabla \cdot \boldsymbol{\tau} - \nabla\, p + \rho\, \boldsymbol{g}$$

Derive the following balance expression for the flow velocity v_x as a function of the pressure drop and position y: **[6 marks]**

$$\mu \frac{\partial^2 v_x}{\partial y^2} = \frac{\partial p}{\partial x}$$

Solution:

Taking the x-component of the Cauchy momentum equation, we can eliminate the time
[1/6] derivative as we are at steady state $\checkmark_1$ and we can eliminate the gravity term as we are con-
[1/6] sidering horizontal flow $\checkmark_1$

$$\rho \cancelto{0}{\frac{\partial v_x}{\partial t}} = -\rho\, v_i \frac{\partial v_x}{\partial r_i} - \frac{\partial \tau_{ix}}{\partial r_i} - \frac{\partial}{\partial x} p + \cancelto{0}{\rho g_x}$$
$$0 = -\rho\, v_i \frac{\partial v_x}{\partial r_i} - \frac{\partial \tau_{ix}}{\partial r_i} - \frac{\partial p}{\partial r_x} \tag{4}$$

We demonstrated in the previous question that $\partial v_x / \partial x = 0$ and the fact that nothing changes in the z-direction (its translationally symmetric) tells us that $\partial v_x / \partial z = 0$. Thus, the only non-zero derivative of v_x is $\partial v_x/\partial y \neq 0$. If we examine the first term of Eq. (4), we find that the only term with a non-zero derivative is

$$\rho\, v_y \frac{\partial v_x}{\partial y}$$

However, $v_y = 0$ and so the whole first term is zero, leaving us with

$$-\frac{\partial \tau_{ix}}{\partial r_i} = \frac{\partial p}{\partial x}. \tag{5}$$

[2/6] $\checkmark_2$ Using the definition of Newton's law (Table.), we can define τ_{ix} as:

$$\tau_{ix} = -\mu\left(\frac{\partial v_i}{\partial x} + \frac{\partial v_x}{\partial r_i}\right)$$

We know that v_y and v_z are zero, and we know that $\partial v_x/\partial x = 0$ (see Eq. 3), therefore the first term is always zero! Inserting this into into our stress balance (Eq. 5) we have

$$\mu\frac{\partial}{\partial r_i}\frac{\partial v_x}{\partial r_i} = \frac{\partial p}{\partial x}$$

[1/6] $\checkmark_1$ We know that $\partial v_x/\partial x = 0$ (see Eq. 3), and we know the velocity doesn't change in the z direction ($\partial v_x/\partial z = 0$). Therefore only the $i = y$ term is non-zero, giving us the final

[1/6] result: $\checkmark_1$

$$\mu\frac{\partial^2 v_x}{\partial y^2} = \frac{\partial p}{\partial x}$$

c) Continuing from the result of the previous question, derive the following expression for the velocity v_x as a function of y using the no-slip boundary condition at the plate surfaces ($v_x = 0$ at $y = 0$ and $y = H$). **[6 marks]**

$$v_x = \frac{p_{out} - p_{in}}{2\,\mu\,L}(y^2 - H\,y)$$

Solution:
Taking the result from the previous question, we can immediately integrate both sides over x:

$$\int_0^L \mu\frac{\partial^2 v_x}{\partial y^2}\mathrm{d}x = \int_0^L \frac{\partial p}{\partial x}\mathrm{d}x$$
$$\left[\mu\frac{\partial^2 v_x}{\partial y^2}x\right]_{x=0}^{x=L} = \int_{p_{in}}^{p_{out}}\mathrm{d}p$$
$$\mu\frac{\partial^2 v_x}{\partial y^2} = \frac{p_{out} - p_{in}}{L}$$

[2/6] $\checkmark_2$ We can now integrate both sides by y, twice, to yield

$$v_x = \frac{p_{out} - p_{in}}{2\,\mu\,L}y^2 + C_1\,y + C_2$$

[2/6] $\checkmark_2$ where C_1 and C_2 are integration constants. From the boundary condition $v_x = 0$ at $y = 0$,

[1/6] we know the last constant $C_2 = 0$. $\checkmark_1$ From the boundary condition $v_x = 0$ at $y = H$, we have

$$C_1 = -\frac{p_{out} - p_{in}}{2\,\mu\,L} H$$

[1/6] $\checkmark_1$ Using this, the final equation becomes

$$v_x = \frac{p_{out} - p_{in}}{2\,\mu\,L}(y^2 - H\,y)$$

d) Integrate the velocity over the plate height and width to prove the following expression for the volumetric flow of liquid through the gap as a function of pressure drop: **[4 marks]**

$$\dot{V}_x = \frac{Z\,H^3}{12\,\mu}\frac{\Delta P}{L}$$

Solution:
For volumetric flow in the x direction, we have:

$$\begin{aligned}
\dot{V}_x &= \int_0^Z \int_0^H v_x \,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^Z \int_0^H \frac{p_{out} - p_{in}}{2\,\mu\,L}(y^2 - H\,y)\,\mathrm{d}y\,\mathrm{d}z \\
&= Z \int_0^H \frac{p_{out} - p_{in}}{2\,\mu\,L}(y^2 - H\,y)\,\mathrm{d}y \\
&= Z\frac{p_{out} - p_{in}}{2\,\mu\,L}\left[\frac{y^3}{3} - H\,\frac{y^2}{2}\right]_{y=0}^{y=H} \\
&= H^3\,Z\frac{p_{in} - p_{out}}{12\,\mu\,L} \\
\dot{V}_x &= \frac{Z\,H^3}{12\,\mu}\frac{\Delta P}{L}
\end{aligned}$$

[4/4] $\checkmark_4$

e) **Extra credit:** Assume that somehow, the top plate is set in motion with a velocity u_{plate} in the x-direction. Derive the following new expression for the velocity between the plates:

$$v_x = \frac{p_{out} - p_{in}}{2\,\mu\,L}(y^2 - H\,y) + \frac{y}{H}\,u_{plate}$$

Solution:
This is just a re-determination of the integration constants from the answer to the previous question using the new boundary condition. We had

$$v_x = \frac{p_{out} - p_{in}}{2\,\mu\,L}y^2 + C_1\,y + C_2$$

Again, from the boundary condition $v_x = 0$ at $y = 0$, we know the last constant $C_2 = 0$. From the boundary condition $v_x = u_{plate}$ at $y = H$, we have

$$\begin{aligned}
u_{plate} &= \frac{p_{out} - p_{in}}{2\,\mu\,L}H^2 + C_1\,H \\
C_1 &= \frac{u_{plate}}{H} - \frac{p_{out} - p_{in}}{2\,\mu\,L}H
\end{aligned}$$

Using this, the final equation becomes

$$v_x = \frac{p_{out} - p_{in}}{2\,\mu\,L}(y^2 - H\,y) + \frac{y}{H}\,u_{plate}$$

[Question total: 20 marks]

Q.15 Question 15

A plate heat exchanger is used to heat water inside a condensing reboiler (a modern central heating boiler). Water flows through both sides of the exchanger. The exchanger consists of 8 channels (4 per side) each with a gap of 1 mm between the plates. Plates may be modelled as 30cm long in the direction of flow and 10 cm wide.

a) If the water pressure drops by 0.06 bar across one side of the exchanger, what is the resultant volumetric flow of water? You may assume an effective viscosity of $\mu \approx 0.5$ mPa s and a density of $\rho = 1000$ kg m^{-3}.

Solution:

In each channel, the volumetic flow is

$$\begin{aligned}\dot{V}_x &= H^3\,W\frac{p_{in} - p_{out}}{12\,\mu\,L} \\ &= \left(1 \times 10^{-3}\right)^3\,0.1\frac{0.06 \times 10^5}{12 \times 0.5 \times 10^{-3} \times 0.3} \\ &\approx 0.00033\ \mathrm{m^3\ s^{-1}} \approx 0.33\ \mathrm{l\ s^{-1}}\end{aligned}$$

The total flowrate over all channels is then $4 \times 0.33 = 1.32\ \mathrm{l\ s^{-1}}$.

b) State all of the assumptions that have you made in this estimate.

Solution:

Assumed steady-state, incompressible, well-developed flow (ignoring the entry ports of the exchanger). Also ignored the effect of changing temperature on the viscosity of the fluid.

c) Is this likely to be an over or under-estimation of the flow rate?

Solution:

The flow rate is likely to be an over-estimation as we have neglected the pressure drop in the entry and developing flow regions, which can be considerable in such a small flow geometry. Realistic flow rates are in the order of 0.04 l s^{-1} for these conditions. Another source of error is that the model does not include the irregular surfaces used to increase the mixing and heat transfer area in plate heat exchangers.

[Question end]

Q.16 Question 16

Water is overflowing a dam and down an inclined slope (see Fig. 3). The surface of the dam can be idealised as a rectangular plane which is symmetric in the z-direction, and (for now) only laminar flow is being considered.

a) Simplify the continuity equation for this system and state any assumptions you make. **[6 marks]**

Solution:

Assuming water is incompressible (ρ = constant) and using Cartesian coordinates (x,y,z),
[2/6] the continuity equation becomes,$\checkmark_2$:

$$\cancelto{0}{\frac{\partial \rho}{\partial t}} + \nabla \cdot \rho\,\vec{v} = 0$$

$$\nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

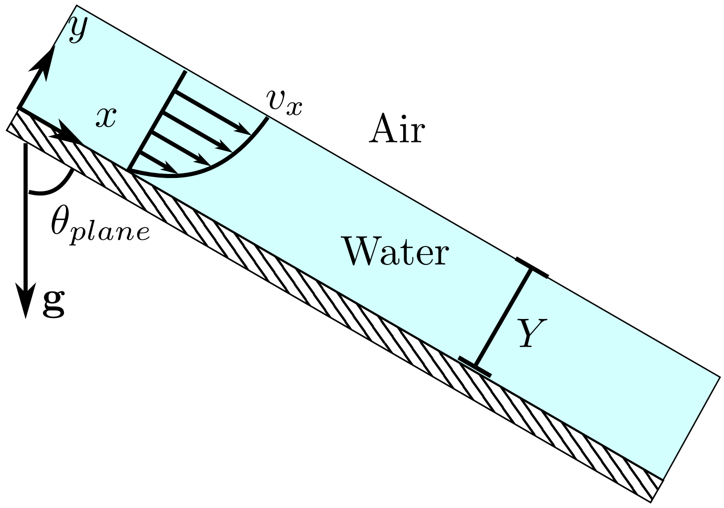


Figure 3: Water flowing down an inclined plane.

[1/6] $\checkmark_1$ As the flow is laminar there will be no flow in the y-direction and we've also been told the system is symmetric in the z-dimension so there is no reason to believe there is flow in the
[2/6] z-direction, so we have $v_z = v_y = 0$ $\checkmark_2$ and the equation becomes:

$$\frac{\partial v_x}{\partial x} = 0 \tag{6}$$

This is a statement that the steady-state velocity profile between the plates does not vary
[1/6] in the x direction. $\checkmark_1$

b) Derive the following results from the Cauchy momentum equation and the general form of Newton's law of viscosity: **[10 marks]**

$$\frac{\partial \tau_{yx}}{\partial y} = \rho\, g_x \qquad\qquad \tau_{yx} = -\mu \frac{\partial v_x}{\partial y}.$$

Solution:

Taking the x-component of the Cauchy momentum equation, the time derivative can be cancelled by assuming we are at steady state and the pressure term also cancels, as the system has a free surface which equalises the pressure along the flow (and we neglect the atmospheric pressure changes).

$$\rho \cancelto{0}{\frac{\partial v_x}{\partial t}} = -\rho\, v_i \frac{\partial v_x}{\partial r_i} - \frac{\partial \tau_{ix}}{\partial r_i} - \cancelto{0}{\frac{\partial p}{\partial x}} + \rho\, g_x \tag{7}$$

[3/10] $\checkmark_3$

Considering the first term and expanding the index notation,

$$\rho\, v_i \frac{\partial v_x}{\partial r_i} = \rho\, v_x \cancelto{0}{\frac{\partial v_x}{\partial x}} + \rho \cancelto{0}{v_y} \frac{\partial v_x}{\partial y} + \rho \cancelto{0}{v_z} \cancelto{0}{\frac{\partial}{\partial z}} v_x$$

The first term cancels from the continuity equation whereas the others cancel as there is no flow in those directions. The final term on the right can also cancel due to symmetry in the
[2/10] z-direction. Thus, this entire term is zero. $\checkmark_2$

The equation now becomes,

$$\frac{\partial \tau_{ix}}{\partial r_i} = \rho\, g_x \tag{8}$$

Using the 3D definition of Newton's law, we can define τ_{ix} as:

$$\tau_{ix} = -\mu\left(\frac{\partial v_i}{\partial x} + \frac{\partial v_x}{\partial r_i}\right) + \delta_{ix}\mu_B \cancel{\nabla\cdot\boldsymbol{v}}^{\,0}$$

[1/10] $^{\checkmark}_{1}$The final term $\nabla\cdot\vec{v} = 0$ cancels from the continuity equation. We also know that v_y and v_z are zero, and we know that $\partial v_x/\partial x = 0$ from the continuity equation, therefore the first term is always zero. Only the $\partial v_x/\partial y$ term is non-zero, thus the expression is

$$\frac{\partial \tau_{yx}}{\partial y} = \rho\, g_x \qquad\qquad \tau_{yx} = -\mu\frac{\partial v_x}{\partial y}$$

[4/10] $^{\checkmark}_{4}$where we have assumed a Newtonian fluid with constant viscosity.

c) Define your boundary conditions and derive the following expression for the velocity profile, **[9 marks]**

$$v_x = \frac{\rho\, g_x}{\mu}\left(Y\,y - \frac{y^2}{2}\right)$$

Solution:
Integrating the equation from the previous slide

$$\tau_{xy} = \rho\, g_x\, y + C_1$$

[1/9] $^{\checkmark}_{1}$At the surface of the flow the stress is negligble due to the low viscosity of air, thus $\tau_{xy}(r = Y) = 0$, and solving for the constant gives the following expression

$$\tau_{xy} = \rho\, g_x\,(y - Y)$$

[3/9] $^{\checkmark}_{3}$Inserting the expression for the stress and integrating again,

$$v_x = -\frac{\rho\, g_x}{\mu}\left(\frac{y^2}{2} - Y\,y\right) + C_2.$$

[1/9] $^{\checkmark}_{1}$The other boundary condition is the no-slip condition, $v_x(r = 0) = 0$. This gives $C_2 = 0$$^{\checkmark}_{1}$

[1/9]

$$v_x = \frac{\rho\, g_x}{\mu}\left(Y\,y - \frac{y^2}{2}\right)$$

[3/9] $^{\checkmark}_{3}$

d) Use an integration of the velocity over the flow area to determine the following expression for the volumetric flow rate, **[6 marks]**

$$\dot{V}_x = \frac{\rho\, g_x\, Y^3\, Z}{3\,\mu}.$$

Solution:
For volumetric flow in the x direction, we have:

$$\begin{aligned}\dot{V}_x &= \int_0^Z \int_0^Y v_x \,\mathrm{d}y\,\mathrm{d}z \\ &= \int_0^Z \int_0^Y \frac{\rho\, g_x\, Y^2}{2\,\mu} \left(2\frac{y}{Y} - \frac{y^2}{Y^2}\right) \mathrm{d}y\,\mathrm{d}z \\ &= Z\frac{\rho\, g_x\, Y^2}{2\,\mu} \left[\frac{y^2}{Y} - \frac{y^3}{3\,Y^2}\right]_0^Y \\ &= \frac{\rho\, g_x\, Y^3\, Z}{2\,\mu} \left[\frac{y^2}{Y^2} - \frac{y^3}{3\,Y^3}\right]_0^Y \\ &= \frac{\rho\, g_x\, Y^3\, Z}{3\,\mu}\end{aligned}$$

[6/6] $\checkmark_{6}$

e) Provide an expression for the maximum flow velocity. **[2 marks]**
Solution:
[2/2] The maximum velocity in the system is at $y = Y$, thus $v_{max} = \rho\, g_x\, Y^2/(2\,\mu)$.$\checkmark_{2}$

[Question total: 33 marks]

Q.17 Question 17

Consider pressure-driven flow along a horizontal pipe, as illustrated in Fig. 4.

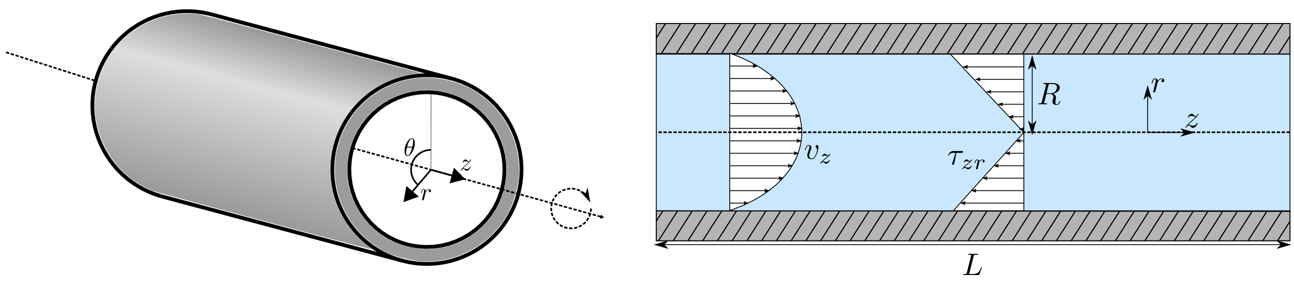


Figure 4: An illustration of pipe flow.

a) Simplify the continuity equation for this system, what does it tell you about the flow? Remember to make your assumptions and their effects clear. **[6 marks]**
Solution:
[1/6] Assuming either steady-state or incompressible fluid, the time derivative can be eliminated.$\checkmark_{1}$

$$\cancelto{0}{\frac{\partial \rho}{\partial t}} - \nabla\cdot\rho\boldsymbol{v} = 0$$
$$\nabla\cdot\rho\boldsymbol{v} = 0$$

[1/6] If the fluid is incompressible, the density can be divided out of the expression.$\checkmark_{1}$

As we're in cylindrical coordinates, we must look up the result of the gradient operator in cylindrical coordinates:

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r}\frac{\partial r v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

[1/6] $\checkmark_1$

[1/6] Assuming the system is rotationally symmetric then $\partial/\partial\theta = 0\checkmark_1$. Assuming well-developed
[1/6] and laminar flow, then $v_r = 0\checkmark_1$. This leaves the final term:

$$\frac{\partial v_z}{\partial z} = 0$$

[1/6] Which states that the flow velocity in the x-direction is constant$\checkmark_1$.

b) Derive the following differential equation from the Cauchy momentum equation.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rz}\right) = -\frac{\partial p}{\partial z}$$

Remember to make your assumptions and their effects clear. **[7 marks]**

Solution:

Starting with the Cauchy momentum equation:

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} = -\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \nabla\cdot\boldsymbol{\tau} - \nabla p + \rho\boldsymbol{g}$$

Assuming steady state,

$$0 = -\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \nabla\cdot\boldsymbol{\tau} - \nabla p + \rho\boldsymbol{g}$$

[1/7] $\checkmark_1$

We're only interested in the z-direction, so

$$0 = -\rho\left[\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right]_z - \left[\nabla\cdot\boldsymbol{\tau}\right]_z - \frac{\partial p}{\partial z} + \rho\, g_z$$

[1/7] $\checkmark_1$

[1/7] As the pipe is horizontal, $g_z = 0\checkmark_1$.

For the first term, we have the following definition from the datasheet for cylindrical coordinates:

$$\left[\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right]_z = v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}$$

[1/7] We have $\partial v_z/\partial z = 0$ from the first question, and $\partial/\partial\theta = 0$ from rotational symmetry$\checkmark_1$. The
[1/7] first term is zero as $v_r = 0$ from laminar well-developed flow$\checkmark_1$, thus this entire term is zero.

Considering the second term, and expanding it from the datasheet:

$$\left[\nabla\cdot\boldsymbol{\tau}\right]_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rz}\right) + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}$$

We can cancel the middle term from the rotational symmetry $\partial/\partial\theta = 0$. The last term can be cancelled as there is no velocity change in the z-direction (thus no stresses can be
[2/7] induced)$\checkmark_2$. Alternatively, each stress term can be individually expanded and eliminated by considering each of the velocities (as is done in a later sub-part of this question for the τ_{rz} term).

Putting this all together yields the final expression.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rz}\right) = -\frac{\partial p}{\partial z}$$

c) Determine the following expression for the stress profile.

$$\tau_{rz} = -\frac{\Delta p}{2\,L}r$$

[3 marks]

Solution:

Performing a definite integral in the z-direction (from $z = 0$ to $z = L$), all terms are constant thanks to the only non-zero velocity being constant, i.e. $\partial v_z/\partial z = 0$. This allows a simple replacement of the pressure drop

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rz}\right) = -\frac{\Delta p}{L}$$

[1/3] where $\Delta p = p\left(z = L\right) - p\left(z = 0\right)\checkmark_1$.

Performing an indefinite integral in the r direction,

$$\int \frac{\partial\, r\,\tau_{rz}}{\partial r}\mathrm{d}r = -\frac{\Delta p}{L}\int r\,\mathrm{d}r$$

$$\tau_{rz} = -\frac{\Delta p}{2\,L}r + \frac{C_1}{r}$$

[1/3] $\checkmark_1$

[1/3] As the stress has to be finite in the centre of the pipe then $C_1 = 0\checkmark_1$. Alternatively, this can also be deduced as the stress must go to zero in the centre of the pipe as it is a line of symmetry in rz. Cancelling the C_1 term gives the final expression:

$$\tau_{rz} = -\frac{\Delta p}{2\,L}r$$

d) Demonstrate that the velocity profile is as given below.

$$v_z = \frac{\Delta p}{4\,\mu\,L}\left(r^2 - R^2\right)$$

[4 marks]

Solution:

Taking a look in the datasheet for the stress:

$$\tau_{zr} = -\mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)$$

[1/4] In this case, $v_r = 0$ due to assuming laminar well-developed flow$\checkmark_1$. Inserting this into the above equation,

$$\tau_{rz} = -\mu\frac{\partial v_z}{\partial r} = -\frac{\Delta p}{2\,L}r$$

[1/4] $\checkmark_1$

Integrating in r,

$$\int \mu\frac{\partial v_z}{\partial r}\mathrm{d}r = \int \frac{\Delta p}{2\,L}r\mathrm{d}r\mu v_z \qquad = \frac{\Delta p}{4\,L}r^2 + C_2$$

[1/4] $\checkmark_1$

As the velocity must go to zero at the walls, C_2 can be determined,

$$v_z = \frac{\Delta p}{4\,\mu\,L}\left(r^2 - R^2\right)$$

[1/4] $\checkmark_1$

[Question total: 20 marks]

Q.18 Question 18

An annulus (see Fig. 5) is a very common flow configuration where a fluid is flowing between two concentric pipes. Real examples of annuli include oil and gas wells and concentric-tube heat-exchangers in air conditioners. A "completed" oil-well may consist of up to 3 annuli around the central "production" pipe. We need design equations to calculate the relationship between pressure drop and volumetric flow-rate.

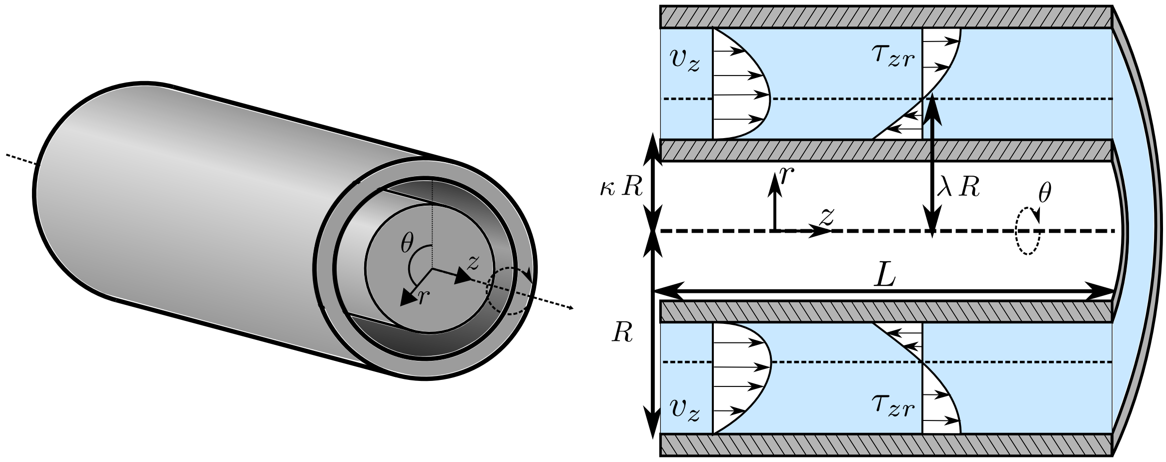


Figure 5: An annular flow geometry.

Assuming we have a steady-state, laminar, incompressible, and well-developed flow inside an annulus:

a) Demonstrate that the continuity equation simplifies to the following expression.

$$\frac{\partial v_z}{\partial z} = 0$$

State your interpretation of this expression.

Solution:

Note: This question covers all parts of the solution in great detail. Please read it carefully and use it as a template to fill in any skipped steps for all later solutions.

If the fluid is incompressible (ρ = constant), we can cancel the time derivative and divide both sides by the density:

$$\overbrace{\cancel{\frac{\partial \rho}{\partial t}}}^{0} + \nabla \cdot \rho\, \boldsymbol{v} = 0$$
$$\nabla \cdot \boldsymbol{v} = 0$$

Using the definition of $\nabla \cdot \boldsymbol{v}$ in cylindrical coordinates, we have:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\, v_r) + \frac{1}{r}\frac{\partial\, v_\theta}{\partial \theta} + \frac{\partial\, v_z}{\partial z} = 0$$

If the flow is laminar and well developed, we have $v_\theta = 0$ and $v_r = 0$ which leaves us with

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\, \overbrace{\cancel{v_r}}^{0}\right) + \frac{1}{r}\frac{\partial\, \overbrace{\cancel{v_\theta}}^{0}}{\partial \theta} + \frac{\partial\, v_z}{\partial z} = 0$$
$$\frac{\partial\, v_z}{\partial z} = 0$$

This is a statement that the steady-state velocity profile does not vary along the pipe axis.

b) Simplify the Cauchy momentum balance equation to yield the following result.

$$0 = -\frac{1}{r}\frac{\partial}{\partial r}(r\,\tau_{rz}) - \frac{\partial p}{\partial z} + \rho\, g_z$$

Solution:
Taking the z-component of the Navier-Stokes equation, we have:

$$\rho\frac{\partial v_z}{\partial t} = -\left[\rho\, \boldsymbol{v}\cdot\nabla \boldsymbol{v}\right]_z - \left[\nabla\cdot\boldsymbol{\tau}\right]_z - \left[\nabla\, p\right]_z + \rho\, g_z$$

we can immediately eliminate the time derivative $\cancelto{0}{\frac{\partial v_z}{\partial t}}$ as we are at steady state to give us

$$0 = -\left[\rho\, \boldsymbol{v}\cdot\nabla \boldsymbol{v}\right]_z - \left[\nabla\cdot\boldsymbol{\tau}\right]_z - \left[\nabla\, p\right]_z + \rho\, g_z$$

We can assume that the first term will disappear as its the advective term and there are no changes in the direction of flow (it has also disappeared every time before), but we must prove this. Looking up the expanded definition of the first term we have:

$$\left[\boldsymbol{v}\cdot\nabla \boldsymbol{v}\right]_z = \cancelto{0}{v_r}\frac{\partial v_z}{\partial r} + \frac{\cancelto{0}{v_\theta}}{r}\frac{\partial v_z}{\partial \theta} + v_z\cancelto{0}{\frac{\partial v_z}{\partial z}}$$

The terms above can be cancelled as we know that $v_r = v_\theta = 0$ as the flow is well developed and the geometry will not allow flow in that direction (so we can immediately delete the first two terms). We also know the last term is zero from the continuity equation. Eliminating this whole term gives us the following:

$$0 = -\left[\nabla\cdot\boldsymbol{\tau}\right]_z - \left[\nabla\, p\right]_z + \rho\, g_z$$

Expanding the left term, we have:

$$\left[\nabla\cdot\boldsymbol{\tau}\right]_z = \frac{1}{r}\frac{\partial}{\partial r}(r\,\tau_{rz}) + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial\,\tau_{zz}}{\partial z}$$

We can insert the definitions of each of the stress terms and cancel the terms with v_r or v_θ in them or derivatives in z. For example:

$$\tau_{\theta z} = -\mu\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)$$

The first term cancels as nothing changes in the θ direction ($\partial v_z/\partial\theta = 0$ as the problem is rotationally symmetric), and the second as $v_\theta = 0$. You should note that the two indicies on the stress always indicate the derivatives and components of the velocity of the two terms. We can then immediately cancel the τ_{zz} term as it is only a function of $\partial v_z/\partial v_z$ or $\nabla\cdot\boldsymbol{v}$, both of which cancel due to the results of the continuity equation. Only the τ_{rz} term remains, inserting this into the balance along with the definition of $\left[\nabla\, p\right]_z$:

$$0 = -\frac{1}{r}\frac{\partial}{\partial r}(r\,\tau_{rz}) - \frac{\partial p}{\partial z} + \rho\, g_z$$

This is the result required.

c) Integrate the equation to express it in terms of the pressure drop over the length of the annulus. Give reasons why the stress term τ_{rz} is independent of z.

$$\frac{\Delta p}{L} = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\tau_{rz}\right) + \rho\,g_z$$

Solution:

Taking the solution to the previous equation

$$0 = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\tau_{rz}\right) - \frac{\partial p}{\partial z} + \rho\,g_z$$

We can rearrange it ready for the integration:

$$\frac{\partial p}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\tau_{rz}\right) + \rho\,g_z$$

Whatever the type of fluid (Newtonian, Power-Law), the viscous stress τ_{rz} is a function of the velocity profile. However, we know that the velocity profile is not a function of the z direction from the continuity equation ($\frac{\partial v_z}{\partial z} = 0$). Therefore, the stress τ_{rz} is not a function of z and neither are its derivatives. Gravity and density are also not a function of z. So we can perform the integration treating the terms on the right as constants, like so

$$\int_{z=0}^{z=L} \frac{\partial p}{\partial z}\mathrm{d}z = \int_{z=0}^{z=L}\left(-\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\tau_{rz}\right) + \rho\,g_z\right)\mathrm{d}z$$

$$\int_{p(0)}^{p(L)} \mathrm{d}p = \left[\left(-\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\tau_{rz}\right) + \rho\,g_z\right) z\right]_{z=0}^{z=L}$$

Note: You should note what just happened on the left hand side. This is how all integrations work, you actually integrate both sides with respect to a variable but if one side is just a derivative then a change of variables takes place! Make sure you understand this and changes of variable before proceeding! Carrying out the integration on the left and substituting in the limits on the right we have:

$$p(z = L) - p(z = 0) = \left(-\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\tau_{rz}\right) + \rho\,g_z\right) L$$

$$\frac{\Delta p}{L} = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\tau_{rz}\right) + \rho\,g_z$$

d) Solve the above equation for the stress profile in an annulus using the assumed boundary condition that the stress is zero at a critical radius $r = \lambda\,R$. Prove that it is the following expression:

$$\tau_{rz} = \frac{1}{2}\left(\rho\,g_z - \frac{\Delta p}{L}\right)\left(r - \frac{\lambda^2\,R_0^2}{r}\right)$$

Note: The critical radius $\lambda\,R$ is the location of the maximum velocity, and will be determined once the viscous model is inserted.

Solution:

Taking the result from the previous question

$$\frac{\Delta p}{L} = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\tau_{rz}\right) + \rho\,g_z$$

rearranging to make it straightforward to integrate

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\tau_{rz}\right)=\frac{\Delta p}{L}-\rho\,g_z$$

Integrating both sides by r:

$$\int\frac{\partial}{\partial r}\left(r\,\tau_{rz}\right)\mathrm{d}r=\int\left(\rho\,g_z-\frac{\Delta p}{L}\right)r\,\mathrm{d}r$$
$$r\,\tau_{rz}=\left(\rho\,g_z-\frac{\Delta p}{L}\right)\frac{r^2}{2}+C$$

Then dividing both sides by r, we have:

$$\tau_{rz}=\left(\rho\,g_z-\frac{\Delta p}{L}\right)\frac{r}{2}+\frac{C}{r}$$

As stated in the question, at a location $r=\lambda\,R$, the stress is zero ($\tau_{rz}=0$). We can then set $r=\lambda\,R$ and set $\tau_{rz}=0$ in the previous equation to find an expression for C.

$$C=-\left(\rho\,g_z-\frac{\Delta p}{L}\right)\frac{\lambda^2\,R^2}{2}$$

Substituting this back into the previous equation we have

$$\tau_{rz}=\frac{1}{2}\left(\rho\,g_z-\frac{\Delta p}{L}\right)\left(r-\frac{\lambda^2\,R_0^2}{r}\right)$$

e) Solve for the velocity profile by assuming the fluid is Newtonian. Try to rearrange the result of the integration into the following convenient form:

$$v_z=-\frac{R^2}{4\,\mu}\left(\rho\,g_z-\frac{\Delta p}{L}\right)\left(\frac{r^2}{R^2}-2\,\lambda^2\ln\left(\frac{r}{R}\right)+C\right)$$

Solution:
Looking up the definition of the τ_{rz} stress from the datasheet tables and substituting it into the expression we have,

$$\tau_{rz}=-\mu\frac{\partial v_z}{\partial r}=\frac{1}{2}\left(\rho\,g_z-\frac{\Delta p}{L}\right)\left(r-\frac{\lambda^2\,R^2}{r}\right)$$

Rearranging the equation to have dimensionless terms (not required, just for neater calculations), we have:

$$\frac{\partial v_z}{\partial r}=-\frac{R}{2\,\mu}\left(\rho\,g_z-\frac{\Delta p}{L}\right)\left(\frac{r}{R}-\lambda^2\frac{R}{r}\right)$$

Performing the integration in r (and skipping over the whole change of variables from before), we have:

$$\begin{aligned}v_z&=-\frac{R}{2\,\mu}\left(\rho\,g_z-\frac{\Delta p}{L}\right)\int\left(\frac{r}{R}-\frac{\lambda^2\,R}{r}\right)\mathrm{d}r\\&=-\frac{R}{2\,\mu}\left(\rho\,g_z-\frac{\Delta p}{L}\right)\left(\frac{r^2}{2\,R}-\lambda^2\,R\ln r+C_1\right)\\&=-\frac{R^2}{4\,\mu}\left(\rho\,g_z-\frac{\Delta p}{L}\right)\left(\frac{r^2}{R^2}-2\,\lambda^2\ln r+\frac{2\,C_1}{R}\right)\end{aligned}$$

As C_1 is an unknown integration constant, we can freely write it in terms of another unknown integration constant C_2

$$\frac{2\,C_1}{R} = C_2 + 2\,\lambda^2\,\ln R$$

Note: This is a common "trick", you can pull any constant terms you like out of an unknown constant! Its very useful for tidying up equations.

This allows us to simplify the above equation further to

$$v_z = -\frac{R^2}{4\,\mu}\left(\rho\,g_z - \frac{\Delta p}{L}\right)\left(\frac{r^2}{R^2} - 2\,\lambda^2\,\ln\left(\frac{r}{R}\right) + C_2\right)$$

This is simpler as each term has a **dimensionless** r/R variable. In fact, all logarithmic terms should always have dimensionless arguments!

f) Using the no slip boundary condition at $r = R$ and $r = \kappa\,R$, solve for the unknown constants C and λ in the above equation and generate the final expression.
Solution:
Starting with $v_z = 0$ at $r = R$, we have

$$1 + C = 0$$

thus $C = -1$ and for $v_z = 0$ at $r = \kappa\,R$ we have

$$\kappa^2 - 2\,\lambda^2\ln\kappa - 1 = 0$$

Rearranging we have

$$2\,\lambda^2 = \frac{\kappa^2 - 1}{\ln\kappa}$$

substituting these constants back in to the final result we have

$$v_z = -\frac{R^2}{4\,\mu}\left(\rho\,g_z - \frac{\Delta p}{L}\right)\left(\frac{r^2}{R^2} - \frac{\kappa^2 - 1}{\ln\kappa}\ln\left(\frac{r}{R}\right) - 1\right)$$

[Question end]

Q.19 Question 19

An evaporative cooler is sketched in Fig. 6. The process functions by first pumping water up a vertical pipe and then allowing it to flow down the exterior of the pipe. The properties of the external film flow are essential for the design of such a cooler.

a) Simplify the continuity equation for this system. What are your assumptions and what does your result tell you about the flow along the pipe? **[5 marks]**
Solution:
If the fluid is incompressible, we have

$$\overset{0}{\cancel{\frac{\partial\rho}{\partial t}}} + \nabla\cdot\rho\,\boldsymbol{v} = 0$$

If the fluid is incompressible (ρ = constant), we can divide both sides by the density to yield

$$\nabla\cdot\boldsymbol{v} = 0$$

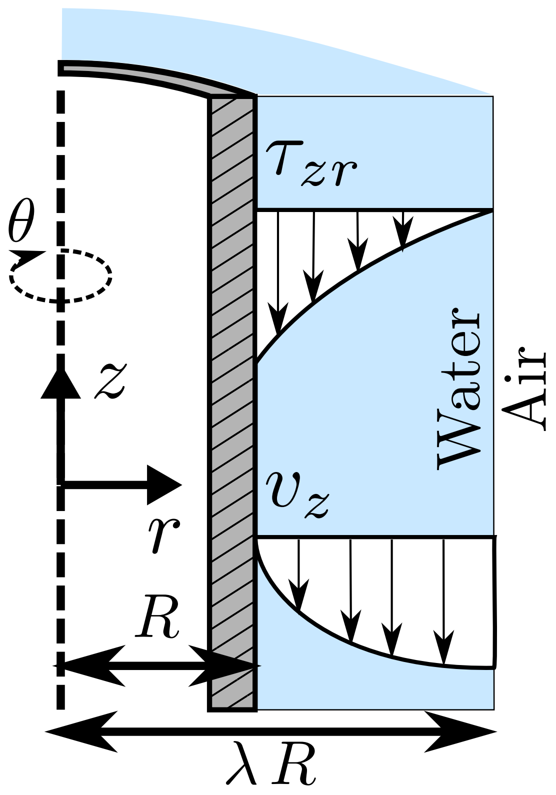


Figure 6: A sketch of the evaporative cooler

It's not straightforward to use index notation in curvelinear coordinates, so we resort to looking up the definitions in the tables in the datasheet. In cylindrical coordinates,

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r}\frac{\partial}{\partial r}(r\, v_r) + \frac{1}{r}\frac{\partial\, v_\theta}{\partial \theta} + \frac{\partial\, v_z}{\partial z}$$

If the flow is laminar and the flow is well developed, the flow in the θ and $\boldsymbol{r}$ directions must be zero, $v_r = v_\theta = 0$. This leaves us with

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\, \cancelto{0}{v_r}\right) + \frac{1}{r}\frac{\partial\, \cancelto{0}{v_\theta}}{\partial \theta} + \frac{\partial\, v_z}{\partial z} = 0$$
$$\frac{\partial v_z}{\partial z} = 0$$

This is a statement that the steady-state velocity profile does not vary along the pipe axis.

b) Derive the following equation for the stress profile from the general momentum balance equation (Eq. (67)). State any additional assumptions you make.

$$\frac{1}{r}\frac{\partial}{\partial r}(r\, \tau_{rz}) = \rho\, g_z$$

[6 marks]

Solution:
We are interested in the flow in the z direction, so we should take the z-component of the Navier-Stokes equation

$$\rho\frac{\partial v_z}{\partial t} = -\rho\,[\boldsymbol{v} \cdot \nabla \boldsymbol{v}]_z - [\nabla \cdot \boldsymbol{\tau}]_z - [\nabla\, p]_z + \rho\, g_z$$

we can immediately eliminate the time derivative $\cancelto{0}{\frac{\partial v_z}{\partial t}}$ as we are at steady state to give us

$$-\rho\,[\boldsymbol{v}\cdot\nabla\boldsymbol{v}]_z - [\nabla\cdot\boldsymbol{\tau}]_z - [\nabla\,p]_z + \rho\,g_z = 0$$

We can also cancel the pressure term as this is film flow, and the system is open to the air.

$$-\rho\,[\boldsymbol{v}\cdot\nabla\boldsymbol{v}]_z - [\nabla\cdot\boldsymbol{\tau}]_z + \rho\,g_z = 0$$

The first term always disappears in this course as we are treating incompressible flow. To demonstrate this, we look up this term in the datasheet:

$$[\boldsymbol{v}\cdot\nabla\boldsymbol{v}]_z = v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta} + v_z\frac{\partial v_z}{\partial z}$$

We know that $v_r = v_\theta = 0$ as the flow is well developed and the geometry will not allow flow in that direction so we can immediately delete the first two terms. We also know from the continuity equation that $\partial v_z/\partial z = 0$, thus this entire term is zero. Next we consider the stress term. Looking it up in the datasheet,

$$[\nabla\cdot\boldsymbol{\tau}]_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\tau_{rz}\right) + \frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{\partial\,\tau_{zz}}{\partial z}$$

Please see the previous question for a full explanation of the steps here. Wherever there is symmetry in well-developed flow, the stresses must be zero. We note that the problem is rotationally symmetric in θ and we have $v_\theta = 0$, thus $\tau_{\theta z} = 0$. From the continuity equation we have $\nabla_z\,v_z = 0$ and $\nabla\cdot\boldsymbol{v}$ thus τ_{zz}. Cancelling those terms leaves

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\tau_{rz}\right) = \rho\,g_z$$

c) Solve the equation for the stress profile to obtain the following velocity profile for the flow.

$$v_z = \frac{\rho\,g\,R^2}{4\,\mu}\left(1 - \left(\frac{r}{R}\right)^2 + 2\,\lambda^2\,\ln\left(\frac{r}{R}\right)\right)$$

[9 marks]

Solution:

Take the answer to the previous question and substitute in the r component of the gradient operator to give

$$\frac{1}{r}\frac{\partial\,r\,\tau_{rz}}{\partial r} = \rho\,g_z$$

We can integrate this expression to yield

$$\tau_{rz} = \frac{\rho\,g_z}{2}r + \frac{C_1}{r}$$

We can solve for this using the boundary condition that the stress is zero at a free surface, $\tau_{rz} = 0$ at $r = \lambda\,R$

$$C_1 = -\frac{\rho\,g_z\,\lambda^2\,R^2}{2}$$

$$\tau_{rz} = \frac{\rho\,g_z}{2}\left(r - \frac{\lambda^2 R^2}{r}\right)$$

Now we substitute in Newton's law of viscosity to obtain

$$-\mu\frac{\partial v_z}{\partial r} = \frac{\rho\, g_z}{2}\left(r - \frac{\lambda^2\, R^2}{r}\right)$$

Integrating we have

$$v_z = -\frac{\rho\, g_z}{2\,\mu}\left(\frac{r^2}{2} - \lambda^2\, R^2 \ln r + C_2\right)$$

To determine the constant, we use the no-slip boundary condition $v_z = 0$ at $r = R$

$$C_2 = \lambda^2\, R^2 \ln R - \frac{R^2}{2}$$

Inserting the expression and tidying up

$$\begin{aligned} v_z &= -\frac{\rho\, g_z}{2\,\mu}\left(\frac{r^2}{2} - \lambda^2\, R^2 \ln r + \lambda^2\, R^2 \ln R - \frac{R^2}{2}\right) \\ &= \frac{\rho\, g\, R^2}{4\,\mu}\left(1 - \left(\frac{r}{R}\right)^2 + 2\,\lambda^2\, \ln\left(\frac{r}{R}\right)\right) \end{aligned}$$

[Question total: 20 marks]

Q.20 Question 20

Example exam question

A Couette viscometer tests the viscous behaviour of a fluid using rotational shear in an annulus (see Fig. 7). The fluid is sheared by rotating the outer wall at an angular velocity of Ω_θ, giving $v_\theta(r = R) = \Omega_\theta\, R$. The inner cylinder is held stationary, giving $v_\theta(r = \kappa\, R) = 0$. There is no flow along the axis of the annulus.

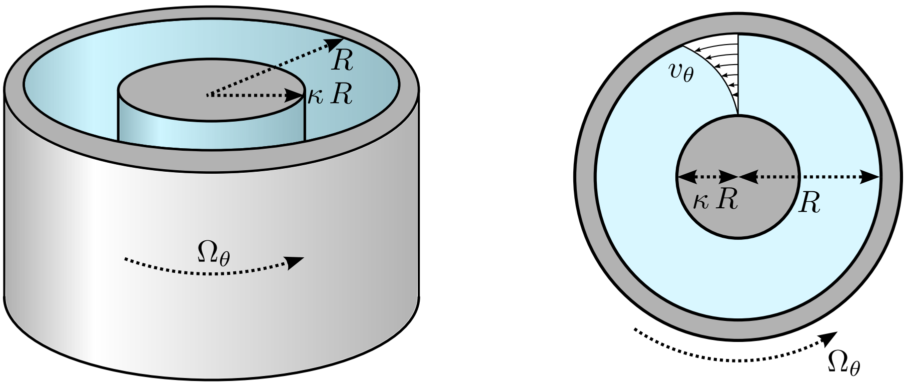


Figure 7: A simplified diagram of a Couette viscometer.

a) Derive the following expression by solving the continuity equation, given in Eq. (65), for this system.

$$\frac{\partial v_\theta}{\partial \theta} = 0 \tag{9}$$

Clearly state any assumptions you make. What does this tell you about the flow? **[5 marks]**

Solution:
The continuity equation is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{v} = 0$$

If we assume the fluid is **incompressible**, we can cancel the first term and divide out the density to yield

$$\nabla \cdot \rho \boldsymbol{v} = 0$$
$$\nabla \cdot \boldsymbol{v} = \frac{1}{r}\frac{\partial}{\partial r}(r\, v_r) + \frac{1}{r}\frac{\partial\, v_\theta}{\partial \theta} + \frac{\partial\, v_z}{\partial z} = 0$$

where we've expanded the gradient operator in cylindrical coordinates. We assume that the flow is **well-developed** and we can cancel any flow in the z and r directions to yield

$$\frac{\partial v_\theta}{\partial \theta} = 0$$

This indicates that the flow does not change in the θ-direction (it is rotationally symmetric).

b) The velocity profile of the system is given by the following expression:

$$v_\theta = \Omega_0\, R \frac{\frac{\kappa\, R}{r} - \frac{r}{\kappa\, R}}{\kappa - 1/\kappa} \tag{10}$$

Derive the following expression for the stress profile in the system.

$$\tau_{r\theta} = 2\frac{\mu\, \Omega_0\, \kappa^2}{\kappa^2 - 1}\frac{R^2}{r^2} \tag{11}$$

[10 marks]

Solution:
From the datasheet, we know the stress is given by

$$\tau_{r\theta} = -\mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)$$

We can cancel the radial velocity term as the flow is well-developed.

$$\tau_{r\theta} = -\mu\, r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)$$

Inserting in Eq. (10), we have

$$\begin{aligned}
\tau_{r\theta} &= -\mu\, r\frac{\partial}{\partial r}\left(\frac{1}{r}\Omega_0\, R\frac{\frac{\kappa\, R}{r} - \frac{r}{\kappa\, R}}{\kappa - 1/\kappa}\right) \\
&= -\frac{\mu\, \Omega_0\, R}{\kappa - 1/\kappa}\, r\frac{\partial}{\partial r}\left(\frac{1}{r}\left(\frac{\kappa\, R}{r} - \frac{r}{\kappa\, R}\right)\right) \\
&= -\frac{\mu\, \Omega_0\, R}{\kappa - 1/\kappa}\, r\frac{\partial}{\partial r}\left(\frac{\kappa\, R}{r^2} - \frac{1}{\kappa\, R}\right) \\
&= -\frac{\mu\, \Omega_0\, R}{\kappa - 1/\kappa}\, r\left(\frac{-2\,\kappa\, R}{r^3} + 0\right) \\
&= 2\frac{\mu\, \Omega_0\, R}{\kappa - 1/\kappa}\frac{\kappa\, R}{r^2} \\
&= 2\frac{\mu\, \Omega_0\, \kappa^2}{\kappa^2 - 1}\frac{R^2}{r^2}
\end{aligned}$$

c) Derive the following expression for the torque exerted on the outer surface ($r = R$) to keep the fluid in motion.

$$\mathcal{T} = 4\pi\,R^2\,L\frac{\mu\,\Omega_0\,\kappa^2}{\kappa^2-1}$$

where L is the length of the viscometer.
Note: The torque is the total magnitude of a tangential force, such as the viscous stress $\tau_{r\theta}$, multiplied by the radial distance at which it acts. **[3 marks]**
Solution:
Take the expression for the stress and calculate it at the outer surface $r = R$, to give

$$\tau_{r\theta} = 2\frac{\mu\,\Omega_0\,\kappa^2}{\kappa^2-1}\frac{\cancelto{1}{R^2}}{\cancel{R^2}}$$

The surface area of the outer cylinder is $2\,\pi\,R\,L$, thus the total force exerted on that face is

$$4\pi\,R\,L\frac{\mu\,\Omega_0\,\kappa^2}{\kappa^2-1}$$

The torque is then

$$\mathcal{T} = 4\pi\,R^2\,L\frac{\mu\,\Omega_0\,\kappa^2}{\kappa^2-1}$$

d) The torque is measured during the operation of the viscometer. How are the viscous properties of the flow determined? **[2 marks]**
Solution:
The torque is directly proportional to the viscosity of the system, thus the answer to the previous question may be used to directly determine it.

Extra credit if the student notes that the stress profile is not linear in the system, as this makes it difficult to solve for the properties of non-Newtonian fluids.

[Question total: 20 marks]

Q.21 Question 21

Example exam question with marks:
Coil-tubing is being removed from an oil and gas well. This may be modelled as a cylindrical rod, radius R_1, moving upwards along the axis of a vertical cylindrical tube with inner radius R_2, at velocity, U (see Fig. 8). Water flows freely in the annular gap between the rod and the tube wall.
Note: You may ignore the effects of pressure gradients in this question.

a) Define the coordinate system you will use and the boundary conditions of the flow. **[3 marks]**
Solution:
A cylindrical coordinate system will be the most convenient for this system as there is an axis of symmetry. There are three coordinates in a cylindrical flow, r, θ, and z. The axial z-direction will be the vertical direction in this case. We will only consider flow in the z-direction.

There are two non-slip boundary conditions for the flow in the z-direction.

$$v_z(r = R_1) = U \qquad\qquad v_z(r = R_2) = 0$$

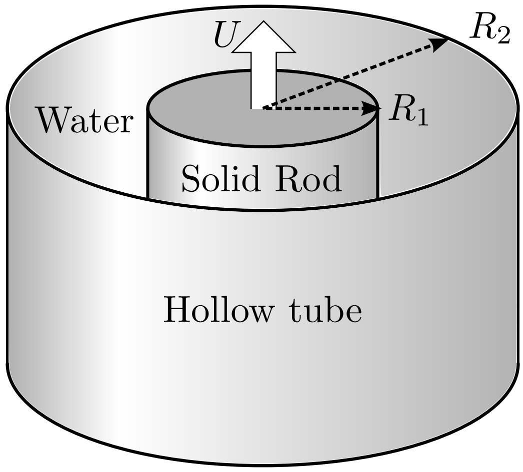


Figure 8: Flow of water within a vertical annulus.

b) Simplify the continuity equation for this system. What are your assumptions and what does your result tell you about the flow along the annulus? **[4 marks]**
Solution:
If the fluid is incompressible, we have

$$\overbrace{\cancel{\frac{\partial \rho}{\partial t}}}^{0} + \nabla \cdot \rho\,\boldsymbol{v} = 0$$

If the fluid is incompressible (ρ = constant), we can also divide both sides by the density to yield

$$\nabla \cdot \boldsymbol{v} = 0$$

It's not straightforward to use index notation in curvelinear coordinates, so we resort to looking up the definitions in the tables in the datasheet. In cylindrical coordinates,

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r}\frac{\partial}{\partial r}(r\, v_r) + \frac{1}{r}\frac{\partial\, v_\theta}{\partial \theta} + \frac{\partial\, v_z}{\partial z}$$

If the flow is laminar and the flow is well developed, the flow in the θ and r directions must be zero, $v_r = v_\theta = 0$. This leaves us with

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\, \overbrace{\cancel{v_r}}^{0}\right) + \frac{1}{r}\frac{\partial\, \overbrace{\cancel{v_\theta}}^{0}}{\partial \theta} + \frac{\partial\, v_z}{\partial z} = 0$$

$$\frac{\partial v_z}{\partial z} = 0$$

This is a statement that the steady-state velocity profile does not vary along the axis.

c) Derive the following balance equation for the momentum. You may assume that water is a Newtonian fluid, the flow is well developed, at steady state, and that any effect of pressure can be ignored. **[5 marks]**

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,\tau_{rz}) = \rho\, g_z$$

Solution:
We are interested in the flow in the z direction, so we should take the z-component of the Navier-Stokes equation

$$\rho \frac{\partial v_z}{\partial t} = -\rho \left[\boldsymbol{v} \cdot \nabla \boldsymbol{v}\right]_z - \left[\nabla \cdot \boldsymbol{\tau}\right]_z - \left[\nabla p\right]_z + \rho g_z$$

we can immediately eliminate the time derivative $\cancelto{0}{\frac{\partial v_z}{\partial t}}$ as we are at steady state AND cancel the pressure term as we are allowed to ignore it in this particular case to give us

$$-\rho \left[\boldsymbol{v} \cdot \nabla \boldsymbol{v}\right]_z - \left[\nabla \cdot \boldsymbol{\tau}\right]_z + \rho g_z = 0$$

The first term always disappears in this course as we are treating incompressible flow. To demonstrate this, we look up this term in the datasheet:

$$\left[\boldsymbol{v} \cdot \nabla \boldsymbol{v}\right]_z = v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}$$

We know that $v_r = v_\theta = 0$ as the flow is well developed and the geometry will not allow flow in that direction so we can immediately delete the first two terms. We also know from the continuity equation that $\partial v_z / \partial z = 0$, thus this entire term is zero.Next we consider the stress term. Looking it up in the datasheet,

$$\left[\nabla \cdot \boldsymbol{\tau}\right]_z = \frac{1}{r} \frac{\partial}{\partial r} \left(r \tau_{rz}\right) + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}$$

Wherever there is symmetry in well-developed flow, the stresses must be zero. We note that the problem is rotationally symmetric in θ and we have $v_\theta = 0$, thus $\tau_{\theta z} = 0$. From the continuity equation we have $\nabla_z v_z = 0$ and $\nabla \cdot \boldsymbol{v} = 0$ thus τ_{zz}. Cancelling those terms leaves

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \tau_{rz}\right) + \frac{1}{r} \frac{\partial \cancelto{0}{\tau_{\theta z}}}{\partial \theta} + \frac{\partial \cancelto{0}{\tau_{zz}}}{\partial z} = -\rho g_z$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \tau_{rz}\right) = \rho g_z$$

d) Derive the following expression for the velocity profile of the fluid within the tube. **[4 marks]**

$$v_z = -\frac{\rho g_z r^2}{4 \mu} + \frac{C_1}{\mu} \ln r + C_2$$

where C_1 and C_2 are unknown integration constants.

Solution:
As the fluid is Newtonian, $\tau_{rz} = -\mu \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right)$. The last term is zero as $v_r = 0$ if the flow is well-developed. Inserting this expression into the result of the previous equation

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \mu \frac{\partial v_z}{\partial r}\right) = -\rho g_z$$

Starting with the equation from the previous question, we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\mu\frac{\partial v_z}{\partial r}\right) = -\rho\,g_z$$
$$\frac{\partial}{\partial r}\left(r\,\mu\frac{\partial v_z}{\partial r}\right) = -r\,\rho\,g_z$$
$$r\,\mu\frac{\partial v_z}{\partial r} = -\frac{r^2}{2}\,\rho\,g_z + C_1$$
$$\frac{\partial v_z}{\partial r} = -\frac{r\,\rho\,g_z}{2\,\mu} + \frac{C_1}{\mu\,r}$$
$$v_z = -\frac{\rho\,g_z\,r^2}{4\,\mu} + \frac{C_1}{\mu}\ln r + C_2$$

e) Using the boundary conditions, solve for the constants C_1 and C_2. **[2 marks]**
Solution:
Using the boundary conditions from the first question, we have

$$v_z(r = R_1) = U \qquad\qquad v_z(r = R_2) = 0$$
$$U = -\frac{R_1^2}{4\,\mu}\,\rho\,g_z + \frac{C_1}{\mu}\ln R_1 + C_2 \qquad\qquad 0 = -\frac{R_2^2}{4\,\mu}\,\rho\,g_z + \frac{C_1}{\mu}\ln R_2 + C_2$$

We can solve these for the constants C_1 and C_2.

$$\begin{aligned}
C_2 &= \frac{R_2^2}{4\,\mu}\,\rho\,g_z - \frac{C_1}{\mu}\ln R_2\\
U &= -\frac{R_1^2}{4\,\mu}\,\rho\,g_z + \frac{C_1}{\mu}\ln R_1 + \frac{R_2^2}{4\,\mu}\,\rho\,g_z - \frac{C_1}{\mu}\ln R_2\\
&= \frac{R_2^2 - R_1^2}{4\,\mu}\,\rho\,g_z + \frac{C_1}{\mu}\ln\left(R_1/R_2\right)\\
C_1 &= \frac{\mu\,U}{\ln\left(R_1/R_2\right)} - \frac{R_2^2 - R_1^2}{4\ln\left(R_1/R_2\right)}\,\rho\,g_z\\
C_2 &= \frac{R_2^2}{4\,\mu}\,\rho\,g_z - \frac{C_1}{\mu}\ln R_2\\
&= \frac{R_2^2}{4\,\mu}\,\rho\,g_z - \frac{U\ln R_2}{\ln\left(R_1/R_2\right)} + \ln R_2\frac{R_2^2 - R_1^2}{4\,\mu\ln\left(R_1/R_2\right)}\,\rho\,g_z
\end{aligned}$$

We will later use the dimensionless variable $\lambda = R_2/R_1$, so it will be convenient to rewrite the constants now using the following identites, $\ln R_1/R_2 = -\ln R_2/R_1 = -\ln\lambda$

$$\begin{aligned}
C_1 &= -\frac{\mu\,U}{\ln\left(R_2/R_1\right)} + \frac{R_2^2 - R_1^2}{4\ln\left(R_2/R_1\right)}\,\rho\,g_z\\
C_2 &= \frac{R_2^2}{4\,\mu}\,\rho\,g_z + \frac{U\ln R_2}{\ln\left(R_2/R_1\right)} - \ln R_2\frac{R_2^2 - R_1^2}{4\,\mu\ln\left(R_2/R_1\right)}\,\rho\,g_z
\end{aligned}$$

Substituting back into the original equation, we have

$$v_z = \frac{\rho\,g_z\left(R_2^2 - r^2\right)}{4\,\mu} - \frac{U}{\ln\left(R_2/R_1\right)}\ln\left(r/R_2\right) + \frac{R_2^2 - R_1^2}{4\,\mu\ln\left(R_2/R_1\right)}\,\rho\,g_z\ln\left(r/R_2\right)$$
$$v_z = \frac{\rho\,g_z}{4\,\mu}\left(R_2^2 - r^2 + \frac{\ln\left(r/R_2\right)\left(R_2^2 - R_1^2\right)}{\ln\left(R_2/R_1\right)}\right) - \frac{U}{\ln\left(R_2/R_1\right)}\ln\left(r/R_2\right)$$

To clean this up, we move to a variable $\lambda = R_2/R_1$.

$$
\begin{aligned}
v_z &= \frac{\rho\, g_z}{4\,\mu}\left(R_2^2 - r^2 + \frac{\ln(r/R_2)\left(R_2^2 - R_1^2\right)}{\ln\lambda}\right) - \frac{U}{\ln\lambda}\ln(r/R_2) \\
&= \frac{\rho\, g_z}{4\,\mu}\left(R_2^2 - r^2 + \frac{\ln((r/R_1)/\lambda)\left(R_2^2 - R_1^2\right)}{\ln\lambda}\right) - \frac{U}{\ln\lambda}\ln((r/R_1)/\lambda) \\
&= \frac{\rho\, g_z}{4\,\mu}\left(R_2^2 - r^2 - R_2^2 + R_1^2 + \frac{\ln(r/R_1)\left(R_2^2 - R_1^2\right)}{\ln\lambda}\right) + U\left(1 - \frac{\ln(r/R_1)}{\ln\lambda}\right) \\
&= \frac{\rho\, g_z\, R_1^2}{4\,\mu}\left(1 - \frac{r^2}{R_1^2} + \left(\lambda^2 - 1\right)\frac{\ln(r/R_1)}{\ln\lambda}\right) + U\left(1 - \frac{\ln(r/R_1)}{\ln\lambda}\right)
\end{aligned}
$$

f) After using the boundary conditions to solve for the constants C_1 and C_2, the velocity profile was determined to be

$$
v_z = \frac{\rho\, g_z\, R_1^2}{4\,\mu}\left(1 - \frac{r^2}{R_1^2} + \left(\lambda^2 - 1\right)\frac{\ln(r/R_1)}{\ln\lambda}\right) + U\left(1 - \frac{\ln(r/R_1)}{\ln\lambda}\right)
$$

where $\lambda = R_2/R_1$. What is the average velocity of water in the annulus?
Note: You may need the integration identity

$$
\int x\,\ln(x)\,\mathrm{d}x = \frac{x^2}{2}\left(\ln(x) - \frac{1}{2}\right)
$$

Solution:
We need to find expression for the volumetric flow rate $\dot{V}_z$ and the velocity U at which the volumetric flow rate is zero. The volumetric flow rate is given by

$$
\begin{aligned}
\dot{V}_z &= \int_{R_1}^{R_2} 2\,\pi\, r\, v_z\,\mathrm{d}r \\
&= \frac{\pi\,\rho\, g_z\, R_1^2}{2\,\mu}\int_{R_1}^{R_2}\left(r - \frac{r^3}{R_1^2} + \left(\lambda^2 - 1\right)\frac{r\ln(r/R_1)}{\ln\lambda}\right)\mathrm{d}r + 2\,\pi\, U\int_{R_1}^{R_2}\left(r - \frac{r\ln(r/R_1)}{\ln\lambda}\right)\mathrm{d}r
\end{aligned}
$$

Making a change of variables $x = r/R_1$, giving $\mathrm{d}r = R_1\,\mathrm{d}x$

$$
\begin{aligned}
\dot{V}_z &= \frac{\pi\,\rho\, g_z\, R_1^4}{2\,\mu}\int_1^{\lambda}\left(x - x^3 + \left(\lambda^2 - 1\right)\frac{x\ln x}{\ln\lambda}\right)\mathrm{d}x + 2\,\pi\, U\, R_1^2\int_1^{\lambda}\left(x - \frac{x\ln x}{\ln\lambda}\right)\mathrm{d}x \\
&= \frac{\pi\,\rho\, g_z\, R_1^4}{4\,\mu}\left[x^2 - \frac{x^4}{2} + \frac{x^2\left(\lambda^2 - 1\right)}{\ln\lambda}\left(\ln x - \frac{1}{2}\right)\right]_1^{\lambda} + \pi\, U\, R_1^2\left[x^2 - \frac{x^2}{\ln\lambda}\left(\ln x - \frac{1}{2}\right)\right]_1^{\lambda}
\end{aligned}
$$

Need more lines!

$$
\begin{aligned}
\dot{V}_z = \frac{\pi\,\rho\, g_z\, R_1^4}{4\,\mu}&\left(\lambda^2 - 1 - \frac{\lambda^4 - 1}{2} + \frac{\lambda^2\left(\lambda^2 - 1\right)}{\ln\lambda}\left(\ln\lambda - \frac{1}{2}\right) + \frac{\lambda^2 - 1}{2\,\ln\lambda}\right) \\
&+ \pi\, U\, R_1^2\left(\lambda^2 - 1 - \frac{\lambda^2}{\ln\lambda}\left(\ln\lambda - \frac{1}{2}\right) - \frac{1}{2\,\ln\lambda}\right)
\end{aligned}
$$

Factoring out a $\lambda^2 - 1$ term in the first term, and simplifying the second...

$$\dot{V}_z = \frac{\pi\,\rho\,g_z\,R_1^4}{4\,\mu}\left(\lambda^2 - 1\right)\left(1 - \frac{\lambda^2+1}{2} + \frac{\lambda^2}{\ln\lambda}\left(\ln\lambda - \frac{1}{2}\right) + \frac{1}{2\,\ln\lambda}\right) + \pi\,U\,R_1^2\left(\frac{\lambda^2-1}{2\,\ln\lambda} - 1\right)$$

Simplifying the first, and back to single line equations

$$\dot{V}_z = \frac{\pi\,\rho\,g_z\,R_1^4}{8\,\mu}\left(\lambda^2 - 1\right)\left(1 + \lambda^2 - \frac{\lambda^2-1}{\ln\lambda}\right) + \pi\,U\,R_1^2\left(\frac{\lambda^2-1}{2\,\ln\lambda} - 1\right)$$

The average velocity is given by the flow-rate divided by the flow area $A = \pi(R_2^2 - R_1^2) = \pi\,R_1^2(\lambda^2 - 1)$

$$\langle v_z\rangle = \frac{\dot{V}_z}{A} = \frac{\rho\,g_z\,R_1^2}{8\,\mu}\left(1 + \lambda^2 - \frac{\lambda^2-1}{\ln\lambda}\right) + \frac{U}{\lambda^2-1}\left(\frac{\lambda^2-1}{2\,\ln\lambda} - 1\right)$$

g) Given a flow system with dimensions of $R_1 = 10$ mm and $R_2 = 11$ mm, at what speed, U, does the rod need to be moved upwards so that there is no net upwards or downwards flow of the fluid? Water has a viscosity of $\mu = 8.9 \times 10^{-4}$ Pa s and a density of $\rho = 1000$ kg m^{-3}. The z-component of gravity is given by $g_z = -9.81$ m s^{-2}. The average flow velocity in the annulus is given by

$$\langle v_z\rangle = \frac{\rho\,g_z\,R_1^2}{8\,\mu}\left(1 + \lambda^2 - \frac{\lambda^2-1}{\ln\lambda}\right) + \frac{U}{\lambda^2-1}\left(\frac{\lambda^2-1}{2\,\ln\lambda} - 1\right)$$

where $\lambda = R_2/R_1$. **[2 marks]**

Solution:

We need to find the velocity where the volumetric flow rate is zero. The volumetric flow rate is given by the average velocity times by the cross-sectional area of the flow $\dot{V} = \langle v_z\rangle\,A$. This means that the average velocity must be zero if the net flow is zero.

Rearranging the above expression for the velocity U and setting $\langle v_z\rangle = 0$, we have

$$\begin{aligned} U &= -\frac{\rho\,g_z\,R_1^2\,\left(\lambda^2-1\right)}{8\,\mu}\left(1 + \lambda^2 - \frac{\lambda^2-1}{\ln\lambda}\right)\left(\frac{\lambda^2-1}{2\,\ln\lambda} - 1\right)^{-1} \\ &= -\frac{1000 \times 9.81 \times 0.01^2\left(1.1^2-1\right)}{8 \times 8.9 \times 10^{-4}}\left(1 + 1.1^2 - \frac{1.1^2-1}{\ln 1.1}\right)\left(\frac{1.1^2-1}{2\,\ln 1.1} - 1\right)^{-1} \\ &\approx 1.90\ \mathrm{m\,s^{-1}} \end{aligned}$$

h) Given a flow system with dimensions of $R_1 = 50$ mm and $R_2 = 51$ mm, at what speed, U, does the rod need to be moved upwards so that there is no net upwards or downwards flow of the fluid? Water has a viscosity of $\mu = 8.9 \times 10^{-4}$ Pa s and a density of $\rho = 1000$ kg m^{-3}. The z-component of gravity is given by $g_z = -9.81$ m s^{-2}. **[2 marks]**

Solution:

As above, but

$$U \approx 1.849\ \mathrm{m\,s^{-1}}$$

[Question total: 22 marks]

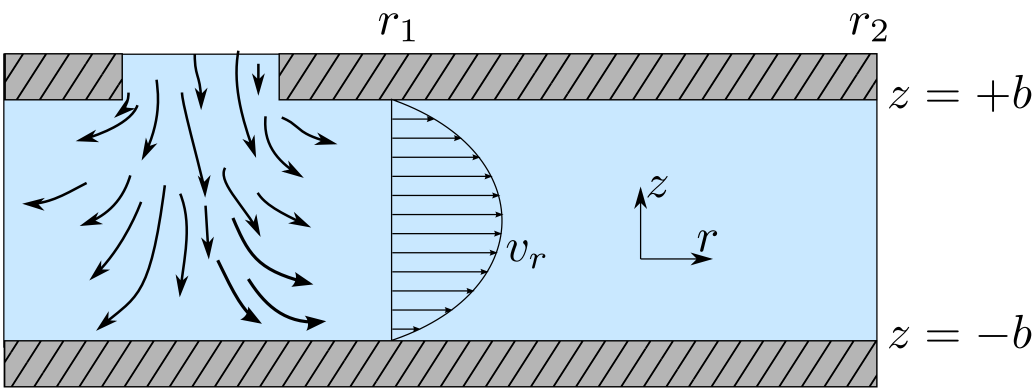


Figure 9: Radial flow between two plates.

Q.22 Question 22

Example exam question (2015)

Oil is used to lubricate two horizontal parallel plates by injecting it and allowing it to flow radially outwards from the point of injection (see Fig. 9). The fluid is flowing radially as there is a pressure difference of $P_1 - P_2$ between the inner and outer radii r_1 and r_2 respectively.

a) Simplify the continuity equation to demonstrate that $r\,v_r$ is a function of z only. **[5 marks]**

Solution:

Assume the oil is incompressible:

$$\cancelto{0}{\frac{\partial \rho}{\partial t}} = -\nabla\cdot\rho\,\boldsymbol{v}$$

$$\nabla\cdot\boldsymbol{v} = 0$$

In cylindrical coordinates:

$$\nabla\cdot\boldsymbol{v} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,v_r\right) + \frac{1}{r}\frac{\partial\,v_\theta}{\partial\theta} + \frac{\partial\,v_z}{\partial z}$$

If the flow is laminar and well-developed by the time it reaches r_1, then we can state that $v_\theta = 0$ and $v_z = 0$. This gives

$$\frac{\partial}{\partial r}\left(r\,v_r\right) = 0$$

Which implies that $r\,v_r$ is a constant of r (i.e., independent of r). Note that this implies that the velocity is proportional to the inverse of the radius (i.e., $v_r \propto r^{-1}$)! There is no reason to believe the system will not also be rotationally symmetric in θ, therefore $r\,v_r$ must only be a function of z.

b) Demonstrate that the stress profile within the channel is a solution of the following equation: **[10 marks]**

$$\rho\,v_r\frac{\partial v_r}{\partial r} = \mu\left(2\frac{\partial^2 v_r}{\partial r^2} + \frac{2}{r}\frac{\partial v_r}{\partial r} - \frac{2\,v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2}\right) - \frac{\partial\,p}{\partial r}$$

Note You must be careful during your derivation and make sure you expand each term of $\boldsymbol{\tau}$ before cancellation.

Solution:
Take the Cauchy momentum equation:

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} = -\rho\, \boldsymbol{v}\cdot\nabla\boldsymbol{v} - \nabla\cdot\boldsymbol{\tau} - \nabla\, p + \rho\, \boldsymbol{g}$$

Assume steady state:

$$\rho\, \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla\cdot\boldsymbol{\tau} - \nabla\, p + \rho\, \boldsymbol{g}$$

Taking the r-component

$$[\rho\, \boldsymbol{v}\cdot\nabla\boldsymbol{v}]_r = -[\nabla\cdot\boldsymbol{\tau}]_r - [\nabla\, p]_r + \rho\, \cancelto{0}{g_r}$$

Where the gravity term is dropped as the plates are horizontal. Inserting the relevant definition for cylindrical flow for the left hand side:

$$\begin{aligned}
[\rho\, \boldsymbol{v}\cdot\nabla\boldsymbol{v}]_r &= \rho\left(v_r\frac{\partial v_r}{\partial r} + \frac{\cancelto{0}{v_\theta}}{r}\frac{\partial v_r}{\partial \theta} - \frac{\cancelto{0}{v_\theta^2}}{r} + \cancelto{0}{v_z}\frac{\partial v_r}{\partial z}\right)\\
&= \rho\, v_r\frac{\partial v_r}{\partial r}
\end{aligned}$$

For the stress term, we have:

$$[\nabla\cdot\boldsymbol{\tau}]_r = \frac{1}{r}\frac{\partial}{\partial r}(r\,\tau_{rr}) + \frac{1}{r}\frac{\partial\,\tau_{r\theta}}{\partial\theta} - \frac{1}{r}\tau_{\theta\theta} + \frac{\partial\,\tau_{rz}}{\partial z}$$

We know that $v_\theta = 0$ and $v_z = 0$ as the flow is assumed to be well developed. From symmetry we also know that the derivative in the θ direction is also zero. We also know that $\nabla\cdot\boldsymbol{v} = 0$ from the continuity equation. Expanding each term of the stress:

$$\begin{aligned}
\tau_{rr} &= -2\,\mu\frac{\partial v_r}{\partial r} + \mu^B\,\cancelto{0}{\nabla\cdot\boldsymbol{v}}\\
\tau_{r\theta} &= -\mu\left(r\frac{\partial}{\partial r}\left(\frac{\cancelto{0}{v_\theta}}{r}\right) + \frac{1}{r}\cancelto{0}{\frac{\partial}{\partial\theta}}\, v_r\right)\\
\tau_{\theta\theta} &= -2\,\mu\left(\frac{1}{r}\frac{\partial \cancelto{0}{v_\theta}}{\partial\theta} + \frac{v_r}{r}\right) + \mu^B\,\cancelto{0}{\nabla\cdot\boldsymbol{v}}\\
\tau_{rz} &= -\mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial \cancelto{0}{v_z}}{\partial r}\right)
\end{aligned}$$

Inserting these definitions back in, we have

$$[\nabla\cdot\boldsymbol{\tau}]_r = -\mu\left(\frac{2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) - \frac{2\,v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2}\right)$$

Placing these back in the stress equation, we have:

$$\rho\, v_r\frac{\partial v_r}{\partial r} = \mu\left(\frac{2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) - \frac{2\,v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2}\right) - \frac{\partial\, p}{\partial r}$$

Performing the product rule:

$$\rho\, v_r \frac{\partial v_r}{\partial r} = \mu \left(2\frac{\partial^2 v_r}{\partial r^2} + \frac{2}{r}\frac{\partial v_r}{\partial r} - \frac{2\, v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2} \right) - \frac{\partial p}{\partial r}$$

End of question solution

The next part is just for revision to show the link to the next section.

We note that:

$$2\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r\, v_r = 2\frac{\partial^2 v_r}{\partial r^2} + \frac{2}{r}\frac{\partial v_r}{\partial r} - \frac{2\, v_r}{r^2}$$

As $r\, v_r$ is only a function of z, then this equation is zero giving:

$$\rho\, v_r \frac{\partial v_r}{\partial r} = \mu\, \frac{\partial^2 v_r}{\partial z^2} - \frac{\partial p}{\partial r}$$

As we know that $r\, v_r = f(z)$, we make the replacement $v_r = f(z)/r$.

$$\rho\, \frac{f}{r} \frac{\partial f\, r^{-1}}{\partial r} = \mu\, \frac{1}{r}\frac{\partial^2 f}{\partial z^2} - \frac{\partial p}{\partial r}$$

The left hand side simplifies:

$$\begin{aligned} \rho\, \frac{f}{r} \frac{\partial f\, r^{-1}}{\partial r} &= \rho\, \frac{f}{r} \left(r^{-1} \cancelto{0}{\frac{\partial f}{\partial r}} - f\, r^{-2} \right) \\ &= -\rho\, \frac{f^2}{r^3} \end{aligned}$$

Which gives:

$$-\rho\, \frac{f^2}{r^3} = \mu\, \frac{1}{r}\frac{\partial^2 f}{\partial z^2} - \frac{\partial p}{\partial r}$$

This equation is difficult to solve, in fact, there is no solution unless we neglect the non-linear term. This is one instance of the creeping flow assumption.

$$r\frac{\partial p}{\partial r} = \mu \frac{\partial^2 f}{\partial z^2}$$

At this point we assume the pressure is only a function of r, and then as both sides are independent of each other they must be constants. Integrating with respect to r:

$$\frac{\Delta P}{\ln(r_2/r_1)} = \mu \frac{\partial^2 f}{\partial z^2}$$

Now integrating twice with respect to z:

$$\begin{aligned} f &= -\frac{\Delta P}{2\,\mu \ln(r_2/r_1)} \left(z^2 + C_1\, z + C_2 \right) \\ v_r &= -r^{-1} \frac{\Delta P}{2\,\mu \ln(r_2/r_1)} \left(z^2 + C_1\, z + C_2 \right) \end{aligned}$$

c) Using the creeping flow assumption, the following expression for the velocity profile was derived **[5 marks]**:

$$v_r = -r^{-1}\frac{\Delta P}{2\,\mu\ln(r_2/r_1)}\left(z^2 + C_1\,z + C_2\right)$$

Determine the integration constants C_1 and C_2, and give the final expression for the velocity profile:

Solution:

As $v_r = 0$ at $z = \pm b$, we have $C_1 = 0$ and $C_2 = -b^2$. The final expression is

$$v_r = r^{-1}\frac{\Delta P}{2\,\mu\ln(r_2/r_1)}\left(b^2 - z^2\right)$$

[Question total: 20 marks]

Q.23 Question 23

Example exam question (2016)

A wire-coating die consists of a cylindrical wire of radius, $\kappa\,R$, moving horizontally at a constant velocity, v_{wire}, along the axis of a cylindrical die of radius, R. You may assume the pressure is constant within the die (it is not pressure driven flow) but the flow is driven by the motion of the wire (it is "axial annular Couette flow"). Neglect end effects and assume an isothermal system.

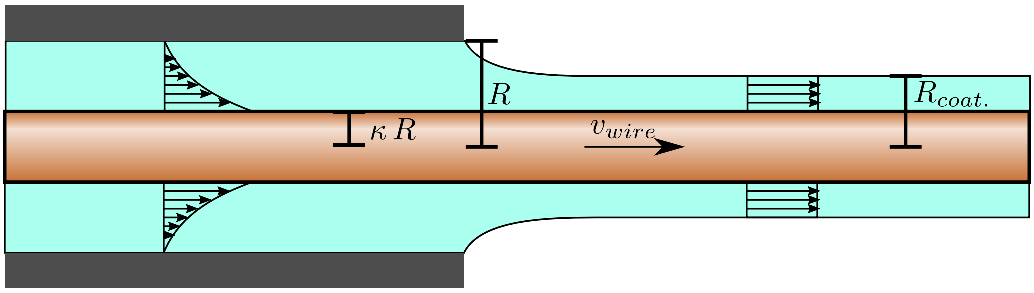


Figure 10: A diagram of a wire coating die for Q. 23.

a) State the two relevant boundary conditions for the flow within the die and how they arise. **[2 marks]**

Solution:

[1/2] Both conditions arise from non-slip conditions of the fluid with a solid boundary. $\checkmark_1$

- $v_z(r = R) = 0$: At the die wall interface.
- $v_z(r = \kappa\,R) = v_{wire}$: At the wire interface.

[1/2] $\checkmark_1$

b) The stress profile for an annular system is of the following form

$$\frac{1}{r}\frac{\partial}{\partial r}r\,\tau_{rz} = -\frac{\partial p}{\partial z} + \rho\,g_z.$$

Derive the following expression for the flow profile

$$v_z = \frac{v_{wire}}{\ln\kappa}\ln\left(\frac{r}{R}\right).$$

[9 marks]

Solution:

There is no driving pressure gradient, and as the flow is horizontal, the two terms on the right hand side are zero

$$\frac{1}{r}\frac{\partial}{\partial r} r\,\tau_{rz} = -\frac{\partial \cancelto{0}{p}}{\partial z} + \rho\, \cancelto{0}{g_z}.$$

[2/9] $\checkmark_2$

Performing the integration of the stress profile,

$$\tau_{rz} = \frac{C_1}{r}.$$

[1/9] $\checkmark_1$

Assuming the fluid is Newtonian, we have

$$-\mu\frac{\partial v_z}{\partial r} = \frac{C_1}{r}.$$

[1/9] $\checkmark_1$

Performing the integration

$$v_z = -\mu^{-1}\, C_1 \ln\, r + C_2.$$

[1/9] $\checkmark_1$

Inserting the two boundary conditions yields the following

$$0 = -\mu^{-1}\, C_1 \ln R + C_2.$$
$$v_{wire} = -\mu^{-1}\, C_1 \ln \kappa\, R + C_2.$$

[1/9] $\checkmark_1$

Solving both equations for the constants,

$$C_2 = \mu^{-1}\, C_1 \ln R$$
$$v_{wire} = \mu^{-1}\, C_1 (\ln R - \ln \kappa\, R)$$
$$C_1 = -\frac{\mu\, v_{wire}}{\ln \kappa}.$$

[2/9] $\checkmark_2$

Inserting these back in gives the final expression

$$v_z = \frac{v_{wire}}{\ln \kappa} \ln\left(\frac{r}{R}\right)$$

[1/9] $\checkmark_1$

c) Derive the following expression for the volumetric flow-rate of liquid through the die

$$\dot{V}_z = -\pi\, R^2\, v_{wire} \left(\kappa^2 + \frac{1-\kappa^2}{2\ln\kappa}\right).$$

[5 marks]

Note: You will need the integration identity

$$\int x\,\ln(x)\,\mathrm{d}x = \frac{x^2}{2}\left(\ln(x) - \frac{1}{2}\right).$$

Solution:
To determine the volumetric flow rate, the following integration is performed

$$\dot{V}_z = 2\,\pi \int_{\kappa R}^{R} r\, v_z\, \mathrm{d}r$$

[1/5] $\checkmark_1$

Performing the integration

$$\begin{aligned}
\dot{V}_z &= 2\,\pi\, R \frac{v_{wire}}{\ln\kappa} \int_{\kappa R}^{R} \frac{r}{R} \ln\left(\frac{r}{R}\right) \mathrm{d}r \\
&= \frac{2\,\pi\, R^2\, v_{wire}}{\ln\kappa} \int_{\kappa}^{1} x\, \ln(x)\, \mathrm{d}x \\
&= \frac{2\,\pi\, R^2\, v_{wire}}{\ln\kappa} \left[\frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)\right]_{\kappa}^{1} \\
&= -\frac{2\,\pi\, R^2\, v_{wire}}{\ln\kappa} \left(\frac{\kappa^2}{2}\left(\ln\kappa - \frac{1}{2}\right) + \frac{1}{4}\right) \\
&= -\pi\, R^2\, v_{wire} \left(\kappa^2 + \frac{1-\kappa^2}{2\ln\kappa}\right)
\end{aligned}$$

[4/5] $\checkmark_4$

d) Derive an expression for the outer radius of the coating, $R_{coat.}$, far away from the die exit.

[4 marks]

Solution:
Solving the stress balance again but for the film coating the wire, the following expression is found again for the stress

$$\tau_{rz} = \frac{C_1}{r}$$

At the exposed surface of the film ($r \neq 0$), the stress is zero (assuming the air exerts close to zero drag). This implies that $C_1 = 0$ as well, as it is the only possible way to set the RHS to zero at finite values of r. As the stress is zero, Newton's law of viscosity then implies the film has a constant velocity which will be the velocity of the wire (note, the diagram gives
[2/4] the student a strong hint that this is true).$\checkmark_2$

The volumetric flowrate of the wire coating is related to the outer radius of the coating, $R_{coat.}$

$$\dot{V}_{z,coating} = v_{wire}\,\pi\left(R_{coat.}^2 - \kappa^2\, R^2\right)$$

[1/4] $\checkmark_1$ This must be equal to the volumetric flowrate of coating through the die

$$v_{wire}\,\pi\left(R_{coating}^2-\kappa^2 R^2\right)=-\pi R^2 v_{wire}\left(\kappa^2+\frac{1-\kappa^2}{2\ln\kappa}\right)$$

$$R_{coating}=R\sqrt{\frac{\kappa^2-1}{2\ln\kappa}}$$

[1/4] $\checkmark_1$

[Question total: 20 marks]

Q.24 Question 24

A solid wire is being used to carry electrical current (see Fig. 11).

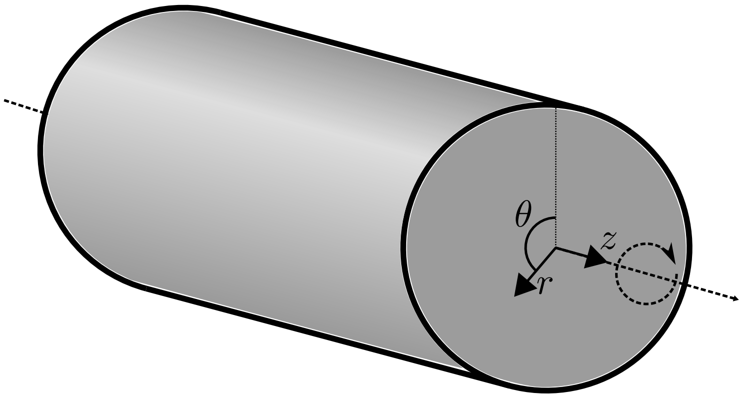


Figure 11: A solid wire.

a) Simplify the differential energy balance equation, state any assumptions you make, and derive the temperature profile in the wire. You may assume that heat is generated constantly within the volume of the wire at the following rate:

$$\sigma_{energy}^{current}=\frac{I^2}{k_e}$$

Solution:

Taking the general balance equation:

$$\rho\,C_p\frac{\partial T}{\partial t}=-\rho\,C_p\,\boldsymbol{v}\cdot\nabla T-\nabla\cdot\boldsymbol{q}-\boldsymbol{\tau}:\nabla\boldsymbol{v}-p\,\nabla\cdot\boldsymbol{v}+\frac{I^2}{k_e}$$

We note that wires are usually made out of solid material (aluminium), so we can state that $\boldsymbol{v}=0$, and cancel all terms with the velocity in them:

$$\rho\,C_p\frac{\partial T}{\partial t}=-\nabla\cdot\boldsymbol{q}+\frac{I^2}{k_e}$$

Assuming the system is at steady state (no severe weather changes or sudden surges in electricity demand), we have

$$\nabla\cdot\boldsymbol{q}=\frac{I^2}{k_e}$$

As our wire is a cylinder, we should use cylindrical coordinates.Our expression becomes

$$\nabla\cdot\boldsymbol{q}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\,q_r\right)+\frac{1}{r}\frac{\partial\,q_\theta}{\partial\theta}+\frac{\partial\,q_z}{\partial z}=\frac{I^2}{k_e}$$

To simplify this problem, we assume that the wire is cooled evenly by the wind so that there is no variance in external temperature with the angle θ or position on the wire z. This makes the problem symmetric in z and θ.

Whenever there is symmetry, there is no transport.This is because there are no gradients or flow across a symmetry as the values of all functions are equal either side of the symmetry. All transport is driven by gradients (convective, Newton's Law, Fourier's Law, and Fick's First Law). Our problem is rotationally symmetric in θ and has translational invariance/symmetry in z so $q_\theta=q_z=0$ and we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,q_r\right)=\frac{I^2}{k_e}$$

Integrating this expression, we have

$$\begin{aligned}r\,q_r&=\frac{I^2}{k_e}\frac{r^2}{2}+C_1'\\ q_r&=\frac{I^2\,R}{k_e}\left(\frac{r}{2\,R}+\frac{C_1\,R}{r}\right)\end{aligned}$$

where the integration constant (C_1') was redefined (to C_1) to bring it inside the parenthesis and the terms were made dimensionless. This is not needed (particularly in this case); however, I often do it as it usually makes the values of the integration constants much simpler.

In this case, the centre of the wire (where $r=0$) the heat flux cannot reach infinity so we must have $C=0$. We could also note that at $r=0$ we are on an axis of symmetry and so $q_r=0$, also requiring $C=0$. Our final expression for the heat flux is then:

$$q_r=\frac{I^2}{2\,k_e}r$$

Selecting the correct cylindrical definition of Fourier's law, we have

$$\frac{\partial T}{\partial r}=-\frac{I^2}{2\,k_e\,k}r$$

Assuming k is constant (it actually depends on the temperature), we can integrate this expression to gives us the temperature profile.

$$\begin{aligned}T&=-\frac{I^2}{4\,k_e\,k}r^2+C_2'\\ &=\frac{I^2\,R^2}{4\,k_e\,k}\left(C_2-\frac{r^2}{R^2}\right)\end{aligned}$$

where again the integration constant was redefined and the terms in parenthesis were made dimensionless. We will assume the simple boundary condition that the exterior of the wire is held at a fixed temperature, i.e., $T(r=R)=T_0$, to solve for the constant,

$$C_2=1+\frac{4\,k_e\,k}{I^2\,R^2}T_0$$

which yields the final expression.

$$T-T_0=\frac{I^2\,R^2}{4\,k_e\,k}\left(1-\frac{r^2}{R^2}\right).$$

b) Discuss if the assumptions you have made are realistic.
Solution:
The assumption that the surface of the wire is held at a constant temperature is unrealistic.

The assumption of steady state is also unlikely as these systems are subject to periodic increases in demand, and the weather causes significant fluctuations. The test of this is if the unsteady response of the wire is slow relative to these fluctuations in power and weather.

c) How might the surface boundary condition be improved?
Solution:
A better boundary condition would be to apply a natural convection coefficient at the surface of the wire to link this problem to the bulk air temperature.

[Question end]

Q.25 Question 25
Example exam question
An electric wire of radius 0.5 mm is made of copper (electrical conductivity $k_e = 5.1 \times 10^7$ ohm^{-1} m^{-1} and thermal conductivity $k = 380$ W m^{-1} K^{-1}). It is insulated to an outer radius of 1.5 mm with plastic (thermal conductivity $k = 0.35$ W m^{-1} K^{-1}). The volumetric heat production σ, is given by $\sigma = I^2/k_e$ where I is the current density A/m^2. The ambient air is at 38°C and the heat transfer coefficient from the outer insulated surface to the surrounding air is 8.5 W m^{-2}K^{-1}.

a) Determine the maximum current in amperes that can flow through the wire if no part of the insulation may exceed 93°C. **[8 marks]**
Solution:
At steady state, all heat produced in the wire must leave. The total heat produced is:

$$Q_{total} = \sigma\, V_{wire} = \frac{I^2\,\pi\,R_{inner}^2\,L}{k_e}$$

To solve for the maximum current density I, we need to examine the hottest location in the insulation, which is at the inner surface of the insulation. The total resistance to heat transfer from the air to this inner surface is:

$$\begin{aligned} R_{total} &= R_{cond} + R_{conv} \\ &= \frac{\ln\left(R_{outer}/R_{inner}\right)}{2\,\pi\,L\,k} + \frac{1}{h_{conv}\,2\pi\,R_{outer}\,L} \end{aligned}$$

Given that, at steady state, all heat which is generated in the wire must leave through the insulation to the air, we have:

$$Q_{total} = \frac{T_{ins./copper} - T_{\infty}}{R_{total}}$$

Setting the two expressions for Q_{total} to be equal, we have:

$$\frac{T_{ins./copper} - T_{\infty}}{\frac{\ln(R_{outer}/R_{inner})}{2\,\pi\,L\,k} + \frac{1}{h_{conv}\,2\pi\,R_{outer}\,L}} = \frac{I^2\,\pi\,R_{inner}^2\,L}{k_e}$$

$$\frac{T_{ins./copper} - T_{\infty}}{\frac{\ln(R_{outer}/R_{inner})}{k} + \frac{1}{h_{conv}\,R_{outer}}} = \frac{I^2\,R_{inner}^2}{2\,k_e}$$

$$I = R_{inner}^{-1}\sqrt{\frac{2\,k_e\left(T_{ins./copper} - T_{\infty}\right)}{\frac{\ln(R_{outer}/R_{inner})}{k} + \frac{1}{h_{conv}\,R_{outer}}}}$$

Placing in the values, we can determine the maximum current density to be:

$$I = 0.0005^{-1}\sqrt{\frac{2\times 5.1\times 10^{7}\,(93-38)}{\frac{\ln(0.0015/0.0005)}{0.35}+\frac{1}{8.5\times 0.0015}}}$$
$$= 1.659\times 10^{7}\,\mathrm{A\,m^{-2}}$$

The total maximum current is

$$I\,\pi\,R_{inner}^{2} = 1.659\times 10^{7}\,\pi\,0.0005^{2} = 13.03\mathrm{A}$$

b) Demonstrate that the heat flux in the copper section of the wire is given by the following expression:

$$q_r = \frac{I^2}{2\,k_e}r$$

[8 marks]

Solution:
Taking our general balance equation, we have

$$\rho\,C_p\frac{\partial T}{\partial t} = -\,\rho\,C_p\,\boldsymbol{v}\cdot\nabla\,T - \nabla\cdot\boldsymbol{q} - \boldsymbol{\tau}:\nabla\boldsymbol{v} - p\,\nabla\cdot\boldsymbol{v} + \frac{I^2}{k_e}$$

The wires are made out of solid material, so we can state that $\boldsymbol{v} = 0$, and cancel all terms with the velocity in them:

$$\rho\,C_p\frac{\partial T}{\partial t} = -\,\nabla\cdot\boldsymbol{q} + \frac{I^2}{k_e}$$

Assuming the system is at steady state, we have

$$\nabla\cdot\boldsymbol{q} = \frac{I^2}{k_e}$$

Using cylindrical coordinates our expression becomes

$$\nabla\cdot\boldsymbol{q} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,q_r\right) + \frac{1}{r}\frac{\partial\,q_\theta}{\partial\theta} + \frac{\partial\,q_z}{\partial z} = \frac{I^2}{k_e}$$

Our problem is rotationally symmetric in θ and has translational invariance/symmetry in z so $q_\theta = q_z = 0$ and we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,q_r\right) = \frac{I^2}{k_e}$$

Integrating this expression, we have:

$$q_r = \frac{I^2}{k_e}\frac{r}{2} + \frac{C}{r}q_r = \frac{I^2}{k_e}\left(\frac{r}{2} + \frac{C}{r}\right)$$

In the centre of the wire where $r = 0$, the heat flux cannot reach infinity so we must have $C = 0$. Alternatively, at $r = 0$ we are on an axis of symmetry and so $q_r = 0$, also requiring $C = 0$. Our final expression for the heat flux is then:

$$q_r = \frac{I^2}{2\,k_e}r$$

c) Solve for the temperature profile within the copper wire, assuming the outer surface of the wire is at $T_{crit.}$. **[4 marks]**

Solution:

Selecting the correct definition of Fourier's law, we have

$$\frac{\partial T}{\partial r} = -\frac{I^2}{2\,k_e\,k}r$$

Assuming k is constant, we can integrate this expression to gives us the temperature profile.

$$T = \frac{I^2}{4\,k_e\,k}\left(C - r^2\right)$$

The exterior of the wire ($r = R$) is at the temperature $T = T_{crit.}$, allowing us to solve for the constant C to give:

$$T - T_{crit.} = \frac{I^2\,R^2}{4\,k_e\,k}\left(1 - \frac{r^2}{R^2}\right)$$

[Question total: 20 marks]

Q.26 Question 26

Again consider that we have a cylindrical wire of length L and radius R, generating heat at a rate of I^2/k_e per unit volume. Using a simple (not differential!) energy balance over the whole volume of the wire, what is the total heat generated Q? Compare this to the expression for the heat flux $q(r)$ evaluated at the surface of the wire ($r = R$) which you derived in Q. 24.

Solution:

At steady state, the total heat flux Q out of the wire must be given by the total heat generated. Assuming heat production is homogeneous (I and k_e are constant) within the wire, we can just multiply the volumetric energy production rate (I^2/k_e) by the volume of the wire:

$$Q = \pi\,R^2\,L\,k_e^{-1}\,I^2 \tag{12}$$

If we divide this by the surface area ($2\,\pi\,R\,L$), we obtain the flux at the surface of the wire (this is because all of the heat generated in the wire must leave by convection from the surface):

$$q_{boundary} = \frac{R\,I^2}{2\,k_e} \tag{13}$$

On comparing with the previous solution(s), it is noted that this could be obtained by setting $r = R$ in the solution derived previously,

$$q(r) = \frac{I^2}{2\,k_e}r. \tag{14}$$

Both approaches give consistent results (as expected).

Only relevant once you've studied non-Newtonian flows:

Here, we see the analogy between electrically heated wires and fluid flow in a pipe continues. Here, the boundary flux of heat is of importance, but in Bingham plastic flows we need to estimate the boundary momentum flux (i.e. stress) to understand if the flow is above or below its yield stress. In both cases the expressions are nearly identical.

[Question end]

Q.27 Question 27

In the lectures, we've derived the following integrated expressions for heat transfer in a plate and a pipe:

$$Q_x = \frac{k}{X} A \left(T_{inner} - T_{outer}\right) \qquad\qquad Q_r = \frac{2\,\pi\,L\,k}{\ln\left(R_{outer}/R_{inner}\right)} \left(T_{in} - T_{out}\right) \tag{15}$$

An equivalent equation is required for spherical geometries.

a) What single assumption was made in the derivation energy balance equation (see Eq. (68))?

Solution:
In the derivation of this equation, the pressure dependency of the internal energy was assumed to be small.

$$\mathrm{d}U = C_p \,\mathrm{d}T + \cancelto{0}{\left(\frac{\partial U}{\partial p}\right)_T \mathrm{d}p}$$

b) Simplify the energy balance equation, Eq. (68), to the following expression:

$$\frac{\partial}{\partial r} r^2\, q_r = 0$$

Clearly state any assumptions you make along the way.

Solution:
As we're considering the derivation of an expression for heat transfer in solids, we can say $\boldsymbol{v} = \boldsymbol{0}$. This greatly simplifies the energy balance equation:

$$\rho\, C_p \frac{\partial T}{\partial t} = -\,\cancelto{0}{\rho\, C_p\, v_j \nabla_j\, T} - \nabla_i\, q_i - \cancelto{0}{\tau_{ji} \nabla_j\, v_i} - \cancelto{0}{p \nabla_i\, v_i} + \sigma_{energy}$$

$$\rho\, C_p \frac{\partial T}{\partial t} = -\,\nabla_i\, q_i + \sigma_{energy}$$

Note: We're using index notation here, which is fine even though this is a curvelinear coordinate system provided we don't actualy start to work with individual components. We're essentially working in cartesian coordinates before changing over to cylindrical.

Which is known as the heat equation. Assuming that there is no source of heat, we can cancel the generation term, $\cancelto{0}{\sigma_{energy}}$. If the system is at steady state, the time-derivative also cancels to yield:

$$\nabla_i\, q_i = 0$$

For spherical systems, we must look up the definition of this term (which is actually $\nabla \cdot \boldsymbol{q}$) which gives:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\, q_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(q_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial\, q_\phi}{\partial \phi} = 0$$

If we assume the system is symmetric in θ and ϕ, we can cancel the gradients in those directions to yield:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\, q_r\right) = 0$$

$$\frac{\partial}{\partial r} r^2\, q_r = 0$$

c) Solve for the following equation for the heat flux in spherical shells.

$$q_r = \frac{k}{r^2 \left(R_{inner}^{-1} - R_{outer}^{-1}\right)} \left(T_{inner} - T_{outer}\right)$$

Solution:
Taking the above equation, we can perform the integration immediately to yield:

$$\begin{aligned} \frac{\partial}{\partial r} r^2\, q_r &= 0 \\ r^2\, q_r &= C_1 \\ q_r &= \frac{C_1}{r^2} \end{aligned} \tag{16}$$

We then need the definition of $q_r = -k\frac{\partial T}{\partial r}$, again taken from the data sheet, we have

$$\begin{aligned} -k\frac{\partial T}{\partial r} &= \frac{C_1}{r^2} \\ -k \int_{T_{inner}}^{T_{outer}} \mathrm{d}T &= \int_{R_{inner}}^{R_{outer}} \frac{C_1}{r^2} \mathrm{d}r \\ -k\left(T_{outer} - T_{inner}\right) &= C_1 \left(R_{inner}^{-1} - R_{outer}^{-1}\right) \\ C_1 &= \frac{k}{R_{inner}^{-1} - R_{outer}^{-1}} \left(T_{inner} - T_{outer}\right) \end{aligned}$$

Reinserting this expression for C_1 into Eq. (16), we have

$$q_r = \frac{k}{r^2 \left(R_{inner}^{-1} - R_{outer}^{-1}\right)} \left(T_{inner} - T_{outer}\right)$$

d) Demonstrate that the resistance to heat transfer, for a spherical shell is given by the following expression:

$$R = \frac{1}{UA} = \frac{R_{inner}^{-1} - R_{outer}^{-1}}{4\,\pi\,k}$$

Note: You will need to derive the expression for the overall heat flux, Q_r, and then isolate the $R = 1(UA)$ term.
Solution:
The heat flux multiplied by the surface area is the overall heat flux. At any point in the shell, the surface area is $A_r = 4\,\pi\,r^2$. We have

$$\begin{aligned} Q_r &= A_r\, q_r \\ &= \frac{4\,\pi\,\cancelto{0}{r^2}\,k}{\cancelto{0}{r^2}\left(R_{inner}^{-1} - R_{outer}^{-1}\right)} \left(T_{inner} - T_{outer}\right) \\ &= \frac{4\,\pi\,k}{R_{inner}^{-1} - R_{outer}^{-1}} \left(T_{inner} - T_{outer}\right) \end{aligned}$$

Here, we can see the expected result that the overall heat flux is constant through the shell.

The terms which correspond to the resistance are:

$$Q_r = U\,A\,(T_{inner} - T_{outer})$$

$$R = \frac{1}{UA} = \frac{R_{inner}^{-1} - R_{outer}^{-1}}{4\,\pi\,k}$$

[Question end]

Q.28 Question 28

A spherical nuclear pellet, with an outer radius of 6 cm, is designed to produce 1kW of heat through fission. The heat transfer from the pellet is limited by a 5 mm pyrolytic graphite coating on the surface, which has a thermal conductivity of 240 W m^{-1} K^{-1}. Underneath the graphite is a 1 mm layer of Silicon Carbide reinforcement, which has a thermal conductivity of 4 W cm^{-1} K^{-1}. As the pellet is cooled by forced convection using a gas, the external convective heat transfer coefficient is around 100 W m^{-2} K^{-1}. If the ambient temperature is 150°C, calculate the surface temperature at the interface between the core and the Silicon Carbide.

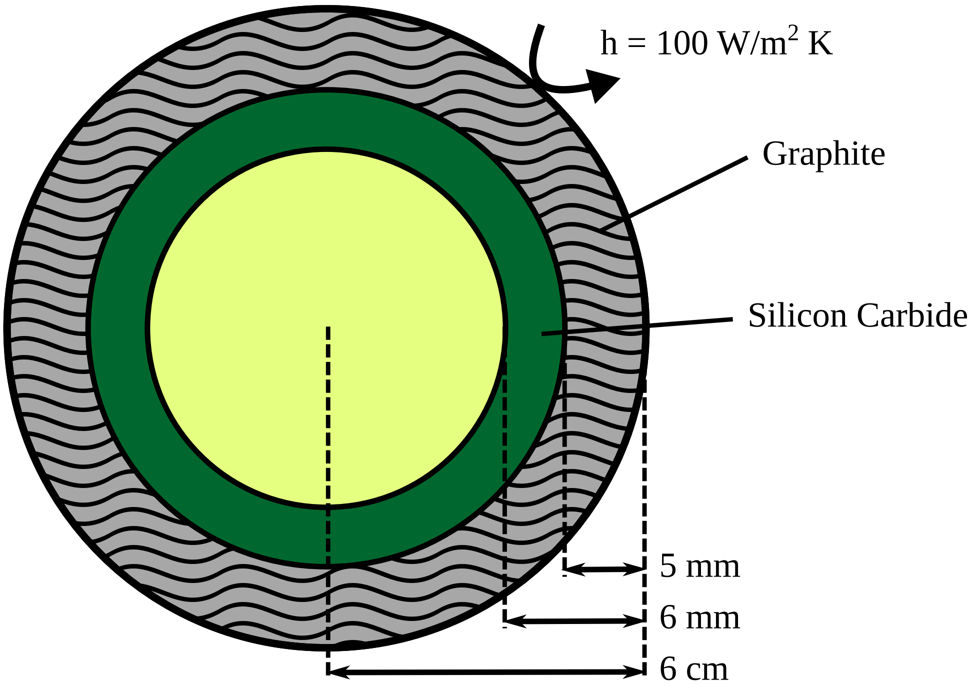


Figure 12: The nuclear pellet described in Q. 28.

Solution:

Here, we have to use the addition of resistances to calculate the internal temperature. There is a resistance resulting from the Silicon Carbide (SiC) layer, from the Graphite (C) layer, and from the convective heat transfer. The overall heat transfer is then given by:

$$Q_r = \frac{1}{R_{SiC} + R_C + R_{conv}}\,(T_{inner} - T_\infty)$$

where R_{SiC} is the resistance (not radius) of the Silicon Carbide layer and R_C is the resistance of the Graphite layer, and $R_{conv} = 1/(h\,A_{outer})$. We can rearrange this expression to make the inner temperature the object

$$T_{inner} = T_\infty + Q_r\,(R_{SiC} + R_C + R_{conv}) \tag{17}$$

The resistance for spheres is given in the datasheet to be:

$$R = \frac{R_{inner}^{-1} - R_{outer}^{-1}}{4\,\pi\,k}$$

For the Graphite layer, we have:

$$R_C = \frac{0.055^{-1} - 0.06^{-1}}{4\,\pi\,240} \approx 5.0 \times 10^{-4}\,\mathrm{W^{-1}K}$$

For the Silicon Carbide layer, we have

$$R_{SiC} = \frac{0.054^{-1} - 0.055^{-1}}{4\,\pi\,400} \approx 6.7 \times 10^{-5}\,\mathrm{W^{-1}K}$$

Given that the surface area of a sphere is $A(r) = 4\,\pi\,r^2$, the convective resistance is

$$R_{conv} = \frac{1}{h_{conv}\,4\,\pi\,R_{outer}^2} = \frac{1}{100 \times 4\,\pi\,0.06^2} \approx 0.221\,\mathrm{W^{-1}K}$$

Inserting these into Eq. (17), we have

$$T_{inner} \approx 150 + 1000\left(6.7 \times 10^{-5} + 5.0 \times 10^{-4} + 0.221\right)$$
$$T_{inner} \approx 370^\circ\mathrm{C}$$

Both layers only provide a small resistance to the heat transfer.
On a related topical note (not part of the course, but part of your embedded safety learning objectives), please read about the Windscale fire, the worst nuclear accident in UK history which occurred when the pyrolytic graphite caught fire in the reactor. This just illustrates the difficulty of controlling heat transfer in complex geometries.

[Question end]

Q.29 Question 29

The temperature profile inside a nuclear fuel rod is needed as part of the design calculations for a reactor. The rod is a cylinder with a radius, R, and is assumed to be composed of a homogeneous fuel which is producing heat with the following profile:

$$\sigma_{heat} = \sigma^0\left(1 + b\left(\frac{r}{R}\right)^2\right) \tag{18}$$

a) What assumption has been made to derive the energy balance equation below?

$$\rho\,C_p\frac{\partial T}{\partial t} = -\rho\,C_p\,\boldsymbol{v}\cdot\nabla\,T - \nabla\cdot\boldsymbol{q} - \boldsymbol{\tau}:\nabla\,\boldsymbol{v} - p\nabla\cdot\boldsymbol{v} + \sigma_{energy}$$

[2 marks]

Solution:

In the derivation of this equation, the pressure dependency of the internal energy was assumed to be small.

$$\mathrm{d}U = C_p\,\mathrm{d}T + \cancelto{0}{\left(\frac{\partial U}{\partial p}\right)_T}\mathrm{d}P$$

[2/2] $\checkmark_2$

b) Simplify the energy balance equation to the following expression:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,q_r) = \sigma_{energy}$$

Clearly state any assumptions you use. **[8 marks]**

Solution:

Starting from the energy balance equation:

$$\rho\,C_p\frac{\partial T}{\partial t} = -\rho\,C_p\,v_j\,\nabla_j\,T - \nabla_i\,q_i - \tau_{ji}\,\nabla_j\,v_i - p\,\nabla_i\,v_i + \sigma_{energy}$$

[1/8] We assume that the system is at **steady state**$\checkmark_1$ to cancel the time derivative.

$$\nabla_i\,q_i = -\rho\,C_p\,v_j\,\nabla_j\,T - \tau_{ji}\,\nabla_j\,v_i - p\,\nabla_i\,v_i + \sigma_{energy}$$

[1/8] $\checkmark_1$ As the nuclear fuel is a **solid**$\checkmark_1$ we can assume $\boldsymbol{v} = 0\checkmark_1$ to cancel most terms, yielding

[1/8]

[1/8]

$$\nabla_i\,q_i = \sigma_{energy}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,q_r) + \frac{1}{r}\frac{\partial\,q_\theta}{\partial\theta} + \frac{\partial\,q_z}{\partial z} = \sigma_{energy}$$

[1/8] $\checkmark_1$ Neglecting **end effects**$\checkmark_1$, we can exploit the **symmetry** of the system to say that any heat

[1/8] transfer in the z and θ directions are zero$\checkmark_1$. This implies that $q_z = 0$ and $q_\theta = 0$, giving the

[1/8] final result

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,q_r) = \sigma_{energy}$$

[1/8] $\checkmark_1$

c) Derive the expression below for the heat flux from the simplified energy balance.

$$q_r = \sigma^0\left(\frac{r}{2} + b\frac{r^3}{4\,R^2}\right) \tag{19}$$

Clearly state any assumptions you use. **[5 marks]**

Solution:

Starting with the result from the previous question, we have

$$\frac{1}{r}\frac{\partial}{\partial r}r\,q_r = \sigma^0\left(1 + b\left(\frac{r}{R}\right)^2\right)$$

$$\frac{\partial}{\partial r}r\,q_r = \sigma^0\left(r + b\frac{r^3}{R^2}\right)$$

$$r\,q_r = \sigma^0\left(\frac{r^2}{2} + b\frac{r^4}{4\,R^2} + C\right)$$

$$q_r = \sigma^0\left(\frac{r}{2} + b\frac{r^3}{4\,R^2} + \frac{C}{r}\right)$$

[3/5] $\checkmark_3$ We know that the heat flux, q_r, at the centre of the rod ($r = 0$) must be finite (it is in fact zero due to the symmetry). Therefore, we must have $C = 0$, which gives the final result

$$q_r = \sigma^0\left(\frac{r}{2} + b\frac{r^3}{4\,R^2}\right)$$

[2/5] $\checkmark_2$

d) Derive the following expression for the temperature profile.

$$T - T_0 = \frac{\sigma^0}{k}\left(\frac{R^2 - r^2}{4} + b\frac{R^4 - r^4}{16\,R^2}\right) \tag{20}$$

You will need to select an appropriate boundary condition and give the meaning of the constant T_0. **[5 marks]**

Solution:

Starting from the answer to the previous question

$$q_r = \sigma^0\left(\frac{r}{2} + b\frac{r^3}{4\,R^2}\right)$$

We insert the definition of the heat flux into the equation to get:

$$\begin{aligned}
-k\frac{\partial T}{\partial r} &= \sigma^0\left(\frac{r}{2} + b\frac{r^3}{4\,R^2}\right)\\
\frac{\partial T}{\partial r} &= -\frac{\sigma^0}{k}\left(\frac{r}{2} + b\frac{r^3}{4\,R^2}\right)\\
T &= -\frac{\sigma^0}{k}\int\left(\frac{r}{2} + b\frac{r^3}{4\,R^2}\right)\mathrm{dr}\\
T &= -\frac{\sigma^0}{k}\left(\frac{r^2}{4} + b\frac{r^4}{16\,R^2}\right) + C
\end{aligned}$$

[2/5] $\checkmark_2$ An appropriate boundary condition for this system is that **the surface of the rod ($r = R$)**
[1/5] **is at a known temperature, T^0.** $\checkmark_1$ Solving for the constant, we have

$$T_0 = -\frac{\sigma^0}{k}\left(\frac{R^2}{4} + b\frac{R^4}{16\,R^2}\right) + C$$

[2/5] $\checkmark_2$ Inserting this expression, we have the final result:

$$T - T_0 = \frac{\sigma^0}{k}\left(\frac{R^2 - r^2}{4} + b\frac{R^4 - r^4}{16\,R^2}\right)$$

[Question total: 20 marks]

Q.30 Question 30

To explore the effect of using a temperature-dependent thermal conductivity, consider heat flowing through an annular (pipe) wall of inside radius R_0 and an outside radius R_1. It is assumed that thermal conductivity varies linearly with temperature from $k_0(T = T_0)$ to $k_1(T = T_1)$ where T_0 and T_1 are the inner and outer wall temperatures respectively.

a) Derive the following energy balance equation

$$\frac{\partial}{\partial r} r\, q_r = 0,$$

and state ALL assumptions required. **[7 marks]**

Solution:

[1/7] Assuming that the pressure dependency of the internal energy of the solid is small $\checkmark_1$, Equation 68 can be used valid.

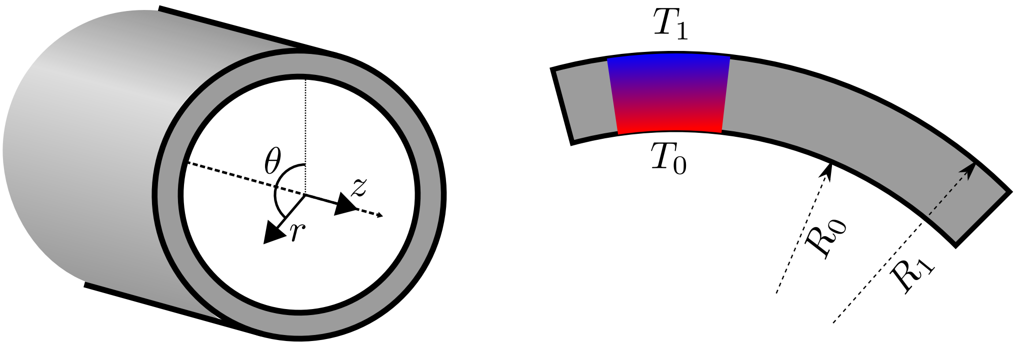


Figure 13: A diagram of conduction through an annular(pipe) wall for Q. 30.

As this is heat transfer in solids, we can set the frame of reference to the wall and $\boldsymbol{v} = \mathbf{0}$. This greatly simplifies the energy balance equation:

$$\rho\, C_p \frac{\partial T}{\partial t} = -\cancelto{0}{\rho\, C_p\, v_j \nabla_j\, T} - \nabla_i\, q_i - \cancelto{0}{\tau_{ji} \nabla_j\, v_i} - \cancelto{0}{p \nabla_i\, v_i} + \sigma_{energy}$$
$$\rho\, C_p \frac{\partial T}{\partial t} = -\nabla_i\, q_i + \sigma_{energy}$$

[1/7] $\checkmark_1$

Assuming the wall does not generate heat:

$$\rho\, C_p \frac{\partial T}{\partial t} = -\nabla_i\, q_i + \cancelto{0}{\sigma_{energy}}$$

[1/7] $\checkmark_1$

And steady state:

$$\cancelto{0}{\rho\, C_p \frac{\partial T}{\partial t}} = -\nabla_i\, q_i$$
$$\nabla_i\, q_i = 0$$

[1/7] $\checkmark_1$

Finally, inserting the cylindrical coordinate system definition of $\nabla_i\, q_i$:

$$\nabla_i\, q_i = \frac{1}{r}\frac{\partial}{\partial r}(r\, q_r) + \frac{1}{r}\frac{\partial\, q_\theta}{\partial \theta} + \frac{\partial\, q_z}{\partial z}$$

[2/7] $\checkmark_2$ Assuming a symmetry of the system ALONG and AROUND the axis, the only remaining derivative is in the r-direction:

$$\nabla_i\, q_i = \frac{1}{r}\frac{\partial}{\partial r}(r\, q_r)$$
$$= \frac{\partial}{\partial r}(r\, q_r) = 0$$

[1/7] $\checkmark_1$ As required.

b) Derive the following expression for the temperature profile

$$Q_r = \frac{2\pi\, L}{\ln\left(\frac{R_0}{R_1}\right)} \frac{k_1 + k_0}{2}(T_1 - T_0),$$

where L is the length of the pipe/annulus. **[10 marks]**
Note: You will need the following identity:

$$T_1^2 - T_0^2 = (T_1 + T_0)(T_1 - T_0).$$

Solution:
Performing the integration, we have

$$r\,q_r = C_1$$
$$q_r = \frac{C_1}{r}$$

[1/10] $\checkmark_1$ Inserting in Fourier's law, we have

$$-k\frac{\partial T}{\partial r} = \frac{C_1}{r}$$

We need to insert the temperature dependent thermal conductivity, which is given by the following linear relationship

$$k = k_0 + (T - T_0)\,\frac{k_1 - k_0}{T_1 - T_0}$$

[1/10] $\checkmark_1$ Inserting this,

$$-\left(k_0 + (T - T_0)\,\frac{k_1 - k_0}{T_1 - T_0}\right)\frac{\partial T}{\partial r} = \frac{C_1}{r}$$

[1/10] $\checkmark_1$ Integrating between the two limits,

$$-\int_{R_0}^{R_1}\left(k_0 + (T - T_0)\,\frac{k_1 - k_0}{T_1 - T_0}\right)\frac{\partial T}{\partial r}\mathrm{d}r = \int_{R_0}^{R_1}\frac{C_1}{r}\mathrm{d}r$$
$$-\int_{T_0}^{T_1}\left(k_0 + (T - T_0)\,\frac{k_1 - k_0}{T_1 - T_0}\right)\mathrm{d}T = C_1\ln\left(\frac{R_1}{R_0}\right)$$
$$-\left(k_0\,(T_1 - T_0) + \left(\frac{T_1^2 - T_0^2}{2} - (T_1 - T_0)\,T_0\right)\frac{k_1 - k_0}{T_1 - T_0}\right) = C_1\ln\left(\frac{R_1}{R_0}\right)$$

[2/10] $\checkmark_2$ Using the identity $T_1^2 - T_0^2 = (T_1 + T_0)(T_1 - T_0)$,

$$-\left(k_0\,(T_1 - T_0) + \frac{T_1 + T_0}{2}(k_1 - k_0) - T_0(k_1 - k_0)\right) = C_1\ln\left(\frac{R_1}{R_0}\right)$$
$$-\left(k_0\,T_1 + \frac{T_1 + T_0}{2}(k_1 - k_0) - T_0\,k_1\right) = C_1\ln\left(\frac{R_1}{R_0}\right)$$

[2/10] $\checkmark_2$ Simple cancellation and factorisation leads to the following

$$\frac{k_1 + k_0}{2\ln\left(\frac{R_0}{R_1}\right)}(T_1 - T_0) = C_1$$

[1/10] $\checkmark_1$ Inserting this back into the expression for the flux, we have

$$q_r = \frac{C_1}{r} = \frac{k_1 + k_0}{2\,r \ln\left(\frac{R_0}{R_1}\right)} (T_1 - T_0)$$

[1/10] $\checkmark_1$ The total flux is given by multiplying by the cylindrical area, $2\,\pi\,r\,L$,

$$Q_r = \frac{2\pi\,L}{\ln\left(\frac{R_0}{R_1}\right)} \frac{k_1 + k_0}{2} (T_1 - T_0)$$

[1/10] $\checkmark_1$

c) Compare this expression to the standard expression for conduction in pipe walls (with constant thermal conductivity), what can you observe? **[3 marks]**

Solution:

The expression for pipes is availabe from the tables in the datasheet, and is as follows

$$Q = \frac{2\pi\,L\,k}{\ln\left(\frac{R_1}{R_0}\right)} \Delta T.$$

[1/3] $\checkmark_1$

On comparision with the derived equation, the only change is to replace the constant thermal conductivity with the average of the thermal conductivity on the inner and outer surfaces.

[1/3] $\checkmark_1$

For small temperature differences (where a linear temperature dependence may be assumed) using the average thermal conductivity is a useful strategy. $\checkmark_1$ **[1/3]**

[Question total: 20 marks]

Q.31 Question 31

Consider the flow of a Newtonian liquid between two plates, similar to Q.14, but now both plates are maintained two different temperatures. We will attempt to take into account the effect of temperature on the flow profile.

You may assume that the viscosity, μ, of the liquid depends on temperature T according to the following relationship:

$$\mu(T) = \frac{\mu_0}{1 + \beta(T - T_0)}$$

where T_0 is a reference temperature, and μ_0 and β are empirical constants. The fluid flows under the influence of a pressure gradient $\Delta P/L$ between two flat plates, as shown in Fig. 14. The walls are at temperatures T_0 and T_1, where T_0 is the reference temperature, and $T_1 > T_0$.

a) Temporarily ignoring the motion of the fluid ($\boldsymbol{v} \approx \vec{0}$), demonstrate that the temperature can be taken to be a linear function of position:

$$T \approx T_0 + (T_1 - T_0)\frac{y}{H}$$

Solution:

Assuming this is an incompressible liquid, we can ignore the pressure dependence of the internal energy and use the energy balance equation. Using rectangular coordinates, we can use the index notation form,

$$\rho\,C_p \frac{\partial T}{\partial t} = -\rho\,C_p\,v_j\,\nabla_j\,T - \nabla_i\,q_i - \tau_{ji}\,\nabla_j\,v_i - p\,\nabla_i\,v_i + \sigma_{energy}.$$

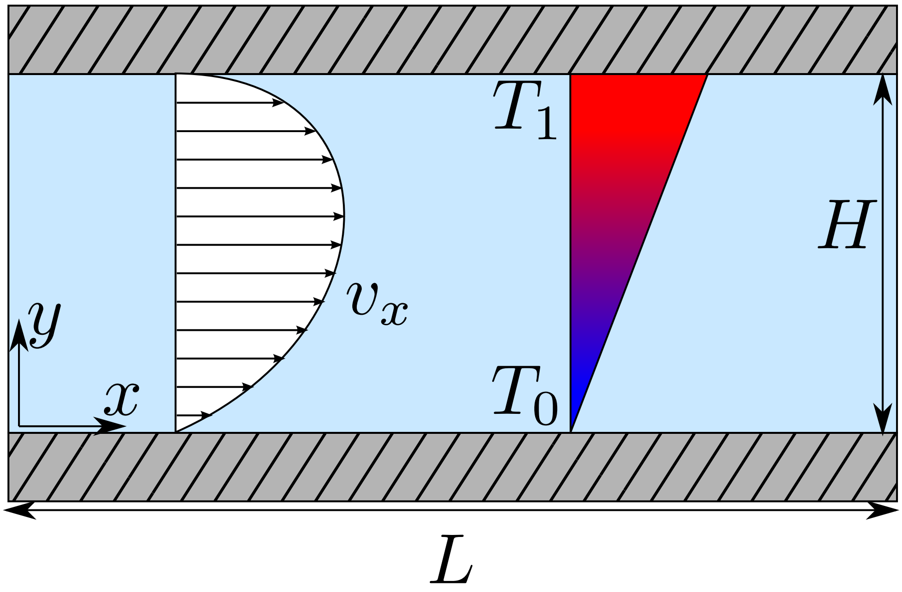


Figure 14: Flow through parallel plates.

There is no "generation" of energy, and the system is at steady state, thus the leftmost and rightmost terms are zero,

$$0 = -\rho\, C_p\, v_j\, \nabla_j\, T - \nabla_i\, q_i - \tau_{ji}\, \nabla_j\, v_i - p\, \nabla_i\, v_i.$$

We are told to assume $\boldsymbol{v} = \mathbf{0}$, thus all terms with the velocity should go to zero as well,

$$0 = \nabla_i\, q_i$$

Assuming the system is symmetric in the x and z directions, we can write,

$$\begin{aligned} \nabla_y\, q_y &= 0, \\ \frac{\partial q_y}{\partial y} &= 0, \\ q_y &= C_1. \end{aligned}$$

Substituting in Fourier's law we have,

$$\begin{aligned} -k\frac{\partial T}{\partial y} &= C, \\ \frac{\partial T}{\partial y} &= -\frac{C}{k}, \\ T &= -\frac{C_1}{k}\, y + C_2. \end{aligned}$$

Noting that $T(y = H) = T_1$ and $T(y = 0) = T_0$, we can determine the constants to give the following equation,

$$T \approx T_0 + (T_1 - T_0)\frac{y}{H}.$$

Additional notes (not assessed/marked):

We are told to assume that motion can be ignored ($\boldsymbol{v} = \mathbf{0}$), but how did we come up with this assumption? If the flow is well-developed, then $v_y = v_z = 0$. However, the term $v_j \nabla_j\, T$ is completely zero if we assume the flow is symmetric in the x-direction (i.e. $\nabla_x\, T = 0$). Also, any terms with $\nabla_i v_x = 0$ are zero using the same assumption, thus all terms with the velocity go to zero if the flow profile does not change along the channel.

b) Derive the stress profile and prove that it is equal to the expression below. Compare this stress profile to the stress profile for flow between two plates, and for film flow on a plate. What is unique about the stress profile?

$$\tau_{yx} = \frac{\Delta p\, H}{L}\left(\frac{y}{H} + C_1\right)$$

Solution:

See Q. 14a-b for the solution, its the same as flow between two unheated plates!

The stress profile is independent of the viscous properties of the fluid. Regardless of if the fluid is Newtonian or has varying viscosity, the steady state stress profile is identical for flow between two stationary plates. Thus what is unique about the stress profile is that its the same form in all three cases.

Additional notes, (not asseessed/marked): Direct force balance

This is an alternative approach to starting with the balance equations and is popular in many text books. Its given here only to demonstrate that you can begin each derivation with a direct force balance; however, it is difficult in curvelinear coordinates to correctly derive it this way so it is recommended that you stick to derivations via the balance equations.

We begin this problem by performing a momentum balance on a thin slab of fluid of thickness dy; the bottom of the slab is located at y.

$$\begin{aligned} 0 &= [\tau_{yx}(y+dy) - \tau_{yx}(y)]L\,Z + [p(0) - p(L)]Z\,\mathrm{d}y \\ &= \left[\frac{\tau_{yx}(y+dy) - \tau_{yx}(y)}{dy}\right] - \frac{\Delta p}{L} \end{aligned} \tag{21}$$

where Z is the width of the plates. Taking the limit that $\mathrm{d}y$ goes to zero, we find

$$\frac{\partial \tau_{yx}}{\partial y} = \frac{\Delta p}{L} \tag{22}$$

Integrating gives

$$\tau_{yx} = \frac{\Delta p}{L} y + C_1' \tag{23}$$

$$= \frac{\Delta p\, H}{L}\left(\frac{y}{H} + C_1\right) \tag{24}$$

c) Assuming the temperature profile is indeed linear, derive the following velocity profile for this system.

$$v_x(y) = -\frac{\Delta p\, H^2}{L\,\mu_0}\frac{y}{H}\left[\beta(T_1 - T_0)\frac{y^2}{3\,H^2} + (1 + C_1\,\beta\,[T_1 - T_0])\,\frac{y}{2\,H} + C_1\right] \tag{25}$$

where C_1 is a dimensionless integration constant which you must determine.

Solution:

In this problem the viscosity depends on position, due to the fact that the temperature depends on position, i.e. we have,

$$\mu(T) = \frac{\mu_0}{1 + \beta(T - T_0)} \qquad\qquad T(y) \approx T_0 + (T_1 - T_0)\frac{y}{H}.$$

Combining these expressions gives viscosity as a function of position

$$\mu(y) = \frac{\mu_0}{1 + \beta(T_1 - T_0)y/H}$$

Using Newton's law of viscosity into the stress equation from the previous question gives the following,

$$\tau_{yx} = \mu(y)\frac{\partial v_x}{\partial y} = -\frac{\Delta p\, H}{L}\left(\frac{y}{H} + C_1\right)$$

Expanding the definition of the viscosity, we have

$$\frac{\partial v_x}{\partial y} = -\frac{\Delta p\, H}{L\,\mu_0}\left[\frac{y}{H} + C_1\right]\left[1 + \beta(T_1 - T_0)\frac{y}{H}\right] \tag{26}$$

Integrating (and redefining the integration constant to bring it inside the brackets), we find

$$v_x(y) = -\frac{\Delta p\, H}{L\,\mu_0}\left[C_1\, y + \frac{y^2}{2\,H} + \beta(T_1 - T_0)\left(\frac{y^3}{3\,H^2} + \frac{y^2\, C_1}{2\,H}\right) + C_2\right] \tag{27}$$

We know that $v_x(y = 0) = 0$, therefore $C_2 = 0$. With $v_x(y = H) = 0$. we find:

$$0 = -\frac{\Delta p\, H}{\mu_0\, L}\left[C_1\, H + \frac{H}{2} + \beta(T_1 - T_0)\left(\frac{H}{3} + \frac{H\, C_1}{2}\right)\right] \tag{28}$$

$$0 = C_1 + \frac{1}{2} + \beta(T_1 - T_0)\left(\frac{1}{3} + \frac{C_1}{2}\right) \tag{29}$$

$$C_1 = -\frac{1 + \frac{2}{3}\beta(T_1 - T_0)}{2 + \beta(T_1 - T_0)} \tag{30}$$

With some rearrangement we find

$$v_x(y) = -\frac{\Delta p\, H^2}{L\,\mu_0}\frac{y}{H}\left[C_1 + \frac{y}{2\,H} + \beta(T_1 - T_0)\left(\frac{y^2}{3\,H^2} + \frac{y\, C_1}{2\,H}\right)\right] \tag{31}$$

$$v_x(y) = -\frac{\Delta p\, H^2}{L\,\mu_0}\frac{y}{H}\left[\beta(T_1 - T_0)\frac{y^2}{3\,H^2} + (1 + C_1\,\beta\,[T_1 - T_0])\,\frac{y}{2\,H} + C_1\right] \tag{32}$$

d) Determine the flow-rate to pressure drop relationship.
Solution:
The average velocity of the fluid between the plates $\bar{v}_x$ is given by

$$\begin{aligned}
\bar{v}_x &= \frac{1}{H\,Z}\int_0^Z\int_0^H v_x(y)\,\mathrm{d}y\,\mathrm{d}z \\
&= \frac{1}{H}\int_0^H v_x(y)\,\mathrm{d}y \\
&= -\frac{1}{H}\frac{\Delta p}{L}\frac{H^2}{\mu_0}\int_0^H\left(\frac{y}{H}\right)\left[\beta(T_1 - T_0)\frac{y^2}{3H^2} + (1 + C_1\,\beta(T_1 - T_0))\,\frac{y}{2H} + C_1\right]\mathrm{d}y \\
&= -\frac{\Delta p}{L}\frac{H^2}{\mu_0}\int_0^1\left[\beta(T_1 - T_0)\frac{\eta^3}{3} + (1 + C_1\,\beta(T_1 - T_0))\,\frac{\eta^2}{2} + C_1\,\eta\right]\mathrm{d}\eta \\
&= -\frac{\Delta p}{L}\frac{H^2}{\mu_0}\left[\beta(T_1 - T_0)\frac{\eta^4}{12} + (1 + C_1\,\beta(T_1 - T_0))\,\frac{\eta^3}{6} + C_1\frac{\eta^2}{2}\right]_0^1 \\
&= -\frac{\Delta p}{L}\frac{H^2}{12\mu_0}\left[(1 + 2\,C_1)\beta(T_1 - T_0) + 2(1 + 3\,C_1)\right] \qquad (33)
\end{aligned}$$

The flowrate is simply the average velocity times by the cross sectional area, i.e., $\dot{V}_x = H\,Z\,\bar{v}_x$.

e) Calculate the x-component of the force of the fluid on the bottom surface $y = 0$ per unit area of the plate and compare it to the value on the top surface.
Solution:
The x-component of the force of the fluid on the bottom surface IS the stress on the plate. Taking the previous expression:

$$\tau_{yx} = \frac{\Delta p\, H}{L}\left(\frac{y}{H} + C_1\right) \tag{34}$$

We have

$$\tau_{yx}(y = 0) = \frac{\Delta p}{L} H\, C_1 \tag{35}$$

and

$$\tau_{yx}(y = H) = \frac{\Delta p}{L} H\,(1 + C_1) \tag{36}$$

Unless the constant C_1 has the value $C_1 = -\frac{1}{2}$, it is clear that the **magnitude** of the stress on each boundary is not equal. Please note, the sign of the stress is opposite on each boundary. The constant C_1 is only $-1/2$ if the temperature difference $T_1 - T_0$ is zero:

$$C_1 = -\frac{1 + \frac{2}{3}\beta\,\cancelto{0}{(T_1 - T_0)}}{2 + \beta\,\cancelto{0}{(T_1 - T_0)}} = -\frac{1}{2} \tag{37}$$

[Question end]

Q.32 Question 32
Perform dimensional analysis on a pendulum of length l, mass m, under gravity g to better understand the period of oscillation, t. How does the pendulum period change with changes in its mass?

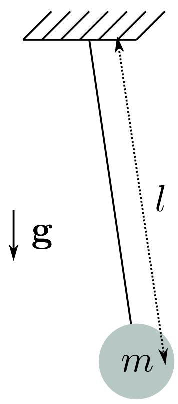


Figure 15: A pendulum with mass m, length l, in gravity of g.

Solution:
We already know that the period of a pendulum doesn't depend on the amplitude of the swing. Its period will be a function of the string length l, gravitational acceleration g and the mass m.

$$t = f(l, m, g)$$

We know that physical systems are independent of units, (the period doesn't change even though we measure it in hours or seconds). We should then make the equation independent of the units by making all terms dimensionless:

$$\frac{t}{T} = f\left(\frac{l}{L}, \frac{m}{M}, \frac{g\,T^2}{L}\right)$$

where L is the unit of length, M is the unit of mass, and T is the unit of time. L, M, and T aren't units in the sense of kilograms or meters, but rather parts of the system we decide to make the unit. For example, in a pipe of radius R, we often use the dimensionless position variable r/R, where R has been chosen as the unit length.
We need to choose a length, mass, and time scale for the pendulum. The unit length of the system can be the length of the pendulum $L = l$, and the unit mass can be it's mass, $M = m$. We can then get a unit of time from the gravitational constant:

$$T = \sqrt{l/g}$$

Inserting these into the expression above, we have

$$\frac{t}{\sqrt{l/g}} = f\left(\frac{l}{l}, \frac{m}{m}, \frac{g\,l}{l\,g}\right) = f\left(1,\,1,\,1\right)$$

The unknown function is now a constant! We now know that the period of a pendulum doesn't depend on its mass as its dimensionless form is equal to a function of constants (also a constant).We could determine the unknown constant either through experimental observation or through a more careful theoretical analysis. The actual result is

$$\frac{t}{\sqrt{l/g}} = 2\,\pi$$

We can see that dimensional analysis has not only simplified the system, but almost solved it. It is clear from dimensional analysis, the period of the oscillator is not affected by the mass of the pendulum. In reality, the pendulum must have sufficient mass to make frictional losses insignificant.

[Question end]

Q.33 Question 33
Example exam question (2016)
Consider laminar flow within a pipe. The only prior knowledge you should assume is that the pressure drop must be a function of pipe diameter D, viscosity μ, density ρ, and average velocity $\langle v_z\rangle$, i.e.,

$$\Delta p/l = f\left(D, \rho, \mu, \langle v_z\rangle\right).$$

a) Perform dimensional analysis on the pressure drop per unit length, $\Delta p/l$, and determine the relevant dimensionless groups. **[12 marks]**
Solution:
The first step is to make the units of each term explicit by dividing out the dimensions of each term

$$\frac{\Delta p}{l}\frac{L^2\,T^2}{M} = f\left(\frac{D}{L}, \frac{\rho\,L^3}{M}, \frac{\mu\,L\,T}{M}, \frac{\langle v_z\rangle\,T}{L}\right).$$

[5/12] Students will recieve FIVE$\checkmark_5$ marks for correctly identifying the units of each term in SI.

[1/12] A convenient length scale is the diameter, $L = D$,$\checkmark_1$ which gives:

$$\frac{\Delta p}{l}\frac{D^2\,T^2}{M} = f\left(1, \frac{\rho\,D^3}{M}, \frac{\mu\,D\,T}{M}, \frac{\langle v_z\rangle\;T}{D}\right)$$

[1/12] $\checkmark_1$

[1/12] A convenient mass scale is $M = \rho\,D^3$,$\checkmark_1$ which gives:

$$\frac{\Delta p}{l}\frac{T^2}{\rho\,D} = f\left(1, 1, \frac{\mu\,T}{\rho\,D^2}, \frac{\langle v_z\rangle\;T}{D}\right)$$

[1/12] $\checkmark_1$

[1/12] Finally, a convenient time scale is $T = D/\langle v_z\rangle$, which gives:$\checkmark_1$

$$\frac{\Delta p}{l}\frac{D}{\rho\,\langle v_z\rangle^2} = f\left(1, 1, \frac{\mu}{\rho\,\langle v_z\rangle\;D}, 1\right)$$

[1/12] Noticing that the dimensionless grouping on the right hand side is the Reynolds number$\checkmark_1$, we have

$$\begin{aligned}\frac{\Delta p}{l}\frac{D}{\rho\,\langle v_z\rangle^2} &= f\left(1, 1, \mathrm{Re}^{-1}, 1\right)\\ &= f\left(\mathrm{Re}\right)\end{aligned}$$

[1/12] $\checkmark_1$

b) Compare this to the exact solution, known as the Hagen-Poiseuille equation, as given below.

$$\dot{V}_z = \pi\left(\frac{-\Delta p}{l} + \rho\,g_z\right)\frac{R^4}{8\,\mu}.$$

Determine the form of the unknown function, f. **[5 marks]**

Solution:

[1/5] Noting that $\langle v_z\rangle = \dot{V}_z/A$$\checkmark_1$ and ignoring gravity$\checkmark_1$, we have

[1/5]

$$\langle v_z\rangle = -\frac{\Delta p}{l}\frac{R^2}{8\,\mu}.$$

[1/5] $\checkmark_1$ Rearranging the equation to make it identical to the LHS of the solution to the previous question, we have

$$\begin{aligned}\frac{\Delta p}{l}\frac{R}{\rho\,\langle v_z\rangle^2} &= -8\frac{\mu}{\rho\,\langle v_z\rangle\;R}\\ \frac{\Delta p}{l}\frac{D}{\rho\,\langle v_z\rangle^2} &= -32\frac{\mu}{\rho\,\langle v_z\rangle\;D}\\ &= -\frac{32}{\mathrm{Re}}\end{aligned}$$

[2/5] $\checkmark_2$ Thus the unknown function is $f = -32\,\mathrm{Re}^{-1}$.

c) Comment on why dimensional analysis is so important. Also comment on why redundant dimensionless groups arise (as an example, consider the relationship between friction factor C_f and the Reynolds number). **[3 marks]**

Solution:

Dimensionless groups are important, and arise so often, as units themselves are an entirely artificial construct and natural phenomena must be independent of the choice of units. For our models/equations to correctly reflect this, units must cancel within expressions and thus
[2/3] our equations must be able to be rearranged into a composition of dimensionless groups. $\checkmark_2$

Redundant dimensionless groups arise as dimensional analysis places no constraints on the functional form of equations, just on the possible groupings of dimensional terms. Thus dimensionless groups (such as the Reynolds number) may appear with arbitrary transformations applied. One example is the friction factor, which is a dimensionless grouping, but is simply a transformation of the Reynolds number dimensionless group, $C_f = 16\,\mathrm{Re}^{-1}$ (and
[1/3] vice-versa). $\checkmark_1$.

[Question total: 20 marks]

Q.34 Question 34

Example exam question (2014)

Carry out a dimensional analysis on the forced convection heat transfer coefficient, h, to determine which are the fundamental dimensionless numbers involved. You may assume the following general dependence

$$h = f\left(d,\, \mu,\, k,\, \langle v\rangle,\, \rho,\, C_p\right)$$

where d is the channel diameter (m), μ is the viscosity (Pa s), k is the thermal conductivity (W m^{-1} K^{-1}), $\langle v\rangle$ is the mean flow velocity (m s^{-1}), ρ is the mass density (kg m^{-3}), and C_p is the specific heat capacity at constant pressure (kJ kg^{-1} K^{-1}). **[10 marks]**

Solution:

Making the expression dimensionless:

$$\frac{h\,\Theta\,T^3}{M} = f\left(\frac{d}{L},\, \frac{\mu\,T\,L}{M},\, \frac{k\,T^3\,\Theta}{M\,L},\, \frac{T\,\langle v\rangle}{L},\, \frac{\rho\,L^3}{M},\, \frac{C_p\,T^2\,\Theta}{L^2}\right)$$

Looking at each term, it is clear that it will simplify if we select $L = d$ as the unit length, $M = \rho\,L^3$ as the unit mass, $T = M/(\mu\,L)$ as the unit time, and $\Theta = M\,L/(T^3\,k)$ as the unit Temperature. Inserting in the temperature unit Θ first, we have:

$$\frac{h\,L}{k} = f\left(\frac{d}{L},\, \frac{\mu\,T\,L}{M},\, 1,\, \frac{T\,\langle v\rangle}{L},\, \frac{\rho\,L^3}{M},\, \frac{C_p\,M}{T\,L\,k}\right)$$

Inserting the time unit T next, we have

$$\frac{h\,L}{k} = f\left(\frac{d}{L},\, 1,\, 1,\, \frac{M\,\langle v\rangle}{\mu\,L^2},\, \frac{\rho\,L^3}{M},\, \frac{C_p\,\mu}{k}\right)$$

Inserting the mass unit M next, we have

$$\frac{h\,L}{k} = f\left(\frac{d}{L},\, 1,\, 1,\, \frac{\rho\,\langle v\rangle\,L}{\mu},\, 1,\, \frac{C_p\,\mu}{k}\right)$$

Finally, inserting in the length unit L, we have:

$$\frac{h\,d}{k} = f\left(1,\, 1,\, 1,\, \frac{\rho\,\langle v\rangle\,d}{\mu},\, 1,\, \frac{C_p\,\mu}{k}\right)$$

You should notice that the left hand side is the Nusselt number, while the two terms inside the unknown function are the Reynolds number and the Prandtl number! We can then write:

$$\mathrm{Nu} = \frac{h\,d}{k} = f\left(\mathrm{Re},\,\mathrm{Pr}\right)$$

[Question total: 10 marks]

Q.35 Question 35

Calculate the dimensionless heat transfer coefficient (Nu) for conductive heat transfer through rectangular walls. **Note:** You will need to rephase the conductive resistance as a heat transfer coefficient h.

Solution:

We have

$$\mathrm{Nu} = \frac{h\,L}{k}$$

But for conduction we have $U = h = k/X$. Choosing our characteristic length as $L = X$ (this is the lengthscale controlling conduction), we have

$$\mathrm{Nu}_{cond.} = \frac{k\,\cancel{L}^{\,1}}{\cancel{k}\,L} = 1$$

[Question end]

Q.36 Question 36

Example exam question (2012)

The heat loss from a pipe which is carrying a hot process fluid must be estimated to evaluate if additional insulation is economically justified.

a) Starting from the general expression for steady-state conduction in cylindrical shells:

$$\frac{\partial}{\partial r} r\,q_r = 0 \tag{38}$$

Derive the following expression for the heat flux in a cylindrical wall:

$$q_r = \frac{k}{r\,\ln\left(R_{outer}/R_{inner}\right)}\left(T_{inner} - T_{outer}\right) \tag{39}$$

[8 marks]

Solution:

Performing the integration, we have

$$r\,q_r = C$$
$$q_r = \frac{C}{r}$$

Inserting in Fourier's law, we have

$$\begin{aligned}
-k\frac{\partial T}{\partial r} &= \frac{C}{r} \\
\frac{\partial T}{\partial r} &= -\frac{C}{k\,r} \\
\int_{T_{inner}}^{T_{outer}} \mathrm{d}T &= -\frac{C}{k}\int_{R_{inner}}^{R_{outer}} \frac{1}{r}\mathrm{d}r \\
T_{outer} - T_{inner} &= -\frac{C}{k}\left[\ln r\right]_{R_{inner}}^{R_{outer}} \\
T_{outer} - T_{inner} &= \frac{C}{k}\ln\frac{R_{inner}}{R_{outer}} \\
C &= \frac{k}{\ln\left(R_{inner}/R_{outer}\right)}\left(T_{outer} - T_{inner}\right)
\end{aligned}$$

Inserting this back into the expression for the flux, we have

$$\begin{aligned}
q_r &= \frac{C}{r} \\
&= \frac{k}{r\,\left(R_{inner}/R_{outer}\right)}\left(T_{outer} - T_{inner}\right) \\
&= \frac{k}{r\,\left(R_{outer}/R_{inner}\right)}\left(T_{inner} - T_{outer}\right)
\end{aligned}$$

b) Derive the following expression for the heat transfer resistance for conduction in a cylindrical wall.

$$R = \frac{\ln\left(R_{outer}/R_{inner}\right)}{2\,\pi\,L\,k} \tag{40}$$

[3 marks]

Solution:
Note: The curved surface of a cylinder has an area of $2\,\pi\,r\,L$.

The surface area for the flux at any radius r is the curved surface area of a cylinder with radius r. The total heat flux is then

$$\begin{aligned}
Q_r &= 2\,\pi\,r\,L\,q_r \\
&= \frac{2\,\pi\,L\,k}{\ln\left(R_{outer}/R_{inner}\right)}\left(T_{inner} - T_{outer}\right)
\end{aligned}$$

If we have $Q = U\,A\,\Delta T = R^{-1}\Delta T$, we can isolate the terms to give the resistance as

$$R = \frac{\ln\left(R_{outer}/R_{inner}\right)}{2\,\pi\,L\,k}$$

c) The pipe carrying the process fluid has an inner diameter of 15 cm and a length of 50 m. The process fluid, flowing at 1 kg s^{-1}, has a density of 800 kg m^{-3}, a viscosity of 2×10^{-3} Pa s, a heat capacity of 1.2 kJ kg^{-1} K^{-1}, and a thermal conductivity of 0.15 W m^{-1} K^{-1}.

i) Is the flow inside the pipe turbulent? **[2 marks]**

Solution:

We must calculate the Reynolds number Re. The volumetric flow rate is

$$\dot{V} = \dot{M}/\rho = 1/800 = 0.00125\ \mathrm{m}^3\ \mathrm{s}^{-1}$$

The average flow velocity is then

$$\langle v\rangle = \dot{V}/A = 0.00125/(\pi\, 0.075^2) \approx 0.0707\ \mathrm{m\ s^{-1}}$$

The Reynolds number is

$$\mathrm{Re} = \frac{\rho\,\langle v\rangle\, D}{\mu} = \frac{800 \times 0.0707 \times 0.15}{2 \times 10^{-3}} \approx 4242$$

The flow is turbulent.

ii) Demonstrate that the forced convection heat transfer coefficient on the inside of the pipe is approximately $h \approx 57$ W m^{-2} K^{-1}. **[2 marks]**

Solution:

The appropriate expression from the data sheet is

$$\mathrm{Nu} \approx \frac{(C_f/2)\mathrm{Re}\,\mathrm{Pr}}{1.07 + 12.7(C_f/2)^{1/2}\left(\mathrm{Pr}^{2/3} - 1\right)}\left(\frac{\mu_b}{\mu_w}\right)^{0.14}$$

Here, we cannot use the viscosity correction as the data is unavailable, so we assume $\mu_b = \mu_w$. Calculating the friction factor, we have

$$C_f = 0.079\,\mathrm{Re}^{-1/4} = 0.00979$$

The Prandtl number is

$$\mathrm{Pr} = \frac{\mu\, C_p}{k} = \frac{2 \times 10^{-3} \times 1.2 \times 10^3}{0.15} = 16$$

Substituting in, we have

$$\begin{aligned}\mathrm{Nu} &\approx \frac{(0.00979/2) \times 4242 \times 16}{1.07 + 12.7(0.00979/2)^{1/2}\left(16^{2/3} - 1\right)}\cancel{\left(\frac{\mu_b}{\mu_w}\right)^{0.14}}\\ &\approx 57\end{aligned}$$

The heat transfer coefficient is obtained from the Nusselt number

$$h = \frac{\mathrm{Nu}\,k}{L} \approx \frac{57 \times 0.15}{0.15} \approx 57\ \mathrm{W\ m^{-2}\ K^{-1}}$$

iii) The pipe has a carbon-steel wall which is 1 cm thick and has a thermal conductivity of 43 W m^{-1} K^{-1}. The pipe is also insulated using a 1 cm layer of rock wool, which has a thermal conductivity of 0.045 W m^{-1} K^{-1}. The external heat transfer coefficient, which includes radiation and natural convection, is estimated to be 5 W m^{-2} K^{-1}. Determine the overall heat flux through the pipe if the process fluid is at 80°C and the surroundings are at 10°C. **[5 marks]**

Solution:

The internal area of the pipe is:

$$A_{inner} = \pi\, D\, L = \pi\,(0.15)50 \approx 23.6\ \mathrm{m}^2$$

The internal resistance to heat transfer is

$$R_{inner} = \frac{1}{h_{inner}\, A_{inner}} = \frac{1}{57 \times 23.6} = 0.000743\ \mathrm{K\ W^{-1}} \quad (0.0372\ \mathrm{K\ W^{-1}\ m})$$

The value in parenthesis is per-metre of pipe. The external area of the pipe is:

$$A_{outer} = \pi\,D\,L = \pi\,(0.15 + 0.02 + 0.02)50 = 29.8\ \mathrm{m}^2$$

The external resistance to heat transfer is

$$R_{outer} = \frac{1}{h_{outer}\,A_{outer}} = \frac{1}{5 \times 29.8} = 0.00671\ \mathrm{K\ W^{-1}}\ (0.336\ \mathrm{K\ W^{-1}\ m})$$

which is more significant than the internal resistance.
The resistance to heat transfer by the wall is

$$\begin{aligned} R_{wall} &= \frac{\ln\left(R_{outer}/R_{inner}\right)}{2\,\pi\,L\,k} \\ &= \frac{\ln\left(0.17/0.15\right)}{2\,\pi\,50 \times 43} \\ &= 9.27 \times 10^{-6}\ \mathrm{K\ W^{-1}}\ (4.63 \times 10^{-4}\ \mathrm{K\ W^{-1}\ m}) \end{aligned}$$

which is negligble compared to the external heat transfer resistance.
The insulation resistance is

$$\begin{aligned} R_{insulation} &= \frac{\ln\left(R_{outer}/R_{inner}\right)}{2\,\pi\,L\,k} \\ &= \frac{\ln\left(0.19/0.17\right)}{2\,\pi\,50 \times 0.045} \\ &= 0.00787\ \mathrm{K\ W^{-1}}\quad (0.393\ \mathrm{K\ W^{-1}\ m}) \end{aligned}$$

which is comparable to the external heat transfer coefficient.
The total resistance is

$$\begin{aligned} R_{total} &= R_{inner} + R_{outer} + R_{wall} + R_{insulation} \\ &= 0.000743 + 0.00671 + 9.27 \times 10^{-6} + 0.00787 \approx 0.0153\ \mathrm{K\ W^{-1}}\quad (0.767\ \mathrm{K\ W^{-1}\ m}) \end{aligned}$$

The total heat flux is

$$Q = R_{total}^{-1}\,(T_{inner} - T_{outer}) = 0.0153^{-1}(80 - 10) \approx 4.58\ \mathrm{kW}\quad (91\ \mathrm{W\ m^{-1}})$$

[Question total: 20 marks]

Q.37 Question 37
Example exam question

a) Chilled water flowing through brass tubes of 0.0126 m inside diameter and 0.0018 m thickness cools a stream of air flowing outside of the tube. The film coefficients for the air and water flows are 176 $\mathrm{W m^{-2} K^{-1}}$ and 5660 $\mathrm{W\,m^{-2}\,K^{-1}}$ respectively and thermal conductivity of the brass is 102 $\mathrm{W\,m^{-1}\,K^{-1}}$ (see Fig. 16).

   i) Calculate overall heat transfer resistance $R_{total} = (UA)_{total}^{-1}$. **[6 marks]**
   Solution:
   The total resistance is given by the sum of the conductive resistance and the two film resistances:

$$R_{total} = R_{cond.} + R_i + R_o$$

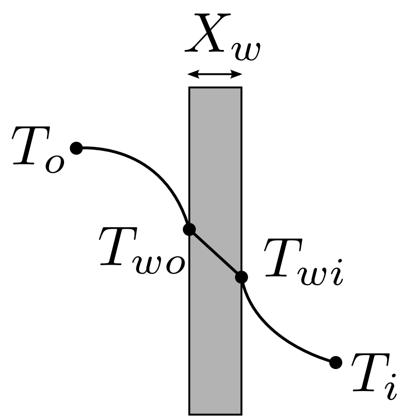


Figure 16: The temperature profile through the pipe wall.

The conductive resistance is given by

$$R_{cond.} = \frac{\ln(R_{outer}/R_{inner})}{2\,\pi\,L\,k}$$
$$= \frac{\ln(0.0162/0.0126)}{2\,\pi\,L\,102} = \frac{3.921\times10^{-4}}{L}\mathrm{K\,W^{-1}\,m^{-1}}$$

We also have

$$R_i = \frac{1}{h_i\,A_i} = \frac{1}{5660\times\pi\times0.0126\,L} \approx \frac{4.463\times10^{-3}}{L}\mathrm{K\,W^{-1}\,m^{-1}}$$
$$R_o = \frac{1}{h_o\,A_o} = \frac{1}{176\times\pi\times0.0162\,L} \approx \frac{0.1116}{L}\mathrm{K\,W^{-1}\,m^{-1}}$$

The total is

$$R_{total} = \frac{3.921\times10^{-4}+0.1116+4.463\times10^{-3}}{L} = \frac{0.1165}{L}\mathrm{K\,W^{-1}\,m^{-1}}$$

ii) State what is the limiting heat resistance (i.e., what is the controlling heat transfer mechanism). **[2 marks]**
Solution:
The convection on the outer surface of the pipe is the dominant heat transfer mechanism as it has the highest heat transfer resistance by several orders of magnitude.

iii) Calculate heat transferred per metre length of tube at the point where the bulk temperatures of the air and water streams are 326°C and 15°C respectively. **[2 marks]**
Solution:
This is given by

$$\frac{Q}{L} = \frac{\Delta T}{R\,L} = \frac{326-15}{0.1165} = 2670\mathrm{W\ m^{-1}}$$

b) The heat transfer coefficient for air flowing over a sphere is to be determined by observing the temperature-time history of a sphere fabricated from pure copper. The diameter of sphere is 17 mm. The sphere is at 86°C before it is inserted into an airstream having a temperature of 22°C. A thermocouple on the outer surface of the sphere indicates 62°C at 116 seconds after the sphere is inserted into the airstream.
Note: The properties of copper at 347K are ρ = 8933 kg m^{-3}, C_p = 389 J kg^{-1} K^{-1}, and k = 398 W m^{-1} K^{-1}.

i) Calculate the heat transfer coefficient by assuming that the lumped capacitance method is valid. **[7 marks]**

Solution:

Using the equation in the datasheet:

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\frac{h\,A_s}{\rho\,V\,C_p}t\right]$$

$T = 62°\mathrm{C}$, $T_i = 86°\mathrm{C}$, $T_\infty = 22°\mathrm{C}$, $A_s = \pi D^2 = 9.079 \times 10^{-4}\ \mathrm{m}^2$, $V_s = 4\pi R^3/3 = \pi D^3/6 = 2.572 \times 10^{-6}\ \mathrm{m}^3$, $t = 116$ s. Substituting in the properties of copper, we have

$$\frac{62 - 22}{86 - 22} = \exp\left[-h \times 116 \frac{9.079 \times 10^{-4}}{8933 \times 2.572 \times 10^{-6} \times 389}\right]$$

$$0.625 = \exp\left[-h \times 116 \frac{9.079 \times 10^{-4}}{8933 \times 2.572 \times 10^{-6} \times 389}\right]$$

$$0.625 = \exp\left[-0.01178\,h\right]$$

$$h = -\frac{\ln(0.625)}{0.01178} = 39.90\,\mathrm{W\,m^{-2}\,K^{-1}}$$

ii) Show that the Biot number supports the application of the lumped capacitance method. **[3 marks]**

Solution:

The Biot number is defined as

$$\mathrm{Bi} = \frac{h\,L_c}{k}$$

For a sphere, the characteristic length is $L_c = V_s/A_s = \frac{1}{6}\pi D^3/\pi D^2 = D/6$. Calculating its value we have:

$$\mathrm{Bi} = \frac{h\,D}{6\,k} = \frac{39.9 \times 0.017}{6 \times 389} = 2.906 \times 10^{-4}$$

As this value is $\mathrm{Bi} < 0.1$, the lumped capacitance assumption is reasonable.

[Question total: 20 marks]

Q.38 Question 38

Example exam question

An electric heater of 0.032 m diameter and 0.85 m in length is used to heat a room. Calculate the electrical input (i.e. the sum of heat transferred by convection and radiation) to the heater when the bulk of the air in the room is at 24°C, the walls are at 12°C, and the surface of the heater is at 532°C. For convective heat transfer from the heater, assume the heater is a horizontal cylinder and the Nusselt number is given by

$$\mathrm{Nu} = 0.38(\mathrm{Gr})^{0.25}$$

where all properties are evaluated at the film temperature. You may assume air is an ideal gas, giving $\beta = T^{-1}$. Take the emissivity of the heater surface as $\epsilon = 0.62$ and assume that the surroundings are black. All other properties should be calculated using the table provided (see Table 1). **[10 marks]**

Solution:

Calculating the film temepature, we have

$$T_f = \frac{532 + 24}{2} = 278°\mathrm{C} = 551\ \mathrm{K}$$

T (K)	μ (kg m^{-1} s^{-1})	k (W m^{-1} K^{-1})	ρ (kg m^{-3})
550	2.849×10^{-5}	4.357×10^{-2}	0.6418
600	3.017×10^{-5}	4.661×10^{-2}	0.5883
700	3.332×10^{-5}	5.236×10^{-2}	0.5043
800	3.624×10^{-5}	5.774×10^{-2}	0.4412
900	3.897×10^{-5}	6.276×10^{-2}	0.3922

Table 1: Physical properties of air at 1 atm for Q.38.

From the tables at 551 K, $\nu = 4.48 \times 10^{-5}$ m^2 s^{-1} and $k = 0.04375$ W m^{-1} K^{-1}. The expansion coefficient is $\beta = 551^{-1}$. Combining these we have:

$$\mathrm{Gr} = \frac{9.81\,(532 - 24)\,0.032^3}{551\,(4.48 \times 10^{-5})^2} \approx 147\,700$$

Calculating the Nusselt number, we have

$$\mathrm{Nu} = 0.38\,(147\,700)^{1/4} \approx 7.45$$

Calculating the convective coefficient we have

$$h = \frac{k\,Nu}{L} = \frac{0.04375 \times 7.45}{0.032} \approx 10.19\,\mathrm{W\,m^{-2}\,K^{-1}}$$

Heat transfer via convection:

$$Q_{conv} = h\,A\,\Delta T = 10.19 \times \pi \times 0.032 \times 0.85\,(532 - 24) \approx 442\mathrm{W}$$

Heat transfer by radiation

$$\begin{aligned} Q_{rad.} &= \sigma\,\epsilon\,A\,\left(T_w^4 - T_\infty^4\right) \\ &= 5.67 \times 10^{-8} \times 0.62 \times \pi \times 0.032 \times 0.85\left(805^4 - 285^4\right) \\ &\approx 1242 \end{aligned}$$

Total energy input is

$$Q_{total} = Q_{rad.} + Q_{conv} = 1242 + 442 = 1684\ \mathrm{W}$$

[Question total: 10 marks]

Q.39 Question 39

A pebble-bed nuclear reactor at 69 bara is used to heat helium (4 g mol^{-1}) as part of the generation of electricity. The helium gas has a heat capacity at constant pressure of C_p = 5190 J kg^{-1} K^{-1}, a dynamic viscosity of $\mu = 5.19 \times 10^{-5}$ Pa s, and a thermal conductivity of $k = 0.405$ W m^{-1} K^{-1} and flows at 15 m s^{-1}. The pebbles have an outer radius of 3 cm which consists of a 0.5 cm coating of graphite around the radioactive core.

a) Assuming helium may be treated as an ideal gas, demonstrate that the density of the gas is 2.83 kg m^{-3}. **[3 marks]**

Solution:

The density of helium from the ideal gas law is,

$$\frac{N}{V} = \frac{p}{R\,T} = \frac{69 \times 10^5}{8.314 \times (900 + 273.15)} = 707\ \mathrm{mol\,m^{-3}}$$

[2/3] $\checkmark_2$ which is $0.004 \times 707 \approx 2.83$ kg m^{-3}.$\checkmark_1$

[1/3]

b) Calculate the surface temperature of the particle if it is emitting 850 W of heat and the surrounding helium is at 900 °C. The following expression for forced convective heat-transfer around a sphere is available,

$$\mathrm{Nu}_D = 2 + 0.47\,\mathrm{Re}_D^{1/2}\,\mathrm{Pr}^{0.36} \qquad \text{for } 3\times10^{-3} < \mathrm{Pr} < 10 \text{ and } 10^2 < \mathrm{Re}_D < 5\times10^4.$$

Radiation is neglible as all pellets have the same surface temperature, and the characteristic length used in the Reynolds and Nusselt number is the sphere diameter. **[12 marks]**

Solution:

The Prandtl and Reynolds number are,

$$\mathrm{Pr} = \frac{5.19\times10^{-5}\times5}{0.405} \approx 0.665$$
$$\mathrm{Re} = \frac{2.83\times15\times0.06}{5.19\times10^{-5}} \approx 49100$$

[4/12] $\checkmark_4$ These are within the range of the expression $\checkmark_1$. The Nusselt number is,

[1/12]

$$\mathrm{Nu}_D = 2 + 0.47\times49100^{1/2}\times0.665^{0.36} \approx 91.9$$

[2/12] $\checkmark_2$ The overall heat transfer coefficient is then,

$$h = \frac{\mathrm{Nu}_D\,k}{D} = \frac{91.9\times0.405}{0.06} \approx 620\ \mathrm{W\,m^{-2}\,K^{-1}}.$$

[2/12] $\checkmark_2$ Solving for the temperature difference,

$$\Delta T = \frac{Q}{A\,h} = \frac{850}{4\,\pi\,0.03^2\times620} \approx 121\ \mathrm{K}.$$

[2/12] $\checkmark_2$ The outer shell temperature is then $900 + 121 \approx 1021$ °C. $\checkmark_1$

[1/12]

[Question total: 15 marks]

Q.40 **Question 40**

Example exam question

A single-pass, counter-flow shell-and-tube heat exchanger is required to operate as an oil cooler with 316 tubes of internal diameter 0.016 m, outer diameter 0.018 m, and length 5.6 m. The oil flows in the tube side entering at a mass flow rate of 32 kg s^{-1} at a temperature of 136°C. Cooling water in the shell side enters at a mass flow rate of 33 kg s^{-1} at a temperature of 10°C. The shell side heat transfer coefficient is 850 W m^{-2} K^{-1}; and the specific heat capacity of water is 4.187 kJ kg^{-1} K^{-1}. The Nusselt number is approximately related to the Reynolds and Prandtl numbers as follows

$$\mathrm{Nu} = 0.025\,\mathrm{Re}^{3/4}\,\mathrm{Pr}^{2/5} \tag{41}$$

and the following property values apply: specific heat capacity of oil: 3.42 kJ kg^{-1} K^{-1}; density of oil: 900 kg m^{-3}; dynamic viscosity of oil: 1.5×10^{-3} kg m^{-1} s^{-1}; thermal conductivity of oil: 0.15 W m^{-1} K^{-1}; thermal conductivity of the steel pipe wall: 54 W m^{-1} K^{-1}. Calculate:

a) The number of transfer units. **[12 marks]**

Solution:

Starting with the oil side, we have:

$$\langle v\rangle = \frac{\dot m}{\rho\,N_{tubes}\,A_{flow,inner}} = \frac{32}{900\times316\times\pi\times0.008^2} \approx 0.5596\ \mathrm{m\,s^{-1}}$$

Calculating the oil side Reynolds number we have

$$\mathrm{Re} = \frac{\rho\,\langle v\rangle\,d}{\mu} = \frac{900\times 0.5596\times 0.016}{1.5\times 10^{-3}} \approx 5372$$

The Prandtl number is

$$\mathrm{Pr} = \frac{\mu\,C_p}{k} = \frac{1.5\times 10^{-3}\times 3.42\times 10^{3}}{0.15} = 34.2$$

The tube-side Nusselt number is then

$$\begin{aligned}\mathrm{Nu} &= 0.025\,\mathrm{Re}^{3/4}\,\mathrm{Pr}^{2/5}\\ &= 0.025\times 5372^{3/4}\times 34.2^{2/5}\approx 64.44\end{aligned}$$

The heat transfer coefficient is then

$$h_{inner} = \frac{k\,\mathrm{Nu}}{L} = \frac{0.15\times 64.44}{0.016}\approx 603.8\mathrm{W\,m^{-2}K^{-1}}$$

The total resistance is

$$\begin{aligned}(U\,A)^{-1}_{total} &= R_{total}\\ &= \frac{1}{h_{inner}\,A_{inner}}+\frac{1}{h_{outer}\,A_{outer}}+\frac{\ln\left(R_{outer}/R_{inner}\right)}{2\,\pi\,L\,k\,N_{tubes}}\\ &= \frac{1}{603.8\times 5.6\times 316\times\pi\times 0.016}+\frac{1}{850\times 5.6\times 316\times\pi\times 0.018}+\frac{\ln\left(0.018/0.016\right)}{2\,\pi\,5.6\times 54\times 316}\\ &\approx 1.862\times 10^{-5}+1.176\times 10^{-5}+1.962\times 10^{-7}\\ &\approx 3.057\times 10^{-5}\approx\left(32710\ \mathrm{W/K}\right)^{-1}\end{aligned}$$

To calculate the NTU, first determine the minimum heat capacity rate. For the oil we have $C_{oil} = \dot{m}\,C_p = 32\times 3.42 = 109.44$ kW K^{-1}. For the water we have $C_{oil} = 33\times 4.18 = 138.2$ kW K^{-1}. The NTU is then

$$\mathrm{NTU} = \frac{U\,A}{C_{min}} = \frac{32710}{109.4\times 10^{3}}\approx 0.2990$$

b) The effectiveness of the heat exchanger. **[3 marks]**

Solution:

The effectiveness of the counter-current flow heat exchanger is given by the following expression from the data-sheet:

$$E = \frac{1-\exp\left[-\mathrm{NTU}(1-C_r)\right]}{1-C_r\exp\left[-\mathrm{NTU}(1-C_r)\right]}$$

where $C_r = C_{min}/C_{max} = 109.44/138.2 = 0.7919$ is the heat capacity ratio. Completing the equation we have

$$\begin{aligned}E &= \frac{1-\exp\left[-0.2990(1-0.7919)\right]}{1-0.7919\exp\left[-0.2990(1-0.7919)\right]}\\ &\approx 0.2358\approx 23.58\%\end{aligned}$$

[Question total: 15 marks]

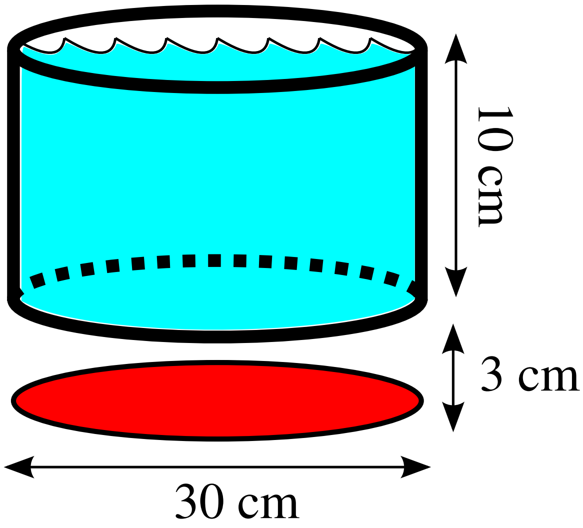


Figure 17: The boiling pot problem.

Q.41 Question 41
Example exam question (2015)
Consider a pot of boiling water placed on a radiant (halogen) cooking hob. As the water is boiling, the surface temperature of the pot will be approximately the boiling temperature. The pot is exposed to the atmosphere and the air/surroundings are at 20°C.

a) Calculate the natural convective heat loss from the sides of the pot given that air has a mean molar mass of $M_W \approx 29$ g mol^{-1}, a dynamic viscosity of $\mu \approx 1.8 \times 10^{-5}$ Pa s, a thermal conductivity of $k_{air} \approx 0.0257$ W m^{-1} K^{-1}, and a Prandtl number of Pr ≈ 0.713. **[11 marks]**

Solution:
Assuming that the surface of the pot is a constant temperature of $100°\mathrm{C}$ as the water is boiling, the film temperature is $(100+20)/2 = 60\ °\mathrm{C} = 333$ K. The density is then

$$\rho = M_W \frac{P}{R\,T} = 29 \frac{10^5}{8.314 \times 333} = 1047\ \mathrm{g\ m^{-3}} = 1.047\ \mathrm{kg\ m^{-3}}$$

Using that $\beta_{ideal\,gas} = T^{-1}$, we can calculate the Grashof number:

$$\frac{\mathrm{Gr}}{L^3} = \frac{g\,\rho^2\,\beta\,(T_w - T_\infty)}{\mu^2} = \frac{9.81 \times 1.047^2 \times 333^{-1}\,(100-20)}{(1.8 \times 10^{-5})^2} \approx 7.974 \times 10^9$$

Calculating this for both length scales in this case we have

$$\mathrm{Gr}_H = 7.974 \times 10^9 \times 0.1^3 = 7.974 \times 10^6 \qquad \mathrm{Gr}_D = 7.974 \times 10^9 \times 0.3^3 = 2.153 \times 10^8$$

Testing if the expression for vertical plates can be used, we have

$$(D/H) \geq 35\,\mathrm{Gr}_H^{-1/4}$$
$$\frac{0.3}{0.1} \geq 35 \times \left(7.974 \times 10^6\right)^{-1/4}$$
$$3 \geq 0.6586$$

As this is true, we do not need the correction factor. The Prandtl number of air is 0.713, thus the Rayleigh number is $\mathrm{Ra} = \mathrm{Gr}_H\,\mathrm{Pr} \approx 5.685 \times 10^6$. In this range of Ra, the Nusselt number for vertical plates is given by

$$\mathrm{Nu}_{plate} = 0.59\,\mathrm{Ra}^{1/4} \approx 28.81$$

The heat transfer coefficient is then:

$$h = \frac{k\,\mathrm{Nu}}{H} \approx \frac{0.0257 \times 28.81}{0.1} \approx 7.40\ \mathrm{W\,m^{-2}\,K^{-1}}$$

The total heat loss is then:

$$\begin{aligned} Q &= h\,A\,\Delta T \\ &= h\,\pi\,D\,H\,\Delta T \\ &= 7.40 \times 3.141 \times 0.3 \times 0.1 \times 80 \\ &= 55.78\ \mathrm{W} \end{aligned}$$

b) Assume that the total heat loss from the pan is 100 W due to evaporation and radiant heat loss to surroundings. Calculate the radiant temperature of the hob/heat-source required to counteract the heat loss. You may assume the pan and heat-source are black-bodies for this calculation.

The view factor between two coaxial discs is

$$F_{1\to 2} = 0.5\left(S - \left(S^2 - 4(r_j/r_i)^2\right)^{0.5}\right)$$

where $S = 1 + \left(1 + R_j^2\right)/R_i^2$, and the reduced radii are $R_i = r_i/L$ and $R_j = r_j/L$. Note L is the gap between the discs, and (r_i, r_j) are the radii of the two discs. **[6 marks]**

Solution:

Radiative heat transfer is given by the following expression:

$$Q = \sigma\,\epsilon\,F_{pot\to hob}\,A_{pot}\left(T_{hob}^4 - T_{pot}^4\right)$$

For this system $R_i = R_j = 0.15/0.03 = 5$. The factor S in the view factor is:

$$\begin{aligned} S &= 1 + \left(1 + R_j^2\right)/R_i^2 \\ &= 1 + \left(1 + 5^2\right)/5^2 \\ &= 2.04 \end{aligned}$$

The view factor is then

$$\begin{aligned} F_{pot\to hob} &= 0.5\left(S - \left(S^2 - 4(r_j/r_i)^2\right)^{0.5}\right) \\ &= 0.5\left(2.04 - \left(2.04^2 - 4\right)^{0.5}\right) \\ &= 0.819 \end{aligned}$$

Solving for the heat source temperature, we have:

$$\begin{aligned} T_{hob} &= \left(\frac{Q}{\sigma\,\epsilon\,F_{pot\to ambient}\,A_{pot}} + T_{pot}^4\right)^{1/4} \\ &= \left(\frac{100}{5.6703 \times 10^{-8} \times 1 \times 0.819 \times 3.141 \times 0.15^2} + 373^4\right)^{1/4} \\ T_{hob} &= 472.5\ \mathrm{K} = 200\ ^\circ\mathrm{C} \end{aligned}$$

c) What fraction of the heat radiated from the heater hits the pot? **[3 marks]**

Solution:

As both the heater and the pot have the same surface area, the view factors are the same (thanks to the reciprocity relationship). Therefore 81.9% of the radiattopm emitted hits the pot!

[Question total: 20 marks]

Q.42 **Question 42**

Consider an unshielded thermometer placed in a room (see Fig. 18). The walls of the house are poorly insulated and the internal surfaces are at a temperature of 5°C. If the thermometer reads 20°C and all surfaces have an emissivity of 0.9, what is the real temperature of the air? You may assume a rough estimate of the natural convective coefficient as $h \approx 10\ \mathrm{W\,m^{-2}\,K^{-1}}$.

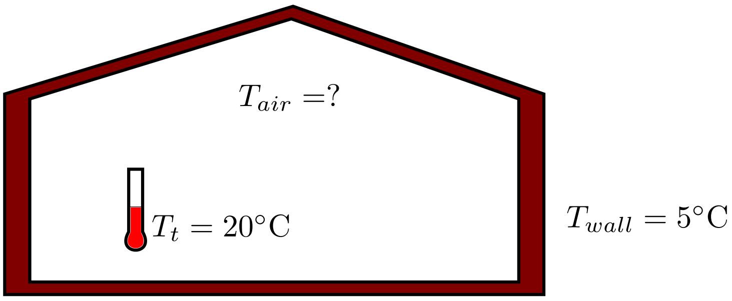


Figure 18: An unshielded thermometer in a room with cold walls.

Solution:

At steady state, the heat gained/lost by convection must equal the heat lost/gained by convection.

$$Q_{conv.} = -Q_{rad.}$$
$$h\,A_t\,(T_{air} - T_t) = -\sigma\,\varepsilon_t\,A_t\left(T_{wall}^4 - T_t^4\right)$$

We assume the radiation to/from the air is negligible compared to the radiation to/from the wall.

Solving for the air temperature difference, we have

$$\begin{aligned} T_{air} - T_T &= -\sigma\,\varepsilon_t\left(T_{wall}^4 - T_t^4\right)/h \\ &= 5.67 \times 10^{-8} \times 0.9 \times \left(293^4 - 278^4\right)/10 \\ &\approx 7.13°\,\mathrm{C} \end{aligned}$$

The air is actually 7.13°C warmer than the thermometer's reading and is at 27.13°C.

[Question end]

Q.43 **Question 43**

The James webb telescope uses a radiation shield to reduce the heat it recieves from the sun, earth, and moon (see Fig. 19). By what factor will the radiation be approximately reduced by? How realistic is this estimate (what approximations are there)? Is this an over or under estimate of the reduction in radiation? **[5 marks]**

Solution:

Radiative heat transfer is reduced by infinitely thin shields as follows:

$$\frac{Q_{shielded}}{Q_{unshielded}} = \frac{1}{1+N}$$

[1/5] $\checkmark_1$ Here $N = 5$, therefore $1/6$th of the radation will reach the telescope $\checkmark_1$. This assumes that the
[1/5] layers are not joined together (but in reality they are at their vertices) and that the shields are infinite (they are not). The finite size of the shield will allow additional radiation to escape to surroundings, and the finite thickness will add an additional conductive resistance. My prediction is that these losses will probably exceed the conduction of heat through the
[3/5] supports, leading the equation above to be an under-estimation of the heat loss. $\checkmark_3$

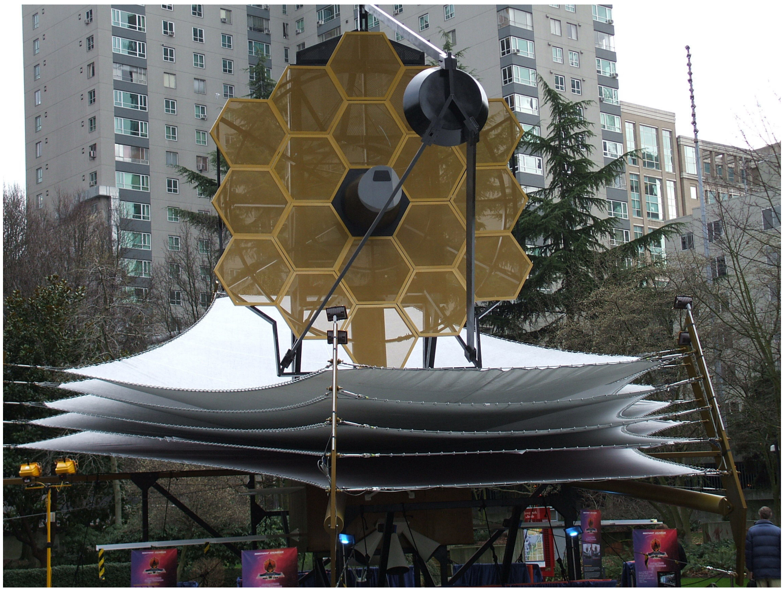


Figure 19: A mock-up of the James Webb telescope, displaying its five-layered sunshield.

[Question total: 5 marks]

Q.44 Question 44

What are the reciprocity relationship and the summation rule with respect to radiative heat transfer? How are these useful?

Solution:

The reciprocity relationship states that the view factors between two objects are related by their area. For example:

$$F_{1\to 2}\,A_1 = F_{2\to 1}\,A_2$$

The summation rule states that the view factors from a single object must sum to unity:

$$F_{1\to 2} + F_{1\to 3} + F_{1\to 4} + ... = 1$$

These rules are useful as they allow a simpler way to calculate view factors in complex geometries. View factors are often the most complex part of radiation calculations.

[Question end]

Q.45 Question 45

Example exam question

A 10 m pipe with a outer-radius of r_{pipe} = 2.5 cm is to be insulated using a layer of insulation with a thermal conductivity of k = 0.18 W m^{-1} K^{-1}. You may assume that the external convective heat transfer coefficient of the insulation is constant at h = 5 W m^{-2} K^{-1} and that these two mechanisms are the only significant heat transfer resistances.

a) Write down the heat transfer equation for this system showing how the overall heat transfer rate Q depends on k, r_{pipe}, the outer radius of the pipe insulation $r_{ins.}$, the pipe length L, and the temperature difference ΔT between the pipe wall and the ambient air. **[4 marks]**
Note: The resistance to heat transfer in a cylindrical shell is:

$$R = \frac{\ln\left(r_{outer}/r_{inner}\right)}{2\,\pi\,k\,L}$$

Solution:
The external convection resistance to heat transfer is

$$R_{conv.} = \frac{1}{2\,\pi\,r_{ins.}\,L\,h}$$

Summing the resistances, we have

$$Q = \Delta T / \left(\frac{\ln\left(r_{ins.}/r_{pipe}\right)}{2\,\pi\,k\,L} + \frac{1}{2\,\pi\,r_{ins.}\,L\,h}\right)$$

b) Calculate the heat transfer rate for three thicknesses of insulation where the outer radius of the insulation is $r_{ins.}$ = 3.0 cm, 3.6 cm, and 4.2 cm). The surface temperature of the pipe is 400°C and ambient conditions are at 10°C. **[3 marks]**
Solution:
Substituting in the known values, we have

$$\begin{aligned} Q &= 2\,\pi\,L\,\Delta T / \left(\frac{\ln\left(r_{ins.}/r_{pipe}\right)}{k} + \frac{1}{r_{ins.}\,h}\right) \\ &= 2\,\pi\,10 \times (400 - 10) / \left(\frac{\ln\left(r_{ins.}/0.025\right)}{0.18} + \frac{1}{r_{ins.} \times 5}\right) \\ &= 24500/\left(\ln\left(r_{ins.}/0.025\right)/0.18 + 0.2/r_{ins.}\right) \\ &= \begin{cases} 3190\ \mathrm{W} & \text{for } r_{ins.} = 3.0 \\ 3231\ \mathrm{W} & \text{for } r_{ins.} = 3.6 \\ 3205\ \mathrm{W} & \text{for } r_{ins.} = 4.2 \end{cases} \end{aligned}$$

c) Explain why you observe a maximum in the heat transfer rate. **[3 marks]**
Solution:
There is a maximum in the heat transfer rate as, at first, the resistance to convection decreases faster than the resistance to conduction increases. Eventually the conduction resistance dominates.

Additional notes (not assessed/marked): The critical radius can be derived as follows:

$$\frac{\partial Q}{\partial r} = 2\,\pi\,L\,\Delta T \frac{\partial}{\partial r} 1 / \left(\frac{\ln\left(r_{ins.}/r_{pipe}\right)}{k} + \frac{1}{r_{ins.}\,h}\right)$$

Using the chain rule

$$\begin{aligned} \frac{\partial}{\partial r} 1 / \left(\frac{\ln\left(r_{ins.}/r_{pipe}\right)}{k} + \frac{1}{r_{ins.}\,h}\right) &= -\left(\frac{\ln\left(r_{ins.}/r_{pipe}\right)}{k} + \frac{1}{r_{ins.}\,h}\right)^{-2} \frac{\partial}{\partial r}\left(\frac{\ln\left(r_{ins.}/r_{pipe}\right)}{k} + \frac{1}{r_{ins.}\,h}\right) \\ &= -\left(\frac{\ln\left(r_{ins.}/r_{pipe}\right)}{k} + \frac{1}{r_{ins.}\,h}\right)^{-2}\left(\frac{1}{r_{ins.}\,k} - \frac{1}{r_{ins.}^2\,h}\right) \end{aligned}$$

Thus the derivative is

$$\frac{\partial Q}{\partial r} = -2\,\pi\,L\,\Delta T \left(\frac{\ln\left(r_{ins.}/r_{pipe}\right)}{k} + \frac{1}{r_{ins.}\,h}\right)^{-2}\left(\frac{1}{r_{ins.}\,k} - \frac{1}{r_{ins.}^2\,h}\right)$$

This derivative is zero when

$$\left(\frac{1}{r_{ins.}\,k} - \frac{1}{r_{ins.}^2\,h}\right) = 0$$

Rearranging for the critical insulation radius gives

$$r_c = \frac{k}{h}$$

In this case the critical radius is $r_c = 0.18/5 = 0.036$ m or 3.6 cm.

[Question total: 10 marks]

Q.46 Question 46

The potential heat loss from a distillation column to the environment must be calculated to determine if lagging (insulation) on the column is required. The proposed design of a distillation column can be modelled to a rough approximation as a 12 m high cylinder with a diameter of 0.7 m. Convection and radiation are assumed to be the limiting heat transfer processes, so it can be assumed the column surface is at the internal operating temperature of 60°C. The minimum ambient air temperature should be used for the calculations in order to design for a worst-case scenario
Aberdeen ambient temperature range: -10°C to 20°C.
Emissivity of oxidised steel: $\varepsilon \approx 0.657$

T (°C)	ρ (kg m^{-3})	C_p (kJ kg^{-1} K^{-1})	k (W m^{-1} K^{-1})	ν ($\times 10^{-6}$ m^2 s^{-1})	Pr
-50	1.534	1.005	0.0204	9.55	0.725
0	1.293	1.005	0.0243	13.30	0.715
20	1.205	1.005	0.0257	15.11	0.713
40	1.127	1.005	0.0271	16.97	0.711
60	1.067	1.009	0.0285	18.90	0.709

Table 2: Properties of Air

a) Describe the physical interpretation of the Grashof number for natural convection. Describe each of its terms and write down an equation for the temperature at which temperature-dependent terms in Gr should be evaluated. **[5 marks]**

Solution:

The Grashof number is the analogue of the Reynolds number for natural convection and is the ratio of bouyancy and viscous forces in the fluid. It is defined as

$$\mathrm{Gr} = \frac{g\,\rho^2\,\beta\,(T_w - T_\infty)\,L^3}{\mu^2},$$

where g is the gravitational acceleration,
ρ is the density of the fluid,
β is the thermal expansion coefficient of the fluid,
T_w is the wall temperature,
T_∞ is the fluid temperature a large distance from the wall (bulk),
L is a characterstic (and often vertical) length scale,
and μ is the fluid viscosity.

The properties of the flow for the Grashof number should be evaluated at the so-called film temperature,

$$T_f = (T_w + T_\infty)/2 \qquad = (60 - 10)/2 = 25^\circ\mathrm{C} \approx 298K$$

b) Show that the thermal expansion coefficient, defined as

$$\beta = \frac{1}{V}\frac{\partial V}{\partial T}$$

reduces to the following expression for an ideal gas

$$\beta_{ig.} = \frac{1}{T}$$

Hint: Use the ideal gas equation! **[2 marks]**

Solution:

We know that $V = N\,R\,T/P$, so

$$\begin{aligned}\beta &= \frac{1}{V}\frac{\partial V}{\partial T}\\ &= \frac{1}{V}\frac{\partial N\,R\,T/P}{\partial T}\\ &= \frac{N\,R}{V\,P}\frac{\partial T}{\partial T}\\ &= \frac{N\,R}{V\,P}\end{aligned}$$

If we rearrange $PV = N\,R\,T$ we have $N\,R/(V\,P) = 1/T$, giving the final result:

$$\begin{aligned}\beta &= \frac{1}{V}\frac{\partial V}{\partial T}\\ \beta_{ig.} &= \frac{1}{V}\frac{\partial N\,R\,T/P}{\partial T}\\ &= \frac{N\,R}{V\,P}\frac{\partial T}{\partial T}\\ &= \frac{1}{T}\end{aligned}$$

c) Calculate the Grashof number and determine the convective flow regime. State any assumptions you make. Remember to use the correct temperature for calculating the properties of the flow! **[5 marks]**

Solution:

Looking at our table of properties of air, we could interpolate for the film temperature $T_f = 25$°C. However, considering the error in convective heat transfer coefficients it is safe enough to take the nearest temperature (20°C).

The thermal expansion coefficient is given by

$$\beta = \frac{1}{V}\frac{\partial V}{\partial T}$$

However air at the ambient temperature and pressure (10°C and 1 atm) behaves similarly to an ideal gas, so we can use the approximation

$$\beta \approx \frac{1}{T_f}$$

Using these values and noting that $\nu = \mu/\rho$ we have

$$\begin{aligned}\mathrm{Gr} &= \frac{g\,(T_w - T_\infty)\,L^3}{T_f\,\nu^2}\\ &= \frac{9.81 \times 70 \times 12^3}{298 \times (15.11 \times 10^{-6})^2}\\ &= 1.74 \times 10^{13}\end{aligned}$$

The critical Grashof number is around $\mathrm{Gr} \approx 4 \times 10^8$, thus the convective regime around the column is turbulent.

d) Calculate the convective heat transfer coefficient for the column surface. Can you use the expression for vertical plates directly? **[9 marks]**

Solution:

The expressions for vertical plates can only be used for cylinders if the boundary layer thickness is small compared to the diameter of the cylinder. The general criterion is given as

$$\begin{aligned}\frac{D}{L} &\geq \frac{35}{\mathrm{Gr}^{1/4}}\\ \frac{0.7}{12} &\geq \frac{35}{(1.74 \times 10^{13})^{1/4}}\\ 0.058 &\geq 0.017\end{aligned}$$

The criterion is satisfied, so we can use the vertical plate expressions for the vertical cylinder.

The value of the Raleigh number is

$$\begin{aligned}\mathrm{Ra} &= \mathrm{Gr}\,\mathrm{Pr} = 1.73 \times 10^{13} \times 0.713\\ &= 1.233 \times 10^{13}\end{aligned}$$

Using Table 5, we find that we can use the following expression for the Nusselt number

$$\begin{aligned}\mathrm{Nu} &= 0.13\,(\mathrm{Gr}\,\mathrm{Pr})^{1/3}\\ &= 0.13\left(1.233 \times 10^{13}\right)^{1/3}\\ &\approx 3000\end{aligned}$$

The heat transfer coefficient is given by

$$\begin{aligned}h &= \frac{k\,\mathrm{Nu}}{L}\\ &= \frac{0.0257 \times 3000}{12}\\ &= 6.425\ \mathrm{W\ m^{-2}\ K^{-1}}\end{aligned}$$

e) Calculate the total heat lost to the environment including radiation. Compare the two losses. **[5 marks]**

Solution:

The total loss of energy is calculated by summing the convective and radiative heat losses.

$$Q = A\,h\,(T_w - T_\infty) + A\,\sigma\,\varepsilon\left(T_w^4 - T_\infty^4\right)$$

The surface area of the cylinder (neglecting top and bottom faces) is

$$A = \pi\,D\,L = \pi \times 0.7 \times 12 \approx 26.4\ m^2$$

substituting all of the known values into the equation we have

$$\begin{aligned}Q &= 26.4 \times 6.425 \times 70 + 26.4 \times 5.67 \times 10^{-8} \times 0.657\left(333^4 - 263^4\right)\\ &= 11873 + 7388\\ &= 19.3\ kW\end{aligned}$$

The convective heat loss is 60% larger than the radiative heat loss.

f) Do you think this heat loss justifies adding insulation or lagging to the outside of the column? **[1 marks]**

Solution:

The answer depends on the requirements of the design and a cost/benefit analysis! In reality, the additional cost of lagging and the extra maintenance cost of removing it for inspection outweighs the energy cost saved by its use. Columns often operate with a boiler and condenser in the 0.1-1 MW range, so this heat loss is negligible.

The actual cost in lost heat is quite small. To an industrial plant, natural gas costs around 2p per kWh in 2010. Assuming there are 8000 plant operating hours in a year, the total cost of this loss is

$$19 \times 8000 \times 2 = 304000 \text{ pence}$$

The insulation alone may cost more than this.

g) Due to strict new environmental legislation, it is decided that the maximum acceptable heat loss to the environment is 10 kW. Roughly calculate the maximum acceptable surface temperature. **[3 marks]**

Solution:

We can assume our heat transfer coefficient remains the same. It only decreases for lower temperatures, so reusing the value will give us a maximum value below the true value. This is also true if we used the radiative heat transfer coefficient analogy in the previous question

$$Q = A\,h\,(T_w - T_\infty) + A\,\sigma\,\varepsilon\left(T_w^4 - T_\infty^4\right)$$

inserting the values we have

$$\begin{aligned} 10000 &= 26.4 \times 6.425\,(T_w - 263) + 26.4 \times 5.67 \times 10^{-8} \times 0.657\left(T_w^4 - 263^4\right) \\ &= 169.62\,(T_w - 263) + 9.835 \times 10^{-7}\left(T_w^4 - 263^4\right) \\ 59300 &= 169.62\,T_w + 9.835 \times 10^{-7}\,T_w^4 \end{aligned}$$

Using excel or Matlab we can solve this equation to find the maximum surface temperature is $T_w = 302$ K.

This is how to solve the above problem the old fashioned way (by hand)

We could also solve this by rearranging the above equation to give

$$T_w = \frac{59300 - 9.835 \times 10^{-7}\,T_w^4}{169.62}$$

This is an expression for a better estimate of T_w, given a current estimate of T_w. We guess the starting value of $T_w = 330$K, insert it into the above equation and calculate a new value then repeat until the new temperature value stops changing.

h) Comment on what steps would be required to improve the accuracy of the surface temperature calculation in Q. g. **[2 marks]**

Solution:

The problem with the above estimate is that the heat transfer coefficient is calculated using the incorrect film temperature T_w. We need to iterate the above calculations!

Explicitly, to obtain a better estimate we need to

i) Take the current estimate for the maximum wall temperature T_w.

ii) Calculate the film temperature $T_f = (T_w + T_\infty)/2$.

iii) Calculate the heat transfer coefficient using this film temperature (as in Q. c–d).

iv) Estimate new maximum surface temperature (as in Q. g).

v) If the new estimate is very different to the current estimate, go back to step hi.

[Question total: 32 marks]

Q.47 Question 47

Write down the expressions for the Prandtl number. Define every term and describe the physical interpretation of the dimensionless numbers.

Solution:

The Prandtl number is defined as

$$\mathrm{Pr} = \frac{\mu\, C_p}{k}$$

where μ is the fluid viscosity, C_p is the fluid heat capacity and k is the thermal conductivity. The Prandtl number is a ratio of the momentum to thermal transport in a fluid.

[Question end]

Q.48 Question 48

Example exam question (2015)

The wall of a furnace comprises three layers as shown in Fig. 20. The first layer is refractory brick (whose maximum allowable temperature is 1400°C) while the second layer is insulation (whose maximum allowable temperature is 1093°C). The third layer is a plate of 6.35 mm thickness of steel (k_{steel} = 45 W m^{-1} K^{-1}). Assume that the layers are thermally bonded.

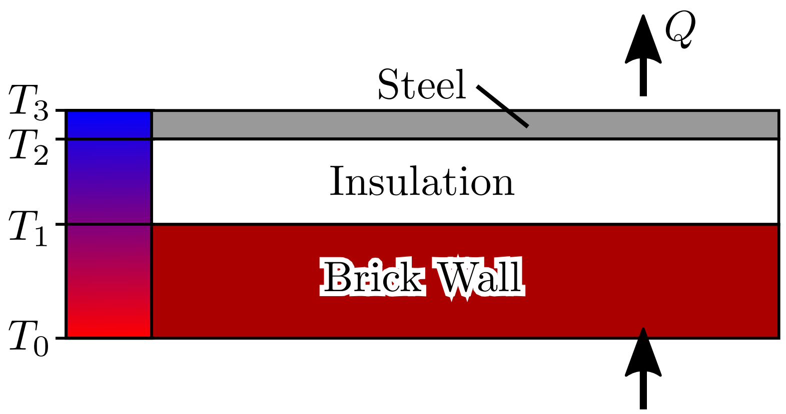


Figure 20: Construction of a furnace wall.

Layer	T = 37.8°C	T = 1093°C
Brick	3.12 W m^{-1} K^{-1}	6.23 W m^{-1} K^{-1}
Insulation	1.56 W m^{-1} K^{-1}	3.12 W m^{-1} K^{-1}

Table 3: Thermal conductivities for Q. 48.

The temperature T_0 on the inside of the refractory is 1370°C, while the temperature on the outside of the steel plate is 37.8°C. the heat loss through the furnace wall is expected to be 15800 W m^{-2}. Determine the thickness of refractory and insulation that results in the minimum total thickness of the wall. You may use the temperature dependent thermal conductivities given in Table 3. **[14 marks]**

Solution:

First, we can work out the temperature T_2:

$$T_2 = \frac{q\,X_{2-3}}{k_{steel}} + T_3 = \frac{15800 \times 0.00635}{45} + 37.8 = 40^{\circ}\mathrm{C}$$

The wall thickness is given by $X_{1-2} + X_{2-3}$. These are calculated using

$$X_{i-j} = \frac{k}{q}(T_i - T_j)$$

We're therefore searching for the minimum of

$$\begin{aligned} X_{1-3} &= q^{-1}\left(k_{brick}\,(T_0 - T_1) + k_{insulation}\,(T_1 - T_2)\right) \\ &= q^{-1}\left(k_{brick}\,T_0 + (k_{insulation} - k_{brick})T_1 - k_{insulation}\,T_2\right) \end{aligned}$$

Clearly, T_1 should be as large as possible as $k_{insulation} - k_{brick}$ is negative. The maximum T_1 can be is $1093^{\circ}\mathrm{C}$ as specified by the insulation limits.

The value of thermal conductivity used for the insulation should span the full temperature range from $40 \to 1093^{\circ}\mathrm{C}$. The most sensible choice given the available information is to use the average $k_{insulation} \approx (1.56 + 3.12)/2 \approx 2.34$ $\mathrm{W\,m^{-1}\,K^{-1}}$.

$$X_{1-2} = \frac{k_{insulation}}{q}(T_1 - T_2) = \frac{2.34}{15800}(1093 - 40) \approx 0.156 \text{ m}$$

The brick temperature is close enough that the single value 6.23 $\mathrm{W\,m^{-1}\,K^{-1}}$ could be used by the students but a better estimate would result from linear extrapolation to give $k_{brick} \approx 7.05$ $\mathrm{W\,m^{-1}\,K^{-1}}$ at $T = 1370^{\circ}\mathrm{C}$. This can be averaged over the operating range to give $k_{brick} \approx (6.23 + 7.05)/2 \approx 6.64$ $\mathrm{W\,m^{-1}\,K^{-1}}$. The brick thickness is then given by

$$X_{0-1} = \frac{k_{brick}}{q}(T_0 - T_1) = \frac{6.64}{15800}(1370 - 1093) \approx 0.116 \text{ m}$$

The total wall thickness is then ≈ 0.278 m.

[Question total: 14 marks]

Q.49 Question 49

In prilling towers, molten fertilizer slurry is dripped to form frozen spherical pellets called prills. As a first approximation to understanding the heat transfer from the falling prills, consider a heated sphere of radius, R, and fixed surface temperature, T_R, suspended in a large, motionless body of fluid.

a) Set up the differential equation describing the temperature, T, in the surrounding fluid as a function of r, the distance from the center of the sphere. The thermal conductivity, k, of the fluid is considered constant. **[14 marks]**

Solution:

[2/14] If we assume there is no pressure dependence of the internal energy of the fluid$\checkmark_2$ we can use the energy balance equation (see Eq.(68)):

$$\rho\,C_p\frac{\partial T}{\partial t} = -\rho\,C_p\,v_j\,\nabla_j\,T - \nabla_i\,q_i - \tau_{ji}\,\nabla_j\,v_i - p\,\nabla_i\,v_i + \sigma_{energy}$$

[1/14] $\checkmark_1$ Assuming the fluid is motionless ($\boldsymbol{v} = 0$) steady state, and no heat generation, we have $\checkmark_3$

[3/14]

$$\cancelto{0}{\rho\, C_p \frac{\partial T}{\partial t}} = -\cancelto{0}{\rho\, C_p\, v_j \nabla_j\, T} - \nabla_i\, q_i - \cancelto{0}{\tau_{ji} \nabla_j\, v_i} - \cancelto{0}{p \nabla_i\, v_i} + \cancelto{0}{\sigma_{energy}}$$

$$\nabla_i\, q_i = 0$$

[1/14] $\checkmark_1$ Using spherical coordinates we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\, q_r\right) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}\left(q_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial\, q_\phi}{\partial \phi} = 0$$

[1/14] $\checkmark_1$ Assuming the system is rotationally symmetric we can state that nothing changes in the θ or ϕ directions to cancel the derivatives OR note that there is no transport in these directions

[2/14] to give: $\checkmark_2$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\, q_r\right) = 0$$

[2/14] $\checkmark_2$ Inserting Fourier's law and noting the thermal conductivity is constant we have

$$-\frac{\partial}{\partial r} r^2\, k \frac{\partial T}{\partial r} = 0$$

$$\frac{\partial}{\partial r} r^2 \frac{\partial T}{\partial r} = 0$$

[2/14] $\checkmark_2$

b) Integrate the differential equation and use these boundary conditions to determine the integration constants: at $r = R$, $T = T_R$; and at $r = \infty$, $T = T_\infty$. **[8 marks]**

Solution:

Integrating the equation once, we have

$$\frac{\partial}{\partial r} r^2 \frac{\partial T}{\partial r} = 0$$

$$\frac{\partial T}{\partial r} = \frac{C_1}{r^2}$$

[2/8] $\checkmark_2$ Integrating again, we have

$$T = \frac{C_1}{r} + C_2$$

[2/8] $\checkmark_2$ Using the boundary conditions, we have at $r = \infty$, $T = T_\infty$ which gives $C_2 = T_\infty$. For $r = R$

[2/8] and $T = T_R$ we have $\checkmark_2$

$$T_R = \frac{C_1}{R} + T_\infty$$

$$C_1 = R\left(T_R - T_\infty\right)$$

$$T = T_\infty + \left(T_R - T_\infty\right)\frac{R}{r}$$

[2/8] $\checkmark_2$

c) From the temperature profile, obtain an expression for the heat flux at the surface. Equate this result to the heat flux given by "Newton's law of cooling" and show that a dimensionless heat transfer coefficient (known as the Nusselt number) is given by,

$$\mathrm{Nu} = \frac{h\,D}{k} = 2,$$

in which D is the sphere diameter. **[12 marks]**

Solution:

[2/12] The heat flux is given by substituting the temperature profile into Fouriers law OR by tracking the constants in the derivation above:$\checkmark_2$

$$q = -k\frac{\partial T}{\partial r}$$
$$q = k\,(T_R - T_\infty)\,\frac{R}{r^2}$$

[2/12] $\checkmark_2$At the surface we have

$$q = \frac{k}{R}\,(T_R - T_\infty)$$

[2/12] $\checkmark_2$Comparing this to Newton's law of cooling

$$q = \frac{Q}{A} = h\,\Delta T$$
$$= \frac{k}{R}\,(T_R - T_\infty)$$
$$h = \frac{k}{R}$$

[3/12] $\checkmark_3$Inserting this into the Nusselt number, we have

$$\mathrm{Nu} = \frac{h\,D}{k}$$
$$= \frac{k\,D}{R\,k} = 2$$

[3/12] $\checkmark_3$

[Question total: 34 marks]

Q.50 Question 50

Example exam question

A black-body car is left in direct sunlight at midday which (at the lattitude of the UK) can be approximated as a constant heat flux q_{sun} = 1000 W m^{-2}. The car's surface temperature reaches steady state with its surroundings and is approximately constant. The car has a surface area of 26 m^2 but only 8 m^2 are exposed to sunlight.

a) Assuming that the ambient temperature is 15°C and that radiation is the only heat transfer mechanism, calculate the surface temperature of the car. Is the estimate realistic? **[5 marks]**

Solution:

At steady state, the flux of energy into the car from the sunlight is equal to the energy lost through radiation:

$$A_{sun}\,q_{sun} = A_{car}\,q_{rad} = \sigma\,A_{car}\,\varepsilon\left(T_{car}^4 - T_\infty^4\right)$$

We have $\varepsilon = 1$ as the car is black and $\sigma = 5.6703 \times 10^{-8}$ W m^{-2} K^{-4} from the data sheet. Substituting in the knowns, we have

$$\begin{aligned}
8 \times 1000 &= 26 \times 5.6703 \times 10^{-8} \left(T_{car}^4 - 288.15^4\right) \\
T_{car}^4 &= \frac{8000}{26 \times 5.6703 \times 10^{-8}} + 288.15^4 \\
T_{car} &= 333\ K = 60\ ^\circ C
\end{aligned}$$

This temperature is fairly realistic for cars in the UK on hot summer days.

b) Using the previous estimate for the surface temperature, estimate the heat flux due to natural convection and comment on its magnitude. You may approximate the sides of the car as a vertical wall 12 m wide and 1.5 m high. You may assume the following properties of air at these conditions. State why natural convection from the top of the car is insignificant when compared to the sides. **[8 marks]**

ρ (kg m^{-3})	k (W m^{-1} K^{-1})	μ (kg m^{-1} s^{-1})	C_p (J mol^{-1} K^{-1})	Avg. Mol. Weight (g mol^{-1})
1.225	0.026	1.827×10^{-5}	29.19	29

Solution:

First we must calculate the Grashof number, but we need the thermal expansion coefficient. We can quickly derive it from the ideal gas equation and the identity in the datasheet

$$\beta = \frac{1}{V}\frac{\partial V}{\partial T} = \frac{1}{V}\frac{\partial}{\partial T}\frac{n\,R\,T}{P} = \frac{n\,R}{P\,V} = \frac{1}{T}$$

Or simply remember that $\beta = 1/T$ for an ideal gas. This must be evaluated at the film temperature $T_f = (T_{wall} + T_\infty)/2 = (60+15)/2 = 37.5$ °C$= 311$ K. The L term in the Grashof number is the plate height, as the height is the characteristic length for convection.

$$\begin{aligned}
\mathrm{Gr} &= \frac{g\,\rho^2\,\beta\,(T_w - T_\infty)\,L^3}{\mu^2} \\
&= \frac{9.81 \times 1.225^2 (60 - 15) 1.5^3}{311 \times (1.827 \times 10^{-5})^2} \\
&\approx 2.1537 \times 10^{10}
\end{aligned}$$

Converting C_p to kJ kg^{-1} K^{-1}, we have $C_p = 29.19/29 = 1.007$ kJ kg^{-1} K^{-1}. Calculating the Prandtl number

$$\mathrm{Pr} = \frac{\mu\,C_p}{k} = \frac{1.827 \times 10^{-5} \times 1.007 \times 10^3}{0.026} \approx 0.71$$

The Rayleigh number is then

$$\mathrm{Ra} = \mathrm{Pr}\,\mathrm{Gr} = 0.71 \times 2.1537 \times 10^{10} = 1.529 \times 10^{10}$$

Looking in the datasheet, this corresponds to the following expression for the Nusselt number

$$\mathrm{Nu} = 0.13\,(\mathrm{Ra})^{1/3} = 0.13 \times (2.1537 \times 10^{10})^{1/3} \approx 361.7$$

This gives a heat transfer coefficient of

$$h = \frac{\mathrm{Nu}\, k}{L} = \frac{361.7 \times 0.026}{1.5} \approx 6.27\ \mathrm{W\ m^{-2}\ K^{-1}}$$

The convective heat transfer is then

$$Q_{conv.} = h\,A\,(T_{wall} - T_\infty) = 6.27 \times 1.5 \times 12\,(60 - 15) = 5079\ \mathrm{W}$$

The natural convective heat transfer is realtively large compared to the radiant heat transfer, therefore this needs needs to be solved implicitly (i.e., via iterations to find the true surface temperature).

This neglects convection from the horizontal surfaces as it is typically much smaller than from vertical surfaces as circulating flow is more difficult to establish in that case.

c) Discuss how you might improve the accuracy of the calculations, and what the effect of setting the car in motion will be. **[2 marks]**
Solution:
The accuracy may be improved by finding a better approximation for the car surface for the convection calculations, specifying realistic emissivities for the car surface, and solving for the radiation and convection fluxes simulateously.

If the car is set in motion, the natural convection will become a forced convection, greatly increasing the heat transfer rate of this mode. It is likely that this will cause the car surface to cool even further.

[Question total: 15 marks]

Q.51 Question 51

The wall of a furnace was measured to be at a temperature of $T_w = 60\ ^\circ\mathrm{C}$ when the ambient air temperature is at $T_\infty = 10\ ^\circ\mathrm{C}$. The wall is 3 m high, 5 m wide, and has a surface emissivity of $\varepsilon = 0.7$. The properties of air are given in the table below.

μ	1.78×10^{-5} Pa s	ρ	1.2 kg m^{-3}
k	0.02685 W m^{-1} K^{-1}	C_p	1.005 kJ kg^{-1} K^{-1}

a) Determine the convective flow regime of the air, noting that the critical Grashof number is $\mathrm{Gr} \approx 4 \times 10^8$.
Solution:
Here we must calculate the Grashof number. The key characteristics are :

- The thermal compressibility β is given by $\beta = 1/T$ for an ideal gas, which is a good approximation for atmospheric air.
- The properties in the Grashof number should be evaluated at the film temperature $T_f = (T_w + T_\infty)/2$.
- The above rule only applies to the thermal compressibility in this question, as the other properties are unavailable.
- For a vertical plate/wall, the characteristic length is the height of the wall.

Using this knowledge we can calculate the thermal compressibility to be

$$\beta \approx \frac{1}{T_f} = \frac{2}{T_w + T_\infty} = \frac{2}{333.15 + 283.15} \approx 0.0032$$

We can now evaluate the Grashof number

$$\begin{aligned}\mathrm{Gr} &= \frac{g\,\rho^2\,\beta\,(T_w - T_\infty)\;L^3}{\mu^2}\\ &= \frac{9.81\times 1.2^2\times 0.0032\,(60-10)\;3^3}{(1.78\times 10^{-5})^2}\\ &\approx 1.93\times 10^{11}\end{aligned}$$

The convective flow is turbulent as $\mathrm{Gr} \gg 4\times 10^8$.

b) Calculate the heat lost through the furnace wall. Remark on the relative magnitudes of the two heat transfer mechanisms involved.
Solution:
For the convective heat transfer, we must calculate a convective heat transfer coefficient using the relations given in the data sheet. The Prandtl number for the flow is

$$\mathrm{Pr} = \frac{C_p\,\mu}{k} = \frac{1.005\times 10^3\times 1.78\times 10^{-5}}{0.02685} \approx 0.666$$

The Rayleigh number of the flow is

$$\mathrm{Ra} = \mathrm{Gr}\,\mathrm{Pr} = 1.93\times 10^{11}\times 0.666 \approx 1.29\times 10^{11}$$

For this Rayleigh number, the relation to the Nusselt number given in the datasheet is

$$\mathrm{Nu} = 0.13\,\mathrm{Ra}^{1/3} = 0.13\times\left(1.29\times 10^{11}\right)^{1/3} \approx 657$$

The heat transfer coefficient is then given by

$$h_{convective} = \frac{k\,\mathrm{Nu}}{L} = \frac{0.02685\times 657}{3} \approx 5.88\mathrm{W\ m^{-2}\,K^{-1}}$$

The heat flux due to convection is

$$\begin{aligned}Q_{convective} &= A\,h_{convective}\,(T_w - T_\infty)\\ &= 3\times 5\times 5.88\,(60-10) \approx 4410\,\mathrm{W}\end{aligned}$$

The heat lost through radiation is given by

$$\begin{aligned}Q_{radiation} &= A\,\sigma\,\varepsilon\left(T_w^4 - T_\infty^4\right)\\ &= 3\times 5\times 5.67\times 10^{-8}\times 0.7\left(333^4 - 283^4\right) \approx 3500\,\mathrm{W}\end{aligned}$$

The heat loss from the furnace wall is mainly lost through convection, but both effects are comparable.

END OF EG40JK QUESTIONS

[Question end]

Q.52 Question 52

A new type of one-coat spray paint is being developed which flows to precisely the minimum thickness required for a uniform coat. To achieve this property, the paint must effectively be a Bingham plastic.

a) Balance the total gravitational force ($\rho\, g_y$) against the viscous force on a vertical plate to derive the following force balance for the stress at the wall surface:

$$\tau_{boundary} = Y \rho\, g_y$$

Solution:

The total force due to gravity on the film of liquid is

$$X\, Y\, Z \rho\, g_y$$

The total stress on the surface of the plate is given by

$$X\, Y\, \tau_{boundary}$$

where $X\, Y$ is the surface area of the vertical plate. If the system is at **steady state**, then these forces are in balance and we have:

$$X\, Y\, \tau_{boundary} = X\, Y\, Z \rho\, g_y$$
$$\tau_{xy} = Z\, \rho\, g_y$$

b) Assuming that the paint has a density of 900 kg m^{-3}, what yield stress (τ_0) is needed to ensure the paint has a maximum static thickness of 2 mm?

Solution:

The stress is at a maximum at the wall, therefore we need a yield stress at the wall which is exactly the stress caused by a 2 mm film of paint.

$$\begin{aligned}\tau_0 = \tau_{boundary} &= Y\, \rho\, g_y \\ &= 0.002 \times 900 \times 9.81 \\ &\approx 17.66\ \mathrm{N\ m^{-2}} = \approx 17.66\ \mathrm{Pa}\end{aligned}$$

Remember to give the correct units! For comparison, here is a table of yeild stresses for real pseudoplastic fluids:

Fluid	τ_0 (Pa)
Ketchup	15
Salad Dressing	30
Mayonnaise	100
Hair Gel	135

[Question end]

Q.53 Question 53

When manufacturing a plastic toy, a polypropylene melt with a density of 739 kg m^{-3} is to be extruded through a pipe with a length of 1 m and a diameter of 2.5 cm into a die. A shear rate of 1000 s^{-1} is expected at the die lips and experiments at this shear rate have measured an apparent viscosity of 10 N s m^{-2}.

a) A Power-Law model with an exponent of $n = 0.35$ is thought to be a suitable model for the viscous behaviour. Assuming this is true, determine the consistency coefficient k and write down the rheological stress-strain equation for the fluid. **[3 marks]**

Solution:

We can equate Newton's law and the Power-law model to find the following expression in terms of the apparent viscosity $\mu_{apparent}$.

$$\tau_{xy} = -\mu_{apparent}\frac{\partial v_x}{\partial y} = -k\left|\frac{\partial v_x}{\partial y}\right|^{n-1}\frac{\partial v_x}{\partial y}$$

Assuming that at a shear rate of $\partial v_x/\partial y = 1000$ s^{-1}, we have an apparent viscosity of $\mu_{apparent} = 10$ N s m^{-2} and the flow index is $n = 0.35$, we have the following expression

$$\mu_{apparent} = k\left|\frac{\partial v_x}{\partial y}\right|^{n-1}$$
$$10 = k\,1000^{-0.65}$$
$$k \approx 891$$

The rheological equation for the fluid is then given by the Power-Law model with the coefficients inserted in

$$\tau_{xy} = -891\left|\frac{\partial v_x}{\partial y}\right|^{-0.65}\frac{\partial v_x}{\partial y}$$

b) What is the type of this fluid and how will it respond to increasing rates of shear? Describe this using the concept of the apparent viscosity. **[2 marks]**

Solution:

The equation above can be rewritten to give an expression for the apparent viscosity $\mu_{apparent}$ as a function of shear rate.

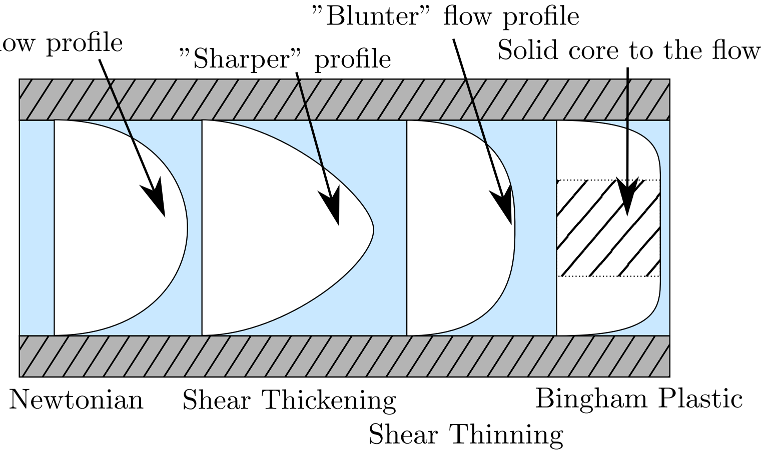
$$\mu_{apparent} = k\left|\frac{\partial v_x}{\partial y}\right|^{n-1}$$

This fluid is shear thinning as $n < 1$, and the apparent viscosity will reduce as the shear rate increases.

c) Sketch two graphs to illustrate the differences between the velocity profile of this fluid and a Newtonian fluid, and between this fluid and a Bingham plastic fluid. **[5 marks]**

Solution:

Here, we're just going to steal the graph from the slides, but in this question you only need to draw the shear thinning, Newtonian and Bingham plastic flow profiles.



The key concepts to highlight are
- The drawings of the velocity profiles
- The parabolic flow profile of a Newtonian fluid
- The blunter flow profile of a shear thinning fluid
- The solid core of a Bingham fluid

d) Derive the following expression for the Reynolds number in Power-Law fluids.

$$\mathrm{Re}_{MR} = \frac{8\,\rho\,\langle v\rangle^{2-n}\,R^n}{k}\left(\frac{n}{3\,n+1}\right)^n$$

Hint: the Metzner-Reed Reynolds number is defined through the friction factor relation,

$$C_f = \frac{16}{\mathrm{Re}_{MR}}.$$

The volumetric flow equation for a laminar power-law fluid is available in the datasheet (see Eq. (69)). **[7 marks]**

Solution:

We need to express the Reynolds number as a function of the desired variables. Take the above definition of the friction factor and substitute it into the Darcy-Wiessbach equation to give

$$-\frac{\Delta p}{L} = \frac{16\,\rho\,\langle v\rangle^2}{\mathrm{Re}_{MR}\,R}$$

Rearranging for the Reynolds number we have

$$\mathrm{Re}_{MR} = -\frac{16\,\rho\,\langle v\rangle^2\,L}{R\,\Delta p}$$

Now we need to eliminate the pressure loss and pipe length terms by expressing it in the desired variables.

The volumetric flow rate for a Power Law fluid is given in the data sheet as

$$\dot{V} = \frac{n\,\pi\,R^3}{3\,n+1}\left(\frac{R}{2\,k}\right)^{\frac{1}{n}}\left(-\frac{\Delta p}{L}\right)^{\frac{1}{n}}$$

The area of the flow is $A = \pi R^2$, therefore the average flow rate is given by

$$\langle v\rangle = \frac{\dot{V}}{A} = \frac{n\,R}{3\,n+1}\left(\frac{R}{2\,k}\right)^{\frac{1}{n}}\left(\frac{-\Delta p}{L}\right)^{\frac{1}{n}}$$

Rearrange this equation to give an expression for the pressure drop in terms of the desired variables

$$-\frac{\Delta p}{L} = \frac{2\,k}{R}\left(\frac{3\,n+1}{n\,R}\right)^{n}\langle v\rangle^{n}$$

We can substitute this into the equation for the Reynolds number to give

$$\mathrm{Re}_{MR} = \frac{16\,\rho\,\langle v\rangle^2}{R}\frac{R}{2\,k}\left(\frac{n\,R}{3\,n+1}\right)^{n}\langle v\rangle^{-n}$$

And cleaning up gives

$$\mathrm{Re}_{MR} = \frac{8\,\rho\,\langle v\rangle^{2-n}\,R^n}{k}\left(\frac{n}{3\,n+1}\right)^{n}$$

e) If a volumetric flow rate of 0.1 $m^3\ h^{-1}$ is required, determine if the flow is laminar in the pipe and calculate the pressure drop. **[3 marks]**

Solution:

The average flow velocity is

$$\langle v\rangle = \frac{\dot{V}}{A} = \frac{0.1}{3600}\frac{1}{\pi\,0.0125^2} \approx 0.057\,\mathrm{m\ s^{-1}}$$

The Reynolds number is then given by

$$\begin{aligned}\mathrm{Re}_{MR} &= \frac{8\,\rho\,\langle v\rangle^{2-n}\,R^n}{k}\left(\frac{n}{3\,n+1}\right)^{n}\\ &= \frac{8\times 739\times 0.057^{1.65}\times 0.0125^{0.35}}{891}\left(\frac{0.35}{3\times 0.35+1}\right)^{0.35}\\ &\approx 6.8\times 10^{-3}\end{aligned}$$

The transition Reynolds number ($\mathrm{Re}_{MR}^{(c)} \approx 2300$) is approximately the same for Power-Law and Newtonian fluids, so this flow is laminar.

The pressure drop can be calculated using the Darcy-Weisbach equation.

$$\begin{aligned}-\Delta p &= \frac{16\,\rho\,\langle v\rangle^2\,L}{\mathrm{Re}_{MR}\,R}\\ &= \frac{16\times 739\times 0.057^2\times 1}{6.8\times 10^{-3}\times 0.0125}\\ &\approx 452\,\mathrm{kPa}\end{aligned}$$

[Question total: 20 marks]

Q.54 Question 54

A non-Newtonian fluid flows through a 20 m length pipe with a diameter of 25 mm. Its apparent viscosity is 0.1 N s m^{-2} at a shear rate of 1000 s^{-1} and its density is estimated to be 1600 kg m^{-3}.

a) If the flow index n is 0.33, show that the consistency k is 10 if the Power Law model applies. Give the rheological equation for the fluid. **[3 marks]**

Solution:

We can equate Newton's law and the Power-law model to find the following expression in terms of the apparent viscosity $\mu_{apparent}$.

$$\mu_{apparent}\left|\frac{\partial v_x}{\partial y}\right| = k\left|\frac{\partial v_x}{\partial y}\right|^n \tag{42}$$

Assuming that at a shear rate of $\partial v_x/\partial y$ = 1000 s^{-1}, we have an apparent viscosity of $\mu_{apparent}$ =0.1 N s m^{-2} and the flow index is $n = 0.33$, we have the following expression

$$0.1 \times 1000 = k\,1000^{0.33}$$
$$k \approx 10.2$$

The rheological equation for the fluid is then given by the Power-Law model with the coefficients inserted in, either expressed in terms of stress magnitude:

$$|\tau_{xy}| = 10.2\left|\frac{\partial v_x}{\partial y}\right|^{0.33}$$

or making the sign of the stress explicit:

$$\tau_{xy} = -10.2\left|\frac{\partial v_x}{\partial y}\right|^{-0.66}\frac{\partial v_x}{\partial y}$$

b) What type of fluid is this and how will it respond to increasing rates of shear? **[3 marks]**

Solution:

Rearranging Eq. (42) to obtain an expression for the apparent viscosity, we have

$$\mu_{apparent} = k\left|\frac{\partial v_x}{\partial y}\right|^{n-1}$$

If $n < 1$, the apparent viscosity will decrease as the shear rate increases. This means the fluid is *shear-thinning*.

c) If a flow-rate of 1 m^3 hr^{-1} is required, show that the flow would be laminar and calculate the pressure drop. **[5 marks]**

Note: The definition of the Metzner-Reed Reynolds number for Power-Law fluids in pipes is given by

$$\mathrm{Re}_{MR} = 8\left(\frac{n}{6\,n+2}\right)^n\frac{\rho\left\langle v\right\rangle^{2-n}D_H^n}{k}$$

Solution:

First, we need the average flow velocity. This is defined as the volumetric flow divided by the cross sectional area of the flow.

The flow rate in standard units is $1/3600$ m^3 s^{-1} and the pipe radius is $R = 0.025/2 = 0.0125$ m. The average velocity is then

$$\begin{aligned}\langle v\rangle &= \frac{\dot{V}}{A_{flow}}\\ &= \frac{1}{3600}\frac{1}{\pi 0.0125^2}\\ &\approx 0.57 m\, s^{-1}\end{aligned}$$

We can then calculate the Reynolds number and we find

$$\begin{aligned}\mathrm{Re}_{MR} &= 8\left(\frac{0.33}{6\times 0.33+2}\right)^{0.33}\frac{1600\times 0.57^{2-0.33}0.025^{0.33}}{10.2}\\ &\approx 65\end{aligned}$$

The fluid becomes turbulent around $\mathrm{Re}_{MR}\approx 2000$, so this flow is certainly laminar.

On to the pressure drop. For laminar flow we have the following definition for the Fanning friction factor

$$\begin{aligned}C_f &= \frac{16}{\mathrm{Re}_{MR}}\\ &\approx 0.25\end{aligned}$$

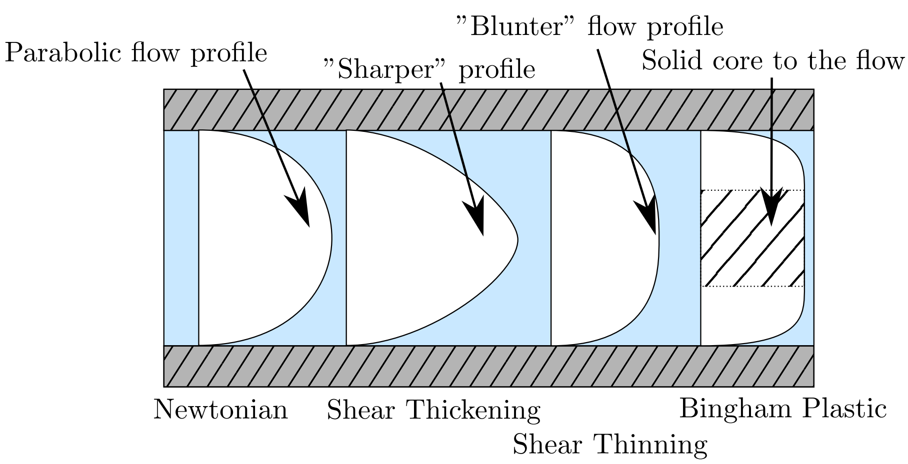
The pressure drop is then given by

$$\begin{aligned}-\Delta p &= \frac{C_f\, L\,\rho\,\langle v\rangle^2}{R}\\ &= \frac{0.25\times 20\times 1600\times 0.57^2}{0.0125}\\ &\approx 207936 \text{ Pa}\end{aligned}$$

The pressure drop is approximately 208 kPa.

d) Roughly sketch the flow profile for this fluid comparing it to the sketch of a Newtonian fluid and a Bingham-plastic fluid. Explain the differences between the profiles. **[3 marks]**

Solution:



The key features are that the Newtonian flow has a parabolic profile, whereas the shear thinning fluid is "blunter" as the apparent viscosity is higher in the centre. The Bingham plastic is different again as it has a solid core in the centre of the flow.

[Question total: 14 marks]

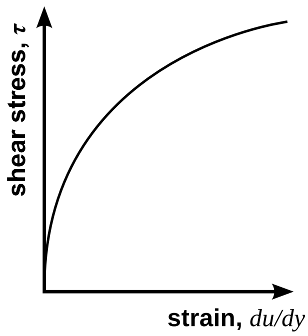
Q.55 Question 55

An incompressible polymeric fluid is to flow through 10 m of 50 mm inner-diameter piping. The flow index, n, for the fluid is 0.3 and the apparent viscosity, μ, at a shear rate of 1000 s^{-1} is 0.1 Pa s.

a) What type of fluid is this? Give a general description of its viscosity and include a sketch of the stress-rate versus strain graph and give the numerical expression for the stress τ_{xy}. **[8 marks]**

Solution:

This is a shear thinning fluid as $n < 1$.



To determine the numerical expression, we must determine the power law parameter k:

$$\mu_{apparent} = k\left|\frac{\partial v_x}{\partial y}\right|^{n-1}$$

Inserting what is known, we have

$$0.1 = k\,(1000)^{0.3-1}$$
$$k = 0.1\,(1000)^{0.7} = 12.59$$

The expression for the stress is then one of the following

$$\tau_{xy} = -k\left|\frac{\partial v_x}{\partial y}\right|^{n-1}\left(\frac{\partial v_x}{\partial y}\right) \qquad \left|\tau_{xy}\right| = k\left|\frac{\partial v_x}{\partial y}\right|^{n}$$

where $k = 12.59$ and $n = 0.3$.

b) Assuming the flow is laminar, what is the frictional pressure loss if the volumetric flow rate required at the end of the pipe is 0.005 $m^3\ s^{-1}$? **[5 marks]**

Solution:

From the data-sheet we have:

$$\dot{V} = \frac{n\,\pi\,R^3}{3\,n+1}\left(\frac{R}{2\,k}\right)^{\frac{1}{n}}\left(-\frac{\Delta p}{L}\right)^{\frac{1}{n}}$$

Rearranging for the pressure loss we have:

$$
\begin{aligned}
\frac{\Delta p}{L} &= -\left(\dot{V}\frac{3\,n+1}{n\,\pi\,R^3}\right)^{n}\left(\frac{R}{2\,k}\right)^{-1} \\
&= -\left(0.005\frac{3\times 0.3+1}{0.3\,\pi\,0.025^3}\right)^{0.3}\left(\frac{0.025}{2\times 12.59}\right)^{-1} \\
&= -7014\ \mathrm{Pa\ m^{-1}}
\end{aligned}
$$

Given the pipe is 10 m long, the total pressure drop is 70140 Pa or 0.7 bar.

c) Using the Metzner-Reed Reynolds number, would you expect the flow in the pipe to be laminar or turbulent? The standard transition value for the Reynolds numer applies and you may assume a fluid density of 1500 kg m^{-3}. **[4 marks]**

Solution:

From the datasheet, we have

$$
\mathrm{Re}_{MR} = -\frac{16\,L\,\rho\,\langle v\rangle^2}{R\,\Delta p}
$$

The flow velocity is

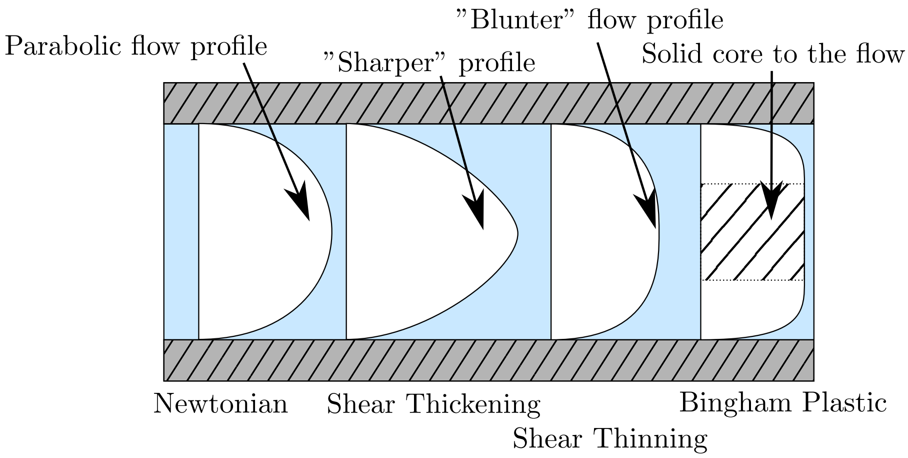
$$
\langle v\rangle = \frac{\dot{V}}{\pi R^2} = \frac{0.005}{\pi 0.025^2} = 2.546\ \mathrm{m\ s^{-1}}
$$

$$
\begin{aligned}
\mathrm{Re}_{MR} &= -\frac{16\,\rho\,\langle v\rangle^2}{R}\frac{L}{\Delta p} \\
&= \frac{16\times 1500\times 2.546^2}{0.025}7014^{-1} \\
&= 887.2
\end{aligned}
$$

This indicates the flow is highly likely to be laminar.

d) How does the velocity profile in this pipe compare to one carrying a Newtonian fluid? Illustrate your answer with an appropriate diagram. **[3 marks]**

Solution:



The key concepts to highlight are
- The drawings of the velocity profiles
- The parabolic flow profile of a Newtonian fluid
- The blunter flow profile of a shear thinning fluid

[Question total: 20 marks]

Q.56 Question 56

Consider the flow profile of a incompressible, Newtonian fluid through a horizontal annnulus (see Fig. 21).

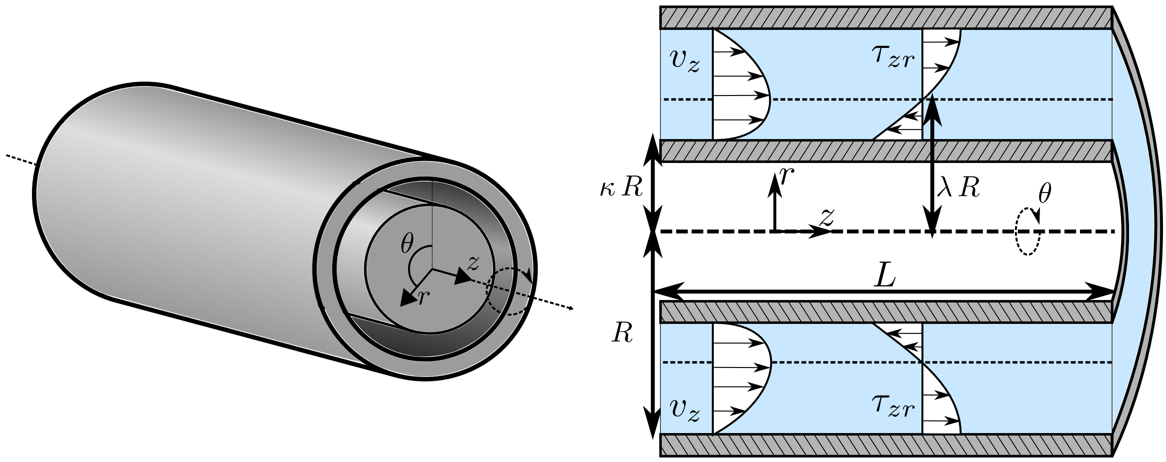


Figure 21: Axial flow in an annulus (pipe in pipe).

The velocity profile was derived in Q.18, and is given by the following equation.

$$v_z = -\frac{\Delta p\, R^2}{4\, L\, \mu}\left(\frac{r^2}{R^2} - \frac{\kappa^2 - 1}{\log\kappa}\log\left(\frac{r}{R}\right) - 1\right)$$

a) Derive the following expression for the volumetric flow rate as a function of pressure drop.

$$\dot{V}_z = \frac{\pi\, \Delta p\, \left(1-\kappa^2\right) R^4}{8\, L\, \mu}\left[1 + \kappa^2 + \frac{\left(1-\kappa^2\right)}{\log\kappa}\right]$$

Hint: You may need the following identity obtained from integration by parts.

$$\int x\, \log(x)\, \mathrm{d}x = \frac{x^2\, \log(x)}{2} - \frac{x^2}{4} + C$$

Solution:

The definition of the volumetric flowrate in cylindrical coordinates is given by integrating the velocity over the cross-sectional area of the flow

$$\begin{aligned}
\dot{V}_z &= \int_{\kappa R}^{R}\int_0^{2\pi} r\, v_z(r)\, \mathrm{d}\theta\, \mathrm{d}r \\
&= 2\,\pi\int_{\kappa R}^{R} r\, v_z(r)\, \mathrm{d}r \\
&= -\frac{\pi\, \Delta p\, R^2}{2\, L\, \mu}\int_{\kappa R}^{R} r\left(\frac{r^2}{R^2} - \frac{\kappa^2 - 1}{\log\kappa}\log\left(\frac{r}{R}\right) - 1\right)\mathrm{d}r
\end{aligned}$$

We can do the integration now, but it's neater to make the change of variables $x = r/R$ (which gives $\mathrm{d}r = R\,\mathrm{d}x$). This gives us

$$\begin{aligned}
\dot{V}_z &= -\frac{\pi\,\Delta p\,R^2}{2\,L\,\mu}\int_{\kappa R}^{R} r\left(\frac{r^2}{R^2} - \frac{\kappa^2-1}{\log\kappa}\log\left(\frac{r}{R}\right) - 1\right)\mathrm{d}r \\
&= -\frac{\pi\,\Delta p\,R^2}{2\,L\,\mu}\int_{\kappa}^{1} R^2\,x\left(x^2 - \frac{\kappa^2-1}{\log\kappa}\log x - 1\right)\mathrm{d}x \\
&= -\frac{\pi\,\Delta p\,R^4}{2\,L\,\mu}\int_{\kappa}^{1}\left(x^3 - \frac{\kappa^2-1}{\log\kappa}x\,\log x - x\right)\mathrm{d}x \\
&= -\frac{\pi\,\Delta p\,R^4}{2\,L\,\mu}\left[\frac{x^4}{4} - \frac{\kappa^2-1}{\log\kappa}\left(\frac{x^2\,\log x}{2} - \frac{x^2}{4}\right) - \frac{x^2}{2}\right]_{\kappa}^{1}
\end{aligned}$$

Substituting in the integration limits, we have

$$\dot{V}_z = -\frac{\pi\,\Delta p\,R^4}{2\,L\,\mu}\left[\frac{1-\kappa^4}{4} + \frac{1-\kappa^2}{\log\kappa}\left(-\frac{\kappa^2\,\log\kappa}{2} - \frac{1-\kappa^2}{4}\right) - \frac{1-\kappa^2}{2}\right]$$

Comparing our current result to the answer, we see there is a factor $1/4$ to be extracted. Taking this out and expanding the terms gives

$$\begin{aligned}
\dot{V}_z &= -\frac{\pi\,\Delta p\,R^4}{8\,L\,\mu}\left[1 - \kappa^4 - 2\,(1-\kappa^2)\kappa^2 - \frac{\left(1-\kappa^2\right)^2}{\log\kappa} - 2(1-\kappa^2)\right] \\
&= -\frac{\pi\,\Delta p\,R^4}{8\,L\,\mu}\left[\kappa^4 - 1 - \frac{\left(1-\kappa^2\right)^2}{\log\kappa}\right]
\end{aligned}$$

Noting that $1-\kappa^4 = (1-\kappa^2)(1+\kappa^2)$, we can write the final form

$$\dot{V}_z = \frac{\pi\,\Delta p\,(1-\kappa^2)\,R^4}{8\,L\,\mu}\left[1 + \kappa^2 + \frac{\left(1-\kappa^2\right)}{\log\kappa}\right]$$

b) Derive the following expression for the mean flow velocity, $\langle v_z\rangle$.

$$\langle v_z\rangle = \frac{\Delta p\,R^2}{8\,L\,\mu}\left[1 + \kappa^2 + \frac{\left(1-\kappa^2\right)}{\log\kappa}\right]$$

Solution:

Straightforward question. Take the previous expression and divide it by the cross sectional area of the flow.

$$\langle v_z\rangle = \frac{\dot{V}_z}{A_{flow}}$$

The cross sectional area of the annulus is

$$A_{flow} = \pi\,R^2\left(1-\kappa^2\right)$$

Straightforward division gives the result

$$\langle v_z\rangle = \frac{\Delta p\,R^2}{8\,L\,\mu}\left[1 + \kappa^2 + \frac{\left(1-\kappa^2\right)}{\log\kappa}\right]$$

c) One method to generalise the definition of the Reynolds number is to use a hydraulic diameter, $D_H = 4\,A_{flow}/P_w$, in place of the diameter:

$$\mathrm{Re}_H \equiv \frac{\rho\,\langle v_z\rangle\,D_H}{\mu}$$

Use this definition to calculate the following expression for the Reynolds number of a incompressible, Newtonian fluid through a horizontal annnulus:

$$\mathrm{Re}_H \equiv \frac{2\,\rho\,\langle v_z\rangle\,R\,(1-\kappa)}{\mu}$$

Solution:
The cross-sectional area of the flow is $A_{flow} = \pi\,R^2\left(1-\kappa^2\right)$, and the wetted perimeter is $P_w = 2\,\pi\,R\,(1+\kappa)$. The hydraulic diameter is then:

$$\begin{aligned}
D_H &= 4\,A_{flow}/P_w \\
&= 4\,\frac{\pi\,R^2\left(1-\kappa^2\right)}{2\,\pi\,R\,(1+\kappa)} \\
&= 2\,R\frac{1-\kappa^2}{1+\kappa} \\
&= 2\,R\frac{(1-\kappa)(1+\kappa)}{1+\kappa} \\
&= 2\,R(1-\kappa) \\
&= D_{outer} - D_{inner}
\end{aligned}$$

Inserting this into the above expression for the Reynolds number, we have

$$\mathrm{Re}_H \equiv \frac{2\,\rho\,\langle v_z\rangle\,R\,(1-\kappa)}{\mu}$$

d) Describe (not derive) how Metzer-Reed generalised the definition of the Reynolds number (what did they do to fix the definiton of Re)?

$$\mathrm{Re}_{MR} = -\frac{8\,\rho\,\langle v\rangle^2\,P_w\,L}{A_{flow}\,\Delta p}$$

Using this approach, derive the following expression for the Metzner-Reed Reynolds number of a incompressible, Newtonian fluid through a horizontal annnulus.

$$\mathrm{Re}_{MR} = -\frac{2\,\rho\,\langle v\rangle\,R}{\mu}\left[\frac{1+\kappa^2}{1-\kappa} + \frac{1+\kappa}{\log\kappa}\right]$$

Solution:
The Metzner-Reed Reynolds number is defined through the fanning friction factor. Specifically, Metzner-Reed declared that for all flow geometries and viscous models, the laminar value of the friction factor is $C_f = 16/\mathrm{Re}_{MR}$. Taking the expression for the Metzner-Reed Reynolds number:

$$\begin{aligned}
\mathrm{Re}_{MR} &= -\frac{8\,\rho\,\langle v\rangle^2\,P_w\,L}{A_{flow}\,\Delta p} \\
&= -\frac{32\,\rho\,\langle v\rangle^2\,L}{\Delta p}\frac{P_w}{4\,A_{flow}}
\end{aligned}$$

Noticing the $4\,A_{flow}/P_w$ factor, which is equal to the hydraulic diameter, we can immediately subsitute in the result derived in the previous question ($D_H = 2\,R(1-\kappa)$).

$$\mathrm{Re}_{MR} = -\frac{16\,\rho\,\langle v\rangle^2}{R(1-\kappa)}\frac{L}{\Delta p}$$

Now we need to substitute in our expression for the pressure drop in terms of the mean flow velocity from Q. b.

$$\langle v_z\rangle = \frac{\Delta p\,R^2}{8\,L\,\mu}\left[1+\kappa^2+\frac{\left(1-\kappa^2\right)}{\log\kappa}\right]$$

Rearranging for the inverse pressure drop, we have

$$\frac{L}{\Delta p} = \frac{R^2}{8\,\langle v_z\rangle\,\mu}\left[1+\kappa^2+\frac{\left(1-\kappa^2\right)}{\log\kappa}\right]$$

Substituting this into the expression for the Reynolds number, we have:

$$\begin{aligned}\mathrm{Re}_{MR} &= -\frac{16\,\rho\,\langle v\rangle^2}{R(1-\kappa)}\frac{L}{\Delta p}\\ &= -\frac{16\,\rho\,\langle v\rangle^2}{R(1-\kappa)}\frac{R^2}{8\,\langle v_z\rangle\,\mu}\left[1+\kappa^2+\frac{\left(1-\kappa^2\right)}{\log\kappa}\right]\\ &= -\frac{2\,\rho\,\langle v\rangle}{(1-\kappa)}\frac{R}{\mu}\left[1+\kappa^2+\frac{1-\kappa^2}{\log\kappa}\right]\\ &= -\frac{2\,\rho\,\langle v\rangle\,R}{\mu}\left[\frac{1+\kappa^2}{1-\kappa}+\frac{1+\kappa}{\log\kappa}\right]\end{aligned}$$

Where again, we factored the term $(1-\kappa^2) = (1-\kappa)(1+\kappa)$.

e) Comment on the two definitions of the Reynolds numbers and discuss which is "better"?
Solution:
No hard and fast "right" answer here, I just want you to demonstrate that you understand the problems of multiple definitions of the Reynolds numbers. My "perfect" answer follows:

Neither Reynolds number is strictly correct, as there are an infinite number of definitions of Re that we can make for flows in annuli. The reason for this is that I have two length scales $D_{inner} = \kappa\,R$ and $D_{outer} = R$, but I only need one for the D term in $\mathrm{Re} = \rho\,\langle v\rangle\,D/\mu$.

I could write $D = D_{inner} + D_{outer}$ or $D = 100\,D_{inner}^2/D_{outer}$ and Re is still dimensionless.

However, the Metzner-Reed has a nice symmetry about it. It ensures that all laminar friction factors have the same definition. If the friction factor is a fundamental property of fluid flow, we might be lucky and find that its more general than the geometry or viscous model. Unfortunately, the research literature indicates that the turbulence transition region is not symmetric (constant) for the Metzner-Reed definiton.

[Question end]

Q.57 Question 57

Two immiscible incompressible Newtonian fluids flow co-currently in a horizontal plane channel, as shown in Fig. 22. The density and viscosity of fluid 1 are ρ_1 and μ_1, respectively; the density and viscosity of fluid 2 are ρ_2 and μ_2. This is the simplest example of **multiphase flow**, and is one of the few systems with an analytical solution. Each phase must be solved separately (two Continuity/Cauchy equations) as they only interact with each other through their boundary conditions. Assuming the two fluids are liquids (not liquid gas), we can apply a no-slip condition between the two phases at $y = h$ (the velocities of the two phases are equal). We also know that the stresses are equal at the interface from Newton's third law of motion.

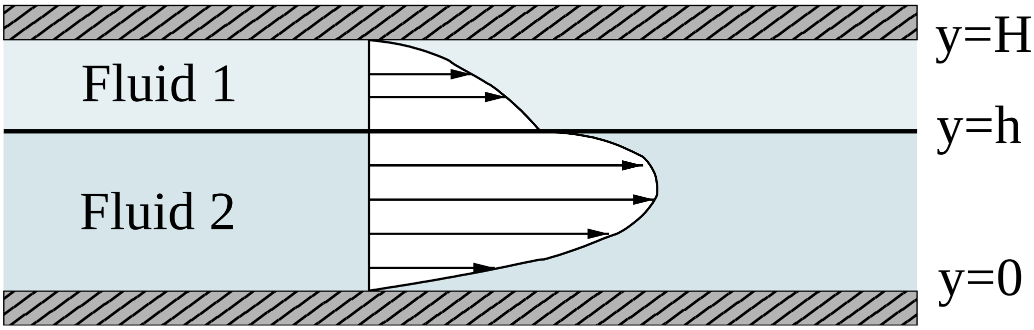


Figure 22: Flow of two immiscible fluids in a planar channel.

a) Derive the following expressions for the velocity distributions in each fluid.

$$v_{x,1}(y) = -\frac{\Delta p H^2}{2\mu_1 L}\left(1 - \frac{y}{H}\right)\left[1 + A_1 + \frac{y}{H}\right] \tag{43}$$

$$v_{x,2}(y) = \frac{\Delta p H^2}{2\mu_2 L}\left(\frac{y}{H}\right)\left(\frac{y}{H} + A_1\right) \tag{44}$$

where

$$A_1 = -\left[1 + \left(\frac{\mu_1}{\mu_2} - 1\right)\frac{h^2}{H^2}\right]\left[1 + \left(\frac{\mu_1}{\mu_2} - 1\right)\frac{h}{H}\right]^{-1} \tag{45}$$

Clearly state what assumptions you make along the way.

Solution:

Standard method to derive the stress equation:

We can start this problem by taking the continuity equation in rectangular coordinates. As with all of the other examples, if we assume well-developed, laminar, and incompressible flow, it simplifies to:

$$\frac{\partial v_x}{\partial x} = 0$$

Note that we don't need to assume steady-state for this!

Noting that the flow is horizontal, and now assuming steady-state flow, the momentum balance equation also simplifies (exactly as before) to

$$\rho\, v_j\, \nabla_j\, v_i = -\nabla_j\, \tau_{ji} - \nabla_i\, p$$

Setting $i = x$ we have

$$\rho\, v_j\, \nabla_j\, v_x = -\nabla_j\, \tau_{jx} - \nabla_x\, p$$

We can eliminate all the v_j terms where $j = [y, z]$ as the velocity is zero in those directions (well-developed laminar-flow approximations) to give

$$\rho\, v_x\, \nabla_x\, v_x = -\nabla_j\, \tau_{jx} - \nabla_x\, p$$

Again, from the continuity equation $\nabla_x\, v_x = 0$ so we have

$$\nabla_j\, \tau_{jx} = -\nabla_x\, p$$

We only have stress in the x-y direction, so we end up with

$$\nabla_y\, \tau_{yx} = -\nabla_x\, p$$

Alternative "balance" method to derive the stress equation

Alternatively, we could start this problem by performing a force balance on a thin slab of fluid within the system. This is a much more common method of derivation in "Transport Phenomenon"; however, it requires some intuition.

The bottom of the slab is located at y, and thickness of the slab is Δy. The system is at steady state, so the various forces that act on the slab will sum to zero. There are two types of forces relevant in the problem: (i) pressure forces and (ii) viscous forces. The force balance is then:

$$\begin{aligned}
0 &= p(x=0)\,W\Delta y - p(x=L)\,W\Delta y + \tau_{yx}(y+\Delta y)\,WL - \tau_{yx}(y)\,WL \\
0 &= \frac{p(x=0) - p(x=L)}{L} + \frac{\tau_{yx}(y+\Delta y) - \tau_{yx}(y)}{\Delta y} \\
\frac{\tau_{yx}(y+\Delta y) - \tau_{yx}(y)}{\Delta y} &= -\frac{\Delta p}{L} \qquad (46)
\end{aligned}$$

where $\Delta p = p(x=0) - p(x=L)$. Taking the limit $\Delta y \to 0$, we find

$$\frac{\partial \tau_{yx}}{\partial y} = -\frac{\Delta p}{L} \qquad (47)$$

Back to solving the stress equation

Carrying on, regardless of which fluid we consider, to determine the velocity distribution we substitute Newton's law of viscosity

$$\tau_{yx} = -\mu \frac{\partial v_x}{\partial y}$$

into the stress balance equation (Eq. (47)), which yields

$$\begin{aligned}
\frac{\partial}{\partial y}\mu \frac{\partial v_x}{\partial y} &= \frac{\Delta p}{L} \\
\frac{\partial^2 v_x}{\partial y^2} &= \frac{\Delta p}{\mu L} \qquad (48)
\end{aligned}$$

This equation can be integrated twice with respect to y to yield the general velocity profile equation, valid for either fluid:

$$v_x(y) = \frac{\Delta p}{2\,\mu\,L} y^2 + A'\,y + B' \tag{49}$$

$$v_x(y) = \frac{\Delta p\,H^2}{2\,\mu\,L}\left(\frac{y^2}{H^2} + \frac{A\,y}{H} + B\right) \tag{50}$$

where A and B are integration constants and were redefined in the second line to make them into dimensionless terms (not needed, just makes what follows much simpler).

In this problem, there are two fluids. In this situation, we need two velocity profile equations, one for each fluid:

$$v_{x,1}(y) = \frac{\Delta p\,H^2}{2\,\mu_1\,L}\left(\frac{y^2}{H^2} + A_1\frac{y}{H} + B_1\right) \tag{51}$$

$$v_{x,2}(y) = \frac{\Delta p\,H^2}{2\,\mu_2\,L}\left(\frac{y^2}{H^2} + A_2\frac{y}{H} + B_2\right) \tag{52}$$

We now have four unknown integration constants. In order to specify these, we need four boundary conditions. These are: (i) $v_{x,1} = 0$ at $y = H$, (ii) $v_{x,2} = 0$ at $y = 0$, (iii) $v_{x,1} = v_{x,2}$ at $y = h$, and (iv) $\tau_{yx,1} = \tau_{yx,2}$ at $y = h$. Boundary conditions (i)–(iii) arise from the no-slip wall and liquid-liquid boundaries of the system. The (iv) boundary condition is Newton's third law (each action has an equal and opposite reaction, so each fluid exerts an equal stress on the other). This boundary condition can also be seen from the observation that the stress profile (Eq. (47)) is independent of the viscous properties of the flow (so the change in viscosity between the fluids has no effect on the stress).

Boundary condition (i) yields:

$$0 = \frac{\Delta p\,H^2}{2\,\mu_1\,L}\left(1 + A_1 + B_1\right)$$

$$B_1 = -1 - A_1 \tag{53}$$

Substituting this back into the velocity profile for fluid 1, we find

$$\begin{aligned} v_{x,1}(y) &= \frac{\Delta p\,H^2}{2\,\mu_1\,L}\left(\frac{y^2}{H^2} + A_1\frac{y}{H} - 1 - A_1\right) \\ &= -\frac{\Delta p\,H^2}{2\,\mu_1\,L}\left[1 - \frac{y^2}{H^2} + A_1\left(1 - \frac{y}{H}\right)\right] \end{aligned} \tag{54}$$

We can obtain the associated expression for the stress in fluid 1 by throwing the last equation back in to the expression for the stress:

$$\begin{aligned} \tau_{yx,1}(y) &= -\mu_1\frac{\partial v_{x,1}}{\partial y} \\ &= -\frac{\Delta p\,H}{2\,L}\left(\frac{2\,y}{H} + A_1\right) \end{aligned} \tag{55}$$

We'll come back to this equation later.

Substituting boundary condition (ii) into the expression for the velocity profile of fluid 2, we find $B_2 = 0$. Therefore, we have

$$v_{x,2}(y) = \frac{\Delta p H^2}{2\mu_2 L}\left(\frac{y^2}{H^2} + A_2\frac{y}{H}\right) \tag{56}$$

The associated expression for the stress is given by

$$\begin{aligned}\tau_{yx,2}(y) &= -\mu_2 \frac{\partial v_{x,2}}{\partial y} \\ &= -\frac{\Delta p H}{2L}\left(\frac{2y}{H} + A_2\right) \qquad (57)\end{aligned}$$

Setting $y = h$, using boundary condition (iv), and combining the two expressions for the stress, we have

$$\begin{aligned}\tau_{yx,1}(y=h) &= \tau_{yx,2}(y=h) \\ -\frac{\Delta p H}{2L}\left(\frac{2h}{H} + A_1\right) &= -\frac{\Delta p H}{2L}\left(\frac{2h}{H} + A_2\right) \\ A_1 &= A_2 \qquad (58)\end{aligned}$$

From boundary condition (iii), we find

$$\begin{aligned}v_{x,1}(h) &= v_{x,2}(h) \\ -\frac{\Delta p H^2}{2\mu_1 L}\left[1 - \frac{h^2}{H^2} + A_1\left(1 - \frac{h}{H}\right)\right] &= \frac{\Delta p H^2}{2\mu_2 L}\left(\frac{h^2}{H^2} + A_2 \frac{h}{H}\right) \\ 1 - \frac{h^2}{H^2} + A_1\left(1 - \frac{h}{H}\right) &= -\frac{\mu_1}{\mu_2}\left(\frac{h^2}{H^2} + A_1 \frac{h}{H}\right) \\ A_1 &= -\left[1 + \left(\frac{\mu_1}{\mu_2} - 1\right)\frac{h^2}{H^2}\right]\left[1 + \left(\frac{\mu_1}{\mu_2} - 1\right)\frac{h}{H}\right]^{-1} \quad (59)\end{aligned}$$

So finally, we find

$$\begin{aligned}v_{x,1}(y) &= -\frac{\Delta p H^2}{2\mu_1 L}\left(1 - \frac{y}{H}\right)\left[1 + A_1 + \frac{y}{H}\right] \\ v_{x,2}(y) &= \frac{\Delta p H^2}{2\mu_2 L}\left(\frac{y}{H}\right)\left(\frac{y}{H} + A_1\right) \qquad (60)\end{aligned}$$

where A_1 is given in Eq. (59).

b) Derive the volumetric flow rate of each phase and give the ratio of the two flow rates. The answer is:

$$\begin{aligned}\frac{\dot{V}_1}{\dot{V}_2} = -\frac{3\mu_2}{\mu_1}\frac{H^3}{h^3}&\left[(1 + A_1)\left(1 - \frac{h}{H}\right) - \frac{A_1}{2}\left(1 - \frac{h^2}{H^2}\right) - \frac{1}{3}\left(1 - \frac{h^3}{H^3}\right)\right] \\ &\times \left[1 + A_1 \frac{3}{2}\frac{H}{h}\right]^{-1} \qquad (61)\end{aligned}$$

Solution:

The volumetric flowrate of fluid 1 ($\dot{V}_1$), per unit width, is given by

$$\begin{aligned}\dot{V}_1 &= \int_h^H v_{x,1}(y)\,\mathrm{d}y \\ &= -\int_h^H \frac{\Delta p H^2}{2\mu_1 L}\left(1 - \frac{y}{H}\right)\left[1 + A_1 + \frac{y}{H}\right]\mathrm{d}y \\ &= -\frac{\Delta p H^3}{2\mu_1 L}\int_{h/H}^1 (1-\eta)(1 + A_1 + \eta)\mathrm{d}\eta \\ &= -\frac{\Delta p H^3}{2\mu_1 L}\left[(1 + A_1)\left(1 - \frac{h}{H}\right) - \frac{A_1}{2}\left(1 - \frac{h^2}{H^2}\right) - \frac{1}{3}\left(1 - \frac{h^3}{H^3}\right)\right] \qquad (62)\end{aligned}$$

The volumetric flowrate of fluid 2, per unit width, ($\dot{V}_2$) is given by

$$\begin{aligned}
\dot{V}_2 &= \int_0^h v_{x,2}(y)\mathrm{d}y \\
&= \int_0^h \frac{\Delta p H^2}{2\mu_2 L}\left(\frac{y}{H}\right)\left(\frac{y}{H}+A_1\right)\mathrm{d}y \\
&= \frac{\Delta p H^3}{2\mu_2 L}\int_0^{h/H}\eta(\eta+A_1)\mathrm{d}\eta \\
&= \frac{\Delta p H^3}{2\mu_2 L}\left[\frac{\eta^3}{3}+A_1\frac{\eta^2}{2}\right]_0^{h/H} \\
&= \frac{\Delta p H^3}{2\mu_2 L}\left[\frac{1}{3}\frac{h^3}{H^3}+\frac{A_1}{2}\frac{h^2}{H^2}\right] \\
&= \frac{\Delta p h^3}{6\mu_2 L}\left[1+A_1\frac{3}{2}\frac{H}{h}\right]
\end{aligned} \tag{63}$$

The ratio of the volumetric flowrates is then

$$\begin{aligned}
\frac{\dot{V}_1}{\dot{V}_2} = &-\frac{3\mu_2}{\mu_1}\frac{H^3}{h^3}\left[(1+A_1)\left(1-\frac{h}{H}\right)-\frac{A_1}{2}\left(1-\frac{h^2}{H^2}\right)-\frac{1}{3}\left(1-\frac{h^3}{H^3}\right)\right] \\
&\times\left[1+A_1\frac{3}{2}\frac{H}{h}\right]^{-1}
\end{aligned} \tag{64}$$

c) Compare the expression above for the ratio of the volumetric flows, to the ratio of the channel occupied by the flow ($(H-h)/h$). Why do these differ? What does this imply for gas-liquid systems?

Solution:

The volumetric flow-rates differ significantly from the occupation of the channel and depends on the gas viscosity. This is because the flow profiles can be quite assymmetric in the two channels.

This also implies that, for gas-liquid systems, the gas volumetric flow-rates can be significantly higher than the liquids, even for mainly liquid-filled channels. Obviously, two-phase flow is difficult to work with.

[Question end]

Q.58 Question 58

Exam question (2011 and 2014)

The Lockhart-Martinelli parameter, X, is a critical parameter in two-phase flow pressure-drop and liquid hold-up calculations. It is defined as the ratio of the frictional pressure drops of each phase, calculated as if each was flowing alone in the pipe.

$$X^2 = \frac{(\partial p/\partial z)_{liq.-only}}{(\partial p/\partial z)_{gas-only}}$$

a) Assuming that the pipe is smooth and that both phases are fully turbulent, derive the following expression for the Martinelli parameter

$$X_{tt} = \left(\frac{1-x}{x}\right)^{0.875}\left(\frac{\mu_{liq.}}{\mu_{gas}}\right)^{0.125}\left(\frac{\rho_{gas}}{\rho_{liq.}}\right)^{0.5}$$

Extra hint: You may need the Darcy-Weissbach equation and a suitable expression for the friction factor (see the datasheet). **[4 marks]**

Solution:

For a single-phase turbulent Newtoninan fluid flowing in a smooth pipe, we can use the Blasisus correlation for the Fanning friction factor in the Darcy-Weisbach equation to yield

$$\frac{\Delta p}{L} = -\frac{0.079\,\mathrm{Re}^{-1/4}\,\rho\,\langle v\rangle^2}{R}$$

We define the mass flux as $G = \rho\,\langle v\rangle$ to yield

$$\frac{\Delta p}{L} = -\frac{0.079\,\mathrm{Re}^{-1/4}\,G^2}{\rho\,R}$$

The Reynolds number is given by

$$\mathrm{Re} = \frac{G\,D}{\mu}$$

Substituting it in to the previous expression, we obtain

$$\frac{\Delta p}{L} = -\frac{0.079\,\mu^{1/4}\,G^{1.75}}{\rho\,R\,D^{1/4}}$$

The proportion of the mass flux in the pipe which is in the gas phase is defined through the quality, x, and we have

$$G_g = x\,G \qquad\qquad G_l = (1-x)\,G$$

We can write the pressure drop in each phase using these mass flow rates and we have

$$\frac{\Delta p_l}{L} = -\frac{0.079\,\mu_l^{1/4}\,(1-x)^{1.75}\,G^{1.75}}{\rho_l\,R\,D^{1/4}}$$
$$\frac{\Delta p_g}{L} = -\frac{0.079\,\mu_g^{1/4}\,x^{1.75}\,G^{1.75}}{\rho_g\,R\,D^{1/4}}$$

Dividing the two equations we have

$$X_{tt}^2 = \frac{\Delta p_l/L}{\Delta p_g/L} = \left(\frac{\mu_l}{\mu_g}\right)^{1/4}\left(\frac{1-x}{x}\right)^{1.75}\frac{\rho_g}{\rho_l}$$

Taking the square root, yields the final expression

$$X_{tt} = \left(\frac{\mu_l}{\mu_g}\right)^{1/8}\left(\frac{1-x}{x}\right)^{0.875}\left(\frac{\rho_g}{\rho_l}\right)^{1/2}$$

b) A mixture of saturated steam at 0.09 kg s^{-1} and water at 1.6 kg s^{-1} is flowing along a horizontal pipe with an internal diameter of 75 mm. The steam has a viscosity of $\mu_g = 0.0113 \times 10^{-3}$ N s m^{-2} and density of 0.788 kg m^{-3}. The water has a viscosity of 0.52×10^{-3} N s m^{-2} and a density of 1000 kg m^{-3}.

i) Determine the flow pattern inside the pipe. **[3 marks]**

Solution:

We need the superficial velocity in each phase.

$$u_l = \frac{M_l}{A\,\rho_l} = \frac{1.6}{\pi\,0.0375^2\,1000} \approx 0.3622\,\mathrm{m\,s^{-1}}$$
$$u_g = \frac{M_g}{A\,\rho_g} = \frac{0.09}{\pi\,0.0375^2\,0.788} \approx 25.85\,\mathrm{m\,s^{-1}}$$

Examining the Chhabra and Richardson flow pattern map it is clearly predicted that the flow is in the Annular flow regime.

ii) Determine the flow regime inside each phase of the pipe. **[4 marks]**

Solution:

For two phase flows, the Reynolds numbers are calculated using the superficial velocity in place of the average velocity.

$$\mathrm{Re}_l = \frac{\rho_l\,u_l\,D}{\mu_l} = \frac{1000 \times 0.3622 \times 0.075}{0.52 \times 10^{-3}} \approx 52240$$
$$\mathrm{Re}_g = \frac{\rho_g\,u_g\,D}{\mu_g} = \frac{0.788 \times 25.85 \times 0.075}{0.0113 \times 10^{-3}} \approx 135198$$

Both phases of the flow are in the turbulent regime ($\mathrm{Re} \gg 2300$)!

iii) Calculate the two phase pressure drop multiplier (for either phase). **[6 marks]**

Solution:

We need to calculate the Martinelli parameter X_{tt} for the flow using the expression given above:

$$X_{tt} = \left(\frac{1-x}{x}\right)^{0.875} \left(\frac{\mu_{liq.}}{\mu_{gas}}\right)^{0.125} \left(\frac{\rho_{gas}}{\rho_{liq.}}\right)^{0.5}$$

The quality, x, is given by the ratio of the gas mass flow-rate to the total mass flow-rate.

$$x = \frac{\dot{M}_g}{\dot{M}_g + \dot{M}_l} = \frac{0.09}{0.09 + 1.6} \approx 0.0533$$

We can now calculate the Martinelli parameter

$$\begin{aligned} X_{tt} &= \left(\frac{1-x}{x}\right)^{0.875} \left(\frac{\mu_{liq.}}{\mu_{gas}}\right)^{0.125} \left(\frac{\rho_{gas}}{\rho_{liq.}}\right)^{0.5} \\ &= \left(\frac{1-0.0533}{0.0533}\right)^{0.875} \left(\frac{0.52}{0.0113}\right)^{0.125} \left(\frac{0.788}{1000}\right)^{0.5} \\ &\approx 0.562 \end{aligned}$$

Some students choose to ignore the formula for X_{tt} but work out the pressure drop in each phase

$$\begin{aligned} \frac{\Delta p_l}{L} &= -\frac{0.079\,\rho_L\,u_L^2}{\mathrm{Re}_L^{1/4}\,R} \\ &= -\frac{0.079 \times 1000 \times 0.3622^2}{52240^{1/4} \times 0.0375} \\ &\approx -18.28\ \mathrm{Pa\,m^{-1}} \end{aligned}$$

$$\begin{aligned} \frac{\Delta p_g}{L} &= -\frac{0.079\,\rho_G\,u_G^2}{\mathrm{Re}_G^{1/4}\,R} \\ &= -\frac{0.079 \times 0.788 \times 25.85^2}{135198^{1/4} \times 0.0375} \\ &\approx -57.85\ \mathrm{Pa\,m^{-1}} \end{aligned}$$

Then X^2 can be obtained directly

$$X_{tt}^2 = \frac{\Delta p_l/L}{\Delta p_g/L} \approx \frac{-18.28}{-57.85} \approx 0.316$$
$$X_{tt} \approx \sqrt{0.316} \approx 0.562$$

Now we need to determine what expression to use for the two phase multiplier $\Phi_{liq.}^2$, or Φ_{gas}^2. Using Chisholm's relation, provided in the data sheet, we have for turbulent flows

$$\Phi_{liq.}^2 = 1 + \frac{20}{X} + \frac{1}{X^2} \qquad \Phi_{gas}^2 = 1 + 20\,X + X^2$$

The two phase multiplier is then

$$\Phi_{liq.}^2 = 1 + \frac{20}{0.562} + \frac{1}{0.562^2} \qquad \Phi_{gas}^2 = 1 + 20 \times 0.562 + 0.562^2$$
$$\Phi_{liq.} \approx \sqrt{39.75} \approx 6.3 \qquad \Phi_{gas} \approx \sqrt{12.56} \approx 3.54$$

iv) Calculate the pressure drop over a 12 m long smooth pipe. **[3 marks]**

Solution:

To use the two-phase multiplier we need an expression for the single-phase pressure drop for the liquid. We can use the Darcy-Weisbach equation provided we use the Blasius correlation for the friction factor in smooth pipes.

$$C_f = 0.079\,\mathrm{Re}_l^{-1/4} = 0.079 \times 52240^{-1/4} \approx 0.00522$$

The single-phase pressure drop is given by

$$\begin{aligned}\Delta p_{lo} &= -\frac{C_f\,L\,\rho_l\,u_l^2}{R}\\ &= -\frac{0.00522 \times 12 \times 1000 \times 0.3622^2}{0.0375}\\ &\approx -219\ \mathrm{Pa}\end{aligned}$$

The multiphase pressure drop is given by

$$\Delta p = \Delta p_{lo}\,\Phi_{liq.}^2 = -219 \times 39.75 \approx -8.5\,\mathrm{kPa}$$

[Question total: 20 marks]

Q.59 Question 59

Example exam question

A mixture of 0.15 kg s^{-1} saturated steam and 1.6 kg s^{-1} water is flowing along a horizontal pipe with an inner diameter of 88.9 mm. At the conditions in the pipe, the steam has a viscosity of $\mu_g = 0.0108 \times 10^{-3}$ N s m^{-2} and density of 0.774 kg m^{-3}. The water has a viscosity of 0.51×10^{-3} N s m^{-2} and a density of 998 kg m^{-3}.

a) Determine the flow pattern inside the pipe. How does this horizontal flow pattern differ from the equivalent vertical flow pattern? **[5 marks]**

Solution:

We need the superficial velocity in each phase.

$$u_l = \frac{M_l}{A\,\rho_l} = \frac{1.6}{\pi\,0.04445^2\,998} \approx 0.258\,\mathrm{m\ s^{-1}}$$
$$u_g = \frac{M_g}{A\,\rho_g} = \frac{0.15}{\pi\,0.04445^2\,0.774} \approx 31.22\,\mathrm{m\ s^{-1}}$$

Looking at the flow pattern map, the flow appears to lie in the annular regime.

Horizontal annular flow differs from vertical annular flow in that the lower film is thicker than the upper film due to the action of gravity.

b) Determine the flow regime for both phases of the flow. **[3 marks]**

Solution:

For two phase flows, the Reynolds numbers are calculated using the superficial velocity in place of the average velocity.

$$\mathrm{Re}_l = \frac{\rho_l\, u_l\, D}{\mu_l} = \frac{998 \times 0.258 \times 0.0889}{0.51 \times 10^{-3}} \approx 44\,900$$
$$\mathrm{Re}_g = \frac{\rho_g\, u_g\, D}{\mu_g} = \frac{0.774 \times 31.22 \times 0.0889}{0.0108 \times 10^{-3}} \approx 199\,000$$

Both phases of the flow are well into the turbulent regime ($\mathrm{Re} \gg 2300$).

c) Calculate the two-phase pressure drop multiplier (for either phase). **[5 marks]**

Solution:

We need to calculate the Martinelli parameter X_{tt} for the flow.

$$X^2 = \frac{\Delta p_{liq.-only}}{\Delta p_{gas-only}}$$

The single phase pressure drops are given using the Darcy-weisbach equation for each phase as if it were flowing alone in the pipe. First, as both phases are turbulent, we calculate the friction factor using the Blasius correlation:

$$C_{f,l} = 0.079\,\mathrm{Re}_l^{-1/4} = 0.00543$$
$$C_{f,g} = 0.079\,\mathrm{Re}_g^{-1/4} = 0.00374$$

Inserting this into the Darcy-Weisbach equation gives:

$$\left(\frac{\Delta p}{L}\right)_l = -\frac{C_{f,l}\,\rho_l\,\langle v_l\rangle^2}{R} = -\frac{0.00543 \times 998 \times 0.258^2}{0.04445} = -8.12\ \mathrm{Pa\ m^{-1}}$$
$$\left(\frac{\Delta p}{L}\right)_g = -\frac{C_{f,g}\,\rho_g\,\langle v_g\rangle^2}{R} = -\frac{0.00374 \times 0.774 \times 31.22^2}{0.04445} = -63.28\ \mathrm{Pa\ m^{-1}}$$

Calculating the Martinelli parameter yeilds

$$X = \sqrt{\frac{-8.12}{-63.28}} \approx 0.358$$

We could also use the following expression from the lecture notes:

$$X_{tt} = \left(\frac{1-x}{x}\right)^{0.875}\left(\frac{\mu_{liq.}}{\mu_{gas}}\right)^{0.125}\left(\frac{\rho_{gas}}{\rho_{liq.}}\right)^{0.5}$$

The quality, x, is given by the ratio of the gas mass flow-rate to the total mass flow-rate.

$$x = \frac{\dot{M}_g}{\dot{M}_g + \dot{M}_l} = \frac{0.15}{0.15 + 1.6} \approx 0.0857$$

We can now calculate the Martinelli parameter

$$\begin{aligned} X_{tt} &= \left(\frac{1-x}{x}\right)^{0.875}\left(\frac{\mu_{liq.}}{\mu_{gas}}\right)^{0.125}\left(\frac{\rho_{gas}}{\rho_{liq.}}\right)^{0.5} \\ &= \left(\frac{1-0.0857}{0.0857}\right)^{0.875}\left(\frac{0.51}{0.0108}\right)^{0.125}\left(\frac{0.774}{998}\right)^{0.5} \\ &\approx 0.358 \end{aligned}$$

Now we need to determine what expression to use for the two phase multiplier $\Phi^2_{liq.}$, or Φ^2_{gas}. Using Chisholm's relation, provided in the data sheet, we have for turbulent flows

$$\Phi^2_{liq.} = 1 + \frac{20}{X} + \frac{1}{X^2} \qquad\qquad \Phi^2_{gas} = 1 + 20\,X + X^2$$

The two phase multipliers are then

$$\Phi^2_{liq.} = 1 + \frac{20}{0.358} + \frac{1}{0.358^2} \qquad\qquad \Phi^2_{gas} = 1 + 20 \times 0.358 + 0.358^2$$

$$\Phi_{liq.} \approx \sqrt{64.67} \approx 8.042 \qquad\qquad \Phi_{gas} \approx \sqrt{8.288} \approx 2.879$$

d) Assuming the Farooqi and Richardson correlation holds for this system, calculate the liquid hold-up and estimate the true average velocities of the gas and liquid phases. **[5 marks]**

Solution:

The Farooqi and Richardson correlation is given by

$$H = \begin{cases} 0.186 + 0.0191\,X & 1 < X < 5 \\ 0.143\,X^{0.42} & 5 < X < 50 \\ 1/(0.97 + 19/X) & 50 < X < 500 \end{cases}$$

Using the value of $X_{tt} = 0.358$ is problematic here, as it is outside the valid range of the Farooqi-Richardson expression. The question says to assume that the correlation holds, so the best thing we can do under the constraints of an exam would be to extrapolate the first expression to lower values of the Martinelli parameter.

$$H = 0.186 + 0.0191 \times 0.358 \approx 0.193$$

A small Martinelli parameter indicates the gas phase flow is dominant in the flowline and is capable of stripping all liquid from the line. Thus we expect the hold-up to continue decreasing with decreasing X.

We note that the first expression has a valid minimum and maximum holdup of

$$H_{max} = 0.186 + 0.0191 \times 5 \approx 0.2815 \qquad\qquad H_{min} = 0.186 + 0.0191 \times 1 \approx 0.2051$$

The predicted value is outside of these values, but not by huge amount. These empirical expressions have large errors, so we can continue with the calculation, but **we must revise this estimate with improved expressions at a later date.**

To calculate the true average velocities, we need to calculate the cross-sectional area of the pipe available for liquid and gas flow. This is given by

$$\begin{aligned} A_l &= A\,H & A_g &= A\,(1-H) \\ &= \pi\,0.04445^2 \times 0.193 & &= \pi\,0.04445^2\,(1-0.193) \\ &\approx 0.00120\ \mathrm{m}^2 & &\approx 0.00501\ \mathrm{m}^2 \end{aligned}$$

Using this available area, we can estimate the true average velocities of the flow.

$$\langle v_L \rangle = \frac{M_l}{A_l\,\rho_l} = \frac{1.6}{0.00120 \times 998} \approx 1.336\,\mathrm{m\ s^{-1}}$$
$$\langle v_L \rangle = \frac{M_g}{A_g\,\rho_g} = \frac{0.15}{0.00501 \times 0.774} \approx 38.68\,\mathrm{m\ s^{-1}}$$

This can be contrasted against the superficial velocities, $u_l = 0.258\,\mathrm{m\,s^{-1}}$ and $u_g = 31.22\,\mathrm{m\,s^{-1}}$.

e) Estimate the average density of the fluid using the liquid hold-up. **[2 marks]**
Solution:
The average density is given by

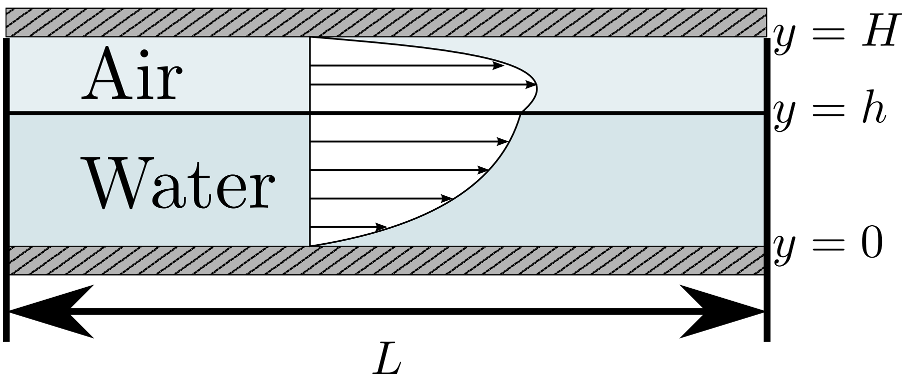
$$\begin{aligned}\rho_{two-phase} &= \rho_l\,H + \rho_g\,(1 - H)\\ &= 998 \times 0.193 + 0.774\,(1 - 0.193)\\ &\approx 192\ \mathrm{kg\ m^{-3}}\end{aligned}$$

[Question total: 20 marks]

Q.60 Question 60

Example exam question with marks

Consider the segregated horizontal flow of water and air between two plates of width Z = 50 cm and length L, spaced H = 5 cm apart. The two fluid phases flow at a rate of $\dot{V}_{water}$ = 10 l min^{-1} and $\dot{V}_{air}$ = 45 l min^{-1}.



At the conditions in the channel, water has a density of ρ_{water} = 985 kg m^{-3} and a viscosity of $\mu_{water} = 0.51 \times 10^{-3}$ Pa s. Air has a density of ρ_{air} = 1.14 kg m^{-3} and a viscosity of $\mu_{air} = 1.89 \times 10^{-5}$ Pa s.

a) Demonstrate that the no-slip liquid hold-up for this system is $h \approx 0.91$ cm. Comment on how realistic this estimation is. **[4 marks]**
Solution:
The no-slip liquid hold-up assumes the phases are flowing at the same velocity in the channel. This implies that the ratio of their flow rates is proportional to the height in the channel. Therefore, we have

$$\frac{h}{H} = \frac{\dot{V}_{water}}{\dot{V}_{water} + \dot{V}_{air}} = \frac{10}{10 + 45} \approx 0.182$$

which means that $h = 0.91$ cm. It is unlikely that this estimate is particularly realistic as there is significant slip of the gas phase in segregated flow.

b) Define and calculate the *superficial* velocity, u, and the *actual* velocity, $\langle v \rangle$, for the each phase, assuming the no-slip liquid holdup estimation is correct. **[5 marks]**

Solution:

The term *superficial* implies that values are calculated assuming that each phase of the multi-phase mixture is flowing alone in the channel. For example, the superficial water velocity is

$$u_{water} = \frac{\dot{V}_{water}}{H\,Z}$$

The actual flow velocity is the average velocity over the actual flow area of the phase. For example, the actual water velocity is

$$\langle v \rangle_{water} = \frac{\dot{V}_{water}}{h\,Z}$$

The flow rates in standard units are

$$\dot{V}_{air} = \frac{45 \times 10^{-3}}{60} = 7.5 \times 10^{-4}\ \mathrm{m^3\ s^{-1}}$$
$$\dot{V}_{water} = \frac{10 \times 10^{-3}}{60} \approx 1.67 \times 10^{-4}\ \mathrm{m^3\ s^{-1}}$$

The corresponding superficial velocities are

$$u_{air} = \frac{7.5 \times 10^{-4}}{0.5 \times 0.05} = 0.03\ \mathrm{m\ s^{-1}}$$
$$u_{water} = \frac{1.67 \times 10^{-4}}{0.5 \times 0.05} = 0.0067\ \mathrm{m\ s^{-1}}$$

The actual velocity is identical for each phase due to the no-slip assumption. E.g.:

$$\langle v \rangle_{air} = \frac{7.5 \times 10^{-4}}{0.5 \times 0.0409} = 0.0367\ \mathrm{m\ s^{-1}}$$
$$\langle v \rangle_{water} = \frac{1.67 \times 10^{-4}}{0.5 \times 0.0091} = 0.0367\ \mathrm{m\ s^{-1}}$$

We could also calculate this through the total volumetric flow rate divided by the full channel flow-area.

c) The Reynolds number for single-phase flow in a pipe is defined as:

$$\mathrm{Re} = \frac{\rho\,\langle v \rangle\,D}{\mu}$$

i) Define and calculate the superficial Reynolds number for the water phase. You should note that the characteristic length for flow between two plates is $2\,H$. **[3 marks]**

Solution:

The superficial Reynolds number is just the standard Reynolds number, but using the superficial velocity.

$$\mathrm{Re} = \frac{2\,\rho\,u_{water}\,H}{\mu} = \frac{2 \times 985 \times 0.0067 \times 0.05}{0.51 \times 10^{-3}} \approx 1294$$

This indicates the flow is laminar if using the single-phase transition value.

ii) Define and calculate the Reynolds number for the actual liquid phase using the hydraulic diameter. You can neglect the effect of the air phase (ignore its wetted perimeter). Is this definition consistent? **[4 marks]**

Solution:

The hydraulic diameter is given in the data sheet as (see Eq. (70))

$$D_H = \frac{4\,A}{P_w}$$

The cross sectional area of the flow of the water phase is $A = Z\,h$ (NOT H), and the wetted perimeter of the water phase is $P_w = Z$ (we can neglect the stress of the air phase); therefore, we have

$$\mathrm{Re} = \frac{4\,\rho\,\langle v\rangle\,h}{\mu} = \frac{4 \times 985 \times 0.0367 \times 0.0091}{0.51 \times 10^{-3}} = 2580$$

This indicates the flow is turbulent.

iii) Comment on the difference between the two results, including the limitations of these expressions. Is one estimate better than the other? **[3 marks]**

Solution:

The superficial Reynolds number is more likely to falsely predict laminar flow (when using a transition region from $2000 \to 2600$) as it underestimates the fluid velocity.

The second Reynolds number is more likely to predict turbulent flow as it uses the non-slip liquid hold-up (which will under-estimate the liquid hold-up due to gas slip).

However, the zero stress (unwetted air-liquid phase) condition makes this problem similar to the bottom half of filled channel solution. In this case, we would make a substitution $h \to h/2$, and recover the first definition of the Reynolds number.

Overall, the first expression is probably better, but a better estimate of the liquid hold-up is required to know for sure.

d) Assuming the no-slip liquid hold-up is correct, use the Chhabra-Richardson flow map to calculate the flow-regime. **[3 marks]**

Solution:

Using the flow map in the data sheet gives that the flow regime is *stratified*, as expected.

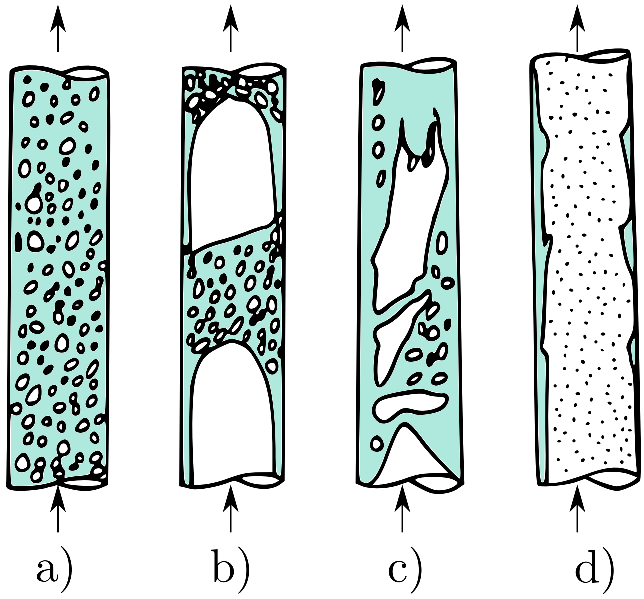
e) Assuming the liquid hold-up remains constant, at what liquid flow-rate does the flow turn intermittent? Why is this flow regime generally avoided? **[3 marks]**

Solution:

The transition point is around u_{water} =0.15 m s^{-1}, which is a volumetric flow rate of $\dot{V}_{water} = 0.15 \times 0.5 \times 0.05 = 3.75 \times 10^{-3}$ m^3 s^{-1}, or 3.75 l s^{-1}, or 225 l min^{-1}.

This flow regime is typically avoided as intermittent flow is hard to control and unsteady-state.

f) Consider the following vertical flow patterns.



i) Name each flow pattern, and identify which you might class as intermittent flow.
Solution:

a) Bubble flow.
b) Slug flow (intermittent).
c) Churn flow (intermittent).
d) Annular flow.

ii) Which is the most desirable flow pattern if the pressure drop is to be minimised and why?
Solution:
The most desirable pattern is annular flow, as this minimises the liquid hold-up, which reduces the hydrostatic pressure loss.

[Question total: 25 marks]

Q.61 Question 61

Consider condensing heat transfer:

a) The film thickness is a critical design parameter of condensing heat transfer. Explain how the thickness of the liquid layer affects the heat transfer and what the optimal conditions are for maximising the condensing rate.
Solution:
In condensation, the surface of the liquid film must be at the dew-point of the liquid. In order to transfer heat through this layer of liquid the rest of the film must be subcooled below this temperature. Therefore the thicker the film, the more energy is required to subcool the liquid before further condensation can take place. A thin liquid film is most desirable for pure condensation.

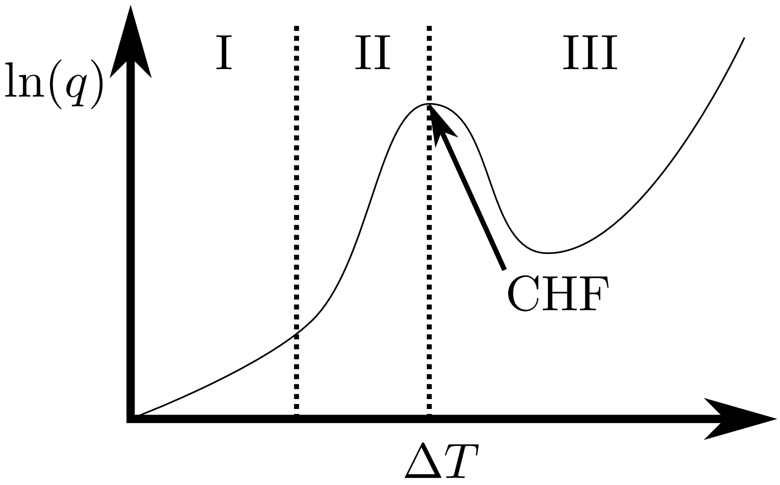
b) Discuss dropwise and film condensation. Which is most likely and why is dropwise condensation more favourable?
Solution:
Dropwise condensation happens only on specially treated and maintained surfaces; however, as sections of the surface are almost bare of any liquid the condensation heat transfer coefficients are extremely high.

The disadvantages of dropwise condensation are that avalanches of droplets cause intermittent surges of condensate which may flood any process equipment after the condenser. Also, when the surface dirties condensation switches back to film-limited transfer.

[Question end]

Q.62 Question 62
Example exam question with marks

a) i) Sketch the pool boiling curve (heat-flux/transfer-coefficient versus excess temperature), identify the key boiling regimes and describe the conditions in each. **[8 marks]**
Solution:



The markings on the graph denote the three main regions:

I Pure convective boiling: The heat transfer is driven by natural convection with evaporation taking place at the surface of the fluid.

II Nucleate boiling: Bubbles begin to form at the hot surface, increasing the heat transfer rate significantly due to increased convection at the surface.

III Film boiling: The high rate of vapour generation causes the bubbles to coalesce and form a vapour film covering the boiling surface.The heat transfer rate continues to increase as radiative heat transfer comes into play .

ii) On your pool boiling curve, indicate the location of the critical heat flux and describe advantages and the danger of operating an electrical boiler at this point. **[2 marks]**
Solution:
The critical heat flux occurs at the maximum in the heat flux just before the onset of film heat transfer. Operation at this location is advantageous as the heat transfer is at its peak;However, if a fluctuation causes the boiler to move into the film boiling, the heat transfer rate drops causing the temperature to rise further and it may cause the boiler to burn out.

iii) Why are electrical boilers vunerable to burnout near the critical heat flux when compared to boilers which use condensing steam as a heat source? **[2 marks]**
Solution:
With electrical heaters the heat flux, Q, is directly controlled, therefore the temperature of the wall may runaway if the actual heat flux is lower; however, steam is supplied to a heat exchanger at a fixed temperature which the boiler cannot exceed.

b) A kettle-type re-boiler operating at a pressure of 0.3 bar is used to boil a fluid of orthodichlorobenzene at a temperature of 120 °C. The properties of the mixture are given in the table below.

μ_L	0.45×10^{-3} Pa s	μ_G	0.01×10^{-3} Pa s
ρ_L	1170 kg m^{-3}	ρ_G	1.31 kg m^{-3}
k_L	0.11 W m^{-1} K^{-1}	$C_{p,L}$	1.25 kJ kg^{-1} K^{-1}
p_c	41 bar	Boiling point	136 °C

i) Assuming 40 m^2 of surface area is available for boiling and neglecting the geometry, calculate the heat transferred due to pure nucleate boiling. **[6 marks]**

Solution:

There are two correlations for the heat transfer coefficient, in the data sheet. However, only the Mostinski correlation is useful as we do not have data on the surface tension, γ.

$$\begin{aligned} h_{nb} &= 0.104\, p_c^{0.69}\, q^{0.7} \left[1.8 \left(\frac{p}{p_c}\right)^{0.17} + 4 \left(\frac{p}{p_c}\right)^{1.2} + 10 \left(\frac{p}{p_c}\right)^{10}\right] \\ &= 0.104 \times 41^{0.69}\, q^{0.7} \left[1.8 \left(\frac{0.3}{41}\right)^{0.17} + 4 \left(\frac{0.3}{41}\right)^{1.2} + 10 \left(\frac{0.3}{41}\right)^{10}\right] \\ &\approx 1.07\, q^{0.7} \end{aligned}$$

If the heat flux is due to pure nucleate boiling then we have

$$q = h_{nb}\,(T_w - T_{fluid})$$

The wall temperature will be at the boiling temperature of the fluid $T_w = 136$ °C. Inserting this expression into the expression for the boiling heat transfer coefficient yields

$$h_{nb} = 1.07 \times (h_{nb}\,[136 - 120])^{0.7}$$

Rearranging for h_{nb}, we have

$$\begin{aligned} h_{nb}^{0.3} &= 1.07 \times ([136 - 120])^{0.7} \\ h_{nb}^{0.3} &= 7.45 \\ h_{nb} &= 808\,\mathrm{W\ m^{-2}\,K^{-1}} \end{aligned}$$

Finally, the total heat transfer rate is given by

$$\begin{aligned} Q = q\,A = h_{nb}\,A\,(T_w - T_{fluid}) \\ &= 808 \times 40\,(136 - 120) \\ &\approx 517\,\mathrm{kW} \end{aligned}$$

ii) Estimate the critical heat flux and determine if the reboiler is operating in a safe region. **[4 marks]**

Solution:

Again we use a Mostinski correlation for the critical heat flux

$$\begin{aligned} q_c &= 3.67 \times 10^4\, p_c \left(\frac{p}{p_c}\right)^{0.35} \left[1 - \frac{p}{p_c}\right]^{0.9} \\ &= 3.67 \times 10^4 \times 41 \left(\frac{0.3}{41}\right)^{0.35} \left[1 - \frac{0.3}{41}\right]^{0.9} \\ &\approx 267\,\mathrm{kW\ m^{-2}} \end{aligned}$$

The total critical heat transfer rate is

$$Q_c = q_c A = 267000 \times 40$$
$$\approx 10.7\,\mathrm{MW}$$

This critical heat flux is well above the operating heat transfer rate, so the reboiler is operating in a safe region.

[Question total: 22 marks]

Q.63 Question 63

Fick's law is often modified to the following form:

$$N_{A,x} = -\left(D_{AB} + E_D\right)\frac{\partial C_A}{\partial x}$$

What is the parameter E_D and what does it represent?

Solution:

E_D is the eddy diffusivity. It represents the additional transport of the species A through B due to small eddies/circulating currents (caused by microscopic differences in temperature/pressure) which causing the fluid to mix and appear to diffuse faster than expected.

[Question end]

Q.64 Question 64

Consider the dimensionless Lewis number:

$$\mathrm{Le} = \frac{k}{\rho\, C_p\, D_{AB}}$$

What two processes are compared through this number and what does the limit $\mathrm{Le} \to \infty$ correspond to?

Solution:

This number is a comparision of the thermal and mass diffusivity. At the limit $\mathrm{Le} \to \infty$, diffusion is negligble when compared to thermal diffusivity (i.e. in a solid).

[Question end]

Q.65 Question 65

Gaseous hydrogen at 10 bar and 27°C is stored in a 140 mm outer-diameter tank having a steel wall 2 mm thick and a height of 850mm. The molar concentration of hydrogen in the steel is 1.5 kmol m^{-3} at the inner surface and neglible at the outer surface, while the diffusion coefficient of hydrogen in steel is approximately 0.3×10^{-12} m^2 s^{-1}. What is the rate of mass loss of hydrogen by diffusion per square meter of tank wall? Assume steady-state, one-dimensional conditions.

a) Assuming the curvature of the tank is negligible (you can use rectangular coordinates), show that the molar flux of hydrogen is constant through the wall.

$$N_{H_2,z} = N_{H_2,0}$$

Solution:
There are two ways we can derive the general balance equation for this problem:

General Balance Equation Approach
The general balance equation for a species a is

$$\frac{\partial C_a}{\partial t} = -\nabla_i N_a$$

As we are using rectangular coordinates and using $a = H_2$, we can write

$$\frac{\partial C_{H_2}}{\partial t} = -\nabla_x N_{H_2,x} - \nabla_y N_{H_2,y} - \nabla_z N_{H_2,z}$$

We are at steady state, and as we assume that the system is one dimensional (symmetric in the x and y dimensions), we can cancel most terms to give

$$\cancelto{0}{\frac{\partial C_{H_2}}{\partial t}} = -\cancelto{0}{\nabla_x N_{H_2,x}} - \cancelto{0}{\nabla_y N_{H_2,y}} - \nabla_z N_{H_2,z}$$

Leaving us with the final equation

$$\nabla_z N_{H_2,z} = \frac{\partial N_{H_2,z}}{\partial z} = 0$$

We can integrate this to obtain

$$N_{H_2,z} = C$$

This is a statement that for a flat plate at steady state the molar flux is a constant value. This constant value is the flux at some point in the system, so we choose $z = 0$ to give $C = N_{H_2,0}$.

$$N_{H_2,z} = N_{H_2,0}$$

Shell Balance Approach
We perform a balance for hydrogen on a thin slab of steel located within the wall of the tank. The bottom of the slab is at z, the thickness of the slab is Δz, and the area of the slab is A. We assume that we are at steady state, so there is no accumulation. Therefore, we expect that the influx of hydrogen should equal the outflux:

$$N_{\mathrm{H}_2,z}(z)\,A - N_{\mathrm{H}_2,z}(z+\Delta z)\,A = 0$$
$$\frac{N_{\mathrm{H}_2,z}(z+\Delta z) - N_{\mathrm{H}_{22},z}(z)}{\Delta z} = 0$$

Taking the limit $\Delta z \to 0$, we find

$$\frac{\partial N_{\mathrm{H}_2,z}}{\partial z} = 0$$

Integrating this equation, we find

$$N_{\mathrm{H}_2,z} = C$$

From this point on the arguments are the same as for the general balance equation approach.

b) Noting that the concentration of hydrogen in the steel wall is very low $x_{H_2} \ll 1$, determine the concentration profile of hydrogen in the wall.
Solution:
If the concentration of hydrogen is small, then we can use Fick's law of diffusion directly.

$$N_{H_2,z} = N_{H_2,0} = -D_{H_2}\frac{\partial C_{H_2}}{\partial z}$$

We can determine the concentration profile using a single integration

$$\begin{aligned}\frac{\partial C_{H_2}}{\partial z} &= -\frac{N_{H_2,0}}{D_{H_2}}\\ C_{H_2} &= -\frac{N_{H_2,0}}{D_{H_2}}\,z + C\\ &= \frac{N_{H_2,0}}{D_{H_2}}\,(C - z)\end{aligned}$$

Now we need to use the boundary conditions to determine the values of the constants $N_{H_2,0}$ and C.

We can set up our coordinate system so that $z = 0$ refers to the inside surface of the steel tank and $z = 2$ mm refers to the outside surface of the tank.

Then our boundary conditions are that, at $z = 2mm$ the concentration of hydrogen is negligible ($C_a(z = 0.002) = 0$). This gives $C = 0.002$.

At $z = 0$ mm the concentration of hydrogen in the steel is 1.5 kmol m^{-3} and we have

$$\begin{aligned}C_{H_2} &= \frac{N_{H_2,0}}{D_{H_2}}\,(0.002 - z)\\ 1.5\times10^{3} &= \frac{N_{H_2,0}}{0.3\times10^{-12}}0.002\\ N_{H_2,0} &= \frac{(1.5\times10^{3})(0.3\times10^{-12})}{0.002} \approx 2.25\times10^{-7}\mathrm{mol\ m^{-2}\,s^{-1}}\end{aligned}$$

c) Calculate the total mass flow rate of hydrogen transported through the side walls of the vessel (consider just the cylindrical sides).
Solution:
The total molar loss of hydrogen from the vessel is given by the surface area of the cylinder times by $N_{H_2,0}$.

$$N_{H_2,0}\,\pi\,D\,L = 2.25\times10^{-7}\,\pi\,0.14\times0.85 \approx 8.4\times10^{-8}\ \mathrm{mol\ s^{-1}}$$

The molar weight of hydrogen gas is 2 g mol^{-1}. This gives us a flow rate of 1.68×10^{-7}g s^{-1} or 6.048×10^{-4}g hr^{-1}.

d) It is determined that the effect of curvature must be included in the estimation of the mass flux (we must use a cylindrical geometry). Derive the following expression for the flux

$$N_{H,r} = \frac{C_1}{r}$$

and derive the following expression for the concentration profile of the hydrogen in the steel wall.

$$C_{H_2} = 5.17\times10^{4}\ln\left(\frac{0.07}{r}\right)$$

Solution:

We just repeat the analysis above but with a cylindrical geometry. The general balance equation for a species a is

$$\frac{\partial C_a}{\partial t} = -\nabla_i N_a$$

As we are using cylindrical coordinates and using $a = H_2$, we can write

$$\frac{\partial C_{H_2}}{\partial t} = -\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\, N_{H_2,r}\right) + \frac{1}{r}\frac{\partial N_{H_2,\theta}}{\partial \theta} + \frac{\partial N_{H_2,z}}{\partial z}\right)$$

We are at steady state, and as we assume that the system is one dimensional (symmetric in the θ and z dimensions), we can cancel most terms to give

$$\cancelto{0}{\frac{\partial C_{H_2}}{\partial t}} = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\, N_{H_2,r}\right)$$

Leaving us with the final equation

$$\frac{\partial\, r\, N_{H_2,r}}{\partial r} = 0$$

We can integrate this to obtain

$$N_{H_2,r} = \frac{C_1}{r}$$

This is a statement that for a curved surface the mass flux changes as the area changes as a function of r.

As the hydrogen is at a low concentration we can use Fick's law directly

$$N_{H_2,r} = -D_{H_2}\frac{\partial C_{H_2}}{\partial r} = \frac{C_1}{r}$$

Integrating once again gives

$$C_{H_2} = -\frac{C_1}{D_{H_2}}\ln(r) + C_2$$

We know that at the inside surface of the cylinder, the concentration of hydrogen is $C_a(r = 0.068) = 1.5$ kmol m^{-3} and at the outside surface of the cylinder the concentration is $C_a(r = 0.07) = 0$.

Using the boundary condition at the outside we have

$$C_2 = \frac{C_1}{D_{H_2}}\ln(0.07)$$

Which gives

$$\begin{aligned} C_{H_2} &= \frac{C_1}{D_{H_2}}\left(\ln(0.07) - \ln(r)\right) \\ &= \frac{C_1}{D_{H_2}}\ln\left(\frac{0.07}{r}\right) \end{aligned}$$

The boundary condition on the inside surface gives

$$\begin{aligned}1.5\times 10^3 &= \frac{C_1}{D_{H_2}}\ln\left(\frac{0.07}{0.068}\right)\\ C_1 &= 1.5\times 10^3\,D_{H_2}\left[\ln\left(\frac{0.07}{0.068}\right)\right]^{-1}\\ &\approx 5.17\times 10^4\,D_{H_2}\end{aligned}$$

Giving the final expression

$$C_{H_2} = 5.17\times 10^4 \ln\left(\frac{0.07}{r}\right)$$

The concentration profile is independent of the diffiusion coefficient! This is analogous to the stress profile which is independent of the viscous behaviour of the fluid.

e) Calculate the mass flux of hydrogen through the wall using the solution to the last question.
Solution:
We need to evaluate the flux at either the inner or outer surface and multiply it by the surface area. For consistency we will use the outer surface.

Using Fick's law, we have

$$\begin{aligned}N_{H_2,r} &= -D_{H_2}\frac{\partial C_{H_2}}{\partial r}\\ &= -5.17\times 10^4\,D_{H_2}\frac{\partial}{\partial r}\ln\left(\frac{0.07}{r}\right)\\ &= \frac{5.17\times 10^4\,D_{H_2}}{r}\end{aligned}$$

The molar flux at the outer surface is

$$\begin{aligned}N_{H_2,r=0.07\ \mathrm{m}} &= \frac{5.17\times 10^4\,D_{H_2}}{0.07}\\ &\approx 2.22\times 10^{-7}\ \mathrm{mol\ m^{-2}\,s^{-1}}\end{aligned}$$

The total mass flux is

$$\begin{aligned}\text{Mass flux} = m_{H_2}\,N_{H_2,r=0.07\ \mathrm{m}}\,\pi\,D\,L = 2\times 2.22\times 10^{-7}\,\pi\,0.14\times 0.85 &\approx 1.66\times 10^{-7}\mathrm{g\ s^{-1}}\\ &\approx 5.98\times 10^{-4}\mathrm{g\ hr^{-1}}\end{aligned}$$

where $m_{H_2} = 2\ \mathrm{g\ mol^{-1}}$ is the molar mass of hydrogen.

This is less than the $6.048\times 10^{-4}\mathrm{g\ hr^{-1}}$ calculated previously but not by a significant amount.

[Question end]

Q.66 Question 66

Example exam question

Helium gas at 100 bar and 20°C is stored in a 140 mm outer-diameter vessel with a pyrex wall 4 mm thick and a height of 850 mm. The molar concentration of helium in the pyrex is 35 mol m^{-3} at the inner surface and negligible at the outer surface, while the diffusion coefficient of helium in pyrex is approximately 0.2×10^{-12} m^2 s^{-1}.

a) Assuming the curvature of the tank is negligible (you can use rectangular coordinates) and steady-state, one-dimensional conditions, show that the molar flux of helium is constant through the wall. **[3 marks]**

$$N_{He,z} = N_{He,0}$$

Solution:

There are two ways we can derive the general balance equation for this problem:

General Balance Equation Approach

The general balance equation for a species a is

$$\frac{\partial C_a}{\partial t} = -\nabla_i N_a$$

As we are using rectangular coordinates and using $a = He$, we can write

$$\frac{\partial C_{He}}{\partial t} = -\nabla_x N_{He,x} - \nabla_y N_{He,y} - \nabla_z N_{He,z}$$

We are at steady state, and as we assume that the system is one dimensional (symmetric in the x and y dimensions), we can cancel most terms to give

$$\cancelto{0}{\frac{\partial C_{He}}{\partial t}} = -\cancelto{0}{\nabla_x N_{He,x}} - \cancelto{0}{\nabla_y N_{He,y}} - \nabla_z N_{He,z}$$

Leaving us with the final equation

$$\nabla_z N_{He,z} = \frac{\partial N_{He,z}}{\partial z} = 0$$

We can integrate this to obtain

$$N_{He,z} = C$$

This is a statement that for a flat plate at steady state the molar flux is a constant value. This constant value is the flux at some point in the system, so we choose $z = 0$ to give $C = N_{He,0}$.

$$N_{He,z} = N_{He,0}$$

Shell Balance Approach

We perform a balance for helium on a thin slab of pyrex located within the wall of the tank. The bottom of the slab is at z, the thickness of the slab is Δz, and the area of the slab is A. We assume that we are at steady state, so there is no accumulation. Therefore, we expect that the influx of helium should equal the outflux:

$$N_{\mathrm{He},z}(z)\,A - N_{\mathrm{He},z}(z+\Delta z)\,A = 0$$
$$\frac{N_{\mathrm{He},z}(z+\Delta z) - N_{\mathrm{He_2},z}(z)}{\Delta z} = 0$$

Taking the limit $\Delta z \to 0$, we find

$$\frac{\partial N_{\text{He},z}}{\partial z} = 0$$

Integrating this equation, we find

$$N_{\text{He},z} = C$$

From this point on the arguments are the same as for the general balance equation approach.

b) The concentration of helium in the pyrex wall is very low $x_{He} \ll 1$, allowing the use of the simple form of Fick's law. Determine the concentration profile of helium in the wall. **[4 marks]**

Solution:

If the concentration of helium is small, then we can use Fick's law of diffusion directly.

$$N_{He,z} = N_{He,0} = -D_{He}\frac{\partial C_{He}}{\partial z}$$

We can determine the concentration profile using a single integration

$$\begin{aligned}
\frac{\partial C_{He}}{\partial z} &= -\frac{N_{He,0}}{D_{He}} \\
C_{He} &= -\frac{N_{He,0}}{D_{He}} z + C \\
&= \frac{N_{He,0}}{D_{He}} (C - z)
\end{aligned}$$

Now we need to use the boundary conditions to determine the values of the constants $N_{He,0}$ and C.

We can set up our coordinate system so that $z = 0$ refers to the inside surface of the pyrex tank and $z = 2$ mm refers to the outside surface of the tank.

Then our boundary conditions are that, at $z = 4mm$ the concentration of helium is negligible ($C_a(z = 0.004) = 0$). This gives $C = 0.004$.

At $z = 0$ mm the concentration of helium in the pyrex is 35 mol m^{-3} and we have

$$\begin{aligned}
C_{He} &= \frac{N_{He,0}}{D_{He}} (0.004 - z) \\
35 &= \frac{N_{He,0}}{0.2 \times 10^{-12}} 0.004 \\
N_{He,0} &= \frac{35 \times 0.2 \times 10^{-12}}{0.004} \approx 1.75 \times 10^{-9} \text{mol m}^{-2}\,\text{s}^{-1}
\end{aligned}$$

c) Calculate the total mass flow-rate of helium transported through the side walls of the vessel (consider just the cylindrical sides). **[3 marks]**

Solution:

The total molar loss of helium from the vessel is given by the surface area of the cylinder multiplied by $N_{He,0}$.

$$N_{He,0}\, \pi\, D\, L = 1.75 \times 10^{-9}\, \pi\, 0.14 \times 0.85 \approx 6.5 \times 10^{-10} \text{ mol s}^{-1}$$

The molar weight of helium gas is 4 g mol^{-1}. This gives us a flow rate of 2.6×10^{-9}g s^{-1} or 9.4×10^{-6}g hr^{-1}.

[Question total: 10 marks]

Q.67 Question 67

To maintain a pressure close to 1 atm, an industrial pipeline containing ammonia gas (17 g mol^{-1}) is vented to ambient air (29 g mol^{-1}). Venting is achieved by tapping the pipe and inserting a 3 mm diameter tube, which extends for 20 m into the atmosphere. With the entire system operating at 25 °C and 1 bar, the ideal gas equation of state predicts a total molar concentration of 40.9 mol m^{-3}. Equimolar counter-diffusion can be assumed, and both the concentration of air in the pipeline and the concentration of ammonia in the atmosphere can be considered negligible. The diffusion coefficient of ammonia through air is approximately 2×10^{-5} m^2 s^{-1}.

a) Determine the mass rate of ammonia (17 g mol^{-1}) lost in to the atmosphere $\mathbf{N}_A$ in kg h^{-1} and the mass rate of contamination of the pipe with air (29 g mol^{-1}), $\mathbf{N}_B$, in the same units.

Solution:

As an interesting aside, the total concentration of the gas was estimated by the ideal gas law like so:

$$\begin{aligned} C_T &= \frac{n}{V} = \frac{p}{RT} \\ &= \frac{101325 \text{ Pa}}{(8.314 \text{ J mol}^{-1}\text{ K}^{-1}))(298.15 \text{ K})} \\ &= 40.9 \text{ mol m}^{-3} \end{aligned}$$

This course will not test you on this knowledge, but in the future you will be expected to be able to do these calculations.

Assuming that the system is one dimensional, we can use rectangular coordinates and we again find that the fluxes of the ammonia and air are constant.

$$N_{A,z} = N_{A,0} \qquad\qquad N_{B,z} = N_{B,0}$$

The boundary conditions are

$$\begin{aligned} C_A(z = 0 \text{ m}) &= 40.9 \text{ mol m}^{-3} & C_A(z = 20 \text{ m}) &= 0 \text{ mol m}^{-3} \\ C_B(z = 0 \text{ m}) &= 0 \text{ mol m}^{-3} & C_B(z = 20 \text{ m}) &= 40.9 \text{ mol m}^{-3} \end{aligned}$$

So ammonia is diffusing up the tube and air is diffusing down the tube. This sounds like equimolar counterdiffusion. If we assume equimolar counterdiffusion, then we have $N_{B,0} = -N_{A,0}$. For equimolar counterdiffusion we can directly use Fick's law again

$$N_{A,z} = N_{A,0} = -D_{AB}\frac{\partial C_A}{\partial z}$$

Integrating this equation, we find:

$$C_A = C - \frac{N_{A,0}}{D_{AB}}z$$

From the first boundary condition in the ammonia ($C_A(z = 0 \text{ m}) = 40.9 \text{ mol m}^{-3}$), we find

$$C = 40.9 \text{ mol m}^{-3}$$

From the second boundary condition we find

$$
\begin{aligned}
0 &= 40.9 - 20\frac{N_{A,0}}{D_{AB}} \\
N_{A,0} &= \frac{40.9\, D_{AB}}{20} \\
&= \frac{40.9 \times 2 \times 10^{-5}}{20} \approx 4.09 \times 10^{-5}\ \mathrm{mol\ m^{-2}\ s^{-1}}
\end{aligned}
$$

If we multiply the flux of ammonia by the cross-sectional area of the tube $\pi D^2/4$ and its molecular weight ($17\ \mathrm{g\ mol^{-1}}$), we will find the mass rate of ammonia lost to the atmosphere:

$$
\begin{aligned}
\text{ammonia lost to atmosphere} &= N_{A,0}\frac{\pi}{4}D^2 M_A \\
&= \left(4.09 \times 10^{-5}\ \frac{\mathrm{mol}}{\mathrm{m^2\ s}}\right)\frac{\pi}{4}(0.003\ \mathrm{m})^2(17\ \mathrm{g\ mol^{-1}}) \\
&\approx 4.91 \times 10^{-9}\ \mathrm{g\ s^{-1}} \\
&\approx 1.77 \times 10^{-8}\ \mathrm{kg\ hr^{-1}}
\end{aligned}
$$

To determine the mass rate of contamination of the pipe with air, we first note the molar flux of air into the pipe is equal and opposite to the molar flux of ammonia into the atmosphere ($N_{A,0} = -N_{B,0}$ due to the assumption of equimolar counterdiffusion). Multiplying this molar flux by the cross-sectional area of the tube and the molecular weight of air ($29\ \mathrm{g\ mol^{-1}}$), we find that the mass flowrate of air into the pipeline is

$$
\begin{aligned}
\text{air entering pipeline} &= -N_{A,0}\frac{\pi}{4}D^2 M_B \\
&= -\left(4.09 \times 10^{-5}\ \mathrm{mol\ m^{-2}\ s^{-1}}\right)\frac{\pi}{4}(0.003\ \mathrm{m})^2(29\ \mathrm{g\ mol^{-1}}) \\
&\approx -8.38 \times 10^{-9}\ \mathrm{g\ s^{-1}} \\
&\approx -3.02 \times 10^{-9}\ \mathrm{kg\ hr^{-1}}
\end{aligned}
$$

b) A new high-tech membrane, which is impermeable to air, is installed at the bottom of the pipe to prevent air polluting the pipeline. The *air* within the tube is now **stationary** and the mole fraction of ammonia at the surface of the membrane is $x_A(z = 0) = 0.9$. Resolve the problem again to determine the flux of ammonia.
Note: Stefan's law in mole fractions is given by.

$$
N_{A,z} = -D_{AB}\frac{C_T}{1 - x_A}\frac{\partial x_A}{\partial z}
$$

Solution:
This problem is similar to diffusion in an Arnold cell.

For equimolar counter-diffusion, we have Stefan's law (as written in C&R vol.1)

$$
N_{A,z} = -D_{AB}\frac{C_T}{C_B}\frac{\partial C_A}{\partial z}
$$

But this question provides you with a more convenient form of Stefan's law, expressed in molar concentrations

$$
N_{A,z} = -D_{AB}\frac{C_T}{1 - x_A}\frac{\partial x_A}{\partial z}
$$

This form is more convenient as the first form is in terms of two **dependent** variables C_A and C_B. The form in terms of the mole fraction is more general and is only in terms of one variable. However, we could also write

$$N_{A,z} = -D_{AB}\frac{C_T}{C_T - C_A}\frac{\partial C_A}{\partial z}$$

and again we have only one variable, C_A. We will solve the mole fraction form of Stefan's law (the one provided in the question).

The flux of ammonia is still constant along the pipe (the balance equation hasn't changed, only the expression for the flux). So we can try integrating Stefan's law

$$N_{A,z} = N_{A,0} = -D_{AB}\frac{C_T}{1 - x_A}\frac{\partial x_A}{\partial z}$$
$$N_{A,0}\int \mathrm{d}z = -D_{AB}\,C_T\int\frac{1}{1-x_A}\mathrm{d}x_A$$
$$N_{A,0}\,z = D_{AB}\,C_T\ln(1 - x_A) + C$$

The boundary condition at the bottom of the pipe, in terms of the mole fraction, is $x_A(z = 0) = 0.9$ which gives

$$0 = D_{AB}\,C_T\ln(0.1) + C$$
$$C = -D_{AB}\,C_T\ln(0.1)$$
$$= -2\times 10^{-5}\times 40.9\times\ln(0.1)\approx 1.88\times 10^{-3}$$

The other boundary condition is that the concentration of ammonia is zero at the exit of the tube $x_A(z = 20\text{ m}) = 0$.

$$20\,N_{A,0} = D_{AB}\,C_T\ln(1) + 1.88\times 10^{-3}$$
$$N_{A,0} = \frac{1.88\times 10^{-3}}{2} = 9.44\times 10^{-5}\text{ mol m}^{-2}\text{ s}^{-1}$$

The total mass flowrate of ammonia is

$$\text{ammonia lost to atmosphere} = N_{A,0}\frac{\pi}{4}D^2 M_A$$
$$= \left(9.44\times 10^{-5}\text{ mol m}^{-2}\text{ s}^{-1}\right)\frac{\pi}{4}(0.003\text{ m})^2(17\text{ g mol}^{-1})$$
$$\approx 1.13\times 10^{-8}\text{ g s}^{-1}$$
$$\approx 4.08\times 10^{-8}\text{ kg hr}^{-1}$$

The flow rate of ammonia has increased from 1.77×10^{-8} kg hr^{-1} (this is a feature of diffusion through a stationary layer), but it is still small.

[Question end]

Q.68 Question 68

A Winkelmann apparatus is used to measure the diffusivity of a substance, A, in air. It is sketched in Fig. 23. To perform the experiment, a quantity of liquid A is placed at the bottom of a test tube. The liquid evaporates to a vapour mole fraction of $x_{A,sat}$ at the liquid surface (which is determined in a separate equilibrium experiment). The vapourised A then diffuses up the tube where it is removed by a steady flow of air. As A is removed, the liquid level in the tube drops and by monitoring it's rate of change the total diffusive flux can be calculated. We can assume the diffusion profile is at steady state if the rate of evaporation is slow. We also assume the vapours of air and A form an ideal gas, so density is constant inside the tube.

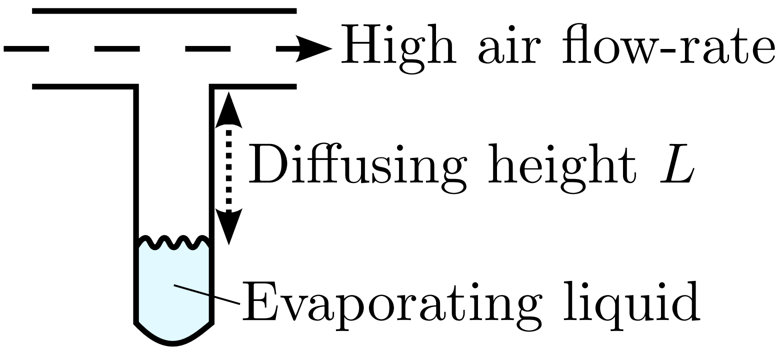


Figure 23: A winklemann experiment.

a) Derive the following differential balance equation governing the diffusion of mass in the system. Remember to state any assumptions you make.

$$\frac{\partial}{\partial z} N_{A,z} = 0$$

[5 marks]

Solution:

[1/5] As the liquid is evaporating slowly, we can assume it is at quasi steady-state.$\checkmark_1$

[1/5] We also assume that the diffusion is one-dimensional and only consider diffusion up the axis of the tube.$\checkmark_1$ We can use either rectangular or cylindrical coordinates, but rectangular coordinates are used here as they're simpler.

From the general balance equation, we have

$$\frac{\partial C_A}{\partial t} = -\nabla_i\, N_{A,i}$$

We choose either a rectangular or cylindrical coordinate system and align the z-axis so that it points up the test tube.

[1/5] At steady state the time derivative is zero$\checkmark_1$ and the flux in the directions perpendicular to
[1/5] z are zero.$\checkmark_1$

$$\cancelto{0}{\frac{\partial C_A}{\partial t}} = -\nabla_x\, \cancelto{0}{N_{A,x}} - \nabla_y\, \cancelto{0}{N_{A,y}} - \nabla_z\, N_{A,z}$$

Substituting in the definition of the z-component of the cylindrical/rectangular gradient operator, we have

$$\nabla_z\, N_{A,z} = \frac{\partial\, N_{A,z}}{\partial z} = 0$$

[1/5] $\checkmark_1$

b) Write down the boundary conditions of the system and state which class of diffusion problem this is. **[3 marks]**

Solution:

The boundary conditions are:

- **[1/3]** The mole fraction at the surface of the liquid is equal to the saturation mole fraction ($x_A = x_{A,sat}$ at $z = 0$). $\checkmark_1$

- At the top of the test tube, the concentration is zero due to the high flow-rate of air

[1/3] ($x_A = 0$ at $z = L$).$\checkmark_1$

[1/3] This is **diffusion through a stagnant layer**.$\checkmark_1$

c) Derive Stefan's law, given below, from the general expression for the diffusive flux. **[4 marks]**

$$N_{A,z} = -D_{A,air}\frac{C}{1-x_A}\frac{\partial x_A}{\partial z}$$

Solution:
Taking the general expression for the diffusive flux from the datasheet, we have

$$N_{A,z} = -D_{AB}\frac{\partial C_A}{\partial z} + x_A\sum_i N_{i,z}$$

There are only two components in this system, the stationary air and the diffusing component (A). Thus, the general expression becomes

$$N_{A,z} = -D_{A,air}\frac{\partial C_A}{\partial z} + x_A\left(N_{A,z} + N_{air,z}\right)$$

[1/4] $\checkmark_1$The air within the test tube must be stationary ($N_{air,z} = 0$), as it is not absorbed or released

[1/4] by the liquid.$\checkmark_1$

$$N_{A,z} = -D_{A,air}\frac{\partial C_A}{\partial z} + x_A\,N_{A,z}$$

Noting that $C_A = x_A\,C$, where C is the total gas concentration in the system, we can write

$$\begin{aligned}
N_{A,z} &= -D_{A,air}\frac{\partial C_A}{\partial z} + x_A\,N_{A,z}\\
&= -D_{A,air}C\frac{\partial x_A}{\partial z} + x_A\,N_{A,z}\\
(1-x_A)N_{A,z} &= -D_{A,air}C\frac{\partial x_A}{\partial z}\\
N_{A,z} &= -D_{A,air}\frac{C}{1-x_A}\frac{\partial x_A}{\partial z}
\end{aligned}$$

[2/4] $\checkmark_2$

d) Derive the following expression for the mole fraction profile x_A in the system. **[8 marks]**

$$x_A = 1-\left(1-x_{A,sat}\right)^{1-z/L}$$

using the identity

$$\frac{\partial N_{A,z}}{\partial z} = 0$$

Solution:
Substituting Stefan's law into the balance equation from the first question, we have

$$\frac{\partial}{\partial z}\left(-D_{A,air}\frac{C}{1-x_A}\frac{\partial x_A}{\partial z}\right) = 0$$

[1/8] $\checkmark_1$Integrating this equation with respect to z, we have

$$D_{A,air}\frac{C}{1-x_A}\frac{\partial x_A}{\partial z}=C_1$$

[1/8] $\checkmark_1$Integrating again we have

$$D_{A,air}\,C\ln\left(1-x_A\right)=C_1\,z+C_2$$

[1/8] $\checkmark_1$The first boundary condition, as $z=0$ we have $x_A=x_{A,sat}$. Which gives

$$C_2=D_{A,air}\,C\ln\left(1-x_{A,sat}\right)$$

[1/8] $\checkmark_1$The second boundary condition is that at $z=L$ we have $x_A=0$. Using these values we find

$$D_{A,air}\,C\cancelto{0}{\ln\left(1-0\right)}=C_1\,L+D_{A,air}\,C\ln\left(1-x_{A,sat}\right)$$
$$C_1=-\frac{D_{A,air}\,C}{L}\ln\left(1-x_{A,sat}\right)$$

[1/8] $\checkmark_1$Substituting these values back in, we find

$$\ln\left(1-x_A\right)=\left(1-\frac{z}{L}\right)\ln\left(1-x_{A,sat}\right)$$
$$\ln\left(\frac{1-x_A}{\left(1-x_{A,sat}\right)^{1-z/L}}\right)=0$$
$$x_A=1-\left(1-x_{A,sat}\right)^{1-z/L}$$

[2/8] $\checkmark_2$

e) The derivative of the mole fraction in position is

$$\frac{\partial x_A}{\partial z}=\frac{\ln\left(1-x_{A,sat}\right)\left(1-x_{A,sat}\right)^{1-z/L}}{L}$$

Derive the following expression for the flux of A, $N_{A,z}$, at any location in the tube. **[3 marks]**

$$N_{A,z}=-D_{A,air}\frac{C}{L}\ln\left(1-x_{A,sat}\right)$$

Solution:

Starting with Stefan's law, we can substitute in the expression for the positional derivative of the mole fraction:

$$N_{A,z}=-D_{A,air}\frac{C}{1-x_A}\frac{\partial x_A}{\partial z}$$
$$=-D_{A,air}\frac{C}{1-x_A}\frac{\ln\left(1-x_{A,sat}\right)\left(1-x_{A,sat}\right)^{1-z/L}}{L}$$

[1/3] $\checkmark_1$Substituting in the expression for the concentration profile, we have

$$N_{A,z}=-D_{A,air}\frac{C}{\left(1-x_{A,sat}\right)^{1-z/L}}\frac{\ln\left(1-x_{A,sat}\right)\left(1-x_{A,sat}\right)^{1-z/L}}{L}$$
$$=-D_{A,air}\frac{C}{L}\ln\left(1-x_{A,sat}\right)$$

[1/3] $\checkmark_1$The flux of the component is constant up the tube (as expected from IN=OUT)$\checkmark_1$.

[1/3]

f) The mysterious ingredient 7X in a popular drinks beverage evaporates to a mole fraction of 0.02 in air at standard temperature and pressure (20 °C and 1 atm). In a Winklemann experiment, the level is dropping at a rate of 1 mm min^{-1} when the diffusing height is 5 cm. Determine the diffusion coefficient of 7X through air. You may assume the vapours of 7X and air form an ideal gas and that liquid 7X has a density of 18 kmol m^{-3}. **[5 marks]**

Solution:
At standard temperature and pressure, the concentration of gas molecules in an ideal gas is given by

$$C = \frac{n}{V} = \frac{P}{RT} = \frac{101300}{8.314 \times 273.15} \approx 44.6 \text{ mol m}^{-3}$$

[1/5] $\checkmark_1$ If the liquid level is lowering by 1 mm min^{-1}, then the volumetric loss of liquid 7X is

$$\dot{V}_{7X} = \frac{1 \times 10^{-3}}{60} \times A_{tube} \text{ m}^3 \text{ s}^{-1}$$

[1/5] $\checkmark_1$ The molar flux is then the volumetric loss multiplied by the density and divided by the cross-sectional area of the tube.

$$N_{7X,z} = \frac{1 \times 10^{-3}}{60} \cancel{A_{tube}} \frac{18 \times 10^3}{\cancel{A_{tube}}} = 0.3 \text{ mol m}^{-2} \text{ s}^{-1}$$

[1/5] $\checkmark_1$

We can now work out the diffusion coefficient in air

$$\begin{aligned} N_{7X,z} &= -D_{7X,air} \frac{C}{L} \ln(1 - x_{7X,sat}) \\ D_{7X,air} &= -\frac{N_{7X,z} L}{C \ln(1 - x_{7X,sat})} \\ &= -\frac{0.3 \times 0.05}{44.6 \ln(1 - 0.02)} \\ &\approx 0.0166 \text{ m}^2 \text{ s}^{-1} \end{aligned}$$

[2/5] $\checkmark_2$

[Question total: 28 marks]

Q.69 Question 69

a) Define the Schmidt number, what does this dimensionless number tell you about the transport processes in a fluid? **[2 marks]**

Solution:
The Schmidt number is defined as

$$\text{Sc} = \frac{\nu}{D}$$

It is the ratio of the rates momentum and mass diffusion in the fluid, and relates to the
[2/2] thickness of the momentum and mass transfer layers$\checkmark_2$.

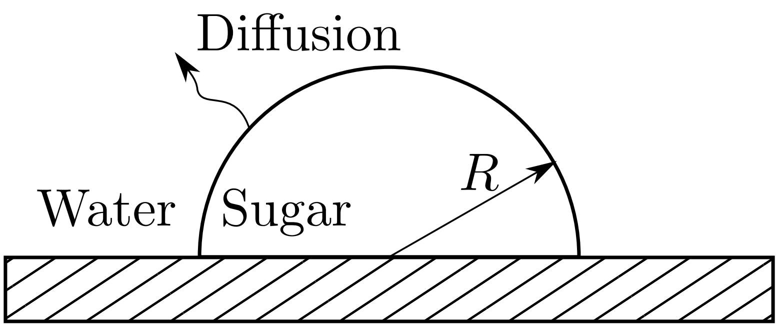


Figure 24: The lump of dissolving sugar.

b) A hemispherical lump of sugar, initially of radius $R = 0.005$ m, is dropped into a cup of tea, quickly coming to rest on the bottom of the cup as shown in Fig. 24. The sugar lump then slowly dissolves into the tea. The diffusion coefficient of sugar in tea is 4×10^{-10} m^2 s^{-1}. The saturation mole fraction of sugar in tea is 0.1 and the total molar density of the system is $c = 55 \times 10^3$ mol m^{-3}.

i) Derive the following differential balance equation for the system.

$$\frac{\partial}{\partial r} r^2 N_{s,r} = 0$$

[5 marks]

Solution:

We start with the general diffusion balance equation. As the sugar lump is dissolving
[1/5] slowly, we can assume it is at quasi steady-state$^{\checkmark}_{1}$.

$$\cancelto{0}{\frac{\partial c_s}{\partial t}} = -\nabla_i N_{s,i}$$

[1/5] We choose a spherical coordinate system due to the symmetry of the system$^{\checkmark}_{1}$. At steady state the time derivative is zero and assume the system is symmetric in the angles θ
[1/5] and $\phi^{\checkmark}_{1}$,

$$\begin{aligned} 0 &= -\nabla_i N_{s,i} \\ &= -\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 N_{s,r}\right) + \frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial}{\partial\theta}}\left(N_{s,\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial}{\partial\phi}} N_{s,\phi}\right) \end{aligned}$$

[2/5] $^{\checkmark}_{2}$This yields the final result:

$$\frac{1}{r^2}\frac{\partial\, r^2 N_{s,r}}{\partial r} = 0$$

ii) Determine the boundary conditions. **[2 marks]**

Solution:

The boundary conditions are

- The concentration is at the precipitation concentration at the surface of the sugar
[1/2] lump ($c_s = 0.1$ at $r = R$)$^{\checkmark}_{1}$.

[1/2]
- We can assume that the concentration is zero at a large distance from the sugar lump, $c_s = 0$ at $r \to \infty \checkmark_1$.

iii) Assuming the tea is stagnant, derive the following expression for the variation of the sugar mole fraction in the water.

$$x_s = 1 - 0.9^{0.005/r}$$

You may need the identity:

$$\int (1-x)^{-1}\,\mathrm{d}x = -\ln(1-x) + C$$

[11 marks]

Solution:

For diffusion through a stationary component we need to use Stefan's law expressed in mole fractions

$$N_{s,r} = -D_{sw}\frac{c}{1-x_s}\frac{\partial x_s}{\partial r}$$

[1/11] $\checkmark_1$Substituting this into the balance equation from the previous question, we have

$$\frac{\partial}{\partial r}\left(r^2\, D_{sw}\frac{c}{1-x_s}\frac{\partial x_s}{\partial r}\right) = 0$$

[1/11] $\checkmark_1$Integrating this equation with respect to r, we have

$$D_{sw}\frac{c}{1-x_s}\frac{\partial x_s}{\partial r} = \frac{C_1}{r^2}$$

[1/11] $\checkmark_1$Integrating again we have

$$D_{sw}\, c\, \ln(1-x_s) = -\frac{C_1}{r} + C_2$$

[2/11] $\checkmark_2$The first boundary condition, as $r \to \infty$ we have $x_s \to 0$. Which gives $C_2 = 0\checkmark_1$.
[1/11] The second boundary condition, at $r = R = 0.005$ we have $x_s = x_{s,sat} = 0.1\checkmark_1$. Rearran-
[1/11] ging the equation we have

$$C_1 = -R\, D_{sw}\, c\, \ln(1-x_{s,sat})$$

[1/11] $\checkmark_1$Substituting in the values, we have

$$\begin{aligned} C_1 &= -0.005 \times 4 \times 10^{-10} \times 55 \times 10^3 \ln(1-0.1) \\ &\approx 1.16 \times 10^{-8} \end{aligned}$$

[1/11] $\checkmark_1$The final solution is given by

$$\begin{aligned} D_{sw}\, c\, \ln(1-x_s) &= \frac{R\, D_{sw}\, c\, \ln(1-x_{s,sat})}{r} \\ \ln(1-x_s) &= \frac{R}{r}\ln(1-x_{s,sat}) \\ x_s &= 1-(1-x_{s,sat})^{R/r} \\ x_s &= 1-0.9^{0.005/r} \end{aligned}$$

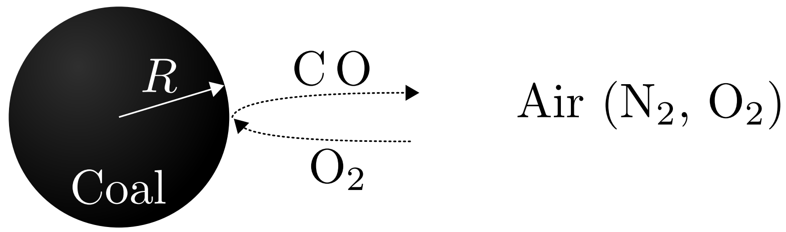
[2/11] $\checkmark_2$

[Question total: 20 marks]

Q.70 Question 70

Consider a spherical coal particle undergoing combustion. Combustion of solids is typically limited by the rate at which oxygen can get to the combusting surface. As the reaction is oxygen limited, we assume that as soon as oxygen reaches the coal surface it is instantly converted to carbon monoxide (CO).

$$2\,C + O_2 \rightarrow 2\,CO$$



You can assume that there is no oxygen at the surface of the coal particle $x_{O_2}(r = R) = 0$, and a oxygen mole fraction of 21% at a large distance from the particle $x_{O_2}(r \rightarrow \infty) = 0.21$. You can also assume steady state conditions, a constant temperature and pressure, and that all gases are ideal gases and mixtures.

a) Specify and simplify the balance equation for the oxygen in this system.

Solution:

The general balance equation, for ANY diffusion problem is

$$\frac{\partial C_A}{\partial t} = -\nabla \cdot \boldsymbol{N}_A + \sigma_A$$

This could be the balance for any of the diffusing species ($A = [\mathrm{O_2}, \mathrm{CO}, \mathrm{N_2}]$), but we'll only need to look at the balance for the oxygen.

$$\frac{\partial C_{O_2}}{\partial t} = -\nabla \cdot \boldsymbol{N}_{O_2} + \sigma_{O_2}$$

We can assume the system is at steady state. We can also state there is no production or consumption of oxygen *in the air*, only at the boundary of the particle. So the generation/consumption of oxygen (σ_{O_2}) is also zero

$$\cancelto{0}{\frac{\partial C_{O_2}}{\partial t}} = -\nabla \cdot \boldsymbol{N}_{O_2} + \cancelto{0}{\sigma_{O_2}}$$

$$\nabla \cdot \boldsymbol{N}_{O_2} = 0$$

We use spherical coordinates as we have a spherical particle. Using spherical coordinates and expanding the dot product above we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 N_{O_2,r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(N_{O_2,\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial N_{O_2,\phi}}{\partial\phi} = 0$$

The particle is symmetric in the θ and ϕ directions, so we can cancel these gradients to give.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 N_{O_2,r}\right) = 0$$

This is the final, simplest balance equation for this system. The steps we went through above are very similar to the steps used to simplify the continuity equation. And, just like in the continuity equation, we almost always make assumptions to reduce it to a single gradient term, as above.

b) Derive the following expression for the oxygen flux.

$$N_{\mathrm{O}_2,r} = -\frac{D\,C_T}{1 + x_{\mathrm{O}_2}} \frac{\partial x_{\mathrm{O}_2}}{\partial r}$$

Solution:

Every flux in *every* diffusion problem is given by the general equation

$$\boldsymbol{N}_A = \boldsymbol{J}_A + x_A \sum_B \boldsymbol{N}_B$$

where $\boldsymbol{J}_A$ is given by Fick's law of diffusion

$$J_{A,x} = -D\frac{\partial C_A}{\partial x}$$

So, considering only oxygen and the flux in the r direction, we have

$$\begin{aligned} N_{\mathrm{O}_2,r} &= J_{\mathrm{O}_2,r} + x_{\mathrm{O}_2} \sum_B N_{B,r} \\ &= -D\frac{\partial C_{\mathrm{O}_2}}{\partial r} + x_{\mathrm{O}_2} \sum_B N_{B,r} \end{aligned}$$

There are three species in the system, O_2, CO, N_2. So the sum on the right can be expanded like so

$$\begin{aligned} N_{\mathrm{O}_2,r} &= -D\frac{\partial C_{\mathrm{O}_2}}{\partial r} + x_{\mathrm{O}_2} \sum_B N_{B,r} \\ &= -D\frac{\partial C_{\mathrm{O}_2}}{\partial r} + x_{\mathrm{O}_2} \left(N_{\mathrm{O}_2,r} + N_{\mathrm{CO},r} + N_{\mathrm{N}_2,r}\right) \end{aligned}$$

In this problem, for every mole of oxygen that reaches the surface, two moles of carbon monoxide are formed.

$$2\,\mathrm{C} + \mathrm{O}_2 \rightarrow 2\,\mathrm{CO}$$

This means that the flux of carbon monoxide must have the opposite sign to the flux of oxygen, and must be twice as large

$$N_{\mathrm{CO},r} = -2\,N_{\mathrm{O}_2,r}$$

The nitrogen is not going anywhere so we have

$$N_{\mathrm{N}_2,r} = 0$$

Substituting this in to the expression for $N_{\mathrm{O}_2,r}$, we have

$$\begin{aligned} N_{\mathrm{O}_2,r} &= -D\frac{\partial C_{\mathrm{O}_2}}{\partial r} + x_{\mathrm{O}_2} \left(N_{\mathrm{O}_2,r} + \cancelto{-2\,N_{\mathrm{O}_2,r}}{N_{\mathrm{CO},r}} + \cancelto{0}{N_{\mathrm{N}_2,r}}\right) \\ &= -D\frac{\partial C_{\mathrm{O}_2}}{\partial r} + x_{\mathrm{O}_2} \left(N_{\mathrm{O}_2,r} - 2\,N_{\mathrm{O}_2,r}\right) \\ N_{\mathrm{O}_2,r} &= -D\frac{\partial C_{\mathrm{O}_2}}{\partial r} - x_{\mathrm{O}_2}\,N_{\mathrm{O}_2,r} \end{aligned}$$

We can rearrange for $N_{O_2,r}$ to give

$$N_{O_2,r} = -\frac{D}{1 + x_{O_2}}\frac{\partial C_{O_2}}{\partial r}$$

The molar concentration is related to the mole fraction by $C_A = x_A\, C_T$, where C_T is the total molar concentration. We assume that the total molar concentration is constant as the temperature and pressure are constant. We can rewrite the equation purely in terms of the mole fraction of oxygen, x_{O_2}

$$\begin{aligned}
N_{O_2,r} &= -\frac{D}{1 + x_{O_2}}\frac{\partial C_{O_2}}{\partial r} \\
&= -\frac{D}{1 + x_{O_2}}\frac{\partial x_{O_2}\, C_T}{\partial r} \\
&= -\frac{D\, C_T}{1 + x_{O_2}}\frac{\partial x_{O_2}}{\partial r}
\end{aligned}$$

This is the final expression. Note that the diffusion is lower by a factor $(1 + x_{O_2})^{-1}$ than just plain Fick's law would give you. This is because the diffusion of carbon monoxide from the surface will hinder the diffusion of oxygen to the surface.

c) Using the expression for the oxygen flux and the balance for the oxygen flux, solve for the concentration profile of oxygen around the particle.
Solution:
Here we have two equations for our system.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\, N_{O_2,r}\right) = 0 \qquad\qquad N_{O_2,r} = -\frac{D\, C_T}{1 + x_{O_2}}\frac{\partial x_{O_2}}{\partial r}$$

We need to solve for the concentration profile, which we can express in terms of x_{O_2}. Just like all of the flow examples, we integrate the balance equation first to find the flux around the particle

$$\begin{aligned}
\frac{1}{r^2}\frac{\partial\, r^2\, N_{O_2,r}}{\partial r} &= 0 \\
\frac{\partial\, r^2\, N_{O_2,r}}{\partial r} &= 0 \\
r^2\, N_{O_2,r} &= C_1 \\
N_{O_2,r} &= \frac{C_1}{r^2}
\end{aligned}$$

Note that the total flux of oxygen into the particle is equal to the flux times by the surface area of a sphere at a radius r, so

$$\text{Total diffusion rate of oxygen onto the particle} = 4\,\pi\, r^2\, N_{O_2,r} = 4\,\pi\, C_1$$

This gives a physical meaning to the constant C_1.

Taking the two expressions for the flux, we have

$$N_{O_2,r} = \frac{C_1}{r^2} = -\frac{D\, C_T}{1 + x_{O_2}}\frac{\partial x_{O_2}}{\partial r}$$

Rearrange both sides ready to integrate and performing the integration we have

$$C_1 \int \frac{1}{r^2} \mathrm{d}r = -D\, C_T \int \frac{1}{1+x_{O_2}} \mathrm{d}x_{O_2}$$
$$-C_1 \frac{1}{r} = -D\, C_T \ln\left(1 + x_{O_2}\right) + C_2$$

Now we need to determine the integration constants using the boundary conditions. As $r \to \infty$ we find $x_{O_2} \to 0.21$, which gives

$$-C_1 \overbrace{\cancel{\frac{1}{\infty}}}^{0} = -D\, C_T \ln(1 + 0.21) + C_2$$
$$C_2 = D\, C_T \ln(1.21)$$

substituting this back in to the previous expression we have

$$-C_1 \frac{1}{r} = D\, C_T \left(\ln(1.21) - \ln\left(1 + x_{O_2}\right)\right)$$

The other boundary condition is at $r = R$, we have $x_{O_2} = 0$ which gives

$$-C_1 \frac{1}{R} = D\, C_T \left(\ln(1.21) - \overbrace{\cancel{\ln(1+0)}}^{0}\right)$$
$$C_1 = -R\, D\, C_T \ln(1.21)$$

Substituting this back in, we have

$$D\, C_T \ln(1.21) \frac{R}{r} = D\, C_T \left(\ln(1.21) - \ln\left(1 + x_{O_2}\right)\right)$$
$$\ln(1.21) \frac{R}{r} = \ln\left(\frac{1.21}{1 + x_{O_2}}\right)$$

We want an expression for the concentration profile, so lets rearrange for x_{O_2}.

$$\ln(1.21) \frac{R}{r} = \ln\left(\frac{1.21}{1 + x_{O_2}}\right)$$
$$\ln\left(1.21^{R/r}\right) = \ln\left(\frac{1.21}{1 + x_{O_2}}\right)$$
$$1.21^{R/r} = \frac{1.21}{1 + x_{O_2}}$$
$$x_{O_2} = \frac{1.21}{1.21^{R/r}} - 1$$
$$x_{O_2} = 1.21^{1-R/r} - 1$$

d) What can you use the information you've derived for?

Solution:

By knowing the rate at which oxygen gets to the particle you can calculate how fast it is burning, from this you can work out.

i) The rate at which you need to add air to the fire.

ii) How much heat is released per second.
iii) How long the coal particle will take to burn.

All of this information is essential if you want to design a coal (or any other solid fuel, e.g., biomass) fired power plant.

Solution:
Extra Notes:

The total concentration of air was calculated using the ideal gas equation of state. If we assume the particle is burning in air at STP, we have

$$\frac{n}{V} = \frac{P}{R\,T} = \frac{10^5}{8.314 \times 293.15} \approx 41 \text{ mol m}^{-3}$$

[Question end]

Q.71 Question 71
Consider a spherical aggregate (or ball) of bacterial cells (assumed to be homogenous) of radius R. Under certain circumstances, the oxygen metabolism rate of the bacterial cells is an almost constant reaction (zero-order) with respect to the oxygen concentration $\sigma_{O_2} = -k_{O_2}$. The diffusion of oxygen within the ball may be described by Fick's law with an effective pseudobinary diffusivity for oxygen in the bacterial medium of D_{O_2-M}. Neglect transient and convection effects because the oxygen solubility is very low in the system. Let $C_{O_2}^{(R)}$ be the oxygen mass concentration at the aggregate surface:

a) Show all of your working and state all assumptions made while demonstrating that the oxygen balance for the system,

$$\frac{\partial C_{O_2}}{\partial t} = -\nabla \cdot \boldsymbol{N}_{O_2} + \sigma_{O_2},$$

simplifies to the following expression,

$$\frac{\partial}{\partial r}\left(r^2 N_{O_2,r}\right) = -k_{O_2}\, r^2.$$

[4 marks]

Solution:
Take the balance for O_2

$$\frac{\partial C_{O_2}}{\partial t} = -\nabla \cdot \boldsymbol{N}_{O_2} + \sigma_{O_2}$$

[1/4] ✓$_1$ Neglecting transient effects (assuming steady state), and inserting the spherical definition of $\nabla \cdot \boldsymbol{N}_{O_2}$ gives

$$\nabla \cdot \boldsymbol{N}_{O_2} = \sigma_{O_2}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 N_{O_2,r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(N_{O_2,\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial N_{O_2,\phi}}{\partial\phi} = \sigma_{O_2}$$

[2/4] ✓$_2$ Assuming the system is rotationally symmetric, we have $\frac{\partial}{\partial\phi} = 0$ and $\frac{\partial}{\partial\theta} = 0$, all terms except the first are zero. Inserting this back in along with the definition of σ_{O_2} gives:

$$\frac{\partial}{\partial r}\left(r^2 N_{O_2,r}\right) = -k_{O_2}\, r^2$$

[1/4] ✓$_1$

b) Demonstrate that the oxygen flux obeys the following relationship:

$$N_{O_2,r} = -k_{O_2} R^2 \left(\frac{r}{3R^2} + \frac{C_1 R}{6r^2}\right)$$

where C_1 is an unknown constant. **[4 marks]**

Solution:

Integrating the previous eqation

$$\frac{\partial}{\partial r}\left(r^2 N_{O_2,r}\right) = -k_{O_2} r^2$$
$$r^2 N_{O_2,r} = -k_{O_2}\frac{r^3}{3} + C_1'$$
$$N_{O_2,r} = -k_{O_2}\frac{r}{3} + \frac{C_1'}{r^2}$$

[1/4] $\checkmark_1$ To match the solution, some rearrangement is needed.

$$N_{O_2,r} = -k_{O_2} R^2 \left(\frac{r}{3R^2} - \frac{C_1'}{r^2 R^2 k_{O_2}}\right)$$

[1/4] $\checkmark_1$ The final step is to redefine the integration constant C_1' in terms of another unknown
[1/4] constant C_1, like so $C_1' = -k_{O_2} C_1 R^3/6\checkmark_1$, giving the final solution

$$N_{O_2,r} = -k_{O_2} R^2 \left(\frac{r}{3R^2} + \frac{C_1 R}{6r^2}\right).$$

[1/4] $\checkmark_1$

c) Demonstrate that the concentration profile obeys the following form in the limit that the O_2 concentration is small:

$$C_{O_2} = \frac{k_{O_2} R^2}{6 D_{O_2-M}}\left(\frac{r^2}{R^2} - C_1\frac{R}{r}\right) + C_2$$

[4 marks]

Solution:

From the datasheet, we have the definition of the diffusive flux:

$$\boldsymbol{N}_{O_2} = \boldsymbol{J}_{O_2} + \cancelto{0}{x_{O_2}}\sum_B \boldsymbol{N}_B$$

[1/4] where the last term cancels due to the low concentration assumption. $\checkmark_1$ Inserting Fick's law from the datasheet (which assumes a ideal mixture):

$$\boldsymbol{N}_{O_2} = \boldsymbol{J}_{O_2} = -D_{O_2-M}\nabla C_{O_2}$$

In particular, for the r-direction, $N_{O_2,r}$ is:

$$N_{O_2,r} = -D_{O_2-M}\frac{\partial C_{O_2}}{\partial r}$$

[1/4] $\checkmark_1$ Inserting this into the balance from the previous equation we have:

$$D_{O_2-M}\frac{\partial C_{O_2}}{\partial r} = k_{O_2} R^2 \left(\frac{r}{3R^2} + \frac{C_1 R}{6r^2}\right)$$

[1/4] $\checkmark_1$ Performing the integral

$$\frac{\partial C_{O_2}}{\partial r} = \frac{k_{O_2}\,R^2}{D_{O_2-M}}\left(\frac{r}{3\,R^2}+\frac{C_1\,R}{6\,r^2}\right)$$
$$C_{O_2} = \frac{k_{O_2}\,R^2}{D_{O_2-M}}\left(\frac{r^2}{6\,R^2}-\frac{C_1\,R}{6\,r}\right)+C_2$$
$$C_{O_2} = \frac{k_{O_2}\,R^2}{6\,D_{O_2-M}}\left(\frac{r^2}{R^2}-C_1\frac{R}{r}\right)+C_2$$

[1/4] $\checkmark_1$

d) Using the only available boundary condition, determine C_2 and demonstrate that the final expression for the concentration is:

$$C_{O_2} = \frac{k_{O_2}\,R^2}{6\,D_{O_2-M}}\left(\frac{r^2}{R^2}+C_1\left(1-\frac{R}{r}\right)-1\right)+C_{O_2}^{(R)}$$

[4 marks]

Solution:

[1/4] For $r = R$ we have $C_{O_2} = C_{O_2}^{(R)}$. $\checkmark_1$

$$C_{O_2}^{(R)} = \frac{k_{O_2}\,R^2}{6\,D_{O_2-M}}\left(1-C_1\right)+C_2$$
$$C_2 = \frac{k_{O_2}\,R^2}{6\,D_{O_2-M}}\left(C_1-1\right)+C_{O_2}^{(R)}$$

[2/4] $\checkmark_2$ which gives

$$C_{O_2} = \frac{k_{O_2}\,R^2}{6\,D_{O_2-M}}\left(\frac{r^2}{R^2}+C_1\left(1-\frac{R}{r}\right)-1\right)+C_{O_2}^{(R)}$$

[1/4] $\checkmark_1$

e) It is possible that the spherical bacterial ball has an oxygen-free core ($C_{O_2} = 0$ for $r \leq r_{core}$). Prove that this only happens for:

$$\frac{k_{O_2}\,R^2}{D_{O_2-M}\,C_{O_2}^{(R)}} \geq 6$$

Hints: As the concentration and diffusive flux are continuous, they both must go to zero at the core radius r_{core}. Use this to solve for C_1, then solve for r_{core} and consider what is required if $r_{core} \geq 0$. **[4 marks]**

Solution:

We have two equations:

$$N_{O_2,r} = -k_{O_2}\,R^2\left(\frac{r}{3\,R^2}+\frac{C_1\,R}{6\,r^2}\right)$$
$$C_{O_2} = \frac{k_{O_2}\,R^2}{6\,D_{O_2-M}}\left(\frac{r^2}{R^2}+C_1\left(1-\frac{R}{r}\right)-1\right)+C_{O_2}^{(R)}$$

[1/4] $\checkmark_1$ At the core radius, we have from the flux equation:

$$\cancelto{0}{N_{O_2,r}} = -k_{O_2}\,R^2\left(\frac{r_{core}}{3\,R^2} + \frac{C_1\,R}{6\,r_{core}^2}\right)$$
$$C_1 = -\frac{2\,r_{core}^3}{R^3}$$

From the concentration equation we have

$$\cancelto{0}{C_{O_2}} = \frac{k_{O_2}\,R^2}{6\,D_{O_2-M}}\left(\frac{r_{core}^2}{R^2} + C_1\left(1 - \frac{R}{r_{core}}\right) - 1\right) + C_{O_2}^{(R)}$$
$$1 - \frac{6\,D_{O_2-M}\,C_{O_2}^{(R)}}{k_{O_2}\,R^2} = \frac{r_{core}^2}{R^2} + C_1\left(1 - \frac{R}{r_{core}}\right)$$

[1/4] $\checkmark_1$ Inserting the definition of C_1, we have

$$1 - \frac{6\,D_{O_2-M}\,C_{O_2}^{(R)}}{k_{O_2}\,R^2} = 3\frac{r_{core}^2}{R^2} - \frac{2\,r_{core}^3}{R^3}$$
$$= \frac{r_{core}^2}{R^2}\left(3 - 2\frac{r_{core}}{R}\right)$$

[1/4] $\checkmark_1$ If $0 < r_{core}/R \leq 1$ (which it must be if it exists) then the right hand side of the equation must be positive. For the left hand side to be equally positive, we must have:

$$\frac{k_{O_2}\,R^2}{D_{O_2-M}\,C_{O_2}^{(R)}} \geq 6$$

[1/4] $\checkmark_1$

[Question total: 20 marks]

DATASHEET

General balance equations:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho\, \boldsymbol{v} \qquad \text{(Mass/Continuity)} \qquad (65)$$

$$\frac{\partial C_A}{\partial t} = -\nabla \cdot \boldsymbol{N}_A + \sigma_A \qquad \text{(Species)} \qquad (66)$$

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} = -\rho\, \boldsymbol{v} \cdot \nabla \boldsymbol{v} - \nabla \cdot \boldsymbol{\tau} - \nabla p + \rho\, \boldsymbol{g} \qquad \text{(Momentum)} \qquad (67)$$

$$\rho\, C_p \frac{\partial T}{\partial t} = -\rho\, C_p\, \boldsymbol{v} \cdot \nabla T - \nabla \cdot \boldsymbol{q} - \boldsymbol{\tau} : \nabla \boldsymbol{v} - p \nabla \cdot \boldsymbol{v} + \sigma_{energy} \qquad \text{(Heat/Energy)} \qquad (68)$$

In Cartesian coordinate systems, ∇ can be treated as a vector of derivatives. In curvelinear coordinate systems, the directions $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\phi}$ depend on the position. For convenience in these systems, look-up tables are provided for common terms involving ∇.

Cartesian coordinates (with index notation examples)
where s is a scalar, $\boldsymbol{v}$ is a vector, and $\boldsymbol{\tau}$ is a tensor.

$$\nabla s = \nabla_i s = \left[\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}, \frac{\partial s}{\partial z}\right]$$

$$\nabla^2 s = \nabla_i \nabla_i s = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2}$$

$$\nabla \cdot \boldsymbol{v} = \nabla_i v_i = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\nabla \cdot \boldsymbol{\tau} = \nabla_i \tau_{ij}$$

$$[\nabla \cdot \boldsymbol{\tau}]_x = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$[\nabla \cdot \boldsymbol{\tau}]_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$

$$[\nabla \cdot \boldsymbol{\tau}]_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} = v_i \nabla_i v_j$$

$$[\boldsymbol{v} \cdot \nabla \boldsymbol{v}]_x = v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}$$

$$[\boldsymbol{v} \cdot \nabla \boldsymbol{v}]_y = v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}$$

$$[\boldsymbol{v} \cdot \nabla \boldsymbol{v}]_z = v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}$$

Cylindrical coordinates
where s is a scalar, $\boldsymbol{v}$ is a vector, and $\boldsymbol{\tau}$ is a tensor. All expressions involving $\boldsymbol{\tau}$ are for symmetrical $\boldsymbol{\tau}$ only.

$$\nabla s = \left[\frac{\partial s}{\partial r}, \frac{1}{r}\frac{\partial s}{\partial \theta}, \frac{\partial s}{\partial z}\right]$$

$$\nabla^2 s = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 s}{\partial \theta^2} + \frac{\partial^2 s}{\partial z^2}$$

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r}\frac{\partial}{\partial r}(r\, v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

$$[\nabla \cdot \boldsymbol{\tau}]_r = \frac{1}{r}\frac{\partial}{\partial r}(r\, \tau_{rr}) + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{1}{r}\tau_{\theta\theta} + \frac{\partial \tau_{rz}}{\partial z}$$

$$[\nabla \cdot \boldsymbol{\tau}]_\theta = \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2}{r}\tau_{r\theta} + \frac{\partial \tau_{\theta z}}{\partial z}$$

$$[\nabla \cdot \boldsymbol{\tau}]_z = \frac{1}{r}\frac{\partial}{\partial r}(r\, \tau_{rz}) + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}$$

$$[\boldsymbol{v} \cdot \nabla \boldsymbol{v}]_r = v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}$$

$$[\boldsymbol{v} \cdot \nabla \boldsymbol{v}]_\theta = v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r\, v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}$$

$$[\boldsymbol{v} \cdot \nabla \boldsymbol{v}]_z = v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}$$

Spherical coordinates
where s is a scalar, $\boldsymbol{v}$ is a vector, and $\boldsymbol{\tau}$ is a tensor. All expressions involving $\boldsymbol{\tau}$ are for symmetrical $\boldsymbol{\tau}$ only.

$$\nabla s = \left[\frac{\partial s}{\partial r}, \frac{1}{r}\frac{\partial s}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial s}{\partial \phi}\right]$$

$$\nabla^2 s = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial s}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial s}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 s}{\partial \phi^2}$$

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\, v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

$$[\nabla \cdot \boldsymbol{\tau}]_r = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\, \tau_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\tau_{r\theta}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial \tau_{r\phi}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r}$$

$$[\nabla \cdot \boldsymbol{\tau}]_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\, \tau_{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\tau_{\theta\theta}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial \tau_{\theta\phi}}{\partial \phi} + \frac{\tau_{r\theta}}{r} - \frac{\cot\theta}{r}\tau_{\phi\phi}$$

$$[\nabla \cdot \boldsymbol{\tau}]_\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\, \tau_{r\phi}\right) + \frac{1}{r}\frac{\partial \tau_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{\tau_{r\theta}}{r} + \frac{2\cot\theta}{r}\tau_{\theta\phi}$$

$$[\boldsymbol{v} \cdot \nabla \boldsymbol{v}]_r = v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}$$

$$[\boldsymbol{v} \cdot \nabla \boldsymbol{v}]_\theta = v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r\, v_\theta - v_\phi^2\cot\theta}{r}$$

$$[\boldsymbol{v} \cdot \nabla \boldsymbol{v}]_\phi = v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r\, v_\phi + v_\theta\, v_\phi\cot\theta}{r}$$

Rectangular		Cylindrical		Spherical	
q_x	$-k\frac{\partial T}{\partial x}$	q_r	$-k\frac{\partial T}{\partial r}$	q_r	$-k\frac{\partial T}{\partial r}$
q_y	$-k\frac{\partial T}{\partial y}$	q_θ	$-k\frac{1}{r}\frac{\partial T}{\partial \theta}$	q_θ	$-k\frac{1}{r}\frac{\partial T}{\partial \theta}$
q_z	$-k\frac{\partial T}{\partial z}$	q_z	$-k\frac{\partial T}{\partial z}$	q_ϕ	$-k\frac{1}{r\,\sin\theta}\frac{\partial T}{\partial \phi}$
τ_{xx}	$-2\,\mu\frac{\partial v_x}{\partial x}+\mu^B\,\nabla\cdot\boldsymbol{v}$	τ_{rr}	$-2\,\mu\frac{\partial v_r}{\partial r}+\mu^B\,\nabla\cdot\boldsymbol{v}$	τ_{rr}	$-2\,\mu\frac{\partial v_r}{\partial r}+\mu^B\,\nabla\cdot\boldsymbol{v}$
τ_{yy}	$-2\,\mu\frac{\partial v_y}{\partial y}+\mu^B\,\nabla\cdot\boldsymbol{v}$	$\tau_{\theta\theta}$	$-2\,\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right)+\mu^B\,\nabla\cdot\boldsymbol{v}$	$\tau_{\theta\theta}$	$-2\,\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right)+\mu^B\,\nabla\cdot\boldsymbol{v}$
τ_{zz}	$-2\,\mu\frac{\partial v_z}{\partial z}+\mu^B\,\nabla\cdot\boldsymbol{v}$	τ_{zz}	$-2\,\mu\frac{\partial v_z}{\partial z}+\mu^B\,\nabla\cdot\boldsymbol{v}$	$\tau_{\phi\phi}$	$-2\,\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_r+v_\theta\cot\theta}{r}\right)$ $+\mu^B\,\nabla\cdot\boldsymbol{v}$
τ_{xy}	$-\mu\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)$	$\tau_{r\theta}$	$-\mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)$	$\tau_{r\theta}$	$-\mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)$
τ_{yz}	$-\mu\left(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right)$	$\tau_{\theta z}$	$-\mu\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta}+\frac{\partial v_\theta}{\partial z}\right)$	$\tau_{\theta\phi}$	$-\mu\left(\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right)$
τ_{xz}	$-\mu\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)$	τ_{zr}	$-\mu\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)$	$\tau_{\phi r}$	$-\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi}+r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right)$

Table 4: Fourier's law for the heat flux and Newton's law for the stress in several coordinate systems. Please remember that the stress is symmetric, so $\tau_{ij}=\tau_{ji}$.

Viscous models:
Power-Law Fluid:

$$\left|\tau_{xy}\right| = k\left|\frac{\partial v_x}{\partial y}\right|^n \tag{69}$$

Bingham-Plastic Fluid:

$$\frac{\partial v_x}{\partial y} = \begin{cases} -\mu^{-1}\left(\tau_{xy}-\tau_0\right) & \text{if } \tau_{xy} > \tau_0 \\ 0 & \text{if } \tau_{xy} \leq \tau_0 \end{cases}$$

Dimensionless Numbers

$$\mathrm{Re} = \frac{\rho\,\langle v\rangle\,D}{\mu} \qquad \mathrm{Re}_H = \frac{\rho\,\langle v\rangle\,D_H}{\mu} \qquad \mathrm{Re}_{MR} = -\frac{16\,L\,\rho\,\langle v\rangle^2}{R\,\Delta p} \tag{70}$$

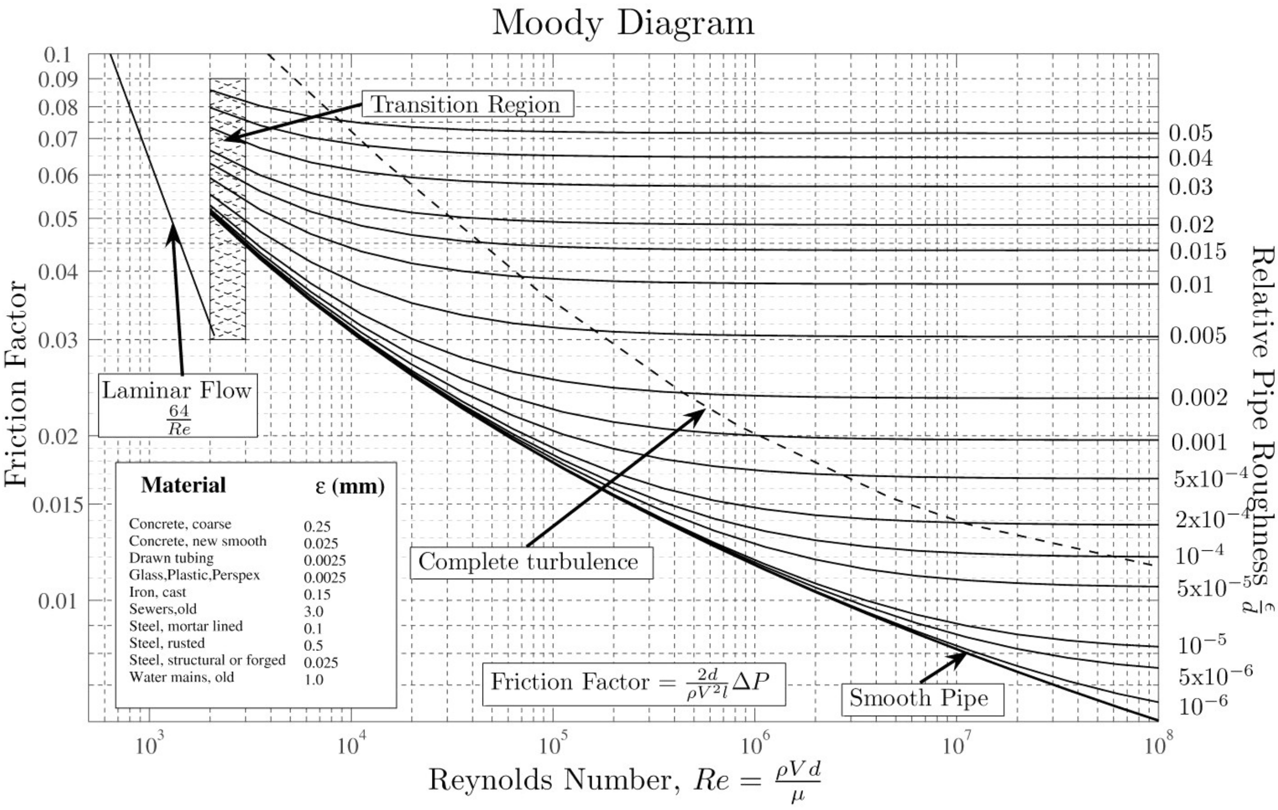
The hydraulic diameter is defined as $D_H = 4\,A/P_w$.
Single phase pressure drop calculations in pipes:
Darcy-Weisbach equation:

$$\frac{\Delta p}{L} = -\frac{C_f\,\rho\,\langle v\rangle^2}{R} \tag{71}$$

where $C_f = 16/Re$ for laminar Newtonian flow. For turbulent flow of Newtonian fluids in smooth pipes, we have the Blasius correlation:

$$C_f = 0.079\,\mathrm{Re}^{-1/4} \qquad \text{for } 2.5\times10^3 < \mathrm{Re} < 10^5 \text{ and smooth pipes.}$$

Otherwise, you may refer to the Moody diagram.



Laminar Power-Law fluid:

$$\dot{V} = \frac{n\,\pi\,R^3}{3\,n+1}\left(\frac{R}{2\,k}\right)^{\frac{1}{n}}\left(-\frac{\Delta p}{L}\right)^{\frac{1}{n}}$$

Two-Phase Flow:
Lockhart-Martinelli parameter:

$$X^2 = \frac{\Delta p_{liq.-only}}{\Delta p_{gas-only}}$$

Pressure drop calculation:

$$\Delta p_{two-phase} = \Phi^2_{liq.}\,\Delta p_{liq.-only} = \Phi^2_{gas}\,\Delta p_{gas-only}$$

Chisholm's relation:

$$\Phi^2_{gas} = 1 + c\,X + X^2$$

$$\Phi^2_{liq.} = 1 + \frac{c}{X} + \frac{1}{X^2} \qquad c = \begin{cases} 20 & \text{turbulent liquid \& turbulent gas} \\ 12 & \text{laminar liquid \& turbulent gas} \\ 10 & \text{turbulent liquid \& laminar gas} \\ 5 & \text{laminar liquid \& laminar gas} \end{cases}$$

Farooqi and Richardson expression for liquid hold-up in co-current flows of Newtonian fluids and air in horizontal pipes:

$$h = \begin{cases} 0.186 + 0.0191\,X & 1 < X < 5 \\ 0.143\,X^{0.42} & 5 < X < 50 \\ 1/(0.97 + 19/X) & 50 < X < 500 \end{cases}$$

Heat Transfer Dimensionless numbers:

$$\mathrm{Nu} = \frac{h\,L}{k} \qquad \mathrm{Pr} = \frac{\mu\,C_p}{k} \qquad \mathrm{Gr} = \frac{g\,\beta\,(T_w - T_\infty)\,L^3}{\nu^2}$$

where $\beta = V^{-1}(\partial V/\partial T)$.

Heat transfer: Resistances

$$Q = U_T\,A_T\,\Delta T = R_T^{-1}\,\Delta T$$

	Conduction Shell Resistances			Radiation
	Rect.	Cyl.	Sph.	
R	$\frac{X}{k\,A}$	$\frac{\ln(R_{outer}/R_{inner})}{2\,\pi\,L\,k}$	$\frac{R_{inner}^{-1} - R_{outer}^{-1}}{4\,\pi\,k}$	$\left[A\,\varepsilon\,\sigma\left(T_j^2 + T_i^2\right)\left(T_j + T_i\right)\right]^{-1}$

Radiation Heat Transfer:

Stefan-Boltzmann constant $\sigma = 5.6703 \times 10^{-8}$ W m^{-2} K^{-4}.
Summation relationship, $\sum_j F_{i\to j} = 1$, and reciprocity relationship, $F_{i\to j}\,A_i = F_{j\to i}\,A_j$. Radiation shielding factor $1/(N+1)$.

$$Q_{rad.,i\to j} = \sigma\,\varepsilon\,F_{i\to j}\,A_i\left(T_j^4 - T_i^4\right) = h_{rad.}\,A\,(T_\infty - T_w)$$

Natural Convection

$\mathrm{Ra} = \mathrm{Gr}\,\mathrm{Pr}$	C	m
$< 10^4$	1.36	1/5
10^4–10^9	0.59	1/4
$> 10^9$	0.13	1/3

Table 5: Natural convection coefficients for isothermal vertical plates in the empirical relation $\mathrm{Nu} \approx C\,(\mathrm{Gr}\,\mathrm{Pr})^m$.

For isothermal vertical cylinders, the above expressions for isothermal vertical plates may be used but must be scaled by a factor, F (i.e., $\mathrm{Nu}_{v.cyl.} = F\,\mathrm{Nu}_{v.plate}$):

$$F = \begin{cases} 1 & \text{for } (D/H) \geq 35\,\mathrm{Gr}_H^{-1/4} \\ 1.3\left[H\,D^{-1}\,\mathrm{Gr}_D^{-1}\right]^{1/4} + 1 & \text{for } (D/H) < 35\,\mathrm{Gr}_H^{-1/4} \end{cases}$$

where D is the diameter and H is the height of the cylinder. The subscript on Gr indicates which length is to be used as the critical length to calculate the Grashof number.
Churchill and Chu expression for natural convection from a horizontal pipe:

$$\mathrm{Nu}^{1/2} = 0.6 + 0.387\left\{\frac{\mathrm{Gr}\,\mathrm{Pr}}{\left[1 + (0.559/\mathrm{Pr})^{9/16}\right]^{16/9}}\right\}^{1/6} \qquad \text{for } 10^{-5} < \mathrm{Gr}\,\mathrm{Pr} < 10^{12}$$

Forced Convection:
Laminar flows:

$$\mathrm{Nu} \approx 0.332\,\mathrm{Re}^{1/2}\,\mathrm{Pr}^{1/3}$$

Well-Developed turbulent flows in smooth pipes:

$$\mathrm{Nu} \approx \frac{(C_f/2)\mathrm{Re}\,\mathrm{Pr}}{1.07 + 12.7(C_f/2)^{1/2}\left(\mathrm{Pr}^{2/3} - 1\right)}\left(\frac{\mu_b}{\mu_w}\right)^{0.14}$$

Boiling:
Forster-Zuber pool-boiling coefficient:

$$h_{nb} = 0.00122\frac{k_L^{0.79}\,C_{p,L}^{0.45}\,\rho_L^{0.49}}{\gamma^{0.5}\,\mu_L^{0.29}\,h_{fg}^{0.24}\,\rho_G^{0.24}}\,(T_w - T_{sat})^{0.24}\,(p_w - p_{sat})^{0.75}$$

Mostinski correlations:

$$h_{nb} = 0.104\,p_c^{0.69}\,q^{0.7}\left[1.8\left(\frac{p}{p_c}\right)^{0.17} + 4\left(\frac{p}{p_c}\right)^{1.2} + 10\left(\frac{p}{p_c}\right)^{10}\right]$$

$$q_c = 3.67 \times 10^4\,p_c\left(\frac{p}{p_c}\right)^{0.35}\left[1 - \frac{p}{p_c}\right]^{0.9}$$

(**Note**: for the Mostinski correlations, the pressures are in units of bar)
Condensing:
Horizontal pipes

$$h = 0.72\left(\frac{k^3\,\rho^2\,g_x\,E_{latent}}{D\,\mu\,(T_w - T_\infty)}\right)^{1/4}$$

Lumped capacitance method:

$$\mathrm{Bi} = \frac{h\,L_c}{\kappa}$$

$$L_c = V/A \qquad \text{for } \mathrm{Bi} < 0.1$$

$$\frac{T(t)-T_\infty}{T_0-T_\infty}=e^{-bt} \qquad b=\frac{hA_s}{\rho V C_p}$$

1-D Transient Heat Conduction:

$$Fo=\frac{\alpha\Delta t}{(\Delta x)^2}=\tau, \qquad \alpha=\kappa\left(\rho C_p\right)^{-1}$$

$$\theta_{\text{wall}}=\frac{T(x,t)-T_\infty}{T_i-T_\infty}=A_1e^{-\lambda_1^2\tau}\cos\left(\frac{\lambda_1 x}{L}\right), \qquad \theta_{\text{cyl}}=\frac{T(r,t)-T_\infty}{T_i-T_\infty}=A_1e^{-\lambda_1^2\tau}\mathbf{J_0}\left(\frac{\lambda_1 r}{r_0}\right)$$

$$\theta_{\text{sph}}=\frac{T(r,t)-T_\infty}{T_i-T_\infty}=A_1e^{-\lambda_1^2\tau}\frac{\sin\left(\frac{\lambda_1 r}{r_0}\right)}{\frac{\lambda_1 r}{r_0}}$$

$$\theta_{0,\text{wall}}=\theta_{0,\text{cyl}}=\theta_{0,\text{sph}}=\frac{T_0-T_\infty}{T_i-T_\infty}=A_1e^{-\lambda_1^2\tau}$$

$$\left(\frac{\mathcal{Q}}{\mathcal{Q}_{\max}}\right)_{\text{wall}}=1-\theta_{0,\text{wall}}\frac{\sin\lambda_1}{\lambda_1}, \qquad \left(\frac{\mathcal{Q}}{\mathcal{Q}_{\max}}\right)_{\text{cyl}}=1-2\theta_{0,\text{cyl}}\frac{\mathbf{J_1}(\lambda_1)}{\lambda_1}$$

$$\left(\frac{\mathcal{Q}}{\mathcal{Q}_{\max}}\right)_{\text{sph}}=1-3\theta_{0,\text{sph}}\frac{\sin\lambda_1-\lambda_1\cos\lambda_1}{\lambda_1^3}$$

Finite-Difference Method:

$$\frac{\partial}{\partial t}(\rho\phi)+\frac{\partial}{\partial x}(\rho\boldsymbol{v}\phi)=\nabla\cdot(\Gamma\nabla\phi)+\mathcal{S} \qquad \text{(1D transport equation)}$$

$$\left(\frac{d\phi}{dx}\right)_i=\frac{\phi_{i+1}-\phi_{i-1}}{2\Delta x} \quad \text{and} \quad \left(\frac{d^2\phi}{dx^2}\right)_i=\frac{\phi_{i-1}+\phi_{i+1}-2\phi_i}{(\Delta x)^2}$$

$$T_i^{j+1}=(1-2\tau)\,T_i^j+\tau\left(T_{i+1}^j+T_{i-1}^j\right)+\frac{\tau(\Delta x)^2}{\kappa}\mathcal{S}_i^j$$

Overall Heat Transfer Coefficient:

$$\dot{\mathcal{Q}}=\frac{\Delta T}{\mathcal{R}}=UA\Delta T=U_iA_i\Delta T=U_oA_o\Delta T$$

$$\mathcal{R}=R_i+R_{\text{wall}}+R_o=\frac{1}{h_iA_i}+\frac{\ln D_o/D_i}{2\pi\kappa L}+\frac{1}{h_oA_o}$$

Fouling Factor:

$$\mathcal{R}=\frac{1}{h_iA_i}+\frac{R_{\text{f,i}}}{A_i}+R_{\text{wall}}+\frac{R_{\text{f,o}}}{A_o}+\frac{1}{h_oA_o}$$

LMTD Method:

$$\dot{Q} = UA_s \Delta T_{\text{lm}} \quad \text{with} \quad \Delta T_{\text{lm}} = \frac{\Delta T_2 - \Delta T_1}{\ln \frac{\Delta T_2}{\Delta T_1}} = \frac{\Delta T_1 - \Delta T_2}{\ln \frac{\Delta T_1}{\Delta T_2}}$$

$$\text{Parallel flows:} \begin{cases} \Delta T_1 = T_{\text{hot,in}} - T_{\text{cold,in}} \\ \Delta T_2 = T_{\text{hot,out}} - T_{\text{cold,out}} \end{cases}$$

$$\text{Counter flows:} \begin{cases} \Delta T_1 = T_{\text{hot,in}} - T_{\text{cold,out}} \\ \Delta T_2 = T_{\text{hot,out}} - T_{\text{cold,in}} \end{cases}$$

ϵ-NTU Method:

$$\epsilon = \frac{\dot{Q}}{\dot{Q}_{\text{max}}}, \quad \text{with } \dot{Q}_{\text{max}} = C_{\text{min}} \left(T_{\text{hot,in}} - T_{\text{cold,in}} \right) \quad \text{and} \quad C_{\text{min}} = Min \left\{ \dot{m}_{\text{hot}} C_{p,\text{hot}}, \dot{m}_{\text{cold}} C_{p,\text{cold}} \right\}$$

$$\text{NTU} = \frac{UA_s}{C_{\text{min}}}$$

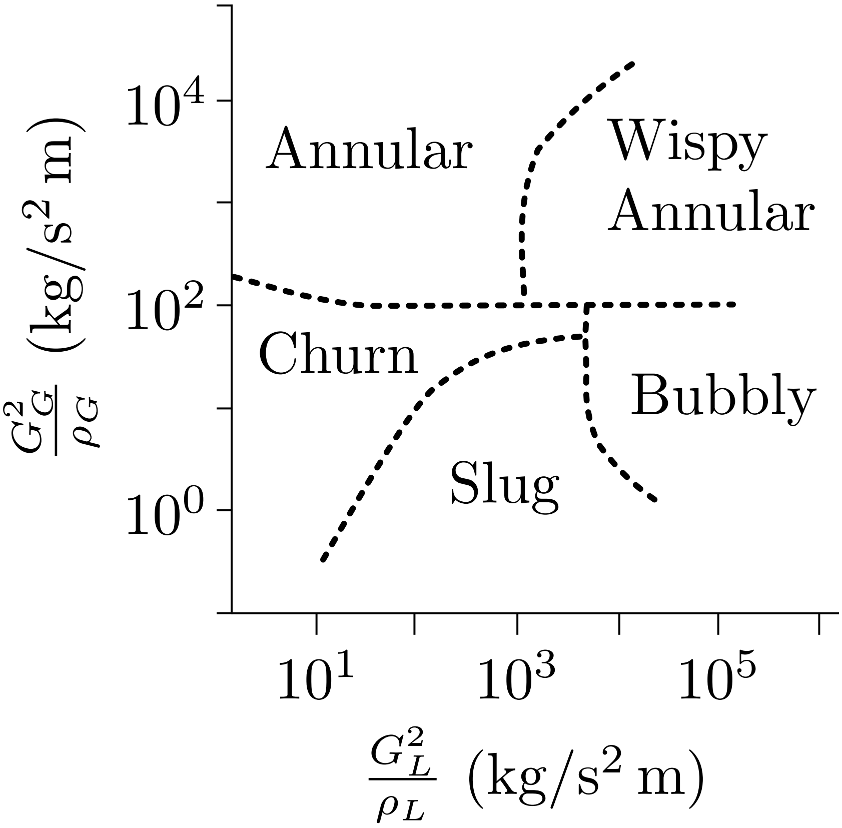


Figure 25: Hewitt-Taylor flow pattern map for multiphase flows in vertical pipes.

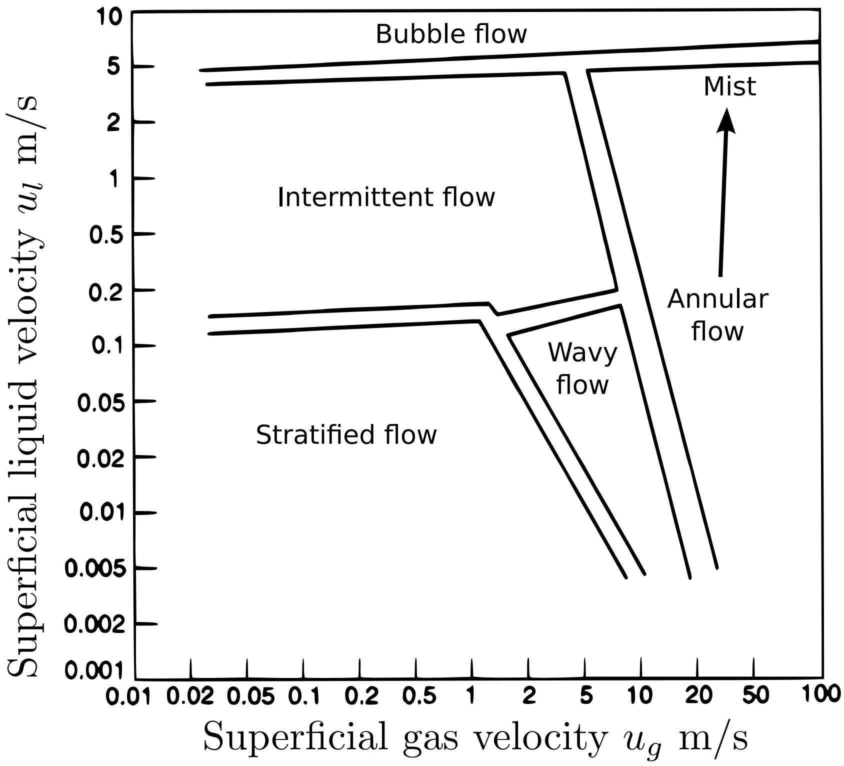


Figure 26: Chhabra and Richardson flow pattern map for horizontal pipes.

TABLE 4–2

Coefficients used in the one-term approximate solution of transient one-dimensional heat conduction in plane walls, cylinders, and spheres (Bi = hL/k for a plane wall of thickness $2L$, and Bi = hr_o/k for a cylinder or sphere of radius r_o)

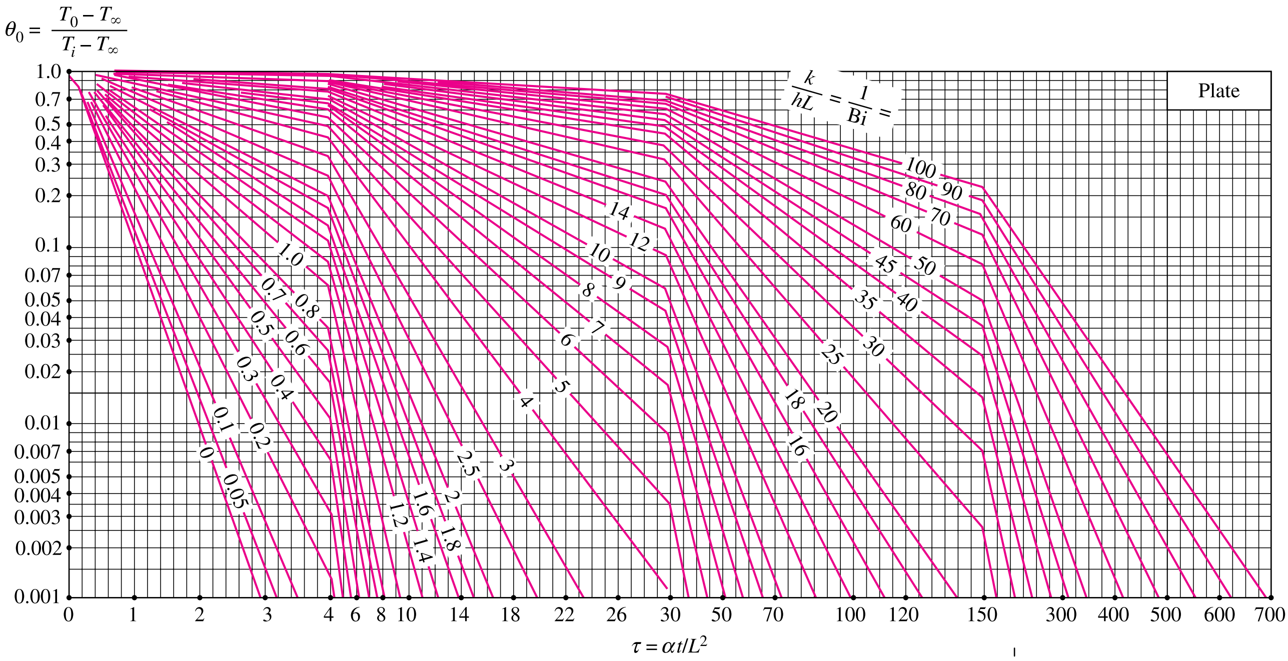
	Plane Wall		Cylinder		Sphere	
Bi	λ_1	A_1	λ_1	A_1	λ_1	A_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.3	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0931	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0918	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2403	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2880	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2732	2.4048	1.6021	3.1416	2.0000

TABLE 4–3

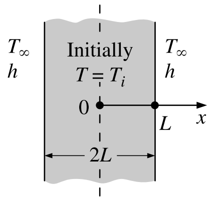
The zeroth- and first-order Bessel functions of the first kind

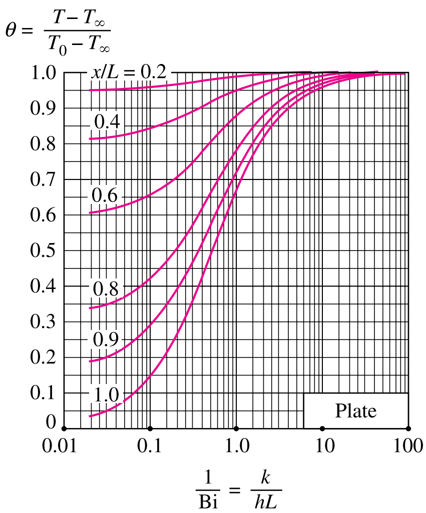
η	$J_0(\eta)$	$J_1(\eta)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419
1.5	0.5118	0.5579
1.6	0.4554	0.5699
1.7	0.3980	0.5778
1.8	0.3400	0.5815
1.9	0.2818	0.5812
2.0	0.2239	0.5767
2.1	0.1666	0.5683
2.2	0.1104	0.5560
2.3	0.0555	0.5399
2.4	0.0025	0.5202
2.6	−0.0968	−0.4708
2.8	−0.1850	−0.4097
3.0	−0.2601	−0.3391
3.2	−0.3202	−0.2613

Figure 27: Coefficients for the 1D transient equations.

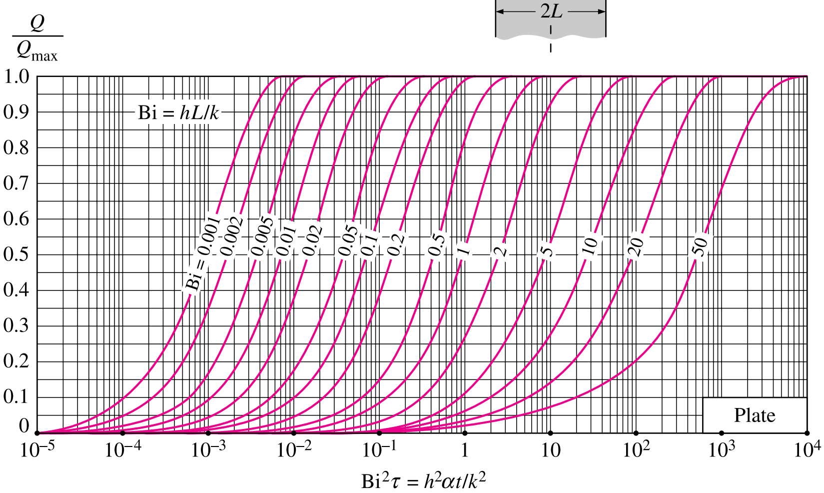


(a) Midplane temperature (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME 69*, 1947, pp. 227–36. Reprinted by permission of ASME International.)





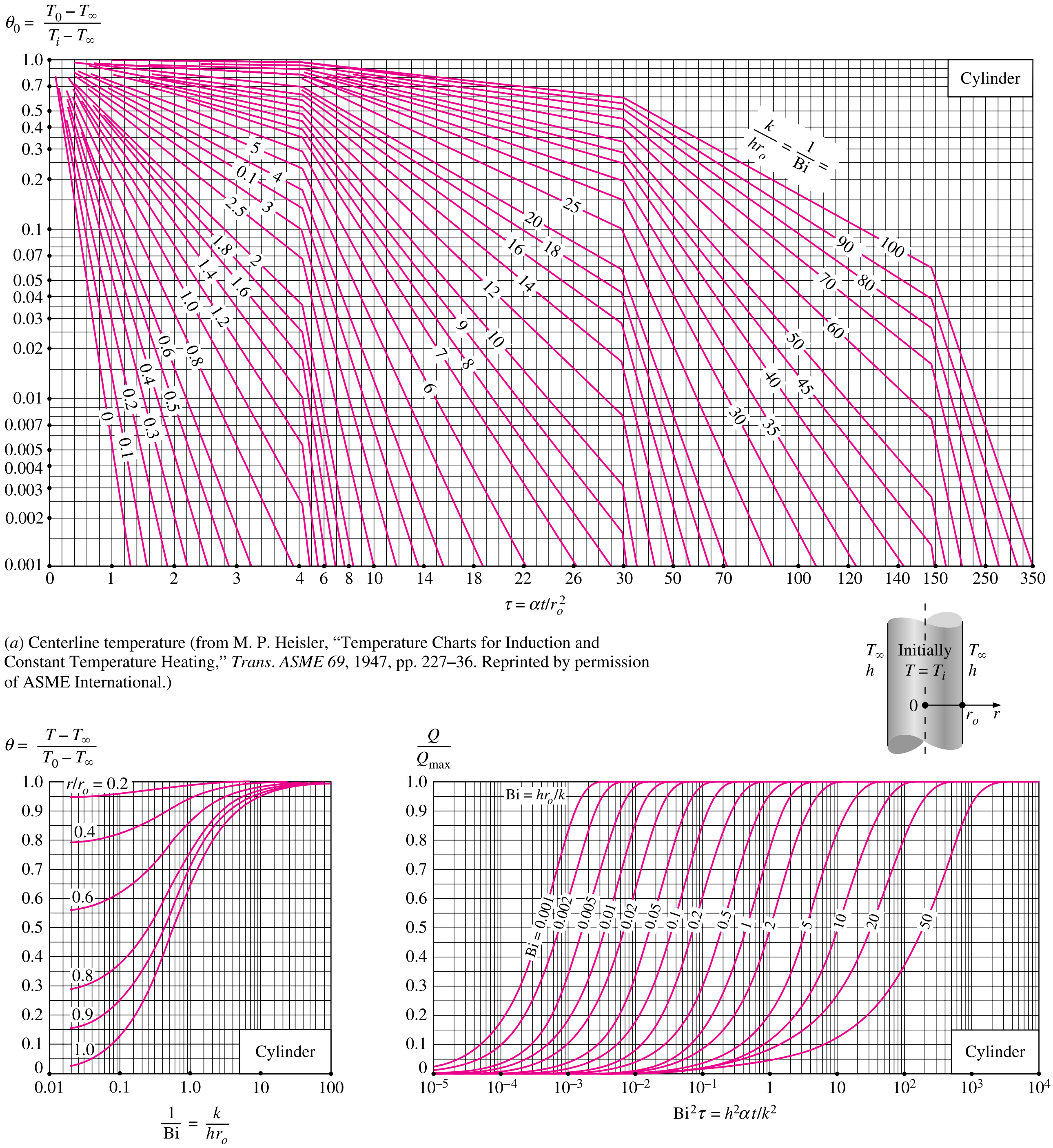
(b) Temperature distribution (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME 69*, 1947, pp. 227–36. Reprinted by permission of ASME International.)



(c) Heat transfer (from H. Gröber et al.)

Transient temperature and heat transfer charts for a plane wall of thickness $2L$ initially at a uniform temperature T_i subjected to convection from both sides to an environment at temperature T_∞ with a convection coefficient of h.

Figure 28:



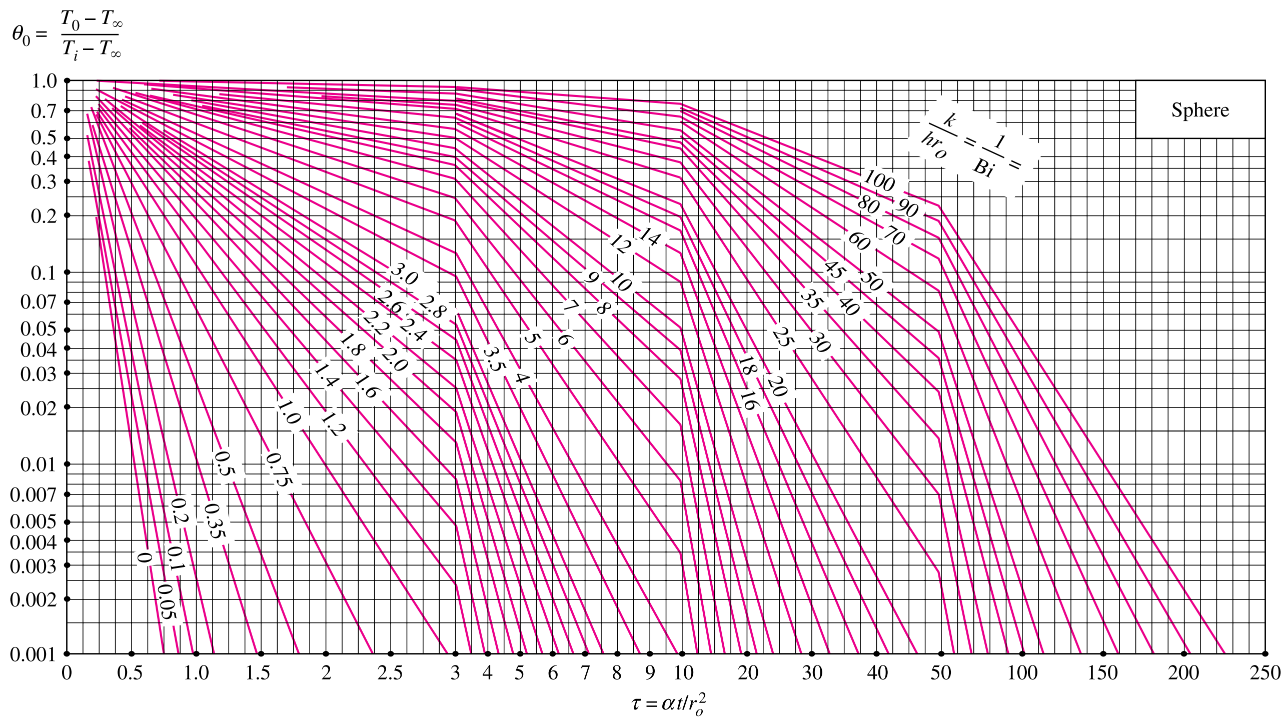
(*a*) Centerline temperature (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME 69*, 1947, pp. 227–36. Reprinted by permission of ASME International.)

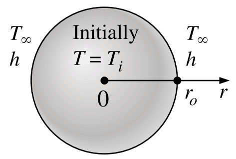
(*b*) Temperature distribution (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME 69*, 1947, pp. 227–36. Reprinted by permission of ASME International.)

(*c*) Heat transfer (from H. Gröber et al.)

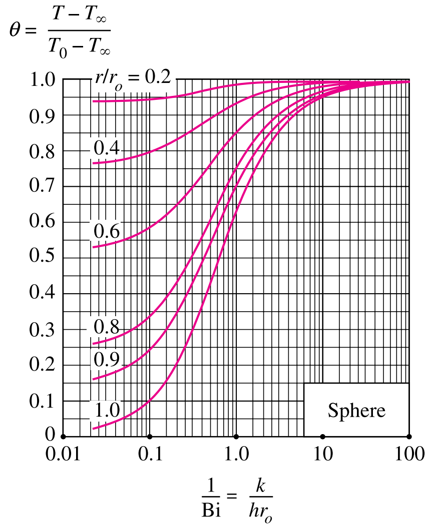
Transient temperature and heat transfer charts for a long cylinder of radius r_o initially at a uniform temperature T_i subjected to convection from all sides to an environment at temperature T_∞ with a convection coefficient of h.

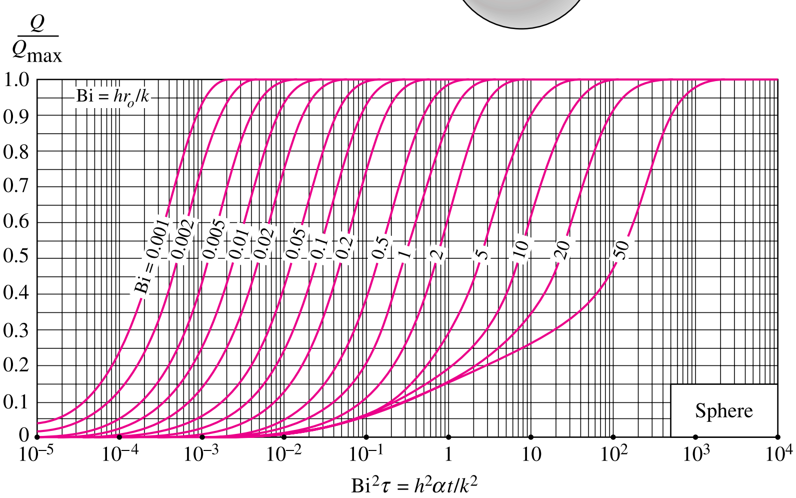
Figure 29:





(*a*) Midpoint temperature (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME 69*, 1947, pp. 227–36. Reprinted by permission of ASME International.)





(*b*) Temperature distribution (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME 69*, 1947,

(*c*) Heat transfer (from H. Gröber et al.)

Transient temperature and heat transfer charts for a sphere of radius r_o initially at a uniform temperature T_i subjected to convection from all sides to an environment at temperature T_∞ with a convection coefficient of h.

Figure 30:

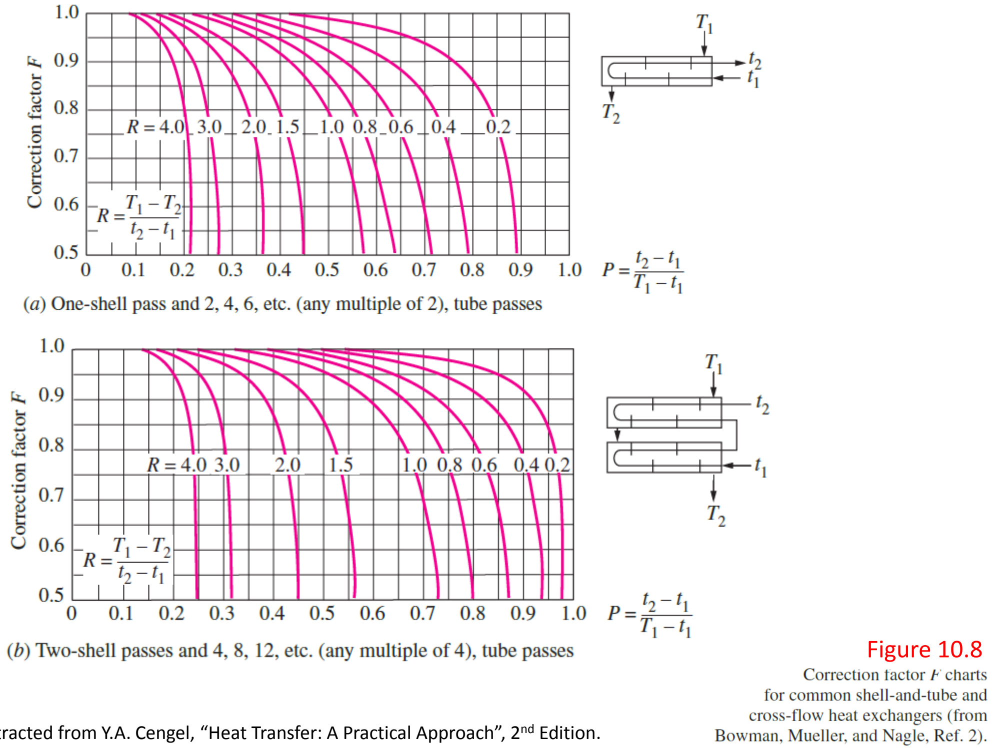


Figure 31: Correction-factors for LMTD Method, extracted from Y. A. Cengel, "Heat transfer:A practical approach", 2nd Ed.

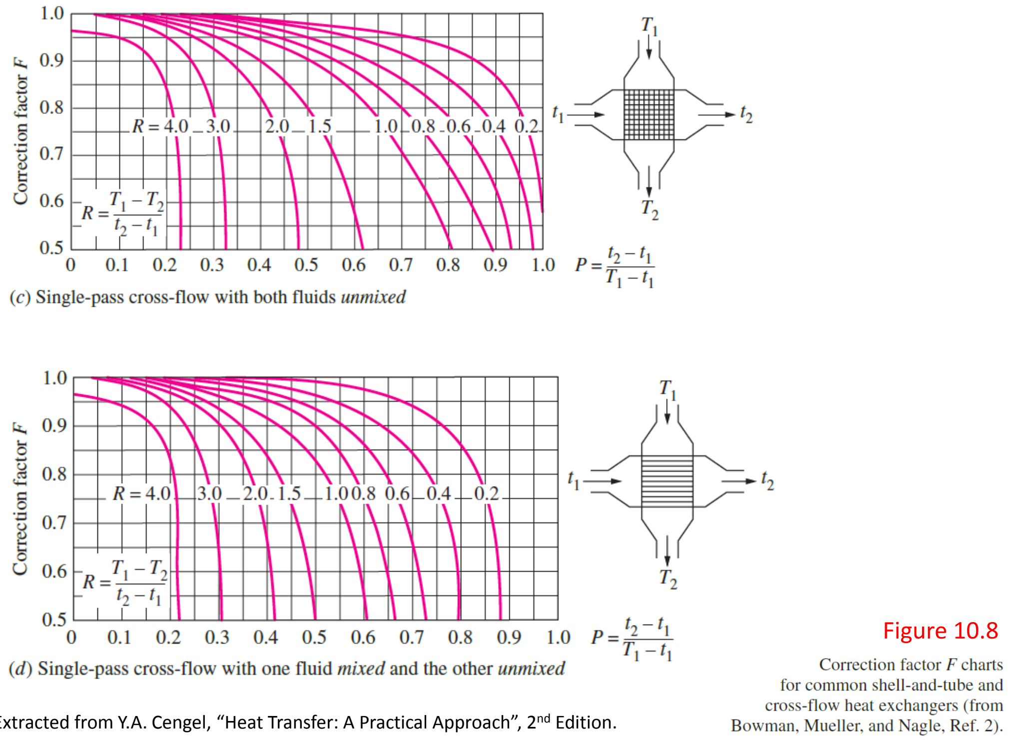


Figure 32: Correction-factors for LMTD Method, extracted from Y. A. Cengel, "Heat transfer:A practical approach", 2nd Ed.

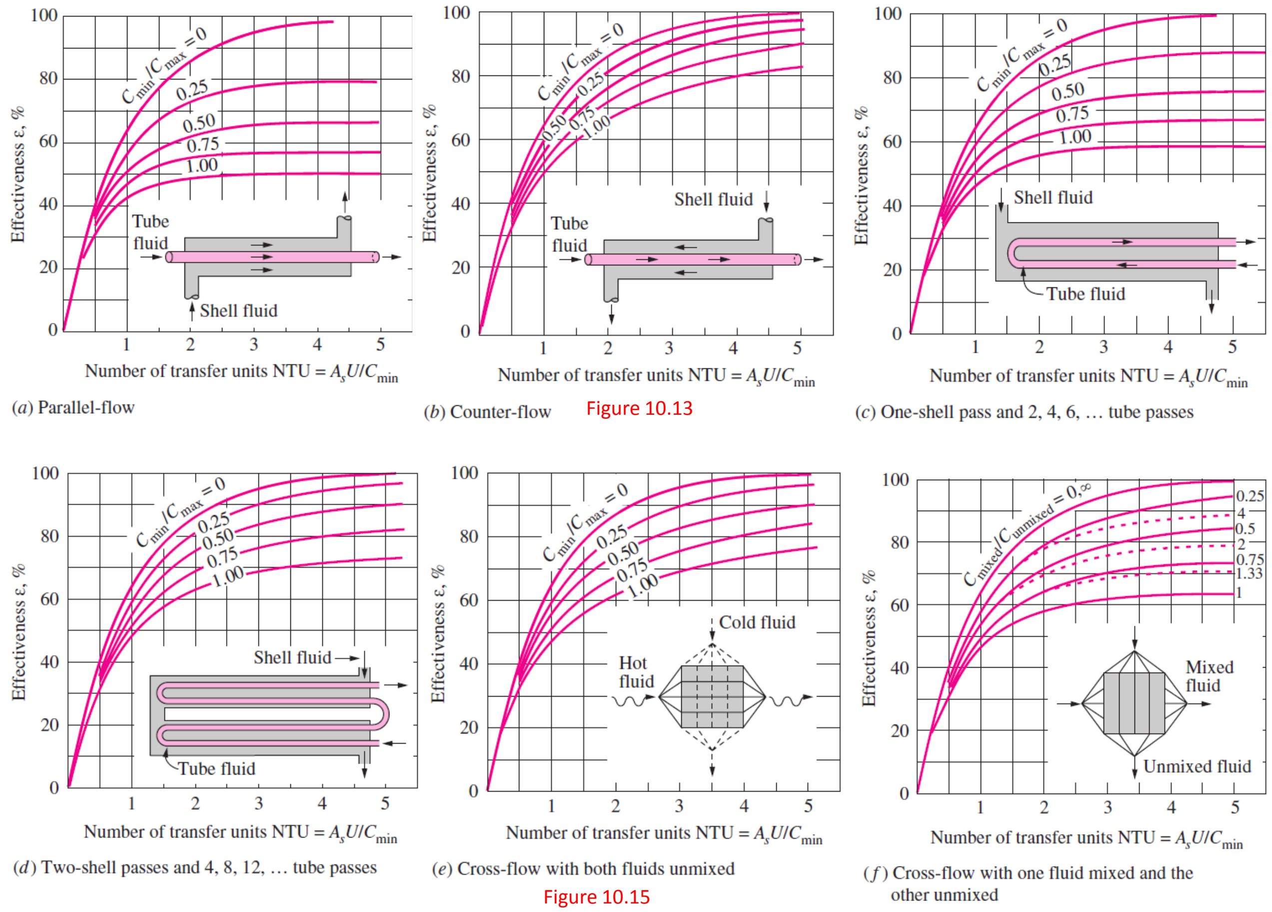
Effectiveness relations for heat exchangers: NTU $= UA_s/C_{min}$ and $c = C_{min}/C_{max} = (\dot{m}C_p)_{min}/(\dot{m}C_p)_{max}$ (Kays and London, Ref. 5.)

Heat exchanger type	Effectiveness relation
1 *Double pipe:*	
Parallel-flow	$\varepsilon = \dfrac{1 - \exp[-\text{NTU}(1 + c)]}{1 + c}$
Counter-flow	$\varepsilon = \dfrac{1 - \exp[-\text{NTU}(1 - c)]}{1 - c\exp[-\text{NTU}(1 - c)]}$
2 *Shell and tube:*	
One-shell pass 2, 4, . . . tube passes	$\varepsilon = 2\left\{1 + c + \sqrt{1+c^2}\,\dfrac{1 + \exp[-\text{NTU}\sqrt{1+c^2}]}{1 - \exp[-\text{NTU}\sqrt{1+c^2}]}\right\}^{-1}$
3 *Cross-flow (single-pass)*	
Both fluids unmixed	$\varepsilon = 1 - \exp\left\{\dfrac{\text{NTU}^{0.22}}{c}[\exp(-c\ \text{NTU}^{0.78}) - 1]\right\}$
C_{max} mixed, C_{min} unmixed	$\varepsilon = \dfrac{1}{c}(1 - \exp\{1 - c[1 - \exp(-\text{NTU})]\})$
C_{min} mixed, C_{max} unmixed	$\varepsilon = 1 - \exp\left\{-\dfrac{1}{c}[1 - \exp(-c\,\text{NTU})]\right\}$
4 *All heat exchangers with $c = 0$*	$\varepsilon = 1 - \exp(-\text{NTU})$

NTU relations for heat exchangers NTU $= UA_s/C_{min}$ and $c = C_{min}/C_{max} = (\dot{m}C_p)_{min}/(\dot{m}C_p)_{max}$ (Kays and London, Ref. 5.)

Heat exchanger type	NTU relation
1 *Double-pipe:* Parallel-flow	$\text{NTU} = -\dfrac{\ln[1 - \varepsilon(1 + c)]}{1 + c}$
Counter-flow	$\text{NTU} = \dfrac{1}{c - 1}\ln\left(\dfrac{\varepsilon - 1}{\varepsilon c - 1}\right)$
2 *Shell and tube:* One-shell pass 2, 4, . . . tube passes	$\text{NTU} = -\dfrac{1}{\sqrt{1+c^2}}\ln\left(\dfrac{2/\varepsilon - 1 - c - \sqrt{1+c^2}}{2/\varepsilon - 1 - c + \sqrt{1+c^2}}\right)$
3 *Cross-flow (single-pass)* C_{max} mixed, C_{min} unmixed	$\text{NTU} = -\ln\left[1 + \dfrac{\ln(1 - \varepsilon c)}{c}\right]$
C_{min} mixed, C_{max} unmixed	$\text{NTU} = -\dfrac{\ln[c\ln(1 - \varepsilon) + 1]}{c}$
4 *All heat exchangers with $c = 0$*	$\text{NTU} = -\ln(1 - \varepsilon)$

Figure 33: NTU relations extracted from Y. A. Cengel, “Heat transfer:A practical approach”, 2nd Ed.



Extracted from Y.A. Cengel, “Heat Transfer: A Practical Approach”, 2nd Edition.

Figure 34: NTU plots extracted from Y. A. Cengel, “Heat transfer:A practical approach”, 2nd Ed.

Diffusion Dimensionless Numbers

$$\mathrm{Sc} = \frac{\mu}{\rho\, D_{AB}} \qquad \mathrm{Le} = \frac{k}{\rho\, C_p\, D_{AB}}$$

Diffusion

General expression for the flux:

$$\boldsymbol{N}_A = \boldsymbol{J}_A + x_A \sum_B \boldsymbol{N}_B$$

Fick's law:

$$\boldsymbol{J}_A = -D_{AB}\,\nabla C_A$$

Stefan's law:

$$N_{s,r} = -D\frac{c}{1-x}\frac{\partial x}{\partial r}$$

Ideal Gas

$$P\,V = n\,R\,T \qquad R \approx 8.314598\ \mathrm{J\,K^{-1}\,mol^{-1}}$$

Geometry

$$P_{\text{circle}} = 2\,\pi\,r \qquad A_{\text{circle}} = \pi\,r^2 \qquad A_{\text{sphere}} = 4\,\pi\,r^2 \qquad V_{\text{sphere}} = \frac{4}{3}\pi\,r^3$$

$$A_{\text{cylinder}} = P_{\text{circle}}\,L \qquad V_{\text{cylinder}} = A_{\text{circle}}\,L$$