Robust Stability for Interval Hopfield Neural Networks with Time Delay

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Abstract—**The conventional Hopfield neural network with time delay is intervalized to consider the bounded effect of deviation of network parameters and perturbations yielding a novel interval dynamic Hopfield neural network (IDHNN) model. A sufficient condition related to the existence of unique equilibrium point and its robust stability is derived.**

Index Terms—**Interval Hopfield neural network, robust stability, time delay.**

I. Introduction

IN models of electronic neural networks, time delays are likely to be present due to the finite switching speed of amplifiers which are the core elements for implementing artificial neurons. A time delay occurred in the interaction between neurons will affect the stability of a network by creating oscillation or unstable phenomena. Recently, the dynamic behavior of Hopfield networks with time delays have been extensively investigated in [2]–[5]. In [2], the analysis is restricted to linear category and requires severe constrain of having symmetric connection weight matrix. In [3] different delay values are allowed in the network and the network dynamics, in particular, oscillations are considered and investigated. In [4] the authors have derived sufficient conditions for global asymptotic stability of equilibria for Hopfield networks with time delays without linearization and not requiring the symmetry of the interconnection. In [5], a generalized sufficient condition which guarantees stability of analog neural networks with time delays is given. In [6] conditions for the existence of equilibrium points, boundedness of trajectories and the global stability of BAM network with asymmetric connection weight matrix and time delay are investigated yielding results of significant generality which are expected being useful in BAM design. However, in deterministic artificial neural networks, vital data such as the neurons fire rate, the synaptic interconnection weight and the signal transmission delays, etc., are usually acquired and processed by means of statistical estimates, estimating errors therefore exist. On the other hand, parameter fluctuation in neural-network implementation on very large scale integration (VLSI) chips is also unavoidable. Nevertheless, it is possible to explore in practice the range of the above-mentioned vital data as well as the bounds of circuit parameters by engineering experience even from incomplete information. This fact paves the way of introducing the theory of interval matrix and interval dynamics [7], [8] to investigate the stability of neural networks. Specifically, we will use this theory to investigate the robust stability of Hopfield network with time delay, this problem has not been solved in related literature reported recently [9], [10].

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II. Formulation of IDHNN Model

By introducing the time-delay τ into the conventional Hopfield neural network, we have its state equations as follows:

$$\frac{du_i(t)}{dt} = -a_i u_i(t) + \sum_{j=1}^{n} w_{ij} f_j(u_j(t)) + \sum_{j=1}^{n} w_{ij}^{\tau} f_j(u_j(t-\tau_{ij})) + I_i \tag{1}$$

where $i = 1, 2, \cdots, n$ nonnegative numbers a_i and τ_{ij} represent the neuron charging time constants, axonal signal transmission delays, respectively, w_{ij} and w_{ij}^{τ} stand for the weights of the neuron interconnections, and f_j and I_i are the activation function (supposed to be continuous, differentiable, monotonically increasing, and bounded) of the neurons, external constant inputs, respectively.

In view of the fact that in implementing the circuit given in Fig. 1, unavoidable parameter fluctuation of the circuitry will lead to some deviation of design values for a_i, τ_{ij} in (1). Besides, in the learning process, the acquisited data-affected interconnection weight w_{ij} and w_{ij}^{τ} will also suffer from the perturbation due to noises as well as some unforced man-made faults. Thus it is essential to investigate the stability and robustness of the network against such intrinsic parameter deviations and external perturbations.

Since in practice these deviations and perturbations are bounded in general, we may correspondingly intervalize the above mentioned quantities, namely, define the following vector and matrix sets:

$$\begin{cases} A_I := \{A = \mathrm{diag}(a_i) : \underline{A} \le A \le \bar{A}, \text{ i.e.} \\ \qquad \underline{a}_i \le a_i \le \overline{a_i}, i = 1, 2, \cdots, n, \forall A \in A_I\} \\ W_I := \{W = (w_{ij})_{n\times n} : \underline{W} \le W \le \bar{W}, \text{ i.e.} \\ \qquad \underline{w}_{ij} \le w_{ij} \le \overline{w_{ij}}, i, j = 1, 2, \cdots, n, \forall W \in W_I\} \\ W_I^{\tau} := \left\{W^{\tau} = \left(w_{ij}^{\tau}\right)_{n\times n} : \underline{W^{\tau}} \le W^{\tau} \le \bar{W}^{\tau}, \text{ i.e.} \\ \qquad \underline{w}_{ij}^{\tau} \le w_{ij}^{\tau} \le \bar{w}_{ij}^{\tau}, i, j, = 1, 2, \cdots, n, \forall W^{\tau} \in W_I^{\tau}\right\} \\ \tau_I := \{\tau = (\tau_{ij})_{n\times n} : \underline{\tau} \le \tau \le \bar{\tau}, \text{ i.e.} \\ \qquad \underline{\tau}_{ij} \le \tau_{ij} \le \overline{\tau_{ij}}, i, j = 1, 2, \cdots, n, \forall \tau \in \tau_I\}. \end{cases} \tag{2}$$

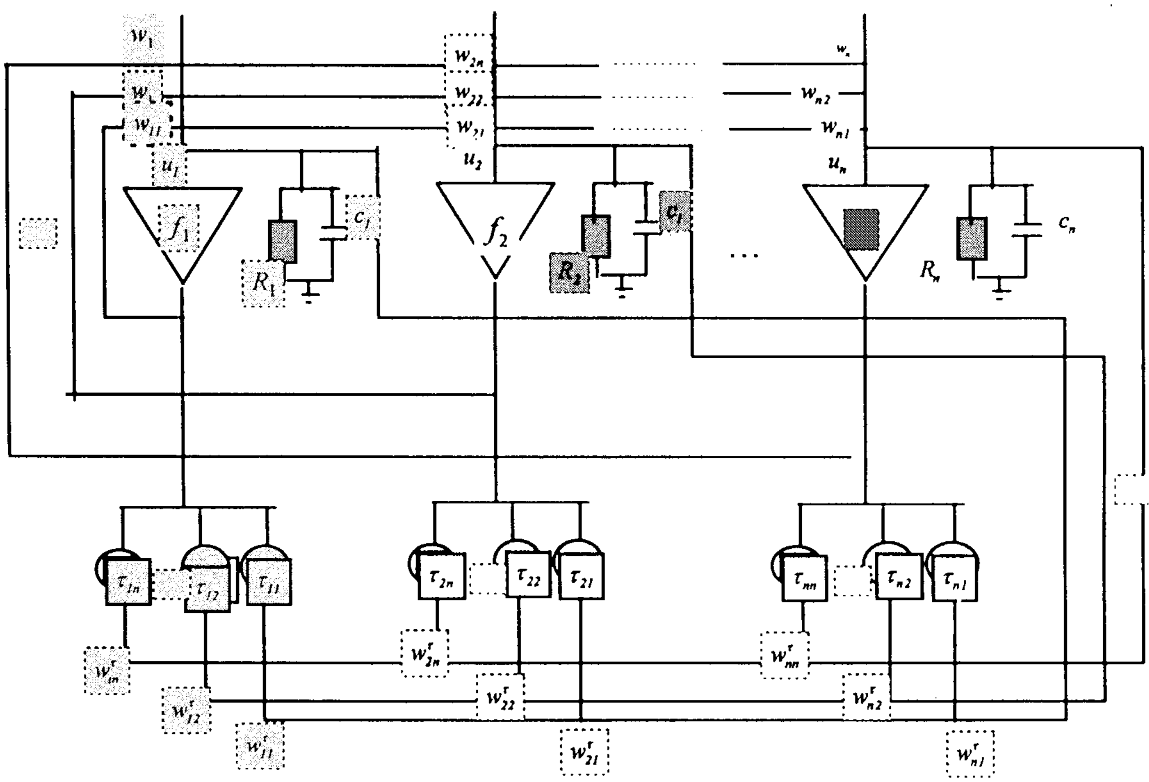

Fig. 1. An analog implementation of IDHNN network $(a_i = R_i)$.

Equations (1) and (2) are the mathematical description of the IDHNN model we suggested.

III. Existence of the Equilibrium and Robust Stability of IDHNN

As regard to the robust stability, at first we have the following.

Definition: IDHNN is called robust stable or globally robust stable if its unique equilibrium $u^* = (u_1^*, u_2^*, \cdots, u_n^*)$ is stable or globally stable for each

$$A = \operatorname{diag}(a_i)_{n\times n} \in A_I, \quad W = (w_{ij})_{n\times n} \in W_I,$$

$$W^\tau = \left(w_{ij}^\tau\right)_{n\times n} \in w_I^\tau, \quad \tau = (\tau_{ij})_{n\times n} \in \tau_I.$$

By interval dynamic approach [7], [8], we can prove the following theorem related to the IDHNN model based on the above definition.

Theorem: Let w_{ij}, w_{ij}^τ, a_i be real constants $(i, j = 1, 2, \cdots, n)$ and assume that

1)

$$w_{ii}^* - \underline{a}_i < 0 \tag{3}$$

2)

$$\begin{aligned} B := &-\operatorname{diag}(w_{11}^* - a_1, w_{22}^* - a_2, \cdots, w_{nn}^* - a_n) \\ &+ ((1-\delta_{ij})w_{ij}^*)_{n\times n}) \text{ is an } M\text{-Matrix} \end{aligned} \tag{4}$$

where

$$w_{ij}^* = \max\left\{\left|\left(\underline{w_{ij}} + \underline{w_{ij}^\tau}\right)\frac{df_i(\sigma)}{d\sigma}\right|, \left(\overline{w_{ij}} + \overline{w_{ij}^\tau}\right)\frac{df_i(\sigma)}{d\sigma}\right\}.$$

Then system (1) with (2) has a unique and robust stable equilibrium $u^* = (u_1^*, u_2^*, \cdots, u_n^*)$ for each constant input $I = [I_1, I_2, \cdots, I_n] \in R^n$.

Proof: **Part I—Existence and Uniqueness of an Equilibrium:**

Suppose that $\beta_i > 0$, let us define

$$F(u) = \begin{bmatrix} -a_1\beta_1 u_1 + \beta_1 \sum_{j=1}^n \left(w_{1j} + w_{1j}^\tau\right) f_j(u_j) \\ \cdots \\ -a_n\beta_n u_n + \beta_n \sum_{j=1}^n \left(w_{nj} + w_{nj}^\tau\right) f_j(u_j) \end{bmatrix}. \tag{5}$$

Since $\mathbf{B}$ is an M-matrix, there exist constants $\beta_i > 0, i = 1, 2, \cdots, n$, $\varepsilon > 0$, such that

$$\beta_i(a_i - w_{ji}^*) - \sum_{\substack{j=1 \\ j\neq i}}^n \beta_j w_{ji}^* \geq \varepsilon > 0. \tag{6}$$

Note the Jacobian matrix of $F(u)$ is quasidiagonally dominant since conditions (6) are satisfied, thus the inverse of this Jacobian matrix exists and $F(u)$ is locally C^1 difffeomorphic.

On the other hand, for any $u=[u_1,u_2,\cdots,u_n]$ we have

$$\begin{aligned}
&\|F(u)\|_1\\
&=\sum_{i=1}^n\left[\beta_i\left|a_iu_i-\sum_{j=1}^n\left(w_{ij}+w_{ij}^\tau\right)f_j(u_j)\right|\right]\\
&\geq\sum_{i=1}^n\left[|\beta_ia_iu_i|-\left|\sum_{j=1}^n\beta_i\left(w_{ij}+w_{ij}^\tau\right)f_j(u_j)\right|\right]\\
&=\sum_{i=1}^n\left[|\beta_ia_iu_i|-\left|\sum_{j=1}^n\beta_j\left(w_{ji}+w_{ji}^\tau\right)f_i(u_i)\right|\right]\\
&=\sum_{i=1}^n\left[\left|\int_0^{u_i}\beta_ia_i\,d\xi\right|-\left|\sum_{j=1}^n\int_0^{u_i}\beta_j\left(w_{ji}+w_{ji}^\tau\right)\frac{df_i(\xi)}{d\xi}\right|d\xi\right]\\
&\geq\sum_{i=1}^n\left[\left|\int_0^{u_i}\beta_ia_i\,d\xi\right|-\left|\sum_{j=1}^n\int_0^{u_i}\beta_j\left(w_{ji}+w_{ji}^\tau\right)\frac{df_i(\xi)}{d\xi}\right|d\xi\right].
\end{aligned}$$

Since $\frac{df_i(\xi)}{d\xi}\geq 0$, it follow from condition (6) that for $u_i\geq 0$

$$\begin{aligned}
&\int_0^{u_i}\beta_ia_i\,d\xi-\sum_{j=1}^n\int_0^{u_i}\beta_i\left|\left(w_{ji}+w_{ji}^\tau\right)\frac{df_i(\xi)}{d\xi}\right|d\xi\\
&\quad\geq\int_0^{u_i}\left[\beta_ia_i-\sum_{j=12}^n\beta_jw_{ij}^*\right]d\xi\\
&\quad\geq\varepsilon u_i=\varepsilon|u_i|.
\end{aligned}\tag{7}$$

Analogously, for $u_i<0$

$$\begin{aligned}
&\int_{u_i}^0\beta_ia_i\,d\xi-\sum_{j=1}^n\int_u^0\beta_i\left|\left(w_{ji}+w_{ji}^\tau\right)\frac{df_i(\xi)}{d\xi}\right|d\xi\\
&\quad\geq\int_{u_i}^0\left[\beta_ia_i-\sum_{j=12}^n\beta_jw_{ji}^*\right]d\xi\\
&\quad\geq-\varepsilon u_i=\varepsilon|u_i|.
\end{aligned}\tag{8}$$

From (7) and (8) we can conclude that for any value of u_i

$$\|F(u)\|_1\geq\varepsilon\|u\|_1\tag{9}$$

that means $\|F(u)\|_1\to\infty$, when $\|u\|_1\to\infty$. Therefore $F(u)$ is C^1 diffeomorphism on R^n and system (1) has a unique equilibrium point [4], [6].

Part II—Robust Stability of the Equilibrium:

Let $u^*=(u_1^*,u_2^*,\cdots,u_n^*)$ be the equilibrium of system (1), rewrite (1) as

$$\begin{aligned}
\frac{d[u_i(t)-u_i^*]}{dt}&=-a_i(u_i(t)-u_i^*)\\
&\quad+\sum_{j=1}^n w_{ij}[f_j(u_j(t))-f_j(u_j^*)]\\
&\quad+\sum_{j=1}^n w_{ij}^\tau[f_j(u_j(t-\tau_{ij}))-f_j(u_j^*)].
\end{aligned}\tag{10}$$

Define a Lyapunov function

$$\begin{aligned}
V(t)=\sum_{i=1}^n\beta_i\Bigg\{&|u_i(t)-u_i^*|+\sum_{i=1}^n\left|w_{ji}^\tau\right|\\
&\times\int_{t-\tau_u}^t|f_j(u_j(\xi))-f_i(u_i^*)|\,d\xi\Bigg\}
\end{aligned}\tag{11}$$

its upper right Dini-derivative along the solution of (1) can be calculated as

$$\begin{aligned}
&D^+V(t)|_{(1)}\\
&=\sum_{i=1}^n\beta_i\Bigg\{\frac{d(u_i(t)-u_i^*)}{dt}\operatorname{sgn}(u_i(t)-u_i^*)+\sum_{j=1}^n\left|w_{ji}^\tau\right|\\
&\quad\times\left(|f_j(u_j(t))-f_j(u_j^*)|-|f_j(u_j(t-\tau_{ij}))-f_j(u_j^*)|\right)\Bigg\}\\
&\leq\sum_{i=1}^n\beta_i\Bigg\{-a_i|u_i(t)-u_i^*|+\sum_{j=1}^n|w_{ij}[f_j(u_j(t))-f_j(u_j^*)]|\\
&\qquad+\sum_{j=1}^n\left|w_{ij}^\tau[f_j(u_j(t-\tau_{ij}))-f_j(u_j^*)]\right|\\
&\qquad+\sum_{j=1}^n\left|w_{ij}^\tau[f_j(u_j(t))-f_j(u_j^*)]\right|\\
&\qquad-\sum_{j=1}^n\left|w_{ij}^\tau[f_j(u_j(t-\tau_{ij}))-f_j(u_j^*)]\right|\Bigg\}\\
&=\sum_{i=1}^n\beta_i\Bigg\{-a_i|u_i(t)-u_i^*|+\sum_{j=1}^n|w_{ij}[f_j(u_j(t))-f_j(u_j^*)]|\\
&\qquad+\sum_{j=1}^n\left|w_{ij}^\tau[f_j(u_j(t)-f_j(u_j^*)]\right|\Bigg\}\\
&=\sum_{i=1}^n\beta_i\Bigg\{-a_i\left|u_i(t)-u_i^*\right|+\sum_{j=1}^n\left|w_{ji}\frac{df_i(\xi)}{d\xi}(u_j(t)-u_i^*)\right|\\
&\qquad+\sum_{j=1}^n\left|w_{ij}^\tau\frac{df_i(\xi)}{d\xi}(u_j(t)-u_j^*)\right|\Bigg\}\\
&=\sum_{i=1}^n\Bigg\{-a_i\beta_i|u_i(t)-u_i^*|+\sum_{j=1}^n\beta_j\left|w_{ji}\frac{df_i(\xi)}{d\xi}\right|\\
&\quad\times|(u_j(t)-u_j^*)|\\
&\quad+\sum_{j=1}^n\beta_j\left|w_{ji}\frac{df_i(\xi)}{d\xi}\right||(u_j(t)-u_j^*)|\Bigg\}\\
&\leq\sum_{i=1}^n\left\{-a_i\beta_i+\sum_{j=1}^n\beta_j\left(|w_{ji}|+\left|w_{ji}^\tau\right|\right)\frac{df_i(\xi)}{d\xi}\right\}\\
&\quad\times|u_j(t)-(u_j^*)|\\
&\leq-\varepsilon\sum_{i=1}^n|u_j(t)-u_j^*|.
\end{aligned}\tag{12}$$

From (11) we have

$$V(t)+\varepsilon\int_0^t\sum_{i=1}^n|u_i(\xi)-u_i^*|\,d\xi\le V(t_0) \tag{13}$$

which means that $V(t)$ is upper bounded on $[0,\infty]$, hence $u_i(t)$ are upper bounded on $[0,\infty]$, and

$$\sum_{i=1}^n|u_i(t)-u_i^*|\in L_1[0,\infty]$$

since $u_i(t)-u_i^*$ are uniformly continuous on $[0,\infty]$. Because $\sum_{i=1}^n|u_i(t)-u_i^*|$ is uniformly continuous on $[0,\infty]$, we have

$$\sum_{i=1}^n|u_i(t)-u_i^*|\to 0, \text{ as } t\to\infty$$

therefore $u_i(t)\to u_i^*$, as $t\to\infty$ for any values of the coefficients in (1), i.e., the system described by (1) is globally robust stable.

From the Theorem we have the following corollaries.

Corollary 1: System (1) with (2) is globally robust stable if

$$a_i-w_{ii}^*>\frac{1}{\beta_i}\sum_{\substack{j=1\\j\ne i}}^n\beta_j|w_{ji}^*|.$$

Proof: Follows directly from the Theorem.

Corollary 2 System (1) with (2) is globally robust stable if

$$a_i-w_{ii}^*>\sum_{\substack{j=1\\j\ne i}}^n w_{ji}^*.$$

Proof: Follows directly from the Theorem.

Note the robust stability becomes to the classical Lyapunov stability if $\underline{A}=A=\bar{A}$, $\underline{W}=W=\bar{W}$, $\underline{W}^\tau=W^\tau=\bar{W}^\tau$ $\underline{\tau}=\tau=\bar{\tau}$, and assume that

1)

$$\left|\left(w_{ii}+w_{ii}^\tau\right)\frac{df_i(\sigma)}{d\sigma}\right|-a_i<0$$

2)

$$\begin{aligned}B:\triangleq -\Bigg(&\mathrm{diag}\bigg(\left(w_{11}+w_{11}^\tau\right)\frac{df_1(\sigma)}{d\sigma}-a_1,\cdots.\\&\left(w_{nn}+w_{nn}^\tau\right)\frac{df_n(\sigma)}{d\sigma}-a_n\bigg)\\&+\left(\left((1-\delta_{ij})\left|\left(w_{ij}+w_{ij}^\tau\right)\frac{df_i(\sigma)}{d\sigma}\right|\right)_{n\times n}\right)\end{aligned}$$

is an M-matrix,

then system (1) is globally stable.

IV. Conclusion

A Hopfield neural-network model with time delay may characterize more precisely the BNN/ANN dynamics. However, its robust stability analysis will also be much more complicated as compared to a system without the time delay; since the system parameter deviation and external perturbation are bounded in practice, to intervalize the HNN model is not only reasonable but also will greatly simplify the dynamic analysis of such new model, namely the IDHNN network. A sufficient condition for the IDHNN being robust stable is derived and can be used to design a robust stable analog neural network in practical applications.

References

[1] J. J. Hopfield, "Neural networks and physics systems with emergent collective computation abilities," in *Proc. Nat. Academy Sci. USA*, 1982, vol. 79, no. 2, pp. 2554–2558.

[2] C. M. Marcus and R. M. Westervelt, "Stability of analogy neural networks with delay," *Phys. Rev. A.*, vol. 39, no. 2, pp. 347–359, 1989.

[3] P. Baldi and A. F. Atiya, "How delays affect neural dynamics and learning," *IEEE Trans. Neural Networks*, vol. 5, pp. 612–621, July 1994.

[4] K. Gopalsamy and X. He, "Stability in asymmetric Hopfield nets with transmmission delays," *Physica D*, vol. 76, pp. 344–358, 1994.

[5] Y. J. Cao and Q. H. Wu, "A note on stability of analog neural networks with time delays," *IEEE Trans. Neural Networks*, vol. 7, pp. 1533–1535, 1996.

[6] X. F. Liao and J. B. Yu, "Qualitative analysis of bidirection associative memory networks with time delays," *Int. J. Circuit Theory and Applicat*, to be published.

[7] J. A. Heinen, "Sufficient conditions for stability of interval matrices," *Int. J. Contr.*, vol. 39, pp. 1323–1328, 1984.

[8] I. Hinrichsen and A. J. Ichmann, "Robustness of stability of time-varying linear systems," *J. Diff. Eqs.*, vol. 82, pp. 219–25, 1989.

[9] Y. Zhang, "Global exponential stability and periodic solutions of delay Hopfield neural network," *Int. J. Syst. Sci.*, vol. 27, no. 2, pp. 227–231, 1996.

[10] ——, "Periodic solutions and stability of Hopfield neural networks with variable delays," *Int. J. Syst. Sci.*, vol. 2, no. 9, pp. 985–901, 1996.

[11] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*. New York: Academic, 1993.