Fractional-Order Hopfield Neural Networks

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Abstract. This paper proposes Fractional-order Hopfield Neural Networks (FHNN). This network is mainly based on the classic well-known Hopfield net in which fractance components with fractional order derivatives, replace capacitors. Stability of FHNN is fully investigated through energy-like function analysis. To show how effective the FHNN network is, an illustrative example for parameter estimation problem of the second-order system is finally considered in the paper. The results of simulation are very promising.

Keywords: Fractional-order, neural networks, parameter estimation.

1 Introduction

The study of the human brain is over thousands years old. With the advent of modern electronics, it was only natural to try to harness this thinking process. The first step toward artificial neural networks came in 1943 when Warren McCulloch, a neurophysiologist, and a young mathematician, Walter Pitts, wrote a paper on how neurons might work [1]. Recent work in neural networks includes Boltzmann machines, Hopfield nets, competitive learning models, multilayer networks, and adaptive resonance theory models.

In the beginning of the 1980s, J. Hopfield designed a new associative memory neural network, named Hopfield. He understood that some models of physical systems could be used to solve computational problems [2]. Such systems could be implemented in hardware by combining common and standard components such as capacitors and resistors. This feature is so significant especially in hardware implementation point of view.

About 300 years ago, calculus was first presented to the fractional world. In the fractional calculus, differential equations have non-integer order. The engineers could understand the importance of the fractional-order equations, especially when they observed that the description of some systems is more accurate, when the fractional derivative is used [3]. A typical example of a fractional order system is the voltage-current relation of a semi-infinite lossy transmission line (Wang.1987).

In the fractional calculus, the main operator is the fractance. The fractance is a generalized capacitor. Actually it is an electrical circuit in which its voltage and current are related by the fractional-order differential equation [4].

From this idea, we use this generalized capacitor in the continuous Hopfield neural network instead of common capacitor and propose a new continuous network, called

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as Fractional-order Hopfield Neural Network (FHNN), in which fractional-order equations can describe its behaviour.

The rest of the paper is organized as follows. In section 2, we review the fractional order systems and their implementation algorithms as well. In section 3, continuous Hopfield neural networks are reviewed. Section 4 fully derives fractional-order Hopfield neural networks and presents it's stability through energy-like function analysis. Section 5 shows capability of FHNN through an illustrative example. Finally, section 6 concludes the paper.

2 Fractional-Order Systems and Implementation Algorithms

We can compactly model, many of physical phenomena, material properties and processes, using uncommon differential equations in which they have fractional differential and integral parts. We name this class of equations, in brief, FDEs. In some researches [5-7], it is noted that many dynamical systems such as electrochemical processes, membranes of cells of biological organism, certain types of electrical noise, and chaos are more adequately described by FODE equations.

The main operator to state FDEs is a fractional differential operator. There are many ways to define this operator. A commonly used definition is the Caputo differintegral operator D_*^{α} as follows [8],

$$\mathrm{D}^{\alpha} y(x) = J^{m-\alpha} y^{(m)}(x) \ . \tag{1}$$

where $m =: \lceil \alpha \rceil$ is the value α rounded up to the nearest integer, y (m) is the ordinary *m*th derivative of y and

$$J^{\beta} z(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} z(t) dt \quad , \beta > 0 \ . \tag{2}$$

is the Riemann − Liouville integral of order β.

The Caputo operator has nice properties. For example it is seen in (3) that the Laplace domain representation of the Caputo derivative uses initial conditions $f^{(k)}(0)$ only for integer k. These initial values typically have a well understood physical meaning and can be measured directly.

$$L\left\{D^{\alpha} f(t)\right\} = S^{\alpha} F(S) - \sum_{k=0}^{m-1} f^{(k)}(0) s^{\alpha-k-1} \ . \tag{3}$$

The next stage toward FDEs is to present their solutions. There are two ways to tackle this problem. One way is to use an approximation method to represent fractional integrator operator $\frac{1}{s^{\alpha}}$ in the Laplace domain, by some pole-zero pairs as a transfer function. This approach uses both singularity functions in the frequency space and Bode diagram [9]. Although this method is good for a specific range of frequencies, as shown in [10], this approximation method may fail in detecting chaos phenomena in fractional-order systems.

Another way is to employ numerical techniques to find FDE solution [11-13]. These numerical methods are based on the approximation integral in (2), by different ways such as two-point trapezoidal quadrature formula. One of the common and reliable numerical methods is the Predictor-Corrector (PC) algorithm based on fractional Adams-Bashforth method. This algorithm is a generalization of the classical Adams-Bashforth-Moulton integrator that is well known for the numerical solution of first-order problems [12]. Knowing the fact that the PC algorithm has been constructed and analysed for the fully general set of equations $D^{\alpha}y(x) = f(x, y(x))$ without any special assumptions with ease of implementation on a computer, this paper employs it as fully discussed below.

3 Continuous Hopfield Neural Networks

The continuous Hopfield model of size N is a fully interconnected neural network with N continuous valued units. It contains linear and non-linear circuit elements, which typically are capacitor, resistor, op-amp, and sources. The topological structure of Hopfield net is shown in Fig.1. The input and output of the net are analog signals. The resistance R_{i0} and capacitor C_i are parallel to simulate the time-delay characteristics of biologic neurons. The resistance $R_{ij}(i, j = 1,2, \dots, N)$ and the op-amps are used to simulate the synapse and the non-linear characteristic of biologic neurons, respectively.

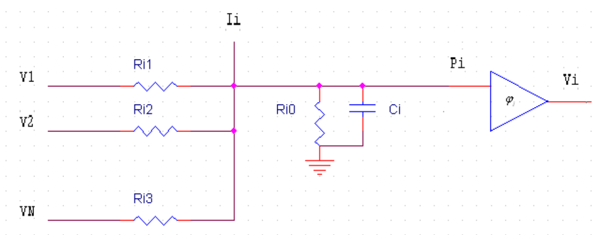

Fig. 1. The dynamic neuron of continuous hopfield

The state and output equations of continuous Hopfield with N neurons are given as follows:

$$C_i \frac{dP_i}{dt} = \sum_{j=1}^{N} W_{ij} V_j - \frac{P_i}{R_i} + I_i \quad , P_i = \left(\frac{1}{\lambda}\right) \varphi^{-1}{}_i (V_i) \quad . \tag{4}$$

where $P_i(t)$ and $V_i(t)$ are the input and output of op-amp, respectively, for the ith neuron at time t. λ is the learning rate and W_{ij} is the conductance between the ith and jth neuron, and the following equations hold.

$$W_{ij} = \frac{1}{R_{ij}}, \quad \frac{1}{R_i} = \frac{1}{R_{i0}} + \sum_{j=1}^{N} W_{ij} \quad . \tag{5}$$

Consider the following energy function as a Lyapunov function,

$$E = -(\frac{1}{2})\sum_i \sum_j W_{ij} V_j V_i - \sum_i I_i V_i + \frac{1}{\lambda}\sum_i (\frac{1}{R_i})\int_0^{V_i} \varphi^{-1}(v)dv \quad . \tag{6}$$

Hopfield showed that, if the weights are symmetric $\mathrm{W_{ij}} = \mathrm{W_{ji}}$, then this energy function has a negative time gradient. This means that the evolution of dynamic system (4) in state space always seek the minima of the energy surface E.

Because of easy structure of Continuous Hopfield neural networks for implementation, they are often used for solving optimization problems.

4 Fractional-Order Hopfield Neural Networks

A new physical component, related to the kind of dielectric, which can be described by Caputo operator, is a fractance [4]. Actually a fractance is a generalized capacitor by fractional-order relationship between the voltage's terminal and the current passing through it. In 1994, Westerlund proposed a new linear capacitor model based on Curie's empirical law that for a general input voltage $\mathrm{u(t)}$, the current is [14],

$$i(t) = F\frac{d^\alpha u(t)}{dt^\alpha} = F.D^\alpha u(t) \quad . \tag{7}$$

From this idea, we use this generalized capacitor in the continuous Hopfield neural network instead of common capacitor and propose a new continuous network, named Fractional-order Hopfield Neural Network (FHNN), which can be described by fractional-order equations. The state and output equations of FHNN are as follows:

$$F_i.D^\alpha P_i = F_i P_i^{(\alpha)} = \sum_{j=1}^{N} W_{ij} V_j - \frac{P_i}{R_i} + I_i, \quad P_i = \left(\frac{1}{\lambda}\right)\varphi^{-1}{}_i(V_i) \quad . \tag{8}$$

where $0 < \alpha < 1$ and $\mathrm{D}^\alpha(.)$ is defined in (1). From the FHNN structure in (8), the energy function for this network is as follows:

$$E = -(\frac{1}{2})\sum_i \sum_j W_{ij} V_j V_i - \sum_i I_i V_i + \frac{1}{\lambda}\sum_i (\frac{1}{R_i})\int_0^{V_i} \varphi^{-1}(v)dv \quad . \tag{9}$$

To show the stability of proposed neural network, we need somehow to generalize the idea of Gradient operator to the fractional one.

Definition 1. The fractional gradient (or fractional gradient vector field) of a scalar function f(x) with respect to a vector variable $x = (x_1, x_2, \dots, x_n)$ is denoted by $\nabla^{(\alpha)} f$, where $\nabla^{(\alpha)}$ shows the fractional gradient operator. The fractional gradient of f contains the vector field whose components are partial fractional differential of function f as follows:

$$\nabla^{(\alpha)} f = \left(\frac{\partial^\alpha f}{\partial x_1} \quad \frac{\partial^\alpha f}{\partial x_2} \quad \cdots \quad \frac{\partial^\alpha f}{\partial x_n}\right) \ . \tag{10}$$

When a function depends also on a parameter such as time, the fractional gradient often refers simply to the vector of its spatial fractional derivatives only.

Theorem 1. For the FHNN described by (8) and (9), if $\varphi_i^{-1}(V_i)$ is a monotone increasing continuous function, and $F_i > 0$, then the FHNN energy function has a negative-definite fractional gradient and the following relationship exists as the net state changes.

$$\nabla^{(\alpha)} E \le 0, \ \ \textit{and} \ \ \nabla^{(\alpha)} E = 0 \ \ \textit{iff} \ \ \nabla^{(\alpha)} V_i = 0 \quad i = (1,\dots,N) \ . \tag{11}$$

Proof: To calculate $\nabla^{(\alpha)}E$, we need some simplifications as follows, (from (1), because $0 < \alpha < 1$, then $m = 1$):

$$\varphi^{-1}(v) := h(v)$$

$$D^{(\alpha)} h(v) = J^{1-\alpha}\left(\frac{\partial h}{\partial v} v'(t)\right) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} . \ v'(\tau) . \frac{\partial h}{\partial v} d\tau \tag{12}$$

$$= \frac{\partial h}{\partial v} . \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} . \ v'(\tau) . \frac{\partial h}{\partial v} d\tau \ .$$

From definition (2), we have

$$D^{(\alpha)} v = J^{1-\alpha}(v'(t)) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} . \ v'(\tau) d\tau \ . \tag{13}$$

From (12) and (13), we can extract the following relationship:

$$D^{(\alpha)} h(v) = D_t^{(\alpha)} v . \frac{\partial}{\partial v}(h(v)) \ . \tag{14}$$

where $D_t^{(\alpha)}(.)$ shows the Caputo differintegral operator with respect to time. Furthermore, partial fractional derivatives of the first and second term in (9) are calculated as:

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(-\frac{1}{2}\sum_i \sum_j W_{ij} V_j V_i\right) = \frac{\partial^\alpha}{\partial t^\alpha}\left(-\frac{1}{2} \underline{V}' W \underline{V}\right) = -\frac{1}{2}\nabla_v \left(\underline{V}' W \underline{V}\right) . D_t^{(\alpha)} \underline{V}$$

$$= -\sum_{i=1}^N D_t^{(\alpha)} V_i \sum_{j=1}^N W_{ij} V_j \ . \tag{15}$$

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(-\sum_{i=1}^N I_i V_i\right) = -\sum_{i=1}^N I_i . D_t^{(\alpha)} V_i \ . \tag{16}$$

Now from (14) to (16), the fractional gradient of the energy function of FHNN will appear as:

$$\nabla_t^{(\alpha)} E = \sum_{i=1}^{N} \frac{\partial E}{\partial V_i} \cdot \frac{\partial^\alpha V_i}{\partial t^\alpha} = -\sum_{i=1}^{N} D_t^{(\alpha)} V_i \sum_{j=1}^{N} W_{ij} V_j - \sum_{i=1}^{N} I_i . D_t^{(\alpha)} V$$

$$+ \frac{1}{\lambda} \sum_{i=1}^{N} (\frac{1}{R_i}) \left(D_t^{(\alpha)} V_i \right) \left(\frac{\partial}{\partial v} (h(v)) \right)$$

$$= -\sum_{i=1}^{N} D_t^{(\alpha)} V_i \cdot \left(\sum_{j=1}^{N} W_{ij} V_j + I_i - \frac{P_i}{R_i} \right) = -\sum_{i=1}^{N} D_t^{(\alpha)} V_i . F_i . D^\alpha P_i$$

$$= -\sum_{i=1}^{N} (D_t^{(\alpha)} V_i)^2 . F_i \cdot \frac{\partial}{\partial v} (\varphi^{-1}(v)) \quad . \tag{17}$$

From (17), if $\varphi_i^{-1}(V_i)$ is a monotonically increasing continuous function and $F_i > 0$, then equation (11) will be satisfied.

5 Illustrative Example

For software implementation of FHNN, the numerical PC algorithm is used assuming that the input impedances R_i, is high enough so that the second term in (8) can be neglected and $F_i = 1, (i = 1,2, \dots, N)$. The PC algorithm is constructed to analyze the following equation.

$$D^{(\alpha)} P_i(t) = \sum_{j=1}^{N} W_{ij} \cdot \frac{1}{2} \left(1 + \tanh \left(\lambda \frac{P_i(t)}{P_0} \right) \right) + I_i \quad . \tag{18}$$

To show application of the preceding network, we want to solve a parameter estimation problem based on optimization using FHNN network.

A second order system with two states and a single input is considered here,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0.3 \\ -0.2 & 0.9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} u \quad . \tag{19}$$

The objective function that would be optimized by the FHNN network is defined based on the parameter estimation error as follows:

$$E = (\frac{1}{k}) \int_0^k (\frac{1}{2}) (\dot{e}_q)^T (\dot{e}_q) dt = (\frac{1}{k}) \int_0^k (\frac{1}{2}) (\dot{x}(t) - \dot{y}(t))^T (\dot{x}(t) - \dot{y}(t)) dt \quad . \tag{20}$$

where $x(t)$ and $y(t)$ are state variables of the system and estimator, respectively. For this problem, there are 6 parameters (four state matrix entries, and two input matrix entries) to estimate, therefore the FHNN should have 6 neurons. By comparing (20) and (9), adjusting laws for W_{ij} and I_i will follow as [15],

$$[W_{ij}] = -(\frac{1}{k})\int_0^k \begin{bmatrix} XX^T & 0 & XU^T & 0 \\ 0 & XX^T & 0 & XU^T \\ UX^T & 0 & UU^T & 0 \\ 0 & UX^T & 0 & UU^T \end{bmatrix} dt \quad .$$

$$[I_i] = \frac{1}{k}\int_0^k [X^T\dot{x}_1, X^T\dot{x}_2, u\dot{X}^T]dt \quad .$$

To show the capability of the FHNN, the index $I_t = t^{-1}\int_{x=0}^{t} \|e(t)\|^2 d\tau$, can be used for the performance index of the parameter estimation error,

Fig.2 shows the simulation results of the parameter estimation by the FHNN network with fractional-order $\alpha = 0.85$, step size of the PC algorithm equal to 0.5 msec and learning rate $\lambda = 1$. Note that this net converges for any arbitrary λ and α, however the step size of the PC algorithm is indeed critical. Therefore, for a given α , it should be chosen by trial and error so that convergence of FHNN is guaranteed.

In Fig.3 the performance index evolving in time is depicted. The performance index will arrive at 0.0073 in 0.5 sec and the FHNN error becomes 2.12×10^{-12}. This value can completely show the convergence of the FHNN that has been proved before.

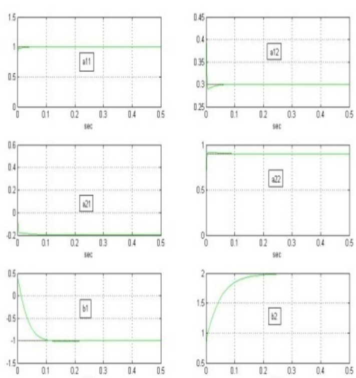

Fig. 2. Parameter estimation by FHNN

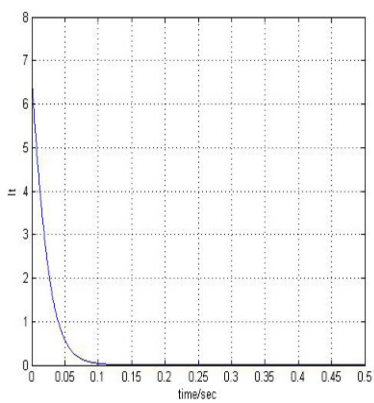

Fig. 3. Performance index of FHNN

6 Conclusion

In this paper, we have proposed fractional-order Hopfield neural networks. Stability of FHNN has been proved via energy-like function analysis. We showed the existence of FHNN and its usage, by an illustrative example in parameter estimation problem. In the further work, it is good to extract significations of FHNN network comparing with the common Hopfield in various fields such as optimization problems.

As fractional order systems are systems that have infinite memory, probably common methods and algorithms by finite memory, may fail. In the future, it seems necessary to extend other methods and algorithms like our work to the fractional-order ones. Surely this approach will increase amount of computations using numerical algorithms for solving FDEs.

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