DEPARTMENT OF MATHEMATICS

MODULE 2: FOURIER SERIES

Introduction:

Fourier series are infinite series designed to represent general periodic functions in terms of simple ones, namely sines and cosines. This trigonometric system is orthogonal allowing the computation of the co-efficients of the Fourier series by the use of well known Euler formulas.

Fourier series are very important to engineers and physicists because they allow the solution of ODEs in connection with forced oscillations and the approximations of periodic functions. Fourier series in certain sense are more universal than the more familiar Taylor's series in calculus because many discontinuous periodic functions that come up in applications can be developed in Fourier series but do not have Taylor's series expansions.

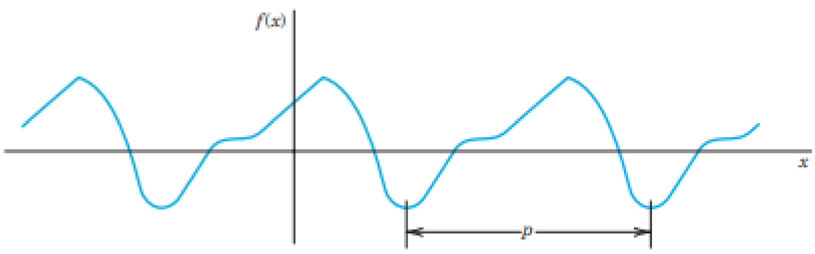
Fourier series can be extended to non-periodic phenomena to obtain Fourier integrals and Fourier transforms, where both play an important application in solving ***PDE***s.

In the digital age , the ***discrete Fourier transform*** plays an important role. Signals, such as voice or music are sampled and analysed. ***Fast Fourier transform*** is the important algorithm in this context.

Fourier series:

Periodic function: Any function which repeats itself at regular intervals of time is called as a periodic function.

Mathematically, a function $f(x), \forall x$ which is real except possibly at some points is said to be periodic with period p if $\boldsymbol{f(x+p)=f(x)}$, $\boldsymbol{p}$ is some positive number.



The smallest positive period is called *fundamental period*.

Examples of periodic functions:

1) Sine 2) Cosine (both are periodic functions of period $p=2\pi$)

DEPARTMENT OF MATHEMATICS

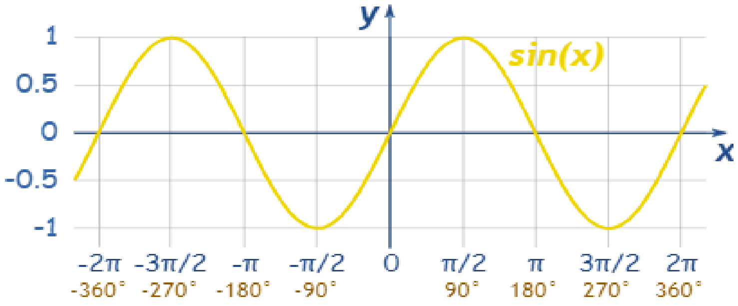


Fig 1: graph of y=sinx

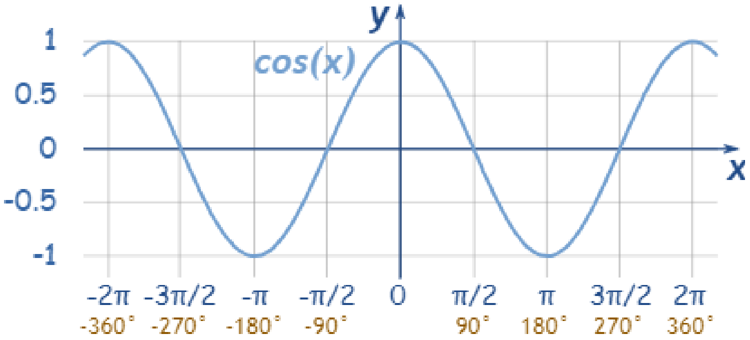


Fig 2: graph of y=cosx

DEPARTMENT OF MATHEMATICS

3) tangent (except at $x = \pm\frac{\pi}{2}, \ \pm\frac{3\pi}{2}$, ...)

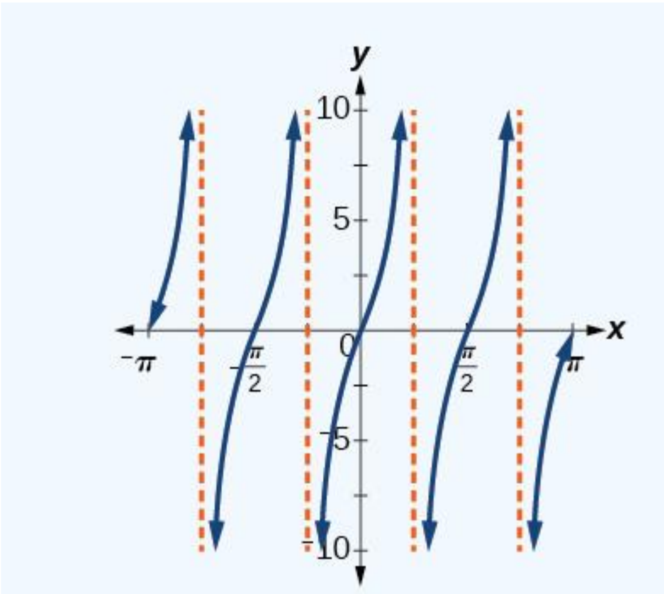


Fig 3: graph of y=tanx

4) cotangent (except at $x = \pm\pi, \ \pm 2\pi, \pm 3\pi,$ ... .)

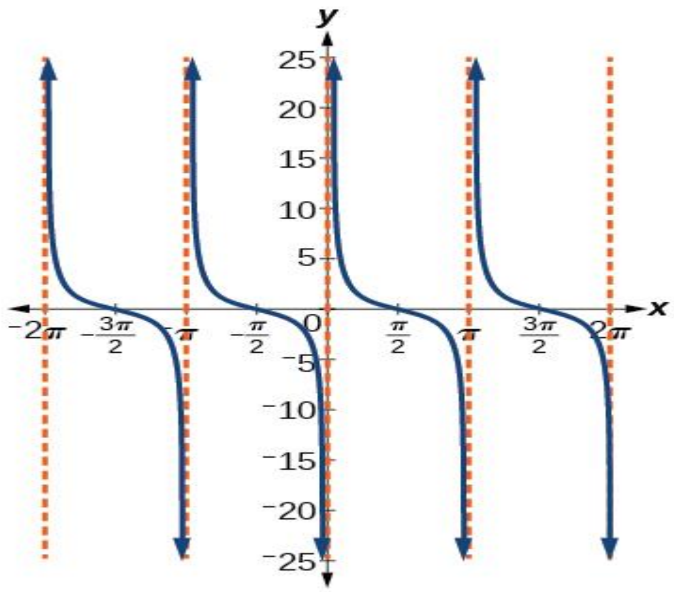


Fig 4: graph of y=cotx

Note: Constant function $\boldsymbol{f(x) = 1}$ is always a periodic function of any period p because

$f(x + p) = f(x)$, where $f(x) = 1$

$\therefore \ f(x + p) = 1 \quad \Rightarrow f(x + p) = f(x) = 1$

DEPARTMENT OF MATHEMATICS

$\therefore$ *Constant function* $f(x) = 1$ *is a periodic function of any period* p but has no fundamental period.

Similarly , $\boldsymbol{f(x) = \pm\, k}$ where k is a constant is a periodic function of any period p but has no fundamental period.

Few examples of non-periodic functions:

1) all polynomial functions like $f(x) = x, x^2, x^3, \dots \dots ..$

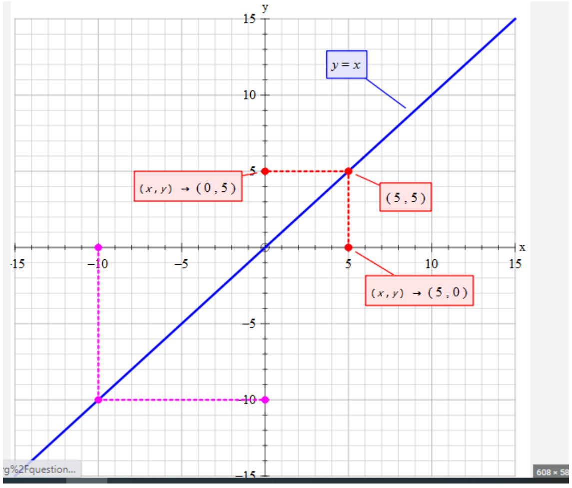


Fig 5: graph of y=x (odd function)

2) exponential function $f(x) = e^x$

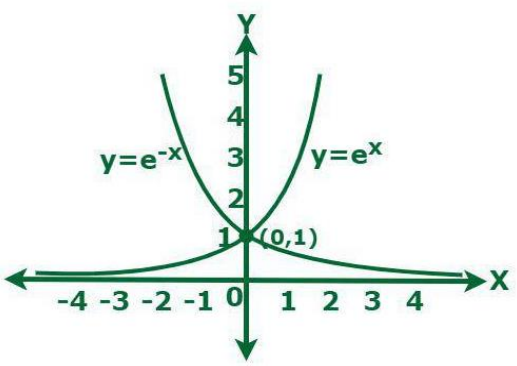


Fig 6: graph of $y = e^x$ & $y = e^{-x}$

3) logarithmic function $f(x) = \ln(x)$

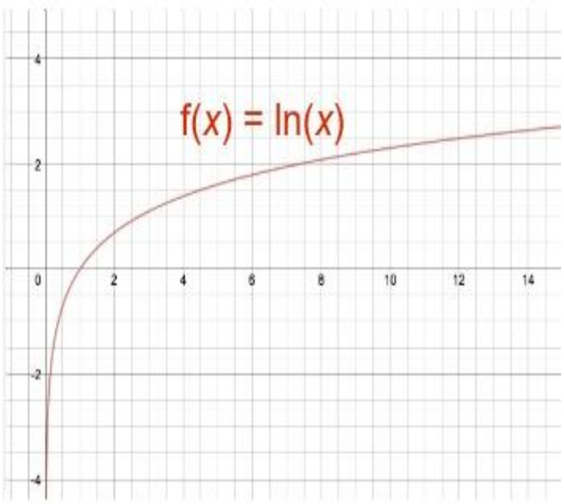
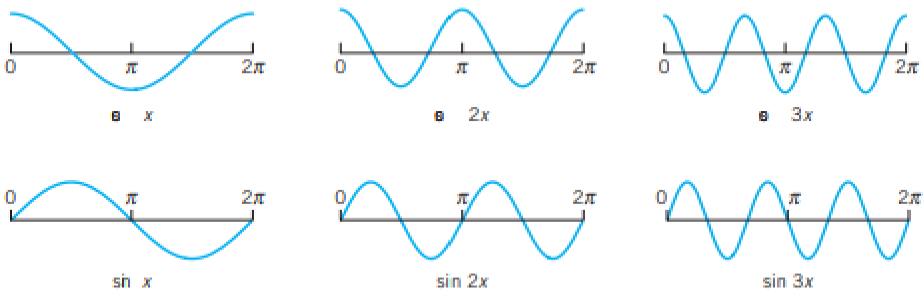


Fig 7: graph of $y = \ln(x)$

Representation of various functions of period $p = 2\pi$

cos(x) & sin(x), cos(2x) & sin(2x), cos(3x) & sin(3x), ……...



Note:

1) $f(x) = cosx\ is\ periodic\ of\ period\ p = 2\pi$

$f(x+p) = cos\{x+p\} = cosx\ cosp - sinx\ sinp$

$Let\ p = 2\pi\ \therefore\ f(x+2\pi) = cosx\cos(2\pi) - sinx\sin(2\pi)$

$\therefore\ f(x+2\pi) = cosx = f(x)$

2) $f(x) = cos2x\ is\ periodic\ of\ period\ p = \pi$

$f(x+p) = cos2\{x+p\} = cos\{2x+2p\} = cos2x\ cos2p - sin2x\ sin2p$

$Let\ p = \pi\ \therefore\ f(x+\pi) = cos2x\cos(2\pi) - sin2x\sin(2\pi)$

$\therefore\ f(x+2\pi) = cos2x = f(x)\quad \therefore\ p = \pi\quad is\ the\ fundamental\ period$

3) $f(x) = sin2x\ is\ periodic\ of\ period\ p = \pi$

$f(x+p) = sin2\{x+p\} = sin\{2x+2p\} = sin2x\ cos2p + cos2x\ sin2p$

$Let\ p = \pi\ \therefore\ f(x+\pi) = sin2x\ cos2\pi + cos2x\ sin2\pi$

$\therefore\ f(x+2\pi) = sin2x = f(x)\quad \therefore\ p = \pi\quad is\ the\ fundamental\ period$

4) $f(x) = cos3x\ is\ periodic\ of\ period\ \ p = \frac{2\pi}{3}$

$f(x+p) = cos3\{x+p\} = cos\{3x+3p\} = cos3x\ cos3p - sin3x\ sin3p$

$Let\ p = \frac{2\pi}{3}\ \therefore\ f\left(x+\frac{2\pi}{3}\right) = cos3x\cos(2\pi) - sin3x\sin(2\pi)$

$\therefore\ f(x+2\pi) = cos3x = f(x)\quad \therefore\ p = \frac{2\pi}{3}\quad is\ the\ fundamental\ period$

5) $f(x) = sin3x\ is\ periodic\ of\ period\ \ p = \frac{2\pi}{3}$

$f(x+p) = sin3\{x+p\} = sin\{3x+3p\} = sin3x\ cos3p + cos3x\ sin3p$

$Let\ p = \frac{2\pi}{3}\ \therefore\ f\left(x+\frac{2\pi}{3}\right) = sin3x\ \cos(2\pi) + cos3x\ \sin(2\pi)$

$\therefore\ f(x+2\pi) = sin3x = f(x)\quad \therefore\ p = \frac{2\pi}{3}\quad is\ the\ fundamental\ period$

In general any trigonometric function say $\cos(nx)$ & $\sin(nx)$ is periodic of period

$$\boldsymbol{p = \frac{2\pi}{n}\ ,\ n = 1, 2, 3, \cdots}$$

The series to be obtained will be a *trigonometric* series which will be of the form

$$\frac{a_0}{2} + \{a_1 cosx + b_1 sinx\} + \{a_2 cosx + b_2 sinx\} + \{a_3 cosx + b_3 sinx\} + \cdots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n\cos(nx) + b_n\sin(nx)\} \qquad \text{..................(1)}$$

Where $a_0, a_1, b_1, a_2, b_2, \ldots$ are all constants called as the **co-efficients** of the series such that every term of the series is periodic of period 2π (if the co-efficients are such that the series converges then its sum will be a function of period 2π).

DEPARTMENT OF MATHEMATICS

Let $f(x)$ be a given function of period 2π such that it can be represented by the infinite series given by eqn(1) i.e eqn(1) converges and moreover has the sum $f(x)$ then using the equality sign we write

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos(nx) + b_n \sin(nx)\}} \qquad \text{.............................(2)}$$

Where eqn(2) is called as the ***Infinite Trigonometric Fourier Series*** expansion of $f(x)$ (or) simply called as the ***Fourier Series of*** $f(x)$.

The co-efficients of eqn(2) are called as ***Fourier co-efficients*** of $f(x)$ which is given by Euler formulas as

$$\boldsymbol{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx} \qquad \text{..............................(3a)}$$

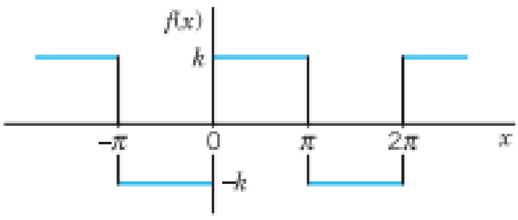
$$\boldsymbol{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \quad n = 1, 2, 3, \ldots..} \qquad \text{.............................(3b)}$$

$$\boldsymbol{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx, \quad n = 1, 2, 3, \ldots..} \qquad \text{..............................(3c)}$$

Note: Leonard Euler(1707-1783) was an enormously great creative Swiss mathematician. He made fundamental contributions to almost all branches of mathematics and its applications to Physics. His important books on algebra and calculus contains numerous basic results of his own research.

Basic Example:

1) Find the Fourier co-efficients of the periodic rectangular wave function $f(x)$ as shown in the figure below $f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$ and $f(x + 2\pi) = f(x)$



(functions of this kind act as external forces acting on mechanical systems, electromotive forces in electric circuits, etc... The value of $f(x)$ at a single point does not affect the integral. Hence we can leave $f(x)$ undefined at $x = 0$ & $x = \pm\pi$)

DEPARTMENT OF MATHEMATICS

Solt: By data , $f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$ ……………………………(1)

$f(x + 2\pi) = f(x)$ comparing with $f(x + p) = f(x)$ we have $p = 2\pi$

i.e the given function $f(x)$ by eqn(1) is periodic of period $p = 2\pi$ ……………………………….(2)

we use Euler's formulas to find the Fourier co-efficients a_0, a_n & b_n

wkt, $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$, using eqn (1) we get

$\Rightarrow \quad a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$

$\Rightarrow \quad a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-k)\,dx + \int_{0}^{\pi}(k)\,dx\right]$

$\Rightarrow \quad a_0 = \frac{1}{2\pi}\left[\{(-k)\,x\}_{-\pi}^{0} + \{(k)\,x\}_{0}^{\pi}\right]$

$\Rightarrow \quad a_0 = \frac{1}{2\pi}\left[(-k)\{0 - (-\pi)\} + (k)\{\pi - 0\}\right]$

$\Rightarrow \quad a_0 = \frac{1}{2\pi}\left[-k\pi + k\pi\right] \quad \therefore \quad \boldsymbol{a_0 = 0}$ ……………………………………………(3)

Wkt, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \quad n = 1, 2, 3, \ldots,$ using eqn. (1) we get

$\Rightarrow \quad a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos(nx)\,dx + \int_{0}^{\pi} f(x)\cos(nx)\,dx\right]$

$\Rightarrow \quad a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos(nx)\,dx + \int_{0}^{\pi}(k)\cos(nx)\,dx\right]$

$\Rightarrow \quad a_n = \frac{1}{\pi}\left[(-k)\left\{\frac{\sin(nx)}{n}\right\}_{-\pi}^{0} + (k)\left\{\frac{\sin(nx)}{n}\right\}_{0}^{\pi}\right]$

$\Rightarrow \quad a_n = \frac{1}{\pi}\left[(-1)\frac{k}{n}\{\sin(nx)\}^{0}{}_{-\pi} + \frac{k}{n}\{\sin(nx)\}^{\pi}{}_{0}\right]$

$\Rightarrow \quad a_n = \frac{1}{\pi} * \left(\frac{k}{n}\right)\left[-\{\sin(nx)\}^{0}{}_{-\pi} + \{\sin(nx)\}^{\pi}{}_{0}\right]$

$\Rightarrow \quad a_n = \frac{1}{\pi} * \left(\frac{k}{n}\right)\left[-\{0 - 0\} + \{0 - 0\}\right] \quad \because \sin(0) = \sin(\pm n\pi) = 0, \; n = 1,2,3 \ldots.$

$\therefore \boldsymbol{a_n = 0}$ …………………………………………………………(4)

Wkt , $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx, \quad n = 1, 2, 3, \ldots.$ using eqn. (1) we get

DEPARTMENT OF MATHEMATICS

$\Rightarrow\ b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin(nx)\,dx + \int_{0}^{\pi} f(x)\sin(nx)\,dx\right]$

$\Rightarrow\ b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin(nx)\,dx + \int_{0}^{\pi}(k)\sin(nx)\,dx\right]$

$\Rightarrow\ b_n = \frac{1}{\pi}\left[(-k)\left\{\frac{-\cos(nx)}{n}\right\}^{0}_{-\pi} + (k)\left\{\frac{-\cos(nx)}{n}\right\}^{\pi}_{0}\right]$

$\Rightarrow\ b_n = \frac{1}{\pi}\left[\left(\frac{k}{n}\right)\{\cos(nx)\}^{0}{}_{-\pi} - \left(\frac{k}{n}\right)\{\cos(nx)\}^{\pi}{}_{0}\right]$

$\Rightarrow\ b_n = \frac{1}{\pi}*\left(\frac{k}{n}\right)[\{1-\cos(n\pi)\} - \{\cos(n\pi)-1\}]$ $\quad\because\ \cos(-n\pi) = \cos(n\pi)$

$\Rightarrow\ b_n = \frac{1}{\pi}*\left(\frac{k}{n}\right)[\{1-\cos(n\pi)\} + \{1-\cos(n\pi)\}]$

$\Rightarrow\ \boldsymbol{b_n = \frac{2}{\pi}*\left(\frac{k}{n}\right)[1-\cos(n\pi)]}$ ....(5a)

Wkt $\cos(\pi) = -1, \cos(2\pi) = 1, \cos(3\pi) = -1, \cos(4\pi) = 1,$ ... ...

Thus $\cos(n\pi) = \begin{cases} -1, & when\ n\ is\ odd \\ 1, & when\ n\ is\ even \end{cases}$ $\quad\therefore\quad 1-\cos(n\pi) = \begin{cases} 2, & when\ n\ is\ odd \\ 0, & when\ n\ is\ even \end{cases}$

Thus eqn. (5a) can be written as

$\Rightarrow\ b_1 = \frac{4k}{\pi}$, $b_2 = 0$, $\Rightarrow\ b_3 = \frac{4k}{3\pi}$, $b_4 = 0$, $b_5 = \frac{4k}{5\pi}$, ...(5b)

Thus substituting eqns. (3), (4), (5b) in Fourier series expansion given by

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n\cos(nx) + b_n\sin(nx)\}$

$\Rightarrow\quad f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$

$\Rightarrow\quad f(x) = 0 + 0 + \{b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x) + b_4\sin(4x) + b_5\sin(5x) + \cdots\}$

$\Rightarrow\quad \boldsymbol{f(x) = \frac{4k}{\pi}\left[sinx + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \cdots\right]}$ ....(6)

gives the Fourier series expansion of the function $f(x)$ given by eqn. (1)

From the above equation the partial sums are:

$S_1 = \frac{4k}{\pi}sinx,\quad S_2 = \frac{4k}{\pi}[sinx + sin3x], \ldots$ ....(7)

As per the figure given in the data indicates that the series is convergent and has the sum (x) , which is the given function.

DEPARTMENT OF MATHEMATICS

Also, we observe that at the points $\boldsymbol{x = 0}$ & $\boldsymbol{at\ x = \pi}$ which are the points of discontinuity of the given function $f(x)$ where all the partial sums have the value *zero* which is the arithmetic mean of the values $-k$ & k of our function at these points.

Note: Let us assume $x = \pi/2$ in eqn. (6), we get

$$\Rightarrow \quad f(\pi/2) = \frac{4k}{\pi}\left[\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(3\frac{\pi}{2}\right) + \frac{1}{5}\sin\left(5\frac{\pi}{2}\right) + \cdots\right]$$

$$\Rightarrow \quad k = \frac{4k}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} + \cdots\right]$$

$$\therefore \quad \boldsymbol{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots} \qquad \text{.......................................(8)}$$

Eqn. (8) is a famous result obtained by *Leibnitz* in 1673 from geometric considerations . It illustrates that the "values of various series with constant terms can be obtained by evaluating Fourier series at specific points".

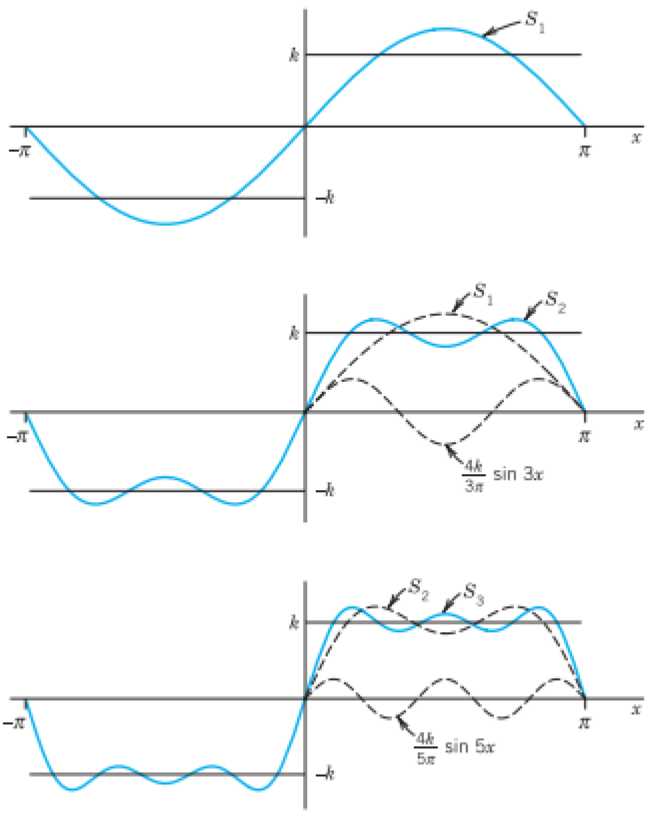


Figure: Graphs of first three partial sums.

DEPARTMENT OF MATHEMATICS

Note:

1) The integral of the product of any two functions (sine and cosine) is ***zero*** over the interval $0 \leq x \leq 2\pi$ (or) $-\pi \leq x \leq \pi$ (or) length of the interval 2π because of periodicity i.e

$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\ dx = 0\ ,\ \ n \neq m$

$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\ dx = 0\ ,\ \ n \neq m$

$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\ dx = 0\ ,\ \ n \neq m\ \ or\ \ n = m$

2) Let $f(x)$ be periodic with period 2π and piecewise continuous over the interval

$-\pi \leq x \leq \pi$ and let $f(x)$ be discontinuous at $x = x_0$ then the sum of the series is the average of the left and right hand limits of $f(x)\ \ at\ \ x = x_0$.

Conditions for the existence of Fourier series:

Only those functions which satisfy a set of conditions over a period can be expressed as an infinite Fourier series i.e as an infinite series of trigonometric sine and cosine functions

($\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$) and the conditions are:

1) $f(x)$ is periodic of period 2π .

2) $f(x)$ is a single valued function and finite over the interval $(c,\ c + 2\pi)$.

3) $f(x)$ has finite number of discontinuities over the interval $(c,\ c + 2\pi)$.

4) $f(x)$ has atmost finite number of maxima and minima over the interval $(c,\ c + 2\pi)$.

Above conditions are called as ***Dirichlet's*** conditions.

Thus if the function $f(x)$ is defined over the interval $(c,\ c + 2\pi)$ and satisfies Dirichlet's conditions then the infinite trigonometric series

DEPARTMENT OF MATHEMATICS

$\boldsymbol{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)}$ is called as **Fourier series** of $f(x)$ over the interval $(c,\ c+2\pi)$.

Note:

1) If $f(x)$ is discontinuous at $x = x_0 \in (c,\ c+2\pi)$ then the infinite trigonometric Fourier series converges to $\frac{1}{2}\,[f(x^+) + f(x^-)]$ where,

$f(x^+) = \lim_{h\to 0} f(x+h)$, the right hand limit

$f(x^-) = \lim_{h\to 0} f(x-h)$, the left hand limit

$h > 0$

2) If $f(x)$ is discontinuous at end points of the interval $(c,\ c+2\pi)$ i.e either at $x = c$ or at $x = c + 2\pi$ then $f(x)\ converges\ to\ \frac{1}{2}\,\{f(c) + f(c+2\pi)\}$

Pre-requisites :

1) $cos\left((2n+1)\frac{\pi}{2}\right) = 0\ ,\quad n = 0, 1, 2, 3, 4, \ldots$

$$\because \quad cos\left(\frac{\pi}{2}\right) = cos\left(3\,\frac{\pi}{2}\right) = cos\left(5\,\frac{\pi}{2}\right) = \cdots = 0$$

2) $sin(n\pi) = 0\ ,\quad n = 0, 1, 2, 3, 4, \ldots \qquad \because \sin(0) = \sin(\pi) = \sin(2\pi) = \cdots = 0$

3) $cos(n\pi) = (-1)^n\ ,\quad n = 0, 1, 2, 3, 4, \ldots \qquad (or)$

$$cos(n\pi) = \begin{cases} +1, & n = 0, 2, 4, 6, \ldots \quad (n = even) \\ -1, & n = 1, 3, 5, \ldots \quad (n = odd) \end{cases}$$

$\cos(0) = \cos(2\pi) = \cos(4\pi) = \cdots = 1 \qquad \& \qquad \cos(\pi) = \cos(3\pi) = \cos(5\pi) = \cdots = -1$

4) $sin\left((2n+1)\frac{\pi}{2}\right) = (-1)^n\ ,\quad n = 0, 1, 2, 3, 4, \ldots$

(or) $sin\left((2n+1)\frac{\pi}{2}\right) = \begin{cases} +1, & n = 0, 2, 4, \ldots \quad (n = even) \\ -1, & n = 1, 3, 5, \ldots \quad (n = odd) \end{cases}$

i.e $sin\left(\frac{\pi}{2}\right) = sin\left(5\,\frac{\pi}{2}\right) = sin\left(9\,\frac{\pi}{2}\right) = \cdots = 1$

& $\quad sin\left(3\frac{\pi}{2}\right) = sin\left(7\frac{\pi}{2}\right) = sin\left(11\frac{\pi}{2}\right) = \cdots = -1$

5) $$\int_{x=-a}^{a} f(x)\, dx = \begin{cases} 2\int_{x=0}^{a} f(x)\, dx\ , & f(-x) = f(x)\ \ i.e\ \ f(x)\ is\ an\ even\ function \\ 0, & f(-x) = -f(x)\ i.e\ f(x)\ is\ an\ odd\ function \end{cases}$$

6) $$\int_{x=0}^{2a} f(x)\, dx = \begin{cases} 2\int_{x=0}^{a} f(x)\, dx\ , & f(2a-x) = f(x)\ \ i.e\ \ f(x)\ is\ an\ even\ function \\ 0, & f(2a-x) = -f(x)\ i.e\ f(x)\ is\ an\ odd\ function \end{cases}$$

7) **Bernoulli's** rule of integration by parts:

$$\int uv = u_0\, v_1 - u_1\, v_2 + u_2\, v_3 - u_3\, v_4 + -\cdots$$

(or) $$\int uv = u_0\, v_1 - u'\, v_2 + u''\, v_3 - u'''\, v_4 + -\cdots$$

Where $u_0\,,\ u_1\,,\ u_2\,,\ u_3\,,\ldots$ i.e all $u's$ represent successive derivatives of u

$v_0\,,\ v_1\,,\ v_2\,,\ v_3\,,\ldots$ i.e all $v's$ represent successive integrals of v

8) $$\int e^{ax} sinbx\, dx = \frac{e^{ax}}{a^2+b^2}\{a\ sinbx - b\ cosbx\},\ \ a\ \&\ b\ \ are\ constants$$

9) $$\int e^{ax} cosbx\, dx = \frac{e^{ax}}{a^2+b^2}\{a\ cosbx + b\ sinbx\},\ \ a\ \&\ b\ \ are\ constants$$

10) length of an open interval $(a, b) = b - a\,,\ \ a < b$

11) $\cos(x)$ is an **even** function whereas $\sin(x)$ is an **odd** function

12) Product of two even functions are always **even**. Similarly , Product of two odd functions are always **even**.

13) The product of an even function with an odd function is always an **odd** function.

14) $$\int_{x=-\pi}^{\pi} \cos(nx)\cos(mx)\ dx = \begin{cases} 0\,, & n \neq m \\ \pi, & n = m > 0 \end{cases}$$

15) $$\int_{x=-\pi}^{\pi} \sin(nx)\sin(mx)\ dx = \begin{cases} 0\,, & n \neq m \\ \pi, & n = m > 0 \end{cases}$$

DEPARTMENT OF MATHEMATICS

Arbitrary period, Odd & Even Functions :

Fourier series of arbitrary period:

Let the function $f(x)$ be defined over the general interval $(c, c+2l)$ whose period length is $2l$ then $f(x)$ after satisfying Dirichlet's conditions in $(c, c+2l)$ we have infinite trigonometric Fourier series (or simply Fourier series) of $f(x)$ given as

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)} \quad \text{..........................(1)}$$

Where a_0, a_n & b_n are constants called as **Fourier co-efficients** which are given by **Euler's** formulas as

$$\boldsymbol{a_0 = \frac{1}{l}\int_{x=c}^{c+2l} f(x)\, dx} \quad \text{.........................(2)}$$

$$\boldsymbol{a_n = \frac{1}{l}\int_{x=c}^{c+2l} f(x)\ cos\left(\frac{n\pi x}{l}\right) dx} \quad \text{.........................(3)}$$

$$\boldsymbol{b_n = \frac{1}{l}\int_{x=c}^{c+2l} f(x)\ sin\left(\frac{n\pi x}{l}\right) dx} \quad \text{.........................(4)}$$

Types of intervals:

There are four types: $(\boldsymbol{0, 2\pi}), (\boldsymbol{-\pi,\ \pi}), (\boldsymbol{0, 2l})$ & $(\boldsymbol{-l,\ l})$

Case(1): $(\boldsymbol{c,\ c+2l}) = (\boldsymbol{0, 2\pi}) \quad \therefore \quad \boldsymbol{c = 0}$ & $\boldsymbol{l = \pi}$

Therefore Fourier series and Euler's formulas given by eqns. (1), (2), (3), (4) reduces to

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)} \quad \text{.........................(5)}$$

$$\boldsymbol{a_0 = \frac{1}{\pi}\int_{x=0}^{2\pi} f(x)\, dx} \quad \text{.........................(6)}$$

$$\boldsymbol{a_n = \frac{1}{\pi}\int_{x=0}^{2\pi} f(x)\ cos(nx) dx} \quad \text{.........................(7)}$$

$$b_n = \frac{1}{\pi}\int_{x=0}^{2\pi} f(x)\ \sin(nx)dx \qquad \text{.........................(8)}$$

Case(2): $(c,\ c+2l) = (-\pi,\ \pi)\quad \therefore\quad c = -l\ \ \&\ \ l = \pi$

Therefore Fourier series and Euler's formulas given by eqns. (1), (2), (3), (4) reduces to

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \qquad \text{.............................(9)}$$

$$a_0 = \frac{1}{\pi}\int_{x=-\pi}^{\pi} f(x)\, dx \qquad \text{........................(10)}$$

$$a_n = \frac{1}{\pi}\int_{x=-\pi}^{\pi} f(x)\ \cos(nx)dx \qquad \text{............................(11)}$$

$$b_n = \frac{1}{\pi}\int_{x=-\pi}^{\pi} f(x)\ \sin(nx)dx \qquad \text{............................(12)}$$

Case(3): $(c,\ c+2l) = (0, 2l)\quad \therefore\quad c = 0$

Therefore Fourier series and Euler's formulas given by eqns. (1), (2), (3), (4) reduces to

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad \text{...........................(13)}$$

$$a_0 = \frac{1}{l}\int_{x=0}^{2l} f(x)\, dx \qquad \text{........................(14)}$$

$$a_n = \frac{1}{l}\int_{x=0}^{2l} f(x)\ \cos\left(\frac{n\pi x}{l}\right) dx \qquad \text{.........................(15)}$$

$$b_n = \frac{1}{l}\int_{x=0}^{2l} f(x)\ \sin\left(\frac{n\pi x}{l}\right) dx \qquad \text{.........................(16)}$$

Case(4): $(c,\ c+2l) = (-l, l)\quad \therefore\quad c = -l$

Therefore Fourier series and Euler's formulas given by eqns. (1), (2), (3), (4) reduces to

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{.........................(17)}$$

$$a_0 = \frac{1}{l}\int_{x=-l}^{l} f(x)\,dx \quad \text{.......................(18)}$$

$$a_n = \frac{1}{l}\int_{x=-l}^{l} f(x)\ \cos\left(\frac{n\pi x}{l}\right) dx \quad \text{........................(19)}$$

$$b_n = \frac{1}{l}\int_{x=-l}^{l} f(x)\ \sin\left(\frac{n\pi x}{l}\right) dx \quad \text{........................(20)}$$

Even & Odd Nature of the Function (x) :

Generally any function $f(x)$ is said to be even when $f(-x) = f(x)$ and odd when $f(-x) = -f(x)$. In Fourier series even and odd nature of the function $f(x)$ will depend upon the interval $(0, 2l)\ (or)\ (0, 2\pi)$ & $(-l,\ l)\ (or)\ (-\pi,\ \pi)$ (refer page no:13, pts (5) & (6))

Even Function:

Case(1) : Le $f(x)$t be an **even** function in the interval $(c,\ c+2l) = (0, 2\pi)$ then

$$a_0 = \frac{2}{\pi}\int_{x=0}^{\pi} f(x)\,dx$$ (from pt (6) page (12), case(1)-page(14))

$$a_n = \frac{2}{\pi}\int_{x=0}^{\pi} f(x)\ \cos(nx)dx$$ (from pt (6) & (12) pg (12), case(1)-pg(14))

$\because\ f(x) \to even\ function\ \ \&\ cos(nx) \to even\ function$

$$b_n = 0$$ (from pt (6) & (13) pg (12, 13), case(1)-pg(14))

$\because\ f(x) \to even\ function\ \ \&\ sin(nx) \to odd\ function$

Case(2) : Le $f(x)$t be an **even** function in the interval $(c,\ c+2l) = (0, 2l)$ then

$$\boldsymbol{a_0 = \frac{2}{l}\int_{x=0}^{l} f(x)\,dx}$$ (from pt (6) page (12), case(3)-page(14))

$$\boldsymbol{a_n = \frac{2}{l}\int_{x=0}^{l} f(x)\ \cos\left(\frac{n\pi x}{l}\right)dx}$$ (from pt (6) & (12) pg (12), case(3)-pg(14))

$\because\ f(x) \to even\,function\ \ \&\ cos\left(\frac{n\pi x}{l}\right) \to even\,function$

$$\boldsymbol{b_n = 0}$$ (from pt (6) & (13) pg (12, 13), case(3)-pg(14))

$\because\ f(x) \to even\,function\ \ \&\ sin(nx) \to odd\,function$

Case(3) : Le $f(x)$t be an **even** function in the interval $(c,\ c+2l) = (-\pi,\ \pi)$ then

$$\boldsymbol{a_0 = \frac{2}{\pi}\int_{x=0}^{\pi} f(x)\,dx}$$ (from pt (5) page (12), case(2)-page(14))

$$\boldsymbol{a_n = \frac{2}{\pi}\int_{x=0}^{\pi} f(x)\ \cos(nx)dx}$$ (from pt (5) & (12) pg (12), case(2)-pg(14))

$\because\ f(x) \to even\,function\ \ \&\ cos(nx) \to even\,function$

$$\boldsymbol{b_n = 0}$$ (from pt (5) & (13) pg (12, 13), case(2)-pg(14))

$\because\ f(x) \to even\,function\ \ \&\ sin(nx) \to odd\,function$

Case(4) : Le $f(x)$t be an **even** function in the interval $(c,\ c+2l) = (-l,\ l)$ then

$$\boldsymbol{a_0 = \frac{2}{l}\int_{x=0}^{l} f(x)\,dx}$$ (from pt (5) page (12), case(4)-page(15))

$$\boldsymbol{a_n = \frac{2}{l}\int_{x=0}^{l} f(x)\ \cos\left(\frac{n\pi x}{l}\right)dx}$$ (from pt (5) & (12) pg (12), case(4)-pg(15))

$\because\ f(x) \to even\,function\ \ \&\ cos\left(\frac{n\pi x}{l}\right) \to even\,function$

$$\boldsymbol{b_n = 0}$$ (from pt (5) & (13) pg (12, 13), case(4)-pg(15))

$\because\ f(x) \to even\ function\ \ \&\ sin(nx) \to odd\ function$

Odd Function:

Case(1) : Le $f(x)$t be an **odd** function in the interval $(c,\ c+2l) = (0, 2\pi)$ then

$$\boldsymbol{a_0 = 0}$$ (from pt (6) page (12), case(1)-page(14))

$$\boldsymbol{a_n = 0}$$ (from pt (6) & (13) pg (12), case(1)-pg(14))

$\because\ f(x) \to odd\ function\ \ \&\ cos(nx) \to even\ function$

$$\boldsymbol{b_n = \frac{2}{\pi}\int_{x=0}^{\pi} f(x)\ \ sin(nx)dx}$$ (from pt (6) & (12) pg (12, 13), case(1)-pg(14))

$\because\ f(x) \to odd\ function\ \ \&\ sin(nx) \to odd\ function$

Case(2) : Le $f(x)$t be an **odd** function in the interval $(c,\ c+2l) = (0, 2l)$ then

$$\boldsymbol{a_0 = 0}$$ (from pt (6) page (12), case(3)-page(14))

$$\boldsymbol{a_n = 0}$$ (from pt (6) & (13) pg (12), case(3)-pg(14))

$\because\ f(x) \to odd\ function\ \ \&\ cos(nx) \to even\ function$

$$\boldsymbol{b_n = \frac{2}{\pi}\int_{x=0}^{\pi} f(x)\ sin\left(\frac{n\pi x}{l}\right) dx}$$ (from pt (6) & (12) pg (12, 13), case(3)-pg(14))

$\because\ f(x) \to odd\ function\ \ \&\ sin(nx) \to odd\ function$

Case(3) : Le $f(x)$t be an **odd** function in the interval $(c,\ c+2l) = (-\pi,\ \pi)$ then

$$\boldsymbol{a_0 = 0}$$ (from pt (5) page (12), case(2)-page(14))

$\boldsymbol{a_n = 0}$ (from pt (5) & (13) pg (12), case(2)-pg(14))

$\because\ f(x) \to 0dd\ function\ \ \&\ cos(nx) \to even\ function$

$\boldsymbol{b_n = \frac{2}{\pi} \int_{x=0}^{\pi} f(x)\ sin(nx) dx}$ (from pt (5) & (12) pg (12, 13), case(2)-pg(14))

$\because\ f(x) \to odd\ function\ \ \&\ sin(nx) \to odd\ function$

Case(4) : Le $f(x)$t be an **odd** function in the interval $(c,\ c + 2l) = (-l,\ l)$ then

$\boldsymbol{a_0 = 0}$ (from pt (5) page (12), case(4)-page(15))

$\boldsymbol{a_n = 0}$ (from pt (5) & (13) pg (12), case(4)-pg(15))

$\because\ f(x) \to odd\ function\ \ \&\ cos\left(\frac{n\pi x}{l}\right) \to even\ function$

$\boldsymbol{b_n = \frac{2}{\pi} \int_{x=0}^{\pi} f(x)\ sin\left(\frac{n\pi x}{l}\right) dx}$ (from pt (5) & (12) pg (12, 13), case(4)-pg(15))

$\because\ f(x) \to odd\ function\ \ \&\ sin\left(\frac{n\pi x}{l}\right) \to odd\ function$

Above obtained results can be formulated in the table as shown below

Interval	Type of fn $f(x)$	$\boldsymbol{a_0}$	$\boldsymbol{a_n}$	$\boldsymbol{b_n}$
$\boldsymbol{(0, 2\pi)}$	**Even**	$\boldsymbol{\frac{2}{\pi} \int_{x=0}^{\pi} f(x)\, dx}$	$\boldsymbol{\frac{2}{\pi} \int_{x=0}^{\pi} f(x)\ cos(nx) dx}$	**0**
$\boldsymbol{(0, 2l)}$	**Even**	$\boldsymbol{\frac{2}{l} \int_{x=0}^{l} f(x)\, dx}$	$\boldsymbol{\frac{2}{l} \int_{x=0}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx}$	**0**
$\boldsymbol{(-\pi, \pi)}$	**Even**	$\boldsymbol{\frac{2}{\pi} \int_{x=0}^{\pi} f(x)\, dx}$	$\boldsymbol{\frac{2}{\pi} \int_{x=0}^{\pi} f(x)\ cos(nx) dx}$	**0**
$\boldsymbol{(-l, l)}$	**Even**	$\boldsymbol{\frac{2}{l} \int_{x=0}^{l} f(x)\, dx}$	$\boldsymbol{\frac{2}{l} \int_{x=0}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx}$	**0**

$(0, 2\pi)$	Odd	0	0	$\frac{2}{\pi}\int_{x=0}^{\pi} f(x)\ \sin(nx)dx$
$(0, 2l)$	Odd	0	0	$\frac{2}{l}\int_{x=0}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)\ dx$
$(-\pi, \pi)$	Odd	0	0	$\frac{2}{\pi}\int_{x=0}^{\pi} f(x)\ \sin(nx)dx$
$(-l, l)$	Odd	0	0	$\frac{2}{l}\int_{x=0}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)\ dx$

Problems on Fourier Series Expansion:

Interval $(\mathbf{0}, \mathbf{2}\boldsymbol{\pi})$ & $(\mathbf{0}, \mathbf{2}\boldsymbol{l})$

Solve the Following:

1) Express $f(x) = (\pi - x)^2$ as a Fourier series of period 2π in the interval $0 < x < 2\pi$. Hence deduce the sum of the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

Solt: By data $f(x) = (\pi - x)^2$ ………………(1) where $0 < x < 2\pi$ $(or)\ x \in (0, 2\pi)$ ………(2)

Comparing $(0, 2\pi)$ $with$ $(c, c + 2l)$ we get $\boldsymbol{c = 0}$ & $\boldsymbol{l = \pi}$ …………………………(3)

Wkt the Fourier series expansion $f(x)$ over the interval $(c, c + 2l)$ is given as

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)} \quad \text{……………………(4)}$$

where $a_0,\ a_n$ & b_n are the Fourier co-efficients which are given by Euler's formulae as

$$\boldsymbol{a_0 = \frac{1}{l}\int_{x=c}^{c+2l} f(x)\, dx} \quad \text{……………………(5)}$$

$$\boldsymbol{a_n = \frac{1}{l}\int_{x=c}^{c+2l} f(x)\ cos\left(\frac{n\pi x}{l}\right) dx} \quad \text{……………………(6)}$$

$$\boldsymbol{b_n = \frac{1}{l}\int_{x=c}^{c+2l} f(x)\ sin\left(\frac{n\pi x}{l}\right) dx} \quad \text{……………………(7)}$$

Finding the nature of the given function $f(x) = (\pi - x)^2$, $x \in (0, 2\pi)$

Wkt

$$\int_{x=0}^{2a} f(x)\, dx = \begin{cases} 2\int_{x=0}^{a} f(x)\, dx\ , & f(2a - x) = f(x)\ \ i.e\ \ f(x)\ is\ an\ even\ function \\ 0, & f(2a - x) = -f(x)\ i.e\ f(x)\ is\ an\ odd\ function \end{cases}$$

$f(2\pi - x) = (\pi - \{2\pi - x\})^2$

$\Rightarrow f(2\pi - x) = (\pi - 2\pi + x)^2$

$\Rightarrow f(2\pi - x) = (-\pi + x)^2 = \{-(\pi - x)\}^2$

$\Rightarrow f(2\pi - x) = (\pi - x)^2 = f(x)$

$\therefore\ f(x) = (\pi - x)^2$ *is an **even** function* $\quad \therefore\ \boldsymbol{b_n = 0}$ ....(8)

Using eqns. (1),(2),(3) & (8) in eqns. (5), (6), (7) we get

$a_0 = \frac{1}{\pi}\int_{x=o}^{2\pi} f(x)\,dx \qquad \Rightarrow a_0 = \frac{2}{\pi}\int_{x=o}^{\pi} f(x)\,dx \quad \because\ f(x)\ is\ an\ even\ fn$

$\Rightarrow a_0 = \frac{2}{\pi}\int_{x=o}^{\pi}(\pi - x)^2\,dx \qquad \Rightarrow a_0 = \frac{2}{\pi}\left\{\frac{(\pi-x)^3}{(-3)}\right\}_0^{\pi}$

$\Rightarrow a_0 = \frac{2}{-3\pi}\{(\pi - x)^3\}_0^{\pi} \qquad \Rightarrow a_0 = \frac{-2}{3\pi}\{(\pi-\pi)^3 - (\pi - 0)^3\}$

$\Rightarrow a_0 = \frac{-2}{3\pi}\{0 - \pi^3\} \qquad \Rightarrow a_0 = \frac{-2}{3\pi}\{-\pi^3\}$

$\boldsymbol{a_0 = \frac{2\pi^2}{3}}$ ---------------------→ (9a) (or) $\boldsymbol{\frac{a_0}{2} = \frac{\pi^2}{3}}$ ------------------------→(9b)

Wkt from eqn.(6)

$a_n = \frac{1}{\pi}\int_{x=0}^{2\pi} f(x)\ cos\left(\frac{n\pi x}{\pi}\right)dx$

$\Rightarrow a_n = \frac{2}{\pi}\int_{x=0}^{\pi}(\pi - x)^2\ cos(nx)dx \quad \because\ f(x)\ \&\cos(nx)\ are\ both\ even\ fns$ ----→(10a)

By using Bernulli's rule of integration by parts, where

$\int uv = u_0v_1 - u_1v_2 + u_2v_3 - u_3v_4 + - \cdots$

$u = u_0 = (\pi - x)^2$	$v = \cos(nx)$
$u_1 = u' = 2(\pi - x)^1 * (-1)$	$\therefore v_1 = \int \cos(nx)\,dx \ \therefore\ v_1 = \frac{\sin(nx)}{n}$
$u_2 = u'' = (-2)(-1) \therefore\ u_2 = u'' = 2$	$\therefore v_2 = \int \frac{\sin(nx)}{n}\,dx \ \therefore\ v_2 = \frac{-\cos(nx)}{n^2}$
$u_3 = u''' = 0$	$\therefore v_3 = \int \frac{-\cos(nx)}{n^2}\,dx \ \therefore\ v_3 = \frac{-\sin(nx)}{n^3}$

Substituting the values obtained from the table in the above Bernoulli's equation we get

$$\int_{x=0}^{\pi}(\pi - x)^2\ cos(nx)dx = \left\{(\pi - x)^2 * \left(\frac{\sin(nx)}{n}\right)\right\}_0^{\pi} - \left\{-2(\pi - x)^1 * \left(\frac{-\cos(nx)}{n^2}\right)\right\}_0^{\pi}$$

$$+ \left\{(2) * \left(\frac{-\sin(nx)}{n^3}\right)\right\}_0^{\pi}$$

$\because \sin(0) = \sin(n\pi) = 0,\ n = 1, 2, 3, 4, \cdots$

$\Rightarrow\ \int_{x=0}^{\pi}(\pi - x)^2\ cos(nx)dx = \{0 - 0\} - \left\{\frac{2}{n^2}\right\}\{(\pi - x) * \cos(nx)\}_0^{\pi} + \left\{\frac{-2}{n^3}\right\}\{0 - 0\}$

$\Rightarrow \int_{x=0}^{\pi}(\pi - x)^2 \ cos(nx)dx = -\left\{\frac{2}{n^2}\right\}[\{(\pi-\pi) * \cos(n\pi)\} - \{(\pi - 0) * \cos(0)\}]$

$\Rightarrow \int_{x=0}^{\pi}(\pi - x)^2 \ cos(nx)dx = -\left\{\frac{2}{n^2}\right\}\{0 - (\pi)\} \quad \because \ \cos(0) = 1$

$\Rightarrow \int_{x=0}^{\pi}(\pi - x)^2 \ cos(nx)dx = -\left\{\frac{2}{n^2}\right\}\{-\pi\}, n = 1, 2, 3, 4, \cdots$

$\Rightarrow \int_{x=0}^{\pi}(\pi - x)^2 \ cos(nx)dx = \frac{2\pi}{n^2}, n = 1, 2, 3, 4, \cdots$ --------------------→(10b)

Substituting eqn. (10b) in the RHS of eqn. (10a) we get

$\Rightarrow a_n = \frac{2}{\pi} * \frac{2\pi}{n^2}, n = 1, 2, 3, 4, \cdots$

$\Rightarrow \boldsymbol{a_n = \frac{4}{n^2}}, \quad \boldsymbol{n = 1, 2, 3, 4, \cdots} \quad \& \ \boldsymbol{n \neq 0}$ --------------------------------→(10c)

Substituting for $a_0, a_n \ \& \ b_n \ from$ eqn.(9b), (5), (10c) & (8) in RHS of eqn. (4) we get

$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} * \cos\left(\frac{n\pi x}{\pi}\right) + 0$

$f(x) = \frac{\pi^2}{3} + 4 * \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$ (or)

$$\boxed{\boldsymbol{(\pi - x)^2 = \frac{\pi^2}{3} + 4 * \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}}}$$ ---------------------------------→ (11)

Thus eqn. (11) gives the Fourier series expansion of $f(x) = (\pi - x)^2$ in the interval $x \in (0, 2\pi)$

To deduce the sum of the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

put $\boldsymbol{x = 0}$ on both the sides of eqn. (11) but $x = 0 \notin (0, 2\pi)$

i.e $x = 0$ is an end point at which $f(x) = (\pi - x)^2$ is discontinuous

$\therefore \ f(x) \to \frac{1}{2}[f(0) + f(2\pi)]$ ....>(12a)

To find $f(0) \ put \ x = 0 \ \& \ to \ find \ f(2\pi) \ put \ x = 2\pi \ in \ f(x) = (\pi - x)^2$ we get

$f(0) = (\pi - 0)^2 \ \therefore \boldsymbol{f(0) = \pi^2}$	$f(2\pi) = (\pi - 2\pi)^2 \Rightarrow f(2\pi) = (-\pi)^2 \ \therefore \boldsymbol{f(2\pi) = \pi^2}$

$\therefore \ LHS \ of \ eqn. (11) \ is \ : \quad LHS = \frac{1}{2}[\pi^2 + \pi^2] \quad \therefore LHS = \pi^2$ in eqn.(11) & put $x = 0$

$\pi^2 = \frac{\pi^2}{3} + 4 * \sum_{n=1}^{\infty} \frac{\cos(0)}{n^2} \quad \Rightarrow \ \pi^2 - \frac{\pi^2}{3} = 4 * \sum_{n=1}^{\infty} \frac{1}{n^2}$

DEPARTMENT OF MATHEMATICS

$\Rightarrow \frac{2\pi^2}{3} = 4 * \sum_{n=1}^{\infty} \frac{1}{n^2}$ $\Rightarrow \frac{2\pi^2}{4*3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

$$\boxed{\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad (or) \quad 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}}$$

2) Express $f(x) = (\pi - x)$ as a Fourier series of period 2π in the interval $0 < x < 2\pi$. Hence deduce the sum of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots$

Solt: By data $f(x) = (\pi - x)$ --------------→(1) where $x \in (0, 2\pi)$ ---------------(2)

Comparing the interval $(c, c + 2l)$ *with* $(0, 2\pi)$ we get

$\boxed{c = 0 \;\&\; l = \pi}$ ------------------------------------(3)

To find the nature of the given function $f(x) = (\pi - x)$ where $x \in (0, 2\pi)$

$f(2\pi - x) = \pi - (2\pi - x) \Rightarrow f(2\pi - x) = \pi - 2\pi + x \Rightarrow f(2\pi - x) = -\pi + x$

$\therefore f(2\pi - x) = -(\pi - x) = -f(x) \quad \therefore f(2\pi - x) = -f(x)$

$\therefore f(x) = (\pi - x)$ *is an* **odd** *function in the interval* $x \in (0, 2\pi)$

$\boxed{\therefore \quad a_0 = 0}$ --------------------------------------→(4)

$\boxed{\therefore \quad a_n = 0}$ --------------------------------------→(5)

Wkt $b_n = \frac{1}{l} \int_{x=c}^{c+2l} f(x) \; sin\left(\frac{n\pi x}{l}\right) dx$, $c = 0 \;\&\; l = \pi$

$\Rightarrow b_n = \frac{1}{\pi} \int_{x=0}^{2\pi} f(x) \; sin\left(\frac{n\pi x}{\pi}\right) dx$, $f(x)$ & *sine fn. both are odd*

$\Rightarrow b_n = \frac{2}{\pi} \int_{x=0}^{\pi} f(x) \; sin(nx) dx$

$\Rightarrow b_n = \frac{2}{\pi} \int_{x=0}^{\pi} (\pi - x) \; sin(nx) dx$ -------------------------------(6a)

By applying Bernoulli's rule of integration by parts,

$$\int uv = u_0v_1 - u_1v_2 + u_2v_3 - u_3v_4 + -\cdots$$

$u = u_0 = (\pi - x)$	$v = \sin(nx)$
$u_1 = u' = -1$	$\therefore v_1 = \int \sin(nx)\,dx \;\therefore\; v_1 = \frac{-\cos(nx)}{n}$
$u_2 = u'' = 0$	$\therefore v_2 = \int \frac{-\cos(nx)}{n}\,dx \;\therefore\; v_2 = \frac{-\sin(nx)}{n^2}$

Substituting above obtained values from the table in the RHS of eqn. (6a) we get

$$\Rightarrow\; b_n = \frac{2}{\pi}\left(\left\{(\pi - x) * \left(\frac{-\cos(nx)}{n}\right)\right\}_0^{\pi} - \left\{(-1) * \left(\frac{-\sin(nx)}{n^2}\right)\right\}_0^{\pi} + 0\right)$$

$$\Rightarrow b_n = \frac{2}{\pi}\left(\frac{-1}{n}\{(\pi - x) * (\cos(nx))\}_0^{\pi} - \{0-0\}\right) \because \sin(0) = \sin(\pi) = \sin(2\pi) = \cdots = 0$$

$$\Rightarrow\; b_n = \frac{2}{\pi} * \left(\frac{-1}{n}\right) * \{(\pi - x) * \cos(nx)\}_0^{\pi}$$

$$\Rightarrow\; b_n = \frac{-2}{\pi} * \left(\frac{1}{n}\right) * \{[(\pi - \pi) * \cos(n\pi)] - [(\pi - 0) * \cos(0)]\}$$

$$\Rightarrow\; b_n = \frac{-2}{\pi} * \left(\frac{1}{n}\right) * \{0 - \pi\} \;\because\; \cos(0) = 1 \qquad \Rightarrow\; b_n = \frac{-2}{\pi} * \left(\frac{1}{n}\right) * (-\pi)$$

$$\Rightarrow\; b_n = 2 * \left(\frac{1}{n}\right) \;\Rightarrow\; b_n = \frac{2}{n}, n = 1, 2, 3, 4, \cdots$$

$$\boxed{\boldsymbol{b_n = \frac{2}{n}, n = 1, 2, 3, 4, \cdots}}$$

------------------------------------→(6b)

Thus substituting the values of a_0, a_n & b_n from eqns. (4), (5) & (6b) respectively in the Fourier series expansion of $f(x)$ over the interval $(c, c + 2l)$ we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \;,\; c = 0 \;\&\; l = \pi$$

$$\Rightarrow f(x) = 0 + 0 + \sum_{n=1}^{\infty}\left\{\left(\frac{2}{n}\right) * \sin\left(\frac{n\pi x}{\pi}\right)\right\}$$

$$\boldsymbol{\therefore (\pi - x) = 2 * \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}}$$ -------------------------------→(7)

DEPARTMENT OF MATHEMATICS

$$\boxed{\therefore (\pi - x) = 2 * \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}}$$

-------------------------------→(7)

Thus eqn.(7) gives the Fourier series expansion of $f(x) = (\pi - x)$ over the interval $(0, 2\pi)$

Deduction:

Put $x = \frac{\pi}{2} \in (0, 2\pi)$ on both the sides of eqn.(7) we get

$$\left(\pi - \frac{\pi}{2}\right) = 2 * \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n}$$

$$\Rightarrow \frac{\pi}{2} = 2 * \left\{\frac{\sin\left(\frac{\pi}{2}\right)}{1} + \frac{\sin\left(\frac{2\pi}{2}\right)}{2} + \frac{\sin\left(3\frac{\pi}{2}\right)}{3} + \frac{\sin\left(4\frac{\pi}{2}\right)}{4} + \frac{\sin\left(5\frac{\pi}{2}\right)}{5} + \frac{\sin\left(6\frac{\pi}{2}\right)}{6} + \frac{\sin\left(7\frac{\pi}{2}\right)}{7} + \cdots\right\}$$

$$\Rightarrow \frac{\pi}{2} = 2 * \left\{1 + 0 + \frac{(-1)}{3} + 0 + \frac{1}{5} + 0 + \frac{(-1)}{7} + - \cdots\right\}$$

$$\Rightarrow \frac{\pi}{2} = 2 * \left\{1 + -\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots\right\} \qquad \Rightarrow \frac{\pi}{4} = 1 + -\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots$$

$$\Rightarrow 1 + -\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}$$

$$\boxed{\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4} \quad \text{(or)} \quad 1 + -\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}}$$

Note:

1) If $f(x) = \begin{cases} \phi(x), & -\pi < x < 0 \\ \psi(x), & 0 < x < \pi \end{cases}$ then we say that $f(x)$ in the interval $(-\boldsymbol{\pi}, \boldsymbol{\pi})$ is

a) **Even** function: when $\phi(-x) = \psi(x)$

b) **Odd** function : when $\phi(-x) = -\psi(x)$

The above definition can be extended to the interval $(-\boldsymbol{l}, \boldsymbol{l})$

DEPARTMENT OF MATHEMATICS

Examples:

1) $f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$ is an **even** function because

Here $\phi(x) = -x$ & $\psi(x) = x$ $\therefore$ $\phi(-x) = -(-x)$ $\Rightarrow$ $\phi(-x) = x$

$\therefore$ $\phi(-x) = \psi(x)$ $\therefore f(x)\ is\ an\ even\ function\ in\ interval\ (-\pi, \pi)$

2) $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi < x < 0 \\ 1 - \frac{2x}{\pi}, & 0 < x < \pi \end{cases}$ is an **even** function because

Here $\phi(x) = 1 + \frac{2x}{\pi}$ & $\psi(x) = 1 - \frac{2x}{\pi}$

$\therefore$ $\phi(-x) = 1 - \frac{2x}{\pi}$ $\Rightarrow$ $\phi(-x) = \psi(x)$

Thus $f(x)$ is an **even** function in the interval $(-\pi, \pi)$

3) $f(x) = \begin{cases} x - \frac{\pi}{2}, & -\pi < x < 0 \\ x + \frac{\pi}{2}, & 0 < x < \pi \end{cases}$ is an **odd** function because

Here $\phi(x) = x - \frac{\pi}{2}$ & $\psi(x) = x + \frac{\pi}{2}$

$\therefore$ $\phi(-x) = -x - \frac{\pi}{2}$ $\Rightarrow$ $\phi(-x) = -\left(x + \frac{\pi}{2}\right)$

$\therefore$ $\phi(-x) = -\psi(x)$

Thus $f(x)$ is an **odd** function in the interval $(-\pi, \pi)$

4) $f(x) = \begin{cases} -k, & -\pi < x < 0 \\ +k, & 0 < x < \pi \end{cases}$ is an **odd** function because

Here $\phi(x) = -k$ & $\psi(x) = +k$

$\therefore$ $\phi(-x) = -k$ $\Rightarrow$ $\phi(-x) = -\psi(x)$

$\therefore$ $\phi(-x) = -\psi(x)$

Thus $f(x)$ is an **odd** function in the interval $(-\pi, \pi)$

DEPARTMENT OF MATHEMATICS

Note: $f(x) = |-x|$ is an even function because $|-x| = |x|$

Wkt $|-x| = \begin{cases} -x, & x < 0 \\ +x, & x > 0 \end{cases}$

Similarly,

If $f(x) = \begin{cases} \phi(x), & 0 < x < \pi \\ \psi(x), & \pi < x < 2\pi \end{cases}$ then we say that $f(x)$ in the interval $(\mathbf{0}, \mathbf{2\pi})$ is

a) **Even** function: when $\phi(2\pi - x) = \psi(x)$

b) **Odd** function : when $\phi(2\pi - x) = -\psi(x)$

The above definition can be extended to the interval $(\mathbf{0},\ \mathbf{2l})$

a) **Even** function: when $\phi(2l - x) = \psi(x)$

b) **Odd** function : when $\phi(2l - x) = -\psi(x)$

3) Obtain the Fourier series for the function $f(x) = 2x - x^2$ in $0 \le x \le 2$

Solt: By data $f(x) = 2x - x^2$ -----------------→(1)

$x \in [0, 2]$ but $(0, 2) \in [0, 2]$ ∴ $x \in (0, 2)$ ----------------------→(2)

Comparing $(0, 2)$ $with$ $(c, c + 2l)$ we get $\boldsymbol{c = 0}$ & $\boldsymbol{l = 1}$

$\boxed{\boldsymbol{c = 0} \ \& \ \boldsymbol{l = 1}}$ **--------------------------→(3)**

Nature of the function: in the interval $(0, 2l)$, $l = 1$ consider

$f(2l - x) = 2(2l - x) - (2l - x)^2,\ l = 1$

$f(2 - x) = 2(2 - x) - (2 - x)^2 \quad \Rightarrow f(2 - x) = (2 - x)\{2 - (2 - x)\}$

$\Rightarrow f(2 - x) = (2 - x)\{2 - 2 + x\} \quad \Rightarrow f(2 - x) = (2 - x)(x)$

$\Rightarrow f(2 - x) = 2x - x^2 \quad \Rightarrow f(2 - x) = f(x)$

∴ $f(x) = 2x - x^2$ is an **even** function in the interval $(0, 2)$

$$\boxed{\therefore\ b_n = 0}$$ --------------------------------→(4)

Wkt $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)dx$, $c = 0$ & $l = 1$

$\therefore a_0 = \frac{1}{1}\int_0^2 f(x)dx$, $f(x) \to even\ fn.$

$\therefore a_0 = 2\int_0^1 f(x)dx \quad \Rightarrow \quad \therefore a_0 = 2\int_0^1 \{2x - x^2\}dx$

$\therefore a_0 = 2\left[2\left(\frac{x^2}{2}\right) - \left(\frac{x^3}{3}\right)\right]_{x=0}^{1}$

$\therefore a_0 = 2\left[\left\{2\left(\frac{1}{2}\right) - \left(\frac{1}{3}\right)\right\} - 0\right] \quad \Rightarrow \quad \therefore a_0 = 2\left[1 - \frac{1}{3}\right]$

$\therefore a_0 = 2 * \frac{2}{3} \qquad \therefore \boldsymbol{a_0 = \frac{4}{3}}$ (or) $\boldsymbol{\frac{a_0}{2} = \frac{2}{3}}$

$$\boxed{\therefore\ \boldsymbol{a_0 = \frac{4}{3}} \quad \textbf{(or)} \quad \boldsymbol{\frac{a_0}{2} = \frac{2}{3}}}$$ ------------------------------→(5)

Wkt $a_n = \frac{2}{l}\int_{x=0}^{l} f(x)\ cos\left(\frac{n\pi x}{l}\right)dx, \quad l = 1$

$$a_n = \frac{2}{1}\int_{x=0}^{1}\{2x - x^2\}\ cos\left(\frac{n\pi x}{1}\right)dx \quad \Rightarrow \quad a_n = 2\int_{x=0}^{1}\{2x - x^2\}\ cos(n\pi x)dx$$

----------------------------→(6a)

By applying Bernoulli's rule of integration by parts,

$\int uv = u_0v_1 - u_1v_2 + u_2v_3 - u_3v_4 + -\cdots$

$u = u_0 = 2x - x^2$	$v = cos(n\pi x)$
$u_1 = u' = 2 - 2x$	$\therefore v_1 = \int cos(n\pi x)\ dx \ \therefore \quad v_1 = \frac{sin(n\pi x)}{n\pi}$
$u_2 = u'' = -2$	$\therefore v_2 = \int \frac{sin(n\pi x)}{n\pi}\ dx \ \therefore \quad v_2 = \frac{-cos(n\pi x)}{(n\pi)^2}$
$u_3 = u''' = 0$	$\therefore v_3 = \int \frac{-cos(n\pi x)}{(n\pi)^2}\ dx \ \therefore \quad v_3 = \frac{-\sin(n\pi x)}{(n\pi)^3}$

Using above obtained values in the RHS of eqn. (6a) we get

$$a_n = 2\left[\left\{(2x-x^2)*\frac{\sin(n\pi x)}{n\pi}\right\}-\left\{(2-2x)*\left(\frac{-\cos(n\pi x)}{(n\pi)^2}\right)\right\}+\left\{(-2)*\left(\frac{-\sin(n\pi x)}{(n\pi)^3}\right)\right\}\right]_0^1$$

$$a_n = 2\left[\{0-0\}+\left(\frac{1}{(n\pi)^2}\right)\{(2-2x)*\cos(n\pi x)\}_0^1+\{0-0\}\right] \quad \because \sin(0)=\sin(n\pi)=0$$

$$a_n = 2\left[\left(\frac{2}{(n\pi)^2}\right)\{(1-x)*\cos(n\pi x)\}\right]_0^1$$

$$a_n = \left(\frac{4}{(n\pi)^2}\right)\{0-1\} \quad \because \cos(0)=1 \qquad \therefore \boldsymbol{a_n = \frac{-4}{n^2\,\pi^2}}$$

$$\boxed{\therefore\ \boldsymbol{a_n = \frac{-4}{n^2\,\pi^2}}}$$ ....>(6b)

Substituting the values of a_0, a_n & b_n from eqns. (5), (6b) & (4) respectively in the Fourier series expansion $f(x)\ over\ interval\ (c, c+2l) = (0,2), c = 0\ \&\ l = 1$

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right) \ ,\ c=0\ \&\ l=1$$

$$\Rightarrow\ f(x) = \frac{2}{3}+\sum_{n=1}^{\infty}\left(\frac{-4}{n^2\,\pi^2}\right)\cos(n\pi x)+0$$

$$\boxed{\therefore\ \boldsymbol{2x-x^2 = \frac{2}{3}-\left(\frac{4}{\pi^2}\right)*\sum_{n=1}^{\infty}\left[\frac{\cos(n\pi x)}{n^2}\right]}}$$

--------------------------------→(7)

Gives the Fourier series expansion of $f(x) = 2x - x^2\ in\ 0 \le x \le 2$

4) Draw the graph of the function $f(x) = \begin{cases}\pi x\ , & 0 \le x \le 1\\ \pi(2-x), & 1 \le x \le 2\end{cases}$ and also

Show that the Fourier expansion of the function $f(x)$ is

$$\frac{\pi}{2}-\frac{4}{\pi}\left\{\frac{cos\pi x}{1^2}-\frac{cos3\pi x}{3^2}+\frac{cos5\pi x}{5^2}+\cdots\right\}$$

Hence show that $\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$

DEPARTMENT OF MATHEMATICS

Solt: By data, $f(x) = \begin{cases} \pi x \,, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$ ------------------------→(1)

Here interval is $(c,\ c+2l) = (0, 2)$ $\therefore$ $c = 0$ & $l = 1$

$$\boxed{\boldsymbol{c = 0 \ \& \ l = 1}}$$

--→(2)

Comparing eqn.(1) to the standard form $f(x) = \begin{cases} \phi(x)\,, & 0 < x < l \\ \psi(x), & l < x < 2l \end{cases}$

we get $\phi(x) = \pi x$ & $\psi(x) = \pi(2-x)$

$\phi(2l - x) = \pi(2l - x)\,,$ $\quad l = 1 \ \Rightarrow \ \phi(2-x) \neq \psi(x)$ & $\phi(2\pi - x) \neq -\psi(x)$

$$\phi(2-x) = \pi(2-x) \quad \Rightarrow \ \phi(2-x) = \psi(x)$$

Therefore the given function $f(x)$ is an **even** function in the interval $(0, 2)$

$$\boxed{\therefore \boldsymbol{b_0 = 0}}$$

------------------------------------→(3)

Euler's co-efficients:

1) $a_0 = \frac{2}{l}\int_{x=0}^{l} f(x)\,dx\,,$ $\quad f(x)$ *is an even function* $c = 0$ & $l = 1$

$\Rightarrow a_0 = \frac{2}{1}\int_{x=0}^{1} f(x)\,dx$

$\Rightarrow a_0 = 2\int_{x=0}^{1} \pi x\,dx$, using eqn.(1)

$\Rightarrow a_0 = 2\pi \left\{\frac{x^2}{2}\right\}_0^1$ $\quad \Rightarrow a_0 = \frac{\pi}{1}\{1 - 0\}$

$$\boxed{\therefore \ \boldsymbol{a_0 = \pi \quad (or) \quad \frac{a_0}{2} = \frac{\pi}{2}}}$$

----------------→(4)

2) $a_n = \frac{1}{l}\int_{x=c}^{c+2l} f(x)\ cos\left(\frac{n\pi x}{l}\right)dx,$ $\quad c = 0$ & $l = 1$

$\Rightarrow a_n = \frac{1}{l}\int_{x=0}^{2l} f(x)\ cos\left(\frac{n\pi x}{l}\right)dx$

$$\Rightarrow a_n = \frac{2}{l}\int_{x=0}^{l} f(x)\ cos\left(\frac{n\pi x}{l}\right)dx \ \because\ f(x)\ is\ an\ even\ fn.$$

$$\Rightarrow a_n = \frac{2}{1}\int_{x=0}^{1}(\pi\, x)\ cos\left(\frac{n\pi x}{1}\right)dx$$

$$\Rightarrow a_n = 2\int_{x=0}^{1}(\pi\, x)\ cos(n\pi x)dx$$

$$\Rightarrow a_n = 2\pi\int_{x=0}^{1}(\,x)\ cos(n\pi x)dx \qquad \text{------------------------}\rightarrow(5a)$$

By Bernoulli's rule of integration of parts

$$\Rightarrow a_n = 2\pi\left\{\left[x\ \left(\frac{\sin(n\pi x)}{n\pi}\right)\right] - \left[(1)\left(\frac{-\cos(n\pi x)}{(n\pi)^2}\right)\right] + 0\right\}_0^1$$

$$\Rightarrow a_n = 2\pi\left\{\left[x\ \left(\frac{\sin(n\pi x)}{n\pi}\right)\right] + \left[(1)\left(\frac{\cos(n\pi x)}{(n\pi)^2}\right)\right]\right\}_0^1$$

$$\Rightarrow a_n = 2\pi\left\{[0-0] + \frac{1}{(n\pi)^2}\ [\cos(n\pi) - 1]\right\} \quad \because \sin(0) = \sin(n\pi) = 0$$

$$\Rightarrow a_n = \frac{2\pi}{n^2\,\pi^2}\ \{\cos(n\pi) - 1\}\ \ \&\, wkt\ \cos(n\pi) = (-1)^n$$

$$\boxed{\therefore\ \ \boldsymbol{a_n = \frac{2}{n^2\,\pi}\ \{(-1)^n - 1\}}} \qquad \text{------------------------------}\rightarrow(5b)$$

Substituting the above obtained values of a_0 , a_n & b_n from equations (4), (5b) and (3) respectively in the Fourier series expansion of $f(x)\ in\ (c, c+2l), c = 0\ \&\ l = 1$ i.e $f(x)\ in\ (0,\ \ 2)$ we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\ \ ,\ c = 0\ \ \&\ \ l = 1$$

$$\Rightarrow\ \ f(x) = \frac{\boldsymbol{\pi}}{\boldsymbol{2}} + \sum_{n=1}^{\infty}\left\{\frac{\boldsymbol{2}}{\boldsymbol{n^2\,\pi}}\ \{(\boldsymbol{-1})^{\boldsymbol{n}} - \boldsymbol{1}\}\right\}\cos(n\pi x) + 0$$

$$\Rightarrow\ \ f(x) = \frac{\boldsymbol{\pi}}{\boldsymbol{2}} + \ \frac{2}{\pi}\ \sum_{n=1}^{\infty}\left\{\frac{(\boldsymbol{-1})^{\boldsymbol{n}}\boldsymbol{-1}}{\boldsymbol{n^2}}\right\}\cos(n\pi x)$$

$$\Rightarrow\ \ f(x) = \frac{\boldsymbol{\pi}}{\boldsymbol{2}} + \ \frac{2}{\pi}\ \left\{\frac{-2}{1^2} * \cos(\pi x) + \ 0 + \frac{-2}{3^2} * \cos(3\pi x) + 0 + \frac{-2}{5^2} * \cos(5\pi x) + \cdots\right\}$$

$$\Rightarrow \; f(x) = \frac{\pi}{2} + \; \frac{2}{\pi}\left\{\frac{-2}{1^2} * \cos(\pi x) + \frac{-2}{3^2} * \cos(3\pi x) + \frac{-2}{5^2} * \cos(5\pi x) + \cdots\right\}$$

$$\Rightarrow \; f(x) = \frac{\pi}{2} + \; \frac{-4}{\pi}\left\{\frac{\cos(\pi x)}{1^2} + \frac{\cos(3\pi x)}{3^2} + \frac{\cos(5\pi x)}{5^2} + \cdots\right\}$$

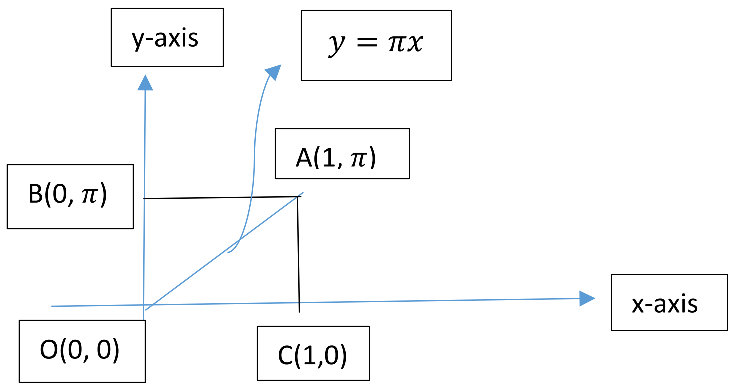
$$\boxed{\therefore \; f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left\{\frac{\cos(\pi x)}{1^2} + \frac{\cos(3\pi x)}{3^2} + \frac{\cos(5\pi x)}{5^2} + \cdots\right\}}$$

-------------→(6)

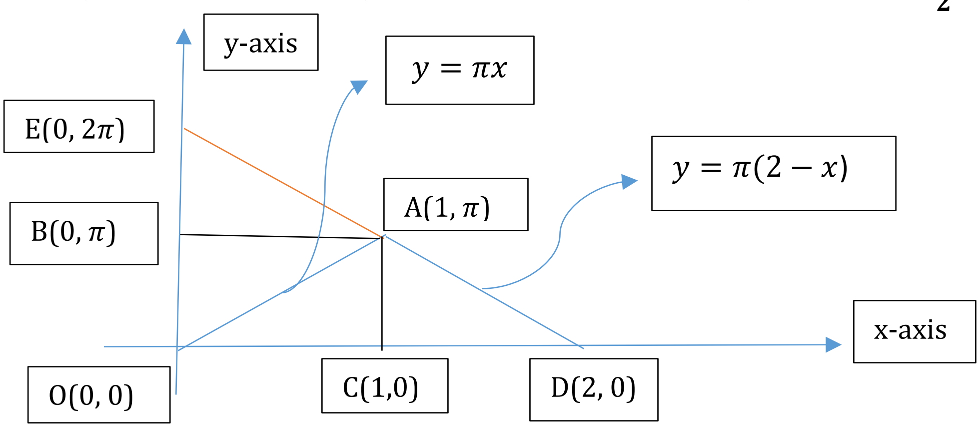
Equation (6) gives the Fourier series expansion of $f(x)$ *in* (0, 2)

Graph:

i) $y = \pi x \quad (y = mx, m = tan\theta = \pi \;\&\; opp = \pi, adj = 1)$



ii) $y = \pi(2 - x) \;\; \Rightarrow y = 2\pi - \pi x \;\; \Rightarrow \;\; \pi x + y = 2\pi \;\; \therefore \; \frac{x}{2} + \frac{y}{2\pi} = 1$



DEPARTMENT OF MATHEMATICS

Note: In the solution for the problem given by eqn.(6)

Let $x = 0 \in [0, 1]$ $\therefore$ $f(x) = 0$, *using eqn.* (1)

Put $f(x) = 0$ *in LHS of eqn.* (6) & $x = 0$ *in RHS of eqn.* (6) *we get*

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left\{\frac{\cos(0)}{1^2} + \frac{\cos(0)}{3^2} + \frac{\cos(0)}{5^2} + \cdots\right\}$$

$$\Rightarrow \; -\frac{\pi}{2} = -\frac{4}{\pi}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right\} \because \; \cos(0) = 1$$

$$\boxed{\therefore \; \boldsymbol{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}} \quad \text{----------------}\!\rightarrow (7)$$

Interval $(-\pi, \pi)$ & $(-l, l)$

5) Obtain the Fourier series expansion of $f(x) = x\cos(x)$ $in \; -\pi < x < \pi$

Soltn: By data we have $f(x) = x\cos(x)$ ----------------→(1)

& $x \in (-\pi, \pi) = (c, c + 2l)$. On comparing we get

$$\boxed{\boldsymbol{c = -\pi \quad \& \quad l = \pi}} \quad \text{-----------------------}\!\rightarrow (2)$$

Nature of the function:

Wkt, $x \; \rightarrow odd\ fn$ & $\cos(x) \rightarrow even\ fn.$

Wkt, the product of an odd fn with an even fn is always an ODD fn.

$\therefore f(x) = x\cos(x)$ *is an* ***ODD*** *function in* $-\pi < x < \pi$

$$\boxed{\boldsymbol{a_0 = 0}} \quad \text{---------------------}\!\rightarrow (3)$$

$$\boxed{\boldsymbol{a_n = 0}} \quad \text{---------------------}\!\rightarrow (4)$$

DEPARTMENT OF MATHEMATICS

To find b_n consider

$$b_n = \frac{1}{l}\int_{x=c}^{c+2l} f(x)\ \sin\left(\frac{n\pi x}{l}\right)dx \ \ ,\ c = -\pi \ \ \&\ l = \pi$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{x=-\pi}^{\pi} f(x)\ \sin\left(\frac{n\pi x}{\pi}\right)dx \ \ \Rightarrow b_n = \frac{1}{\pi}\int_{x=-\pi}^{\pi} f(x)\ \sin(nx)dx$$

Where, $f(x) = x\cos(x)$ is an ODD function

$$b_n = \frac{2}{\pi}\int_{x=0}^{\pi} f(x)\ \sin(nx)dx$$

$$b_n = \frac{2}{\pi}\int_{x=0}^{\pi} x\cos(x)\,sin(nx)dx \qquad \text{-----------------}\rightarrow(5a)$$

Wkt $\cos A\, sinB = \frac{1}{2}\{sin(A+B) - \sin(A-B)\},\ A = x\ \&\ B = nx$

$A + B = x + nx = x(1+n)\ (or)\ (n+1)x$

$A - B = x - nx = x(1-n)\ (or) - (n-1)x$

$\cos x \sin(nx) = \frac{1}{2}\{sin(n+1)x - \sin(-(n-1)x)\}$

(Note: $\cos x \sin(nx) = \sin(nx)\cos(x) \because angles\ for\ cosine\ \&\ sine\ are\ same\ on\ both\ sides\ \ but\ \ \cos x \sin(nx) \neq \sin x \cos(nx)$

$cosA * sinB = sinB * cosA \ \ but\ \ cosA * sinB \neq sinA * cosB$)

$\cos x \sin(nx) = \frac{1}{2}\{sin(n+1)x + \sin((n-1)x)\} \because \sin(-\theta) = \sin(\theta)$ ----------$\rightarrow$(5b)

Substitute eqn. (5b) in RHS of eqn.(5a) we get

$$b_n = \frac{2}{\pi}\int_{x=0}^{\pi} x\frac{1}{2}\{sin(n+1)x + \sin((n-1)x)\}dx$$

$$\Rightarrow\ b_n = \frac{2}{2\pi}\int_{x=0}^{\pi} x\{sin(n+1)x + \sin((n-1)x)\}dx$$

$$\Rightarrow\ b_n = \frac{1}{\pi}\int_{x=0}^{\pi} x\{sin(n+1)x + \sin((n-1)x)\}dx$$

$$\Rightarrow\ b_n = \frac{1}{\pi}\left\{\int_{x=0}^{\pi} x * sin(n+1)xdx \ + \int_{x=0}^{\pi} x * sin(n+1)x\,dx\right\} \qquad \text{---------}\rightarrow(5c)$$

DEPARTMENT OF MATHEMATICS

By applying Bernoullli's rule of integration by parts,

$\int_{x=0}^{\pi} x * sin(n+1)xdx$

$$= \left[\left\{x \, . \, \frac{-\cos(n+1)x}{(n+1)}\right\} - \left\{(1). \frac{-\sin(n+1)x}{(n+1)^2}\right\} + 0\right]_0^{\pi}$$

$$= \left[\left\{x \, . \, \frac{-\cos(n+1)x}{(n+1)}\right\}\right]_0^{\pi} + 0 \;\because\; \sin(n+1)\,\pi = \sin(0) = 0$$

$$= \frac{-1}{(n+1)} \left[\{x \, . \; \cos(n+1)\, x\}\right]_0^{\pi}$$

$$= \frac{-1}{(n+1)} \left[\{\pi \, . \; \cos((n+1)\pi)\} - \{0\}\right]$$

$$= \frac{-1}{(n+1)} \{\pi \; * \; (-1)^{n+1}\} \quad \because \; \cos(n\pi) = (-1)^n$$

$$= \frac{-1}{(n+1)} \{\pi \; * \; (-1)\,(-1)^n\} \quad (or) \;\; \pi \; \frac{(-1)^n}{(n+1)}$$

$$\therefore \;\; \int_{x=0}^{\pi} x * sin(n+1)xdx \;\; = -\pi \; \frac{(-1)^{n+1}}{(n+1)} \qquad \text{--------------}\!\rightarrow\!(5d)$$

$$\sim ly \quad \therefore \;\; \int_{x=0}^{\pi} x * sin(n-1)xdx \;\; = -\pi \; \frac{(-1)^{n-1}}{(n-1)} \qquad \text{--------------}\!\rightarrow\!(5e)$$

Substituting eqns. (5d) & (5e) in the RHS of eqn.(5c) we get

$$b_n = \frac{1}{\pi}\left\{\left[-\pi \; \frac{(-1)^{n+1}}{(n+1)}\right] + \left[-\pi \; \frac{(-1)^{n-1}}{(n-1)}\right]\right\}$$

$$b_n = \frac{-\pi}{\pi}\left\{\left[\frac{(-1)^{n+1}}{(n+1)}\right] + \left[\frac{(-1)^{n-1}}{(n-1)}\right]\right\}$$

$$b_n = -1\left\{\left[\frac{(-1)(-1)^n}{(n+1)}\right] + \left[\frac{(-1)(-1)^n}{(n-1)}\right]\right\}$$

$$b_n = (-1)\,(-1)(-1)^n\left\{\frac{1}{(n+1)} + \frac{1}{(n-1)}\right\}$$

$$b_n = (-1)^n\left\{\frac{1}{(n+1)} + \frac{1}{(n-1)}\right\}$$

$$b_n = (-1)^n\left\{\frac{2n}{n^2-1}\right\} \quad (or) \quad b_n = 2\left\{\frac{(-1)^n * n}{n^2-1}\right\}$$

$$\therefore \boldsymbol{b_n = 2\left\{\frac{(-1)^n * n}{n^2-1}\right\}}\ ,\ n \neq 1 \qquad \text{----------------------------}\rightarrow(6)$$

To find b_n at $n = 1$ $i.e\ b_1$ put $n = 1$ in eqn.(5a)

$$b_1 = \frac{2}{\pi}\int_{x=0}^{\pi} x\cos(x)\,sin(x)dx$$

Wkt $sin(2x) = 2sin(x)cos(x) = 2cos(x)\,sin(x) \therefore \cos(x)\sin(x) = \frac{\sin(2x)}{2}$

$$\Rightarrow b_1 = \frac{2}{\pi}\int_{x=0}^{\pi} x\frac{\sin(2x)}{2}dx \qquad \Rightarrow b_1 = \frac{2}{2\pi}\int_{x=0}^{\pi} x * sin(2x)dx$$

By applying Bernoulli's rule of integration by parts , $u = u_0 = x$ & $v = \sin(2x)$

$$\Rightarrow b_1 = \frac{1}{\pi}\left[\left\{x * \frac{-\cos(2x)}{2}\right\} - \left\{1 * \frac{-\sin(2x)}{4}\right\} + 0\right]_0^{\pi}$$

$$\Rightarrow b_1 = \frac{1}{\pi}\left[\frac{-1}{2}\{\pi * \cos(2\pi) - 0\} + \frac{1}{4}\{0-0\}\right] \because \sin(2\pi) = \sin(0) = 0$$

$$\Rightarrow b_1 = \frac{1}{\pi}\left[\frac{-1}{2}\{\pi * 1\}\right] \because \cos(2\pi) = 1$$

$$\Rightarrow b_1 = \frac{1}{\pi} * \left(\frac{-\pi}{2}\right) \quad \therefore b_1 = -\left(\frac{1}{2}\right)$$

$$\boxed{\boldsymbol{b_1 = -\left(\frac{1}{2}\right)}} \qquad \text{------------------------------}\rightarrow(7)$$

Thus substituting the above obtained values of a_0 , a_n , b_n & b_1 from eqns.(3), (4), (6) & (7) respectively in the Fourier series expansion of

$f(x) = x * cosx\ \ in\ (-\pi, \pi), c = -\pi\ \&\ l = \pi$ we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)\ ,\ c = -\pi\ \&\ l = \pi$$

$x * cosx = 0 + \sum_{n=1}^{\infty} 0 + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\pi}\right)$

$x * cosx = \sum_{n=1}^{\infty} b_n \sin(nx)$

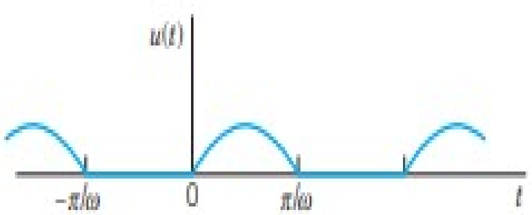
$\Rightarrow x * cosx = b_1 \sin(x) + \sum_{n=2}^{\infty} b_n \sin(nx)$

$$\boxed{\therefore \boldsymbol{x * cosx = \left(\frac{-1}{2}\right)\sin(x) + 2\sum_{n=2}^{\infty}\left\{\frac{(-1)^n * n}{n^2-1}\right\}\sin(nx)}} \quad \text{--------}\rightarrow(8)$$

Thus eqn.(8) gives the Fourier series expansion of $f(x) = x * \cos(x) \ \ in \ -\pi < x < \pi$

6) A sinusoidal voltage $n(\omega t)$, t is time is passed through a half wave rectifier that clips the negative portion of the wave as shown in the figure below. Find the Fourier series of the resulting periodic function

$$u(t) = \begin{cases} 0, & -L < t < 0 \\ Esin(\omega t), & 0 < t < L \end{cases} \quad \text{where period} \ \ p = \frac{2\pi}{\omega}$$



Soltn: By data , $u(t) = \begin{cases} 0, & -L < t < 0 \\ Esin(\omega t), & 0 < t < L \end{cases}$

For simplicity let $u(t) \ be \ replaced \ by \ f(t)$

$$\therefore \quad f(t) = \begin{cases} 0, & -L < t < 0 \\ Esin(\omega t), & 0 < t < L \end{cases} \quad \text{-------------------}\rightarrow(1)$$

Here $p = 2L = \frac{2\pi}{\omega} \;\therefore\; L = \frac{\pi}{\omega} \quad \because length\ of\ (-L,\ L) = 2L$

Also, $(c, c + 2L) = (-L, L) \quad (or) \quad (c, c + 2L) = \left(-\frac{\pi}{\omega}, \frac{\pi}{\omega}\right) \;\therefore\; c = \frac{\pi}{\omega}$

$$\boxed{\boldsymbol{c = \frac{\pi}{\omega} \ \&\ L = \frac{\pi}{\omega}}} \quad \text{--------------------------------}\rightarrow(2)$$

Nature: **Neither even nor odd**

Euler's Co-efficients:

$$a_0 = \frac{1}{L} \int_{t=c}^{c+2L} f(t)\, dt$$

$$\Rightarrow \ a_0 = \frac{1}{\frac{\pi}{\omega}} \int_{t=-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(t)\, dt$$

$$\Rightarrow \ a_0 = \frac{\omega}{\pi} \left[\int_{-\frac{\pi}{\omega}}^{0} f(t)\ dt + \int_{0}^{\frac{\pi}{\omega}} f(t)\ dt\right]$$

$$\Rightarrow \ a_0 = \frac{\omega}{\pi} \left[0 + \int_{0}^{\frac{\pi}{\omega}} Esin(\omega t)\ dt\right]$$

$$\Rightarrow \ a_0 = \frac{E\omega}{\pi} \int_{0}^{\frac{\pi}{\omega}} sin(\omega t)\ dt$$

$$\Rightarrow \ a_0 = \frac{E\omega}{\pi} \left[\frac{-\cos(\omega t)}{\omega}\right]_0^{\frac{\pi}{\omega}}$$

$$\Rightarrow \ a_0 = \frac{-E\omega}{\omega\pi} \left[\frac{\cos(\omega t)}{1}\right]_0^{\frac{\pi}{\omega}} \quad \Rightarrow \ a_0 = \frac{-E}{\pi} \left\{cos\left(\frac{\omega\pi}{\omega}\right) - \cos(0)\right\}$$

$$\Rightarrow \ a_0 = \frac{-E}{\pi} \{-1 - 1\} \ \because \ \cos(\pi) = -1 \ \& \ \cos(0) = 1$$

$$\boxed{\therefore \ \boldsymbol{a_0 = \frac{2E}{\pi} \quad (or) \quad \frac{a_0}{2} = \frac{E}{\pi}}} \quad \text{---------------------}\rightarrow (3)$$

wkt

$$a_n = \frac{1}{L}\int_{t=c}^{c+2L} f(t)\ \cos\left(\frac{n\pi t}{L}\right)\ dt$$

$$\Rightarrow\ a_n = \frac{1}{\frac{\pi}{\omega}}\int_{t=-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(t)\ \cos\left(\frac{n\pi t}{\frac{\pi}{\omega}}\right)\ dt$$

$$\Rightarrow\ a_n = \frac{\omega}{\pi}\int_{t=-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(t)\ \cos(n\omega t)\ dt$$

$$\Rightarrow\ a_n = \frac{\omega}{\pi}\left[\int_{-\frac{\pi}{\omega}}^{0} f(t)\cos(n\omega t)\ dt\ +\ \int_{0}^{\frac{\pi}{\omega}} f(t)\ \cos(n\omega t)\,dt\right]$$

$$\Rightarrow\ a_n = \frac{\omega}{\pi}\left[0\ +\ \int_{0}^{\frac{\pi}{\omega}} Esin(\omega t)\ \cos(n\omega t)\,dt\right]$$

$$\Rightarrow\ a_n = \frac{E\omega}{\pi}\left[\int_{0}^{\frac{\pi}{\omega}} sin(\omega t)\ \cos(n\omega t)\,dt\right] \quad \text{------------------------}\rightarrow(4a)$$

Wkt $sinA * cosB = \frac{1}{2}\ [\sin(A+B)+\sin(A-B)]$

$A = \omega t\ \&\ B = n\omega t\ \therefore\ A+B = (n+1)\omega t\ \&\ A-B = -(n-1)\omega t$

$\sin\omega t\ *\cos n\omega t = \frac{1}{2}\ \left[\sin\big((n+1)\omega t\big)+\sin(-(n-1)\omega t)\right]$

$\therefore \sin\omega t\ *\cos n\omega t = \frac{1}{2}\ \left[\sin\big((n+1)\omega t\big)-\sin((n-1)\omega t)\right]$

$$\Rightarrow\ a_n = \frac{E\omega}{\pi}\left[\int_{0}^{\frac{\pi}{\omega}} \frac{1}{2}\left[\sin\big((n+1)\omega t\big)-\sin\big((n-1)\omega t)\right]dt\right]$$

$$\Rightarrow\ a_n = \frac{E\omega}{2\pi}\left[\left\{\frac{-\cos\big((n+1)\omega t\big)}{(n+1)\omega}\right\}-\left\{\frac{-\cos\big((n-1)\omega t\big)}{(n-1)\omega}\right\}\right]_0^{\frac{\pi}{\omega}}$$

$$\Rightarrow\ a_n = \frac{E\omega}{2\pi\omega}\left[\left\{\frac{-\cos\big((n+1)\omega t\big)}{(n+1)}\right\}+\left\{\frac{\cos\big((n-1)\omega t\big)}{(n-1)}\right\}\right]_0^{\frac{\pi}{\omega}}$$

$$\Rightarrow\ a_n = \frac{E}{2\pi}\left[\frac{-1}{n+1}\left\{\cos\left((n+1)\omega\frac{\pi}{\omega}\right) - \cos(0)\right\} + \right.$$
$$\left. + \frac{1}{n-1}\left\{\cos\left((n-1)\omega\frac{\pi}{\omega}\right) - \cos(0)\right\}\right]$$

$$\Rightarrow\ a_n = \frac{E}{2\pi}\left[\frac{-1}{n+1}\{\cos((n+1)\pi) - 1\} + \frac{1}{n-1}\{\cos((n-1)\pi) - 1\}\right]$$

$$\Rightarrow\ a_n = \frac{E}{2\pi}\left[\frac{-1}{n+1}\{(-1)^{n+1} - 1\} + \frac{1}{n-1}\{(-1)^{n-1} - 1\}\right]$$

$$\because\ \cos(n\pi) = (-1)^n$$

$$\Rightarrow\ a_n = \frac{E}{2\pi}\left[\frac{-1}{n+1}\{(-1)^n(-1) - 1\} + \frac{1}{n-1}\{(-1)^n(-1) - 1\}\right]$$

$$\Rightarrow\ a_n = \frac{E}{2\pi}\left[\frac{-1(-1)}{n+1}\{(-1)^n + 1\} + \frac{(-1)}{n-1}\{(-1)^n + 1\}\right]$$

$$\Rightarrow\ a_n = \frac{E\{(-1)^n+1\}}{2\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$\Rightarrow\ a_n = \frac{E\{(-1)^n+1\}}{2\pi}\left[\frac{-2}{n^2-1}\right], n \neq 1$$

$$\Rightarrow\ a_n = \frac{-E}{\pi}\left[\frac{\{(-1)^n+1\}}{n^2-1}\right], n \neq 1$$

$$\boxed{\therefore\ \boldsymbol{a_n} = \begin{cases} -\dfrac{\mathbf{2E}}{\boldsymbol{\pi}} * \left(\dfrac{\mathbf{1}}{\boldsymbol{n^2 - 1}}\right), & \boldsymbol{n = even = 2, 4, 6, \cdots} \\ \mathbf{0}, & \boldsymbol{n = odd = 3, 5, 7, \cdots} \end{cases}} \quad \text{----}\rightarrow(4b)$$

When $n = 1\ in\ eqn.(4a) \Rightarrow\ a_1 = \frac{E\omega}{\pi}\left[\int_0^{\frac{\pi}{\omega}} sin(\omega t)\ \cos(\omega t)\, dt\right]$

$$\Rightarrow\ a_1 = \frac{E\omega}{\pi}\left[\int_0^{\frac{\pi}{\omega}} \frac{\sin(2\omega t)}{2}\ dt\right]$$

DEPARTMENT OF MATHEMATICS

$$\Rightarrow a_1 = \frac{E\omega}{2\pi}\left[\frac{-\cos(2\omega t)}{2\omega}\right]_0^{\frac{\pi}{\omega}}$$

$$\Rightarrow a_1 = \frac{-E\omega}{4\pi\omega}\left[\cos(2\omega t)\right]_0^{\frac{\pi}{\omega}}$$

$$\Rightarrow a_1 = \frac{-E}{4\pi}\{\cos(2\pi) - \cos(0)\}$$

$$\Rightarrow a_1 = 0, \because \cos(2\pi) = \cos(0) = 1$$

$\therefore$ $\boldsymbol{a_1 = 0}$ ----------------------------→(4c)

$$b_n = \frac{1}{L}\int_{t=c}^{c+2L} f(t)\ sin\left(\frac{n\pi t}{L}\right)\ dt$$

$$\Rightarrow b_n = \frac{1}{\frac{\pi}{\omega}}\int_{t=-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(t)\ \sin\left(\frac{n\pi t}{\frac{\pi}{\omega}}\right)\ dt$$

$$\Rightarrow b_n = \frac{\omega}{\pi}\int_{t=-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(t)\ \sin(n\omega t)\ dt$$

$$\Rightarrow b_n = \frac{\omega}{\pi}\left[\int_{-\frac{\pi}{\omega}}^{0} f(t)\sin(n\omega t)\ dt + \int_0^{\frac{\pi}{\omega}} f(t)\ \sin(n\omega t)\, dt\right]$$

$$\Rightarrow b_n = \frac{\omega}{\pi}\left[0 + \int_0^{\frac{\pi}{\omega}} Esin(\omega t)\ \sin(n\omega t)\, dt\right]$$

$\Rightarrow b_n = \frac{E\omega}{\pi}\int_0^{\frac{\pi}{\omega}} sin(\omega t)\ \sin(n\omega t)\, dt$ ----------------→(5a)

Wkt $sinA * sinB = -\frac{1}{2}[\cos(A+B) - cos(A-B)]$

$A = \omega t$ & $B = n\omega t$ $\therefore$ $A + B = (n+1)\omega t$ & $A - B = -(n-1)\omega t$

$\sin\omega t * \sin n\omega t = -\frac{1}{2}[\cos(n+1)\omega t - \cos(-(n-1)\omega t)]$

$\sin(\omega t) * \sin(n\omega t) = -\frac{1}{2}\left[\cos\big((n+1)\omega t\big) - \cos\big((n-1)\omega t\big)\right] \because \cos(-\theta) = cos(\theta)$

DEPARTMENT OF MATHEMATICS

$$\Rightarrow \ b_n = -\frac{E\omega}{\pi}\int_0^{\frac{\pi}{\omega}}\frac{1}{2}\left[\cos\big((n+1)\omega t\big)-\cos\big((n-1)\omega t\big)\right]dt$$

$$\Rightarrow \ b_n = -\frac{E\omega}{2\pi}\left[\left\{\frac{\sin\big((n+1)\omega t\big)}{(n+1)\omega}\right\}-\left\{\frac{\sin\big((n-1)\omega t\big)}{(n-1)\omega}\right\}\right]_0^{\frac{\pi}{\omega}}$$

$$\Rightarrow \ b_n = -\frac{E\omega}{2\pi\omega}\left[\frac{1}{(n+1)}\{\sin\big((n+1)\omega t\big)\}-\frac{1}{(n-1)}\{\sin\big((n-1)\omega t\big)\}\right]_0^{\frac{\pi}{\omega}}$$

-------------------------→(5b)

$$\Rightarrow \ b_n = \frac{E}{2\pi}[0-0]$$

$$\Rightarrow \ b_n = 0 \ \because \ \sin(n+1)\pi = \sin(n-1)\pi = \sin 0 = 0$$

$$\Rightarrow \ b_n = 0\,, n \neq 1 \quad i.e \quad n > 1$$ --------------------→(5c)

$n = 1 \Rightarrow \ b_1 = \frac{E\omega}{\pi}\int_0^{\frac{\pi}{\omega}} sin(\omega t)\ \sin(\omega t)\,dt$, using eqn. (5a)

$$\Rightarrow \ b_1 = \frac{E\omega}{\pi}\int_0^{\frac{\pi}{\omega}}\sin^2(\omega t)\ dt$$

Wkt $\sin^2(t) = \frac{1-\cos(2t)}{2}$ $\therefore \ \sin^2(\omega t) = \frac{1-\cos(2\omega t)}{2}$

$$\Rightarrow \ b_1 = \frac{E\omega}{\pi}\int_0^{\frac{\pi}{\omega}}\left[\frac{1-\cos(2\omega t)}{2}\right]dt$$

$$\Rightarrow \ b_1 = \frac{E\omega}{2\pi}\left[t-\frac{\sin(2\omega t)}{2\omega}\right]_0^{\frac{\pi}{\omega}}$$

$$\Rightarrow \ b_1 = \frac{E\omega}{2\pi}\left[\left\{\frac{\pi}{\omega}-0\right\}-\frac{1}{2\omega}\{0-0\}\right] \ \because \ \sin(2\pi) = \sin(0) = 0$$

$$\Rightarrow \ b_1 = \frac{E\omega}{2\pi} * \frac{\pi}{\omega}$$

$$\therefore \ b_1 = \frac{E}{2}$$ -------------------→(5d)

From eqns. (5c) & (5d) we have

$$\boxed{\therefore \ \boldsymbol{b_n = \begin{cases}\frac{E}{2}, & n = 1\\ 0, & n > 1\end{cases}}}$$ --------------------→(6)

Thus substituting the above obtained values of a_0 , a_n & b_n from eqns.(3), (4b) & (6) respectively in the Fourier series expansion of

$f(t)\ in\ the\ interval\ \left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right), c=-\frac{\pi}{\omega}$ & $l=\frac{\pi}{\omega}$ we get

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi t}{l}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi t}{l}\right)\ ,\qquad c=-\frac{\pi}{\omega}\ \&\ l=\frac{\pi}{\omega}$$

$$f(t)=\frac{E}{\pi}+\sum_{n=1,3,5,\cdots}^{\infty}a_n\cos\left(\frac{n\pi t}{l}\right)+\sum_{n=2,4,6,\cdots}^{\infty}a_n\cos\left(\frac{n\pi t}{l}\right)+$$

$$+b_1\sin\left(\frac{\pi t}{l}\right)+\sum_{n=2}^{\infty}b_n\sin\left(\frac{n\pi t}{l}\right)$$

$$f(t)=\frac{E}{\pi}+0+\sum_{n=2,4,6,\cdots}^{\infty}a_n\cos\left(\frac{n\pi t}{l}\right)+\frac{E}{2}\sin\left(\frac{\pi t}{l}\right)+0$$

$$f(t)=\frac{E}{\pi}+\sum_{n=2,4,6,\cdots}^{\infty}-\frac{2E}{\pi}*\left(\frac{1}{n^2-1}\right)\cos\left(\frac{n\pi t}{\frac{\pi}{\omega}}\right)+\frac{E}{2}\sin\left(\frac{\pi t}{\frac{\pi}{\omega}}\right)+0$$

$$f(t)=\frac{E}{\pi}-\frac{2E}{\pi}\sum_{n=2,4,6,\cdots}^{\infty}\left(\frac{1}{n^2-1}\right)\cos(n\omega t)+\frac{E}{2}\sin(\omega t)$$

$$\boxed{\therefore\ \boldsymbol{f(t)=\frac{E}{\pi}-\frac{2E}{\pi}\left[\frac{\cos(2\omega t)}{3}+\frac{\cos(4\omega t)}{15}+\frac{\cos(6\omega t)}{35}+\cdots\right]+\frac{E}{2}\sin(\omega t)}}$$

DEPARTMENT OF MATHEMATICS

---------------------------------------→(7)

Thus eqn.(7) gives the Fourier series of the periodic function passed through a half wave rectifier in the interval $(-L, L) = \left(-\frac{\pi}{\omega}, \frac{\pi}{\omega}\right)$

7) Obtain the Fourier series expansion of the function

$$f(x) = x^2 \ \textit{in the interval} \ \left(-\frac{3}{2},\ \frac{3}{2}\right)$$

Soltn: By data, $f(x) = x^2$ -------------------→(1)

$$(c,\ c+2l) = \left(-\frac{3}{2},\ \frac{3}{2}\right) \ \therefore \ c = -\frac{3}{2} \ \& \ l = \frac{3}{2}$$

$$\boxed{\therefore \ \boldsymbol{c = -\frac{3}{2}} \ \& \ \boldsymbol{l = \frac{3}{2}}}$$ ----------------------→(2)

Nature of the function: Here interval is $\left(-\frac{3}{2},\ \frac{3}{2}\right) = (-l, l)$

$f(x) = x^2 \quad \therefore \ f(-x) = (-x)^2 \ \Rightarrow \ f(-x) = x^2 \ \therefore \ \boldsymbol{f(-x) = f(x)}$

$\therefore \ f(x) = x^2$ *is an* ***even*** *function in the interval* $\left(-\frac{3}{2},\ \frac{3}{2}\right)$

$$\boxed{\therefore \ \boldsymbol{b_n = 0}}$$ -----------------------------→(3)

wkt when the function $f(x) = x^2$ is even in the interval $\left(-\frac{3}{2},\ \frac{3}{2}\right) = (-l, l)$

$a_0 = \frac{2}{l} * \int_{x=0}^{l} f(x)\ dx\,,\ l = \frac{3}{2}$

$\Rightarrow \ a_0 = \frac{2}{3/2} * \int_{x=0}^{3/2} x^2\ dx$

$\Rightarrow \ a_0 = \frac{4}{3} * \left\{\frac{x^3}{3}\right\}_0^{\frac{3}{2}} \qquad \Rightarrow \ a_0 = \frac{4}{9} * \left\{\left(\frac{3}{2}\right)^3 - 0\right\}$

$\Rightarrow \quad a_0 = \frac{4}{9} * \ \frac{27}{8} \quad \therefore \ a_0 = \frac{3}{2} \quad (or) \ \frac{a_0}{2} = \frac{3}{4}$

$$\boxed{\therefore \ \boldsymbol{a_0 = \frac{3}{2} \quad (or) \ \frac{a_0}{2} = \frac{3}{4}}} \qquad \text{-----------------------------}\rightarrow(4)$$

$a_n = \frac{2}{l} * \int_{x=0}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) \ dx, \ \ l = \frac{3}{2}$

$\Rightarrow \ a_n = \frac{2}{(3/2)} * \int_{x=0}^{3/2} x^2 \cos\left(\frac{n\pi x}{3/2}\right) \ dx \quad \Rightarrow \ a_n = \frac{4}{3} * \int_{x=0}^{3/2} x^2 \cos\left(\frac{2n\pi x}{3}\right) \ dx$

By applying Bernoulli's rule of integration by parts,

$$u = u_0 = x^2 \ \& \ v = \cos\left(\frac{2n\pi x}{3}\right), \qquad \int uv = u_0 v_1 - u_1 v_2 + u_2 v_3 - u_3 v_4 + - \cdots$$

$\Rightarrow \ a_n = \frac{4}{3} * \int_{x=0}^{3/2} x^2 \cos\left(\frac{2n\pi x}{3}\right) \ dx$

$$\Rightarrow \ a_n = \frac{4}{3} * \left[\left\{x^2 * \left(\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right)\right\} - \left\{2x * \left(\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right)\right\} + \left\{2 * \left(\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right)\right\} - 0\right]_0^{\frac{3}{2}}$$

$$\Rightarrow \ a_n = \frac{4}{3} * \left[\{0-0\} + \left(\frac{3}{2n\pi}\right)^2 * 2 * \left\{x * \ \cos\left(\frac{2n\pi x}{3}\right)\right\}_0^{\frac{3}{2}} + \{0-0\}\right]$$

$$\because \ \sin(0) = 0 = \sin(n\pi), n = 0,1,2,3,4\cdots$$

$$\Rightarrow \ a_n = \frac{4}{3} * \left(\frac{3}{2n\pi}\right)^2 * 2 \left[\left\{\left(\frac{3}{2}\right) * \right\} - 0\right]$$

$$\Rightarrow \ a_n = \frac{4}{3} * \left(\frac{3}{2n\pi}\right)^2 * 2 * \left(\frac{3}{2}\right) \ * \cos(n\pi)$$

$$\boxed{\therefore \ \boldsymbol{a_n = \left(\frac{3}{n\pi}\right)^2 * (-1)^n, \ \ n = 1, 2, 3, 4, \cdots}} \qquad \text{---------------------------}\rightarrow(5)$$

Thus substituting the above obtained values of a_0 , a_n & b_n from eqns.(4), (5) & (3) respectively in the Fourier series expansion of $f(x) = x^2$ *in the interval* $\left(-\frac{3}{2}, \frac{3}{2}\right)$ we get ,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \; , \; l = \frac{3}{2}$$

$$x^2 = \frac{\mathbf{3}}{\mathbf{4}} + \left(\frac{\mathbf{3}}{\boldsymbol{\pi}}\right)^{\mathbf{2}} \sum_{n=1}^{\infty} \frac{1}{n^2} * \cos\left(\frac{2n\pi x}{3}\right) + 0$$ and the required Fourier series is

$$\boxed{\therefore \;\; \boldsymbol{x^2 = \frac{3}{4} + \left(\frac{3}{\pi}\right)^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} * \cos\left(\frac{2n\pi x}{3}\right)}} \qquad \text{-------------}\rightarrow(6)$$

DEPARTMENT OF MATHEMATICS

Practice Problems:

1) Obtain the Fourier series expansion of $f(x) = \begin{cases} x^2 , & x \in (0,\pi) \\ -(2\pi - x)^2 , & x \in (\pi, 2\pi) \end{cases}$

Soltn: $f(x)$ *is an* **odd** fn *in* $0 < x < 2\pi$

$\therefore\ a_0 = 0 = a_n \quad \& \quad b_n = 2\pi * \frac{(-1)^{n+1}}{n} + \frac{4}{\pi}\left\{\frac{(-1)^n - 1}{n^3}\right\}$

$f(x) = \sum_{n=1}^{\infty}\left\{2\pi * \frac{(-1)^{n+1}}{n} + \frac{4}{\pi}\left\{\frac{(-1)^n-1}{n^3}\right\}\right\}\sin(nx)$

2) Obtain the Fourier series expansion of $f(x) = x\sin(x)$ $in\ -\pi < x < \pi$ and hence deduce that $\frac{1}{1*3} - \frac{1}{3*5} + \frac{1}{5*7} - \cdots = \frac{\pi - 2}{4}$ $in\ -\pi < x < \pi$

Solt: $f(x) = x\sin(x)$ *is an* **even** fn *in* $-\pi < x < \pi$

$b_n = 0\ ,\ a_0 = 2\ \ (or)\ \frac{a_0}{2} = 1\ ,\ a_n = 2\ \frac{(-1)^{n+1}}{n^2-1}\ , n \neq 1\ \ \&\ \ a_1 = -\frac{1}{2}$

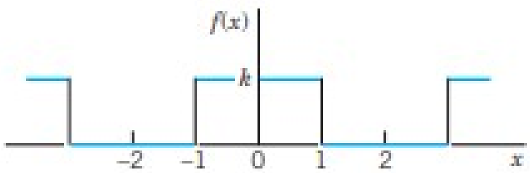
$$\therefore \boldsymbol{x * sinx = 1 - \frac{1}{2}\cos(x) + 2\sum_{n=2}^{\infty}\left\{\frac{(-1)^{n+1}}{n^2-1}\right\}\cos(nx)}$$

Deduction: put $x = \frac{\pi}{2}$

3) Find the Fourier series expansion of the function $f(x) = \begin{cases} 0\ , & -2 < x < -1 \\ k\ , & -1 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$

(or) Find the Fourier series expansion of the rectangular wave in the interval

$-2 < x < 2$



Soltn: $a_0 = k \ \ (or) \ \frac{a_0}{2} = \frac{k}{2}$

$a_n = \frac{2k}{n\pi}$, $n = 1, 5, 9, \cdots$ & $a_n = \frac{-2k}{n\pi}$, $n = 3, 7, 11, \cdots$

$a_n = 0, \ \ n = even \ i.e \ n = 2, 4, 6, \cdots$

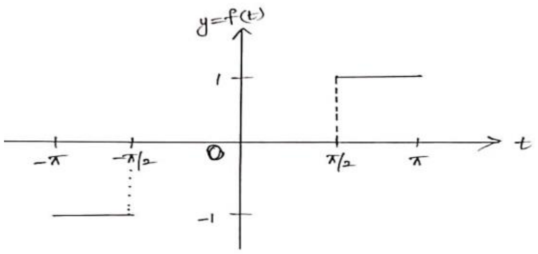
$b_n = 0, \ \because sin\left(\frac{n\pi x}{2}\right) \ is \ an \ odd \ fn \ in \ (-1, 1)$

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left\{cos\left(\frac{\pi x}{2}\right) - \frac{1}{3} * cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5} * cos\left(\frac{5\pi x}{2}\right) - + \cdots\right\}$$

4) Find the Fourier series for the function $f(t) = \begin{cases} -1, & -\pi < t < -\pi/2 \\ 0, & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < t < \pi \end{cases}$

Plot the graph for the given function.

(or) Obtain the Fourier series expansion for the function given in the graph below

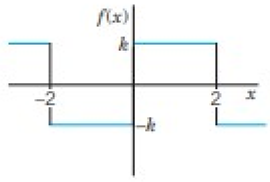


Soltn: $f(t) = \begin{cases} -1, & -\pi < t < -\pi/2 \\ 0, & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < t < \pi \end{cases}$

$a_0 = 0 = a_n \quad , \quad b_n = \frac{2}{\pi}\left[\frac{1}{n}\left\{\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right\}\right]$

$$f(t) = \frac{2}{\pi}\left[\sin t - \sin 2t + \frac{\sin 3t}{3} - \cdots\right]$$

5) Obtain the Fourier series expansion of the periodic rectangular wave function



Soltn: $f(x) = \begin{cases} -k, & -2 < x < 0 \\ k, & 0 < x < 2 \end{cases}$ $\quad (c, c+2l) = (-2, 2) \therefore \quad c = -2 \;\&\; l = 2$

Nature of fn: **ODD** in $(-2, 2)$ $\quad \boldsymbol{a_0 = 0 = a_n}$

$b_n = -\frac{2k}{n\pi} * \,[(-1)^n - 1],\;\; n = 1, 2, 3, 4, 5, \cdots$

$$\boldsymbol{b_n = -\frac{2k}{n\pi} * \,[-2] \quad (or) \quad b_n = \frac{4k}{n\pi}\;,\;\; n = 1, 3, 5, \cdots}$$

$$\boldsymbol{f(x) = \frac{4k}{\pi} * \left\{sin\left(\frac{\pi x}{2}\right) + \frac{1}{3} * sin\left(\frac{3\pi x}{2}\right) + \frac{1}{5} * sin\left(\frac{5\pi x}{2}\right) + \cdots\right\}}$$

6) Obtain the Fourier series expansion of the function

$f(x) = x^2 \;\; in\ the\ interval\ (-l,\;\; l)$

Solt: Nature: Even $\quad \therefore\; b_n = 0\;,\quad a_0 = \frac{2l^2}{3} \quad \& \quad a_n = \left(\frac{2l}{\pi}\right)^2 * \frac{(-1)^n}{n^2}$

$$\boldsymbol{\therefore\; x^2 = \frac{l^2}{3} + \left(\frac{2l}{\pi}\right)^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} * \cos\left(\frac{n\pi x}{l}\right)}$$

7) Obtain the Fourier series expansion of the function

$$f(x) = \begin{cases} \pi x \quad\quad\;\;, & 0 \le x \le 1 \\ \pi(2 - x), & 1 \le x \le 2 \end{cases}$$

Solt: Nature: even. $\quad \therefore\; b_n = 0 \quad \& \quad (0,\; 2) = (0, 2l) \;\therefore\; l = 1 \;\;\&\;\; c = 0$

$a_0 = \pi \quad , \quad a_n = \frac{2}{\pi} * \left\{\frac{(-1)^n - 1}{n^2}\right\}$

$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,7,\cdots}^{\infty} \left\{\frac{1}{n^2}\right\} * \cos(n\pi x)$

8) Obtain the Fourier series expansion of the function

$f(x) = -x^2 \;\; in\ the\ interval\ (-l,\ \ l)$

Solt: Nature: Even $\therefore\ a_0 = 0 = a_n$,

$b_n = \frac{2l^2}{\pi} * \left[\frac{1}{n} * \left\{(-1)^n - \frac{2}{\pi^2 n^2}\{(-1)^n - 1\}\right\}\right]$

9) Obtain the Fourier series expansion of the function $f(x) = e^x$, $0 < x < 2$

Soltn: $(0, 2) = (0, 2l) \therefore \quad l = 1$ Nature: Neither Even nor Odd

$a_0 = e^2 - 1$, $a_n = \frac{a(e^2-1)}{1+n^2\pi^2}$ & $b_n = \frac{b(1-e^2)}{1+n^2\pi^2}$

$$\boldsymbol{e^x = \frac{e^2-1}{2} + a(e^2-1) \sum_{n=1}^{\infty} \frac{\cos(n\pi x)}{1+n^2\pi^2} + b(1-e^2) \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{1+n^2\pi^2}}$$