Due by September 10th at 11:59pm EST on Gradescope

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There are 4 questions to answer for a total of 30 points:

1. [**3 points**] Suppose x and y are real, and $x \leq y + \epsilon$ for every $\epsilon > 0$. Show that $x \leq y$.

2. [**4 points**] (Rudin, Chapter 1, #5): Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

3. [**8 points**] Show that:

   (a) $k \leq 2^k$ for every natural number $k \in \mathbb{N}$ by induction.

   (b) The *dyadic rationals* are the rationals of the form

$$\left\{ \frac{j}{2^k} : j \in \mathbb{Z},\ k \in \mathbb{N} \right\}.$$

   Prove that for any two real numbers $x < y$, there exists a dyadic rational p such that $x < p < y$.

4. [**15 points**] (Rudin, Chapter 1, #6): Fix $b > 1$.

   (a) If m, n, p, q are integers, $n > 0, q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

   Hence it makes sense to define $b^r = (b^m)^{1/n}$.

   (b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

   (c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

   when r is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

   for every real x

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y following the steps outlined below:

i. First prove that $b^x b^y \leq b^{x+y}$ directly.

ii. Show that if $b > 1$ and $n \in \mathbb{N}$, then $b^{\frac{1}{n}} > 1$.

iii. Show that if $x > 0$ and $x \in \mathbb{R}$, then $(1+x)^n \geq 1 + nx$ for every $n \in \mathbb{N}$.

iv. Show that if $b > 1$ and $n \in \mathbb{N}$, then $1 < b^{\frac{1}{n}} < 1 + \frac{b-1}{n}$.

v. Show that if $u, v \in \mathbb{Q}$, $u > v$, and $b > 1$, then $b^u > b^v$.

vi. Note that, by the law of exponential for integers and #4(b), we have

$$1 = b^0 = b^{\frac{1}{n}+(-\frac{1}{n})} = b^{\frac{1}{n}} b^{-\frac{1}{n}}.$$

vii. Let $\epsilon > 0$ be arbitrary, and let $n \in \mathbb{N}$ be small enough (you need to specify this n).

viii. By the density of rationals, we can choose $r, s \in \mathbb{Q}$ such that

$$x - \frac{1}{2n} < r < x, \quad y - \frac{1}{2n} < s < y,$$

and thus

$$x + y - \frac{1}{n} < r + s < x + y.$$

ix. Now let $t \in \mathbb{Q}$ be such that $t \leq x + y$. By trichotomy, we have three different cases:

$$t < r + s, \quad t = r + s, \quad t > r + s.$$

x. If $t \leq r + s$, show that $b^t \leq b^{x+y}$.

xi. If $r + s < t$, then show that

$$t - \frac{1}{n} < r + s < t \leq x + y,$$

and using this and the above lemmas, show that

$$b^t < b^x b^y + \epsilon.$$

xii. Thus, in any case, we have

$$b^t < b^x b^y + \epsilon.$$

xiii. Show that if $b^{x+y} > b^x b^y$, then we get a contradition, and thus, $b^{x+y} \leq b^x b^y$.

Additional problems (for your own practice)

Do not turn in these, but I am happy to discuss:

- Rudin, Chapter 1, #1, #7