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A Series of Modern Surveys in Mathematics

Joram Lindenstrauss   Lior Tzafriri

Classical Banach Spaces I

Sequence Spaces

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Joram Lindenstrauss
Lior Tzafriri
Department of Mathematics, The Hebrew University of Jerusalem
Jerusalem, Israel

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To Naomi and Marianne

Preface

The appearance of Banach's book [8] in 1932 signified the beginning of a systematic study of normed linear spaces, which have been the subject of continuous research ever since.

In the sixties, and especially in the last decade, the research activity in this area grew considerably. As a result, Banach space theory gained very much in depth as well as in scope. Most of its well known classical problems were solved, many interesting new directions were developed, and deep connections between Banach space theory and other areas of mathematics were established.

The purpose of this book is to present the main results and current research directions in the geometry of Banach spaces, with an emphasis on the study of the structure of the classical Banach spaces, that is $C(K)$ and $L_p(\mu)$ and related spaces. We did not attempt to write a comprehensive survey of Banach space theory, or even only of the theory of classical Banach spaces, since the amount of interesting results on the subject makes such a survey practically impossible.

A part of the subject matter of this book appeared in outline in our lecture notes [96]. In contrast to those notes, most of the results presented here are given with complete proofs. We therefore hope that it will be possible to use the present book both as a text book on Banach space theory and as a reference book for research workers in the area. It contains much material which was not discussed in [96], a large part of which being the result of very recent research work. An indication to the rapid recent progress in Banach space theory is the fact that most of the many problems stated in [96] have been solved by now.

In the present volume we also state some open problems. It is reasonable to expect that many of these will be solved in the not too far future. We feel, however, that most of the topics discussed here have reached a relatively final form, and that their presentation will not be radically affected by the solution of the open problems. Among the topics discussed in detail in this volume, the one which seems to us to be the least well understood and which might change the most in the future, is that of the approximation property.

We divided our book into four volumes. The present volume deals with sequence spaces. The notion of a Schauder basis plays a central role here. The classical spaces which are in the most natural way sequence spaces are c_0 and l_p, $1 \leqslant p \leqslant \infty$. Volumes II and III will deal with function spaces. In Volume II we shall present the general theory of Banach lattices with an emphasis on those notions concerning lattices which are related to $L_p(\mu)$ spaces. Volume III will be devoted to a study of the structure of the spaces $L_p(0, 1)$, $C(K)$ and general preduals of

$L_1(\mu)$ spaces. The division of the common Banach spaces into sequence and function spaces is made according to the usual practice. It should be remembered, however, that several spaces have natural representations both as sequence and function spaces. The best known example is the separable Hilbert space, which can be represented both as the sequence space l_2 and as the function space $L_2(0, 1)$. A less trivial example is the space l_p, $1 \leqslant p \leqslant \infty$, which is isomorphic to the function space $H_p(D)$ of the analytic functions on the disc $D = \{z;\ |z| < 1\}$ with $\|f\| = (\iint |f(z)|^p \, \mathrm{d}x \, dy)^{1/p} < \infty$ (cf. [88]). Also, the spaces $C(0, 1)$ and $L_p(0, 1)$, $1 \leqslant p < \infty$, have Schauder bases, and thus it is convenient sometimes to use their representations as sequence spaces.

In Volume IV we intend to present the local theory of Banach spaces. This theory deals with the structure of finite-dimensional Banach spaces and the relation between an infinite-dimensional Banach space and its finite-dimensional subspaces. A central part in this approach to Banach space theory is played by the evaluation of various parameters of finite-dimensional Banach spaces. The role of the classical finite-dimensional spaces, that is of the spaces l_p^n, $1 \leqslant p \leqslant \infty$, $n = 1, 2, \ldots$ in the local theory of Banach spaces is even more central than the role of the classical spaces in the general theory of Banach sequence spaces and function spaces.

We sketch now briefly the contents of this volume. Chapter 1 contains a quite complete account of the main results on Schauder bases in general Banach spaces. Several notions related to Schauder bases—the various approximation properties, general biorthogonal systems and Schauder decompositions—as well as some examples are discussed in detail.

Chapter 2 is devoted to a study of the spaces c_0 and l_p, $1 \leqslant p < \infty$, and to some extent also of l_∞. Section *a* is devoted to an examination of the basic properties of these spaces, some of which are shown to characterize these spaces among general Banach spaces. The other sections of Chapter 2 are basically independent of each other and can thus be read in any order. In Sections *b* and *c* we discuss certain ideals of operators on general Banach spaces and show how they can be used in the study of the structure of the classical sequence spaces. Section *d* contains a structure theorem for "nice" subspaces of c_0 and l_p as well as examples of subspaces which are not "nice" (i.e. subspaces which fail to have the approximation property). This section contains also a discussion of general results related to the approximation property which complement the treatment of this property in Section *e* of Chapter 1. Section *f* contains an example of an infinite-dimensional Banach space which fails to have any of the classical sequence spaces as a subspace and also criteria for general Banach spaces to have subspaces isomorphic to c_0 and especially to l_1. The final section of Chapter 2 deals with the extension properties of c_0 and l_∞, the lifting property of l_1, and the closely related topic of the automorphisms of these spaces.

In Chapter 3 we discuss the special properties of symmetric bases and the relation between symmetric bases and general unconditional bases. A large part of this chapter is devoted to results and examples related to the possible characterizations of c_0 and l_p, $1 \leqslant p < \infty$, in the class of all spaces with a symmetric basis. The final chapter of this volume is devoted to a detailed study of the structure of some particular classes of spaces with symmetric bases, mainly Orlicz sequence spaces. The main emphasis is again on the relation between these spaces and the spaces c_0 and l_p. Several examples given there demonstrate how much more complicated the structure of general Orlicz sequence spaces is, as compared to that of l_p spaces. In section d it is shown that Orlicz sequence spaces enter naturally into the study of spaces like $l_p \oplus l_r$ with $p \neq r$. In Vol. III it will be shown that Orlicz sequence spaces arise naturally in the study of the structure of subspaces of $L_1(0, 1)$.

We assume that the reader is familiar with the basic results of real analysis and functional analysis which are usually covered in first year graduate courses in these subjects. An acquaintance with the main results in chapters I–VI of [33] will certainly suffice (much less is actually needed for being able to read this book).

The bibliography contains only those papers which are actually quoted in the text. We tried to indicate in the text the source of the main results which we present. The reference list is, however, far from being complete. Reference to papers where the basic results in Banach space theory were first proved can be found, for example, in [28] and [33]. Further references on bases may be found in [135]. References to further literature on Orlicz spaces may be found in [75].

The overlap between this book and existing books on related topics is very small.

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Jerusalem
January 1977

Joram Lindenstrauss
Lior Tzafriri

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Standard Definitions, Notations and Conventions

For most of the results presented in this book it does not matter whether the field of scalars is real or complex. In the isometric theory there are some differences (usually minor) between real and complex spaces. As a rule we shall work with real scalars and, in a few places, we shall indicate the changes needed in the complex case. In a few instances e.g. where spaces of analytic functions are involved or where spectral theory is used we shall use complex scalars.

By $L_p(\mu)=L_p(\Omega, \Sigma, \mu)$, $1\leqslant p\leqslant\infty$ we denote the Banach space of equivalence classes of measurable functions on (Ω, Σ, μ) whose p'th power is integrable (respectively, which are essentially bounded if $p=\infty$). The norm in $L_p(\mu)$ is defined by $\|f\|=(\int|f(\omega)|^p\, d\mu(\omega))^{1/p}$ (ess sup $|f(\omega)|$ if $p=\infty$). If (Ω, Σ, μ) is the usual Lebesgue measure space on $[0, 1]$ we denote $L_p(\mu)$ by $L_p(0, 1)$. If (Γ, Σ, μ) is the discrete measure space on a set Γ, with $\mu(\{\gamma\})=1$ for every $\gamma\in\Gamma$, we denote $L_p(\mu)$ by $l_p(\Gamma)$. If Γ is the set of positive integers we denote $l_p(\Gamma)$ also by l_p while if $\Gamma=\{1, 2, \ldots, n\}$, for some $n<\infty$, we denote $l_p(\Gamma)$ by l_p^n. The subspace of $l_\infty(\Gamma)$, of those functions which vanish at ∞, is denoted by $c_0(\Gamma)$ (if Γ is the set of positive integers we denote this space by c_0). The subspace of l_∞ consisting of convergent sequences is denoted by c. For a compact Hausdorff space K we denote by $C(K)$ the Banach space of all continuous scalar-valued functions on K with the supremum norm. If K is the unit interval $[0, 1]$ in its usual topology we denote $C(K)$ by $C(0, 1)$.

In a Banach space X we denote the ball with center x and radius r, i.e. $\{y;\ \|y-x\|\leqslant r\}$, by $B_X(x, r)$. If the space X is clear from the context, we simply write $B(x, r)$. The unit ball $B_X(0, 1)$ of X is denoted also by B_X. For a sequence $\{x_n\}_{n=1}^\infty$ of elements of X we denote by span $\{x_n\}_{n=1}^\infty$ the algebraic linear span of $\{x_n\}_{n=1}^\infty$ i.e. the set of all finite linear combinations of $\{x_n\}_{n=1}^\infty$. The closure of span $\{x_n\}_{n=1}^\infty$ is denoted by $[x_n]_{n=1}^\infty$. A similar notation is used for the span of a set other than a sequence. For a set $A\subset X$ its norm closure is denoted by $\overline{A}$, e.g. $[x_n]_{n=1}^\infty=\overline{\text{span}}\ \{x_n\}_{n=1}^\infty$. The convex hull of a sequence $\{x_n\}_{n=1}^\infty$ is denoted by conv $\{x_n\}_{n=1}^\infty$; the closed convex hull by $\overline{\text{conv}}\ \{x_n\}_{n=1}^\infty$.

The term "operator" means a bounded linear operator unless specified otherwise. The space of all operators from X to Y with the usual operator norm is denoted by $L(X, Y)$. An operator $T\in L(X, Y)$ is called *compact* if $\overline{TB_X}$ is a norm compact subset of Y. The identity operator of a Banach space X is denoted by I_X (or simply by I if X is clear from the context). For an operator $T\in L(X, Y)$ the notation $T_{|Z}$ denotes the restriction of T to the subspace Z of X.

Two Banach spaces X and Y are called *isomorphic* (denoted by $X\approx Y$) if there exists an invertible operator from X onto Y. *The Banach-Mazur distance coefficient*

$d(X, Y)$ is defined by $\inf \|T\|\,\|T^{-1}\|$, the infimum being taken over all invertible operators from X onto Y (if X is not isomorphic to Y we put $d(X, Y)=\infty$). Notice that $d(X, Y)\geq 1$, for every X and Y, and that $d(X, Y)\, d(Y, Z)\geq d(X, Z)$, for every X, Y and Z. If there exists an invertible operator T from X onto Y so that $\|T\|=\|T^{-1}\|=1$ (i.e. $\|Tx\|=\|x\|$, for every $x\in X$) we say that *X is isometric to Y*. In this case $d(X, Y)=1$ (the converse is false in general; it is possible that $d(X, Y)=1$ but that the infimum in the definition of $d(X, Y)$ is not attained i.e. X is not isometric to Y). An operator $T\in L(X, Y)$ is said to be an isomorphism into Y if there is some constant $C>0$ so that $\|Tx\|\geq C\|x\|$ for every $x\in X$. In this case T^{-1} is a well defined element in $L(TX, X)$.

A closed linear subspace Y of a Banach space X is said to be a *complemented subspace* of X if there is a bounded linear projection from X onto Y, or what is the same, if there exists a closed linear subspace Z of X so that X is the direct sum of Y and Z, i.e. $X=Y\oplus Z$. We shall also use some direct sums of infinite sequences of Banach spaces. If $\{X_n\}_{n=1}^\infty$ is a sequence of Banach spaces we define the direct sum of these spaces in the sense of l_p, $1\leqslant p<\infty$, namely $\left(\sum_{n=1}^\infty \oplus X_n\right)_p$, as the space of all sequences $x=(x_1, x_2, \dots)$, with $x_n\in X_n$ for all n, for which $\|x\|=\left(\sum_{n=1}^\infty \|x_n\|^p\right)^{1/p}<\infty$. Similarly, $\left(\sum_{n=1}^\infty \oplus X_n\right)_0$ denotes the direct sum of $\{X_n\}_{n=1}^\infty$ in the sense of c_0 i.e. the space of all sequences $x=(x_1, x_2, \dots)$, with $x_n\in X_n$ for all n, for which $\lim_n \|x_n\|=0$. The norm in this direct sum is taken as $\|x\|=\max_n \|x_n\|$. We shall occasionally use also other types of infinite direct sums. These will be defined in the proper places in the text.

Besides the norm (or strong) topology of a Banach space X we often use some other topologies. If Y is a subspace of the dual X^* of X then the Y-topology of X is the weakest topology making all the elements of Y continuous. A basis for the Y topology is obtained by taking all the sets of the form $V(x, \varepsilon, A)=\{u;\ |x^*(u)-x^*(x)|<\varepsilon,\ x^*\in A\}$, where $x\in X$, $\varepsilon>0$ and A is a finite subset of Y. If $Y=X^*$ the Y topology is called the weak topology (w topology). If $X=Z^*$ and we take as Y the canonical image of Z in $Z^{**}=X^*$ we obtain the w^* topology induced by Z (if Z is clear from the context we simply talk of the w^* topology). Convergence of sequences in the w topology (resp. w^* topology) is denoted by $x_n \xrightarrow{w} x$ or $w \lim_n x_n=x$ (resp. $x_n \xrightarrow{w^*} x$ or $w^* \lim_n x_n=x$). An operator $T\in L(X, Y)$ is said to be *w compact* if $\overline{TB_X}$ is a compact set in Y, in its w topology (i.e. a w compact set in Y).

Whenever we consider a Banach space X as a subspace of its second dual X^{**} we assume that it is embedded canonically. For a subset $A\subset X$ we denote by $A^\perp$ the subspace $\{x^*;\ x^*(x)=0,\ x\in A\}$ of X^*. For a subset $A\subset X^*$ we denote by $A^\top$ the subspace $\{x;\ x^*(x)=0,\ x^*\in A\}$ of X. For every subset $A\subset X$ we have $A^{\perp\top}\supset A$ and equality holds if and only if A is a closed linear subspace.

Besides subspaces of Banach spaces we shall also study quotient spaces. An operator $T\colon X\to Y$ is called a *quotient map* if $\overline{TB_X}=B_Y$. A Banach space Y is isomorphic to a quotient space of a space X if and only if there exists an operator T from X onto Y. If such a T exists then $Y\approx X/\ker T$, where $\ker T=\{x;\ Tx=0\}$, and Y^* is isomorphic to the subspace $(\ker T)^\perp$ of X^*. Similarly, if Z is a subspace of X then Z^* is isometric to the quotient space $X/(Z^\perp)$.

Among the general notations used in this book we want to single out the following. For a positive number S we denote by $[S]$ the largest integer $\leqslant S$. For a set A we denote by $\overline{\overline{A}}$ the cardinality of A. If A and B are sets we put $A \sim B = \{x, x \in A, x \notin B\}$.

1. Schauder Bases

a. Existence of Bases and Examples

The aim of this volume is to describe some results concerning sequence spaces, i.e. those Banach spaces which can be presented in some natural manner as spaces of sequences. In general, such a representation is achieved by introducing in the space a sort of "coordinate system". There are, obviously, many different ways of giving a precise meaning to the terms "Banach sequence spaces" and "coordinate systems". The best known and most useful approach is by using the notion of a Schauder basis.

Definition 1.a.1. A sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space X is called a *Schauder basis* of X if for every $x \in X$ there is a unique sequence of scalars $\{a_n\}_{n=1}^{\infty}$ so that $x = \sum_{n=1}^{\infty} a_n x_n$. A sequence $\{x_n\}_{n=1}^{\infty}$ which is a Schauder basis of its closed linear span is called a *basic sequence*.

In this book we shall not consider any type of bases in infinite-dimensional Banach spaces besides Schauder bases. We shall therefore often omit the word Schauder. In addition to Schauder bases we shall only encounter algebraic bases in finite-dimensional spaces. This should not cause any confusion. As a matter of fact, quantitative notions concerning Schauder bases (like the basis constant defined below) have a meaning and will be used also in the context of algebraic bases in finite dimensional spaces.

Evidently, a space X with a Schauder basis $\{x_n\}_{n=1}^{\infty}$ can be considered as a sequence space by identifying each $x = \sum_{n=1}^{\infty} a_n x_n$ with the unique sequence of coefficients $(a_1, a_2, a_3, \ldots)$. It is important to note that for describing a Schauder basis one has to define the basis vectors not only as a set but as an ordered sequence.

Let $(X, \|\ \|)$ be a Banach space with a basis $\{x_n\}_{n=1}^{\infty}$. For every $x = \sum_{n=1}^{\infty} a_n x_n$ in X the expression $|||x||| = \sup_n \left\| \sum_{i=1}^{n} a_i x_i \right\|$ is finite. Evidently, $|||\cdot|||$ is a norm on X and $\|x\| \leq |||x|||$ for every $x \in X$. A simple argument shows that X is complete also with respect to $|||\cdot|||$ and thus, by the open mapping theorem, the norms $\|\cdot\|$ and $|||\cdot|||$ are equivalent. These remarks prove the following proposition [8].

Proposition 1.a.2. *Let X be a Banach space with a Schauder basis $\{x_n\}_{n=1}^{\infty}$. Then the* *projections* $P_n: X \to X$, *defined by* $P_n\left(\sum_{i=1}^{\infty} a_i x_i\right) = \sum_{i=1}^{n} a_i x_i$, *are bounded linear operators and* $\sup_n \|P_n\| < \infty$.

The projections $\{P_n\}_{n=1}^{\infty}$ are called the natural projections associated to $\{x_n\}_{n=1}^{\infty}$; the number $\sup_n \|P_n\|$ is called the *basis constant* of $\{x_n\}_{n=1}^{\infty}$. A basis whose basis constant is 1 is called a *monotone basis*. In other words, a basis is monotone if, for every choice of scalars $\{a_n\}_{n=1}^{\infty}$, the sequence of numbers $\left\{\left\|\sum_{i=1}^{n} a_i x_i\right\|\right\}_{n=1}^{\infty}$ is non-decreasing. Every Schauder basis $\{x_n\}_{n=1}^{\infty}$ is monotone with respect to the norm $|||x||| = \sup_n \|P_n x\|$ which was already used above. Indeed,

$$|||P_n x||| = \sup_m \|P_m P_n x\| = \sup_{1 \leqslant m \leqslant n} \|P_m x\| \leqslant |||x||| .$$

Thus, given any Schauder basis $\{x_n\}_{n=1}^{\infty}$ of X, we can pass to an equivalent norm in X for which the given basis is monotone.

There is a simple and useful criterion for checking whether a given sequence is a Schauder basis.

Proposition 1.a.3. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence of vectors in* X. *Then* $\{x_n\}_{n=1}^{\infty}$ *is a Schauder basis of* X *if and only if the following three conditions hold.*

(i) $x_n \neq 0$ *for all* n.

(ii) *There is a constant* K *so that, for every choice of scalars* $\{a_i\}_{i=1}^{\infty}$ *and integers* $n < m$, *we have*

$$\left\|\sum_{i=1}^{n} a_i x_i\right\| \leqslant K \left\|\sum_{i=1}^{m} a_i x_i\right\| .$$

(iii) *The closed linear span of* $\{x_n\}_{n=1}^{\infty}$ *is all of* X.

The proof is easy. The necessity of (i) and (iii) is clear from the definition, while that of (ii) follows from 1.a.2. Conversely, if (i) and (ii) hold then $\sum_{n=1}^{\infty} a_n x_n = 0$ implies that $a_n = 0$ for all n. This proves the uniqueness of the expansion in terms of $\{x_n\}_{n=1}^{\infty}$. In order to prove that every $x \in X$ has such an expansion it is enough, in view of (iii), to show that the space of all elements of the form $\sum_{n=1}^{\infty} a_n x_n$ is a closed linear space. This latter fact can be easily proved by using (ii). □

Obviously conditions (i) and (ii) of 1.a.3, by themselves, form a necessary and sufficient condition for a sequence $\{x_n\}_{n=1}^{\infty}$ to be a basic sequence. It is also worthwhile to observe that in case we can take $K = 1$ it is enough to verify (ii) for $m = n + 1$.

A basis $\{x_n\}_{n=1}^{\infty}$ is called *normalized* if $\|x_n\| = 1$ for all n. Clearly, whenever $\{x_n\}_{n=1}^{\infty}$ is a Schauder basis of X, the sequence $\{x_n / \|x_n\|\}_{n=1}^{\infty}$ is a normalized basis in X.

Before proceeding with the general discussion we present some examples of bases. The unit vectors $e_n=(0, 0, 0,\dots, \overset{n}{1}, 0,\dots)$ form a monotone and normalized basis in each of the spaces c_0 and l_p, $1\leqslant p<\infty$. An example of a basis in the space c, of convergent sequences of scalars, is given by

$$x_1=(1, 1, 1,\dots) \quad \text{and, for } n>1,\ x_n=e_{n-1}\,.$$

The expansion of $x=(a_1, a_2,\dots)\in c$ with respect to this basis is

$$x=(\lim_n a_n)x_1+(a_1-\lim_n a_n)x_2+(a_2-\lim_n a_n)x_3+\cdots\,.$$

An important example of a Schauder basis is the Haar system in $L_p(0, 1)$, $1\leqslant p<\infty$.

Definition 1.a.4. The sequence of functions $\{\chi_n(t)\}_{n=1}^\infty$ defined by $\chi_1(t)\equiv 1$ and, for $k=0, 1, 2,\dots, l=1, 2,\dots, 2^k$,

$$\chi_{2^k+l}(t)=\begin{cases} 1 & \text{if } t\in[(2l-2)2^{-k-1}, (2l-1)2^{-k-1}] \\ -1 & \text{if } t\in((2l-1)2^{-k-1}, 2l\cdot 2^{-k-1}] \\ 0 & \text{otherwise}\end{cases}$$

is called the *Haar system.*

The Haar system is (in its given order) a monotone (but obviously not normalized) basis of $L_p(0, 1)$ for every $1\leqslant p<\infty$. Indeed, since the linear span of the Haar system contains all the characteristic functions of dyadic intervals (i.e. intervals of the form $[l\cdot 2^{-k}, (l+1)\cdot 2^{-k}]$), it is clear that (iii) of 1.a.3 holds. We have only to verify that (ii) holds with $K=1$. Let $\{a_i\}_{i=1}^\infty$ be any sequence of scalars, let n be an integer and let $f(t)=\sum_{i=1}^n a_i\chi_i(t)$ and $g(t)=\sum_{i=1}^{n+1} a_i\chi_i(t)$. The only difference between f and g is that on some dyadic interval I where f has the constant value b, say, g has the value $b+a_{n+1}$ on the first half of I and $b-a_{n+1}$ on the second half. Since, for every $p\geqslant 1$, $|b+a_{n+1}|^p+|b-a_{n+1}|^p\geqslant 2|b|^p$ we get that $||f||\leqslant||g||$.

By integrating the Haar system or more precisely by putting

$$\varphi_1(t)\equiv 1; \qquad \varphi_n(t)=\int_0^t \chi_{n-1}(u)\,du, \quad n>1$$

we obtain another famous and important basis. The sequence $\{\varphi_n\}_{n=1}^\infty$ is called the *Schauder system.* The Schauder system is a monotone basis of $C(0, 1)$. Indeed, the linear span of the $\{\varphi_n\}_{n=1}^\infty$ consists exactly of the continuous piecewise linear functions on $[0, 1]$ whose nodes are dyadic points. This shows that (iii) of 1.a.3 is satisfied. Since, for every integer n, the interval on which the function $\varphi_{n+1}(t)$ is different from 0 is such that on it all the functions $\{\varphi_i(t)\}_{i=1}^n$ are linear it follows immediately that (ii) of 1.a.3 holds with $K=1$.

Schauder bases have been constructed in many other important Banach spaces appearing in analysis. Of particular interest in this direction are the results of Z. Ciesielski and J. Domsta [18] and S. Schonefeld [132] who proved the existence of a basis in $C^k(I^n)$ (=the space of all real functions $f(t_1, t_2,\dots, t_n)$, $t_i \in [0, 1]$ which are k times continuously differentiable, with the obvious norm) and the result of S. V. Botschkariev [13] who proved the existence of a basis in the disc algebra A (=the space consisting of all the functions $f(z)$ which are analytic on $|z|<1$ and continuous on $|z|\leqslant 1$, with the sup norm). In these papers an important role is played by the *Franklin system.* The Franklin system consists of the sequence $\{f_n(t)\}_{n=1}^\infty$ of functions on [0, 1] which are obtained from the Schauder system $\{\varphi_n\}_{n=1}^\infty$ by applying the Gram–Schmidt orthogonalization procedure (with respect to the Lesbegue measure on [0, 1]). The Franklin system is (by definition) an orthonomal sequence which turns out to be also a Schauder basis of $C(0, 1)$. For a detailed study of the Franklin system we refer to the above mentioned papers as well as to [17].

The fact that in the common spaces there exists a Schauder basis led Banach to pose the question whether every separable Banach space has a basis. This problem (known as the basis problem) remained open for a long time and was solved in the negative by P. Enflo [37]. We shall present later on in this book (in Section 2.d) a variant of Enflo's solution.

The question whether every infinite-dimensional Banach space contains a basic sequence has, however, a positive answer. This simple fact was known already to Banach.

Theorem 1.a.5. *Every infinite dimensional Banach space contains a basic sequence.*

The proof, due to S. Mazur, is based on the following lemma.

Lemma 1.a.6. *Let X be an infinite dimensional Banach space. Let $B\subset X$ be a finite-dimensional subspace and let $\varepsilon>0$. Then there is an $x\in X$ with $||x||=1$ so that $||y||\leqslant(1+\varepsilon)||y+\lambda x||$ for every $y\in B$ and every scalar λ.*

Proof of 1.a.6. We may clearly assume that $\varepsilon<1$. Let $\{y_i\}_{i=1}^m$ be elements of norm 1 in B such that for every $y\in B$ with $||y||=1$ there is an i for which $||y-y_i||<\varepsilon/2$. Let $\{y_i^*\}_{i=1}^m$ be elements of norm 1 in X^* so that $y_i^*(y_i)=1$ for all i, and let $x\in X$ with $||x||=1$ and $y_i^*(x)=0$ for all i. This x has the desired property. Indeed, let $y\in Y$ with $||y||=1$, let i be such that $||y-y_i||\leqslant\varepsilon/2$ and let λ be a scalar. Then

$$||y+\lambda x||\geqslant||y_i+\lambda x||-\varepsilon/2\geqslant y_i^*(y_i+\lambda x)-\varepsilon/2=1-\varepsilon/2\geqslant||y||/(1+\varepsilon). \quad \square$$

Proof of 1.a.5. Let ε be any positive number and let $\{\varepsilon_n\}_{n=1}^\infty$ be positive numbers such that $\prod_{n=1}^\infty (1+\varepsilon_n)\leqslant 1+\varepsilon$. Let x_1 be any element in X with norm 1. By 1.a.6 we can construct inductively a sequence of unit vectors $\{x_n\}_{n=2}^\infty$ so that for every $n\geqslant 1$

$$||y||\leqslant(1+\varepsilon_n)||y+\lambda x_{n+1}|| \quad \text{for all } y\in \operatorname{span}\{x_1,\dots, x_n\} \text{ and every scalar } \lambda\,.$$

The sequence $\{x_n\}_{n=1}^\infty$ is a basic sequence in X whose basis constant is $\leqslant 1+\varepsilon$ (observe that $\|P_n\| \leqslant \prod_{i=n}^\infty (1+\varepsilon_i)$, $n=1, 2,\ldots$). $\square$

Remark. It is useful to note that in the proof of 1.a.6 it is enough to take a vector x with $\|x\|=1$ and $|y_i^*(x)|<\varepsilon/4$ for $i=1,\ldots, m$. Indeed, if $\|y\|=1$ and $|\lambda|\geqslant 2$ then $\|y+\lambda x\|\geqslant\|y\|$ while if $|\lambda|<2$ the computation in the proof of 1.a.6 gives that $\|y+\lambda x\|>(1-\varepsilon)\|y\|$. It follows from this observation and the proof of 1.a.5 that *if* $\{x_n\}_{n=1}^\infty$ *is a sequence of vectors in* X *such that* $\liminf_n \|x_n\|>0$ *and* $x_n \xrightarrow{w} 0$ *then* $\{x_n\}_{n=1}^\infty$ *has a subsequence* $\{x_{n_k}\}_{k=1}^\infty$ *which is a basic sequence.*

Once it is known that a Banach space has a Schauder basis it is natural to raise the question of its uniqueness. In order to study this question properly we introduce first the notion of equivalence of bases.

Definition 1.a.7. Two bases, $\{x_n\}_{n=1}^\infty$ of X and $\{y_n\}_{n=1}^\infty$ of Y, are called *equivalent* provided a series $\sum_{n=1}^\infty a_n x_n$ converges if and only if $\sum_{n=1}^\infty a_n y_n$ converges.

Thus the bases are equivalent if the sequence space associated to X by $\{x_n\}_{n=1}^\infty$ is identical to the sequence space associated to Y by $\{y_n\}_{n=1}^\infty$. It follows immediately from the closed graph theorem that $\{x_n\}_{n=1}^\infty$ is equivalent to $\{y_n\}_{n=1}^\infty$ if and only if there is an isomorphism T from X onto Y for which $Tx_n=y_n$ for all n.

Using the notion of equivalence, the uniqueness question can be given a meaningful formulation. It turns out however that even up to equivalence bases, if they exist at all, are never unique.

Theorem 1.a.8 [120]. *Let X be an infinite dimensional Banach space with a Schauder basis. Then there are uncountably many mutually non-equivalent normalized bases in X.*

We shall discuss in detail some aspects of uniqueness of bases in Chapters 2 and 3 below. We shall show there that if we restrict the discussion to bases which have some nice properties then it is possible to have uniqueness in some interesting special cases. In this context we shall also present a proof of a weak version of 1.a.8 (namely that there are at least two non-equivalent normalized bases in every space having a basis).

Schauder bases have certain stability properties. If we perturb each element of a basis by a sufficiently small vector we still get a basis. The perturbed basis is equivalent to the original one. The simplest result in this direction is the following useful proposition [76].

Proposition 1.a.9. (i) *Let $\{x_n\}_{n=1}^\infty$ be a normalized basis of a Banach space X with basis constant K. Let $\{y_n\}_{n=1}^\infty$ be a sequence of vectors in X with $\sum_{n=1}^\infty \|x_n-y_n\|<1/2K$. Then $\{y_n\}_{n=1}^\infty$ is a basis of X which is equivalent to $\{x_n\}_{n=1}^\infty$ (if $\{x_n\}_{n=1}^\infty$ is just a basic sequence then $\{y_n\}_{n=1}^\infty$ will also be a basic sequence which is equivalent to $\{x_n\}_{n=1}^\infty$).*

(ii) *Let* $\{x_n\}_{n=1}^\infty$ *be a normalized basic sequence in a Banach space* X *with a basis constant* K. *Assume that there is a projection* P *from* X *onto* $[x_n]_{n=1}^\infty$. *Let* $\{y_n\}_{n=1}^\infty$ *be a sequence of vectors in* X *such that* $\sum_{n=1}^\infty \|x_n-y_n\| \leqslant 1/8K\|P\|$. *Then* $Y=[y_n]_{n=1}^\infty$ *is complemented in* X.

Proof. For $x=\sum_{n=1}^\infty a_n x_n \in X$ define $Tx=\sum_{n=1}^\infty a_n y_n$. The series converges and

$$(*) \qquad \|x-Tx\| \leqslant \sum_{n=1}^\infty |a_n|\,\|x_n-y_n\| \leqslant \max_n |a_n| \sum_{n=1}^\infty \|x_n-y_n\| \leqslant 2K\|x\| \sum_{n=1}^\infty \|x_n-y_n\| .$$

To prove (i) we have just to observe that under its assumptions $\|I-T\|<1$ and hence T is an automorphism of X.

To prove (ii) we have to observe that if we put $y=Tx$, then $\|y-x\|<\|x\|/4$ and in particular $\|x\|<2\|y\|$ and $\|T\|<2$. Thus

$$\|TPy-y\| = \Big\|TP\Big(\sum_{n=1}^\infty a_n(y_n-x_n)\Big)\Big\| < 8K\|P\|\,\|y\| \sum_{n=1}^\infty \|x_n-y_n\| = \delta\|y\|$$

for some $\delta<1$. Thus $S=TP_{|Y}$ is an invertible operator on Y and $S^{-1}TP$ is a projection from X onto Y. □

A very useful method to obtain new basic sequences, starting from a given basis or basic sequence, is by considering block bases.

Definition 1.a.10. Let $\{x_n\}_{n=1}^\infty$ be a basic sequence in a Banach space X. A sequence of non-zero vectors $\{u_j\}_{j=1}^\infty$ in X of the form $u_j=\sum_{n=p_j+1}^{p_{j+1}} a_n x_n$, with $\{a_n\}_{n=1}^\infty$ scalars and $p_1<p_2<\cdots$ an increasing sequence of integers, is called a *block basic sequence* or briefly a *block basis* of the $\{x_n\}_{n=1}^\infty$.

It is obvious that a block basis $\{u_j\}_{j=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ is a basic sequence whose basis constant does not exceed that of $\{x_n\}_{n=1}^\infty$. The usefulness of the notion of block basis rests very much on the following simple observation [9].

Proposition 1.a.11. *Let* X *be a Banach space with a Schauder basis* $\{x_n\}_{n=1}^\infty$. *Let* Y *be a closed infinite dimensional subspace of* X. *Then there is a subspace* Z *of* Y *which has a basis which is equivalent to a block basis of* $\{x_n\}_{n=1}^\infty$.

Proof. Observe first that since Y is infinite dimensional there is, for every integer p, an element $y \in Y$ with $\|y\|=1$ of the form $y=\sum_{n=p+1}^\infty a_n x_n$. We construct the block basis of $\{x_n\}_{n=1}^\infty$ inductively. Pick any $y_1=\sum_{n=1}^\infty a_{n,1} x_n \in Y$ with $\|y_1\|=1$. Let p_1 be an integer so that $||y_1-u_1||<1/4K$ where $u_1=\sum_{n=1}^{p_1} a_{n,1}x_n$ and K is the basis constant of $\{x_n\}_{n=1}^\infty$. Next we pick a $y_2=\sum_{n=p_1+1}^{\infty} a_{n,2}x_n \in Y$ with $||y_2||=1$ and an integer p_2 so that $||y_2-u_2||<1/4^2K$ where $u_2=\sum_{n=p_1+1}^{p_2} a_{n,2}x_n$. We continue in an obvious manner. The sequence $\{u_j\}_{j=1}^\infty$ obtained in this way is a block basis of $\{x_n\}_{n=1}^\infty$. Since $\sum_{j=1}^\infty ||y_j-u_j||<1/3K$ it follows by 1.a.9 that $\{y_j\}_{j=1}^\infty$ is a basic sequence which is equivalent to $\{u_j\}_{j=1}^\infty$. The space $Z=[y_j]_{j=1}^\infty$ has the desired property. □

In the proof of 1.a.11 we used the vectors $y_j \in Y$ whose expansions with respect to the basis $\{x_n\}_{n=1}^\infty$ started arbitrarily far. In some instances it is important to be able to choose a basic sequence out of a subset Y of X which is not a subspace. For this purpose it is of interest to observe that what we actually need in the proof of 1.a.11 is the following. For every $\varepsilon>0$ and every integer p there is a $y=\sum_{n=1}^\infty a_nx_n$ in Y with $||y||\geqslant 1$ and $\left\|\sum_{n=1}^p a_nx_n\right\|\leqslant\varepsilon$. This remark proves the following.

Proposition 1.a.12. *Let $\{x_n\}_{n=1}^\infty$ be a Schauder basis of a Banach space X. Let $y_k=\sum_{n=1}^\infty a_{n,k}x_n$, $k=1, 2,\dots$, be a sequence of vectors such that $\limsup_k ||y_k||>0$ and $\lim_k a_{n,k}=0$ for every n (this is the case in particular if $y_k \xrightarrow{w} 0$ and $||y_k|| \nrightarrow 0$). Then there is a subsequence $\{y_{k_j}\}_{j=1}^\infty$ of $\{y_k\}_{k=1}^\infty$ which is equivalent to a block basis of $\{x_n\}_{n=1}^\infty$.*

Proposition 1.a.11 enables us to give an alternative proof of 1.a.5. It is clearly enough to prove 1.a.5 for separable Banach spaces X. Every such X is isometric to a subspace of $C(0, 1)$. Hence, by 1.a.11, X has a subspace with a basis which is equivalent to a block basis of the Schauder system in $C(0, 1)$.

b. Schauder Bases and Duality

Let X be a Banach space with a Schauder basis $\{x_n\}_{n=1}^\infty$. For every integer n the linear functional x_n^* on X defined by $x_n^*\left(\sum_{i=1}^\infty a_ix_i\right)=a_n$ is, by 1.a.2, a bounded linear functional. In fact $||x_n^*||\leqslant 2K/||x_n||$ where K is the basis constant of $\{x_n\}_{n=1}^\infty$. These functionals $\{x_n^*\}_{n=1}^\infty$, which are characterized by the relation $x_n^*(x_m)=\delta_n^m$, are called the *biorthogonal functionals* associated to the basis $\{x_n\}_{n=1}^\infty$. Let $\{P_n\}_{n=1}^\infty$ be the natural projections associated to the basis, i.e. $P_n\left(\sum_{i=1}^\infty a_ix_i\right)=\sum_{i=1}^n a_ix_i$. For every choice of scalars $\{a_i\}_{i=1}^\infty$ and for all integers $n<m$ we have $P_n^*\left(\sum_{i=1}^m a_ix_i^*\right)=\sum_{i=1}^n a_ix_i^*$.

Hence, by 1.a.3, the sequence $\{x_n^*\}_{n=1}^\infty$ is a basic sequence in X^* whose basis constant is identical to that of $\{x_n\}_{n=1}^\infty$. Since $\lim_n \|P_n x - x\| = 0$, for every $x \in X$, we get that, in the sense of convergence in the w^* topology, $x^* = \sum_{n=1}^\infty x^*(x_n)x_n^*$ for every $x^* \in X^*$. In general, this expansion does not converge in norm. We have convergence in norm for every $x^* \in X^*$ if and only if the sequence $\{x_n^*\}_{n=1}^\infty$ is a basis of X^* i.e., (by 1.a.3) if and only if the closed linear span of $\{x_n^*\}_{n=1}^\infty$ is all of X^*. For this to happen X^* must in particular be separable; thus, for $X = l_1$ or $X = C(0, 1)$ this cannot happen for any basis. On the other hand this is always the case if X is reflexive. The following proposition gives a very simple but useful criterion for checking whether $\{x_n^*\}_{n=1}^\infty$ is a basis of X^*.

Proposition 1.b.1. *Let $\{x_n\}_{n=1}^\infty$ be a basis of a Banach space X. The biorthogonal functionals $\{x_n^*\}_{n=1}^\infty$ form a basis of X^* if and only if, for every $x^* \in X^*$, the norm of $x^*_{|[x_i]_{i=n}^\infty}$ (= the restriction of x^* to the span of $\{x_i\}_{i=n}^\infty$) tends to 0 as $n \to \infty$. A basis $\{x_n\}_{n=1}^\infty$ which has this property is called shrinking.*

Proof. If $\{x_n^*\}_{n=1}^\infty$ is a basis of X^* then, for every $x^* \in X^*$, $\|P_n^* x^* - x^*\| \to 0$. Since $(P_{n-1}^* x^*)_{|[x_i]_{i=n}^\infty} = 0$ it follows that $\lim_n \|x^*{}_{|[x_i]_{i=n}^\infty}\| = 0$. Conversely, assume that $\|x^*{}_{|[x_i]_{i=n}^\infty}\| \to 0$ and let $x \in X$ be any element of norm 1. Then,

$$(x^* - P_n^* x^*)(x) = x^*((I - P_n)x) \leqslant \|x^*{}_{|[x_i]_{i=n+1}^\infty}\|(K+1)$$

where K is the basis constant of $\{x_n\}_{n=1}^\infty$. Hence, $\|P_n^* x^* - x^*\| \to 0$ $\square$

If X has a shrinking basis it is possible to give a convenient representation of X^{**} by using the basis.

Proposition 1.b.2. *Let $\{x_n\}_{n=1}^\infty$ be a shrinking basis of a Banach space X. Then X^{**} can be identified with the space of all sequences of scalars $\{a_n\}_{n=1}^\infty$ such that $\sup_n \left\|\sum_{i=1}^n a_i x_i\right\| < \infty$. This correspondence is given by $x^{**} \leftrightarrow (x^{**}(x_1^*), x^{**}(x_2^*), \ldots)$. The norm of x^{**} is equivalent (and in case the basis constant is 1 even equal) to $\sup_n \left\|\sum_{i=1}^n x^{**}(x_i^*)x_i\right\|$.*

Proof. We may clearly assume that the basis constant of $\{x_n\}_{n=1}^\infty$ is 1. If $x^{**} \in X^{**}$ and $\{P_n\}_{n=1}^\infty$ are the projections on X associated to the basis then $P_n^{**}x^{**} = \sum_{i=1}^n x^{**}(x_i^*)x_i$ and $\|x^{**}\| = \lim_n \|P_n^{**}x^{**}\| = \sup_n \|P_n^{**}x^{**}\|$. Conversely, if $\{a_n\}_{n=1}^\infty$ is such that $\sup_n \left\|\sum_{i=1}^n a_i x_i\right\| < \infty$ then any w^* limit point x^{**} of the bounded set $\left\{\sum_{i=1}^n a_i x_i\right\}_{n=1}^\infty$ satisfies $x^{**}(x_i^*) = a_i$ for all i. (In particular, the uniqueness of x^{**} implies that $\left\{\sum_{i=1}^n a_i x_i\right\}_{n=1}^\infty$ must tend w^* to x^{**}.) $\square$

Observe that the canonical image of X in X^{**} corresponds to those sequences $\{a_n\}_{n=1}^\infty$ for which $\left\{\sum_{i=1}^n a_i x_i\right\}_{n=1}^\infty$ is not only bounded but converges in norm.

Another important notion concerning bases, which is in a sense dual to "shrinking", is that of "boundedly complete".

Definition 1.b.3. A basis $\{x_n\}_{n=1}^\infty$ of a Banach space is called *boundedly complete* if, for every sequence of scalars $\{a_n\}_{n=1}^\infty$ such that $\sup_n \left\|\sum_{i=1}^n a_i x_i\right\| < \infty$, the series $\sum_{n=1}^\infty a_n x_n$ converges.

A typical example of a non-boundedly complete basis is the unit vector basis of c_0. The unit vector basis is boundedly complete in all the l_p spaces, $1 \leqslant p < \infty$.

If $\{x_n\}_{n=1}^\infty$ is a shrinking basis in X then $\{x_n^*\}_{n=1}^\infty$ is a boundedly complete basis in X^*. Indeed, if $\sup_n \left\|\sum_{i=1}^n a_i x_i^*\right\| < \infty$ then the expansion of a w^* limit point x^* of $\left\{\sum_{i=1}^n a_i x_i^*\right\}_{n=1}^\infty$ with respect to the basis $\{x_n^*\}_{n=1}^\infty$ must be $\sum_{n=1}^\infty a_n x_n^*$. The converse of this remark is also valid:

Proposition 1.b.4. *A Banach space X with a boundedly complete basis $\{x_n\}_{n=1}^\infty$ is isomorphic to a conjugate space. More precisely, X is isomorphic to the dual of the subspace $[x_n^*]_{n=1}^\infty$ of X^*.*

Proof. Let $Z = [x_n^*]_{n=1}^\infty$ and let J be the canonical map from X to Z^* defined by $Jx(z) = z(x)$. We claim that J is an isomorphism onto. Indeed, let $x \in \operatorname{span}\{x_i\}_{n=1}^\infty$, for some integer n, and $x^* \in X^*$ be such that $\|x^*\| = 1$ and $x^*(x) = \|x\|$. Then, $P_n^* x^*(x) = x^*(x)$, $P_n^* x^* \in Z$ and $\|P_n^* x^*\| \leqslant K$, where K is the basis constant of $\{x_n\}_{n=1}^*$. Hence, $\|x\|/K \leqslant \|Jx\| \leqslant \|x\|$ and this shows that J is an isomorphism. To show that J is onto observe that $\{Jx_n\}_{n=1}^\infty$ are functionals biorthogonal to $\{x_n^*\}_{n=1}^*$ in Z^*. Let $z^* \in Z^*$. The sequence $\left\{\sum_{i=1}^n z^*(x_i^*) J x_i\right\}_{n=1}^\infty$ is bounded in norm (by $K\|z^*\|$) and thus, since $\{x_n\}_{n=1}^\infty$ is boundedly complete, the series $\sum_{n=1}^\infty z^*(x_n^*) x_n$ converges in X to an element x. Clearly $z^* = Jx$. □

By combining the notions of shrinking and boundedly complete we get the following elegant characterization of reflexivity in terms of bases.

Theorem 1.b.5 [53]. *Let X be a Banach space with a Schauder basis $\{x_n\}_{n=1}^\infty$. Then X is reflexive if and only if $\{x_n\}_{n=1}^\infty$ is both shrinking and boundedly complete.*

Proof. Assume first that X is reflexive. We already observed above that $\{x_n\}_{n=1}^\infty$ must be shrinking. Also if $\sup_n \left\|\sum_{i=1}^n a_i x_i\right\| < \infty$ then any w limit point x of $\left\{\sum_{i=1}^n a_i x_i\right\}_{n=1}^\infty$ must be of the form $\sum_{n=1}^\infty a_n x_n$ and in particular this series converges.

The converse assertion follows immediately from 1.b.2. We shall give an additional proof of the converse assertion since it is somewhat more convenient to use it in more general situations in which an analogue of 1.b.5 is valid. Let $\{y_k\}_{k=1}^\infty$ be a sequence of vectors of norm 1 in X. By the diagonal procedure we can find a subsequence $\{y_{k_j}\}_{j=1}^\infty$ of $\{y_k\}_{k=1}^\infty$ so that $a_n = \lim_j x_n^*(y_{k_j})$ exists for every n. We have that $\sum_{i=1}^n a_i x_i = \lim_j P_n y_{k_j}$ and hence, $\sup_n \left\| \sum_{i=1}^n a_i x_i \right\| \leq K$. Since $\{x_n\}_{n=1}^\infty$ is boundedly complete $y = \sum_{n=1}^\infty a_n x_n$ exists and, by definition, $\lim_j x_n^*(y_{k_j}) = x_n^*(y)$, for every n. Since the basis $\{x_n\}_{n=1}^\infty$ is also shrinking $[x_n^*]_{n=1}^\infty = X^*$ and therefore $w \lim_j y_{k_j} = y$. This proves that X is reflexive. □

In 1.b.5 we considered a single basis in X. In [147] M. Zippin showed that if we consider all the bases in a given space it is enough to use only one of the two properties appearing in 1.b.5. More precisely: a Banach space X with a basis is reflexive if (and clearly only if) every basis in X is shrinking or, alternatively, if every basis in X is boundedly complete.

We pass now to questions relating existence of bases and duality. These questions are non-trivial (and thus of interest) only for non-reflexive Banach spaces. If a Banach space X has a basis its dual X^* need not have a basis even if X^* is separable (cf. Section e below). An interesting and rather deep result of W. B. Johnson, H. P. Rosenthal and M. Zippin [61] shows that the existence of a basis in X^* does imply that also X has a basis.

Theorem 1.b.6. *Let X be a Banach space such that X^* has a basis. Then X has a shrinking basis and therefore X^* has a boundedly complete basis.*

The proof of this theorem is closely related to the local theory of Banach spaces and therefore we shall give it only in Vol. IV.

In Section a we proved that every Banach space X contains a basic sequence. For a reflexive X this result implies that X has a quotient space with a basis. W. B. Johnson and H. P. Rosenthal proved in [60] that the same holds for general separable Banach spaces.

Theorem 1.b.7. *Every separable infinite dimensional Banach space has an infinite-dimensional quotient space with a basis.*

For the proof of 1.b.7 we introduce first the following.

Definition 1.b.8. A basic sequence $\{x_n^*\}_{n=1}^\infty$ in the dual X^* of a Banach space X is called a w^* *basic sequence* if there exists a sequence $\{x_n\}_{n=1}^\infty$ in X for which $x_n^*(x_m) = \delta_n^m$ and such that, for every x^* in the w^* closure of span $\{x_n^*\}_{n=1}^\infty$, we have

$$x^* = w^* \lim_n \sum_{i=1}^n x^*(x_i) x_i^*.$$

The next proposition clarifies the meaning of the notion of a w^* basic sequence and its relation to 1.b.7.

Proposition 1.b.9. *A sequence $\{x_n^*\}_{n=1}^\infty \in X^*$ is a w^* basic sequence if and only if there is a basis $\{y_n\}_{n=1}^\infty$ of $Y = X/([x_n^*]_{n=1}^\infty)^\top$ so that $x_n^* = T^* y_n^*$, $n=1, 2, \ldots$, where $T\colon X \to Y$ is the quotient map and $\{y_n^*\}_{n=1}^\infty$ are the functionals biorthogonal to $\{y_n\}_{n=1}^\infty$.*

This proposition is proved by a straightforward verification. Note that T^* is an isometry from Y^* onto the w^* closure of $[x_n^*]_{n=1}^\infty$ since it is w^* continuous.

In the proof of 1.b.7 we shall use also the following fact which is a direct consequence of the w^* density of the unit ball of a Banach space X in the unit ball of X^{**}. Let B be a finite dimensional subspace of X^* and let $\varepsilon > 0$. Then, there exists a finite set F of elements of norm 1 in X so that, for every $f \in B^*$ with $\|f\| = 1$, there is an $x \in F$ which satisfies $|f(x^*) - x^*(x)| \leqslant \varepsilon \|x^*\|$, for all $x^* \in B$.

Proof of 1.b.7. For every infinite-dimensional Banach space X the set $\{x^* \in X^*;\ \|x^*\| = 1\}$ is w^* dense in the unit ball of X^*. Since X is separable the unit ball in X^* is w^* metrizable and hence there is a sequence $\{x_k^*\}_{k=1}^\infty$ in X^* so that $\|x_k^*\| = 1$ for all k and $w^* \lim x_k^* = 0$. We shall construct a subsequence of $\{x_k^*\}_{k=1}^\infty$ which is w^* basic (this will conclude the proof in view of 1.b.9).

Let $\{\varepsilon_n\}_{n=1}^\infty$ be a sequence of numbers so that $0 < \varepsilon_n < 1$ and $\sum_{n=1}^\infty \varepsilon_n < \infty$. By the remark above and the separability of X we can choose inductively a sequence of integers $k_1 < k_2 < \cdots$ and a sequence of finite sets $F_1 \subset F_2 \subset \cdots$ of elements of norm 1 in X so that

(i) $X = \overline{\operatorname{span}} \bigcup_{n=1}^\infty F_n$

(ii) For every $f \in ([x_{k_i}^*]_{i=1}^n)^*$ with $\|f\| = 1$ there is an $x \in F_n$ so that $|f(x^*) - x^*(x)| < \varepsilon_n \|x^*\|/3$ for every $x^* \in [x_{k_i}^*]_{i=1}^n$.

(iii) $|x_{k_{n+1}}^*(x)| \leqslant \varepsilon_n/3$ for all $x \in F_n$.

We claim that $\{x_{k_n}^*\}_{n=1}^\infty$ is a w^* basic sequence. Note first that, by the proof of 1.a.5 (and the remark following it), this is a basic sequence. Moreover, if $\{P_n\}_{n=1}^\infty$ denote the natural projections (on $[x_{k_n}^*]_{n=1}^\infty$) which are associated to this basic sequence, then

$$(*) \qquad \|P_n\| \leqslant \prod_{j=n}^\infty (1-\varepsilon_j)^{-1} \xrightarrow[n\to\infty]{} 1\,.$$

Let $\{y_n\}_{n=1}^\infty \subset ([x_{k_n}^*]_{n=1}^\infty)^*$ be the functionals biorthogonal to $\{x_{k_n}^*\}_{n=1}^\infty$. In view of 1.b.9 it suffices to show that the operator $T\colon X \to ([x_{k_n}^*]_{n=1}^\infty)^*$ defined by $Tx(x^*) = x^*(x)$, $x^* \in [x_{k_n}^*]_{n=1}^\infty$ maps X onto $[y_n]_{n=1}^\infty$ and thus, in particular, is a quotient map (note that the kernel of T is $([x_{k_n}^*]_{n=1}^\infty)^\top$).

We show first that $TX \subset [y_n]_{n=1}^\infty$. This follows from (i) and (iii) since if $x \in F_n$ for some n then $\sum_{i=1}^\infty |x_{k_i}^*(x)| < \infty$ and hence $Tx = \sum_{i=1}^\infty x_{k_i}^*(x) y_i \in [y_n]_{n=1}^\infty$.

To prove that TX exhausts all of $[y_n]_{n=1}^\infty$ it suffices to show that, for every

$y \in \operatorname{span}\{y_n\}_{n=1}^{\infty}$ of norm 1 and every $\varepsilon > 0$, there exists an $x \in X$ with $\|x\| = 1$ and $\|Tx - y\| < 4\varepsilon$ (the desired result will then follow by successive approximation). We may assume that $y \in \operatorname{span}\{y_i\}_{i=1}^{n}$ and that n is so large that $\varepsilon > \sum_{i=n}^{\infty} \varepsilon_i$ and $\|P_m\| < 1 + \varepsilon$ for $m \geq n$ (use (*)). For $u \in \operatorname{span}\{y_i\}_{i=1}^{n}$ we denote $\|u_{|[x^*_{k_i}]_{i=1}^n}\|$ by $\|u\|_1$. Observe that for every such u

$$\|u\|_1 \leq \|u\| \leq \|P_n\| \, \|u\|_1 \leq (1+\varepsilon)\|u\|_1 \, .$$

Choose an $x \in F_n$ which satisfies (ii) for $z = y/\|y\|_1$. We get that $\left\| \sum_{i=1}^{n} x^*_{k_i}(x) y_i - z \right\|_1 < \varepsilon_n/3 < \varepsilon/3$ and hence $\left\| \sum_{i=1}^{n} x^*_{k_i}(x) y_i - z \right\| < 2\varepsilon/3$. Since $\|y_i\| = \|P_i - P_{i-1}\| \leq 4$ for $i \geq n$ we deduce from (iii) that $\left\| \sum_{i=n+1}^{\infty} x^*_{k_i}(x) y_i \right\| < 4\varepsilon/3$. Hence

$$\begin{aligned}
\|Tx - y\| &\leq \|Tx - z\| + \|y - z\| \leq \left\| \sum_{i=1}^{\infty} x^*_{k_i}(x) y_i - z \right\| + 2\varepsilon \\
&\leq \left\| \sum_{i=1}^{n} x^*_{k_i}(x) y_i - z \right\| + \left\| \sum_{i=n+1}^{\infty} x^*_{k_i}(x) y_i \right\| + 2\varepsilon \\
&\leq 2\varepsilon/3 + 4\varepsilon/3 + 2\varepsilon \leq 4\varepsilon \, .
\end{aligned}$$

This concludes the proof of 1.b.7. $\square$

It is not known whether 1.b.7 is true without the separability assumption on X. This question is clearly equivalent to the following: Does every infinite-dimensional Banach space have an infinite dimensional and separable quotient space?

Another open problem which arises naturally in view of 1.a.5 and 1.b.7 is the following

Problem 1.b.10. *Let X be an infinite-dimensional separable Banach space. Does there exist a subspace Y of X so that both Y and X/Y have a Schauder basis?*

In Section g below we shall present a partial positive answer to 1.b.10.

The construction used in the proof of 1.b.7 yields under an additional assumption a boundedly complete basis. Before stating this precisely we prove a useful renorming result due to Kadec [65] and Klee [71]. The proof we present is taken from [25].

Proposition 1.b.11. *Let X be a separable Banach space and let Y be a separable subspace of X^*. Then, there is an equivalent norm $|||\cdot|||$ on X so that, whenever $x_n^* \xrightarrow{w^*} x^*$ with $\{x_n^*\}_{n=1}^{\infty} \subset X^*$, $x^* \in Y$ and $|||x_n^*||| \to |||x^*|||$, we have also that $|||x_n^* - x^*||| \to 0$. ($|||\cdot|||$ denotes here the new norm in X as well as the new norm induced by it in X^*.)*

Proof. Let $B_1 \subset B_2 \subset \cdots$ be a sequence of finite dimensional subspaces of Y whose union is dense in Y. Define a new norm on X^* by putting

$$|||x^*||| = ||x^*|| + \sum_{n=1}^{\infty} 2^{-n} d(x^*, B_n)$$

where $d(x^*, B_n)$ denotes the distance with respect to $||\cdot||$ of x^* from B_n (i.e., the norm of the canonical image of x^* in the quotient space X^*/B_n). Clearly $|||\cdot|||$ is an equivalent norm on X^* and since its unit ball is w^* closed (observe that each B_n is w^* closed) it is induced by an equivalent norm in X which is also denoted by $|||\cdot|||$.

Assume now that $x_n^* \xrightarrow{w^*} x^*$ with $x^* \in Y$ and $|||x_n^*||| \to |||x^*|||$. Then, since $\liminf_n d(x_n^*, B_k) \geqslant d(x^*, B_k)$ for all k and $\liminf_n ||x_n^*|| \geqslant ||x^*||$, we deduce that $d(x^*, B_k) = \lim_n d(x_n^*, B_k)$. From this and the fact that $\lim_k d(x^*, B_k) = 0$ it follows that for every $\varepsilon > 0$ there exists an integer k and elements $u_n^* \in B_k$, $n = 1, 2, \ldots$, such that $||x_n^* - u_n^*|| < \varepsilon/4$ for n sufficiently large. Since B_k is finite-dimensional, by passing to a subsequence if needed, we can assume that $u_n^* \to u^*$ for some $u^* \in B_k$. Thus, for n large enough, $||x_n^* - u^*|| < \varepsilon/2$ which, by taking the w^*-limit, gives $||x^* - u^*|| \leqslant \varepsilon/2$ i.e., $||x^* - x_n^*|| \leqslant \varepsilon$. This proves that $||x_n^* - x^*|| \to 0$ or, equivalently, that $|||x_n^* - x^*||| \to 0$. □

Observe that if X^* is separable we can take $Y = X^*$. In this case we obtain a renorming of X so that in X^*, w^* convergence on the boundary of the new unit ball is equivalent to norm convergence.

We can now prove the result ensuring the existence of boundedly complete basic sequences [60] at which we hinted above.

Proposition 1.b.12. *Let X be a Banach space whose dual X^* is separable. Then every sequence $\{x_k^*\}_{k=1}^\infty$ in X^* such that $x_k^* \xrightarrow{w^*} 0$ and $\limsup_k ||x_k^*|| > 0$ has a boundedly complete basic subsequence $\{x_{k_n}^*\}_{n=1}^\infty$.*

Proof. By 1.b.11 there is no loss of generality to assume that in X^*, $y_n^* \xrightarrow{w^*} y^*$ and $||y_n^*|| \to ||y^*|| \Rightarrow ||y_n^* - y^*|| \to 0$. In the proof of 1.b.7 we showed that $\{x_k^*\}_{k=1}^\infty$ has a w^* basic subsequence $\{x_{k_n}^*\}_{n=1}^\infty$ for which $||P_n|| \to 1$ (see (*)). Thus, if $y_n^* = \sum_{i=1}^n a_i x_{k_i}^*$, $n = 1, 2, \ldots$, is a bounded sequence it follows from the fact that $\{x_{k_n}^*\}_{n=1}^\infty$ is w^* basic that y_n^* converges w^* to a limit y^*. Since

$$||y^*|| \leqslant \liminf_n ||y_n^*|| \leqslant \limsup_n ||y_n^*|| = \limsup_n ||P_n^* y^*|| \leqslant ||y^*||$$

it follows that $||y_n^* - y^*|| \to 0$ i.e., $\sum_{n=1}^\infty a_n x_{k_n}^*$ converges. □

The dual result to 1.b.12 is also true. This was observed in [30].

Proposition 1.b.13. *Let X be an infinite-dimensional Banach space with a separable dual. Then X contains a shrinking basic sequence.*

Proof. Let $\{y_k^*\}_{k=1}^\infty$ be a sequence which is norm dense in the unit ball of X^*. The construction of $\{x_n\}_{n=1}^\infty$ in the proof of 1.a.5 can clearly be carried out so that in addition to the properties required there we have also that $y_k^*(x_n)=0$ for $n>k$. The basic sequence $\{x_n\}_{n=1}^\infty$ obtained in this manner is obviously shrinking. □

The preceding propositions imply an interesting result whose statement has nothing to do with bases. This result, stated by V. D. Milman [106] and first proved by W. B. Johnson and H. P. Rosenthal [60], is a very good illustration of the use of bases in the investigation of linear topological properties of Banach spaces.

Theorem 1.b.14. (i) *Let X be a Banach space whose dual X^* is separable. Let Y be an infinite dimensional subspace of X^* with a separable dual Y^*. Then Y has an infinite-dimensional reflexive subspace.*

(ii) *Let X be an infinite-dimensional Banach space whose second dual X^{**} is separable. Then every infinite dimensional subspace of X or of X^* contains an infinite-dimensional reflexive subspace.*

Proof [60]. (i). By 1.b.13 Y contains a shrinking basic sequence $\{y_k\}_{k=1}^\infty$ with $||y_k||=1$ for all k. Clearly, $y_k \xrightarrow{w} 0$ and thus also $y_k \xrightarrow{w*} 0$ (as an element of X^*). By 1.b.11 there is a subsequence $\{y_{k_n}\}_{n=1}^\infty$ of $\{y_k\}_{k=1}^\infty$ which is a boundedly complete basic sequence. Since $\{y_{k_n}\}_{n=1}^\infty$ is of course also shrinking we deduce from 1.b.5 that $[y_{k_n}]_{n=1}^\infty$ is reflexive.

(ii) Both assertions are immediate consequences of (i). For example, let Y be an infinite-dimensional subspace of X. Then Y^* is separable and Y is a subspace of the separable conjugate X^{**} so (i) applies to Y. □

We conclude this section by mentioning the following result [24].

Theorem 1.b.15. *Let X be a Banach space whose dual X^* is separable. Then there is a Banach space Y with a shrinking basis which has X as a quotient space. In particular, X^* is isomorphic to a subspace of a space with a boundedly complete basis, namely Y^*.*

The proof of this theorem is quite long. Since the theorem will not be used in the sequel we omit its proof and refer the interested reader to [24].

Note that every separable Banach space is a quotient space of l_1 which has a boundedly complete basis. Thus, in one sense, the dual to 1.b.15 is trivially true. In another sense the dualization of 1.b.15 leads to the following open problem.

Problem 1.b.16. *Let X be a Banach space with a separable dual. Is X isomorphic to a subspace of a space with a shrinking basis?*

c. Unconditional Bases

The existence of a Schauder basis in a Banach space does not give very much information on the structure of the space. If one wants to study in more detail the structure of a Banach space by using bases one is led to consider bases with various special properties. In Section b we encountered already two such useful types of bases, namely shrinking and boundedly complete bases. Undoubtedly, the most useful and widely studied special class of bases is that of unconditional bases.

Before studying unconditional bases we present some general facts concerning unconditional convergence.

Proposition 1.c.1. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of vectors in a Banach space X. Then the following conditions are equivalent.*

(i) *The series $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges for every permutation π of the integers.*

(ii) *The series $\sum_{i=1}^{\infty} x_{n_i}$ converges for every choice of $n_1 < n_2 < n_3 \ldots$.*

(iii) *The series $\sum_{n=1}^{\infty} \theta_n x_n$ converges for every choice of signs θ_n (i.e. $\theta_n = \pm 1$).*

(iv) *For every $\varepsilon > 0$ there exists an integer n so that $\left\|\sum_{i \in \sigma} x_i\right\| < \varepsilon$ for every finite set of integers σ which satisfies* $\min\{i \in \sigma\} > n$.

A series $\sum_{n=1}^{\infty} x_n$ which satisfies one, and thus all of the above conditions, is said to be unconditionally convergent.

Proof. The equivalence of (ii) and (iii) is obvious. If (iv) holds then the partial sums of the series appearing in (i) and in (ii) satisfy the Cauchy condition and thus (iv) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii). Assume that (iv) is not satisfied. Then, there is an $\varepsilon > 0$ and finite sets $\{\sigma_n\}_{n=1}^{\infty}$ of integers so that

$$q_n = \max\{i, i \in \sigma_n\} < p_{n+1} = \min\{i; i \in \sigma_{n+1}\}$$

and $\left\|\sum_{i \in \sigma_n} x_i\right\| \geq \varepsilon$, for all n. It is clear that $\sigma = \bigcup_{n=1}^{\infty} \sigma_n$ is a subsequence of the integers for which $\sum_{i \in \sigma} x_i$ does not converge (hence (ii) $\Rightarrow$ (iv)). Also if π is a permutation of the integers which, for every n, maps the set $\{i; p_n \leq i \leq q_n\}$ onto itself in such a manner that $\pi^{-1}(\sigma_n) = \{p_n, p_n + 1, \ldots, p_n + k_n\}$, where k_n is the cardinality of σ_n, then $\sum_{i=1}^{\infty} x_{\pi(i)}$ does not converge (and hence (i) $\Rightarrow$ (iv)). $\square$

It is easily verified that if $\sum_{n=1}^{\infty} x_n$ converges unconditionally then the sum of $\sum_{n=1}^{\infty} x_{\pi(n)}$ does not depend on the permutation π. The set of vectors of the form

$\sum_{n=1}^{\infty} \theta_n x_n$, $\theta_n = \pm 1$ forms a norm compact set (by (iv) the map from $\{-1, 1\}^N$ into X which assigns to $\{\theta_n\}_{n=1}^{\infty}$ the point $\sum_{n=1}^{\infty} \theta_n x_n$ is continuous). It is also easy to verify that if $\sum_{n=1}^{\infty} x_n$ converges unconditionally then, for every bounded sequence of scalars $\{a_n\}_{n=1}^{\infty}$, the series $\sum_{n=1}^{\infty} a_n x_n$ converges and the operator $T: l_\infty \to X$, defined by $T(a_1, a_2, \ldots) = \sum_{n=1}^{\infty} a_n x_n$, is a bounded linear operator.

In finite-dimensional spaces a series $\sum_{n=1}^{\infty} x_n$ converges unconditionally if and only if it *converges absolutely*, i.e. $\sum_{n=1}^{\infty} ||x_n|| < \infty$. In every infinite-dimensional space there exists a series $\sum_{n=1}^{\infty} x_n$ which converges unconditionally but not absolutely. More precisely we have the following result of Dvoretzky and Rogers [34].

Theorem 1.c.2. *Let X be an infinite-dimensional Banach space. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$. Then, there is an unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ in X such that $||x_n|| = \lambda_n$, for every n.*

For the proof of 1.c.2 we need the following result, due to Auerbach, which is useful in many contexts.

Proposition 1.c.3. *Let B be a Banach space of dimension n. Then, there exist n vectors $\{x_i\}_{i=1}^{n}$ of norm 1 in B and n vectors $\{x_i^*\}_{i=1}^{n}$ of norm 1 in X^* so that $x_j^*(x_i) = \delta_i^j$.*

Proof. Introduce a coordinate system in B and, for $y_1, \ldots, y_n$ in the unit ball of B, let $V(y_1, y_2, \ldots, y_n)$ be the determinant of $(a_{i,j})_{i,j=1}^{n}$, where $(a_{i,1}, a_{i,2}, \ldots, a_{i,n})$ denote the coordinates of y_i, $1 \leqslant i \leqslant n$. The function V attains its maximum at an n-tuple $\{x_1, x_2, \ldots, x_n\}$ of vectors of norm 1. Put

$$x_i^*(x) = V(x_1, x_2, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) / V(x_1, x_2, \ldots, x_n).$$

The n-tuples $\{x_i\}_{i=1}^{n}$ and $\{x_i^*\}_{i=1}^{n}$ have the desired property. □

The n-tuples of vectors $\{x_i\}_{i=1}^{n}$, whose existence is ensured by 1.c.3, is called an *Auerbach system.*

The main step in the proof of 1.c.2 is the next lemma (the proof we present is taken from [43]).

Lemma 1.c.4. *Let B be a Banach space of dimension n^2 and norm $||\cdot||$. Then there is an n-dimensional subspace C of B and an inner product norm $|||\ |||$ on C so that $||y|| \leqslant |||y|||$, for all $y \in C$, and an orthonormal (with respect to $|||\cdot|||$) basis $\{y_i\}_{i=1}^{n}$ of C with $||y_i|| \geqslant 1/8$, for every i.*

Proof. Let $\{x_j\}_{j=1}^{n^2}$ be an Auerbach system in B and put $|||x|||_1=n\left(\sum_{j=1}^{n^2} x_j^*(x)^2\right)^{1/2}$. Then, $|||\cdot|||_1$ is an inner product norm on B and

$$|||x|||_1/n^2\leqslant\max_j |x_j^*(x)|\leqslant||x||\leqslant\sum_{j=1}^{n^2} |x_j^*(x)|\leqslant|||x|||_1 .$$

Consider the following statement

(†) Every subspace C of B with $\dim C>\dim B/2$ contains a vector y with $|||y|||_1=1$ and $||y||>1/8$.

If (†) is true we can construct inductively at least $n^2/2-1$ elements y_i which are orthonormal with respect to $|||\cdot|||_1$ and satisfy $||y_i||\geqslant 1/8$, for every i. In this case there is nothing more to prove.

If (†) does not hold there is a subspace B_2 of B with $\dim B_2>\dim B/2$ and an inner product $|||\cdot|||_2=|||\cdot|||_1/8$ on B_2 so that $8|||x|||_2/n^2\leqslant||x||\leqslant|||x|||_2$, for every $x\in B_2$. Consider now the statement obtained from (†) by replacing B by B_2 and $|||\cdot|||_1$ by $|||\cdot|||_2$. If the statement we get is true we have nothing more to prove. If it fails there is a subspace B_3 of B_2 with $\dim B_3>\dim B_2/2$ and an inner product norm $|||\cdot|||_3$ on B_3 so that $8^2|||x|||_3/n^2\leqslant||x||\leqslant|||x|||_3$, for every $x\in B_3$.

We continue in an obvious way. The process must end after $l-1$ steps for an integer l such that $8^l\leqslant n^2$. The space B_l will be of dimension $\geqslant n^2/2^{l-1}$. In this space we can find at least $\dim B_l/2-1$ vectors y_i which are orthonormal with respect to $|||\cdot|||_l$ and satisfy $||y_i||>1/8$, for every i. Since $n^2\geqslant 8^l$ we get that $n^2\cdot 2^{-l}-1>n$ and this concludes the proof. □

Observe that the unit vectors $u_i=y_i/||y_i||$, $1\leqslant i\leqslant n$, whose existence was proved in 1.c.4, satisfy

$$\left\|\sum_{i=1}^n a_iu_i\right\|\leqslant\left|\left|\left|\sum_{n=1}^n a_iu_i\right|\right|\right|=\left(\sum_{i=1}^n |a_i|^2|||u_i|||^2\right)^{1/2}\leqslant 8\left(\sum_{i=1}^n |a_i|^2\right)^{1/2}$$

for every choice of scalars $\{a_i\}_{i=1}^n$.

Proof of 1.c.2. Let λ_i be positive numbers such that $\sum_{i=1}^\infty \lambda_i^2<\infty$. Choose an increasing sequence of integers $\{n_k\}_{k=1}^\infty$ such that $\sum_{i=n_k}^\infty \lambda_i^2\leqslant 2^{-2k}$, $k=1, 2,\dots$. By the preceding lemma we can find in any Banach space of dimension $\geqslant(n_{k+1}-n_k)^2$ (and thus in every infinite-dimensional Banach space) unit vectors $\{u_i\}_{i=n_k}^{n_{k+1}-1}$ so that if we put $x_i=\lambda_iu_i$ then, for every choice of signs θ_i,

$$\left\|\sum_{i=n_k}^{n_{k+1}-1} \theta_ix_i\right\|\leqslant 8\left(\sum_{i=n_k}^{n_{k+1}-1} \lambda_i^2\right)^{1/2}\leqslant 8\cdot 2^{-k} .$$

For $i<n_1$ we take as x_i any vector in X of norm λ_i. The series $\sum_{i=1}^\infty x_i$ converges unconditionally and clearly $||x_i||=\lambda_i$, for all i. □

Theorem 1.c.2 is the strongest possible general result in this direction. It follows readily from the parallelogram identity in Hilbert space that if $\{x_i\}_{i=1}^n$ are any n vectors in l_2 then the average of $\left\|\sum_{i=1}^n \theta_i x_i\right\|^2$ taken over all 2^n choices of signs $\{\theta_i\}_{i=1}^n$ is equal to $\sum_{i=1}^n ||x_i||^2$. Consequently, if $\sum_{i=1}^\infty x_i$ is an unconditionally convergent series in l_2 then $\sum_{i=1}^\infty ||x_i||^2<\infty$. We shall discuss possible converse statements to 1.c.2 in other examples of Banach spaces and their relation to uniform convexity in Vol. II (see also Remark 2 following 2.b.9 below).

We pass now to unconditional bases.

Definition 1.c.5. A basis $\{x_n\}_{n=1}^\infty$ of a Banach space X is said to be *unconditional* if for every $x \in X$, its expansion in terms of the basis $\sum_{n=1}^\infty a_n x_n$ converges unconditionally.

The following proposition is an immediate consequence of 1.c.1.

Proposition 1.c.6. *A basic sequence $\{x_n\}_{n=1}^\infty$ is unconditional if and only if any of the following conditions holds.*

(i) *For every permutation π of the integers the sequence $\{x_{\pi(n)}\}_{n=1}^\infty$ is a basic sequence.*

(ii) *For every subset σ of the integers the convergence of $\sum_{n=1}^\infty a_n x_n$ implies the convergence of $\sum_{n\in\sigma} a_n x_n$.*

(iii) *The convergence of $\sum_{n=1}^\infty a_n x_n$ implies the convergence of $\sum_{n=1}^\infty b_n x_n$ whenever $|b_n|\leqslant |a_n|$, for all n.*

It follows from (ii) and the closed graph theorem that if $\{x_n\}_{n=1}^\infty$ is an unconditional basic sequence and σ is a subset of the integers then there is a bounded linear projection P_σ defined on $[x_n]_{n=1}^\infty$ by $P_\sigma\left(\sum_{n=1}^\infty a_n x_n\right)=\sum_{n\in\sigma} a_n x_n$. These projections are called the *natural projections associated to the unconditional basic sequence.* For the sets σ of the form $\sigma=\{1, 2,\ldots, n\}$ we get that the projections P_σ coincide with the projections P_n which are the natural projections associated to the basic sequence $\{x_n\}_{n=1}^\infty$. Similarly, for every choice of signs $\theta=\{\theta_n\}_{n=1}^\infty$, we have a bounded linear operator M_θ on $[x_n]_{n=1}^\infty$ defined by $M_\theta\left(\sum_{n=1}^\infty a_n x_n\right)=\sum_{n=1}^\infty a_n\theta_n x_n$. Observe that if $\sigma=\{n;\ \theta_n=1\}$ then $P_\sigma=(I+M_\sigma)/2$, and if $\eta=\{\eta_n\}_{n=1}^\infty$ is another sequence of signs then $M_\theta M_\eta=M_{\theta\eta}$ where $(\theta\eta)_n=\theta_n\eta_n$. The uniform boundedness principle implies that $\sup_\sigma ||P_\sigma||$ and $\sup_\sigma ||M_\theta||$ are finite. These numbers are related by the inequality

$$\sup_\sigma ||P_\sigma||\leqslant \sup_\theta ||M_\theta||\leqslant 2 \sup_\sigma ||P_\sigma|| .$$

The number $\sup_\theta ||M_\theta||$ is called *the unconditional constant* of $\{x_n\}_{n=1}^\infty$. Observe that the unconditional constant of a basis is always larger or equal to the basis constant. If $\{x_n\}_{n=1}^\infty$ is an unconditional basis of X we can always define on X an equivalent norm so that the unconditional constant becomes 1. We have simply to take as a new norm the expression $|||x|||=\sup_\theta \|M_\theta x\|$. Every block basis of an unconditional basis is again unconditional. The unconditional constant of a block basis is smaller or equal to the unconditional constant of the original basis. If $\{x_n\}_{n=1}^\infty$ is an unconditional basis of X then the biorthogonal functionals $\{x_n^*\}_{n=1}^\infty$ form an unconditional basic sequence in X^* whose unconditional constant is the same as that of $\{x_n\}_{n=1}^\infty$.

Another often used trivial observation concerning the unconditional constant is the following.

Proposition 1.c.7. *Let $\{x_n\}_{n=1}^\infty$ be an unconditional basic sequence with an unconditional constant K. Then, for every choice of scalars $\{a_n\}_{n=1}^\infty$ such that $\sum_{n=1}^\infty a_n x_n$ converges and every choice of bounded scalars $\{\lambda_n\}_{n=1}^\infty$, we have*

$$\left\|\sum_{n=1}^\infty \lambda_n a_n x_n\right\| \leqslant 2K \sup_n |\lambda_n| \left\|\sum_{n=1}^\infty a_n x_n\right\|$$

(in the real case we can take K instead of 2K).

Proof. Assume the scalars are real and pick an $x^* \in X^*$, with $\|x^*\|=1$, so that $\sum_{n=1}^\infty \lambda_n a_n x^*(x_n)=\left\|\sum_{n=1}^\infty \lambda_n a_n x_n\right\|$. Let $\{\theta_n\}_{n=1}^\infty$ be defined by $\theta_n=1$ if $a_n x^*(x_n)\geqslant 0$ and $\theta_n=-1$ if $a_n x^*(x_n)<0$. Then

$$\begin{aligned}\left\|\sum_{n=1}^\infty \lambda_n a_n x_n\right\| &\leqslant \sum_{n=1}^\infty |\lambda_n|\,|a_n x^*(x_n)| \leqslant \sup_n |\lambda_n| \sum_{n=1}^\infty \theta_n a_n x^*(x_n)\\ &\leqslant \sup_n |\lambda_n| x^*\left(M_\theta\left(\sum_{n=1}^\infty a_n x_n\right)\right) \leqslant \sup_n |\lambda_n|\cdot K\left\|\sum_{n=1}^\infty a_n x_n\right\|.\end{aligned}$$

If the scalars are complex we get the desired result by considering separately the real and imaginary parts of $\sum_{n=1}^\infty a_n x^*(x_n)$. □

The simplest examples of unconditional bases are the unit vector bases in c_0 or in l_p, $1\leqslant p<\infty$. A much more interesting example of an unconditional basis is the Haar system in $L_p(0, 1)$, $1<p<\infty$. We shall prove the unconditionality of this basis in Vol. II. There is a general simple procedure of constructing an unconditional basis. Let $\{x_n\}_{n=1}^\infty$ be a sequence of non-zero vectors in a Banach space X. Let X_0 be the completion of the space of all sequences of scalars $y=(a_1, a_2, \ldots)$, which are eventually zero, with respect to the norm

$$\|y\|=\sup\left\{\left\|\sum_{n=1}^\infty \theta_n a_n x_n\right\|,\ \theta_n=\pm 1,\ n=1, 2, \ldots\right\}.$$

The unit vectors form an unconditional basis of X_0 with unconditional constant 1.

Obviously, this basis is equivalent to $\{x_n\}_{n=1}^\infty$ if and only if $\{x_n\}_{n=1}^\infty$ is itself an unconditional basic sequence.

A simple and important example of a basis which is not unconditional is the *summing basis* in c. It consists of the vectors

$$x_n=(\overbrace{0, 0,\ldots, 0}^{n-1}, 1, 1,\ldots), \quad n=1, 2, 3,\ldots.$$

The norm of $\sum_{n=1}^m a_n x_n$ is $\sup_{1\leqslant n\leqslant m}\left|\sum_{i=1}^n a_i\right|$. The basis $\{x_n\}_{n=1}^\infty$ is a monotone and normalized basis of c which is not unconditional since $\left\|\sum_{i=1}^n x_i\right\|=n$ while $\left\|\sum_{i=1}^n (-1)^i x_i\right\|=1$, for all n. It is of interest to note that if we apply the previously described general procedure of constructing unconditional bases to the sequence $\{x_n\}_{n=1}^\infty$ in c we get as X_0 the space l_1. As we shall see in the next section, the Schauder system in $C(0, 1)$ and the Haar basis of $L_1(0, 1)$ are not unconditional. In the next section we shall present several other examples of bases which are not unconditional.

Bounded linear operators which map one sequence space into another sequence space have a natural representation by an infinite matrix. If $\{x_i\}_{i=1}^\infty$ is a basis of X and $\{y_j\}_{j=1}^\infty$ is a basis of Y, the matrix $A=(\alpha_{i,j})$ corresponding to a bounded linear operator T: $X\to Y$ is defined by the relation $Tx_i=\sum_{j=1}^\infty \alpha_{i,j}y_j$. This representation is especially useful in the case when both bases are unconditional. We shall prove here only one simple fact (cf. [139]), concerning matrix representation, which will be applied in the sequel. A study of some related and deeper questions may be found in [78].

Proposition 1.c.8. *Let the matrix $A=(\alpha_{i,j})$ represent a bounded linear operator T from a Banach space X into a Banach space Y with unconditional bases $\{x_i\}_{i=1}^\infty$ and $\{y_j\}_{j=1}^\infty$, respectively. Then the diagonal of A (i.e. the matrix $(\delta_j^i\alpha_{i,j})$) also represents a bounded linear operator D from X into Y. If the unconditional constants of $\{x_i\}_{i=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ are 1 then $\|D\|\leqslant\|T\|$.*

Proof. It is clearly enough to prove only the second part of the statement. Assume therefore that the unconditional constants are 1. Notice that the matrices

$$A_1=\begin{pmatrix} -\alpha_{1,1} & -\alpha_{1,2} & -\alpha_{1,3} & \cdots \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \cdots \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad A_2=\begin{pmatrix} -\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \cdots \\ -\alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \cdots \\ -\alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

represent operators with the same norm as that of T. Hence, the matrix $(A_1+A_2)/2$ i.e.,

$$\begin{pmatrix} -\alpha_{1,1} & 0 & 0 & \cdots \\ 0 & \alpha_{2,2} & \alpha_{2,3} & \cdots \\ 0 & \alpha_{3,2} & \alpha_{3,3} & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

represents an operator of norm $\leqslant ||T||$. Applying a similar procedure to this matrix, using the second column and row, we get that

$$\begin{pmatrix} -\alpha_{1,1} & 0 & 0 & 0 & \cdots \\ 0 & -\alpha_{2,2} & 0 & 0 & \cdots \\ 0 & 0 & \alpha_{3,3} & \alpha_{3,4} & \cdots \\ 0 & 0 & \alpha_{4,3} & \alpha_{4,4} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

also represents an operator of norm $\leqslant ||T||$. By continuing inductively we obtain the desired result. ▯

Remarks 1. The same method of proof shows that 1.c.8 is also valid for "block diagonal" matrices. More precisely, if $\{m_k\}_{k=1}^\infty$ and $\{n_k\}_{k=1}^\infty$ are increasing sequences of integers and

$$d_{i,j}=\begin{cases}\alpha_{i,j} & \text{for } m_k\leqslant i<m_{k+1},\ n_k\leqslant j<n_{k+1},\ k=1,2,\dots\\ 0 & \text{otherwise}\end{cases}$$

then $(d_{i,j})$ represents a bounded linear operator (whose norm does not exceed that of T if the unconditional constants are 1).

2. A variant of 1.c.8 which is valid even when only one of the spaces has an unconditional basis will also be used later on. Let Z be a Banach space with an unconditional basis $\{z_n\}_{n=1}^\infty$ and let $\{y_n\}_{n=1}^\infty$ be a sequence of non-zero vectors in Z. Let V be the completion of the space of all finite sequences of scalars $v=(a_1, a_2,\dots)$, which are eventually zero, equipped with the norm

$$|||v|||=\sup\left\{\left\|\sum_{n=1}^\infty a_n\theta_n y_n\right\|,\ \theta_n=\pm 1,\ n=1,2,\dots\right\}.$$

Put $y_i=\sum_{j=1}^\infty \alpha_{i,j}z_j$ and observe that the operator $T\colon V\to Z$, defined by $Tv=\sum_{n=1}^\infty a_n y_n$, has norm $\leqslant 1$. Since the unit vectors form an unconditional basis of V it follows from 1.c.8 that the diagonal of $(\alpha_{i,j})_{i,j=1}^\infty$ defines a bounded operator from V into Z. More precisely, for each sequence of scalars $\{a_n\}_{n=1}^\infty$, we have

$$\left\|\sum_{n=1}^\infty a_n\alpha_{n,n}z_n\right\|\leqslant K\sup_{\theta_n=\pm 1}\left\|\sum_{n=1}^\infty a_n\theta_n y_n\right\|,$$

where K denotes the unconditional constant of $\{z_n\}_{n=1}^\infty$.

For Banach spaces having an unconditional basis there are two fundamental structure theorems which are due to R. C. James [53].

Theorem 1.c.9. *Let X be a Banach space with an unconditional basis $\{x_n\}_{n=1}^\infty$. Then $\{x_n\}_{n=1}^\infty$ is shrinking if and only if X does not have a subspace isomorphic to l_1.*

Proof. If l_1 is isomorphic to a subspace of X then X^* is non-separable and therefore no basis of X can be shrinking. Conversely, if $\{x_n\}_{n=1}^\infty$ is not shrinking there is an $x^* \in X^*$ with $\|x^*\|=1$, an $\varepsilon>0$ and a normalized block basis $\{u_j\}_{j=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ so that $x^*(u_j)\geqslant\varepsilon$ for every j. Hence, for every choice of positive $\{a_j\}_{j=1}^m$,

$$\left\|\sum_{j=1}^m a_j u_j\right\| \geqslant x^*\left(\sum_{j=1}^m a_j u_j\right) \geqslant \varepsilon \sum_{j=1}^m a_j\,.$$

It follows that, for every choice of scalars $\{a_j\}_{j=1}^m$, $\left\|\sum_{j=1}^m a_j u_j\right\| \geqslant \varepsilon \sum_{j=1}^m |a_j|/K$, where K is the unconditional constant of $\{x_n\}_{n=1}^\infty$. The $\{u_j\}_{j=1}^\infty$ are thus equivalent to the unit vector basis of l_1. $\square$

Theorem 1.c.10. *Let X be a Banach space with an unconditional basis $\{x_n\}_{n=1}^\infty$. Then, the following three assertions are equivalent.*

(i) *The basis is boundedly complete.*

(ii) *X is weakly sequentially complete (i.e. if $\{y_i\}_{i=1}^\infty \subset X$ are such that* $\lim_i x^*(y_i)$ *exists for every $x^* \in X^*$ then there is a $y \in X$ such that* $x^*(y)=\lim_i x^*(y_i)$ *for every $x^* \in X^*$).*

(iii) *X does not have a subspace isomorphic to c_0.*

Proof. We assume, as we clearly may, that the unconditional constant of $\{x_n\}_{n=1}^\infty$ is 1. Since c_0 is not w sequentially complete it is clear that (ii) $\Rightarrow$ (iii). We shall prove now that (iii) $\Rightarrow$ (i). Assume that the basis is not boundedly complete. Then there exist scalars $\{a_n\}_{n=1}^\infty$ such that $\left\|\sum_{i=1}^n a_i x_i\right\| \leqslant 1$, for every n, but $\sum_{n=1}^\infty a_n x_n$ does not converge. It follows that there is an $\varepsilon>0$ and a sequence of integers $p_1<q_1<p_2<q_2\ldots$ so that if $u_j=\sum_{i=p_j}^{q_j} a_i x_i$ then $\|u_j\|\geqslant\varepsilon$, for every j. It follows from 1.c.7 that for every choice of $\{\lambda_j\}_{j=1}^m$,

$$\left\|\sum_{j=1}^m \lambda_j u_j\right\| \leqslant 2\sup_j|\lambda_j| \left\|\sum_{j=1}^m u_j\right\| \leqslant 2\sup_j|\lambda_j|\,.$$

On the other hand, $\left\|\sum_{j=1}^m \lambda_j u_j\right\| \geqslant \varepsilon \sup_j |\lambda_j|$ and hence, $\{u_j\}_{j=1}^\infty$ is equivalent to the unit vector basis in c_0.

In order to prove the remaining implication in 1.c.10 (i.e. that (i) $\Rightarrow$ (ii)) we need the following lemma.

Lemma 1.c.11. *Let $\{x_n\}_{n=1}^\infty$ be an unconditional basis of a Banach space X with biorthogonal functionals $\{x_n^*\}_{n=1}^\infty$. Let $\{y_i\}_{i=1}^\infty$ be a bounded sequence in X such that* $\lim_i x^*(y_i)$ *exists for every $x^* \in X^*$, and such that* $\lim_i x_n^*(y_i)=0$ *for every n. Then,* $\lim_i x^*(y_i)=0$ *for every $x^* \in X^*$.*

Proof. Assume that, for some $x^* \in X^*$ and some $\varepsilon > 0$, $x^*(y_i) \geqslant \varepsilon$ for all i. Since $\lim_i x_n^*(y_i) = 0$ for every n, there is a subsequence $\{y_{i_k}\}_{k=1}^\infty$ of $\{y_i\}_{i=1}^\infty$ so that $\|y_{i_k} - u_k\| < 2^{-k}$, for some block basis $\{u_k\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$. Since $x^*(u_k) > \varepsilon/2$, for sufficiently large k, the proof of 1.c.9 shows that $\{u_k\}_{k=1}^\infty$, and therefore also $\{y_{i_k}\}_{k=1}^\infty$, are equivalent to the unit vector basis of l_1. Thus, there is an element $y^* \in X^*$ such that $y^*(y_{i_k}) = (-1)^k$, for every k. This contradicts the assumption that $\lim_i y^*(y_i)$ exists. □

We now prove (i) ⇒ (ii) of 1.c.10. Assume that $\{x_n\}_{n=1}^\infty$ is boundedly complete and that $\lim_i x^*(y_i)$ exists for every $x^* \in X^*$. Put $a_n = \lim_i x_n^*(y_i)$, $n = 1, 2, \ldots$. For every integer m, $\left\|\sum_{n=1}^m a_n x_n\right\| = \lim_i \|P_m y_i\| \leqslant \sup_i \|y_i\|$. Hence, $\sum_{n=1}^\infty a_n x_n$ converges to an element $y \in X$. By applying 1.c.11 to $\{y_i - y\}_{i=1}^\infty$ we get that $y_i \xrightarrow{w} y$. □

The following theorem is an immediate consequence of 1.c.9, 1.c.10 and the results 1.b.4, 1.b.5 of the previous section.

Theorem 1.c.12. (a) *A Banach space X with an unconditional basis which does not have subspaces isomorphic to c_0 or l_1 must be reflexive. In particular, if X has an unconditional basis and X^{**} is separable then X is reflexive.*

(b) *A weakly sequentially complete Banach space with an unconditional basis is isomorphic to a conjugate space.*

(c) *If X has an unconditional basis and X^* is separable then X^* has an unconditional basis.*

In connection with assertion (c) above and Theorem 1.b.6 it is worthwhile to remark that the existence of an unconditional basis in X^* does not imply that X has an unconditional basis. For example, let K be the set of ordinals $\leqslant \omega^\omega$, endowed with their usual order topology. We shall prove in Vol. III that $C(K)$ does not have an unconditional basis. On the other hand $C(K)^*$, which is isometric to l_1, has an unconditional basis.

The preceding results were generalized by Bessaga and Pelczynski [10, 11], as follows

Theorem 1.c.13. *Let Y be a closed subspace of a Banach space X with an unconditional basis. Then,*

(a) *Y is weakly sequentially complete if and only if Y contains no subspace isomorphic to c_0.*

(b) *Each norm bounded set in Y is weakly conditionally compact (i.e. each bounded sequence in Y contains a subsequence which is Cauchy in the weak sense) if and only if Y contains no subspace isomorphic to l_1.*

(c) *Y is reflexive if and only if Y contains no subspace isomorphic to c_0 or l_1.*

We shall prove 1.c.13 in Vol. II in the more general setting of spaces Y which are subspaces of suitable Banach lattices. H. P. Rosenthal has proved that part

(b) above holds without any assumption on Y (i.e. for this we do not have to assume the existence of $X \supset Y$ with an unconditional basis). This is a deeper result and we shall discuss it in detail in Section 2.e below.

d. Examples of Spaces Without an Unconditional Basis

Since the notion of an unconditional basis is much stronger than that of a basis it is naturally much easier to exhibit examples of separable spaces which fail to have an unconditional basis than to exhibit spaces which fail to have a basis altogether. The most common non-reflexive classical function spaces, i.e. $C(0, 1)$ (cf. [68]) and $L_1(0, 1)$, fail to have an unconditional basis. That $L_1(0, 1)$ fails to have an unconditional basis can be deduced from 1.c.12(b) since $L_1(0, 1)$ is w sequentially complete and not isomorphic to a conjugate space (this latter fact will be proved in Vol. III). We present here a short proof of the fact, due to Pelczynski [115], that $L_1(0, 1)$ is not even isomorphic to a subspace of a space with an unconditional basis. The proof we give is due to V. D. Milman [107].

Proposition 1.d.1. *The space $L_1(0, 1)$ is not isomorphic to a subspace of a space with an unconditional basis.*

Proof. Let $\{r_n(t)\}_{n=1}^\infty$ be the Rademacher functions on $[0, 1]$ defined by $r_n(t)=$ sign sin $2^n\pi t$. Then, for every $x \in L_1(0, 1)$, we have

$$x(t)r_n(t) \xrightarrow{w} 0; \qquad \|x(t)+x(t)r_n(t)\| \to \|x(t)\|$$

(to check, e.g., the second statement observe that if x is the characteristic function of an interval $[k2^{-n}, (k+1)2^{-n}]$ then $\|x+r_m x\|=\|x\|$ for $m>n$).

Assume now that $L_1(0, 1) \subset Y$ and that Y has an unconditional basis $\{y_i\}_{i=1}^\infty$. Pick any $x_1 \in L_1(0, 1)$ with $\|x_1\|=1$. By the observation above we can define inductively vectors $\{x_n\}_{n=2}^\infty$ in $L_1(0, 1)$ of the form

$$x_2=x_1 \cdot r_{k_1}, \qquad x_3=(x_1+x_2)r_{k_2},\ldots, x_n=\left(\sum_{j=1}^{n-1} x_j\right) r_{k_{n-1}},\ldots$$

so that $\frac{1}{2} \leqslant \|x_n\|=\|x_1+x_2+\cdots+x_{n-1}\| \leqslant 2$, for all n, and so that $\|x_n-u_n\| \leqslant 2^{-n}$, where $\{u_n\}_{n=1}^\infty$ is a suitable block basis of $\{y_i\}_{i=1}^\infty$. It follows from these relations that $\{u_n\}_{n=1}^\infty$, and therefore $\{x_n\}_{n=1}^\infty$, is equivalent to the unit vector basis of c_0. This is impossible since $L_1(0, 1)$ is w-sequentially complete. $\square$

Since every separable Banach space is isometric to a subspace of $C(0, 1)$ it follows from 1.d.1 that also $C(0, 1)$ cannot be embedded in a space with an unconditional basis.

We shall consider now another example of a separable Banach space which, among its other interesting properties, cannot be embedded in a space with an unconditional basis. This example, which is due to R. C. James [53, 54], had an important role in the development of Banach space theory and is still now a source of inspiration to many constructions in the theory.

Example 1.d.2. *The space J: A Banach space with a Schauder basis whose canonical image is of codimension* 1 *in its second dual J^{**} and is also isometric to J^{**}.*

Observe that, in spite of the fact that J is isometric to J^{**}, the space J is not reflexive. Since J^{**} is separable it cannot have a subspace isomorphic to c_0 or l_1. By 1.c.13, J is not isomorphic to a subspace of a space with an unconditional basis.

The space J consists of all sequences of scalars $x=(a_1, a_2, \ldots, a_n, \ldots)$ for which

(i) $$\|x\|=\sup \frac{1}{\sqrt{2}}\,[(a_{p_1}-a_{p_2})^2+(a_{p_2}-a_{p_3})^2+\cdots+(a_{p_{m-1}}-a_{p_m})^2+ +(a_{p_m}-a_{p_1})^2]^{1/2}<\infty$$

and

(ii) $$\lim_n a_n=0\,.$$

The supremum in (i) is taken over all choices of m and $p_1<p_2<\cdots<p_m$. It is useful to note that

$$|||x|||=\sup\,[(a_{p_1}-a_{p_2})^2+(a_{p_2}-a_{p_3})^2+\cdots+(a_{p_{m-1}}-a_{p_m})^2]^{1/2}$$

is an equivalent norm on J (the special form of $\|\cdot\|$ is of importance only for proving that J^{**} is isometric and not merely isomorphic to J).

It is easy to verify that J is a Banach space and that the unit vectors $\{e_n\}_{n=1}^\infty$ form a monotone basis with respect to both norms. The vectors $\{e_1+e_2+\cdots+e_n\}_{n=1}^\infty$ have all norm 1 but they have no w limit point in J; therefore, J is not reflexive.

The unit vector basis is a shrinking basis of J. Indeed, assume that for some $x^*\in J^*$, some $\varepsilon>0$ and a normalized block basis $\{u_k\}_{k=1}^\infty$ of $\{e_n\}_{n=1}^\infty$ we have $x^*(u_k)\geq\varepsilon$, for all k. It is easily verified (by using $|||\cdot|||$) that $\sum_{k=1}^\infty u_k/k$ converges in J. Since $\sum_{k=1}^\infty x^*(u_k)/k$ does not converge we arrived at a contradiction. By 1.b.2, the space J^{**} consists of all sequences $(a_1, a_2, \ldots, a_n, \ldots)$ for which $\sup_n \left\|\sum_{i=1}^n a_ie_i\right\|<\infty$, i.e. all sequences for which (i) in the definition of J holds. Since (i) implies the existence of $\lim_n a_n$ we infer that J^{**} is the linear span of J (or, more precisely, of the canonical image of J in J^{**}) and the functional x_0^{**} defined by $x_0^{**}(e_n^*)=1$ for all n (i.e. the functional which corresponds to the sequence $(1, 1, 1, \ldots)$). The map $U: J^{**}\to J$ defined by

$$Ux^{**}=(-\lambda,\, x^{**}(e_1^*)-\lambda,\, x^{**}(e_2^*)-\lambda, \ldots)\,,$$

where $\lambda = \lim_n x^{**}(e_n^*)$, is an isometry of J^{**} onto J. (Recall that the norm in J^{**} is given by $\|x^{**}\| = \sup_n \left\| \sum_{i=1}^n x^{**}(e_i^*)e_i \right\|$.)

It is noteworthy to remark that the basis $\{e_n^*\}_{n=1}^\infty$ of J^* has the property that $\left\| \sum_{n=1}^\infty c_n e_n^* \right\| = \sum_{n=1}^\infty c_n$, whenever all the c_n are non-negative. Since J^* does not contain a subspace isomorphic to l_1 no subsequence of $\{e_n^*\}_{n=1}^\infty$ can be unconditional.

A useful variant of 1.d.2 shows that not only the one-dimensional space but every separable Banach space X can be realized as Z^{**}/Z for a suitable Z. This was proved in [55] under some restrictions on X and in the general form in [85].

Theorem 1.d.3. *Let X be a separable Banach space. Then there is a separable Banach space Z such that Z^{**} has a boundedly complete basis and Z^{**}/Z is isomorphic to X.*

For the proof of 1.d.3 we refer to [85]; we just mention here the definition of Z. Let $\{x_n\}_{n=1}^\infty$ be a sequence which is dense on the boundary of the unit ball of X. The space Z consists of all the sequences $z = (a_1, a_2, \ldots)$ of scalars for which

(i) $$\|z\| = \sup \left(\sum_{j=1}^m \left\| \sum_{i=p_{j-1}+1}^{p_j} a_i x_i \right\|^2 \right)^{1/2} < \infty$$

and

(ii) $$\sum_{i=1}^\infty a_i x_i = 0 .$$

The supremum in (i) is taken over all choices of integers m and $0 = p_0 < p_1 < \cdots < p_m$. The main point in the proof is to show that Z^{**} can be identified with the space of all sequences $(a_1, a_2, \ldots)$ for which condition (i) above holds. Observe that condition (i) implies that $\sum_{i=1}^\infty a_i x_i$ converges. The operator T from Z^{**} to X defined by $T(a_1, a_2, \ldots) = \sum_{i=1}^\infty a_i x_i$ is bounded and, by the density of $\{x_i\}_{i=1}^\infty$, it is actually a quotient map. The kernel of T is, by definition, the canonical image of Z in Z^{**} and thus Z^{**}/Z is isomorphic to X. From the description of Z^{**} it is obvious that the unit vectors $\{e_n\}_{n=1}^\infty$ form a boundedly complete basis of Z^{**}. From this fact and 1.b.6 it follows that Z has a shrinking basis.

In view of 1.b.2 it is possible to phrase 1.d.3 as follows. Let X be a Banach space. Then, there is a norm $\|\cdot\|$ on the space of the sequences of scalars which are eventually 0 so that X is isometric to Z_1/Z_2, where Z_1 consists of all sequences $(a_1, a_2, \ldots)$ such that $\sup_n \left\| \sum_{i=1}^n a_i e_i \right\| < \infty$ while Z_2 consists of all those sequences for which $\left\{ \sum_{i=1}^n a_i e_i \right\}_{n=1}^\infty$ is a Cauchy sequence.

Another construction of a space Z, having the properties required in 1.d.3, is given in [24]. This paper contains a construction of spaces Z satisfying $Z^{**}/Z \approx X$

also for a large natural class of non-separable spaces X. It seems to be unknown whether an arbitrary non-separable X can be represented as Z^{**}/Z, for a suitable Z.

Every space Z obtained in 1.d.3 is non-reflexive but has a separable second dual. Thus, by 1.c.13, it is not isomorphic to a subspace of a space with an unconditional basis. In all examples considered thus far the non-reflexivity of the space played a crucial role in establishing the non-existence of an unconditional basis. The first example of a separable reflexive space which cannot be embedded in a space with an unconditional basis was given in [78]. Later it was shown in [88] that no space with an unconditional basis can contain isometric copies of $L_p(0, 1)$, with p arbitrarily close to 1 or to ∞ (we shall present the proof of this fact in Vol. II). Thus, for every sequence $\{p_n\}_{n=1}^{\infty}$ of numbers so that $1<p_n<\infty$ for every n and either $\inf_n p_n=1$ or $\sup_n p_n=\infty$, the reflexive space $\left(\sum_{n=1}^{\infty} \oplus L_{p_n}(0, 1)\right)_2$ is not isomorphic to a subspace of a space with an unconditional basis.

The previous examples cannot be embedded isomorphically into a Banach space having an unconditional basis. There are also examples of spaces which fail to have an unconditional basis but do embed into a space having such a basis. The first and perhaps simplest example of this kind (cf. [82]) is the subspace D of l_1, spanned by $x_n=e_n-(e_{2n}+e_{2n+1})/2$, $n=1, 2,\ldots$, where $\{e_n\}_{n=1}^{\infty}$ are the unit vectors in l_1. The sequence $\{x_n\}_{n=1}^{\infty}$ forms a monotone basis of D. The fact that D does not have an unconditional basis follows from 1.c.12 since D is weakly sequentially complete without being isomorphic to a conjugate space (this latter fact will be proved in Vol. IV).

More complicated but also more interesting examples are obtained by using Enflo's solution to the basis problem (and its modification in [20] and [40]); there exist subspaces of c_0 and l_p, $2<p<\infty$, which fail even to have a basis. We shall discuss this in detail in the next chapter.

We mention now two questions related to the existence of unconditional bases which are still open.

Problem 1.d.4. *Let X be a Banach space with an unconditional basis and let Y be a complemented subspace of X. Does Y have an unconditional basis?*

The second question asks whether Banach's theorem on the existence of basic sequences can be strengthened in the following sense.

Problem 1.d.5. *Does every infinite dimensional Banach space X contain an unconditional basic sequence?*

The answer to 1.d.5 is positive whenever X is isomorphic to a subspace of a space with an unconditional basis (use 1.a.11), or more generally, to a subspace of a nice Banach lattice (this will be made precise and proved in Vol. II). In a Banach space which is a subspace of a space with an unconditional basis an unconditional basic sequence can be obtained from every sequence of vectors $\{u_n\}_{n=1}^{\infty}$ with $||u_n||=1$ and $u_n \xrightarrow{w} 0$, by passing to a subsequence. A natural approach to 1.d.5 is therefore to investigate whether in an arbitrary Banach space a sequence $\{u_n\}_{n=1}^{\infty}$, for which $||u_n||=1$ for all n and $u_n \xrightarrow{w} 0$, must have an unconditional basic subsequence. B. Maurey and H. P. Rosenthal [104] proved recently that this is not the case.

Example 1.d.6. *There is a Banach space having a Schauder basis* $\{e_n\}_{n=1}^\infty$ *which converges weakly to* 0 *but which has no unconditional subsequence.*

Let $1>\varepsilon>0$ and let $M=\{m_n\}_{n=1}^\infty$ be an increasing sequence of integers with $m_1=1$ so that

$$(*)\qquad \sum_{i=1}^\infty \sum_{j\neq i} \inf\left(\sqrt{\frac{m_i}{m_j}}, \sqrt{\frac{m_j}{m_i}}\right) \leqslant \varepsilon/2\,.$$

We shall construct a collection Δ of sequences $\delta=\{\sigma_n\}_{n=1}^\infty$ of finite subsets σ_n of the integers N whose main properties are:

1. If $\delta=\{\sigma_n\}_{n=1}^\infty \in \Delta$ then, for every n, the largest integer in σ_n is smaller than the smallest integer in σ_{n+1} (i.e. $\max \sigma_n < \min \sigma_{n+1}$).
2. For $\delta=\{\sigma_n\}_{n=1}^\infty \in \Delta$ the cardinalities $\bar{\bar{\sigma}}_n$ of the sets σ_n satisfy $1=\bar{\bar{\sigma}}_1<\bar{\bar{\sigma}}_2<\bar{\bar{\sigma}}_3<\cdots$ and $\bar{\bar{\sigma}}_n \in M$, $n=1, 2,\ldots$.
3. If, for a pair of sequences $\delta^1=\{\sigma_n^1\}_{n=1}^\infty$ and $\delta^2=\{\sigma_n^2\}_{n=1}^\infty$ in Δ, we have $\bar{\bar{\sigma}}_{j+1}^1=\bar{\bar{\sigma}}_{k+1}^2$, for some integers j and k, then necessarily $j=k$ and $\sigma_i^1=\sigma_i^2$ for $1\leqslant i\leqslant k$.
4. For every infinite subset N_1 of the integers there is a $\delta=\{\sigma_n\}_{n=1}^\infty$ in Δ so that $\sigma_n \subset N_1$ for all n.

The family Δ is defined as follows: Pick a one to one mapping ψ from the set of all finite subsets of the integers into M which satisfies $\psi(\sigma)>\bar{\bar{\sigma}}$, for every σ, and take as Δ the family of all $\delta=\{\sigma_n\}_{n=1}^\infty$ which satisfy 1 above and for which $\bar{\bar{\sigma}}_1=1$ and $\bar{\bar{\sigma}}_{n+1}=\psi(\sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_n)$, $n=1, 2,\ldots$. With this definition of Δ it is obvious that 1, 2 and 4 hold. Let us verify that also 3 holds. We observe first that the growth condition (*) on $\{m_n\}_{n=1}^\infty$ is such that the following is true. If

$$m_{n_1}+m_{n_2}+\cdots+m_{n_k}=m_{s_1}+m_{s_2}+\cdots+m_{s_h}$$

for some choice of $n_1<n_2<\cdots<n_k$ and $s_1<s_2<\cdots<s_h$ then $k=h$ and $n_i=s_i$ for $1\leqslant i\leqslant k$. Assume now that, as in 3, $\bar{\bar{\sigma}}_{j+1}^1=\bar{\bar{\sigma}}_{k+1}^2$. By the definition of Δ and the fact that ψ is one to one it follows that $\bigcup_{i=1}^{j} \sigma_i^1=\bigcup_{i=1}^{k} \sigma_i^2$ and hence, $\sum_{i=1}^{j} \bar{\bar{\sigma}}_i^1=\sum_{i=1}^{k} \bar{\bar{\sigma}}_i^2$. By 2 and the remark made above on M it follows that $j=k$ and $\bar{\bar{\sigma}}_i^1=\bar{\bar{\sigma}}_i^2$ for $1\leqslant i\leqslant k$. From this and the fact that ψ is one to one we get that $\sigma_i^1=\sigma_i^2$ for $1\leqslant i\leqslant k$, as desired.

We pass now to the construction of the space E of Maurey and Rosenthal. It is the completion of the space of sequences $x=(a_1, a_2,\ldots)$ of scalars, which are eventually 0, with respect to the norm

$$||x||=\sup\left|\sum_{n=1}^\infty\left(\sum_{i\in\sigma_n} a_i\right)\bar{\bar{\sigma}}_n^{-1/2}\right|$$

where the supremum is taken over all sequences $\delta=\{\sigma_n\}_{n=1}^\infty$ in Δ. It is clear that the unit vectors $\{e_n\}_{n=1}^\infty$ form a monotone basis of E. Let $\delta=\{\sigma_j\}_{j=1}^\infty \in \Delta$ and put $u_j^{(\delta)}=\left(\sum_{i\in\sigma_j} e_i\right)/\bar{\sigma}_j^{1/2}, j=1, 2, \ldots$. Let $\{c_j\}_{j=1}^\infty$ be scalars and let n be an integer. By using in the definition of $\|\cdot\|$ elements $\eta^k=\{\tau_j^k\}_{j=1}^\infty \in \Delta$, $1\leqslant k\leqslant n$, with $\tau_j^k=\sigma_j$ if $1\leqslant j\leqslant k$ and $\min \tau_{k+1}^k > \max \sigma_n$, it follows that

$$\left\|\sum_{j=1}^n c_j u_j^{(\delta)}\right\| \geqslant \sup_{1\leqslant k\leqslant n}\left|\sum_{j=1}^k c_j\right| .$$

It follows from (*) and condition 3 on Δ, by an easy computation, that

$$\left\|\sum_{j=1}^n c_j u_j^{(\delta)}\right\| \leqslant (1+\varepsilon) \sup_{1\leqslant k\leqslant n}\left|\sum_{j=1}^k c_j\right| .$$

Hence, $\{u_j^{(\delta)}\}_{j=1}^\infty$ is equivalent to the summing basis in c. Consequently, by 4, every subsequence of $\{e_n\}_{n=1}^\infty$ has a block basis which is equivalent to the summing basis and thus no subsequence of $\{e_n\}_{n=1}^\infty$ can be unconditional.

It remains to show that $\{e_n\}_{n=1}^\infty$ tends weakly to 0. This follows from the fact that, for every increasing sequence $\{n_k\}_{k=1}^\infty$ of integers, we have

$$\lim_{k\to\infty} \|e_{n_1}+e_{n_2}+\cdots+e_{n_k}\|/k=0 .$$

Indeed, by 2 and the definition of $\|\cdot\|$, we obtain that $\|e_{n_1}+e_{n_2}+\cdots+e_{n_k}\| \leqslant \sum_{i=1}^k \alpha_i$, where $\alpha_i=m_j^{-1/2}$ whenever $\sum_{h=1}^{j-1} m_h < i \leqslant \sum_{h=1}^{j} m_h$. Obviously, $\lim_k \left(\sum_{i=1}^k \alpha_i\right)/k=0$.

The space E described here is not reflexive. In Vol. II we shall show how to modify the construction of E in order to get even a uniformly convex space with a basis which tends weakly to 0 but has no unconditional subsequence. Thus, there is also a uniformly convex space which is not isomorphic to a subspace of a space with an unconditional basis.

e. The Approximation Property

A result which goes back to the beginnings of functional analysis asserts that the compact operators on a Hilbert space are exactly those operators which are limits in norm of operators of finite rank. One part of this assertion, namely that every $T\in L(X, Y)$ for which $\|T-T_n\|\to 0$ for suitable $\{T_n\}_{n=1}^\infty \in L(X, Y)$ with $\dim T_n X<\infty$ is compact, is trivially true for every pair of Banach spaces X and Y. It was realized long ago that the converse assertion is also true for many examples of spaces X and Y besides Hilbert spaces. For example, if Y has a Schauder basis $\{y_n\}_{n=1}^\infty$ then, for every compact $T\in L(X, Y)$, $\|T-P_nT\|\to 0$, where the $\{P_n\}_{n=1}^\infty$ are the projections associated to the basis $\{y_n\}_{n=1}^\infty$. The question whether the converse assertion is true for arbitrary Banach spaces X and Y (which was called for obvious reasons the approximation problem) was open for a long time. This problem was solved (in the negative) by P. Enflo [37]. The observation above shows that this solution provides also a negative solution to the basis problem. We shall present Enflo's solution or, more precisely, a simplified version of it, due to Davie [20], in the next chapter. Our purpose in this section is to investigate those Banach spaces which have one of the many variants of the "approximation property", i.e. those spaces X for which every compact T in $L(X, Y)$, or $L(Y, X)$, (Y arbitrary) is a limit in norm of a suitable sequence of finite rank operators. The investigation of the various variants of the approximation property and the relations between them was initiated by Grothendieck [48]. Many of the results presented here are taken from Grothendieck's memoir. The real impetus to the investigation of the approximation properties was however given by Enflo's result which ensured that the class of spaces which fail to have any of the approximation properties is not void. Much progress has been done in this study but, generally speaking, the situation is still very far from being clear. We shall mention below several of the many natural open problems concerning the approximation property. What is clear by now is that there are many examples of spaces which fail to have the approximation property even among spaces which are "nice" in other respects (there is, for example, a Banach lattice which fails to have the approximation property [138]. We shall present this example in Vol. II). It is also clear that the study of the approximation property is important in many contexts in Banach space theory and even in some areas of analysis outside the framework of this theory.

Definition 1.e.1. A Banach space X is said to have the *approximation property* (A.P. in short) if, for every compact set K in X and every $\varepsilon>0$, there is an operator $T: X\to X$ of finite rank (i.e. $Tx=\sum_{i=1}^{n} x_i^*(x)x_i$, for some $\{x_i\}_{i=1}^n\subset X$ and $\{x_i^*\}_{i=1}^n\subset X^*$) so that $\|Tx-x\|\leqslant\varepsilon$, for every $x\in K$.

Every space with a Schauder basis has the A.P. Indeed, for every compact K and every $\varepsilon>0$, we have $\|P_nx-x\|<\varepsilon$ for every $x\in K$ provided $n>n(\varepsilon, K)$, where $\{P_n\}_{n=1}^\infty$ are the projections associated to the basis.

In order to study the A.P. we need two general facts—one concerns the structure of compact sets in a Banach space and the other gives a concrete representation of the dual of some space of operators.

Proposition 1.e.2. *A closed subset K of a Banach space X is compact if and only if there is a sequence $\{x_n\}_{n=1}^\infty$ in X such that $\|x_n\|\to 0$ and $K\subset\overline{\operatorname{conv}}\,\{x_n\}_{n=1}^\infty$.*

Proof. It is easily checked that if $\|x_n\|\to 0$ then $\left\{\sum_{n=1}^\infty \lambda_n x_n;\ \lambda_n\geqslant 0,\ \sum_{n=1}^\infty \lambda_n\leqslant 1\right\}$ is compact and coincides with $\overline{\operatorname{conv}}\,\{x_n\}_{n=1}^\infty$. This proves the "if" part. We prove now the "only if" part. Let K be compact. Let $\{x_{i,1}\}_{i=1}^{n_1}$ be a finite set of elements of X so that $2K\subset\bigcup_{i=1}^{n_1} B(x_{i,1}, 1/4)$. Put

$$K_2=\bigcup_{i=1}^{n_1}\{(B(x_{i,1}, 1/4)\cap 2K)-x_{i,1}\}\,.$$

Then K_2 is a compact subset of $B(0, 1/4)$. Pick next $\{x_{i,2}\}_{i=1}^{n_2}$ in $B(0, 1/2)$ so that $2K_2 \subset \bigcup_{i=1}^{n_2} B(x_{i,2}, 1/4^2)$ and put

$$K_3 = \bigcup_{i=1}^{n_2} \{(B(x_{i,2}, 1/4^2) \cap 2K_2) - x_{i,2}\}.$$

We continue the inductive construction of $\{x_{i,j}\}_{i=1}^{n_j}$, $j=1, 2, 3,\ldots$ in an obvious way.

For every $x \in K$ there is an $1 \leqslant i_1 \leqslant n_1$ so that $2x - x_{i_1,1} \in K_2$; hence an $1 \leqslant i_2 \leqslant n_2$ so that $4x - 2x_{i_1,1} - x_{i_2,2} \in K_3$ and, in general,

$$x - (x_{i_1,1}/2 + x_{i_2,2}/2^2 + \cdots + x_{i_k,k}/2^k) \in 2^{-k} K_{k+1}.$$

It follows that $x \in \overline{\operatorname{conv}}\,\{x_{i,j};\ 1 \leqslant i \leqslant n_j, j=1, 2,\ldots\}$. Since $\|x_{i,j}\| \leqslant 2 \cdot 4^{-j+1}$, for $j>1$ and every $i \leqslant n_j$, our assertion is proved. $\square$

Proposition 1.e.3. *Let X and Y be Banach spaces and put on $L(X, Y)$ the topology τ of uniform convergence on compact sets in X (this is the locally convex topology generated by the seminorms of the form $\|T\|_K = \sup\{\|Tx\|, x \in K\}$, where K ranges over the compact subsets of X). Then, the continuous linear functionals on $(L(X, Y), \tau)$ consist of all functionals φ of the form*

$$\varphi(T) = \sum_{i=1}^{\infty} y_i^*(Tx_i), \quad \{x_i\}_{i=1}^{\infty} \subset X, \quad \{y_i^*\}_{i=1}^{\infty} \subset Y^*, \qquad \sum_{i=1}^{\infty} \|x_i\|\,\|y_i^*\| < \infty.$$

Proof. Assume that φ has such a representation. We may clearly assume that $x_i \neq 0$ for every i. Let $\{\eta_i\}_{i=1}^{\infty}$ be a sequence of positive scalars tending to ∞ so that $\sum_{i=1}^{\infty} \eta_i \|x_i\|\,\|y_i^*\| = C < \infty$. Put $K = \{x_i/\|x_i\|\eta_i\}_{i=1}^{\infty} \cup \{0\}$. Then K is compact and

$$|\varphi(T)| \leqslant \sum_{i=1}^{\infty} \|y_i^*\|\,\|x_i\|\eta_i \|T(x_i/\|x_i\|\eta_i)\| \leqslant C\|T\|_K.$$

Conversely, assume that φ is a linear functional on $L(X, Y)$ so that $|\varphi(T)| \leqslant C\|T\|_K$, for some constant C and some compact set $K \subset X$. By 1.e.2 we may assume without loss of generality that $K = \overline{\operatorname{conv}}\,\{x_n\}_{n=1}^{\infty}$, where $\|x_n\| \to 0$. Let $S: L(X, Y) \to (Y \oplus Y \oplus \cdots)_0$ be defined by $S(T) = (Tx_1, Tx_2, \ldots)$. Since $|\varphi(T)| \leqslant C\|S(T)\|$ it follows that there is a linear functional ψ defined on the closure of $SL(X, Y)$ so that $\varphi(T) = \psi(S(T))$. By the Hahn–Banach theorem we may extend ψ to a continuous linear functional on $(Y \oplus Y \oplus \cdots)_0$, i.e. to an element of $(Y^* \oplus Y^* \oplus \cdots)_1$. In other words there exist $\{y_n^*\}_{n=1}^{\infty}$ in Y^* so that $\sum_{n=1}^{\infty} \|y_n^*\| < \infty$ and $\varphi(T) = \sum_{n=1}^{\infty} y_n^* T(x_n)$. $\square$

The next theorem, due to Grothendieck [48], clarifies the relation between the approximation property and the question of approximating compact operators, with which we started this section.

Theorem 1.e.4. *Let X be a Banach space. The following five assertions are equivalent.*

(i) *X has the approximation property.*

(ii) *For every Banach space Y the finite rank operators are dense in $L(Y, X)$, in the topology τ of uniform convergence on compact sets.*

(iii) *For every Banach space Y the finite rank operators are dense in $L(X, Y)$, in the topology τ of uniform convergence on compact sets.*

(iv) *For every choice of $\{x_n\}_{n=1}^\infty \subset X$, $\{x_n^*\}_{n=1}^\infty \subset X^*$ such that $\sum_{n=1}^\infty \|x_n^*\|\,\|x_n\| < \infty$ and $\sum_{n=1}^\infty x_n^*(x)x_n = 0$, for all $x \in X$, we have $\sum_{n=1}^\infty x_n^*(x_n) = 0$.*

(v) *For every Banach space Y, every compact $T \in L(Y, X)$ and every $\varepsilon > 0$ there is a finite rank operator $T_1 \in L(Y, X)$ with $\|T - T_1\| < \varepsilon$.*

Proof. The equivalence of (i) and (iv) is a consequence of 1.e.3. Indeed, by definition, (i) means that the identity operator is in the τ closure of the space of finite rank operators in $L(X, X)$. This happens if and only if every τ continuous linear functional φ on $L(X, X)$, which vanishes on operators of rank 1, vanishes also on the identity operator. By 1.e.3 this is exactly what (iv) means.

It is clear that (ii) or (iii) (with $Y = X$) imply (i). We shall show that (i) implies (ii) and (iii). Let $T \in L(Y, X)$. For every compact set $K \subset Y$ the set TK is compact in X. Hence, given $\varepsilon > 0$, we have by (i) a finite rank operator T_1 on X so that $\|T_1Ty - Ty\| \leqslant \varepsilon$, for $y \in K$. Since T_1T is of finite rank we proved (ii). Let now $0 \neq T \in L(X, Y)$, let K be a compact set in X and let $\varepsilon > 0$. By (i) there is a finite rank operator T_1 on X so that $\|T_1x - x\| \leqslant \varepsilon/\|T\|$, for $x \in K$. Then, $\|TT_1x - Tx\| \leqslant \varepsilon$ for $x \in K$ and this proves (iii).

It remains to prove the equivalence of (i) and (v). Assume that (i) holds and let $T \in L(Y, X)$ be a compact operator. The set $K = \overline{TB_Y(0, 1)}$ is compact and hence, for every $\varepsilon > 0$, there is a finite rank operator T_1 on X so that $\|T_1x - x\| \leqslant \varepsilon$ for $x \in K$. Then, $\|T_1T - T\| \leqslant \varepsilon$ and thus (v) holds.

Assume that (v) holds and let K be a compact subset of X and $\varepsilon > 0$. By 1.e.2 we may assume without loss of generality that $K = \overline{\text{conv}}\,\{x_n\}_{n=1}^\infty$, with $\|x_n\| \downarrow 0$ and $\|x_1\| \leqslant 1$. Put $U = \overline{\text{conv}}\,\{\pm x_n/\|x_n\|^{1/2}\}_{n=1}^\infty$. Clearly, U is a compact convex set in X which is symmetric with respect to the origin. Let Y be the linear span of U in X, i.e. $Y = \bigcup_{n=1}^\infty nU$, and introduce in Y a norm $|||\cdot|||$ which makes U its unit ball (i.e. $|||y||| = \inf\{\lambda > 0;\ y/\lambda \in U\}$). A routine argument shows that $(Y, |||\cdot|||)$ is a Banach space (in particular, it is complete). The formal identity map from Y to X is compact and hence, by (v), there are $\{y_i^*\}_{i=1}^m \subset Y^*$ and $\{u_i\}_{i=1}^m \subset X$ so that $\left\|\sum_{i=1}^m y_i^*(x)u_i - x\right\| \leqslant \varepsilon/2$, for every $x \in U$, and hence $x \in K$. The $\{y_i^*\}_{i=1}^m$ are continuous with respect to $|||\cdot|||$ but need not be continuous with respect to $\|\cdot\|$ (and thus are not in general restrictions of elements of X^* to Y). In order to conclude the proof it is enough to verify the following statement. Given any $y^* \in Y^*$ and $\delta > 0$ (in our case we take $\delta = \varepsilon/2m \cdot \max_i \|u_i\|$) there is an $x^* \in X^*$ such that $|y^*(x) - x^*(x)| < \delta$ for $x \in K$ i.e. $|y^*(x_n) - x^*(x_n)| < \delta$, for every n.

Observe that, since $x_n/\|x_n\|^{1/2} \in U$, we have $|||x_n||| \leqslant \|x_n\|^{1/2}$, for every n, and thus $|||x_n|||\to 0$. For $n\geqslant n_0$ we have therefore $|y^*(x_n)|<\delta/2$. Put $K_0=2\delta^{-1}\overline{\operatorname{conv}\{\pm x_n\}_{n=n_0+1}^\infty}$ (notice that the closures in $||\cdot||$ and $|||\cdot|||$ are the same) and $F=\{x;\, x\in\operatorname{span}\{x_n\}_{n=1}^{n_0},\, y^*(x)=1\}$. Then, F is $||\cdot||$ closed, K_0 is $||\cdot||$ compact and $K_0\cap F=\varnothing$. By the geometric version of the Hahn–Banach theorem there is a $||\cdot||$ closed hyperplane $\hat{F}$ in X so that $F\subset\hat{F}$ and $\hat{F}\cap K_0=\varnothing$. Let $x^*\in X^*$ be such that $\hat{F}=\{x;\, x^*(x)=1\}$. Then, $x^*(x_n)=y^*(x_n)$ for $n\leqslant n_0$ and $|x^*(x_n)|<\delta/2$ for $n>n_0$. Consequently, $|x^*(x_n)-y^*(x_n)|<\delta$ for every n, as desired. □

In view of (ii), (iii) and (v) of 1.e.4 it is natural to ask what is the situation if we reverse the roles of X and Y in (v). The answer is given by the following result which is also due to Grothendieck [48].

Theorem 1.e.5. *Let X be a Banach space. Then, X^* has the approximation property if and only if, for every Banach space Y, every $\varepsilon>0$ and every compact $T\in L(X, Y)$, there is a finite rank operator $T_1\in L(X, Y)$ such that $||T-T_1||\leqslant\varepsilon$.*

Proof. Assume that, for every Y, every compact $T\in L(X, Y)$ can be approximated as above. Let Z be any Banach space, let $T\in L(Z, X^*)$ be compact and let $\varepsilon>0$. By considering the compact operator $T^*_{|X}\colon X\to Z^*$ it follows from our assumption that there are $\{x_i^*\}_{i=1}^n\subset X^*$, $\{z_i^*\}_{i=1}^n\subset Z^*$ so that, for every x with $||x||\leqslant 1$,

$$\left\|T^*x-\sum_{i=1}^n x_i^*(x)z_i^*\right\|\leqslant\varepsilon\,.$$

Hence, for every $z\in Z$ with $||z||\leqslant 1$,

$$\left\|Tz(x)-\sum_{i=1}^n z_i^*(z)x_i^*(x)\right\|\leqslant\varepsilon\,,$$

i.e. $\left\|Tz-\sum_{i=1}^n z_i^*(z)x_i^*\right\|\leqslant\varepsilon$. By 1.e.4, X^* has the A.P.

Assume, conversely, that X^* has the A.P. Let $T\in L(X, Y)$ be compact and $1/2>\varepsilon>0$. The operator $T^*\colon Y^*\to X^*$ is also compact and hence there are $\{y_i^{**}\}_{i=1}^n$ in Y^{**} and $\{x_i^*\}_{i=1}^n$ in X^* so that $\left\|T^*y^*-\sum_{i=1}^n y_i^{**}(y^*)x_i^*\right\|\leqslant\varepsilon$, whenever $||y^*||\leqslant 1$. It follows that $\left\|Tx-\sum_{i=1}^n x_i^*(x)y_i^{**}\right\|\leqslant\varepsilon$, whenever $||x||\leqslant 1$. This does not conclude the proof since the $\{y_i^{**}\}_{i=1}^n$ are not necessarily contained in Y. We have to "push" the $\{y_i^{**}\}_{i=1}^n$ into Y. This is done by using the following lemma.

Lemma 1.e.6 [90]. *Let X be a Banach space, let D be a finite-dimensional subspace of X^{**} and let $\varepsilon>0$. Then, there is an operator $S\colon D\to X$ such that $||S||\leqslant 1+\varepsilon$ and $S_{|D\cap X}$ is the identity.*

Let us first see how 1.e.6 is used to conclude the proof of 1.e.5. By the compactness of T there are $\{x_j\}_{j=1}^m$ in the unit ball $B_X(0, 1)$ of X so that $TB_X(0,1)\subset\bigcup_{j=1}^m B_Y(Tx_j, \varepsilon)$. Apply 1.e.6 to $D=\operatorname{span}\{Tx_j\}_{j=1}^m\cup\{y_i^{**}\}_{i=1}^n$. We claim that

$\left\|Tx-\sum_{i=1}^{n} x_i^*(x)Sy_i^{**}\right\| \leqslant 4\varepsilon$, for every $x \in B_X(0,1)$. Indeed, fix $x \in B_X(0,1)$ and pick a j such that $\|Tx_j-Tx\| \leqslant \varepsilon$. Then, $\left\|Tx_j-\sum_{i=1}^{n} x_i^*(x)y_i^{**}\right\| \leqslant 2\varepsilon$. By applying S we get that $\left\|Tx_j-\sum_{i=1}^{n} x_i^*(x)Sy_i^{**}\right\| \leqslant 2\varepsilon(1+\varepsilon) \leqslant 3\varepsilon$ and this proves our assertion and concludes the proof of 1.e.5. □

Lemma 1.e.6 is a special instance of a result which plays a central role in the local theory of Banach spaces and which will be discussed in detail in Vol. IV. For the sake of completeness we present its proof (cf. [29]) also here.

Proof of 1.e.6. First we notice that $L(l_1^n, X^{**})=L(l_1^n, X)^{**}$. This follows from the fact that the correspondence $T \to \{y_i=Te_i\}_{i=1}^n$ (where $\{e_i\}_{i=1}^n$ are the unit vectors of l_1^n) is an isometry from $L(l_1^n, X)$ onto $(\overbrace{X \oplus \cdots \oplus X}^{\text{n times}})_\infty$.

Let $I\colon D \to X^{**}$ be the identity mapping and let $\varepsilon>0$. There exist an n and vectors $\{u_j\}_{j=1}^n$ in D of norm $\leqslant 1+\varepsilon$ so that conv $\{u_j\}_{j=1}^n \supset B_D(0,1)$. Hence, there is an operator $V\colon l_1^n \to D$, for which $\|V\| \leqslant 1+\varepsilon$ and $VB_{l_1^n}(0,1) \supset B_D(0,1)$. Since $IV \in L(l_1^n, X)^{**}$ there is a net $\{S_\alpha\} \subset L(l_1^n, X)$ with $\|S_\alpha\| \leqslant \|IV\| \leqslant 1+\varepsilon$ for all α and $\{S_\alpha\}$ converges to IV in the w^* topology of $L(l_1^n, X)^{**}$. Any pair $e \in l_1^n$, $x^* \in X^*$ defines a functional $(e, x^*) \in L(l_1^n, X)^*$, by setting $(e, x^*)(S)=x^*Se$. This implies that, for every $e \in l_1^n$, $S_\alpha e \xrightarrow{w^*} IVe$. Put $B=\{e \in l_1^n;\ Ve \in D \cap X\}$ and observe that $S_\alpha e \xrightarrow{w} IVe$ for all $e \in B$. Thus, by taking a suitable convex combination of S_α's and by using a standard perturbation argument, we can construct an operator $T\colon l_1^n \to X$ such that $T_{|B}=IV_{|B}$ and $\|T\|<1+2\varepsilon$. If $D \ni d=Vv_1=Vv_2$, for some $v_1, v_2 \in l_1^n$, then $v_1-v_2 \in B$ and therefore $Tv_1=Tv_2$. Hence, by setting $Sd=Tv$, where $v \in l_1^n$ is any vector satisfying $Vv=d$, we define an operator $S \in L(D, X)$ for which $\|S\|<1+2\varepsilon$ and $S_{|D\cap X}=I_{|D\cap X}$. This completes the proof. □

The relation between the properties appearing in 1.e.4 and 1.e.5 is clarified in the following result.

Theorem 1.e.7. (a) *Let X be a Banach space. If X^* has the* A.P. *then X has the* A.P. *In particular, if X is reflexive then X has the* A.P. *if and only if X^* has the* A.P.

(b) *There is a separable Banach space having a Schauder basis whose dual is separable but fails to have the* A.P.

Proof. Assertion (a) follows immediately from the equivalence (i) $\Leftrightarrow$ (iv) of 1.e.4. If $\{x_n\}_{n=1}^\infty \subset X$ and $\{x_n^*\}_{n=1}^\infty \subset X^*$ are such that $\sum_{n=1}^\infty \|x_n\|\,\|x_n^*\|<\infty$ and $\sum_{n=1}^\infty x_n^*(x)x_n=0$, for every $x \in X$, then also $\sum_{n=1}^\infty x^*(x_n)x_n^*=0$, for every $x^* \in X^*$. In other words $\sum_{n=1}^\infty Jx_n(x^*)x_n^*=0$, where $J\colon X \to X^{**}$ is the canonical embedding. Since X^* has the A.P. $\sum_{n=1}^\infty Jx_n(x_n^*)=\sum_{n=1}^\infty x_n^*(x_n)=0$.

The proof of assertion (b) uses of course the fact that there is a Banach space which fails to have the A.P. (this will be proved in Section 2.d below). Let X be a separable Banach space which does not have the A.P. By 1.d.3 there is a space Z so that Z^{**} has a basis and Z^{**}/Z is isomorphic to X. By passing to the duals we get that $Z^{***} \approx Z^* \oplus X^*$ (observe that for every Banach space Z there is a projection of norm 1 from Z^{***} onto Z^*. The projection is the map which assigns to every functional on Z^{**} its restriction to Z). Since X fails to have the A.P. the same is true for X^*, by assertion (a). It is trivial to verify that a complemented subspace of a space having the A.P. has also the A.P. Hence Z^{***} (which is a dual of a space Z^{**} with a basis) fails to have the A.P. If X is such that X^* is separable (e.g. if X is isomorphic to a subspace of c_0; see the remark following 1.e.8) then Z^{***} is separable. □

While investigating the approximation problem Grothendieck found many nice equivalent formulations of this problem. We give two of those formulations here. As stated, this result is of course only of historical interest. However, from a different point of view it is still useful since it shows the connection between the A.P. and some problems arising in classical analysis. Moreover, the simple and explicit proof of the result enables us to transfer each counterexample to one of the versions of the approximation problem to a counterexample to the others.

Proposition 1.e.8. *The following three assertions are equivalent.*

(i) *Every Banach space has the* A.P.

(ii) *Every matrix $A=(a_{i,j})_{i,j=1}^{\infty}$ of scalars, for which $\lim_j a_{i,j}=0$, $i=1, 2,\ldots$,*

$$\sum_{i=1}^{\infty} \max_j |a_{i,j}|<\infty \textit{ and } A^2=0, \textit{ satisfies } \operatorname{trace} A=\sum_{n=1}^{\infty} a_{nn}=0.$$

(iii) *Every continuous function $K(s, t)$ on $[0, 1)\times[0, 1]$, for which $\int_0^1 K(s, t)K(t, u)\,dt=0$ for every s and u, satisfies $\int_0^1 K(t, t)\,dt=0$.*

Proof. (ii) ⇒ (i). Let X be a Banach space which fails to have the A.P. Then, by 1.e.4, there are $\{x_n\}_{n=1}^{\infty}\subset X$, $\{x_n^*\}_{n=1}^{\infty}\subset X^*$ with $\sum_{n=1}^{\infty}||x_n||\,||x_n^*||<\infty$ and $\sum_{n=1}^{\infty} x_n^*(x)x_n=0$ for every $x\in X$ but $\sum_{n=1}^{\infty} x_n^*(x_n)\neq 0$. There is clearly no loss of generality to assume that $||x_n||\to 0$ and $\sum_{n=1}^{\infty}||x_n^*||<\infty$. The matrix $A=(x_i^*(x_j))_{i,j=1}^{\infty}$ satisfies $\lim_j x_i^*(x_j)=0$, $\sum_{i=1}^{\infty}\max_j |x_i^*(x_j)|<\infty$ and also $A^2=0$ (the entries of A^2 are expressions of the form $\sum_{n=1}^{\infty} x_n^*(x_i)x_k^*(x_n)$). However, $\operatorname{trace} A=\sum_{n=1}^{\infty} x_n^*(x_n)\neq 0$.

(iii) ⇒ (ii). Assume that (ii) fails and that $A=(a_{i,j})$ is a counterexample. Put $\alpha_i=\max_j |a_{i,j}|$ and choose a sequence of positive numbers $\{\eta_i\}_{i=1}^{\infty}$ such that $\eta_i\to\infty$ and $\sum_{i=1}^{\infty}\alpha_i\eta_i<1$. Put $b_{i,j}=a_{i,j}/\alpha_i\eta_i$; then $\lim_{i,j} b_{i,j}=0$ (in the sense that, for every

$\varepsilon>0$, there are only finitely many pairs (i,j) for which $|b_{i,j}|>\varepsilon$). Let $1=t_1>s_1>t_2>s_2\ldots$ be numbers such that $\lim_n t_n=0$ and $t_i-s_i>\alpha_i\eta_i$, for every i (this is possible since $\sum_{i=1}^{\infty}\alpha_i\eta_i<1$). Let $\{\varphi_i\}_{i=1}^{\infty}$ be continuous functions on $[0,1]$ such that $0\leqslant\varphi_i\leqslant 1$, φ_i vanishes outside $[s_i,t_i]$ and $\int_{s_i}^{t_i}\varphi_i(t)\,dt=\alpha_i\eta_i$. It is easy to verify that $K(s,t)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sqrt{\varphi_i(s)\varphi_j(t)}\,b_{i,j}$ is continuous on $[0,1]\times[0,1]$ (at each point the sum consists of at most one summand). The fact that $A^2=0$ implies that $\int_0^1 K(s,t)K(t,u)\,dt=0$ for every s and u while

$$\int_0^1 K(t,t)\,dt=\sum_{n=1}^{\infty}\int_0^1\varphi_n(t)\,dt\cdot b_{n,n}=\sum_{n=1}^{\infty}\alpha_n\eta_n b_{nn}=\sum_{n=1}^{\infty}a_{n,n}\neq 0\,.$$

(i) $\Rightarrow$ (iii). Assume that there is a counterexample $K(s,t)$ to (iii). For every $s\in[0,1]$, let $f_s(t)\in C(0,1)$ be defined by $f_s(t)=K(s,t)$. The continuity of $K(s,t)$ implies that $K_0=\{f_s\}_{0\leqslant s\leqslant 1}$ is a compact subset of $C(0,1)$. We claim that $X_0=\overline{\text{span}}\,K_0$ is a subspace of $C(0,1)$ which does not have the A.P. Indeed, assume that $\varepsilon>0$ is such that there is an operator T of finite rank $Tg=\sum_{i=1}^{n}x_i^*(g)f_i$ on X_0 such that $\|Tg-g\|<\varepsilon$ for all $g\in K_0$. Since span K_0 is dense in X_0 there is no loss of generality to assume (by increasing n) that $f_i\in K_0$ for all i, i.e. $f_i=f_{s_i}$, for suitable $s_i\in[0,1]$. The functionals x_i^* extend to functionals on $C(0,1)$ and can thus be considered as measures on $[0,1]$. Since the measures with finite support are w^* dense in the set of all measures it follows easily that we may assume that each x_i^* is a finitely supported measure and thus (by increasing n again) that each x_i^* is of the form $x_i^*(g)=\lambda_i g(u_i)$, for suitable $u_i\in[0,1]$ and scalars $\{\lambda_i\}_{i=1}^{n}$. In conclusion we get that, for every $s\in[0,1]$, $t\in[0,1]$,

$$\left|K(s,t)-\sum_{i=1}^{n}\lambda_i K(s,u_i)K(s_i,t)\right|\leqslant\varepsilon$$

and, in particular,

$$\left|K(t,t)-\sum_{i=1}^{n}\lambda_i K(s_i,t)K(t,u_i)\right|\leqslant\varepsilon\,.$$

This however is impossible for small enough ε since $\int_0^1 K(t,t)\,dt\neq 0$ while

$$\int_0^1 K(s_i,t)K(t,u_i)\,dt=0,\quad\text{for every } i.\quad\square$$

It is also instructive to note that it is very simple to verify directly that (i) $\Rightarrow$ (ii) in 1.e.8. Indeed, let $A=(a_{i,j})$ be a matrix which fails (ii) and put $x_i=(a_{i,1},a_{i,2},\ldots)\in c_0$, for $i=1,2,\ldots$. Then, the closed linear span X of $\{x_i\}_{i=1}^{\infty}$ in c_0 is a space which fails the A.P. (apply 1.e.4 to the vectors $\{x_i\}_{i=1}^{\infty} \in X$ and $\{e_i\}_{i=1}^{\infty} \in X^*$ where e_i is the restriction to X of the i'th unit vector in l_1). Thus, the analysis of Grothendieck of the approximation property shows that if there is a Banach space which does not have the A.P. then there is also a subspace of c_0 which does not have the A.P. In the next chapter we shall show that not only c_0 but also the sequence spaces l_p, for $p>2$, have subspaces which fail to have the A.P.

We mention now some of the open problems concerning the A.P.

Problem 1.e.9. *Let X be a Banach space such that every compact $T:X\to X$ is a limit in norm of finite rank operators from X into itself. Does X have the* A.P.*?*

Problem 1.e.10. *Does the space $L(l_2, l_2)$, of all bounded operators on l_2 with the usual operator norm, have the* A.P.*? Does the space H_∞, of all the bounded analytic functions on $\{z; |z|<1\}$ with the supremum norm, have the* A.P.*?*

Observe that the two concrete spaces appearing in 1.e.10, for which the A.P. has not yet been verified, are non-separable. The common separable spaces which appear in analysis have the A.P. and, as a matter of fact, as mentioned already in Section a they even have a Schauder basis. We would like to point out that it is usually much easier to verify that a given space has the A.P. than to construct a basis in this space. Let us illustrate this by considering the disc algebra A. As mentioned in Section a it is known by now that A has a basis. However, it is not easy to construct such a basis (and its existence was open for a long time). On the other hand it is very easy to verify that A has the A.P. Indeed, for $f(x)=a_0+a_1z+a_2z^2+\cdots \in A$ put $S_n f=a_0+a_1z+\cdots+a_nz^n$ and $\sigma_n f=(S_1 f+S_2 f+\cdots+S_n f)/n$, $n=1, 2, \ldots$. The classical result of Fejer states that $\|\sigma_n\| \leqslant 1$ for all n and that $\|\sigma_n f - f\| \to 0$, for every $f \in A$. This shows that the disc algebra has the A.P. (even the M.A.P. defined below).

In the definition 1.e.1 of the A.P. we imposed no requirement on the norm of the operator T. For a Banach space with a basis the operator T required in 1.e.1 can be chosen to be bounded by a constant independent of the compact set K (namely, by the basis constant). We shall study now this stronger version of the A.P.

Definition 1.e.11. Let X be a Banach space and let $1 \leqslant \lambda < \infty$. We say that X has the *λ-approximation property* (λ-A.P. in short) if, for every $\varepsilon>0$ and every compact set K in X, there is a finite rank operator T in X so that $\|Tx-x\| \leqslant \varepsilon$, for every $x \in K$, and $\|T\| \leqslant \lambda$. A Banach space is said to have the *bounded approximation property* (B.A.P. in short) if it has the λ-A.P., for some λ. A Banach space is said to have the *metric approximation property* (M.A.P. in short) if it has the 1-A.P.

Observe that in 1.e.11 it is enough to take instead of a general compact set K finite sets only. Indeed, given K and ε we find $\{x_i\}_{i=1}^{n}$ so that $K \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon/3\lambda)$. If $\|T\| \leqslant \lambda$ and $\|Tx_i - x_i\| \leqslant \varepsilon/3$, for every i, then $\|Tx-x\| \leqslant \varepsilon$, for every $x \in K$ (in 1.e.1 it is of course essential that K is infinite; for finite sets K a T satisfying the requirements in 1.e.1 always exists trivially).

As we already observed, a space with a basis has the B.A.P. (and a space with a monotone basis has the M.A.P.). It is not known whether the converse is true.

Problem 1.e.12. *Does there exist a separable Banach space which has the* B.A.P. *but fails to have a basis?*

It is likely that the answer to 1.e.12 is negative and that the "right" relation between bases and the B.A.P. is the one given by the following result of [118] and [61].

Theorem 1.e.13. *A separable Banach space X has the* B.A.P. *if and only if X is isomorphic to a complemented subspace of a space with a basis.*

Proof [118]. The "if" part is trivial and so we have merely to prove the "only if" part. We start by making two observations.

1. Assume that X is separable and has the λ-A.P. Then there exists a sequence of finite rank operators $\{S_n\}_{n=1}^\infty$ on X so that $x=\sum_{n=1}^\infty S_n x$, for every $x \in X$, and $\left\|\sum_{i=1}^n S_i\right\| \leqslant \lambda$, for every n. Indeed, let $\{y_i\}_{i=1}^\infty$ be a dense sequence in X. There exist, for $n=1, 2,\ldots$, $T_n \in L(X, X)$ of finite rank such that $\|T_n\| \leqslant \lambda$ and $\|T_n y_i - y_i\| \leqslant n^{-1}$ for $1 \leqslant i \leqslant n$. The operators $\{S_n\}_{n=1}^\infty$ defined by $S_1=T_1$ and $S_n=T_n-T_{n-1}$, for $n>1$, have the desired property.

2. Let B be a Banach space with $\dim B=n$. Then, there are operators $\{U_i\}_{i=1}^{n^2}$ in $L(B, B)$ such that $\dim U_k B=1$, $\left\|\sum_{i=1}^k U_i\right\| \leqslant 2$, for every $1 \leqslant k \leqslant n^2$, and $\sum_{i=1}^{n^2} U_i x = x$ for every $x \in B$. Indeed, let $\{x_j\}_{j=1}^n$ and $\{x_j^*\}_{j=1}^n$ be an Auerbach system for X (see 1.c.3). For $i=rn+j$, $0 \leqslant r < n$, $1 \leqslant j \leqslant n$ put $U_i x = x_j^*(x) x_j / n$. Then, for every $k=rn+j$, we get

$$\left\|\sum_{i=1}^k U_i\right\| \leqslant \left\|\sum_{i=1}^{rn} U_i\right\| + \sum_{i=1}^j \|U_{rn+i}\| = \|rI/n\| + \sum_{i=1}^j n^{-1} \leqslant 2$$

(I denotes the identity operator on B).

Let now X be a separable space having the B.A.P. and choose $\{S_n\}_{n=1}^\infty$ as in observation 1. Since every space $S_n X$ is finite dimensional we can construct, for each n, operators $\{U_{i,n}\}_{i=1}^{m_n}$ on $S_n X$, as in observation 2, where $m_n = (\dim S_n X)^2$. Put $V_j = U_{i,n} S_n$ if $j = m_1 + m_2 + \cdots + m_{n-1} + i$, $1 \leqslant i \leqslant m_n$, $n=1, 2,\ldots$. Then, for every $x \in X$, $x = \sum_{j=1}^\infty V_j x$, $\left\|\sum_{j=1}^k V_j\right\| \leqslant 5\lambda$ for every k and $\dim V_j X = 1$ for every j. Let v_j be a vector of norm 1 in $V_j X$, $j=1, 2,\ldots$. Let Y be the space consisting of all sequences of scalars $y=(a_1, a_2,\ldots)$ such that $\sum_{j=1}^\infty a_j v_j$ converges and put

$$\|y\| = \sup_k \left\|\sum_{j=1}^k a_j v_j\right\|.$$

The unit vectors form clearly a monotone basis of Y. Define $V: X \to Y$ by $Vx=(a_1, a_2,\ldots)$, where a_j is the scalar determined by $V_j x = a_j v_j$, $j=1, 2,\ldots$. Clearly, $\|V\| \leq 5\lambda$ and $\|V^{-1}\| \leq 1$, i.e. V is an isomorphism (into). Let $U: Y \to X$ be the operator defined by $U(a_1, a_2,\ldots)=\sum_{j=1}^{\infty} a_j v_j$. Clearly, UV is the identity operator of X and hence VX is complemented in Y. □

A part of Theorem 1.e.4 can be generalized to the setting of B.A.P. or M.A.P. without any change in the proof. We state this for the M.A.P.

Proposition 1.e.14. *Let X be a Banach space. The following four assertions are equivalent.*

(i) *X has the* M.A.P.

(ii) *For every Banach space Y the finite rank operators of norm ≤ 1 are dense in the unit ball of $L(Y, X)$ in the topology τ.*

(iii) *For every Banach space Y the finite rank operators of norm ≤ 1 are dense in the unit ball of $L(X, Y)$ in the topology τ.*

(iv) *For every choice of $\{x_n\}_{n=1}^{\infty} \subset X$, $\{x_n^*\}_{n=1}^{\infty} \subset X^*$ such that $\sum_{n=1}^{\infty} \|x_n\|\,\|x_n^*\| < \infty$ and $\left|\sum_{n=1}^{\infty} x_n^*(Tx_n)\right| \leq \|T\|$, for every operator T of finite rank in $L(X, X)$, we have $\left|\sum_{n=1}^{\infty} x_n^*(x_n)\right| \leq 1$.*

Condition (v) of 1.e.4 does not generalize to this setting since the T_1 given there satisfies automatically $\|T_1\| \leq \|T\| + \varepsilon$ and it can actually be chosen always so that $\|T_1\| = \|T\|$.

We conclude this section with a discussion of the relation between the A.P. and the M.A.P. Grothendieck [48] proved the surprising result that in many cases the A.P. implies the M.A.P.

Theorem 1.e.15. *Let X be a separable space which is isometric to a dual space and which has the* A.P. *Then X has the* M.A.P.

For the proof of 1.e.15 we need two simple lemmas.

Lemma 1.e.16. *Let X be separable and let $\varepsilon > 0$. Then there exists a sequence of functions $\{f_i\}_{i=1}^{\infty}$ on the unit ball B_X of X so that $x=\sum_{i=1}^{\infty} f_i(x)$, for every x in B_X, each $f_i(x)$ is of the form $\sum_{j=1}^{\infty} \chi_{E_{i,j}}(x) x_{i,j}$, where $\{E_{i,j}\}_{j=1}^{\infty}$ are disjoint Borel sets of B_X, $\{x_{i,j}\}_{j=1}^{\infty} \subset B_X$ and $\sum_{i=1}^{\infty} \|f_i\|_{\infty} < 1+\varepsilon$ where $\|f_i\|_{\infty} = \sup_x \|f_i(x)\| = \sup_j \|x_{i,j}\|$.*

Proof. We construct the $\{f_i\}_{i=1}^{\infty}$ inductively. Choose first an f_1 of the suitable form so that $\|f_1\|_{\infty} \leq 1$ and $\|x-f_1(x)\|_{\infty} \leq \varepsilon/2$, then an f_2 so that $\|f_2\|_{\infty} \leq \varepsilon/2$ and $\|x-f_1(x)-f_2(x)\|_{\infty} \leq \varepsilon/4$ and continue in an obvious manner. □

Lemma 1.e.17. *Let $X = Y^*$. The space of all operators T of the form*

$$(*)\qquad Tx = \sum_{i=1}^{n} x(y_i)x_i, \quad \textit{with } \{x_i\}_{i=1}^{n} \subset X \quad \textit{and} \quad \{y_i\}_{i=1}^{n} \subset Y,$$

is τ-dense in the space of all finite rank operators from X into itself.

Proof. It is enough to note that every $x^* \in X^* = Y^{**}$ is a limit (in the sense of uniform convergence on compact sets of X) of elements from $JY \subset Y^{**}$. $\square$

Proof of 1.e.15. Let $X = Y^*$ be a space having the A.P. By 1.e.14 we have to show that if φ is a τ-continuous linear functional on $L(X, X)$ such that $|\varphi(T)| \leq \|T\|$, for finite rank operators, then $|\varphi(T)| \leq \|T\|$, for every $T \in L(X, X)$. We shall prove this in the following manner. For every $\varepsilon > 0$ we shall construct a τ continuous linear functional ψ_ε on $L(X, X)$ such that $\psi_\varepsilon(T) = \varphi(T)$, for T of the form (*), and for which it will be evident that $|\psi_\varepsilon(T)| \leq (1+\varepsilon)\|T\|$ for every $T \in L(X, X)$. By the assumption that X has the A.P. and 1.e.17 it follows from this that $\psi_\varepsilon(T) = \varphi(T)$, for all $T \in L(X, X)$, and thus $|\varphi(T)| \leq (1+\varepsilon)\|T\|$ for every T. Since $\varepsilon > 0$ is arbitrary this gives the desired result.

For the construction of ψ_ε we use 1.e.16. First we let $K = B_X \times B_{X^*}$. This is a compact metric space if we endow B_X with the w^* topology induced by Y and B_{X^*} with the w^* topology induced by X. To every T of the form (*) we assign a function $g_T \in C(K)$ by $g_T(x, x^*) = x^*(Tx)$. The special form of T ensures that g_T is continuous. The map $T \to g_T$ is an isometry. By the Hahn–Banach and the Riesz representation theorems it follows that there is a measure μ of norm 1 on K so that

$$\varphi(T) = \int_K x^*(Tx)\, d\mu, \quad T \text{ of the form } (*)\,.$$

Apply now 1.e.16. Then for T of the form (*), we get

$$\binom{*}{*}\qquad \varphi(T) = \sum_{i=1}^{\infty} \int_K x^*T(f_i(x))d\mu = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{i,j}^* T x_{i,j}$$

where $x_{i,j}^*$ is the functional on X defined by $x_{i,j}^*(x) = \int_{E_{i,j} \times B_{X^*}} x^*(x) d\mu$. Clearly $\|x_{i,j}^*\| \leq |\mu|(E_{i,j} \times B_{X^*})$ and hence $\sum_{j=1}^{\infty} \|x_{i,j}^*\| \leq \|\mu\| \leq 1$, for every i. Also $\sum_{i=1}^{\infty} \sup_j \|x_{i,j}\| \leq 1+\varepsilon$. It follows that the right-hand side of $\binom{*}{*}$, which we denote by $\psi_\varepsilon(T)$, is a τ continuous functional on $L(X, X)$ which satisfies

$$|\psi_\varepsilon(T)| \leq \|T\| \cdot \sum_{i,j=1}^{\infty} \|x_{i,j}^*\| \|x_{i,j}\| \leq (1+\varepsilon)\|T\|. \quad \square$$

It follows from 1.e.15 that, for separable reflexive spaces, the A.P. implies the M.A.P. The same is true for nonseparable reflexive spaces. This follows e.g. from the fact (cf. [83]) that if X is reflexive and X_0 is a separable subspace of X then there is a separable space Z, $X_0 \subset Z \subset X$ so that there is a projection of norm 1 from X onto Z.

In general, the A.P. does not imply the B.A.P. and the B.A.P. does not imply the M.A.P. This was shown by T. Figiel and W. B. Johnson [41]. In order to present their example it is convenient to use the following variant of the λ-A.P. A Banach space X is said to satisfy the (ε, λ)-A.P. if, for every finite dimensional subspace B of X and every $\delta > 0$, there is a finite rank operator T on X so that $||Tx - x|| \leqslant (\varepsilon + \delta)||x||$ for $x \in B$ and $||T|| \leqslant \lambda + \delta$. It is easily checked that the λ-A.P. is the same property as the $(0, \lambda)$-A.P. The (ε, λ)-A.P. implies the λ'-A.P, for some $\lambda' = \lambda'(\lambda, \varepsilon)$. More precisely,

Lemma 1.e.18. *A Banach space which has the* (ε, λ)-A.P. *with* $0 < \varepsilon < 1$ *has also the* $(1-\varepsilon)^{-1}\lambda$-A.P.

Proof. Let $\delta > 0$ be such that $\varepsilon + \delta < 1$ and let $B \subset X$ with $\dim B < \infty$. By our assumption we can find inductively a sequence $\{T_n\}_{n=1}^{\infty}$ of finite rank operators of norm $\leqslant \lambda + \delta$ on X so that $||T_1 x - x|| \leqslant (\varepsilon + \delta)||x||$, for $x \in B$, and

$$||T_{n+1}x - x|| \leqslant (\varepsilon + \delta)||x||, \quad \text{for } x \in \operatorname{span}\left\{B \cup \bigcup_{i=1}^{n} T_i X\right\}.$$

For $n \geqslant 1$ let $S_n \in L(X, X)$ be defined by $(I - S_n) = (I - T_n)(I - T_{n-1})\dots(I - T_1)$. Then. for $x \in B$, $||(I - S_n)x|| \leqslant (\varepsilon + \delta)^n ||x||$. Also,

$$S_n = (I - T_n)(I - T_{n-1})\dots(I - T_2)T_1 + (I - T_n)\dots(I - T_3)T_2 + \dots + T_n\,.$$

Hence, $||S_n|| = (\lambda + \delta)((\varepsilon + \delta)^{n-1} + (\varepsilon + \delta)^{n-2} + \dots + (\varepsilon + \delta) + 1) \leqslant (\lambda + \delta)/(1 - \varepsilon - \delta)$. Since δ can be taken arbitrarily small this proves our assertion. ☐

The main step in the construction of Figiel and Johnson is contained in the next lemma.

Lemma 1.e.19 *Let X be a Banach space and $\lambda \geqslant 1$ be a number such that X has the* λ-A.P. *in every equivalent norm. Then, for every $\varepsilon > 0$, the dual X^* of X has the* $(\varepsilon, \lambda(1 + 2\varepsilon^{-1}\lambda))$-A.P.

Proof. Let B be a finite dimensional subspace of X^*, let $\delta > 0$ and $\beta = \lambda + \delta$. Let C be a finite-dimensional subspace of X such that, for every $x^* \in B$, $||x^*|| \leqslant (1 + \delta) \sup \{|x^*(x)|;\, x \in C,\, ||x|| = 1\}$. We fix $\varepsilon > 0$ and introduce a new norm $|||\cdot|||$ on X^* by $|||x^*||| = ||x^*|| + 2\varepsilon^{-1}\beta d(x^*, B)$. Since the unit ball of $|||\cdot|||$ is w^* compact this norm is induced by an equivalent norm in X which is also denoted by $|||\cdot|||$.

Since, by our assumption, $(X, |||\cdot|||)$ has the λ-A.P. there is a finite rank operator T on X so that $||Tx - x|| \leqslant \delta ||x||$, for $x \in C$, and $|||T||| \leqslant \beta$. Passing to the dual we get, for $x^* \in X^*$

$$(\dagger) \qquad ||T^*x^*|| + 2\varepsilon^{-1}\beta d(T^*x^*, B) \leqslant \beta(||x^*|| + 2\varepsilon^{-1}\beta d(x^*, B))\,.$$

It follows that $||T^*x^*||\leqslant\beta(1+2\varepsilon^{-1}\beta)||x^*||$ and hence, $||T||\leqslant\beta(1+2\varepsilon^{-1}\beta)$. For $x^*\in B$ we get from (†) that $d(T^*x^*, B)\leqslant\varepsilon||x^*||/2$, i.e. there is a $y^*\in B$ such that $||T^*x^*-y^*||\leqslant\varepsilon||x^*||/2$. Let $y\in C$ be any element of norm 1. Then, $|T^*x^*(y)-x^*(y)|=|x^*(Ty-y)|\leqslant\delta||x^*||$. Hence,

$$\begin{aligned}||x^*-y^*||&\leqslant(1+\delta)\sup\{|x^*(y)-y^*(y)|,\ y\in C,\ ||y||=1\}\\&\leqslant(1+\delta)(\delta+\varepsilon/2)||x^*||\,,\end{aligned}$$

and consequently, $||T^*x^*-x^*||\leqslant((1+\delta)(\delta+\varepsilon/2)+\varepsilon/2)||x^*||$. Since $\delta>0$ is arbitrary the lemma is proved. □

Example 1.e.20. *There is a separable Banach space X (which has even a separable dual) such that X has the* A.P. *but not the* B.A.P.

Proof. Let Z be a space such that Z has the A.P. but Z^* fails to have the A.P. and is separable (see 1.e.7(b)). By 1.e.18 and 1.e.19 we can find, for every integer n, an equivalent norm $|||\cdot|||_n$ on Z so that $(Z, |||\cdot|||_n)$ fails to have the n-A.P. The space $\left(\sum_{n=1}^{\infty}\oplus(Z, |||\cdot|||_n)\right)_2=X$ clearly has the A.P., fails to have the B.A.P. and has a separable dual. □

Note that if Z has the B.A.P. and Z^* fails to have the A.P. then $(Z, |||\cdot|||_n)$, for $n>1$, is an example of a space which has the B.A.P. but not the M.A.P.

In connection with 1.e.15 and 1.e.20 we mention the following open problem.

Problem 1.e.21. *Let X be a Banach space having the* B.A.P. *Does there exist an equivalent norm $|||\cdot|||$ on X so that $(X, |||\cdot|||)$ has the* M.A.P.*?*

The investigation of the approximation property will be continued in Section 2.d. This section (starting from Theorem 2.d.3) can be read directly after the present section.

f. Biorthogonal Systems

The existence of separable Banach spaces which fail to have a basis motivates the attempts to try to use some weaker forms of coordinate systems. One approach, which has been studied for a long time and for which strong existence theorems are now available, is that of using biorthogonal systems.

Definition 1.f.1. Let X be a Banach space. A pair of sequences $\{x_n\}_{n=1}^{\infty}$ in X and $\{x_n^*\}_{n=1}^{\infty}$ in X^* is called a *biorthogonal system* if $x_m^*(x_n)=\delta_n^m$. A sequence $\{x_n\}_{n=1}^{\infty}$ in X is called a *minimal system* if there exists a sequence $\{x_n^*\}_{n=1}^{\infty}$ in X such that $(\{x_n\}_{n=1}^{\infty}, \{x_n^*\}_{n=1}^{\infty})$ is a biorthogonal system.

It is clear that a sequence $\{x_n\}_{n=1}^{\infty}$ is minimal if and only if, for every integer n, $x_n\notin[x_i]_{i=1, i\neq n}^{\infty}$. Observe that if $(\{x_n\}_{n=1}^{\infty}, \{x_n^*\}_{n=1}^{\infty})$ forms a biorthogonal system then both $\{x_n\}_{n=1}^\infty$ and $\{x_n^*\}_{n=1}^\infty$ are minimal systems. Every basic sequence $\{x_n\}_{n=1}^\infty$ is a minimal system. The functionals $\{x_n^*\}_{n=1}^\infty$ are in this case the biorthogonal functionals discussed in Section b (more precisely, extensions of those functionals from $[x_n]_{n=1}^\infty$ to all of X). Observe that, unlike the notion of a basis, the notions of a minimal system and of a biorthogonal system involve no natural ordering and they should therefore be considered as countable sets of elements rather than sequences. There are important examples of minimal systems which do not form a basic sequence in any ordering. For example, take $x_n(t)=e^{int}$, $n=0, \pm 1, \pm 2, \ldots$ in $\tilde{C}(0, 2\pi)$ (= the subspace of $C(0, 2\pi)$ consisting of those functions f for which $f(0)=f(2\pi)$). The corresponding functionals x_n^* in $\tilde{C}(0, 2\pi)^*$ are in this case the measures given by $x_n^*=e^{-int}\,dt$, $n=0, \pm 1, \pm 2, \ldots$. The fact that $\{x_n\}_{n=-\infty}^\infty$ does not form a basis under any ordering follows e.g. from a result of P. Cohen [19].

In Section c above we proved the existence of a nice biorthogonal system in any finite dimensional space (called there an Auerbach system). We shall present now existence theorems in the infinite dimensional case. Since we want to construct a biorthogonal system which has to play a role similar to that of a basis rather than a basic sequence we introduce first a definition of a suitable notion of completeness of a biorthogonal system.

Definition 1.f.2. A minimal system $\{x_n\}_{n=1}^\infty$ is called *fundamental* if $[x_n]_{n=1}^\infty$ is all of X (i.e. $x^*(x_n)=0$ for all $n \Rightarrow x^*=0$). A minimal system $\{x_n^*\}_{n=1}^\infty$ in X^* is called *total* if $x_n^*(x)=0$ for all $n \Rightarrow x=0$ (i.e. X^* is the w^* closed linear span of $\{x_n^*\}_{n=1}^\infty$).

If $\{x_n\}_{n=1}^\infty$ is a basis in X then it is clearly fundamental and its biorthogonal functionals are total. The trigonometric system mentioned above is total and fundamental in $\tilde{C}(0, 2\pi)$. The following general existence theorem is known for a long time (it was proved first by Markushevich [101]).

Proposition 1.f.3. *Let X be a separable Banach space. Then X contains a fundamental minimal system whose biorthogonal functionals are total (or more briefly, with a slight abuse of language, X contains a total and fundamental biorthogonal system).*

Proof. Let $\{y_n\}_{n=1}^\infty$ be a sequence of non-zero elements in X such that $[y_n]_{n=1}^\infty=X$ and let $\{y_n^*\}_{n=1}^\infty$ be a sequence in X^* such that $y_n^*(x)=0$ for all n implies $x=0$. We shall construct inductively elements $\{x_n\}_{n=1}^\infty\subset X$, $\{x_n^*\}_{n=1}^\infty\subset X^*$ such that $x_m^*(x_n)=\delta_n^m$, span $\{x_n\}_{n=1}^\infty\supset$ span $\{y_n\}_{n=1}^\infty$ and span $\{x_n^*\}_{n=1}^\infty\supset$ span $\{y_n^*\}_{n=1}^\infty$.

We start by taking $x_1=y_1$ and put $x_1^*=y_{k_1}^*/y_{k_1}^*(y_1)$, where k_1 is any integer such that $y_{k_1}^*(y_1)\neq 0$. Next, we take the smallest integer h_2 such that $y_{h_2}^* \notin$ span x_1^*. Put $x_2^*=y_{h_2}^*-x_1^*\cdot y_{h_2}^*(x_1)$ and let $x_2=(x_{k_2}-x_1\cdot x_1^*(x_{k_2}))/x_2^*(x_{k_2})$, where k_2 is any index such $x_2^*(x_{k_2})\neq 0$. It is easily checked that with this choice $x_n^*(x_m)=\delta_n^m$ for $1\leqslant n, m\leqslant 2$. In the next step we let h_3 be the smallest integer such that $y_{h_3}\notin$ span $\{x_1, x_2\}$. We put

$$x_3=y_{h_3}-x_1\cdot x_1^*(y_{h_3})-x_2\cdot x_2^*(y_{h_3})$$

and

$$x_3^*=(y_{k_3}^*-x_1^*\cdot y_{k_3}^*(x_1)-x_2^*\cdot y_{k_3}^*(x_2))/y_{k_3}^*(x_3)\ ,$$

where k_3 is such that $y^*_{k_3}(x_3) \neq 0$. We continue in an obvious way. In the step $2n$ we start in X^* and construct first the element x^*_{2n} while in the step $2n+1$ we start by constructing x_{2n+1}. It is clear that span $\{x_i\}_{i=1}^{2n} \supset$ span $\{y_i\}_{i=1}^{n}$ and span $\{x_i^*\}_{i=1}^{2n} \supset$ span $\{y_i^*\}_{i=1}^{n}$, for every n, and that $x_j^*(x_i) = \delta_i^j$. $\square$

Remark. The proof given above shows that we can assure that the $\{x_n^*\}_{n=1}^\infty$ are not only total but also that $[x_n^*]_{n=1}^\infty$ is norming, i.e. that for every $x \in X$, $||x|| = \sup\{|x^*(x)|;\ ||x^*|| \leq 1,\ x^* \in [x_n^*]_{n=1}^\infty\}$. Indeed, we have simply to start the construction with a sequence $\{y_n^*\}_{n=1}^\infty$ so that $[y_n^*]_{n=1}^\infty$ is norming. Similarly, we can assure that $[x_n^*]_{n=1}^\infty = X^*$ if X^* is separable.

The simple proposition 1.f.3 can be used for replacing bases (which may not exist) in several situations. There is, however, one obvious drawback to the construction given in 1.f.3. We have no control on $||x_n||$ and $||x_n^*||$; in general, $\sup_n ||x_n||\,||x_n^*|| = \infty$ (of course we can always normalize the system so that, e.g. $||x_n|| = 1$, for every n, but then $\sup_n ||x_n^*||$ may be ∞). If $\{x_n\}_{n=1}^\infty$ is a basis then clearly $\sup_n ||x_n^*||\,||x_n|| < \infty$. Solving a problem which was open for a long time, R. Ovsepian and A. Pelczynski [112] proved that 1.f.3 can be strengthened to ensure that we get also $\sup_n ||x_n^*||\,||x_n|| < \infty$.

Theorem 1.f.4. *In every separable and infinite-dimensional Banach space X there is a fundamental and total biorthogonal system $(\{x_n\}_{n=1}^\infty, \{x_n^*\}_{n=1}^\infty)$ so that $||x_n|| \cdot ||x_n^*|| \leq 20$, for every n. If X^* is separable the system may be chosen so that, in addition, $[x_n^*]_{n=1}^\infty = X^*$.*

Proof. The proof is divided into two steps. The first step is a slight refinement of the proof of 1.f.3 which shows that it is possible to choose a fundamental and total biorthogonal system $(\{u_n\}_{n=1}^\infty, \{u_n^*\}_{n=1}^\infty)$ so that, for some subsequence $\{n_k\}_{k=1}^\infty$ of the integers, $||u_{n_k}|| \cdot ||u_{n_k}^*|| \leq 3$, $k = 1, 2, \ldots$. The second step shows how, by using this well behaved subsequence, it is possible to replace the biorthogonal system by another one in which the entire sequence behaves well.

We begin with the proof of the first step. As in the proof of 1.f.3 we construct $\{u_n\}_{n=1}^\infty$ and $\{u_n^*\}_{n=1}^\infty$ inductively. The inductive construction will depend on n (mod 3). If $n = 3j+1$, resp. $n = 3j+2$, we do exactly the same which we did in the proof of 1.f.3 for n odd, resp. n even. This, by itself, will ensure that span $\{u_n\}_{n=1}^\infty \supset$ span $\{y_n\}_{n=1}^\infty$ and span $\{u_n^*\}_{n=1}^\infty \supset$ span $\{y_n^*\}_{n=1}^\infty$. The inductive construction for $n = 3j$ will be such that $||u_{3j}^*||\,||u_{3j}|| \leq 3$. Assume that the u_n and u_n^* have been chosen for $n < 3j$. By 1.a.6 there is a vector u_{3j} in X such that $||u_{3j}|| = 1$, $u_n^*(u_{3j}) = 0$ if $n < 3j$ and $||x|| \leq 2||x + \lambda u_{3j}||$, $x \in$ span $\{u_n\}_{n=1}^{3j-1}$, λ scalar. The functional on span $\{u_n\}_{n=1}^{3j}$, which assigns the value 0 to u_n with $n < 3j$ and the value 1 to u_{3j}, has norm ≤ 3. We take as u_{3j}^* any Hahn–Banach extension of it to all of X^*. This concludes the proof of the first step.

For the second step we need the following lemma

Lemma 1.f.5. *Let X be a Banach space and let $\{u_i\}_{i=1}^{2n} \subset X$ and $\{u_i^*\}_{i=1}^{2n} \subset X^*$ be such* *that* $u_j^*(u_i)=\delta_i^j$. *Then, there exists a real unitary matrix* $A=(a_{k,i})$ *of order* $2^n\times 2^n$ *so that if*

$$x_k=\sum_{i=1}^{2^n} a_{k,i}u_i, \qquad x_k^*=\sum_{i=1}^{2^n} a_{k,i}u_i^*, \quad k=1,2,\ldots,2^n$$

then

1. $\max_{1\leqslant k\leqslant 2^n} \|x_k\|\leqslant(1+\sqrt{2})\max_{1\leqslant i<2^n}\|u_i\|+2^{-n/2}\|u_{2^n}\|$

2. $\max_{1\leqslant k\leqslant 2^n} \|x_k^*\|\leqslant(1+\sqrt{2})\max_{1\leqslant i<2^n}\|u_i^*\|+2^{-n/2}\|u_{2^n}^*\|$

3. $x_j^*(x_i)=\delta_i^j, \quad 1\leqslant i,j\leqslant 2^n$

4. $\operatorname{span}\{x_k\}_{k=1}^{2^n}=\operatorname{span}\{u_k\}_{k=1}^{2^n}; \qquad \operatorname{span}\{x_k^*\}_{k=1}^{2^n}=\operatorname{span}\{u_k^*\}_{k=1}^{2^n}$.

Proof. The relations 3 and 4 hold for any choice of a unitary matrix A. We get that also 1 and 2 hold if we choose $A=(a_{k,i})$ so that, for $k=1,\ldots,2^n$,

$$\sum_{i=1}^{2^n}|a_{k,i}|\leqslant 1+\sqrt{2} \quad\text{and}\quad |a_{k,2^n}|\leqslant 2^{-n/2}.$$

Such an orthogonal matrix A exists; put for $0\leqslant s\leqslant n-1$, $0\leqslant r\leqslant 2^s-1$

$$a_{k,2^s+r}=\begin{cases}2^{(s-n)/2}, & 2^{n-s-1}2r<k\leqslant 2^{n-s-1}(2r+1)\\ -2^{(s-n)/2}, & 2^{n-s-1}(2r+1)<k\leqslant 2^{n-s-1}(2r+2)\\ 0 & \text{otherwise}\end{cases}$$

and $a_{k,2^n}=2^{-n/2}$ for every k. □

We give now the proof of step 2 of 1.f.4. We take the biorthogonal sequence which was constructed in step 1 and reorder it in such a manner that in the new order "most" elements are nicely bounded. More precisely, we give this sequence new indices such that with the new indices the following holds. There is a sequence of integers $\{n_j\}_{j=1}^\infty$ so that if k is not of the form $2^{n_1}+2^{n_2}+\cdots+2^{n_i}$ for some i then $\|u_k\|\leqslant 1$ and $\|u_k^*\|\leqslant 3$ while, for $k=2^{n_1}+2^{n_2}+\cdots+2^{n_i}$, $\|u_k\|2^{-n_i/2}<1/20$, $\|u_k^*\|2^{-n_i/2}<1/20$. It is clear that such a choice of indices can be made. Indeed, pick the first element in the original sequence, say an element $u\in X$ with a corresponding $u^*\in X^*$. We find an n_1 so that $\|u\|2^{-n_1/2}<1/20$ and $\|u^*\|2^{-n_1/2}<1/20$ and give this element the index 2^{n_1}. For $1\leqslant k<2^{n_1}$ we take out of the original sequence the first $2^{n_1}-1$ elements whose norms are 1 and whose biorthogonal functionals have norms $\leqslant 3$. We consider now the first element in the original sequence which was not chosen so far and use it to determine n_2. We continue in an obvious manner.

For every integer i we apply 1.f.5 to construct $x_k\in X$ and $x_k^*\in X^*$ for $2^{n_1}+2^{n_2}+\cdots+2^{n_{i-1}}<k\leqslant 2^{n_1}+2^{n_2}+\cdots+2^{n_i}$ so that the span of the x_k with these indices

(resp. of the x_k^* with these indices) coincides with that of the u_k (resp. the u_k^*) with the same indices, $x_k^*(x_j)=\delta_k^j$, and

$$\|x_k\| \leqslant (1+\sqrt{2})\cdot 1+1/20<2.5$$
$$\|x_k^*\| \leqslant (1+\sqrt{2})\cdot 3+1/20<7.5 . \quad \square$$

The proof gives clearly a constant which is better than the constant 20 appearing in the statement of 1.f.4. By modifying the proof, Pelczynski [119] was able to show that 1.f.4 remains true if 20 is replaced by any constant >1. It seems to be open whether it is actually possible to get $\|x_n\|\,\|x_n^*\|=1$, for every n (in the finite dimensional case this is possible; this is exactly the assertion of Auerbach's Lemma 1.c.3).

Many of the notions we encountered in Sections a and b can be defined in a meaningful way also in the context of general biorthogonal systems. A minimal system $\{x_n\}_{n=1}^\infty$ is said to be equivalent to a minimal system $\{y_n\}_{n=1}^\infty$ if there is an isomorphism T of $[x_n]_{n=1}^\infty$ onto $[y_n]_{n=1}^\infty$ such that $Tx_n=y_n$ for all n. In analogy to 1.a.9 it is easy to prove the following stability theorem. If $\{x_n\}_{n=1}^\infty \subset X$ and $\{x_n^*\}_{n=1}^\infty \subset X^*$ form a biorthogonal system and if $\{y_n\}_{n=1}^\infty \subset X$ satisfy $\sum_{n=1}^\infty \|x_n-y_n\|\,\|x_n^*\|<1$ then $\{y_n\}_{n=1}^\infty$ is a minimal system which is equivalent to $\{x_n\}_{n=1}^\infty$. A minimal system $\{x_n\}_{n=1}^\infty$ is said to be shrinking if the following relation holds $[x_n^*]_{n=1}^\infty=([x_n]_{n=1}^\infty)^*$ (here the x_n^* are considered as functionals on $[x_n]_{n=1}^\infty$ only). A minimal system $\{x_n\}_{n=1}^\infty$ is said to be boundedly complete if, for every bounded sequence $\{y_i\}_{i=1}^\infty$ in $[x_n]_{n=1}^\infty$, the existence of $\lim_i x_n^*(y_i)=a_n$ for every n implies the existence of a vector y in $[x_n]_{n=1}^\infty$ such that $x_n^*(y)=a_n$ for every n. Clearly, if $\{x_n\}_{n=1}^\infty$ is a basic sequence these notions agree with those defined in Section b. The natural generalization of 1.b.5 is true in the present setting (with the same proof); let $(\{x_n\}_{n=1}^\infty, \{x_n^*\}_{n=1}^\infty)$ be a fundamental and total biorthogonal system for a Banach space X. Then X is reflexive if and only if $\{x_n\}_{n=1}^\infty$ is both shrinking and boundedly complete.

A detailed discussion of minimal systems and their applications is given in [106]. In this paper V. D. Milman uses, e.g. biorthogonal systems for proving 1.b.14. (In the original proof of 1.b.14 given by Milman there is however a wrong statement. He stated that a block minimal system (in the obvious definition of this notion) of a boundedly complete minimal system is again boundedly complete. As observed in [26] this is not true even for bases. What is true however is that every block minimal system of a boundedly complete system has a boundedly complete subsequence. This is a direct consequence of 1.b.12.)

We conclude this section by showing that there is no non-trivial generalization of the notion of an unconditional basis to the setting of biorthogonal systems. This was proved by various authors in several degrees of generality (see, e.g. [7] and [23]). We present here, following [98], a very simple variant.

Proposition 1.f.6. *Let $\{x_n\}_{n=1}^\infty$ be a fundamental minimal system in a Banach space X such that $\{x_n^*\}_{n=1}^\infty$ is total. Assume that, for every $x \in X$ and every subset σ of the* *integers, there is an element x_σ in X such that $x_n^*(x_\sigma)=x_n^*(x)$ for $n\in\sigma$ and $x_n^*(x_\sigma)=0$ for $n\notin\sigma$. Then, $\{x_n\}_{n=1}^\infty$ is already an unconditional basis of X.*

Proof. It is clearly enough to show that $\{x_n\}_{n=1}^\infty$ is a basis of X (the assumptions are independent of the order). By 1.a.3 we have to show that the operators $\{P_n\}_{n=1}^\infty$, defined by $P_nx=\sum_{i=1}^n x_i^*(x)x_i$, are uniformly bounded. Observe first that it follows from our assumption and the closed graph theorem that, for every subset σ of the integers, there is a bounded linear operator P_σ on X defined by $P_\sigma x=x_\sigma$. If the $\{P_n\}_{n=1}^\infty$ are not uniformly bounded we can construct inductively a sequence of integers $1=p_1<q_1<p_2<q_2\cdots$ and vectors $\{u_j\}_{j=1}^\infty$ so that $||u_j||=2^{-j}$, $u_j\in\operatorname{span}\{x_i\}_{i=p_j}^{p_{j+1}-1}$ and $||P_{q_j}u_j||\geqslant 1$ for $j=1,2,\ldots$. Put $\sigma=\bigcup_{j=1}^\infty\{i;\,p_j\leqslant i\leqslant q_j\}$. Then $\sum_{j=1}^\infty u_j$ converges but $\sum_{j=1}^\infty P_\sigma u_j$ fails to converge and this contradicts the continuity of P_σ. □

g. Schauder Decompositions

A Schauder basis decomposes, in a sense, a Banach space into a sum of one-dimensional spaces. It is sometimes useful to consider cruder decompositions where the components into which we decompose a given Banach space are subspaces of dimension larger than 1.

Definition 1.g.1. Let X be a Banach space. A sequence $\{X_n\}_{n=1}^\infty$ of closed subspaces of X is called a *Schauder decomposition* of X if every $x\in X$ has a unique representation of the form $x=\sum_{n=1}^\infty x_n$, with $x_n\in X_n$ for every n.

Observe that if $\dim X_n=1$ for every n, i.e. $X_n=\operatorname{span}\{x_n\}$ then $\{X_n\}_{n=1}^\infty$ is a Schauder decomposition of X if and only if $\{x_n\}_{n=1}^\infty$ is a Schauder basis of X. Many results concerning bases generalize trivially to the setting of Schauder decompositions. Every Schauder decomposition $\{X_n\}_{n=1}^\infty$ of a Banach space X determines a sequence of projections $\{P_n\}_{n=1}^\infty$ on X by putting $P_n\sum_{i=1}^\infty x_i=\sum_{i=1}^n x_i$. These projections are bounded linear operators and $\sup_n ||P_n||<\infty$. The number $\sup_n ||P_n||$ is called the *decomposition constant* of $\{X_n\}_{n=1}^\infty$. Conversely, every sequence of bounded projections $\{P_n\}_{n=1}^\infty$ on X such that $P_nP_m=P_{\min(n,m)}$ and $\lim_n P_nx=x$ for every $x\in X$ determines a unique Schauder decomposition of X by putting $X_1=P_1X$ and $X_n=(P_n-P_{n-1})X$ for $n>1$. As in 1.a.3 it is easily seen that it is possible to replace the condition $\lim_n P_nx=x$ by the apparently weaker conditions $\sup_n ||P_n||<\infty$ and $\overline{\bigcup_{n=1}^\infty P_nX}=X$.

A decomposition $\{X_n\}_{n=1}^\infty$ is called *boundedly complete* if, for every sequence $\{x_n\}_{n=1}^\infty$ with $x_n \in X_n$, $n=1, 2, 3,\dots$ for which $\sup_n \left\|\sum_{i=1}^n x_i\right\| < \infty$, the series $\sum_{i=1}^\infty x_i$ converges. The decomposition is called *shrinking* if, for every $x^* \in X^*$, we have $\|P_n^* x^* - x^*\| \to 0$. If this is the case the sequence $\{P_n^*\}_{n=1}^\infty$ determines a boundedly complete Schauder decomposition of X^*. A decomposition $\{X_n\}_{n=1}^\infty$ is *unconditional* if, for every $x \in X$, the series $\sum_{n=1}^\infty x_n$, which represents x, converges unconditionally. In this case, for every sequence $\theta=(\theta_1, \theta_2,\dots)$ of signs, the operator M_θ defined by $M_\theta \sum_{n=1}^\infty x_n = \sum_{n=1}^\infty \theta_n x_n$ is a bounded linear operator. The constant $\sup_\theta \|M_\theta\|$ is called the *unconditional constant* of the decomposition.

The decompositions, which are most useful in applications, are those in which $\dim X_n < \infty$ for all n ($\sup_n \dim X_n$ need not be finite). Such decompositions are called *finite dimensional Schauder decompositions* or F.D.D. in short. The same proof as that of 1.b.5 shows that if $\{X_n\}_{n=1}^\infty$ is an F.D.D. of a Banach space X then X is reflexive and only if $\{X_n\}_{n=1}^\infty$ is boundedly complete and shrinking.

Of course, the interest in Schauder decompositions does not stem from results which generalize trivially theorems on bases. Their importance stems from the fact that there are results on F.D.D.'s whose analogues for bases are not known (and perhaps false) or do not even have a meaningful analogue in terms of bases. We shall illustrate this by proving here two results on F.D.D.'s which will be applied in the next chapter. The first theorem shows that the answer to Problem 1.b.10 concerning bases has a positive answer in the setting of F.D.D.'s. The second result is a theorem which makes sense only in the setting of F.D.D.

Theorem 1.g.2 [60]. *Let X be a separable infinite-dimensional Banach space. Then there exists a subspace Y of X such that both Y and X/Y have a* F.D.D. *Moreover, if X^* is separable Y may be chosen so that Y and X/Y have a shrinking* F.D.D.

Proof. By 1.f.3 (and the remark following it) there is a biorthogonal system $\{x_n\}_{n=1}^\infty \subset X$ and $\{x_n^*\}_{n=1}^\infty \subset X^*$ so that $[x_n]_{n=1}^\infty = X$ and $[x_n^*]_{n=1}^\infty$ is norming over X. We can therefore choose inductively finite sets $\sigma_1 \subset \sigma_2 \subset \cdots$ and $\eta_1 \subset \eta_2 \subset \cdots$ so that $\sigma = \bigcup_{n=1}^\infty \sigma_n$ and $\eta = \bigcup_{n=1}^\infty \eta_n$ are complementary infinite subsets of the positive integers and, for $n=1, 2,\dots,$

$$\|x\| \leq (1+n^{-1}) \sup\{|x^*(x)|;\ \|x^*\|=1,\ x^* \in [x_i^*]_{i \in \sigma_n \cup \eta_n}\}, \text{ for every } x \in [x_i]_{i \in \sigma_n}.$$
$$\|x^*\| \leq (1+n^{-1}) \sup\{|x^*(x)|,\ \|x\|=1,\ x \in [x_i]_{i \in \eta_n \cup \sigma_{n+1}}\}, \text{ for every } x^* \in [x_i^*]_{i \in \eta_n}.$$

For every n let S_n and T_n be the projections on X defined by $S_n x = \sum_{i \in \sigma_n} x_i^*(x) x_i$, respectively $T_n x = \sum_{i \in \eta_n} x_i^*(x) x_i$. We claim that

(i) $\quad \|T^*_{n|[x_i]^{\perp}_{i \in \sigma_{n+1}}}\| \leq 1+n^{-1}$

(ii) $\quad \|S_{n|[x_i^*]^{\top}_{i \in \eta_n}}\| \leq 1+n^{-1}$.

Indeed, let $x^* \in [x_i]^{\perp}_{i \in \sigma_{n+1}}$ and pick an $x \in [x_i]_{i \in \eta_n \cup \sigma_{n+1}}$ so that $||x||=1$ and $||T_n^* x^*|| \leqslant (1+n^{-1})|T_n^* x^*(x)|$. Since $T_n x - x \in [x_i]_{i \in \sigma_{n+1}}$ and $|T_n^* x^*(x)| = |x^*(x) + x^*(T_n x - x)| = |x^*(x)|$ we get that $||T_n^* x^*|| \leqslant (1+n^{-1})||x^*||$. This proves (i); the proof of (ii) is similar.

We show next that for $x^* \in [x_i]^{\perp}_{i \in \sigma}$, $T_n^* x^* \xrightarrow{w^*} x^*$. By (i) the sequence $\{T_n^* x^*\}_{n=1}^{\infty}$ is bounded. Let y^* be any w^* limit point of $\{T_n^* x^*\}_{n=1}^{\infty}$. Then, clearly $y^*(x_i) = x^*(x_i) = 0$ for $i \in \sigma$ and $y^*(x_i) = x^*(x_i)$ for $i \in \eta$. Since $\sigma \cup \eta$ is the set of all positive integers we deduce that $y^* = x^*$ and thus, indeed, $T_n^* x^* \xrightarrow{w^*} x^*$. It follows also that $[x_i]^{\perp}_{i \in \sigma}$ is the w^* closure of $[x_i^*]_{i \in \eta}$. Put $Y = [x_i^*]^{\top}_{i \in \eta} = [x_i]_{i \in \sigma}$. By the analogue of 1.b.9 for F.D.D.'s it follows that X/Y has an F.D.D. From (ii) it follows that $\{S_{n|Y}\}_{n=1}^{\infty}$ determines an F.D.D. in Y. This concludes the proof of the first assertion of the theorem.

Assume now that X^* is separable. Then the $\{x_n\}_{n=1}^{\infty}$ can be chosen so that, in addition, $[x_n^*]_{n=1}^{\infty} = X^*$. Also, we may assume that the norm in X is such that $\{y_n^*\}_{n=1}^{\infty} \subset X^*$, $y_n^* \xrightarrow{w^*} y^*$ and $||y_n^*|| \to ||y^*|| \Rightarrow ||y_n^* - y_n|| \to 0$ (use 1.b.11). The proof given above shows that in this case $||T_n^* x^* - x^*|| \to 0$ for every $x^* \in [x_i]^{\perp}_{i \in \sigma}$ (it is here that we use the factor $1+n^{-1}$ in (i); for the proof of the first assertion of 1.g.2 it would have been enough to replace $1+n^{-1}$ by 2, say). This shows that $[x_i^*]_{i \in \eta}$ is w^* closed and that the decomposition of X/Y constructed above is shrinking. Similarly, it follows from $[x_n^*]_{n=1}^{\infty} = X^*$ that $\{S_{n|Y}\}_{n=1}^{\infty}$ determines a shrinking decomposition of Y. $\square$

For stating the next result we need first a definition.

Definition 1.g.3. Let $\{X_n\}_{n=1}^{\infty}$ be a Schauder decomposition of X. Let $1 = k_1 < k_2 < k_3 < \cdots$ be an increasing sequence of integers and put $Y_i = X_{k_i} \oplus X_{k_i+1} \oplus \cdots \oplus X_{k_{i+1}-1}$, $i = 1, 2, \ldots$. Then, the decomposition $\{Y_i\}_{i=1}^{\infty}$ of X is said to be a *blocking* of the decomposition $\{X_n\}_{n=1}^{\infty}$.

If $\{P_n\}_{n=1}^{\infty}$ is the sequence of projection associated to an F.D.D. $\{X_n\}_{n=1}^{\infty}$ then the subsequences of $\{P_n\}_{n=1}^{\infty}$ are exactly the sequences of projections associated to the blockings of $\{X_n\}_{n=1}^{\infty}$. The decomposition (resp. the unconditional decomposition) constant of $\{Y_i\}_{i=1}^{\infty}$ is smaller or equal to the decomposition (resp. the unconditional decomposition) constant of $\{X_n\}_{n=1}^{\infty}$. One word of caution should be said concerning this definition. The notion of the blocking of a decomposition is not the obvious generalization of the notion of a block basic sequence. The direct generalization of the notion of a block basic sequence to the setting of Schauder decomposition would be to consider decompositions $\{Y_i\}_{i=1}^{\infty}$ of subspaces of X such that $Y_i \subset X_{k_i} \oplus X_{k_i+1} \oplus \cdots \oplus X_{k_{i+1}-1}$ for every i.

The following result concerning blockings of F.D.D.'s was proved in [63] and [59]. It turns out to be very useful in the study of the structure of subspaces of L_p spaces (and other spaces as well). In view of its general nature we state and prove it here; however, its significance will become clear only in view of its applications (for example, in Section 2.d below).

Proposition 1.g.4. (a) *Let T: $X \to Y$ be a bounded linear operator. Let $\{B_n\}_{n=1}^{\infty}$ be a shrinking* F.D.D. *of X and let $\{C_n\}_{n=1}^{\infty}$ be an* F.D.D. *of Y. Let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a sequence* *of positive numbers tending to* 0. *Then there are blockings* $\{B_i'\}_{i=1}^\infty$ *of* $\{B_n\}_{n=1}^\infty$ *and* $\{C_i'\}_{i=1}^\infty$ *of* $\{C_n\}_{n=1}^\infty$ *so that, for every* $x \in B_i'$, *there is a* $y \in C_{i-1}' \oplus C_i'$ *so that* $||Tx-y|| \leqslant \varepsilon_i ||x||$.

(b) *Let* $T\colon X \to Y$ *be a quotient map. Let* $\{B_n\}_{n=1}^\infty$ *be an* F.D.D. *of* X *and* $\{C_n\}_{n=1}^\infty$ *a shrinking* F.D.D. *of a subspace of* Y. *Let* $\{\varepsilon_i\}_{i=1}^\infty$ *be a sequence of positive numbers tending to* 0. *Then there is a constant* K *and blockings* $\{B_i'\}_{i=1}^\infty$ *of* $\{B_n\}_{n=1}^\infty$ *and* $\{C_i'\}_{i=1}^\infty$ *of* $\{C_n\}_{n=1}^\infty$ *so that, for every* $y \in C_i'$, *there is an* $x \in B_i' \oplus B_{i+1}'$ *such that* $||Tx-y|| \leqslant \varepsilon_i ||y||$ *and* $||x|| \leqslant K||y||$.

Both parts of 1.g.4 (which are in a sense dual to each other) assert that after a suitable blocking the given operator is close to being diagonal with respect to the blockings; e.g. in (a) TB_i' is "essentially" contained in $C_{i-1}' \oplus C_i'$.

Proof of (a). Let $\{P_n\}_{n=1}^\infty$, resp. $\{Q_n\}_{n=1}^\infty$, be the projections associated to the given decomposition of X, resp. Y. We note first that, for every $\varepsilon>0$ and integer n, there is an integer m such that if $x \in X$ with $P_m x=0$ then $||Q_n Tx|| \leqslant \varepsilon ||x||$. Indeed, otherwise there would be an n, an $\varepsilon>0$ and a sequence of vectors $\{x_k\}_{k=1}^\infty$ in X so that $||x_k||=1$, $||Q_n Tx_k|| \geqslant \varepsilon$ for all k and $\lim_k P_m x_k=0$ for every m. Since $Q_n Y$ is finite-dimensional we may assume (by passing to a subsequence if necessary) that, for some $y^* \in Y^*$ with $||y^*||=1$, $T^*y^*(x_k) \geqslant \varepsilon/2$ for every k. This however contradicts the assumption that $\{B_n\}_{n=1}^\infty$ is shrinking.

Using this observation we construct two sequences of integers $1=m_1<m_2<m_3<\cdots$ and $1=k_1<k_2<k_3\cdots$ as follows. We pick m_2 so that if $P_{m_2}x=0$ then $||Q_{k_1}Tx|| \leqslant \varepsilon_1 ||x||/2$. Next, we let k_2 be such that, for every $x \in P_{m_2}X$, $||Tx-Q_{k_2}Tx|| \leqslant \varepsilon_1||x||/2$. Then, we pick m_3 so that if $P_{m_3}x=0$ then $||Q_{k_2}Tx|| \leqslant \varepsilon_2||x||/2$ and k_3 so that $||Tx-Q_{k_3}Tx|| \leqslant \varepsilon_2||x||/2$ for every $x \in P_{m_3}X$. We continue in an obvious manner. The sequences $\{1, m_2+1, m_3+1, \ldots\}$ and $\{1, k_2+1, k_3+1, \ldots\}$ determine blockings with the desired properties.

Proof of (b). Let again $\{P_n\}_{n=1}^\infty$ and $\{Q_n\}_{n=1}^\infty$ be the projections which correspond to the given decompositions (note that the $\{Q_n\}_{n=1}^\infty$ are defined only on the subspace $Y_0=[C_n]_{n=1}^\infty$ of Y). Let $K=4+4\sup_n ||P_n||$. As above, we shall define the suitable sequences of integers $1=m_1<m_2<\cdots$ and $1=k_1<k_2<\cdots$ inductively. It will be clear how to choose the $\{m_i\}_{i=1}^\infty$ and $\{k_i\}_{i=1}^\infty$ once we show the following. For every $\varepsilon>0$ and every integer n there is an integer k so that if $y \in Y_0$ with $Q_k y=0$ there is an $x \in X$ with $P_n x=0$, $||x|| \leqslant K||y||$ and $||Tx-y|| \leqslant \varepsilon||y||$. Suppose this were false for some $\varepsilon>0$ and integer n. Then, there is an $\varepsilon>0$ and a sequence $\{y_j\}_{j=1}^\infty$ of vectors of norm 1 in Y_0 so that $d(y_j, TU) \geqslant \varepsilon$, $j=1,2,\ldots$ (where $U=\{x \in X;\, P_n x=0,\ ||x|| \leqslant K\}$) and $\lim_j Q_k y_j=0$ for every k (and thus, since $\{Q_n\}_{n=1}^\infty$ is shrinking, $y_j \xrightarrow{w} 0$). Since T is a quotient map there are $\{x_j\}_{j=1}^\infty$ in X such that $||x_j|| \leqslant 2$ and $Tx_j=y_j$, $j=1,2,\ldots$. Put $v_j=P_n x_j$ and $u_j=x_j-v_j$. Clearly, $||u_j|| \leqslant K/2$ for every j. Since $P_n X$ is finite dimensional we may assume (by passing to a subsequence if necessary) that $||v_j-v_1|| \leqslant \varepsilon/2$ for every j. Hence,

$$d(y_1-y_j, TU) \leqslant ||(y_1-y_j)-T(u_1-u_j)|| = ||T(v_1-v_j)|| \leqslant \varepsilon/2\,.$$

Since $y_j \xrightarrow{w} 0$ the point y_1 belongs to the closed convex hull of $\{y_1 - y_j\}_{j=2}^{\infty}$ and consequently, $d(y_1, TU) \leqslant \varepsilon/2$. This contradicts the choice of y_1 and concludes the proof. □

Remark. The proof of 1.g.4 ensures that not only the specified blockings have the desired property but that the same is true for suitable blockings of these blockings. More precisely, assume for simplicity that $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 \ldots$. Then, the blockings chosen in (a) have the following property. For every blocking $\{B_j''\}_{j=1}^{\infty}$ of $\{B_i'\}_{i=1}^{\infty}$ there is a blocking $\{C_j''\}_{j=1}^{\infty}$ of $\{C_i'\}_{i=1}^{\infty}$ so that, for every $x \in B_j''$, there is a $y \in C_{j-1}'' \oplus C_j''$ with $\|Tx - y\| \leqslant \varepsilon_j \|x\|$. Similarly, the blockings chosen in (b) have the following property. For every blocking $\{C_j''\}_{j=1}^{\infty}$ of $\{C_i'\}_{i=1}^{\infty}$ there is a blocking $\{B_j''\}_{j=1}^{\infty}$ of $\{B_i'\}_{i=1}^{\infty}$ so that, for every $y \in C_j''$, there is an $x \in B_j'' \oplus B_{j+1}''$ with $\|Tx - y\| \leqslant \varepsilon_j \|y\|$ and $\|x\| \leqslant K \|y\|$.

We turn now to a discussion of the relation between the existence of F.D.D.'s and that of existence of bases.

Since obviously the existence of an F.D.D. of a Banach space X implies that X has the B.A.P. we note first that, by Enflo's example, there are separable Banach spaces which fail to have an F.D.D. It is not known whether the existence of an F.D.D. implies the existence of a basis (this is a special instance of problem 1.e.12). What is trivially true is the following fact: Let $\{B_n\}_{n=1}^{\infty}$ be an F.D.D. of a Banach space X. Assume that every B_n has a basis $\{x_{i,n}\}_{i=1}^{k_n}$ with basis constant K_n and $\sup_n K_n < \infty$. Then, the sequence

$$x_{1,1}, x_{2,1}, \ldots, x_{k_1,1}, x_{1,2}, \ldots, x_{k_2,2}, x_{1,3}, \ldots$$

forms a basis of X whose basis constant is $\leqslant K \cdot \sup_n K_n$, where K is the decomposition constant of $\{B_n\}_{n=1}^{\infty}$.

For unconditional bases the situation is more involved. Let, for example, X be the space of all compact operators T on l_2 which have a triangular representing matrix with respect to the unit vector basis (i.e. $Te_n = \sum_{m=1}^{n} a_{n,m} e_m$, $n = 1, 2, \ldots$). Let B_n be the subspace of X consisting of those $T \in X$ such that $Te_j = 0$ for $j \neq n$ (i.e. for which $a_{j,m} = 0$ unless $j = n$). It is clear that B_n is isometric to l_2^n, $n = 1, 2, \ldots$. Moreover, it is trivial to check that $\{B_n\}_{n=1}^{\infty}$ forms an unconditional decomposition of X. Nevertheless, it follows from the results of [47] that X does not have an unconditional basis and it is not even complemented in a space with an unconditional basis. This space does however embed in a space with an unconditional basis. This is a special case of the following general result.

Theorem 1.g.5. *Let X be a Banach space admitting an unconditional* F.D.D. $\{B_n\}_{n=1}^{\infty}$. *Then X is isomorphic to a subspace of a space with an unconditional basis.*

Proof. Without loss of generality we may assume that the unconditional constant of $\{B_n\}_{n=1}^{\infty}$ is 1. For each n we choose a set of non-zero elements $\{x_{i,n}^*\}_{i=1}^{k_n}$ in B_n^* such that $\|x_{i,n}^*\| \leqslant 1$ for all i and so that, for every x^* in the unit ball of B_n^*, there is an i such that $\|x^* - x_{i,n}^*\| \leqslant 4^{-n}$.

Define a map T from $X_0=\operatorname{span}\{B_n\}_{n=1}^\infty$ into the space Y_0 of sequences of scalars which are eventually 0 by

$$T\sum_{n=1}^m x_n=(x_{1,1}^*(x_1),\, x_{2,1}^*(x_1),\dots,\, x_{k_1,1}^*(x_1),\, x_{1,2}^*(x_2),\dots,\, x_{k_2,2}^*(x_2),\dots)\,.$$

Let U be the unit ball of X_0 and let V be the convex hull of $\bigcup_\theta M_\theta TU$, where the union is taken over all sequences of signs $\theta=(\theta_1,\theta_2,\dots)$ and M_θ is the operator on Y_0 defined by $M_\theta(a_1,a_2,\dots)=(\theta_1a_1,\theta_2a_2,\dots)$. We introduce a norm in Y_0, by putting $\|y\|=\inf\{\lambda>0;\, y/\lambda\in V\}$, and we let Y be the completion of $(Y_0,\|\cdot\|)$. It is clear that the unit vectors form an unconditional basis of Y and that T extends to an operator of norm 1 from X into Y. We have to show that T is an isomorphism into.

Let $u=\sum_{n=1}^m u_n$ be an element of norm 1 in X_0. By the Hahn–Banach theorem and the fact that the unconditional constant of the decomposition is 1 there is an $x^*=\sum_{n=1}^m x_n^*$ in X^* with $\|x^*\|=1$, $x_n^*\in B_n^*$ and $1=x^*(u)=\sum_{n=1}^m x_n^*(u_n)$ (we identify in an obvious manner B_n^* with a suitable subspace of X^*). For each $1\leqslant n\leqslant m$ let i_n be such that $\|x_n^*-x_{i_n,n}^*\|\leqslant 4^{-n}$. Let φ be the linear functional on Y_0 defined by

$$\varphi(a_1,a_2,a_3,\dots)=\sum_{n=1}^m a_{j_n}$$

where j_n is the index assigned to $x_{i_n,n}^*$ by T (i.e. $j_n=k_1+k_2+\dots+k_{n-1}+i_n$). By the definition of φ we get that $\varphi(Tu)=\sum_{n=1}^m x_{i_n,n}^*(u_n)\geqslant\sum_{n=1}^m x_n^*(u_n)-\sum_{n=1}^m 4^{-n}\geqslant 1/2$. Also, for every $x=\sum_{n=1}^\infty x_n\in X_0$ with $\|x\|\leqslant 1$ and for every θ,

$$\varphi(M_\theta Tx)=\sum_{n=1}^m \theta_n x_{i_n,n}^*(x_n)\leqslant\sum_{n=1}^m\theta_n x_n^*(x_n)+\sum_{n=1}^\infty 4^{-n}=$$
$$x^*\left(\sum_{n=1}^m\theta_n x_n\right)+\sum_{n=1}^\infty 4^{-n}\leqslant 3/2\,.$$

Hence, by the definition of the norm in Y, $\varphi\in Y^*$ and $\|\varphi\|\leqslant 3/2$. Consequently, $\|Tu\|\geqslant 2\varphi(Tu)/3\geqslant 1/3$ and this concludes the proof. □

2. The Spaces c_0 and l_p

a. Projections in c_0 and l_p and Characterizations of these Spaces

The simplest examples of infinite-dimensional Banach spaces are l_p, $1 \leqslant p \leqslant \infty$, and c_0. These spaces appeared in many problems in analysis much before a systematic theory of normed linear spaces was developed, and, as a result of continuous efforts, their geometry is quite well known today.

The unit vectors $\{e_n\}_{n=1}^\infty$ form an unconditional basis (with unconditional constant 1) of c_0 and l_p, $1 \leqslant p < \infty$. Some very simple but important properties of this basis are exhibited in the following proposition.

Proposition 2.a.1. *Let X be either c_0 or l_p, $1 \leqslant p < \infty$, and let $\{u_j\}_{j=1}^\infty$ be a normalized block basis of the unit vector basis $\{e_n\}_{n=1}^\infty$. Then,*

(i) *$\{u_j\}_{j=1}^\infty$ is equivalent to $\{e_n\}_{n=1}^\infty$ and $[u_j]_{j=1}^\infty$ is isometric to X.*
(ii) *There is a projection of norm 1 from X onto $[u_j]_{j=1}^\infty$.*

Proof. We present the proof in the case of l_p. The proof for c_0 is the same but the notation is somewhat different. Let $u_j = \sum_{i=m_j+1}^{m_{j+1}} \lambda_i e_i$ with $\sum_{i=m_j+1}^{m_{j+1}} |\lambda_i|^p = 1$, $j = 1, 2, \ldots$. For every sequence of scalars $\{a_j\}_{j=1}^\infty$ with $\sum_{j=1}^\infty |a_j|^p < \infty$ we have

$$\left\| \sum_{j=1}^\infty a_j u_j \right\| = \left(\sum_{j=1}^\infty |a_j|^p \sum_{i=m_j+1}^{m_{j+1}} |\lambda_i|^p \right)^{1/p} = \left(\sum_{j=1}^\infty |a_j|^p \right)^{1/p}$$

and this proves (i).

To prove (ii) choose, for every j, an element $u_j^* \in \operatorname{span} \{e_i\}_{i=m_j+1}^{m_{j+1}} \subset l_p^*$ so that $\|u_j^*\| = u_j^*(u_j) = 1$. Then $u_j^*(u_k) = 0$ for $k \neq j$ and the operator P defined by $Px = \sum_{j=1}^\infty u_j^*(x) u_j$ is a projection of norm 1 from X onto $[u_j]_{j=1}^\infty$. Indeed, if $x = \sum_{i=1}^\infty a_i e_i \in l_p$ then $|u_j^*(x)|^p \leqslant \sum_{i=m_j+1}^{m_{j+1}} |a_i|^p$ for every j and thus $\|Px\|^p = \sum_{j=1}^\infty |u_j^*(x)|^p \leqslant \|x\|^p$. $\square$

From 2.a.1 we get immediately (by using 1.a.9) the following result.

Proposition 2.a.2. *Let X be either c_0 or l_p, $1 \leqslant p < \infty$. Then every infinite dimensional subspace Y of X contains a subspace Z which is isomorphic to X and complemented in X (and therefore also in Y).*

Another interesting fact which follows immediately from 2.a.1 (and 1.a.12) is that *no space of the family c_0 and l_p, $1 \leqslant p < \infty$, is isomorphic to a subspace of another member of this family.* By using 2.a.2 A. Pelczynski gave in [114] a complete characterization of the complemented subspaces of c_0 and l_p, $1 \leqslant p < \infty$.

Theorem 2.a.3. *Let X be either c_0 or l_p, $1 \leqslant p < \infty$. Then every infinite dimensional complemented subspace of X is isomorphic to X.*

Proof. Let Y be an infinite-dimensional complemented subspace of X; then $X = Y \oplus X_1$ for some Banach space X_1. By 2.a.2, $Y = Z \oplus Y_1$ with $Z \approx X$ and a suitable Banach space Y_1. Then,

$$X \oplus Y \approx X \oplus (Z \oplus Y_1) \approx (X \oplus X) \oplus Y_1 \approx X \oplus Y_1 \approx Y,$$

since $X \oplus X \approx X$. Furthermore, observe that if $X = l_p$, $1 \leqslant p < \infty$ (resp. $X = c_0$) then X is isometric to $(X \oplus X \oplus \cdots)_p$ (resp. $(X \oplus X \oplus \cdots)_0$); in other words, we can write for every such X, $X = (X \oplus X \oplus \cdots)_X$. Consequently,

$$\begin{aligned} X \oplus Y = (X \oplus X \oplus \cdots)_X \oplus Y &\approx ((Y \oplus X_1) \oplus (Y \oplus X_1) \oplus \cdots)_X \oplus Y \\ &\approx (X_1 \oplus X_1 \oplus \cdots)_X \oplus (Y \oplus Y \oplus \cdots)_X \oplus Y \\ &\approx (X_1 \oplus X_1 \oplus \cdots)_X \oplus (Y \oplus Y \oplus \cdots)_X \\ &\approx ((Y \oplus X_1) \oplus (Y \oplus X_1) \oplus \cdots)_X = X. \end{aligned}$$

Hence $X \approx X \oplus Y \approx Y$ and this concludes the proof (the verification that all the computations done here with infinite direct sums are valid, is straightforward). $\square$

The elegant method of proof of 2.a.3 is called Pelczynski's *decomposition method.* This method (as well as several variants of it) is very useful in many contexts. Observe that all that we used in the computations above was that X and Y are each isomorphic to a complemented subspace of the other space and that X is isomorphic to an infinite direct sum of itself with respect to a suitable norm (the fact that $X \oplus X$ is also isomorphic to X is a consequence of this assumption). Let us point out that it is unknown whether the assumption that X and Y are each isomorphic to a complemented subspace of the other space suffices to ensure that X is isomorphic to Y.

The decomposition method has however one drawback: it is hard to give an explicit form of an isomorphism whose existence is proved by the decomposition method. From the practical point of view (though not from the formal theoretical point of view) the decomposition method is basically an existence proof. In simple cases where other methods are available they usually give more information than the decomposition method. We shall illustrate this by considering projections of norm 1 in c_0 or l_p, $1 \leqslant p < \infty$. In this case it is possible to give explicit representations of the projections and their ranges. In particular, if X is c_0 or l_p, $1 \leqslant p < \infty$ and P is a projection of norm 1 on X with infinite dimensional range than PX is isometric to X (this is a special case of a result due essentially to Ando [5] cf. also [79]). We shall prove this here only in the case $1 < p < \infty$, $p \neq 2$. For $p = 2$ the result is of course trivial. The cases l_1 and c_0 are somewhat different; the proofs in these cases are simpler.

Theorem 2.a.4. *Let $X=l_p$, for some $1<p<\infty$, $p\neq 2$ and let P be a projection of norm 1 in X. Then, there exist vectors $\{u_j\}_{j=1}^m$ of norm 1 in X (where $m=\dim PX$ is either an integer or ∞) of the form*

$$(*)\qquad u_j=\sum_{i\in\sigma_j}\lambda_i e_i,\ 1\leqslant j\leqslant m,\quad \text{with } \sigma_j\cap\sigma_k=\varnothing \text{ for } k\neq j$$

so that $Px=\sum_{j=1}^{m}u_j^*(x)u_j$, where $\{u_j^*\}_{j=1}^m\in X^*$ satisfy $\|u_j^*\|=u_j^*(u_j)=1$, $j=1,\dots,m$. In particular, $PX=[u_j]_{j=1}^m$ is isometric to l_p^m.

Observe that the proof of 2.a.1(ii) shows that, conversely, every P defined as above is a projection of norm 1 on X. Note also that the $\{u_j\}_{j=1}^n$ do not necessarily form a block basis of $\{e_i\}_{i=1}^\infty$ according to definition 1.a.10; first, since the sets $\{\sigma_j\}_{j=1}^m$ may be infinite and even if all the sets $\{\sigma_j\}_{j=1}^m$ are finite the $\{u_j\}_{j=1}^m$ are then only a block basis of a suitable permutation of $\{e_i\}_{i=1}^\infty$. What 2.a.4 asserts is that the obvious generalization of 2.a.1(ii) to the setting in which a block basis of $\{e_n\}_{n=1}^\infty$ is replaced by vectors satisfying (*) gives the most general projection of norm 1 on l_p, $1<p<\infty$, $p\neq 2$.

Proof. We introduce first some notations. For $0\neq x\in l_p$ the support $\operatorname{supp} x$ of x is the set of integers i such that $x(i)$ (the i'th coordinate of x) is $\neq 0$. The function $\operatorname{sign} x$ on the integers is defined by $\operatorname{sign} x(i)=1$ if $x(i)>0$, $=-1$ if $x(i)<0$ and $=0$ if $x(i)=0$ (we give the proof in the case where the scalars are real; the same argument with some slight changes works also in the complex case). For any $0\neq x\in l_p$ let $\psi_p(x)$ be the unique element in $l_q=l_p^*$ for which $\psi_p(x)(x)=\|\psi_p(x)\|_q\|x\|_p=\|x\|_p^p$. The element $\psi_p(x)$ is given explicitly by $\psi_p(x)=|x|^{p-1}\operatorname{sign} x$ (i.e. $\psi_p(x)(i)=|x(i)|^{p-1}\operatorname{sign} x(i)$ for $1\leqslant i<\infty$); for $x=0$ put $\psi_p(x)=0$. Observe that $\psi_q(\psi_p(x))=x$ for every $x\in l_p$.

Let now P be a projection of norm 1 in $X=l_p$ with $2<p<\infty$ (by duality it is enough to prove the theorem in this case) and let $Y=PX$. Note first that if $x=Px$ then $P^*\psi_p(x)(x)=\psi_p(x)(Px)=\psi_p(x)(x)$ and hence, by the uniqueness of $\psi_p(x)$, we get that $P^*\psi_p(x)=\psi_p(x)$. By duality we get conversely that $P^*\psi_p(x)=\psi_p(x)\Rightarrow Px=x$. Thus,

$$\ker P=\{x^*;\ P^*x^*=x^*\}^\top=\{\psi_p(y);\ y\in Y\}^\top=\psi_p(Y)^\top .$$

Hence, since Y determines both PX and $\ker P$, it follows that Y determines P uniquely, i.e. on a subspace of l_p there is at most one projection of norm 1. The preceding remarks show also that $\psi_p(Y)=P^*X^*$ is a linear subspace of l_q. We shall exploit this fact in proving the following lemma.

Lemma 2.a.5. *Let Y be a subspace of l_p, $2<p<\infty$, on which there is a projection of norm 1. Let $y, z\in Y$ be two non-zero vectors. Then $r_z(y)\in Y$, where $r_z(y)$ is the* *restriction of* y *to the support of* z, *i.e.* $r_z(y)(i)=y(i)$ *if* $i \in \operatorname{supp} z$ *and* $r_z(y)(i)=0$ *if* $i \notin \operatorname{supp} z$.

Proof. Since $\psi_p(Y)$ is a linear subspace of l_q we get that, for every real t, $\psi_p(z+ty)-\psi_p(z) \in \psi_p(Y)$. A simple computation shows that $\lim_{t\to\infty} (\psi_p(z+ty)-\psi_p(z))/t$ exists (in norm) and is equal to $(p-1)|z|^{p-2}y$ and thus $|z|^{p-2}y \in \psi_p(Y)$. Hence, $u_1=|z|^{1-(q-1)}|y|^{q-1} \operatorname{sign} y=\psi_q(|z|^{p-2}y) \in \psi_q(\psi_p(Y))=Y$.

By repeating the same argument we get that

$$u_2=|z|^{1-(q-1)}|u_1|^{q-1} \operatorname{sign} u_1=|z|^{1-(q-1)^2}|y|^{(q-1)^2} \operatorname{sign} y \in Y\,,$$

and, in general, $u_n=|z|^{1-(q-1)^n}|y|^{(q-1)^n} \operatorname{sign} y \in Y$, $n=1, 2,\ldots$. Since $p>2$ it follows that $q-1<1$ and hence, $u=\lim_n u_n=|z| \operatorname{sign} y \in Y$. Consequently,

$$r_z(y)=|y| \operatorname{sign} (|z| \operatorname{sign} y)=|y| \operatorname{sign} u \in Y\,. \quad \square$$

We return to the proof of 2.a.4. We show first the following. Let i_0 be an integer which belongs to the support of some element in Y. Then, among all $0\neq y \in Y$ such that $i_0 \in \operatorname{supp} y$, there is an element y_0 whose support is minimal. In order to show this it is enough to verify that if $\operatorname{supp} y_1 \supset \operatorname{supp} y_2 \supset \cdots \ni i_0$, with $\{y_j\}_{j=1}^\infty \subset Y$, then there is a $\hat{y} \in Y$ such that $\operatorname{supp} \hat{y}=\bigcap_{j=1}^\infty \operatorname{supp} y_j$. Such an element is $\lim_j r_{y_j}(y_1)$, which belongs to Y by 2.a.5.

The subspace of Y consisting of those elements y for which $\operatorname{supp} y \subset \operatorname{supp} y_0$ is one-dimensional. Otherwise, there would exist $y, z \in Y$ such that $\operatorname{supp} y \subset \operatorname{supp} y_0$, $\operatorname{supp} z \subset \operatorname{supp} y_0$, $z(i_0)=y(i_0)=1$ and $z(i)\neq y(i)$ for some $i \in \operatorname{supp} y_0 \sim \{i_0\}$. Then, $i_0 \in \operatorname{supp} (z(i)y-y(i)z) \subset \operatorname{supp} y_0 \sim \{i\}$ and this contradicts the minimality of $\operatorname{supp} y_0$.

It follows from the preceding observation that there is a set $\{\sigma_j\}_{j=1}^m$ (m finite or infinite) of disjoint subsets of the integers such that every element of Y vanishes outside $\bigcup_{j=1}^m \sigma_j$ and elements $\{u_j\}_{j=1}^m$ of norm 1 in Y such that $\sigma_j=\operatorname{supp} u_j$ and $\{y;\, y \in Y, \operatorname{supp} y \subset \sigma_j\}$ is the one dimensional space spanned by u_j; $j=1, 2,\ldots, m$. From this and 2.a.5 we deduce that $Y=[u_j]_{j=1}^m$. The fact that P has the desired form follows from the fact that P is determined uniquely by Y. $\square$

In terms of the Banach-Mazur distance, 2.a.4 (and its analogue for c_0 and l_1) states that if X is either c_0 or l_p, for $1\leqslant p<\infty$, and P is a projection of norm 1 on X with $\dim PX=\infty$ then $d(X, PX)=1$. From 2.a.3 it is easy to deduce that there is a function $f(\lambda)$ defined on $\lambda\geqslant 1$ (and independent of p) such that if P is a projection on X with $\dim PX=\infty$ then $d(X, PX)\leqslant f(\|P\|)$. Since $f(1)\neq 1$ this does not reduce to 2.a.4 if $\|P\|=1$ and leaves open the question whether the following "perturbed form" of 2.a.4 is valid. Does there exist a function $g(\lambda)$ defined on $\lambda\geqslant 1$ such that $\lim_{\lambda\to 1} g(\lambda)=1$ and such that, for every projection P on X with $\dim PX=\infty$, $d(X, PX)\leqslant g(\|P\|)$?

The property of c_0 and l_p, $1\leqslant p<\infty$ exhibited in 2.a.3 is of enough interest to justify a special terminology.

Definition 2.a.6. An infinite-dimensional Banach space X is said to be *prime* if every infinite-dimensional complemented subspace of X is isomorphic to X.

Besides c_0 and l_p, $1 \leqslant p < \infty$ the only known example of a prime space is l_∞. It is very likely that there are many other examples of prime spaces. In Section 4.c we shall give some examples of Orlicz sequences spaces which are conjectured to be prime. We present now the proof that l_∞ is prime. It is clear that this case requires a proof different from that of 2.a.3 since l_∞ is not separable and thus does not have a Schauder basis.

Theorem 2.a.7 [84]. *The space l_∞ is prime.*

Proof. Let Y be an infinite dimensional complemented subspace of l_∞. The main point in the proof is to show that Y has a subspace Z isomorphic to l_∞. Such a subspace Z is necessarily complemented in Y. Indeed, if $T: Z \to l_\infty$ is an isomorphism define an extension $\hat{T}$ of T, from Y into l_∞, by $\hat{T}y = (y_1^*(y), y_2^*(y), \ldots)$, where y_i^* is a Hahn–Banach extension of the functional $z \to Tz(i)$, $i = 1, 2, \ldots$, (for $u \in l_\infty$, $u(i)$ denotes the i'th coordinate of u). The operator $Q = T^{-1}\hat{T}$ is a projection from Y onto Z. We are now in a position to apply Pelczynski's decomposition method to conclude that $Y \approx l_\infty$.

The proof that Y contains a subspace isomorphic to l_∞ is based on the following two general facts concerning $C(K)$ spaces which will be proved in Vol. III.

(i) *Let T be an operator from a $C(K)$ space into a Banach space X which does not have a subspace isomorphic to c_0. Then, T is weakly compact.*
(ii) *If T: $C(K) \to C(K)$ is weakly compact then T^2 is compact.*

By applying (i) and (ii) to a projection operator P on a $C(K)$ space we see that if the range of P is infinite dimensional then P is not compact, hence not even weakly compact (note that $P = P^2$). Thus, the range of P contains a subspace isomorphic to c_0. Since l_∞ is a $C(K)$ space (K = the Stone–Cech compactification of the integers) it follows that the complemented subspace Y of l_∞ contains a subspace isomorphic to c_0. Thus, there exist vectors $\{y_n\}_{n=1}^\infty$ in Y and a constant M such that

$$(\dagger) \qquad \sup_n |a_n| \leqslant \left\| \sum_{n=1}^{\infty} a_n y_n \right\| \leqslant M \cdot \sup_n |a_n|, \quad \text{if } \lim_n a_n = 0\,.$$

It follows from (†) that, for every integer i, $\sum_{n=1}^{\infty} |y_n(i)| \leqslant M$. Hence, the series $\sum_{n=1}^{\infty} a_n y_n$ is w^* convergent for every bounded sequence $\{a_n\}_{n=1}^\infty$ of scalars $\left(\text{i.e. } \sum_{n=1}^{\infty} a_n y_n(i) \text{ converges for every } i\right)$. The right-hand inequality of (†) remains valid for every choice of a bounded sequence of scalars $\{a_n\}_{n=1}^\infty$. It is not clear however that the same is true for the left-hand inequality of (†). Moreover, since Y is not necessarily w^* closed the expression $\sum_{n=1}^{\infty} a_n y_n$ need not be in Y if $\{a_n\}_{n=1}^\infty$ does not converge to 0. We shall see that by passing to a subsequence we can overcome these two difficulties.

Lemma 2.a.8. *Let $\{y_n\}_{n=1}^\infty$ be a sequence of vectors in l_∞ which satisfies (†). Then there is a subsequence $\{y_{n_k}\}_{k=1}^\infty$ of $\{y_n\}_{n=1}^\infty$ so that*

$$(\ddagger)\qquad \tfrac{1}{2}\sup_k |a_k| \leqslant \left\| \sum_{k=1}^\infty a_k y_{n_k} \right\| \leqslant M \sup_k |a_k|; \quad \text{if } \sup_k |a_k| < \infty .$$

Proof. For $x \in l_\infty$ and $\varepsilon > 0$ we put $N(x, \varepsilon) = \{i; |x(i)| < \varepsilon\}$. We observe first that, for every ε, there is an integer n_0 so that, for infinitely many indices n, the restriction of y_n to the subset $N(y_{n_0}, \varepsilon)$ of the integers N has norm $\geqslant 1$. This follows from the fact that $\sum_{n=1}^\infty |y_n(i)| \leqslant M$ and thus, if $r > M/\varepsilon$, $\bigcup_{n=1}^r N(y_n, \varepsilon) = N$. Hence, we can take as n_0 one of the integers from 1 to r.

Using this observation we can find an integer n_1 and an infinite sequence of integers N_1 so that if $n \in N_1$ the restriction of y_n to $N(y_{n_1}, 1/8)$ has norm $\geqslant 1$. Pick i_1 such that $|y_{n_1}(i_1)| \geqslant 3/4$. By passing to a subsequence of N_1, if necessary, we may assume also that $\sum_{n \in N_1} |y_n(i_1)| < 1/8$.

Next we pick an $n_2 \in N_1$ and an infinite subsequence $N_2 \subset N_1$ so that, for $n \in N_2$, the norm of the restriction of y_n to $N(y_{n_1}, 1/8) \cap N(y_{n_2}, 1/8^2)$ is $\geqslant 1$. By our assumption on N_1 there is an $i_2 \in N(y_{n_1}, 1/8)$ so that $|y_{n_2}(i_2)| > 3/4$. By passing to an infinite subsequence of N_2, if necessary, we may assume also that $\sum_{n \in N_2} |y_n(i_2)| < 1/8^2$.

We continue this inductive construction in an obvious manner and obtain subsequences $\{n_k\}_{k=1}^\infty$ and $\{i_k\}_{k=1}^\infty$ of the integers so that $|y_{n_k}(i_k)| > 3/4$ for every k and

$$\sum_{\substack{j=1 \\ j \neq k}}^\infty |y_{n_j}(i_k)| \leqslant 1/8 + 1/8^2 + \cdots = 1/7 .$$

The subsequence $\{y_{n_k}\}_{k=1}^\infty$ satisfies (‡). Indeed, if $\sup_k |a_k| = 1$ pick a k_0 such that $|a_{k_0}| > 9/10$. Then,

$$\left\| \sum_{k=1}^\infty a_k y_{n_k} \right\| \geqslant \left| \sum_{k=1}^\infty a_k y_{n_k}(i_{k_0}) \right| \geqslant |a_{k_0}|\,|y_{n_{k_0}}(i_{k_0})| - \sum_{k \neq k_0} |y_{n_k}(i_{k_0})|$$
$$\geqslant (3/4)\cdot(9/10) - 1/7 > 1/2 .$$

This proves the left-hand side of (‡). We observed already that the right-hand side of (‡) is automatically true. $\square$

We return to the proof of 2.a.7. For simplicity of notation we assume, as we may, that $\{y_n\}_{n=1}^\infty$ itself satisfies (‡) For every infinite subset N_0 of the integers we let X_{N_0} be the subspace of l_∞ consisting of all vectors of the form $\sum_{n=1}^\infty a_n y_n$, where $\{a_n\}_{n=1}^\infty$ is bounded and $a_n = 0$ if $n \notin N_0$. By (‡) each such X_{N_0} is isomorphic to l_∞. We shall show now that there is an N_0 such that $X_{N_0} \subset Y$ and this will conclude the proof. Let $\{N_\gamma\}_{\gamma \in \Gamma}$ be an uncountable collection of infinite subsets of the integers such that $N_{\gamma_1} \cap N_{\gamma_2}$ is finite for every $\gamma_1 \neq \gamma_2$ (to prove the existence of such a collection assign to each real number t a sequence of rational numbers converging to t. In this way we get an uncountable collection of subsets of a countable set (the rationals) such that the intersection of every two different sets in the collection is finite). If, for each $\gamma \in \Gamma$, the space X_{N_γ} is not contained in Y we can find, for each γ, an $x_\gamma \in X_{N_\gamma}$ with $\|x_\gamma\|=1$ and $Tx_\gamma \neq 0$, where $T\colon l_\infty \to l_\infty/Y$ is the quotient map. By our assumption on the collection $\{N_\gamma\}_{\gamma\in\Gamma}$ and the fact that all the $\{y_n\}_{n=1}^\infty$ belong to Y it follows that, for every choice of scalars $\{b_k\}_{k=1}^m$ and every choice of distinct indices $\{\gamma_k\}_{k=1}^m$ in Γ, $\left\|\sum_{k=1}^m b_k T x_{\gamma_k}\right\| \leqslant 2M \sup_k |b_k|$. Hence, for every $\varphi \in (l_\infty/Y)^*$ and every $\varepsilon>0$, there are only finitely many $\gamma \in \Gamma$ such that $|\varphi(Tx_\gamma)|>\varepsilon$ and thus, there are only countably many $\gamma\in\Gamma$ such that $\varphi(Tx_\gamma)\neq 0$.

Since Y is complemented in l_∞ the space l_∞/Y is isomorphic to a subspace of l_∞. Hence, there is a sequence of functionals $\{\varphi_j\}_{j=1}^\infty$ in $(l_\infty/Y)^*$ which is total (i.e, if $u\in l_\infty/Y$ is such that $\varphi_j(u)=0$, for every j, then $u=0$). Since Γ is uncountable we deduce that there is a $\gamma\in\Gamma$ such that $\varphi_j(Tx_\gamma)=0$ for $j=1,2,\ldots$. This however contradicts the assumption that the $\{\varphi_j\}_{j=1}^\infty$ are total and $Tx_\gamma\neq 0$. □

We already noted above that it seems likely that the fact that c_0 and l_p, $1\leqslant p<\infty$ are prime does not characterize these spaces among the class of all separable spaces and probably not even among all spaces having an unconditional basis. The main ingredients used in the proof of 2.a.3, namely the two parts of the simple Proposition 2.a.1, do however characterize c_0 and l_p, $1\leqslant p<\infty$ up to isomorphism. This (and its analogues in the function space and "local" settings) explains why the structure of $L_p(\mu)$ and $C(K)$ spaces is far simpler than that of general function spaces. We present first a result of M. Zippin [146] which shows that 2.a.1(i) characterizes in a very strong sense the unit vectors of c_0 or l_p.

Theorem 2.a.9. *Let X be a Banach space with a normalized basis $\{x_n\}_{n=1}^\infty$. Assume that $\{x_n\}_{n=1}^\infty$ is equivalent to all its normalized block bases. Then, $\{x_n\}_{n=1}^\infty$ is equivalent to the unit vector basis in c_0 or in some l_p, $1\leqslant p<\infty$.*

Proof. First notice that since $\{x_n\}_{n=1}^\infty$ is equivalent to $\{\theta_n x_n\}_{n=1}^\infty$, for every choice of signs $\{\theta_n\}_{n=1}^\infty$, we get that $\{x_n\}_{n=1}^\infty$ is unconditional. Hence, we can assume without loss of generality that the unconditional constant of $\{x_n\}_{n=1}^\infty$ is 1. Next, using a uniform boundedness argument, the following is proved: there exists a constant M so that, for every normalized block basis $\{u_j\}_{j=1}^\infty$ of $\{x_n\}_{n=1}^\infty$, the operator T which exhibits the equivalence of these basic sequences satisfies $\|T\|, \|T^{-1}\|\leqslant M$ or, equivalently,

$$(*)\qquad M^{-1}\left\|\sum_{j=1}^\infty a_j x_j\right\| \leqslant \left\|\sum_{j=1}^\infty a_j u_j\right\| \leqslant M\left\|\sum_{j=1}^\infty a_j x_j\right\|$$

for all choices of scalars $\{a_j\}_{j=1}^\infty$ such that $\sum_{j=1}^\infty a_j x_j$ converges. Taking in (*) $u_j=x_{m_j}$, where $m_1<m_2<\cdots$, $a_j=1$ for $j=1,\ldots,n$ and $a_j=0$ for $j>n$, we get

$$M^{-1}\left\|\sum_{j=1}^n x_j\right\| \leqslant \left\|\sum_{j=1}^n x_{m_j}\right\| \leqslant M\left\|\sum_{j=1}^n x_j\right\|,\quad n=1,2,\ldots.$$

Let n and k be integers and construct blocks $\{u_j\}_{j=1}^\infty$ as follows:

$$u_1=\sum_{i=1}^{n^k-1} x_i\Big/\Big\|\sum_{i=1}^{n^k-1} x_i\Big\|,\qquad u_2=\sum_{i=n^k-1+1}^{2n^k-1} x_i\Big/\Big\|\sum_{i=n^k-1+1}^{2n^k-1} x_i\Big\|,\ldots.$$

By applying (*) to these $\{u_j\}_{j=1}^\infty$ and suitably chosen $\{a_j\}_{j=1}^\infty$ we get that

$$M^{-2}\lambda(n)\lambda(n^{k-1})\leqslant\lambda(n^k)\leqslant M^2\lambda(n)\lambda(n^{k-1}),\quad n,k=1,2,\ldots,$$

where $\lambda(n)=\Big\|\sum_{i=1}^n x_i\Big\|$. It follows easily by induction that

$$M^{-2k}\lambda(n)^k\leqslant\lambda(n^k)\leqslant M^{2k}\lambda(n)^k\,.$$

Let m, n and k be integers and denote by $[x]$ the integer part of a positive real number x. By using the preceding inequality we get that

$$\begin{aligned}M^{-2[k\log m]}\lambda(n)^{[k\log m]}&\leqslant\lambda(n^{[k\log m]})\leqslant\lambda(m^{[k\log n]+1})\\&\leqslant M^{2[k\log n]+2}\lambda(m)^{[k\log n]+1}\,.\end{aligned}$$

It follows from this, after some easy computations and letting $k\to\infty$, that

$$\left|\frac{\log\lambda(n)}{\log n}-\frac{\log\lambda(m)}{\log m}\right|\leqslant 2\log M\left(\frac{1}{\log n}+\frac{1}{\log m}\right)\cdot$$

Hence, $c=\lim_n\log\lambda(n)/\log n$ exists. Passing to the limit, as $m\to\infty$, in the preceding inequality we get

(**) $$M^{-2}n^c\leqslant\lambda(n)\leqslant M^2n^c,\quad n=1,2,\ldots.$$

Since $1\leqslant\lambda(n)\leqslant n$ it follows that $0\leqslant c\leqslant 1$. If $c=0$ we get that $\lambda(n)\leqslant M^2$ and thus

$$\sup_n|a_n|\leqslant\Big\|\sum_{n=1}^\infty a_nx_n\Big\|\leqslant M^2\sup_n|a_n|$$

for every sequence of scalars $\{a_n\}_{n=1}^\infty$ which are eventually 0, i.e. $\{x_n\}_{n=1}^\infty$ is equivalent to the unit vector basis of c_0.

If $c>0$ we put $p=1/c$. To prove the equivalence of $\{x_n\}_{n=1}^\infty$ with the unit vector basis of l_p we let $\{r_j\}_{j=1}^m$ be positive rational numbers with $r_j=k_j/k$, where $\{k_j\}_{j=1}^m$ and k are positive integers. It follows from (**) that

$$\Big\|\sum_{j=1}^m r_j^{1/p}x_j\Big\|=k^{-1/p}\Big\|\sum_{j=1}^m k_j^{1/p}x_j\Big\|\geqslant M^{-2}k^{-1/p}\Big\|\sum_{j=1}^m\lambda(k_j)x_j\Big\|\,.$$

Using again (*) with $u_j = \sum_{i=s_j+1}^{s_{j+1}} x_i \Big/ \Big\| \sum_{i=s_j+1}^{s_{j+1}} x_i \Big\|$, $a_j = \Big\| \sum_{i=s_j+1}^{s_{j+1}} x_i \Big\|$, $j=1,\ldots,m$, where $s_j = k_1 + k_2 + \cdots + k_{j-1}$, we get that

$$\Big\| \sum_{j=1}^{m} \lambda(k_j) x_j \Big\| \geqslant M^{-1} \Big\| \sum_{j=1}^{m} a_j x_j \Big\| \geqslant M^{-2} \Big\| \sum_{j=1}^{m} a_j u_j \Big\| = M^{-2} \lambda\Big(\sum_{j=1}^{m} k_j\Big)$$

$$\geqslant M^{-4} \Big(\sum_{j=1}^{m} k_j\Big)^{1/p} = M^{-4} k^{1/p} \Big(\sum_{j=1}^{m} r_j\Big)^{1/p},$$

and thus, $\Big\| \sum_{j=1}^{m} r_j^{1/p} x_j \Big\| \geqslant M^{-6} \Big(\sum_{j=1}^{m} r_j\Big)^{1/p}$. A similar argument shows that

$$\Big\| \sum_{j=1}^{m} r_j^{1/p} x_j \Big\| \leqslant M^{6} \Big(\sum_{j=1}^{m} r_j\Big)^{1/p}$$

and this completes the proof. □

Remarks. 1. The following terminology is used in several places. A normalized basis $\{x_n\}_{n=1}^\infty$ is said to be *perfectly homogeneous* if it is equivalent to any of its normalized block bases. Since 2.a.9 states that the perfectly homogeneous bases are exactly those which are equivalent to the unit vector basis of c_0 or some l_p, $1 \leqslant p < \infty$ this terminology is no longer very useful; it is just an abbreviation of "being equivalent to the unit vector basis of c_0 or some l_p, $1 \leqslant p < \infty$".

2. The proof of 2.a.9 uses only the fact that $\{x_n\}_{n=1}^\infty$ is equivalent to each of its normalized block bases with constant coefficients, i.e. blocks having the form $u_j = \sum_{n=m_j+1}^{m_{j+1}} x_n \Big/ \Big\| \sum_{n=m_j+1}^{m_{j+1}} x_n \Big\|$, for some increasing sequence of integers $\{m_j\}_{j=1}^\infty$. The significance of this remark will become clear in the next chapter.

We show now that a modified version of 2.a.1(ii) also characterizes the unit vector basis of c_0 or l_p.

Theorem 2.a.10 [92]. *Let $\{x_n\}_{n=1}^\infty$ be a normalized unconditional basis of a Banach space X. Assume that, for every permutation π of the integers and for every block basis $\{u_j\}_{j=1}^\infty$ of $\{x_{\pi(n)}\}_{n=1}^\infty$, the subspace $[u_j]_{j=1}^\infty$ is complemented. Then $\{x_n\}_{n=1}^\infty$ is equivalent to the unit vector basis of c_0 or l_p for some $1 \leqslant p < \infty$.*

The proof of 2.a.10 is based on a lemma which will be useful also in other contexts.

Lemma 2.a.11. *Let X be a Banach space with an unconditional basis $\{x_n\}_{n=1}^\infty$. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of scalars tending to 0 and let $v_j = \sum_{n\in\delta_j} a_n x_n$, $w_j = \sum_{n\in\sigma_j} a_n x_n$, $j=1,2,\ldots$ be normalized block bases of a permutation of $\{x_n\}_{n=1}^\infty$ such that $\delta_i \cap \sigma_j = \varnothing$ for every i and j. Assume that $[v_j + \lambda_j w_j]_{j=1}^\infty$ is complemented in X. Then, for every choice of scalars $\{\eta_j\}_{j=1}^\infty$ such that $\sum_{j=1}^\infty \eta_j v_j$ converges, the series $\sum_{j=1}^\infty \lambda_j \eta_j w_j$ converges too.*

Proof. Let $u_j=v_j+\lambda_j w_j$, $j=1,2,\ldots$ and let Q be a projection from X onto $[u_j]_{j=1}^{\infty}$. Put

$$Qv_i=\sum_{j=1}^{\infty} b_{i,j}u_j, \qquad Qw_i=\sum_{j=1}^{\infty} c_{i,j}u_j, \quad i=1,2,\ldots.$$

Then, for every i, $b_{i,i}+\lambda_i c_{i,i}=1$ and, since $\sup_i |c_{i,i}|<\infty$, we get that $\lim_i b_{i,i}=1$. Let P_σ be the projection associated to the unconditional basis $\{x_n\}_{n=1}^{\infty}$ and the set $\sigma=\bigcup_{j=1}^{\infty}\sigma_j$ (i.e. $P_\sigma x_n=x_n$ if $n\in\sigma$ and $P_\sigma x_n=0$ if $n\notin\sigma$). Then, $P_\sigma Qv_i=\sum_{j=1}^{\infty} b_{i,j}\lambda_j w_j$, $i=1,2,\ldots$. We regard $P_\sigma Q$ as an operator from $V=[v_i]_{i=1}^{\infty}$ into $W=[w_j]_{j=1}^{\infty}$. The matrix corresponding to $P_\sigma Q$ with respect to the given bases is $(b_{i,j}\lambda_j)$. By 1.c.8 the operator $D\colon V\to W$, which corresponds to the diagonal of this matrix (i.e. defined by $Dv_j=\lambda_j b_{j,j}w_j$, $j=1,2,\ldots$), is also bounded. Thus the convergence of $\sum_{j=1}^{\infty}\eta_j v_j$ implies that of $\sum_{j=1}^{\infty}\eta_j Dv_j$ and therefore also that of $\sum_{j=1}^{\infty}\lambda_j\eta_j w_j$ (recall that $\lim_j b_{j,j}=1$). □

Proof of 2.a,10. Let $\{v_j\}_{j=1}^{\infty}$ and $\{w_j\}_{j=1}^{\infty}$ be normalized block bases of $\{x_{2n}\}_{n=1}^{\infty}$, respectively $\{x_{2n+1}\}_{n=1}^{\infty}$. We shall show that $\{v_j\}_{j=1}^{\infty}$ is equivalent to $\{w_j\}_{j=1}^{\infty}$. This will imply that $\{x_{2n}\}_{n=1}^{\infty}$ and $\{x_{2n+1}\}_{n=1}^{\infty}$ are both perfectly homogeneous and equivalent to each other. An application of 2.a.9 will thus conclude the proof.

Let $\{\eta_j\}_{j=1}^{\infty}$ be a sequence of scalars such that $\sum_{j=1}^{\infty}\eta_j v_j$ converges. By the assumption in 2.a.10 and Lemma 2.a.11 we get that, for every sequence $\{\lambda_j\}_{j=1}^{\infty}$ tending to 0, the series $\sum_{j=1}^{\infty}\lambda_j\eta_j w_j$ converges. If $\sum_{j=1}^{\infty}\eta_j w_j$ fails to converge we can find disjoint finite sets of integers $\{\sigma_i\}_{i=1}^{\infty}$ and an $\varepsilon>0$ such that $\left\|\sum_{j\in\sigma_i}\eta_j w_j\right\|\geqslant\varepsilon$ for every i. Put $u_i=\sum_{j\in\sigma_i}\eta_j w_j$, $i=1,2,\ldots$. Since $\sum_{i=1}^{\infty}\lambda_i u_i$ converges whenever $\lambda_i\to 0$ we get that $\{u_i\}_{i=1}^{\infty}$ is equivalent to the unit vector basis of c_0. By using again 2.a.11 it follows that in this case $\{v_j\}_{j=1}^{\infty}$ is also equivalent to the unit vector basis of c_0 and (apply 2.a.11 once more) so is $\{w_j\}_{j=1}^{\infty}$. Thus, $\sum_{j=1}^{\infty}\eta_j w_j$ must converge. □

The use of permutations in the statement of 2.a.10 is necessary. This follows from the next proposition.

Proposition 2.a.12 [15]. *Let X be a Banach space with an unconditional basis $\{x_n\}_{n=1}^{\infty}$. Let $\{m_j\}_{j=1}^{\infty}$ be an increasing sequence of integers with $m_1=0$ and put $B_j=\operatorname{span}\{x_n\}_{n=m_j+1}^{m_{j+1}}$, $j=1,2,\ldots$. If every block basis of $\{x_n\}_{n=1}^{\infty}$ spans a complemented subspace of X then the same is true for the natural basis of $Y=\left(\sum_{j=1}^{\infty}\oplus B_j\right)_p$, where $p=0$ or $1\leqslant p<\infty$.*

Proof. We may assume without loss of generality that the unconditional constant of $\{x_n\}_{n=1}^{\infty}$ is 1. By the "natural basis" of Y we mean the vectors $\{y_n\}_{n=1}^{\infty}$ defined as follows: if $m_j<n\leqslant m_{j+1}$ we let y_n be the element in the direct sum whose only non-

zero component is x_n in the j'th place. It is obvious that $\{y_n\}_{n=1}^\infty$ is an unconditional basis of Y whose unconditional constant is 1. For the proof we need the following trivial observation. If $\{P_j\}_{j=1}^\infty$ are projections in $\{B_j\}_{j=1}^\infty$ such that $\sup_j \|P_j\|<\infty$ then $Q(z_1, z_2,\dots)=(P_1z_1, P_2z_2,\dots)$ is a projection on Y with $\|Q\|=\sup_j \|P_j\|$ (here $z_j \in B_j$ for every j). Q is called the projection determined by $\{P_j\}_{j=1}^\infty$.

Let now $u_k=\sum_{n=q_k+1}^{q_{k+1}} a_n y_n$, $k=1, 2,\dots$ be a normalized block basis of $\{y_n\}_{n=1}^\infty$. Let N_1 be the set of those integers k for which there exists a j such that $u_k \in B_j$ (i.e. $m_j \leqslant q_k < q_{k+1} \leqslant m_{j+1}$). By our assumption on $\{x_n\}_{n=1}^\infty$ there is a bounded linear projection P from X onto $\left[\sum_{i=q_k+1}^{q_{k+1}} a_n x_n\right]_{k\in N_1}$. By 1.c.8 (and the remark thereafter) there is no loss of generality to assume that if $q_k<n\leqslant q_{k+1}$ then $Px_n=\lambda_n \sum_{i=q_k+1}^{q_{k+1}} a_i x_i$, where $\lambda_n=0$ whenever $k\notin N_1$. Hence, $PB_j\subset B_j$ for every j. Let Q_1 be the projection on Y determined by $\{P_{|B_j}\}_{j=1}^\infty$. Then $Q_1 Y=[u_k]_{k\in N_1}$ and $Q_1u_k=0$ if $k\notin N_1$.

We divide now $N\sim N_1$ into two sets $N_2 \cup N_3$ by taking every second element in $N\sim N_1$ (i.e. the first, third,...) into N_2 and the rest (i.e. the second, fourth,...) into N_3. By this choice of N_2 we ensure that if $k<h$ are two integers in N_2 then there is a j such that $q_{k+1}\leqslant m_j<q_h$ and a similar statement holds for N_3. Thus, the sequence $\{u_k\}_{k\in N_2}$ is equivalent to the unit vector basis of l_p, resp. c_0 (if $p=0$). (If $N\sim N_1$ is finite then N_2 is finite and we get only the unit vector basis of l_p^r for some finite r.) Also, there exist $\{u_k^*\}_{k\in N_2}$ such that $\|u_k^*\|=u_k^*(u_k)=1$ for every $k\in N_2$, $u_k^*(u_h)=0$ for every $h\neq k$, and $\{u_k^*\}_{k\in N_2}$ is equivalent to the unit vector basis in l_p^*, resp. c_0^*. The operator Q_2 on Y defined by $Q_2u=\sum_{k\in N_2} u_k^*(y)u_k$ is a projection of norm 1 onto $[u_k]_{k\in N_2}$ so that $Q_2u_k=0$ if $k\notin N_2$. In a similar manner we define Q_3. Then $Q=Q_1+Q_2+Q_3$ is a bounded linear projection from Y onto $[u_k]_{k=1}$. □

It follows from 2.a.12 that, e.g. the natural basis of $X=\left(\sum_{n=1}^\infty \oplus l_s^n\right)_p$ has, for every choice of s and p, the property that all its block bases span a complemented subspace of X. For $p\neq s$ the natural basis of this space is not perfectly homogeneous.

b. Absolutely Summing Operators and Uniqueness of Unconditional Bases

In this section we shall present some of the basic facts concerning the class of p-absolutely summing operators. These operators are, by definition, closely connected to the spaces $L_p(\mu)$. They are of importance in the geometric theory of general Banach spaces and, in particular, in the study of the structure of the classical spaces. In this section we apply them in proving that c_0 and l_1 have up to equivalence, only one normalized unconditional basis (namely the unit vector basis). Further applications of p-absolutely summing operators to the study of classical spaces will be presented in Vol. II.

Definition 2.b.1. Let X and Y be Banach spaces and let $p\geqslant 1$. An operator

$T \in L(X, Y)$ is called *p-absolutely summing* if there is a constant K so that, for every choice of an integer n and vectors $\{x_i\}_{i=1}^n$ in X, we have

$$\left(\sum_{i=1}^n \|Tx_i\|^p\right)^{1/p} \leqslant K \sup_{\|x^*\| \leqslant 1} \left(\sum_{i=1}^n |x^*(x_i)|^p\right)^{1/p}$$

The smallest possible constant K is denoted by $\pi_p(T)$. The class of all p-absolutely summing operators in $L(X, Y)$ is denoted by $\Pi_p(X, Y)$.

For $T \notin \Pi_p(X, Y)$ we shall put $\pi_p(T) = \infty$. The 1-absolutely summing operators will be simply called absolutely summing operators. A straightforward verification shows that, for every p, $\Pi_p(X, Y)$ is a linear subspace of $L(X, Y)$ and $\pi_p(T)$ defines a norm on $\Pi_p(X, Y)$ in which this space is even complete (i.e. a Banach space). It is also trivial to verify that if S and T are bounded linear operators whose composition is defined then $\pi_p(ST) \leqslant \pi_p(S) \cdot \|T\|$ and $\pi_p(ST) \leqslant \|S\| \pi_p(T)$. The name "absolutely summing" will become clear if we consider the definition for $p=1$. Observe that $\sup_{\|x^*\|=1} \sum_{i=1}^n |x^*(x_i)| = \sup\left\{\left\|\sum_{i=1}^n \theta_i x_i\right\|,\ \theta_i = \pm 1,\ 1 \leqslant i \leqslant n\right\}$. This implies that *an operator $T \in L(X, Y)$ is absolutely summing if and only if, for every sequence $\{x_n\}_{n=1}^\infty$ in X such that $\sum_{n=1}^\infty x_n$ converges unconditionally, the series $\sum_{n=1}^\infty Tx_n$ converges absolutely* $\left(\text{i.e. } \sum_{n=1}^\infty \|Tx_n\| < \infty\right)$.

The following factorization theorem, due to A. Pietsch [121], clarifies the notion of p-absolutely summing operators for a general p and is a basic tool in several applications.

Theorem 2.b.2. *An operator $T \in L(X, Y)$ is p-absolutely summing for some $1 \leqslant p < \infty$ if and only if there is a regular probability measure μ on the unit ball B_{X^*} of X^* (in its w^* topology) and a constant K so that $\|Tx\| \leqslant K\left(\int_{B_{X^*}} |x^*(x)|^p \, d\mu(x^*)\right)^{1/p}$. Moreover, the smallest possible constant K for which such a measure μ exists is equal to $\pi_p(T)$.*

This theorem can be interpreted as follows. Let $I: X \to C(B_{X^*})$ be the canonical isometry defined by $Ix(x^*) = x^*(x)$. For a probability measure μ on B_{X^*} let J_μ be the formal identity map from $C(B_{X^*})$ into $L_p(B_{X^*}, \mu)$. Then, $T: X \to Y$ is p-absolutely summing if and only if there is a measure μ and a bounded linear operator S from $\overline{J_\mu IX} = Z$ into Y such that the following diagram commutes

$$\begin{tikzcd} C(B_{X^*}) \arrow[r, "J_\mu"] & L_p(B_{X^*}, \mu) \supset Z \arrow[d, "S"] \\ X \arrow[u, "I"] \arrow[ur, "J_\mu I"] \arrow[r, "T"'] & Y \end{tikzcd}$$

In general, S cannot be extended to an operator from $L_p(B_{X^*}, \mu)$ into Y (of course, if $p=2$ this is always possible since Z is then complemented in the Hilbert space $L_2(B_{X^*}, \mu)$).

Proof of 2.b.2. Assume first that such μ and K exist. Let $\{x_i\}_{i=1}^n$ be elements of X; then

$$\sum_{i=1}^n ||Tx_i||^p \leqslant K^p \int_{B_{X^*}} \sum_{i=1}^n |x^*(x_i)|^p \, d\mu(x^*) \leqslant K^p \sup_{||x^*|| \leqslant 1} \sum_{i=1}^n |x^*(x_i)|^p$$

and thus $\pi_p(T) \leqslant K < \infty$. Assume conversely, that $T \in \Pi_p(X, Y)$ and $\pi_p(T)=1$. Consider the following subsets of $C(B_{X^*})$

$$F_1 = \left\{f \in C(B_{X^*});\ \sup_{x^* \in B_{X^*}} f(x^*) < 1\right\},$$

$$F_2 = \operatorname{conv}\{f; f(x^*) = |x^*(x)|^p,\ ||Tx|| = 1\}.$$

The sets F_1 and F_2 are convex, F_1 is open and the assumption that $\pi_p(T)=1$ implies that $F_1 \cap F_2 = \varnothing$. By the Hahn–Banach and the Riesz representation theorems there exists a positive constant λ and a regular measure μ on B_{X^*} such that $f \in F_1 \Rightarrow \int_{B_{X^*}} f(x^*) \, d\mu(x^*) \leqslant \lambda$ and $f \in F_2 \Rightarrow \int_{B_{X^*}} f(x^*) \, d\mu(x^*) \geqslant \lambda$. Since F_1 contains all the negative functions the measure μ must be a positive measure and thus we may assume without loss of generality that it is a probability measure. Since F_1 contains the open unit ball of $C(B_{X^*})$ we get that $\lambda \geqslant 1$. Hence if $x \in X$ with $||Tx|| = 1$ then $\int_{B_{X^*}} |x^*(x)|^p \, d\mu(x^*) \geqslant 1$, i.e. for every $x \in X$, $||Tx||^p \leqslant \int_{B_{X^*}} |x^*(x)|^p \, d\mu(x^*)$. □

Several interesting facts can be read off directly from 2.b.2.

1. Since, for a probability measure μ, the norm of a function in $L_p(\mu)$ is always smaller than its norm in $L_r(\mu)$, if $p < r$, we get that for all Banach spaces X and Y, *$\Pi_p(X, Y) \subset \Pi_r(X, Y)$ whenever $1 \leqslant p < r < \infty$ and, moreover, $\pi_p(T) \geqslant \pi_r(T)$ for every $T \in L(X, Y)$.*

2. *Every p-absolutely summing operator is weakly compact.* For $p > 1$ this is evident from the factorization diagram (since Z is reflexive). For $p = 1$ use the preceding observation.

3. Theorem 2.b.2 provides a proof of a weak version of the Dvoretzky–Rogers theorem 1.c.2. In every infinite dimensional Banach space X there is a sequence $\{x_i\}_{i=1}^\infty$ such that $\sum_{i=1}^\infty x_i$ converges unconditionally but not absolutely. Indeed, assume that X is an infinite dimensional Banach space in which unconditional convergence implies absolute converges. Then the identity operator of X is 1-absolutely summing and hence also 2-absolutely summing. By 2.b.2 there is a Hilbert space H and operators $S_1: X \to H$, $S_2: H \to X$ such that $S_2 S_1$ is the identity of X. Thus, X is isomorphic to a subspace of H and is therefore isomorphic to a Hilbert space. It is however, obvious that in an infinite-dimensional Hilbert space there are unconditionally converging series which are not absolutely converging.

Before proceeding with the study of absolutely summing operators we prove a classical inequality of Khintchine which has very many applications in the study of the spaces l_p and $L_p(\mu)$, in general.

Theorem 2.b.3. *Let $r_n(t)=\operatorname{sign}\sin 2^n\pi t$, $n=0, 1, 2,\ldots$ be the Rademacher functions on* $[0, 1]$. *For every $1\leqslant p<\infty$ there exist positive constants A_p and B_p so that*

$$A_p\Big(\sum_{n=1}^{m}|a_n|^2\Big)^{1/2}\leqslant\Big(\int_0^1\Big|\sum_{n=1}^{m}a_nr_n(t)\Big|^p\,dt\Big)^{1/p}\leqslant B_p\Big(\sum_{n=1}^{m}|a_n|^2\Big)^{1/2}$$

for every choice of scalars $\{a_n\}_{n=1}^m$.

Proof. It is trivial that we can take $A_2=B_2=1$ and also $A_p=1$ if $p\geqslant 2$ and $B_p=1$ if $1\leqslant p<2$. It is clearly enough to show that we can find a suitable B_p if p is an even integer and a suitable A_p if $p=1$. By considering the real and purely imaginary parts of the $\{a_n\}_{n=1}^m$ separately we see immediately that it is enough to consider only real coefficients.

Observe that $\int_0^1 r_{n_1}^{k_1}(t)r_{n_2}^{k_2}(t)\ldots r_{n_s}^{k_s}(t)\,dt$, where $n_1<n_2<\cdots<n_s$, is 0 unless all the $\{k_i\}_{i=1}^s$ are even in which case the integral is equal to 1. A direct calculation shows therefore that $\int_0^1\Big(\sum_{n=1}^m a_nr_n(t)\Big)^{2k}\,dt$ is equal to $\Sigma\gamma(2k_1, 2k_2,\ldots, 2k_s)a_{n_1}^{2k_1}\,a_{n_2}^{2k_2}\ldots a_{n_s}^{2k_s}$, where the sum is taken over all choices of $n_1, n_2,\ldots, n_s$ between 1 and m and all choices of positive integers $\{k_i\}_{i=1}^s$ such that $k=\sum_{i=1}^s k_i$. The explicit form of γ is given by $\gamma(2k_1, 2k_2,\ldots, 2k_s)=(2k_1+\cdots+2k_s)!/(2k_1)!(2k_2)!\ldots(2k_s)!$. By expanding $\Big(\sum_{n=1}^m a_n^2\Big)^k$ we get a very similar expression the only change being that $\gamma(2k_1, 2k_2,\ldots, 2k_s)$ is replaced by $\gamma(k_1, k_2,\ldots, k_s)$. Thus, the desired inequality for $p=2k$ holds if we take $B_{2k}^{2k}=\sup\gamma(2k_1, 2k_2,\ldots, 2k_s)/\gamma(k_1, k_2,\ldots, k_s)$. A short computation shows that this gives $B_{2k}\leqslant k^{1/2}$. It remains to verify the existence of A_1. Put $f(t)=\sum_{n=1}^m a_nr_n(t)$. Then, by Holder's inequality,

$$\int_0^1|f(t)|^2\,dt=\int_0^1|f(t)|^{2/3}|f(t)|^{4/3}\,dt\leqslant\Big(\int_0^1|f(t)|\,dt\Big)^{2/3}\Big(\int_0^1|f(t)|^4\,dt\Big)^{1/3}$$

$$\leqslant\Big(\int_0^1|f(t)|\,dt\Big)^{2/3}\cdot B_4^{4/3}\Big(\int_0^1|f(t)|^2\,dt\Big)^{2/3}.$$

This gives the desired result (with $A_1=B_4^{-2}$). □

We shall apply now Khintchine's inequality for identifying the class of p-absolutely summing operators in $L(X, X)$ for the simplest possible case, i.e. when $X=l_2$. It turns out that in this special case the space $\Pi_p(X, X)$ does not depend on p.

Theorem 2.b.4 [116]. *For every $1 \leqslant p < \infty$ the space $\Pi_p(l_2, l_2)$ consists exactly of the Hilbert–Schmidt operators.*

We recall that an operator T in $L(l_2, l_2)$ is called a Hilbert–Schmidt operator if $\sum_{n=1}^{\infty} ||Te_n||^2 < \infty$, where $\{e_n\}_{n=1}^{\infty}$ is the unit vector basis in l_2. Also, if $\{y_n\}_{n=1}^{\infty}$ is any orthonormal basis in l_2 then

$$\sum_{n=1}^{\infty} ||Ty_n||^2 = \sum_{n,m=1}^{\infty} |(Ty_n, e_m)|^2 = \sum_{m=1}^{\infty} ||T^* e_m||^2 ,$$

hence, $\sum_{n=1}^{\infty} ||Ty_n||^2 = \sum_{n=1}^{\infty} ||T^* y_n||^2$ and this expression is independent of the particular choice of $\{y_n\}_{n=1}^{\infty}$ and is denoted by $||T||_{\mathrm{HS}}^2$.

Every Hilbert–Schmidt operator T is compact. Hence, as for any compact T in $L(l_2, l_2)$, there is an orthonormal basis $\{x_n\}_{n=1}^{\infty}$ of l_2, an orthonormal system $\{y_n\}_{n=1}^{\infty}$ in l_2 and scalars $\{\lambda_n\}_{n=1}^{\infty}$ such that $Tx_n = \lambda_n y_n$ for all n. An operator T given in this form is Hilbert–Schmidt if and only if $\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty$.

Proof of 2.b.4. We show first that every Hilbert–Schmidt operator T is 1-absolutely summing. By observation 1 following 2.b.2 this will prove that T is p-absolutely summing for every $p > 1$.

Let T be given by $Tx_n = \lambda_n y_n$, $n = 1, 2, \ldots$ and let $\lambda = \left(\sum_{n=1}^{\infty} |\lambda_n|^2 \right)^{1/2} < \infty$. Let $\{u_j\}_{j=1}^{m}$ be any m-tuple of vectors in l_2 and, for $0 \leqslant t \leqslant 1$, let $v(t) = \lambda^{-1} \sum_{n=1}^{\infty} r_n(t) \lambda_n x_n \in l_2$, where $\{r_n\}_{n=1}^{\infty}$ are the Rademacher functions. Then by 2.b.3,

$$\begin{aligned} \sup_{||x|| \leqslant 1} \sum_{j=1}^{m} |(u_j, x)| &\geqslant \sup_{0 \leqslant t \leqslant 1} \sum_{j=1}^{m} |(u_j, v(t))| \\ &\geqslant \lambda^{-1} \sum_{j=1}^{m} \int_0^1 \left| \sum_{n=1}^{\infty} r_n(t) \bar{\lambda}_n (u_j, x_n) \right| dt \\ &\geqslant A_1 \lambda^{-1} \sum_{j=1}^{m} \left(\sum_{n=1}^{\infty} |\lambda_n (u_j, x_n)|^2 \right)^{1/2} = A_1 \lambda^{-1} \sum_{j=1}^{m} ||Tu_j|| \end{aligned}$$

and thus $\pi_1(T) \leqslant A_1^{-1} \cdot \lambda$.

Assume, conversely, that T is p-absolutely summing for some $p \geqslant 2$. It follows that, for every orthonormal basis $\{u_n\}_{n=1}^{\infty}$ of l_2,

$$\left(\sum_{n=1}^{n} ||Tu_n||^p \right)^{1/p} \leqslant K \sup_{||x||=1} \left(\sum_{n=1}^{\infty} |(u_n, x)|^p \right)^{1/p} \leqslant K$$

and hence $\lim_n ||Tu_n|| = 0$. This implies easily that T is compact and thus there are orthonormal bases $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in l_2 and scalars $\{\lambda_n\}_{n=1}^{\infty}$ such that $Tx_n = \lambda_n y_n$.

Let m be an integer and put $w(t)=\sum_{n=1}^{m} r_n(t)x_n$, $0\leqslant t\leqslant 1$, where $\{r_n\}_{n=1}^m$ are the Rademacher functions. Then, by 2.b.3,

$$\begin{aligned}\Big(\sum_{n=1}^{m}|\lambda_n|^2\Big)^{1/2}&=\Big(\int_0^1\|Tw(t)\|^p\,dt\Big)^{1/p}\\&\leqslant \pi_p(T)\sup_{\|x\|=1}\Big(\int_0^1|(w(t),x)|^p\,dt\Big)^{1/p}\\&=\pi_p(T)\sup_{\|x\|=1}\Big(\int_0^1\Big|\sum_{n=1}^{m}r_n(t)(x_n,x)\Big|^p\,dt\Big)^{1/p}\\&\leqslant \pi_p(T)B_p\sup_{\|x\|=1}\Big(\sum_{n=1}^{m}|(x,x_n)|^2\Big)^{1/2}\leqslant \pi_p(T)\cdot B_p\,.\end{aligned}$$

Hence, $\sum_{n=1}^{\infty}|\lambda_n|^2<\infty$ and T is a Hilbert–Schmidt operator. □

Many interesting results concerning p-absolutely summing operators as well as several applications of these operators are based on the following inequality due to Grothendieck [49].

Theorem 2.b.5. *Let $(\alpha_{i,j})_{i,j=1}^n$ be a matrix of scalars such that $\left|\sum_{i,j=1}^{n}\alpha_{i,j}t_is_j\leqslant 1\right|$ for every choice of scalars $\{t_i\}_{i=1}^n$ and $\{s_j\}_{j=1}^n$ satisfying $|t_i|\leqslant 1$, $|s_j|\leqslant 1$. Then, for any choice of vectors $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^n$ in a Hilbert space,*

$$\Big|\sum_{i,j=1}^{n}\alpha_{i,j}(x_i,y_j)\Big|\leqslant K_G\max_i\|x_i\|\max_j\|y_j\|\,,$$

where K_G is Grothendieck's universal constant (in case the scalars are real $K_G\leqslant(e^{\pi/2}-e^{-\pi/2})/2$).

Proof. It is clearly enough to prove the theorem for real scalars. Also, it is easily seen that it is enough to consider vectors $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^n$ so that $\|x_i\|=\|y_i\|=1$ for every i. Since every finite dimensional subspace of a Hilbert space is isometric to l_2^k for some k we may assume that $\{x_i\}_{i=1}^n$, $\{y_j\}_{j=1}^n\subset l_2^k$ for some k. Let $S=\{u;\,u\in l_2^k,\ \|u\|=1\}$ and let μ be the unique probability measure on S which is rotation invariant. A simple two-dimensional computation shows that, for every choice of $x,y\in S$,

$$\int_S \operatorname{sign}(x,u)\operatorname{sign}(y,u)\,d\mu(u)=1-2\theta[x,y]/\pi$$

where $\theta[x,y]$ is the angle between x and y (i.e. $0\leqslant\theta[x,y]\leqslant\pi$ and $\cos\theta[x,y]=(x,y)$).

In view of our assumption on $(\alpha_{i,j})$ we have, for every $u \in S$ and for every $\{t_i\}_{i=1}^n$, $\{s_j\}_{j=1}^n$ with $|t_i| \leqslant 1$, $|s_j| \leqslant 1$, that

$$-1 \leqslant \sum_{i,j=1} \alpha_{i,j} s_i t_j \operatorname{sign}(x_i, u) \operatorname{sign}(y_j, u) \leqslant 1 .$$

By integrating with respect to μ we get that

$$-1 \leqslant \sum_{i,j=1}^{n} \alpha_{i,j} s_i t_j (1 - 2\theta[x_i, y_j]/\pi) \leqslant 1 .$$

Hence, the matrix $(\alpha_{i,j}(1-2\theta[x_i, y_j]/\pi)$ also satisfies the assumptions made on $(\alpha_{i,j})$. Thus, by iterating the same argument, we get for every integer m that

$$\left| \sum_{i,j=1}^{n} \alpha_{i,j} (1 - 2\theta[x_i, y_j]/\pi)^m \right| \leqslant 1 .$$

Since, for every $1 \leqslant i, j \leqslant n$,

$$\begin{aligned}(x_i, y_j) &= \cos \theta[x_i, y_j] = \sin(\pi/2 - \theta[x_i, y_j]) \\ &= \sum_{m=0}^{\infty} (-1)^m (\pi/2 - \theta[x_i, y_i])^{2m+1}/(2m+1)!\end{aligned}$$

we conclude that

$$\left| \sum_{i,j=1}^{n} \alpha_{i,j} (x_i, y_j) \right| \leqslant \sum_{m=0}^{\infty} (\pi/2)^{2m+1}/(2m+1)! = (e^{\pi/2} - e^{-\pi/2})/2 . \quad \square$$

Several other proofs of 2.b.5 can be found in the literature (cf. [12, 103, 123]). Some give also a better estimate for K_G. The best possible value of K_G seems to be unknown.

The next result, due to Grothendieck [49], is actually a restatement of 2.b.5 in terms of absolutely summing operators.

Theorem 2.b.6. *Every bounded linear operator T from l_1 into l_2 is absolutely summing and $\pi_1(T) \leqslant K_G \|T\|$.*

Proof. Let $\{e_j\}_{j=1}^{\infty}$ be the unit vector basis of l_1 and $u_i = \sum_{j=1}^{m} \alpha_{i,j} e_j$, $i = 1, 2, \ldots, n$ be vectors in l_1^m, for some m, such that $\sum_{i=1}^{n} |x^*(u_i)| \leqslant \|x^*\|$, for every $x^* \in l_1^*$. Let $\{s_j\}_{j=1}^m$ be scalars of absolute value $\leqslant 1$ and let $x_s^* \in l_1^*$ be defined by $x_s^*(e_j) = s_j$ if $1 \leqslant j \leqslant m$ and $x^*(e_j) = 0$ if $j > m$. For every choice of $\{t_i\}_{i=1}^n$ such that $|t_i| \leqslant 1$, $1 \leqslant i \leqslant m$,

$$\left| \sum_{j=1}^{m} \sum_{i=1}^{n} \alpha_{i,j} t_i s_j \right| \leqslant \sum_{i=1}^{n} |t_i| \left| \sum_{j=1}^{m} \alpha_{i,j} s_j \right| \leqslant \sum_{i=1}^{n} |x_s^*(u_i)| \leqslant 1 .$$

For every $1 \leqslant i \leqslant n$ let $y_i \in l_2$ be such that $||y_i||=1$ and $(Tu_i, y_i)=||Tu_i||$. Then, by 2.b.5

$$\sum_{i=1}^{n} ||Tu_i|| = \sum_{i=1}^{n} (Tu_i, y_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{i,j}(Te_j, y_i) \leqslant K_G ||T|| . \quad \square$$

Let us point out that 2.b.6 is in a sense a joint characterization of l_1 and l_2. It is proved in [87] that *if X has an unconditional basis and, for some Banach space Y, $\Pi_1(X, Y)=L(X, Y)$ then X is isomorphic to l_1 and Y is isomorphic to a Hilbert space.* If we do not assume that X has an unconditional basis then there are more spaces which satisfy $\Pi_1(X, l_2)=L(X, l_2)$. For example, $X=L_1(0, 1)$ has this property.

There are many examples of pairs of spaces X and Y such that $L(X, Y)=\Pi_p(X, Y)$, for some $p \geqslant 2$. For instance, we have the following:

Theorem 2.b.7 [87]. *Every bounded linear operator T from c_0 into l_p, with $1 \leqslant p \leqslant 2$, is 2-absolutely summing and $\pi_2(T) \leqslant K_G ||T||$.*

Proof. Let $\{e_i\}_{i=1}^{\infty}$ and $\{f_j\}_{j=1}^{\infty}$ be the unit vector bases of c_0, respectively l_p, and let $T \in L(c_0, l_p)$. Let $(\alpha_{i,j})_{i,j=1}^{\infty}$ be the matrix defined by $Te_i = \sum_{j=1}^{\infty} \alpha_{i,j} f_j$, $i=1, 2, \ldots$. For every choice of $y^*=(a_1, a_2, \ldots) \in l_p^*$ with $||y^*||=1$ and every choice of scalars $\{t_i\}_{i=1}^{\infty}$, $\{s_j\}_{j=1}^{\infty}$ of absolute value $\leqslant 1$ such that $\lim_i t_i=0$ we have

$$(*) \qquad \left| \sum_{i,j=1}^{\infty} \alpha_{i,j} a_j t_i s_j \right| = \left| y_s^* \left(\sum_{i=1}^{\infty} T t_i e_i \right) \right| \leqslant ||T|| \left\| \sum_{i=1}^{\infty} t_i e_i \right\| \leqslant ||T|| ,$$

where $y_s^*=(s_1 a_1, s_2 a_2, \ldots) \in l_p^*$. Let $\{x_k\}_{k=1}^{n}$ be n vectors in c_0 of the form $x_k = \sum_{i=1}^{m} b_{k,i} e_i$, for some integer m, which satisfy $\sum_{k=1}^{n} |x^*(x_k)|^2 \leqslant 1$ whenever $x^* \in c_0^*$ with $||x^*|| \leqslant 1$. In particular, by taking as x^* the unit vectors in l_1 we get that, for $1 \leqslant i \leqslant m$, $\sum_{k=1}^{n} b_{k,i}^2 \leqslant 1$. By considering the m vectors $u_i=(b_{1,i}, b_{2,i}, \ldots, b_{n,i})$, $1 \leqslant i \leqslant m$ in l_2^n it follows from (*) and 2.b.5 that

$$\sum_{j=1}^{\infty} \left\| \sum_{i=1}^{m} a_j \alpha_{i,j} u_i \right\|_2 = \sum_{j=1}^{\infty} a_j \left(\sum_{k=1}^{n} \left(\sum_{i=1}^{m} b_{k,i} \alpha_{i,j} \right)^2 \right)^{1/2} \leqslant K_G ||T|| .$$

Since this holds whenever $||(a_1, a_2, \ldots)||=1$ (in l_p^*) we get that

$$(\text{**}) \qquad \left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{n} \left(\sum_{i=1}^{m} b_{k,i} \alpha_{i,j} \right)^2 \right)^{p/2} \right)^{1/p} \leqslant K_G ||T|| .$$

Put $c_{j,k} = \left| \sum_{i=1}^{m} b_{k,i} \alpha_{i,j} \right|^p$. By using the triangle inequality in $l_{2/p}$ (recall that $p \leqslant 2$), i.e

$$\left(\sum_{k=1}^{n} \left(\sum_{j=1}^{\infty} c_{j,k} \right)^{2/p} \right)^{p/2} \leqslant \sum_{j=1}^{\infty} \left(\sum_{k=1}^{n} c_{j,k}^{2/p} \right)^{p/2}$$

we deduce from $\binom{*}{*}$ that

$$\begin{pmatrix}*\\*\\*\end{pmatrix} \qquad \left(\sum_{k=1}^{n}\left(\sum_{j=1}^{\infty}\left|\sum_{i=1}^{m} b_{k,i}\alpha_{i,j}\right|^{p}\right)^{2/p}\right)^{1/2} \leqslant K_G\|T\|\,.$$

Since $Tx_k=\sum_{i=1}^{m} b_{k,i}\, Te_i=\sum_{j=1}^{\infty}\sum_{i=1}^{m} b_{k,i}\alpha_{i,j}f_j$, $k=1, 2,\ldots, n$, $\begin{pmatrix}*\\*\\*\end{pmatrix}$ states that

$$\left(\sum_{k=1}^{n} \|Tx_k\|^2\right)^{1/2} \leqslant K_G\|T\|$$

and thus $\pi_2(T)\leqslant K_G\|T\|$. □

We state now without proof a result of L. Schwartz [133] and S. Kwapien [77] which shows what happens if we take $p>2$ in 2.b.7.

Theorem 2.b.8. *Let* $2<p<r<\infty$; *then* $L(c_0, l_p)=\Pi_r(c_0, l_p)$. *There are however operators in* $L(c_0, l_p)$ *which are not p-absolutely summing.*

There are of course many other situations which can be investigated, e.g. the structure of $\Pi_r(l_p, l_s)$ for arbitrary r, p, and s. The only operators T which are relatively easy to investigate in the general case are the diagonal operators (i.e. operators of the form $Te_i=\lambda_i f_i$, $i=1, 2,\ldots$, where $\{e_i\}_{i=1}^{\infty}$, resp. $\{f_i\}_{i=1}^{\infty}$, are the unit vector bases in l_p, resp. l_s. For a discussion of these results and several of their variants we refer to [122].

We turn now to the question of uniqueness of unconditional bases in the spaces c_0 and l_p, $1\leqslant p<\infty$. We consider first the simplest case, i.e. $p=2$. It follows from the parallelogram identity in Hilbert space that the average of $\left\|\sum_{i=1}^{n} \theta_i x_i\right\|^2$, over all choices of signs $\{\theta_i\}_{i=1}^{n}$, is equal to $\sum_{i=1}^{n} \|x_i\|^2$. Thus, *if* $\{u_n\}_{n=1}^{\infty}$ *is a normalized unconditional basis in* l_2 *then* $\sum_{n=1}^{\infty} a_n u_n$ *converges if and only if* $\sum_{n=1}^{\infty} a_n^2<\infty$. In other words, every normalized unconditional basis of l_2 is equivalent to the unit vector basis (this observation goes back to G. Köthe [72] and E. R. Lorch [98]). The situation in c_0 and l_1 is similar to that in l_2 but the proof is more difficult (since it is based on 2.b.5).

Proposition 2.b.9 [87]. *Every normalized unconditional basis in* l_1 *or in* c_0 *is equivalent to the unit vector basis of the space.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a normalized unconditional basis of l_1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of scalars such that $\sum_{n=1}^{\infty} a_n x_n$ converges. Let S be the operator from c_0 into l_1 defined by $S(\lambda_1, \lambda_2,\ldots)=\sum_{n=1}^{\infty} \lambda_n a_n x_n$. By 2.b.7 $\pi_2(S)\leqslant K_G\|S\|\leqslant K_G M\left\|\sum_{n=1}^{\infty} a_n x_n\right\|$, where M is the unconditional constant of $\{x_n\}_{n=1}^{\infty}$. Hence, $\left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}\leqslant K_G M\left\|\sum_{n=1}^{\infty} a_n x_n\right\|$.

It follows that the operator $T: l_1 \to l_2$, defined by $T\left(\sum_{n=1}^{\infty} a_n x_n\right) = (a_1, a_2, \ldots)$, is a bounded linear operator with $\|T\| \leqslant K_G M$. By 2.b.6, $\pi_1(T) \leqslant K_G \|T\| \leqslant K_G^2 M$. Hence, for every $x = \sum_{n=1}^{\infty} a_n x_n$ in l_1,

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \|T a_n x_n\| \leqslant K_G^2 M \sup_{\theta_n = \pm 1} \left\|\sum_{n=1}^{\infty} \theta_n a_n x_n\right\| \leqslant K_G^2 M^2 \left\|\sum_{n=1}^{\infty} a_n x_n\right\| .$$

Since obviously $\left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leqslant \sum_{n=1}^{\infty} |a_n|$ we deduce that $\{x_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis in l_1.

Let now $\{y_n\}_{n=1}^{\infty}$ be normalized unconditional basis in c_0. By 1.c.9 the basis $\{y_n\}_{n=1}^{\infty}$ is shrinking and thus the biorthogonal functionals $\{y_n^*\}_{n=1}^{\infty}$ associated to $\{y_n\}_{n=1}^{\infty}$ form an unconditional basis of l_1 such that $1 \leqslant \|y_n^*\| \leqslant M$, for some M and $n = 1, 2, \ldots$. By the first part of the proposition the sequence $\{y_n^*\}_{n=1}^{\infty}$ is equivalent to the unit vector basis in l_1. This implies immediately that $\{y_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis of c_0. □

Remarks. 1. The same argument as that used in the proof of 2.b.9 shows that if $\{X_n\}_{n=1}^{\infty}$ is an unconditional Schauder decomposition of l_1, respectively c_0, then there is a constant M so that, for every choice of $x_n \in X_n$, $n = 1, 2, \ldots$ with $\sum_{n=1}^{\infty} x_n$ converging, we have $\sum_{n=1}^{\infty} \|x_n\| \leqslant M \left\|\sum_{n=1}^{\infty} x_n\right\|$, respectively $\left\|\sum_{n=1}^{\infty} x_n\right\| \leqslant M \sup_n \|x_n\|$.

2. The arguments used in the beginning of the proofs of 2.b.9 and 2.b.7 show also the following. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of elements in l_p, $1 \leqslant p < 2$ such that $\sum_{n=1}^{\infty} u_n$ converges unconditionally; then $\sum_{n=1}^{\infty} \|u_n\|^2 < \infty$. This fact was first proved by Orlicz [110] and his proof is simpler than the one given here (which relies on 2.b.5). Similarly, we deduce from 2.b.8 that if $\sum_{n=1}^{\infty} u_n$ converges unconditionally in l_p, for $p > 2$, then $\sum_{n=1}^{\infty} \|u_n\|^r < \infty$ for every $r > p$. This however is not the best possible result. We shall see in Vol. II that we can take also $r = p$.

In contrast to the cases of c_0, l_1 and l_2 there are normalized unconditional bases in l_p, $1 < p < \infty$, $p \neq 2$ which are not equivalent to the unit vector basis [114]. In order to see this we observe that the Khintchine inequality 2.b.3 shows that the mapping $T: l_2 \to L_p(0, 1)$, defined by $T(a_1, a_2, \ldots) = \sum_{n=1}^{\infty} a_n r_n(t)$, is an isomorphism for every $1 \leqslant p < \infty$.

Let now $p \geqslant 2$ and let P be the orthogonal projection from $L_p(0, 1)$ (which is, under our assumption on p, a linear subspace of $L_2(0, 1)$) onto $[r_n]_{n=1}^{\infty}$. The projection P is defined by

$$Pf(t) = \sum_{n=1}^{\infty} \int_0^1 f(s) r_n(s)\, ds \cdot r_n(t) .$$

It is bounded in L_p since

$$||Pf||_p \leq B_p ||Pf||_2 \leq B_p ||f||_2 \leq B_p ||f||_p \, .$$

For every n let F_n be the subspace of $L_p(0, 1)$ spanned by the characteristic functions of the intervals $[k/2^n, (k+1)/2^n]$, $k=0, 1,\ldots, 2^n-1$. Clearly, $F_n \supset E_n = [r_i]_{i=1}^n$, F_n is isometric to $l_p^{2^n}$ and $PF_n = E_n$ for every n. In other words, there is a constant K_p so that, for every n, there exist a subspace C_n of $l_p^{2^n}$ with $d(C_n, l_2^n) \leq K_p$ and a projection from $l_p^{2^n}$ onto C_n of norm $\leq K_p$. Consequently, the space $\left(\sum_{n=1}^\infty \oplus l_2^n\right)_p$ is, for $2 \leq p < \infty$, isomorphic to a complemented subspace of $\left(\sum_{n=1}^\infty \oplus l_p^{2^n}\right)_p = l_p$. By 2.a.3, $\left(\sum_{n=1}^\infty \oplus l_2^n\right)_p$ is isomorphic to l_p for $2 \leq p < \infty$ and, by duality, also for $1 < p < 2$. The obvious unit vector basis in $\left(\sum_{n=1}^\infty \oplus l_2^n\right)_p$ is therefore, for $1 < p < \infty, p \neq 2$, an example of a normalized unconditional basis in l_p (more precisely, in a space isomorphic to l_p) which is not equivalent to the unit vector basis of l_p. Observe that the basis we have just discussed has, by 2.a.12, the property that all its block bases span a complemented subspace of l_p. Observe also that by 2.b.9 the space l_1 is not isomorphic to $\left(\sum_{n=1}^\infty \oplus l_2^n\right)_1$ and, similarly, c_0 is not isomorphic to $\left(\sum_{n=1}^\infty \oplus l_2^n\right)_0$.

The preceding discussion concerning uniqueness of unconditional bases proves a part of the following theorem.

Theorem 2.b.10 [97]. *A Banach space has, up to equivalence, a unique unconditional basis if and only if it is isomorphic to one of the following three spaces*: c_0, l_1 *or* l_2.

We shall conclude the proof of 2.b.10 in Section 3.a below.

We would like to show now how the weak form of 1.a.8 (namely, that every infinite-dimensional Banach space with a basis has at least two non-equivalent normalized bases) can be deduced from 2.b.10. Let $\{x_n\}_{n=1}^\infty$ be a normalized basis of a Banach space X. If $\{x_n\}_{n=1}^\infty$ is not unconditional then, for some sequence of signs $\{\theta_n\}_{n=1}^\infty$, the normalized basis $\{\theta_n x_n\}_{n=1}^\infty$ of X is not equivalent to $\{x_n\}_{n=1}^\infty$. If $\{x_n\}_{n=1}^\infty$ is unconditional then, by 2.b.10, X has a normalized unconditional basis which is not equivalent to $\{x_n\}_{n=1}^\infty$ unless X is c_0, l_1 or l_2. Thus, we have only to show that these three spaces have conditional (i.e. not unconditional) bases. For l_1 and c_0 this assertion is trivial. The vectors $e_1, e_2-e_1, e_3-e_2,\ldots$ form a conditional basis of l_1 ($\{e_n\}_{n=1}^\infty$ denotes as usual the unit vector basis). The summing basis is a conditional basis of c (which is of course isomorphic to c_0 via the map $T(a_1, a_2,\ldots) = (\lim_n a_n, a_1 - \lim_n a_n, a_2 - \lim_n a_n,\ldots)$, from c onto c_0). For l_2 the assertion is by no means obvious.

Proposition 2.b.11. *The space* l_2 *has a conditional basis.*

This proposition was proved first by Babenko [6] who used harmonic analysis to construct concrete examples of conditional bases in Hilbert function spaces. For example, he showed that, for $0 < \alpha < 1/2$, the sequence $|t|^\alpha e^{int}$, $n=0, \pm 1, \pm 2,\ldots$

is a conditional basis of $L_2(-\pi, \pi)$. The proof we present here is completely different and is due to C. A. McCarthy and J. Schwartz [105].

Proof. It is clearly enough to construct, for every even integer n, a set $\{Q_i\}_{i=1}^n$ of projections in l_2^n so that $\dim Q_i l_2^n = 1$ for every i, $Q_i Q_j = 0$ if $i \neq j$, $\left\|\sum_{i=1}^k Q_i\right\| \leqslant 2$ for $1 \leqslant k \leqslant n$ and $\lim_n \|Q_1 + Q_3 + \cdots + Q_{n-1}\| = \infty$. Indeed, these assumptions on $\{Q_i\}_{i=1}^n$ imply that $\sum_{i=1}^n Q_i = I$ and thus if $0 \neq x_i \in Q_i l_2^n$, $1 \leqslant i \leqslant n$, then $\{x_i\}_{i=1}^n$ is a basis of l_2^n whose basis constant is $\leqslant 2$ and whose unconditional constant tends to ∞ with n. From this it is obvious how to construct a conditional basis in $l_2 = \left(\sum_{n=1}^\infty \oplus l_2^n\right)_2$.

In order to construct suitable Q_i we pick a sequence $\{\alpha_k\}_{k=1}^\infty$ of positive numbers such that

$$\sum_{k=1}^\infty k\alpha_k^2 \leqslant 1 \quad \text{and} \quad \lambda_n = \frac{1}{n} \sum_{k=1}^n \left(\sum_{i=1}^k \alpha_i\right)^2 \to \infty, \quad \text{as } n \to \infty$$

(take, e.g. $\alpha_k = \delta/k \log k$, for $k > 1$ and a suitable $\delta > 0$).

Let now $n = 2m$ and let Q_i be the operator on l_2^n whose matrix with respect to the unit vector basis is defined as follows:

If $i = 2j-1$, $j = 1, \ldots, m$ the matrix which represents Q_i has all its columns equal to 0 except the i'th column which is equal to $(0, 0, \ldots, 0, 1, \alpha_1, 0, \alpha_2, 0, \ldots, \alpha_{m-j+1})$ (for typographical reasons we write this column here as a row). If $i = 2j$, $j = 1, \ldots, m$ the matrix which represents Q_i has all rows equal to 0 except the i'th row which is equal to $(-\alpha_j, 0, -\alpha_{j-1}, 0, \ldots, -\alpha_1, \overset{i}{1}, 0, 0, \ldots, 0)$. It is clear that each Q_i is a rank one operator and also that $Q_i^2 = Q_i$ (observe that the matrices representing the Q_i are triangular with a diagonal which is equal to 0 except that i'th element which is equal to 1). It is also easy to verify that $Q_{i_1} \cdot Q_{i_2} = 0$ for $i_1 \neq i_2$.

In order to estimate the norm of $Q_1 + Q_2 + \cdots + Q_i$, $1 \leqslant i \leqslant n$, we write this operator in the form of $D_i + R_i$ where D_i is the operator which is represented by the diagonal of the matrix representing $Q_1 + Q_2 + \cdots + Q_i$. (Thus the matrix representing D_i is a diagonal matrix whose diagonal is $(1, 1, \ldots, \overset{i}{1}, 0, \ldots, 0)$.) Observe that the non-zero entries in the matrix representing R_i (i.e. the off diagonal entries in the matricial representation of $Q_1 + Q_2 + \cdots + Q_i$) consist of at most k times the term α_k, $k = 1, 2, \ldots, m$. Thus the Hilbert Schmidt norm of R_i is $\leqslant \left(\sum_{k=1}^m k\alpha_k^2\right)^{1/2}$. Hence,

$$\|Q_1 + Q_2 + \cdots + Q_i\| \leqslant \|D_i\| + \|R_i\| \leqslant 1 + \left(\sum_{k=1}^m k\alpha_k^2\right)^{1/2} \leqslant 2\,.$$

In order to estimate the norm of $Q_1 + Q_3 + \cdots + Q_{2m-1}$ let x be the vector $(1, 0, 1, \ldots, 1, 0)$ in l_2^n. Then $\|x\| = m^{1/2}$ and

$$(Q_1 + Q_3 + \cdots + Q_{2m-1})x = (1, \alpha_1, 1, \alpha_1 + \alpha_2, \ldots, 1, \alpha_1 + \alpha_2 + \cdots + \alpha_m)\,.$$

Hence

$$\|Q_1+Q_3+\cdots+Q_{2m-1}\|^2 \geqslant \|(Q_1+Q_3+\cdots+Q_{2m-1})x\|^2/\|x\|^2$$
$$\geqslant m^{-1}\sum_{k=1}^{m}\left(\sum_{i=1}^{k}\alpha_i\right)^2=\lambda_m . \quad \square$$

c. Fredholm Operators, Strictly Singular Operators and Complemented Subspaces of $l_p \oplus l_r$

We have proved in Section a that each infinite-dimensional complemented subspace of c_0 or l_p, $1<p<\infty$ is isomorphic to the entire space. In this section we are going to present a result of I. S. Edelstein and P. Wojtaszczyk [35] which shows that the only isomorphism types of infinite dimensional complemented subspaces of $l_p \oplus l_r$ are the obvious ones, i.e. l_p, l_r and $l_p \oplus l_r$. The proof of this result makes use of some notions (like strictly singular and Fredholm operators) which enter into many other contexts in functional analysis. A large part of this section will be devoted to proving the basic results concerning these important notions. Our presentation will go a little beyond the material which is strictly needed for the proof of the result of Edelstein and Wojtaszczyk. Concerning the proof itself let us point out already here one interesting feature. It uses spectral theory in an essential way. This topic had so far only few applications in the structure theory of Banach spaces. The section concludes with a study of the unconditional bases of the spaces $l_p \oplus l_r$.

The fact that makes the approach of Edelstein and Wojtaszczyk work is that, for $p \neq r$, the space l_p does not have an infinite-dimensional subspace isomorphic to a subspace of l_r (see the remark following 2.a.2). Let us give this property a formal name.

Definition 2.c.1. Two infinite-dimensional Banach spaces X and Y are called *totally incomparable* if there exists no infinite dimensional Banach space Z which is isomorphic to a subspace of X and to a subspace of Y.

As mentioned above, any two different spaces of the set $\{c_0\} \cup \{l_p;\, 1 \leqslant p<\infty\}$ are totally incomparable. The notion of "totally incomparable spaces" was introduced by H. P. Rosenthal [126]. In this paper it is proved that *X and Y are totally incomparable if and only if, whenever U is a Banach space with subspaces X_1 and Y_1 which are isomorphic to X, respectively Y, the algebraic sum X_1+Y_1 is closed in U.*

Definition 2.c.2. An operator $T\colon X \to Y$ is called *strictly singular* if the restriction of T to any infinite-dimensional subspace of X is not an isomorphism.

If X and Y are totally incomparable any operator from X to Y is strictly singular. For general spaces X and Y, every compact operator is strictly singular. The formal identity map from l_p to l_r, if $r>p$, is an example of a strictly singular operator which is not compact. In this connection it is worthwhile to mention the following result of H. R. Pitt.

Proposition 2.c.3. *Let $1 \leqslant p < r < \infty$. Then, every bounded linear operator from l_r into l_p is compact. The same is true for every linear operator from c_0 into l_p.*

Proof. Assume that T is a non-compact operator from l_r into l_p. Then there is a sequence $\{x_n\}_{n=1}^\infty$ in l_r so that $x_n \xrightarrow{w} 0$ and $\|Tx_n\| \geqslant \varepsilon$ for some $\varepsilon>0$ and all integers n. By passing to a subsequence we may assume by 1.a.9 and 2.a.1 that $\{x_n\}_{n=1}^\infty$ is equivalent to the unit vector basis in l_r and $\{Tx_n\}_{n=1}^\infty$ is equivalent to the unit vector basis in l_p. Since the formal identity map from l_r into l_p is not bounded (recall that $r>p$) we arrived at a contradiction. The proof in the case where c_0 replaces l_r is the same. $\square$

Observe that the proof of 2.c.3 shows also that a $T \in L(l_p, l_p)$ is strictly singular if and only if it is compact.

We present now two simple propositions concerning strictly singular operators due to T. Kato [69].

Proposition 2.c.4. *Let X and Y be infinite-dimensional Banach spaces. Assume that T: $X \to Y$ is an operator such that the restriction of T to any subspace of X of finite codimension is not an isomorphism (this is the case in particular if TX is not closed in Y). Then, for every $\varepsilon>0$ there is an infinite-dimensional subspace Z of X so that $T_{|Z}$ is compact and $\|T_{|Z}\| \leqslant \varepsilon$.*

Proof. For every $\delta>0$ and every finite set $\{x_i^*\}_{i=1}^m$ of elements in X^* there is, by our assumption, an element $x \in X$ with $\|x\|=1$, $\|Tx\|<\delta$ and $x_i^*(x)=0$, $1 \leqslant i \leqslant m$. Hence, for every $\varepsilon>0$ (see the proof of 1.a.5), there exists a normalized basic sequence $\{x_n\}_{n=1}^\infty$ in X, with basis constant $\leqslant 2$, so that $\|Tx_n\| \leqslant \varepsilon \cdot 8^{-n}$, $n=1, 2, \ldots$. The space $Z=[x_n]_{n=1}^\infty$ has the desired property. $\square$

Observe that if, for some $X_0 \subset X$ of finite codimension, $T_{|X_0}$ is an isomorphism then TX_0, and thus also TX, are closed in Y.

Proposition 2.c.5. (i) *The sum of two strictly singular operators is strictly singular.*

(ii) *The composition of a strictly singular and a bounded operator is strictly singular.*

Proof. (i) Let T and S be strictly singular operators from X to Y and let Z be an infinite-dimensional subspace of X. By 2.c.4 there is an infinite dimensional subspace Z_1 of Z such that $T_{|Z_1}$ is compact. By applying 2.c.4 again we get an infinite-dimensional subspace Z_2 of Z_1 such that $S_{|Z_2}$ is compact. Clearly, $(S+T)_{|Z_2}$ is compact and thus $S+T$ is not an isomorphism on Z. The verification of (ii) is straightforward. $\square$

Another way of formulating 2.c.5 is to say that the strictly singular operators form an operator ideal.

It should be pointed out that, in contrast to the ideal of compact (or weakly compact) operators, the dual operator T^* of a strictly singular operator T need not be strictly singular. For example, let X be a separable Banach space which does not contain a subspace isomorphic to l_1 (e.g. $X=l_p$, for $p>1$, or $X=c_0$). Let T be a quotient map from l_1 onto X (cf. 2.f). By 2.a.2 T is strictly singular. The operator $T^*: X \to l_\infty$ is an isometry (into) and thus is not strictly singular.

We introduce now another important notion, namely that of the index of an operator.

Definition 2.c.6. Let $T: X \to Y$ be a bounded linear operator for which TX is closed. Put

$$\alpha(T)=\dim \ker T, \qquad \beta(T)=\dim\ Y/TX\,.$$

If either $\alpha(T)<\infty$ or $\beta(T)<\infty$ we define the *index* $i(T)$ of T by $i(T)=\alpha(T)-\beta(T)$. If $\alpha(T)$ and $\beta(T)$ are both finite (i.e. if $i(T)$ is defined and is finite) then T is called a *Fredholm operator*.

Observe that if the algebraic dimension of Y/TX is finite then, by the open mapping theorem, it follows that TX is closed in Y.

A typical example of a Fredholm operator of index $k>0$ is the operator, in c_0 or l_p, which sends $(a_1, a_2, \ldots)$ into $(a_{k+1}, a_{k+2}, \ldots)$.

We shall prove now several general results concerning Fredholm and strictly singular operators which are due to I. E. Gohberg and M. G. Krein [46] and to T. Kato [69]. Observe first that $T: X \to Y$ is a Fredholm operator if and only if there exist subspaces X_1 and B of X; Y_1 and C of Y, so that $X=X_1 \oplus B$, $Y=Y_1 \oplus C$, $T_{|B}=0$, $T_{|X_1}$ is an isomorphism onto Y_1 and $\dim B<\infty$, $\dim C<\infty$. With this notation we have $\alpha(T)=\dim B$ and $\beta(T)=\dim C$. This observation and simple finite-dimensional linear algebra prove the following proposition.

Proposition 2.c.7. (i) *Let $T: X \to Y$ be a Fredholm operator and let $S: X \to Y$ be a finite rank operator. Then, $T+S$ is a Fredholm operator and $i(T+S)=i(T)$.*

(ii) *Let $T_1: X \to Y$, $T_2: Y \to Z$ be Fredholm operators. Then, T_2T_1 is a Fredholm operator and $i(T_2T_1)=i(T_2)+i(T_1)$.*

(iii) *Let $T: X \to Y$ be a Fredholm operator. Then $T^*: Y^* \to X^*$ is also a Fredholm operator for which $\alpha(T^*)=\beta(T)$, $\beta(T^*)=\alpha(T)$ and therefore, $i(T^*)=-i(T)$.*

In the sequel we shall need the following finite dimensional lemma.

Lemma 2.c.8. *Let X be a Banach space and let B and C be subspaces of X with* $\dim B<\infty$ *and* $\dim C>\dim B$. *Then, there is an $x \in C$ such that $\|x\|=d(x, B)=1$.*

Proof. Assume first that X is a strictly convex Banach space. For every $x \in C$ with $\|x\|=1$ let $f(x)$ be the unique point in B for which $\|x-f(x)\|=d(x, B)$. Clearly, f is continuous and $f(-x)=-f(x)$. By the antipodal map theorem of

Borsuk [32, p. 347] and the fact that $\dim C > \dim B$ we infer that there is an x with $||x||=1$ such that $f(x)=0$, i.e. $d(x, B)=1$.

In the general case we observe that we may assume without loss of generality that $\dim C = \dim B + 1 < \infty$ and that $X = \operatorname{span}\{B, C\}$. Hence, X is finite-dimensional and thus has an equivalent strictly convex norm $|||\cdot|||$. For every integer n let $||\cdot||_n$ be the strictly convex norm on X defined by $||x||_n = ||x|| + |||x|||/n$, where $||\cdot||$ is the given norm in X. By the first part of the proof we can find, for every integer n, an $x_n \in C$ with $||x_n||_n = 1$ and $\inf\{||x_n - y||_n;\ y \in B\} = 1$. If x is any limiting point of the sequence $\{x_n\}_{n=1}^{\infty}$ then $||x|| = d(x, B) = 1$, as desired. $\square$

Proposition 2.c.9. *Let T: $X \to Y$ be an operator for which $i(T)$ is defined. There is a number $\lambda(T) > 0$ so that if S: $X \to Y$ satisfies $||S|| < \lambda(T)$ then*

(i) *$\alpha(T+S) \leqslant \alpha(T)$.*
(ii) *$T+S$ has a closed range and $\beta(T+S) \leqslant \beta(T)$.*
(iii) *$i(T+S) = i(T)$.*

Proof. Every operator T with a closed range admits a unique factorization of the form

$$X \xrightarrow{\pi_T} X/\ker T \xrightarrow{\tilde{T}} TX \xrightarrow{i_T} Y$$

where π_T is the natural quotient map, $\tilde{T}$ is an isomorphism and i_T is the inclusion map. Put $\lambda(T) = ||\tilde{T}^{-1}||^{-1}$. Since $\lambda(T) = \lambda(T^*)$ we may assume without loss of generality that $\alpha(T) < \infty$ (otherwise, we work with T^*).

Let $||S|| < \lambda(T)$ and suppose that $\alpha(T+S) > \alpha(T)$. Then, by 2.c.8, there is an $x \in \ker(T+S)$ such that $1 = ||x|| = d(x, \ker T) = ||\pi_T x||$. Hence,

$$||Tx|| = ||\tilde{T}\pi_T x|| \geqslant \lambda(T)\,.$$

On the other hand, $Sx = -Tx$ and thus $||Tx|| = ||Sx|| \leqslant ||S||$. This contradicts the assumption that $||S|| < \lambda(T)$ and (i) is thus proved.

Assume that $T+S$ does not have a closed range. Let $0 < \varepsilon < \lambda(T) - ||S||$. By 2.c.4 there is an infinite dimensional subspace Z of X so that $||(T+S)_{|Z}|| < \varepsilon$. By 2.c.8 there is an $z \in Z$ such that $1 = ||z|| = ||\pi_T z||$. Hence,

$$\lambda(T) > ||S|| + \varepsilon \geqslant ||Sz|| + ||(S+T)z|| \geqslant ||Tz|| = ||\tilde{T}\pi_T z|| \geqslant \lambda(T)\,,$$

and we arrived at a contradiction. If $\beta(T) = \infty$ this concludes the proof of (ii). If $\beta(T) < \infty$ we can apply (i) to T^* and thereby conclude the proof of (ii).

In order to prove (iii) it is enough to verify it with some $\varepsilon > 0$ instead of the $\lambda(T)$ defined above. Indeed, once we show that $||U|| \leqslant \varepsilon_T$ implies $i(T+U) = i(T)$, we can apply this fact (by (i) and (ii)) to $T+tS$ for every $t \in [0, 1]$ provided of course $||S|| < \lambda(T)$. The function $i(T+tS)$ is thus a continuous function of t which takes only discrete values; hence, it must be constant, i.e. $i(T) = i(T+S)$.

We prove the existence of such an $\varepsilon > 0$ first in the case $\alpha(T) = 0$, i.e. when T is an isomorphism. Let U: $X \to Y$ satisfy $||U|| < \lambda(T)/2$ and let $x \in X$ with $||x|| = 1$.

Then $\|(T+U)x\| \geqslant \lambda(T) - \|Ux\| > \lambda(T)/2$ and hence, $\lambda(T+U) > \lambda(T)/2$. By (i) and (ii) we have $\alpha(T+U)=0$ and $\beta(T+U) \leqslant \beta(T)$. Since $\|U\| < \lambda(T+U)$ we may apply (ii) also to $T+U$ and deduce that $\beta(T) \leqslant \beta(T+U)$. Consequently, $\beta(T) = \beta(T+U)$ and thus also $i(T)=i(T+U)$.

The general case (i.e. $\alpha(T)>0$ but finite) reduces easily to the previous case. Indeed, let $X=X_1 \oplus \ker T$ and let $W= Y \oplus \ker T$. Let $T_1: X \to W$ be such that $T_{1|X_1} = T_{|X_1}$ and T_1 is the identity on $\ker T$. Then, $\alpha(T_1)=0$ and $\beta(T_1)=\beta(T)$, i.e. $i(T_1)=i(T)-\alpha(T)$. Let P be the natural projection from W onto Y and let $U: X \to Y$. Since $\alpha(T_1)=0$ we have that if $\|U\| < \lambda(T_1)/2$ then $i(T_1+U)=i(T_1)=i(T)-\alpha(T)$. By 2.c.7(i) the index of $T+U=P(T_1+U)$, as an operator from X to W, is the same as that of T_1+U. Hence, the index of $T+U$, as an operator from X to Y, is $i(T_1+U)+\alpha(T)$, i.e. is equal to $i(T)$. □

Proposition 2.c.10. *Let $T: X \to Y$ be an operator with closed range for which $\alpha(T) < \infty$. Let $S: X \to Y$ be strictly singular. Then, $\alpha(T+S) < \infty$, $T+S$ has a closed range and $i(T+S)=i(T)$.*

Proof. Let $X_1 \subset X$ be such that $X=X_1 \oplus \ker T$. Then, $T_{|X_1}$ is an isomorphism. If $\dim \ker (T+S)=\infty$ there would exist an infinite dimensional subspace Z of X_1 on which $T+S=0$, i.e. $S=-T$ is invertible. This contradicts the assumption that S is strictly singular. If $(T+S)X$ were not closed we could, by 2.c.4, find an infinite dimensional subspace Z of X_1 so that $\|(T+S)_{|Z}\| < \|T_{|X_1}^{-1}\|^{-1}$. But then $S_{|Z}$ is an isomorphism and we reached again a contradiction.

By what we have already proved the number $i(T+tS)$ is defined for all $t \in [0,1]$. By 2.c.9, $i(T+tS)$ is a continuous function of t and thus a constant, i.e. $i(T)=i(T+S)$. □

Proposition 2.c.11. *Let $T: X \to X$ be a Fredholm operator. There is an $\varepsilon > 0$ so that $\alpha(T+\lambda I)$ is constant on the set $\{\lambda;\, 0 < |\lambda| \leqslant \varepsilon\}$.*

Proof. All the iterates of T have a closed range and thus $Y = \bigcap_{n=1}^{\infty} T^n X$ is a closed subspace of X (which may consist of the vector 0 only). Clearly, $TY \subset Y$; we shall prove that $TY=Y$. Observe first that since $\alpha(T)<\infty$ there is an integer k so that $T^k X \cap \ker T = T^n X \cap \ker T$, for every $n>k$. Let $y \in Y$ and let $u \in T^k X$ and $v \in T^n X$ (for some $n>k$) be such that $Tu=y=Tv$. Then, $u-v \in T^k X \cap \ker T \subset T^n X$, i.e. $u \in T^n X$. Since this is true for every $n>k$ we get that $u \in Y = \bigcap_{n=1}^{\infty} T^n X$.

Let $T_1 = T_{|Y}$. We have shown that $\beta(T_1)=0$ and thus, by 2.c.9, $\beta(T_1+\lambda I_Y)=0$ for λ sufficiently small. By using 2.c.9 again, we deduce that $i(T_1+\lambda I_Y)=i(T_1)$ and thus also $\alpha(T_1+\lambda I_Y)=\alpha(T_1)$ if λ is sufficiently small. To conclude the proof we have only to observe that, for $\lambda \neq 0$, $\ker (T+\lambda I) = \ker (T_1+\lambda I_Y)$. Indeed, if $Tx=-\lambda x$, with $\lambda \neq 0$, then $x \in \bigcap_{n=1}^{\infty} T^n X = Y$. □

The next lemma involves spectral properties of operators and it therefore requires that we work with a complex Banach space.

Lemma 2.c.12. *Let X be a complex Banach space and let P be a projection in X. Let $S: X \to X$ be a strictly singular operator and let $Q = P + S$. Then, the spectrum $\sigma(Q)$ of Q is countable and its only possible limit points are* 0 *and* 1.

Proof. Let C denote the complex plane. By 2.c.10 the operator $Q - \lambda I$ is a Fredholm operator of index 0 for every $\lambda \in C \sim \{0, 1\}$. By 2.c.11, any two points in $C \sim \{0, 1\}$ can be connected by a finite chain of discs such that $\alpha(Q - \lambda I)$ is constant on each disc (with the possible exception of the center of the disc). Consequently, $\alpha(Q - \lambda I)$ is constant on $C \sim \{0, 1\}$ except for a set of isolated points. This constant value is 0 since $Q - \lambda I$ is invertible for $|\lambda| > \|Q\|$. Since $i(Q - \lambda I) = 0$ it follows that also $\beta(Q - \lambda I) = 0$ except for a set of isolated points in $C \sim \{0, 1\}$. $\square$

We are now ready to prove the theorem of I. S. Edelstein and P. Wojtaszczyk [35] and P. Wojtaszczyk [143].

Theorem 2.c.13. *Let X and Y be two Banach spaces so that every operator from Y into X in strictly singular. Let P be a projection of $X \oplus Y$ onto an infinite-dimensional subspace Z. Then there exist an automorphism τ_0 of $X \oplus Y$ and complemented subspaces X_0 of X and Y_0 of Y such that $\tau_0 Z = X_0 \oplus Y_0$.*

Proof. We shall first assume that X and Y are complex Banach spaces. As every operator from $X \oplus Y$ into itself, the projection P has a natural representation as a matrix $P = \begin{pmatrix} S_1 & S_3 \\ S_2 & S_4 \end{pmatrix}$, where $S_1: X \to X$, $S_2: X \to Y$, $S_3: Y \to X$ and $S_4: Y \to Y$. Since S_3 is strictly singular the same is true for $P - Q$, where $Q = \begin{pmatrix} S_1 & 0 \\ S_2 & S_4 \end{pmatrix}$. We want to replace Q by a projection $\tilde{P}$ having a similar matricial representation (i.e. zero in the upper right corner). We achieve this by putting

$$\tilde{P} = \frac{1}{2\pi i} \int_\Gamma R(\lambda, Q)\, d\lambda\,,$$

where $R(\lambda, Q)$ is the resolvent of Q and Γ is a closed simple curve which does not intersect $\sigma(Q)$, has the number 1 in its interior and 0 and 2 in its exterior. Such a Γ exists by 2.c.12.

The operator $\tilde{P}$ is a projection in $X \oplus Y$ so that $\sigma(Q_{|\tilde{P}(X \oplus Y)}) \subset$ interior of Γ and $\sigma(Q_{|(I-\tilde{P})(X \oplus Y)}) \subset$ exterior of Γ (cf. [33, VII.3.11]). Moreover, it is easily checked that $\tilde{P} = \begin{pmatrix} P_1 & 0 \\ P_2 & P_4 \end{pmatrix}$, where P_1 is a projection in X, P_4 a projection in Y and P_2 an operator from X into Y.

The operator $I + \tilde{P} - Q$ on $X \oplus Y$ leaves $\tilde{P}(X \oplus Y)$ invariant and is invertible there (since $(I + \tilde{P} - Q)_{|\tilde{P}(X \oplus Y)} = (2I - Q)_{|\tilde{P}(X \oplus Y)}$ and 2 is in the exterior of Γ). This operator also leaves $(I - \tilde{P})(X \oplus Y)$ invariant and is invertible on this subspace (since 1 is in the interior of Γ). Hence, $I + \tilde{P} - Q$ is an invertible operator on $X \oplus Y$ and thus, by 2.c.10, $T_1 = I + \tilde{P} - P = (I + \tilde{P} - Q) + (Q - P)$ is a Fredholm operator of index 0 on $X \oplus Y$.

We shall show next that $T_1 Z$ is a subspace of finite codimension in $\tilde{P}(X \oplus Y)$ Since $T_1 P = \tilde{P} P$ it follows that $T_1 Z = T_1 P(X \oplus Y) \subset \tilde{P}(X \oplus Y)$. To show that $T_1 Z$ is of finite codimension in $\tilde{P}(X \oplus Y)$ is the same as to show that Z is of finite codimension in $T_1^{-1}(\tilde{P}(X \oplus Y))$ (this follows from the fact that T_1 is a Fredholm operator; in particular, it is obvious that $T_1 Z$ is closed). If $\dim T_1^{-1}(\tilde{P}(X \oplus Y))/Z = \infty$ there would exist an infinite dimensional subspace W of $X \oplus Y$ so that $P_{|W} = 0$, $T_{1|W}$ is an isomorphism and $T_1 W \subset \tilde{P}(X \oplus Y)$. For $w \in W$ we have $T_1 w = w + \tilde{P} w$ and thus $w = T_1 w - \tilde{P} w \in \tilde{P}(X \oplus Y)$, i.e. $W \subset \tilde{P}(X \oplus Y)$. On $\tilde{P}(X \oplus Y)$ the operator Q is invertible (since 0 is in the exterior of Γ). But $Q_{|W} = (Q-P)_{|W}$ and, since $Q-P$ is strictly singular, we arrived at a contradiction.

Consider now the projection $\tilde{\tilde{P}} = \begin{pmatrix} P_1 & 0 \\ 0 & P_4 \end{pmatrix}$ on $X \oplus Y$. The next step in the proof is to construct an automorphism T_2 of $X \oplus Y$ so that $T_2 \tilde{P} T_2^{-1} = \tilde{\tilde{P}}$. We put $S = \tilde{P} - \tilde{\tilde{P}}$ and notice that $S^2 = S\tilde{P}S = \tilde{P}S\tilde{P} = 0$. Hence, $\tilde{S} = S\tilde{P} - \tilde{P}S$ satisfies

$$\tilde{S}^2 = S\tilde{P}S\tilde{P} - S\tilde{P}S - \tilde{P}S^2\tilde{P} + \tilde{P}S\tilde{P}S = 0\,.$$

It follows that $T_2 = I - \tilde{S}$ is an automorphism of $X \oplus Y$ with $T_2^{-1} = I + \tilde{S}$ and we have

$$T_2 \tilde{P} T_2^{-1} = (I - \tilde{S})\tilde{P}(I + \tilde{S}) = (\tilde{P} - S)\tilde{P}(\tilde{P} - S) = \tilde{\tilde{P}}\tilde{P}\tilde{\tilde{P}} = \tilde{\tilde{P}}\,.$$

The operator $T = T_2 T_1 T_2^{-1}$, which is also a Fredholm operator of index 0 in $X \oplus Y$, maps Z into a subspace of finite codimension of $\tilde{\tilde{P}}(X \oplus Y) = X_1 \oplus Y_1$, where $X_1 = P_1 X$ and $Y_1 = P_4 Y$. Since T is a Fredholm operator of index 0 there is a finite rank operator R on $X \oplus Y$ so that $\tau_1 = T + R$ is an automorphism of $X \oplus Y$. There are finite dimensional subspaces B and C of $X \oplus Y$ so that $\tau_1(Z) \oplus B = X_1 \oplus Y_1 \oplus C$. In particular, $\tau_1(Z) \subset X_2 \oplus Y_2$ with $X_2 = X_1 \oplus C_X$ and $Y_2 = Y_1 \oplus C_Y$ where C_X and C_Y are suitable finite dimensional subspaces of X, respectively Y.

We recall now the fact that if V_1 and V_2 are two closed subspaces of a Banach space U with $\dim U/V_1 = \dim U/V_2 < \infty$ then there is an automorphism of U which maps V_1 onto V_2 (the automorphism can be chosen to be the identity on $V_1 \cap V_2$ since $\dim V_1/(V_1 \cap V_2) = \dim V_2/(V_1 \cap V_2) < \infty$).

Hence, if X_0 and Y_0 are subspaces of finite codimension in X_2, respectively Y_2, so that $\dim (X_2 \oplus Y_2)/\tau_1(Z) = \dim (X_2 \oplus Y_2)/(X_0 \oplus Y_0)$ then there is an automorphism τ_2 of $X_2 \oplus Y_2$ onto itself which maps $\tau_1(Z)$ onto $X_0 \oplus Y_0$. Since $X_2 \oplus Y_2$ is complemented in $X \oplus Y$, τ_2 can be extended to an automorphism $\hat{\tau}_2$ of $X \oplus Y$. Consequently, $\tau_0 = \hat{\tau}_2 \tau_1$ is an automorphism of $X \oplus Y$ so that $\tau_0(Z) = X_0 \oplus Y_0$. Since X_0 is complemented in X and Y_0 is complemented in Y this concludes the proof in the complex case.

We point out now the modification needed in the case where the scalars are real. If X and Y are real Banach spaces we pass to their natural "complexifications" $\hat{X}$, respectively $\hat{Y}$. For example, $\hat{X} = \{(u, v);\ u, v \in X\}$, where we put $(a+ib)(u, v) = (au - bv, av + bu)$ and $\|(u, v)\| = \max\{(\|au - bv\|^2 + \|av + bu\|^2)^{1/2};\ a^2 + b^2 = 1\}$. The given projection P on $X \oplus Y$ induces in an obvious way a projection $\hat{P}$ on $\hat{X} \oplus \hat{Y}$.

The proof proceeds as in the complex case. We have only to ensure that the operator $(2\pi i)^{-1}\int_\Gamma R(\lambda, \hat{Q})\,d\lambda$, used in the proof, is of the form $\hat{\tilde{P}}$, for a suitable operator $\tilde{P}$ on $X\oplus Y$. This is ensured if, e.g. we take as Γ a rectangle symmetric with respect to the real axis. □

Remarks. 1. Observe that the assumptions on X and Y in 2.c.13 are satisfied in particular if X and Y are totally incomparable. In this case the proof 2.c.13, can be simplified somewhat. Indeed, we could take here $Q=\begin{pmatrix} S_1 & 0 \\ 0 & S_4 \end{pmatrix}$ and get that $\tilde{P}=\begin{pmatrix} P_1 & 0 \\ 0 & P_4 \end{pmatrix}$. So the step of producing $\tilde{\tilde{P}}$ becomes unnecessary in this case.

2. It follows from 2.c.13 that if Y is an arbitrary Banach space and X is l_p, $1\leqslant p<\infty$ or c_0 then any complemented subspace of $X\oplus Y$ is of the form Y_0 or $X\oplus Y_0$ for some complemented subspace Y_0 of Y. Indeed, if there is a non-strictly singular operator from Y into X then, by 2.a.2, Y has a complemented subspace isomorphic to X and thus $Y\oplus X\approx Y$.

From 2.c.13 and 2.a.3 we get, by an obvious induction argument, the following result

Theorem 2.c.14. *Let $\{X_i\}_{i=1}^m$ be distinct spaces out of the set $\{c_0\}\cup\{l_p;\ 1\leqslant p<\infty\}$. Then any infinite-dimensional complemented subspace of $X_1\oplus X_2\oplus\cdots\oplus X_m$ is isomorphic to $X_{i_1}\oplus X_{i_2}\oplus\cdots\oplus X_{i_k}$, for some $1\leqslant i_1<i_2<\cdots<i_k\leqslant m$.*

There is no analogue to 2.c.14 if we consider general subspaces instead of complemented subspaces. We shall discuss this situation in Section 4.d.

We shall apply now 2.c.14 to describe the structure of unconditional bases in spaces of the form $l_{p_1}\oplus l_{p_2}\oplus\cdots\oplus l_{p_m}$.

Theorem 2.c.15 [35]. *Let $\{X_i\}_{i=1}^m$ be distinct spaces out of the set $\{c_0\}\cup\{l_p;\ 1\leqslant p<\infty\}$. Let $\{z_n\}_{n=1}^\infty$ be an unconditional basis of $X_1\oplus X_2\oplus\cdots\oplus X_m$. Then we can partition the integers N into m disjoint sets $\{N_i\}_{i=1}^m$ such that $[z_n]_{n\in N_i}$ is isomorphic to X_i, $i=1,\ldots,m$.*

This theorem reduces the question of describing the unconditional bases in the direct sum $\sum_{i=1}^m \oplus\, l_{p_i}$ to that of describing the unconditional bases in each component. In particular, we get by 2.c.15 and 2.b.9 (cf. also the remark preceding it) the following result.

Theorem 2.c.16. *Let Z be one of the spaces, $l_1\oplus l_2$, $c_0\oplus l_1$, $c_0\oplus l_2$ or $c_0\oplus l_1\oplus l_2$. Let $\{z_n'\}_{n=1}^\infty$ and $\{z_n''\}_{n=1}^\infty$ be two normalized unconditional bases of Z. Then there is a permutation π of the positive integers such that $\{z_n'\}_{n=1}^\infty$ is equivalent to $\{z_{\pi(n)}''\}_{n=1}^\infty$.*

The proof of 2.c.15 is by induction on m. The argument needed in the general induction step is very similar to that used in the case $m=2$. We present here only the proof for $m=2$. Also, there is some difference between the reflexive and non-reflexive case; we shall treat here only the reflexive case. The main step of the proof consists of the following lemma.

Lemma 2.c.17. *Let $1<p<r<\infty$ and let $z_n=(x_n, y_n)$, $n=1, 2,\ldots$ be a normalized unconditional basis of $l_p \oplus l_r$ with unconditional constant K ($l_p \oplus l_r$ is normed by $\|(x, y)\|=\|x\|+\|y\|$ for $x \in l_p$, $y \in l_r$). Let $\{n_k\}_{k=1}^\infty$ be a subsequence of the integers so that $\alpha=\sup_k \|x_{n_k}\|<K^{-1}$. Then, $[z_{n_k}]_{k=1}^\infty$ is isomorphic to l_r.*

Proof. The proof we give here is very similar to that of 2.a.11. We claim first that, for every sequence of scalars $\{\lambda_k\}_{k=1}^\infty$ such that $\sum_{k=1}^\infty \lambda_k y_{n_k}$ converges unconditionally, the series $\sum_{k=1}^\infty \lambda_k z_{n_k}$ converges too. To prove this let $\{a_n^k\}_{n=1}^\infty$ and $\{b_n^k\}_{n=1}^\infty$ be scalars so that

$$(x_{n_k}, 0)=\sum_{n=1}^\infty a_n^k z_n, \qquad (0, y_{n_k})=\sum_{n=1}^\infty b_n^k z_n, \quad k=1, 2,\ldots.$$

Then, $a_{n_k}^k+b_{n_k}^k=1$ and, since $|a_{n_k}^k| \leq K\|x_{n_k}\|$, we get that $b_{n_k}^k \geq 1-K\alpha>0$ for every k. By Remark 2 following 1.c.8 we deduce that

$$\left\|\sum_{k=1}^m \lambda_k b_{n_k}^k z_{n_k}\right\| \leq K \sup_{\pm} \left\|\sum_{j=1}^m \pm \lambda_j y_{n_j}\right\|.$$

Thus, if $\{\lambda_k\}_{k=1}^\infty$ is such that $\sum_{k=1}^\infty \lambda_k y_{n_k}$ converges unconditionally the series $\sum_{k=1}^\infty \lambda_k b_{n_k}^k z_{n_k}$, and therefore also $\sum_{k=1}^\infty \lambda_k z_{n_k}$, converge too.

If $[z_{n_k}]_{k=1}^\infty$ is not isomorphic to l_r it would follow from 2.c.14 that $[z_{n_k}]_{k=1}^\infty$ contains a subspace isomorphic to l_p. Hence, by 1.a.9, there is a normalized block basis $w_j=\sum_{k \in \sigma_j} \alpha_k z_{n_k}$, $j=1, 2,\ldots$ of $\{z_{n_k}\}_{k=1}^\infty$ which is equivalent to the unit vector basis of l_p (here $\{\sigma_j\}_{j=1}^\infty$ denotes a sequence of disjoint finite subsets of the integers). Let Q be the natural projection from $l_p \oplus l_r$ onto l_r and let $\{P_m\}_{m=1}^\infty$ be the projections on l_r which are associated to the unit vector basis. Since $z_n \xrightarrow{w} 0$ we get that, for every m, $\lim_{n\to\infty} \|P_m Q_{|[z_k]_{k=n}^\infty}\|=0$. Hence, by passing to a subsequence if necessary, we may assume that there exists a sequence $\{m_j\}_{j=1}^\infty$ of integers so that $\|(P_{m_{j+1}}-P_{m_j})Qz-Qz\| \leq 2^{-j}\|z\|$, for every $z \in B_j=[z_{n_k}]_{k \in \sigma_j}$. In particular, if $u_j \in B_j$ for $j=1, 2,\ldots, s$ then

$$\left\|\sum_{j=1}^s Qu_j\right\| \leq \sum_{j=1}^s 2^{-j}\|u_j\|+\left(\sum_{j=1}^s \|Qu_j\|^r\right)^{1/r}.$$

Let $\{\beta_j\}_{j=1}^\infty$ be a sequence of scalars so that $\sum_{j=1}^\infty |\beta_j|^r<\infty$ and let $\{\theta_k\}_{k=1}^\infty$ be a sequence of signs. Put $u_j=\sum_{k\in\sigma_j}\theta_k\alpha_k z_{n_k}$, $j=1,2,\dots$. Then, $\|u_j\|\leqslant K\|w_j\|\leqslant K$ and hence,

$$\Big\|\sum_{j=1}^{\infty}\sum_{k\in\sigma_j}\beta_j\theta_k\alpha_k y_{n_k}\Big\|=\Big\|\sum_{j=1}^{\infty}\beta_j Qu_j\Big\|\leqslant\sum_{j=1}^{\infty}K2^{-j}|\beta_j|+K\Big(\sum_{j=1}^{\infty}|\beta_j|^r\Big)^{1/r}.$$

In other words, $\sum_{j=1}^{\infty}\sum_{k\in\sigma_j}\beta_j\alpha_k y_{n_k}$ converges unconditionally. By the first part of the proof this implies that $\sum_{j=1}^{\infty}\beta_j w_j=\sum_{j=1}^{\infty}\sum_{k\in\sigma_j}\beta_j\alpha_k z_{n_k}$ converges. This however contradicts the assumption that $\{w_j\}_{j=1}^{\infty}$ is equivalent to the unit vector basis of l_p and that $p<r$. □

Lemma 2.c.18. *Let $1<p<r<\infty$ and let $z_n=(x_n,y_n)$, $n=1,2,\dots$, be a normalized unconditional basis of $l_p\oplus l_r$. Let $\{n_k\}_{k=1}^{\infty}$ be a subsequence of the integers so that $\|x_{n_k}\|\geqslant\alpha$, for some $\alpha>0$ and every integer k. Then $[z_{n_k}]_{k=1}^{\infty}$ is isomorphic to l_p.*

Proof. Every subsequence $\{n'_k\}_{k=1}^{\infty}$ of $\{n_k\}_{k=1}^{\infty}$ has, by 1.a.12 and 2.a.1, a subsequence $\{n''_k\}_{k=1}^{\infty}$ such that $\{x_{n''_k}\}_{k=1}^{\infty}$ and $\{y_{n''_k}/\|y_{n''_k}\|\}_{k=1}^{\infty}$ are equivalent to the unit vector bases of l_p, respectively l_r (if $y_{n''_k}=0$ the situation is even simpler). For such a sequence $\{n''_k\}_{k=1}^{\infty}$ the basic sequence $\{z_{n''_k}\}_{k=1}^{\infty}$ is also equivalent to the unit vector basis of l_p. It follows from this remark and 2.c.14 that $[z_{n_k}]_{k=1}^{\infty}$ is isomorphic to either l_p or $l_p\oplus l_r$.

Assume that $[z_{n_k}]_{k=1}^{\infty}$ is isomorphic to $l_p\oplus l_r$ and let $z^*_{n_k}=(y^*_{n_k},x^*_{n_k})$, $k=1,2,\dots$ be the functionals biorthogonal to z_{n_k} in $(l_p\oplus l_r)^*=l_r^*\oplus l_p^*$. Since $[z^*_{n_k}]_{k=1}^{\infty}\approx([z_{n_k}]_{k=1}^{\infty})^*\approx l_r^*\oplus l_p^*$ it follows by 2.c.17, that $\limsup_k\|y^*_{n_k}\|>0$ and hence, by the first part of this proof, there is a subsequence $\{n'_k\}_{k=1}^{\infty}$ of $\{n_k\}_{k=1}^{\infty}$ such that $[z^*_{n'_k}]_{k=1}^{\infty}$ is isomorphic to l_r^*. Hence, $[z_{n'_k}]_{k=1}^{\infty}$ is isomorphic to l_r. Using again the first part of this proof we arrive at a contradiction ($\{z_{n'_k}\}_{k=1}^{\infty}$ cannot have a subsequence equivalent to the unit vector basis of l_p). □

It is now obvious how to conclude the *proof of* 2.c.15 (for $l_p\oplus l_r$ with $1<p<r<\infty$). We simply take $N_1=\{n;\ \|x_n\|>1/2K\}$ and $N_2=\{n;\ \|x_n\|\leqslant 1/2K\}$. Then, $[z_n]_{n\in N_1}$ is isomorphic to l_p and $[z_n]_{n\in N_2}$ is isomorphic to l_r. □

A stronger version of 2.c.15 was proved recently by P. Wojtaszczyk [143].

We conclude this section by mentioning (an admittedly vaguely stated) problem arising from 2.c.16.

Problem 2.c.19. *Describe all the separable Banach spaces which have, up to equivalence and to a permutation, a unique normalized unconditional basis.*

d. Subspaces of c_0 and l_p and the Approximation Property, Complementably Universal Spaces

This section is devoted mainly to the study of subspaces of c_0 and l_p, $1<p<\infty$. We present first a result of W. B. Johnson and M. Zippin on the structure of subspaces of c_0 or l_p, $1<p<\infty$ having a shrinking basis. The main part of this section is devoted to Davie's proof of the fact that c_0 and l_p, $2<p<\infty$ have subspaces which fail to have the approximation property. The section concludes with some general results concerning universal spaces and the approximation property.

We start with the result of Johnson and Zippin [63]. This result gives actually more information than indicated in the preceding paragraph. It describes even the structure of subspaces of quotients of c_0 or l_p, $1<p<\infty$ and the assumptions concerning the existence of a basis can be somewhat relaxed. Before stating this result let us observe that the notions of a quotient of a subspace and that of a subspace of a quotient coincide. Let X be a subspace of a quotient space Y of Z and let $T: Z\to Y$ be the quotient map. Then X is a quotient of the subspace $T^{-1}X$ of Z. Conversely, assume that X is a quotient space of a subspace Y of Z and let $T: Y\to X$ be the quotient map. Then X is isometric to $Y/\ker T$ which is a subspace of the quotient space $Z/\ker T$ of Z.

Theorem 2.d.1. *Let $\{B_n\}_{n=1}^\infty$ be a sequence of finite-dimensional Banach spaces. Let X be an infinite dimensional subspace of a quotient space of $\left(\sum_{n=1}^\infty \oplus B_n\right)_p$, $1<p<\infty$ $\left(\text{resp. } \left(\sum_{n=1}^\infty \oplus B_n\right)_0\right)$ having a shrinking* F.D.D. *Then X is isomorphic to $\left(\sum_{k=1}^\infty \oplus D_k\right)_p$ $\left(\text{resp. } \left(\sum_{k=1}^\infty \oplus D_k\right)_0\right)$, for a suitable sequence of finite dimensional spaces $\{D_k\}_{k=1}^\infty$.*

Proof. We prove the theorem first in the case $1<p<\infty$. Let $\{C_k\}_{k=1}^\infty$ be an F.D.D. of X and let $T: \left(\sum_{n=1}^\infty \oplus B_n\right)_p \to Y$ be a quotient map, where $Y\supset X$. By 1.g.4(b) there exists a constant K, a blocking $\{B'_n\}_{n=1}^\infty$ of $\{B_n\}_{n=1}^\infty$ and a blocking $\{C'_k\}_{k=1}^\infty$ of $\{C_k\}_{k=1}^\infty$ so that, for every $x\in C'_k$, there is a $u\in B'_k\oplus B'_{k+1}$ with $\|u\|\leqslant K\|x\|$ so that $\|Tu-x\|\leqslant 2^{-k}\|x\|$. Let $x=\sum_{k=1}^\infty x_k$ be an element in X with $x_k\in C'_k$, for every k, and assume that $\sum_{k=1}^\infty \|x_k\|^p<\infty$. For $k=1, 2,\dots$ let $u_k\in B'_k\oplus B'_{k+1}$ satisfy $\|u_k\|\leqslant K\|x_k\|$ and $\|Tu_k-x_k\|\leqslant 2^{-k}\|x_k\|$. Put $u=u_1+u_3+u_5+\cdots$. Then, $\|u\|=\left(\sum_{i=1}^\infty \|u_{2i-1}\|^p\right)^{1/p}\leqslant K\left(\sum_{k=1}^\infty \|x_k\|^p\right)^{1/p}$. Since $\left\|Tu-\sum_{i=1}^\infty x_{2i-1}\right\|\leqslant \sum_{k=1}^\infty 2^{-k}\|x_k\|$ we get that $\left\|\sum_{i=1}^\infty x_{2i-1}\right\|\leqslant \|u\|+\sum_{k=1}^\infty 2^{-k}\|x_k\|$ and thus $\left\|\sum_{i=1}^\infty x_{2i-1}\right\|\leqslant M\left(\sum_{k=1}^\infty \|x_k\|^p\right)^{1/p}$, for a suitable constant M. The same calculation shows that this estimate holds also for $\left\|\sum_{i=1}^\infty x_{2i}\right\|$ and thus

$$(*)\qquad \left\|\sum_{k=1}^\infty x_k\right\|\leqslant 2M\left(\sum_{k=1}^\infty \|x_k\|^p\right)^{1/p}.$$

In order to obtain an estimate on $\|x\|$ from below we pass to the dual. The space X^* is a subspace of a quotient space of $\left(\sum_{k=1}^\infty \oplus (B'_n)^*\right)_q$, where $q^{-1}+p^{-1}=1$. By applying 1.g.4(b) once more and repeating the argument above it follows that there is a blocking $\{D_k\}_{k=1}^\infty$ of $\{C'_k\}_{k=1}^\infty$ so that, whenever $x^* = \sum_{k=1}^\infty x_k^*$ with $x_k^* \in D_k^*$, $k=1, 2, \ldots$, we have

$$(**) \qquad ||x^*|| \leqslant M' \left(\sum_{k=1}^\infty ||x_k^*||^q \right)^{1/q}$$

for a suitable constant M'. Clearly, (*) remains true if we consider decompositions with respect to the blocking $\{D_k\}_{k=1}^\infty$, i.e. if $x = \sum_{k=1}^\infty x_k$ with $x_k \in D_k$, $k=1, 2, \ldots$ (see the remark following 1.g.4). From (*) and (**) it follows that X is isomorphic to $\left(\sum_{k=1}^\infty \oplus D_k \right)_p$.

The proof for the case $\left(\sum_{n=1}^\infty \oplus B_n \right)_0$ is simpler. In this case it is enough to do the first step. The inequality (*) reads $\left\| \sum_{k=1}^\infty x_k \right\| \leqslant 2M \sup_k ||x_k||$ while the inequality $\left\| \sum_{k=1}^\infty x_k \right\| \geqslant \sup_k ||x_k||/M'$, for some M', follows from the fact that $\{C_k\}_{k=1}^\infty$, and thus $\{C'_k\}_{k=1}^\infty$, are F.D.D.'s. We remark that only in this case is the assumption that $\{C_k\}_{k=1}^\infty$ is shrinking a real restriction. In the case $1 < p < \infty$ this assumption holds automatically since X is reflexive. □

Remarks. 1. Theorem 2.d.1 is no longer true if $p=1$. This is completely obvious if we consider subspaces of quotients or even only quotients since every separable Banach space is a quotient space of l_1. If we consider only subspaces of l_1, Theorem 2.d.1 is valid for a trivial reason—no infinite dimensional subspace of l_1 has a shrinking F.D.D. Without the requirement that the F.D.D. be shrinking the theorem fails even for subspaces. We mentioned already in Section 1.d that $\{e_n - (e_{2n} + e_{2n+1})/2\}_{n=1}^\infty$ is a basic sequence in l_1 whose span is not isomorphic to a conjugate space and thus is not of the form $\left(\sum_{n=1}^\infty \oplus D_n \right)_1$.

2. It is unknown whether the assumption of the existence of a F.D.D. can be replaced by the weaker assumption that X has the approximation property. This is a special instance of Problem 1.e.12.

Theorem 2.d.1 does not answer of course all the natural questions on subspaces of l_p, $1 < p < \infty$ or c_0, even on those subspaces which have a basis. For example, the answer to the following problem is unknown even under the assumption of the existence of a basis.

Problem 2.d.2. *Let X be an infinite-dimensional Banach space and let $1 < p < \infty$ be such that X is a subspace, as well as a quotient space, of l_p. Is X isomorphic to l_p?*

We pass now to the construction of subspaces of c_0 and l_p, $2 < p < \infty$ which do not have the approximation property. We reproduce here the proof due to A. M.

Davie [20, 21]. Our approach will be to construct an infinite matrix of the type appearing in 1.e.8 (this is the main step) and then deduce from the existence of such a matrix the existence of the desired subspaces.

Theorem 2.d.3. *There exists an infinite matrix $A=(a_{i,j})_{i,j=1}^{\infty}$ of scalars such that for every i, $a_{i,j}\neq 0$ only for finitely many indices j, $\sum_{i=1}^{\infty}(\max_j |a_{i,j}|)^r<\infty$ for every $r>2/3$, $A^2=0$ and* trace $A=\sum_{i=1}^{\infty} a_{i,i}\neq 0$.

Proof. We shall work with complex scalars since this is somewhat more convenient.

For every $k=0, 1, 2,\dots$ we let U_k be a unitary matrix of order $3\cdot 2^k$ (the specific choice of U_k will be made later). We partition U_k as follows $\begin{pmatrix} 2^{(k+1)/2}P_k \\ 2^{k/2}Q_k \end{pmatrix}=U_k$, where $2^{(k+1)/2}P_k$ is the $2^{k+1}\times 3\cdot 2^k$ matrix consisting of the first 2^{k+1} rows of U_k and $2^{k/2}Q_k$ is the $2^k\times 3\cdot 2^k$ matrix consisting of the last 2^k rows of U_k. Since $U_kU_k^*=I_{3\cdot 2^k}$ (where I_m denotes the identity matrix of order m) we get that

$$P_kP_k^*=2^{-(k+1)}I_{2^{k+1}},\qquad Q_kQ_k^*=2^{-k}I_{2^k},\qquad P_kQ_k^*=Q_kP_k^*=0,$$
$$k=0, 1, 2,\dots.$$

Consider now the matrix

$$A=(a_{i,j})$$
$$=\begin{pmatrix} P_0^*P_0 & P_0^*Q_1 & 0 & 0 & 0 & \dots \\ -Q_1^*P_0 & P_1^*P_1-Q_1^*Q_1 & P_1^*Q_2 & 0 & 0 & \dots \\ 0 & -Q_2^*P_1 & P_2^*P_2-Q_2^*Q_2 & P_2^*Q_3 & 0 & \dots \\ 0 & 0 & -Q_3^*P_2 & P_3^*P_3-Q_3^*Q_3 & P_3^*Q_4 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

It is easily verified that $A^2=0$. Clearly, trace $(P_k^*P_k-Q_k^*Q_k)=0$ for $k=1, 2,\dots$ and thus trace $A=$ trace $P_0^*P_0=1$. We shall construct the U_k's so that each element in the k'th block of rows in A (i.e. each element of the matrices $-Q_k^*P_{k-1}$, $P_k^*P_k-Q_k^*Q_k$ and $P_k^*Q_{k+1}$) is $\leq C(k+1)^{1/2}2^{-3k/2}$, where C is a constant. Since the k'th block contains $3\cdot 2^k$ rows this will imply that, for $r>2/3$,

$$\sum_{i=1}^{\infty}\left(\max_j |a_{i,j}|\right)^r\leq \sum_{k=0}^{\infty} 3\cdot 2^kC^r(k+1)^{r/2}2^{-3rk/2}<\infty.$$

Observe that, since $P_k^*Q_{k+1}=(Q_{k+1}^*P_k)^*$, it is enough to examine only the matrices $Q_k^*P_{k-1}$ and $P_k^*P_k-Q_k^*Q_k$. For the construction of the $\{U_k\}_{k=0}^{\infty}$ we need two lemmas.

Lemma 2.d.4. (a) *Let $\{\alpha_j\}_{j=1}^{n}$ be complex numbers and let $\{\theta_j\}_{j=1}^{n}$ be independent* *random variables each taking the values* $+1$ *and* -1 *with probability* $1/2$. *Then there is an absolute constant* K *so that*

$$\text{Probability}\left\{\left|\sum_{j=1}^{n} \theta_j\alpha_j\right| > K\left(\sum_{j=1}^{n} |\alpha_j|^2 \log n\right)^{1/2}\right\} < Kn^{-3}.$$

(b) *The same assertion as* (a) *with the only difference that each* θ_j *takes the value* 2 *with probability* $1/3$ *and* -1 *with probability* $2/3$.

Proof. We shall prove assertion (a) only; the proof of (b) is very similar. By considering separately the real and the imaginary parts it follows that it is enough to prove (a) for real $\{\alpha_j\}_{j=1}^n$. Also, there is no loss of generality to assume that $\sum_{j=1}^{n} \alpha_j^2 = 1$. We write $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ and $f(\theta) = \left|\sum_{j=1}^{n} \theta_j\alpha_j\right|$. We denote by E the expectation with respect to the probability distribution of θ (i.e. the average over all 2^n possible choices of θ). Then, for every $\lambda > 0$,

$$E(e^{\lambda f(\theta)}) \leq E(e^{\lambda \sum_j \theta_j\alpha_j} + e^{-\lambda \sum_j \theta_j\alpha_j}) = 2\prod_{j=1}^{n} (e^{\lambda\alpha_j} + e^{-\lambda\alpha_j})/2\,.$$

Since $(e^x + e^{-x})/2 \leq e^{x^2}$ for real x we get that

$$\text{Probability}\,\{\lambda f(\theta) - \lambda^2 - 3\log n > 0\} \leq E(e^{\lambda f(\theta) - \lambda^2 - 3\log n}) \leq 2n^{-3}\,.$$

The desired result follows by taking, e.g. $\lambda = (3\log n)^{1/2}$. □

Lemma 2.d.5. *Let* G *be an Abelian group of order* $3\cdot 2^k$ *for some integer* k. *Then it is possible to divide the characters of* G *into two sets: one consisting of* 2^{k+1} *elements, denoted by* $\{\tau_j\}_{j=1}^{2^{k+1}}$, *and another consisting of* 2^k *elements, denoted by* $\{\sigma_j\}_{j=1}^{2^k}$, *so that for every* $g \in G$

$$\left|2\sum_{j=1}^{2^k} \sigma_j(g) - \sum_{j=1}^{2^{k+1}} \tau_j(g)\right| \leq L(k+1)^{1/2}2^{k/2}$$

where L *is an absolute constant.*

Recall that a character γ of an Abelian group G is a homomorphism from G into the multiplicative group $\{z;\, |z| = 1\}$ in the plane. An Abelian group of order m has exactly m characters and any two different characters are orthogonal, i.e. $\sum_{g\in G} \gamma(g)\overline{\gamma'(g)} = 0$ if $\gamma \neq \gamma'$. For our purpose it is enough to consider a cyclic group G of order m. In this case the characters of G are given by $\gamma_k(g_0^l) = e^{-2\pi ikl/m}$, $0 \leq k < m$, $0 \leq l < m$, where g_0 is a generator of G.

Proof. Let $\{\gamma_j\}_{j=1}^{3\cdot 2^k}$ be an enumeration of all the characters of G. By 2.d.4(b) there is a choice of $\{\theta_j\}_{j=1}^{3\cdot 2^k}$ such that θ_j is either 2 or -1, for every j, and so that, for some constant L,

$$\left|\sum_{j=1}^{3\cdot 2^k} \theta_j \gamma_j(g)\right| \leqslant L(k+1)^{1/2} 2^{k/2}, \quad \text{for every } g \in G\,.$$

The number of these inequalities is $n=3\cdot 2^k$ and thus such $\{\theta_j\}_{j=1}^n$ exist, by 2.d.4(b), whenever $n<n^3/K$, i.e. for $n>n_0=\sqrt{K}$. (For $n<n_0$ there is nothing to prove.) By taking, in particular, g to be the identity element of G we get that $\left|\sum_{j=1}^{3\cdot 2^k} \theta_j\right| \leqslant L(k+1)^{1/2}2^{k/2}$. Thus, by changing if necessary at most $2L(k+1)^{1/2}2^{k/2}$ of the θ_j and replacing L by $2L$, we may assume that $\sum_{j=1}^{3\cdot 2^k} \theta_j=0$, i.e. exactly 2^{k+1} of the θ_j are equal to -1 and 2^k of the θ_j are equal to 2. Those γ_j for which the corresponding θ_j are equal to -1 are denoted by $\{\tau_j\}_{j=1}^{2^{k+1}}$ and the rest by $\{\sigma_j\}_{j=1}^{2^k}$. From the choice of $\{\theta_j\}_{j=1}^{3\cdot 2^k}$ it is clear that this partition of the characters into two sets has the required properties. □

We return to the proof of 2.d.3. We make the set $\{1, 2, \ldots, 3\cdot 2^k\}$ into an Abelian group G_k and define the τ_j^k and σ_j^k as in 2.d.5. We let the rows of P_k be $3^{-1/2}2^{-(2k+1)/2}\tau_j^k$, $1\leqslant j\leqslant 2^{k+1}$ (i.e. the entries of P_k are $3^{-1/2}2^{-(2k+1)/2}\tau_j^k(i)$, where $i\in G_k$, i.e. $1\leqslant i\leqslant 3\cdot 2^k$) and let the rows of Q_k be $3^{-1/2}2^{-k}\theta_j^k\sigma_j^k$, $1\leqslant j\leqslant 2^k$, where the θ_j^k are either $+1$ or -1 and will be determined below. Whatever is the choice of $\{\theta_j^k\}_{j=1}^{2^k}$ the matrix U_k defined in this manner is unitary (this follows from the orthogonality property of the characters).

We now write down explicitly the elements of $Q_k^*P_{k-1}$ and $P_k^*P_k-Q_k^*Q_k$. They are

$$(1)\qquad 3^{-1}\cdot 2^{1/2-2k}\sum_{j=1}^{2^k} \theta_j^k\overline{\sigma_j^k(h)}\tau_j^{k-1}(g), \quad h\in G_k,\ g\in G_{k-1}$$

$$(2)\qquad 3^{-1}\cdot 2^{-2k}\left(\frac{1}{2}\sum_{j=1}^{2^{k+1}} \tau_j^k(h)-\sum_{j=1}^{2^k}\sigma_j^k(h)\right), \quad h\in G_k\,.$$

In the derivation of (2) we used the fact that the τ_j^k and σ_j^k are characters, i.e. that $\tau_j^k(h_1\circ h_2)=\tau_j^k(h_1)\tau_j^k(h_2)$, where $h_1\circ h_2$ denotes the multiplication in G_k. We have to show that, for suitable choices of $\{\theta_j^k\}_{j=1}^{2^k}$, $k=0, 1, 2, \ldots$ all these terms are in absolute value $\leqslant C(k+1)^{1/2}2^{-3k/2}$. The expressions appearing in (2) are independent of θ_j^k and they are of the right order of magnitude by Lemma 2.d.5. Observe that the number of terms written in (1) is $3\cdot 2^{k-1}\cdot 3\cdot 2^k\leqslant 5\cdot n^2$, where $n=2^k$. Hence, by Lemma 2.d.4(a), we infer that, whenever $5\cdot 2^{2k}<2^{3k}/K$, the $\{\theta_j^k\}_{j=1}^{2^k}$ can be chosen so that all terms in (1) are in absolute value of the required order of magnitude (for the finitely many k for which $5\cdot 2^{2k}>2^{3k}/K$ there is of course nothing to prove). This concludes the construction of $\{U_k\}_{k=1}^\infty$ and thus also the proof of 2.d.3. □

The matrix A, constructed in 2.d.3, satisfies in particular $\sum_i \max_j |a_{i,j}|<\infty$. Thus, by 1.e.8, the rows of A span a subspace of c_0 which does not have the A.P. The fact that the matrix of 2.d.3 has a stronger property enables us to show, by the same argument which was used in the proof of 1.e.8, that also the spaces l_p, for $2<p<\infty$ have subspaces which fail to have the A.P.

Theorem 2.d.6. *The spaces c_0 and l_p, $2<p<\infty$, have subspaces which do not have the approximation property.*

Proof. We have to prove the theorem only in the case of l_p. Let A be a matrix satisfying the requirements of 2.d.3 (the assumption that for every i only finitely many j satisfy $a_{i,j}\neq 0$ will not be used here). Put $\lambda_i=\max_j |a_{i,j}|$, $1\leq i<\infty$ and $b_{i,j}=(\lambda_j/\lambda_i)^{1/(p+1)}a_{i,j}$, $1\leq i,j<\infty$. The matrix $B=(b_{i,j})_{i,j=1}^\infty$ satisfies $B^2=0$ and trace $B=$trace $A\neq 0$. Let $y_i=(b_{i,1}, b_{i,2},\ldots)$. Then, since $p/(p+1)>2/3$, $y_i\in l_p$ and

$$\|y_i\|=\left(\sum_{j=1}^\infty |b_{i,j}|^p\right)^{1/p}\leq \lambda_i^{p/(p+1)}\left(\sum_{j=1}^\infty \lambda_j^{p/(p+1)}\right)^{1/p}\leq L\lambda_i^{p/(p+1)},$$

for some constant L. Consequently, $\sum_{i=1}^\infty \|y_i\|<\infty$. Denote by $\{e_i\}_{i=1}^\infty$ the unit unit vector basis of l_q. For every y in $[y_i]_{i=1}^\infty\subset l_p$ we have $\sum_{i=1}^\infty y_ie_i(y)=0$, but $\sum_{i=1}^\infty e_i(y_i)=$trace $B\neq 0$. Hence, by 1.e.4, the space $[y_i]_{i=1}^\infty$ does not have the A.P. $\square$

Remarks. 1. Since every subspace of l_2 has the A.P. the argument used in the preceding proof shows that if $A=(a_{i,j})_{i,j=1}^\infty$ is a matrix such that $A^2=0$ and $\sum_{i=1}^\infty (\sup_j |a_{i,j}|)^{2/3}<\infty$ then trace $A=0$. Thus, the exponent 2/3 appearing in 2.d.3, is the smallest possible one.

2. In vol II we shall present a proof of the fact that the spaces l_p for $1\leq p<2$ also have subspaces which fail to have the A.P.

3. In 1.e.8 we presented a result of Grothendieck which shows that the approximation problem has also an equivalent formulation in terms of continuous functions $K(s,t)$ on $[0,1]\times[0,1]$. We can now give a precise answer also to this problem. By using the proof of (iii) $\Rightarrow$ (ii) of 1.e.8 and the matrix given in 2.d.3 we obtain a function $K(s,t)$ on $[0,1]\times[0,1]$ which satisfies a Lipschitz condition of every order $<1/2$ and so that $\int_0^1 K(s,t)K(t,u)\,dt=0$, for every s and u, while $\int_0^1 K(t,t)\,dt\neq 0$. On the other hand it can be shown that there is no function which satisfies a Lipschitz condition of order 1/2 and has these properties.

It is not easy to construct an infinite dimensional subspace X of l_p, for $2<p<\infty$, which is not isomorphic to l_p. In fact, prior to the work of Davie [20] and Figiel [40] who exhibited such X which fail to have the A.P., no such examples were known. Once we know of the existence of such X it is possible to show that there are "many" such examples. We show first that, for $2<p<\infty$ (and, in fact, for every

$1 \leqslant p < \infty, p \neq 2$), there is an infinite dimensional subspace X of l_p so that X has the A.P. but it is not isomorphic to l_p.

Proposition 2.d.7. (a) *Let $1<p<\infty$ and let Y be a Banach space which is not isomorphic to a complemented subspace of $L_p(0,1)$. Assume that $Y=\overline{\bigcup_{n=1}^{\infty} B_n}$ with $B_n \subset B_{n+1}$ and* dim $B_n<\infty$ *for every n. Then, $\left(\sum_{n=1}^{\infty} \oplus B_n\right)_p$ is not isomorphic to l_p.*

(b) *For every $1 \leqslant p < \infty, p \neq 2$ there is a subspace of l_p which is isomorphic to $\left(\sum_{n=1}^{\infty} \oplus B_n\right)_p$, for suitable finite dimensional $\{B_n\}_{n=1}^{\infty}$, and which is not isomorphic to l_p itself.*

Outline of proof. The proof uses arguments whose natural setting is the local theory of Banach spaces. These arguments will be discussed in detail in Vol. IV; here we only outline them. If $\left(\sum_{n=1}^{\infty} \oplus B_n\right)_p$ is isomorphic to l_p then there is a λ and operators $U_n: B_n \to l_p$, $V_n: l_p \to B_n$, $n=1,2,\ldots$ so that $V_n U_n = I_{B_n}$ and $\|U_n\|\,\|V_n\| \leqslant \lambda$, for every n. Using this fact and a compactness argument we get that Y must be isomorphic to a complemented subspace of $L_p(0,1)$, contradicting the assumption in (a).

To prove part (b), let first $2<p<\infty$ and let Y be a subspace of l_p which does not have the A.P. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence which spans Y and put $B_n=[y_i]_{i=1}^n$, $n=1,2,\ldots$. It is easily seen that $\left(\sum_{n=1}^{\infty} \oplus B_n\right)_p$ is isomorphic to a subspace of l_p which, by part (a), is not isomorphic to l_p itself.

For $1 \leqslant p < 2$ we take $p<r<2$ and apply part (a) to $Y=l_r$. It will be shown in Vol. II that $\left(\sum_{n=1}^{\infty} \oplus l_r^n\right)_p$ is isomorphic to a subspace of l_p but that l_r is not isomorphic to a complemented subspace of $L_p(0,1)$. Thus, $\left(\sum_{n=1}^{\infty} \oplus l_r^n\right)_p$ is not isomorphic to l_p. $\square$

The examples constructed for $1 \leqslant p < 2$ have an unconditional basis. Also for $2<p<\infty$, l_p has a subspace with an unconditional basis which is not isomorphic to l_p itself. This is a consequence of Szankowski's result [138] that a certain lattice does not have the A.P. We shall discuss this fact in detail in Vol. IV.

We conclude this section with some results involving universal spaces. These results show that there are "many" spaces which fail to have the A.P. even if we consider only subspaces of l_p, for some fixed $2<p<\infty$. It is well known that $C(0,1)$ is a universal Banach space in the sense that every separable Banach space Y is isometric to a subspace of $C(0,1)$. In general, Y is not isomorphic to a complemented subspace of $C(0,1)$. The question whether there is a separable Banach space X which contains isomorphic copies of all separable spaces as complemented subspaces turns out to be closely related to the approximation property. The following two theorems answer this question.

Theorem 2.d.8 [118, 66]. *There is a Banach space X having a basis such that every* *separable Banach space having the* B.A.P. *is isomorphic to a complemented subspace of* X.

Theorem 2.d.9 [62]. *There is no separable Banach space X so that every separable Banach space Y is isomorphic to a complemented subspace of X. Moreover, such an X fails to exist even if we consider only those separable spaces Y which have the* A.P. *or, alternatively, all the subspaces Y of l_p which fail to have the* A.P. *for any fixed* p, $2<p<\infty$.

Since for every sequence $\{Y_n\}_{n=1}^\infty$ of separable Banach spaces there is trivially a separable Banach space X which contains all the Y_n as complemented subspaces $\left(\text{e.g. } X=\left(\sum_{n=1}^\infty \oplus Y_n\right)_p\right)$, the last statement in 2.d.9 asserts in particular that, for every $2<p<\infty$, there are uncountably many mutually non-isomorphic subspaces of l_p all of which fail to have the A.P.

We prove first Theorem 2.d.8. This theorem is an immediate consequence of Theorem 1.e.13 and the second assertion of the following theorem due essentially to Pelczynski [117].

Theorem 2.d.10. (a) *There exists a separable Banach space U_1 having an unconditional basis $\{x_i\}_{i=1}^\infty$ such that every unconditional basic sequence (in an arbitrary separable Banach space) is equivalent to a subsequence of $\{x_i\}_{i=1}^\infty$.*

(b) *There exists a separable Banach space U_2 having a Schauder basis $\{x_n\}_{n=1}^\infty$ such that, for any basic sequence $\{y_k\}_{k=1}^\infty$, there is a subsequence $\{n_k\}_{k=1}^\infty$ of the integers such that $\{y_k\}_{k=1}^\infty$ is equivalent to $\{x_{n_k}\}_{k=1}^\infty$ and the natural projection P on $[x_{n_k}]_{k=1}^\infty$ (defined by $Px_{n_k}=x_{n_k}$, $k=1, 2,\ldots$ and $Px_n=0$ if $n\notin\{n_k\}_{k=1}^\infty$) is bounded.*

The spaces U_1 and U_2 are determined uniquely, up to isomorphism, by the properties appearing in (a), *resp.* (b).

Proof. We shall present a proof due to Schechtman [131] which is considerably shorter than the original proof. We start with the construction of U_1. Let $\{u_n\}_{n=1}^\infty$ be a sequence which is dense in $C(0, 1)$. We introduce a norm $|||\cdot|||_1$ in the space U_0 of all sequences $x=(a_1, a_2, a_3,\ldots)$ of scalars which are eventually 0, by putting,

$$|||x|||_1=\sup\left\{\left\|\sum_{n=1}^\infty \theta_n a_n u_n\right\|;\ \theta_n=\pm 1,\ n=1, 2,\ldots\right\}.$$

Let $x_n=(0, 0,\ldots, \overset{n}{1}, 0,\ldots)$, $n=1,\ldots$. The sequence $\{x_n\}_{n=1}^\infty$ is an unconditional basis of the completion U_1 of U_0 with respect to $|||\cdot|||_1$. Let $\{y_k\}_{k=1}^\infty$ be any unconditional basis of a separable Banach space. By the universality of $C(0, 1)$ we may assume that $y_k\in C(0, 1)$, for every k. Let K be the unconditional constant of $\{y_k\}_{k=1}^\infty$. Choose integers $\{n_k\}_{k=1}^\infty$ so that $\|y_k-u_{n_k}\|\leqslant\|y_k\|/K\cdot 2^{k+2}$ for every k. By 1.a.9 the sequence $\{u_{n_k}\}_{k=1}^\infty$ is an unconditional basic sequence equivalent to $\{y_k\}_{k=1}^\infty$. From the definition of $|||\cdot|||_1$ it follows that $\{x_{n_k}\}_{k=1}^\infty$ is equivalent to $\{u_{n_k}\}_{k=1}^\infty$ and thus to $\{y_k\}_{k=1}^\infty$.

We pass now to the proof of the existence of U_2. Without the requirement of the existence of P we could have proceeded in a way very similar to that used in the preceding case. In the case of an unconditional basis the natural projection is of course defined and bounded for every subsequence $\{n_k\}_{k=1}^\infty$. The main point in our construction now will be that, in spite of the fact that we deal with a conditional basis, the natural projection on a large collection of subsequences of the integers (called "branches") will be bounded.

Let $\{u_n\}_{n=1}^\infty$ be a sequence which is dense in $C(0, 1)$. Let φ be a one to one function from the set of all finite sequences of positive integers onto the positive integers so that $\varphi(i_1, i_2, \ldots, i_k) < \varphi(i_1, i_2, \ldots, i_k, i_{k+1})$ for every choice of k and $\{i_j\}_{j=1}^{k+1}$. A subsequence of the integers is called a branch if it has the form

$$\varphi(i_1),\ \varphi(i_1, i_2),\ \varphi(i_1, i_2, i_3), \ldots,$$

for a suitable choice of integers $\{i_j\}_{j=1}^\infty$. The set of all branches will be denoted by $\mathscr{B}$. For $n=\varphi(i_1, i_2, \ldots, i_k)$ we put $v_n = u_{i_k}$. We define now a norm $|||\cdot|||_2$ on the space U_0 of sequences of scalars $x=(a_1, a_2, a_3, \ldots)$ which are eventually 0, by putting,

$$|||x|||_2 = \sup\left\{\left\|\sum_{j=1}^n \chi_B(j) a_j v_j\right\|;\ n=1, 2, \ldots,\ B \in \mathscr{B}\right\}$$

where $\chi_B(j)$ is 1 if $j \in B$ and is 0 if $j \notin B$. Let $x_n = (0, 0, \ldots, 0, \overset{n}{1}, 0, \ldots)$, $n=1, 2, \ldots$. Then $\{x_n\}_{n=1}^\infty$ forms a basis of the completion U_2 of U_0 with respect to $|||\cdot|||_2$. It is clear that if $B = \{n_k\}_{k=1}^\infty \in \mathscr{B}$ then the natural projection from U_2 onto $[x_{n_k}]_{k=1}^\infty$ is bounded (in fact, it has norm 1).

Let $\{y_k\}_{k=1}^\infty$ be a basic sequence in $C(0, 1)$ and let K be its basis constant. Choose integers $\{i_k\}_{k=1}^\infty$ so that $\|u_{i_k} - y_k\| \leq \|y_k\|/2^{k+2}K$, and let $n_k = \varphi(i_1, i_2, \ldots, i_k)$, $k=1,2,\ldots$. Then, $\{n_k\}_{k=1}^\infty \in \mathscr{B}$ and, by 1.a.9, $\{y_k\}_{k=1}^\infty$ is equivalent to $\{u_{i_k}\}_{k=1}^\infty$, i.e. to $\{v_{n_k}\}_{k=1}^\infty$. It follows immediately from the definition of $|||\cdot|||_2$ and the fact that the intersection of two different branches is an initial segment of these branches that $\{x_{n_k}\}_{k=1}^\infty$ is equivalent to the basic sequence $\{v_{n_k}\}_{k=1}^\infty$ and thus to $\{y_k\}_{k=1}^\infty$. This proves that U_2 has the desired properties.

We prove now the uniqueness of U_1 and U_2. We actually prove the uniqueness in the following stronger sense: every separable Banach space which has an unconditional basis (respectively, a basis) and which contains isomorphic copies of all separable spaces with an unconditional basis (resp. with a basis) as complemented subspaces must be isomorphic to U_1 (resp. U_2). We present the proof in the case of U_1 (the proof for U_2 is identical). Let V be another space which has the above mentioned property of U_1. Then, there exist Banach spaces X and Y such that $V \approx U_1 \oplus X$ and $U_1 \approx V \oplus Y$. Also, there is a Banach space Z so that $U_1 \approx (U_1 \oplus U_1 \oplus \cdots)_2 \oplus Z$ and thus,

$$U_1 \oplus U_1 \approx U_1 \oplus (U_1 \oplus U_1 \oplus \cdots)_2 \oplus Z \approx (U_1 \oplus U_1 \oplus \cdots)_2 \oplus Z \approx U_1\,.$$

Similarly, $V \oplus V \approx V$. Thus, we get that

$$U_1 \approx V \oplus Y \approx V \oplus V \oplus Y \approx V \oplus U_1 \approx U_1 \oplus U_1 \oplus X \approx U_1 \oplus X \approx V. \quad \square$$

Remarks. 1. Some further interesting properties of the space U_1 will be presented in Section 3.b.

2. In connection with 2.d.10 (a) it is worthwhile to mention the following result of Schechtman [131] concerning unconditional basic sequences in l_p. *Let* $1<p<\infty$; *then, there exists an unconditional basic sequence* $\{x_n\}_{n=1}^\infty$ *in* l_p *so that every unconditional basic sequence in* l_p *is equivalent to a subsequence* $\{x_{n_k}\}_{k=1}^\infty$ *of* $\{x_n\}_{n=1}^\infty$. The proof of this fact is based on 2.d.1.

We shall now give a partial *proof of* 2.d.9. We have seen that for every $2<p<\infty$ there is a subspace Y_p of l_p which does not have the A.P. Actually, the spaces obtained in Davie's construction fail to have an apparently weaker property namely, the *compact approximation property* (C.A.P. in short). A Banach space Y is said to have the C.A.P. if the identity operator on Y is in the closure of the set of compact operators from Y into itself with respect to the topology τ of uniform convergence on compact subsets of Y. That the spaces Y_p fail to have the C.A.P. is evident from the argument in [20] but it is not as apparent from the approach presented in this section via infinite matrices. In [20] Davie defines a sequence of τ continuous linear functionals $\beta_n(T)$, $n=1,\dots$ so that $\beta(T)=\lim_n \beta_n(T)$ exists for every T and is a τ continuous linear functional. The $\{\beta_n\}_{n=1}^\infty$ satisfy $\beta_n(I)=1$ for every n while $\lim_n \beta_n(T)=0$ for every compact operator $T\colon Y_p\to Y_p$. Thus, β is a τ continuous functional which vanishes on the compact operators and is equal to 1 on the identity operator.

Assume now that X is a separable space such that, for every $2<p<\infty$, there is a subspace Z_p of X which is isomorphic to Y_p and so that there is a bounded linear projection Q_p from X onto Z_p. There is an uncountable subset A of $(2,\infty)$ and a $\lambda<\infty$ so that $\|Q_p\|\leqslant\lambda$ for every $p\in A$.

Since each Z_p does not have the C.A.P. it follows that there are finite sets $\{z_{i,p}\}_{i=1}^{n(p)}$ of unit vectors in Z_p and an $\varepsilon_p>0$ so that, whenever T is a compact operator on Z_p for which $\|z_{i,p}-Tz_{i,p}\|<\varepsilon_p$ for $1\leqslant i\leqslant n(p)$, then $\|T\|>\lambda^2$. Let B be an uncountable subset of A so that $n(p)$ is constant (say $=n$) on B and $\inf_{p\in B}\varepsilon_p=\varepsilon>0$. Since B is uncountable and X is separable there exist $p<r$ in B so that $\|z_{i,p}-z_{i,r}\|<(\lambda+\lambda^2)^{-1}\varepsilon$ for $1\leqslant i\leqslant n$.

The proof of 2.c.3 shows also that every operator from a subspace of l_r into l_p is compact. Thus, every operator from Z_r to Z_p is compact. In particular, $T=Q_rQ_{p|Z_r}$ is a compact operator from Z_r into itself with $\|T\|\leqslant\lambda^2$. Note that, for $1\leqslant i\leqslant n$,

$$\begin{aligned}\|z_{i,r}-Tz_{i,r}\|&=\|z_{i,r}-Q_rQ_pz_{i,r}\|\leqslant\|z_{i,r}-Q_rz_{i,p}\|+\|Q_rz_{i,p}-Q_rQ_pz_{i,r}\|\\&=\|Q_r(z_{i,r}-z_{i,p})\|+\|Q_rQ_p(z_{i,p}-z_{i,r})\|\\&\leqslant(\lambda+\lambda^2)\|z_{i,r}-z_{i,p}\|<\varepsilon,\end{aligned}$$

but this contradicts the choice of $\{z_{i,r}\}_{i=1}^n$. This proves the first assertion in the statement of 2.d.9. The proofs of the other two statements of 2.d.9 are more complicated and we do not reproduce them here (we refer the reader to [62]). We just remark that the proof of the fact that there is no separable space which contains isomorphic copies of all separable spaces having the A.P. as complemented subspaces is based on the construction presented in 1.e.20 of a space which has the A.P. but fails to have the B.A.P. The proof of the fact that there is no separable space which contains isomorphic copies of all subspaces of l_p (for a fixed $2<p<\infty$) which fail to have the A.P., as complemented subspaces is obtained by modifying Davie's construction. (Instead of constructing one space the same method is used to construct a suitable uncountable family of spaces.)

e. Banach Spaces Containing l_p or c_0

A long standing open problem going back to Banach's book was the following: *Does every infinite-dimensional Banach space have a subspace isomorphic to either c_0 or l_p, for some $1\leqslant p<\infty$?* For the common examples of Banach spaces it was easy to show that the answer is positive while, in general, the problem seemed to be quite difficult. Therefore, it came as a surprise when a rather simple counterexample was constructed by B. S. Tsirelson [140]. Tsirelson constructed an example of a reflexive space with an unconditional basis which contains no isomorphic copy of any l_p space, $1<p<\infty$ (actually, his example contains no uniformly convexifiable subspace). We shall present here the dual of Tsirelson's original example which also solves the question stated above. In our presentation we follow T. Figiel and W. B. Johnson [42].

Example 2.e.1. *There is a reflexive Banach space T with an unconditional basis which contains no isomorphic copy of any l_p space, $1\leqslant p<\infty$.*

Proof. We start by defining a sequence of norms on T_0, the space of all sequences of scalars which are eventually zero. We denote by $\{t_n\}_{n=1}^\infty$ the unit vector basis of T_0 and set, for $x=\sum_{n=1}^\infty a_n t_n \in T_0$, $||x||_0=\max_n |a_n|$, and for $m\geqslant 0$

$$||x||_{m+1}=\max\left\{||x||_m,\ 2^{-1}\max\sum_{j=1}^k\left\|\sum_{n=p_j+1}^{p_{j+1}} a_n t_n\right\|_m\right\},$$

where the inner max is taken over all choices of $k\leqslant p_1<p_2<\cdots<p_{k+1}$, $k=1, 2,\ldots$. Obviously, $||x||=\lim_{n\to\infty} ||x||_m$ exists for all $x\in T_0$ and defines a norm on T_0. The unit vectors $\{t_n\}_{n=1}^\infty$ form a normalized unconditional basis of the completion T of T_0. It is also clear that

$$||x||=\max\left\{\max_n |a_n|,\ 2^{-1}\sup\left(\sum_{j=1}^k\left\|\sum_{n=p_j+1}^{p_{j+1}} a_n t_n\right\|,\right.\right.$$
$$\left.\left. k\leqslant p_1<p_2<\cdots<p_{k+1},\ k=1, 2,\ldots\right)\right\},$$

for every $x=\sum_{n=1}^{\infty} a_n t_n \in T$. Consequently, for any k and any sequence of k normalized blocks $u_j=\sum_{n=p_j+1}^{p_{j+1}} a_n t_n$, $1 \leqslant j \leqslant k$ with $k \leqslant p_1 < p_2 < \cdots < p_{k+1}$, we have

$$(*)\qquad \sum_{j=1}^{k} |c_j| \geqslant \left\| \sum_{j=1}^{k} c_j u_j \right\| \geqslant 2^{-1} \sum_{j=1}^{k} |c_j| ,$$

for every choice of scalars $\{c_j\}_{j=1}^{k}$. This fact and 1.a.12 show that T contains no subspace isomorphic to c_0 or to some l_p, $1<p<\infty$.

To show that T contains no subspace isomorphic to l_1 is more difficult. To this end we have to apply a result of R. C. James (cf. Proposition 2.e.3 below) which ensures that if T has a subspace isomorphic to l_1 then there is a normalized block basis $\{v_j\}_{j=0}^{\infty}$ of $\{t_n\}_{n=1}^{\infty}$ so that, for every choice of scalars $\{b_j\}_{j=0}^{\infty}$,

$$\sum_{j=0}^{\infty} |b_j| \geqslant \left\| \sum_{j=0}^{\infty} b_j v_j \right\| \geqslant (8/9) \sum_{j=0}^{\infty} |b_j| ,$$

and, in particular,

$$(\overset{*}{*})\qquad \|v_0+r^{-1}(v_1+v_2+\cdots+v_r)\| \geqslant 16/9, \quad r=1,2,\ldots .$$

Consider now integers $k \leqslant p_1 < p_2 < \cdots < p_{k+1}$ and let $\{P_j\}_{j=1}^{k}$ be the projections associated to the basis $\{t_n\}_{n=1}^{\infty}$ so that $P_j t_n = t_n$ if $p_j < n \leqslant p_{j+1}$ and $P_j t_n = 0$, otherwise. Let n_0 be the largest integer for which t_n belongs to the support of v_0. If $k \geqslant n_0$ then

$$\sum_{j=1}^{k} \|P_j(v_0+r^{-1}(v_1+\cdots+v_r))\| = \sum_{j=1}^{k} \|P_j r^{-1}(v_1+\cdots+v_r)\| \leqslant 2 .$$

If $k<n_0$ we set

$$\delta=\{i;\ \|P_j v_i\| \neq 0 \text{ for at least two values of } j\} ,$$
$$\sigma=\{i;\ \|P_j v_i\| \neq 0 \text{ for at most one value of } j\} .$$

Then, since δ has at most $k-1$ elements, we get that

$$\begin{aligned}
&\sum_{j=1}^{k} \|P_j(v_0+r^{-1}(v_1+\cdots+v_r))\| \\
&\leqslant \sum_{j=1}^{k} \|P_j v_0\| + r^{-1}\Big(\sum_{i\in\delta} \sum_{j=1}^{k} \|P_j v_i\| + \sum_{i\in\sigma} \sum_{j=1}^{k} \|P_j v_i\| \Big) \\
&\leqslant 2\|v_0\| + r^{-1}\Big(2 \sum_{i\in\delta} \|v_i\| + \sum_{i\in\sigma} \|v_i\|\Big) \\
&\leqslant 2+r^{-1}(2(k-1)+r-k+1) \\
&\leqslant 3+(k-1)r^{-1} \leqslant 3+(n_0-1)r^{-1} .
\end{aligned}$$

Therefore, by taking $r \geqslant 2n_0$, we get that

$$\sum_{j=1}^{k} \|P_j(v_0 + r^{-1}(v_1 + \cdots + v_r))\| \leqslant 7/2 .$$

It follows from the definition of the norm in T that $\|v_0 + r^{-1}(v_1 + \cdots + v_r)\| \leqslant 7/4$ and this contradicts $(^*_*)$.

Since T has an unconditional basis but does not contain subspaces isomorphic to c_0 or l_1 we deduce from 1.c.12 that T is reflexive. $\square$

Remark. The relation (*) actually shows that T has no uniformly convexifiable infinite-dimensional subspace. Figiel and Johnson show in [42] how to modify T so as to obtain an example of a uniformly convex space with an unconditional basis containing no isomorphic copy of any l_p; $1 \leqslant p < \infty$ (we shall discuss this matter in Vol. II).

Closely related to the problem with which we started this section and which was solved by Tsirelson's example is the problem whether a space which contains an isomorphic copy of some l_p must actually contain almost isometric copies of this space. More precisely,

Problem 2.e.2. *Let X be the space l_p for some $1 < p < \infty$, with the usual norm $\|\cdot\|$. Let $|||\cdot|||$ be an equivalent norm on X. Given $\varepsilon > 0$, does there exist a subspace Y of X so that $d((Y, |||\cdot|||), (X, \|\cdot\|)) < 1 + \varepsilon$?*

This problem, which is called for obvious reasons the "distortion problem", is still open. Some partial positive answers to it will be described in Vol. III and Vol. IV. Some negative results concerning the analogous distortion problem in $L_p(0, 1)$, $p \neq 2$, are given in [88]. The most interesting case in 2.e.2 is the case $p = 2$.

Notice that in the statement of 2.e.2 we excluded the cases $X = l_1$ and $X = c_0$. In these cases the answer to the problem is known to be positive. This is a result of R. C. James [56] which has already been used in the proof of 2.e.1.

Proposition 2.e.3. *Let $(X, \|\cdot\|)$ be the space l_1 or c_0 with its usual norm. Let $|||\cdot|||$ be an equivalent norm on X. Then, for every $\varepsilon > 0$, there is a subspace Y of X with $d((Y, |||\cdot|||), (X, \|\cdot\|)) < 1 + \varepsilon$.*

Proof. Assume that $X = l_1$ and that $\alpha |||x||| \leqslant \|x\| \leqslant |||x|||$, for some $\alpha > 0$ and all $x \in X$. Let $\varepsilon > 0$ and let $\{P_n\}_{n=1}^\infty$ be the natural projections induced by the unit vector basis. For every n put $\lambda_n = \sup\{\|x\|;\ |||x||| = 1, P_n x = 0\}$. Clearly, $\lambda_n \downarrow \lambda$, for some $1 \geqslant \lambda \geqslant \alpha$. Let n_0 be such that $\lambda_{n_0} < \lambda(1 + \varepsilon)$. By the definition of the $\{\lambda_n\}_{n=1}^\infty$ there is a block basis $\{y_k\}_{k=1}^\infty$ of the unit vector basis so that, for all k, $|||y_k||| = 1$, $P_{n_0} y_k = 0$ and $\|y_k\| > \lambda/(1 + \varepsilon)$. For every choice of scalars $\{a_k\}_{k=1}^\infty$ we have $P_{n_0}\left(\sum_{k=1}^\infty a_k y_k\right) = 0$ and hence

$$\left|\left|\left|\sum_{k=1}^\infty a_k y_k\right|\right|\right| \geqslant \lambda_{n_0}^{-1} \left\|\sum_{k=1}^\infty a_k y_k\right\| = \lambda_{n_0}^{-1} \sum_{k=1}^\infty |a_k|\, \|y_k\|$$

$$\geqslant \lambda_{n_0}^{-1}(1 + \varepsilon)^{-1} \lambda \sum_{k=1}^\infty |a_k| \geqslant (1 + \varepsilon)^{-2} \sum_{k=1}^\infty |a_k| .$$

On the other hand, by the triangle inequality, $\left|\left|\left|\sum_{k=1}^{\infty} a_k y_k\right|\right|\right| \leqslant \sum_{k=1}^{\infty} |a_k| \, |||y_k||| \leqslant \sum_{k=1}^{\infty} |a_k|$ and thus $d(([y_k]_{k=1}^{\infty}, |||\cdot|||), l_1) \leqslant (1+\varepsilon)^2$.

The proof for c_0 is similar. By replacing the "sup" in the definition of λ_n, by "inf", we get that, for some constant λ, there is a block basis $\{y_k\}_{k=1}^{\infty}$ of the unit vector basis of c_0 with $|||y_k|||=1$, $||y_k||<\lambda(1+\varepsilon)$ for every k, and $\left|\left|\left|\sum_{k=1}^{\infty} a_k y_k\right|\right|\right| \leqslant (1+\varepsilon)^2 \cdot \max_k |a_k|$ for every sequence of scalars $\{a_k\}_{k=1}^{\infty}$ tending to 0. An estimate from below can again be deduced from the triangle inequality. Indeed, assume that $|a_{k_0}| = \max_k |a_k|$; then

$$\left|\left|\left|\sum_{k=1}^{\infty} a_k y_k\right|\right|\right| \geqslant |||2a_{k_0} y_{k_0}||| - \left|\left|\left|\sum_{k=1}^{\infty} a_k y_k - 2a_{k_0} y_{k_0}\right|\right|\right|$$
$$\geqslant 2|a_{k_0}| - (1+\varepsilon)^2 \cdot \max_k |a_k| = (1-2\varepsilon-\varepsilon^2) \max_k |a_k| . \quad \square$$

The preceding proof does not work for $1<p<\infty$ since the triangle inequality cannot be used to get one of the desired inequalities automatically. The only thing which can obviously be done is to choose a block basis $\{y_k\}_{k=1}^{\infty}$ of the unit vector basis so that $|||y_k|||=1$ for every k and $\left|\left|\left|\sum_{k=1}^{\infty} a_k y_k\right|\right|\right| \geqslant \left(\sum_{k=1}^{\infty} |a_k|^p\right)^{1/p} \Big/ (1+\varepsilon)$ for every choice of scalars $\{a_k\}_{k=1}^{\infty}$, and a block basis $\{z_k\}_{k=1}^{\infty}$ of the unit vector basis so that $|||z_k|||=1$ for every k and $\left|\left|\left|\sum_{k=1}^{\infty} a_k z_k\right|\right|\right| \leqslant (1+\varepsilon)\left(\sum_{k=1}^{\infty} |a_k|^p\right)^{1/p}$ for every choice of scalars $\{a_k\}_{k=1}^{\infty}$.

We return to the discussion centered around the question with which we started this section. While there are no known good criteria for a space to contain subspaces isomorphic to l_p, for some $1<p<\infty$, the situation is different for c_0 and especially for l_1. We shall first present a simple characterization for spaces containing c_0 and then present a deep result characterizing spaces containing l_1.

Proposition 2.e.4 [9]. *A Banach space X has a subspace isomorphic to c_0 if and only if there is a sequence $\{x_n\}_{n=1}^{\infty}$ in X so that $\sum_{n=1}^{\infty} |x^*(x_n)|<\infty$ for every $x^* \in X^*$ but $\sum_{n=1}^{\infty} x_n$ fails to converge.*

Proof. The "only if" part is trivial: we simply take as $\{x_n\}_{n=1}^{\infty}$ a basic sequence which is equivalent to the unit vector basis of c_0. To prove the "if" part let $\{x_n\}_{n=1}^{\infty}$ be such that $\sum_{n=1}^{\infty} |x^*(x_n)|<\infty$ for every $x^* \in X^*$ and $\sum_{n=1}^{\infty} x_n$ diverges. It follows from the uniform boundedness principle that there is a constant M so that $\sum_{n=1}^{\infty} |x^*(x_n)| \leqslant M||x^*||$ for every $x^* \in X^*$. Since $\sum_{n=1}^{\infty} x_n$ diverges there is an $\varepsilon>0$ and integers $p_1<q_1<p_2<q_2<\cdots$ so that $\left|\left|\sum_{n=p_k}^{q_k} x_n\right|\right| \geqslant \varepsilon$ for every k. Put $y_k = \sum_{n=p_k}^{q_k} x_n$, $k=1, 2, \ldots$.

Since $\sum_{k=1}^{\infty} |x^*(y_k)| < \infty$ for every $x^* \in X^*$ it follows that $y_k \xrightarrow{w} 0$. By 1.a.12 (and the remark following it) we may assume without loss of generality that $\{y_k\}_{k=1}^{\infty}$ forms a basic sequence with basis constant K (otherwise, pass to a subsequence). For every finite sequence of scalars $\{a_k\}_{k=1}^{m}$ we have $\left\|\sum_{k=1}^{m} a_k y_k\right\| \geq \varepsilon \max_k |a_k|/2K$ and also

$$\left\|\sum_{k=1}^{m} a_k y_k\right\| = \sup\left\{\left|\sum_{k=1}^{\infty} a_k x^*(y_k)\right|;\ \|x^*\| \leq 1\right\} \leq M \max_k |a_k| .$$

Thus, $\{y_k\}_{k=1}^{\infty}$ is equivalent to the unit vector basis of c_0. □

Remark. A series $\sum_{n=1}^{\infty} x_n$ for which $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty$ for every $x^* \in X^*$ is said to be *weakly unconditionally convergent* (*w.u.c.*). It is easily seen that $\sum_{n=1}^{\infty} x_n$ is a w.u.c. series in an arbitrary Banach space X if and only if $\sum_{n=1}^{\infty} a_n x_n$ converges unconditionally whenever $a_n \to 0$.

We pass now to a fundamental result due to H. P. Rosenthal [129].

Theorem 2.e.5. *Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in a Banach space X. Then, $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ satisfying one of the two mutually exclusive alternatives*:

(i) $\{x_{n_i}\}_{i=1}^{\infty}$ *is equivalent to the unit vector basis of* l_1.
(ii) $\{x_{n_i}\}_{i=1}^{\infty}$ *is a weak Cauchy sequence.*

Consequently, the unit ball of X is weakly conditionally compact if and only if no closed subspace of X is isomorphic to l_1.

The proof given in [129] is valid only for real spaces. L. Dor [31] adapted the proof to the complex case. We shall reproduce here a simplified proof due to J. Farahat [38].

Proof. By using the canonical embedding of X into X^{**} we can consider each x_n as an affine continuous function f_n on the unit ball S of X^*. In this setting we have to prove that if $\{f_n\}_{n=1}^{\infty}$ does not have a subsequence which converges pointwise then it has a subsequence which is equivalent, in the sup norm on S, to the unit vector basis of l_1.

Suppose that no subsequence of $\{f_n\}_{n=1}^{\infty}$ converges pointwise on S. Let $\mathscr{D} = \{D_k^1, D_k^2\}_{k=1}^{\infty}$ be the countable family of all pairs of open discs in the complex plane for which both centers and radii are rational and such that

$$\operatorname{diam} D_k^1 = \operatorname{diam} D_k^2 < d(D_k^1, D_k^2)/2, \quad k = 1, 2, \ldots .$$

Then there exists an index k_0 and an infinite subsequence $\{f_n\}_{n \in M}$, $M \subset N$, such that for every subsequence $\{f_n\}_{n \in L}$ with $L \subset M$ there is an $s_L \in S$ for which the sequence of scalars $\{f_n(s_L)\}_{n\in L}$ has points of accumulation in both $D^1_{k_0}$ and $D^2_{k_0}$. Indeed, if this were false we could construct a sequence of infinite subsequences of the integers $N\supset M_1\supset M_2\cdots\supset M_k\supset\cdots$ such that, for every k and every $s\in S$, the sequence $\{f_n(s)\}_{n\in M_k}$ does not have points of accumulation in both discs D^1_k and D^2_k. Let $L=\{m_k\}_{k=1}^\infty$ be a subsequence of the integers so that $m_k\in M_k$ for every k. In view of our assumption, $\{f_n\}_{n\in L}$ does not converge pointwise on S and thus there is an $s_0\in S$ for which $\{f_n(s_0)\}_{n\in L}$ has at least two distinct points of accumulation d_1 and d_2. This, however, contradicts our assumption concerning M_k, for k chosen so that $d_1\in D^1_k$ $d_2\in D^2_k$.

Let α be the center of $D^1_{k_0}$ and β the center of $D^2_{k_0}$. There is no loss of generality to assume that $\beta-\alpha$ is real and positive (otherwise, replace $\{f_n\}_{n\in M}$, $D^1_{k_0}$ and $D^2_{k_0}$ by $\{\gamma f_n\}_{n\in M}$, $\gamma D^1_{k_0}$ and $\gamma D^2_{k_0}$, respectively, where $\gamma=|\beta-\alpha|/(\beta-\alpha)$).

To continue the proof we need the following lemma.

Lemma 2.e.6. *Let $\{A_j, B_j\}_{j=1}^\infty$ be a sequence of pairs of subsets of a set S so that $A_j\cap B_j=\varnothing$, $j=1, 2,\ldots$. Assume that there is no subsequence $\{A_{j_h}, B_{j_h}\}_{h=1}^\infty$ so that, for every $s\in S$, either $\lim_h \chi_{A_{j_h}}(s)=0$ or $\lim_h \chi_{B_{j_h}}(s)=0$ (here χ_A denotes the characteristic function of A). Then there exists an infinite subsequence $\{A_j, B_j\}_{j\in J}$ so that*

$$\Big(\bigcap_{j\in\delta} A_j\Big)\cap\Big(\bigcap_{j\in\sigma} B_j\Big)\neq\varnothing$$

for any pair of disjoint finite sets δ, $\sigma\subset J$ (such a sequence $\{A_j, B_j\}_{j\in J}$ is called Boolean independent).

Once 2.e.6 is proved, the proof of 2.e.5 can be completed as follows.

Let k_0 and $M=\{n_j\}_{j=1}^\infty$ be as above. Set

$$A_j=\{s;f_{n_j}(s)\in D^1_{k_0}\},\qquad B_j=\{s;f_{n_j}(s)\in D^2_{k_0}\},\quad j=1, 2,\ldots.$$

The properties of k_0 and M show that the assumptions in 2.e.6 are satisfied. Thus, there exists a Boolean independent infinite subsequence $\{A_j, B_j\}_{j\in J}$. We shall show that if $d=d(D^1_{k_0}, D^2_{k_0})$ and $c_j=a_j+ib_j$, $j\in J$ are arbitrary complex scalars then

$$\Big\|\sum_{j\in J} c_j f_{n_j}\Big\|\geqslant(d/8)\sum_{j\in J}|c_j|\,,$$

i.e. that $\{f_{n_j}\}_{j\in J}$ is equivalent to the unit vector basis of l_1. Let σ be any finite subset of J and assume, for simplicity, that $\sum_{j\in\sigma}|a_j|\geqslant\sum_{j\in\sigma}|b_j|$. Set $\sigma_+=\{j;j\in\sigma, a_j\geqslant 0\}$, $\sigma_-=\sigma\sim\sigma_+$ and choose

$$s_1\in\Big(\Big(\bigcap_{j\in\sigma_+} B_j\Big)\cap\Big(\bigcap_{j\in\sigma_-} A_j\Big)\Big);\qquad s_2\in\Big(\Big(\bigcap_{j\in\sigma_+} A_j\Big)\cap\Big(\bigcap_{j\in\sigma_-} B_j\Big)\,.$$

Since, for $z_1\in D^1_{k_0}$ and $z_2\in D^2_{k_0}$, we have

$$\operatorname{Re}(z_2-z_1)\geqslant d\quad\text{and}\quad\operatorname{Im}(z_2-z_1)\leqslant\operatorname{diam} D^1_{k_0}<d/2,$$

it follows that

$$\Big\|\sum_{j\in\sigma} c_j f_{n_j}\Big\| = \sup_{s\in S}\Big|\sum_{j\in\sigma} c_j f_{n_j}(s)\Big| \geqslant \operatorname{Re}\sum_{j\in\sigma} c_j f_{n_j}((s_1-s_2)/2)$$

$$\geqslant \frac{1}{2}\sum_{j\in\sigma} a_j \operatorname{Re}(f_{n_j}(s_1)-f_{n_j}(s_2)) - \frac{1}{2}\sum_{j\in\sigma}|b_j \operatorname{Im}(f_{n_j}(s_1)-f_{n_j}(s_2))|$$

$$\geqslant \frac{d}{2}\sum_{j\in\sigma}|a_j| - \frac{d}{4}\sum_{j\in\sigma}|b_j| \geqslant \frac{d}{4}\sum_{j\in\sigma}|a_j| \geqslant \frac{d}{8}\sum_{j\in\sigma}|c_j|\,. \quad \square$$

Proof of 2.e.6. The proof is based on the following combinatorial result, due to C. St. J. A. Nash-Williams [108] (for a simpler proof, as well as a more general result, see [36]): *Let $\mathscr{P}_\infty(L)$ denote the set of all infinite subsets of a countable set L and let $\mathscr{P}\subset\mathscr{P}_\infty(N)$ be a closed subset ($\mathscr{P}_\infty(N)$ is identified with $\{0,1\}^N$, endowed with the product topology). Then $\mathscr{P}$ is a Ramsey set, i.e. for every $M\in\mathscr{P}_\infty(N)$, there exists an $L\in\mathscr{P}_\infty(M)$ such that either $\mathscr{P}_\infty(L)\subset\mathscr{P}$ or $\mathscr{P}_\infty(L)\subset\mathscr{P}_\infty(N)\sim\mathscr{P}$.*

Let the sequence $\{A_j, B_j\}_{j=1}^\infty$ satisfy the assumptions of 2.e.6 and, for notational convenience, denote B_j by $-A_j$, $j=1,2,\ldots$. For each integer k consider the subset $\mathscr{P}_k\subset\mathscr{P}_\infty(N)$ consisting of all $M=\{n_h\}_{h=1}^\infty$ for which $\bigcap_{h=1}^k(-1)^h A_{n_h}\neq\varnothing$. Each of the sets $\mathscr{P}_k$ is closed and so is $\mathscr{P}=\bigcap_{k=1}^\infty\mathscr{P}_k$. Thus $\mathscr{P}$ is a Ramsey set; this means that there exists an $L=\{m_p\}_{p=1}^\infty\in\mathscr{P}_\infty(N)$ so that either $\mathscr{P}_\infty(L)\subset\mathscr{P}$ or $\mathscr{P}_\infty(L)\subset\mathscr{P}_\infty(N)\sim\mathscr{P}$. In our case we must have $\mathscr{P}_\infty(L)\subset\mathscr{P}$. Indeed, in view of our assumption, there is an $s_0\in S$ so that both sets $\{n\in L;\, s_0\in A_n\}$ and $\{n\in L;\, s_0\in B_n\}$ are infinite which implies the existence of an infinite subsequence $L_0=\{n_h\}_{h=1}^\infty$ of L so that $s_0\in(-1)^h A_{n_h}$, $h=1,2,\ldots$. Thus, $L_0\in\mathscr{P}$ and therefore $\mathscr{P}_\infty(L)\subset\mathscr{P}$.

Let $J=\{m_{2p}\}_{p=1}^\infty$; we claim that $\{A_j, B_j\}_{j\in J}$ is Boolean independent. Indeed, let $\{\theta_p\}_{p=1}^k$ be a finite sequence of signs. Construct a subset $L_1=\{n_h\}_{h=1}^\infty$ of L which contains the integers $m_2, m_4,\ldots, m_{2k}$, scattered among $n_1, n_2,\ldots, n_{2k}$, so that if $n_h=m_{2p}$ then $\theta_p=(-1)^h$. We have $\bigcap_{p=1}^k\theta_p A_{m_{2p}}\supset\bigcap_{h=1}^{2k}(-1)^h A_{n_h}\neq\varnothing$. $\square$

We pass now to another result which characterizes Banach spaces containing l_1. This result is due to Odell and Rosenthal [109] and Rosenthal [130].

Theorem 2.e.7. *Let X be a separable Banach space. Then, the following assertions are equivalent*

(i) *X does not contain a subspace isomorphic to l_1.*
(ii) *Every element in X^{**} is the w^*-limit of a sequence of elements in X (i.e. in the canonical image of X in X^{**}).*
(iii) *The cardinality of X^{**} is equal to that of X (i.e. the cardinality of the continuum).*
(iv) *Every bounded sequence in X^{**} has a w^* convergent subsequence.*

We shall outline the proof of a part of this theorem; namely of the equivalence of (i), (ii) and (iii). Assume that every element of X^{**} is a w^* limit of a sequence of elements in X. This sequence can obviously be taken out of any fixed countable dense set in X. Hence, the cardinality of X^{**} is at most the cardinality of the set of all subsets of the integers, i.e. that of the continuum. Thus (ii) $\Rightarrow$ (iii). It is also trivial that (iii) $\Rightarrow$ (i) since l_1^{**} has a cardinality larger than that of the continuum (observe, e.g. that if Γ is a set of the cardinality of the continuum then $l_1(\Gamma)\subset C(0, 1)^*\subset l_\infty = l_1^*$ and thus l_1^{**} has $l_\infty(\Gamma)$ as a quotient space).

Outline of the proof of (i) $\Rightarrow$ (ii). Assume that there is an $x_0^{**}\in X^{**}$ with $\|x_0^{**}\|=1$ which is not a w^* limit of a sequence of elements in X. Let S be the unit ball of X^* with the w^* topology. Since X is separable S is a compact metric space. The function $x_0^{**}(x^*)$ on S is not the pointwise limit of a sequence of continuous functions on S, i.e. does not belong to the first Baire class on S. This fact is however not obvious; our assumption on x_0^{**} implies immediately only that it is not the pointwise limit of a sequence of affine continuous functions on S. A direct proof of this fact is given in [109]; it can also be deduced from a general result of Choquet concerning functions of Baire class 1 (cf. [1, p. 16]).

Once we know that $x_0^{**}(x^*)$ is not in Baire class 1 on S it follows from the classical characterization of Baire class 1 functions (cf. [51, p. 288]) that there is a closed non-empty set K in S so that the restriction of x_0^{**} to K has no points of continuity. Let $\mathscr{D}=\{D_k^1, D_k^2\}_{k=1}^\infty$ be the family of discs used in the proof of 2.e.5 and let, for $k=1, 2,\ldots,$

$$F_k=\{x^*\in K;\ \text{in any } w^* \text{ neighbourhood } G \text{ of } x^* \text{ there are } y^*\in K\cap G \text{ and } z^*\in K\cap G \text{ so that } x_0^{**}(y^*)\in D_k^1,\ x_0^{**}(z^*)\in D_k^2\}\,.$$

Then, clearly, each F_k is a closed subset of K and $K=\bigcup_{k=1}^\infty F_k$. By the Baire category theorem there is an integer k_0 so that F_{k_0} has a non-empty interior, say G_0, relative to K. Thus, for every non-empty open subset G of G_0, there are $y^*, z^*\in G$ so that $x_0^{**}(y^*)\in D_{k_0}^1$ and $x_0^{**}(z^*)\in D_{k_0}^2$.

We shall prove that there is a sequence $\{x_n\}_{n=1}^\infty$ in the unit ball of X so that the sequence $\{A_n, B_n\}_{n=1}^\infty$, where

$$A_n=\{x^*\in G_0;\ x^*(x_n)\in D_{k_0}^1\},\qquad B_n=\{x^*\in G_0;\ x^*(x_n)\in D_{k_0}^2\}\,,$$

is Boolean independent. Once this is proved it follows, as in the proof of 2.e.5, that $\{x_n\}_{n=1}^\infty$ i.e. equivalent to the unit vector basis in l_1.

We choose the $\{x_n\}_{n=1}^\infty$ inductively. By the definition of G_0 there are $y^*, z^*\in G_0$ so that $x_0^{**}(y^*)\in D_{k_0}^1$ and $x_0^{**}(z^*)\in D_{k_0}^2$. By the w^* density of the unit ball of X in the unit ball of X^{**} there is an $x_1\in X$ with $\|x_1\|\leqslant 1$ so that $y^*(x_1)\in D_{k_0}^1$ and $z^*(x_1)\in D_{k_0}^2$. With this choice of x_1 we have that $A_1\neq\varnothing$, $B_1\neq\varnothing$ and obviously $A_1\cap B_1=\varnothing$. Assume that $\{x_n\}_{n=1}^m$ have already been chosen in the unit ball of X so that, for every choice of signs $\theta=(\theta_1, \theta_2,\ldots, \theta_m)$, $\bigcap_{n=1}^m \theta_n A_n\neq\varnothing$ (where for notational convenience we put $-A_n=B_n$). Since $\bigcap_{n=1}^m \theta_n A_n$ is an open subset of G_0 there are, for every such θ, elements y_θ^* and z_θ^* in $\bigcap_{n=1}^m \theta_n A_n$ so that $x_0^{**}(y_\theta^*)\in D_{k_0}^1$ and

$x_0^{**}(z_\theta^*) \in D_{k_0}^2$. By the w^* density of the unit ball of X in that of X^{**} we get that there is an x_{m+1} with $\|x_{m+1}\| \leq 1$ so that $y_\theta^*(x_{m+1}) \in D_{k_0}^1$ and $z_\theta^*(x_{m+1}) \in D_{k_0}^2$ for all the 2^m possible choices of $(\theta_1, \theta_2, \ldots, \theta_m)$. With this choice of x_{m+1} it is clear that $\bigcap_{n=1}^{m+1} \theta_n A_n \neq \varnothing$ for all $(m+1)$-tuples of signs $(\theta_1, \theta_2, \ldots, \theta_{m+1})$. □

There are also other interesting theorems characterizing Banach spaces containing l_1. These theorems involve the function spaces $C(0, 1)$ and $L_1(0, 1)$ and we shall discuss them in Vol. III.

A question which goes back to Banach is whether every separable Banach space X whose dual is non-separable must contain a subspace isomorphic to l_1. This question was answered negatively by two, independently constructed, counterexamples. One (cf. [57] and also [91]), denoted by JT and called the *James tree*, is obtained from the space J (cf. Example 1.d.2) by replacing its index set (i.e. the integers) by an infinite tree. The second example (cf. [91]), denoted by JF and called the *James function space*, is the continuous analogue of J. This space is easier to define (but more difficult to analyse) than JT. The space JF is the completion of the linear span of characteristic functions of subintervals of [0, 1] with respect to the norm

$$\|f\| = \sup \left(\sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} f(t)\, dt \right)^2 \right)^{1/2}$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \cdots < t_n = 1$ of [0, 1]. For both spaces the verification that their duals are non-separable is trivial while the proof that they do not contain l_1 is more difficult. Since the discovery of these two examples several other counterexamples have been found. We mention, in particular, an example due to Hagler [50] of a space X which, among its other interesting properties, satisfies the following: X is separable, X^* is non-separable, every infinite-dimensional subspace Y of X has a subspace isomorphic to c_0 and every infinite-dimensional subspace Z of X^* has a subspace isomorphic to l_1. We shall present in Vol. IV still another counterexample to Banach's question which is obtained by using a counterexample of James [58] to an important question in the local theory of Banach spaces.

We pass now to some questions involving duality which concern Banach spaces containing c_0 or l_1.

Proposition 2.e.8 [9]. *Let X be a Banach space such that X^* contains a subspace isomorphic to c_0. Then, X has a complemented subspace isomorphic to l_1. Consequently, X^* has a subspace isomorphic to l_∞.*

Proof. Let $T: c_0 \to X^*$ be an isomorphism into and let $\{e_n\}_{n=1}^\infty$ denote the unit vector basis of c_0. The map $x \to Sx = (Te_1(x), Te_2(x), \ldots)$ is the restriction of T^* to $X \subset X^{**}$ and thus it maps X into l_1. Since T^* maps X^{**} onto l_1 and the unit ball of X is w^* dense in the unit ball of X^{**} there exists a constant K such that, for every n, there is an $x_n \in X$ with $\|x_n\| \leq K$, $Te_n(x_n) = 1$ and $\sum_{i=1}^{n-1} |Te_i(x_n)| < 1/n$. The sequence $\{Sx_n\}_{n=1}^\infty$ has, by 1.a.9, 1.a.12 and 2.a.1, a subsequence $\{Sx_{n_k}\}_{k=1}^\infty$ which is equivalent to the unit vector basis of l_1 and whose span is complemented in l_1 by a projection P. Hence, for some constant M and every choice of scalars $\{a_k\}_{k=1}^\infty$, we have

$$\left\|\sum_{k=1}^\infty a_k x_{n_k}\right\| \leqslant K\sum_{k=1}^\infty |a_k| \leqslant KM\left\|\sum_{k=1}^\infty a_k Sx_{n_k}\right\| \leqslant KM\|S\|\left\|\sum_{k=1}^\infty a_k x_{n_k}\right\|.$$

Thus, S is invertible on $Y=[x_{n_k}]_{k=1}^\infty$ which implies that Y is isomorphic to l_1 and $S^{-1}PS$ is a projection from X onto Y. □

Proposition 2.e.8 shows in particular that c_0 *is not isomorphic to a subspace of a separable conjugate space.*

We cannot interchange the roles of c_0 and l_1 in 2.e.8. If, e.g. $X=l_1$, then X^* contains l_1 as a subspace without X having c_0 as a subspace. It is also possible for X^* to contain l_1 as a subspace without X having c_0 as a quotient space. This is e.g. the case for $X=l_\infty$ (it will be proved in Vol. II that every separable quotient space of l_∞ is reflexive (cf. Proposition 2.f.4 below)). However, for separable spaces, we have the following result [60].

Proposition 2.e.9. *Let X be a separable space such that X^* contains a subspace isomorphic to l_1. Then c_0 is isomorphic to a quotient space of X.*

Proof. Let $\{x_n^*\}_{n=1}^\infty$ be a sequence in X^* which is equivalent to the unit vector basis of l_1. Since X is separable there is a subsequence $\{x_{n_k}^*\}_{k=1}^\infty$ which converges w^*. The sequence $y_k^*=x_{n_{2k+1}}^*-x_{n_{2k}}^*$, $k=1,2,\ldots$ is also equivalent to the unit vector basis of l_1 and converges w^* to 0. By the proof of 1.b.7 there is a subsequence $\{y_{k_j}^*\}_{j=1}^\infty$ of $\{y_k^*\}_{k=1}^\infty$ which is a w^* basic sequence. Hence, by 1.b.9, the space $X/([y_{k_j}^*]_{j=1}^\infty)^\top$ is isomorphic to c_0. □

We conclude this section with an open problem which is closely related to the results discussed above.

Problem 2.e.10. *Does every infinite-dimensional Banach space X contain an infinite dimensional subspace which is either reflexive or isomorphic to c_0 or to l_1?*

By 1.c.12 the answer to 2.e.10 is positive if X has a subspace with an unconditional basis (thus, 2.e.10 is a weak version of Problem 1.d.5). Another partial answer to 2.e.10 is given in 1.b.14.

f. Extension and Lifting Properties, Automorphisms of l_∞, c_0 and l_1

The spaces l_∞ and c_0 have important "extension properties" which characterize them while l_1 is characterized by a "lifting property". Some of these properties were mentioned briefly in the previous sections. Here we shall treat them in detail. The main part of this section will be devoted to the study of automorphisms (i.e. invertible linear operators) of l_∞, c_0 and l_1. It turns out that these spaces are surprisingly rich in automorphisms.

We start by considering the extension property of l_∞.

Definition 2.f.1. A Banach space X is said to be *injective* if, for every Banach space Y containing X as a subspace, there is a bounded linear projection from Y onto X.

Injective spaces can be characterized by extension properties for operators, as shown in the following simple proposition.

Proposition 2.f.2. *The following three assertions concerning a Banach space X are equivalent.*

(i) *X is injective.*
(ii) *For every Banach space $Y \supset X$, every Banach space Z and every $T \in L(X, Z)$ there is a $\hat{T} \in L(Y, Z)$ which extends T.*
(iii) *For every pair of Banach spaces $Z \supset Y$ and every $T \in L(Y, X)$ there is a $\hat{T} \in L(Z, X)$ which extends T.*

Proof. Assertion (i) is a special case of both (ii) and (iii) (take e.g. in (ii) $Z=X$ and T the identity operator) and thus (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i). Assume now that (i) holds and let Y, Z and T be given as in (ii). Let P be a projection from Y onto X. The operator $\hat{T}=TP$ has the desired property. It remains to prove that (i) $\Rightarrow$ (iii). Let $\{x_\gamma^*\}_{\gamma \in \Gamma}$ be a set of functionals of norm 1 on X such that $\|x\|=\sup_\gamma |x_\gamma^*(x)|$ for every $x \in X$ (if X is separable Γ can of course be taken to be countable). Define an isometry S, from X into $l_\infty(\Gamma)$, by $Sx(\gamma)=x_\gamma^*(x)$, $\gamma \in \Gamma$. For every $\gamma \in \Gamma$ let $z_\gamma^* \in Z^*$ be a norm preserving extension of $T^*x_\gamma^*$ from Y to Z and let $\hat{T}_0 \in L(Z, l_\infty(\Gamma))$ be defined by $\hat{T}_0 z(\gamma)=z_\gamma^*(z)$, $\gamma \in \Gamma$. Let P be a projection from $l_\infty(\Gamma)$ onto SX (here we use (i)). Then, $\hat{T}=S^{-1}P\hat{T}_0$ has the desired property. $\square$

The proof of (i) $\Rightarrow$ (iii) above shows in particular that, for every set Γ, *the space $l_\infty(\Gamma)$ is injective* (there is even a projection of norm 1 from any $Z \supset l_\infty(\Gamma)$ onto $l_\infty(\Gamma)$). The spaces $l_\infty(\Gamma)$ are not the only injective spaces. We shall discuss in detail the question of the structure of injective spaces in Vol. III. We shall show here only that l_∞ is characterized by the fact that it is the "smallest" injective space. Theorem 2.a.7 states that every injective infinite-dimensional subspace of l_∞ is isomorphic to l_∞ and, in particular, that there are no separable infinite-dimensional injective spaces. Essentially the same argument as that used in 2.a.7 proves however the following stronger version of 2.a.7.

Theorem 2.f.3 [125]. *Every infinite-dimensional injective Banach space has a subspace isomorphic to l_∞.*

This theorem is a consequence of

Proposition 2.f.4. *Let Γ be a set and T a non-weakly compact operator from $l_\infty(\Gamma)$ into some Banach space Z. Then, there exists a subspace U of $l_\infty(\Gamma)$ which is isomorphic to l_∞ so that $T_{|U}$ is an isomorphism.*

We show first that 2.f.4 implies 2.f.3. Let X be an infinite-dimensional injective space and let Γ be such that X is isometric to a subspace X_0 of $l_\infty(\Gamma)$. Let P be a projection from $l_\infty(\Gamma)$ onto X_0. By fact (ii) stated in the proof of 2.a.7 P is not w compact and thus, by 2.f.4, X_0 has a subspace isomorphic to l_∞.

Proof of 2.f.4. The starting point of the proof is a stronger variant of fact (i) which was used in the proof of 2.a.7: every bounded non-w compact operator T, defined on a $C(K)$ space, is an isomorphism on a suitable subspace which is isomorphic to c_0. This result is due to Pelczynski [114] and will be proved in Vol. III.

Let T: $l_\infty(\Gamma)\to Z$ be non-w compact. Then, by the preceding remark, there are vectors $\{y_i\}_{i=1}^\infty$ in $l_\infty(\Gamma)$ which are equivalent to the unit vector basis of c_0 and so that $T_1=T_{|[y_i]_{i=1}^\infty}$ is an isomorphism. Since $[y_i]_{i=1}^\infty$ is separable we may consider it as a subspace of l_∞. Since l_∞ and $l_\infty(\Gamma)$ are injective there are bounded linear operators $S_1\colon l_\infty\to l_\infty(\Gamma)$ and $S_2\colon Z\to l_\infty$ so that $S_{1|[y_i]_{i=1}^\infty}=$identity and $S_{2|[Ty_i]_{i=1}^\infty}=T_1^{-1}$, i.e. the following diagram commutes

$$\begin{array}{ccccccc} l_\infty & \xrightarrow{S_1} & l_\infty(\Gamma) & \xrightarrow{T} & Z & \xrightarrow{S_2} & l_\infty \\ \cup & & \cup & & \cup & & \cup \\ [y_i]_{i=1}^\infty & \xrightarrow{id} & [y_i]_{i=1}^\infty & \xrightarrow{T_1} & [Ty_i]_{i=1}^\infty & \xrightarrow{T_1^{-1}} & [y_i]_{i=1}^\infty \end{array}$$

Let I be the identity operator of l_∞. The operator $I-S_2TS_1\colon l_\infty\to l_\infty$ vanishes on $[y_i]_{i=1}^\infty$. By the proof of 2.a.7, $I-S_2TS_1$ must vanish on a subspace V of l_∞ which is isomorphic to l_∞. Hence, S_2TS_1 is the identity on V and therefore $U=S_1V$ is isomorphic to l_∞ and $T_{|U}$ is an isomorphism. □

We pass now to the extension property of c_0. The space c_0 is, by 2.a.7 (as every other separable space), not injective. If we consider however only separable spaces containing c_0 then c_0 behaves as an injective space. More precisely, we have the following theorem due to Sobczyk [136].

Theorem 2.f.5. *Let Y be a separable Banach space containing c_0. Then there is a projection of norm $\leqslant 2$ from Y onto c_0.*

Proof (due to Veech [142]). Let $\{e_n^*\}_{n=1}^\infty$ denote the unit vector basis in $l_1=c_0^*$. For every n let $y_n^*\in Y^*$ be a norm preserving extension of e_n^* from c_0 to Y. Let $F=B_{Y^*}\cap(c_0^\perp)$ and let d be a translation invariant metric on Y^* which induces on B_{Y^*} the w^* topology (here, the separability of Y is used). Since every w^* limit point of the set $\{y_n^*\}_{n=1}^\infty$ belongs to F it follows that $d(y_n^*, F)\to 0$. Let $\{z_n^*\}_{n=1}^\infty$ be a sequence of elements in F such that $\lim_n d(y_n^*, z_n^*)=0$, i.e. $w^*\lim(y_n^*-z_n^*)=0$. The operator P: $Y\to c_0$, defined by $Py=(y_1^*(y)-z_1^*(y), y_2^*(y)-z_2^*(y),\ldots)$, is a projection of norm $\leqslant 2$. □

The property of c_0, which was exhibited in 2.f.5, actually characterizes c_0 among the separable Banach spaces. More precisely, a separable infinite-dimensional Banach space which is complemented in every separable space containing it must be isomorphic to c_0. This is a difficult result, due to M. Zippin [149], and will be presented in Vol. III.

We shall present now a result on quotient spaces of c_0 whose proof uses 2.f.5.

Theorem 2.f.6 [64]. *Every quotient space of c_0 is isomorphic to a subspace of c_0.*

Proof. Let X be a quotient space of c_0. Since X^* is clearly separable we deduce from 1.g.2 that there is a closed subspace Y of X so that both Y and X/Y have shrinking F.D.D.'s. By 2.d.1 there exist sequences $\{B_n\}_{n=1}^\infty$ and $\{C_n\}_{n=1}^\infty$ of finite dimensional Banach spaces so that Y is isomorphic to $\left(\sum_{n=1}^\infty \oplus B_n\right)_0$ and X/Y is isomorphic to $\left(\sum_{n=1}^\infty \oplus C_n\right)_0$. Since $\left(\sum_{n=1}^\infty \oplus B_n\right)_0$ is clearly isomorphic to a subspace of $(c_0 \oplus c_0 \oplus \cdots)_0 = c_0$ we get that Y, and similarly X/Y, are isomorphic to subspaces of c_0. Let $T\colon Y \to c_0$ and $S\colon X/Y \to c_0$ be isomorphisms into and let $Q\colon X \to X/Y$ be the quotient map. The operator T can be extended to a bounded linear operator $\hat{T}\colon X \to c_0$. Indeed, by 2.f.2, T can be extended to an operator $\hat{T}_0\colon X \to l_\infty$. Since $\hat{T}_0 X$ is separable there is, by 2.f.5, a projection P from span $\{\hat{T}_0 X, c_0\}$ onto c_0. The operator $\hat{T} = P\hat{T}_0$ maps X into c_0 and extends T. It is easily verified that the operator $R\colon X \to c_0 \oplus c_0$, defined by $Rx = (\hat{T}x, SQx)$, is an isomorphism into. $\square$

We turn now to the study of the space l_1 which has a property dual to the extension property, namely the "lifting property".

Proposition 2.f.7. *Let X and Y be Banach spaces so that there is an operator S from Y onto X. Then, for every $T \in L(l_1, X)$, there is a $\hat{T} \in L(l_1, Y)$ for which $S\hat{T} = T$. Moreover, if S is a quotient map then, for every $\varepsilon > 0$, $\hat{T}$ may be chosen so that $\|\hat{T}\| \leqslant (1+\varepsilon)\|T\|$.*

$$\begin{tikzcd} & Y \arrow[d, "S"] \\ l_1 \arrow[ur, "\hat{T}"] \arrow[r, "T"'] & X \end{tikzcd}$$

Proof. Let $\{e_n\}_{n=1}^\infty$ be the unit vector basis of l_1 and put $x_n = Te_n$, $n = 1, 2, \ldots$. Then, $\sup_n \|x_n\| < \infty$. It follows from the open mapping theorem that there is a bounded sequence $\{y_n\}_{n=1}^\infty$ in Y such that $Sy_n = x_n$, $n = 1, 2, \ldots$. The operator $\hat{T}$, defined by $\hat{T}(a_1, a_2, \ldots) = \sum_{n=1}^\infty a_n y_n$, has the desired property. If S is a quotient map then, for every $\varepsilon > 0$, we can choose the $\{y_n\}_{n=1}^\infty$ so that $\|y_n\| \leqslant \|x_n\|(1+\varepsilon)$, for all n. This proves the second assertion. $\square$

The property of l_1 exhibited in 2.f.7 characterizes l_1. We say that a Banach space Z has the *lifting property* if, for every operator S from a Banach space Y onto a space X and for every $T \in L(Z, X)$, there is a $\hat{T} \in L(Z, Y)$ so that $T = S\hat{T}$. Every separable infinite dimensional Banach space with the lifting property is isomorphic to l_1. Indeed, if we take $Y = l_1$, $X = Z$, S a quotient map from l_1 onto Z and T = identity of Z, in the definition of the lifting property we deduce that there is a $\hat{T} \in L(Z, l_1)$ so that $S\hat{T} = I_Z$. Consequently, $\hat{T}$ is an isomorphism into and $\hat{T}S$ is a projection from l_1 onto $\hat{T}Z$. Thus, by 2.a.3, $Z \approx l_1$.

The spaces having the lifting property have been characterized also in the non-separable case. It is clear from the proof of 2.f.7 that, for every set Γ, the space $l_1(\Gamma)$ has the lifting property. Köthe [73] generalized 2.a.3 to the non-separable case and showed that, conversely, *every space with the lifting property is isomorphic to $l_1(\Gamma)$, for some set Γ.*

We used in the preceding paragraph (and in several places in the preceding sections) the fact that *every separable Banach space X is a quotient space of l_1.* Let us recall the proof of this simple fact. We let $\{x_n\}_{n=1}^\infty$ be a dense sequence in the unit ball of X and define $T: l_1 \to X$ by $T(a_1, a_2 \ldots) = \sum_{n=1}^{\infty} a_n x_n$. It is clear that $\|T\| \leqslant 1$ and, since the image under T of the unit ball of l_1 is dense in the unit ball of X, we get that T is a quotient map. This proof shows that T is highly non-unique; it depends on the choice of the sequence $\{x_n\}_{n=1}^\infty$ (actually the $\{x_n\}_{n=1}^\infty$ need not even be dense in the unit ball of X. It is easily seen that a necessary and sufficient condition for T to be a quotient map is that $B_X = \overline{\operatorname{conv}}\, \{x_n\}_{n=1}^\infty$). It is therefore somewhat surprising that, up to an automorphism of l_1, T is actually unique.

Theorem 2.f.8 [89]. *Let T_1 and T_2 be two linear operators mapping l_1 onto the same Banach space X which is not isomorphic to l_1. Then there exists an automorphism τ of l_1 (i.e. an invertible linear operator from l_1 onto l_1) so that $T_1 = T_2\tau$. In particular,* $\ker T_1$ *is isomorphic to* $\ker T_2$.

Proof. By 2.a.2 and 2.a.3 there are subspaces U and V of l_1 so that $l_1 = U \oplus V$, $U \subset \ker T_1$ and $U \approx V \approx l_1$. Let P be the projection of l_1 onto V which maps U to 0. Then, $T_1 = T_1 P$ and thus $T_{1|V}$ maps V onto X. Using twice the lifting property for $T_{1|V}$ (acting on $V \approx l_1$) and T_2 we conclude that there are operators $\hat{T}_1: V \to l_1$ and $\hat{T}_2: l_1 \to V$ so that $T_2\hat{T}_1 = T_{1|V}$ and $T_1\hat{T}_2 = T_2$. Let R be an isomorphism from U onto l_1 normalized so that $\|R^{-1}\| > 1$ and put $S = (I - \hat{T}_1\hat{T}_2)R(I-P) + \hat{T}_1 P$, where I is the identity on l_1. The operator S maps l_1 into l_1 and satisfies

$$T_2 S = (T_2 - T_2\hat{T}_1\hat{T}_2)R(I-P) + T_2\hat{T}_1 P = T_1 P = T_1 .$$

Passing to the duals we get that $S^* = (I - P^*)R^*(I - \hat{T}_2^*\hat{T}_1^*) + P^*\hat{T}_1^*$ and therefore,

$$\|S^* x^*\| \geqslant \tfrac{1}{2}\|P\|^{-1} \max\left(\|(I-P^*)R^*(I-\hat{T}_2^*\hat{T}_1^*)x^*\|,\ \|P^*\hat{T}_1^* x^*\|\right)$$

for every $x^* \in l_1^*$. Since $\|P^*\hat{T}_1^* x^*\| \geqslant \|\hat{T}_1^* x^*\|$ and, similarly,

$$\|(I-P^*)R^*(I-\hat{T}_2^*\hat{T}_1^*)x^*\| \geqslant \|R^{-1}\|^{-1}(\|x^*\| - \|\hat{T}_2\|\,\|\hat{T}_1^* x^*\|)$$

we get that

$$\begin{aligned}||S^*x^*|| &\geq \tfrac{1}{2}||P||^{-1}||R^{-1}||^{-1} \max(||x^*|| - ||\hat{T}_2|| \, ||\hat{T}_1^* x^*||, ||\hat{T}_1^* x^*||) \\ &\geq (4||P|| \, ||R^{-1}||(1+||\hat{T}_2||))^{-1}||x^*|| .\end{aligned}$$

Hence, S^* is an isomorphism and thus S is an operator from l_1 onto l_1. By using again the lifting property of l_1 we deduce that there is an $\tilde{I} \in L(l_1, l_1)$ such that $S\tilde{I} = I$. The operator $I - \tilde{I}S$ is a projection from l_1 onto ker S.

Let Q be a projection in l_1 such that $Ql_1 \subset \ker T_2$ and $Ql_1 \approx (I-Q)l_1 \approx l_1$. Then l_1 can be decomposed into the direct sum $l_1 = \ker S \oplus W_1 \oplus W_2$ so that $SW_1 = Ql_1$ and $SW_2 = (I-Q)l_1$. Let $\tau \in L(l_1, l_1)$ be an operator which maps $\ker S \oplus W_1$ isomorphically onto Ql_1 and whose restriction to W_2 is equal to $S_{|W_2}$. Then, τ is an automorphism and $T_1 = T_2\tau$ since $\ker S \oplus W_1 \subset \ker T_1$ and $T_1 = T_2 S$. $\square$

Remarks. 1. The requirement that X is not isomorphic to l_1 was used in the proof only to ensure that ker T_1 and ker T_2 are both infinite-dimensional. If $X = l_1$ and if dim ker $T_1 \neq$ dim ker T_2 (i.e. if one of them is finite and the second infinite or if both are finite but different) then, clearly, there does not exist an automorphism τ so that $T_1 = T_2\tau$.

2. Theorem 2.f.8 shows, in particular, that to every separable Banach space X (with $X \not\approx l_1$) there corresponds an, up to isomorphism unique, subspace of l_1, namely the kernel of any quotient map from l_1 onto X. This correspondence is not however, one to one. For example, if T is a quotient map from l_1 onto some Banach space X then $T: l_1 \oplus l_1 \to X \oplus l_1$, defined by $\tilde{T}(u, v) = (Tu, v)$, is also a quotient map. Clearly, ker T = ker $\tilde{T}$ though, in general, X is not isomorphic to $X \oplus l_1$.

It is possible that 2.f.8 or even some weaker versions of it characterize the space l_1. For example, the following problem is open.

Problem 2.f.9. *Let X be an infinite-dimensional separable Banach space. Assume that, for every pair of quotient maps T_1 and T_2 from X onto the same space Y (with $Y \not\approx X$), there is an automorphism τ of X so that $T_1 = T_2\tau$. Is X isomorphic to either l_1 or l_2?*

Notice that, for $X = l_2$, the assumptions in 2.f.9 are trivially satisfied.

Theorem 2.f.8 can be dualized to theorems concerning extension of isomorphisms in c_0 and l_∞. We state first the result for c_0.

Theorem 2.f.10 [89]. *Let Y be a closed subspace of c_0 and let T be an isomorphism from Y into c_0 so that* dim c_0/Y = dim $c_0/TY = \infty$. *Then, there is an automorphism τ of c_0 so that $\tau_{|Y} = T$. In particular, $c_0/Y \approx c_0/TY$.*

The proof of 2.f.10 is very similar to the proof of the first part of the corresponding result for l_∞ (Theorem 2.f.12 below) and therefore we omit it. Theorem 2.f.10 enables us to associate to every Banach space X isomorphic to a subspace of c_0 (with $X \not\approx c_0$) a, unique up to isomorphism, quotient space of c_0 (denoted by c_0/X). In analogy to 2.f.9 we have the following problem.

Problem 2.f.11. *Let X be a separable infinite-dimensional Banach space. Assume that, for every pair Y, Z of isomorphic subspaces of X of infinite-codimension, there is an automorphism τ of X so that $\tau Y = Z$. Is X isomorphic to c_0 or l_2?*

We pass now to the space l_∞.

Theorem 2.f.12 [89]. *Let Y be a subspace of l_∞ and let T be an isomorphism from Y into l_∞ so that* $\dim l_\infty/Y = \dim l_\infty/TY = \infty$. *Then,*

(i) *If l_∞/Y and l_∞/TY are both non-reflexive there is an automorphism τ of l_∞ which extends T (i.e. $\tau_{|Y} = T$).*
(ii) *If only one of the spaces l_∞/Y and l_∞/TY is reflexive T cannot be extended to an automorphism of l_∞.*
(iii) *If l_∞/Y and l_∞/TY are both reflexive then every extension $\hat{T}$ of T to an operator from l_∞ into itself is a Fredholm operator. The index $i(\hat{T})$ of $\hat{T}$ does not depend on the particular choice of $\hat{T}$ and thus defines an integer valued invariant of T denoted by $\hat{\imath}(T)$. The operator T can be extended to an automorphism of l_∞ if and only if $\hat{\imath}(T) = 0$.*

Since assertion (ii) is trivial only (i) and (iii) require a proof.

Proof of 2.f.12(i). We show first that if Y is a subspace of l_∞ with l_∞/Y non-reflexive then there is a projection P on l_∞ with $PY = \{0\}$ and $\dim Pl_\infty = \infty$ (i.e. $Pl_\infty \approx l_\infty$). Indeed, since the quotient map $\varphi: l_\infty \to l_\infty/Y$ is not w compact there is, by 2.f.4, a subspace U of l_∞ with $U \approx l_\infty$ so that $\varphi_{|U}$ is an isomorphism. Since l_∞ is injective there is a projection P_0 from l_∞/Y onto φU. The projection $P = (\varphi_{|U})^{-1} P_0 \varphi$ has the desired properties.

Let T: $Y \to l_\infty$ be an isomorphism such that l_∞/Y and l_∞/TY are both non-reflexive. By the preceding remark there are projections P and Q on l_∞ so that $PY = QTY = \{0\}$ and $Pl_\infty \approx Ql_\infty \approx l_\infty$. Let I denote the identity operator of l_∞. The subspace $(I-Q)l_\infty$ contains TY and hence, by the injectivity of l_∞, there is an $S_1: l_\infty \to (I-Q)l_\infty$ such that $S_{1|Y} = T$. Similarly, there is an $S_2: l_\infty \to (I-P)l_\infty$ so that $S_{2|TY} = T^{-1}$. Let R be an isomorphism from l_∞ onto Ql_∞ normalized so that $\|R^{-1}\| > 1$ and define $\hat{T}: l_\infty \to l_\infty$ by $\hat{T} = S_1 + R(I - S_2 S_1)$. Since $(I - S_2 S_1)_{|Y} = 0$ it follows that $\hat{T}_{|Y} = T$. We show next that $\hat{T}$ is an isomorphism into l_∞. Indeed, let $x \in l_\infty$; then

$$\begin{aligned}
\|\hat{T}x\| &\geq \max(\|Q\hat{T}x\|/\|Q\|, \|(I-Q)\hat{T}x\|/\|I-Q\|) \\
&\geq \tfrac{1}{2}\|Q\|^{-1} \max(\|R(I-S_2S_1)x\|, \|S_1x\|) \\
&\geq \tfrac{1}{2}\|Q\|^{-1}\|R^{-1}\|^{-1} \max(\|x\| - \|S_2\|\,\|S_1x\|, \|S_1x\|) \\
&\geq \|x\|/2\|Q\|\,\|R^{-1}\|(\|S_2\|+1)\,.
\end{aligned}$$

The operator $\hat{T}$ is not necessarily an automorphism of l_∞ since $\hat{T}$ need not map l_∞ onto l_∞. In order to replace $\hat{T}$ by an automorphism consider the subspace $\hat{T}(I-P)l_\infty$ of l_∞. This is a complemented subspace of infinite-codimension in l_∞ (since $(I-P)l_\infty$ is of infinite codimension in l_∞). Hence, by 2.a.7, there is a subspace W of l_∞ so that $l_\infty = W \oplus \hat{T}(I-P)l_\infty$ and $W \approx l_\infty$. Let R_0 be an isomorphism from

Pl_∞ onto W. The operator $\tau = R_0 P + \hat{T}(I-P)$ is an automorphism of l_∞ and $\tau_{|Y} = R_0 P_{|Y} + \hat{T}(I-P)_{|Y} = \hat{T}_{|Y} = T$. □

Proof of 2.f.12(iii). We remark first that if K is a w compact operator on l_∞ then $I+K$ is a Fredholm operator of index 0. This follows from the fact that K^2 is a compact operator (cf. fact (ii) in the proof of 2.a.7) and that the classical Riesz theory for compact operators applies also to operators which have a compact power (cf. [33, VII.4.6]). Alternatively, we could apply 2.c.10 and use the fact that K is strictly singular. Indeed, let $V \subset l_\infty$ be such that $K_{|V}$ is an isomorphism. By the injectivity of l_∞ there is a $K_0 \colon l_\infty \to l_\infty$ such that $K_{0|KV} = K^{-1}$. Then $K_0 K_{|V} =$identity and since $(K_0 K)^2$ is compact it follows that dim $V < \infty$.

Let now R and S be operators from l_∞ into itself such that $R_{|Y} = T$, $S_{|TY} = T^{-1}$ (R and S exist since l_∞ is injective). The operator $I - SR$ vanishes on Y and thus it factors through the reflexive space l_∞ / Y. Consequently, $I - SR$ is w compact and hence SR is a Fredholm operator of index 0. Similarly, since l_∞ / TY is reflexive, RS is a Fredholm operator of index 0. Hence, both R and S are Fredholm operators and $i(R) + i(S) = 0$. This proves that $i(R)$ does not depend on the choice of R and, thus, $\hat{\imath}(T)$ is well defined. If there is an automorphism τ which extends T then $\hat{\imath}(T) = i(\tau) = 0$. Conversely, if $\hat{\imath}(T) = 0$ then T can be extended to an $R \colon l_\infty \to l_\infty$ with index 0. By adding to R a finite rank operator (which maps ker R isomorphically on a subspace U of l_∞ such that $U \oplus Rl_\infty = l_\infty$) we get an automorphism τ of l_∞ which extends T. □

Remarks. There are subspaces Y of l_∞ such that l_∞ / Y is reflexive and infinite-dimensional. For example, *l_2 is isomorphic to a quotient space of l_∞*. This follows from the following facts. By 2.b.3 $L_1(0, 1)$ has a subspace isomorphic to l_2 and thus l_2 is a quotient space of $L_\infty(0, 1)$. *The space $L_\infty(0, 1)$ is isomorphic to l_∞* [113]. Indeed, $L_\infty(0, 1)$ as a dual of a separable space is isometric to a subspace of l_∞. The usual proof of the Hahn–Banach theorem can be used to show that, for every $Y \supset X$ and every $T \in L(X, L_\infty(0, 1))$, there is a $\hat{T} \in L(Y, L_\infty(0, 1))$ with $\|\hat{T}\| = \|T\|$ and $\hat{T}_{|X} = T$. Thus, $L_\infty(0, 1)$ is injective and its isomorphism with l_∞ follows by using the decomposition method or, alternatively, from 2.a.7.

Let us verify now that cases (ii) and (iii) of 2.f.12 can actually occur. Let l_∞ / Y be isomorphic to l_2. Since $l_\infty \oplus l_\infty = l_\infty$ there is a subspace Y_0 of l_∞ so that Y_0 is isometric to Y and $l_\infty / Y_0 \approx l_\infty / Y \oplus l_\infty$, i.e. l_∞ / Y_0 is non-reflexive. By letting T be an isometry from Y to Y_0 we get an instance where case (ii) occurs. Let now $S \colon l_\infty \to l_\infty$ be the shift operator defined by $S(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots)$. For a positive integer k let $T_k = S^k_{|Y}$. Then, l_∞ / Y and $l_\infty / T_k Y$ are both reflexive and $\hat{\imath}(T_k) = k$, and $\hat{\imath}(T_k^{-1}) = -k$.

In connection with 2.f.12 it is of interest to study further the subspaces Y of l_∞ so that l_∞ / Y is reflexive. In this direction we have the following.

Proposition 2.f.13 [89]. *Let Y be a subspace of l_∞ such that l_∞ / Y is reflexive. Then Y has a subspace isomorphic to l_∞.*

Proof. We show first that Y has a subspace isomorphic to c_0. Otherwise, Y and c_0

would be totally incomparable and thus (cf. the result quoted following 2.c.1) $Y+c_0$ would be a closed subspace of l_∞ with $\dim(Y\cap c_0)<\infty$. Consequently, the quotient map $l_\infty\to l_\infty/Y$ would be an isomorphism on an infinite-dimensional subspace of c_0 which contradicts the reflexivity to l_∞/Y.

By using 2.a.8 we get that there is a sequence $\{y_n\}_{n=1}^\infty$ in Y which is equivalent to the unit vector basis of c_0 so that, for every infinite subset M of the integers, the w^* closure X_M of $[y_n]_{n\in M}$ is isomorphic to l_∞. Let $\{N_\gamma\}_{\gamma\in\Gamma}$ be an uncountable set of subsets of the integers so that $N_{\gamma_1}\cap N_{\gamma_2}$ is finite for every $\gamma_1\neq\gamma_2$. We claim that, for some $\gamma\in\Gamma$, $X_{N_\gamma}\subset Y$. Assume that, for every γ, there is an $x_\gamma\in X_{N_\gamma}$ with $\|x_\gamma\|=1$ and $x_\gamma\notin Y$. Let $T\colon l_\infty\to l_\infty/Y$ be the quotient map. Since Γ is uncountable there is a sequence $\{\gamma_i\}_{i=1}^\infty$ of elements in Γ and a $\delta>0$ such that $\|Tx_{\gamma_i}\|\geqslant\delta$ for every i. Since $[y_n]_{n=1}^\infty\subset Y$ it follows that, for every choice of scalars $\{a_i\}_{i=1}^j$, $\left\|\sum_{i=1}^j a_iTx_{\gamma_i}\right\|\leqslant\max_{1\leqslant i\leqslant j}|a_i|$ and thus $S\colon c_0\to l_\infty/Y$, defined by $S(a_1,a_2,\ldots)=\sum_{i=1}^\infty a_iTx_{\gamma_i}$, is a bounded operator, and $Tx_{\gamma_i}\to 0$ weakly in l_∞/Y. Since l_∞/Y is reflexive S is weakly compact and therefore also compact (recall that in l_1 a sequence converges w if and only if it converges in norm and that an operator is compact, resp. w compact, if and only if its conjugate is such an operator). Thus, $\{Tx_{\gamma_i}\}_{i=1}^\infty$ has a subsequence which converges in norm. The limit of this subsequence must be 0 but this contradicts the fact that $\|Tx_{\gamma_i}\|\geqslant\delta$ for all i. □

It follows from 2.f.12 and 2.f.13 that if Y is separable then, for any pair of isomorphisms T_1 and T_2 of Y into l_∞, there is an automorphism τ of l_∞ so that $\tau T_1=T_2$ and, in particular, $l_\infty/T_1Y\approx l_\infty/T_2Y$. It makes sense therefore to talk of the space l_∞/Y.

Theorems 2.f.8, 2.f.10 and 2.f.12 show that there are many automorphisms of l_1, c_0 and l_∞. The automorphisms constructed in these theorems are in general not isometries. The isometries form a rather restricted class and it is easy to determine their most general form.

Proposition 2.f.14. *Let τ be an isometry of c_0 or l_p, $1\leqslant p\leqslant\infty$, $p\neq 2$ onto itself. Then there is a sequence $\{\theta_i\}_{i=1}^\infty$ of signs and a permutation π of the integers so that $\tau(a_1,a_2,\ldots)=(\theta_1a_{\pi(1)},\theta_2a_{\pi(2)},\theta_3a_{\pi(3)},\ldots)$.*

Proof. Let us first consider the spaces l_p, $1\leqslant p<\infty$, $p\neq 2$. It is not hard to verify (and for $p=1$ it is trivial) that if $x=(a_1,a_2,\ldots)$ and $y=(b_1,b_2,\ldots)$ then $\|x+y\|^p=\|x-y\|^p=\|x\|^p+\|y\|^p$ if and only if x and y have disjoint supports (i.e. $a_ib_i=0$ for every i). Hence, if τ is an isometry of l_p into itself and $\{e_i\}_{i=1}^\infty$ denotes the unit vector basis of l_p then $\{\tau e_i\}_{i=1}^\infty$ have mutually disjoint supports. Consequently, if τ is, in addition, onto then there must exist a permutation π of the integers and signs $\{\theta_i\}_{i=1}^\infty$ so that $\tau e_i=\theta_ie_{\pi(i)}$ for every i. The case of c_0 can be proved similarly or by using the fact that if τ is an isometry of c_0 onto c_0 then τ^* is an isometry of l_1 onto l_1. In the case of l_∞ the simplest way to prove 2.f.14 is to deduce it as a special case of the general result on the structure of isometries of $C(K)$ spaces (the classical Banach–Stone theorem which will be discussed in Vol. III). □

3. Symmetric Bases

a. Properties of Symmetric Bases, Examples and Special Block Bases

It is a trivial observation that the unit vector basis $\{e_n\}_{n=1}^{\infty}$ of c_0 or l_p, $p \geqslant 1$, besides being unconditional, is equivalent to any of its permutations. Moreover, the unit ball of each of these spaces is a symmetric body in the sense that it contains a sequence $(a_1, a_2, \ldots)$ if and only if it contains the sequence $(a_{\pi(1)}, a_{\pi(2)}, \ldots)$, for every permutation π of the integers. The purpose of this chapter is to study those Banach spaces which share this property with c_0 and l_p.

Definition 3.a.1 [134]. A basis $\{x_n\}_{n=1}^{\infty}$ of a Banach space X is said to be *symmetric* if, for any permutation π of the integers, $\{x_{\pi(n)}\}_{n=1}^{\infty}$ is equivalent to $\{x_n\}_{n=1}^{\infty}$.

It is interesting to compare this definition with 1.c.6 where it is shown that a basis $\{x_n\}_{n=1}^{\infty}$ is unconditional if and only if, for each permutation π, $\{x_{\pi(n)}\}_{n=1}^{\infty}$ is a basic sequence. For $\{x_n\}_{n=1}^{\infty}$ to be a symmetric basis one has to require that, in addition, the basic sequence $\{x_{\pi(n)}\}_{n=1}^{\infty}$ be equivalent to $\{x_n\}_{n=1}^{\infty}$ for each such π. This remark shows that *every symmetric basis is also unconditional.*

Let $\{x_n\}_{n=1}^{\infty}$ be a symmetric basis of a Banach space X. Then, for each permutation π of the integers, the operator $V_\pi \colon X \to X$ defined by $V_\pi\left(\sum_n a_n x_n\right) = \sum_n a_n x_{\pi(n)}$ is evidently an automorphism of X. Moreover, the family $\{V_\pi\}$ of all these automorphisms induced by permutations forms a uniformly bounded group. Indeed, if this were not true then, by the uniform boundedness principle, we could produce a vector $x = \sum_{n=1}^{\infty} a_n x_n \in X$, a sequence $\{\pi_j\}_{j=1}^{\infty}$ of permutations of the integers and a sequence $\{\sigma_j\}_{j=1}^{\infty}$ of finite subsets of the integers with

$$\sigma_j \cap \sigma_k = \varnothing \quad \text{and} \quad \pi_j(\sigma_j) \cap \pi_k(\sigma_k) = \varnothing\ ,$$

for $j \neq k$, such that $\left\|\sum_{n \in \sigma_j} a_n x_{\pi_j(n)}\right\| \geqslant 1$ for all j. Then, for any permutation π_0 such that $\pi_0(n) = \pi_{2j}(n)$ for $n \in \sigma_{2j}$, $j = 1, 2, \ldots$, it would follow that $\sum_{n=1}^{\infty} a_n x_{\pi_0(n)}$ diverges, contrary to our assumption.

Since, as remarked above, $\{x_n\}_{n=1}^{\infty}$ is also unconditional, we get that $K = \sup_{\theta, \pi} \|M_\theta V_\pi\| < \infty$ (recall that, for every choice of signs $\theta = \{\theta_n\}_{n=1}^{\infty}$, the operator M_θ is defined by $M_\theta\left(\sum_{n=1}^{\infty} a_n x_n\right) = \sum_{n=1}^{\infty} a_n \theta_n x_n$). The number K, thus defined, is called the *symmetric constant* of $\{x_n\}_{n=1}^{\infty}$. Notice that the symmetric constant of any symmetric basis is always larger than or equal to the unconditional constant of that basis. If, for $x=\sum_{n=1}^{\infty} a_n x_n \in X$, we put

$$||x||_0=\sup_{\theta_n=\pm 1} \sup_{\pi} \left\|\sum_{n=1}^{\infty} a_n\theta_n x_{\pi(n)}\right\|$$

then $||\cdot||_0$ is a new norm on X such that $||x||\leqslant||x||_0\leqslant K||x||$ for all $x\in X$. This new norm satisfies $\left\|\sum_{n=1}^{\infty} a_n\theta_n x_{\pi(n)}\right\|_0=\left\|\sum_{n=1}^{\infty} a_n x_n\right\|_0$ for every permutation π of the integers and every choice of signs $\theta_n=\pm 1$. If X is equipped with such a norm (called sometimes a symmetric norm) then its unit ball is a symmetric body with respect to all permutations of the coordinates and all changes of signs.

We introduce now a notion which is slightly weaker than that of a symmetric basis.

Definition 3.a.2. A basis $\{x_n\}_{n=1}^{\infty}$ of a Banach space X is called *subsymmetric* if it is unconditional and, for every increasing sequence of integers $\{n_i\}_{i=1}^{\infty}$, $\{x_{n_i}\}_{i=1}^{\infty}$ is equivalent to $\{x_n\}_{n=1}^{\infty}$.

The requirement that $\{x_n\}_{n=1}^{\infty}$ be unconditional is not redundant: a basis which is equivalent to each of its subsequences need not be unconditional. For instance, the summing basis of the space c, introduced in Section 1.c, is subsequence equivalent but not unconditional.

Let $\{x_n\}_{n=1}^{\infty}$ be a subsymmetric basis of a Banach space X. It is easily checked that, for any increasing sequence of integers $\{n_i\}_{i=1}^{\infty}$, the operator $S_{\{n_i\}}; X\to X$, defined by $S_{\{n_i\}}\left(\sum_{n=1}^{\infty} a_n x_n\right)=\sum_{i=1}^{\infty} a_i x_{n_i}$, is an isomorphism from X onto $[x_{n_i}]_{i=1}^{\infty}$. Moreover, the family of all these isomorphisms is uniformly bounded; hence, we can define the *subsymmetric constant* of $\{x_n\}_{n=1}^{\infty}$ as the number

$$K_1=\sup_{\theta,\{n_i\}} ||M_\theta S_{\{n_i\}}||<+\infty.$$

If, for $x=\sum_{n=1}^{\infty} a_n x_n\in X$, we put $||x||_0=\sup_{\theta_i=\pm 1}\sup_{\{n_i\}}\left\|\sum_{i=1}^{\infty} a_i\theta_i x_{n_i}\right\|$ then $||\cdot||_0$ is a new norm on X which satisfies $||x||\leqslant||x||_0\leqslant K_1||x||$ and $\left\|\sum_{i=1}^{\infty} a_i\theta_i x_{n_i}\right\|_0=\left\|\sum_{n=1}^{\infty} a_n x_n\right\|_0$ for all $x\in X$. Such a norm is called a subsymmetric norm.

The relationship between symmetric and subsymmetric bases is described in the following simple result.

Proposition 3.a.3 [134]. *Every symmetric basis is subsymmetric.*

Proof. Let X be a Banach space with a symmetric basis $\{x_n\}_{n=1}^{\infty}$ whose symmetric constant is 1. For every increasing sequence of integers $\{n_i\}_{i=1}^{\infty}$ and every integer k there is a permutation π such that $\pi(i)=n_i$ for $1\leqslant i\leqslant k$. Then, for any choice of scalars $(a_1,\ldots, a_k)$, we have $\left\|\sum_{i=1}^{k} a_i x_i\right\| = \left\|\sum_{i=1}^{k} a_i x_{\pi(i)}\right\| = \left\|\sum_{i=1}^{k} a_i x_{n_i}\right\|$, which completes the proof since k is arbitrary. □

It follows easily from 3.a.3 and its proof that, for any symmetric basis, the symmetric constant is larger than or equal to its subsymmetric constant.

That the notions of a symmetric and of a subsymmetric basis do not coincide is shown by the following example, due to D. J. H. Garling [44]. Let Y be the space of all sequences of scalars $y=(a_1, a_2,\ldots)$ for which

$$||y|| = \sup \sum_{i=1}^{\infty} |a_{n_i}| i^{-1/2} < \infty,$$

where the supremum is taken over all increasing sequences of integers $\{n_i\}_{i=1}^\infty$. It is easily checked that Y, endowed with the norm defined above, is a Banach space whose unit vectors $\{e_n\}_{n=1}^\infty$ form a subsymmetric basis. However, the unit vectors do not form a symmetric basis in Y. Indeed, for each fixed k, the vector $y^{(k)}=(1, 2^{-1/2},\ldots, k^{-1/2}, 0, 0,\ldots)$ is obtained from $z^{(k)}=(k^{-1/2},(k-1)^{-1/2},\ldots, 1, 0, 0,\ldots)$ by a suitable permutation of the integers but, in spite of this, $||y^{(k)}|| = \sum_{n=1}^{k} n^{-1}$ and $||z^{(k)}|| = \sum_{n=1}^{k} (k-n+1)^{-1/2} n^{-1/2}$, i.e. $\sup_k ||y^{(k)}|| = \infty$ while $\sup_k ||z^{(k)}|| < \infty$.

In the past, spaces with a subsymmetric basis were studied only very little. However, quite recently, there has been a growing interest in the notion of a subsymmetric basis especially since it has turned out that such bases arise naturally in the local theory of Banach spaces. We shall discuss this in more detail in Vol. III.

The most interesting class of spaces with a symmetric basis is perhaps that of Orlicz sequence spaces which were originally introduced by W. Orlicz [111]. An Orlicz function M is a convex, non-decreasing continuous function on $[0,\infty)$ such that $M(0)=0$ and $\lim_{t\to\infty} M(t)=\infty$. To any Orlicz function M one can associate the space l_M of all the sequences $x=(a_1, a_2,\ldots)$ of scalars such that $\sum_{n=1}^{\infty} M(|a_n|/\rho)<\infty$ for some $\rho>0$. The conditions imposed on M make l_M into a Banach space when it is equipped with the norm

$$||x|| = \inf\left\{\rho>0;\ \sum_{n=1}^{\infty} M(|a_n|/\rho) \leq 1\right\}.$$

The space l_M is called an Orlicz sequence space and it can be easily seen that, for $M(t)=t^p$ with $p\geq 1$, l_M coincides with the classical space l_p. The unit vectors $\{e_n\}_{n=1}^\infty$ form clearly a symmetric basis of $h_M=[e_n]_{n=1}^\infty$ and, in general, h_M does not coincide with l_M. We shall discuss this matter, as well as many other questions regarding the structure of Orlicz spaces, in the next chapter.

Another class of spaces with a symmetric basis related to the l_p spaces is that of Lorentz sequence spaces. For every $p\geq 1$ and every non-increasing sequence of positive numbers $w=\{w_n\}_{n=1}^\infty$ we consider the space $d(w, p)$ of all sequences of scalars $x=(a_1, a_2, \ldots)$ for which $\|x\|=\sup\left(\sum_{n=1}^{\infty}|a_{\pi(n)}|^p w_n\right)^{1/p}<\infty$, the supremum being taken over all permutations π of the integers. It is easily checked that $d(w, p)$, endowed with the norm $\|\cdot\|$ defined above, is a Banach space. If $\inf_n w_n>0$ then $d(w, p)$ is isomorphic to l_p. Another trivial case occurs when $\sum_{n=1}^{\infty} w_n<\infty$: then, $d(w, p)\approx l_\infty$. We shall exclude these two trivial cases, i.e. we shall assume that $\lim_{n\to\infty} w_n=0$, $\sum_{n=1}^{\infty} w_n=\infty$ and $w_1=1$ (to normalize the unit vectors). When these conditions are satisfied $d(w, p)$ is called a Lorentz sequence space. We shall discuss these spaces in Section 4.e.

There is also a general procedure to construct symmetric bases starting with an arbitrary sequence of vectors. Let X be a Banach space and let $\{x_n\}_{n=1}^{\infty}$ be an arbitrary sequence of non-zero vectors in X. Denote by Z the completion of the space of all sequences of scalars $z=(a_1, a_2, \ldots)$ which are eventually equal to zero, with respect to the norm $\|z\|_Z=\sup_{\theta_n=\pm 1}\sup_{\pi}\left\|\sum_{n=1}^{\infty} a_n\theta_n x_{\pi(n)}\right\|$. The space Z, equipped with the norm $\|\cdot\|_Z$, is a Banach space in which the unit vector basis is a symmetric basis.

New spaces with a symmetric basis can be constructed from old ones by some averaging processes. Sometimes, we obtain in this way new interesting spaces. For instance, let μ be a probability measure on the interval $[1, 2]$ and let X_μ be the space of all sequences of scalars $x=(a_1, a_2, \ldots)$ with the norm $\|x\|_\mu=\int_1^2\left(\sum_{n=1}^{\infty}|a_n|^p\right)^{1/p} d\mu(p)<\infty$. We shall see in Vol. II that X_μ is a subspace of $L_1(0, 1)$. These spaces, as well as spaces obtained by averaging suitable Orlicz norms, play a role in the investigation of the structure of $L_1(0, 1)$.

Let X be a Banach space with a normalized symmetric basis $\{x_n\}_{n=1}^{\infty}$ whose symmetric constant is 1. In the study of subspaces of X a special role is played by block bases with constant coefficients (with respect to $\{x_n\}_{n=1}^{\infty}$). Let $\sigma=\{\sigma_j\}_{j=1}^{\infty}$ be a sequence of consecutive disjoint finite subsets of the integers (i.e. the greatest integer in σ_j precedes the smallest integer in σ_{j+1}, for all j), let $\bar{\bar{\sigma}}_j$ denote the number of elements in σ_j and define the operator P_σ on X, as follows.

$$P_\sigma(x)=\sum_{j=1}^{\infty}\left(\left(\sum_{n\in\sigma_j} a_n\right)\Big/\bar{\bar{\sigma}}_j\right)\left(\sum_{n\in\sigma_j} x_n\right), \quad x=\sum_{n=1}^{\infty} a_n x_n\in X.$$

The operator P_σ, thus defined, is called *the averaging projection* or *the conditional expectation with respect to* σ and has the following property.

Proposition 3.a.4. *For every X, $\{x_n\}_{n=1}^{\infty}$ and σ as above, the averaging projection P_σ is a norm-one linear projection in X whose range is the closed linear span of the vectors $u_j=\sum_{n\in\sigma} x_n$, $j=1, 2, \ldots$. In other words, the closed linear span of any block basis with constant coefficients is complemented.*

If $\{x_n\}_{n=1}^{\infty}$ is only a normalized subsymmetric basis whose subsymmetric constant is 1, *the same holds with the exception that $\|P_\sigma\|\leqslant 2$.*

Proof. Fix an integer k and let $\{\pi_i\}_{i=1}^{m(k)}$, with $m(k)=\bar{\bar{\sigma}}_1\cdot\bar{\bar{\sigma}}_2\ldots\bar{\bar{\sigma}}_k$, be the set of all distinct permutations π such that π leaves invariant all the integers outside $\sigma_1\cup\sigma_2\cup\cdots\cup\sigma_k$ and, for $1\leqslant j\leqslant k$, $\pi(\sigma_j)=\sigma_j$ and π restricted to σ_j acts as a cyclic permutation on σ_j (by cyclic permutation τ on a set $\{n_i\}_{i=1}^{k}$, where $n_1<n_2<\cdots<n_k$, we mean a permutation of the form $\tau_j(n_i)=n_{i+j(\mathrm{mod}\, k)}$, $1\leqslant i\leqslant k$, $1\leqslant j\leqslant k$). For every vector $y=\sum_{n=1}^{\infty} a_n x_n\in X$ and every $1\leqslant i\leqslant m(k)$ we have $\left\|\sum_{n=1}^{\infty} a_{\pi_i(n)}x_n\right\|=\|y\|$, in the symmetric case, and $\left\|\sum_{n=1}^{\infty} a_{\pi_i(n)}x_n\right\|\leqslant 2\|y\|$, in the subsymmetric case.

By averaging over $i=1, 2,\ldots, m(k)$ we get that

$$\left\|\sum_{j=1}^{k}\left(\left(\sum_{n\in\sigma_j} a_n\right)\Big/\bar{\bar{\sigma}}_j\right)u_j+\sum_{n\notin\bigcup_{j=1}^{k}\sigma_j} a_n x_n\right\|,$$

and therefore also $\left\|\sum_{j=1}^{k}\left(\left(\sum_{n\in\sigma_j} a_n\right)\Big/\bar{\bar{\sigma}}_j\right)u_j\right\|$, are $\leqslant\|y\|$, in the symmetric case, and $\leqslant 2\|y\|$, in the subsymmetric case. Since k is arbitrary it follows that P_σ is a well-defined operator on X whose norm is $\leqslant 1$, respectively $\leqslant 2$. Obviously, P_σ is a projection whose range is $[u_j]_{j=1}^{\infty}$. □

In the case $\{x_n\}_{n=1}^{\infty}$ is a symmetric basis 3.a.4 clearly remains true even if the u_j's are taken to be a block basis with constant coefficients of some permutation of $\{x_n\}_{n=1}^{\infty}$. As pointed out by J. T. Woo [145], there exist also non-symmetric unconditional bases which have this property, i.e. in which every block basis with constant coefficients of any permutation of the basis spans a complemented subspace. Such a basis is, for example, the natural basis of the space $X_{\infty,r}$, $r\geqslant 1$ (see the remark following 4.d.7 below).

As a first application of 3.a.4, we bring the following.

Proposition 3.a.5. *Let X be a Banach space with a symmetric basis $\{x_n\}_{n=1}^{\infty}$. For every $\sigma=\{\sigma_j\}_{j=1}^{\infty}$ denote by U the closed linear span of the vectors $u_j=\sum_{n\in\sigma_j} x_n$, $j=1, 2,\ldots$. Then X is isomorphic to $X\oplus U$.*

Proof. We shall use a simplified argument due to W. B. Johnson. Let $\{v_n\}_{n=1}^{\infty}$ be a normalized block basis with constant coefficients of $\{x_n\}_{n=1}^{\infty}$ so that every possible normalized block with constant coefficients of $\{x_n\}_{n=1}^{\infty}$ appears infinitely many times in the sequence $\{v_n\}_{n=1}^{\infty}$. This property of $\{v_n\}_{n=1}^{\infty}$ ensures that $V=[v_n]_{n=1}^{\infty}$ satisfies $V\oplus V\approx V$ and $V\oplus U\approx V$ for every U as above. It follows that we can apply the decomposition method for the spaces $X\oplus V$ and X since $X\oplus X\approx X$, $X\oplus V\approx(X\oplus V)\oplus(X\oplus V)$, and, by 3.a.4, $X\oplus V$ is a complemented subspace of X. We conclude that

$$X\approx X\oplus V\approx X\oplus V\oplus U\approx X\oplus U\,. \quad \square$$

We are now prepared to complete the proof of the fact that c_0, l_1 and l_2 are the only Banach spaces which have, up to equivalence, a unique unconditional basis.

Proof of 2.b.10. Assume that a Banach space X has, up to equivalence, a unique normalized unconditional basis $\{x_n\}_{n=1}^\infty$. Then, for each permutation π of the integers, $\{x_{\pi(n)}\}_{n=1}^\infty$ is equivalent to $\{x_n\}_{n=1}^\infty$ and therefore $\{x_n\}_{n=1}^\infty$ is a symmetric basis of X. Let $\{u_j\}_{j=1}^\infty$ be any normalized block basis with constant coefficients of $\{x_n\}_{n=1}^\infty$ and let $U=[u_j]_{j=1}^\infty$. By 3.a.5, $X\approx X\oplus U$, which implies that $\{x_1, u_1, x_2, u_2,\dots, x_n, u_n,\dots\}$ is a normalized unconditional basis of X. Thus, $\{u_j\}_{j=1}^\infty$ is equivalent to $\{x_n\}_{n=1}^\infty$ and, by the second remark following 2.a.9, it follows that $\{x_n\}_{n=1}^\infty$ is a perfectly homogeneous basis, i.e. $\{x_n\}_{n=1}^\infty$ is equivalent to the unit vector basis of c_0 or l_p for some $p\geq 1$. Since each of the spaces l_p with $p\neq 1, 2$ has an unconditional basis which is not equivalent to the unit vector basis (see the discussion preceding 2.b.10), the only remaining possibilities are c_0, l_1 and l_2. $\square$

It is interesting to point out that *a Banach space with an unconditional basis can have either a unique normalized unconditional basis, as in the case of l_1, l_2 and c_0, or uncountably many non-equivalent unconditional bases.* The construction of infinitely many unconditional bases is quite simple (cf. [52]). Let X be a space having a least two non-equivalent normalized unconditional bases. Then, we can easily check that X admits a normalized unconditional basis, say $\{x_n\}_{n=1}^\infty$, which is not symmetric. For such a basis there is a partition of the integers into disjoint infinite subsets $\{N_j\}_{j=1}^\infty$ so that, for each j, $\{x_n\}_{n\in N_j}$ is not a symmetric basis sequence. Let $\pi_{j,1}$ be a permutation of N_j such that $\{x_n\}_{n\in N_j}$ and $\{x_{\pi_{j,1}(n)}\}_{n\in N_j}$ are not equivalent. For each sequence $\{\eta(j)\}_{j=1}^\infty$ of zeros and ones, we consider the basis $\bigcup_{j=1}^\infty \{x_{\pi_{j,\eta(j)}(n)}\}_{n\in N_j}$, where $\pi_{j,0}$ denotes simply the identity on N_j. It is easily seen that distinct sequences of zeros and ones give rise to non-equivalent unconditional bases of X.

Proposition 3.a.4 is also useful if we desire to study the behavior of the sums of vectors of a normalized symmetric or subsymmetric basis.

Proposition 3.a.6. *Let $\{x_n\}_{n=1}^\infty$ be a normalized symmetric (subsymmetric) basis of a Banach space X with symmetric (subsymmetric) constant equal to* 1. *Let $\{x_n^*\}_{n=1}^\infty$ be the sequence of the functionals biorthogonal to $\{x_n\}_{n=1}^\infty$ and, for each n, put $\lambda(n)=\|x_1+x_2+\cdots+x_n\|$ and $\mu(n)=\|x_1^*+x_2^*+\cdots+x_n^*\|$. Then,*

$$\lambda(n)\mu(n)=n \qquad (n\leq\lambda(n)\mu(n)\leq 2n)\,,$$

for all integers n.

Proof. Since $\left(\sum_{k=1}^n x_k^*\right)\left(\sum_{k=1}^n x_k\right)=n$ the inequality $\lambda(n)\mu(n)\geq n$ is true for every space with a basis. By 3.a.4, applied to a single block σ, we obtain the reverse inequality (i.e. $\lambda(n)\mu(n)\leq n$ in the symmetric case or $\lambda(n)\mu(n)\leq 2n$ in the subsymmetric case). Indeed,

$$\mu(n)=\left\|\sum_{k=1}^n x_k^*\right\|=\sup_{\{a_k\}}\left(\sum_{k=1}^n a_k\right)\Big/\left\|\sum_{k=1}^n a_kx_k\right\|\leq n/\lambda(n),$$

(respectively, $2n/\lambda(n)$). $\square$

For every space with a normalized symmetric basis $\{x_n\}_{n=1}^\infty$ with symmetric constant equal to 1, $\{\lambda(n)\}_{n=1}^\infty$ is a non-decreasing sequence which tends to ∞ unless $\{x_n\}_{n=1}^\infty$ is equivalent to the unit vector basis of c_0 (see e.g. the proof of 2.a.9). In most of the known examples $\lambda(\cdot)$ is even a "concave" function on the integers in the sense that $\{\lambda(n+1)-\lambda(n)\}_{n=1}^\infty$ is a non-increasing sequence. This fact is not true in general but we have the following result (cf. [148]).

Proposition 3.a.7. *Let $(X, \|\cdot\|)$ be a Banach space with a symmetric basis $\{x_n\}_{n=1}^\infty$ whose symmetric constant is equal to 1. Then there exists a new norm $\|\cdot\|_0$ on X such that:*

(i) $\|x\| \leqslant \|x\|_0 \leqslant 2\|x\|$ *for all* $x \in X$,

(ii) *The symmetric constant of* $\{x_n\}_{n=1}^\infty$ *with respect to* $\|\cdot\|_0$ *is equal to* 1,

(iii) *if we put* $\lambda_0(n) = \left\|\sum_{i=1}^n x_i\right\|_0$, $n=1, 2, \ldots$ *then* $\{\lambda_0(n+1)-\lambda_0(n)\}_{n=1}^\infty$ *is a non-increasing sequence, i.e.* $\lambda_0(\cdot)$ *is a concave function on the integers.*

Proof. Let $\{x_n^*\}_{n=1}^\infty$ denote again the sequence of the functionals biorthogonal to $\{x_n\}_{n=1}^\infty$ and let $\lambda(n) = \left\|\sum_{i=1}^n x_i\right\|$, $\mu(n) = \left\|\sum_{i=1}^n x_i^*\right\|$ for $n \geqslant 1$ and $\lambda(0)=0$. For each integer n, we put further $\lambda_0(n) = \sup \sum_{i=1}^n [\lambda(k_i)-\lambda(k_i-1)]$, where the supremum is taken over all n-tuples of integers $\{k_1 < k_2 < \cdots < k_n\}$. By 3.a.6 we get, for every $n \geqslant 1$,

$$\begin{aligned}\lambda(n) \leqslant \lambda_0(n) &= \sup\Big\{\sum_{k_i \leqslant n} (\lambda(k_i)-\lambda(k_i-1)) + \sum_{k_i > n} (k_i/\mu(k_i)-(k_i-1)/\mu(k_i-1))\Big\}\\ &\leqslant \lambda(n) + n/\mu(n) = 2\lambda(n)\,.\end{aligned}$$

Now, fix $\varepsilon > 0$ and n and choose suitable integers so that

$$\lambda_0(n-1) < \varepsilon/2 + \sum_{i=1}^{n-1} (\lambda(h_i)-\lambda(h_i-1)),$$

$$\lambda_0(n+1) < \varepsilon/2 + \sum_{i=1}^{n+1} (\lambda(j_i)-\lambda(j_i-1))\,.$$

It follows immediately that $\lambda_0(n-1)+\lambda_0(n+1) \leqslant \varepsilon + 2\lambda_0(n)$ which shows that $\lambda_0(n+1)-\lambda_0(n) \leqslant \lambda_0(n)-\lambda_0(n-1)$ since $\varepsilon > 0$ is arbitrary. The sequence $\lambda_0(n)$ is therefore concave. Observe that a similar argument shows also that $k\lambda_0(n) \leqslant n\lambda_0(k)$ for all $1 \leqslant k \leqslant n$. In order to complete the proof it suffices to construct a new norm $\|\cdot\|_0$ satisfying (i), (ii) and such that $\left\|\sum_{i=1}^n x_i\right\|_0 = \lambda_0(n)$ for all n. This is achieved by setting,

$$\|x\|_0 = \max\Big\{\|x\|, \sup_n \sup_\pi \lambda_0(n) \sum_{i=1}^n |a_{\pi(i)}|/n\Big\} \quad \text{for } x = \sum_{n=1}^\infty a_n x_n \in X\,,$$

where the inner supremum ranges over all the permutations of the integers. The norm $\|\cdot\|_0$ has indeed all the desired properties since, by 3.a.4,

$$\lambda(n) \sum_{i=1}^{n} |a_{\pi(i)}|/n \leqslant \left\| \sum_{i=1}^{n} |a_{\pi(i)}| x_i \right\| \leqslant \|x\|$$

for all x, n and π as above. □

The converse to 3.a.7 is also true in the sense that, for every concave non-decreasing sequence of positive numbers $\{\lambda_n\}_{n=1}^\infty$, there exists at least one Banach space X having a symmetric basis $\{x_n\}_{n=1}^\infty$ with symmetric constant equal to 1 such that $\left\|\sum_{i=1}^{n} x_i\right\| = \lambda_n$ for every n. For instance, we can take the space $d(w, 1)$ with the sequence $w = \{w_n\}_{n=1}^\infty$ defined as follows: $w_1 = \lambda_1$ and, for $n > 1$, $w_n = \lambda_n - \lambda_{n-1}$. The sequence $\{\lambda_n\}_{n=1}^\infty$ does not determine the equivalence class of $\{x_n\}_{n=1}^\infty$ except for the following two extreme cases: if $\sup_n \lambda_n < \infty$ then $\{x_n\}_{n=1}^\infty$ must be equivalent to the unit vector basis of c_0 and, if $\limsup_n \lambda(n)/n > 0$ then $\{x_n\}_{n=1}^\infty$ is necessarily equivalent to the unit vector basis of l_1 (this last fact is an immediate consequence of 3.a.4 or of 3.a.6). If we exclude these two cases there are always many mutually non-isomorphic spaces with a symmetric basis $\{x_n\}_{n=1}^\infty$ such that $\left\|\sum_{i=1}^{n} x_i\right\| = \lambda_n$, for all n. For example, in addition to the space $d(w, 1)$ considered above, we can take the closed linear span of the functionals biorthogonal to the unit vector basis of $d(w', 1)$, where $w' = \{w'_n\}_{n=1}^\infty$ is defined as follows: $w'_1 = 1/\lambda_1$ and, for $n > 1$, $w'_n = n/\lambda_n - (n-1)/\lambda_{n-1}$ (see the discussion following 4.e.4 below).

Besides block bases with constant coefficients there is another type of special block bases which is of interest in the study of symmetric bases.

Definition 3.a.8 [4]. Let X be a Banach space with a symmetric basis $\{x_n\}_{n=1}^\infty$ and let $N_i = \{n_{i,1} < n_{i,2} < \cdots\}$, $i = 1, 2, \ldots$ be any sequence of disjoint infinite subsets of the integers. For every $0 \neq \alpha = \sum_{n=1}^{\infty} a_n x_n \in X$, the sequence $\{u_i^{(\alpha)}\}_{i=1}^\infty$, defined by $u_i^{(\alpha)} = \sum_{j=1}^{\infty} a_j x_{n_{i,j}}$, $i = 1, 2, \ldots$, is called a *block basis generated by the vector* α.

Clearly, $\{u_i^{(\alpha)}\}_{i=1}^\infty$ is a symmetric basic sequence and its equivalence type does not depend on the particular choice of the sets $\{N_i\}_{i=1}^\infty$. Observe also that we use the term "block basis" though, strictly speaking, the $u_i^{(\alpha)}$'s do not form a block basis of $\{x_n\}_{n=1}^\infty$.

The following elegant result concerning block bases generated by one vector was proved in [15].

Theorem 3.a.9. *Let X be a Banach space with a symmetric basis $\{x_n\}_{n=1}^\infty$. Let $\alpha = \sum_{n=1}^{\infty} a_n x_n \neq 0$ and let $\{u_i^{(\alpha)}\}_{i=1}^\infty$ be a block basis generated by the vector α. Then $\{u_i^{(\alpha)}\}_{i=1}^\infty$ is equivalent to $\{x_n\}_{n=1}^\infty$ if and only if $[u_i^{(\alpha)}]_{i=1}^\infty$ is a complemented subspace of X.*

Proof. Fix an integer h so that $a_h \neq 0$. Then, for every $x \in X$, put

$$Px = \sum_{i=1}^{\infty} (b_{i,h}/a_h) u_i^{(\alpha)},$$

where $b_{i,h}$ is the coefficient of $x_{n_{i,h}}$ in the expansion of x with respect to the basis $\{x_n\}_{n=1}^{\infty}$. If $\{u_i^{(\alpha)}\}_{i=1}^{\infty}$ is equivalent to $\{x_n\}_{n=1}^{\infty}$ then P is a bounded linear projection from X onto $[u_i^{(\alpha)}]_{i=1}^{\infty}$. Indeed, if K_α is chosen so that $\left\|\sum_{i=1}^{\infty} c_i u_i^{(\alpha)}\right\| \leqslant K_\alpha \left\|\sum_{i=1}^{\infty} c_i x_i\right\|$ for every choice of $\{c_i\}_{i=1}^{\infty}$ then $\|P\| \leqslant K_\alpha/|a_h|$.

To prove the converse we first notice that whenever $\sum_{i=1}^{\infty} c_i u_i^{(\alpha)}$ converges so does $\sum_{i=1}^{\infty} c_i a_h x_{n_{i,h}}$, and thus also $\sum_{i=1}^{\infty} c_i x_i$. Assume now that $[u_i^{(\alpha)}]_{i=1}^{\infty}$ is a complemented subspace of X and consider a series $\sum_{i=1}^{\infty} c_i x_i$ which converges in X. Choose an increasing sequence of integers $\{k_m\}_{m=1}^{\infty}$ so that $\left\|\sum_{i=k_m}^{\infty} c_i x_i\right\| \leqslant 2^{-m}$ for all m and, for $k_m \leqslant i < k_{m+1}$, set $v_i^{(\alpha)} = \sum_{j=1}^{m} a_j x_{n_{i,j}}$, $\lambda_i = \|u_i^{(\alpha)} - v_i^{(\alpha)}\|$ and $w_i^{(\alpha)} = (u_i^{(\alpha)} - v_i^{(\alpha)})/\lambda_i$. Then $u_i^{(\alpha)} = v_i^{(\alpha)} + \lambda_i w_i^{(\alpha)}$ for all i (notice that this is true even if $\lambda_i = 0$ in which case $w_i^{(\alpha)}$ is not well-defined and can be taken to be an arbitrary unit vector). We have,

$$\left\|\sum_{i=k_1}^{\infty} c_i v_i^{(\alpha)}\right\| = \left\|\sum_{m=1}^{\infty} \sum_{i=k_m}^{k_{m+1}-1} c_i \sum_{j=1}^{m} a_j x_{n_{i,j}}\right\| = \left\|\sum_{j=1}^{\infty} a_j \sum_{m=j}^{\infty} \sum_{i=k_m}^{k_{m+1}-1} c_i x_{n_{i,j}}\right\| \leqslant \sum_{j=1}^{\infty} |a_j|/2^{j-1},$$

which shows that $\sum_{i=1}^{\infty} c_i v_i^{(\alpha)}$ converges. In order to conclude the proof we have to prove that $\sum_{i=1}^{\infty} c_i \lambda_i w_i^{(\alpha)}$ converges too. This however is an immediate consequence of 2.a.11. □

In the spaces c_0 and l_p, $p \geqslant 1$ all the block bases generated by one vector are equivalent to the unit vector basis. This can also happen in other spaces with a symmetric basis, for example, in the Lorentz sequence spaces $d(w, p)$ with $w = \{n^{-1}\}_{n=1}^{\infty}$ and $p \geqslant 1$ arbitrary (see 4.e.5). However, if this situation occurs both in the space and in its dual then the underlying space has to be isomorphic to c_0 or to l_p for some $p \geqslant 1$. This was proved by Z. Altshuler [2].

Theorem 3.a.10 . *Let $\{x_n\}_{n=1}^{\infty}$ be a symmetric basis of a Banach space X and let $\{x_n^*\}_{n=1}^{\infty}$ be the functionals biorthogonal to $\{x_n\}_{n=1}^{\infty}$. Then, X is isomorphic to either c_0 or to l_p for some $p \geqslant 1$ if (and only if) all the block bases of $\{x_n\}_{n=1}^{\infty}$ and of $\{x_n^*\}_{n=1}^{\infty}$, which are generated by one vector, are equivalent to $\{x_n\}_{n=1}^{\infty}$, respectively $\{x_n^*\}_{n=1}^{\infty}$.*

Proof. We assume, as we may without loss of generality, that the symmetric constant of $\{x_n\}_{n=1}^{\infty}$ is equal to 1. Next, we observe that if $\alpha = \sum_{n=1}^{\infty} a_n x_n$ and $\beta = \sum_{n=1}^{\infty} b_n x_n$

are two non-zero vectors in X then $\left\|\sum_{i=1}^{\infty} a_i u_i^{(\beta)}\right\| = \left\|\sum_{i=1}^{\infty} b_i u_i^{(\alpha)}\right\|$. Using this fact and the uniform boundedness principle it follows immediately that if all block bases of $\{x_n\}_{n=1}^{\infty}$, which are generated by one-vector, are equivalent to $\{x_n\}_{n=1}^{\infty}$ then there exists a constant K so that $\left\|\sum_{i=1}^{\infty} b_i u_i^{(\alpha)}\right\| \leqslant K \cdot ||\alpha|| \cdot ||\beta||$ for all $\alpha, \beta \in X$. Using the similar inequality for $\{x_n^*\}_{n=1}^{\infty}$ and a standard duality argument we get that $K^{-1}||\alpha|| \cdot ||\beta|| \leqslant \left\|\sum_{i=1}^{\infty} b_i u_i^{(\alpha)}\right\|$.

Put $\lambda(n) = \left\|\sum_{i=1}^{n} x_i\right\|$ and use the preceding inequalities for $\alpha = \sum_{i=1}^{n} x_i$ and $\beta = \sum_{i=1}^{k} x_i$, with n and k being arbitrary integers. We get that

$$K^{-1} \leqslant \lambda(nk)/\lambda(n) \cdot \lambda(k) \leqslant K, \quad n, k = 1, 2, \ldots.$$

Hence (see the proof of 2.a.9), either $\sup_n \lambda(n) < \infty$ or $K^{-1} \leqslant \lambda(n)/n^{1/p} \leqslant K$ for some $p \geqslant 1$. If $\sup_n \lambda(n) < \infty$ then X is isomorphic to c_0. If $p=1$ then X is isomorphic to l_1, by the remark following 3.a.7. We can therefore assume that $p > 1$. By duality and 3.a.6 it suffices to prove that, for every choice of scalars $\{b_i\}_{i=1}^{n}$, we have

$$\left\|\sum_{i=1}^{n} b_i x_i\right\| \leqslant K\left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p}; \quad n = 1, 2, \ldots.$$

We shall prove this fact only for $n=2$; it will become clear from the proof how to generalize the argument for a general n.

Put $\beta_1 = b_1 x_1 + b_2 x_2$. By the inequalities established above we have

$$||\beta_1||^2 \leqslant K||b_1 u_1^{(\beta_1)} + b_2 u_2^{(\beta_1)}|| = K||\beta_2||,$$

where $\beta_2 = b_1^2 x_1 + b_1 b_2 (x_2 + x_3) + b_2^2 x_4$. Similarly, we get that

$$||\beta_1|| \cdot ||\beta_2|| \leqslant K||b_1 u_1^{(\beta_2)} + b_2 u_2^{(\beta_2)}|| = K||\beta_3||,$$

where $\beta_3 = b_1^3 x_1 + b_1^2 b_2 (x_2 + x_3 + x_4) + b_1 b_2^2 (x_5 + x_6 + x_7) + b_2^3 x_8$. Continuing inductively we get that

$$||\beta_1|| \cdot ||\beta_{m-1}|| \leqslant K||\beta_m||, \quad m > 1,$$

where $\beta_m = \sum_{j=0}^{m} b_1^{m-j} b_2^j \sum_{i \in \sigma_j} x_i$ with $\{\sigma_j\}_{j=0}^{m}$ being disjoint subsets of integers for which $\bar{\bar{\sigma}}_j = \binom{m}{j}$. Hence,

$$||\beta_1||^m \leqslant K^{m-1}||\beta_m|| \leqslant K^{m-1} \sum_{j=0}^{m} |b_1|^{m-j}|b_2|^j \lambda(\bar{\bar{\sigma}}_j) \leqslant K^m \sum_{j=0}^{m} |b_1|^{m-j}|b_2|^j (\bar{\bar{\sigma}}_j)^{1/p}$$
$$= K^m \sum_{j=0}^{m} \binom{m}{j}^{1/p} |b_1|^{m-j}|b_2|^j.$$

By Holder's inequality we deduce that

$$||\beta_1||^m \leqslant K^m(m+1)^{(p-1)/p}(|b_1|^p+|b_2|^p)^{m/p}, \quad m=1, 2,\dots$$

and thus, by taking the m'th root and letting $m\to\infty$, we obtain the desired inequality, i.e. $||\beta_1||\leqslant K(|b_1|^p+|b_2|^p)^{1/p}$. □

b. Subspaces of Spaces with a Symmetric Basis

The properties of the spaces with a symmetric basis might create the impression that their subspaces, or perhaps their complemented subspaces, should inherit some of the symmetric structure. The next result (cf. [86]) shows that this is not the case.

Theorem 3.b.1. *Every Banach space with an unconditional basis is isomorphic to a complemented subspace of a space with a symmetric basis.*

Proof. We may assume without loss of generality that X is a Banach space with a normalized unconditional basis $\{x_n\}_{n=1}^\infty$ whose unconditional constant is equal to 1. We denote by X_0 the algebraic span of $\{x_n\}_{n=1}^\infty$ and by Y_0 the vector space of all sequences of scalars which are eventually equal to zero.

Let $\{\alpha_i\}_{i=1}^\infty, \{\beta_i\}_{i=1}^\infty$ be decreasing sequences of positive numbers and $\{n_i\}_{i=1}^\infty$ an increasing sequence of integers such that $\alpha_1=\beta_1=n_1=1$ and, for $i>1$, $\alpha_i/\alpha_{i-1}<2^{-i}$, $\beta_i N_{i-1}<2^{-i-1}$ and $\alpha_i\beta_i n_i=1$, where $N_i=\sum_{j=1}^{i} n_j$. We consider now the linear operator T from X_0 into Y_0 which maps a vector $x=\sum_{i=1}^{k}\lambda_i x_i\in X_0$ into the sequence $y=Tx=(a_1, a_2,\dots)\in Y_0$ so that $a_1=\lambda_1\alpha_1$, $a_j=\lambda_i\alpha_i$ for $N_{i-1}<j\leqslant N_i$, $i=2, 3,\dots, k$ and $a_j=0$ for $j>N_k$.

Let K be the convex hull of all sequences $z=(c_1, c_2,\dots)\in Y_0$ such that $|c_n|\leqslant|a_{\pi(n)}|$, $n=1, 2,\dots$, for some permutation π of the integers and some sequence $y=(a_1, a_2,\dots)$ which belongs to the image, under T, of the unit ball of X_0. With the aid of K we introduce a norm on Y_0 by putting

$$||z||=\inf\{\rho>0;\ \rho^{-1}z\in K\},\quad z\in Y_0.$$

Then the closure of K in the normed space Y_0 coincides with its unit ball and therefore T becomes a norm-one operator. It follows that T has a unique continuous extension to a norm-one operator from X into the norm completion Y of Y_0. We shall show that T is an isomorphism. Fix an $x=\sum_{i=1}^{k}\lambda_i x_i\in X_0$ with $||x||=1$. By the Hahn–Banach theorem there exist scalars $\{t_i\}_{i=1}^k$ such that $\sum_{i=1}^{k} t_i\lambda_i=1$ and $\sum_{i=1}^{k}|t_i\mu_i|\leqslant 1$

whenever $\left\|\sum_{i=1}^{k} \mu_i x_i\right\| \leqslant 1$ (in particular, $|t_i| \leqslant 1$ for $i=1, 2,\ldots, k$). We define now a linear functional y^* on Y_0 by putting

$$y^*(z)=\sum_{i=1}^{k} \sum_{j=N_{i-1}+1}^{N_i} t_i\beta_i c_j, \quad z=(c_1, c_2,\ldots) \in Y_0 .$$

Since $\alpha_i\beta_i n_i=1$ for all i we get that

$$y^*(Tx)=\sum_{i=1}^{k} \sum_{j=N_{i-1}+1}^{N_i} t_i\beta_i\lambda_i\alpha_i=\sum_{i=1}^{k} t_i\lambda_i=1 ,$$

i.e. $||Tx|| \geqslant 1/||y^*||$. Hence, T would be an isomorphism provided we could show that $||y^*||$, which a priori depends on x, is bounded by a number independent of the choice of x.

Let π be a permutation of the integers, let $u=\sum_{i=1}^{\infty} \mu_i x_i$ be a vector in X_0 with norm $\leqslant 1$ and put $Tu=(b_1, b_2,\ldots) \in Y_0$. For every $1 \leqslant i \leqslant k$ we split the set $\sigma_i=\{j;\ N_{i-1}<j\leqslant N_i\}$ into three disjoint (possibly empty) subsets: $\sigma_{i,1}=\{j \in \sigma_i;\ \pi(j) \in \sigma_i\}$, $\sigma_{i,2}=\{j \in \sigma_i;\ \pi(j)\leqslant N_{i-1}\}$ and $\sigma_{i,3}=\{j \in \sigma_i;\ \pi(j)>N_i\}$. Then, by the choice of $\{\alpha_i\}_{i=1}^\infty$, $\{\beta_i\}_{i=1}^\infty$ and $\{n_i\}_{i=1}^\infty$, we have for every $i \leqslant k$

$$\sum_{j \in \sigma_{i,1}} |t_i\beta_i b_{\pi(j)}| \leqslant n_i\beta_i\alpha_i|t_i\mu_i|=|t_i\mu_i| ,$$

$$\sum_{j \in \sigma_{i,2}} |t_i\beta_i b_{\pi(j)}| \leqslant N_{i-1}\beta_i \leqslant 2^{-i-1} ,$$

$$\sum_{j \in \sigma_{i,3}} |t_i\beta_i b_{\pi(j)}| \leqslant n_i\beta_i\alpha_{i+1} \leqslant 2^{-i-1} .$$

Hence, if $\hat{\pi}Tu$ denotes the vector $(b_{\pi(1)}, b_{\pi(2)},\ldots) \in Y_0$ then

$$|y^*(\hat{\pi}Tu)| \leqslant \sum_{i=1}^{k} \sum_{j=N_{i-1}+1}^{N_i} |t_i\beta_i b_{\pi(j)}| \leqslant \sum_{i=1}^{k} (|t_i\mu_i|+2\cdot 2^{-i-1}) \leqslant 2 ,$$

i.e. $||y^*|| \leqslant 2$. This implies that $||Tx|| \geqslant 1/2$, for all $x \in X$ with $||x||=1$, i.e. T is an isomorphism from X into Y. This completes the proof since the unit vectors of Y_0 form a symmetric basis in Y and TX is complemented in Y in view of 3.a.4 (TX is clearly the closed linear span of a block basis with constant coefficients). □

Two variants of 3.b.1 are presented in the following theorem.

Theorem 3.b.2. *Every reflexive (uniformly convex) Banach space with an unconditional basis is isomorphic to a complemented subspace of some reflexive (uniformly convex) space with a symmetric basis.*

The reflexive case has been originally proved by A. Szankowski [137] and the uniformly convex one by W. J. Davis [22]. We present here the approach of Davis which, besides proving the uniformly convex case, provides an alternative proof of 3.b.1, as well as of the reflexive case. The approach of Davis, which in essence is an interpolation method (based on some ideas from [24]) proceeds as follows.

Let E and F be two Banach sequence spaces so that the unit vectors $\{e_k\}_{k=1}^\infty$ form a symmetric basis with symmetric constant equal to 1 in both E and F. We assume that

(i) For each sequence of scalars $\alpha=(a_1, a_2,\ldots)\in F$, $\|\alpha\|_E \leqslant \|\alpha\|_F$ and

(ii) $\lim_{n\to\infty} \lambda_E(n)/\lambda_F(n)=0$, where, for $n\geqslant 1$, we set $\lambda_E(n)=\left\|\sum_{k=1}^n e_k\right\|_E$ and $\lambda_F(n)=\left\|\sum_{k=1}^n e_k\right\|_F$.

For every number $m\geqslant 1$ we define a new norm on E by putting

$$\|\alpha\|_m=\inf\{(\|\beta\|_E^2+\|\gamma\|_F^2)^{1/2};\ \alpha=m^{-1}\beta+m\gamma \text{ with } \beta\in E \text{ and } \gamma\in F\}\,.$$

Let $\alpha=m^{-1}\beta+m\gamma$ be as above; then $\|\alpha\|_E\leqslant(m^{-2}+m^2)^{1/2}(\|\beta\|_E^2+\|\gamma\|_F^2)^{1/2}\leqslant 2m(\|\beta\|_E^2+\|\gamma\|_F^2)^{1/2}$, which implies that $\|\alpha\|_E\leqslant 2m\|\alpha\|_m$ for every $\alpha\in E$. On the other hand, we clearly have $\|\alpha\|_m\leqslant m\|\alpha\|_E$, $\alpha\in E$. A better estimate can be obtained for $\gamma\in F$, namely $\|\gamma\|_m\leqslant m^{-1}\|\gamma\|_F$.

It follows from this discussion that, for every $m\geqslant 1$, the space $E_m=(E, \|\cdot\|_m)$ is isomorphic to E and the identity mapping of F into E_m has norm $\leqslant m^{-1}$.

We compute now the value of $\lambda_{E_m}(n)=\left\|\sum_{k=1}^n e_k\right\|_m$, $n\geqslant 1$.

Lemma 3.b.3. *With λ_E, λ_F and λ_{E_m} as above we have*

$$\lambda_{E_m}(n)=(m^{-2}\lambda_E^{-2}(n)+m^2\lambda_F^{-2}(n))^{-1/2}$$

for every integer n and every $m\geqslant 1$. For a fixed n the maximal value of $\lambda_{E_m}(n)$ is equal to

$$\max_{m\geqslant 1}\lambda_{E_m}(n)=(2^{-1}\lambda_E(n)\cdot\lambda_F(n))^{1/2}$$

and is attained at $m=(\lambda_F(n)/\lambda_E(n))^{1/2}$.

Proof. Since the norm of any sequence in E or in F does not increase when the sequence is replaced by its restriction to the first n components or by its average over all possible permutations of the integers $\{1, 2,\ldots, n\}$ it follows that

$$\lambda_{E_m}(n)=\inf\{(B^2\lambda_E^2(n)+C^2\lambda_F^2(n))^{1/2};\ Bm^{-1}+Cm=1\}\,.$$

However, for any constants B and C so that $Bm^{-1}+Cm=1$, we have

$$1\leqslant(m^{-2}\lambda_E^{-2}(n)+m^2\lambda_F^{-2}(n))^{1/2}\cdot(B^2\lambda_E^2(n)+C^2\lambda_F^2(n))^{1/2}\,.$$

This proves the first assertion of the lemma. The second assertion is an immediate consequence of the first. □

Let now X be a Banach space with a normalized unconditional basis $\{x_n\}_{n=1}^\infty$ whose unconditional constant is equal to 1. For every increasing sequence of numbers $\{1 \leqslant m_1 < m_2 < \cdots\}$ so that $\sum_{n=1}^\infty m_n^{-1} < \infty$ we let $Y = Y(E, F, X, \{m_n\}_{n=1}^\infty)$ be the space of all sequences of scalars $\alpha \in E$ for which

$$\|\alpha\|_Y = \left\| \sum_{n=1}^\infty \|\alpha\|_{m_n} x_n \right\|_X < \infty .$$

The condition imposed on the sequence $\{m_n\}_{n=1}^\infty$ ensures that the identity mapping of F into Y is a bounded operator. Indeed, for $\gamma \in F$, we have

$$\|\gamma\|_Y = \left\| \sum_{n=1}^\infty \|\gamma\|_{m_n} x_n \right\|_X \leqslant \|\gamma\|_F \cdot \left\| \sum_{n=1}^\infty m_n^{-1} x_n \right\|_X \leqslant \|\gamma\|_F \cdot \sum_{n=1}^\infty m_n^{-1} .$$

In particular, it follows that the unit vectors belong to Y and, as easily checked, they form there a symmetric basis with symmetric constant equal to 1. We can prove now the embedding result (cf. [22]).

Proposition 3.b.4. *For every E, F and X as above there exists an increasing sequence of numbers $\{m_n\}_{n=1}^\infty$ with $\sum_{n=1}^\infty m_n^{-1} < \infty$ such that $Y = Y(E, F, X, \{m_n\}_{n=1}^\infty)$ contains a complemented subspace isomorphic to X. Moreover, every infinite dimensional subspace of Y contains an infinite dimensional subspace which is isomorphic to a subspace of E or to a subspace of X.*

Proof. We recall that by 3.b.3, for each fixed n, the maximal value of $\lambda_{E_m}(n)$ is attained at $m(n) = (\lambda_F(n)/\lambda_E(n))^{1/2}$. The condition (ii) imposed on E and F therefore implies that $m(n) \to \infty$ as $n \to \infty$. Thus, we can construct an increasing sequence of integers $\{n_j\}_{j=1}^\infty$ so that the corresponding numbers $m_j = m(n_j)$, $j = 1, 2, \ldots$ satisfy

$$m_j^{-1} \sum_{i=1}^{j-1} m_i + m_j \cdot \sum_{i=j+1}^\infty m_i^{-1} < 2^{-j-1} \quad \text{for all } j .$$

Next, we choose a sequence $\{\sigma_j\}_{j=1}^\infty$ of disjoint subsets of integers so that $\bar{\bar{\sigma}}_j = n_j$ and put

$$u_j = \Big(\sum_{k \in \sigma_j} e_k \Big) / (2^{-1} \lambda_E(n_j) \cdot \lambda_F(n_j))^{1/2}, \quad j = 1, 2, \ldots .$$

Then, by 3.b.3 and the definition of the m_j's, we have $\|u_j\|_{m_j} = 1$ for all j and, for $i \neq j$,

$$\begin{aligned} \|u_j\|_{m_i} &= \lambda_{E_{m_i}}(n_j) / (2^{-1} \lambda_E(n_j) \cdot \lambda_F(n_j))^{1/2} \\ &= 2^{1/2} / (m_i^{-2} \cdot \lambda_E^{-1}(n_j) \cdot \lambda_F(n_j) + m_i^2 \cdot \lambda_E(n_j) \cdot \lambda_F^{-1}(n_j))^{1/2} \\ &= 2^{1/2} / (m_i^{-2} m_j^2 + m_i^2 m_j^{-2})^{1/2} \leqslant 2^{1/2} \min (m_i/m_j, m_j/m_i) . \end{aligned}$$

The lacunarity condition imposed on the sequence $\{m_j\}_{j=1}^\infty$ implies that $\sum_{j\neq i} \|u_j\|_{m_i} \leqslant 2^{1/2}\cdot 2^{-i-1}$ for all i. Hence, for any choice of $\{c_j\}_{j=1}^\infty$, we have

$$0 \leqslant \left\| \sum_{j=1}^\infty c_j u_j \right\|_{m_i} - |c_i| \leqslant \sum_{j\neq i} |c_j|\, \|u_j\|_{m_i} \leqslant 2^{1/2}\cdot 2^{-i-1} \max_j |c_j| ,$$

which further implies that in $Y = Y(E, F, X, \{m_i\}_{i=1}^\infty)$ we have

$$\left\| \sum_{j=1}^\infty c_j u_j \right\|_Y = \left\| \sum_{i=1}^\infty \left\| \sum_{j=1}^\infty c_j u_j \right\|_{m_i} x_i \right\|_X \leqslant \left\| \sum_{i=1}^\infty c_i x_i \right\|_X + 2^{-1/2} \max_i |c_i| ,$$

i.e. $\left\| \sum_{j=1}^\infty c_j u_j \right\|_Y \leqslant 2 \left\| \sum_{i=1}^\infty c_i x_i \right\|_X$. On the other hand, we clearly have

$$\left\| \sum_{j=1}^\infty c_j u_j \right\|_Y \geqslant \left\| \sum_{i=1}^\infty |c_i| x_i \right\|_X = \left\| \sum_{i=1}^\infty c_i x_i \right\|_X .$$

This shows that $\{x_n\}_{n=1}^\infty$ is equivalent to the block basis $\{u_j\}_{j=1}^\infty$ in Y and, since the u_j's have constant coefficients, $[u_j]_{j=1}^\infty$ is complemented there, by 3.a.4.

Let now V be an infinite dimensional subspace of Y and suppose that the formal identity mapping of V into E is a strictly singular operator. Then, there exists a basic sequence $\{w_j\}_{j=1}^\infty$ in V with $\|w_j\|_Y = 1$ for all j and $\|w_j\|_E \to 0$. Since each of the norms $\|\cdot\|_{m_i}$, $i = 1, 2, \ldots$ used in the construction of Y is equivalent to the norm of E we can find an increasing sequence of integers $\{q_k\}_{k=1}^\infty$ and a subsequence $\{w_{j_k}\}_{k=1}^\infty$ of $\{w_j\}_{j=1}^\infty$ so that $\left\| \sum_{i=q_k}^\infty \|w_{j_k}\|_{m_i} x_i \right\|_X < 2^{-k-1}$, $k \geqslant 1$ and

$$\left\| \sum_{i=1}^{q_{k-1}} \|w_{j_k}\|_{m_i} x_i \right\|_X < 2^{-k-1}, \quad k > 1 .$$

The basic sequence $\{w_{j_k}\}_{k=1}^\infty$ is easily seen to be equivalent to the block basis $y_k = \sum_{i=q_{k-1}}^{q_k - 1} \|w_{j_k}\|_{m_i} x_i$, $k = 1, 2, \ldots$ of $\{x_n\}_{n=1}^\infty$. □

We note that 3.b.4 already provides an alternative proof for 3.b.1. We can also give now the

Proof of 3.b.2. Let X be a space with an unconditional basis $\{x_n\}_{n=1}^\infty$. We may assume without loss of generality that $\{x_n\}_{n=1}^\infty$ is normalized and has unconditional constant equal to 1. Take $E = l_q$ and $F = l_p$ with $1 < p < q < \infty$ and observe that conditions (i) and (ii) are satisfied with this choice of E and F. Using 3.b.4 we construct the space $Y = Y(E, F, X, \{m_n\}_{n=1}^\infty)$ so that it contains a complemented subspace isomorphic to X. We recall that the unit vectors $\{e_n\}_{n=1}^\infty$ form a symmetric basis in Y.

Assume now that X is reflexive. Then, by the last part of 3.b.4, Y contains no subspace isomorphic to l_1 or to c_0. Consequently, by 1.c.12, Y is a reflexive space too.

We consider now the case when X is a uniformly convex space. Uniform convexity will be studied quite in detail in Vol. II. For the present proof we just recall that the *modulus of convexity* δ_U of a Banach space U is defined by

$$\delta_U(\varepsilon)=\inf\{1-||u+v||/2;\ ||u||=||v||=1,\ ||u-v||\geqslant\varepsilon\},\quad 0<\varepsilon<2\,.$$

A Banach space U is *uniformly convex* provided $\delta_U(\varepsilon)>0$ for every $0<\varepsilon<2$.

We note that in our case the spaces E_m, $m\geqslant 1$ are uniformly uniformly convex in the sense that their moduli of convexity δ_{E_m} satisfy $\inf\limits_m \delta_{E_m}(\varepsilon)>0$ for every $0<\varepsilon<2$. Indeed, the mapping $Q_m\colon (E\oplus F)_2\to E_m$, defined by $Q_m(\beta,\gamma)=m^{-1}\beta+m\gamma$ for $\beta\in E$ and $\gamma\in F$, maps the unit ball of $(E\oplus F)_2$ onto the unit ball of E_m. Thus, for any $m\geqslant 1$ and $0<\varepsilon<2$, we have $\delta_{E_m}(\varepsilon)\geqslant\delta_{(E\oplus F)_2}(\varepsilon)$. By using an argument of M. M. Day [27] we shall show that, under the present assumptions, the direct sum $Z=\left(\sum\limits_{n=1}^{\infty}\oplus E_{m_n}\right)_X$ is also a uniformly convex space. This direct sum is defined as the space of all the sequences $\{\alpha^{(n)}\}_{n=1}^{\infty}$ with $\alpha^{(n)}\in E_{m_n}$ for all n for which $||\{\alpha^{(n)}\}_{n=1}^{\infty}||_Z=\left\|\sum\limits_{n=1}^{\infty}||\alpha^{(n)}||_{m_n}x_n\right\|_X<\infty$. Since Y is clearly a subspace of Z (more precisely, the "diagonal" of Z) the uniform convexity of Z would imply that Y is also a uniformly convex space, thus completing the proof.

Let $\{\alpha^{(n)}\}_{n=1}^{\infty}$ and $\{\beta^{(n)}\}_{n=1}^{\infty}$ be two elements of Z so that

$$||\{\alpha^{(n)}\}_{n=1}^{\infty}||_Z=||\{\beta^{(n)}\}_{n=1}^{\infty}||_Z=1\quad\text{and}\quad ||\{\alpha^{(n)}-\beta^{(n)}\}_{n=1}^{\infty}||_Z\geqslant\varepsilon$$

for some $0<\varepsilon<2$. We first consider the case where $||\alpha^{(n)}||_{m_n}=||\beta^{(n)}||_{m_n}$ for all n. Put $a_n=||\alpha^{(n)}||_{m_n}$, $c_n=||\alpha^{(n)}-\beta^{(n)}||_{m_n}$, $n=1,2,\ldots$, $\sigma=\{n;\ c_n/a_n>\varepsilon/2\}$ and $\delta(\varepsilon)=\inf\limits_m\delta_{E_m}(\varepsilon)$. Since $||\alpha^{(n)}+\beta^{(n)}||_{m_n}\leqslant 2a_n(1-\delta(c_n/a_n))$ we get that

$$\begin{aligned}||\{\alpha^{(n)}+\beta^{(n)}\}_{n=1}^{\infty}||_Z&\leqslant 2\left\|\sum_{n=1}^{\infty}a_n(1-\delta(c_n/a_n))x_n\right\|_X\\&\leqslant 2\left\|(1-\delta(\varepsilon/2))\sum_{n\in\sigma}a_nx_n+\sum_{n\notin\sigma}a_nx_n\right\|_X.\end{aligned}$$

We also have $1\geqslant\left\|\sum\limits_{n\notin\sigma}a_nx_n\right\|_X\geqslant(2/\varepsilon)\left\|\sum\limits_{n\notin\sigma}c_nx_n\right\|_X$, which implies that $\left\|\sum\limits_{n\in\sigma}a_nx_n\right\|_X\geqslant\frac{1}{2}\left\|\sum\limits_{n\in\sigma}c_nx_n\right\|_X\geqslant\varepsilon/4$. Thus, by the uniform convexity of X, we get that

$$||\{\alpha^{(n)}+\beta^{(n)}\}_{n=1}^{\infty}||_Z\leqslant 2(1-\eta(\varepsilon))\,,$$

where $\eta(\varepsilon)=\delta_X(\varepsilon\delta(\varepsilon/2)/8)$.

In the general case put $\gamma^{(n)}=||\alpha^{(n)}||_{m_n}\cdot\beta^{(n)}/||\beta^{(n)}||_{m_n}$, $n=1,2,\ldots$. Since $||\gamma^{(n)}||_{m_n}=||\alpha^{(n)}||_{m_n}$ for all n we have $||\{\gamma^{(n)}\}_{n=1}^{\infty}||_Z=1$. We observe that

$$\begin{aligned}||\{\alpha^{(n)}-\gamma^{(n)}\}_{n=1}^{\infty}||_Z&\geqslant||\{\alpha^{(n)}-\beta^{(n)}\}_{n=1}^{\infty}||_Z-||\{\beta^{(n)}-\gamma^{(n)}\}_{n=1}^{\infty}||_Z\\&\geqslant\varepsilon-\left\|\sum_{n=1}^{\infty}||\beta^{(n)}-\gamma^{(n)}||_{m_n}x_n\right\|_X\\&=\varepsilon-\left\|\sum_{n=1}^{\infty}\big|\,||\alpha^{(n)}||_{m_n}-||\beta^{(n)}||_{m_n}\big|x_n\right\|_X.\end{aligned}$$

If $\left\|\sum_{n=1}^{\infty} |\,||\alpha^{(n)}||_{m_n} - ||\beta^{(n)}||_{m_n}|x_n\right\|_X \geqslant \eta(\varepsilon/2)$ then, by the uniform convexity of X, we get that

$$||\{\alpha^{(n)}+\beta^{(n)}\}_{n=1}^{\infty}||_Z \leqslant \left\|\sum_{n=1}^{\infty} (||\alpha^{(n)}||_{m_n} + ||\beta^{(n)}||_{m_n})x_n\right\|_X \leqslant 2(1-\delta_X(\eta(\varepsilon/2)))\,.$$

Otherwise, $||\{\alpha^{(n)}-\gamma^{(n)}\}_{n=1}^{\infty}||_Z \geqslant \varepsilon - \eta(\varepsilon/2) \geqslant \varepsilon/2$ which, by the first part of this proof, implies that

$$\begin{aligned}||\{\alpha^{(n)}+\beta^{(n)}\}_{n=1}^{\infty}||_Z &\leqslant ||\{\alpha^{(n)}+\gamma^{(n)}\}_{n=1}^{\infty}||_Z + ||\{\beta^{(n)}-\gamma^{(n)}\}_{n=1}^{\infty}||_Z \\ &\leqslant 2(1-\eta(\varepsilon/2)) + \eta(\varepsilon/2) \leqslant 2(1-\eta(\varepsilon/2)/2)\,.\end{aligned}$$

Hence, Z and therefore also Y, are uniformly convex spaces. □

As an immediate application of 3.b.2 we get that each of the spaces $L_p(0, 1)$, $1<p<\infty$, $p\neq 2$ is isomorphic to a complemented subspace of a uniformly convex space with a symmetric basis. It will be shown in Vol. III that none of these spaces themselves has a symmetric basis.

An interesting application of 3.b.1 is connected with the universal space U_1 of Pelczynski, which was introduced in 2.d.10(a). We recall that U_1 is a space with an unconditional basis $\{u_n\}_{n=1}^{\infty}$, which is universal in the sense that every other space with an unconditional basis is isomorphic to a complemented subspace of U_1. The space U_1 is determined uniquely, up to isomorphism, by this universality property.

It turns out that U_1 *has a symmetric basis*. Indeed, by 3.b.1, this universal space U_1 is isomorphic to a complemented subspace of a space Y with a symmetric basis. It follows that Y itself is a universal space for all spaces with an unconditional basis and therefore, by the uniqueness of U_1, we get that Y is isomorphic to U_1. Hence, U_1 has a symmetric basis. This fact is somewhat surprising if we recall that, by its construction, the natural basis $\{u_n\}_{n=1}^{\infty}$ of U_1 has the property that every other unconditional basis is equivalent to a subsequence $\{u_{n_j}\}_{j=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$.

We pass now to the question of uniqueness, up to equivalence, of a symmetric basis. This property, which is shared by a relatively large class of spaces with a symmetric basis, is quite useful in applications and it is applied mostly in the following typical situation. Let X and Y be two Banach spaces with symmetric bases $\{x_n\}_{n=1}^{\infty}$, respectively $\{y_n\}_{n=1}^{\infty}$, and assume that $\{x_n\}_{n=1}^{\infty}$ is, up to equivalence, the unique symmetric basis of X. In this case, if we wish to check whether X is isomorphic to Y or not it suffices to check whether $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are equivalent which, in general, is much easier.

Proposition 3.b.5. *The spaces c_0 and l_p with $1\leqslant p<\infty$ have, up to equivalence, a unique symmetric basis.*

Proof. Let X be either c_0 or l_p for some $1\leqslant p<\infty$ and let $\{x_n\}_{n=1}^{\infty}$ be a symmetric basis of X. Let $\{e_n^*\}_{n=1}^{\infty}$ denote the sequence of the functionals biorthogonal to the unit vector basis $\{e_n\}_{n=1}^{\infty}$ of X. Choose a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that

$\lim_{i\to\infty} e_m^* x_{n_i}$ exists for each m. If all these limits are equal to 0 it follows from 1.a.12 and 2.a.1(ii) that $\{x_{n_i}\}_{i=1}^\infty$, and thus also $\{x_n\}_{n=1}^\infty$, are equivalent to the unit vector basis of X. If $\lim_{i\to\infty} e_m^* x_{n_i} \neq 0$ for some m then a subsequence of $\{x_{n_i}\}_{i=1}^\infty$, and therefore also $\{x_n\}_{n=1}^\infty$, are equivalent to the unit vector basis of l_1. □

Remark. The proof actually shows that every symmetric basic sequence in X is equivalent to the unit vector basis.

In contrast to the case of c_0 and l_p, the universal space U_1 (for all the spaces with an unconditional basis) *has uncountably many mutually non-equivalent symmetric bases.* Moreover, for every $p \geqslant 1$, the space U_1 has a symmetric basis $\{u_n^{(p)}\}_{n=1}^\infty$ such that, for every $\varepsilon > 0$, there is a constant $K_\varepsilon > 0$ with the property that

$$K_\varepsilon\Big(\sum_n |a_n|^{p+\varepsilon}\Big)^{1/(p+\varepsilon)} \leqslant \Big\|\sum_n a_n u_n^{(p)}\Big\| \leqslant \Big(\sum_n |a_n|^p\Big)^{1/p},$$

for every choice of $\{a_n\}$. Indeed, take $F = l_p$ and $E = l_{M_p}$, where $M_p(t) = t^p/(1 + |\log t|)$, and construct the space $Y_p = Y(E, F, U_1, \{m_n\}_{n=1}^\infty)$ so that Y_p contains a complemented subspace isomorphic to U_1 and $\sum_{n=1}^\infty m_n^{-1} < 1$. This is possible in view of 3.b.4 and it implies that $U_1 \approx Y_p$, by the uniqueness of the universal space U_1. To complete the proof we observe that

$$\|\alpha\|_p \geqslant \Big(\sum_{n=1}^\infty m_n^{-1}\Big)\|\alpha\|_F \geqslant \|\alpha\|_{Y_p} \geqslant \|\alpha\|_{m_1} \geqslant K\|\alpha\|_{l_{M_p}},$$

for some constant $K > 0$ and every sequence $\alpha \in l_p$. This means that the unit vector basis of Y_p is mapped, by the isomorphism between Y_p and U_1, into a symmetric basis $\{u_n^{(p)}\}_{n=1}^\infty$ of U_1 having all the desired properties.

Unlike the case of uniqueness of unconditional bases (cf. 2.b.10) it does not seem likely that there is a way to describe those spaces having a unique symmetric basis. It is of interest to point out that from the isometric point of view a symmetric basis is unique. More precisely, we have the following result due to A. Pelczynski and S. Rolewicz (cf. [124, Th. IX.8.3]). Let X and Y be Banach spaces having normalized bases $\{x_n\}_{n=1}^\infty$, resp. $\{y_n\}_{n=1}^\infty$, whose symmetric constants are equal to 1. Then, X is isometric to Y if and only if the basis $\{x_n\}_{n=1}^\infty$ is isometrically equivalent to $\{y_n\}_{n=1}^\infty$, i.e. the map $T: X \to Y$ defined by $T\Big(\sum_{n=1}^\infty a_n x_n\Big) = \sum_{n=1}^\infty a_n y_n$ is an isometry.

Another problem related to the uniqueness of symmetric bases involves the possible number of mutually non-equivalent symmetric bases of one space. In all the examples studied so far (see also the Section 4.b, 4.c and 4.e for the cases of Orlicz and Lorentz sequences spaces) either there is a unique symmetric basis or there are uncountably many non-equivalent symmetric bases. Hence, we have the following question.

Problem 3.b.6. *Is there any Banach space with exactly* 2 (*or any other finite number, or* $\aleph_0$) *non-equivalent symmetric bases?*

In 2.a.6 we have introduced the notion of a prime space, i.e. a Banach space in which all infinite-dimensional complemented subspaces are isomorphic to each other. There are relatively few prime spaces (the only known examples are c_0 and l_p, $1\leqslant p\leqslant\infty$). We introduce now a related class of spaces which is however much larger.

Definition 3.b.7. A Banach space X is said to be *primary* if, for every bounded projection Q on X, either QX or $(I-Q)X$ is isomorphic to X.

In Vol. II we shall prove that the common classical function spaces $C(K)$ (with K being a compact metric topological space) and $L_p(0, 1)$, $1\leqslant p\leqslant\infty$ are all primary. We shall show now that the universal space U_1 (of 2.d.10(a)) is primary. This fact is proved by using the decomposition method of Pelczynski described in the proof of 2.a.3 (observe that this method can be applied since

$$U_1\approx(U_1\oplus U_1\oplus\cdots\oplus U_1\oplus\cdots)_2)$$

and the following result of P. G. Casazza and Bor-Luh Lin [16].

Proposition 3.b.8. *Let X be a Banach space with a subsymmetric basis $\{x_n\}_{n=1}^\infty$. Then, for every bounded projection Q on X, either QX or $(I-Q)X$ contains a subspace isomorphic to X which is complemented in X.*

Proof. We may assume that $\{x_n\}_{n=1}^\infty$ is a normalized basis with subsymmetric constant equal to 1. We can also assume that $x_n\xrightarrow{w}0$ since, otherwise, X is isomorphic to l_1 which is prime. By putting $Qx_n=\sum_{k=1}^\infty\lambda_{n,k}x_k$, $n=1, 2,\ldots$ we can find an infinite subset N_1 of the integers so that either $|\lambda_{n,n}|\geqslant 1/2$ or $|1-\lambda_{n,n}|\geqslant 1/2$ for all $n\in N_1$. If, for instance, the first alternative holds then, by 1.a.12, there are an infinite subset $N_2=\{n_1<n_2<\cdots<n_j<\cdots\}$ of N_1 and a block basis $\{u_j\}_{j=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that $\{Qx_{n_j}\}_{j=1}^\infty$ satisfies $\|Qx_{n_j}-u_j\|<1/5\cdot 2^{j+3}\cdot\|Q\|$ for all j.

Let $\{x_n^*\}_{n=1}^\infty$ be the sequence of the functionals biorthogonal to $\{x_n\}_{n=1}^\infty$. Then, $|x_{n_j}^*u_j|\geqslant|x_{n_j}^*Qx_{n_j}|-1/2^{j+3}\geqslant|\lambda_{n_j,n_j}|-1/2^4\geqslant 1/4$ for all j. It follows that the convergence of a series $\sum_{j=1}^\infty a_jQx_{n_j}$, which is equivalent to that of $\sum_{j=1}^\infty a_ju_j$, implies the convergence of $\sum_{j=1}^\infty a_jx_{n_j}$. On the other hand, it is obvious that $\sum_{j=1}^\infty a_jQx_{n_j}$ converges whenever $\sum_{j=1}^\infty a_jx_{n_j}$ is a convergent series. Hence, since $\{x_n\}_{n=1}^\infty$ is subsymmetric, we get that $[Qx_{n_j}]_{j=1}^\infty$ is a subspace of QX which is isomorphic to X. Using the equivalence between $\{u_j\}_{j=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ it is easy to show that the operator $P\colon X\to X$, defined by

$$P\Big(\sum_{n=1}^\infty a_nx_n\Big)=\sum_{j=1}^\infty(a_{n_j}/x_{n_j}^*u_j)u_j,\quad\sum_{n=1}^\infty a_nx_n\in X,$$

is a bounded projection onto $[u_j]_{j=1}^\infty$ with $\|P\|\leqslant 5\|Q\|$. Thus, by 1.a.9(ii), $[Qx_{n_j}]_{j=1}^\infty$ is also complemented in X.

If the second alternative holds, i.e. if $|1-\lambda_{n,n}|\geqslant 1/2$ for all $n\in N_1$ we use $I-Q$ instead of Q. □

Proposition 3.b.8 suggests the following

Problem 3.b.9. *Is every Banach space X, with a symmetric or even subsymmetric basis, primary?*

Observe that, by the decomposition method, this problem would have a positive answer provided we could show that the factor containing a complemented subspace isomorphic to X, say QX, satisfies $QX\approx QX\oplus QX$. In general, however, a space with a symmetric basis might have complemented subspaces which are not isomorphic to their square (this follows from T. Figiel [39] and 3.b.1).

Let us mention another space which is primary; the space J of James, introduced in 1.d.2 (cf. P. G. Casazza [14]).

We conclude this section by discussing some examples of spaces with a symmetric basis which have some special properties. T. Figiel and W. B. Johnson [42] have constructed a Banach space with a symmetric basis which contains no subspace isomorphic to c_0 or to l_p, $p\geqslant 1$. Their example disproved the feeling that, by some fixed point argument, it might be possible to come up with a positive answer to the question whether every space with a symmetric basis contains c_0 or l_p (such a method works for Orlicz sequence spaces, as shown in 4.a.9 below). The construction of Figiel and Johnson is based on the procedure described in 3.b.4 and the proof of 3.b.2. More precisely, in the notation used there, their example is just the space $Y(c_0, l_1, T, \{m_n\}_{n=1}^\infty)$, where $\{m_n\}_{n=1}^\infty$ is any increasing sequence satisfying $m_n\geqslant 1$ and $\sum_{n=1}^\infty m_n^{-1}<\infty$, and T is the dual of the space of Tsirelson introduced in 2.e.1. We recall that T is a reflexive space with an unconditional basis $\{t_n\}_{n=1}^\infty$. The proof of 2.e.1. actually shows that T *contains no subsymmetric basic sequence.* Instead of discussing here in detail the example of Figiel and Johnson we present a modified example, due to Z. Altshuler [3], which has some additional interesting properties.

Example 3.b.10. *A Banach space Y with a symmetric basis $\{e_n\}_{n=1}^\infty$ in which all symmetric basic sequences are equivalent to $\{e_n\}_{n=1}^\infty$ but which contains no subspace isomorphic to c_0 or l_p, $p\geqslant 1$.*

The additional interest in this example stems from the fact, proved in 3.a.10, that a space X with a symmetric basis $\{x_n\}_{n=1}^\infty$ has to be isomorphic to c_0 or to l_p for some $p\geqslant 1$ if all symmetric basic sequences in X are equivalent to each other and the same holds in X^*. The example 3.b.10 shows that it is not enough to assume that this property holds only in X.

We first define a sequence of symmetric norms on c_0 as follows. For any integer n and any $\alpha=(a_1, a_2,\ldots)\in c_0$ we put

$$||\alpha||_n=\sup_k \sup_\pi \left(\sum_{j=1}^k |a_{\pi(j)}|\,j^{-1}\right)/(2^n+2^{-n}s_k)\,,$$

where $s_k=\sum_{j=1}^{k} j^{-1}$, $k=1, 2,\ldots$ and the inner supremum ranges over all permutations of the integers. It is easily checked that

$$2^{-n-1}\|\alpha\|_0 \leqslant \|\alpha\|_n \leqslant 2^n\|\alpha\|_0\,,$$

for all n and all $\alpha\in c_0$, where $\|\cdot\|_0$ denotes the norm in c_0.

Let Y be the space of all sequences $\alpha\in c_0$ for which $\|\alpha\|_Y=\left\|\sum_{n=1}^{\infty}\|\alpha\|_n t_n\right\|_T<\infty$, where $\{t_n\}_{n=1}^\infty$ is the unconditional basis of the space T introduced in 2.e.1. The unit vectors $\{e_i\}_{i=1}^\infty$ belong to Y since, for each i,

$$\|e_i\|_Y=\left\|\sum_{n=1}^{\infty}\|e_i\|_n t_n\right\|_T=\left\|\sum_{n=1}^{\infty} t_n/(2^n+2^{-n})\right\|_T\leqslant 1\,.$$

Moreover, they form a symmetric basis of Y with symmetric constant equal to 1. We also note that $\lim_{k\to\infty}\left\|\sum_{i=1}^{k} e_i\right\|_Y=\infty$. Indeed, let n and k be such that $s_k/2<2^{2n}\leqslant 2s_k$. Then, $\left\|\sum_{i=1}^{k} e_i\right\|_Y\geqslant\left\|\sum_{i=1}^{k} e_i\right\|_n\geqslant s_k/(2^n+2^{-n}s_k)\geqslant s_k^{1/2}/3$, which proves our assertion since $\lim_{k\to\infty} s_k=\infty$. It follows from 3.b.5 that Y is not isomorphic to c_0.

In order to prove that Y has the desired properties we first remark that every symmetric basic sequence in Y is equivalent to a (symmetric) block basis of $\{e_i\}_{i=1}^\infty$ (apply the argument used in the proof of 3.b.5). We begin by treating some particular cases of such block bases.

Lemma 3.b.11. *Let $u_m=\sum_{i=q_m+1}^{q_{m+1}} c_ie_i$, $m=1, 2,\ldots$ be a normalized block basis of the unit vector basis $\{e_i\}_{i=1}^\infty$ of Y. If $\lim_{i\to\infty} c_i=0$ then there exists a subsequence $\{u_{m_j}\}_{j=1}^\infty$ of $\{u_m\}_{m=1}^\infty$ which is equivalent to a block basis $\{x_j\}_{j=1}^\infty$ of $\{t_n\}_{n=1}^\infty$, the natural basis of T.*

Proof. The relation between the norms $\|\cdot\|_n$ and $\|\cdot\|_0$ of c_0 shows that, for each fixed m and N, we have

$$\sum_{n=1}^{N-1}\|u_m\|_n\leqslant\sum_{n=1}^{N-1}2^n\|u_m\|_0\leqslant 2^N\max\{|c_i|\,;\,q_m<i\leqslant q_{m+1}\}\,.$$

Therefore, we can construct inductively two increasing sequences of integers, $\{m_j\}_{j=1}^\infty$ and $\{N_j\}_{j=1}^\infty$, such that

$$\left\|\sum_{n=1}^{N_{j-1}-1}\|u_{m_j}\|_n t_n+\sum_{n=N_j}^{\infty}\|u_{m_j}\|_n t_n\right\|<2^{-j-1}\quad\text{for all } j \text{ (we take } N_0=1)\,.$$

As easily checked, this implies that the basic sequence $\{u_{m_j}\}_{j=1}^\infty$ is equivalent to the block basis $x_j=\sum_{n=N_{j-1}}^{N_j-1}\|u_{m_j}\|_n t_n$, $j=1, 2,\ldots$ of $\{t_n\}_{n=1}^\infty$. $\square$

It follows from 3.b.11 and the fact that T contains no subsymmetric basic sequences that there is no symmetric block basis $u_m = \sum_{i=q_m+1}^{q_{m+1}} c_i e_i$, $m=1,2,\dots$ of $\{e_i\}_{i=1}^\infty$ so that the sequence $\{c_i\}_{i=1}^\infty$ tends to zero. This implies that $\{e_i\}_{i=1}^\infty$ is not equivalent to the unit vector basis of l_p, $p\geq 1$ and thus, by 3.b.5, Y is not isomorphic to any l_p with $p\geq 1$.

We consider next block bases of the unit vector basis of Y which are generated by one vector (defined in 3.a.8).

Proposition 3.b.12. *Every block basis $\{u_i^{(\alpha)}\}_{i=1}^\infty$ of $\{e_i\}_{i=1}^\infty$, which is generated by a vector $\alpha\in Y$, is equivalent to $\{e_i\}_{i=1}^\infty$.*

Proof. We first recall that it suffices to show that, for any fixed $0\neq\alpha=\sum_{i=1}^\infty a_i e_i\in Y$, a series $\sum_{i=1}^\infty b_i u_i^{(\alpha)}$ converges whenever $\beta=\sum_{i=1}^\infty b_i e_i\in Y$. We further observe that it is actually enough to prove that $\sum_{i=1}^\infty a_i u_i^{(\alpha)}$ is a convergent series for every $0\neq\alpha=\sum_{i=1}^\infty a_i e_i\in Y$. Indeed, if this were the case then, for α and β as above with $a_i\geq 0$ and $b_i\geq 0$ for all i, we would get that $\sum_{i=1}^\infty (a_i+b_i)u_i^{(\alpha+\beta)}$, and therefore also $\sum_{i=1}^\infty b_i u_i^{(\alpha)}$, is a convergent series.

Fix $\alpha=\sum_{i=1}^\infty a_i e_i\in Y$ with $1\geq a_1\geq a_2\geq\cdots\geq a_i\geq\cdots\geq 0$ and notice that in order to check whether $\sum_{i=1}^\infty a_i u_i^{(\alpha)}$ converges in Y we have to compute the $\|\cdot\|_n$-norms of the double sequence $\{a_i a_j\}_{i,j=1}^\infty$ (the numbers $a_i a_j$, $i,j=1,2,\dots$ are the coefficients in the expansion of $\sum_{i=1}^\infty a_i u_i^{(\alpha)}$ with respect to $\{e_i\}_{i=1}^\infty$). Let $a(t)$ be a non-increasing function on $[1,\infty)$ such that $a(i)=a_i$ for all integers i. If, for some integer m, $i\cdot j=m$ then at least one of the integers i or j is $\geq m^{1/2}$ and therefore $a_i a_j\leq a(m^{1/2})$. It follows that the non-increasing rearrangement of $\{a_i a_j\}_{i,j=1}^\infty$ (as a single sequence) is majorated by the sequence $\beta=(b_1,b_2,\dots)$ whose explicit form is

$$\beta=(\overbrace{a(1^{1/2})}^{\tau(1)\text{—times}},\overbrace{a(2^{1/2}),a(2^{1/2}),\dots}^{\tau(2)\text{—times}},\overbrace{a(m^{1/2}),\dots,a(m^{1/2})}^{\tau(m)\text{—times}},\dots)\,,$$

where $\tau(m)$ is the number of distinct divisors of m. Thus, for every n, we have $\left\|\sum_{i=1}^\infty a_i u_i^{(\alpha)}\right\|_n\leq\|\beta\|_n$.

For each integer m let $\varphi(m)$ be the first place where $a(m)$ appears in the sequence β. Then, for $\varphi(m)\leq k<\varphi(m+1)$, we have

$$\begin{aligned}\Big(\sum_{i=1}^k b_i i^{-1}\Big)/(2^n+2^{-n}s_k) &\leq \Big(\sum_{j=1}^m\sum_{i=\varphi(j)}^{\varphi(j+1)-1} b_i i^{-1}\Big)/(2^n+2^{-n}s_k)\\ &\leq\Big(\sum_{j=1}^m b_{\varphi(j)}\varphi(j)^{-1}(\varphi(j+1)-\varphi(j))\Big)/(2^n+2^{-n}s_{\varphi(m)})\\ &\leq\Big(\sum_{j=1}^m a_j\varphi(j)^{-1}(\varphi(j+1)-\varphi(j))\Big)/(2^n+2^{-n}s_{\varphi(m)})\,.\end{aligned}$$

Since $s_{\varphi(m)} \geqslant \log \varphi(m)$ we get that

$$||\beta||_n \leqslant \sup_m \left(\sum_{j=1}^{m} a_j \varphi(j)^{-1}(\varphi(j+1)-\varphi(j))\right)/(2^n+2^{-n}\log \varphi(m)),$$
$$n=1, 2, \dots .$$

To estimate further the norm of β we use the fact that $\sum_{i=1}^{k} \tau(i) = k \log k + (2\gamma - 1)k + O(k^{1/2})$, where $\gamma = 0.57721\dots$ is the constant of Euler (this formula is proved in many books on number theory; e.g. see [80]). It follows that there are constants C_1 and C_2 so that, for $j \geqslant 1$, we have

$$\varphi(j) = 1 + \sum_{i=1}^{j^2-1} \tau(i) \geqslant 1 + C_1 j^2 \log j ,$$

and $\varphi(j+1) - \varphi(j) \leqslant C_2(1 + j \log j)$. Since s_m behaves asymptotically as $\log m$ we get, by substituting the estimates of $\varphi(j)$ and $\varphi(j+1)-\varphi(j)$ in that of $||\beta||_n$, that

$$||\beta||_n \leqslant C_3 \sup_m \left(\sum_{j=1}^{m} a_j j^{-1}\right)/(2^n+2^{-n}s_m) = C_3 ||\alpha||_n ,$$

for all n and for some constant $C_3 < \infty$. This implies that $||\beta||_Y \leqslant C_3 ||\alpha||_Y$, i.e. $\left\|\sum_{i=1}^{\infty} a_i u_i^{(\alpha)}\right\|_Y \leqslant C_3 ||\alpha||_Y$ for all $\alpha \in Y$. $\square$

We pass now to the study of general symmetric block bases of $\{e_i\}_{i=1}^{\infty}$. Let $u_m = \sum_{i=q_m+1}^{q_{m+1}} c_i e_i$, $m=1, 2, \dots$ be a normalized symmetric block basis of $\{e_i\}_{i=1}^{\infty}$ in Y. We can assume without loss of generality that in each block u_m the coefficients c_i are arranged in non-increasing order and that they are non-negative.

Suppose first that, for every $\varepsilon > 0$, there exists an integer $r = r(\varepsilon)$ such that $\left\|\sum_{i=q_m+r}^{q_{m+1}} c_i e_i\right\|_Y < \varepsilon$ for all m for which $q_{m+1} - q_m \geqslant r$. In this case $\{u_m\}_{m=1}^{\infty}$ is equivalent to a block basis generated by one vector and thus, by 3.b.12, it is equivalent to $\{e_i\}_{i=1}^{\infty}$. Indeed, by putting $v_m = \sum_{i=1}^{q_{m+1}-q_m} c_{i+q_m} e_i$, $m=1, 2, \dots$ and using a standard diagonal argument we can find a subsequence $\{v_{m_j}\}_{j=1}^{\infty}$ of $\{v_m\}_{m=1}^{\infty}$ such that $a_i = \lim_{j\to\infty} c_{i+q_{m_j}}$ exists for every i. For $\varepsilon > 0$ and $r = r(\varepsilon)$ we have $\left\|\sum_{i=r}^{q_{m_j+1}-q_{m_j}} c_{i+q_{m_j}} e_i\right\|_Y < \varepsilon$ for all j which implies that $\left\|\sum_{i=r}^{s} a_i e_i\right\|_Y \leqslant \varepsilon$, for every $s > r$. Hence, $\alpha = \sum_{i=1}^{\infty} a_i e_i \in Y$ and

$$||\alpha - v_{m_j}|| \leqslant 2\varepsilon + \left\|\sum_{i=1}^{r-1} (a_i - c_{i+q_{m_j}}) e_i\right\|_Y \quad \text{for all } j .$$

It follows that a suitable subsequence of $\{u_{m_j}\}_{j=1}^{\infty}$ is equivalent to a block basis generated by α (recall that each v_{m_j} is a "translation" of u_{m_j}).

Finally, suppose that there exist an $\varepsilon>0$ and an increasing sequence of integers $\{m_h\}_{h=1}^{\infty}$ such that $q_{m_h+1}-q_{m_h}>h$ and $\left\|\sum_{i=q_{m_h}+h+1}^{q_{m_h+1}} c_ie_i\right\|_Y\geqslant\varepsilon$ for all h. Put $v_h=\sum_{i=q_{m_h}+1}^{q_{m_h}+h} c_ie_i$ and $w_h=\sum_{i=q_{m_h}+h+1}^{q_{m_h+1}} c_ie_i$. Then, $u_{m_h}=v_h+w_h$ and $\|w_h\|_Y\geqslant\varepsilon$ for every integer h. Notice also that $c_{q_{m_h}+h}\geqslant c$, for some constant $c>0$ and every h, would imply $1\geqslant\|v_h\|_Y\geqslant c\left\|\sum_{i=1}^{h} e_i\right\|_Y$, $h=1, 2,\ldots$, i.e. $c=0$. Thus, $\lim_{h\to\infty} c_{q_{m_h}+h}=0$ which means that $\{w_h\}_{h=1}^{\infty}$ is a block of $\{e_i\}_{i=1}^{\infty}$ with coefficients tending to zero.

By using 3.b.11 and passing to a subsequence if necessary we can assume that $\{w_h\}_{h=1}^{\infty}$ is equivalent to a block basis $\{x_h\}_{h=1}^{\infty}$ of $\{t_n\}_{n=1}^{\infty}$, the unit vector basis of T. The definition of the norm in T implies the existence of a constant $A_1>0$ such that, for every k, we have $\left\|\sum_{h=k+1}^{2k} x_h\right\|_T\geqslant A_1k$. It follows that, for every integer k and some constant A_2,

$$\left\|\sum_{h=1}^{2k} u_{m_h}\right\|_Y\geqslant\left\|\sum_{h=1}^{2k} w_h\right\|_Y\geqslant A_2\left\|\sum_{h=k+1}^{2k} x_h\right\|_T\geqslant A_1A_2k\,.$$

Since $\{u_{m_h}\}_{h=1}^{\infty}$ is a symmetric basic sequence we get, by the discussion following 3.a.7, that $\{u_{m_h}\}_{h=1}^{\infty}$, and thus also $\{u_m\}_{m=1}^{\infty}$, are equivalent to the unit vector basis of l_1. This would imply that there exists in Y a block basis of $\{e_i\}_{i=1}^{\infty}$ with coefficients tending to zero which is also equivalent to the unit vector basis of l_1. This however is impossible in view of 3.b.11 and the discussion thereafter. □

Remark. It is interesting to compare the construction of Y in 3.b.10 with the general method of constructing spaces with a symmetric basis which has been described in the proof of 3.b.2. It can be shown that the space Y of 3.b.10 coincides with the space denoted, in the notation of 3.b.4, by $Y(c_0, d(w, 1), T, \{2^n\}_{n=1}^{\infty})$, where $w=\{n^{-1}\}_{n=1}^{\infty}$.

4. Orlicz Sequence Spaces

a. Subspaces of Orlicz Sequence Spaces which have a Symmetric Basis

Most of this chapter is devoted to a quite detailed study of Orlicz sequence spaces, of particular interest being the relation between these spaces and the l_p spaces. Later on in this chapter we study the subspaces and the quotients of subspaces of the direct sum $l_p \oplus l_r$. It turns out that this topic is closely connected to the study of Orlicz sequence spaces. In the last section of this chapter we present some results on another class of spaces with a symmetric basis, namely Lorentz sequence spaces.

The introduction of Orlicz functions has been inspired by the obvious role played by the functions t^p in the definition of the spaces l_p or, more generally $L_p(\mu)$. It is quite natural to try to replace t^p by a more general function M and then to consider the set of all sequences of scalars $\{a_n\}_{n=1}^\infty$ for which the series $\sum_{n=1}^\infty M(|a_n|)$ converges. W. Orlicz [111] has checked the restrictions which have to be imposed on the function M in order to make this set of sequences into a suitable Banach space. His study led to the following definition of the so-called Orlicz functions and Orlicz sequence spaces (for basic material on Orlicz spaces the reader is referred also to [75]).

Definition 4.a.1. An *Orlicz function* M is a continuous non-decreasing and convex function defined for $t \geqslant 0$ such that $M(0)=0$ and $\lim_{t\to\infty} M(t)=\infty$. If $M(t)=0$ for some $t>0$, M is said to be a *degenerate Orlicz function.*

To any Orlicz function M we associate the space l_M of all sequences of scalars $x=(a_1, a_2,\dots)$ such that $\sum_{n=1}^\infty M(|a_n|/\rho)<\infty$ for some $\rho>0$. The space l_M equipped with the norm

$$\|x\|=\inf\left\{\rho>0;\ \sum_{n=1}^\infty M(|a_n|/\rho)\leqslant 1\right\}$$

is a Banach space usually called an Orlicz sequence space.

Of particular interest is the subspace h_M of l_M consisting of those sequences $x=(a_1, a_2,\dots)\in l_M$ for which $\sum_{n=1}^\infty M(|a_n|/\rho)<\infty$ *for every* $\rho>0$. Some basic properties of h_M are collected in the following proposition.

Proposition 4.a.2. *Let M be an Orlicz function. Then h_M is a closed subspace of l_M and the unit vectors $\{e_n\}_{n=1}^{\infty}$ form a symmetric basis of h_M.*

Proof. It is clear that the unit vectors form a symmetric basic sequence in l_M. Therefore, both assertions of the proposition will be proved if we show that h_M coincides with $[e_n]_{n=1}^{\infty}$. An element $x=(a_1, a_2, \ldots)$ belongs to $[e_n]_{n=1}^{\infty}$ if and only if, for every $\rho>0$, there exists an integer $N=N(\rho)$ such that $\left\|\sum_{n=N}^{\infty} a_n e_n\right\| \leqslant \rho$, i.e. if and only if $\sum_{n=N}^{\infty} M(|a_n|/\rho) \leqslant 1$. $\square$

It is easily verified that if M is a degenerate Orlicz function then $l_M \approx l_\infty$ and $h_M \approx c_0$. Since this case is not interesting in the present context *we shall assume from now on that all the Orlicz functions considered in the sequel are non-degenerate unless specified otherwise.*

In general, the spaces l_M and h_M are distinct. In order to give conditions for l_M to coincide with h_M we need the following definition.

Definition 4.a.3. An Orlicz function M is said to satisfy the *Δ_2-condition* at zero if $\limsup_{t \to 0} M(2t)/M(t) < \infty$.

It is easily checked that the Δ_2-condition at 0 implies that, for every positive number Q, $\limsup_{t \to 0} M(Qt)/M(t) < \infty$ (this condition is sometimes called the Δ_Q-condition). The importance of the Δ_2-condition is illustrated by the following result.

Proposition 4.a.4. *For an Orlicz function M the following conditions are equivalent.*

(i) *M satisfies the Δ_2-condition at* 0.
(ii) *$l_M = h_M$.*
(iii) *The unit vectors form a boundedly complete symmetric basis of l_M.*
(iv) *l_M is separable.*
(v) *l_M contains no subspace isomorphic to l_∞.*

Proof. The fact that the convergence of a series $\sum_{n=1}^{\infty} M(|a_n|/\rho)$ implies that of $\sum_{n=1}^{\infty} M(|a_n|/\eta)$ follows easily from the Δ_Q-condition at zero with $Q=\rho/\eta$. This proves the implication (i) $\Rightarrow$ (ii). In order to prove that (ii) $\Rightarrow$ (iii) we use 4.a.2 and the fact that $\sup_n \left\|\sum_{i=1}^{n} a_i e_i\right\| \leqslant 1$, for some sequence $\{a_i\}_{i=1}^{\infty}$, implies that $\sum_{i=1}^{\infty} M(|a_i|) \leqslant 1$, i.e. $(a_1, a_2, \ldots) \in l_M = h_M$ and thus $\sum_{i=1}^{\infty} a_i e_i$ converges. It is obvious that (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v). Assume now that an Orlicz function M does not satisfy the Δ_2-condition at zero. Then, we can find a sequence $\{t_n\}_{n=1}^{\infty}$ such that $M(2t_n)/M(t_n) > 2^{n+1}$ and $M(t_n) \leqslant 2^{-n}$. Let k_n be integers chosen so that $2^{-(n+1)} <$

$k_n M(t_n) \leqslant 2^{-n}$ for all n. Then $\sum_{n=1}^{\infty} k_n M(t_n) \leqslant 1$ while $k_n M(2t_n) > 1$. Thus, for any choice of scalars $\{a_n\}_{n=1}^{\infty}$, we have that

$$2^{-1} \sup_n |a_n| \leqslant \|(\overbrace{a_1 t_1, \ldots, a_1 t_1}^{k_1 \text{ times}}, \overbrace{a_2 t_2, \ldots, a_2 t_2}^{k_2 \text{ times}}, \ldots, \overbrace{a_n t_n, \ldots, a_n t_n}^{k_n \text{ times}}, \ldots)\| \leqslant \sup_n |a_n|$$

and, therefore, (v) $\Rightarrow$ (i). $\square$

The definitions of l_M and of h_M show that, up to an isomorphism, what really matters is the behavior of M in the neighborhood of $t=0$: if two Orlicz functions M_1 and M_2 coincide on an interval $0 \leqslant t \leqslant t_0$ then l_{M_1} and l_{M_2} consist of the same sequences and the norms induced by M_1 and M_2 are equivalent. The same is true for h_{M_1} and h_{M_2}. More generally, we have the following result.

Proposition 4.a.5. *Let M_1 and M_2 be two Orlicz functions. Then, the following assertions are equivalent.*

(i) *$l_{M_1} = l_{M_2}$ (i.e. both spaces consist of the same sequences) and the identity mapping is an isomorphism between l_{M_1} and l_{M_2}.*
(ii) *The unit vector bases of h_{M_1} and h_{M_2} are equivalent.*
(iii) *M_1 and M_2 are equivalent at zero, i.e. there exist constants $k>0$, $K>0$ and $t_0>0$ such that, for all $0 \leqslant t \leqslant t_0$, we have*

$$K^{-1} M_2(k^{-1}t) \leqslant M_1(t) \leqslant K M_2(kt) .$$

The proof of this proposition is very simple. The implication (ii) $\Rightarrow$ (iii), for instance, is proved by comparing the norms of $\{e_1 + \cdots + e_n\}_{n=1}^{\infty}$ in l_{M_1} and l_{M_2}.

If at least one of the functions satisfies the Δ_2-condition at zero then the equivalence at zero of M_1 and M_2 can be expressed in a simpler form: there exist constants K and $t_0 > 0$ such that $K^{-1} \leqslant M_1(t)/M_2(t) \leqslant K$ for all $0 < t \leqslant t_0$.

There are many instances where an Orlicz function M is defined only in a neighborhood of zero. In this situation the function M can be extended for $t > t_0$ so that it becomes an Orlicz function on the entire positive line. By 4.a.5 the corresponding spaces l_M and h_M will be the same regardless of the way we have extended M. The norms associated to two distinct extensions might be different but always equivalent.

Every Orlicz function M, being non-decreasing and convex, has a right-derivative $p(t)$ for every $t>0$ and $M(t) = \int_0^t p(s)\, ds$. Since the function p is non-negative and non-decreasing it follows that $1 \leqslant tp(t)/M(t)$ for all $t>0$. In particular, this implies by differentiation that $M(t)/t$ is a non-decreasing function. This fact can be also obtained directly from the convexity of M since, for $0<s<t$, we have $M(s) \leqslant (s/t)M(t) + (1-s/t)M(0) = (s/t)M(t)$.

For every Orlicz function M there exists an Orlicz function M_0 which is equivalent to M and has a continuous derivative. We simply put $M_0(t) = \int_0^t (M(s)/s)\, ds$ and get $M(t) \geqslant M_0(t) \geqslant \int_{t/2}^t (M(s)/s)\, ds \geqslant M(t/2)$ for all $t>0$.

The ratio $tp(t)/M(t)$ is also related to the Δ_2-condition. More precisely, *an Orlicz function M satisfies the Δ_2-condition at zero if and only if*

$$\limsup_{t\to 0} tp(t)/M(t)<\infty .$$

Indeed, if $M(2t)\leqslant KM(t)$, for some constant K and for $0\leqslant t\leqslant t_0$, then

$$tp(t)\leqslant \int_t^{2t} p(s)\,ds=M(2t)-M(t)\leqslant KM(t)\,.$$

Conversely, if $tp(t)/M(t)\leqslant K_1$, $0<t\leqslant t_0$ then

$$\log (M(2t)/M(t))=\int_t^{2t} (p(s)/M(s))\,ds\leqslant K_1 \log 2,\ 0<t\leqslant t_0/2\,,$$

i.e. $M(2t)\leqslant 2^{K_1}M(t)$.

The fact that Orlicz sequence spaces are a natural generalization of l_p spaces might suggest that they have a structure almost as simple as that of l_p spaces. Already the study of subspaces of Orlicz sequence spaces shows that this is not the case. For instance, there is no simple description of a general subspace with a basis of an Orlicz sequence space similar to that for l_p spaces (cf. 2.d.1). As we shall see in the next section, the structure of complemented subspaces of l_M is much more involved. It is however possible to describe quite satisfactorily those subspaces of an h_M space which themselves possess a symmetric basis. To present this approach we need the following lemma which was proved in [81] and [93] under the assumption that M satisfies the Δ_2-condition; the fact that this lemma, as well as 4.a.7 and 4.a.8 below, are valid without the Δ_2-condition was noticed in [70].

Lemma 4.a.6. *Let M be an Orlicz function and consider the following subsets of $C(0, \frac{1}{2})$*

$$E_{M,\Lambda}=\overline{\{M(\lambda t)/M(\lambda);\, 0<\lambda<\Lambda\}},\qquad 0<\Lambda\leqslant\infty,\qquad E_M=\bigcap_{\Lambda>0} E_{M,\Lambda},$$

$$C_{M,\Lambda}=\overline{\mathrm{conv}}\, E_{M,\Lambda},\qquad C_M=\bigcap_{\Lambda>0} C_{M,\Lambda}\,.$$

where the closure is taken in the norm topology of $C(0, \frac{1}{2})$. Then $E_{M,\Lambda}$, E_M, $C_{M,\Lambda}$ and C_M are non-void norm compact subsets of $C(0, \frac{1}{2})$ consisting entirely of Orlicz functions which might be degenerate.

Proof. Let p be the right-derivative of M. Then, for $\lambda>0$, we have

$$M(\lambda)=\int_0^{\lambda} p(s)\,ds\geqslant \int_{\lambda/2}^{\lambda} p(s)\,ds\geqslant \tfrac{1}{2}\lambda p(\lambda/2)\,.$$

Hence, for $0\leqslant t_1,\ t_2\leqslant 1/2$ and any $\lambda>0$, we get that

$$\left|\frac{M(\lambda t_1)}{M(\lambda)}-\frac{M(\lambda t_2)}{M(\lambda)}\right|\leqslant |t_1-t_2|\,\frac{\lambda p(\lambda/2)}{M(\lambda)}\leqslant 2|t_1-t_2|$$

which shows that the functions in $E_{M,\infty}$, considered as elements of $C(0, \frac{1}{2})$, are equi-continuous. In addition, these functions are uniformly bounded by 1. Thus, $E_{M,\infty}$ is a norm compact subset of $C(0, \frac{1}{2})$ and so are all the other sets defined above. It is quite clear that every $N \in C_{M,\infty}$ is an Orlicz function on $[0, \frac{1}{2}]$ (as a uniform limit of Orlicz functions) though it is even possible that $N(t)=0$ for every $t \in [0, \frac{1}{2}]$. The functions of $C_{M,\infty}$ will be extended, for convenience, to Orlicz functions defined on $[0, \infty)$. □

Remark. It is easily checked that the statement of 4.a.6 remains valid if, instead of $[0, \frac{1}{2}]$, we use any other interval $[0, t_0]$ with $0<t_0<1$. If the Δ_2-condition at zero does hold for M then, for any $0<\Lambda<\infty$, $\sup_{0<\lambda\leq\Lambda} (\lambda p(\lambda)/M(\lambda))<\infty$ which implies that the set $C_{M,\Lambda}$ is compact also when considered as a subset of $C(0, 1)$, the closure being subsequently taken also in $C(0, 1)$. We shall use this observation later on in this section.

Let M be an Orlicz function, $\{e_n\}_{n=1}^{\infty}$ the unit vectors in l_M and $u_j = \sum_{i=n_{j-1}+1}^{n_j} c_i e_i$, $j=1,2,\ldots$ $(n_0=0)$ any normalized block basis of $\{e_n\}_{n=1}^{\infty}$. To every vector u_j we associate the function $M_j(t) = \sum_{i=n_{j-1}+1}^{n_j} M(|c_i|t)$. Since $\sum_{i=n_{j-1}+1}^{n_j} M(|c_i|)=1$ it follows immediately that the functions $\{M_j\}_{j=1}^{\infty}$, as elements of $C(0, \frac{1}{2})$, belong to the set $C_{M,1}$. By 4.a.6 there exists a subsequence $\{M_{j_k}\}_{k=1}^{\infty}$ of $\{M_j\}_{j=1}^{\infty}$ and an Orlicz function $N \in C_{M,1}$, which might be degenerate, so that $|M_{j_k}(t)-N(t)| \leq 2^{-k}$, $0 \leq t \leq \frac{1}{2}$, $k=1, 2, \ldots$. Assume that N is not degenerate; then, we get that $\sum_{k=1}^{\infty} M_{j_k}(|a_k|)<\infty$ if and only if $\sum_{k=1}^{\infty} N(|a_k|)<\infty$, i.e. the subsequence $\{u_{j_k}\}_{k=1}^{\infty}$ is equivalent to the unit vector basis of h_N and $[u_{j_k}]_{k=1}^{\infty} \approx h_N$. Moreover, the map $(a_1, a_2, \ldots) \to (a_1c_1, \ldots, a_1c_{n_1}, a_2c_{n_1+1}, \ldots, a_2c_{n_2}, \ldots)$ is an isomorphism from l_N into l_M. If $N(t)=0$ for some $t>0$ then $\{u_{j_k}\}_{k=1}^{\infty}$ is equivalent to the unit vector basis of c_0 which, in this case, is isomorphic to h_N.

We also note that every subsymmetric basic sequence in h_M is equivalent to some normalized block basis of $\{e_n\}_{n=1}^{\infty}$ (see the proof of 3.b.5 and the remark thereafter). These facts and 1.a.11 prove the following result (cf. [81]).

Proposition 4.a.7. *For every Orlicz function M the following assertions are true.*

(i) *Every infinite-dimensional subspace Y of h_M contains a closed subspace Z which is isomorphic to some Orlicz sequence space h_N.*
(ii) *Let X be a subspace of h_M which has a subsymmetric basis $\{x_n\}_{n=1}^{\infty}$. Then X is isomorphic to some Orlicz sequence space h_N and $\{x_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis of h_N.*

The functions N appearing in (i) and (ii) might be degenerate. The discussion above shows that the function N appearing in 4.a.7 belongs to the compact convex set $C_{M,1}$ introduced in 4.a.6. The next result proved in [93, 94] shows that $C_{M,1}$ actually "coincides" with the collection of all subspaces of h_M which have a subsymmetric (or a symmetric) basis.

Theorem 4.a.8. *Let M be any Orlicz function. An Orlicz sequence space h_N, where N might be a degenerate Orlicz function, is isomorphic to a subspace of h_M if and only if N is equivalent to some function in $C_{M,1}$.*

Proof. We have to prove only the "if" part. Let $N \in C_{M,1}$ and observe that the extreme points of $C_{M,1}$ are contained in the compact set $E_{M,1}$. The correspondence $\lambda \to M(\lambda t)/M(\lambda)$ is a continuous map from the interval $I_0=(0, 1]$ into $E_{M,1}$ and, therefore, it may be extended uniquely to a map $\omega \to M_\omega$ from βI_0, the Stone–Čech compactification of I_0, onto $E_{M,1}$. By the Krein–Milman theorem there exists a probability measure μ on βI_0 so that

$$N(t)=\int_{\beta I_0} M_\omega(t)\, d\mu(\omega), \quad 0 \leqslant t \leqslant \tfrac{1}{2}\,.$$

For every integer n put $\Lambda_n=1$ if $\mu(I_0)>0$ and $\Lambda_n=1/2^{n+1}$ if $\mu(I_0)=0$. Then, choose a sequence of probability measures $\{\mu_n\}_{n=1}^\infty$ on βI_0 so that μ_n is supported by the interval $(0, \Lambda_n)$ and

$$|N(t)-\int_0^{\Lambda_n} (M(\lambda t)/M(\lambda))\, d\mu_n(\lambda)|<1/2^{n+1}$$

for all n and for all $t \in [0, \frac{1}{2}]$. Fix $0<\tau<1$ and, for every n and j, set $\alpha_{j,n}=\int_{\tau^j \Lambda_n}^{\tau^{j-1}\Lambda_n} d\mu_n(\lambda)/M(\lambda)$. Then,

$$\sum_{j=1}^\infty [\alpha_{j,n}]M(\tau^j \Lambda_n t)-1/2^{n+1} \leqslant N(t) \leqslant \sum_{j=1}^\infty [\alpha_{j,n}]M(\tau^{j-1}\Lambda_n t)$$
$$+M(t)\Lambda_n/(1-\tau)+1/2^{n+1}$$

for all n and all $t \in [0, \frac{1}{2}]$. Choosing integers k_n so that

$$\sum_{j=k_n+1}^\infty [\alpha_{j,n}]M(\tau^{j-1}\Lambda_n/2) \leqslant 1/2^{n+1}, \quad n=1, 2,\ldots$$

we get that

$$(*) \qquad F_n(\tau t)-1/2^{n+1} \leqslant N(t) \leqslant F_n(t)+M(t)\Lambda_n/(1-\tau)+1/2^n$$

for all n and all $t \in [0, \frac{1}{2}]$, where $F_n(t)=\sum_{j=1}^{k_n} [\alpha_{j,n}]M(\tau^{j-1}\Lambda_n t)$. We also observe that, in the case $\mu(I_0)>0$, $N(t) \geqslant \gamma M(\lambda_0 t)$ for some constants $1>\gamma$, $\lambda_0>0$ and for every $t \in [0, \frac{1}{2}]$. Thus, in this case

$$\left({*\atop *}\right) \qquad \tfrac{1}{2}(F_n(\tau t)+M(\lambda_0 t)\gamma-1/2^n) \leqslant N(t) \leqslant F_n(t)+M(t)/(1-\tau)+1/2^n\,.$$

Let $\{\eta_n\}_{n=1}^\infty$ be disjoint subsets of integers and let $\{\eta_{j,n}\}_{j=0}^\infty$ be a disjoint splitting of η_n so that $\eta_{0,n}$ consists only of one element m_n and $\eta_{j,n}$ has $[\alpha_{j,n}]$ elements for $j=1, 2,\dots, k_n$ and is void for $j>k_n$. Then, the vectors

$$u_n=\Delta_n\left[\sum_{j=1}^{k_n}\tau^j\sum_{i\in\eta_{j,n}}e_i+e_{m_n}\right]\in h_M;\quad n=1, 2,\dots$$

form a basic sequence which, by the inequalities (*) and (**), is equivalent to the unit vector basis of h_N ($\{e_i\}_{i=1}^\infty$ denotes here the unit vector basis of h_M). □

We have presented in 2.e.1 the negative solution of Tsirelson to the question whether every Banach space contains a subspace isomorphic to c_0 or to l_p for some $1\leqslant p<\infty$. Nevertheless, there are some important classes of Banach spaces which are not connected a priori to some l_p or to c_0 but for which this problem has a positive answer. The next theorem gives a positive answer in the case of Orlicz sequences spaces (and in view of 4.a.7(i) also for their subspaces) in very precise terms (cf. [93] and [95]).

Theorem 4.a.9. *The space l_p, or c_0 if $p=\infty$, is isomorphic to a subspace of an Orlicz sequence space h_M if and only if $p\in[\alpha_M, \beta_M]$ where*

$$\alpha_M=\sup\left\{q;\ \sup_{0<\lambda,t\leqslant 1} M(\lambda t)/M(\lambda)t^q<\infty\right\}\quad\textit{and}$$

$$\beta_M=\inf\left\{q;\ \inf_{0<\lambda,t\leqslant 1} M(\lambda t)/M(\lambda)t^q>0\right\}.$$

Proof. It is easily checked that we always have $1\leqslant\alpha_M\leqslant\beta_M\leqslant\infty$ and $\beta_M<\infty$ if and only if M satisfies the Δ_2-condition at zero. It follows from 4.a.4 that the Δ_2-condition holds for M if and only if the unit vectors of l_M form a boundedly complete basis. Hence, by 1.e.10, $\beta_M=\infty$ if and only if c_0 is isomorphic to a subspace of h_M. From now on we consider only finite values of p. For $p\notin[\alpha_M, \beta_M]$ it is easily seen that the function t^p is not equivalent to any function in $C_{M,1}$ and therefore, by 4.a.8, l_p is not isomorphic to a subspace of h_M. If $\alpha_M=\beta_M<\infty$ the Δ_2-condition holds for M and using the remark following 4.a.6 we can consider $C_{M,1}$ as a convex compact subset of $C(0, 1)$. Let $0<\tau<1$ and consider the map T_τ on $C_{M,1}$ defined by $T_\tau N(t)=N(\tau t)/N(\tau)$. Since $C_{M,1}\subset C(0, 1)$ T_τ is well-defined, continuous and it maps $C_{M,\Lambda}$ into $C_{M,\Lambda\tau}$ for every $0<\Lambda\leqslant 1$. Hence, by the Schauder–Tychonoff fixed point theorem [33, V.10.5] T_τ has a fixed point $N_\tau\in C_M$ which, by definition, satisfies $N_\tau(\tau t)=N_\tau(\tau)N_\tau(t)$, $0\leqslant t\leqslant 1$. Putting $q_\tau=\log N_\tau(\tau)/\log\tau$, we get that $N_\tau(\tau^n)=\tau^{nq_\tau}$ for all n. Let $t\in(0, 1)$ and choose an integer n such that $\tau^n<t\leqslant\tau^{n-1}$. Then,

$$\begin{aligned}|N_\tau(t)-t^{q_\tau}|&\leqslant|N_\tau(t)-N_\tau(\tau^n)|+|\tau^{nq_\tau}-t^{q_\tau}|\leqslant 2(\tau^{(n-1)q_\tau}-\tau^{nq_\tau})\\&\leqslant 2q_\tau(1-\tau),\quad 0\leqslant t\leqslant 1\,.\end{aligned}$$

By letting $\tau\to 1$ and using the compactness of C_M we get that $t^q\in C_M$ for some $q\geqslant 1$ (observe that $\sup_\tau q_\tau<\infty$ because of the Δ_2-condition). As remarked above, this q must be equal to $\alpha_M=\beta_M$.

In the case $\alpha_M<\beta_M$ we choose $p\in(\alpha_M,\beta_M)$. By the definition of α_M and β_M there are numbers $0<u_n<v_n<w_n\leq 1$ with $w_n\to 0$, $u_n/v_n\to 0$ such that $n\varphi(u_n)<\varphi(v_n/2)$, $n\varphi(w_n)<\varphi(v_n/2)$, where $\varphi(t)=M(t)/t^p$. Put

$$M_n(t)=A_n^{-1}\int_{u_n/w_n}^{1} M(tsw_n)s^{-p-1}\,ds\,,$$

where $A_n=\int_{u_n/w_n}^{1} M(sw_n)s^{-p-1}\,ds$. Clearly, $M_n\in C_{M,w_n}$ for all n. Let $a_n=u_n/w_n$ and $b_n=v_n/w_n$; by substituting $y=ts$ we get that $M_n(t)=A_n^{-1}t^p\int_{a_nt}^{t} M(yw_n)y^{-p-1}\,dy$. Since $\int_{a_nt}^{t}=\int_{a_n}^{1}+\int_{a_nt}^{a_n}-\int_{t}^{1}$ it follows that $M_n(t)=t^p+f_n(t)-g_n(t)$, where

$$f_n(t)=A_n^{-1}t^p\int_{a_nt}^{a_n} M(yw_n)y^{-p-1}\,dy\leq A_n^{-1}a_n^{-p}M(u_n)$$

and

$$g_n(t)=A_n^{-1}t^p\int_{t}^{1} M(yw_n)y^{-p-1}\,dy\leq A_n^{-1}M(w_n)\,.$$

On the other hand, since $b_n/a_n=v_n/u_n\to\infty$, we get, for n sufficiently large, that

$$A_n\geq\int_{b_n/2}^{b_n} M(sw_n)s^{-p-1}\,ds\geq 2^{-1}M(v_n/2)b_n^{-p}\,.$$

It follows that

$$f_n(t)\leq 2^{p+1}\varphi(u_n)/\varphi(v_n/2)\leq 2^{p+1}/n \quad\text{and}\quad g_n(t)\leq 2^{p+1}\varphi(w_n)/\varphi(v_n/2)\leq 2^{p+1}/n\,,$$

i.e. $M_n(t)\to t^p$ uniformly on $[0,\frac{1}{2}]$. Hence, $t^p\in C_M$ and this concludes the proof in view of 4.a.8 and the fact that C_M is closed. □

Let us make a few comments on 4.a.9. The proof above shows that, for any finite value of p in $[\alpha_M,\beta_M]$, we have $t^p\in C_M$; hence, as easily checked in the proof of 4.a.8, we get that h_M actually contains almost isometric copies of l_p. In the case $p=\infty$ the same is true for c_0 (use 2.e.3).

For the actual computation of the interval $[\alpha_M,\beta_M]$ it is of some interest to point out (cf. [95 p. 374]) that $\alpha_M=\sup a_N$ and $\beta_M=\inf b_N$ if $\beta_M<\infty$, where the supremum, respectively the infimum, is taken over all Orlicz functions N which are equivalent to M at zero, and

$$a_N=\liminf_{t\to 0} tN'(t)/N(t),\qquad b_N=\limsup_{t\to 0} tN'(t)/N(t)\,.$$

Theorems 4.a.8 and 4.a.9 give a complete description of those Orlicz sequence spaces and, in particular, of those l_p spaces which embed isomorphically into a given Orlicz sequence space. There are, however, some interesting questions of a more specific nature on mappings between Orlicz sequence spaces. We conclude this section by presenting some results, due to N. J. Kalton [67], which give necessary and sufficient conditions for the identity mapping between two Orlicz sequence spaces to be an isomorphism on some infinite dimensional subspace.

Theorem 4.a.10. *Let M and $\tilde{M}$ be two Orlicz functions satisfying the Δ_2-condition at zero so that the formal identity mapping $T: l_M \to l_{\tilde{M}}$ is bounded (i.e. $M(t) \geqslant A\tilde{M}(t)$ for some constant $A>0$ and for every $t \in [0, 1]$). Then T is strictly singular if and only if, for every constant $B<\infty$, there exists a finite sequence $\{\tau_i\}_{i=1}^m \subset (0, 1]$ so that*

$$\sum_{i=1}^m M(\tau_i t) \geqslant B \sum_{i=1}^m \tilde{M}(\tau_i t)$$

for every $t \in [0, 1]$.

Proof. We shall prove here only the necessity since this is the part which will be used in Section 2.c below. Fix $B<\infty$ and suppose that the formal identity mapping T from l_M into $l_{\tilde{M}}$ is a strictly singular operator. By using the Δ_2-condition and the fact that $M(t) \geqslant A\tilde{M}(t)$ for all $t \in [0, 1]$, it is easily checked that the set

$$D = D(M, \tilde{M}, B) = \overline{\operatorname{conv}}\,\{(M(\lambda t) - B\tilde{M}(\lambda t))/M(\lambda);\ 0<\lambda\leqslant 1\}$$

is a norm compact subset of $C(0, 1)$.

We shall show now that the set D contains no non-positive function. Assume the contrary, i.e. that there is $f \in D$ with $f(t) \leqslant 0$ for all $t \in [0, 1]$. Let $\omega \to M_\omega$ and $\omega \to \tilde{M}_\omega$ denote the continuous mappings from βI_0, the Stone–Čech compactification of the interval $I_0 = (0, 1]$, onto $E_{M,1}$, respectively $E_{\tilde{M},1}$, such that $M_\lambda(t) = M(\lambda t)/M(\lambda)$ and $\tilde{M}_\lambda(t) = \tilde{M}(\lambda t)/\tilde{M}(\lambda)$ for $\lambda \in I_0$. Let R_ω be the unique extension to βI_0 of the (bounded) function $\tilde{M}(\lambda)/M(\lambda)$, $\lambda \in I_0$. Then, by the definition of D, there is a probability measure μ on βI_0 so that $f(t) = \int_{\beta I_0} (M_\omega(t) - BR_\omega \tilde{M}_\omega(t))\, d\mu(\omega)$, $0 \leqslant t \leqslant 1$. It follows that

$$A \int_{\beta I_0} \tilde{M}_\omega(t) R_\omega\, d\mu(\omega) \leqslant \int_{\beta I_0} M_\omega(t)\, d\mu(\omega) \leqslant B \int_{\beta I_0} \tilde{M}_\omega(t) R_\omega\, d\mu(\omega), \quad 0 \leqslant t \leqslant 1\,.$$

Put $N(t) = \int_{\beta I_0} M_\omega(t)\, d\mu(\omega)$, $\tilde{N}(t) = \int_{\beta I_0} \tilde{M}_\omega(t) R_\omega\, d\mu(\omega)$ and notice that $N \in C_{M,1}$ and $\tilde{N}(t)/\tilde{N}(1) \in C_{\tilde{M},1}$. Hence, by 4.a.8, the unit vectors in l_N (respectively in $l_{\tilde{N}}$) form a basis equivalent to a block basis $\{u_n\}_{n=1}^\infty$ (respectively $\{\tilde{u}_n\}_{n=1}^\infty$) of the unit vectors in l_M (respectively in $l_{\tilde{M}}$). Reviewing the proof of 4.a.8 it is easy to check that the blocks $\tilde{u}_n$, $n=1, 2, \ldots$ can be constructed as to actually coincide with u_n, $n=1, 2, \ldots$. Indeed, if we keep the notation of 4.a.8 and approximate μ by the same sequence of measures $\{\mu_n\}_{n=1}^\infty$ on $(0, \Lambda_n)$, for both N and $\tilde{N}$, the forms of u_n and $\tilde{u}_n$ depend only on the numbers

$$\alpha_{j,n} = \int_{\tau^j \Lambda_n}^{\tau^{j-1}\Lambda_n} d\mu_n(\lambda)/M(\lambda), \quad \text{respectively } \tilde{\alpha}_{j,n} = \int_{\tau^j \Lambda_n}^{\tau^{j-1}\Lambda_n} (R_\lambda/\tilde{M}(\lambda))\, d\mu_n(\lambda)\,.$$

But $\alpha_{j,n}=\tilde{\alpha}_{j,n}$ for all j and n which proves that $u_n=\tilde{u}_n$. Hence, $T_{|[u_n]_{n=1}^\infty}$ is an isomorphism and this contradicts our assumption.

Since the compact set D is disjoint from $\{f; f(t)\leqslant 0\}$, by the geometric form of the Hahn–Banach theorem and by the Riesz representation theorem for functionals in $C(0, 1)$, we get that there exists a probability measure ν on $[0, 1]$ such that $\inf\left\{\int_0^1 f(t)\,d\nu(t); f\in D\right\}>0$. This measure ν can be approximated by a convex combination, with rational coefficients, of point-mass measures. Thus, there exist integers $\{n_j\}_{j=1}^k$ and reals $\{t_j\}_{j=1}^k$, with $0<t_j\leqslant 1$ for all j, so that $\sum_{j=1}^k n_j(M(\lambda t_j)-B\tilde{M}(\lambda t_j))>0$, $0<\lambda\leqslant 1$. To complete the proof of the necessity we just take as $\{\tau_i\}_{i=1}^m$ the points $\{t_j\}_{j=1}^k$, each t_j being repeated n_j times. □

The condition appearing in 4.a.10 takes a particularly simple form if $M(t)=t^p$.

Corollary 4.a.11 [67]. *Let M be an Orlicz function satisfying the Δ_2-condition at zero and such that $M(t)\geqslant At^p$ for some $A>0$, $1\leqslant p<\infty$ and for every $0\leqslant t\leqslant 1$. Then the identity mapping from l_M into l_p is a strictly singular operator if and only if*

$$\liminf_{\varepsilon\to 0}\ \inf_{0<s\leqslant 1}\frac{1}{\log 1/\varepsilon}\int_\varepsilon^1\frac{M(st)}{s^pt^{p+1}}\,dt=\infty\,.$$

Proof. Again, as in 4.a.10, we shall prove only the necessity so we assume that the identity mapping from l_M into l_p is strictly singular. By 4.a.10, for every $B>0$, there are $\{\tau_i\}_{i=1}^m\subset(0, 1]$ so that $\sum_{i=1}^m M(\tau_i su)\geqslant Bs^pu^p\sum_{i=1}^m\tau_i^p$ for every s, $u\in[0, 1]$. Put $\tau=\min_{1\leqslant i\leqslant m}\tau_i$. Then, for $0<\varepsilon<\tau^2$, we have

$$\begin{aligned}B\sum_{i=1}^m\tau_i^p\log\frac{\tau}{\varepsilon}&=B\sum_{i=1}^m\tau_i^p\int_{\varepsilon/\tau}^1\frac{du}{u}\leqslant\sum_{i=1}^m\int_{\varepsilon/\tau}^1\frac{M(\tau_i su)}{s^pu^{p+1}}\,du\\&=\sum_{i=1}^m\tau_i^p\int_{\tau_i\varepsilon/\tau}^{\tau_i}\frac{M(st)}{s^pt^{p+1}}\,dt\leqslant\sum_{i=1}^m\tau_i^p\int_\varepsilon^1\frac{M(st)}{s^pt^{p+1}}\,dt\,.\end{aligned}$$

Thus, for $0<\varepsilon<\tau^2$ and for every $s\in(0, 1]$, we get that

$$\frac{1}{\log 1/\varepsilon}\int_\varepsilon^1\frac{M(st)}{s^pt^{p+1}}\,dt\geqslant B\frac{\log\tau/\varepsilon}{\log 1/\varepsilon}\geqslant\frac{B}{2}$$

which proves our assertion since B is arbitrary. □

Using 4.a.11 it is easy to check that the identity operator from l_{M_p}, with $M_p(t)=t^p(1+|\log t|)$, into l_p is strictly singular in spite of the fact that l_{M_p} contains complemented subspaces isomorphic to l_p (cf. 4.c.1 below).

b. Duality and Complemented Subspaces

Let M be a non-degenerate Orlicz function whose right-derivative p satisfies $p(0)=0$ and $\lim_{t\to\infty} p(t)=\infty$. These restrictions exclude only the case when $M(t)$ is equivalent to t, i.e. $l_M \approx l_1$. Consider the right-inverse q of p which is defined by $q(u)=\sup\{t; p(t)\leqslant u\}$, $u\geqslant 0$. It is easily verified that q is a right-continuous non-decreasing function such that $q(0)=0$ and $q(u)>0$ whenever $u>0$. Put $M^*(u)=\int_0^u q(v)\,dv$ for $u\geqslant 0$. Then, M^* is also a non-degenerate Orlicz function and q is its right-derivative. The function M^*, defined in this way, is called *the function complementary to M*. It is clear that M is the function complementary to M^*, i.e. $M^{**}=M$.

A quick glance at the graph of p shows that, for any t and $u\geqslant 0$, we have the so-called *Young inequality*, namely

$$tu \leqslant M(t)+M^*(u),$$

with equality holding if $u=p(t)$ (or $t=q(u)$). In other words, for any $u\geqslant 0$, we have

$$uq(u)=M(q(u))+M^*(u),$$

i.e. M^* satisfies

$$M^*(u)=\max\{tu-M(t);\ 0<t<\infty\}.$$

With the aid of the complementary function M^* we can introduce a new norm on l_M by putting,

$$|||x|||_M=\sup\left\{\sum_{n=1}^{\infty} a_n b_n;\ \sum_{n=1}^{\infty} M^*(|b_n|)\leqslant 1\right\}$$

for $x=(a_1, a_2,\ldots)\in l_M$. The norm $|||\cdot|||_M$ satisfies

$$\|x\|_M \leqslant |||x|||_M \leqslant 2\|x\|_M$$

for every $x\in l_M$. The right-hand side inequality follows directly from Young's inequality. In order to prove the left-hand side inequality let $x=(a_1, a_2,\ldots)\in l_M$ be such that $|||x|||_M=1$ and take $b_n=p(|a_n|)$. Then $|a_n|b_n=M(|a_n|)+M^*(b_n)$ for all n. If $\sum_{i=1}^{\infty} M^*(b_i)>1$ it would follow by the convexity of M^* that $M^*\left(b_n\Big/\sum_{i=1}^{\infty} M^*(b_i)\right)\leqslant M^*(b_n)\Big/\sum_{i=1}^{\infty} M^*(b_i)$ and thus, $\sum_{n=1}^{\infty} M^*\left(b_n\Big/\sum_{i=1}^{\infty} M^*(b_i)\right)\leqslant 1$. Using the fact that $|||x|||_M=1$ we would get that

$$\sum_{i=1}^{\infty} M^*(b_i)\geqslant \sum_{n=1}^{\infty} |a_n|b_n=\sum_{n=1}^{\infty} M(|a_n|)+\sum_{n=1}^{\infty} M^*(b_n)$$

and this is a contradiction. Thus, $\sum_{i=1}^{\infty} M^*(b_i) \leqslant 1$ and therefore $1 \geqslant \sum_{n=1}^{\infty} |a_n| b_n \geqslant \sum_{n=1}^{\infty} M(|a_n|)$, i.e. $\|x\|_M \leqslant 1$.

The complementary function can be used to describe the dual space of an Orlicz sequence space.

Proposition 4.b.1. *Let M and M^* be complementary Orlicz functions. Then $h_M^* \approx l_{M^*}$, $l_M^* \approx h_{M^*}^{**}$ and if, in addition, M^* satisfies the Δ_2-condition at zero then $h_M^{**} \approx l_M$.*

Proof. Let $x^* \in h_M^*$ and put $c_n = x^*(e_n)$, $n=1, 2, \ldots$. Then,

$$\|(c_1, c_2, \ldots)\|_{M^*} = \sup\left\{\sum_{n=1}^{\infty} a_n c_n;\ \sum_{n=1}^{\infty} M(|a_n|) \leqslant 1\right\}$$

$$= \sup\left\{x^*\left(\sum_{n=1}^{\infty} a_n e_n\right);\ \|(a_1, a_2, \ldots)\|_M \leqslant 1\right\} = \|x^*\| .$$

As easily checked, this implies that the map $x^* \to (x^*(e_1), x^*(e_2), \ldots)$ defines an isometry from h_M^* onto l_{M^*}, endowed with $|||\cdot|||_{M^*}$, and therefore an isomorphism from h_M^* onto l_{M^*}. The other two assertions follow immediately from the existence of this isomorphism and 4.a.4. $\square$

Combining 4.b.1 with 4.a.4 we get the following criterion for reflexivity of Orlicz sequence spaces.

Proposition 4.b.2. *Let M and M^* be complementary Orlicz function. Then h_M (or l_M) is reflexive if and only if both M and M^* satisfy the Δ_2-condition at zero.*

We have already remarked that M satisfies the condition Δ_2 at zero if and only if $b_M = \limsup_{t\to 0} tp(t)/M(t) < \infty$. If we assume, in addition, that both p and q are continuous functions then M^* satisfies the Δ_2-condition if and only if $1 < a_M = \liminf_{t\to 0} tp(t)/M(t)$. More precisely, we have $a_M = b_{M^*}/(b_{M^*}-1)$. Indeed, if $uq(u)/M^*(u) \leqslant b_{M^*} + \varepsilon$ for some $\varepsilon > 0$ and $0 < u \leqslant u_0$ then

$$uq(u)/M(q(u)) \geqslant (b_{M^*}+\varepsilon)/(b_{M^*}+\varepsilon-1) .$$

Because of the continuity of q we further get $tp(t)/M(t) \geqslant (b_{M^*}+\varepsilon)/(b_{M^*}+\varepsilon-1)$ in some neighborhood of $t=0$. This shows that $a_M \geqslant b_{M^*}/(b_{M^*}-1)$. The opposite inequality is proved in a similar manner.

Sometimes the following terminology is used: an Orlicz function M is said to satisfy the *Δ_2^*-condition* at zero if $\liminf_{t\to 0} tp(t)/M(t) > 1$. We can thus restate 4.b.2 in the following form.

Proposition 4.b.2′. *Let M be an Orlicz function with a continuous strictly-increasing* *derivative. Then* l_M *the (or* h_M*) is reflexive if and only if* $1<a_M\leqslant b_M<\infty$ *(i.e. both the* Δ_2*- and the* Δ_2^**-conditions hold for* M*).*

Observe that M' is strictly increasing if and only if q, the derivative of M^*, is continuous.

For reflexive Orlicz sequence spaces the duality between subspaces and quotient spaces is reflected by the following result.

Theorem 4.b.3. *Let* l_M *be a reflexive Orlicz sequence space and let* M^* *be the function complementary to* M*. Then the following assertions hold.*

(i) *An Orlicz sequence space* l_N *is isomorphic to a quotient space of* l_M *if and only if* N^* *is equivalent to a function in* $C_{M^*,1}$.
(ii) *An Orlicz function* N *is equivalent at zero to a function in* $E_{M,1}$ *if and only if* N^* *is equivalent to a function in* $E_{M^*,1}$.
(iii) $\alpha_M^{-1}+\beta_{M^*}^{-1}=1,\ \alpha_{M^*}^{-1}+\beta_M^{-1}=1.$
(iv) l_M *contains a subspace isomorphic to* l_p *for some* $p\geqslant 1$ *if and only if* l_M *has a quotient space isomorphic to* l_p.

Before proving the theorem let us mention that Examples 4.c.1 and 4.c.2, to be presented in the sequel, show that, in general, property (ii) is not shared by the sets $C_{M,1}$ and $C_{M^*,1}$ or by C_M and C_{M^*}. In other words, condition (iv) above is not necessarily valid when the space l_p is replaced by a general Orlicz sequence space.

Proof of 4.b.3. The assertion (i) is just a restatement of 4.a.8. To prove (ii) we notice that for any $0<\lambda$, u

$$\begin{aligned}(M(\lambda t)/M(\lambda))^*(u)&=\max\{tu-M(\lambda t)/M(\lambda);\,0<t<\infty\}\\&=M^*(\lambda^{-1}M(\lambda)u)/M(\lambda)\\&=(M^*(\mu u)/M^*(\mu))\cdot(M^*(\mu)/M(\lambda))\,,\end{aligned}$$

where $\mu=\lambda^{-1}M(\lambda)$. It is easily checked that the reflexivity of l_M and 4.b.2′ imply that $\mu\to 0$, while the ratio $M^*(\mu)/M(\lambda)$ remains bounded and bounded away from zero, as $\lambda\to 0$. This completes the proof of (ii), and therefore also that of (iii), which is an immediate consequence of (ii). Finally, (iv) follows from (iii) and 4.a.9. □

A simple application of 4.a.9 and 4.b.3 is the following generalization of 2.c.3. *Let* M *and* N *be two Orlicz functions satisfying the* Δ_2*-condition at zero. Then, every bounded linear operator from* l_M *into* l_N *is compact if and only if* $\alpha_M>\beta_N$. The "if" part of this assertion is proved by using arguments similar to those used in 2.c.3. In order to prove the "only if" part assume that $\alpha_M\leqslant\beta_N$. Let T_1 be a quotient map from l_M onto l_{α_M}, let I be the formal identity map from l_{α_M} into l_{β_N} and let T_2 be an isomorphism from l_{β_N} into l_N. Then, the operator $T=T_2IT_1$ is a non-compact operator from l_M into l_N.

We turn now to the study of complemented subspaces of an Orlicz sequence space. The results proved so far yield immediately a necessary condition as well as a sufficient condition on an Orlicz function N, for l_N to be isomorphic to a complemented subspace of l_M. Assume that l_M is reflexive; then, by 4.b.3, a necessary condition is

(*) N is equivalent to a function in $C_{M,1}$ and N^* is equivalent to a function in $C_{M^*,1}$.

In order to derive a sufficient condition observe that in an Orlicz space l_M normalized block bases of the unit vector basis with constant coefficients correspond (via the general correspondence between blocks and functions in $C_{M,1}$, which was described in 4.a.8) to functions in $E_{M,1}$. Hence, by 3.a.4, a sufficient condition is

($\substack{*\\*}$) N is equivalent to a function in $E_{M,1}$.

We shall present in the next section examples which show that (*) is not a sufficient condition and that ($\substack{*\\*}$) is not a necessary condition (in both cases l_M will be reflexive and $N(t)$ equivalent to t^p for some p).

There is, however, a weaker version of ($\substack{*\\*}$) which is already a necessary condition for l_N to be isomorphic to a complemented subspace of l_M. To explain the definition below we first write down the negation of ($\substack{*\\*}$) in an explicit manner: N is not equivalent to any function in $E_{M,1}$ if and only if

(†) For every $K \geqslant 1$ there exist m_K points $t_i \in (0, 1/2)$ such that, for every $\lambda \in (0, 1)$, there is at least one index i, $1 \leqslant i \leqslant m_K$ for which

$$M(\lambda t_i)/M(\lambda)N(t_i) \notin [K^{-1}, K]\,.$$

Definition 4.b.4. Let M be an Orlicz function. A function N is said to be *strongly non-equivalent* to $E_{M,1}$ if (†) holds with the additional requirement that m_K can be chosen so that $m_K = o(K^\alpha)$ as $K \to \infty$, for every $\alpha > 0$.

Theorem 4.b.5 [94]. *Let l_M be a separable Orlicz sequence space and N an Orlicz function which is strongly non-equivalent to $E_{M,1}$. Then l_N is not isomorphic to a complemented subspace of l_M.*

Proof. Suppose that l_N is isomorphic to a complemented subspace of l_M and let $\{e_n\}_{n=1}^\infty$ denote the unit vector basis of l_M. By 1.a.12 and 1.a.9(ii) there exists a normalized block basis $w_j = \sum_{i \in \sigma_j} a_i e_i$, $j = 1, 2, \ldots$ of $\{e_n\}_{n=1}^\infty$ such that $\{w_j\}_{j=1}^\infty$ is equivalent to the unit vector basis of l_N and there exists a projection P from l_M onto $[w_j]_{j=1}^\infty$. By passing to a subsequence and changing the signs of the coefficients, if necessary, we may assume that $a_i > 0$ for all $i \in \sigma_j$ and that the functions $N_j(t) = \sum_{i \in \sigma_j} M(a_i t)$ satisfy $|N_j(t) - \tilde{N}(t)| \leqslant 2^{-j}$, $j = 1, 2, \ldots$ for some Orlicz function $\tilde{N} \in C_{M,1}$ equivalent to N and for every $t \in [0, 1]$. We also observe that, since M satisfies the Δ_2-condition at zero, there exists a $p < \infty$ so that $M(st) \leqslant s^p M(t)$ for $s > 1$ and every $t > 0$.

Assume now that N is strongly non-equivalent to $E_{M,1}$. Then, there are a number K and m_K points $t_h \in (0, 1)$, $h = 1, 2, \ldots, m_K$ so that

$$m_K / K^{1/p} \leqslant \min\,(2^{-1} \cdot 4^{-1/p} \|P\|^{-1}, 2^{-p-1} \|P\|^{-p})$$

and, for every $\lambda \in (0, 1)$, there exists at least one h, $1 \leqslant h \leqslant m_K$ for which $M(\lambda t_h)/M(\lambda)\tilde{N}(t_h) \notin [K^{-1}, K]$. By passing to a subsequence of $\{N_j\}_{j=1}^{\infty}$, if necessary, we may assume without loss of generality that $|N_j(t) - \tilde{N}(t)| \leqslant \min \{\tilde{N}(t_h);\ 1 \leqslant h \leqslant m_K\}$ for all $t \in [0, 1]$.

We split now each of the sets σ_j into $2m_K$ disjoint subsets of integers δ_j^h and η_j^h so that, for every $1 \leqslant h \leqslant m_K$, we have $M(a_i t_h)/M(a_i)\tilde{N}(t_h) < K^{-1}$ if $i \in \delta_j^h$ and $M(a_i t_h)/M(a_i)\tilde{N}(t_h) > K$ for $i \in \eta_j^h$. Then, for $j \geqslant 1$ and $1 \leqslant h \leqslant m_K$, we have

$$K\tilde{N}(t_h) \sum_{i \in \eta_j^h} M(a_i) < \sum_{i \in \eta_j^h} M(a_i t_h) \leqslant N_j(t_h) \leqslant 2\tilde{N}(t_h)$$

which implies that $\sum_{i \in \eta_j^h} M(a_i) \leqslant 2/K$. Thus, $\sum_{h=1}^{m_K} \sum_{i \in \eta_j^h} M(a_i) \leqslant 2m_K/K$ for $j \geqslant 1$.

Every function $F \in C_{M,1}$ satisfies $F(st)/F(t) \leqslant s^p$ for all $s > 1$ and $t > 0$. In particular, if we put $F_j(t) = \sum_{h=1}^{m_K} \sum_{i \in \eta_j^h} M(a_i t)$ then $F_j(t)/F_j(1) \in C_{M,1}$ and therefore

$$F_j(2\|P\|) \leqslant 2^p \|P\|^p F_j(1) \leqslant 2^{p+1} \|P\|^p m_K/K, \quad j \geqslant 1\ .$$

The condition imposed on the ratio m_K/K implies that $F_j(2\|P\|) \leqslant 1$, i.e. the vectors $v_j = \sum_{h=1}^{m_K} \sum_{i \in \eta_j^h} a_i e_i$, $j \geqslant 1$, have norms $\leqslant 1/2\|P\|$.

Put, for $1 \leqslant h \leqslant m_K$ and $j = 1, 2, \ldots$, $u_j^h = \sum_{i \in \delta_j^h} a_i e_i$ and let Q_j be the norm one projection from l_M onto $[e_i]_{i \in \sigma_j}$. Then, $w_j = Q_j P w_j = \sum_{h=1}^{m_K} Q_j P u_j^h + Q_j P v_j$, $j = 1, 2, \ldots$ which implies that

$$\sum_{h=1}^{m_K} \|Q_j P u_j^h\| \geqslant \|w_j\| - \|Q_j P v_j\| \geqslant 1 - \|P\| \cdot \|v_j\| \geqslant \tfrac{1}{2}\ .$$

Hence, for every $j \geqslant 1$, there exists at least one index $1 \leqslant h_j \leqslant m_K$ such that $\|Q_j P u_j^{h_j}\| \geqslant 1/2m_K$. It follows that if we put $P u_j^{h_j} = \sum_{i=1}^{\infty} d_{i,j} w_i$ then

$$|d_{j,j}| = \|Q_j P u_j^{h_j}\| \geqslant 1/2m_K$$

for all $j \geqslant 1$. By 1.c.8 the linear operator $D\colon [u_j^{h_j}]_{j=1}^{\infty} \to [w_j]_{j=1}^{\infty}$, defined by $D u_j^{h_j} = d_{j,j} w_j$, $j \geqslant 1$ (i.e. the "diagonal" of P), is bounded and $\|D\| \leqslant \|P\|$. Consequently, for any set of coefficients $\{b_j\}_{j=1}^{J}$, we have

$$\left\|\sum_{j=1}^{J} b_j w_j\right\| \leqslant 2m_K \left\|\sum_{j=1}^{J} b_j d_{j,j} w_j\right\| \leqslant 2m_K \|P\| \cdot \left\|\sum_{j=1}^{J} b_j u_j^{h_j}\right\| \cdot$$

Choosing an integer J so that $2 \leqslant \sum_{j=1}^{J} \tilde{N}(t_{h_j}) \leqslant 3$ we get that $1 \leqslant \sum_{j=1}^{J} N_j(t_{h_j}) \leqslant 4$. Thus, in view of the correspondence between the functions $\{N_j\}_{j=1}^J$ and the blocks $\{w_j\}_{j=1}^J$, we have

$$1 \leqslant \left\|\sum_{j=1}^{J} t_{h_j} w_j\right\| \leqslant 2m_K\|P\| \cdot \left\|\sum_{j=1}^{J} t_{h_j} u_j^{h_j}\right\| \cdot$$

On the other hand,

$$\sum_{j=1}^{J} \sum_{i \in \delta_j^{h_j}} M(a_i t_{h_j}) \leqslant K^{-1} \sum_{j=1}^{J} \tilde{N}(t_{h_j}) \sum_{i \in \delta_j^{h_j}} M(a_i) \leqslant 3K^{-1}.$$

Therefore, by the fact that $F(st)/F(t) \leqslant s^p$, for all $F \in C_{M,1}$ and $t>0$ we conclude that $\left\|\sum_{j=1}^{J} t_{h_j} u_j^{h_j}\right\| \leqslant 3^{1/p} K^{-1/p}$. This implies that $1/2m_K\|P\| \leqslant 3^{1/p}K^{-1/p}$, which contradicts the choice of K and m_K. $\square$

The results on subspaces of Orlicz sequence spaces (and especially the method used to prove 4.a.9) put in evidence the mappings $T_\lambda \colon C_{M,1} \to C_{M,1}$ defined by $(T_\lambda N)(t) = N(\lambda t)/N(\lambda)$. The pair $(C_{M,1}, \{T_\lambda\})$ forms what is called a flow in topological dynamics. Some standard notions and reasonings from topological dynamics yield interesting facts on Orlicz sequence spaces if we consider this particular flow. For instance, the notion of a minimal set from topological dynamics has some applications in our context. The notion of a minimal Orlicz function M will be defined only for those M which satisfy the Δ_2-condition. Therefore, using the remark following 4.a.6, we can and shall consider the sets $E_{M,1}$, $C_{M,1}$, etc. as compact subsets of $C(0, 1)$ (rather than subsets of $C(0, \frac{1}{2})$). In this way the mapping T_λ is well-defined for every $0<\lambda<1$.

Definition 4.b.6. An Orlicz function M satisfying the Δ_2-condition at zero is called *minimal* if the set $E_{M,1}$ has no proper closed subsets which are invariant under the flow $(T_\lambda; 0<\lambda<1)$. In other words, if $E_{N,1} = E_{M,1}$ for every $N \in E_{M,1}$.

Let M be any Orlicz function satisfying the Δ_2-condition at zero. A standard application of Zorn's lemma to the set $E_{M,1}$, endowed with the order $F \prec G \Leftrightarrow F \in E_{G,1}$, shows that $E_{M,1}$ *contains at least one minimal Orlicz function.*

Minimal Orlicz sequence spaces have the following property.

Proposition 4.b.7 [94]. *Let M be a minimal Orlicz function. Then every block basis with constant coefficients of the unit vector basis of l_M spans a subspace which is isomorphic to l_M itself.*

Proof. It follows from 3.a.5 that if U is a subspace of l_M which is spanned by a block basis with constant coefficients then $l_M \approx l_M \oplus U$. On the other hand, by 4.a.7 and its proof, $U \approx l_N \oplus V$ for some Orlicz function $N \in E_{M,1}$ and some Banach space V. Since M is minimal we have $M \in E_{N,1}$ and thus, by (*), $U \approx l_M \oplus W$ for some space W. Consequently,

$$l_M \approx l_M \oplus U \approx l_M \oplus l_M \oplus W \approx l_M \oplus W \approx U. \quad \square$$

It is worthwhile to compare 4.b.7 with the second remark following 2.a.9: for a minimal Orlicz function M which is not equivalent to any t^p the isomorphism between U and l_M cannot be always induced by mapping the n'th block to the n'th unit vector.

We shall present later on in this chapter some examples of minimal Orlicz functions which are not equivalent to any t^p. Actually, we will see that for any interval $[\alpha, \beta]$ there is a minimal Orlicz function M with $\alpha_M=\alpha$ and $\beta_M=\beta$; the only restriction being that $\alpha>1$ (it is easily verified that if M is minimal and $\alpha_M=1$ then $M(t)=ct$).

The proposition above states that for minimal Orlicz sequence spaces the "obvious" complemented subspaces of l_M (i.e. those spanned by block bases with constant coefficients) are necessarily isomorphic to l_M. It is possible that this is also true for any other complemented subspaces. We thus formulate.

Problem 4.b.8. *Assume that M is a minimal Orlicz function. Is then l_M a prime Banach space?*

It would be actually of interest to decide whether l_M is prime even for a single example of a minimal function M other than t^p.

In Section 3.b we have seen that the universal space U_1 of Pelczynski has uncountably many mutually non-equivalent symmetric bases. Among the Orlicz sequence spaces there are also many examples of spaces having at least two non-equivalent symmetric bases. The construction of such spaces is based on the following remark: *if l_M is isomorphic to l_N but M is not equivalent to N then l_M has at least two non-equivalent symmetric bases, namely the unit vector bases of l_M and of l_N.*

This observation can be used in the case of minimal Orlicz functions in order to prove the next result.

Theorem 4.b.9 [95]. *Let M be a minimal Orlicz function which is not equivalent to any t^p, $1 \leqslant p<\infty$. Then l_M has uncountably many mutually non-equivalent symmetric bases.*

Proof. It follows from the definition of minimality, condition (⁑) and the decomposition method of Pelczynski that, for every $N \in E_{M,1}$, the space l_N is isomorphic to l_M. Therefore, in view of the preceding remark, it suffices to show that $E_{M,1}$ contains uncountably many mutually non-equivalent functions.

Assume that there are only countably many equivalence classes in $E_{M,1}$ and denote by $\{M_n\}_{n=1}^{\infty}$ their representatives. Then, by Baire's category theorem, one of the sets

$$F_{n,k}=\{N \in E_{M,1};\, k^{-1} \leqslant N(t)/M_n(t) \leqslant k,\ 0<t \leqslant 1\},$$

whose union covers entirely the set $E_{M,1}$, contains a relatively open set G. By minimality there exists, for every $N \in E_{M,1}$, a $\lambda \in (0, 1)$ such that $N(\lambda t)/N(\lambda) \in G$

and thus, $E_{M,1}$ consists of exactly one equivalence class. In order to complete the proof it suffices to show that all the functions in $E_{M,1}$ are uniformly equivalent to M. For $0<\lambda<1$ we put $G_\lambda=\{N\in E_{M,1};\ N(\lambda t)/N(\lambda)\in G\}$. The sets G_λ are open and, as remarked above, $E_{M,1}=\bigcup\ \{G_\lambda;\ 0<\lambda\leqslant 1\}$. Hence, by the compactness of $E_{M,1}$, there is a $\lambda_0>0$ such that $E_{M,1}=\bigcup\{G_\lambda;\ \lambda_0\leqslant\lambda\leqslant 1\}$. It follows that, for every $0<\mu\leqslant 1$, there is a λ with $\lambda_0\leqslant\lambda\leqslant 1$ such that $M(\lambda\mu t)/M(\lambda\mu)\in G$, i.e. $k^{-1}\leqslant M(\lambda\mu t)/M(\lambda\mu)M(t)\leqslant k$ (if we assume, as we may, that $G\subset F_{n,k}$ and $M_n=M$). Since $\lambda\geqslant\lambda_0$ the Δ_2-condition implies the existence of a constant $A>0$ so that, for all $0<\mu,\ t\leqslant 1$,

$$A^{-1}\leqslant M(\mu t)/M(\mu)M(t)\leqslant A\ .$$

This implies that M is equivalent to t^p for some $1\leqslant p<\infty$, contrary to the assumption. □

The concept of a minimal Orlicz sequence space can be extended in a natural manner to the more general setting of spaces with a symmetric basis. A symmetric basis $\{x_n\}_{n=1}^\infty$ of a Banach space X is said to be *minimal symmetric* provided every block basis of $\{x_n\}_{n=1}^\infty$ with constant coefficients spans a subspace which is isomorphic to the whole space X.

In view of 4.b.9 it is natural to ask whether the only spaces with a minimal symmetric basis which is unique, up to equivalence, are c_0 and l_p, $1\leqslant p<\infty$.

There are many interesting examples of Orlicz sequence spaces which do have, up to equivalence, a unique symmetric basis. A sufficient condition for this to happen is given in the next proposition (cf. [93]).

Proposition 4.b.10. *Let M be an Orlicz function for which the set C_M contains no Orlicz function equivalent to M itself. Then the unit vector basis is, up to equivalence, the unique symmetric basis of h_M.*

Proof. If h_M has in addition to the unit vector basis $\{e_n\}_{n=1}^\infty$ another symmetric basis $\{f_n\}_{n=1}^\infty$ then each of these two bases is equivalent to a block basis of the other. It is easily seen that if, in both block basis representations, the coefficients do not tend to zero then $\{f_n\}_{n=1}^\infty$ is equivalent to $\{e_n\}_{n=1}^\infty$. If this is not the case then $\{e_n\}_{n=1}^\infty$ is equivalent to a block basis of itself with coefficients tending to zero. In view of 4.a.8 and its proof this implies that C_M contains a function which is equivalent to M itself. □

A simple consequences of 4.b.10 is that *Orlicz sequence spaces h_M for which $\lim_{t\to 0} tM'(t)/M(t)$ exists (i.e. $a_M=b_M$) have, up to equivalence, a unique symmetric basis.* This follows from the fact that, in this case, C_M consists only of one Orlicz function, namely t^p, provided $p=\lim_{t\to 0} tM'(t)/M(t)$ is finite or $f(t)=0$ in $[0, 1/2]$ if $p=\infty$.

Besides minimal Orlicz functions it is of interest to consider some maximal ones. More precisely, we shall construct Orlicz functions $U_{c,d}$ which are universal for the class of all Orlicz functions M with $[\alpha_M, \beta_M] \subset (c, d)$. We need first the following lemma.

Lemma 4.b.11. *Let F and G be two continuous non-decreasing convex functions, defined on an interval $[\tau, 1]$ with $0<\tau<1$. Assume that*

(i) *$F(1)=G(1)=1$, $0<F(\tau)<1$, $0<G(\tau)<1$ and,*
(ii) *for some numbers c and d such that $1<c<d$, $F'(1)=G'(1)=c$ and $c \leqslant tF'(t)/F(t) \leqslant d$, $c \leqslant tG'(t)/G(t) \leqslant d$ for every $t \in [\tau, 1]$ (here F' and G' stand for the right-derivatives except for $t=1$ where they mean the left-derivatives).*

Then the function

$$H(t)=\begin{cases} F(t), & \tau \leqslant t \leqslant 1 \\ F(\tau)G(t/\tau), & \tau^2 \leqslant t < \tau \end{cases}$$

is continuous, convex, non-decreasing and $H'(1)=c \leqslant tH'(t)/H(t) \leqslant d$ for every $t \in [\tau^2, 1]$.

The proof is straightforward.

Theorem 4.b.12. *For every $1 \leqslant c<d<\infty$ there exists an Orlicz function $U=U_{c,d}$ such that*

(i) *$c \leqslant tU'(t)/U(t) \leqslant d$ for all $t \in [0, 1]$,*
(ii) *for every Orlicz function M with $c \leqslant tM'(t)/M(t) \leqslant d$ for all $t \in (0, 1]$ there exists in E_U a function equivalent to M,*
(iii) *there exists a constant $K_{c,d}$ such that, for every M with $c \leqslant tM'(t)/M(t) \leqslant d$, there is a norm-one projection P_M in l_U such that $d(P_M l_U, l_M) \leqslant K_{c,d}$.*

Proof. Assume first that $c>1$ and choose a sequence $\{N_n(t)\}_{n=1}^\infty$ of Orlicz functions which is dense in the set $\mathscr{F}$ of all Orlicz functions N satisfying $N(1)=1$, $N'(1)=c$ and $c \leqslant tN'(t)/N(t) \leqslant d$ for all $t \in (0, 1]$. Put $\tau_n=2^{-2^{n-1}}$, $n=1, 2,\dots$ and define

$$U(t)=\begin{cases} N_1(t), & \tau_1 \leqslant t \leqslant 1 \\ N_n(t/\tau_n)U(\tau_n), & \tau_{n+1} \leqslant t < \tau_n,\ n=1, 2,\dots \\ 0, & t=0\,. \end{cases}$$

In view of 4.b.11, U is an Orlicz function defined on the entire interval $[0,1]$ and such that $c \leqslant tU'(t)/U(t) \leqslant d$ for all $0<t \leqslant 1$, Moreover, for every n and every $\tau_n \leqslant t \leqslant 1$, we get that $U(\tau_n t)/U(\tau_n)=N_n(t)$ which implies that E_U contains all the functions of $\mathscr{F}$.

Let now M be any Orlicz function such that $M(1)=1$ and $c \leqslant tM'(t)/M(t) \leqslant d$ for all $0<t \leqslant 1$. Choose $t_1=t_{c,d}$ so that $t_1c(d-1)/(c-1)d=1/2$ and t_2 so that $c(M'(t_1)(t_2-t_1)+M(t_1))=M'(t_1)t_2$. It is easily verified that

$$t_1 \leqslant t_2 \leqslant t_1c(d-1)/(c-1)d=1/2\,.$$

Define

$$M_0(t)=\begin{cases} M(t), & 0\leqslant t\leqslant t_1 \\ M(t_1)+M'(t_1)(t-t_1), & t_1<t\leqslant t_2 \\ (M(t_1)+M'(t_1)(t_2-t_1))(t/t_2)^c, & t_2<t\leqslant 1\,. \end{cases}$$

Then the function $M_1(t)=M_0(t)/M_0(1)\in\mathscr{F}$ and $K^{-1}M_1(t)\leqslant M(t)\leqslant KM_1(t)$ for all $t\in[0, 1]$ and for some constant $K=K_{c,d}$, independent of M. This concludes the proof in the case $c>1$.

The case $c=1$ can be reduced to $c>1$ in the following manner. Fix $d>1$ and let $U=U_{2,d+1}$ be a universal function corresponding to the numbers 2 and $d+1$ (whose existence follows from the previous case). Put $U_0(t)=U(t)/t$. In general, U_0 need not be convex so, instead, we consider the function $U_1(t)=\int_0^t (U_0(s)/s)\,ds$. It is easily checked that U_1 is an Orlicz function satisfying $1\leqslant tU_1'(t)/U_1(t)\leqslant d$ and $U_1(t)\leqslant U_0(t)\leqslant dU_1(t)$ for all $0<t\leqslant 1$. We shall show that $U_1=U_{1,d}$ has all the desired properties. Let N be any Orlicz function such that $N(1)=1$ and $1\leqslant tN'(t)/N(t)\leqslant d$ for all $0<t\leqslant 1$. Then, $M(t)=tN(t)$ satisfies $M(1)=1$ and $2\leqslant tM'(t)/M(t)\leqslant d+1$ for all $0<t\leqslant 1$; hence, by the first part of the proof, E_U contains a function M_1 such that $K^{-1}M_1(t)\leqslant M(t)\leqslant KM_1(t)$, $0\leqslant t\leqslant 1$, with K being a constant independent of M. This means that, for some sequence $\lambda_n\to 0$, $U(\lambda_n t)/U(\lambda_n)\to M_1(t)$ for all $0<t\leqslant 1$. By passing to a subsequence of $\{\lambda_n\}_{n=1}^\infty$, if needed, we can assume with no loss of generality that, for some $N_1\in E_{U_1}$, $U_1(\lambda_n t)/U_1(\lambda_n)\to N_1(t)$ uniformly for $0\leqslant t\leqslant 1$. It is easily verified that $(dK)^{-1}N(t)\leqslant N_1(t)\leqslant dKN(t)$ for all $0\leqslant t\leqslant 1$. □

Remarks. It follows immediately from ($\substack{*\\*}$) and Pelczynski's decomposition method that $l_{U_{c,d}}$ is determined uniquely, up to an isomorphism, by c and d. If we choose c and d so that $c^{-1}+d^{-1}=1$ then $U_{c,d}^*$ is also universal for the same c and d. Consequently, $l_{U_{c,d}}$ is isomorphic to $l_{U_{c,d}}^*$. This constitutes a non-trivial example, i.e. different from l_2, of *a Banach space with a symmetric basis which is isomorphic to its own conjugate*. The fact that all the universal spaces corresponding to the same c and d are isomorphic to each other shows that $l_{U_{c,d}}$ is another example of a space with infinitely many mutually non-equivalent symmetric bases. Indeed, it suffices to observe that, by proper rearrangements of the sequence $\{N_n\}_{n=1}^\infty$ in the construction of $U_{c,d}$, we obtain uncountably many universal functions $\{U_{c,d}^{(\alpha)}\}_\alpha$ such that, for $\alpha\neq\beta$, $U_{c,d}^{(\alpha)}$ is not equivalent at zero to $U_{c,d}^{(\beta)}$.

c. Examples of Orlicz Sequence Spaces

The difference between the structure of Orlicz sequence spaces and that of l_p spaces is best illustrated by considering suitable examples. In the beginning of this section we present some examples of Orlicz functions which are given by concrete formulas. In addition to these examples we shall describe a general method of constructing Orlicz functions by using sequences of zeros and ones.

Example 4.c.1. *A reflexive Orlicz sequence space l_M with a unique symmetric basis such that the sets $C_{M^*,1}$ and $C^*_{M,1}=\{N^*;\, N\in C_{M,1}\}$ have a totally different structure (recall that, up to equivalence at zero, the set $E_{M^*,1}$ coincides with $E^*_{M,1}$ in view of 4.b.3.*(ii)).

Let $M(t)=t^p\,|\log t|^\alpha$ with $1<p<\infty$ and $\alpha>0$. It is easily checked that M is an Orlicz function on some interval $[0, t_0]$ with $t_0>0$. Thus, M can be extended to an Orlicz function on $[0,\infty)$ but for the present discussion the values of M outside a neighborhood of $t=0$ are of no importance. A trivial computation shows that $\lim_{t\to 0} tM'(t)/M(t)=p$. Hence, by 4.b.2′, l_M is reflexive and, by 4.b.10, the unit vectors are, up to equivalence, the unique symmetric basis of l_M. We also have

$$\lim_{\lambda\to 0} M(\lambda t)/M(\lambda)=\lim_{\lambda\to 0} t^p(1+\log t/\log\lambda)^\alpha=t^p,\quad 0<t\leqslant t_0\,.$$

Therefore, the sets E_M and C_M consist both of only one function, namely t^p. It follows that the set $E_{M,1}$ has exactly two equivalence classes: t^p and functions equivalent to M itself. Also $C_{M,1}$ consists of these two equivalent classes. Indeed, let $N(t)=\int_0^{t_0} (M(\lambda t)/M(\lambda))\, d\mu(\lambda)$, for some probability measure μ on $[0, t_0]$ (where $M(0t)/M(0)$ stands for t^p). Observe that, for a fixed $0<t\leqslant t_0$, $M(\lambda t)/M(\lambda)$ is an increasing function of λ. Hence, for every $0<t_1<t_0$,

$$M(t_0t)/M(t_0)\geqslant N(t)\geqslant \mu([t_1, t_0])M(t_1t)/M(t_1)$$

and, unless μ is concentrated in the origin, N is equivalent to M. We conclude that, up to an isomorphism, the only Orlicz sequence subspaces of l_M are l_p and l_M itself, and both are also complemented subspaces of l_M.

In order to study the quotient spaces of l_M we have to compute the complementary function M^*. The exact computation of M^* is quite complicated but, for our purposes, it suffices to find a function which is equivalent to M^*. Observe first that M is equivalent at zero to $M_1(t)=\int_0^t f(s)\, ds$, where $f(s)=s^{p-1}\,|\log s|^\alpha$. Take q so that $p^{-1}+q^{-1}=1$ and put $g(t)=t^{q-1}\,|\log t|^{\alpha(1-q)}$. Since

$$f(g(t))=g^{p-1}(t)\,|\log g(t)|^\alpha=t\,|\log t|^{-\alpha}|(q-1)\log t+\alpha(1-q)\log|\log t||^\alpha$$

we get that $\lim_{t\to 0} f(g(t))/t=(q-1)^\alpha$. This implies that g is equivalent at zero to the inverse function of f and thus, M^* is equivalent to $\int_0^t g(s)\, ds$ which, in turn, is equivalent to the function $t^q\,|\log t|^{\alpha(1-q)}$. By abuse of notation we shall put $M^*(t)=t^q\,|\log t|^{\alpha(1-q)}$.

It is easily verified that E_{M^*} and C_{M^*} consist only of one function, namely t^q. However, the set $C_{M^*,1}$ turns out to contain infinitely many equivalence classes: for example, for every $0<\varepsilon<\alpha(q-1)$, $C_{M^*,1}$ contains a function equivalent to

$t^q\,|\log t|^{-\varepsilon}$. Indeed, the function $N_\varepsilon(t)=\int_0^{e^{-1}}(M^*(\lambda t)/M^*(\lambda))\lambda^{-1}\,|\log\lambda|^{-1-\varepsilon}\,d\lambda$ clearly satisfies $N_\varepsilon(t)/N_\varepsilon(1)\in C_{M^*,1}$ and

$$\begin{aligned}N_\varepsilon(t)&=t^q\int_0^{e^{-1}}\left(\frac{\log\lambda}{\log\lambda+\log t}\right)^{\alpha(q-1)}\lambda^{-1}\,|\log\lambda|^{-1-\varepsilon}d\lambda\\&=t^q\int_1^{\infty}\left(\frac{u}{u+|\log t|}\right)^{\alpha(q-1)}u^{-1-\varepsilon}\,du\\&=t^q\,|\log t|^{-\varepsilon}\int_{|\log t|^{-1}}^{\infty}\left(\frac{v}{1+v}\right)^{\alpha(q-1)}v^{-1-\varepsilon}\,dv\\&=Ct^q\,|\log t|^{-\varepsilon}+t^q\cdot O(|\log t|^{\alpha(1-q)})\quad\text{as } t\to 0\,,\end{aligned}$$

for some constant $C>0$. It follows that N_ε is equivalent to $t^q\,|\log t|^{-\varepsilon}$.

It is clear from the discussion in Section b above that, for any $N\in C_{M^*,1}$ such that N is not equivalent to a function in $E_{M^*,1}$ (i.e. to M^* itself or to t^q), l_N is isomorphic to a subspace of l_{M^*} (or, equivalently, l_{N^*} is isomorphic to a quotient space of l_M) but not to a complemented subspace of l_{M^*}. □

We consider next an example introduced in [81] and investigated in [81] and [94].

Example 4.c.2. *A reflexive Orlicz sequence space l_M with a unique symmetric basis for which $\alpha_M<\beta_M$ and C_{M^*} is different from $C_M^*=\{N^*;N\in C_M\}$ (recall that, in 4.c.1, $C_{M^*}=C_M^*=\{t^q\}$).*

Let $M(t)=t^{p+\sin(\log|\log t|)}$; it is easily checked that, for $p-\sqrt{2}>1$, M is an Orlicz function in some neighborhood of $t=0$. Put

$$U(\lambda)=\lambda M'(\lambda)/M(\lambda)=p+\sin(\log|\log\lambda|)+\cos(\log|\log\lambda|)\,.$$

Then, for any $1>\delta>0$, $\lim_{\lambda\to 0}(U(\lambda s)-U(\lambda))=0$ uniformly for $s\in[\delta,1]$. Suppose now that $\lim_{n\to\infty}U(\lambda_n)=r$ for some $r\in[a_M,b_M]$ and some sequence $\lambda_n\to 0$. By using the uniformity of the limit above and by passing to a subsequence, if needed, we can assume that

$$r-n^{-1}\leqslant U(\lambda_n s)\leqslant r+n^{-1}\quad\text{for } 2^{-n}\leqslant s\leqslant 1\,.$$

Integrating this inequality between t and 1 we get that

$$t^{r+n^{-1}}\leqslant M(\lambda_n t)/M(\lambda_n)\leqslant t^{r-n^{-1}}\quad\text{for } 2^{-n}\leqslant t\leqslant 1\,,$$

i.e. $M(\lambda_n t)/M(\lambda_n)$ tends to t^r. It follows that $E_M=\{t^r;p-\sqrt{2}\leqslant r\leqslant p+\sqrt{2}\}$ while C_M consists of all the functions N which can be represented as

$$N(t)=\int_{p-\sqrt{2}}^{p+\sqrt{2}}t^r\,d\mu(r)$$

for some probability measure μ on $[p-\sqrt{2}, p+\sqrt{2}]$. By taking, for instance, μ to be uniformly distributed on the interval $[r, p+\sqrt{2}]$ we get a function equivalent to $t^r/|\log t|$. A simple computation shows that if $N\in C_M$ is represented by a measure μ and r is the smallest number in the support of μ then $\lim_{\lambda\to 0} N(\lambda t)/N(\lambda)=t^r$. Consequently, C_M contains no function equivalent to M itself and, by 4.b.10, l_M has, up to equivalence, a unique symmetric basis.

In order to prove that the sets C_M^* and C_{M^*} are different we shall show that every Orlicz function, which is simultaneously equivalent to a function in C_M and to a function in $C_{M^*}^*$, is already equivalent to a function in E_M, i.e. to t^{r_0} for some $p-\sqrt{2}\leqslant r_0\leqslant p+\sqrt{2}$. Indeed, let $N\in C_M$ and let r_0 be the smallest number in the support of the measure μ representing N. Then, as remarked above, $E_N=\{t^{r_0}\}$ and

$$N(t)/t^{r_0}=\int_{r_0}^{p+\sqrt{2}} t^{r-r_0}\,d\mu(r)\leqslant \mu([r_0, r])+t^{r-r_0}\quad \text{for } r_0<r\leqslant p+\sqrt{2}\,.$$

If N is not equivalent to t^{r_0} then $\mu(\{r_0\})=0$ and hence, $\lim_{t\to 0} N(t)/t^{r_0}=0$. It follows that $\lim_{t\to 0} N^*(t)/t^{q_0}=\infty$, where $q_0^{-1}+r_0^{-1}=1$.

On the other hand, if N^* is equivalent at zero to a function in

$$C_{M^*}=\overline{\operatorname{conv}}\,\{t^s;\ s_1\leqslant s\leqslant s_2\}\,,$$

where $s_1^{-1}+(p+\sqrt{2})^{-1}=1$ and $s_2^{-1}+(p-\sqrt{2})^{-1}=1$, then, up to equivalence,

$$N^*(t)=\int_{s_1}^{s_2} t^s\,d\nu(s)$$

for some probability measure ν on $[s_1, s_2]$. Since $E_{N^*}=\{t^{q_0}\}$ the smallest number in the support of ν must be q_0. This however is a contradiction since it implies that $\limsup_{t\to 0} N^*(t)/t^{q_0}\leqslant 1$. □

The following example, constructed by N. J. Kalton [67], shows that Theorem 4.b.5 is no longer valid if we replace "strong non-equivalence" by "non-equivalence".

Example 4.c.3. *A separable Orlicz sequence space l_M which contains complemented subspaces isomorphic to l_p for some $1\leqslant p<\infty$ but t^p is not equivalent to any function in $E_{M,1}$.*

We first define a sequence of functions $\{f_n\}_{n=1}^{\infty}$ on $[0,\infty)$ in the following way: for each integer n, f_n is the function of period $P_n=2^{2^{2n}}$ such that

$$f_n(t)=\begin{cases}0, & 0\leqslant t\leqslant P_n-4\cdot 2^n\\ \frac{1}{2}(t-P_n)+2\cdot 2^n, & P_n-4\cdot 2^n<t\leqslant P_n-2\cdot 2^n\\ \frac{1}{2}(P_n-t), & P_n-2\cdot 2^n<t\leqslant P_n\,.\end{cases}$$

It is easily seen that f_n is a continuous function on $[0,\infty)$ whose maximal value is equal to 2^n. Let $f(t)=\max\{f_n(t);\, n=1,2,\ldots\}$ and observe that $|f(t_1)-f(t_2)|\leqslant \frac{1}{2}|t_1-t_2|$ for any $t_1, t_2>0$. Fix $p>3/2$ and put $M(t)=t^p e^{f(-\log t)}$, for $0<t\leqslant 1$, and $M(0)=0$. The function M is continuous but not necessarily convex on $[0,1]$. However, it is easily checked that, with M' standing for the right-derivative of M, we have

$$p-\tfrac{1}{2}\leqslant tM'(t)/M(t)=p-f'(-\log t)\leqslant p+\tfrac{1}{2}$$

for every $t\in(0,1]$. By the condition imposed on p we get that $M(t)/t$ is an increasing function and thus, M is equivalent at zero to an Orlicz function M_1, defined on $[0,\infty)$ (take, e.g. $M_1(t)=\int_0^t (M(u)/u)\,du$). Moreover, M_1 satisfies the Δ_2-condition at zero since $tM'(t)/M(t)\leqslant p+\frac{1}{2}$.

It is evident from the definition of M that the formal identity mapping T from l_{M_1} into l_p is a bounded operator. We shall show that T is not strictly singular. This would imply that l_{M_1} contains a complemented subspace isomorphic to l_p. Indeed, if W is an infinite-dimensional subspace of l_{M_1} for which $T_1=T_{|W}$ is an isomorphism then, by 2.a.2, TW contains a subspace $V\approx l_p$ so that there exists a bounded projection P from l_p onto V. It is easily checked that $Q=T_1^{-1}PT$ is a bounded projection from l_{M_1} onto its subspace $T_1^{-1}V$, which is clearly isomorphic to l_p.

To prove that T is not a strictly singular operator we use 4.a.11. Put $\varepsilon_n=e^{-P_n}$, $n=1,2,\ldots$. Then,

$$\frac{1}{\log 1/\varepsilon_n}\int_{\varepsilon_n}^{1}\frac{M(t)}{t^{p+1}}\,dt=\frac{1}{P_n}\int_{\varepsilon_n}^{1}\frac{e^{f(-\log t)}}{t}\,dt=\frac{1}{P_n}\int_0^{P_n} e^{f(u)}\,du\,.$$

Notice that, for $t\in[0,P_n]$, $f(t)=\max\{f_i(t);\,1\leqslant i\leqslant n\}$. Hence, by using the definition of f_i, we get that

$$\frac{1}{\log 1/\varepsilon_n}\int_{\varepsilon_n}^{1}\frac{M(t)}{t^{p+1}}\,dt\leqslant\frac{1}{P_n}\sum_{i=1}^{n}\int_0^{P_n} e^{f_i(u)}\,du\leqslant\frac{1}{P_n}\sum_{i=1}^{n}\frac{P_n}{P_i}4\cdot 2^i\cdot e^{2^i}$$

$$\leqslant 4\sum_{i=1}^{\infty}2^{-2^{2i}}\cdot 2^i\cdot e^{2^i}<\infty\,,$$

for every integer n, and this proves that T is strictly singular.

It remains to show that t^p is not equivalent to any function in $E_{M_1,1}$. Assume to the contrary that there exists a constant $K>1$ such that, for any $\tau>0$, there is a $u=u(\tau)$ for which $e^{-K}t^p\leqslant M(e^{-u}t)/M(e^{-u})\leqslant e^K t^p$, $e^{-\tau}\leqslant t\leqslant 1$ or, equivalently $|f(u+v)-f(u)|\leqslant K$, $0\leqslant v\leqslant\tau$. Thus, for $u\leqslant t_1$, $t_2\leqslant u+\tau$, we have

$$|f(t_2)-f(t_1)|\leqslant 2K\,.$$

For each $t>0$ let $n(t)$ be the least integer for which $f_n(t)=f(t)$. Choose $\tau>3\cdot 2^{64K^2}$ and observe that if $0<f(t)<2^{n(t)}-2K$ for some $u+\tau/3\leqslant t\leqslant u+2\tau/3$ then there exists a t_1 so that $f(t)+3K>f_{n(t)}(t_1)>f(t)+2K$ and $|t_1-t|=2(f_{n(t)}(t_1)-f(t))<6K<\tau/3$. It follows that $u<t_1<u+\tau$ and this contradicts the fact that $f(t_1)-f(t)>2K$. Thus, for every $u+\tau/3\leqslant t\leqslant u+2\tau/3$, either $f(t)=0$ or $f(t)\geqslant 2^{n(t)}-2K$. Since $\tau/3\geqslant P_2$ there exists a $t\in[u+\tau/3, u+2\tau/3]$ so that $n(t)>1$. We also notice that $n(t)$ is not a constant on $[u+\tau/3, u+2\tau/3]$ since this would imply that the variation of f on this interval exceeds $\tau/12>2K$. Hence, if $t_1, t\in[u+\tau/3, u+2\tau/3]$ are chosen so that $n(t)-n(t_1)\geqslant 1$ then $f(t)\neq 0$ (otherwise $n(t)=1$) and therefore $f(t)\geqslant 2^{n(t)}-2K\geqslant 2\cdot 2^{n(t_1)}-2K\geqslant 2f(t_1)-2K$, i.e. $f(t_1)\leqslant 2K+f(t)-f(t_1)\leqslant 4K$. Consequently,

$$2^{n(t)}\leqslant 2K+f(t)\leqslant 2K+f(t)-f(t_1)+f(t_1)\leqslant 8K\,.$$

It follows that, for every $t\in[u+\tau/3, u+2\tau/3]$, we have $f(t)=\max\{f_i(t);\ 1\leqslant i\leqslant n_0\}$, where n_0 is the largest integer satisfying $2^{n_0}\leqslant 8K$. Hence, on this interval, f has period $P_{n_0}=2^{2^{2n_0}}\leqslant 2^{64K^2}<\tau/3$ which means that f takes both values 0 and 2^{n_0} there. This however implies that the variation of f on $[u+\tau/3, u+2\tau/3]$ is equal to $2^{n_0}\geqslant 4K$, which leads to a contradiction. $\square$

We present now a general procedure of constructing (or representing) Orlicz functions M in a form in which the set $E_{M,1}$ can be easily described (cf. [95]).

Fix $0<\tau<1$ and let F and G be two strictly increasing continuous convex functions on the interval $[\tau, 1]$ such that

(i) $F(1)=G(1)=1,\ 0<F(\tau)<1,\ 0<G(\tau)<1$.
(ii) $F'(1)=G'(1)$, $F'(1)\leqslant tF'(t)/F(t)$ and $G'(1)\leqslant tG'(t)/G(t)$ for all $t\in[\tau, 1]$.
(iii) $F(\tau)=\tau^{p_1}$ and $G(\tau)=\tau^{p_2}$ for some $1<p_1<p_2$.

For every sequence of digits $\eta=\{\eta(n)\}_{n=1}^{\infty}$ with $\eta(n)$ equal to 0 or to 1 for each n we define a function M_η on $[0, 1]$ in the following way. We put $M_\eta(1)=1$, $M_\eta(0)=0$ and, for $\tau^n\leqslant t<\tau^{n-1}$, $n=1, 2,\ldots$

$$M_\eta(t)=\begin{cases} M_\eta(\tau^{n-1})F(t/\tau^{n-1}) & \text{if } \eta(n)=0 \\ M_\eta(\tau^{n-1})G(t/\tau^{n-1}) & \text{if } \eta(n)=1\,. \end{cases}$$

Using 4.b.11 it is easily verified that M_η is indeed an Orlicz function on $[0, 1]$ which satisfies the Δ_2-condition at zero.

The function M_η has the following quite obvious properties.

(a) $M_\eta(\tau^k)=\tau^{p_1k+(p_2-p_1)\sum_{n=1}^{k}\eta(n)}$, $k=1, 2,\ldots$ which implies that, up to equivalence, M_η is solely determined by p_1, p_2 and η and does not depend on the particular choice of F and G.

(b) For the same τ, p_1 and p_2 and for two different sequences of digits $\eta=\{\eta(n)\}_{n=1}^{\infty}$ and $\rho=\{\rho(n)\}_{n=1}^{\infty}$, the functions M_η and M_ρ are equivalent if and only if

$$\sup_k\left|\sum_{n=1}^{k}\eta(n)-\sum_{n=1}^{k}\rho(n)\right|<\infty.$$

(c) For fixed τ, p_1 and p_2, the set of all the functions of the form M_η, with η being a sequence of zeros and ones, is a norm compact subset of $C(0, 1)$ and the map $\eta \to M_\eta$ is a homeomorphism from $\{0, 1\}^{\aleph_0}$, equipped with the product topology, into $C(0, 1)$.

(d) Consider the map T defined by $(TN)(t) = N(\tau t)/N(\tau)$ and let Φ be the shift by one to the left, i.e. $(\Phi\eta)(n) = \eta(n+1)$. Then $TM_\eta = M_{\Phi\eta}$.

(e) Up to equivalence, $E_{M_\eta, 1}$ consists of functions of the form M_ρ (for the same τ, p_1 and p_2) with ρ being a pointwise limit of sequences having the form $\{\Phi^{k_j}\eta\}_{j=1}^\infty$.

The following proposition describes the interval $[\alpha_{M_\eta}, \beta_{M_\eta}]$ corresponding to an Orlicz function of the type M_η (see 4.a.9).

Proposition 4.c.4. *Let τ, p_1, p_2, η and M_η be as above. Then,*

$$\alpha_{M_\eta} = p_1 + (p_2 - p_1) \liminf_{k \to \infty} \inf_n \delta(n+1, n+k)\,,$$

$$\beta_{M_\eta} = p_1 + (p_2 - p_1) \limsup_{k \to \infty} \sup_n \delta(n+1, n+k)\,,$$

where $\delta(n+1, n+k)$ denotes the density of ones between the numbers $n+1$ and $n+k$, i.e. $\delta(n+1, n+k) = \sum_{i=n+1}^{n+k} \eta(i)/k$.

Proof. Since M_η satisfies the Δ_2-condition at zero it suffices to consider in the definition of α_{M_η} only expressions of the form

$$M_\eta(\tau^n \cdot \tau^k)/M_\eta(\tau^n)\tau^{pk} = \tau^{(p_1 - p)k + (p_2 - p_1) \sum_{i=n+1}^{n+k} \eta(i)}\,.$$

Hence, α_{M_η} is the supremum of all numbers p for which

$$\inf_{n, k} \left((p_1 - p)k + (p_2 - p_1) \sum_{i=n+1}^{n+k} \eta(i) \right) > -\infty\,.$$

A simple argument shows that α_{M_η} is therefore equal to the expression given in the statement. The proof for β_{M_η} is similar. □

We present now a characterization of minimal Orlicz function of the form M_η.

Proposition 4.c.5. *Let M_η be as above. The function M_η is equivalent to a minimal Orlicz function if and only if there exists a constant K such that, for every integer k, there is an integer $n = n(k)$ with the following property: for every integer h there is an $m \leq n$ such that*

$$\left| \sum_{i=h+m+1}^{h+m+j} \eta(i) - \sum_{i=1}^{j} \eta(i) \right| \leq K \quad \textit{for } j = 1, 2, \ldots, k\,.$$

Proof. Assume first that such K and $n(k)$ do exist. Let $\{h_l\}_{l=1}^\infty$ be an increasing sequence of integers for which the pointwise limit of $\{\Phi^{h_l}\eta\}_{l=1}^{\infty}$ exists. Denote this limit sequence by ρ and fix k. Then there exists an index h_l so that

$$\rho(i)=(\Phi^{h_l}\eta)(i)=\eta(h_l+i) \quad \text{for } 1\leqslant i\leqslant k+n(k)\,.$$

By our assumption, there is an $1\leqslant m\leqslant n(k)$ so that

$$\left|\sum_{i=m+1}^{m+j}\rho(i)-\sum_{i=1}^{j}\eta(i)\right|=\left|\sum_{i=h_l+m+1}^{h_l+m+j}\eta(i)-\sum_{i=1}^{j}\eta(i)\right|\leqslant K, \quad j=1,2,\ldots,h\,.$$

Hence, by using the observations (e) and (b) above, it follows that $E_{M_\rho,1}$ contains a function equivalent to M_η. By the remark preceding 4.b.7 it follows that M_η is equivalent to a minimal Orlicz function.

In order to prove the converse we suppose that there exists a minimal function N such that $A^{-1}\leqslant N(t)/M_\eta(t)\leqslant A$ for some constant $A>0$ and every $0<t\leqslant 1$. Assume now that the condition described in the statement is not satisfied for any K and $n(k)$. This means that, for every integer K, there exist an integer $k(K)$ and sequences $\{h_1(n,K)\}_{n=1}^{\infty}$, $\{h_2(n,K)\}_{n=1}^{\infty}$ for which $h_2(n,K)-h_1(n,K)=n$, $n=1,2,\ldots$ and such that, for every $h_1(n,K)\leqslant h<h_2(n,K)$, there is a $j\leqslant k(K)$ with

$$\left|\sum_{i=h+1}^{h+j}\eta(i)-\sum_{i=1}^{j}\eta(i)\right|>K\,.$$

For a fixed K let η_K be any limit point of the sequence $\{\Phi^{h_1(n,K)}\eta\}_{n=1}^{\infty}$. Then $M_{\eta_K}\in E_{M_\eta,1}$ and, for every integer m, there is a $j\leqslant k(K)$ so that

$$\left|\sum_{i=m+1}^{m+j}\eta_K(i)-\sum_{i=1}^{j}\eta(i)\right|>K\,.$$

It follows from the observations (a), (b) and (e) that, for any $N_0\in E_{M_{\eta_K}}$, there exists $0<t\leqslant 1$ for which the ratio $N_0(t)/M_\eta(t)$ is not in the interval $[B^{-1}\tau^{K(p_2-p_1)}, B\tau^{-K(p_2-p_1)}]$, where B is a constant depending only on F and G. This however contradicts the minimality of N when K is chosen so that $B\tau^{-K(p_2-p_1)}>A^4$. $\square$

The previous propositions will now be used to study some concrete minimal Orlicz sequence spaces.

Example 4.c.6 [94, 95]. *A minimal Orlicz sequence space l_M whose interval $[\alpha_M,\beta_M]$ reduces to a single point p and which does not have any complemented subspace isomorphic to l_p.*

Let $0<\tau<1$ and $1<p_1<p_2$ be arbitrary. We construct simultaneously two sequences of zeros and ones, $\eta=\{\eta(i)\}_{i=1}^{\infty}$ and $\rho=\{\rho(i)\}_{i=1}^{\infty}$, as follows. Put $\eta(1)=0$, $\rho(1)=1$ and, for $n=0,1,2,\ldots$,

$$\begin{aligned}
&\eta(2^{3n}+i)=\rho(i) &&\text{and}\quad \rho(2^{3n}+i)=\rho(i) &&\text{for } 1\leqslant i\leqslant 2^{3n},\\
&\eta(2^{3n+1}+i)=\eta(i) &&\text{and}\quad \rho(2^{3n+1}+i)=\rho(i) &&\text{for } 1\leqslant i\leqslant 2^{3n+1},\\
&\eta(2^{3n+2}+i)=\eta(i) &&\text{and}\quad \rho(2^{3n+2}+i)=\eta(i) &&\text{for } 1\leqslant i\leqslant 2^{3n+2}.
\end{aligned}$$

Thus, these two sequences begin as follows

$$\eta=(0,\overbrace{1},\overbrace{0,1},\overbrace{0,1,0,1},\overbrace{1,1,1,1,0,1,0,1},\ldots)\,,$$
$$\rho=(1,\overbrace{1},\overbrace{1,1},\overbrace{0,1,0,1},\overbrace{1,1,1,1,0,1,0,1},\ldots)\,.$$

We shall prove first that M_η (and also M_ρ) is equivalent to a minimal Orlicz function. For every n let A_n (respectively B_n) be the block consisting of the first 2^{3n} digits in η (respectively ρ). By the inductive definition of η and ρ both A_{n+1} and B_{n+1} contain a block equal to A_n and a block equal to B_n. Since ρ can be written as a succession of blocks $C_1C_2C_3\ldots$, where each block C_j is equal either to A_{n+1} or to B_{n+1}, it follows that every block of ρ of length $\geqslant 3\cdot 2^{3(n+1)}$ contains in it either A_{n+1} or B_{n+1} and thus, both A_n and B_n. This implies that the condition 4.c.5 holds for M_η and also for M_ρ, with $K=0$ and $n(k)=3\cdot 2^{3(n+1)}$ whenever $k\leqslant 2^{3n}$. This proves that both M_η and M_ρ are equivalent to minimal functions.

The functions M_η and M_ρ are not equivalent at zero because

$$\sum_{i=1}^{2^{3n}}(\rho(i)-\eta(i))=2^n \quad \text{for all } n\,.$$

On the other hand, η is the pointwise limit of the sequence $\{\Phi^{2^{3n+2}}\rho\}_{n=1}^\infty$ and ρ that of $\{\Phi^{2^{3n}}\eta\}_{n=1}^\infty$. Hence, $M_\eta\in E_{M_\rho,1}$ and $M_\rho\in E_{M_\eta,1}$ which, by condition $(\ast)$ of the preceding section, implies that $l_{M_\rho}\approx l_{M_\eta}$. We therefore conclude that neither M_η nor M_ρ is equivalent to t^p for some $p\geqslant 1$. We note in passing that, by 4.b.9, l_{M_η} has uncountably many mutually non-equivalent symmetric bases.

In order to determine the interval associated to M_η let us denote by a_n (respectively b_n) the number of times the digit 1 appears in the block A_n (respectively B_n). Then, $a_0=0$, $b_0=1$ and

$$a_{n+1}=4a_n+4b_n,\qquad b_{n+1}=2a_n+6b_n,\quad n=0,1,2,\ldots\,.$$

The density of ones in the block A_n (respectively B_n) is equal to $a_n/2^{3n}$ (respectively $b_n/2^{3n}$). Easy computations show that $(b_n-a_n)/2^{3n}=1/4^n$ and $a_n/2^{3n}=\sum_{i=0}^{n-1}1/2\cdot 4^i$ for all n. It follows immediately that $\lim_{n\to\infty}a_n/2^{3n}=\lim_{n\to\infty}b_n/2^{3n}=2/3$. Let $\varepsilon>0$ be given and choose an integer n so that $|a_n/2^{3n}-2/3|<\varepsilon$ and $|b_n/2^{3n}-2/3|<\varepsilon$. It is easily checked that the density of ones in any block of ρ of length $2^{3n}\cdot l$ is between $(l-2)(2/3-\varepsilon)/l$ and $((l-2)(2/3+\varepsilon)+2)/l$. Therefore, for l large enough, the density is between $2/3-2\varepsilon$ and $2/3+2\varepsilon$. By 4.c.3 it follows that $\alpha_{M_\rho}=\beta_{M_\rho}=p_1+2(p_2-p_1)/3$. Since l_{M_ρ} is isomorphic to l_{M_η} we get that M_η has the same interval.

It remains to show that, for $p=p_1+2(p_2-p_1)/3$, l_{M_ρ} contains no complemented subspace isomorphic to l_p. In order to prove this we use 4.b.5 and show that the function t^p is strongly non-equivalent to $E_{M_\rho,1}$. For an integer n put $m(n)=3\cdot 2^{3(n+1)}+2^{3n}$ and assume the existence of an integer k and of a constant $K>0$ so that

$$K^{-1}\tau^{pi}\leqslant M_\rho(\tau^k\cdot\tau^i)/M_\rho(\tau^k)\leqslant K\tau^{pi},\quad i=1,2,\ldots,m(n)\,.$$

Let $1 \leqslant j \leqslant 3 \cdot 2^{3(n+1)}$; by using the above inequality with $i=j$ and $i=j+2^{3n}$ we get

$$K^{-2}\tau^{p \cdot 2^{3n}} \leqslant M_\rho(\tau^{k+j}\tau^{2^{3n}})/M_\rho(\tau^{k+j}) \leqslant K^2 \tau^{p \cdot 2^{3n}}, \quad 1 \leqslant j \leqslant 3 \cdot 2^{3(n+1)}.$$

Recall now that every block of ρ of length $\geqslant 3 \cdot 2^{3(n+1)}$ contains a block equal to A_n and a block equal to B_n. This implies that there are integers j_1, j_2, $1 \leqslant j_1, j_2 < 3 \cdot 2^{3(n+1)}$ for which

$$M_\rho(\tau^{k+j_1}\tau^{2^{3n}})/M_\rho(\tau^{k+j_1}) = M_\rho(\tau^{2^{3n}}), \quad M_\rho(\tau^{k+j_2}\tau^{2^{3n}})/M_\rho(\tau^{k+j_2}) = M_\eta(\tau^{2^{3n}}).$$

It follows that $K^{-4} \leqslant M_\rho(\tau^{2^{3n}})/M_\eta(\tau^{2^{3n}}) \leqslant K^4$. On the other hand,

$$M_\rho(\tau^{2^{3n}})/M_\eta(\tau^{2^{3n}}) = \tau^{(p_2-p_1)\left(\sum_{i=1}^{2^{3n}} \rho(i) - \sum_{i=1}^{2^{3n}} \eta(i)\right)} = \tau^{(p_2-p_1)2^n}.$$

This means that for $\tau^{-(p_2-p_1)2^{n-1}} \leqslant K < \tau^{-(p_2-p_1)2^{n-2}}$ it suffices to take $m_K = 3 \cdot 2^{3(n+1)} + 2^{3n}$ points in order to prove that condition (†) of 4.b.4 holds. Since, for any $\alpha > 0$, $m_K = o(K^\alpha)$ we get that t^p is strongly non-equivalent to $E_{M_\rho, 1}$. □

The next example is similar to 4.c.6 with the exception that its interval can be chosen arbitrarily.

Example 4.c.7 [95]. *A minimal Orlicz sequence space l_M with $\alpha_M < \beta_M$ which contains no complemented subspace isomorphic to l_p for $p \geqslant 1$.*

Let $0 < \tau < 1$ and $1 < p_1 < p_2$ be arbitrarily chosen. For every $0 < \alpha < \beta < 1$ construct a sequence of positive integers $\{n_j\}_{j=1}^\infty$ such that $\sum_{j=1}^\infty 1/n_j \leqslant \alpha$ and $\prod_{j=1}^\infty (1 - 1/n_j) \geqslant \beta$. We define two sequences ρ and η of zeros and ones, as follows. Let $m_j = n_1 \cdot n_2 \ldots n_{j-1} (m_1 = 1)$ and let A_j (respectively B_j) denote the block of the first m_j digits of ρ (respectively η). A_1 consists of the digit 1 and B_1 of the digit 0 while, for $j > 1$, A_j and B_j are defined inductively by

$$A_{j+1} = \overbrace{A_j A_j \ldots A_j}^{n_j - 1 \text{ times}} B_j, \qquad B_{j+1} = \overbrace{B_j B_j \ldots B_j}^{n_j - 1 \text{ times}} A_j.$$

The same argument as in 4.c.6 shows that, for this ρ, the Orlicz function M_ρ is minimal (the condition in the statement of 4.c.5 is satisfied with $K=0$ and $n(m_{j-1}) = 3m_j$). To estimate the size of the interval of M_ρ we remark that the density of ones in A_j is larger than $\prod_{i=1}^{j-1} (1 - 1/n_i) \geqslant \beta$ while the density of ones in B_j is less than $\sum_{i=1}^{j-1} 1/n_i \leqslant \alpha$. It follows from 4.c.4 that $\alpha_{M_\rho} \leqslant p_1 + \alpha(p_2 - p_1)$ and $\beta_{M_\rho} \geqslant p_1 + \beta(p_2 - p_1)$, i.e. $\beta_{M_\rho} - \alpha_{M_\rho} = (\beta - \alpha)(p_2 - p_1) > 0$.

In spite of the fact that l_{M_ρ} has subspaces isomorphic to l_p for an entire interval of p's it does not admit any l_p as a complemented subspace. Since the proof is similar to that presented in 4.c.6 we do not reproduce it here. □

We conclude the study of functions of the form M_η by proving that they actually represent all reflexive Orlicz sequence spaces.

Proposition 4.c.8. *For every Orlicz function M such that l_M is reflexive there exist $1<p_1<p_2$, $0<\tau<1$ and a sequence $\eta=\{\eta(n)\}_{n=1}^\infty$ of zeros and ones such that the corresponding function M_η is equivalent to M.*

Proof. We may assume that $M(1)=1$. Since l_M is reflexive it follows from 4.b.2′ that, for some $1<p_1<p_2$ and all $0<t\leqslant 1$, $p_1\leqslant tM'(t)/M(t)\leqslant p_2$. Choose $0<\tau<1$ so that $p_1\tau-p_1+1=\tau^{p_2}$. Then, the functions $F(t)=t^{p_1}$ and $G(t)=p_1t-p_1+1$ satisfy in the interval $[\tau, 1]$ all the assumptions appearing in the definition of functions of the type M_η.

Now, we construct inductively a sequence $\eta=\{\eta(n)\}_{n=1}^\infty$ as follows. We set $\eta(1)=1$ and if $M_\eta(\tau^n)\tau^{p_1}\leqslant M(\tau^{n+1})$ we put $\eta(n+1)=0$; otherwise, $\eta(n+1)=1$. It is easily verified that

$$M_\eta(\tau^n)\leqslant M(\tau^n)\leqslant \tau^{p_1-p_2}M_\eta(\tau^n) \quad \text{for all } n\geqslant 1 ,$$

i.e. M_η is equivalent to M. □

Remark. The construction above does not work for a non-reflexive space l_M satisfying the Δ_2-condition because, in this case, p_1 is necessarily equal to 1 and thus, 4.b.11 cannot be used. However, the same construction can be still performed with $F(t)=t$, $G(t)=t^{p_2}$ and $0<\tau<1$ arbitrarily chosen. The function $M_\eta(t)$ obtained in this way is equivalent to $M(t)$ and $M_\eta(t)/t$ in an increasing function but, in general, $M_\eta(t)$ need not be convex.

d. Modular Sequence Spaces and Subspaces of $l_p \oplus l_r$

In the study of Orlicz sequence spaces we have already encountered spaces which are generated by a sequence of Orlicz functions. More precisely, we have seen that, for any block basis $\{u_n\}_{n=1}^\infty$ of the unit vector basis of an Orlicz sequence space l_M, there exists a sequence of Orlicz functions $\{M_n\}_{n=1}^\infty$ such that a series $\sum_{n=1}^\infty a_nu_n$ converges if and only if $\sum_{n=1}^\infty M_n(|a_n|/\rho)<\infty$ for some $\rho>0$. It is therefore natural to consider the following class of sequence spaces.

Definition 4.d.1. Let $\{M_n\}_{n=1}^\infty$ be a sequence of Orlicz functions. The space $l_{\{M_n\}}$, is the Banach space of all sequences $x=(a_1, a_2,\dots)$ with $\sum_{n=1}^\infty M_n(|a_n|/\rho)<\infty$ for some $\rho>0$, equipped with the norm

$$||x||=\inf\left\{\rho>0;\ \sum_{n=1}^\infty M_n(|a_n|/\rho)\leqslant 1\right\}.$$

The space $l_{\{M_n\}}$ is called a *modular sequence space*.

An important subspace of $l_{\{M_n\}}$ is $h_{\{M_n\}}$ which consists of those sequences $x=(a_1, a_2,\dots)\in l_{\{M_n\}}$ such that $\sum_{n=1}^\infty M_n(|a_n|/\rho)<\infty$ *for every* $\rho>0$.

The notion of equivalence between sequences of Orlicz functions is defined in the following way: *two sequences of Orlicz functions $\{M_n\}_{n=1}^{\infty}$ and $\{N_n\}_{n=1}^{\infty}$ are said to be equivalent if $l_{\{M_n\}}$ and $l_{\{N_n\}}$ are equal as sets, i.e. they consist of the same sequences.* It is easily checked, using the closed graph theorem, that if $\{M_n\}_{n=1}^{\infty}$ and $\{N_n\}_{n=1}^{\infty}$ are equivalent then the identity map from $l_{\{M_n\}}$ onto $l_{\{N_n\}}$ is an isomorphism. Analytic conditions for the equivalence of two sequences of Orlicz functions are in general quite unnatural. A sufficient condition, used often in the sequel, is the following: *$\{M_n\}_{n=1}^{\infty}$ and $\{N_n\}_{n=1}^{\infty}$ are equivalent provided there exist numbers $K>0$, $t_n \geq 0$, $n=1, 2,\dots$ and an integer n_0 so that*

(a) $K^{-1}M_n(t) \leq N_n(t) \leq KM_n(t)$ *for all* $n \geq n_0$ *and* $t \geq t_n$

(b) $\sum_{n=1}^{\infty} M_n(t_n) < \infty$.

The proof is straightforward.

In order to avoid technical difficulties which arise in some non-interesting cases we assume, unless stated otherwise, that the functions $\{M_n\}_{n=1}^{\infty}$, $\{N_n\}_{n=1}^{\infty}$ etc., are nondegenerate, strictly increasing and have a derivative for every $t \geq 0$. A sequence of Orlicz functions $\{M_n\}_{n=1}^{\infty}$ is said to be normalized if $M_n(1)=1$ for all n. If $\{M_n\}_{n=1}^{\infty}$ is not normalized we put $N_n(t)=M_n(\tau_n t)$, where $M_n(\tau_n)=1$. Then, $\{N_n\}_{n=1}^{\infty}$ is normalized and $l_{\{N_n\}}$ is isomorphic to $l_{\{M_n\}}$. Since we consider here only linear topological properties of modular sequence spaces we can and shall assume that $\{M_n\}_{n=1}^{\infty}$ is a normalized sequence.

In the study of Orlicz sequence spaces a crucial role is played by the Δ_2-condition at zero. What we need in the present case is a "uniform" Δ_2-condition.

Definition 4.d.2. A sequence of Orlicz functions $\{M_n\}_{n=1}^{\infty}$ is said to satisfy the *uniform Δ_2-condition (Δ_2^*-condition)* at zero if there exist a number $p>1$ $(r>1)$ and an integer n_0 such that, for all $t \in (0, 1)$ and $n \geq n_0$, we have $tM_n'(t)/M_n(t) \leq p$ $(tM_n'(t)/M_n(t) > r)$.

An equivalent definition of the uniform Δ_2-condition at zero is obtained by requiring the existence of a constant $K<\infty$ and of an integer n_0 such that $M_n(2t)/M_n(t) \leq K$ for all $n \geq n_0$ and $t \in (0, 1/2]$. The uniform Δ_2-condition is not preserved by equivalence. For example, let $\{M_n\}_{n=1}^{\infty}$ be any sequence of Orlicz functions for which the uniform Δ_2-condition at zero holds. Choose real numbers $t_n>0$, $n=1, 2,\dots$ so that $\sum_{n=1}^{\infty} M_n(t_n)<\infty$; then, we can easily construct Orlicz functions N_n, $n=1, 2,\dots$ which do not satisfy the Δ_2-condition (even individually) but such that $N_n(t)=M_n(t)$ for all n and $t \geq t_n$. As observed above, $\{M_n\}_{n=1}^{\infty}$ and $\{N_n\}_{n=1}^{\infty}$ are equivalent.

The importance of the uniform Δ_2-condition is illustrated by the following result.

Proposition 4.d.3 [144]. *For any sequence of Orlicz functions $\{M_n\}_{n=1}^{\infty}$ the following conditions are equivalent.*

(i) *The sequence $\{M_n\}_{n=1}^{\infty}$ is equivalent to a sequence $\{N_n\}_{n=1}^{\infty}$ which satisfies the uniform Δ_2-condition at zero.*

(ii) $l_{\{M_n\}}=h_{\{M_n\}}$.
(iii) *The unit vectors form a boundedly complete normalized unconditional basis of* $l_{\{M_n\}}$.
(iv) $l_{\{M_n\}}$ *is separable.*
(v) $l_{\{M_n\}}$ *contains no subspace isomorphic to* l_∞.

Notice that the unit vectors are normalized for it is assumed that $M_n(1)=1$ for all n. The proof of 4.d.3 is quite similar to that of 4.a.4 and we do not reproduce it here.

We pass now to another routine matter, namely that of duality of modular spaces. Let $\{M_n\}_{n=1}^\infty$ be a sequence of Orlicz functions such that none of which is equivalent to t. Let $\{M_n^*\}_{n=1}^\infty$ be the corresponding sequence of complementary functions. In general, the functions M_n^* do not satisfy $M_n^*(1)=1$ for all n but we still can consider the space $l_{\{M_n^*\}}$. For every $y=(a_1, a_2, \dots) \in l_{\{M_n^*\}}$ we put

$$|||y|||=\sup\left\{\sum_{n=1}^\infty a_n b_n;\ \sum_{n=1}^\infty M(|b_n|)\leqslant 1\right\}.$$

It follows from Young's inequality that $||y||\leqslant |||y|||\leqslant 2||y||$ for every $y \in l_{\{M_n^*\}}$.

Let $x^* \in h_{\{M_n\}}^*$ and put $c_n=x^*(e_n)$, $n=1, 2, \dots$, where $\{e_n\}_{n=1}^\infty$ denotes, as usual, the sequence of the unit vectors. Then, we have

$$|||(c_1, c_2, \dots)|||=\sup\left\{x^*\left(\sum_{n=1}^\infty b_n e_n\right);\ \sum_{n=1}^\infty M(|b_n|)\leqslant 1\right\}=||x^*||\,,$$

i.e. the mapping $x^* \to (x^*(e_1), x^*(e_2), \dots)$ defines an isomorphism from $h_{\{M_n\}}^*$ into $l_{\{M_n^*\}}$. It is easily verified that this isomorphism is onto, i.e. $h_{\{M_n\}}^* \approx l_{\{M_n^*\}}$.

We present now without proof a result which generalizes 4.b.2′ to the case of modular sequence spaces.

Proposition 4.d.4 [144]. *A modular sequence space* $l_{\{M_n\}}$ *is reflexive if and only if* $\{M_n\}_{n=1}^\infty$ *is equivalent to a sequence of Orlicz functions* $\{N_n\}_{n=1}^\infty$ *for which the uniform* Δ_2- *and* Δ_2^*-*conditions hold.*

In the sequel we shall be interested only in modular sequence spaces satisfying the uniform Δ_2-condition at zero. These spaces are related to Orlicz sequence spaces in a very simple way.

Theorem 4.d.5. *Let* $1\leqslant q<s<\infty$. *A Banach space is a modular sequence space* $l_{\{M_n\}}$, *with* M_n *satisfying* $q\leqslant tM_n'(t)/M_n(t)\leqslant s$ *for all* n *and* $t \in (0, 1)$, *if and only if it is isomorphic to the closed linear span of a block basis of the unit vectors in some Orlicz sequence space* l_M, *with* M *satisfying* $q\leqslant tM'(t)/M(t)\leqslant s$ *for every* $t>0$.

Proof. The "if" part is immediate (use, e.g. the argument preceding 4.a.7). Conversely, let $\{M_n\}_{n=1}^\infty$ be a sequence of Orlicz functions satisfying our hypotheses for some $1\leqslant q<s<\infty$. Let $U=U_{q,s}$ be the universal Orlicz function whose existence is ensured by 4.b.13. The function U has the property that there are a constant $K=K_{q,s}$ and functions $N_n \in E_U$, $n=1, 2, \dots$ such that $K^{-1}\leqslant N_n(t)/M_n(t)\leqslant K$ for all n and

$t \in (0, 1)$. The arguments already used in the proof of 4.a.8 show that there is a block basis $\{u_n\}_{n=1}^\infty$ (with constant coefficients) of the unit vector basis of l_U such that $[u_n]_{n=1}^\infty \approx l_{\{N_n\}} \approx l_{\{M_n\}}$. We remark that $[u_n]_{n=1}^\infty$ is actually a complemented subspace of l_U. □

We turn now our attention to a special class of modular sequence spaces. Fix $1 \leqslant r < p < \infty$ and let $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ be the unit vector bases of l_p, respectively l_r. For a sequence $w = \{w_n\}_{n=1}^\infty$ of positive reals put $e_n = f_n + w_n g_n$, $n = 1, 2, \ldots$ and let $X_{p,r,w}$ be the closed linear span of $\{e_n\}_{n=1}^\infty$ in $(l_p \oplus l_r)_\infty$. In many cases, the space $X_{p,r,w}$ is isomorphic to either l_p, l_r or to $l_p \oplus l_r$. A non-trivial case which is of importance occurs when the sequence $w = \{w_n\}_{n=1}^\infty$ satisfies the condition

$$(*) \qquad \sum_{n=1}^\infty w_n^{pr/(p-r)} = \infty, \quad w_n \to 0 \text{ and } w_n < 1 \text{ for all } n\,.$$

The spaces $X_{p,2,w}$ were introduced by H. P. Rosenthal [127]. These spaces play an important role in the study of complemented subspaces of $L_p(0, 1)$ and they will be further investigated in Vol. II. Some of the results, to be presented in this section, have been originally proved by probabilistic methods in the case when $r = 2$ (cf. [128]). The probabilistic methods do not work for other values of r and therefore we present here an approach based on modular and Orlicz sequence spaces. This method is due to J. T. Woo [145].

Before describing these results we prove that $X_{p,r,w}$ does not really depend on the sequence w provided condition (*) holds.

Proposition 4.d.6 [127, 145]. *Let $w = \{w_n\}_{n=1}^\infty$ and $w' = \{w'_n\}_{n=1}^\infty$ be two sequences both satisfying the condition (*). Then the spaces $X_{p,r,w}$ and $X_{p,r,w'}$ are isomorphic.*

Proof. Since w satisfies (*) there are disjoint finite subsets $\{\sigma_j\}_{j=1}^\infty$ of integers so that the numbers $v_j = \left(\sum_{n \in \sigma_j} w_n^{pr/(p-r)}\right)^{(p-r)/pr}$ satisfy $w'_j \leqslant v_j \leqslant 2w'_j$ for all j.

Let $\{e_n\}_{n=1}^\infty$ be the natural basis of $X_{p,r,w}$ and put

$$h_j = \left(\sum_{n \in \sigma_j} w_n^{r/(p-r)} e_n\right) \Big/ \left(\sum_{n \in \sigma_j} w_n^{pr/(p-r)}\right)^{1/p}, \quad j = 1, 2, \ldots.$$

Then, for every choice of scalars $\{a_j\}_{j=1}^\infty$, we have

$$\begin{aligned}\left\|\sum_{j=1}^\infty a_j h_j\right\| &= \max\left\{\left[\sum_{j=1}^\infty \left(\sum_{n \in \sigma_j} |a_j|^p w_n^{pr/(p-r)}\right) \Big/ \left(\sum_{n \in \sigma_j} w_n^{pr/(p-r)}\right)\right]^{1/p},\right.\\
&\qquad \left.\left[\sum_{j=1}^\infty \left(\sum_{n \in \sigma_j} |a_j|^r w_n^{r^2/(p-r)} w_n^r\right) \Big/ \left(\sum_{n \in \sigma_j} w_n^{pr/(p-r)}\right)^{r/p}\right]^{1/r}\right\}\\
&= \max\left\{\left(\sum_{j=1}^\infty |a_j|^p\right)^{1/p}, \left(\sum_{j=1}^\infty |a_j|^r \left(\sum_{n \in \sigma_j} w_n^{pr/(p-r)}\right)^{(p-r)/p}\right)^{1/r}\right\}\\
&= \max\left\{\left(\sum_{j=1}^\infty |a_j|^p\right)^{1/p}, \left(\sum_{j=1}^\infty |a_j|^r v_j^r\right)^{1/r}\right\}.\end{aligned}$$

This shows that $\{h_j\}_{j=1}^\infty$ is equivalent to the natural basis of $X_{p,r,w'}$. We shall prove now that $[h_j]_{j=1}^\infty$ is a complemented subspace of $X_{p,r,w}$. For any sequence of scalars $\{a_n\}_{n=1}^\infty$ which are eventually equal to zero we set

$$P\Big(\sum_{n=1}^\infty a_n e_n\Big)=\sum_{j=1}^\infty\Big[\Big(\sum_{n\in\sigma_j} a_n w_n^{(pr-r)/(p-r)}\Big)\Big/\Big(\sum_{n\in\sigma_j} w_n^{pr/(p-r)}\Big)^{(p-1)/p}\Big]h_j\,.$$

We have

$$\Big\|P\Big(\sum_{n=1}^\infty a_n e_n\Big)\Big\|=\max\Big\{\Big[\sum_{j=1}^\infty\Big|\sum_{n\in\sigma_j} a_n w_n^{(pr-r)/(p-r)}\Big|^p\Big/\Big(\sum_{n\in\sigma_j} w_n^{pr/(p-r)}\Big)^{p-1}\Big]^{1/p},$$
$$\Big[\sum_{j=1}^\infty\Big|\sum_{n\in\sigma_j} a_n w_n^{(pr-r)/(p-r)}\Big|^r\Big/\Big(\sum_{n\in\sigma_j} w_n^{pr/(p-r)}\Big)^{r-1}\Big]^{1/r}\Big\}$$

and, using twice Holder's inequality, we obtain

$$\Big\|P\Big(\sum_{n=1}^\infty a_n e_n\Big)\Big\|\leqslant\Big\|\sum_{j=1}^\infty\sum_{n\in\sigma_j} a_n e_n\Big\|\leqslant\Big\|\sum_{n=1}^\infty a_n e_n\Big\|\,.$$

Hence, the operator P extends to a norm-one projection from $X_{p,r,w}$ onto $[h_j]_{j=1}^\infty$. We remark that the proof that $[h_j]_{j=1}^\infty$ is complemented in $X_{p,r,w}$ is valid for any choice of $\{\sigma_j\}_{j=1}^\infty$ (i.e. it does not depend on the particular relation between v_j and w_j' which was taken into account when the sets $\{\sigma_j\}_{j=1}^\infty$ were constructed). The existence of the projection P shows that $X_{p,r,w'}$ is isomorphic to a complemented subspace of $X_{p,r,w}$ and, by symmetry, also that $X_{p,r,w}$ is isomorphic to a complemented subspace of $X_{p,r,w'}$. Thus, by Pelczynski's decomposition method (see the proof of 2.d.10), it would follow that $X_{p,r,w}\approx X_{p,r,w'}$ provided we prove that each of the spaces $X_{p,r,w}$ and $X_{p,r,w'}$ is isomorphic to its own square. This means that in order to complete the proof it suffices to prove the following lemma.

Lemma 4.d.7. *For each sequence $w=\{w_n\}_{n=1}^\infty$ satisfying the condition* (*) *we have $X_{p,r,w}\oplus X_{p,r,w}\approx X_{p,r,w}$.*

Proof. We first split the integers N into disjoint infinite subsets $N_1, N_2,\ldots, N_k,\ldots$ so that, for each k, $\sum_{n\in N_k} w_n^{pr/(p-r)}=\infty$. Put $w^{(k)}=\{w_n\}_{n\in N_k}$. By the first part of the proof of 4.d.6 there exists a constant $A<\infty$ such that, for each k, there is a block basis $\{u_j^{(k)}\}_{j\in N_k}$ of the natural basis of $X_{p,r,w^{(k)}}$, considered as a subspace of $X_{p,r,w}$, which is equivalent to the natural basis of $X_{p,r,w}$, $d([u_j^{(k)}]_{j\in N_k}, X_{p,r,w})\leqslant A$, and whose span $[u_j^{(k)}]_{j\in N_k}$ is the range of a norm-one projection in $X_{p,r,w^{(k)}}$.

Using the remark made in the proof of 4.d.6 we get that the space $Y=[u_j^{(k)}]_{j\in N_k, k\in N}$ is also complemented in $X_{p,r,w}$, i.e. $X_{p,r,w}=Y\oplus Z$ for some Banach space Z. Every $y\in Y$ can be represented uniquely as $y=\sum_{k=1}^\infty y^{(k)}$ with

$y^{(k)} = \sum_{j \in N_k} c_j^{(k)} u_j^{(k)} \in [u_j^{(k)}]_{j \in N_k}$ for all k. Moreover, for some constant C depending only on A, we have for every $y \in Y$ that

$$C^{-1}\|y\| \leqslant \max\left\{\left(\sum_{k=1}^{\infty}\sum_{j\in N_k}|c_j^{(k)}|^p\right)^{1/p}, \left(\sum_{k=1}^{\infty}\sum_{j\in N_k}|c_j^{(k)}|^r w_j^r\right)^{1/r}\right\} \leqslant C\|y\| .$$

This implies that $Y \approx \left\{y = \sum_{k=1}^{\infty} y^{(k)};\ y^{(k)}=0 \text{ for } k>1\right\} \oplus \left\{y=\sum_{k=1}^{\infty} y^{(k)};\ y^{(1)}=0\right\}$, i.e. $Y \approx X_{p,r,w} \oplus Y$. It follows that $X_{p,r,w} \oplus X_{p,r,w} \approx X_{p,r,w} \oplus Y \oplus Z \approx Y \oplus Z = X_{p,r,w}$ and this completes the proof. $\square$

Until now we have considered only the spaces $X_{p,r,w}$ with $1 \leqslant r < p < \infty$. If, in the definition of $X_{p,r,w}$, we replace the space l_p by c_0 we obtain a subspace of $(c_0 \oplus l_r)_\infty$, denoted by $X_{\infty,r,w}$. The condition (*) is replaced in this case by

$$(*)_\infty \qquad \sum_{n=1}^{\infty} w_n^r = \infty, \quad w_n \to 0 \text{ and } w_n < 1 \text{ for all } n .$$

By the same method as in 4.d.6 it can be shown that, for any two sequences $w=\{w_n\}_{n=1}^\infty$ and $w'_n=\{w'_n\}_{n=1}^\infty$ which both satisfy the condition $(*)_\infty$, the spaces $X_{\infty,r,w}$ and $X_{\infty,r,w'}$ are isomorphic. The blocks $\{h_j\}_{j=1}^\infty$ used in the proof are in this case blocks with constant coefficients. The proof that P is a norm-one projection shows in this case that every block basis with constant coefficients of any permutation of the natural basis of $X_{\infty,r,w}$ spans a complemented subspace.

In view of 4.d.6 and the preceding remark it is justified to use the notation $X_{p,r}$, $1 \leqslant r < p \leqslant \infty$ instead of $X_{p,r,w}$ without reference to the particular sequence w (satisfying the condition (*), respectively $(*)_\infty$).

We shall show now that, for any fixed $1 \leqslant r < p \leqslant \infty$ and $w=\{w_n\}_{n=1}^\infty$ satisfying (*), respectively $(*)_\infty$, the natural basis of $X_{p,r,w}$ is equivalent to the unit vector basis of some modular space.

We first consider the case $1 \leqslant r < p < \infty$. For every n we define the function

$$M_{w_n}(t) = \max\{t^p, w_n^r t^r\} = \begin{cases} w_n^r t^r, & 0 \leqslant t \leqslant w_n^{r/(p-r)} \\ t^p, & w_n^{r/(p-r)} < t < \infty . \end{cases}$$

It is clear that each M_{w_n} is an Orlicz function satisfying $r \leqslant tM'_{w_n}(t)/M_{w_n}(t) \leqslant p$ for every $t>0$ for which the derivative exists. Moreover, it is easily checked that a series $\sum_{n=1}^{\infty} M_{w_n}(|a_n|)$ converges if and only if $\max\left\{\left(\sum_{n=1}^{\infty}|a_n|^p\right)^{1/p}, \left(\sum_{n=1}^{\infty}|a_n|^r w_n^r\right)^{1/r}\right\} < \infty$. This implies that the natural basis of $X_{p,r,w}$ is equivalent to the unit vector basis of $l_{\{M_{w_n}\}}$. Consequently, $X_{p,r} \approx l_{\{M_{w_n}\}}$.

In the case when $p=\infty$ we put

$$M_{w_n}(t) = \max\{t^n, w_n^r t^r\} .$$

Then, for any $\rho>0$, a series $\sum_{n=1}^{\infty} M_{w_n}(|a_n|/\rho)$ converges if and only if

$$\max\left\{\sum_{n=1}^{\infty}|a_n|^n\rho^{-n},\ \left(\sum_{n=1}^{\infty}|a_n|^r w_n^r\rho^{-r}\right)^{1/r}\right\}<\infty .$$

It follows from this fact that the natural basis of $X_{\infty,r,w}$ is equivalent to the unit vector basis of $h_{\{M_{w_n}\}}$. Obviously, in this case the sequence $\{M_{w_n}\}_{n=1}^{\infty}$ is not equivalent to any sequence satisfying the uniform Δ_2-condition at zero.

In order to determine the dual of $X_{p,r}$ we restrict ourselves to the case $1<r<p\leqslant\infty$ (for $r=1$, the functions complementary to M_{w_n} are degenerate). We take q and s so that $p^{-1}+q^{-1}=1$ and $r^{-1}+s^{-1}=1$ ($q=1$ if $p=\infty$). For $p<\infty$ the function $M_{w_n}^*$, complementary to M_{w_n}, can be computed directly from the definition and we get

$$M_{w_n}^*(t)=\begin{cases} t^s/sr^{s-1}w_n^s, & 0\leqslant t\leqslant rw_n^{s/(s-q)}\\ tw_n^{(sq-s)/(s-q)}-w_n^{sq/(s-q)}, & rw_n^{s/(s-q)}<t\leqslant pw_n^{s/(s-q)}\\ t^q/qp^{q-1}, & pw_n^{s/(s-q)}<t<\infty .\end{cases}$$

For computations it is however more convenient to use the non-convex function N_{w_n}, defined by

$$N_{w_n}(t)=\min\{t^q,\ t^s/w_n^s\},$$

which has the property that $A^{-1}\leqslant N_{w_n}(t)/M_{w_n}^*(t)\leqslant A$ for every $t>0$ and for some constant $A>0$ which depends only on p and r but not on t and n. This formula is valid also for $p=\infty$, i.e. $q=1$.

It follows from this discussion that the dual $Y_{q,s}$, $1\leqslant q<s<\infty$, of $X_{p,r}$, consists of all sequences $y=(a_1, a_2,\dots)$ for which $\sum_{n=1}^{\infty} N_{w_n}(|a_n|)<\infty$.

Before we state the main result of this section we prove an elementary technical lemma which will be used in the sequel.

Lemma 4.d.8. [102] *Let Q_0 be a continuous function on an interval* $[0, t_0]$ *so that*

(i) $Q_0(0)=0$ *and* $Q_0(t)>0$ for $0<t\leqslant t_0$,
(ii) Q_0 *has a right derivative which satisfies* $0\leqslant tQ_0'(t)/Q_0(t)\leqslant 1$ *for* $0<t<t_0$.

Then there exists a concave increasing twice continuously differentiable function Q so that $Q_0(t)/8\leqslant Q(t)\leqslant Q_0(t)$ for $0\leqslant t\leqslant t_0$.

Proof. Let $\varphi(t)$ be a continuously differentiable function on $[0, t_0]$ so that $1/4\leqslant\varphi(t)\leqslant 1/2$, $\varphi'(t)\geqslant 0$, $4t\varphi'(t)+tQ_0'(t)/Q_0(t)\leqslant 1$ for $0<t<t_0$ and so that $\varphi'(t)>0$ whenever t is an interior point of $\{t;\ Q_0'(t)=0\}$. These properties of φ ensure that $Q_1=Q_0\varphi$ is a continuous strictly increasing function on $[0, t_0]$ satisfying $tQ_1'(t)/Q_1(t)\leqslant 1$ on that interval. Therefore, the inverse function N_1 of Q_1 is also strictly increasing and satisfies $tN_1'(t)/N(t)\geqslant 1$, in a certain neighbourhood of 0. It follows that $N_2(t)=\int_0^t (N_1(u)/u)du$ is a continuously differentiable convex strictly increasing function such that $N_1(t/2) \leqslant N_2(t) \leqslant N_1(t)$ in some neighborhood of 0. In order to get a twice continuously differentiable function we take $N(t) = \int_0^t (N_2(u)/u)\,du$ which satisfies $N_1(t/4) \leqslant N(t) \leqslant N_1(t)$. The inverse function Q of N is a twice continuously differentiable concave function on $[0, t_0]$ which satisfies $Q(t)/4 \leqslant Q_1(t) \leqslant Q(t)$ and thus also $Q_0(t)/8 \leqslant Q(t) \leqslant Q_0(t)$ for $t \in [0, t_0]$. □

We are now ready to prove the theorem of J. T. Woo [145].

Theorem 4.d.9. *Let $1 \leqslant q < s < \infty$. A Banach space X with a symmetric basis is isomorphic to a subspace of $Y_{q,s}$ if and only if X is isomorphic to an Orlicz sequence space l_M with M satisfying $q \leqslant tM'(t)/M(t) \leqslant s$ for every $t > 0$.*

Proof. Let X be a Banach space with a symmetric basis $\{x_n\}_{n=1}^\infty$ which is isomorphic to a subspace of $Y_{q,s}$. Since $Y_{q,s} \approx l_{\{M^*_{w_n}\}}$ we can assume with no loss of generality that $\{x_n\}_{n=1}^\infty$ is equivalent to a normalized block basis $\{u_j\}_{j=1}^\infty$ of the unit vector basis of $l_{\{M^*_{w_n}\}}$. As in the case of Orlicz sequence spaces (see the discussion preceding 4.a.7), there exists a sequence of Orlicz functions $\{N_j\}_{j=1}^\infty$, with each N_j being a normalized convex combination of the $M^*_{w_n}$'s, such that a series $\sum_{j=1}^\infty a_j u_j$ converges if and only if $\sum_{j=1}^\infty N_j(|a_j|) < \infty$. The functions $\{M^*_{w_n}\}_{n=1}^\infty$, and thus also $\{N_j\}_{j=1}^\infty$, satisfy the uniform Δ_2-condition at zero; more precisely, we have $q \leqslant tN_j'(t)/N_j(t) \leqslant s$ for all j and $t > 0$. Hence, for any N which is a uniform limit of a subsequence of $\{N_j\}_{j=1}^\infty$, $\{u_j\}_{j=1}^\infty$ is equivalent to the unit vector basis of l_N. Thus, $X \approx l_N$ and $q \leqslant tN'(t)/N(t) \leqslant s$ for every $t > 0$. (Instead of repeating the argument used in 4.a.7 we could have also applied 4.d.5.)

In order to prove the converse we consider an Orlicz function $M(t)$ which satisfies the assumptions of the theorem, is not equivalent to t^q and for which $M(1) = 1$. We can exclude the case when $M(t)$ is equivalent at zero to t^q since $Y_{q,s}$ contains even a complemented subspace isomorphic to l_q. Indeed, by taking a subsequence of integers $\{n_i\}_{i=1}^\infty$ for which $\sum_{i=1}^\infty w_{n_i}^{pr/(p-r)} < \infty$ if $p < \infty$, or $\sum_{i=1}^\infty w_{n_i}^r < \infty$ if $p = \infty$, we get that $l_{\{M^*_{w_{n_i}}\}}$ is a complemented subspace of $l_{\{M^*_{w_n}\}} \approx Y_{q,s}$ which is isomorphic to l_q.

Put $Q_0(t) = M(t^{1/(s-q)})/t^{q/(s-q)}$. In view of the conditions imposed on M we get that Q_0 satisfies all the requirements of 4.d.8. Hence, there is a concave increasing function Q with two continuous derivatives and $Q(0) = 0$ which is equivalent to Q_0. Let $w = \{w_n\}_{n=1}^\infty$ be a sequence satisfying (*) if $p < \infty$ or $(*)_\infty$ if $p = \infty$, let $a_n = w_n^{s/(s-q)}$ and $b_n = a_n^{2(s-q)/(2s-q)}$. Put

$$G_n(t) = C_n^{-1} \int_{a_n}^1 N_{w_n}(b_n tu)[-(s-q)u^{q-2s-1} Q''(a_n^{s-q} b_n^{q-s} u^{q-s})]\,du\,,$$

where

$$C_n = \int_{a_n}^1 N_{w_n}(b_n u)[-(s-q)u^{q-2s-1} Q''(a_n^{s-q} b_n^{q-s} u^{q-s})]\,du\,.$$

This formula is inspired by the method used in the proof of 4.a.9. By substituting the explicit value of N_{w_n} and putting $v=a_n^{s-q}b_n^{q-s}u^{q-s}$ we get that

$$\begin{aligned}
G_n(t)&=C_n^{-1}\Big[t^s\int_{a_n}^{a_n/b_nt}-(s-q)a_n^{q-s}b_n^s u^{q-s-1}Q''(a_n^{s-q}b_n^{q-s}u^{q-s})\,du\\
&\quad+t^q\int_{a_n/b_nt}^{1}-(s-q)b_n^q u^{2q-2s-1}Q''(a_n^{s-q}b_n^{q-s}u^{q-s})\,du\Big]\\
&=C_n^{-1}\Big[t^s\int_{b_n^{q-s}}^{t^{s-q}}Q''(v)\,dv+t^q\int_{t^{s-q}}^{(a_n/b_n)^{s-q}}vQ''(v)\,dv\Big]\\
&=C_n^{-1}\Big\{t^s[Q'(t^{s-q})-Q'(b_n^{q-s})]+t^q[(a_n/b_n)^{s-q}Q'((a_n/b_n)^{s-q})\\
&\quad-t^{s-q}Q'(t^{s-q})]+t^q\int_{(a_n/b_n)^{s-q}}^{t^{s-q}}Q'(v)\,dv\Big\}\\
&=C_n^{-1}\{-t^sQ'(b_n^{q-s})+t^q(a_n/b_n)^{s-q}Q'((a_n/b_n)^{s-q})\\
&\quad+t^qQ(t^{s-q})-t^qQ((a_n/b_n)^{s-q})\}\,.
\end{aligned}$$

Since the behavior of M at ∞ is not important we may suppose that $\lim_{t\to\infty} Q'(t)=0$. This implies that $Q'(b_n^{q-s})\to 0$ as $n\to\infty$. Moreover, from the fact that $\lim_{t\to 0} tQ'(t)=\lim_{t\to 0} Q(t)=0$ it follows that also the second term and the fourth term in the parenthesis tend to zero since $a_n/b_n=a_n^{q/(2s-q)}\to 0$ as $n\to\infty$. Thus, for $t=1$, we get that $C_nG_n(1)=C_n\to Q(1)$. Consequently, the following limit exists uniformly for $t\in[0,1]$

$$G(t)=\lim_{n\to\infty} G_n(t)=t^qQ(t^{s-q})/Q(1),$$

and, since Q is equivalent to Q_0, we get that G is equivalent to M. From the definition of G_n it follows that, for every integer k, there is an integer $n(k)$ and real numbers $\{u_{i,k}\}_{i=1}^{j(k)}$ and $\{\lambda_{i,k}\}_{i=1}^{j(k)}$ in $[0,1]$ with $\sum_{i=1}^{j(k)}\lambda_{i,k}=1$ such that

$$\Big|G(t)-\sum_{i=1}^{j(k)}\lambda_{i,k}N_{w_{n(k)}}(b_{n(k)}tu_{i,k})\Big|\le 2^{-k},\quad t\in[0,1]\,.$$

Since $\lambda N_{w_n}(bt)=N_{v_n}(ct)$ for suitable v_n and c we can write the previous inequality also in the form

$$\Big|G(t)-\sum_{i=1}^{j(k)}N_{v_{i,k}}(c_{i,k}t)\Big|\le 2^{-k},\quad t\in[0,1]$$

for some $v_{i,k}$ and $c_{i,k}$. This clearly shows that l_G, or equivalently l_M, is isomorphic to a subspace of the modular sequence space $l_{\{N_{v_{i,k}}\}}$ (recall that the non-convex functions $N_{v_{i,k}}$ can be always replaced by proper Orlicz functions). In order to conclude the proof we remark that the sequence $\{v_{i,k}\}_{i=1}^{j(k)}{}_{k=1}^{\infty}$ tends to zero and if

$\sum_{k=1}^{\infty}\sum_{i=1}^{j(k)} v_{i,k}^{pr/(p-r)}=\infty$ in the case when $p<\infty$, or $\sum_{k=1}^{\infty}\sum_{i=1}^{j(k)} v_{i,k}^{r}=\infty$ in the case when $p=\infty$, then $l_{\{N_{v_{i,k}}\}}\approx Y_{q,s}$. If, on the other hand, $\sum_{k=1}^{\infty}\sum_{i=1}^{j(k)} v_{i,k}^{pr/(p-r)}<\infty$, respectively $\sum_{k=1}^{\infty}\sum_{i=1}^{j(k)} v_{i,k}^{r}<\infty$, then by applying Holder's inequality we get that the unit vectors of $l_{\{N_{v_{i,k}}\}}$ form a basis equivalent to the unit vector basis of l_q. Thus, $l_M\approx l_q$ and we have already proved that $Y_{q,s}$ contains a subspace isomorphic to l_q. □

Using 4.d.9 together with 4.d.5 we obtain the following corollary.

Theorem 4.d.10. *Let $1\leqslant q<s<\infty$ and let $\{M_n\}_{n=1}^{\infty}$ be a sequence of Orlicz functions satisfying $q\leqslant tM_n'(t)/M_n(t)\leqslant s$ for all n and $t\in(0,1)$. Then $Y_{q,s}$ contains a subspace isomorphic to $l_{\{M_n\}}$.*

By its construction the space $Y_{q,s}$ is a quotient of $l_q\oplus l_s$. Thus, every Orlicz sequence space "between" q and s is isomorphic to a subspace of a quotient of $l_q\oplus l_s$. Comparing 4.d.9 and 4.d.10 with 2.c.14 and 2.d.1 we note the marked difference between the behavior of subspaces of quotients of $l_q\oplus l_s$ from that of complemented subspaces of $l_q\oplus l_s$ and also from that of subspaces of quotients of l_p.

e. Lorentz Sequence Spaces

The Lorentz sequence spaces $d(w,p)$ which have already been mentioned in Section 3.a, as well as the Lorentz function spaces (cf. [99]), were introduced in connection with some problems of harmonic analysis and interpolation theory. We do not study here this aspect; instead, we present briefly some results regarding their geometric structure. Let us first recall the definition of a Lorentz sequence space.

Definition 4.e.1. Let $1\leqslant p<\infty$ and let $w=\{w_n\}_{n=1}^{\infty}$ be a non-increasing sequence of positive numbers such that $w_1=1$, $\lim_{n\to\infty} w_n=0$ and $\sum_{n=1}^{\infty} w_n=\infty$. The Banach space of all sequences of scalars $x=(a_1,a_2,\ldots)$ for which

$$||x||=\sup_{\pi}\left(\sum_{n=1}^{\infty}|a_{\pi(n)}|^p w_n\right)^{1/p}<\infty,$$

where π ranges over all the permutations of the integers, is denoted by $d(w,p)$ and it is called a *Lorentz sequence space.*

If $\{a_n^*\}_{n=1}^{\infty}$ is a non-increasing rearrangement of the sequence $\{a_n\}_{n=1}^{\infty}$, i.e. $\{a_n^*\}_{n=1}^{\infty}$

is a non-increasing sequence obtained from $\{|a_n|\}_{n=1}^\infty$ by a suitable permutation of the integers then $||x||=\left(\sum_{n=1}^\infty a_n^{*p}w_n\right)^{1/p}$ for $x=(a_1, a_2,\ldots)\in d(w,p)$.

Lorentz sequence spaces are, in general, different from the previously defined classes of spaces with a symmetric basis. For instance, the uniqueness of the symmetric basis in l_p (see 3.b.5) easily implies that no Lorentz sequence space is isomorphic to an l_p space. In some cases a Lorentz sequence space is isomorphic to an Orlicz sequence space. An easy computation shows that this is the case, for example, if $w_n=1/(1+\log n)$ for $n\geqslant 1$ and $p\geqslant 1$ arbitrary: the corresponding space $d(w,p)$ is isomorphic to l_M, where $M(t)=t^p/(1+|\log t|)$. By the uniqueness of the symmetric basis in any Lorentz sequence space (see 4.e.4 below), a space $d(w,p)$ is isomorphic to an Orlicz sequence space l_M if and only if they are identical, i.e. they consist of the same sequences. In other words, $d(w,p)\approx l_M$ if and only if

$$(\dagger) \qquad \sum_{n=1}^\infty \lambda_n^p w_n<\infty \Leftrightarrow \sum_{n=1}^\infty M(\lambda_n)<\infty,$$

whenever $\{\lambda_n\}_{n=1}^\infty$ is a non-increasing sequence of reals tending to zero.

G. G. Lorentz [100] has found necessary and sufficient conditions on w for the existence of an Orlicz function M for which (†) holds and also necessary and sufficient conditions on M so that there exists a sequence $w=\{w_n\}_{n=1}^\infty$ for which (†) is satisfied. The conditions given by G. G. Lorentz were actually stated in the more general context of function spaces but they can be easily translated into the sequence spaces language, as follows.

Let $w=\{w_n\}_{n=1}^\infty$ be a strictly decreasing sequence of reals such that $w_1=1$, $\lim_{n\to\infty} w_n=0$ and $\sum_{n=1}^\infty w_n=\infty$ (the requirement that w be strictly decreasing sequence is not really a restriction since every $d(w,p)$ space is isomorphic to a space $d(w',p)$ for which w' is a strictly decreasing sequence). Construct two continuous functions $W(t)$ and $S(t)$, defined both on $[1,\infty)$, such that W is strictly decreasing, S is concave and strictly increasing, $W(n)=w_n$ and $S(n)=s_n=\sum_{i=1}^n w_i$ for $n=1,2,\ldots$. Since S^{-1} is a strictly increasing convex function on $[1,\infty)$ it follows easily that $F(t)=1/S^{-1}(1/t)$ satisfies $1\leqslant tF'(t)/F(t)$ for all $t\in(0,1)$ for which the derivative exists. This implies that $F(t)/t$ is non-decreasing and therefore, F is equivalent at zero to the Orlicz function $N(t)=\int_0^t (F(u)/u)\,du$. With these notations we are prepared to state the results of G. G. Lorentz [100].

Theorem 4.e.2. *A Lorentz sequence space $d(w,1)$ is isomorphic to an Orlicz sequence space l_M (i.e. $d(w,1)=l_M$ as sets) if and only if*

(i) *there exists a constant $\gamma>0$ such that $\sum_{n=1}^\infty 1/W^{-1}(\gamma w_n)<\infty$.*

In this case M is equivalent to the Orlicz function N, defined above.

We also have

Theorem 4.e.2′. *An Orlicz sequence space l_M is isomorphic to a Lorentz sequence space $d(w, 1)$ if and only if*

(i′) *there exists a constant $\delta>0$ so that $\int_1^\infty M^*(\delta M^{*-1}(1/t))\,dt<\infty$.*

If this is the case the sequence w may be taken to be $w_n=M^{*-1}(1/n)$, $n=1, 2,\ldots$.

Since these two theorems are not used in the sequel we do not reproduce their proofs here. The fact that both 4.e.2 and 4.e.2′ apply only in the case $p=1$ is not really a restriction. It is relatively easy to check that, for any $p>1$, we have $d(w, p)\approx l_{M_p}$ if and only if $d(w, 1)\approx l_{M_1}$, where the Orlicz functions M_p and M_1 are connected by the relation $M_p(t)=M_1(t^p)$.

Proposition 4.e.3. *Let $\{e_n\}_{n=1}^\infty$ be the unit vector basis of a Lorentz sequence space $d(w, p)$ with $p\geqslant 1$. Then every normalized block basis $u_n=\sum_{i=q_n+1}^{q_{n+1}} a_i e_i$, $n=1, 2,\ldots$ such that $\lim_{i\to\infty} a_i=0$ contains, for every $\varepsilon>0$, a subsequence $\{u_{n_j}\}_{j=1}^\infty$ which is $1+\varepsilon$-equivalent to the unit vector basis of l_p and so that $[u_{n_j}]_{j=1}^\infty$ is complemented in $d(w, p)$.*

Consequently, every infinite dimensional subspace of $d(w, p)$ contains complemented subspaces which are nearly isometric to l_p.

Proof. Since every change of signs and every permutation of the integers induces an isometry in $d(w, p)$ we may assume, by switching to a subsequence if necessary, that $\{a_i\}_{i=1}^\infty$ is a non-increasing sequence of positive numbers.

Fix $\varepsilon>0$ and construct by induction two increasing sequences of integers $\{n_j\}_{j=1}^\infty$ and $\{r_j\}_{j=1}^\infty$ such that $q_{n_j}<r_j<q_{n_j+1}$, $Q_{j-1}=\sum_{k=1}^{j-1}(q_{n_k+1}-q_{n_k})\leqslant r_j-q_{n_j}$ and $\left(\sum_{i=q_{n_j}+1}^{r_j} a_i^p w_{i-q_{n_j}}\right)^{1/p}<\varepsilon/2^{j+1}$ for all j. Then, for any set of coefficients $\{\lambda_j\}_{j=1}^\infty$, we have

$$\begin{aligned}
\left\|\sum_{j=1}^\infty \lambda_j u_{n_j}\right\| &\geqslant \left\|\sum_{j=1}^\infty \lambda_j \sum_{i=r_j+1}^{q_{n_j+1}} a_i e_i\right\| - \sum_{j=1}^\infty |\lambda_j| \left\|\sum_{i=q_{n_j}+1}^{r_j} a_i e_i\right\| \\
&\geqslant \left(\sum_{j=1}^\infty |\lambda_j|^p \sum_{i=r_j+1}^{q_{n_j+1}} |a_i|^p w_{Q_{j-1}-r_j+i}\right)^{1/p} - \frac{\varepsilon}{2}\max_j |\lambda_j| \\
&\geqslant \left(\sum_{j=1}^\infty |\lambda_j|^p \sum_{i=r_j+1}^{q_{n_j+1}} |a_i|^p w_{i-q_{n_j}}\right)^{1/p} - \frac{\varepsilon}{2}\max_j |\lambda_j| \\
&\geqslant (1-\varepsilon)\left(\sum_{j=1}^\infty |\lambda_j|^p\right)^{1/p}.
\end{aligned}$$

On the other hand, it is easily checked that, for every normalized block basis and therefore in particular for $\{u_{n_j}\}_{j=1}^\infty$, we have $\left\|\sum_{j=1}^\infty \lambda_j u_{n_j}\right\|\leqslant\left(\sum_{j=1}^\infty |\lambda_j|^p\right)^{1/p}$.

In order to prove that $[u_{n_j}]_{j=1}^{\infty}$ is complemented in $d(w,p)$ we set

$$P\Big(\sum_{n=1}^{\infty} c_n e_n\Big)=\sum_{j=1}^{\infty}\Big\{\Big(\sum_{i=r_j+1}^{q_{n_j+1}} c_i a_i^{p-1} w_{i-q_{n_j}}\Big)\Big/\Big(\sum_{i=r_j+1}^{q_{n_j+1}} a_i^p w_{i-q_{n_j}}\Big)\Big\}u_{n_j}.$$

Then, using Holder's inequality we get

$$\begin{aligned}\Big\|P\Big(\sum_{n=1}^{\infty} c_n e_n\Big)\Big\|^p &\leq \sum_{j=1}^{\infty}\Big|\sum_{i=r_j+1}^{q_{n_j+1}} c_i a_i^{p-1} w_{i-q_{n_j}}\Big|^p\Big/\Big(\sum_{i=r_j+1}^{q_{n_j+1}} a_i^p w_{i-q_{n_j}}\Big)^p\\ &\leq \sum_{j=1}^{\infty}\Big(\sum_{i=r_j+1}^{q_{n_j+1}} |c_i|^p w_{i-q_{n_j}}\Big)\Big/\Big(\sum_{i=r_j+1}^{q_{n_j+1}} a_i^p w_{i-q_{r_j}}\Big)\\ &\leq (1+\varepsilon)^p\Big\|\sum_{n=1}^{\infty} c_n e_n\Big\|^p,\end{aligned}$$

i.e. P is a bounded linear projection from $d(w,p)$ onto $[u_{n_j}]_{j=1}^{\infty}$. The last assertion follows easily from 1.a.11. □

The following result from [4] is an immediate corollary of 4.e.3.

Theorem 4.e.4. *Let X be a subspace of a Lorentz sequence space $d(w,p)$, $p\geq 1$ which has a symmetric basis $\{x_n\}_{n=1}^{\infty}$. Then, up to equivalence, $\{x_n\}_{n=1}^{\infty}$ is the unique symmetric basis of this subspace.*

Proof. Assume that $\{y_n\}_{n=1}^{\infty}$ is another symmetric basis of X. If $X\approx l_p$ then, by 3.b.5, $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are equivalent. Otherwise, by 1.a.12, $\{x_n\}_{n=1}^{\infty}$ is equivalent to a block basis $u_n=\sum_{i=q_n+1}^{q_{n+1}} a_i e_i$, $n=1,2,\ldots$ of $\{e_n\}_{n=1}^{\infty}$, the unit vector basis of $d(w,p)$, and $\{y_m\}_{m=1}^{\infty}$ is equivalent to a block basis $v_m=\sum_{n=r_m+1}^{r_{m+1}} b_n x_n$, $m=1,2,\ldots$ of $\{x_n\}_{n=1}^{\infty}$, with $\limsup_{n\to\infty} b_n\neq 0$ (for, by 4.e.3, $\lim_{n\to\infty} b_n=0$ implies that $\{y_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis of l_p, contrary to our assumption). Hence, by the symmetry of $\{y_n\}_{n=1}^{\infty}$, we may assume without loss of generality that there is an $\varepsilon>0$ so that, for every m, $|b_n|\geq\varepsilon$ for some $r_m<n<r_{m+1}$. Consequently, if a series $\sum_{n=1}^{\infty}\lambda_n y_n$ converges so does $\sum_{n=1}^{\infty}\lambda_n x_n$. By interchanging the roles of $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ we deduce the equivalence of these two bases. □

It follows from 4.e.3 and 1.c.12(a) that a Lorentz sequence space $d(w,p)$ is reflexive if and only if $p>1$. The dual $d^*(w,p)$ of $d(w,p)$ is never isomorphic to a Lorentz sequence space. Indeed, assume that $d^*(w,p)=d(w',q)$ for some sequence w' and for some $q\geq 1$. By 4.e.3 we must have $p^{-1}+q^{-1}=1$ (if $p=1$ the assertion is entirely obvious so we assume that $p>1$). Let $\{e_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ be the unit vector bases of $d(w',q)$, respectively $d^*(w,p)$. Then, a series $\sum_{n=1}^{\infty}\lambda_n e_n$ converges whenever $\lambda=(\lambda_1, \lambda_2,\dots)\in l_q$ while there is a $\lambda\notin l_q$ for which $\sum_{n=1}^{\infty} \lambda_n e_n$ converges. On the other hand, the convergence of $\sum_{n=1}^{\infty} \lambda_n f_n$ implies that $\lambda\in l_q$ while there is a $\lambda\in l_q$ for which $\sum_{n=1}^{\infty} \lambda_n f_n$ fails to converge. This contradicts 4.e.4.

It is easy to give an explicit representation of $d^*(w,p)$. In the case $p>1$ this space consists of all sequences $x=(a_1, a_2,\dots)$ for which

$$||x||=\inf \sup_n \left(\sum_{i=1}^{n} a_i^*\right) \Big/ \left(\sum_{i=1}^{n} |y_i| w_i^{1/p}\right) < \infty ,$$

where the infimum is taken over all $y=(y_1, y_2,\dots)\in l_q$ with $||y||=1$, where $p^{-1}+q^{-1}=1$ and $\{a_i^*\}_{i=1}^{\infty}$ denotes a non-increasing rearrangement of $\{|a_i|\}_{i=1}^{\infty}$ (cf. [45]).

In the remainder of this section we present, without proofs, some results concerning symmetric basis sequences in $d(w,p)$ spaces. These sequences have been studied in [4]. It has been proved there that *every symmetric basic sequence in a Lorentz sequence space $d(w,p)$ is equivalent either to the unit vector basis of l_p or to a block basis $\{u_n^{(\alpha)}\}_{n=1}^{\infty}$ generated by a vector $0\neq\alpha=\sum_{n=1}^{\infty} a_n e_n\in d(w,p)$* (recall definition 3.a.8).

In many cases, block bases generated by one vector are themselves equivalent to the unit vector basis of some Lorentz sequence space but this is not true in general. For example, it has been shown in [4] that in $d(w,p)$, with $w_n=1/n^{1/2}(1+\log n)^2$, $n=1, 2,\dots$ and $p\geq 1$ arbitrary, the block basis generated by the vector $\alpha=\sum_{n=1}^{\infty} e_n/n^{1/2p}$ spans a subspace (with a symmetric basis) which is not isomorphic to any Lorentz sequence space or to any l_p space. On the other hand, there are cases where all block bases generated by one vector are equivalent to each other. These Lorentz sequence spaces have been characterized in [4].

Theorem 4.e.5. *In a Lorentz sequence space $d(w,p)$ there are exactly two non-equivalent symmetric basic sequences (namely, the unit vector basis of l_p and that of $d(w,p)$ itself) if and only if $\sup_{n,k} s_{nk}/s_n s_k<\infty$, where $s_n=\sum_{i=1}^{n} w_i$.*

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