2025

INVERSE PENDULUM

PHASE I

VIBRATIONS SPRING 1404

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THE CURRENT DOCUMENT IS A REPORT OF THE FIRST VIBRATIONS PROJECT REGARDING INVERTED PENDULUM

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Introduction

The current document is a report of the first Vibrations Project, prepared for the class Vibrations-ME 8106-116-0.

Knowledge of '**Dynamics**" and '**Vibrations**" is required for understanding this document.

For further insights, the reader can study "Theory of Vibration with Applications" written by William T. Thomson.

Systems Equations

Problem description

In the Inverse Pendulum shown, l = 0.2 m, M = 0.4 kg and m = 0.1 kg. Knowing that the force applied to the cart is F = 3N, Obtain the motion equations of this system.

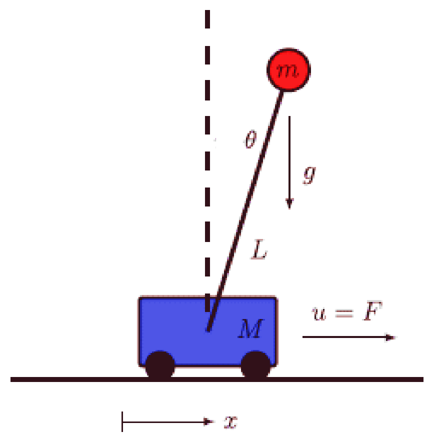

Figure 1. Inverse Pendulum

Given

$\boldsymbol{l = 0.4\ (m)}$

$\boldsymbol{M = 0.4\ (kg)}$

$\boldsymbol{m = 0.1\ (kg)}$

Required

Equations of Motion

Under the assumptions that the surface is frictionless, the components are rigid and the joints are ideal.

Solution

Knowledge of '**Lagrange's Equation**" is required for understanding this section.

For further insights, the reader can refer to chapter 7.3 of "Theory of Vibration with Applications" written by William T. Thomson.

Solution: Lagrange's Equation

Lagrange

Let the Lagrangian coordinates be θ and x as shown in figure 1. Let L be the Lagrangian. Let T be the kinetic energy of the system and let U be the potential energy. Hence

$$L = T - U$$

And

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}mv^2$$

Where v is the linear velocity of the blob m relative to the inertial system.

Since $v = l\dot{\theta}$, we obtain:

$$\begin{aligned} v^2 &= (\dot{x} + v_x)^2 + v_y^2 \\ &= (\dot{x} + v\cos\theta)^2 + (v\sin\theta)^2 \\ &= \left(\dot{x} + l\dot{\theta}\cos\theta\right)^2 + \left(l\dot{\theta}\sin\theta\right)^2 \\ &= \dot{x}^2 + l^2\dot{\theta}^2\cos^2\theta + 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2\sin^2\theta \\ &= \dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}l\dot{\theta}\cos\theta \end{aligned}$$

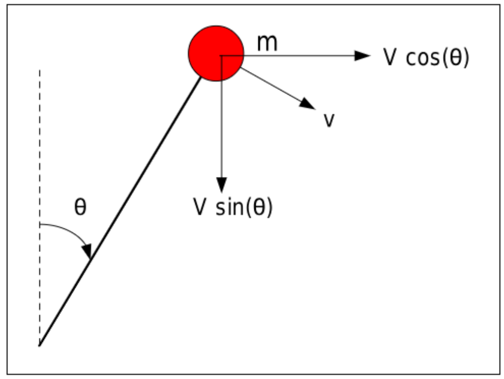

Figure 2. Velocity diagram

Hence T becomes

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}l\dot{\theta}cos\theta)$$

And since the blob is losing potential energy as it move downwards, we obtain ***U*** as (assuming zero potential energy is the ground level)

$$U = mglcos\theta$$

Therefore the Lagrangian is

$$L = T - U$$
$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}l\dot{\theta}cos\theta\right) - mglcos\theta$$

To obtain the equations of motion, we need to evaluate $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$ for each Lagrangian coordinate q_i and Q_i is the generalized force for that coordinate. Hence for θ we obtain:

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

$$\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + ml\dot{\theta}cos\theta$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = (M+m)\ddot{x} + ml\ddot{\theta}cos\theta - ml\dot{\theta}^2sin\theta$$

$$\frac{\partial L}{\partial x} = 0 \;\rightarrow (M+m)\ddot{x} + ml\ddot{\theta}cos\theta - ml\dot{\theta}^2sin\theta = F$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml\dot{x}cos\theta + ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml\ddot{x}cos\theta - ml\dot{x}\dot{\theta}sin\theta + ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -ml\dot{x}\dot{\theta}sin\theta + mglsin\theta \rightarrow ml\ddot{x}cos\theta + ml^2\ddot{\theta} - mglsin\theta = 0$$

And thus the equations of Motion are:

$$(M+m)\ddot{x} + ml\ddot{\theta}cos\theta + ml\dot{\theta}^2 sin\theta = F$$

$$ml\ddot{x}cos\theta + ml^2\ddot{\theta} - mglsin\theta = 0$$

Notice that here the rod connecting the pendulum is assumed to be of negligble mass and the mass m is thought to be concentrated at the end of the pendulum. If we were to model **the inverse pendulum more precisely**, we could neglect the ball at the end of the pendulum and instead model the pendulum with a rotating rod (figure 3). By Carring out the Lagrangian method (about the pendulums center of mass (at l/2)) we would obtain:

$$(M+m)\ddot{x} + ml\ddot{\theta}cos\theta - ml\dot{\theta}^2 sin\theta = F$$

$$ml\ddot{x}cos\theta + (I + ml^2)\ddot{\theta} - mglsin\theta = 0 \rightarrow I = \frac{1}{3}ml^2$$

Note that here l = l/2 = 0.2 m

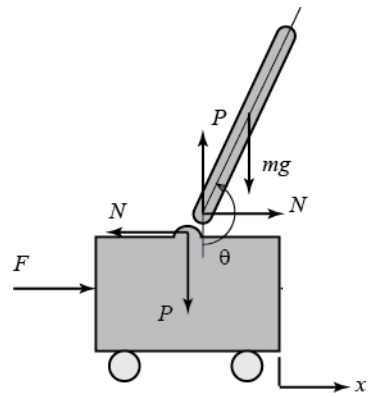

Figure 3. Inverse Pendulum (Modeled more precisely)

In the coming sections $(I + ml^2)$ is represented by Aml^2 where A can be either 4/3 or 1 depending on the model studied (figure 3 or figure 1 respectively). Note that when the model in figure 3 is used, l represents half the length of the rod (Since equations are obtained with respect to the pendulum's center of mass).

Solving the Equations of Motion

After extracting the dynamic equations of the cart and inverted pendulum system using the Lagrange method, it is necessary to solve these equations numerically for accurate analysis of the system's behavior. In this project, the equations are solved using two distinct methods to allow for verification of the accuracy and precision of the simulations.

Matlab | ODE45

State Space Representation

In order to use ODE45, we need to obtain state space of the EOM. Below is the state-space representation of the system:

$$\begin{aligned}
\dot{x}_1 &= \dot{x} = x_2 \\
\dot{x}_2 &= F - ml\ddot{x}cosx_3 + mlx_4^2 sinx_3 \\
\dot{x}_3 &= \dot{\theta} = x_4 \\
\dot{x_4} &= \frac{mglsinx_3 - ml\dot{x_2}cosx_3}{Aml^2}
\end{aligned}$$

To obtain the state space equations, $\dot{x_2}$ and $\dot{x_4}$ must be independent from any differential variable. Hence after the substitution, the SS equations are derived as such:

$$\begin{aligned}
\dot{x}_1 &= \dot{x} = x_2 \\
\dot{x}_2 &= \frac{F + mlx_4^2 sinx_3 - mgcosx_3 sinx_3}{M + m(1 - \frac{cosx_3^2}{A})} \\
\dot{x}_3 &= \dot{\theta} = x_4 \\
\dot{x_4} &= \frac{(M+m)gsinx_3 - Fcosx_3 - mlx_4^2 sinx_3 cosx_3}{M + ml(1 - \frac{cosx_3^2}{A})}
\end{aligned}$$

Matlab|ODE45

First Model

Below is the code used to solve the EOMs for the model represented in figure 1(where the rod is of negligible mass and A = 1):

```
% SOLVE (M+m)dx2/dt2 + mldthdt2cos(th) - mldthdt^2sin(th) = F
% SOLVE mldx2/dt2costh + Aml^2dth/dt2 - mglsin(th) = 0
% initial conditions x(0) = 0, x'(0) = 0, th(0) = 0, th'(0) = 0
% mass concentrated at the end of the pendulum : A = 1

t = 0: 0.001 : 5;

initial_x1 = 0;
initial_x2 = 0;
initial_dx1dt = 0;
initial_dx2dt = 0;
```

```
[t, x] = ode45(@inverse_pen, t, [initial_x1 initial_dx1dt initial_x2
initial_dx2dt]);

% ploting the results
plot(t, x(:, 1)); % X
title('Inverse Pendulum:Model 1');
plot(t, x(:, 3)); % theta
title('Inverse Pendulum:Model 1');
xlabel('t'); ylabel('x');


function dxdt = inverse_pen(t,x)
    M = 0.4;
    m = 0.1;
    l = 0.4;
    g = 9.8;
    F = 3;
    dxdt_1 = x(2);
    dxdt_2 = m*g*sin(x(3))*cos(x(3))+m*l*(x(4))^2*sin(x(3)) + F)/
              ((M+m) - m*cos(x(3))^2);

    dxdt_3 = x(4);
    dxdt_4 = ( (M+m)*g*sin(x(3)) - l*m*l*(x(4))^2*sin(x(3))*cos(x(3)) –
              F*cos(x(3)))/((M+m)*l - m*l*(cos(x(3))^2));


    dxdt=[dxdt_1; dxdt_2; dxdt_3; dxdt_4];
end
```

Output

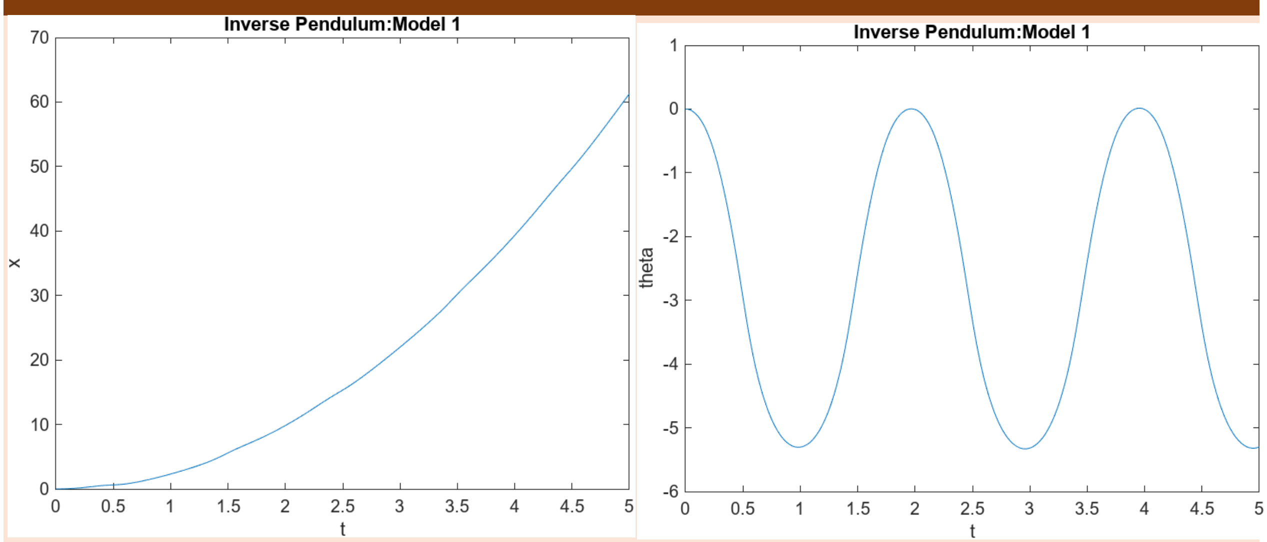

Figure 4. X and theta for model 1

The angle theta oscillates as expected. Please note that the angle is reported in radians.

Model 2

Output

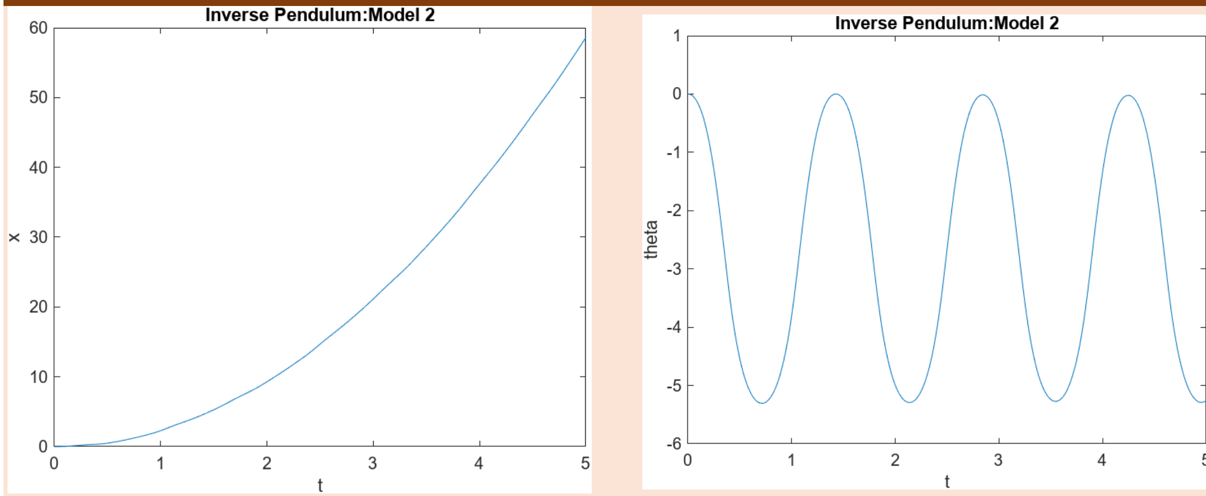

Figure 5. X and theta for model 2

As expected X(t) peaks less as the rod's oscillations slows the velocity in x direction but theta oscillates more due to the change of structure of the pendulum.

It is worth mentioning that **the second model (figure 3) behaves differently** when the force applied **F= 1N**. But the first model keeps oscillating.

Output

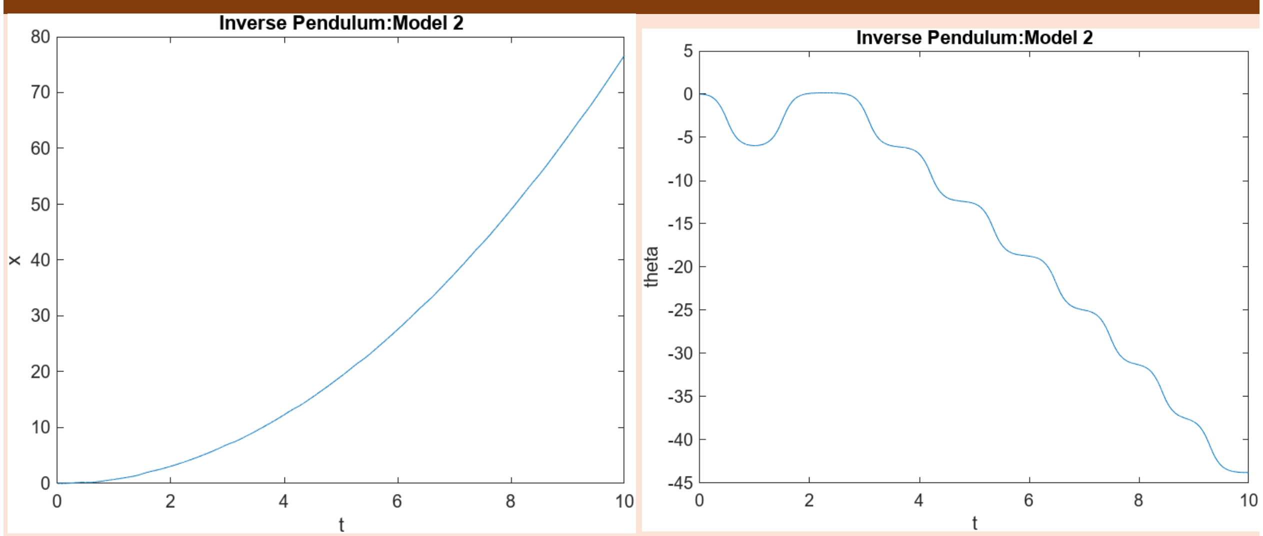

Figure 6. X and theta for model 2 when F = 1N

Simulink

First Model(A=1):

In order to draw the block diagram of the above equations, we linearized them (since when using these blocks, we're technically taking a Laplace transform from the equation terms such as $\dot{\theta}^2 sin\theta$ has no specific Laplace format). Notice how all the initial conditions are zero.

Linearization around $\theta = 0\ and\ thus\ cos\theta = 1, sin\theta = \theta$:

$(M + m)\ddot{x} + ml\ddot{\theta}cos\theta + ml\dot{\theta}^2 sin\theta = F \rightarrow (M + m)\ddot{x} + ml\ddot{\theta} = F$

$ml\ddot{x}cos\theta + ml^2\ddot{\theta} - mglsin\theta = 0 \rightarrow ml\ddot{x} + ml^2\ddot{\theta} - mgl\theta = 0$

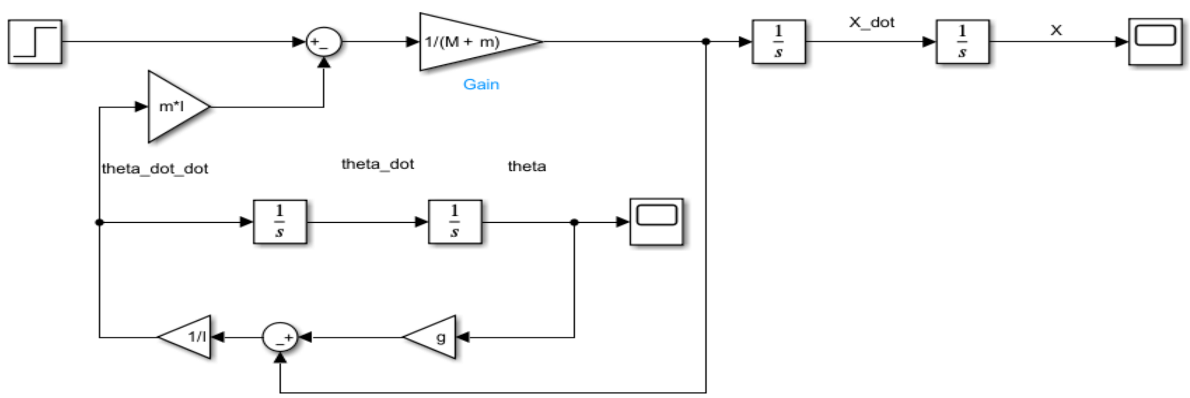

Figure 7. Block diagram of the equations

Output

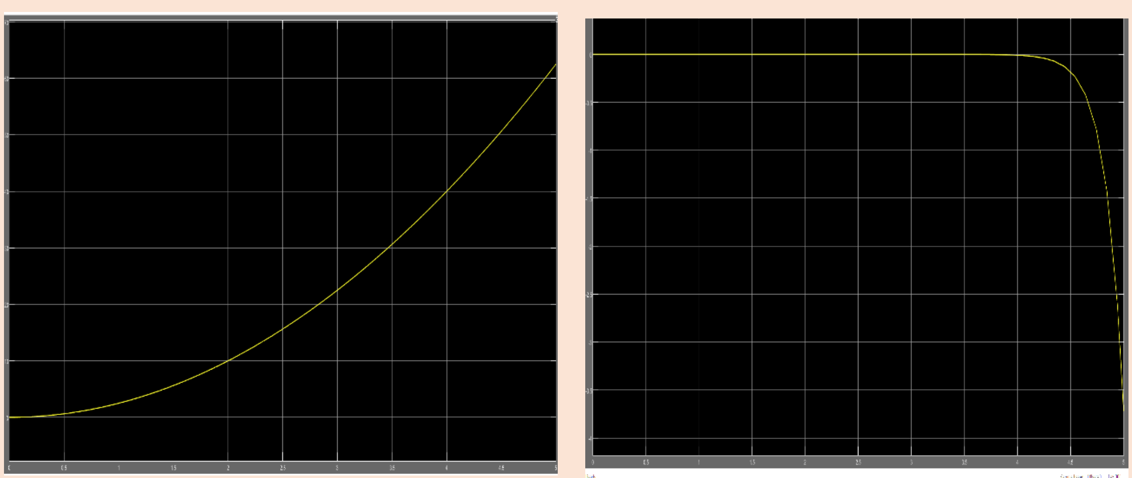

Figure 5. X and theta for model 1

The two graphs obtained above are only valid for a proximity of zero. And it does make sense as the x value changes, theta decreases (increases in magnitude). It is not analogous to the graphs obtained before as these are the output to the linearized EOM.

ADAMS

In this section model 2 is simulated as it is closer to reality.

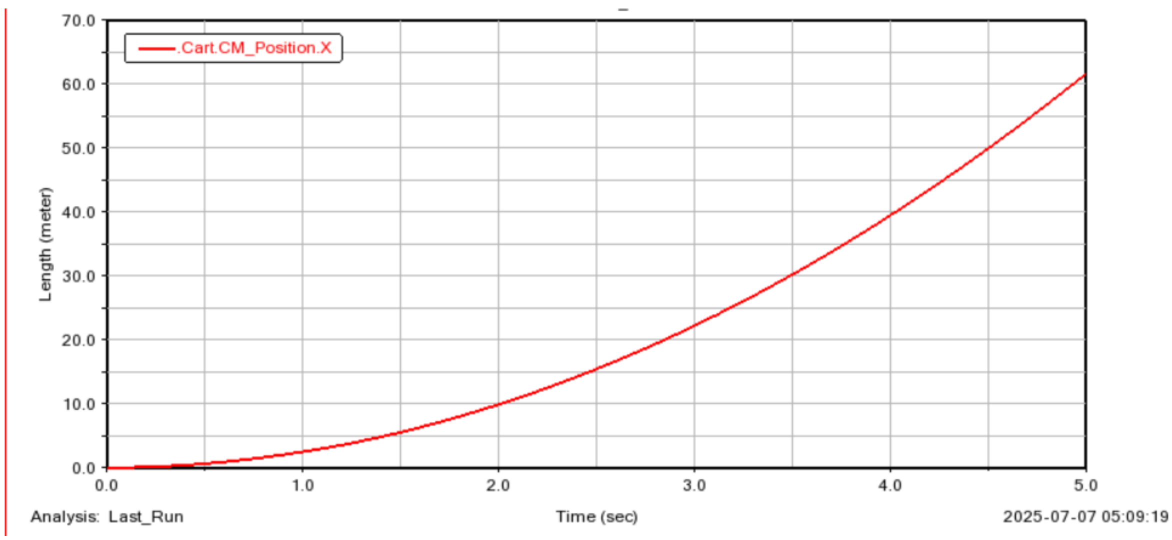

Figure 8. X(t) of the inverted pendulum

It is evident that the system behaves quite analogously in x direction to the behavior predicted by figure 5. But in the simulation, for the angle, damping is observed:

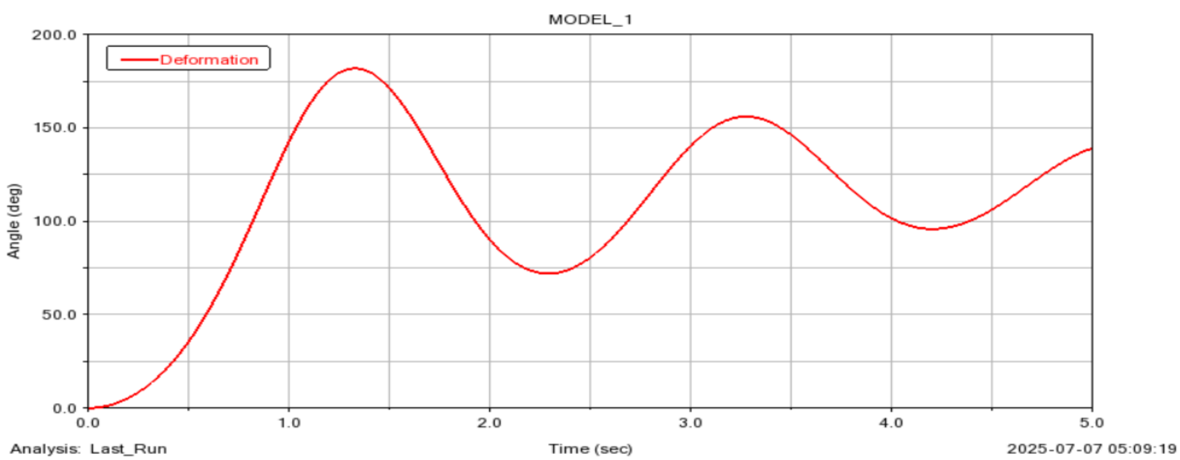

Figure 9. theta(t) of the inverted pendulum

For calculating theta, a **deforming spring** is used. Plus in in Adams an object **cannot have zero mass,** and thus the **geometry is quite different** from what we've studied theoretically. The rod goes through the cart which is also strange and unrealistic. Notice how figure 6 is conceptually close to figure 6.

Damping is perhaps caused by these constraints.

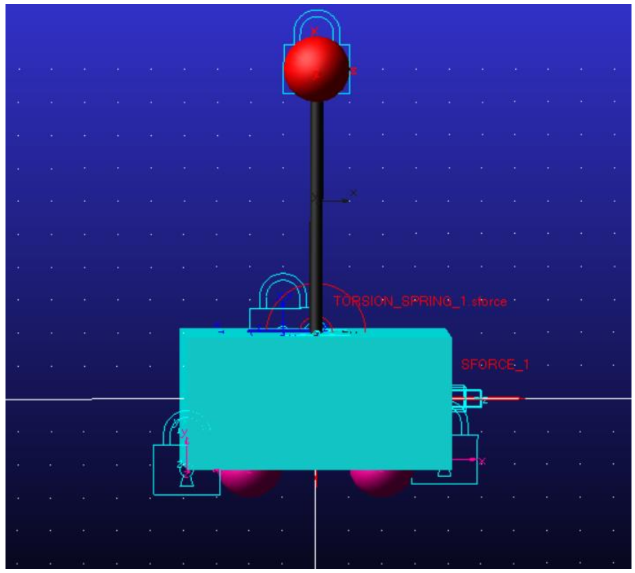

Figure 10. The inverted Pendulum designed in ADAMS

Below is another version of inverted pendulum that we designed:

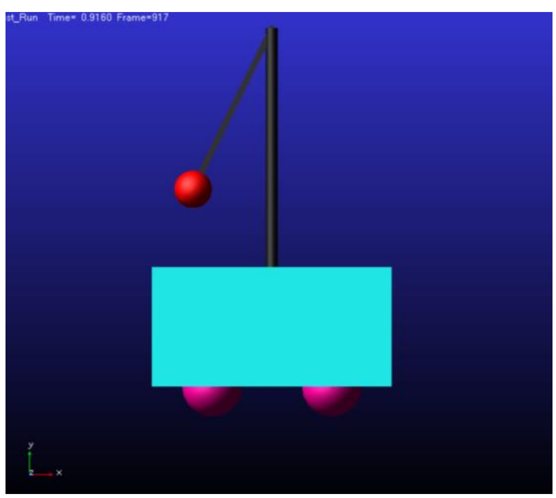

Figure 11. Another version of inverted pendulum.

Damping is not observed for this simulation.

Inverted Pendulums

Inverted pendulums are used in a variety of applications, primarily due to their unstable nature, which makes them excellent for testing and demonstrating control systems. They are used in robotics, biomechanics, and structural monitoring. Apparently there are other types of inverted pendulums as well, for example double inverted or triple inverted and etc.

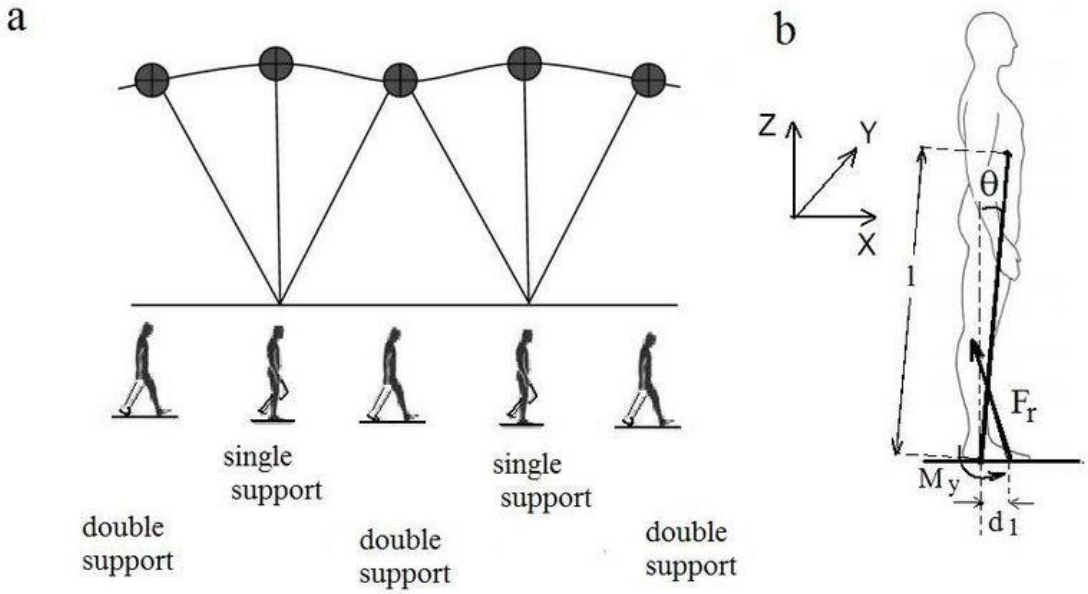

Figure 6 Use In biomechanics

In biomechanics Inverted Pendulums are used to model the stability of human body! In Civil engineering this mechanism is used to build buildings that are stable under the occurance of earthquakes.