#### Economics 103 – Statistics for Economists

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Lecture # 9

# Discrete RVs - Part II

#### Variance and Standard Deviation of a RV

The Defs are the same for continuous RVs, but the method of calculating will differ.

Variance (Var)

$$\sigma^2 = Var(X) = E[(X - \mu)^2] = E[(X - E[X])^2]$$

Standard Deviation (SD)

$$\sigma = \sqrt{\sigma^2} = SD(X)$$

### **Key Point**

Variance and std. dev. are expectations of functions of a RV

#### It follows that:

- 1. Variance and SD are constants
- 2. To derive facts about them you can use the facts you know about expected value

#### How To Calculate Variance for Discrete RV?

Remember: it's just a function of X!

Recall that 
$$\mu = E[X] = \sum_{\mathsf{all} \ x} x p(x)$$

$$Var(X) = E[(X - \mu)^2] = \sum_{\text{all } x} (x - \mu)^2 p(x)$$

#### Shortcut Formula For Variance

This is *not* the definition, it's a shortcut for doing calculations:

$$Var(X) = E[(X - \mu)^2] = E[X^2] - (E[X])^2$$

We'll prove this in an upcoming lecture.



Let  $X \sim \text{Bernoulli}(1/2)$ .

1. What is E[X]?



1. What is 
$$E[X]$$
?  $E[X] = 0 \times 1/2 + 1 \times 1/2 = 1/2$ 



- 1. What is E[X]?  $E[X] = 0 \times 1/2 + 1 \times 1/2 = 1/2$
- 2. What is  $E[X^2]$ ?



- 1. What is E[X]?  $E[X] = 0 \times 1/2 + 1 \times 1/2 = 1/2$
- 2. What is  $E[X^2]$ ?  $E[X^2] = 0^2 \times 1/2 + 1^2 \times 1/2 = 1/2$



- 1. What is E[X]?  $E[X] = 0 \times 1/2 + 1 \times 1/2 = 1/2$
- 2. What is  $E[X^2]$ ?  $E[X^2] = 0^2 \times 1/2 + 1^2 \times 1/2 = 1/2$
- 3. What is Var(X)?



- 1. What is E[X]?  $E[X] = 0 \times 1/2 + 1 \times 1/2 = 1/2$
- 2. What is  $E[X^2]$ ?  $E[X^2] = 0^2 \times 1/2 + 1^2 \times 1/2 = 1/2$
- 3. What is Var(X)?  $E[X^2] (E[X])^2 = 1/2 (1/2)^2 = 1/4$

#### Variance of Bernoulli RV – via the Shortcut Formula

Step 
$$1 - E[X]$$
  
 $\mu = E[X] = \sum_{x \in \{0,1\}} p(x) \cdot x = (1 - p) \cdot 0 + p \cdot 1 = p$   
Step  $2 - E[X^2]$ 

$$E[X^{2}] = \sum_{x \in \{0,1\}} x^{2} p(x) = 0^{2} (1-p) + 1^{2} p = p$$

Step 3 - Combine with Shortcut Formula

$$\sigma^2 = Var[X] = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$$

#### Variance of Bernoulli RV – Without Shortcut

You will fill in the missing steps on Problem Set 5.

$$\sigma^{2} = Var(X) = \sum_{x \in \{0,1\}} (x - \mu)^{2} p(x)$$

$$= \sum_{x \in \{0,1\}} (x - p)^{2} p(x)$$

$$\vdots$$

$$= p(1 - p)$$

#### Variance of a Linear Function



Suppose X is a random variable with  $Var(X) = \sigma^2$  and a, b are constants. What is Var(a + bX)?

- (a)  $\sigma^2$
- (b)  $a + \sigma^2$
- (c)  $b\sigma^2$
- (d)  $a + b\sigma^2$
- (e)  $b^2 \sigma^2$

#### Variance and SD are NOT Linear

$$Var(a+bX) = b^2\sigma^2$$

$$SD(a+bX) = |b|\sigma$$

These should look familiar from the related results for sample variance and std. dev. that you worked out on an earlier problem set.

#### Variance of a Linear Transformation

$$Var(a + bX) = E \left[ \{ (a + bX) - E(a + bX) \}^{2} \right]$$

$$= E \left[ \{ (a + bX) - (a + bE[X]) \}^{2} \right]$$

$$= E \left[ (bX - bE[X])^{2} \right]$$

$$= E[b^{2}(X - E[X])^{2}]$$

$$= b^{2}E[(X - E[X])^{2}]$$

$$= b^{2}Var(X) = b^{2}\sigma^{2}$$

The key point here is that variance is defined in terms of expectation and expectation is linear.

# Binomial Random Variable

What we get if we sum a bunch of indep. Bernoulli RVs

#### Binomial Random Variable

Let X = the sum of n independent Bernoulli trials, each with probability of success p. Then we say that:  $X \sim \text{Binomial}(n, p)$ 

#### **Parameters**

p = probability of "success," n = # of trials

#### Support

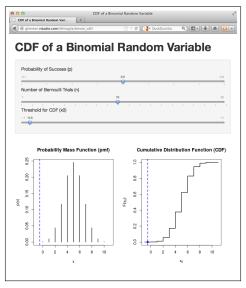
 $\{0, 1, 2, \ldots, n\}$ 

Probability Mass Function (pmf)

$$p(x) = \binom{n}{x} p^{x} (1-p)^{n-x}$$

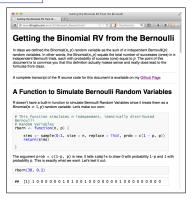
### http://fditraglia.shinyapps.io/binom\_cdf/

Try playing around with all three sliders. If you set the second to 1 you get a Bernoulli.



#### $http://fditraglia.github.com/Econ103Public/Rtutorials/Bernoulli\_Binomial.html \\$

Source Code on my Github Page



#### Don't forget this!

Binomial RV counts up the *total* number of successes (ones) in n indep. Bernoulli trials, each with prob. of success p.



#### Question

Suppose we flip a fair coin 3 times. What is the probability that we get exactly 2 heads?



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Suppose we flip a fair coin 3 times. What is the probability that we get exactly 2 heads?

#### Answer

Three basic outcomes make up this event: {HHT, HTH, THH}, each has probability  $1/8 = 1/2 \times 1/2 \times 1/2$ . Basic outcomes are mutually exclusive, so sum to get 3/8 = 0.375

#### Question

Suppose we flip an *unfair* coin 3 times, where the probability of heads is 1/3. What is the probability that we get exactly 2 heads?

#### Answer

No longer true that *all* basic outcomes are equally likely, but those with exactly two heads *still are* 

$$P(HHT) = (1/3)^2(1 - 1/3) = 2/27$$
  
 $P(THH) = 2/27$   
 $P(HTH) = 2/27$ 

Summing gives  $2/9 \approx 0.22$ 

Starting to see a pattern?

Suppose we flip an unfair coin 4 times, where the probability of heads is 1/3. What is the probability that we get exactly 2 heads?

Six equally likely, mutually exclusive basic outcomes make up this event:

$$\binom{4}{2}(1/3)^2(2/3)^2$$

### Multiple RVs at once - Definition of Joint PMF

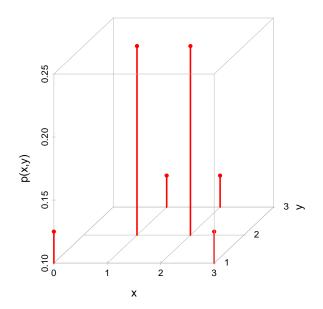
Let X and Y be discrete random variables. The joint probability mass function  $p_{XY}(x,y)$  gives the probability of each pair of realizations (x,y) in the support:

$$p_{XY}(x,y) = P(X = x \cap Y = y)$$

### Example: Joint PMF in Tabular Form

			Y	
		1	2	3
	0	1/8	0	0
V	1	0	1/4	1/8
X	2	0	1/4	1/8
	3	1/8	0	0

### Plot of Joint PMF



## What is $p_{XY}(1,2)$ ?



		Y	
	1	2	3
0	1/8	0	0
1	0	1/4	1/8
2	0	1/4	1/8
3	1/8	0	0
	1 2	0 1/8 1 0 2 0	1 2 0 1/8 0 1 0 1/4 2 0 1/4

### What is $p_{XY}(1,2)$ ?



			Y	
		1	2	3
	0	1/8	0	0
V	1	0	1/4	1/8
X	2	0	1/4	1/8
	3	1/8	0	0

$$p_{XY}(1,2) = P(X = 1 \cap Y = 2) = \frac{1}{4}$$

# What is $p_{XY}(2,1)$ ?



		Y	
	1	2	3
0	1/8	0	0
1	0	1/4	1/8
2	0	1/4	1/8
3	1/8	0	0
	1 2	0 1/8 1 0 2 0	1 2 0 1/8 0 1 0 1/4 2 0 1/4

### What is $p_{XY}(2,1)$ ?



			Y	
		1	2	3
	0	1/8	0	0
\ \ <u>\</u>	1	0	1/4	1/8
X	2	0	1/4	1/8
	3	1/8	0	0

$$p_{XY}(2,1) = P(X = 2 \cap Y = 1) = 0$$

### Properties of Joint PMF

- 1.  $0 \le p_{XY}(x, y) \le 1$  for any pair (x, y)
- 2. The sum of  $p_{XY}(x, y)$  over all pairs (x, y) in the support is 1:

$$\sum_{x}\sum_{y}p(x,y)=1$$

### Does this satisfy the properties of a joint pmf?



$$(A = YES, B = NO)$$

			Y	
		1	2	3
	0	1/8	0	0
X	1	0	1/4	1/8
^	2	0	1/4	1/8
	3	1/8	0	0

### Does this satisfy the properties of a joint pmf?



$$(A = YES, B = NO)$$

			Y	
		1	2	3
	0	1/8	0	0
\ \	1	0	1/4	1/8
X	2	0	1/4	1/8
	3	1/8	0	0

- 1.  $p(x,y) \ge 0$  for all pairs (x,y)
- 2.  $\sum_{x} \sum_{y} p(x,y) = 1/8 + 1/4 + 1/8 + 1/4 + 1/8 + 1/8 = 1$

### Joint versus Marginal PMFs

#### Joint PMF

$$p_{XY}(x,y) = P(X = x \cap Y = y)$$

#### Marginal PMFs

$$p_X(x) = P(X = x)$$

$$p_Y(y) = P(Y = y)$$

You can't calculate a joint pmf from marginals alone but you *can* calculate marginals from the joint!

### Marginals from Joint

$$p_X(x) = \sum_{\text{all } y} p_{XY}(x, y)$$

$$p_Y(y) = \sum_{\mathsf{all}\ x} p_{XY}(x,y)$$

Why?

$$p_Y(y) = P(Y = y) = P\left(\bigcup_{\text{all } x} \{X = x \cap Y = y\}\right)$$
$$= \sum_{\text{all } x} P(X = x \cap Y = y) = \sum_{\text{all } x} p_{XY}(x, y)$$

			Y		
		1	2	3	
	0	1/8	0	0	
X	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	

			Y		
		1	2	3	
	0	1/8	0	0	1/8
V	1	0	1/4	1/8	
Χ	2	0	1/4	1/8	
	3	1/8	0	0	

$$p_X(0) = 1/8 + 0 + 0 = 1/8$$

			Y		
		1	2	3	
	0	1/8	0	0	1/8
Χ	1	0	1/4	1/8	3/8
^	2	0	1/4	1/8	
	3	1/8	0	0	

$$p_X(0) = 1/8 + 0 + 0 = 1/8$$
  
 $p_X(1) = 0 + 1/4 + 1/8 = 3/8$ 

			Y		
		1	2	3	
	0	1/8	0	0	1/8
_	1	0	1/4	1/8	3/8
X	2	0	1/4	1/8	3/8
	3	1/8	0	0	

$$p_X(0) = 1/8 + 0 + 0 = 1/8$$
  
 $p_X(1) = 0 + 1/4 + 1/8 = 3/8$   
 $p_X(2) = 0 + 1/4 + 1/8 = 3/8$ 

			Y		
		1	2	3	
	0	1/8	0	0	1/8
Χ	1	0	1/4	1/8	3/8
^	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
					1

$$p_X(0) = 1/8 + 0 + 0 = 1/8$$
  
 $p_X(1) = 0 + 1/4 + 1/8 = 3/8$   
 $p_X(2) = 0 + 1/4 + 1/8 = 3/8$   
 $p_X(3) = 1/8 + 0 + 0 = 1/8$ 



			Y		
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8 1/8	
	3	1/8	0	0	



			Y		
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	
		1/4			

$$\rho_Y(1) = 1/8 + 0 + 0 + 1/8 = 1/4$$



			Y		
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	
		1/4	1/2		

$$p_Y(1) = 1/8 + 0 + 0 + 1/8 = 1/4$$
  
 $p_Y(2) = 0 + 1/4 + 1/4 + 0 = 1/2$ 



			Y		
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	
		1/4	1/2	1/4	1

$$p_Y(1) = 1/8 + 0 + 0 + 1/8 = 1/4$$
  
 $p_Y(2) = 0 + 1/4 + 1/4 + 0 = 1/2$   
 $p_Y(3) = 0 + 1/8 + 1/8 + 0 = 1/4$