Economics 103 – Statistics for Economists

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Lecture # 9

Discrete RVs – Part II

Linearity of Expectation

Holds for Continuous RVs as well, but proof is different.

Let X be a RV and a, b be constants. Then:

$$E[a + bX] = a + bE[X]$$

This is one of the most important facts in the course: the special case in which $E[g(X)] = g(E[X])$ is $g = a + bX$.

Example: Linearity of Expectation

Let $X \sim \text{Bernoulli}(1/3)$ and define $Y = 3X + 2$

1. What is $E[X]$? $E[X] = 0 \times 2/3 + 1 \times 1/3 = 1/3$
2. What is $E[Y]$? $E[Y] = E[3X + 2] = 3E[X] + 2 = 3$

Proof: Linearity of Expectation For Discrete RV

$$
\begin{aligned}
E[a + bX] &= \sum_{\text{all } x} (a + bx)p(x) \\[1em]
&= \sum_{\text{all } x} p(x) \cdot a + \sum_{\text{all } x} p(x) \cdot bx \\[1em]
&= a \sum_{\text{all } x} p(x) + b \sum_{\text{all } x} x \cdot p(x) \\[1em]
&= a + bE[X]
\end{aligned}
$$

Variance and Standard Deviation of a RV

The Defs are the same for continuous RVs, but the method of calculating will differ.

Variance (Var)

$$\sigma^2 = Var(X) = E\left[(X - \mu)^2\right] = E\left[(X - E[X])^2\right]$$

Standard Deviation (SD)

$$\sigma = \sqrt{\sigma^2} = SD(X)$$

Key Point

Variance and std. dev. are *expectations of functions of a RV*

It follows that:

1. Variance and SD are constants

2. To derive facts about them you can use the facts you know about expected value

How To Calculate Variance for Discrete RV?

Remember: it's just a function of X!

Recall that $\mu = E[X] = \displaystyle\sum_{\text{all } x} x p(x)$

$$Var(X) = E\left[(X - \mu)^2\right] = \sum_{\text{all } x} (x - \mu)^2 p(x)$$

Shortcut Formula For Variance

This is *not* the definition, it's a shortcut for doing calculations:

$$Var(X) = E\left[(X - \mu)^2\right] = E[X^2] - (E[X])^2$$

We'll prove this in an upcoming lecture.

Example: The Shortcut Formula

Let $X \sim$ Bernoulli$(1/2)$.

1. What is $E[X]$? $E[X] = 0 \times 1/2 + 1 \times 1/2 = 1/2$

2. What is $E[X^2]$? $E[X^2] = 0^2 \times 1/2 + 1^2 \times 1/2 = 1/2$

3. What is $Var(X)$? $E[X^2] - (E[X])^2 = 1/2 - (1/2)^2 = 1/4$

Variance of Bernoulli RV – via the Shortcut Formula

Step 1 – $E[X]$

$$\mu = E[X] = \sum_{x \in \{0,1\}} p(x) \cdot x = (1-p) \cdot 0 + p \cdot 1 = p$$

Step 2 – $E[X^2]$

$$E[X^2] = \sum_{x \in \{0,1\}} x^2 p(x) = 0^2(1-p) + 1^2 p = p$$

Step 3 – Combine with Shortcut Formula

$$\sigma^2 = Var[X] = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$$

Variance of Bernoulli RV – Without Shortcut

You will fill in the missing steps on Problem Set 5.

$$
\begin{aligned}
\sigma^2 \;=\; Var(X) &= \sum_{x \in \{0,1\}} (x - \mu)^2 p(x) \\
&= \sum_{x \in \{0,1\}} (x - p)^2 p(x) \\
&\;\vdots \\
&= p(1 - p)
\end{aligned}
$$

Variance of a Linear Function

Suppose X is a random variable with $Var(X) = \sigma^2$ and a, b are constants. What is $Var(a + bX)$?

(a) σ^2

(b) $a + \sigma^2$

(c) $b\sigma^2$

(d) $a + b\sigma^2$

(e) $b^2\sigma^2$

Variance and SD are *NOT* Linear

$$Var(a + bX) = b^2\sigma^2$$

$$SD(a + bX) = |b|\sigma$$

These should look familiar from the related results for sample variance and std. dev. that you worked out on an earlier problem set.

Variance of a Linear Transformation

$$
\begin{aligned}
Var(a + bX) &= E\left[\{(a + bX) - E(a + bX)\}^2\right] \\
&= E\left[\{(a + bX) - (a + bE[X])\}^2\right] \\
&= E\left[(bX - bE[X])^2\right] \\
&= E[b^2(X - E[X])^2] \\
&= b^2 E[(X - E[X])^2] \\
&= b^2 Var(X) = b^2 \sigma^2
\end{aligned}
$$

The key point here is that variance is defined in terms of expectation and expectation is linear.

Binomial Random Variable

What we get if we sum a bunch of indep. Bernoulli RVs

Binomial Random Variable

Let $X =$ the sum of n independent Bernoulli trials, each with probability of success p. Then we say that: $X \sim \text{Binomial}(n, p)$

Parameters

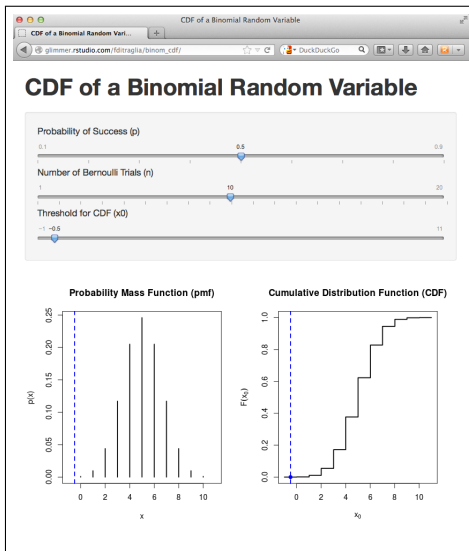$p =$ probability of "success," $n = \#$ of trials

Support

$\{0, 1, 2, \ldots, n\}$

Probability Mass Function (pmf)

$$p(x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

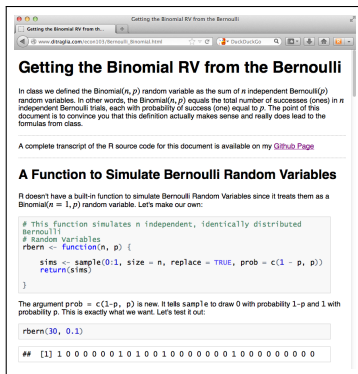# http://fditraglia.shinyapps.io/binom_cdf/

Try playing around with all three sliders. If you set the second to 1 you get a Bernoulli.

Source Code on my | Github Page |

Don't forget this!

Binomial RV counts up the *total* number of successes (ones) in n indep. Bernoulli trials, each with prob. of success p.

Where does the Binomial pmf come from?

Question

Suppose we flip a fair coin 3 times. What is the probability that we get exactly 2 heads?

Answer

Three basic outcomes make up this event: $\{HHT, HTH, THH\}$, each has probability $1/8 = 1/2 \times 1/2 \times 1/2$. Basic outcomes are mutually exclusive, so sum to get $3/8 = 0.375$

Where does the Binomial pmf come from?

Question

Suppose we flip an *unfair* coin 3 times, where the probability of heads is $1/3$. What is the probability that we get exactly 2 heads?

Answer

No longer true that *all* basic outcomes are equally likely, but those with exactly two heads *still are*

$$
\begin{aligned}
P(HHT) &= (1/3)^2(1 - 1/3) = 2/27 \\
P(THH) &= 2/27 \\
P(HTH) &= 2/27
\end{aligned}
$$

Summing gives $2/9 \approx 0.22$

Where does the Binomial pmf come from?

Starting to see a pattern?

Suppose we flip an unfair coin *4* times, where the probability of heads is $1/3$. What is the probability that we get exactly 2 heads?

HHTT   TTHH
HTHT   THTH
HTTH   THHT

Six equally likely, mutually exclusive basic outcomes make up this event:

$$\binom{4}{2}(1/3)^2(2/3)^2$$

Multiple RVs *at once* - Definition of Joint PMF

Let X and Y be discrete random variables. The joint probability mass function $p_{XY}(x, y)$ gives the probability of each pair of realizations (x, y) in the support:

$$\boxed{p_{XY}(x, y) = P(X = x \cap Y = y)}$$

Example: Joint PMF in Tabular Form

		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

Plot of Joint PMF
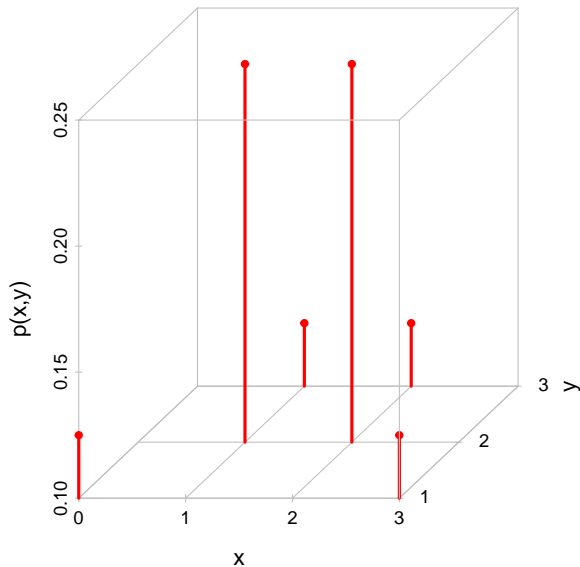
What is $p_{XY}(1, 2)$?

		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

$$p_{XY}(1, 2) = P(X = 1 \cap Y = 2) = 1/4$$

What is $p_{XY}(2, 1)$?

		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

$$p_{XY}(2, 1) = P(X = 2 \cap Y = 1) = 0$$

Properties of Joint PMF

1. $0 \leq p_{XY}(x, y) \leq 1$ for any pair (x, y)

2. The sum of $p_{XY}(x, y)$ over all pairs (x, y) in the support is 1:

$$\sum_x \sum_y p(x, y) = 1$$

Does this satisfy the properties of a joint pmf?

(A = YES, B = NO)

		Y		
		1	2	3
	0	1/8	0	0
X	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

1. $p(x, y) \geq 0$ for all pairs (x, y)
2. $\sum_x \sum_y p(x, y) = 1/8 + 1/4 + 1/8 + 1/4 + 1/8 + 1/8 = 1$

Joint versus Marginal PMFs

Joint PMF

$p_{XY}(x, y) = P(X = x \cap Y = y)$

Marginal PMFs

$p_X(x) = P(X = x)$
$p_Y(y) = P(Y = y)$

You can't calculate a joint pmf from marginals alone but you *can* calculate marginals from the joint!

Marginals from Joint

$$p_X(x) = \sum_{\text{all } y} p_{XY}(x, y)$$

$$p_Y(y) = \sum_{\text{all } x} p_{XY}(x, y)$$

Why?

$$
\begin{aligned}
p_Y(y) &= P(Y = y) = P\left(\bigcup_{\text{all } x} \{X = x \cap Y = y\} \right) \\
&= \sum_{\text{all } x} P(X = x \cap Y = y) = \sum_{\text{all } x} p_{XY}(x, y)
\end{aligned}
$$

To get the marginals sum "into the margins" of the table.

		Y			
		1	2	3	
	0	1/8	0	0	1/8
X	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
					1

$$p_X(0) = 1/8 + 0 + 0 = 1/8$$
$$p_X(1) = 0 + 1/4 + 1/8 = 3/8$$
$$p_X(2) = 0 + 1/4 + 1/8 = 3/8$$
$$p_X(3) = 1/8 + 0 + 0 = 1/8$$

What is $p_Y(2)$?

		Y			
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	
		1/4	1/2	1/4	1

$$p_Y(1) = 1/8 + 0 + 0 + 1/8 = 1/4$$

$$p_Y(2) = 0 + 1/4 + 1/4 + 0 = 1/2$$

$$p_Y(3) = 0 + 1/8 + 1/8 + 0 = 1/4$$