

# Economics 103 – Statistics for Economists

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Lecture # 10

# Discrete RVs – Part III

# Overview

## So Far

Consider one RV at a time.

## Today

Consider relationships *between* RVs.

## Definition of Joint PMF

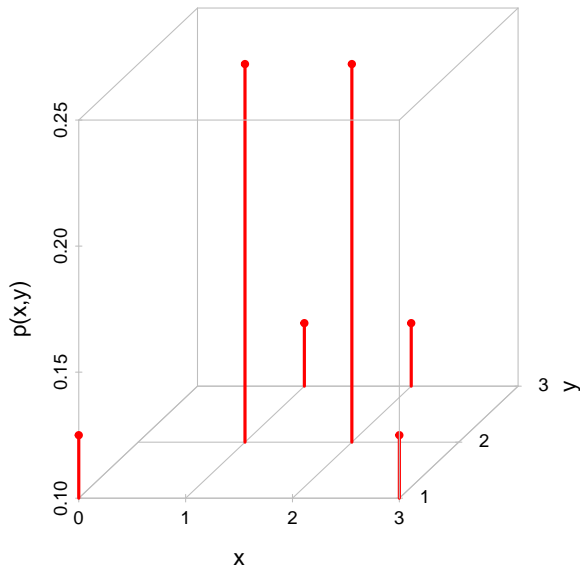
Let  $X$  and  $Y$  be discrete random variables. The joint probability mass function  $p_{XY}(x, y)$  gives the probability of each pair of realizations  $(x, y)$  in the support:

$$p_{XY}(x, y) = P(X = x \cap Y = y)$$

## Example: Joint PMF in Tabular Form

		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

## Plot of Joint PMF



What is  $p_{XY}(1, 2)$ ?



		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

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	2	0	1/4	1/8
	3	1/8	0	0

$$p_{XY}(1, 2) = P(X = 1 \cap Y = 2) = 1/4$$



What is  $p_{XY}(2, 1)$ ?



		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

What is  $p_{XY}(2, 1)$ ?



		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

$$p_{XY}(2, 1) = P(X = 2 \cap Y = 1) = 0$$

# Properties of Joint PMF

1.  $0 \leq p_{XY}(x, y) \leq 1$  for any pair  $(x, y)$
2. The sum of  $p_{XY}(x, y)$  over all pairs  $(x, y)$  in the support is 1:

$$\sum_x \sum_y p(x, y) = 1$$

Does this satisfy the properties of a joint pmf?



(A = YES, B = NO)

		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

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	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

1.  $p(x, y) \geq 0$  for all pairs  $(x, y)$
2.  $\sum_x \sum_y p(x, y) = 1/8 + 1/4 + 1/8 + 1/4 + 1/8 + 1/8 = 1$

# Joint versus Marginal PMFs

## Joint PMF

$$p_{XY}(x, y) = P(X = x \cap Y = y)$$

## Marginal PMFs

$$p_X(x) = P(X = x)$$

$$p_Y(y) = P(Y = y)$$

You can't calculate a joint pmf from marginals alone but you *can* calculate marginals from the joint!

## Marginals from Joint

$$p_X(x) = \sum_{\text{all } y} p_{XY}(x, y)$$

$$p_Y(y) = \sum_{\text{all } x} p_{XY}(x, y)$$

Why?

$$\begin{aligned} p_Y(y) &= P(Y = y) = P\left(\bigcup_{\text{all } x} \{X = x \cap Y = y\}\right) \\ &= \sum_{\text{all } x} P(X = x \cap Y = y) = \sum_{\text{all } x} p_{XY}(x, y) \end{aligned}$$

To get the marginals sum “into the margins” of the table.

		Y			
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	



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	2	0	1/4	1/8	
	3	1/8	0	0	

$$p_X(0) = 1/8 + 0 + 0 = 1/8$$

To get the marginals sum “into the margins” of the table.

		Y			
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X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	
	3	1/8	0	0	

$$p_X(0) = 1/8 + 0 + 0 = 1/8$$

$$p_X(1) = 0 + 1/4 + 1/8 = 3/8$$

To get the marginals sum “into the margins” of the table.

		Y			
		1	2	3	
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	3	1/8	0	0	

$$p_X(0) = 1/8 + 0 + 0 = 1/8$$

$$p_X(1) = 0 + 1/4 + 1/8 = 3/8$$

$$p_X(2) = 0 + 1/4 + 1/8 = 3/8$$

To get the marginals sum “into the margins” of the table.

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
					1

$$p_X(0) = 1/8 + 0 + 0 = 1/8$$

$$p_X(1) = 0 + 1/4 + 1/8 = 3/8$$

$$p_X(2) = 0 + 1/4 + 1/8 = 3/8$$

$$p_X(3) = 1/8 + 0 + 0 = 1/8$$

What is  $p_Y(2)$ ?



		Y			
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	

What is  $p_Y(2)$ ?



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	2	0	1/4	1/8	
	3	1/8	0	0	
		1/4			

$$p_Y(1) = 1/8 + 0 + 0 + 1/8 = 1/4$$

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		Y			
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	2	0	1/4	1/8	
	3	1/8	0	0	
		1/4	1/2		

$$p_Y(1) = 1/8 + 0 + 0 + 1/8 = 1/4$$

$$p_Y(2) = 0 + 1/4 + 1/4 + 0 = 1/2$$

What is  $p_Y(2)$ ?



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	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	
		1/4	1/2	1/4	1

$$p_Y(1) = 1/8 + 0 + 0 + 1/8 = 1/4$$

$$p_Y(2) = 0 + 1/4 + 1/4 + 0 = 1/2$$

$$p_Y(3) = 0 + 1/8 + 1/8 + 0 = 1/4$$



# Definition of Conditional PMF

How does the distribution of  $y$  change with  $x$ ?

$$p_{Y|X}(y|x) = P(Y = y|X = x)$$

Which of these is the formula for  $p_{Y|X}(y|x)$ ?



You can figure this out from what you already know about probability, using the definition  $p_{Y|X}(y|x) = P(Y = y|X = x)$

- (a)  $p_X(x)/p_Y(y)$
- (b)  $p_{XY}(x, y)/p_X(x)$
- (c)  $p_X(x)p_{XY}(x, y)$
- (d)  $p_{XY}(x, y)/p_Y(y)$
- (e)  $p_Y(y)/p_X(x)$

## Conditional PMF from Joint and Marginal

$$p_{Y|X}(y|x) = P(Y = y|X = x) = \frac{P(Y = y \cap X = x)}{P(X = x)} = \frac{p_{XY}(x, y)}{p_X(x)}$$

Hence,

$$p_{Y|X}(y|x) = \frac{p_{XY}(x, y)}{p_X(x)}$$

and similarly,

$$p_{X|Y}(x|y) = \frac{p_{XY}(x, y)}{p_Y(y)}$$

## Conditional PMF of $Y$ given $X = 2$

		$Y$			
		1	2	3	
$X$	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8

$$p_{Y|X}(1|2) =$$

## Conditional PMF of $Y$ given $X = 2$

		$Y$			
		1	2	3	
$X$	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8

$$p_{Y|X}(1|2) = \frac{p_{XY}(2,1)}{p_X(2)} =$$

## Conditional PMF of $Y$ given $X = 2$

		$Y$			
		1	2	3	
$X$	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8

$$p_{Y|X}(1|2) = \frac{p_{XY}(2,1)}{p_X(2)} = \frac{0}{3/8} = 0$$

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		$Y$			
		1	2	3	
$X$	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8

$$p_{Y|X}(1|2) = \frac{p_{XY}(2,1)}{p_X(2)} = \frac{0}{3/8} = 0$$

$$p_{Y|X}(2|2) = \frac{p_{XY}(2,2)}{p_X(2)} = \frac{1/4}{3/8} = 2/3$$

## Conditional PMF of $Y$ given $X = 2$

		$Y$			
		1	2	3	
$X$	0	1/8	0	0	1/8
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$$p_{Y|X}(3|2) = \frac{p_{XY}(2,3)}{p_X(2)} = \frac{1/8}{3/8} = 1/3$$



What is  $p_{X|Y}(1|2)$ ?



		Y			
		1	2	3	
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$$p_{X|Y}(0|2) = \frac{p_{XY}(0,2)}{p_Y(2)} = \frac{0}{1/2} = 0$$

What is  $p_{X|Y}(1|2)$ ?



		Y			
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What is  $p_{X|Y}(1|2)$ ?



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$$p_{X|Y}(1|2) = \frac{p_{XY}(1,2)}{p_Y(2)} = \frac{1/4}{1/2} = 1/2$$

$$p_{X|Y}(2|2) = \frac{p_{XY}(2,2)}{p_Y(2)} = \frac{1/4}{1/2} = 1/2$$

$$p_{X|Y}(3|2) = \frac{p_{XY}(3,2)}{p_Y(2)} = \frac{0}{1/2} = 0$$

# Independent RVs

## Definition

We say that two discrete RVs are **independent** if and only if their joint pmf equals the product of their marginal pmfs:

$$p_{XY}(x, y) = p_X(x)p_Y(y)$$

for all pairs  $(x, y)$  in the support.

## In Terms of Conditional PMF

From the previous slide, it follows that an equivalent definition of independence is that both conditional pmfs equal the corresponding marginal pmfs:  $p_{Y|X}(y|X) = p_Y(y)$  and  $p_{X|Y}(x|y) = p_X(x)$  for all  $(x, y)$  in the support.

# Are $X$ and $Y$ Independent?



(A = YES, B = NO)

		$Y$			
		1	2	3	
$X$	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

# Are $X$ and $Y$ Independent?



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	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

$$p_{XY}(2, 1) = 0$$



# Are $X$ and $Y$ Independent?



(A = YES, B = NO)

		$Y$			
		1	2	3	
$X$	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

$$p_{XY}(2, 1) = 0$$

$$p_X(2) \times p_Y(1) =$$

## Are $X$ and $Y$ Independent?



(A = YES, B = NO)

		$Y$			
		1	2	3	
$X$	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

$$p_{XY}(2, 1) = 0$$

$$p_X(2) \times p_Y(1) = (3/8) \times (1/4) \neq 0$$

Therefore  $X$  and  $Y$  are *not* independent.

# Conditional Expectation

## Intuition

$E[Y|X]$  is our “best guess” of the realization that  $Y$  will take on having observed the realization of  $X$ .

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## $E[Y|X = x]$ is a Constant

To get a “best guess” for  $Y$ , we plug in the realization we observed for  $X$ :  $E[Y|X = x]$  is a constant, our guess of the realization of  $Y$ .

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## Calculating $E[Y|X = x]$

Take the mean of the conditional pmf of  $Y$  given  $X = x$ .

## Conditional Expectation: $E[Y|X = 2]$

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

We showed above that the conditional pmf of  $Y|X = 2$  is:

$$p_{Y|X}(1|2) = 0 \quad p_{Y|X}(2|2) = 2/3 \quad p_{Y|X}(3|2) = 1/3$$

Hence

$$E[Y|X = 2] = 2 \times 2/3 + 3 \times 1/3 = 7/3$$

## Conditional Expectation: $E[Y|X = 0]$

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

The conditional pmf of  $Y|X = 0$  is

$$p_{Y|X}(1|0) = 1 \quad p_{Y|X}(2|0) = 0 \quad p_{Y|X}(3|0) = 0$$

Hence  $E[Y|X = 0] = 1$



Calculate  $E[Y|X = 3]$

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

The conditional pmf of  $Y|X = 3$  is

$$p_{Y|X}(1|3) = 1 \quad p_{Y|X}(2|3) = 0 \quad p_{Y|X}(3|3) = 0$$

Hence  $E[Y|X = 3] = 1$

Calculate  $E[Y|X = 1]$



		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

Calculate  $E[Y|X = 1]$



		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

The conditional pmf of  $Y|X = 1$  is

$$p_{Y|X}(1|1) = 0 \quad p_{Y|X}(2|1) = 2/3 \quad p_{Y|X}(3|1) = 1/3$$

Hence

$$E[Y|X = 1] = 2 \times 2/3 + 3 \times 1/3 = 7/3$$

## $E[Y|X]$ is a Random Variable

In this particular example we have seen that:

$$E[Y|X] = \begin{cases} 1 & X = 0 \\ 7/3 & X = 1 \\ 7/3 & X = 2 \\ 1 & X = 3 \end{cases}$$

But from above we know the marginal distribution of  $X$  :

$$P(X = 0) = 1/8 \quad P(X = 1) = 3/8$$

$$P(X = 2) = 3/8 \quad P(X = 3) = 1/8$$

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In this particular example we have seen that:

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$$P(X = 2) = 3/8 \quad P(X = 3) = 1/8$$

Therefore,  $E[Y|X]$  is a RV that takes on the value 1 with probability 1/4 and the value 7/3 with probability 3/4.

# The Law of Iterated Expectations

Since  $E[Y|X]$  is a random variable, we can ask what its expectation is. It turns out that, for any RVs  $X$  and  $Y$

$$E[E[Y|X]] = E[Y]$$

and this is called the **Law of Iterated Expectations**. I've posted a proof [HERE](#) for those who want are interested.

This will be helpful in Econ 104...

# Law of Iterated Expectations for Our Example

Marginal pmf of  $Y$

$$P(Y = 1) = 1/4$$

$$P(Y = 2) = 1/2$$

$$P(Y = 3) = 1/4$$

# Law of Iterated Expectations for Our Example

Marginal pmf of  $Y$

$$P(Y = 1) = 1/4$$

$$P(Y = 2) = 1/2$$

$$P(Y = 3) = 1/4$$

$$\begin{aligned} E[Y] &= 1 \times 1/4 + 2 \times 1/2 + 3 \times 1/4 \\ &= 2 \end{aligned}$$



# Law of Iterated Expectations for Our Example

Marginal pmf of  $Y$

$$P(Y = 1) = 1/4$$

$$P(Y = 2) = 1/2$$

$$P(Y = 3) = 1/4$$

$E[Y|X]$

$$E[Y|X] = \begin{cases} 1 & \text{w/ prob. } 1/4 \\ 7/3 & \text{w/ prob. } 3/4 \end{cases}$$

$$\begin{aligned} E[Y] &= 1 \times 1/4 + 2 \times 1/2 + 3 \times 1/4 \\ &= 2 \end{aligned}$$

# Law of Iterated Expectations for Our Example

Marginal pmf of  $Y$

$$P(Y = 1) = 1/4$$

$$P(Y = 2) = 1/2$$

$$P(Y = 3) = 1/4$$

$$\begin{aligned} E[Y] &= 1 \times 1/4 + 2 \times 1/2 + 3 \times 1/4 \\ &= 2 \end{aligned}$$

$E[Y|X]$

$$E[Y|X] = \begin{cases} 1 & \text{w/ prob. } 1/4 \\ 7/3 & \text{w/ prob. } 3/4 \end{cases}$$

$$\begin{aligned} E[E[Y|X]] &= 1 \times 1/4 + 7/3 \times 3/4 \\ &= 2 \end{aligned}$$

## Expectation of Function of Two Discrete RVs

$$E[g(X, Y)] = \sum_x \sum_y g(x, y) p_{XY}(x, y)$$

# Some Extremely Important Examples

Same For Continuous Random Variables

Let  $\mu_X = E[X], \mu_Y = E[Y]$

Covariance

$$\sigma_{XY} = \text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

Correlation

$$\rho_{XY} = \text{Corr}(X, Y) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

## Shortcut Formula for Covariance

Much easier for calculating:

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

We'll talk more about this in an upcoming lecture...

## Calculating $\text{Cov}(X, Y)$

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

$$E[X] = 3/8 + 2 \times 3/8 + 3 \times 1/8 = 3/2$$

$$E[Y] = 1/4 + 2 \times 1/2 + 3 \times 1/4 = 2$$

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$$\begin{aligned} E[XY] &= 1/4 \times (2 + 4) + 1/8 \times (3 + 6 + 3) \\ &= 3 \end{aligned}$$

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$$\text{Corr}(X, Y) = \text{Cov}(X, Y) / [SD(X)SD(Y)] = 0$$

Hence, zero covariance (correlation)  
does *not* imply independence!

However, independence *does* imply  
zero covariance (correlation)

$X, Y$  Independent  $\Rightarrow \text{Cov}(X, Y) = 0$

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$X, Y$  Independent  $\Rightarrow \text{Cov}(X, Y) = 0$

$$\begin{aligned}\text{Cov}(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\ &= \sum_x \sum_y (x - \mu_X)(y - \mu_Y)p(x, y) \\ &= \end{aligned}$$

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