Problem Set #5

Econ 103

Part I – Problems from the Textbook

Chapter 4: 1, 3, 5, 7, 9, 11, 13, 15, 25, 27, 29
Chapter 5: 1, 3, 5, 9, 11, 13, 17

Part II – Additional Problems

1. Suppose X is a random variable with support $\{-1, 0, 1\}$ where $p(-1) = q$ and $p(1) = p$.
   (a) What is $p(0)$?
   (b) Calculate the CDF, $F(x_0)$, of X.
   (c) Calculate $E[X]$.
   (d) What relationship must hold between p and q to ensure $E[X] = 0$?
2. Fill in the missing details from class to calculate the variance of a Bernoulli Random Variable *directly*, that is *without* using the shortcut formula.
3. Prove that the Bernoulli Random Variable is a special case of the Binomial Random variable for which $n = 1$. (Hint: compare pmfs.)
4. Suppose that X is a random variable with support $\{1, 2\}$ and Y is a random variable with support $\{0, 1\}$ where X and Y have the following joint distribution:

$$\begin{aligned} p_{XY}(1,0) &= 0.20, \qquad p_{XY}(1,1) = 0.30 \\ p_{XY}(2,0) &= 0.25, \qquad p_{XY}(2,1) = 0.25 \end{aligned}$$

   (a) Express the joint distribution in a 2×2 table.
   (b) Using the table, calculate the marginal probability distributions of X and Y.
   (c) Calculate the conditional probability distribution of $Y|X = 1$ and $Y|X = 2$.
   (d) Calculate $E[Y|X]$.
   (e) What is $E[E[Y|X]]$?

(f) Calculate the covariance between X and Y using the shortcut formula.

5. Let X and Y be discrete random variables and a, b, c, d be constants. Prove the following:
   (a) $Cov(a + bX, c + dY) = bdCov(X, Y)$
   (b) $Corr(a + bX, c + dY) = Corr(X, Y)$

6. Fill in the missing steps from lecture to prove the shortcut formula for covariance:

$$Cov(X, Y) = E[XY] - E[X]E[Y]$$

7. Let X_1 be a random variable denoting the returns of stock 1, and X_2 be a random variable denoting the returns of stock 2. Accordingly let $\mu_1 = E[X_1]$, $\mu_2 = E[X_2]$, $\sigma_1^2 = Var(X_1)$, $\sigma_2^2 = Var(X_2)$ and $\rho = Corr(X_1, X_2)$. A *portfolio*, Π, is a linear combination of X_1 and X_2 with weights that sum to one, that is $\Pi(\omega) = \omega X_1 + (1 - \omega)X_2$, indicating the proportions of stock 1 and stock 2 that an investor holds. In this example, we require $\omega \in [0, 1]$, so that *negative* weights are not allowed. (This rules out short-selling.)
   (a) Calculate $E[\Pi(\omega)]$ in terms of ω, μ_1 and μ_2.
   (b) If $\omega \in [0, 1]$ is it possible to have $E[\Pi(\omega)] > \mu_1$ *and* $E[\Pi(\omega)] > \mu_2$? What about $E[\Pi(\omega)] < \mu_1$ and $E[\Pi(\omega)] < \mu_2$? Explain.
   (c) Express $Cov(X_1, X_2)$ in terms of ρ and σ_1, σ_2.
   (d) What is $Var[\Pi(\omega)]$? (Your answer should be in terms of ρ, σ_1^2 and σ_2^2.)
   (e) Using part (d) show that the value of ω that minimizes $Var[\Pi(\omega)]$ is

$$\omega^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

   In other words, $\Pi(\omega^*)$ is the *minimum variance portfolio.*
   (f) If you want a challenge, check the second order condition from part (e).