Economics 103 – Statistics for Economists

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Lecture # 12

Continuous Distributions – Part II

Last Time: Continuous RVs, Probability As Area

Probability Density Function (pdf)

- $\int_a^b f(x)\, dx = P(a \leq X \leq b)$
- $f(x) \geq 0$ for all x in the support
- $f(x) \neq P(X = x)$, can be greater than one

Cumulative Distribution Function

- $F(x_0) \equiv P(X \leq x_0) = \int_{-\infty}^{x_0} f(x)\, dx$
- First Fundamental Theorem of Calculus: $f(x) = F'(x)$

Last Time: Uniform$(0, 1)$ RV

Intuition

Equally likely to take on any value on its support: $[0, 1]$

Probability Density Function

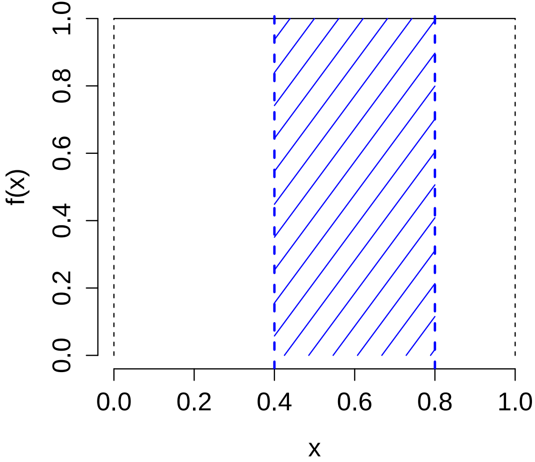
$f(x) = 1$ for $x \in [0, 1]$, zero otherwise

Cumulative Distribution Function

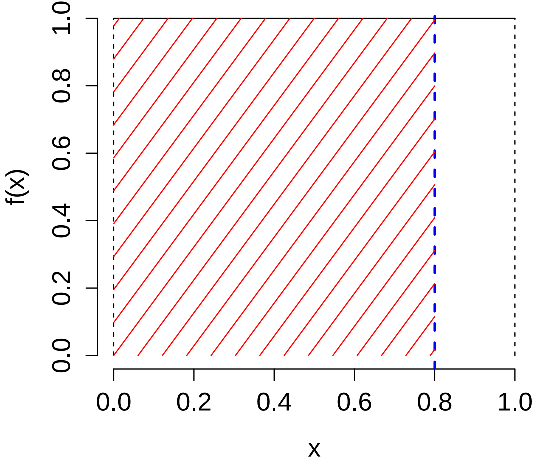
$$F(x_0) = \begin{cases} 0, x_0 < 0 \\ x_0, 0 \leq x_0 \leq 1 \\ 1, x_0 > 1 \end{cases}$$

Key Idea: Probability of Intervals

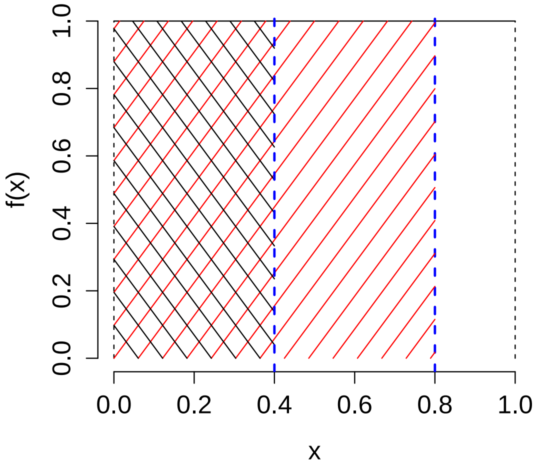
What is $P(0.4 \leq X \leq 0.8)$ if $X \sim \text{Uniform}(0, 1)$?



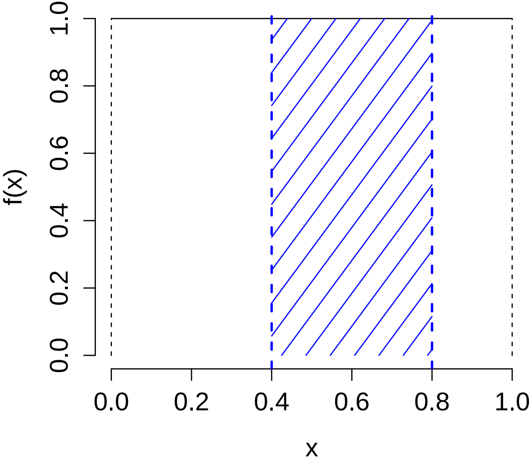
$$F(0.8) = P(X \leq 0.8)$$



$F(0.8) - F(0.4) = ?$



$$F(0.8) - F(0.4) = P(0.4 \leq X \leq 0.8) = 0.4$$



Probability of Interval for Continuous RV

$$\boxed{P(a \leq X \leq b) = \int_a^b f(x)\, dx = F(b) - F(a)}$$

This is just the Second Fundamental Theorem of Calculus.

Expected Value for Continuous RVs

$$\boxed{\int_{-\infty}^{\infty} xf(x)\ dx}$$

Remember: Integrals Replace Sums!

Example: Uniform(0,1) Random Variable

$$E[X] = \int_{-\infty}^{\infty} xf(x)\,dx =$$

Example: Uniform(0,1) Random Variable

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^1 x \cdot 1\, dx \\ &= \left. \frac{x^2}{2} \right|_0^1 = 1/2 - 0 = 1/2 \end{aligned}$$

Expected Value of a Function of a Continuous RV

$$\boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\, dx}$$

Example: Uniform(0, 1) RV

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^1 x^2 \cdot 1\, dx \\ &= \left.\frac{x^3}{3}\right|_0^1 = 1/3 \end{aligned}$$

Once we have defined expected value for continuous RVs, we can use everything we know about variance, covariance, etc. from discrete RVs!

Variance of Continuous RV

$$\boxed{Var(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx}$$

where

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

Shortcut formula still holds for continuous RVs!

$$Var(X) = E[X^2] - (E[X])^2$$

Example: Uniform(0,1) Random Variable

$$Var(X) \;=\; E\left[(X - E[X])^2\right] = E[X^2] - (E[X])^2$$

Example: Uniform(0,1) Random Variable

$$
\begin{aligned}
Var(X) &= E\left[(X - E[X])^2\right] = E[X^2] - (E[X])^2 \\
&= 1/3 - (1/2)^2 \\
&= 1/12 \\
&\approx 0.083
\end{aligned}
$$

Much More Complicated Without the Shortcut Formula!

$$
\begin{aligned}
Var(X) &= E\left[(X - E[X])^2\right] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx \\
&= \int_0^1 (x - 1/2)^2 \cdot 1\, dx = \int_0^1 (x^2 - x + 1/4)\, dx \\
&= \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right)\Big|_0^1 = 1/3 - 1/2 + 1/4 \\
&= 4/12 - 6/12 + 3/12 = 1/12
\end{aligned}
$$

We're Won't Say More About These, But Just So You're Aware of Them...

Joint Density

$$P(a \leq X \leq b \cap c \leq Y \leq d) = \int_c^d \int_a^b f(x, y)\, dxdy$$

Marginal Densities

$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy, \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$

Independence in Terms of Joint and Marginal Densities

$f_{XY}(x, y) = f_X(x) f_Y(y)$

Conditional Density

$f_{Y|X} = f_{XY}(x, y)/f_X(x)$

We've now covered everything on the

Random Variables Handout

The Most Important RV of All

Normal Random Variable

Notation: $X \sim N(\mu, \sigma^2)$

Parameters: $\mu = E[X]$, $\sigma^2 = Var(X)$

Support: $(-\infty, +\infty)$

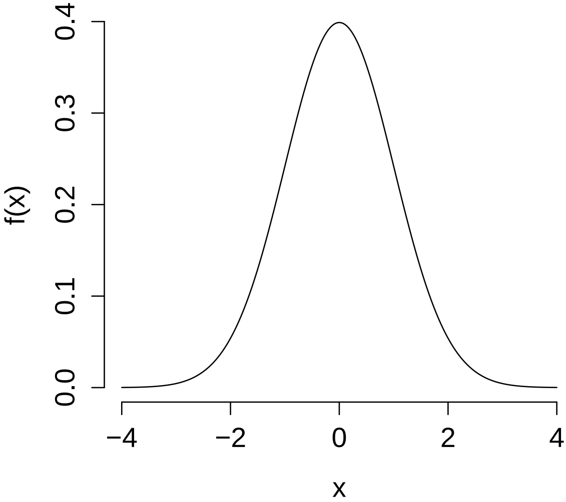
Probability Density Function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$$

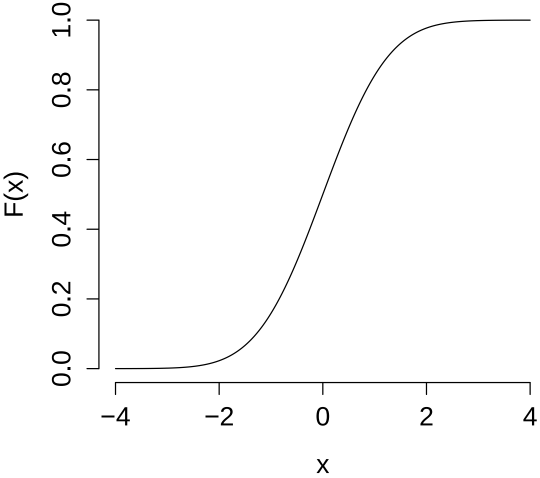
No Explicit Formula for CDF (use computer instead)

$$F(x_0) = \int_{-\infty}^{x_0} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} dx$$

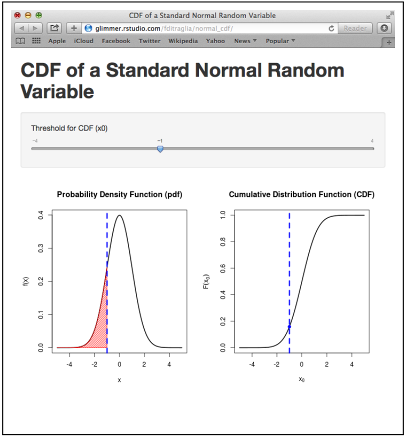
Normal PDF Centered at the Mean (Here $\mu = 0$, $\sigma^2 = 1$)



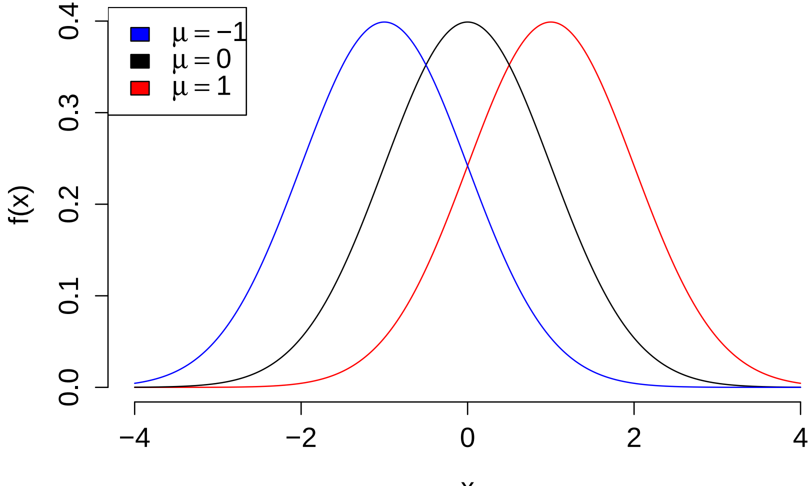
Normal CDF ($\mu = 0$, $\sigma^2 = 1$)



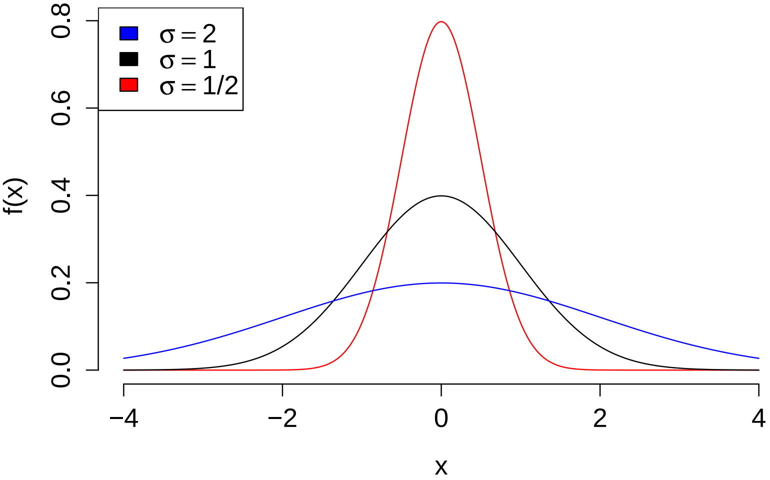
http://glimmer.rstudio.com/fditraglia/normal_cdf/



Different Means, Same Variance



Same Mean, Different Variances



Linear Function of Normal RV is a Normal RV

Suppose that $X \sim N(\mu, \sigma^2)$. Then if a and b constants,

$$\boxed{a + bX \sim N(a + b\mu, b^2\sigma^2)}$$

Important

- Using what know know about expectations of linear functions, no surprise what mean and variance are.
- Surprise is that the linear combination is *normal*
- Linear trans. does not preserve, e.g., Bernoulli or Binomial.

Example

Suppose $X \sim N(\mu, \sigma^2)$ and let $Z = (X - \mu)/\sigma$. What is the distribution of Z?

(a) $N(\mu, \sigma^2)$

(b) $N(\mu, \sigma)$

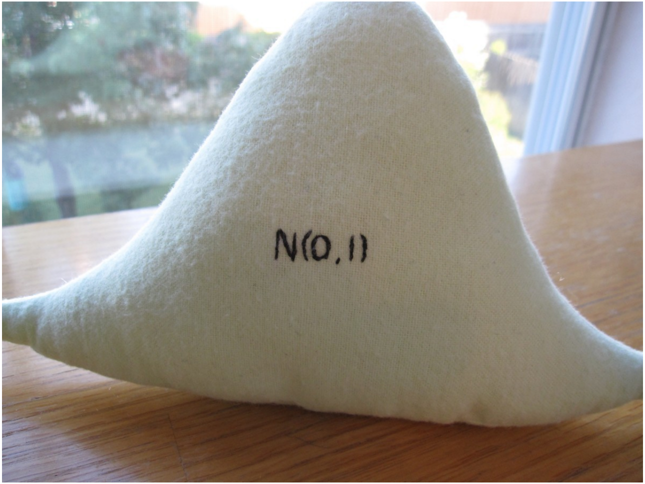
(c) $N(0, \sigma^2)$

(d) $N(0, \sigma)$

(e) $N(0, 1)$

Figure : Standard Normal Distribution (PDF)

Standard Normal Distribution: $N(0, 1)$



Standard Normal Distribution: $N(0, 1)$

Mean $= 0$, Variance $=$ Standard Deviation $= 1$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Special symbol for Standard Normal CDF (no closed form):

$$\Phi(x_0) = \int_{-\infty}^{x_0} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

R Command: $\Phi(x_0) =$`pnorm()`

Where does the Empirical Rule come from?

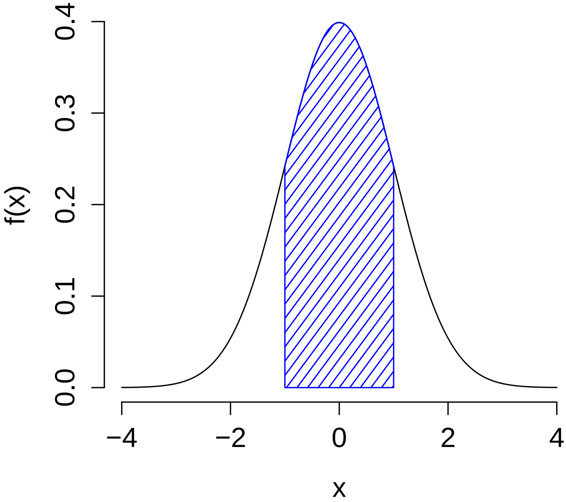
Empirical Rule

Approximately 68% of observations within $\mu \pm \sigma$

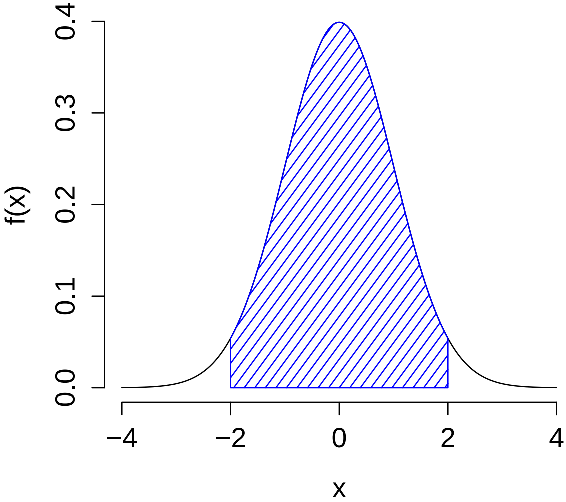
Approximately 95% of observations within $\mu \pm 2\sigma$

Nearly all observations within $\mu \pm 3\sigma$

Middle 68% of $N(0,1) \Rightarrow$ approx. $(-1,1)$



Middle 95% of $N(0, 1) \Rightarrow$ approx. $(-2, 2)$



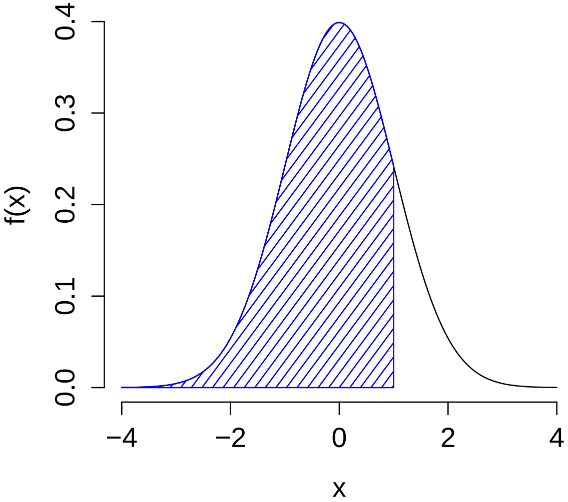
More Formally...

$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \approx 0.68$$

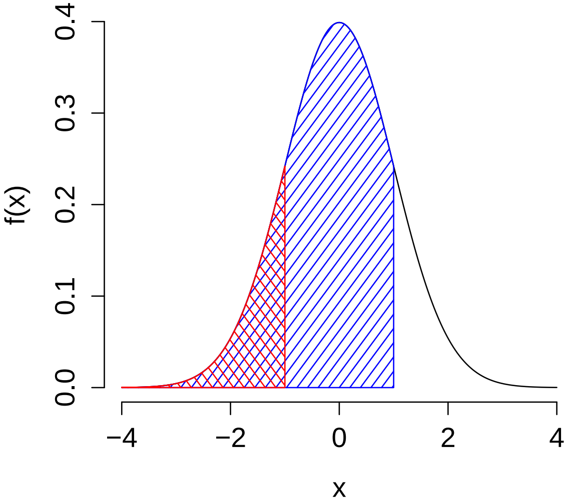
$$\int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \approx 0.95$$

But how do we know this?

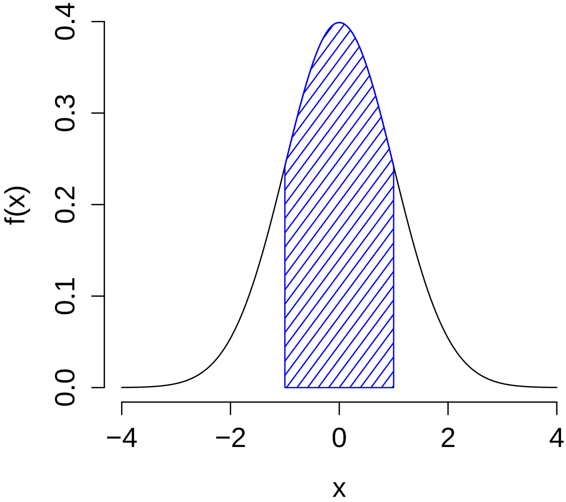
$\Phi(1) = \texttt{pnorm(1)} \approx 0.84$



$\Phi(1) - \Phi(-1) =$`pnorm(1) - pnorm(-1)`$\approx 0.84 - 0.16$



$\Phi(1) - \Phi(-1) =$ `pnorm(1) - pnorm(-1)` ≈ 0.68



Suppose $X \sim N(0,1)$

$$
\begin{aligned}
P(-2 \leq X \leq 2) &= \Phi(2) - \Phi(-2) \\
&= \texttt{pnorm(2) - pnorm(-2)} \\
&\approx 0.95
\end{aligned}
$$

Suppose $X \sim N(0, 1)$

$$
\begin{aligned}
P(-2 \leq X \leq 2) &= \Phi(2) - \Phi(-2) \\
&= \texttt{pnorm(2) - pnorm(-2)} \\
&\approx 0.95
\end{aligned}
$$

$$
\begin{aligned}
P(-3 \leq X \leq 3) &= \Phi(3) - \Phi(-3) \\
&= \texttt{pnorm(3) - pnorm(-3)} \\
&\approx 1
\end{aligned}
$$

What if $X \sim N(\mu, \sigma^2)$?

$$P(X \le a) \;=$$

What if $X \sim N(\mu, \sigma^2)$?

$$
\begin{aligned}
P(X \leq a) &= P(X - \mu \leq a - \mu) \\
&=
\end{aligned}
$$

What if $X \sim N(\mu, \sigma^2)$?

$$
\begin{aligned}
P(X \leq a) &= P(X - \mu \leq a - \mu) \\
&= P\left(\frac{X - \mu}{\sigma} \leq \frac{a - \mu}{\sigma}\right) \\
&=
\end{aligned}
$$

What if $X \sim N(\mu, \sigma^2)$?

$$
\begin{aligned}
P(X \leq a) &= P(X - \mu \leq a - \mu) \\
&= P\left(\frac{X - \mu}{\sigma} \leq \frac{a - \mu}{\sigma}\right) \\
&= P\left(Z \leq \frac{a - \mu}{\sigma}\right)
\end{aligned}
$$

Where Z is a standard normal random variable, i.e. $N(0, 1)$.

Which of these equals $P\left(Z \leq (a-\mu)/\sigma\right)$ if $Z \sim N(0,1)$?

(a) $\Phi(a)$

(b) $1 - \Phi(a)$

(c) $\Phi(a)/\sigma - \mu$

(d) $\Phi\left(\frac{a-\mu}{\sigma}\right)$

(e) None of the above.

What if $X \sim N(\mu, \sigma^2)$?

$$
\begin{aligned}
P(X \le a) &= P(X - \mu \le a - \mu) \\
&= P\left(\frac{X-\mu}{\sigma} \le \frac{a-\mu}{\sigma}\right) \\
&= P\left(Z \le \frac{a-\mu}{\sigma}\right) \\
&= \Phi\left(\frac{a-\mu}{\sigma}\right) \\
&= \texttt{pnorm}((a-\mu)/\sigma)
\end{aligned}
$$

Where Z is a standard normal random variable, i.e. $N(0, 1)$.

Suppose $X \sim N(\mu, \sigma^2)$

Which of these is $P(X \geq b)$?

(a) $\Phi(b)$

(b) $1 - \Phi\left(\frac{b-\mu}{\sigma}\right)$

(c) $1 - \Phi(b)$

(d) $1 - (\Phi(b)/\sigma - \mu)$

Suppose $X \sim N(\mu, \sigma^2)$

$$P(X \geq b) \quad =$$

Suppose $X \sim N(\mu, \sigma^2)$

$$P(X \geq b) \;=\; 1 - P(X \leq b) =$$

Suppose $X \sim N(\mu, \sigma^2)$

$$
\begin{aligned}
P(X \geq b) &= 1 - P(X \leq b) = 1 - P\left(\frac{X-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}\right) \\
&=
\end{aligned}
$$

Suppose $X \sim N(\mu, \sigma^2)$

$$
\begin{aligned}
P(X \geq b) &= 1 - P(X \leq b) = 1 - P\left(\frac{X-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}\right) \\
&= 1 - P\left(Z \leq \frac{b-\mu}{\sigma}\right) = 1 - \Phi\left(\frac{b-\mu}{\sigma}\right) \\
&=
\end{aligned}
$$

Suppose $X \sim N(\mu, \sigma^2)$

$$
\begin{aligned}
P(X \geq b) &= 1 - P(X \leq b) = 1 - P\left(\frac{X-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}\right) \\
&= 1 - P\left(Z \leq \frac{b-\mu}{\sigma}\right) = 1 - \Phi\left(\frac{b-\mu}{\sigma}\right) \\
&= 1 - \texttt{pnorm}((b-\mu)/\sigma)
\end{aligned}
$$

Where Z is a standard normal random variable.

Suppose $X \sim N(\mu, \sigma^2)$

$$P(a \leq X \leq b) \quad =$$

Suppose $X \sim N(\mu, \sigma^2)$

$$
\begin{aligned}
P(a \le X \le b) &= P\left(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right) \\
&=
\end{aligned}
$$

Suppose $X \sim N(\mu, \sigma^2)$

$$
\begin{aligned}
P(a \le X \le b) &= P\left(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right) \\
&= P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right) \\
&=
\end{aligned}
$$

Suppose $X \sim N(\mu, \sigma^2)$

$$
\begin{aligned}
P(a \leq X \leq b) &= P\left(\frac{a-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}\right) \\
&= P\left(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right) \\
&= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \\
&=
\end{aligned}
$$

Suppose $X \sim N(\mu, \sigma^2)$

$$\begin{aligned}
P(a \le X \le b) &= P\left(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right) \\
&= P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right) \\
&= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \\
&= \texttt{pnorm}((b-\mu)/\sigma) - \texttt{pnorm}((a-\mu)/\sigma)
\end{aligned}$$

Where Z is a standard normal random variable.

Suppose $X \sim N(\mu, \sigma^2)$

What is $P(\mu - \sigma \leq X \leq \mu + \sigma)$?

Suppose $X \sim N(\mu, \sigma^2)$

$$\begin{aligned}
P(\mu - \sigma \le X \le \mu + \sigma) &= P\left(-1 \le \frac{X-\mu}{\sigma} \le 1\right) \\
&= P\left(-1 \le Z \le 1\right) \\
&= \Phi(1) - \Phi(-1) \\
&= \texttt{pnorm(1)} - \texttt{pnorm(-1)} \\
&\approx 0.68
\end{aligned}$$

Suppose $X \sim N(\mu, \sigma^2)$

What is $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$?

Suppose $X \sim N(\mu, \sigma^2)$

$$
\begin{aligned}
P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) &= P\left(-2 \leq \frac{X-\mu}{\sigma} \leq 2\right) \\
&= P\left(-2 \leq Z \leq 2\right) \\
&= \Phi(2) - \Phi(-2) \\
&= \texttt{pnorm(2)} - \texttt{pnorm(-2)} \\
&\approx 0.95
\end{aligned}
$$