Economics 103 – Statistics for Economists

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Lecture # 11

Continuous RVs – Part I

What Changes?

1. Probability Density Functions replace Probability Mass Functions (aka Probability Distributions)
2. Integrals Replace Sums

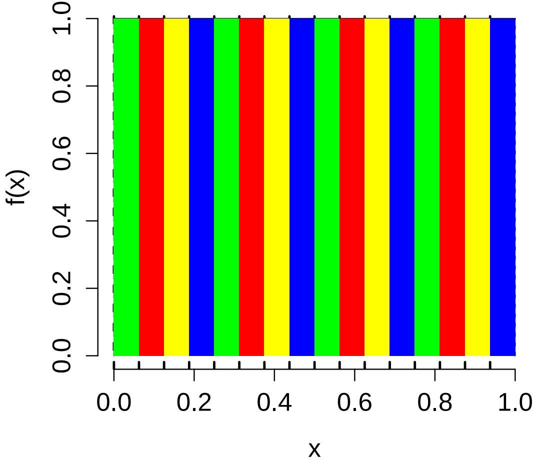
Everything Else is Essentially Unchanged!

What is the probability of “Yellow?”

What is the probability of “Right Hand Blue?”

What is the probability that the spinner lands in any *particular* place?

From Twister to Density – Probability as *Area*



Continuous Random Variables

For continuous RVs, probability is a matter of finding the area of *intervals*. Individual *points* have *zero* probability.

Probability Density Function (PDF)

For a continuous random variable X,

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

where $f(x)$ is the *probability density function* for X.

Extremely Important

For any realization x, $P(X = x) = 0 \neq f(x)$!

Properties of PDFs

1. $\int_{-\infty}^{\infty} f(x)\, dx = 1$
2. $f(x) \geq 0$ for all x
3. $f(x)$ is *not* a probability and can be greater than one!
4. $P(X \leq x_0) = F(x_0) = \int_{-\infty}^{x_0} f(x)\, dx$

We'll start with the simplest possible example: the Uniform$(0, 1)$ RV.

Uniform$(0, 1)$ Random Variable

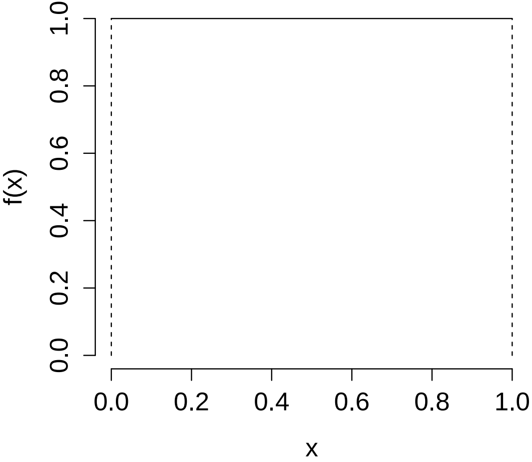
$X \sim \text{Uniform}(0, 1)$

We say that X follows a Uniform(0,1) distribution, if it is equally likely to take on *any value* in the range $[0, 1]$ and never takes on a value outside this range.

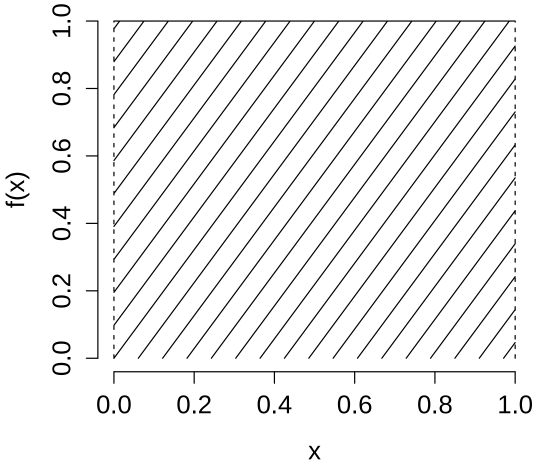
Uniform PDF

$f(x) = 1$ for $0 \leq x \leq 1$, zero elsewhere.

Uniform$(0, 1)$ PDF



What is the area of the shaded region?



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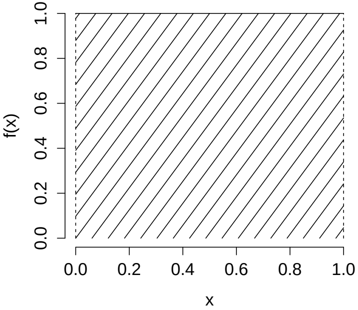
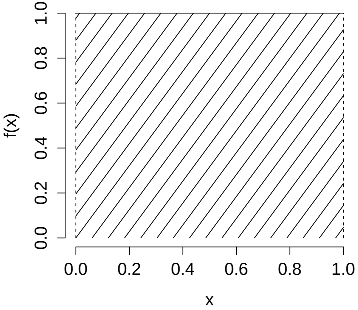


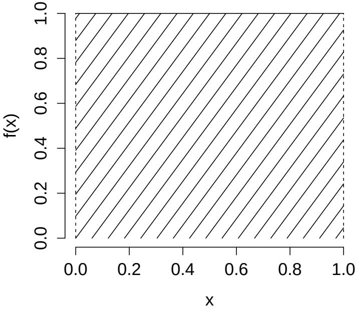
$$\int_{-\infty}^{\infty} f(x)\, dx =$$

What is the area of the shaded region?



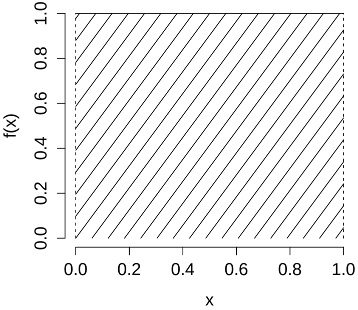
$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^1 1\, dx =$$

What is the area of the shaded region?



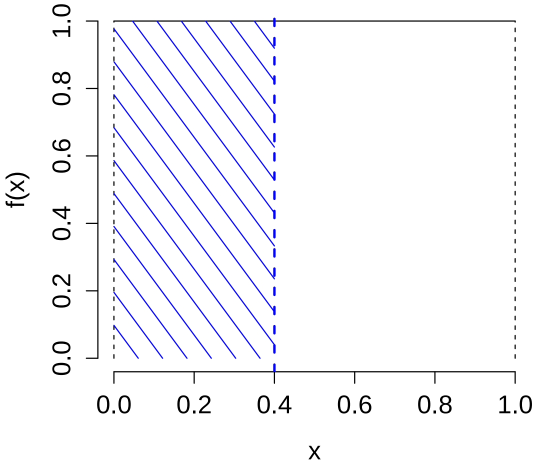
$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^1 1\, dx = x\big|_0^1 =$$

What is the area of the shaded region?

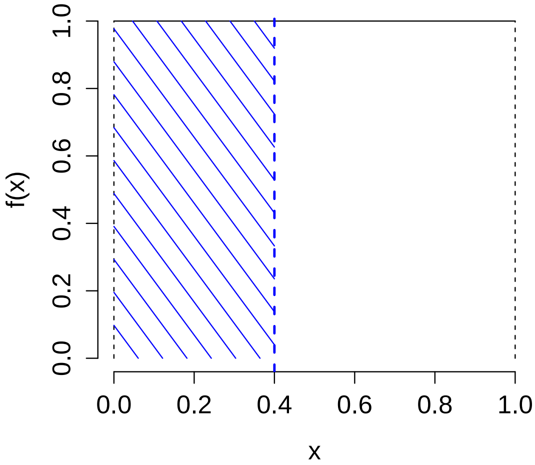


$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^1 1\, dx = x\big|_0^1 = 1 - 0 = 1$$

What is the area of the shaded region?



$$F(0.4) = P(X \leq 0.4) = 0.4$$



Relationship between PDF and CDF

Integrate the pdf to get the CDF

$$F(x_0) = P(X \leq x_0) = \int_{-\infty}^{x_0} f(x)\, dx$$

Differentiate the CDF to get the pdf

$$f(x) = \frac{d}{dx} F(x)$$

This is just the Fundamental Theorem of Calculus.

Example: Uniform$(0, 1)$ RV

Integrate the pdf, $f(x) = 1$, to get the CDF

$$F(x_0) = \int_{-\infty}^{x_0} f(x)\, dx =$$

Example: Uniform$(0, 1)$ RV

Integrate the pdf, $f(x) = 1$, to get the CDF

$$F(x_0) = \int_{-\infty}^{x_0} f(x)\, dx = \int_0^{x_0} 1\, dx =$$

Example: Uniform(0, 1) RV

Integrate the pdf, $f(x) = 1$, to get the CDF

$$F(x_0) = \int_{-\infty}^{x_0} f(x)\, dx = \int_0^{x_0} 1\, dx = \left. x \right|_0^{x_0} =$$

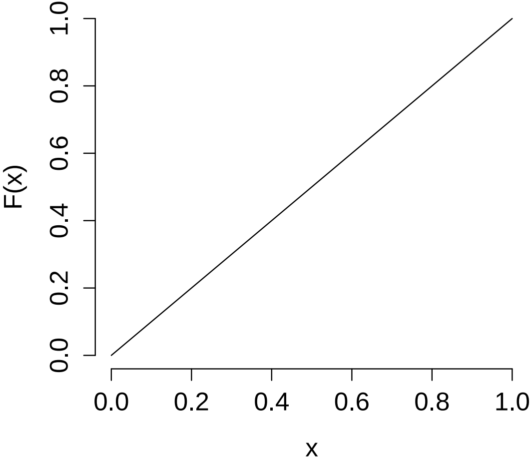
Example: Uniform(0, 1) RV

Integrate the pdf, $f(x) = 1$, to get the CDF

$$F(x_0) = \int_{-\infty}^{x_0} f(x)\, dx = \int_{0}^{x_0} 1\, dx = \left. x \right|_0^{x_0} = x_0 - 0 = x_0$$

$$F(x_0) = \begin{cases} 0, x_0 < 0 \\ x_0, 0 \leq x_0 \leq 1 \\ 1, x_0 > 1 \end{cases}$$

Uniform(0, 1) CDF



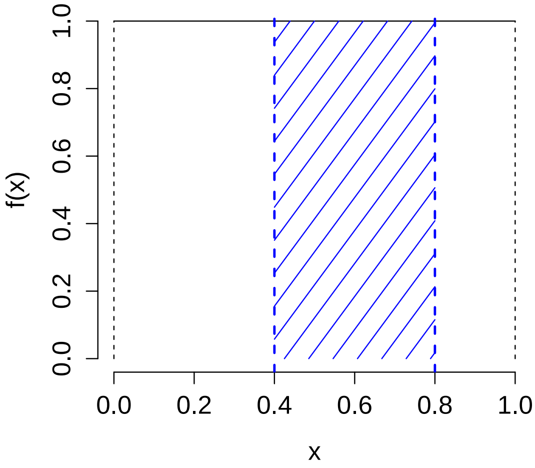
Example: Uniform$(0, 1)$ RV

Differentiate the CDF, $F(x_0) = x_0$, to get the pdf

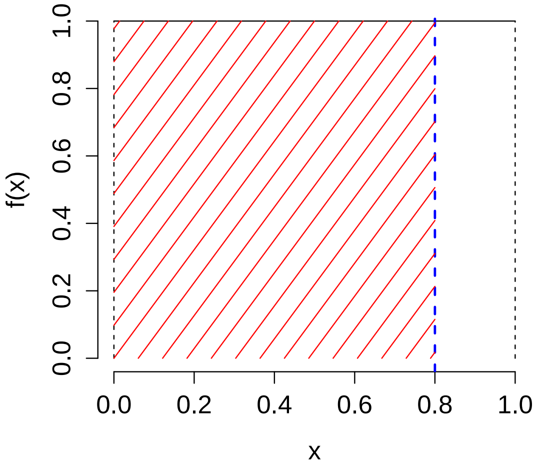
$$\frac{d}{dx}F(x) = 1 = f(x)$$

Key Idea: Probability of Intervals

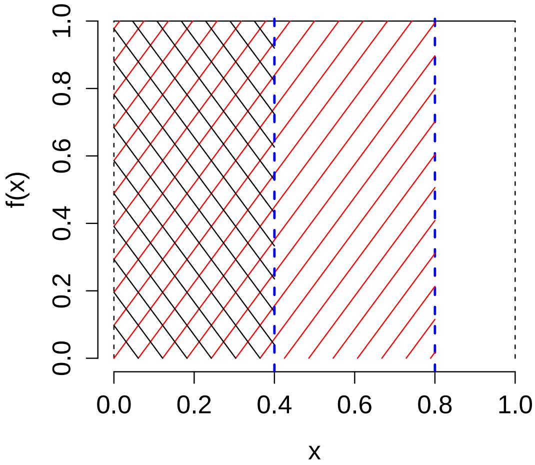
What is $P(0.4 \leq X \leq 0.8)$ if $X \sim \text{Uniform}(0, 1)$?



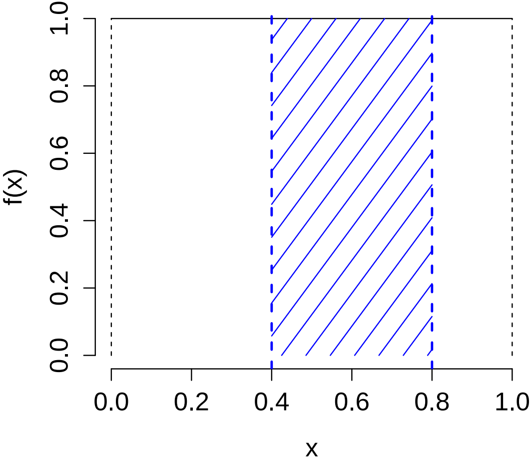
$$F(0.8) = P(X \leq 0.8)$$



$F(0.8) - F(0.4) = ?$



$$F(0.8) - F(0.4) = P(0.4 \leq X \leq 0.8) = 0.4$$



Probability of Interval for Continuous RV

$$\boxed{P(a \leq X \leq b) = \int_a^b f(x)\, dx = F(b) - F(a)}$$

This is just the Second Fundamental Theorem of Calculus.

Expected Value for Continuous RVs

$$\boxed{\int_{-\infty}^{\infty} x f(x)\, dx}$$

Remember: Integrals Replace Sums!

Example: Uniform(0,1) Random Variable

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx =$$

Example: Uniform(0,1) Random Variable

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^1 x \cdot 1\, dx \\
&= \left.\frac{x^2}{2}\right|_0^1 = 1/2 - 0 = 1/2
\end{aligned}
$$

Expected Value of a Function of a Continuous RV

$$\boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) \, dx}$$

Example: Uniform(0, 1) RV

$$
\begin{aligned}
E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^1 x^2 \cdot 1\, dx \\
&= \left.\frac{x^3}{3}\right|_0^1 = 1/3
\end{aligned}
$$

Once we have defined expected value for continuous RVs, we can use everything we know about variance, covariance, etc. from discrete RVs!

Variance of Continuous RV

$$\boxed{Var(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx}$$

where

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

Shortcut formula still holds for continuous RVs!

$$Var(X) = E[X^2] - (E[X])^2$$

Example: Uniform(0,1) Random Variable

$$Var(X) = E\left[(X - E[X])^2\right] = E[X^2] - (E[X])^2$$

Example: Uniform(0,1) Random Variable

$$\begin{aligned}
Var(X) &= E\left[(X - E[X])^2\right] = E[X^2] - (E[X])^2 \\
&= 1/3 - (1/2)^2 \\
&= 1/12 \\
&\approx 0.083
\end{aligned}$$

Much More Complicated Without the Shortcut Formula!

$$
\begin{aligned}
Var(X) &= E\left[(X - E[X])^2\right] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx \\
&= \int_0^1 (x - 1/2)^2 \cdot 1\, dx = \int_0^1 (x^2 - x + 1/4)\, dx \\
&= \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right)\Big|_0^1 = 1/3 - 1/2 + 1/4 \\
&= 4/12 - 6/12 + 3/12 = 1/12
\end{aligned}
$$

We're Won't Say More About These, But Just So You're Aware of Them...

Joint Density

$$P(a \leq X \leq b \cap c \leq Y \leq d) = \int_c^d \int_a^b f(x, y)\, dxdy$$

Marginal Densities

$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy, \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$

Independence in Terms of Joint and Marginal Densities

$f_{XY}(x, y) = f_X(x) f_Y(y)$

Conditional Density

$f_{Y|X} = f_{XY}(x, y)/f_X(x)$

We've now covered everything on the

$\boxed{\text{Random Variables Handout}}$