Economics 103 – Statistics for Economists

Francis J. DiTraglia

University of Pennsylvania

Lecture # 10

Discrete RVs – Part III

Overview

So Far

Consider one RV at a time.

Today

Consider relationships *between* two RVs.

Definition of Joint PMF

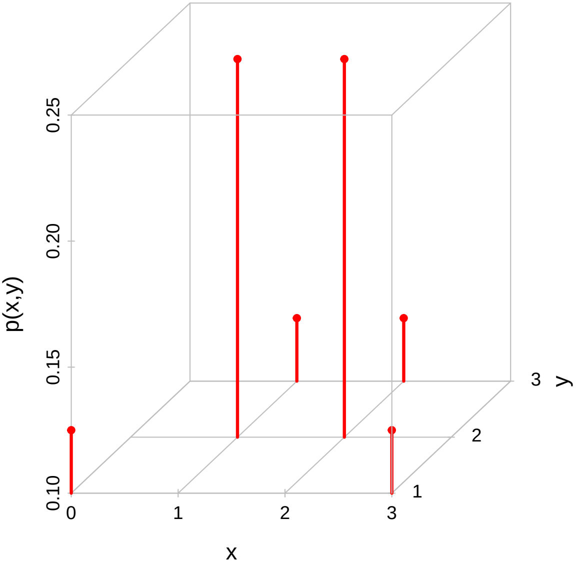
Let X and Y be discrete random variables. The joint probability mass function $p_{XY}(x, y)$ gives the probability of each pair of realizations (x, y) in the support:

$$\boxed{p_{XY}(x, y) = P(X = x \cap Y = y)}$$

Example: Joint PMF in Tabular Form

		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

Plot of Joint PMF



What is $p_{XY}(1,2)$?

		Y		
		1	2	3
	0	1/8	0	0
X	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

$$p_{XY}(1,2) = P(X = 1 \cap Y = 2) = 1/4$$

What is $p_{XY}(2,1)$?

		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

$$p_{XY}(2,1) = P(X = 2 \cap Y = 1) = 0$$

Properties of Joint PMF

1. $0 \leq p_{XY}(x, y) \leq 1$ for any pair (x, y)
2. The sum of $p_{XY}(x, y)$ over all pairs (x, y) in the support is 1:

$$\sum_{x} \sum_{y} p(x, y) = 1$$

Does this satisfy the properties of a joint pmf?

(A = YES, B = NO)

		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

1. $p(x, y) \geq 0$ for all pairs (x, y)
2. $\sum_x \sum_y p(x, y) = 1/8 + 1/4 + 1/8 + 1/4 + 1/8 + 1/8 = 1$

Joint versus Marginal PMFs

Joint PMF

$p_{XY}(x, y) = P(X = x \cap Y = y)$

Marginal PMFs

$p_X(x) = P(X = x)$

$p_Y(y) = P(Y = y)$

You can't calculate a joint pmf from marginals alone but you *can* calculate marginals from the joint!

Marginals from Joint

$$\boxed{p_X(x) = \sum_{\text{all } y} p_{XY}(x,y)}$$

$$\boxed{p_Y(y) = \sum_{\text{all } x} p_{XY}(x,y)}$$

Why?

$$\begin{aligned} p_Y(y) &= P(Y=y) = P\left(\bigcup_{\text{all } x} \{X = x \cap Y = y\}\right) \\ &= \sum_{\text{all } x} P(X = x \cap Y = y) = \sum_{\text{all } x} p_{XY}(x,y) \end{aligned}$$

To get the marginals sum "into the margins" of the table.

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
					1

$$
\begin{aligned}
p_X(0) &= 1/8 + 0 + 0 = 1/8 \\
p_X(1) &= 0 + 1/4 + 1/8 = 3/8 \\
p_X(2) &= 0 + 1/4 + 1/8 = 3/8 \\
p_X(3) &= 1/8 + 0 + 0 = 1/8
\end{aligned}
$$

What is $p_Y(2)$?

		Y			
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	
		1/4	1/2	1/4	1

$$
\begin{aligned}
p_Y(1) &= 1/8 + 0 + 0 + 1/8 = 1/4 \\
p_Y(2) &= 0 + 1/4 + 1/4 + 0 = 1/2 \\
p_Y(3) &= 0 + 1/8 + 1/8 + 0 = 1/4
\end{aligned}
$$

Definition of Conditional PMF

How does the distribution of y change with x?

$$\boxed{p_{Y|X}(y|x) = P(Y = y|X = x)}$$

Which of these is the formula for $p_{Y|X}(y|x)$?

You can figure this out from what you already know about probability, using the definition $p_{Y|X}(y|x) = P(Y = y|X = x)$

(a) $p_X(x)/p_Y(y)$

(b) $p_{XY}(x, y)/p_X(x)$

(c) $p_X(x)p_{XY}(x, y)$

(d) $p_{XY}(x, y)/p_Y(y)$

(e) $p_Y(y)/p_X(x)$

Conditional PMF from Joint and Marginal

$$p_{Y|X}(y|x) = P(Y = y|X = x) = \frac{P(Y = y \cap X = x)}{P(X = x)} = \frac{p_{XY}(x, y)}{p_X(x)}$$

Hence,

$$\boxed{p_{Y|X}(y|x) = \frac{p_{XY}(x, y)}{p_X(x)}}$$

and similarly,

$$\boxed{p_{X|Y}(x|y) = \frac{p_{XY}(x, y)}{p_Y(y)}}$$

Conditional PMF of Y given $X = 2$

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8

$$p_{Y|X}(1|2) = \frac{p_{XY}(2,1)}{p_X(2)} = \frac{0}{3/8} = 0$$

$$p_{Y|X}(2|2) = \frac{p_{XY}(2,2)}{p_X(2)} = \frac{1/4}{3/8} = 2/3$$

$$p_{Y|X}(3|2) = \frac{p_{XY}(2,3)}{p_X(2)} = \frac{1/8}{3/8} = 1/3$$

What is $p_{X|Y}(1|2)$?

		Y			
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	
		1/4	1/2	1/4	

$$
\begin{aligned}
p_{X|Y}(0|2) &= \frac{p_{XY}(0,2)}{p_Y(2)} = \frac{0}{1/2} = 0 \\
p_{X|Y}(1|2) &= \frac{p_{XY}(1,2)}{p_Y(2)} = \frac{1/4}{1/2} = 1/2 \\
p_{X|Y}(2|2) &= \frac{p_{XY}(2,2)}{p_Y(2)} = \frac{1/4}{1/2} = 1/2 \\
p_{X|Y}(3|2) &= \frac{p_{XY}(3,2)}{p_Y(2)} = \frac{0}{1/2} = 0
\end{aligned}
$$

Independent RVs

Definition

We say that two discrete RVs are independent if and only if their joint pmf equals the product of their marginal pmfs:

$$p_{XY}(x, y) = p_X(x)p_Y(y)$$

for all pairs (x, y) in the support.

In Terms of Conditional PMF

From the previous slide, it follows that an equivalent definition of independence is that both conditional pmfs equal the corresponding marginal pmfs: $p_{Y|X}(y|x) = p_Y(y)$ and $p_{X|Y}(x|y) = p_X(x)$ for all (x, y) in the support.

Are X and Y Independent?

(A = YES, B = NO)

		Y = 1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

$$\begin{aligned} p_{XY}(2,1) &= 0 \\ p_X(2) \times p_Y(1) &= (3/8) \times (1/4) \neq 0 \end{aligned}$$

Therefore X and Y are *not* independent.

Conditional Expectation

Intuition

$E[Y|X]$ is our "best guess" of the realization that Y will take on having observed the realization of X.

$E[Y|X]$ is a Random Variable

Unlike $E[Y]$ which is a constant, $E[Y|X]$ is a function of X, hence it is a Random Variable.

$E[Y|X = x]$ is a Constant

To get a "best guess" for Y, we plug in the realization we observed for X: $E[Y|X = x]$ is a constant, our guess of the realization of Y.

Calculating $E[Y|X = x]$

Take the mean of the conditional pmf of Y given $X = x$.

Conditional Expectation: $E[Y|X=2]$

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

We showed above that the conditional pmf of $Y|X=2$ is:

$$\boxed{p_{Y|X}(1|2) = 0 \qquad p_{Y|X}(2|2) = 2/3 \qquad p_{Y|X}(3|2) = 1/3}$$

Hence

$$E[Y|X=2] = 2 \times 2/3 + 3 \times 1/3 = 7/3$$

Conditional Expectation: $E[Y|X=0]$

		Y 1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

The conditional pmf of $Y|X=0$ is

$$\boxed{p_{Y|X}(1|0)=1 \qquad p_{Y|X}(2|0)=0 \qquad p_{Y|X}(3|0)=0}$$

Hence $E[Y|X=0]=1$

Calculate $E[Y|X = 3]$

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

The conditional pmf of $Y|X = 3$ is

$$\boxed{p_{Y|X}(1|3) = 1 \qquad p_{Y|X}(2|3) = 0 \qquad p_{Y|X}(3|3) = 0}$$

Hence $E[Y|X = 3] = 1$

Calculate $E[Y|X=1]$

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

The conditional pmf of $Y|X=1$ is

$$\boxed{p_{Y|X}(1|1) = 0 \qquad p_{Y|X}(2|1) = 2/3 \qquad p_{Y|X}(3|1) = 1/3}$$

Hence

$$E[Y|X=1] = 2 \times 2/3 + 3 \times 1/3 = 7/3$$

$E[Y|X]$ is a Random Variable

In this particular example we have seen that:

$$E[Y|X] = \begin{cases} 1 & X = 0 \\ 7/3 & X = 1 \\ 7/3 & X = 2 \\ 1 & X = 3 \end{cases}$$

But from above we know the marginal distribution of X :

$$P(X = 0) = 1/8 \qquad P(X = 1) = 3/8$$
$$P(X = 2) = 3/8 \qquad P(X = 3) = 1/8$$

Therefore, $E[Y|X]$ is a RV that takes on the value 1 with probability $1/4$ and the value $7/3$ with probability $3/4$.

The Law of Iterated Expectations

Since $E[Y|X]$ is a random variable, we can ask what its expectation is. It turns out that, for any RVs X and Y

$$\boxed{E\left[E\left[Y|X\right]\right] = E[Y]}$$

and this is called the Law of Iterated Expectations. I've posted a proof HERE for those who want are interested.

This will be helpful in Econ 104...

Law of Iterated Expectations for Our Example

Marginal pmf of Y

$$
\begin{aligned}
P(Y=1) &= 1/4 \\
P(Y=2) &= 1/2 \\
P(Y=3) &= 1/4
\end{aligned}
$$

$$
\begin{aligned}
E[Y] &= 1 \times 1/4 + 2 \times 1/2 + 3 \times 1/4 \\
&= 2
\end{aligned}
$$

$E[Y|X]$

$$
E[Y|X] = \begin{cases} 1 & \text{w/ prob. } 1/4 \\ 7/3 & \text{w/ prob. } 3/4 \end{cases}
$$

$$
\begin{aligned}
E\left[E\left[Y|X\right]\right] &= 1 \times 1/4 + 7/3 \times 3/4 \\
&= 2
\end{aligned}
$$

Expectation of Function of Two Discrete RVs

$$\boxed{E[g(X, Y)] = \sum_x \sum_y g(x, y) p_{XY}(x, y)}$$

Some Extremely Important Examples

Same For Continuous Random Variables

Let $\mu_X = E[X], \mu_Y = E[Y]$

Covariance

$$\sigma_{XY} = Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

Correlation

$$\rho_{XY} = Corr(X, Y) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Shortcut Formula for Covariance

Much easier for calculating:

$$\boxed{Cov(X, Y) = E[XY] - E[X]E[Y]}$$

We'll talk more about this in an upcoming lecture...

Calculating $Cov(X, Y)$

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

$$E[X] = 3/8 + 2 \times 3/8 + 3 \times 1/8 = 3/2$$

$$E[Y] = 1/4 + 2 \times 1/2 + 3 \times 1/4 = 2$$

$$\begin{aligned} E[XY] &= 1/4 \times (2+4) + 1/8 \times (3+6+3) \\ &= 3 \end{aligned}$$

$$\begin{aligned} Cov(X, Y) &= E[XY] - E[X]E[Y] \\ &= 3 - 3/2 \times 2 = 0 \end{aligned}$$

$$Corr(X, Y) = Cov(X, Y)/\left[SD(X)SD(Y)\right] = 0$$

Hence, zero covariance (correlation) does *not* imply independence!

However, independence *does* imply zero covariance (correlation)

X, Y Independent $\Rightarrow Cov(X, Y) = 0$

$$
\begin{aligned}
Cov(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\
&= \sum_x \sum_y (x - \mu_X)(y - \mu_Y)p(x, y) \\
&= \sum_x \sum_y (x - \mu_X)(y - \mu_Y)p(x)p(y) \\
&= \sum_x (x - \mu_X)p(x) \left[\sum_y (y - \mu_Y)p(y) \right] \\
&= E[Y - \mu_Y] \sum_x (x - \mu_X)p(x) \\
&= E[Y - \mu_Y]E[X - \mu_X] \\
&= 0
\end{aligned}
$$