Problem Set #6

Econ 103

Part I – Problems from the Textbook

Chapter 5: 1, 3, 5, 9, 11, 13, 17
Chapter 4: 19, 21, 23
(*When necessary, use R rather than the Normal tables in the front of the textbook.*)

Part II – Additional Problems

1. Let X and Y be discrete random variables and a, b, c, d be constants. Prove the following:

   (a) $Cov(a + bX, c + dY) = bdCov(X, Y)$

Solution: Let $\mu_X = E[X]$ and $\mu_Y = E[Y]$. By the linearity of expectation,

$$\begin{aligned} E[a + bX] &= a + b\mu_X \\ E[c + dY] &= c + d\mu_Y \end{aligned}$$

Thus, we have

$$\begin{aligned} (a + bx) - E[a + bX] &= b(x - \mu_X) \\ (c + dy) - E[c + dY] &= d(y - \mu_Y) \end{aligned}$$

Substituting these into the formula for the covariance between two discrete random variables,

$$\begin{aligned} Cov(a + bX, c + dY) &= \sum_x \sum_y \left[b(x - \mu_X)\right]\left[d(y - \mu_Y)\right] p(x, y) \\ &= bd \sum_x \sum_y (x - \mu_X)(y - \mu_Y) p(x, y) \\ &= bdCov(X, Y) \end{aligned}$$

(b) $Corr(a + bX, c + dY) = Corr(X, Y)$

Solution:

$$\begin{aligned} Corr(a + bX, c + dY) &= \frac{Cov(a + bX, c + dY)}{\sqrt{Var(a + bX)Var(c + dY)}} \\ &= \frac{bdCov(X, Y)}{\sqrt{b^2Var(X)d^2Var(Y)}} \\ &= \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}} \\ &= Corr(X, Y) \end{aligned}$$

2. Fill in the missing steps from the lecture to prove the shortcut formula for covariance:

$$Cov(X, Y) = E[XY] - E[X]E[Y]$$

Solution: By the Linearity of Expectation,

$$\begin{aligned} Cov(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\ &= E[XY - \mu_X Y - \mu_Y X + \mu_X\mu_Y] \\ &= E[XY] - \mu_x E[Y] - \mu_Y E[X] + \mu_X\mu_Y \\ &= E[XY] - \mu_X\mu_Y - \mu_Y\mu_X + \mu_X\mu_Y \\ &= E[XY] - \mu_X\mu_Y \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

3. Let X_1 be a random variable denoting the returns of stock 1, and X_2 be a random variable denoting the returns of stock 2. Accordingly let $\mu_1 = E[X_1]$, $\mu_2 = E[X_2]$, $\sigma_1^2 = Var(X_1)$, $\sigma_2^2 = Var(X_2)$ and $\rho = Corr(X_1, X_2)$. A *portfolio*, Π, is a linear combination of X_1 and X_2 with weights that sum to one, that is $\Pi(\omega) = \omega X_1 + (1 - \omega)X_2$, indicating the proportions of stock 1 and stock 2 that an investor holds. In this example, we require $\omega \in [0, 1]$, so that *negative* weights are not allowed. (This rules out short-selling.)

   (a) Calculate $E[\Pi(\omega)]$ in terms of ω, μ_1 and μ_2.

Solution:

$$\begin{aligned} E[\Pi(\omega)] &= E[\omega X_1 + (1-\omega)X_2] = \omega E[X_1] + (1-\omega)E[X_2] \\ &= \omega\mu_1 + (1-\omega)\mu_2 \end{aligned}$$

(b) If $\omega \in [0,1]$ is it possible to have $E[\Pi(\omega)] > \mu_1$ *and* $E[\Pi(\omega)] > \mu_2$? What about $E[\Pi(\omega)] < \mu_1$ and $E[\Pi(\omega)] < \mu_2$? Explain.

Solution: No. If short-selling is disallowed, the portfolio expected return must be between μ_1 and μ_2.

(c) Express $Cov(X_1, X_2)$ in terms of ρ and σ_1, σ_2.

Solution: $Cov(X, Y) = \rho\sigma_1\sigma_2$

(d) What is $Var[\Pi(\omega)]$? (Your answer should be in terms of ρ, σ_1^2 and σ_2^2.)

Solution:

$$\begin{aligned} Var[\Pi(\omega)] &= Var[\omega X_1 + (1-\omega)X_2] \\ &= \omega^2 Var(X_1) + (1-\omega)^2 Var(X_2) + 2\omega(1-\omega)Cov(X_1, X_2) \\ &= \omega^2\sigma_1^2 + (1-\omega)^2\sigma_2^2 + 2\omega(1-\omega)\rho\sigma_1\sigma_2 \end{aligned}$$

(e) Using part (d) show that the value of ω that minimizes $Var[\Pi(\omega)]$ is

$$\omega^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

In other words, $\Pi(\omega^*)$ is the *minimum variance portfolio.*

Solution: The First Order Condition is:

$$2\omega\sigma_1^2 - 2(1-\omega)\sigma_2^2 + (2-4\omega)\rho\sigma_1\sigma_2 = 0$$

Dividing both sides by two and rearranging:

$$\begin{aligned} \omega\sigma_1^2 - (1-\omega)\sigma_2^2 + (1-2\omega)\rho\sigma_1\sigma_2 &= 0 \\ \omega\sigma_1^2 - \sigma_2^2 + \omega\sigma_2^2 + \rho\sigma_1\sigma_2 - 2\omega\rho\sigma_1\sigma_2 &= 0 \\ \omega(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2) &= \sigma_2^2 - \rho\sigma_1\sigma_2 \end{aligned}$$

So we have

$$\omega^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

(f) If you want a challenge, check the second order condition from part (e).

Solution: The second derivative is

$$2\sigma_1^2 - 2\sigma_2^2 - 4\rho\sigma_1\sigma_2$$

and, since $\rho = 1$ is the largest possible value for ρ,

$$2\sigma_1^2 - 2\sigma_2^2 - 4\rho\sigma_1\sigma_2 \geq 2\sigma_1^2 - 2\sigma_2^2 - 4\sigma_1\sigma_2 = 2(\sigma_1 - \sigma_2)^2 \geq 0$$

so the second derivative is positive, indicating a minimum. This is a global minimum since the problem is quadratic in ω.

4. Suppose that X is a random variable with the following PDF

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2-x & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Graph the PDF of X.

Solution: It's an isosceles triangle with base from (0,0) to (0,2) and height 1.

(b) Show that $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

Solution:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\, dx &= \int_0^1 x\, dx + \int_1^2 (2-x)\, dx = \left.\frac{x^2}{2}\right|_0^1 + \left.\left(2x - \frac{x^2}{2}\right)\right|_1^2 \\ &= 1/2 + (4-2) - (2 - 1/2) = 1 \end{aligned}$$

(c) What is $P(0.5 < X < 1.5)$?

Solution:

$$\begin{aligned}
P(0.5 < X < 1.5) &= \int_{0.5}^{1.5} f(x)\,dx = \int_{0.5}^{1} x\,dx + \int_{1}^{1.5} (2-x)\,dx \\
&= \left.\frac{x^2}{2}\right|_{0.5}^{1} + \left.\left(2x - \frac{x^2}{2}\right)\right|_{1}^{1.5} \\
&= (1/2 - 1/8) + (3 - 9/8) - (2 - 1/2) \\
&= 3/8 + 15/8 - 2 + 1/2 = 18/8 - 16/8 + 4/8 \\
&= 6/8 = 3/4 = 0.75
\end{aligned}$$

5. A random variable is said to follow a Uniform(a, b) distribution if it is equally likely to take on any value in the range $[a, b]$ and never takes a value outside this range. Suppose that X is such a random variable, i.e. $X \sim \text{Uniform}(a, b)$.

   (a) What is the support of X?

   Solution: $[a, b]$

   (b) Explain why the PDF of X is $f(x) = 1/(b-a)$ for $a \leq x \leq b$, zero elsewhere.

   Solution: This simply generalizes the Uniform$(0, 1)$ random variable from class. To capture the idea that X is equally likely to take on any value in the range $[a, b]$, the PDF must be constant. To ensure that it integrates to 1, the denominator must be $b - a$.

   (c) Using the PDF from part (b), calculate the CDF of X.

   Solution:

$$F(x_0) = \int_{-\infty}^{x_0} f(x)\,dx = \int_{a}^{x_0} \frac{dx}{b-a} = \left.\frac{x}{b-a}\right|_{a}^{x_0} = \frac{x_0 - a}{b-a}$$

   (d) Verify that $f(x) = F'(x)$ for the present example.

   Solution:

$$F'(x) = \frac{d}{dx}\left(\frac{x-a}{b-a}\right) = \frac{1}{b-a} = f(x)$$

(e) Calculate $E[X]$.

Solution:

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \int_a^b \frac{x}{b-a}\, dx = \left.\frac{x^2}{2(b-a)}\right|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$

(f) Calculate $E[X^2]$. *Hint:* recall that $b^3 - a^3$ can be factorized as $(b-a)(b^2+a^2+ab)$.

Solution:

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_a^b \frac{x^2}{b-a} = \left.\frac{x^3}{3(b-a)}\right|_a^b = \frac{b^3-a^3}{3(b-a)} \\ &= \frac{(b-a)(b^2+a^2+ab)}{3(b-a)} = \frac{b^2+a^2+ab}{3} \end{aligned}$$

(g) Using the shortcut formula and parts (e) and (f), show that $Var(X) = (b-a)^2/12$.

Solution:

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 = \frac{b^2+a^2+ab}{3} - \left(\frac{a+b}{2}\right)^2 \\ &= \frac{b^2+a^2+ab}{3} - \frac{a^2+2ab+b^2}{4} \\ &= \frac{4b^2+4a^2+4ab-3a^2-6ab-3b^2}{12} \\ &= \frac{b^2+a^2-2ab}{12} = \frac{(b-a)^2}{12} \end{aligned}$$

6. Suppose that $X \sim N(0, 16)$ independent of $Y \sim N(2, 4)$. Recall that when our convention is to express the normal distribution in terms of its mean and variance, i.e. $N(\mu, \sigma^2)$. Hence, X has a mean of zero and variance of 16, while Y has a mean of 2 and a variance of 4. In completing some parts of this question you will need to use the R function `pnorm` described in class. In this case, please write down the command you used as well as the numeric result.

(a) Calculate $P(-8 \le X \le 8)$.

Solution:

$$P(-8 \le X \le 8) = P(-8/4 \le X/4 \le 8/4) = P(-2 \le Z \le 2) \approx 0.95$$

where Z is a standard normal random variable.

(b) Calculate $P(0 \le Y \le 4)$.

Solution:

$$P(0 \le Y \le 4) = P\left(\frac{0-2}{2} \le \frac{Y-2}{2} \le \frac{4-2}{2}\right) = P(-1 \le Z \le 1) \approx 0.68$$

where Z is a standard normal random variable.

(c) Calculate $P(-1 \le Y \le 6)$.

Solution:

$$\begin{aligned} P(-1 \le Y \le 6) &= P\left(\frac{-1-2}{2} \le \frac{Y-2}{2} \le \frac{6-2}{2}\right) \\ &= P(-1.5 \le Z \le 2) \\ &= \Phi(2) - \Phi(-1.5) \\ &= \texttt{pnorm(2) - pnorm(-1.5)} \\ &\approx 0.91 \end{aligned}$$

where Z is a standard normal random variable.

(d) Calculate $P(X \ge 10)$.

Solution:

$$\begin{aligned} P(X \ge 10) &= 1 - P(X \le 10) = 1 - P(X/4 \le 10/4) = 1 - P(Z \le 2.5) \\ &= 1 - \Phi(2.5) = 1 - \texttt{pnorm(2.5)} \\ &\approx 0.006 \end{aligned}$$