

# Problem Set #6

Econ 103

## Part I – Problems from the Textbook

Chapter 5: 1, 3, 5, 9, 11, 13, 17

Chapter 4: 19, 21, 23

(When necessary, use  $R$  rather than the Normal tables in the front of the textbook.)

## Part II – Additional Problems

1. Let  $X$  and  $Y$  be discrete random variables and  $a, b, c, d$  be constants. Prove the following:

(a)  $Cov(a + bX, c + dY) = bdCov(X, Y)$

**Solution:** Let  $\mu_X = E[X]$  and  $\mu_Y = E[Y]$ . By the linearity of expectation,

$$E[a + bX] = a + b\mu_X$$

$$E[c + dY] = c + d\mu_Y$$

Thus, we have

$$(a + bx) - E[a + bX] = b(x - \mu_X)$$

$$(c + dy) - E[c + dY] = d(y - \mu_Y)$$

Substituting these into the formula for the covariance between two discrete random variables,

$$\begin{aligned} Cov(a + bX, c + dY) &= \sum_x \sum_y [b(x - \mu_X)] [d(y - \mu_Y)] p(x, y) \\ &= bd \sum_x \sum_y (x - \mu_X)(y - \mu_Y) p(x, y) \\ &= bdCov(X, Y) \end{aligned}$$

(b)  $Corr(a + bX, c + dY) = Corr(X, Y)$

**Solution:**

$$\begin{aligned}
 Corr(a + bX, c + dY) &= \frac{Cov(a + bX, c + dY)}{\sqrt{Var(a + bX)Var(c + dY)}} \\
 &= \frac{bdCov(X, Y)}{\sqrt{b^2Var(X)d^2Var(Y)}} \\
 &= \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}} \\
 &= Corr(X, Y)
 \end{aligned}$$

2. Fill in the missing steps from the lecture to prove the shortcut formula for covariance:

$$Cov(X, Y) = E[XY] - E[X]E[Y]$$

**Solution:** By the Linearity of Expectation,

$$\begin{aligned}
 Cov(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\
 &= E[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y] \\
 &= E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_X \mu_Y \\
 &= E[XY] - \mu_X \mu_Y - \mu_Y \mu_X + \mu_X \mu_Y \\
 &= E[XY] - \mu_X \mu_Y \\
 &= E[XY] - E[X]E[Y]
 \end{aligned}$$

3. Let  $X_1$  be a random variable denoting the returns of stock 1, and  $X_2$  be a random variable denoting the returns of stock 2. Accordingly let  $\mu_1 = E[X_1]$ ,  $\mu_2 = E[X_2]$ ,  $\sigma_1^2 = Var(X_1)$ ,  $\sigma_2^2 = Var(X_2)$  and  $\rho = Corr(X_1, X_2)$ . A *portfolio*,  $\Pi$ , is a linear combination of  $X_1$  and  $X_2$  with weights that sum to one, that is  $\Pi(\omega) = \omega X_1 + (1 - \omega)X_2$ , indicating the proportions of stock 1 and stock 2 that an investor holds. In this example, we require  $\omega \in [0, 1]$ , so that *negative* weights are not allowed. (This rules out short-selling.)
- (a) Calculate  $E[\Pi(\omega)]$  in terms of  $\omega$ ,  $\mu_1$  and  $\mu_2$ .

**Solution:**

$$\begin{aligned}E[\Pi(\omega)] &= E[\omega X_1 + (1 - \omega)X_2] = \omega E[X_1] + (1 - \omega)E[X_2] \\&= \omega\mu_1 + (1 - \omega)\mu_2\end{aligned}$$

- (b) If  $\omega \in [0, 1]$  is it possible to have  $E[\Pi(\omega)] > \mu_1$  and  $E[\Pi(\omega)] > \mu_2$ ? What about  $E[\Pi(\omega)] < \mu_1$  and  $E[\Pi(\omega)] < \mu_2$ ? Explain.

**Solution:** No. If short-selling is disallowed, the portfolio expected return must be between  $\mu_1$  and  $\mu_2$ .

- (c) Express  $Cov(X_1, X_2)$  in terms of  $\rho$  and  $\sigma_1, \sigma_2$ .

**Solution:**  $Cov(X, Y) = \rho\sigma_1\sigma_2$

- (d) What is  $Var[\Pi(\omega)]$ ? (Your answer should be in terms of  $\rho, \sigma_1^2$  and  $\sigma_2^2$ .)

**Solution:**

$$\begin{aligned}Var[\Pi(\omega)] &= Var[\omega X_1 + (1 - \omega)X_2] \\&= \omega^2 Var(X_1) + (1 - \omega)^2 Var(X_2) + 2\omega(1 - \omega)Cov(X_1, X_2) \\&= \omega^2\sigma_1^2 + (1 - \omega)^2\sigma_2^2 + 2\omega(1 - \omega)\rho\sigma_1\sigma_2\end{aligned}$$

- (e) Using part (d) show that the value of  $\omega$  that minimizes  $Var[\Pi(\omega)]$  is

$$\omega^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

In other words,  $\Pi(\omega^*)$  is the *minimum variance portfolio*.

**Solution:** The First Order Condition is:

$$2\omega\sigma_1^2 - 2(1 - \omega)\sigma_2^2 + (2 - 4\omega)\rho\sigma_1\sigma_2 = 0$$

Dividing both sides by two and rearranging:

$$\begin{aligned}\omega\sigma_1^2 - (1 - \omega)\sigma_2^2 + (1 - 2\omega)\rho\sigma_1\sigma_2 &= 0 \\ \omega\sigma_1^2 - \sigma_2^2 + \omega\sigma_2^2 + \rho\sigma_1\sigma_2 - 2\omega\rho\sigma_1\sigma_2 &= 0 \\ \omega(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2) &= \sigma_2^2 - \rho\sigma_1\sigma_2\end{aligned}$$

So we have

$$\omega^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

- (f) If you want a challenge, check the second order condition from part (e).

**Solution:** The second derivative is

$$2\sigma_1^2 - 2\sigma_2^2 - 4\rho\sigma_1\sigma_2$$

and, since  $\rho = 1$  is the largest possible value for  $\rho$ ,

$$2\sigma_1^2 - 2\sigma_2^2 - 4\rho\sigma_1\sigma_2 \geq 2\sigma_1^2 - 2\sigma_2^2 - 4\sigma_1\sigma_2 = 2(\sigma_1 - \sigma_2)^2 \geq 0$$

so the second derivative is positive, indicating a minimum. This is a global minimum since the problem is quadratic in  $\omega$ .

4. Suppose that  $X$  is a random variable with the following PDF

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

- (a) Graph the PDF of  $X$ .

**Solution:** It's an isosceles triangle with base from (0,0) to (2,0) and height 1.

- (b) Show that  $\int_{-\infty}^{\infty} f(x) dx = 1$ .

**Solution:**

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \int_0^1 x dx + \int_1^2 (2 - x) dx = \frac{x^2}{2} \Big|_0^1 + \left( 2x - \frac{x^2}{2} \right) \Big|_1^2 \\ &= 1/2 + (4 - 2) - (2 - 1/2) = 1 \end{aligned}$$

- (c) What is  $P(0.5 < X < 1.5)$ ?

**Solution:**

$$\begin{aligned}P(0.5 < X < 1.5) &= \int_{0.5}^{1.5} f(x) dx = \int_{0.5}^1 x dx + \int_1^{1.5} (2 - x) dx \\&= \left. \frac{x^2}{2} \right|_{0.5}^1 + \left( 2x - \frac{x^2}{2} \right) \Big|_1^{1.5} \\&= (1/2 - 1/8) + (3 - 9/8) - (2 - 1/2) \\&= 3/8 + 15/8 - 2 + 1/2 = 18/8 - 16/8 + 4/8 \\&= 6/8 = 3/4 = 0.75\end{aligned}$$

5. A random variable is said to follow a  $\text{Uniform}(a, b)$  distribution if it is equally likely to take on any value in the range  $[a, b]$  and never takes a value outside this range. Suppose that  $X$  is such a random variable, i.e.  $X \sim \text{Uniform}(a, b)$ .

- (a) What is the support of  $X$ ?

**Solution:**  $[a, b]$

- (b) Explain why the PDF of  $X$  is  $f(x) = 1/(b - a)$  for  $a \leq x \leq b$ , zero elsewhere.

**Solution:** This simply generalizes the  $\text{Uniform}(0, 1)$  random variable from class. To capture the idea that  $X$  is equally likely to take on any value in the range  $[a, b]$ , the PDF must be constant. To ensure that it integrates to 1, the denominator must be  $b - a$ .

- (c) Using the PDF from part (b), calculate the CDF of  $X$ .

**Solution:**

$$F(x_0) = \int_{-\infty}^{x_0} f(x) dx = \int_a^{x_0} \frac{dx}{b - a} = \frac{x}{b - a} \Big|_a^{x_0} = \frac{x_0 - a}{b - a}$$

- (d) Verify that  $f(x) = F'(x)$  for the present example.

**Solution:**

$$F'(x) = \frac{d}{dx} \left( \frac{x - a}{b - a} \right) = \frac{1}{b - a} = f(x)$$

(e) Calculate  $E[X]$ .

**Solution:**

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_a^b \frac{x}{b-a} dx = \frac{x^2}{2(b-a)} \Big|_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$$

(f) Calculate  $E[X^2]$ . *Hint:* recall that  $b^3 - a^3$  can be factorized as  $(b-a)(b^2 + a^2 + ab)$ .

**Solution:**

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x) dx = \int_a^b \frac{x^2}{b-a} = \frac{x^3}{3(b-a)} \Big|_a^b = \frac{b^3 - a^3}{3(b-a)} \\ &= \frac{(b-a)(b^2 + a^2 + ab)}{3(b-a)} = \frac{b^2 + a^2 + ab}{3} \end{aligned}$$

(g) Using the shortcut formula and parts (e) and (f), show that  $Var(X) = (b-a)^2/12$ .

**Solution:**

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 = \frac{b^2 + a^2 + ab}{3} - \left(\frac{a+b}{2}\right)^2 \\ &= \frac{b^2 + a^2 + ab}{3} - \frac{a^2 + 2ab + b^2}{4} \\ &= \frac{4b^2 + 4a^2 + 4ab - 3a^2 - 6ab - 3b^2}{12} \\ &= \frac{b^2 + a^2 - 2ab}{12} = \frac{(b-a)^2}{12} \end{aligned}$$

6. Suppose that  $X \sim N(0, 16)$  independent of  $Y \sim N(2, 4)$ . Recall that when our convention is to express the normal distribution in terms of its mean and variance, i.e.  $N(\mu, \sigma^2)$ . Hence,  $X$  has a mean of zero and variance of 16, while  $Y$  has a mean of 2 and a variance of 4. In completing some parts of this question you will need to use the R function `pnorm` described in class. In this case, please write down the command you used as well as the numeric result.

- (a) Calculate  $P(-8 \leq X \leq 8)$ .

**Solution:**

$$P(-8 \leq X \leq 8) = P(-8/4 \leq X/4 \leq 8/4) = P(-2 \leq Z \leq 2) \approx 0.95$$

where  $Z$  is a standard normal random variable.

- (b) Calculate  $P(0 \leq Y \leq 4)$ .

**Solution:**

$$P(0 \leq Y \leq 4) = P\left(\frac{0-2}{2} \leq \frac{Y-2}{2} \leq \frac{4-2}{2}\right) = P(-1 \leq Z \leq 1) \approx 0.68$$

where  $Z$  is a standard normal random variable.

- (c) Calculate  $P(-1 \leq Y \leq 6)$ .

**Solution:**

$$\begin{aligned} P(-1 \leq Y \leq 6) &= P\left(\frac{-1-2}{2} \leq \frac{Y-2}{2} \leq \frac{6-2}{2}\right) \\ &= P(-1.5 \leq Z \leq 2) \\ &= \Phi(2) - \Phi(-1.5) \\ &= \text{pnorm}(2) - \text{pnorm}(-1.5) \\ &\approx 0.91 \end{aligned}$$

where  $Z$  is a standard normal random variable.

- (d) Calculate  $P(X \geq 10)$ .

**Solution:**

$$\begin{aligned} P(X \geq 10) &= 1 - P(X \leq 10) = 1 - P(X/4 \leq 10/4) = 1 - P(Z \leq 2.5) \\ &= 1 - \Phi(2.5) = 1 - \text{pnorm}(2.5) \\ &\approx 0.006 \end{aligned}$$