Economics 103 – Statistics for Economists

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Lecture # 4

Introduction to Regression

How to fairly account for missing midterm score?

- In my first semester at Penn, several students missed Midterm 2 because of illness so I decided to up-weight their finals.
- Problem: Midterm 2 turned out easier than Midterm 1 and this put the students who had missed the second midterm at a disadvantage when I curved the class.
- In order to correct for this, I needed a way to *fill in* a score for the missing midterm.
- How could I do this fairly?

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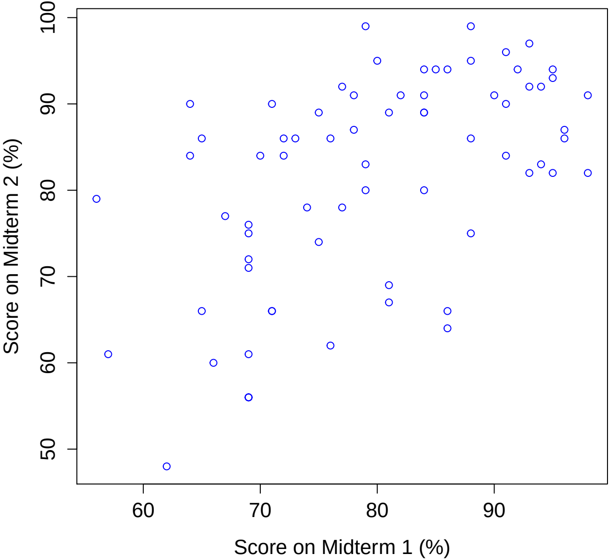
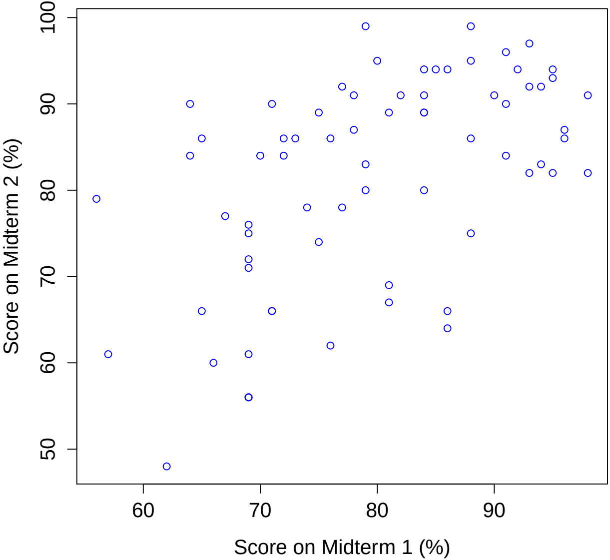
- How could I do this fairly?
  - Just fill in mean score on second exam?
  - Use performance on first midterm to predict?

Data for students who took both midterms:



Predict Second Midterm given 81 on First



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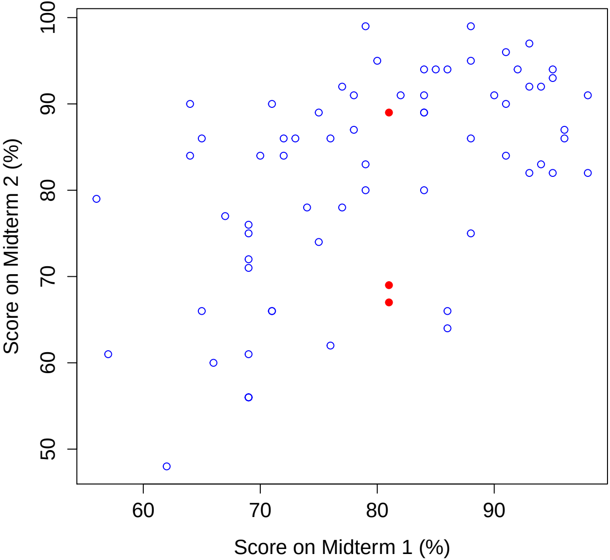

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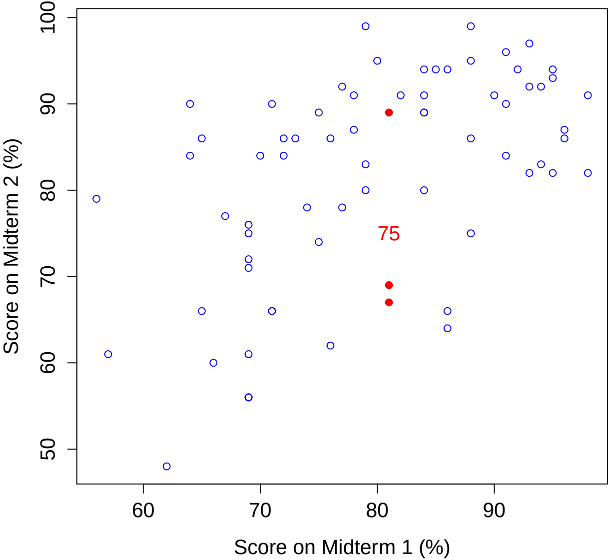
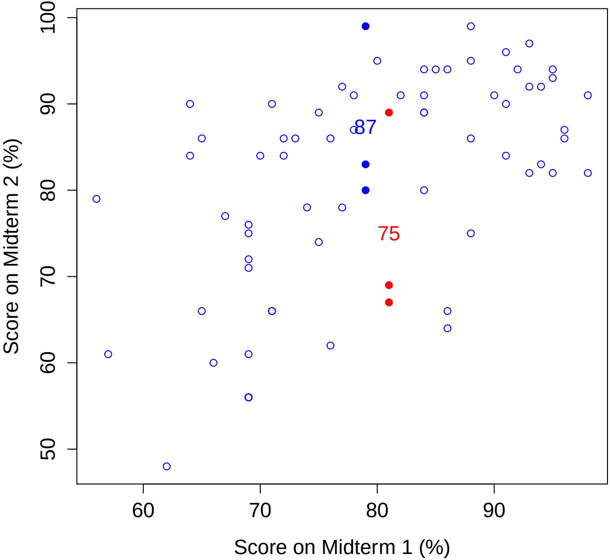
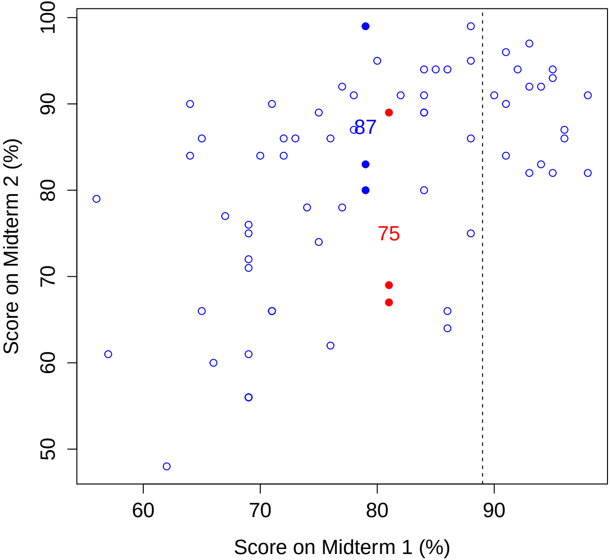


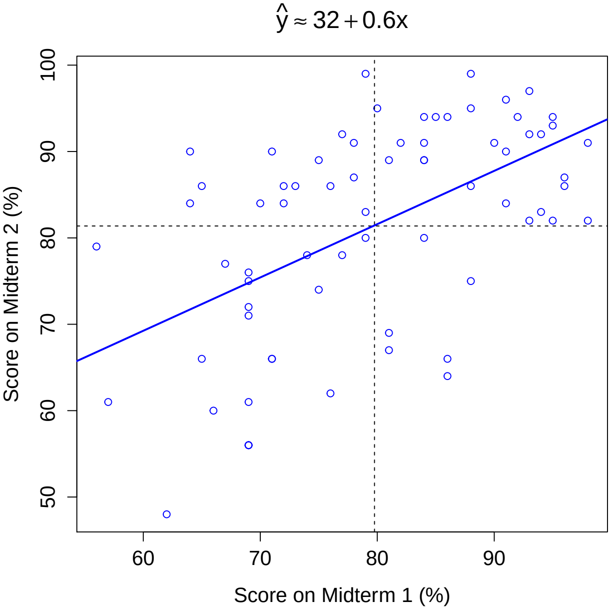
But if they'd only gotten 79 we'd predict higher?!



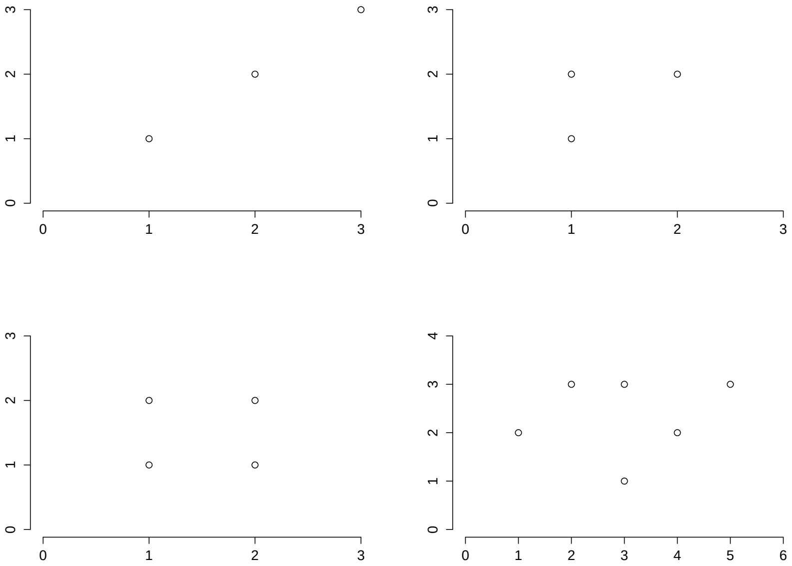
No one who took both exams got 89 on the first!

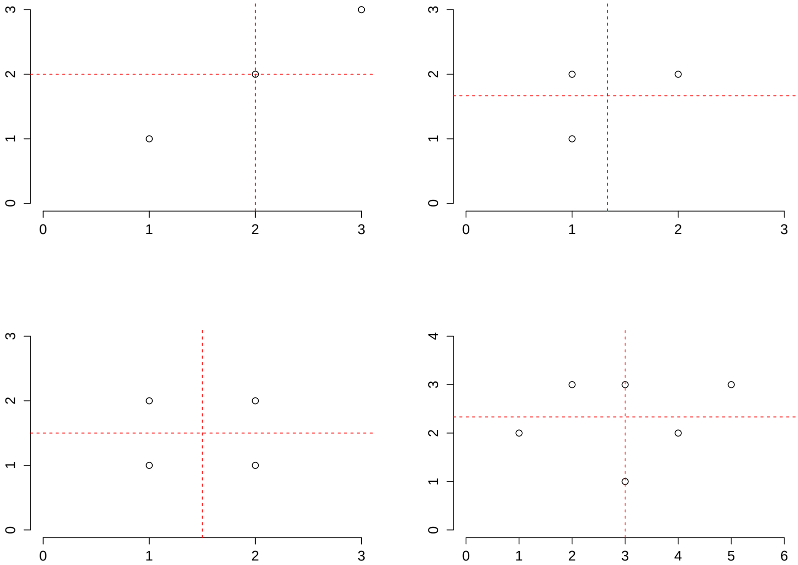


Regression: "Best Fitting" Line Through Cloud of Points



Fitting a Line by Eye


0 1 2 3
0 1 2 3
0 1 2 3
0 1 2 3
0 1 2 3
0 1 2 3
0 1 2 3 4
0 1 2 3 4 5 6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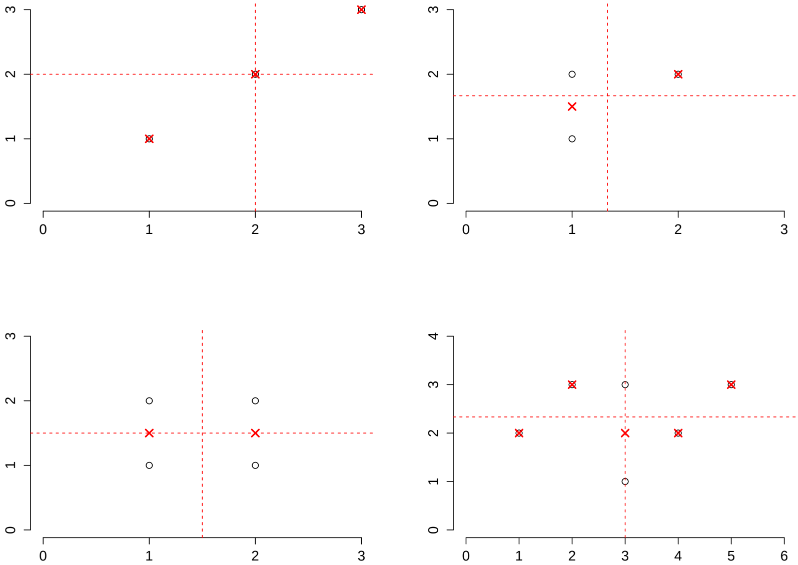
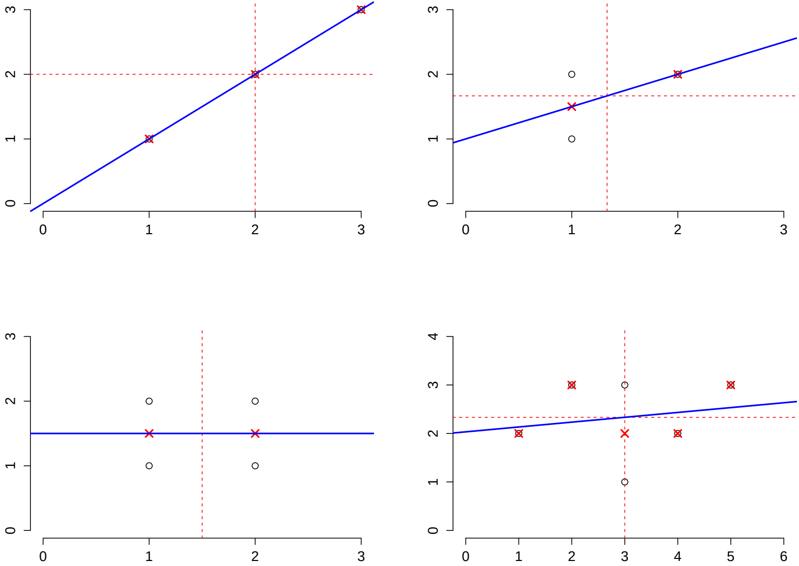
1
2
3
4


0 1 2 3
0 1 2 3
0 1 2 3
0 1 2 3
0 1 2 3
0 1 2 3
0 1 2 3 4 5 6
0 1 2 3 4


0 1 2 3
0 1 2 3
0 1 2 3
0 1 2 3
0 1 2 3
0 1 2 3
0 1 2 3 4
0 1 2 3 4 5 6

But How to Do this Formally?

Least Squares Regression – Predict Using a Line

Linear Model

$\hat{y} = a + bx$

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$\hat{y} = a + bx$

Choose a, b to Minimize Sum of Squared Vertical Deviations

$$\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

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The Prediction

Predict score $\hat{y} = a + bx$ on second midterm for someone with score x on first.

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The Prediction

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Why Vertical Deviations? Why Squared Deviations?

Important Point About Notation

$$\boxed{\text{minimize } \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (y_i - a - bx_i)^2}$$

$$\boxed{\hat{y} = a + bx}$$

- $(x_i, y_i)_{i=1}^{n}$ are the observed data
- $\widehat{y}$ is our prediction for a given value of x
- Neither x nor $\widehat{y}$ needs to be in out dataset!

Key Point

- Each choice of a, b defines a line

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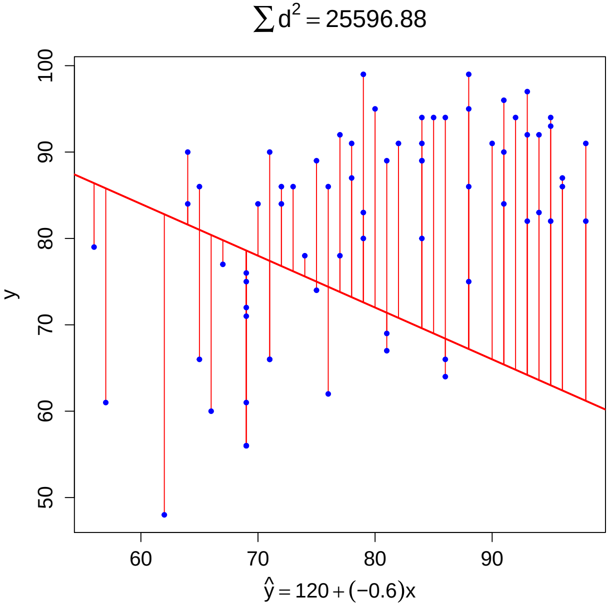
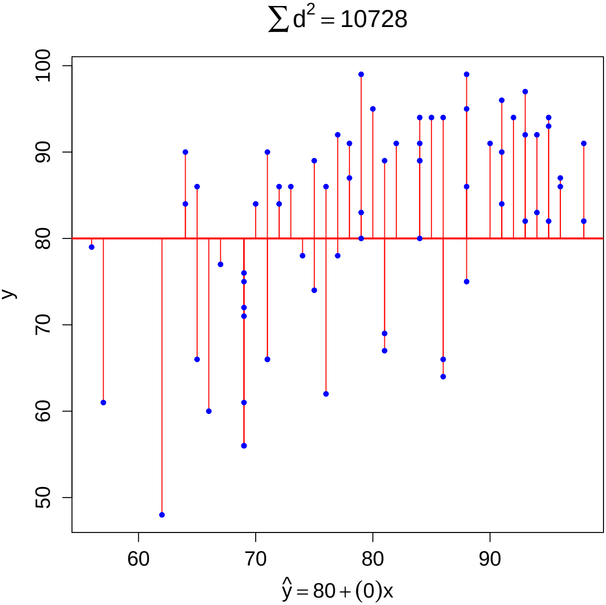
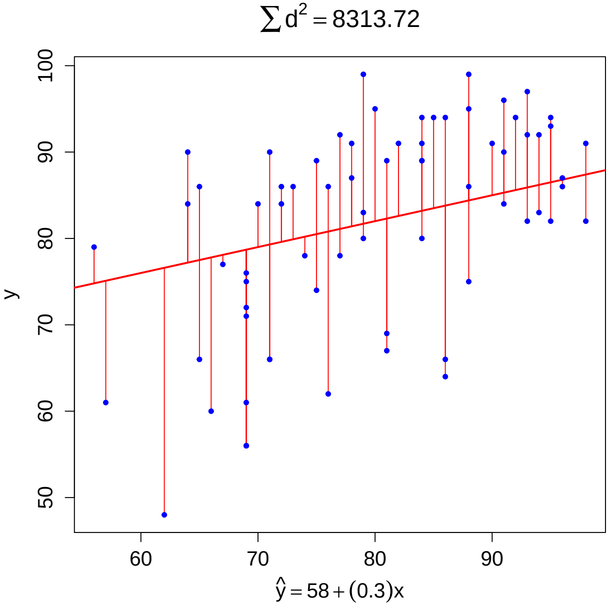
- Each choice of a, b defines a line
- Given the data, each line defines collection of vertical devs.
  $d_i = y_i - a - bx_i$ for $i = 1, \ldots, n$
- Each collection of vertical devs. gives sum of squares $\sum_{i=1}^{n} d_i^2$
- We choose a, b to minimize $\sum_{i=1}^{n} d_i^2$

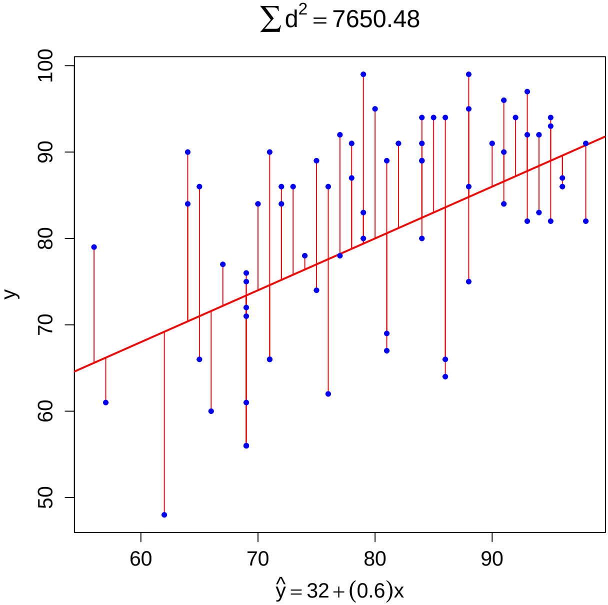


$\sum d^2 = 25596.88$
y
100
90
80
70
60
50
60
70
80
90
$\hat{y} = 120 + (-0.6)x$


$\sum d^2 = 10728$
y
100
90
80
70
60
50
60
70
80
90
$\hat{y} = 80 + (0)x$

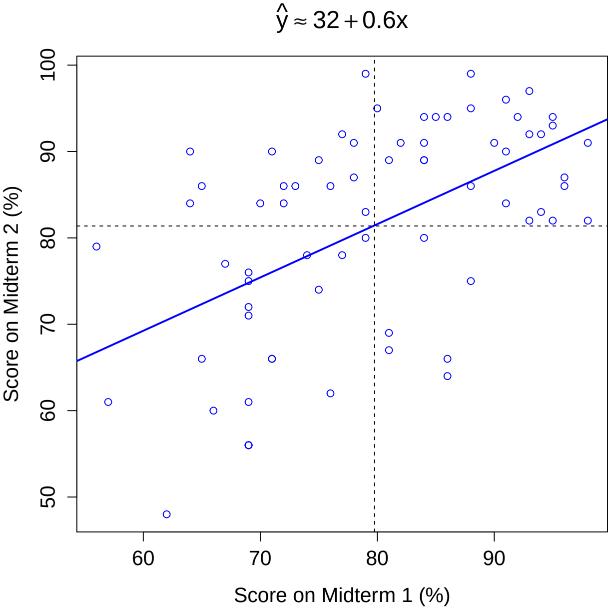


$\sum d^2 = 8313.72$
y
100
90
80
70
60
50
60
70
80
90
$\hat{y} = 58 + (0.3)x$

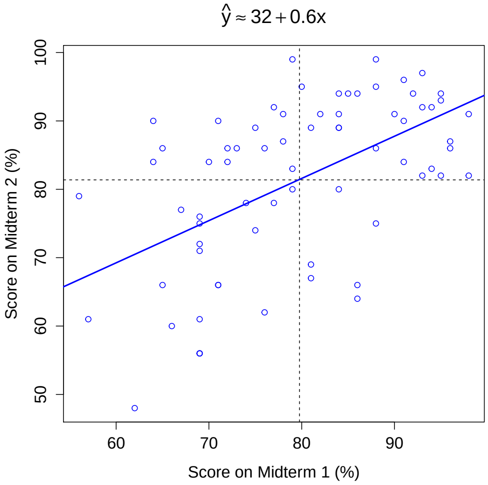


$\sum d^2 = 7650.48$
y
100
90
80
70
60
50
60
70
80
90
$\hat{y} = 32 + (0.6)x$

Prediction given 89 on Midterm 1?



Prediction given 89 on Midterm 1?



$$32 + 0.6 \times 89 = 32 + 53.4 = 85.4$$

You Need to Know How To Derive This

Minimize the sum of squared vertical deviations from the line:

$$\min_{a,b} \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

How should we proceed?

(a) Differentiate with respect to x

(b) Differentiate with respect to y

(c) Differentiate with respect to x, y

(d) Differentiate with respect to a, b

(e) Can't solve this with calculus.

Objective Function

$$\min_{a,b} \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

FOC with respect to a

Objective Function

$$\min_{a,b} \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

FOC with respect to a

$$-2 \sum_{i=1}^{n} (y_i - a - bx_i) = 0$$

Objective Function

$$\min_{a,b} \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

FOC with respect to *a*

$$\begin{aligned} -2 \sum_{i=1}^{n} (y_i - a - bx_i) &= 0 \\ \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} a - b \sum_{i=1}^{n} x_i &= 0 \end{aligned}$$

Objective Function

$$\min_{a,b} \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

FOC with respect to a

$$\begin{aligned}
-2\sum_{i=1}^{n}(y_i - a - bx_i) &= 0 \\
\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} a - b\sum_{i=1}^{n} x_i &= 0 \\
\frac{1}{n}\sum_{i=1}^{n} y_i - \frac{na}{n} - \frac{b}{n}\sum_{i=1}^{n} x_i &= 0
\end{aligned}$$

Objective Function

$$\min_{a,b} \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

FOC with respect to a

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\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} a - b\sum_{i=1}^{n} x_i &= 0 \\
\frac{1}{n}\sum_{i=1}^{n} y_i - \frac{na}{n} - \frac{b}{n}\sum_{i=1}^{n} x_i &= 0 \\
\bar{y} - a - b\bar{x} &= 0
\end{aligned}$$

Regression Line Goes Through the Means!

$$\boxed{\bar{y} = a + b\bar{x}}$$

Substitute: Eliminate a from Objective Function

$$\boxed{a = \bar{y} - b\bar{x}}$$

$$\sum_{i=1}^{n}(y_i - a - bx_i)^2 \quad =$$

Substitute: Eliminate a from Objective Function

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$$\begin{aligned} \sum_{i=1}^{n}(y_i - a - bx_i)^2 &= \sum_{i=1}^{n}(y_i - \bar{y} + b\bar{x} - bx_i)^2 \\ &= \end{aligned}$$

Substitute: Eliminate a from Objective Function

$$\boxed{a = \bar{y} - b\bar{x}}$$

$$\begin{aligned}
\sum_{i=1}^{n}(y_i - a - bx_i)^2 &= \sum_{i=1}^{n}(y_i - \bar{y} + b\bar{x} - bx_i)^2 \\
&= \sum_{i=1}^{n}\left[(y_i - \bar{y}) - b\,(x_i - \bar{x})\right]^2
\end{aligned}$$

Objective Function Without a

$$\boxed{\sum_{i=1}^{n} \left[(y_i - \bar{y}) - b(x_i - \bar{x})\right]^2}$$

FOC with respect to b

Objective Function Without a

$$\boxed{\sum_{i=1}^{n} \left[(y_i - \bar{y}) - b(x_i - \bar{x})\right]^2}$$

FOC with respect to b

$$-2\sum_{i=1}^{n} \left[(y_i - \bar{y}) - b\,(x_i - \bar{x})\right](x_i - \bar{x}) \;=\; 0$$

Objective Function Without a

$$\boxed{\sum_{i=1}^{n} \left[(y_i - \bar{y}) - b(x_i - \bar{x})\right]^2}$$

FOC with respect to b

$$\begin{aligned}
-2\sum_{i=1}^{n} \left[(y_i - \bar{y}) - b\,(x_i - \bar{x})\right](x_i - \bar{x}) &= 0 \\
\sum_{i=1}^{n} (y_i - \bar{y})\,(x_i - \bar{x}) - b\sum_{i=1}^{n} (x_i - \bar{x})^2 &= 0
\end{aligned}$$

Objective Function Without a

$$\boxed{\sum_{i=1}^{n} \left[(y_i - \bar{y}) - b(x_i - \bar{x}) \right]^2}$$

FOC with respect to b

$$\begin{aligned}
-2 \sum_{i=1}^{n} \left[(y_i - \bar{y}) - b\,(x_i - \bar{x}) \right] (x_i - \bar{x}) &= 0 \\
\sum_{i=1}^{n} (y_i - \bar{y})\,(x_i - \bar{x}) - b \sum_{i=1}^{n} (x_i - \bar{x})^2 &= 0
\end{aligned}$$

$$b = \frac{\sum_{i=1}^{n} (y_i - \bar{y})\,(x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

Simple Linear Regression

Problem

$$\min_{a,b} \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

Solution

$$b = \frac{\sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$a = \bar{y} - b\bar{x}$$

Relating Regression to Covariance and Correlation

$$b = \frac{\sum_{i=1}^{n} \left(y_i - \bar{y}\right)\left(x_i - \bar{x}\right)}{\sum_{i=1}^{n} \left(x_i - \bar{x}\right)^2} =$$

Relating Regression to Covariance and Correlation

$$b = \frac{\sum_{i=1}^{n}\left(y_i - \bar{y}\right)\left(x_i - \bar{x}\right)}{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2} = \frac{\frac{1}{n-1}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)\left(x_i - \bar{x}\right)}{\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2} =$$

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$$r = \frac{s_{xy}}{s_x s_y} =$$

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$$r = \frac{s_{xy}}{s_x s_y} = \frac{s_{xy}}{s_x s_y} \cdot \frac{s_x}{s_x} =$$

Relating Regression to Covariance and Correlation

$$b = \frac{\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{\frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x})}{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{s_{xy}}{s_x^2}$$

$$r = \frac{s_{xy}}{s_x s_y} = \frac{s_{xy}}{s_x s_y} \cdot \frac{s_x}{s_x} = \frac{s_{xy}}{s_x^2} \cdot \frac{s_x}{s_y} =$$

Relating Regression to Covariance and Correlation

$$b = \frac{\sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = \frac{\frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x})}{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2} = \frac{s_{xy}}{s_x^2}$$

$$r = \frac{s_{xy}}{s_x s_y} = \frac{s_{xy}}{s_x s_y} \cdot \frac{s_x}{s_x} = \frac{s_{xy}}{s_x^2} \cdot \frac{s_x}{s_y} = b \frac{s_x}{s_y}$$

Relating Regression to Covariance and Correlation

$$b = \frac{\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{\frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x})}{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{s_{xy}}{s_x^2}$$

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$$b = r\frac{s_y}{s_x}$$

Comparing Regression, Correlation and Covariance

Units

Correlation is unitless, covariance and regression coefficients (a, b) are not. (What are the units of these?)

Symmetry

Correlation and covariance are symmetric, regression isn't.

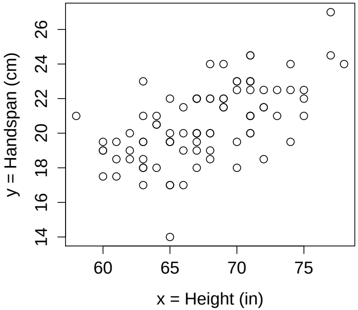
(Switching x and y axes changes the slope and intercept.)

On the Homework

Regression with z-scores rather than raw data gives $a = 0, b = r_{xy}$

$$s_{xy} = 6, \quad s_x = 5, \quad s_y = 2, \quad \bar{x} = 68, \quad \bar{y} = 21$$

What is the sample correlation between height (x) and handspan (y)?



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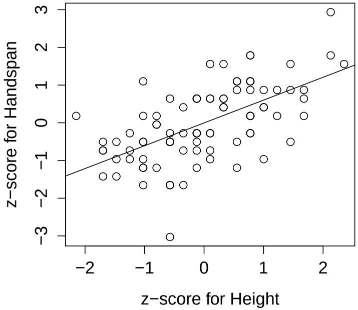


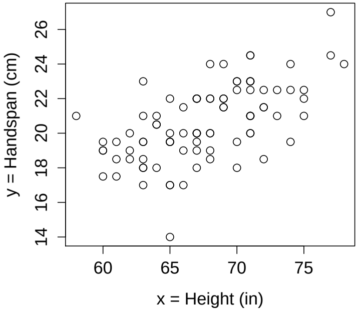
$$r = \frac{s_{xy}}{s_x s_y} = \frac{6}{5 \times 2} = 0.6$$

$$s_{xy} = 6, \quad s_x = 5, \quad s_y = 2, \quad \bar{x} = 68, \quad \bar{y} = 21$$

What is the value of b for the regression:

$$\hat{y} = a + bx$$

where x is height and y is handspan?



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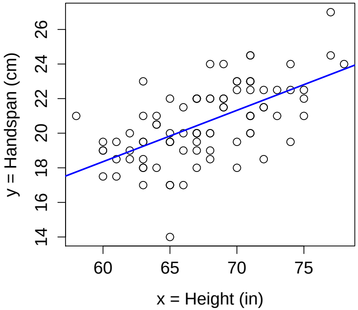


$$b = \frac{s_{xy}}{s_x^2} = \frac{6}{5^2} = 6/25 = 0.24$$

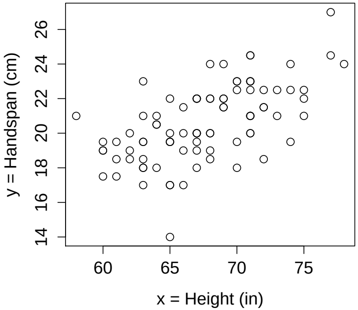
$$s_{xy} = 6, \quad s_x = 5, \quad s_y = 2, \quad \bar{x} = 68, \quad \bar{y} = 21$$

What is the value of a for the regression:

$$\hat{y} = a + bx$$

where x is height and y is handspan?

(prev. slide $b = 0.24$)



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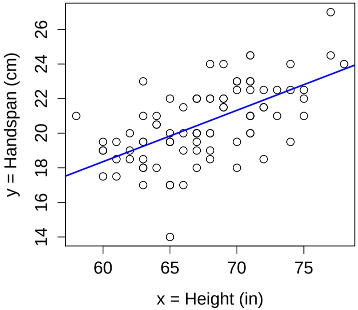


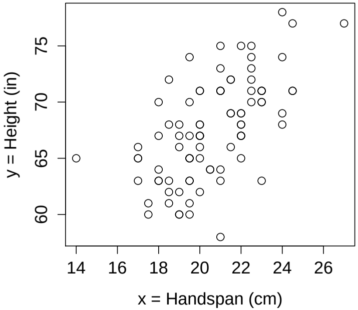
$$a = \bar{y} - b\bar{x} = 21 - 0.24 \times 68 = 4.68$$

$s_{xy} = 6, \quad s_y = 5, \quad s_x = 2, \quad \bar{y} = 68, \quad \bar{x} = 21$

What is the value of b for the regression:

$$\hat{y} = a + bx$$

where x is handspan and y is height?



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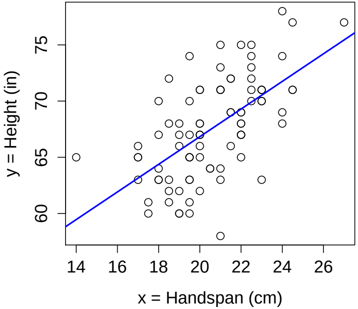
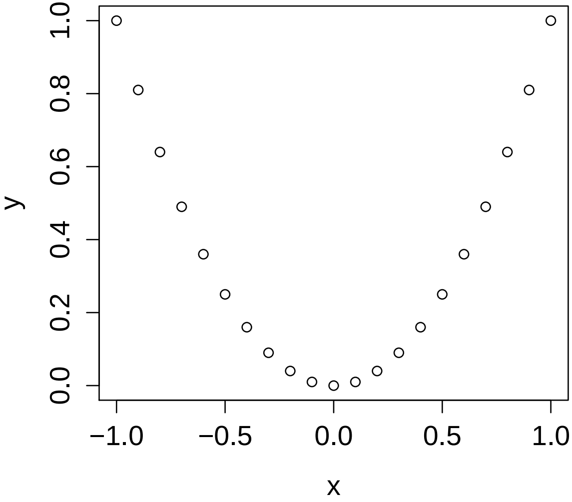


$$b = \frac{s_{xy}}{s_x^2} = 6/2^2 = 1.5$$

EXTREMELY IMPORTANT

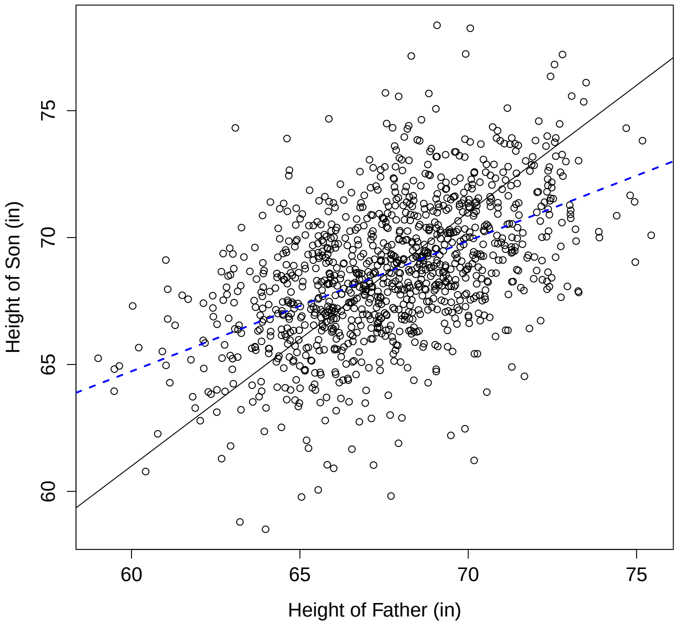
- Regression, Covariance and Correlation: linear association.
- Linear association $\neq$ causation.
- Linear is not the only kind of association!

Correlation = 0



Why is it called “regression?”

Pearson Dataset



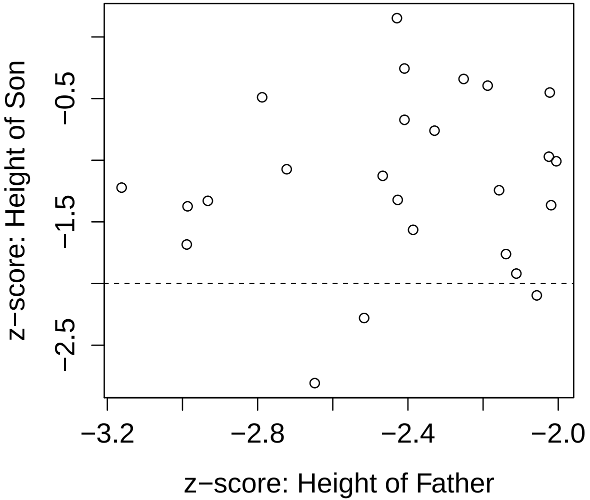
Suppose a father is very short compared to other fathers (very negative z-score). Would you expect his son to be:

(a) Shorter

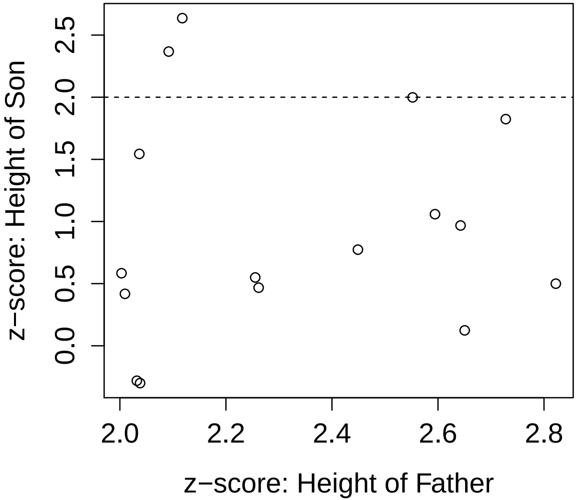
(b) About as short

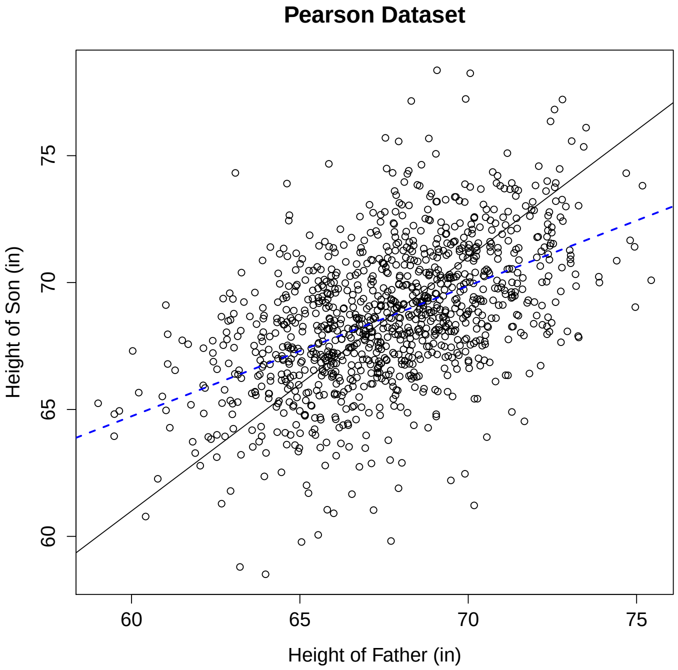
(c) Taller

Very Short Fathers and Their Sons



Very Tall Fathers and Their Sons




Pearson Dataset
Height of Son (in)
Height of Father (in)
60
65
70
75
60
65
70
75

Regression to the Mean

Skill and Luck / Genes and Random Environmental Factors

Unless $r_{xy} = 1$, There Is Regression to the Mean

$$\frac{\hat{y} - \bar{y}}{s_y} = r_{xy} \frac{x - \bar{x}}{s_x}$$

Least-squares Prediction $\hat{y}$ closer to $\bar{y}$ than x is to $\bar{x}$

You will derive the above formula in this week's homework.

Regression Fallacy

For More, See the Document Posted on Piazza

Pre-test

Which students are strongest, which are weakest?

Intervention

Put the best performing in an enrichment program and the worst performing in a remedial class

Post-test

The weak students did better than on their first test, but the strong students did *worse*.

Mistaken Conclusion

Remedial classes are beneficial, enrichment programs are harmful