Economics 103 – Statistics for Economists

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Lecture # 6

Basic Probability – Part II

Derive Rules for Computing Probabilities from Axioms

Recall: Axioms of Probability

Let S be the sample space. With each event $A \subseteq S$ we associate a real number $P(A)$ called the probability of A, satisfying the following conditions:

Axiom 1 $0 \leq P(A) \leq 1$

Axiom 2 $P(S) = 1$

Axiom 3 If $A_1, A_2, A_3, \ldots$ are mutually exclusive events, then
$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \ldots$$

Key Point

The axoims of probability are out *starting assumptions* – they are a complete description what we *mean* when we say "probability." We use the axioms to derive various results for *computing* probabilities.

The Complement Rule: $P(A^c) = 1 - P(A)$

Since A, A^c are mutually exclusive and collectively exhaustive:

$$P(A \cup A^c) = P(A) + P(A^c) = P(S) = 1$$

Rearranging:

$$P(A^c) = 1 - P(A)$$

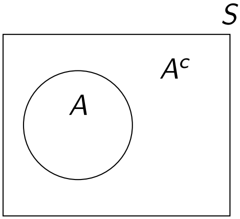


Figure : $A \cap A^c = \emptyset$, $A \cup A^c = S$

Another Important Rule – Equivalent Events

If A and B are Logically Equivalent, then $P(A) = P(B)$.

In other words, if A and B contain exactly the same basic outcomes, then $P(A) = P(B)$.

Although this seems obvious it's important to keep in mind, especially later in the course...

The Logical Consequence Rule

If B Logically Entails A, then $P(B) \leq P(A)$

In other words, $B \subseteq A \Rightarrow P(B) \leq P(A)$

Why is this so?

If $B \subseteq A$, then all the basic outcomes in B are also in A.

Deriving The Logical Consequence Rule

Since $B \subseteq A$, we have $B = A \cap B$ and $A = B \cup (A \cap B^c)$. Combining these,

$$A = (A \cap B) \cup (A \cap B^c)$$

Now since $(A \cap B) \cap (A \cap B^c) = \emptyset$,

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^c) \\ &= P(B) + P(A \cap B^c) \\ &\geq P(B) \end{aligned}$$

because $0 \leq P(A \cap B^c) \leq 1$.

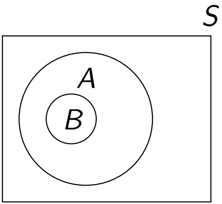


Figure : $B = A \cap B$, and $A = B \cup (A \cap B^c)$

“Odd Question” # 2

Pia is thirty-one years old, single, outspoken, and smart. She was a philosophy major. When a student, she was an ardent supporter of Native American rights, and she picketed a department store that had no facilities for nursing mothers. Rank the following statements in order from most probable to least probable.

(a) Pia is an active feminist.

(b) Pia is a bank teller.

(c) Pia works in a small bookstore.

(d) Pia is a bank teller and an active feminist.

(e) Pia is a bank teller and an active feminist who takes yoga classes.

(f) Pia works in a small bookstore and is an active feminist who takes yoga classes.

"Odd Question" # 2 – Seven *Events*

Write events D, E, and F in terms of A, B, C, and Y.

A $=$ Pia is an active feminist.

B $=$ Pia is a bank teller.

C $=$ Pia works in a small bookstore.

Y $=$ Pia takes yoga classes.

D $=$ Pia is a bank teller and an active feminist $= A \cap B$

E $=$ Pia is a bank teller and an active feminist who takes yoga classes $= A \cap B \cap Y$

F $=$ Pia works in a small bookstore and is an active feminist who takes yoga classes $= A \cap C \cap Y$

"Odd Question" # 2 – Which Events are Subsets?

A $=$ Pia is an active feminist.

B $=$ Pia is a bank teller.

C $=$ Pia works in a small bookstore.

Y $=$ Pia takes yoga classes.

D $= A \cap B \Rightarrow D \subseteq A, D \subseteq B$

E $= A \cap B \cap Y \Rightarrow E \subseteq D$

F $= A \cap C \cap Y \Rightarrow F \subseteq A, F \subseteq C$

"Odd Question" # 2 – Apply Logical Consequence Rule

A $=$ Pia is an active feminist.

B $=$ Pia is a bank teller.

C $=$ Pia works in a small bookstore.

Y $=$ Pia takes yoga classes.

D $= A \cap B \Rightarrow D \subseteq A, D \subseteq B \Rightarrow P(D) \leq P(A), P(D) \leq P(B)$

E $= A \cap B \cap Y \Rightarrow E \subseteq D \Rightarrow P(E) \leq P(D)$

F $= A \cap C \cap Y \Rightarrow F \subseteq A, F \subseteq C \Rightarrow P(F) \leq P(A), P(F) \leq P(C)$

"Odd Question" # 2 – Putting These Together...

(a) Pia is an active feminist.

(b) Pia is a bank teller.

(c) Pia works in a small bookstore.

(d) Pia is a bank teller and an active feminist.

(e) Pia is a bank teller and an active feminist who takes yoga classes.

(f) Pia works in a small bookstore and is an active feminist who takes yoga classes.

Any Correct Ranking Must Satisfy:

$$(a) \geq (d) \geq (e)$$

$$(b) \geq (d) \geq (e)$$

$$(a) \geq (f)$$

$$(c) \geq (f)$$

Throw a Fair Die Once

$E =$ roll an even number

What are the basic outcomes?

$\{1, 2, 3, 4, 5, 6\}$

What is $P(E)$?

$E = \{2, 4, 6\}$ and the basic outcomes are equally likely (and mutually exclusive), so

$$P(E) = 1/6 + 1/6 + 1/6 = 3/6 = 1/2$$

Throw a Fair Die Once

$E =$ roll an even number $\qquad M =$ roll a 1 or a prime number

What is $P(E \cup M)$?

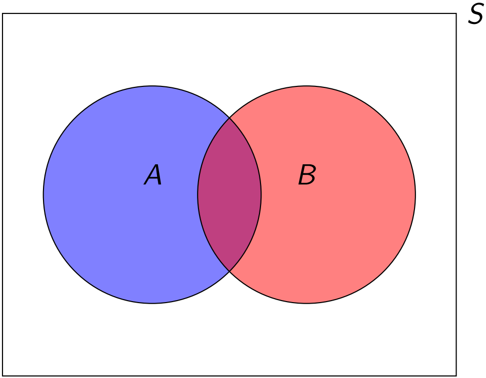
Key point: E and M are not mutually exclusive!

$$\begin{aligned} P(E \cup M) &= P(\{1,2,3,4,5,6\}) = 1 \\ P(E) &= P(\{2,4,6\}) = 1/2 \\ P(M) &= P(\{1,2,3,5\}) = 4/6 = 2/3 \end{aligned}$$

$$P(E) + P(M) = 1/2 + 2/3 = 7/6 \neq P(E \cup M) = 1$$

The Addition Rule – Don't Double-Count!

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Construct a formal proof as an optional homework problem.

Who's on the other side?

Three Cards, Each with a Face on the Front and Back

1. Gaga/Gaga
2. Obama/Gaga
3. Obama/Obama

I draw a card at random and look at one side: it's Obama.

What is the probability that the other side is also Obama?

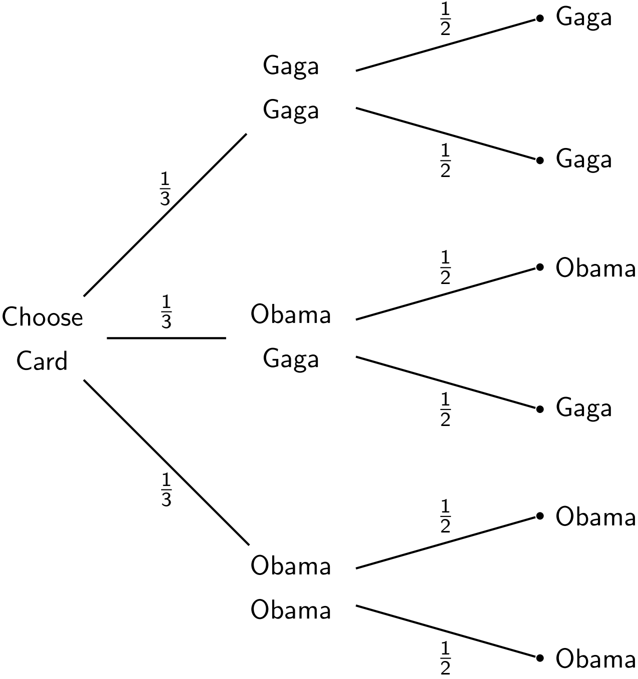
Let's Try The Method of Monte Carlo...

When you don't know how to calculate, simulate.

Procedure

1. Close your eyes and thoroughly shuffle your cards.
2. Keeping eyes closed, draw a card and place it on your desk.
3. Stand if Obama is face-up on your chosen card.
4. We'll count those standing and call the total N
5. Of those standing, sit down if Obama is *not* on the back of your chosen card.
6. We'll count those *still* standing and call the total m.

Monte Carlo Approximation of Desired Probability $= \dfrac{m}{N}$


Choose
Card
1/3
1/3
1/3
Gaga
Gaga
Obama
Gaga
Obama
Obama
1/2
1/2
1/2
1/2
1/2
1/2
Gaga
Gaga
Obama
Gaga
Obama
Obama

Conditional Probability – Reduced Sample Space

Set of relevant outcomes restricted by condition

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) > 0$$

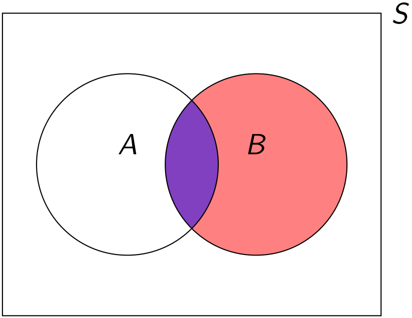


Figure : B becomes the "new sample space" so we need to re-scale by $P(B)$ to keep probabilities between zero and one.

Who's on the other side?

Let O_F be the event that Obama is on the front of the card of the card we draw and O_B be the event that he is on the back.

$$P(O_B|O_F) = \frac{P(O_B \cap O_F)}{P(O_F)} = \frac{1/3}{1/2} = 2/3$$