Economics 103 – Statistics for Economists

Francis J. DiTraglia

University of Pennsylvania

Lecture # 9

Discrete RVs - Part II

Variance and Standard Deviation of a RV

The Defs are the same for continuous RVs, but the method of calculating will differ.

Variance (Var)

$$\sigma^2 = Var(X) = E[(X - \mu)^2] = E[(X - E[X])^2]$$

Standard Deviation (SD)

$$\sigma = \sqrt{\sigma^2} = SD(X)$$

Key Point

Variance and std. dev. are expectations of functions of a RV

It follows that:

- 1. Variance and SD are constants
- 2. To derive facts about them you can use the facts you know about expected value

How To Calculate Variance for Discrete RV?

Remember: it's just a function of X!

Recall that
$$\mu = E[X] = \sum_{\text{all } x} xp(x)$$

$$Var(X) = E[(X - \mu)^2] = \sum_{\text{all } x} (x - \mu)^2 p(x)$$

Shortcut Formula For Variance

This is *not* the definition, it's a shortcut for doing calculations:

$$Var(X) = E[(X - \mu)^2] = E[X^2] - (E[X])^2$$

We'll prove this in an upcoming lecture.

Example: The Shortcut Formula



Let $X \sim \text{Bernoulli}(1/2)$.

- 1. What is E[X]? $E[X] = 0 \times 1/2 + 1 \times 1/2 = 1/2$
- 2. What is $E[X^2]$? $E[X^2] = 0^2 \times 1/2 + 1^2 \times 1/2 = 1/2$
- 3. What is Var(X)? $E[X^2] (E[X])^2 = 1/2 (1/2)^2 = 1/4$

Variance of Bernoulli RV – via the Shortcut Formula

Step
$$1 - E[X]$$

 $\mu = E[X] = \sum_{x \in \{0,1\}} p(x) \cdot x = (1 - p) \cdot 0 + p \cdot 1 = p$
Step $2 - E[X^2]$

$$E[X^{2}] = \sum_{x \in \{0,1\}} x^{2} p(x) = 0^{2} (1-p) + 1^{2} p = p$$

Step 3 - Combine with Shortcut Formula

$$\sigma^2 = Var[X] = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$$

Variance of Bernoulli RV – Without Shortcut

You will fill in the missing steps on Problem Set 5.

$$\sigma^{2} = Var(X) = \sum_{x \in \{0,1\}} (x - \mu)^{2} p(x)$$

$$= \sum_{x \in \{0,1\}} (x - p)^{2} p(x)$$

$$\vdots$$

$$= p(1 - p)$$

Variance of a Linear Function



Suppose X is a random variable with $Var(X) = \sigma^2$ and a, b are constants. What is Var(a + bX)?

- (a) σ^2
- (b) $a + \sigma^2$
- (c) $b\sigma^2$
- (d) $a + b\sigma^2$
- (e) $b^2 \sigma^2$

Variance and SD are NOT Linear

$$Var(a+bX) = b^2\sigma^2$$

$$SD(a+bX) = |b|\sigma$$

These should look familiar from the related results for sample variance and std. dev. that you worked out on an earlier problem set.

Variance of a Linear Transformation

$$Var(a + bX) = E \left[\{ (a + bX) - E(a + bX) \}^{2} \right]$$

$$= E \left[\{ (a + bX) - (a + bE[X]) \}^{2} \right]$$

$$= E \left[(bX - bE[X])^{2} \right]$$

$$= E[b^{2}(X - E[X])^{2}]$$

$$= b^{2}E[(X - E[X])^{2}]$$

$$= b^{2}Var(X) = b^{2}\sigma^{2}$$

The key point here is that variance is defined in terms of expectation and expectation is linear.

Binomial Random Variable

What we get if we sum a bunch of indep. Bernoulli RVs

Binomial Random Variable

Let X = the sum of n independent Bernoulli trials, each with probability of success p. Then we say that: $X \sim \text{Binomial}(n, p)$

Parameters

p = probability of "success," n = # of trials

Support

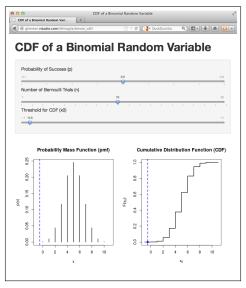
 $\{0, 1, 2, \ldots, n\}$

Probability Mass Function (pmf)

$$p(x) = \binom{n}{x} p^{x} (1-p)^{n-x}$$

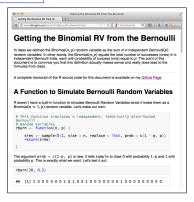
http://fditraglia.shinyapps.io/binom_cdf/

Try playing around with all three sliders. If you set the second to 1 you get a Bernoulli.



$http://fditraglia.github.com/Econ103Public/Rtutorials/Bernoulli_Binomial.html \\$

Source Code on my Github Page



Don't forget this!

Binomial RV counts up the *total* number of successes (ones) in n indep. Bernoulli trials, each with prob. of success p.

Where does the Binomial pmf come from?



Question

Suppose we flip a fair coin 3 times. What is the probability that we get exactly 2 heads?

Answer

Three basic outcomes make up this event: {HHT, HTH, THH}, each has probability $1/8 = 1/2 \times 1/2 \times 1/2$. Basic outcomes are mutually exclusive, so sum to get 3/8 = 0.375

Where does the Binomial pmf come from?

Question

Suppose we flip an *unfair* coin 3 times, where the probability of heads is 1/3. What is the probability that we get exactly 2 heads?

Answer

No longer true that *all* basic outcomes are equally likely, but those with exactly two heads *still are*

$$P(HHT) = (1/3)^2(1 - 1/3) = 2/27$$

 $P(THH) = 2/27$
 $P(HTH) = 2/27$

Summing gives $2/9 \approx 0.22$

Where does the Binomial pmf come from?

Starting to see a pattern?

Suppose we flip an unfair coin 4 times, where the probability of heads is 1/3. What is the probability that we get exactly 2 heads?

Six equally likely, mutually exclusive basic outcomes make up this event:

$$\binom{4}{2}(1/3)^2(2/3)^2$$

Multiple RVs at once - Definition of Joint PMF

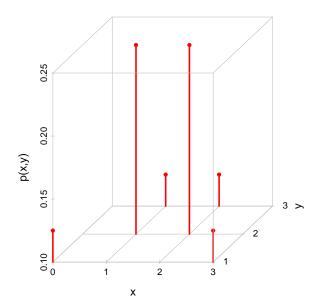
Let X and Y be discrete random variables. The joint probability mass function $p_{XY}(x,y)$ gives the probability of each pair of realizations (x,y) in the support:

$$p_{XY}(x,y) = P(X = x \cap Y = y)$$

Example: Joint PMF in Tabular Form

			Y	
		1	2	3
Х	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

Plot of Joint PMF



What is $p_{XY}(1,2)$?



			Y	
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
^	2	0	1/4	1/8
	3	1/8	0	0

$$p_{XY}(1,2) = P(X = 1 \cap Y = 2) = \frac{1}{4}$$

What is $p_{XY}(2,1)$?



			Y	
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

$$p_{XY}(2,1) = P(X = 2 \cap Y = 1) = 0$$

Properties of Joint PMF

- 1. $0 \le p_{XY}(x, y) \le 1$ for any pair (x, y)
- 2. The sum of $p_{XY}(x, y)$ over all pairs (x, y) in the support is 1:

$$\sum_{x}\sum_{y}p(x,y)=1$$

Does this satisfy the properties of a joint pmf?



$$(A = YES, B = NO)$$

			Y	
		1	2	3
V	0	1/8	0	0
	1	0	1/4	1/8
X	2	0	1/4	1/8
	3	1/8	0	0

- 1. $p(x,y) \ge 0$ for all pairs (x,y)
- 2. $\sum_{x} \sum_{y} p(x,y) = 1/8 + 1/4 + 1/8 + 1/4 + 1/8 + 1/8 = 1$

Joint versus Marginal PMFs

Joint PMF

$$p_{XY}(x,y) = P(X = x \cap Y = y)$$

Marginal PMFs

$$p_X(x) = P(X = x)$$

$$p_Y(y) = P(Y = y)$$

You can't calculate a joint pmf from marginals alone but you *can* calculate marginals from the joint!

Marginals from Joint

$$p_X(x) = \sum_{\text{all } y} p_{XY}(x, y)$$

$$p_Y(y) = \sum_{\mathsf{all} \ x} p_{XY}(x, y)$$

Why?

$$p_Y(y) = P(Y = y) = P\left(\bigcup_{\text{all } x} \{X = x \cap Y = y\}\right)$$
$$= \sum_{\text{all } x} P(X = x \cap Y = y) = \sum_{\text{all } x} p_{XY}(x, y)$$

To get the marginals sum "into the margins" of the table.

			Y		
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
					1

$$p_X(0) = 1/8 + 0 + 0 = 1/8$$

 $p_X(1) = 0 + 1/4 + 1/8 = 3/8$
 $p_X(2) = 0 + 1/4 + 1/8 = 3/8$
 $p_X(3) = 1/8 + 0 + 0 = 1/8$

What is $p_Y(2)$?



			Y		
		1	2	3	
	0	1/8	0	0	
X	1	0	1/4	1/8	
^	2	0	1/4	1/8	
	3	1/8	0	0	
		1/4	1/2	1/4	1

$$p_Y(1) = 1/8 + 0 + 0 + 1/8 = 1/4$$

 $p_Y(2) = 0 + 1/4 + 1/4 + 0 = 1/2$
 $p_Y(3) = 0 + 1/8 + 1/8 + 0 = 1/4$