Economics 103 – Statistics for Economists

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Lecture 17

Last Time

Confidence Interval for Population Mean:

$$\boxed{\bar{X}_n \pm \texttt{qnorm}(1 - \alpha/2) \times \sigma/\sqrt{n}}$$

Based on Assumptions:

1. The population standard deviation σ was known.
2. The population is normally distributed (bell-shaped).

Today

What if population is normal but σ is unknown?

We Don't know σ. What to use instead?

$$\boxed{\bar{X}_n \pm \mathtt{qnorm}(1-\alpha/2) \times \sigma/\sqrt{n}}$$

What about Sample Standard Deviation S?

$$P\left(-2 \leq \frac{\bar{X}_n - \mu}{S/\sqrt{n}} \leq 2\right) = 0.95\ ???$$

Not Quite!

Although $(\bar{X}_n - \mu)/(\sigma/\sqrt{n}) \sim N(0,1)$, $S \neq \sigma$. In fact, S is an estimator of σ so it is a random variable!

What is the sampling distribution?

Suppose $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$

$$\boxed{\frac{\bar{X}_n - \mu}{S/\sqrt{n}} \sim \ ???}$$

First Step

What is the sampling distribution of S?

What is the Distribution?

Suppose $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$. What is the distribution of this sum?

$$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2$$

(a) $\chi^2(n)$

(b) $N(\mu, \sigma^2)$

(c) $N(0, 1)$

(d) $N(\mu, \sigma^2/n)$

(e) $\chi^2(1)$

Towards the Sampling Dist. of S

If $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$, then

$$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2 \sim \chi^2(n)$$

Now:

$$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2 = \left(\frac{n-1}{\sigma^2} \right) \left[\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \mu)^2 \right] \sim \chi^2(n)$$

Anything look familiar?

Sampling Distribution of Sample Variance

Suppose $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$. Then whereas

$$\left(\frac{n-1}{\sigma^2}\right)\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\mu)^2\right] \sim \chi^2(n)$$

Replacing μ with $\bar{X}$ "loses" a degree of freedom

$$\left(\frac{n-1}{\sigma^2}\right)\left[\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2\right] = \left(\frac{n-1}{\sigma^2}\right)S^2 \sim \chi^2(n-1)$$

Ultimately, we will use this fact to work out the sampling distribution of $\sqrt{n}(\bar{X}_n - \mu)/S$, but for now let's take a detour...

95% CI for Variance of Normal Population

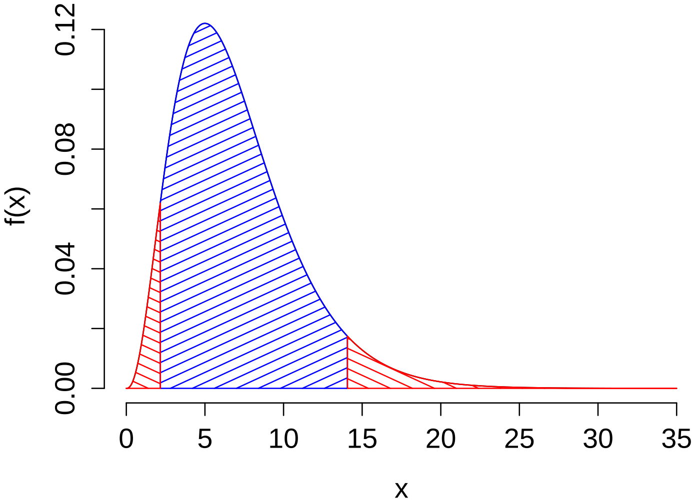
We know that:

$$\left(\frac{n-1}{\sigma^2}\right) S^2 \sim \chi^2(n-1)$$

First Step: find a, b such that

$$P\left[a \leq \left(\frac{n-1}{\sigma^2}\right) S^2 \leq b\right] = 0.95$$

Although there are many choices for a, b that would work, a sensible idea is to put 2.5% in each tail...


0.00
0.04
0.08
0.12
f(x)
0
5
10
15
20
25
30
35
x

What R command should I use to calculate a?

$$P\left[a \leq \left(\frac{n-1}{\sigma^2}\right) S^2 \leq b\right] = 0.95$$

(a) $\texttt{qchisq}(0.95, \texttt{df = } n-1)$

(b) $\texttt{qchisq}(0.025, \texttt{df = } n)$

(c) $\texttt{qchisq}(0.975, \texttt{df = } n-1)$

(d) $\texttt{qchisq}(0.025, \texttt{df = } n-1)$

(e) $\texttt{qchisq}(0.975, \texttt{df = } n)$

What R command should I use to calculate b?

$$P\left[a \leq \left(\frac{n-1}{\sigma^2}\right) S^2 \leq b\right] = 0.95$$

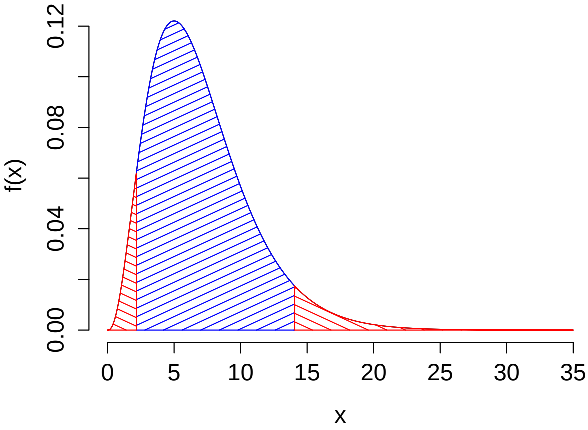
(a) `qchisq`$(0.95,$ `df =` $n-1)$

(b) `qchisq`$(0.025,$ `df =` $n)$

(c) `qchisq`$(0.975,$ `df =` $n-1)$

(d) `qchisq`$(0.025,$ `df =` $n-1)$

(e) `qchisq`$(0.975,$ `df =` $n)$



a = qchisq(0.025, df = n - 1)

b = qchisq(0.975, df = n - 1)

Step 2: After Finding a, b Rearrange

$$
\begin{aligned}
P\left[a \leq \left(\frac{n-1}{\sigma^2}\right) S^2 \leq b\right] &= 0.95 \\
P\left[\frac{a}{(n-1)S^2} \leq \frac{1}{\sigma^2} \leq \frac{b}{(n-1)S^2}\right] &= 0.95 \\
P\left[\frac{(n-1)S^2}{b} \leq \sigma^2 \leq \frac{(n-1)S^2}{a}\right] &= 0.95
\end{aligned}
$$

This CI is *not* symmetric: it *doesn't* take the form $\widehat{\theta} \pm ME$!

Example: 95% Confidence Interval for Normal Variance

$X_1, \ldots, X_{100} \sim N(\mu, \sigma^2)$. Here $n-1 = 99$, hence

$$a = \texttt{qchisq(0.025, df = 99)} \approx 73$$

$$b = \texttt{qchisq(0.975, df = 99)} \approx 128$$

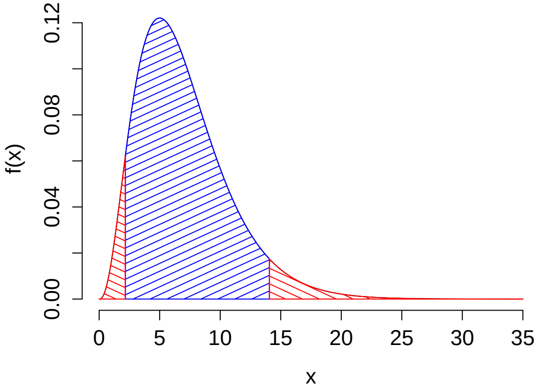
From the sample data, $s^2 = 4.3$

$$\begin{aligned} LCL &= (n-1)s^2/b = 99 \times 4.3/128 \approx 3.3 \\ UCL &= (n-1)s^2/a = 99 \times 4.3/73 \approx 5.8 \end{aligned}$$

95% CI for σ^2 is $[3.3, 5.8]$. What values are plausible?

The actual population variance in this case was 4

Arbitrary Confidence Level: $(1-\alpha)$



a $=$ `qchisq(`$\alpha/2$`, df = n - 1)`

b $=$ `qchisq(`$1-\alpha/2$`, df = n - 1)`

CI for Normal Variance

$\texttt{a} = \texttt{qchisq}(\alpha/2\texttt{, df = n - 1)}$

$\texttt{b} = \texttt{qchisq}(1-\alpha/2\texttt{, df = n - 1)}$

$$
\begin{aligned}
P\left[a \leq \left(\frac{n-1}{\sigma^2}\right) S^2 \leq b\right] &= 1-\alpha \\
P\left[\frac{a}{(n-1)S^2} \leq \frac{1}{\sigma^2} \leq \frac{b}{(n-1)S^2}\right] &= 1-\alpha \\
P\left[\frac{(n-1)S^2}{b} \leq \sigma^2 \leq \frac{(n-1)S^2}{a}\right] &= 1-\alpha
\end{aligned}
$$

CI for Normal Variance

Suppose $X_1, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$ and let:

$$
\begin{aligned}
a &= \texttt{qchisq(}\alpha/2\texttt{, df = n - 1)} \\
b &= \texttt{qchisq(}1-\alpha/2\texttt{, df = n - 1)}
\end{aligned}
$$

Then,

$$\left[\frac{(n-1)S^2}{b}, \frac{(n-1)S^2}{a}\right]$$

is a $100 \times (1-\alpha)\%$ confidence interval for σ^2.

End of Detour

We want to know the Sampling Distribution of $\sqrt{n}(\bar{X}_n - \mu)/S$ and we just saw that:

$$\boxed{\left(\frac{n-1}{\sigma^2}\right) S^2 \sim \chi^2(n-1)}$$

How can we use this fact to help us?

What is the Sampling Distribution of $\sqrt{n}(\bar{X}_n - \mu)/S$?

This slide is just algebra:

$$
\begin{aligned}
\frac{\bar{X}_n - \mu}{S/\sqrt{n}} &= \frac{\bar{X}_n - \mu}{S/\sqrt{n}} \cdot \left(\frac{\sigma/\sqrt{n}}{\sigma/\sqrt{n}}\right) = \left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right)\left(\frac{\sigma/\sqrt{n}}{S/\sqrt{n}}\right) \\
&= \left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right)\left(\frac{\sigma}{S}\right) = \left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right)\left(\sqrt{\frac{n-1}{n-1}} \cdot \sqrt{\frac{\sigma^2}{S^2}}\right) \\
&= \left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right)\left(\sqrt{\frac{(n-1)\sigma^2}{(n-1)S^2}}\right) \\
&= \frac{\left(\dfrac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right)}{\sqrt{\left[\frac{(n-1)S^2}{\sigma^2}\right]/(n-1)}}
\end{aligned}
$$

Distribution of $\sqrt{n}(\bar{X}_n - \mu)/\sigma$

Suppose $X_1, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$ and $\bar{X}_n$ is ths sample mean. Then the sampling distribution of $\sqrt{n}(\bar{X}_n - \mu)/\sigma$ is

(a) $t(n)$

(b) $t(n-1)$

(c) $\chi^2(n)$

(d) $\chi^2(n-1)$

(e) $N(\mu, \sigma^2)$

(f) $N(0, 1)$

(g) $N(\mu, \sigma^2/n)$

(h) $F(n, n-1)$

Distribution of $(n-1)S^2/\sigma^2$

Suppose $X_1, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$ and S^2 is the sample variance. Then the sampling distribution of $(n-1)S^2/\sigma^2$ is

(a) $t(n)$

(b) $t(n-1)$

(c) $\chi^2(n)$

(d) $\chi^2(n-1)$

(e) $N(\mu, \sigma^2)$

(f) $N(0, 1)$

(g) $N(\mu, \sigma^2/n)$

(h) $F(n, n-1)$

What is the Sampling Distribution?

Suppose $Z \sim N(0,1)$ independent of $Y \sim \chi^2(n-1)$. Then the sampling distribution of $Z/\sqrt{Y/(n-1)}$ is

(a) $t(n)$

(b) $t(n-1)$

(c) $\chi^2(n)$

(d) $\chi^2(n-1)$

(e) $N(\mu, \sigma^2)$

(f) $N(0,1)$

(g) $N(\mu, \sigma^2/n)$

(h) $F(n, n-1)$

What is the Sampling Distribution of $\sqrt{n}(\bar{X}_n - \mu)/S$?

From three slides back:

$$\begin{aligned}
\frac{\bar{X}_n - \mu}{S/\sqrt{n}} &= \frac{\left(\dfrac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right)}{\sqrt{\left[\frac{(n-1)S^2}{\sigma^2}\right]/(n-1)}} \\
&= \frac{N(0,1)}{\sqrt{\dfrac{\chi^2(n-1)}{n-1}}} \\
&\sim t(n-1)
\end{aligned}$$

Strictly speaking, need to show that numerator and denominator are independent, but you can take my word for it!

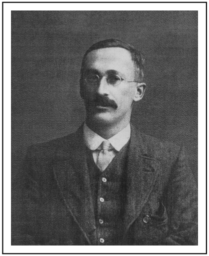
Punchline: Sampling Distribution of $\sqrt{n}(\bar{X}_n - \mu)/S$

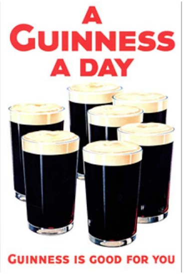
If $X_1, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$, then

$$\boxed{\frac{\bar{X}_n - \mu}{S/\sqrt{n}} \sim t(n-1)}$$

Who was "Student?"

"Guinnessometrics: The Economic Foundation of Student's t"





"Student" is the pseudonym used in 19 of 21 published articles by William Sealy Gosset, who was a chemist, brewer, inventor, and self-trained statistician, agronomer, and designer of experiments ... [Gosset] worked his entire adult life ... as an experimental brewer for one employer: Arthur Guinness, Son & Company, Ltd., Dublin, St. Jamess Gate. Gosset was a master brewer and rose in fact to the top of the top of the brewing industry: Head Brewer of Guinness.

Three Key Sampling Distributions

Suppose that $X_1, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$. Then:

$$\begin{aligned}
\left(\frac{n-1}{\sigma^2}\right) S^2 &\sim \chi^2(n-1) \\
\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} &\sim N(0,1) \\
\frac{\bar{X}_n - \mu}{S/\sqrt{n}} &\sim t(n-1)
\end{aligned}$$

CI for Mean of Normal Distribution, Popn. Var. Unknown

Same argument as we used when the variance was known, except with $t(n-1)$ rather than standard normal distribution:

$$P\left(-c \leq \frac{\bar{X}_n - \mu}{S/\sqrt{n}} \leq c\right) = 1-\alpha$$

$$P\left(\bar{X}_n - c\frac{S}{\sqrt{n}} \leq \mu \leq \bar{X}_n + c\frac{S}{\sqrt{n}}\right) = 1-\alpha$$

$c = \mathtt{qt}(1-\alpha/2, \mathtt{df} = n-1)$

$$\boxed{\bar{X}_n \pm \mathtt{qt}(1-\alpha/2, \mathtt{df} = n-1)\ \frac{S}{\sqrt{n}}}$$

Comparison of CIs for Mean of Normal Distribution

$100 \times (1-\alpha)\%$ Confidence Level

$$\boxed{X_1, \ldots, X_n \sim \text{iid } N(\mu, \sigma^2)}$$

Known Population Std. Dev. (σ)

$$\bar{X}_n \pm \texttt{qnorm}(1-\alpha/2)\,\frac{\sigma}{\sqrt{n}}$$

Unknown Population Std. Dev. (σ)

$$\bar{X}_n \pm \texttt{qt}(1-\alpha/2, \texttt{df} = n-1)\,\frac{S}{\sqrt{n}}$$

Standard Error vs. Estimator of Standard Error

Standard Error

Recall that the standard deviation of the sampling distribution of an estimator is called the *standard error (SE)* of that estimator.

Example: Standard Error of the Mean

$$SE(\bar{X}_n) = \sqrt{Var\left(\bar{X}_n\right)} = \sigma/\sqrt{n}$$

Estimator of Standard Error of the Mean

Whereas $\sigma/\sqrt{n}$ *is* the standard error of the mean, $S/\sqrt{n}$ is an *estimator* of this quantity: $\widehat{SE}(\bar{X}_n) = S/\sqrt{n}$

Writing the CIs in terms of Actual and Estimated SE

$100 \times (1 - \alpha)\%$ Confidence Level

$$\boxed{X_1, \ldots, X_n \sim \text{iid } N(\mu, \sigma^2)}$$

Known Population Std. Dev. (σ)

$$\bar{X}_n \pm \texttt{qnorm}(1 - \alpha/2)\, SE(\bar{X}_n)$$

Unknown Population Std. Dev. (σ)

$$\bar{X}_n \pm \texttt{qt}(1 - \alpha/2, \texttt{df} = n - 1)\, \widehat{SE}(\bar{X}_n)$$

Comparison of Normal and t CIs

Table : Values of $\texttt{qt}(1-\alpha/2, \texttt{df} = n-1)$ for various choices of n and α.

n	1	5	10	30	100	∞
$\alpha = 0.10$	6.31	2.02	1.81	1.70	1.66	1.64
$\alpha = 0.05$	12.71	2.57	2.23	2.04	1.98	1.96
$\alpha = 0.01$	63.66	4.03	3.17	2.75	2.63	2.58

Recall that as $n \to \infty$, $t(n-1) \to N(0,1)$

In a sense, using the t-distribution involves making a "small-sample correction." In other words, it is only when n is fairly small that this makes a practical difference for our confidence intervals.

Am I Taller Than The Average American Male?

Source: Centers for Disease Control (pg. 16)

Sample Mean	69 inches
Sample Std. Dev.	6 inches
Sample Size	5647
My Height	73 inches

$$\begin{aligned} \widehat{SE}(\bar{X}_n) &= s/\sqrt{n} \\ &= 6/\sqrt{5647} \\ &\approx 0.08 \end{aligned}$$

Assuming the population is normal,

$$\boxed{\bar{X}_n \pm \texttt{qt}(1-\alpha/2, \texttt{df} = n-1)\ \widehat{SE}(\bar{X}_n)}$$

What is the approximate value of

`qt(1-0.05/2, df = 5646)?`

For large n, $t(n-1) \approx N(0,1)$, so the answer is approximately 2

What is the ME for the 95% CI?

$ME \approx 0.16 \implies 69 \pm 0.16$