# Economics 103 – Statistics for Economists

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Lecture 15

# Sampling Distributions and Estimation – Part II

# Unbiased means “Right on Average”

## Bias of an Estimator

Let $\widehat{\theta}_n$ be a sample estimator of a population parameter $\theta_0$. The *bias* of $\widehat{\theta}_n$ is $E[\widehat{\theta}_n] - \theta_0$.

## Unbiased Estimator

A sample estimator $\widehat{\theta}_n$ of a population parameter $\theta_0$ is called *unbiased* if $E[\widehat{\theta}_n] = \theta_0$

# A Different Estimator of the Population Variance

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$$

$$E[\widehat{\sigma}^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2\right] = \frac{1}{n}E\left[\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2\right] = \frac{(n-1)\sigma^2}{n}$$

Bias of $\widehat{\sigma}^2$

$$E[\widehat{\sigma}^2] - \sigma^2 = \frac{(n-1)\sigma^2}{n} - \sigma^2 = \frac{(n-1)\sigma^2}{n} - \frac{n\sigma^2}{n} = -\sigma^2/n$$

# Procedure versus Result of the Procedure

## Procedure = Random Variable

- $X_1, \ldots, X_n$ represents procedure of taking a random sample.
- $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ represents procedure of taking sample mean

## Sampling Dist. = Probabilistic Behavior of Procedure

If I repeat the procedure of taking the mean of a random sample over and over for many samples, what relative frequencies do I get for the sample means?

## Result of Procedure = Constant

- $x_1, \ldots, x_n$ is the result of sampling, the observed data.
- $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$ is the result of taking sample mean

# Procedure? Long-Run Relative Freqencies?

Why would I advise you not to play the lottery?

- You may sometimes win, but if you play the lottery many times, on average you will lose money.
- Let $X$ be a random variable representing lottery winnings. I am arguing that $E[X]$ - Cost of Ticket $< 0$

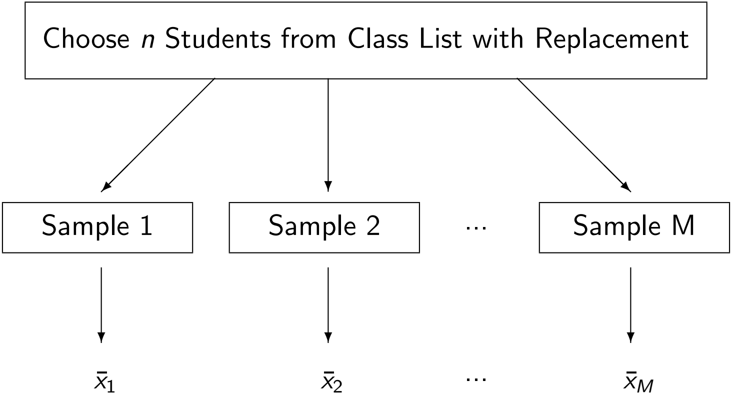
## Procedure = Random Variable

Making a habit of playing the lottery. Expectation is negative.

## Result of that Procedure = Constant

How much you win in a *particular* lottery. Could be greater than or less than cost of ticket in any *individual* instance.

# Sampling Distribution of $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$




Repeat $M$ times $\rightarrow$ get $M$ different sample means

Sampling Dist: long run relative frequencies of the $\bar{x}_i$

# Height of Econ 103 Students

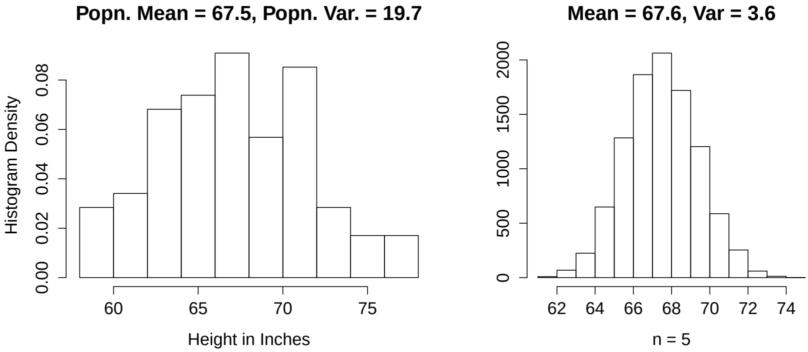



Figure : Left: Population, Right: Sampling distribution of $\bar{X}_5$

# How Large is the Average Family?

How many brothers and sisters are in your family, including yourself?

The average number of children per family was about 2.0 twenty years ago.

# What's Going On Here?

Biased Sample!

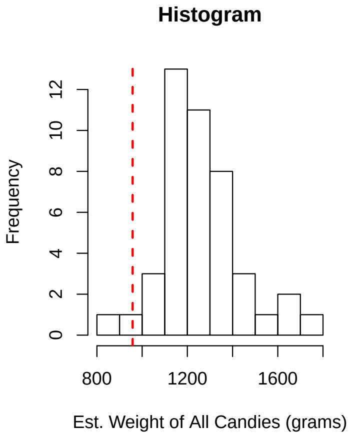
- Zero children $\Rightarrow$ didn't send any to college
- Sampling by *children* so large families oversampled

# Candy Weighing: 44 Estimates, Each With $n = 5$

$$\widehat{\theta} = 20 \times (X_1 + \ldots + X_5)$$

| Summary of Sampling Dist. | |
|---|---|
| Overestimates | 42 |
| Exactly Correct | 0 |
| Underestimates | 2 |
| $E[\hat{\theta}]$ | 1269 grams |
| $\mathrm{SD}(\widehat{\theta})$ | 189 grams |

Actual Mass: $\theta_0 =$958 grams



# What was in the bag?

100 Candies Total:

- 20 Fun Size Snickers Bars (large)
- 30 Reese's Miniatures (medium)
- 50 Tootsie Roll "Midgees" (small)

## So What Happened?

Not a random sample! The Snickers bars were *oversampled*.

## Could we have avoided this? How?

Let $X_1, X_2, \ldots X_n \sim iid$ mean $\mu$, variance $\sigma^2$ and define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. True or False:

$\bar{X}_n$ *is an unbiased estimator of* $\mu$

(a) True

(b) False

TRUE!

Let $X_1, X_2, \ldots X_n \sim iid$ mean $\mu$, variance $\sigma^2$. True or False:

*$X_1$ is an unbiased estimator of $\mu$*

(a) True

(b) False

TRUE!

# How to choose between two unbiased estimators?

Suppose $X_1, X_2, \ldots X_n \sim \textit{iid}$ with mean $\mu$ and variance $\sigma^2$

$$
\begin{aligned}
E[\bar{X}_n] &= E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \mu \\
E[X_1] &= \mu \\
Var(\bar{X}_n) &= Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \sigma^2/n \\
Var(X_1) &= \sigma^2
\end{aligned}
$$

# Efficiency - Compare Unbiased Estimators by Variance

Let $\widehat{\theta}_1$ and $\widehat{\theta}_2$ be unbiased estimators of $\theta_0$. We say that $\widehat{\theta}_1$ is *more efficient* than $\widehat{\theta}_2$ if $Var(\widehat{\theta}_1) < Var(\widehat{\theta}_2)$.

# Mean-squared Error

Except in very simple situations, unbiased estimators are hard to come by. In fact, in many interesting applications there is a *tradeoff* between bias and variance:

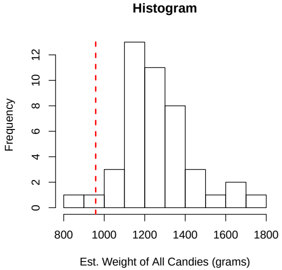
- Low bias estimators often have a high variance
- Low variance estimators often have high bias

Mean Squared Error (MSE): Squared Bias plus Variance

$$\begin{aligned} MSE(\widehat{\theta}) &= \text{Bias}(\widehat{\theta})^2 + Var(\widehat{\theta}) \\ &= \left(E[\widehat{\theta}] - \theta_0\right)^2 + Var(\widehat{\theta}) \end{aligned}$$

# Calculate MSE for Candy Experiment

| | |
|---|---|
| $E[\hat{\theta}]$ | 1269 grams |
| $\theta_0$ | 958 grams |
| $SD(\widehat{\theta})$ | 189 grams |




$$
\begin{aligned}
\text{Bias} &= 1269 \text{ grams} - 958 \text{ grams} \\
&= 311 \text{ grams} \\
\text{MSE} &= \text{Bias}^2 + \text{Variance} \\
&= (311^2 + 189^2) \text{ grams}^2 \\
&= 1.3244 \times 10^5 \text{ grams}^2 \\
\text{RMSE} &= \sqrt{\text{MSE}} = 364 \text{ grams}
\end{aligned}
$$

# Finite Sample versus Asymptotic Properties of Estimators

## Finite Sample Properties

For *fixed sample size $n$* what are the properties of the sampling distribution of $\widehat{\theta}_n$? (E.g. bias and variance.)

## Asymptotic Properties

What happens to the sampling distribution of $\widehat{\theta}_n$ *as the sample size $n$ gets larger and larger?* (That is, $n \to \infty$).

# Why Asymptotics?

### Law of Large Numbers

Make precise what we mean by "bigger samples are better."

### Central Limit Theorem

As $n \to \infty$ *nearly all* sampling distributions behave behave like a normal random variable!

# Consistency

## Consistency

If an estimator $\widehat{\theta}_n$ (which is a RV) *converges* to $\theta_0$ (a constant) as $n \to \infty$, we say that $\widehat{\theta}_n$ *is consistent for* $\theta_0$.

What does it mean for a *RV* to converge to a *constant?*

For this course we'll use *MSE Consistency*:

$$\lim_{n \to \infty} \text{MSE}(\widehat{\theta}_n) = 0$$

This makes sense since $\text{MSE}(\widehat{\theta}_n)$ is a *constant*, so this is just an ordinary limit from calculus.

# Law of Large Numbers (aka Law of Averages)

Let $X_1, X_2, \ldots X_n \sim iid$ mean $\mu$, variance $\sigma^2$. Then the sample mean

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

is consistent for the population mean $\mu$.

# Law of Large Numbers (aka Law of Averages)

Let $X_1, X_2, \ldots X_n \sim iid$ mean $\mu$, variance $\sigma^2$.

$$\begin{aligned}
E[\bar{X}_n] &= E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \mu \\
Var(\bar{X}_n) &= Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \sigma^2/n \\
\text{MSE}(\bar{X}_n) &= \text{Bias}(\bar{X}_n)^2 + Var(\bar{X}_n) \\
&= \left(E[\bar{X}_n] - \mu\right)^2 + Var(\bar{X}_n) \\
&= 0 + \sigma^2/n \\
&\to 0
\end{aligned}$$

Hence $\bar{X}_n$ is consistent for $\mu$

# Important!

An estimator *can* be biased but still consistent, as long as the bias disappears as $n \to \infty$

$$\widehat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2$$

## Bias of $\widehat{\sigma}^2$

$$E[\widehat{\sigma}^2] - \sigma^2 = \frac{(n-1)\sigma^2}{n} - \sigma^2 = -\sigma^2/n \to 0$$

Suppose $X_1, X_2, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$. What is the sampling distribution of $\bar{X}_n$?

(a) $\chi^2(n)$

(b) $t(n)$

(c) $F(n, n)$

(d) $N(\mu, \sigma^2/n)$

(e) Not enough information to determine.

But still, how can something random converge to something constant?

# Sampling Distribution of $\bar{X}_n$ Collapses to $\mu$

Look at an example where we can directly calculate not only the mean and variance of the sampling distribution of $\bar{X}_n$, but the *sampling distribution itself*:

$$X_1, X_2, \ldots, X_n \sim \text{iid } N(\mu, \sigma^2) \Rightarrow \bar{X}_n \sim N(\mu, \sigma^2/n)$$

# Sampling Distribution of $\bar{X}_n$ Collapses to $\mu$

$X_1, X_2, \ldots, X_n \sim$ iid $N(\mu, \sigma^2 \Rightarrow \bar{X}_n \sim N(\mu, \sigma^2/n)$.

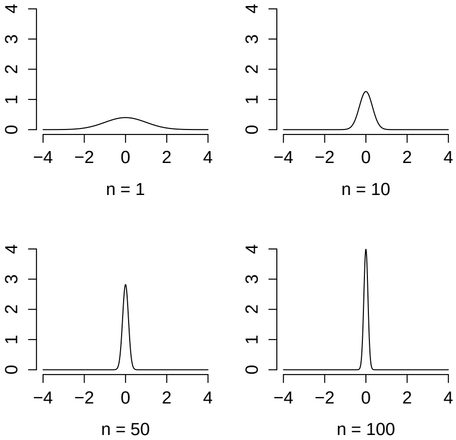



Figure : Sampling Distributions for $\bar{X}_n$ where $X_i \sim$ iid $N(0, 1)$

# Another Visualization: Keep Adding Observations

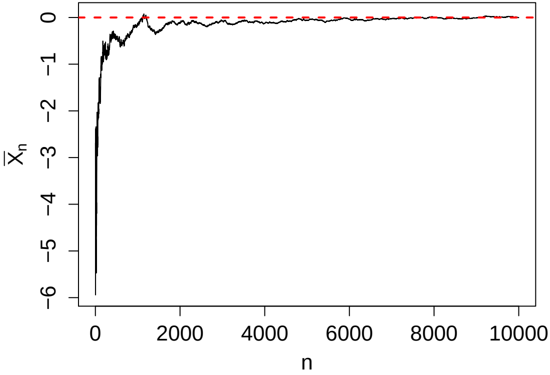



Figure : Running sample means: $X_i \sim$ iid $N(0, 100)$

| $n$ | $\bar{X}_n$ |
|---|---|
| 1 | -2.69 |
| 2 | -3.18 |
| 3 | -5.94 |
| 4 | -4.27 |
| 5 | -2.62 |
| 10 | -2.89 |
| 20 | -5.33 |
| 50 | -2.94 |
| 100 | -1.58 |
| 500 | -0.45 |
| 1000 | -0.13 |
| 5000 | -0.05 |
| 10000 | 0.00 |

# Important!

Although I showed two examples involving normal RVs, the LLN holds IN GENERAL!