Economics 103 – Statistics for Economists

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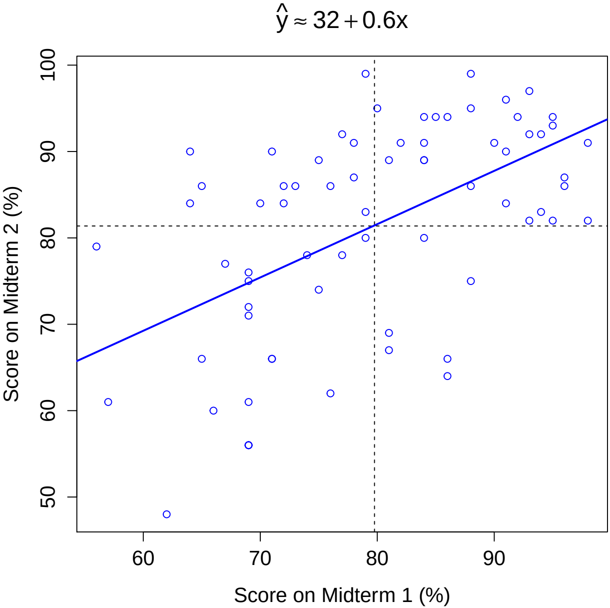
Lecture 24

Regression – Part II

Recall: "Best Fitting" Line Through Cloud of Points



Recall: Regression as a Data Summary

Linear Model

$\hat{y} = a + bx$

Choose a, b to Minimize Sum of Squared Vertical Deviations

$$\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

The Prediction

Predict score $\hat{y} = a + bx$ on second midterm for someone with score x on first.

Recall: Regression as a Data Summary

Problem

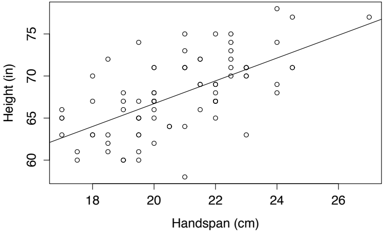
$$\min_{a,b} \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

Solution

$$\begin{aligned} b &= \frac{\sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = \frac{s_{xy}}{s_x^2} = r\frac{s_y}{s_x} \\ a &= \bar{y} - b\bar{x} \end{aligned}$$

Beyond Regression as a Data Summary

Based on a sample of Econ 103 students, we made the following graph of handspan against height, and fitted a linear regression:



The estimated slope was about 1.4 inches/cm and the estimated intercept was about 40 inches.

What if anything does this tell us about the relationship between height and handspan *in the population*?

The Population Regression Model

How is Y (height) related to X (handspan) in the population?

Assumption I: Linearity

The random variable Y is linearly related to X according to

$$Y = \beta_0 + \beta_1 X + \epsilon$$

β_0, β_1 are two unknown population parameters (constants).

Assumption II: Error Term ϵ

$E[\epsilon] = 0$, $Var(\epsilon) = \sigma^2$ and ϵ is indpendent of X. The error term ϵ measures the unpredictability of Y *after controlling for* X

Predictive Interpretation of Regression

Under Assumptions I and II

$$E[Y|X] = \beta_0 + \beta_1 X$$

- "Best guess" of Y having observed $X = x$ is $\beta_0 + \beta_1 x$
- If $X = 0$, we predict $Y = \beta_0$
- If two people differ by one unit in X, we predict that they will differ by β_1 units in Y.

The only problem is, we don't know β_0, β_1...

Estimating β_0, β_1

Suppose we observe an iid sample $(Y_1, X_1), \ldots, (Y_n, X_n)$ from the population: $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$. Then we can *estimate* β_0, β_1:

$$\widehat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}$$

$$\widehat{\beta}_0 = \bar{Y}_n - \widehat{\beta}_1 \bar{X}_n$$

Once we have estimators, we can think about sampling uncertainty...

Sampling Uncertainty: Pretend the Class is our Population

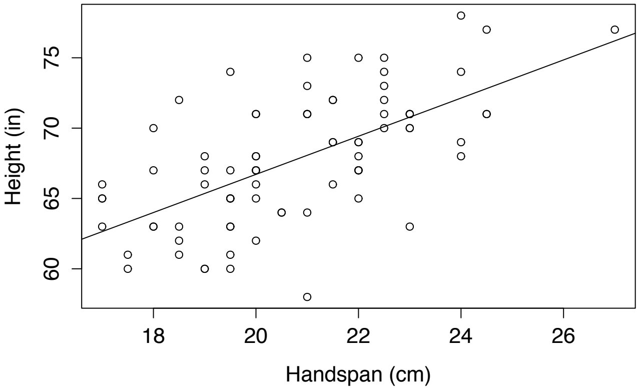
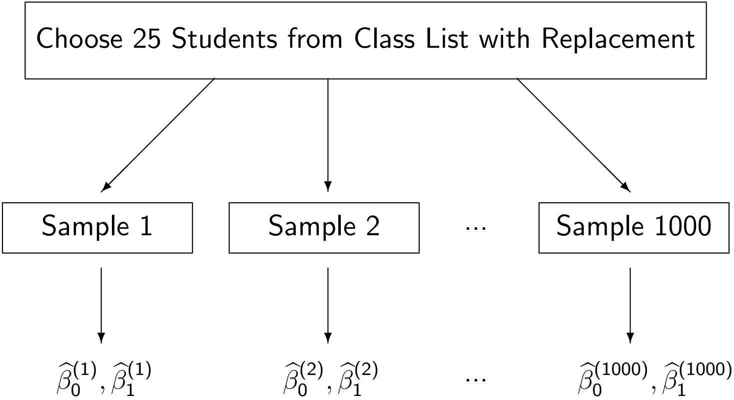


Figure : Estimated Slope = 1.4, Estimated Intercept = 40

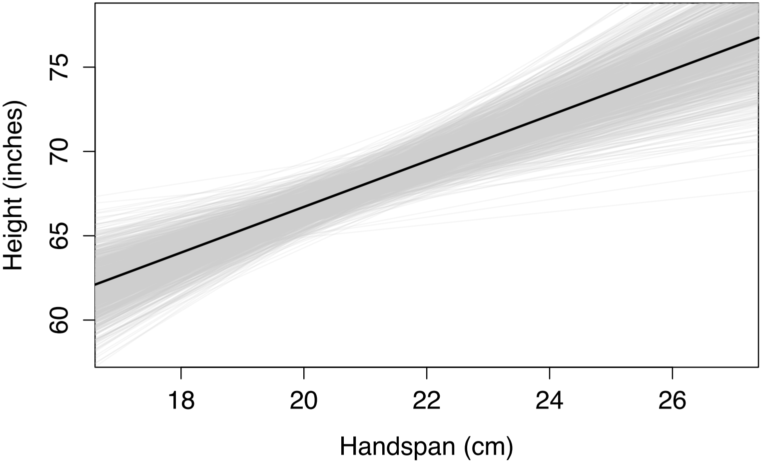
Sampling Distribution of Regression Coefficients $\widehat{\beta}_0$ and $\widehat{\beta}_1$



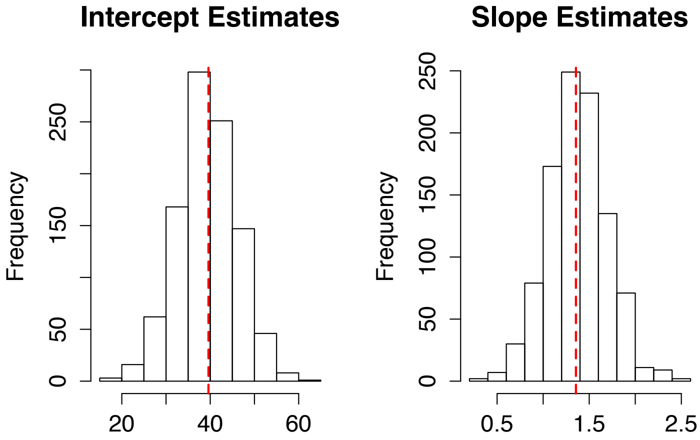
Repeat 1000 times $\rightarrow$ get 1000 different pairs of estimates

Sampling Distribution: long-run relative frequencies

1000 Replications, $n = 25$



Population: Intercept = 40, Slope = 1.4



Based on 1000 Replications, $n = 25$

Inference for Linear Regression

Central Limit Theorem

$$\frac{\widehat{\beta} - \beta}{\widehat{SE}(\widehat{\beta})} \approx N(0,1)$$

How to calculate $\widehat{SE}$?

- Complicated
  - Depends on variance of errors ϵ and all predictors in regression.
  - We'll look at a few simple examples
  - R does this calculation for us
- Requires assumptions about population errors ϵ_i
  - Simplest (and R default) is to assume $\epsilon_i \sim iid(0, \sigma^2)$
  - Weaker assumptions in Econ 104

Intuition for What Effects $SE(\widehat{\beta}_1)$ for Simple Regression

$$SE(\widehat{\beta}_1) \approx \frac{\sigma}{\sqrt{n}} \cdot \frac{1}{s_X}$$

- $\sigma = SD(\epsilon)$ – inherent variability of the Y, even after controlling for X
- n is the sample size
- s_X is the sampling variability of the X observations.

I treated the class as our population for the purposes of the simulation experiment but it makes more sense to think of the class as a sample from some population. We'll take this perspective now and think about various inferences we can draw from the height and handspan data using regression.

Height $= \beta_0 + \epsilon$

```
lm(formula = height ~ 1, data = student.data)
            coef.est coef.se
(Intercept) 67.74     0.51
---
n = 80, k = 1
```

Height $= \beta_0 + \epsilon$

```
lm(formula = height ~ 1, data = student.data)
            coef.est coef.se
(Intercept) 67.74     0.51
---
n = 80, k = 1

> mean(student.data$height)
[1] 67.7375
```

Height $= \beta_0 + \epsilon$

```
lm(formula = height ~ 1, data = student.data)
            coef.est coef.se
(Intercept) 67.74    0.51
---
n = 80, k = 1
```

```
> mean(student.data$height)
[1] 67.7375
```

```
> sd(student.data$height)/sqrt(length(student.data$height))
[1] 0.5080814
```

Dummy Variable (aka Binary Variable)

A predictor variable that takes on only two values: 0 or 1. Used to represent two categories, e.g. Male/Female.

Height $= \beta_0 + \beta_1$ Male $+\epsilon$

```
lm(formula = height ~ sex, data = student.data)
            coef.est coef.se
(Intercept) 64.46     0.56
sexMale      6.10     0.76
---
n = 80, k = 2
residual sd = 3.38, R-Squared = 0.45
```

Height $= \beta_0 + \beta_1$ Male $+\epsilon$

```
lm(formula = height ~ sex, data = student.data)
            coef.est coef.se
(Intercept) 64.46     0.56
sexMale      6.10     0.76
---
n = 80, k = 2
residual sd = 3.38, R-Squared = 0.45

> mean(male$height) - mean(female$height)
[1] 6.09868
```

Height $= \beta_0 + \beta_1$ Male $+\epsilon$

```
lm(formula = height ~ sex, data = student.data)
            coef.est coef.se
(Intercept) 64.46     0.56
sexMale      6.10     0.76
---
n = 80, k = 2
residual sd = 3.38, R-Squared = 0.45

> mean(male$height) - mean(female$height)
[1] 6.09868

> sqrt(var(male$height)/length(male$height) +
  var(female$height)/length(female$height))
[1] 0.7463796
```

Height $= \beta_0 + \beta_1$ Male $+\epsilon$

What is the ME for an approximate 95% confidence interval for the difference of population means of height: (men - women)?

```
lm(formula = height ~ sex, data = student.data)
            coef.est coef.se
(Intercept) 64.46     0.56
sexMale      6.10     0.76
---
n = 80, k = 2
residual sd = 3.38, R-Squared = 0.45
```

Height $= \beta_0 + \beta_1$ Handspan $+\epsilon$

```
lm(formula = height ~ handspan, data = student.data)
            coef.est coef.se
(Intercept) 39.60     3.96
handspan     1.36     0.19
---
n = 80, k = 2
residual sd = 3.56, R-Squared = 0.40
```

Height $= \beta_0 + \beta_1$ Handspan $+\epsilon$

What is the ME for an approximate 95% CI for β_1?

```
lm(formula = height ~ handspan, data = student.data)
            coef.est coef.se
(Intercept) 39.60    3.96
handspan     1.36    0.19
---
n = 80, k = 2
residual sd = 3.56, R-Squared = 0.40
```

Simple vs. Multiple Regression

Terminology

Y is the "outcome" and X is the "predictor."

Simple Regression

One predictor variable: $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$

Multiple Regression

More than one predictor variable:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \ldots + \beta_k X_{ik} + \epsilon_i$$

- In both cases $\epsilon_1, \epsilon_2, \ldots, \epsilon_n \sim \text{iid}(0, \sigma^2)$
- Multiple regression coefficient estimates $\widehat{\beta}_1, \widehat{\beta}_1, \ldots, \widehat{\beta}_k$ calculated by minimizing sum of squared vertical deviations, but formula requires linear algebra so we won't cover it.

Interpreting Multiple Regression

Predictive Interpretation

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \ldots + \beta_k X_{ik} + \epsilon_i$$

β_j is the difference in Y that we would predict between two individuals who differed by one unit in predictor X_j *but who had the same values for the other X variables.*

What About an Example?

In a few minutes, we'll work through an extended example of multiple regression using real data.

Inference for Multiple Regression

In addition to estimating the coefficients $\widehat{\beta}_1, \widehat{\beta}_1, \ldots, \widehat{\beta}_k$ for us, R will calculate the corresponding standard errors. It turns out that

$$\frac{\widehat{\beta}_j - \beta_j}{\widehat{SE}(\widehat{\beta})} \approx N(0,1)$$

for *each* of the $\widehat{\beta}_j$ by the CLT provided that the sample size is large.

Height $= \beta_0 + \beta_1$ Handspan $+\epsilon$

What are residual sd and R-squared?

```
lm(formula = height ~ handspan, data = student.data)
            coef.est coef.se
(Intercept) 39.60    3.96
handspan     1.36    0.19
---
n = 80, k = 2
residual sd = 3.56, R-Squared = 0.40
```

Fitted Values and Residuals

Fitted Value $\widehat{y}_i$

The value of the Y-variable that we would *predict* for person i using our estimated regression coefficients, given her values for all of the X-variables: $\widehat{y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_{i1} + \ldots + \widehat{\beta}_k x_{ik}$

Residual $\widehat{\epsilon}_i$

The difference between the actual value of the Y-variable that we observed for person i (y_i) and the value predicted from our regression model: $\widehat{\epsilon}_i = y_i - \widehat{y}_i$. The residuals are *stand-ins* for the errors ϵ_i, which we don't observe.

Residual Standard Deviation: $\widehat{\sigma}$

$$\widehat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n} \widehat{\epsilon}_i^2}{n-k}}$$

- Average distance observations fall from regression line.
- Summarizes scale of the residuals $\widehat{\epsilon}_i$
  - Suppose model predicting children's test scores has $\widehat{\sigma} = 16$.
    This model predicts to an accuracy of about 16 points.
- A measure of "unexplained variation" in Y
  - Higher values means there is more unexplained variation in Y
- Same units as Y (Exam practice: verify this)
- Denominator $(n-k)$ = "degrees of freedom" of regression
  - (# Datapoints - # of X variables)

Proportion of Variance Explained: R^2

aka Coefficient of Determination

$$R^2 = 1 - \left[\frac{\sum_{i=1}^n \widehat{\epsilon}_i^2}{\sum_{i=1}^n (y_i - \bar{y})^2}\right] \approx 1 - \frac{\widehat{\sigma^2}}{s_y^2}$$

- Measures proportion of variance in Y explained by the model.
  - Higher value means greater proportion explained
- Unitless, between 0 and 1
- People often claim "you want R^2 to be as high as possible" but this is not quite accurate...
- For simple linear regression $R^2 = (r_{xy})^2$ and this where its name comes from!

Height $= \beta_0 + \beta_1$ Handspan $+\epsilon$

```
lm(formula = height ~ handspan, data = student.data)
            coef.est coef.se
(Intercept) 39.60    3.96
handspan     1.36    0.19
---
n = 80, k = 2
residual sd = 3.56, R-Squared = 0.40
> cor(student.data$height, student.data$handspan)^2
[1] 0.3954669
```

Which Gives Better Predictions: Sex (a) or Handspan (b)?

```
lm(formula = height ~ sex, data = student.data)
            coef.est coef.se
(Intercept) 64.46     0.56
sexMale      6.10     0.76
---
n = 80, k = 2
residual sd = 3.38, R-Squared = 0.45

lm(formula = height ~ handspan, data = student.data)
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(Intercept) 39.60     3.96
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---
n = 80, k = 2
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```

Bring Your Laptop Next Time: We'll be Using R