

# Economics 103 – Statistics for Economists

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Lecture # 8

# Random Variables

*A random variable is neither random nor a variable.*

## Random Variable (RV): $X$

A deterministic (i.e. non-random) function that assigns a numeric value to each basic outcome of a random experiment.

## Realization: $x$

A particular numeric value that an RV could take on. We write  $\{X = x\}$  to refer to the *event* that the RV  $X$  took on the value  $x$ .

## Support Set (aka Support)

The set of all possible realizations of a RV.

# Random Variables (continued)

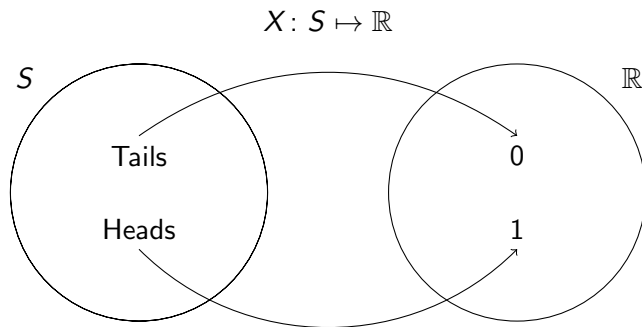
## Notation

Capital latin letters for RVs, e.g.  $X$ ,  $Y$ ,  $Z$ , and the corresponding lowercase letters for their realizations, e.g.  $x$ ,  $y$ ,  $z$ .

## Intuition

You can think of an RV as a machine that spits out random numbers: although the machine is deterministic, its inputs, the outcomes of a random experiment, are not.

## Example: Coin Flip Random Variable



**Figure :** This random variable assigns numeric values to the random experiment of flipping a fair coin once: Heads is assigned 1 and Tails 0.

Which of these is a realization of the Coin Flip RV?



- (a) Tails
- (b) 2
- (c) 0
- (d) Heads
- (e)  $1/2$

# What is the support set of the Coin Flip RV?



- (a) {Heads, Tails}
- (b)  $1/2$
- (c) 0
- (d)  $\{0, 1\}$
- (e) 1

Let  $X$  denote the Coin Flip RV



What is  $P(X = 1)$ ?

- (a) 0
- (b) 1
- (c)  $1/2$
- (d) Not enough information to determine

## Two Kinds of RVs: Discrete and Continuous

**Discrete** support set is finite or countable , e.g.  $\{0, 1\}$ ,  
 $\{\dots, -2, -1, 0, 1, 2, \dots\}$

**Continuous** support set is *uncountable* e.g.  $[-1, 1]$ ,  $\mathbb{R}$ .

Start with the discrete case since it's easier, but most of the ideas we learn will carry over to the continuous case.



# Discrete Random Variables I

## Probability Mass Function (pmf)

A function that gives  $P(X = x)$  for any realization  $x$  in the support set of a discrete RV  $X$ . We use the following notation for the pmf:

$$p(x) = P(X = x)$$

Plug in a realization  $x$ , get out a probability  $p(x)$ .

## Probability Mass Function for Coin Flip RV

$$X = \begin{cases} 0, \text{Tails} \\ 1, \text{Heads} \end{cases}$$

$$p(0) = 1/2$$

$$p(1) = 1/2$$

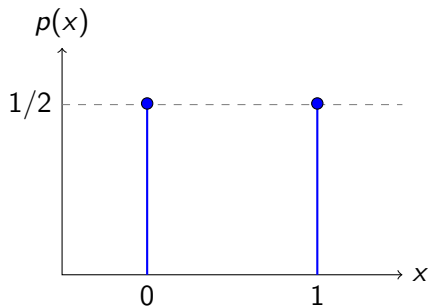


Figure : Plot of pmf for Coin Flip Random Variable

## Important Note about Support Sets

Whenever you write down the pmf of a RV, it is **crucial** to also write down its Support Set. Recall that this is the set of *all possible realizations for a RV*. Outside of the support set, all probabilities are zero. In other words, the pmf is **only defined** on the support.

# Properties of Probability Mass Functions

If  $p(x)$  is the pmf of a random variable  $X$ , then

(i)  $0 \leq p(x) \leq 1$  for all  $x$

(ii)  $\sum_{\text{all } x} p(x) = 1$

where “all  $x$ ” is shorthand for “all  $x$  in the support of  $X$ .”

# Cumulative Distribution Function (CDF)

This Def. is **the same** for continuous RVs.

The CDF gives the probability that a RV  $X$  **does not exceed** a specified threshold  $x_0$ , as a function of  $x_0$

$$F(x_0) = P(X \leq x_0)$$

**Important!**

The threshold  $x_0$  is allowed to be *any real number*. In particular, it doesn't have to be in the support of  $X$ !

## Discrete RVs: Sum the pmf to get the CDF

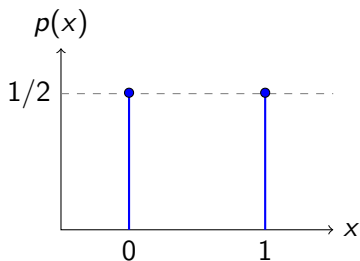
$$F(x_0) = \sum_{x \leq x_0} p(x)$$

Why?

The events  $\{X = x\}$  are mutually exclusive, so we sum to get the probability of their union for all  $x \leq x_0$ :

$$F(x_0) = P(X \leq x_0) = P\left(\bigcup_{x \leq x_0} \{X = x\}\right) = \sum_{x \leq x_0} P(X = x) = \sum_{x \leq x_0} p(x)$$

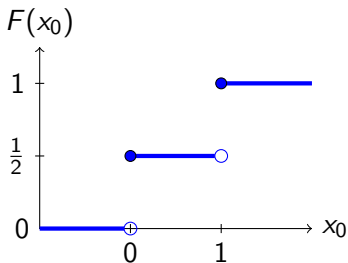
## Probability Mass Function



$$p(0) = 1/2$$

$$p(1) = 1/2$$

## Cumulative Dist. Function



$$F(x_0) = \begin{cases} 0, & x_0 < 0 \\ \frac{1}{2}, & 0 \leq x_0 < 1 \\ 1, & x_0 \geq 1 \end{cases}$$



# Properties of CDFs

These are also true for continuous RVs.

1.  $\lim_{x_0 \rightarrow \infty} F(x_0) = 1$
2.  $\lim_{x_0 \rightarrow -\infty} F(x_0) = 0$
3. Non-decreasing:  $x_0 < x_1 \Rightarrow F(x_0) \leq F(x_1)$
4. Right-continuous (“open” versus “closed” on prev. slide)

Since  $F(x_0) = P(X \leq x_0)$ , we have  $0 \leq F(x_0) \leq 1$  for all  $x_0$

# Bernoulli Random Variable – Generalization of Coin Flip

## Support Set

$\{0, 1\}$  – 1 traditionally called “success,” 0 “failure”

## Probability Mass Function

$$p(0) = 1 - p$$

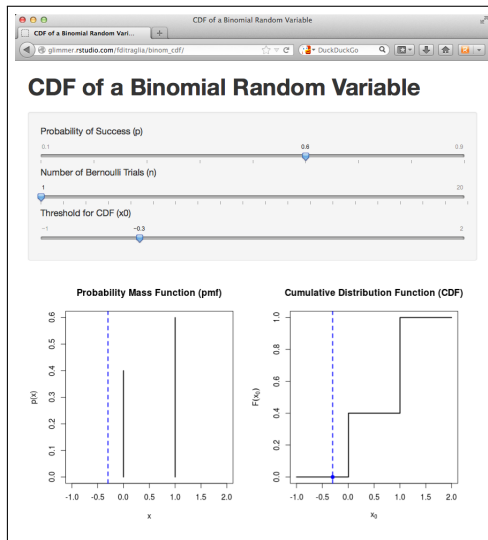
$$p(1) = p$$

## Cumulative Distribution Function

$$F(x_0) = \begin{cases} 0, & x_0 < 0 \\ 1 - p, & 0 \leq x_0 < 1 \\ 1, & x_0 \geq 1 \end{cases}$$

[http://fditraglia.shinyapps.io/binom\\_cdf/](http://fditraglia.shinyapps.io/binom_cdf/)

Set the second slider to 1 and play around with the others.



## Average Winnings Per Trial



If the realizations of the coin-flip RV were **payoffs**, how much would you expect to win per play *on average* in a long sequence of plays?

$$X = \begin{cases} \$0, \text{ Tails} \\ \$1, \text{ Heads} \end{cases}$$

## Expected Value (aka Expectation)

The expected value of a discrete RV  $X$  is given by

$$E[X] = \sum_{\text{all } x} x \cdot p(x)$$

In other words, the expected value of a discrete RV is the *probability-weighted average of its realizations*.

### Notation

We sometimes write  $\mu$  as shorthand for  $E[X]$ .

## Expected Value of Bernoulli RV

$$X = \begin{cases} 0, \text{Failure: } 1 - p \\ 1, \text{Success: } p \end{cases}$$

$$\sum_{\text{all } x} x \cdot p(x) = 0 \cdot (1 - p) + 1 \cdot p = p$$

## Your Turn to Calculate an Expected Value



Let  $X$  be a random variable with support set  $\{1, 2, 3\}$  where  $p(1) = p(2) = 1/3$ . Calculate  $E[X]$ .

## Your Turn to Caculate an Expected Value



Let  $X$  be a random variable with support set  $\{1, 2, 3\}$  where  $p(1) = p(2) = 1/3$ . Calculate  $E[X]$ .

$$E[X] = \sum_{\text{all } x} x \cdot p(x) = 1 \times 1/3 + 2 \times 1/3 + 3 \times 1/3 = 2$$



# Random Variables and Parameters

Notation:  $X \sim \text{Bernoulli}(p)$

Means  $X$  is a Bernoulli RV with  $P(X = 1) = p$  and  $P(X = 0) = 1 - p$ . The tilde is read “distributes as.”

Parameter

Any constant that appears in the definition of a RV, here  $p$ .

# Constants Versus Random Variables

This is a crucial distinction that students sometimes miss:

## Random Variables

- ▶ Suppose  $X$  is a RV – the values it takes on are random
- ▶ A function  $g(X)$  of a RV is itself a RV as we'll learn today.

## Constants

- ▶  $E[X]$  is a constant (you should convince yourself of this)
- ▶ Realizations  $x$  are constants. What is random is *which* realization the RV takes on.
- ▶ Parameters are constants (e.g.  $p$  for Bernoulli RV)
- ▶ Sample size  $n$  is a constant

# The St. Petersburg Game

## How Much Would You Pay?



How much would you be willing to pay for the right to play the following game?

*Imagine a fair coin. The coin is tossed once. If it falls heads, you receive a prize of \$2 and the game stops. If not, it is tossed again. If it falls heads on the second toss, you get \$4 and the game stops. If not, it is tossed again. If it falls heads on the third toss, you get \$8 and the game stops, and so on. The game stops after the first head is thrown. If the first head is thrown on the  $x^{\text{th}}$  toss, the prize is  $\$2^x$*

$X =$  Trial Number of First Head

$x$	$2^x$	$p(x)$	$2^x \cdot p(x)$
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$$E[Y] = \sum_{\text{all } x} 2^x \cdot p(x) =$$

$X =$  Trial Number of First Head

$x$	$2^x$	$p(x)$	$2^x \cdot p(x)$
1	2	1/2	1
2	4	1/4	1
3	8	1/8	1

$$E[Y] = \sum_{\text{all } x} 2^x \cdot p(x) =$$

$X =$  Trial Number of First Head

$x$	$2^x$	$p(x)$	$2^x \cdot p(x)$
1	2	$1/2$	1
2	4	$1/4$	1
3	8	$1/8$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n$	$2^n$	$1/2^n$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$E[Y] = \sum_{\text{all } x} 2^x \cdot p(x) =$$

$X$  = Trial Number of First Head

$x$	$2^x$	$p(x)$	$2^x \cdot p(x)$
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2	4	$1/4$	1
3	8	$1/8$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n$	$2^n$	$1/2^n$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$E[Y] = \sum_{\text{all } x} 2^x \cdot p(x) = 1 + 1 + 1 + \dots$$



$X$  = Trial Number of First Head

$x$	$2^x$	$p(x)$	$2^x \cdot p(x)$
1	2	$1/2$	1
2	4	$1/4$	1
3	8	$1/8$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n$	$2^n$	$1/2^n$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$E[Y] = \sum_{\text{all } x} 2^x \cdot p(x) = 1 + 1 + 1 + \dots = \infty$$

# Functions of Random Variables are Themselves Random Variables

## Example: Function of Bernoulli RV

Let  $Y = e^X$  where  $X \sim \text{Bernoulli}(p)$

Support of  $Y$

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Support of  $Y$

$$\{e^0, e^1\} = \{1, e\}$$

Probability Mass Function for  $Y$

## Example: Function of Bernoulli RV

Let  $Y = e^X$  where  $X \sim \text{Bernoulli}(p)$

Support of  $Y$

$$\{e^0, e^1\} = \{1, e\}$$

Probability Mass Function for  $Y$

$$p_Y(y) = \begin{cases} p & y = e \\ 1 - p & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

## Expectation: Function of Bernoulli RV

Let  $Y = e^X$  where  $X \sim \text{Bernoulli}(p)$

### Probability Mass Function for $Y$

$$p_Y(y) = \begin{cases} p & y = e \\ 1 - p & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

### Expectation of $Y = e^X$

## Expectation: Function of Bernoulli RV

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$$p_Y(y) = \begin{cases} p & y = e \\ 1 - p & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

### Expectation of $Y = e^X$

$$\sum_{y \in \{1, e\}} y \cdot p_Y(y) =$$

## Expectation: Function of Bernoulli RV

Let  $Y = e^X$  where  $X \sim \text{Bernoulli}(p)$

### Probability Mass Function for $Y$

$$p_Y(y) = \begin{cases} p & y = e \\ 1 - p & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

### Expectation of $Y = e^X$

$$\sum_{y \in \{1, e\}} y \cdot p_Y(y) = (1 - p) \cdot 1 + p \cdot e = 1 + p(e - 1)$$



## Expectation: Function of Bernoulli RV

Let  $Y = e^X$  where  $X \sim \text{Bernoulli}(p)$

### Expectation of the Function

$$\sum_{y \in \{1, e\}} y \cdot p_Y(y) = (1 - p) \cdot 1 + p \cdot e = 1 + p(e - 1)$$

### Function of the Expectation

$$e^{E[X]} = e^p$$

$$E[g(X)] \neq g(E[X])$$

(Expected value of Function  $\neq$  Function of Expected Value)

## Expectation of a Function of a Discrete RV

Let  $X$  be a random variable and  $g$  be a function. Then:

$$E[g(X)] = \sum_{\text{all } x} g(x)p(x)$$

This is how we proceeded in the St. Petersburg Game Example

Your Turn: Calculate  $E[X^2]$



$X$  has support  $\{-1, 0, 1\}$ ,  $p(-1) = p(0) = p(1) = 1/3$ .

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$$E[X^2] = \sum_{\text{all } x} x^2 p(x) = \sum_{x \in \{-1, 0, 1\}} x^2 p(x)$$

## Your Turn: Calculate $E[X^2]$



$X$  has support  $\{-1, 0, 1\}$ ,  $p(-1) = p(0) = p(1) = 1/3$ .

$$\begin{aligned} E[X^2] &= \sum_{\text{all } x} x^2 p(x) = \sum_{x \in \{-1, 0, 1\}} x^2 p(x) \\ &= (-1)^2 \cdot (1/3) + (0)^2 \cdot (1/3) + (1)^2 \cdot (1/3) \end{aligned}$$

## Your Turn: Calculate $E[X^2]$



$X$  has support  $\{-1, 0, 1\}$ ,  $p(-1) = p(0) = p(1) = 1/3$ .

$$\begin{aligned} E[X^2] &= \sum_{\text{all } x} x^2 p(x) = \sum_{x \in \{-1, 0, 1\}} x^2 p(x) \\ &= (-1)^2 \cdot (1/3) + (0)^2 \cdot (1/3) + (1)^2 \cdot (1/3) \\ &= 1/3 + 1/3 \\ &= 2/3 \approx 0.67 \end{aligned}$$