

# Problem Set #6

Econ 103

## Part I – Problems from the Textbook

Chapter 5: 1, 3, 5, 9, 11, 13, 17

Chapter 4: 19, 21, 23

(When necessary, use  $R$  rather than the Normal tables in the front of the textbook.)

## Part II – Additional Problems

1. Let  $X$  and  $Y$  be discrete random variables and  $a, b, c, d$  be constants. Prove the following:
  - (a)  $Cov(a + bX, c + dY) = bdCov(X, Y)$
  - (b)  $Corr(a + bX, c + dY) = Corr(X, Y)$

2. Fill in the missing steps from the lecture to prove the shortcut formula for covariance:

$$Cov(X, Y) = E[XY] - E[X]E[Y]$$

3. Let  $X_1$  be a random variable denoting the returns of stock 1, and  $X_2$  be a random variable denoting the returns of stock 2. Accordingly let  $\mu_1 = E[X_1]$ ,  $\mu_2 = E[X_2]$ ,  $\sigma_1^2 = Var(X_1)$ ,  $\sigma_2^2 = Var(X_2)$  and  $\rho = Corr(X_1, X_2)$ . A *portfolio*,  $\Pi$ , is a linear combination of  $X_1$  and  $X_2$  with weights that sum to one, that is  $\Pi(\omega) = \omega X_1 + (1 - \omega)X_2$ , indicating the proportions of stock 1 and stock 2 that an investor holds. In this example, we require  $\omega \in [0, 1]$ , so that *negative* weights are not allowed. (This rules out short-selling.)
  - (a) Calculate  $E[\Pi(\omega)]$  in terms of  $\omega$ ,  $\mu_1$  and  $\mu_2$ .
  - (b) If  $\omega \in [0, 1]$  is it possible to have  $E[\Pi(\omega)] > \mu_1$  and  $E[\Pi(\omega)] > \mu_2$ ? What about  $E[\Pi(\omega)] < \mu_1$  and  $E[\Pi(\omega)] < \mu_2$ ? Explain.
  - (c) Express  $Cov(X_1, X_2)$  in terms of  $\rho$  and  $\sigma_1$ ,  $\sigma_2$ .
  - (d) What is  $Var[\Pi(\omega)]$ ? (Your answer should be in terms of  $\rho$ ,  $\sigma_1^2$  and  $\sigma_2^2$ .)

- (e) Using part (d) show that the value of  $\omega$  that minimizes  $Var[\Pi(\omega)]$  is

$$\omega^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

In other words,  $\Pi(\omega^*)$  is the *minimum variance portfolio*.

- (f) If you want a challenge, check the second order condition from part (e).

4. Suppose that  $X$  is a random variable with the following PDF

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

- (a) Graph the PDF of  $X$ .
  - (b) Show that  $\int_{-\infty}^{\infty} f(x) dx = 1$ .
  - (c) What is  $\mathbb{P}(0.5 < X < 1.5)$ ?
5. A random variable is said to follow a  $\text{Uniform}(a, b)$  distribution if it is equally likely to take on any value in the range  $[a, b]$  and never takes a value outside this range. Suppose that  $X$  is such a random variable, i.e.  $X \sim \text{Uniform}(a, b)$ .
- (a) What is the support of  $X$ ?
  - (b) Explain why the PDF of  $X$  is  $f(x) = 1/(b - a)$  for  $a \leq x \leq b$ , zero elsewhere.
  - (c) Using the PDF from part (b), calculate the CDF of  $X$ .
  - (d) Verify that  $f(x) = F'(x)$  for the present example.
  - (e) Calculate  $E[X]$ .
  - (f) Calculate  $E[X^2]$ . *Hint:* recall that  $b^3 - a^3$  can be factorized as  $(b - a)(b^2 + a^2 + ab)$ .
  - (g) Using the shortcut formula and parts (e) and (f), show that  $Var(X) = (b - a)^2/12$ .
6. Suppose that  $X \sim N(0, 16)$  independent of  $Y \sim N(2, 4)$ . Recall that when our convention is to express the normal distribution in terms of its mean and variance, i.e.  $N(\mu, \sigma^2)$ . Hence,  $X$  has a mean of zero and variance of 16, while  $Y$  has a mean of 2 and a variance of 4. In completing some parts of this question you will need to use the R function `pnorm` described in class. In this case, please write down the command you used as well as the numeric result.
- (a) Calculate  $P(-8 \leq X \leq 8)$ .
  - (b) Calculate  $P(0 \leq Y \leq 4)$ .
  - (c) Calculate  $P(-1 \leq Y \leq 6)$ .
  - (d) Calculate  $P(X \geq 10)$ .