Economics 103 – Statistics for Economists

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Lecture # 8

Recall: Properties of Probability Mass Functions

If $p(x)$ is the pmf of a random variable X, then

(i) $0 \leq p(x) \leq 1$ for all x

(ii) $\displaystyle\sum_{\text{all } x} p(x) = 1$

where "all x" is shorthand for "all x in the support of X."

Cumulative Distribution Function (CDF)

This Def. is the same for continuous RVs.

The CDF gives the probability that a RV X does not exceed a specified threshold x_0, as a function of x_0

$$F(x_0) = P(X \leq x_0)$$

Important!

The threshold x_0 is allowed to be *any real number*. In particular, it doesn't have to be in the support of X!

Discrete RVs: Sum the pmf to get the CDF

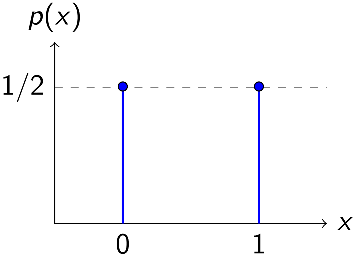
$$\boxed{F(x_0) = \sum_{x \leq x_0} p(x)}$$

Why?

The events $\{X = x\}$ are mutually exclusive, so we sum to get the probability of their union for all $x \leq x_0$:

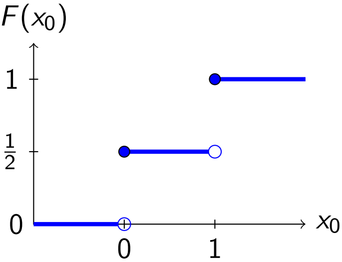
$$F(x_0) = P(X \leq x_0) = P\left(\bigcup_{x \leq x_0} \{X = x\}\right) = \sum_{x \leq x_0} P(X = x) = \sum_{x \leq x_0} p(x)$$

Probability Mass Function



$$
\begin{aligned}
p(0) &= 1/2 \\
p(1) &= 1/2
\end{aligned}
$$

Cumulative Dist. Function



$$
F(x_0) = \begin{cases} 0, & x_0 < 0 \\ \frac{1}{2}, & 0 \le x_0 < 1 \\ 1, & x_0 \ge 1 \end{cases}
$$

Properties of CDFs

These are also true for continuous RVs.

1. $\lim_{x_0 \to \infty} F(x_0) = 1$
2. $\lim_{x_0 \to -\infty} F(x_0) = 0$
3. Non-decreasing: $x_0 < x_1 \Rightarrow F(x_0) \leq F(x_1)$
4. Right-continuous ("open" versus "closed" on prev. slide)

Since $F(x_0) = P(X \leq x_0)$, we have $0 \leq F(x_0) \leq 1$ for all x_0

Bernoulli Random Variable – Generalization of Coin Flip

Support Set

$\{0, 1\}$ – 1 traditionally called "success," 0 "failure"

Probability Mass Function

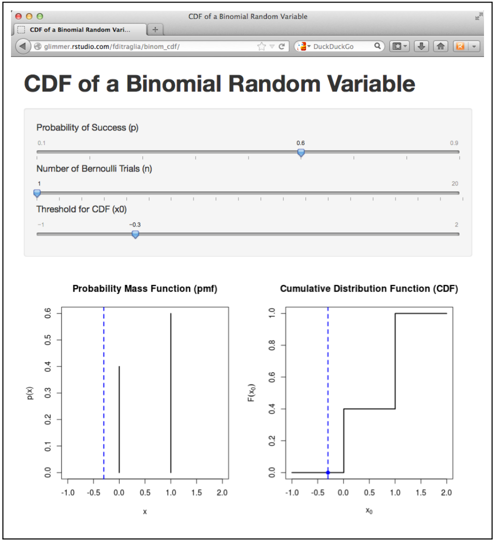
$$
\begin{aligned}
p(0) &= 1 - p \\
p(1) &= p
\end{aligned}
$$

Cumulative Distribution Function

$$
F(x_0) = \begin{cases} 0, & x_0 < 0 \\ 1 - p, & 0 \leq x_0 < 1 \\ 1, & x_0 \geq 1 \end{cases}
$$

http://fditraglia.shinyapps.io/binom_cdf/

Set the second slider to 1 and play around with the others.



Average Winnings Per Trial

If the realizations of the coin-flip RV were payoffs, how much would you expect to win per play *on average* in a long sequence of plays?

$$X = \begin{cases} \$0, \text{Tails} \\ \$1, \text{Heads} \end{cases}$$

Expected Value (aka Expectation)

The expected value of a discrete RV X is given by

$$E[X] = \sum_{\text{all } x} x \cdot p(x)$$

In other words, the expected value of a discrete RV is the *probability-weighted average of its realizations*.

Notation

We sometimes write μ as shorthand for $E[X]$.

Expected Value of Bernoulli RV

$$X = \begin{cases} 0, \text{Failure: } 1 - p \\ 1, \text{Success: } p \end{cases}$$

$$\sum_{\text{all } x} x \cdot p(x) = 0 \cdot (1 - p) + 1 \cdot p = p$$

Your Turn to Caculate an Expected Value

Let X be a random variable with support set $\{1, 2, 3\}$ where $p(1) = p(2) = 1/3$. Calculate $E[X]$.

$$E[X] = \sum_{\text{all } x} x \cdot p(x) = 1 \times 1/3 + 2 \times 1/3 + 3 \times 1/3 = 2$$

Random Variables and Parameters

Notation: $X \sim \text{Bernoulli}(p)$

Means X is a Bernoulli RV with $P(X = 1) = p$ and $P(X = 0) = 1 - p$. The tilde is read "distributes as."

Parameter

Any constant that appears in the definition of a RV, here p.

Constants Versus Random Variables

This is a crucial distinction that students sometimes miss:

Random Variables

- Suppose X is a RV – the values it takes on are random
- A function $g(X)$ of a RV is itself a RV as we'll learn today.

Constants

- $E[X]$ is a constant (you should convince yourself of this)
- Realizations x are constants. What is random is *which* realization the RV takes on.
- Parameters are constants (e.g. p for Bernoulli RV)
- Sample size n is a constant

The St. Petersburg Game

How Much Would You Pay?

How much would you be willing to pay for the right to play the following game?

> *Imagine a fair coin. The coin is tossed once. If it falls heads, you receive a prize of \$2 and the game stops. If not, it is tossed again. If it falls heads on the second toss, you get \$4 and the game stops. If not, it is tossed again. If it falls heads on the third toss, you get \$8 and the game stops, and so on. The game stops after the first head is thrown. If the first head is thrown on the x^{th} toss, the prize is* $\$2^x$

X = Trial Number of First Head

x	2^x	$p(x)$	$2^x \cdot p(x)$
1	2	1/2	1
2	4	1/4	1
3	8	1/8	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	2^n	$1/2^n$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$E[Y] = \sum_{\text{all } x} 2^x \cdot p(x) = 1 + 1 + 1 + \ldots = \infty$$

Functions of Random Variables are Themselves Random Variables

Example: Function of Bernoulli RV

$$\boxed{\text{Let } Y = e^X \text{ where } X \sim \text{Bernoulli}(p)}$$

Support of Y

$\{e^0, e^1\} = \{1, e\}$

Probability Mass Function for Y

$$p_Y(y) = \begin{cases} p & y = e \\ 1 - p & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

Expectation: Function of Bernoulli RV

$\boxed{\text{Let } Y = e^X \text{ where } X \sim \text{Bernoulli}(p)}$

Probability Mass Function for Y

$$p_Y(y) = \begin{cases} p & y = e \\ 1 - p & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

Expectation of $Y = e^X$

$$\sum_{y \in \{1, e\}} y \cdot p_Y(y) = (1 - p) \cdot 1 + p \cdot e = 1 + p(e - 1)$$

Expectation: Function of Bernoulli RV

$\boxed{\text{Let } Y = e^X \text{ where } X \sim \text{Bernoulli}(p)}$

Expectation of the Function

$$\sum_{y \in \{1,e\}} y \cdot p_Y(y) = (1-p) \cdot 1 + p \cdot e = 1 + p(e-1)$$

Function of the Expectation

$$e^{E[X]} = e^p$$

$$E[g(X)] \neq g(E[X])$$

(Expected value of Function $\neq$ Function of Expected Value)

Expectation of a Function of a Discrete RV

Let X be a random variable and g be a function. Then:

$$\boxed{E[g(X)] = \sum_{\text{all } x} g(x)p(x)}$$

This is how we proceeded in the St. Petersburg Game Example

Your Turn: Calculate $E[X^2]$

X has support $\{-1, 0, 1\}$, $p(-1) = p(0) = p(1) = 1/3$.

$$
\begin{aligned}
E[X^2] &= \sum_{\text{all } x} x^2 p(x) = \sum_{x \in \{-1,0,1\}} x^2 p(x) \\
&= (-1)^2 \cdot (1/3) + (0)^2 \cdot (1/3) + (1)^2 \cdot (1/3) \\
&= 1/3 + 1/3 \\
&= 2/3 \approx 0.67
\end{aligned}
$$

Linearity of Expectation

Holds for Continuous RVs as well, but proof is different.

Let X be a RV and a, b be constants. Then:

$$E[a + bX] = a + bE[X]$$

This is one of the most important facts in the course: the special case in which $E[g(X)] = g(E[X])$ is $g = a + bX$.

Example: Linearity of Expectation

Let $X \sim \text{Bernoulli}(1/3)$ and define $Y = 3X + 2$

1. What is $E[X]$? $E[X] = 0 \times 2/3 + 1 \times 1/3 = 1/3$
2. What is $E[Y]$? $E[Y] = E[3X + 2] = 3E[X] + 2 = 3$

Proof: Linearity of Expectation For Discrete RV

$$
\begin{aligned}
E[a + bX] &= \sum_{\text{all } x} (a + bx)p(x) \\
&= \sum_{\text{all } x} p(x) \cdot a + \sum_{\text{all } x} p(x) \cdot bx \\
&= a \sum_{\text{all } x} p(x) + b \sum_{\text{all } x} x \cdot p(x) \\
&= a + bE[X]
\end{aligned}
$$