

# Economics 103 – Statistics for Economists

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Lecture # 10

# Discrete RVs – Part III

# Definition of Conditional PMF

How does the distribution of  $y$  change with  $x$ ?

$$p_{Y|X}(y|x) = P(Y = y|X = x)$$

Which of these is the formula for  $p_{Y|X}(y|x)$ ?



You can figure this out from what you already know about probability, using the definition  $p_{Y|X}(y|x) = P(Y = y|X = x)$

- (a)  $p_X(x)/p_Y(y)$
- (b)  $p_{XY}(x, y)/p_X(x)$
- (c)  $p_X(x)p_{XY}(x, y)$
- (d)  $p_{XY}(x, y)/p_Y(y)$
- (e)  $p_Y(y)/p_X(x)$

## Conditional PMF from Joint and Marginal

$$p_{Y|X}(y|x) = P(Y = y|X = x) = \frac{P(Y = y \cap X = x)}{P(X = x)} = \frac{p_{XY}(x, y)}{p_X(x)}$$

Hence,

$$p_{Y|X}(y|x) = \frac{p_{XY}(x, y)}{p_X(x)}$$

and similarly,

$$p_{X|Y}(x|y) = \frac{p_{XY}(x, y)}{p_Y(y)}$$

## Conditional PMF of $Y$ given $X = 2$

		$Y$			
		1	2	3	
$X$	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8

$$p_{Y|X}(1|2) = \frac{p_{XY}(2,1)}{p_X(2)} = \frac{0}{3/8} = 0$$

$$p_{Y|X}(2|2) = \frac{p_{XY}(2,2)}{p_X(2)} = \frac{1/4}{3/8} = 2/3$$

$$p_{Y|X}(3|2) = \frac{p_{XY}(2,3)}{p_X(2)} = \frac{1/8}{3/8} = 1/3$$

What is  $p_{X|Y}(1|2)$ ?



		Y			
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	
		1/4	1/2	1/4	

$$p_{X|Y}(0|2) = \frac{p_{XY}(0,2)}{p_Y(2)} = \frac{0}{1/2} = 0$$

$$p_{X|Y}(1|2) = \frac{p_{XY}(1,2)}{p_Y(2)} = \frac{1/4}{1/2} = 1/2$$

$$p_{X|Y}(2|2) = \frac{p_{XY}(2,2)}{p_Y(2)} = \frac{1/4}{1/2} = 1/2$$

$$p_{X|Y}(3|2) = \frac{p_{XY}(3,2)}{p_Y(2)} = \frac{0}{1/2} = 0$$

# Independent RVs

## Definition

We say that two discrete RVs are **independent** if and only if their joint pmf equals the product of their marginal pmfs:

$$p_{XY}(x, y) = p_X(x)p_Y(y)$$

for all pairs  $(x, y)$  in the support.

## In Terms of Conditional PMF

From the previous slide, it follows that an equivalent definition of independence is that both conditional pmfs equal the corresponding marginal pmfs:  $p_{Y|X}(y|x) = p_Y(y)$  and  $p_{X|Y}(x|y) = p_X(x)$  for all  $(x, y)$  in the support.



# Are $X$ and $Y$ Independent?



(A = YES, B = NO)

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

$$p_{XY}(2, 1) = 0$$

$$p_X(2) \times p_Y(1) = (3/8) \times (1/4) \neq 0$$

Therefore  $X$  and  $Y$  are *not* independent.

# Conditional Expectation

## Intuition

$E[Y|X]$  is our “best guess” of the realization that  $Y$  will take on having observed the realization of  $X$ .

## $E[Y|X]$ is a Random Variable

Unlike  $E[Y]$  which is a constant,  $E[Y|X]$  is a function of  $X$ , hence it is a **Random Variable**.

## $E[Y|X = x]$ is a Constant

To get a “best guess” for  $Y$ , we plug in the realization we observed for  $X$ :  $E[Y|X = x]$  is a constant, our guess of the realization of  $Y$ .

## Calculating $E[Y|X = x]$

Take the mean of the conditional pmf of  $Y$  given  $X = x$ .

## Conditional Expectation: $E[Y|X = 2]$

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

We showed above that the conditional pmf of  $Y|X = 2$  is:

$$p_{Y|X}(1|2) = 0 \quad p_{Y|X}(2|2) = 2/3 \quad p_{Y|X}(3|2) = 1/3$$

Hence

$$E[Y|X = 2] = 2 \times 2/3 + 3 \times 1/3 = 7/3$$

## Conditional Expectation: $E[Y|X = 0]$

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

The conditional pmf of  $Y|X = 0$  is

$$p_{Y|X}(1|0) = 1 \quad p_{Y|X}(2|0) = 0 \quad p_{Y|X}(3|0) = 0$$

Hence  $E[Y|X = 0] = 1$

Calculate  $E[Y|X = 3]$

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

The conditional pmf of  $Y|X = 3$  is

$$p_{Y|X}(1|3) = 1 \quad p_{Y|X}(2|3) = 0 \quad p_{Y|X}(3|3) = 0$$

Hence  $E[Y|X = 3] = 1$

Calculate  $E[Y|X = 1]$



		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

The conditional pmf of  $Y|X = 1$  is

$$p_{Y|X}(1|1) = 0 \quad p_{Y|X}(2|1) = 2/3 \quad p_{Y|X}(3|1) = 1/3$$

Hence

$$E[Y|X = 1] = 2 \times 2/3 + 3 \times 1/3 = 7/3$$

## $E[Y|X]$ is a Random Variable

In this particular example we have seen that:

$$E[Y|X] = \begin{cases} 1 & X = 0 \\ 7/3 & X = 1 \\ 7/3 & X = 2 \\ 1 & X = 3 \end{cases}$$

But from above we know the marginal distribution of  $X$  :

$$P(X = 0) = 1/8 \quad P(X = 1) = 3/8$$

$$P(X = 2) = 3/8 \quad P(X = 3) = 1/8$$

Therefore,  $E[Y|X]$  is a RV that takes on the value 1 with probability 1/4 and the value 7/3 with probability 3/4.

# The Law of Iterated Expectations

Since  $E[Y|X]$  is a random variable, we can ask what its expectation is. It turns out that, for any RVs  $X$  and  $Y$

$$E[E[Y|X]] = E[Y]$$

and this is called the **Law of Iterated Expectations**. I've posted a proof [HERE](#) for those who are interested.

This will be helpful in Econ 104...



# Law of Iterated Expectations for Our Example

Marginal pmf of  $Y$

$$P(Y = 1) = 1/4$$

$$P(Y = 2) = 1/2$$

$$P(Y = 3) = 1/4$$

$$\begin{aligned} E[Y] &= 1 \times 1/4 + 2 \times 1/2 + 3 \times 1/4 \\ &= 2 \end{aligned}$$

$E[Y|X]$

$$E[Y|X] = \begin{cases} 1 & \text{w/ prob. } 1/4 \\ 7/3 & \text{w/ prob. } 3/4 \end{cases}$$

$$\begin{aligned} E[E[Y|X]] &= 1 \times 1/4 + 7/3 \times 3/4 \\ &= 2 \end{aligned}$$

## Expectation of Function of Two Discrete RVs

$$E[g(X, Y)] = \sum_x \sum_y g(x, y) p_{XY}(x, y)$$

# Some Extremely Important Examples

Same For Continuous Random Variables

Let  $\mu_X = E[X]$ ,  $\mu_Y = E[Y]$

Covariance

$$\sigma_{XY} = \text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

Correlation

$$\rho_{XY} = \text{Corr}(X, Y) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

## Shortcut Formula for Covariance

Much easier for calculating:

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

We'll talk more about this in an upcoming lecture...

## Calculating $\text{Cov}(X, Y)$

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

$$E[X] = 3/8 + 2 \times 3/8 + 3 \times 1/8 = 3/2$$

$$E[Y] = 1/4 + 2 \times 1/2 + 3 \times 1/4 = 2$$

$$\begin{aligned} E[XY] &= 1/4 \times (2 + 4) + 1/8 \times (3 + 6 + 3) \\ &= 3 \end{aligned}$$

$$\begin{aligned} \text{Cov}(X, Y) &= E[XY] - E[X]E[Y] \\ &= 3 - 3/2 \times 2 = 0 \end{aligned}$$

$$\text{Corr}(X, Y) = \text{Cov}(X, Y) / [SD(X)SD(Y)] = 0$$

Hence, zero covariance (correlation)  
does *not* imply independence!

However, independence *does* imply  
zero covariance (correlation)

I've posted a proof [HERE](#) for those who are interested.

# Linearity of Expectation, Again

Holds for Continuous RVs as well, but different proof.

In general,  $E[g(X, Y)] \neq g(E[X], E[Y])$ . The key exception is when  $g$  is a linear function:

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

where  $X, Y$  are random variables and  $a, b, c$  are constants. I've posted a proof [HERE](#) for those who are interested.



## Application: Shortcut Formula for Variance

By the Linearity of Expectation,

$$\begin{aligned}\text{Var}(X) &= E[(X - \mu)^2] = E[X^2 - 2\mu X + \mu^2] \\ &= E[X^2] - 2\mu E[X] + \mu^2 \\ &= E[X^2] - 2\mu^2 + \mu^2 \\ &= E[X^2] - \mu^2\end{aligned}$$

We saw in a previous lecture that it's typically much easier to calculate variances using the shortcut formula.

## Another Application: Shortcut Formula for Covariance

Similar to Shortcut for Variance: in fact  $\text{Var}(X) = \text{Cov}(X, X)$

$$\begin{aligned}\text{Cov}(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\ &= E[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y] \\ &\quad \vdots \\ &= E[XY] - E[X]E[Y]\end{aligned}$$

You'll fill in the details for homework...

## Expected Value of Sum = Sum of Expected Values

Repeatedly applying the linearity of expectation,

$$E[X_1 + X_2 + \dots + X_n] = E[X_1] + E[X_2] + \dots + E[X_n]$$

regardless of how the RVs  $X_1, \dots, X_n$  are related to each other. In particular it **doesn't matter if they're dependent or independent.**

# Independent and Identically Distributed (iid) RVs

## Example

$$X_1, X_2, \dots, X_n \sim \text{iid Bernoulli}(p)$$

## Independent

Joint pmf equals product of marginal pmfs (see last lecture):  
Knowing the realization of one of the RVs gives no information about the others.

## Identically Distributed

Each  $X_i$  is the same kind of RV, with the same values for any parameters. (Hence same pmf, cdf, mean, variance, etc.)

# Binomial( $n, p$ ) Random Variable

## Definition

Sum of  $n$  independent Bernoulli RVs, each with probability of “success,” i.e. 1, equal to  $p$

## Parameters

$p$  = probability of “success,”  $n$  = # of trials

## Support

$\{0, 1, 2, \dots, n\}$

## Using Our New Notation

Let  $X_1, X_2, \dots, X_n \sim \text{iid Bernoulli}(p)$ ,  $Y = X_1 + X_2 + \dots + X_n$ .

Then  $Y \sim \text{Binomial}(n, p)$ .

Which of these is the PMF of a Binomial( $n, p$ ) RV?



(a)  $p(x) = p^x(1 - p)^{n-x}$

(b)  $p(x) = \binom{n}{x} p^x(1 - p)^{n-x}$

(c)  $p(x) = \binom{x}{n} p^x$

(d)  $p(x) = \binom{n}{x} p^{n-x}(1 - p)^x$

(e)  $p(x) = p^n(1 - p)^x$

$$p(x) = \binom{n}{x} p^x(1 - p)^{n-x}$$

## Expected Value of Binomial RV

Use the fact that a Binomial( $n, p$ ) RV is defined as the sum of  $n$  iid Bernoulli( $p$ ) Random Variables and the Linearity of Expectation:

$$\begin{aligned} E[Y] &= E[X_1 + X_2 + \dots + X_n] = E[X_1] + E[X_2] + \dots + E[X_n] \\ &= p + p + \dots + p \\ &= np \end{aligned}$$

Extremely Important:

Variance of Sum  $\neq$  Sum of Variances!



## Variance of a Sum

$$\begin{aligned}\text{Var}(aX + bY) &= E \left[ \{(aX + bY) - E[aX + bY]\}^2 \right] \\&= E \left[ \{a(X - \mu_X) + b(Y - \mu_Y)\}^2 \right] \\&= E \left[ a^2(X - \mu_X)^2 + b^2(Y - \mu_Y)^2 + 2ab(X - \mu_X)(Y - \mu_Y) \right] \\&= a^2 E[(X - \mu_X)^2] + b^2 E[(Y - \mu_Y)^2] + 2ab E[(X - \mu_X)(Y - \mu_Y)] \\&= a^2 \text{Var}(X) + b^2 \text{Var}(Y) + 2ab \text{Cov}(X, Y)\end{aligned}$$

Since  $\sigma_{XY} = \rho\sigma_X\sigma_Y$ , this is sometimes written as:

$$\text{Var}(aX + bY) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y$$

$$\text{Independence} \Rightarrow \text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

As explained above, if  $X$  and  $Y$  are independent,  $\text{Cov}(X, Y) = 0$ .

Hence, independence implies

$$\begin{aligned}\text{Var}(X + Y) &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) \\ &= \text{Var}(X) + \text{Var}(Y)\end{aligned}$$

This is also true for more than two RVs

If  $X_1, X_2, \dots, X_n$  are independent, then

$$\text{Var}(X_1 + X_2 + \dots + X_n) = \text{Var}(X_1) + \text{Var}(X_2) + \dots + \text{Var}(X_n)$$

# Crucial Distinction

## Expected Value

It is **always** true that

$$E[X_1 + X_2 + \dots + X_n] = E[X_1] + E[X_2] + \dots + E[X_n]$$

## Variance

It is **not true in general** that

$$\text{Var}[X_1 + X_2 + \dots + X_n] = \text{Var}[X_1] + \text{Var}[X_2] + \dots + \text{Var}[X_n]$$

but it **is true** in the special case where  $X_1, \dots, X_n$  are independent.

# Variance of Binomial Random Variable

## Definition from Sequence of Bernoulli Trials

If  $X_1, X_2, \dots, X_n \sim \text{iid Bernoulli}(p)$  then

$$Y = X_1 + X_2 + \dots + X_n \sim \text{Binomial}(n, p)$$

## Using Independence

$$\begin{aligned} \text{Var}[Y] &= \text{Var}[X_1 + X_2 + \dots + X_n] \\ &= \text{Var}[X_1] + \text{Var}[X_2] + \dots + \text{Var}[X_n] \\ &= p(1 - p) + p(1 - p) + \dots + p(1 - p) \\ &= np(1 - p) \end{aligned}$$