Problem Set #6

Econ 103

Part I – Problems from the Textbook

Chapter 5: 1, 3, 5, 9, 11, 13, 17
Chapter 4: 19, 21, 23
(*When necessary, use R rather than the Normal tables in the front of the textbook.*)

Part II – Additional Problems

1. Let X and Y be discrete random variables and a, b, c, d be constants. Prove the following:
   (a) $Cov(a + bX, c + dY) = bdCov(X, Y)$
   (b) $Corr(a + bX, c + dY) = Corr(X, Y)$
2. Fill in the missing steps from the lecture to prove the shortcut formula for covariance:
$$Cov(X, Y) = E[XY] - E[X]E[Y]$$
3. Let X_1 be a random variable denoting the returns of stock 1, and X_2 be a random variable denoting the returns of stock 2. Accordingly let $\mu_1 = \mathbb{E}[X_1]$, $\mu_2 = \mathbb{E}[X_2]$, $\sigma_1^2 = Var(X_1)$, $\sigma_2^2 = Var(X_2)$ and $\rho = Corr(X_1, X_2)$. A *portfolio*, Π, is a linear combination of X_1 and X_2 with weights that sum to one, that is $\Pi(\omega) = \omega X_1 + (1 - \omega)X_2$, indicating the proportions of stock 1 and stock 2 that an investor holds. In this example, we require $\omega \in [0, 1]$, so that *negative* weights are not allowed. (This rules out short-selling.)
   (a) Calculate $\mathbb{E}[\Pi(\omega)]$ in terms of ω, μ_1 and μ_2.
   (b) If $\omega \in [0, 1]$ is it possible to have $\mathbb{E}[\Pi(\omega)] > \mu_1$ *and* $\mathbb{E}[\Pi(\omega)] > \mu_2$? What about $\mathbb{E}[\Pi(\omega)] < \mu_1$ and $\mathbb{E}[\Pi(\omega)] < \mu_2$? Explain.
   (c) Express $Cov(X_1, X_2)$ in terms of ρ and σ_1, σ_2.
   (d) What is $Var[\Pi(\omega)]$? (Your answer should be in terms of ρ, σ_1^2 and σ_2^2.)

(e) Using part (d) show that the value of ω that minimizes $Var[\Pi(\omega)]$ is

$$\omega^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

In other words, $\Pi(\omega^*)$ is the *minimum variance portfolio.*

(f) If you want a challenge, check the second order condition from part (e).

4. Suppose that X is a random variable with the following PDF

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Graph the PDF of X.
   (b) Show that $\int_{-\infty}^{\infty} f(x)\, dx = 1$.
   (c) What is $\mathbb{P}(0.5 < X < 1.5)$?

5. A random variable is said to follow a Uniform(a, b) distribution if it is equally likely to take on any value in the range $[a, b]$ and never takes a value outside this range. Suppose that X is such a random variable, i.e. $X \sim$ Uniform(a, b).

   (a) What is the support of X?
   (b) Explain why the PDF of X is $f(x) = 1/(b - a)$ for $a \leq x \leq b$, zero elsewhere.
   (c) Using the PDF from part (b), calculate the CDF of X.
   (d) Verify that $f(x) = F'(x)$ for the present example.
   (e) Calculate $E[X]$.
   (f) Calculate $E[X^2]$. *Hint:* recall that $b^3 - a^3$ can be factorized as $(b-a)(b^2+a^2+ab)$.
   (g) Using the shortcut formula and parts (e) and (f), show that $Var(X) = (b-a)^2/12$.

6. Suppose that $X \sim N(0, 16)$ independent of $Y \sim N(2, 4)$. Recall that when our convention is to express the normal distribution in terms of its mean and variance, i.e. $N(\mu, \sigma^2)$. Hence, X has a mean of zero and variance of 16, while Y has a mean of 2 and a variance of 4. In completing some parts of this question you will need to use the R function `pnorm` described in class. In this case, please write down the command you used as well as the numeric result.

   (a) Calculate $P(-8 \leq X \leq 8)$.
   (b) Calculate $P(0 \leq Y \leq 4)$.
   (c) Calculate $P(-1 \leq Y \leq 6)$.
   (d) Calculate $P(X \geq 10)$.