

Economics 103 – Statistics for Economists

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Lecture # 4

Introduction to Regression

How to fairly account for missing midterm score?

- ▶ In my first semester at Penn, several students missed Midterm 2 because of illness so I decided to up-weight their finals.
- ▶ Problem: Midterm 2 turned out easier than Midterm 1 and this put the students who had missed the second midterm at a disadvantage when I curved the class.
- ▶ In order to correct for this, I needed a way to *fill in* a score for the missing midterm.
- ▶ How could I do this fairly?

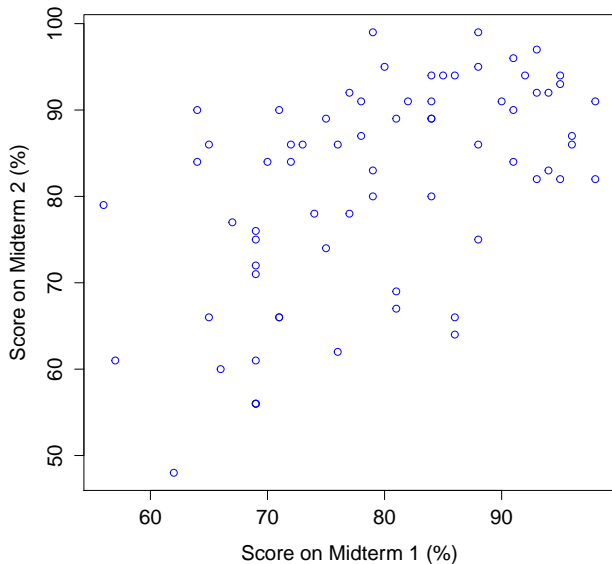
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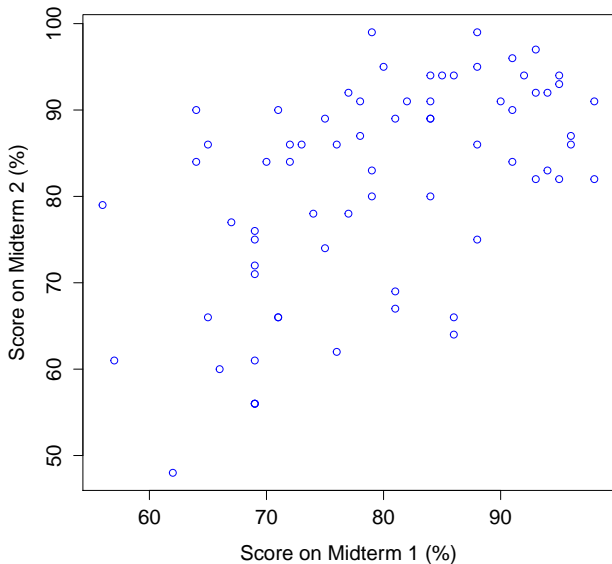
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- ▶ How could I do this fairly?
 - ▶ Just fill in mean score on second exam?
 - ▶ Use performance on first midterm to predict?

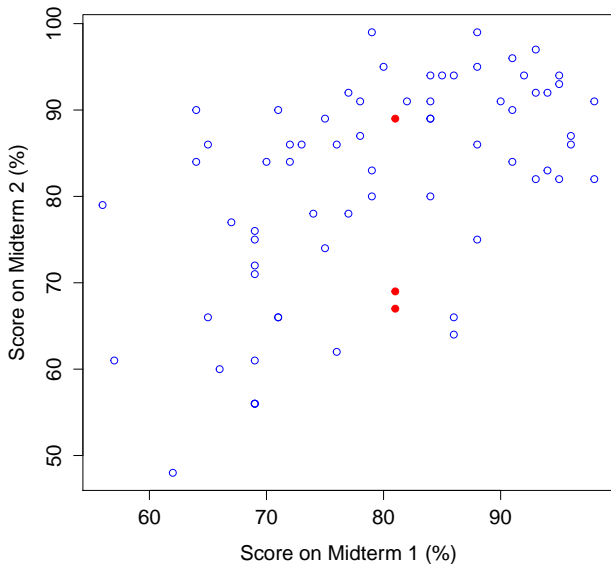
Data for students who took both midterms:



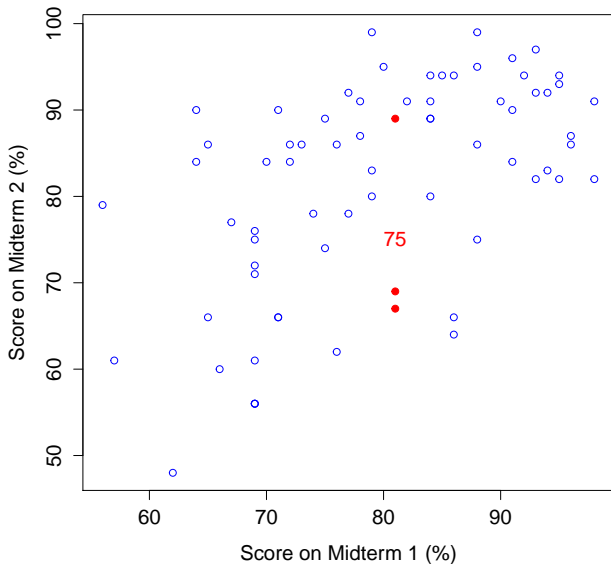
Predict Second Midterm given 81 on First



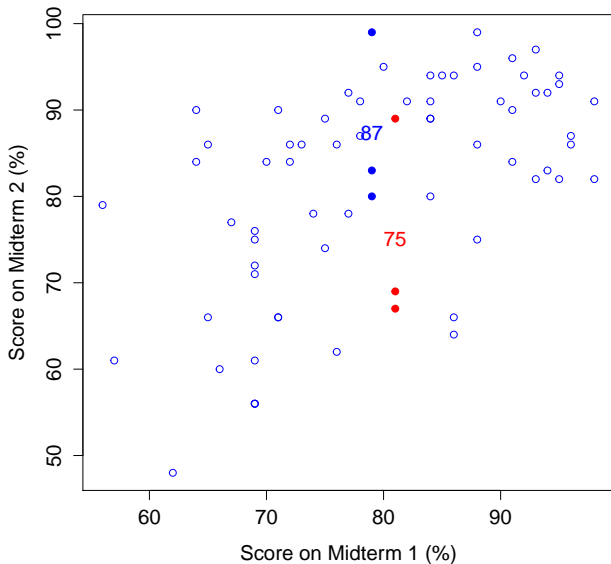
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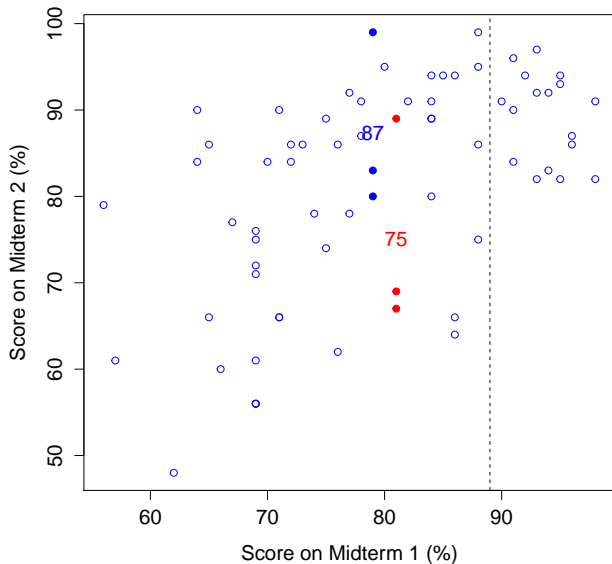
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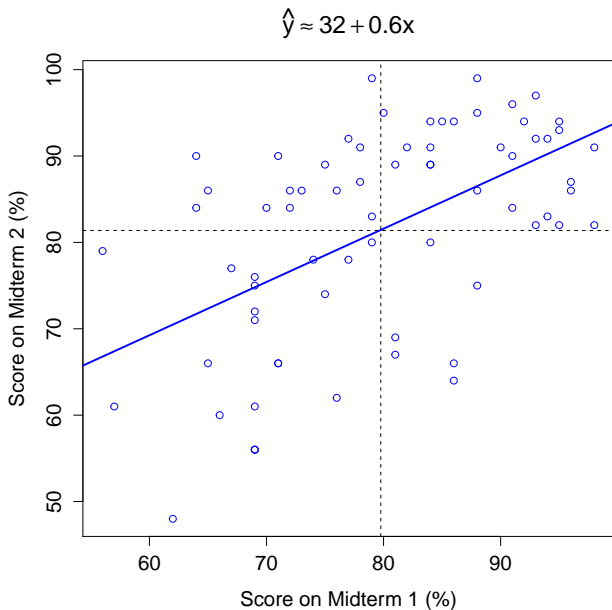
But if they'd only gotten 79 we'd predict higher?!



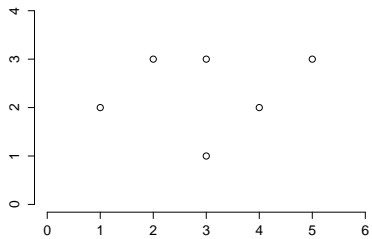
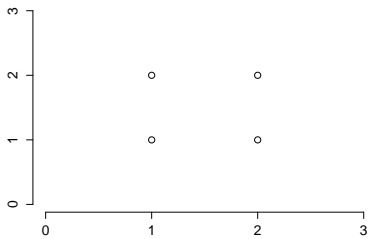
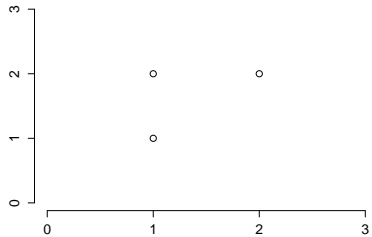
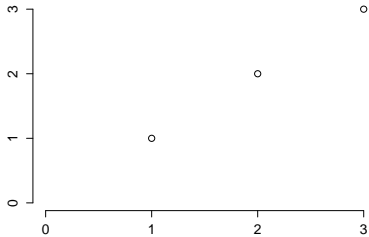
No one who took both exams got 89 on the first!

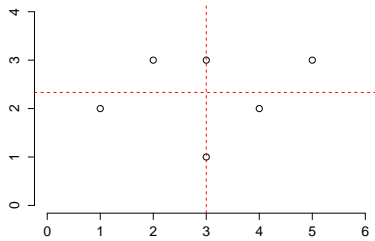
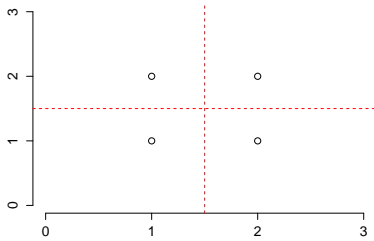
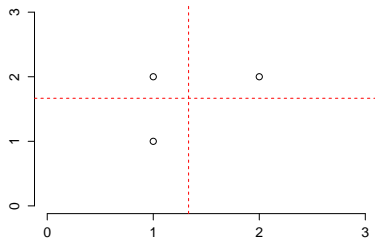
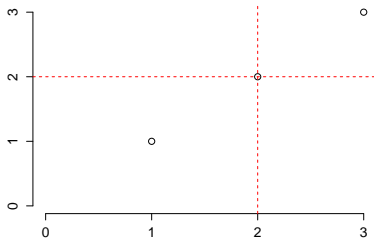


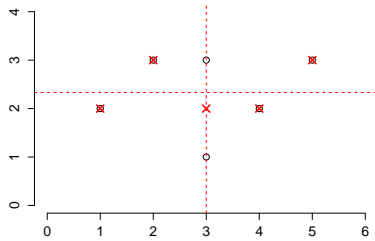
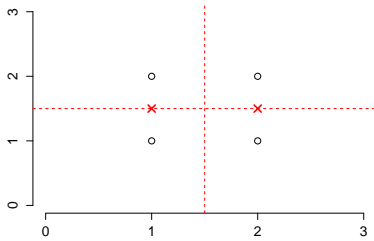
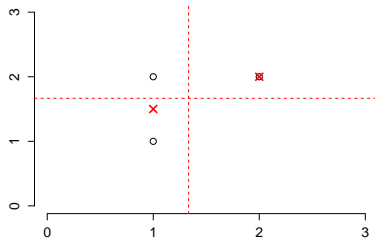
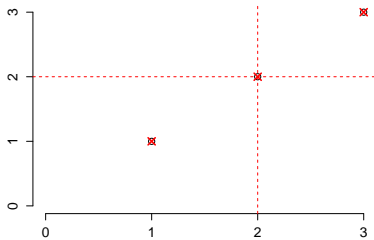
Regression: “Best Fitting” Line Through Cloud of Points

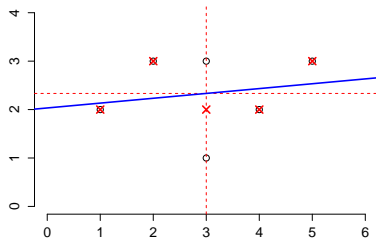
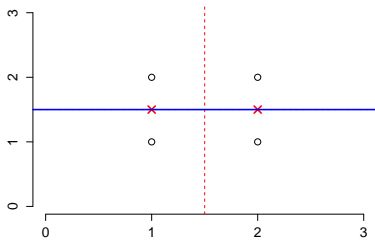
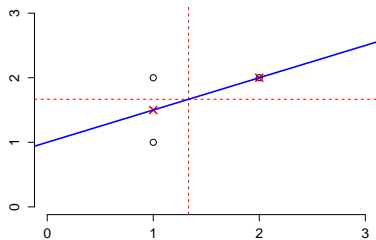
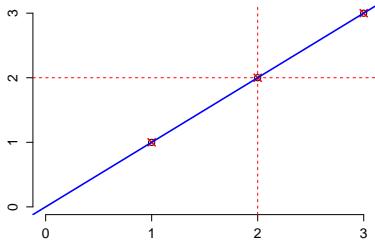


Fitting a Line by Eye









But How to Do this Formally?

Least Squares Regression – Predict Using a Line

Linear Model

$$\hat{y} = a + bx$$

Least Squares Regression – Predict Using a Line

Linear Model

$$\hat{y} = a + bx$$

Choose a, b to Minimize Sum of Squared Vertical Deviations

$$\sum_{i=1}^n d_i^2 = \sum_{i=1}^n (y_i - a - bx_i)^2$$

Least Squares Regression – Predict Using a Line

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The Prediction

Predict score $\hat{y} = a + bx$ on second midterm for someone with score x on first.

Least Squares Regression – Predict Using a Line

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Why Vertical Deviations? Why Squared Deviations?

Important Point About Notation

$$\text{minimize } \sum_{i=1}^n d_i^2 = \sum_{i=1}^n (y_i - a - bx_i)^2$$

$$\hat{y} = a + bx$$

- ▶ $(x_i, y_i)_{i=1}^n$ are the **observed data**
- ▶ \hat{y} is our **prediction** for a given value of x
- ▶ Neither x nor \hat{y} needs to be in our dataset!

Key Point

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 $d_i = y_i - a - bx_i$ for $i = 1, \dots, n$

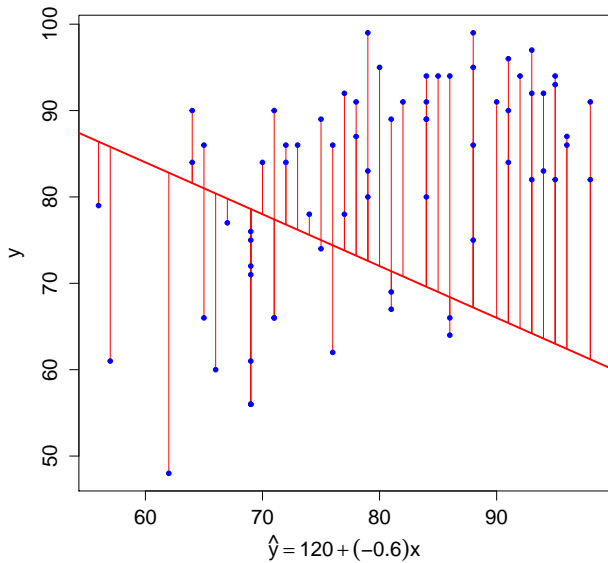
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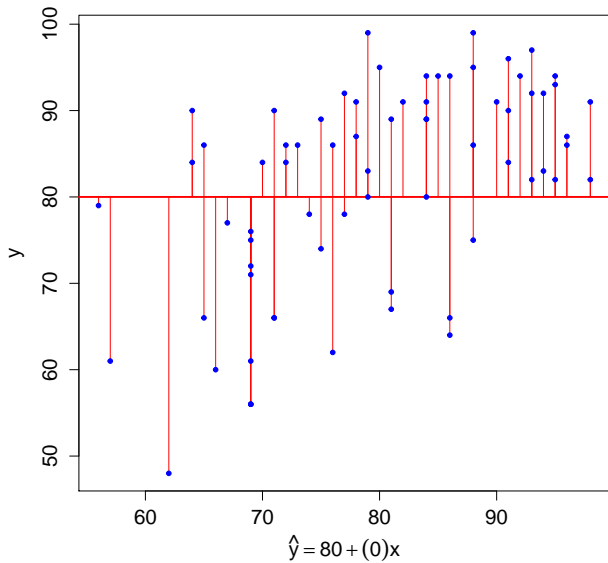
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 $d_i = y_i - a - bx_i$ for $i = 1, \dots, n$
- ▶ Each collection of vertical devs. gives sum of squares $\sum_{i=1}^n d_i^2$
- ▶ We choose a, b to minimize $\sum_{i=1}^n d_i^2$

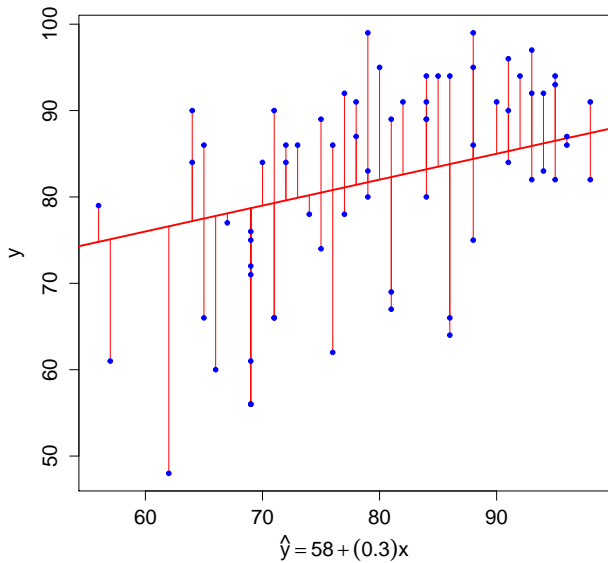
$$\sum d^2 = 25596.88$$



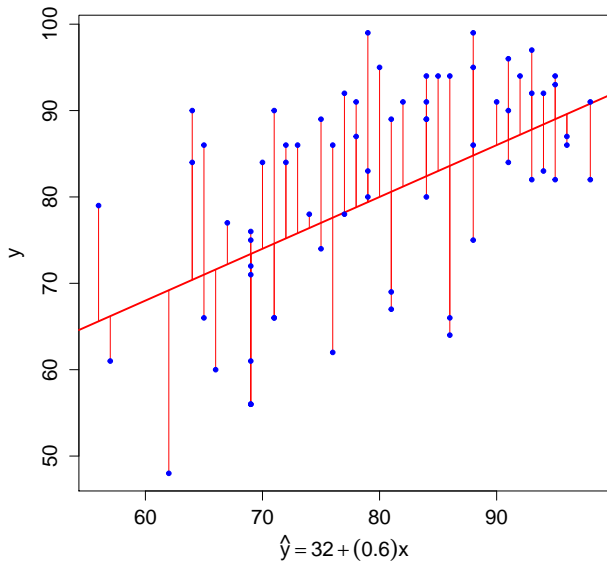
$$\sum d^2 = 10728$$



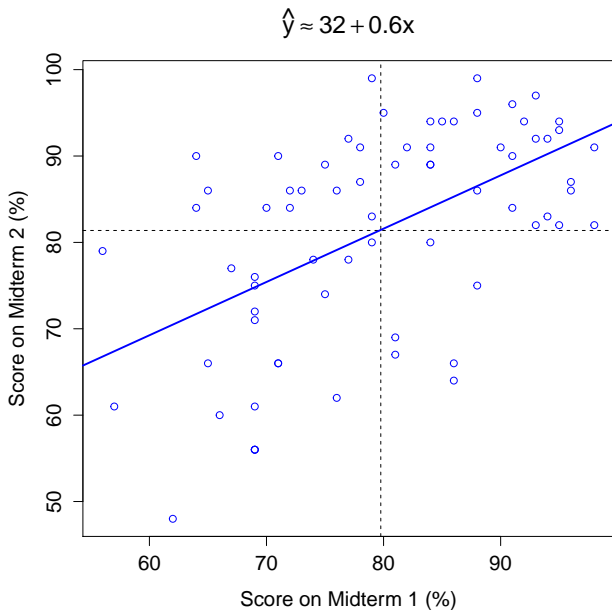
$$\sum d^2 = 8313.72$$



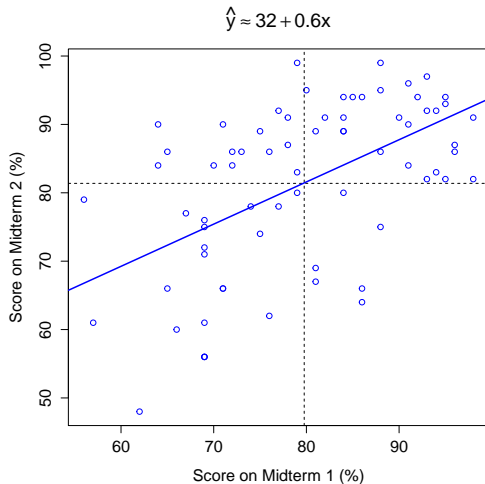
$$\sum d^2 = 7650.48$$



Prediction given 89 on Midterm 1?



Prediction given 89 on Midterm 1?



$$32 + 0.6 \times 89 = 32 + 53.4 = 85.4$$

You Need to Know How To Derive This



Minimize the sum of squared vertical deviations from the line:

$$\min_{a,b} \sum_{i=1}^n (y_i - a - bx_i)^2$$

How should we proceed?

- (a) Differentiate with respect to x
- (b) Differentiate with respect to y
- (c) Differentiate with respect to x, y
- (d) Differentiate with respect to a, b
- (e) Can't solve this with calculus.

Objective Function

$$\min_{a,b} \sum_{i=1}^n (y_i - a - bx_i)^2$$

FOC with respect to a

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FOC with respect to a

$$-2 \sum_{i=1}^n (y_i - a - bx_i) = 0$$

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$$\frac{1}{n} \sum_{i=1}^n y_i - \frac{na}{n} - \frac{b}{n} \sum_{i=1}^n x_i = 0$$

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$$\bar{y} - a - b\bar{x} = 0$$

Regression Line Goes Through the Means!

$$\bar{y} = a + b\bar{x}$$

Substitute: Eliminate a from Objective Function

$$a = \bar{y} - b\bar{x}$$

$$\sum_{i=1}^n (y_i - a - bx_i)^2 =$$

Substitute: Eliminate a from Objective Function

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$$\begin{aligned}\sum_{i=1}^n (y_i - a - bx_i)^2 &= \sum_{i=1}^n (y_i - \bar{y} + b\bar{x} - bx_i)^2 \\ &= \end{aligned}$$

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Objective Function Without a

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FOC with respect to b

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$$\sum_{i=1}^n [(y_i - \bar{y}) - b(x_i - \bar{x})]^2$$

FOC with respect to b

$$-2 \sum_{i=1}^n [(y_i - \bar{y}) - b(x_i - \bar{x})] (x_i - \bar{x}) = 0$$

Objective Function Without a

$$\sum_{i=1}^n [(y_i - \bar{y}) - b(x_i - \bar{x})]^2$$

FOC with respect to b

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$$\sum_{i=1}^n (y_i - \bar{y}) (x_i - \bar{x}) - b \sum_{i=1}^n (x_i - \bar{x})^2 = 0$$

Objective Function Without a

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$$\sum_{i=1}^n (y_i - \bar{y}) (x_i - \bar{x}) - b \sum_{i=1}^n (x_i - \bar{x})^2 = 0$$

$$b = \frac{\sum_{i=1}^n (y_i - \bar{y}) (x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

Simple Linear Regression

Problem

$$\min_{a,b} \sum_{i=1}^n (y_i - a - bx_i)^2$$

Solution

$$b = \frac{\sum_{i=1}^n (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

$$a = \bar{y} - b\bar{x}$$

Relating Regression to Covariance and Correlation

$$b = \frac{\sum_{i=1}^n (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2} =$$

Relating Regression to Covariance and Correlation

$$b = \frac{\sum_{i=1}^n (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{\frac{1}{n-1} \sum_{i=1}^n (y_i - \bar{y})(x_i - \bar{x})}{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2} =$$

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$$r = \frac{s_{xy}}{s_x s_y} =$$

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Relating Regression to Covariance and Correlation

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Relating Regression to Covariance and Correlation

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Relating Regression to Covariance and Correlation

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$$b = r \frac{s_y}{s_x}$$

Comparing Regression, Correlation and Covariance

Units

Correlation is unitless, covariance and regression coefficients (a , b) are not. (What are the units of these?)

Symmetry

Correlation and covariance are symmetric, regression isn't. (Switching x and y axes changes the slope and intercept.)

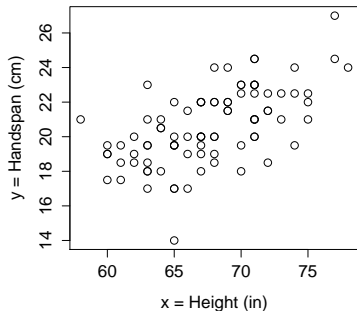
On the Homework

Regression with z-scores rather than raw data gives $a = 0$, $b = r_{xy}$



$$s_{xy} = 6, \quad s_x = 5, \quad s_y = 2, \quad \bar{x} = 68, \quad \bar{y} = 21$$

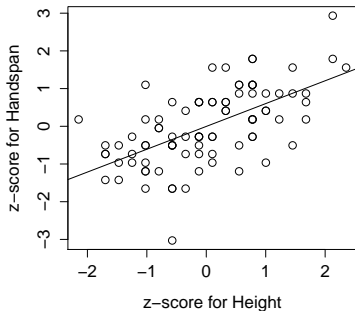
What is the sample correlation between height (x) and handspan (y)?





$$s_{xy} = 6, \quad s_x = 5, \quad s_y = 2, \quad \bar{x} = 68, \quad \bar{y} = 21$$

What is the sample correlation between height (x) and handspan (y)?



$$r = \frac{s_{xy}}{s_x s_y} = \frac{6}{5 \times 2} = 0.6$$



$$s_{xy} = 6, \quad s_x = 5, \quad s_y = 2, \quad \bar{x} = 68, \quad \bar{y} = 21$$

What is the value of b for the regression:

$$\hat{y} = a + bx$$

where x is height and y is handspan?



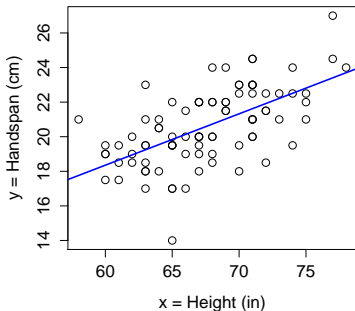


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$$b = \frac{s_{xy}}{s_x^2} = \frac{6}{5^2} = 6/25 = 0.24$$

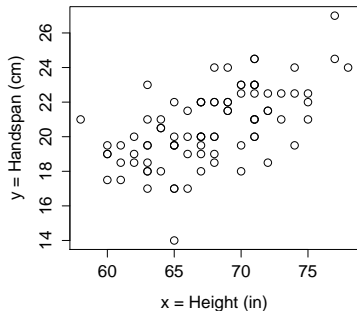


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(prev. slide $b = 0.24$)



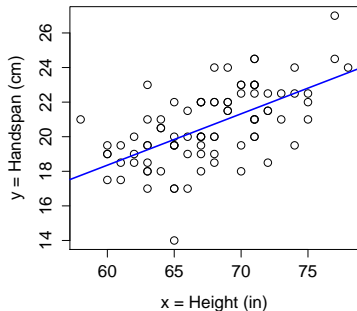


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What is the value of a for the regression:

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where x is height and y is handspan?
(prev. slide $b = 0.24$)



$$a = \bar{y} - b\bar{x} = 21 - 0.24 \times 68 = 4.68$$

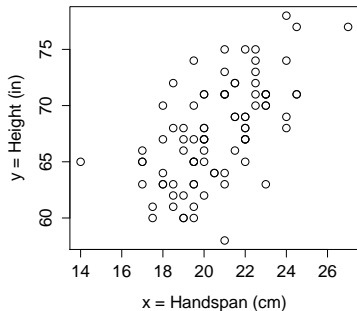


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What is the value of b for the regression:

$$\hat{y} = a + bx$$

where x is handspan and y is height?



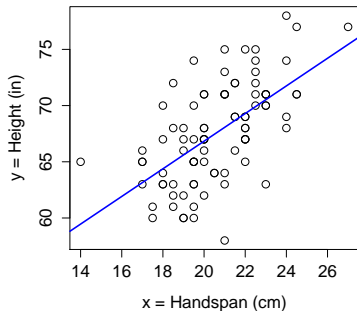


$$s_{xy} = 6, \quad s_y = 5, \quad s_x = 2, \quad \bar{y} = 68, \quad \bar{x} = 21$$

What is the value of b for the regression:

$$\hat{y} = a + bx$$

where x is handspan and y is height?

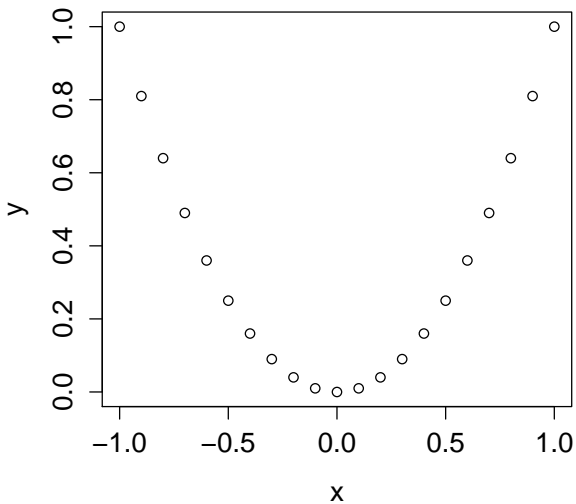


$$b = \frac{s_{xy}}{s_x^2} = 6/2^2 = 1.5$$

EXTREMELY IMPORTANT

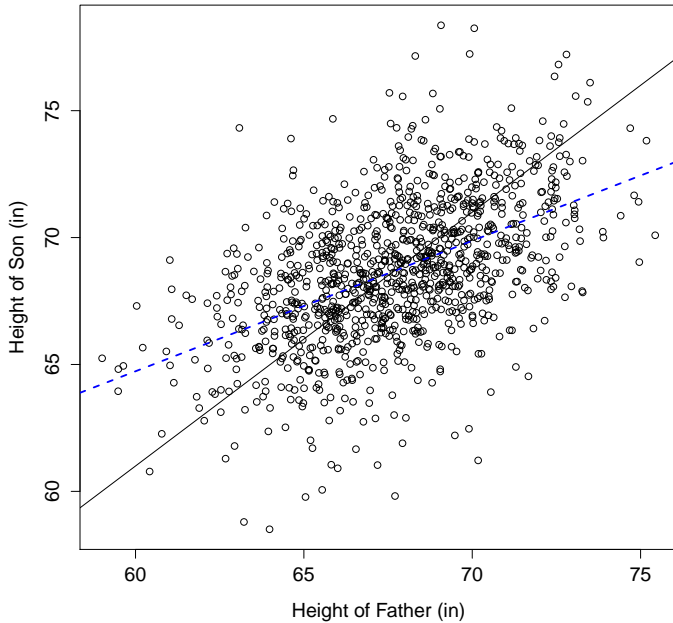
- ▶ Regression, Covariance and Correlation: linear association.
- ▶ Linear association \neq causation.
- ▶ Linear is not the only kind of association!

Correlation = 0



Why is it called “regression?”

Pearson Dataset

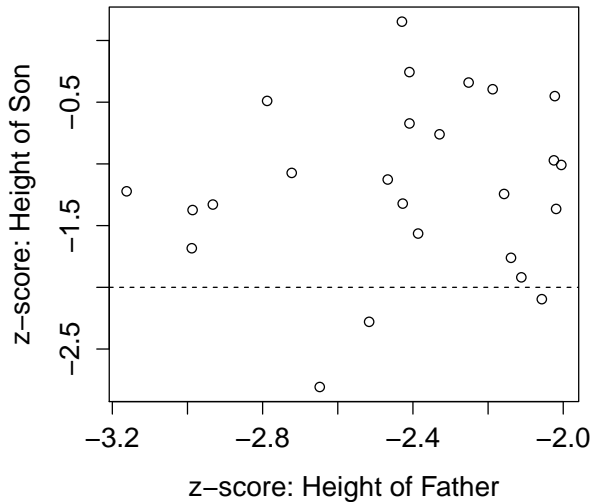




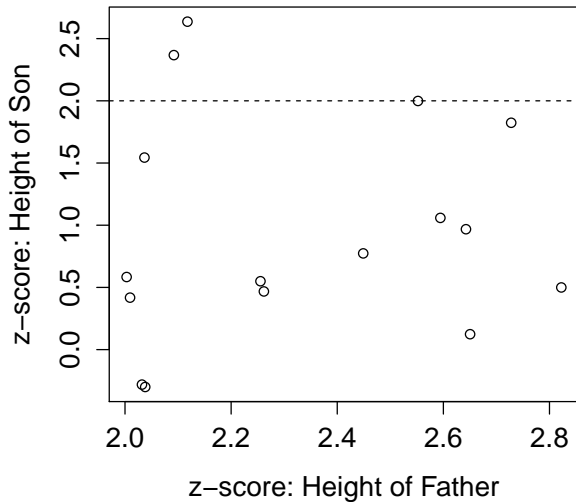
Suppose a father is very short compared to other fathers (very negative z-score). Would you expect his son to be:

- (a) Shorter
- (b) About as short
- (c) Taller

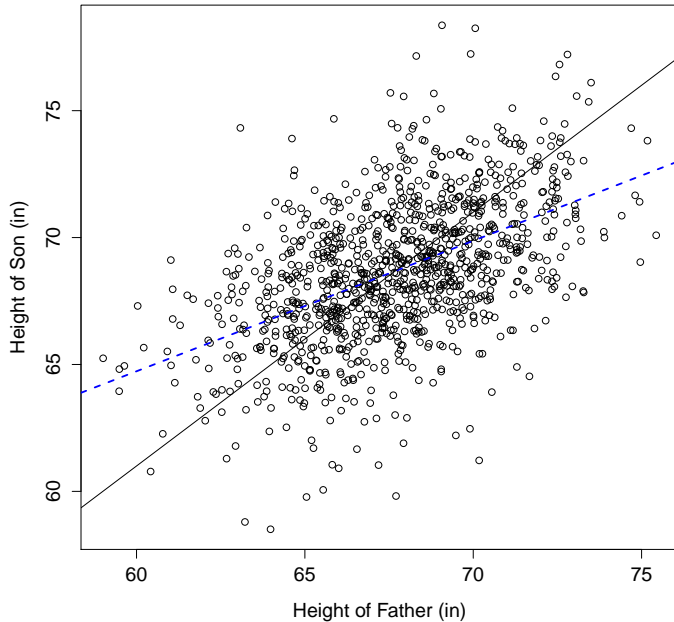
Very Short Fathers and Their Sons



Very Tall Fathers and Their Sons



Pearson Dataset



Regression to the Mean

Skill and Luck / Genes and Random Environmental Factors

Unless $r_{xy} = 1$, There Is Regression to the Mean

$$\frac{\hat{y} - \bar{y}}{s_y} = r_{xy} \frac{x - \bar{x}}{s_x}$$

Least-squares Prediction \hat{y} closer to \bar{y} than x is to \bar{x}

You will derive the above formula in this week's homework.

Regression Fallacy

For More, See the Document Posted on Piazza

Pre-test

Which students are strongest, which are weakest?

Intervention

Put the best performing in an enrichment program and the worst performing in a remedial class

Post-test

The weak students did better than on their first test, but the strong students did *worse*.

Mistaken Conclusion

Remedial classes are beneficial, enrichment programs are harmful