Economics 103 – Statistics for Economists

Francis J. DiTraglia

University of Pennsylvania

Lecture 18

Two-sample Problem

Suppose $X_1, \ldots, X_n \sim$ iid $N(\mu_x, \sigma_x^2)$ independently of $Y_1, \ldots, Y_m \sim$ iid $N(\mu_y, \sigma_y^2)$. What is $E[\bar{X}_n - \bar{Y}_m]$, the expectation of the sampling distribution of the difference of sample means?

(a) μ_x

(b) $\mu_x - \mu_y$

(c) μ_y

(d) $\mu_x + \mu_y$

(e) 0

$$E[\bar{X}_n - \bar{Y}_m] = E[\bar{X}_n] - E[\bar{Y}_m] = \mu_x - \mu_y$$

Two-sample Problem

Suppose $X_1, \ldots, X_n \sim$ iid $N(\mu_x, \sigma_x^2)$ independently of $Y_1, \ldots, Y_m \sim$ iid $N(\mu_y, \sigma_y^2)$. What is $Var[\bar{X}_n - \bar{Y}_m]$, the variance of the sampling distribution of the difference of sample means?

(a) $\sigma_x^2 - \sigma_y^2$

(b) $\sigma_x^2 + \sigma_y^2$

(c) $\sigma_x^2/n + \sigma_y^2/m$

(d) $\sigma_x^2/n - \sigma_y^2/m$

(e) 1

By independence: $Var[\bar{X}_n - \bar{Y}_m] = Var[\bar{X}_n] + Var[\bar{Y}_m] = \dfrac{\sigma_x^2}{n} + \dfrac{\sigma_y^2}{m}$

Two-sample Problem

Suppose $X_1, \ldots, X_n \sim$ iid $N(\mu_x, \sigma_x^2)$ independently of $Y_1, \ldots, Y_m \sim$ iid $N(\mu_y, \sigma_y^2)$. What is the sampling distribution of $\bar{X}_n - \bar{Y}_m$, the difference of sample means?

(a) χ^2

(b) t

(c) F

(d) Normal

Normal, by independence and linearity property of normal distributions.

Sampling Distribution of $\bar{X}_n - \bar{Y}_m$

Suppose $X_1, \ldots, X_n \sim$ iid $N(\mu_x, \sigma_x^2)$ independently of $Y_1, \ldots, Y_m \sim$ iid $N(\mu_y, \sigma_y^2)$. Then,

$$(\bar{X}_n - \bar{Y}_m) \sim N\left(\mu_x - \mu_y, \frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}\right)$$

$$\frac{(\bar{X}_n - \bar{Y}_m) - (\mu_x - \mu_y)}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}} \sim N(0,1)$$

Shorthand: $SE(\bar{X}_n - \bar{Y}_m) = \sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}$

CI for Difference of Population Means, σ_x^2, σ_y^2 Known

$$\frac{(\bar{X}_n - \bar{Y}_m) - (\mu_x - \mu_y)}{SE(\bar{X}_n - \bar{Y}_m)} \sim N(0, 1)$$

Thus, we construct a $100 \times (1 - \alpha)\%$ CI for $\mu_x - \mu_y$ as follows:

$$(\bar{X}_n - \bar{Y}_m) \pm \texttt{qnorm}(1 - \alpha/2)\ SE(\bar{X}_n - \bar{Y}_m)$$

Where $SE(\bar{X}_n - \bar{Y}_m) = \sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}$

Calculate the ME for the Difference of Means

I generated independent random samples of size 25 from two normal distributions in R. One had a population standard deviation of 4 and the other had a population standard deviation of 3. The sample means were approximately 4.2 and 3.1.

Calculate the ME for a 95% confidence interval for the difference of population means.

$$SE = \sqrt{\frac{3^2}{25} + \frac{4^2}{25}} = \frac{\sqrt{9 + 16}}{5} = 1$$

$$ME = \texttt{qnorm(1 - 0.05/2)} \times SE \approx 2 \times SE = 2$$

Calculate the LCL for the Difference of Means

I generated independent random samples of size 25 from two normal distributions in R. One had a population standard deviation of 4 and the other had a population standard deviation of 3. The sample means were approximately 4.2 and 3.1.

Calculate the LCL for a 95% confidence interval for the difference of population means.

$$LCL = (4.2 - 3.1) - ME = 1.1 - 2 = -0.9$$

Calculate the UCL for the Difference of Means

I generated independent random samples of size 25 from two normal distributions in R. One had a population standard deviation of 4 and the other had a population standard deviation of 3. The sample means were approximately 4.2 and 3.1.

Calculate the UCL for a 95% confidence interval for the difference of population means.

$$UCL = (4.2 - 3.1) + ME = 1.1 + 2 = 3.1$$

$\boxed{\text{95\% Confidence Interval: } (-0.9, 3.1)}$

The actual population means were 4 and 3, respectively

What if σ_x^2, σ_y^2 are Unknown?

Suppose $X_1, \ldots, X_n \sim$ iid $N(\mu_x, \sigma_x^2)$ independently of $Y_1, \ldots, Y_m \sim$ iid $N(\mu_y, \sigma_y^2)$. Then,

$$\frac{\left(\bar{X}_n - \bar{Y}_m\right) - (\mu_x - \mu_y)}{\sqrt{\dfrac{S_x^2}{n} + \dfrac{S_y^2}{m}}} \sim t(\nu)$$

Formula for ν is Complicated and You Don't Need to Know it

Two possibilities:

1. Have R find the correct value of ν for us
2. If m, n are large enough, approximately standard normal.

Case of Equal, Unknown Variances

The book considers a case where $\sigma_x^2 = \sigma_y^2 = \sigma^2$, that is a common unknown variance. This is a very dangerous assumption. It is almost certainly false and can throw off our results in a serious way. You are not responsible for this case.

Sampling Distributions Under Normality: One-sample

Suppose that $X_1, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$. Then:

$$
\begin{aligned}
\left(\frac{n-1}{\sigma^2}\right) S^2 &\sim \chi^2(n-1) \\
\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} &\sim N(0,1) \\
\frac{\bar{X}_n - \mu}{S/\sqrt{n}} &\sim t(n-1)
\end{aligned}
$$

Sampling Distributions Under Normality: Two-sample

Suppose $X_1, \ldots, X_n \sim$ iid $N(\mu_x, \sigma_x^2)$ independently of $Y_1, \ldots, Y_m \sim$ iid $N(\mu_y, \sigma_y^2)$. Then:

$$\frac{(\bar{X}_n - \bar{Y}_n) - (\mu_x - \mu_y)}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}} \quad \sim \quad N(0,1)$$

$$\frac{(\bar{X}_n - \bar{Y}_m) - (\mu_x - \mu_y)}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}} \quad \sim \quad t(\nu)$$

But what if the population isn't Normal?

The Central Limit Theorem

Suppose that $X_1, \ldots, X_n$ are a random sample from a population with unknown mean μ. Then, provided that n is *sufficiently large*, the sampling distribution of $\bar{X}_n$ is approximately $N\left(\mu, \widehat{SE}(\bar{X}_n)^2\right)$, even if the even if the underlying population is non-normal.

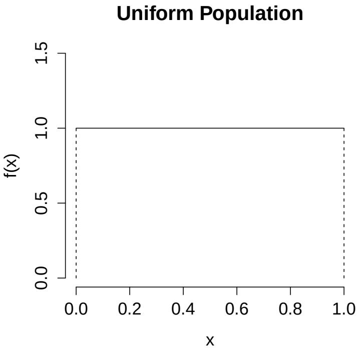
In Other Words...

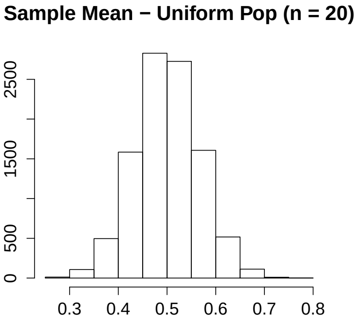
$$\frac{\bar{X}_n - \mu}{\widehat{SE}(\bar{X}_n)} \approx N(0, 1)$$

Use this to create *approximate* CIs for population mean!

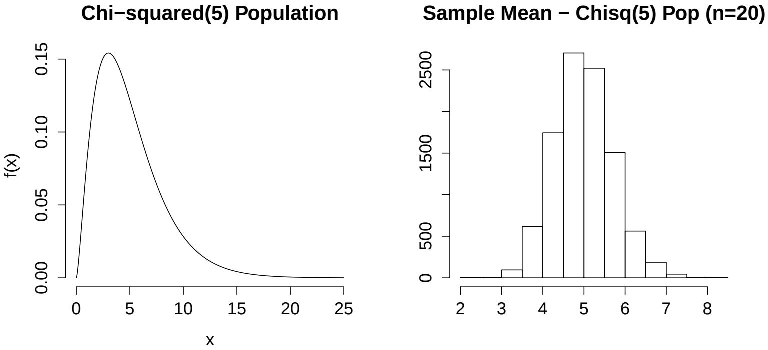
You should be amazed by this.

Example: Uniform(0,1) Population, $n = 20$

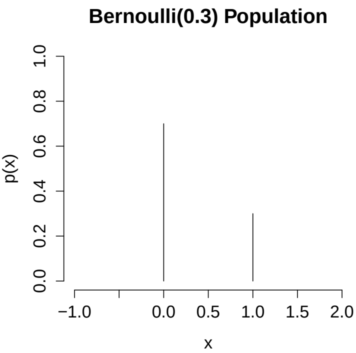


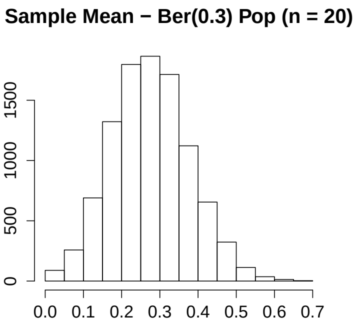


Example: $\chi^2(5)$ Population, $n = 20$



Example: Bernoulli(0.3) Population, $n = 20$





Who is the Chief Justice of the US Supreme Court?

(a) Harry Reid

(b) John Roberts

(c) William Rehnquist

(d) Stephen Breyer

Are US Voters Really That Ignorant?

Pew: "What Voters Know About Campaign 2012"

The Data

Of 771 registered voters polled, only 39% correctly identified John Roberts as the current chief justice of the US Supreme Court.

Research Question

Is the majority of voters unaware that John Roberts is the current chief justice, or is this just sampling variation?

Assume Random Sampling...

Confidence Interval for a Proportion

What is the appropriate probability model for the sample?

$X_1, \ldots, X_n \sim$ iid Bernoulli(p), $1 =$ Know Roberts is Chief Justice

What is the parameter of interest?

$p =$ Proportion of voters *in the population* who know Roberts is Chief Justice.

What is our estimator?

Sample Proportion: $\widehat{p} = (\sum_{i=1}^{n} X_i)/n$

Sample Proportion *is* the Sample Mean!

$X_1, \ldots, X_n \sim$ iid Bernoulli(p)

$$\widehat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}_n$$

$$\begin{aligned}
E[\widehat{p}] &= E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{np}{n} = p \\
Var(\widehat{p}) &= Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n} \\
SE(\widehat{p}) &= \sqrt{Var(\widehat{p})} = \sqrt{\frac{p(1-p)}{n}} \\
\widehat{SE}(\widehat{p}) &= \sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}
\end{aligned}$$

Central Limit Theorem Applied to Sample Proportion

Central Limit Theorem: Intuition

Sample means are approximately normally distributed provided the sample size is large even if the population is non-normal.

CLT For Sample Mean

$$\frac{\bar{X}_n - \mu}{\widehat{SE}(\bar{X}_n)} \approx N(0, 1)$$

CLT for Sample Proportion

$$\frac{\widehat{p} - p}{\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}} \approx N(0, 1)$$

In this example, the population is Bernoulli(p) rather than normal.

The sample mean is $\widehat{p}$ and the population mean is p.

Approximate 95% CI for Population Proportion

$$\frac{\widehat{p} - p}{\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}} \approx N(0,1)$$

$$\begin{aligned} P\left(-2 \le \frac{\widehat{p} - p}{\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}} \le 2\right) &\approx 0.95 \\ P\left(\widehat{p} - 2\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}} \le p \le \widehat{p} + 2\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}\right) &\approx 0.95 \end{aligned}$$

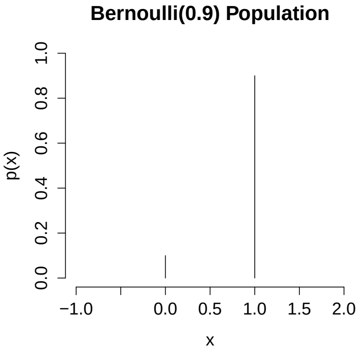
$100 \times (1-\alpha)$ CI for Population Proportion (p)

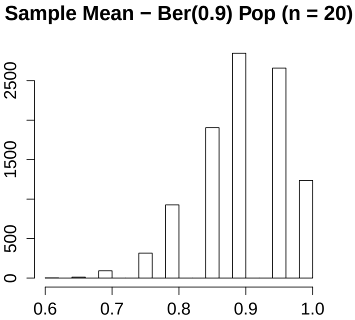
$X_1, \ldots, X_n \sim$ iid Bernoulli(p)

$$\widehat{p} \pm \texttt{qnorm}(1-\alpha/2)\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}$$

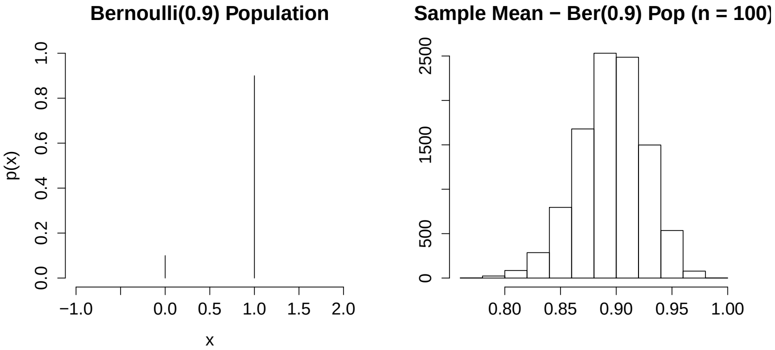
Approximation based on the CLT. Works well provided n is large and p isn't too close to zero or one.

Example: Bernoulli(0.9) Population, $n = 20$





Example: Bernoulli(0.9) Population, $n = 100$



Approximate 95% CI for Population Proportion

39% of 771 Voters Polled Correctly Identified Chief Justice Roberts

$$\begin{aligned}\widehat{SE}(\widehat{p}) &= \sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}} = \sqrt{\frac{(0.39)(0.61)}{771}} \\ &\approx 0.018\end{aligned}$$

What is the ME for an approximate 95% confidence interval?

$$ME \approx 2 \times \widehat{SE}(\bar{X}_n) \approx 0.04$$

What can we conclude?

$$\boxed{\text{Approximate 95\% CI: } (0.35, 0.43)}$$

Are Republicans Better Informed Than Democrats?

Pew: "What Voters Know About Campaign 2012"

Of the 239 Republicans surveyed, 47% correctly identified John Roberts as the current chief justice. Only 31% of the 238 Democrats surveyed correctly identified him. Is this difference meaningful or just sampling variation?

Again, assume random sampling.

Confidence Interval for a Difference of Proportions

What is the appropriate probability model for the sample?

$X_1, \ldots, X_n \sim$ iid Bernoulli(p) independently of

$Y_1, \ldots, Y_m \sim$ iid Bernoulli(q)

What is the parameter of interest?

The difference of population proportions $p - q$

What is our estimator?

The difference of sample proportions: $\widehat{p} - \widehat{q}$ where:

$$\widehat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \widehat{q} = \frac{1}{m}\sum_{i=1}^{m} Y_i$$

Difference of Sample Proportions $\widehat{p} - \widehat{q}$ and the CLT

What We Have

Approx. sampling dist. for *individual* sample proportions from CLT:

$\widehat{p} \approx N\left(p, \widehat{SE}(\widehat{p})^2\right), \quad \widehat{q} \approx N\left(q, \widehat{SE}(\widehat{q})^2\right)$

What We Want

Sampling Distribution of the *difference* $\widehat{p} - \widehat{q}$

Use Independence of the Two Samples

$\widehat{p} - \widehat{q} \approx N\left(p - q, \widehat{SE}(\widehat{p})^2 + \widehat{SE}(\widehat{q})^2\right)$

$$\implies \widehat{SE}(\widehat{p} - \widehat{q}) = \sqrt{\widehat{SE}(\widehat{p})^2 + \widehat{SE}(\widehat{q})^2} = \sqrt{\frac{\widehat{p}(1-\widehat{p})}{n} + \frac{\widehat{q}(1-\widehat{q})}{m}}$$

Approx. 95% CI for Difference of Population Proportions

$$\frac{(\widehat{p}-\widehat{q})-(p-q)}{\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}+\frac{\widehat{q}(1-\widehat{q})}{m}}} \approx N(0,1)$$

$$P\left(-2 \leq \frac{(\widehat{p}-\widehat{q})-(p-q)}{\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}+\frac{\widehat{q}(1-\widehat{q})}{m}}} \leq 2\right) \approx 0.95$$

$$(\widehat{p}-\widehat{q}) \pm \texttt{qnorm}(1-\alpha/2)\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}+\frac{\widehat{q}(1-\widehat{q})}{m}}$$

$100 \times (1-\alpha)$ CI for Diff. of Popn. Proportions $(p-q)$

$X_1, \ldots, X_n \sim$ iid Bernoulli(p) indep. $Y_1, \ldots, Y_n \sim$ iid Bernoulli(q)

$$(\widehat{p} - \widehat{q}) \pm \texttt{qnorm}(1-\alpha/2)\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n} + \frac{\widehat{q}(1-\widehat{q})}{m}}$$

Approximation based on the CLT. Works well provided n, m large and p, q aren't too close to zero or one.

ME for approx. 95% for Difference of Proportions

47% of 239 Republicans vs. 31% of 238 Democrats identified Roberts

Republicans

$$\begin{aligned}
\widehat{p} &= 0.47 \\
n &= 239 \\
\widehat{SE}(\widehat{p}) &= \sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}} \approx 0.032
\end{aligned}$$

Democrats

$$\begin{aligned}
\widehat{q} &= 0.31 \\
m &= 238 \\
\widehat{SE}(\widehat{q}) &= \sqrt{\frac{\widehat{q}(1-\widehat{q})}{m}} \approx 0.030
\end{aligned}$$

Difference: (Republicans - Democrats)

$$\begin{aligned}
\widehat{p}-\widehat{q} &= 0.47-0.31 = 0.16 \\
\widehat{SE}(\widehat{p}-\widehat{q}) &= \sqrt{\widehat{SE}(\widehat{p})^2 + \widehat{SE}(\widehat{q})^2} \approx 0.044 \implies ME \approx 0.09
\end{aligned}$$

Approximate 95% CI $(0.07, 0.25)$ What can we conclude?