

Economics 103 – Statistics for Economists

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Lecture # 14

Sampling Distributions and Estimation – Part I

Weighing a Random Sample

Bag Contains 100 Candies

Estimate total weight of candies by weighing a random sample of size 5 and multiplying the result by 20.

Your Chance to Win

The bag of candies and a digital scale will make their way around the room **during the lecture**. Each team (2 students) gets a chance to draw 5 candies and weigh them.

Team with closest estimate wins the bag of candy!

Weighing a Random Sample

Procedure

When the bag and scale reach your team, do the following:

1. Fold the top of the bag over and shake to randomize.
2. Randomly draw 5 candies **without replacement**.
3. Weigh your sample and record the result **in grams**.
4. Rodrigo will enter your result into his spreadsheet and multiply it by 20 to estimate the weight of the bag.
5. Replace your sample and shake again to re-randomize.
6. Pass bag and scale to next team.

What is this all about?

(Sample) Statistic

Any function of the data *alone*, e.g. sample mean $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$

Statistics

1. Estimation – Use data to construct educated guess (Estimate) about true value of population parameter.
2. Inference – Quantify uncertainty about estimate using:
 - ▶ Confidence Intervals
 - ▶ Hypothesis Testing

How does Probability connect to Statistics?

Random Sampling!

Random Sampling

(Simple) Random Sample

Select a sample of n objects from a population in such a way that:

1. Each member of the population has the same probability of being selected
2. The fact that one individual is selected does not affect the chance that any other individual is selected
3. Each sample of size n is equally likely to be selected

Random Sampling

In other words:

$$X_1, X_2, \dots, X_n \sim \text{iid } f(x)$$

is a **Random Sample**

Statistics

Sample is drawn randomly, so sample statistics are *also random*.

Use what we know about probability theory to analyze the *distribution* of a statistic under random sampling.

Sample with or Without Replacement?

Strictly speaking, random samples should be drawn *with replacement*, otherwise there is *dependence*. In practice, this doesn't matter much so long as the population is large relative to the sample.

Candy Example In Progress: 100 is large relative to 5.

Estimator versus Estimate

Estimator

An estimator is a function $T(X_1, \dots, X_n)$ of the random variables we use to represent the random sampling procedure. Hence, it is a random variable itself.

Estimator versus Estimate

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Sampling Distribution

The probability distribution of an Estimator is called a *sampling distribution*.

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Sampling Distribution

The probability distribution of an Estimator is called a *sampling distribution*.

Estimate

An estimate is a function $T(x_1, \dots, x_n)$ of the *observed data*, i.e. the *realizations* of the random variables we use to represent random sampling. An estimate is a *constant* since the observed data are *constants*

Population: $f(x)$

Probability Distribution

Population: $f(x)$

Probability Distribution



Random Sample of Size n

Random Variables

$X_1, X_2, \dots, X_n \sim \text{iid } f(x)$

Population: $f(x)$

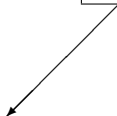
Probability Distribution



Random Sample of Size n

Random Variables

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$x_1^{(1)}, \dots, x_n^{(1)}$

Population: $f(x)$

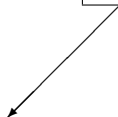
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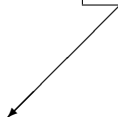
Probability Distribution



Random Sample of Size n

Random Variables

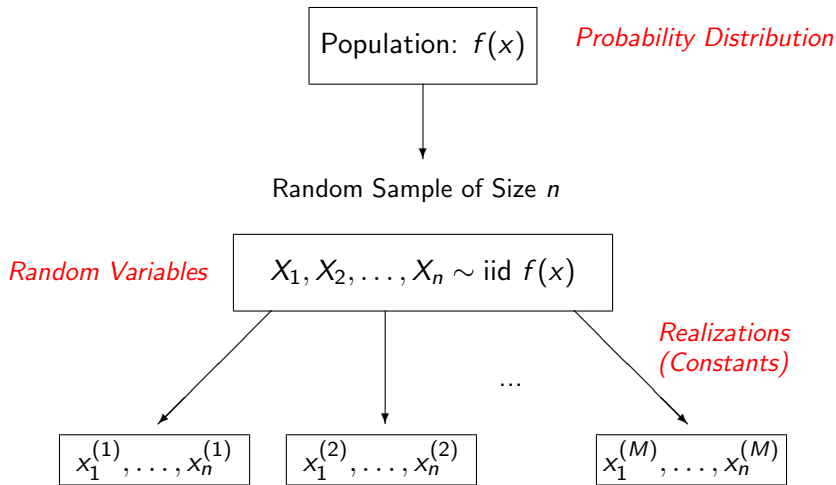
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...

$x_1^{(1)}, \dots, x_n^{(1)}$

$x_1^{(2)}, \dots, x_n^{(2)}$



M Replications, each containing n Observations

Random Sample of Size n

*Random
Variables*

$$X_1, X_2, \dots, X_n \sim \text{iid } f(x)$$

*Random
Variables*

Random Sample of Size n

$$X_1, X_2, \dots, X_n \sim \text{iid } f(x)$$



Estimator

$$T(X_1, \dots, X_n)$$

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

*Random
Variables*

Random Sample of Size n

$$X_1, X_2, \dots, X_n \sim \text{iid } f(x)$$

Estimator

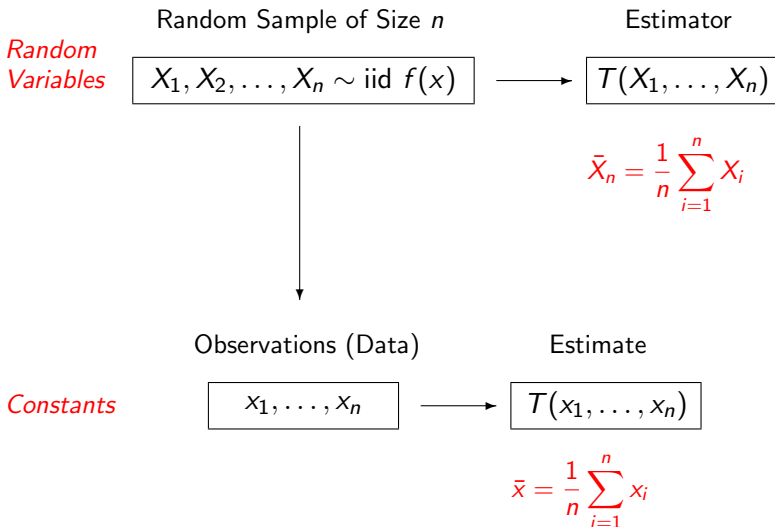
$$T(X_1, \dots, X_n)$$

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

Observations (Data)

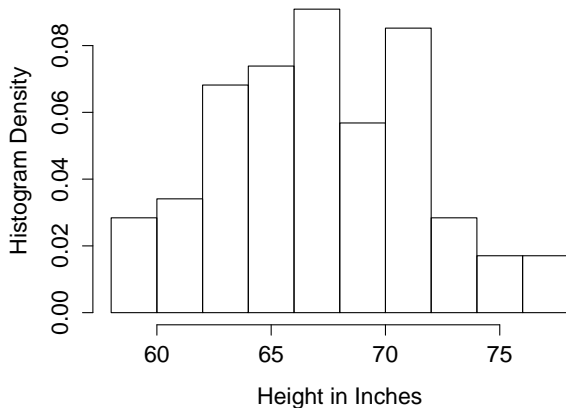
Constants

$$x_1, \dots, x_n$$



Population: All Students in the Class

Popn. Mean = 67.5, Popn. Var. = 19.7

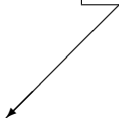


Random Sample of Size n

$$X_1, X_2, \dots, X_n \sim \text{iid } f(x)$$

Random Sample of Size n

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$$x_1^{(1)}, \dots, x_n^{(1)}$$

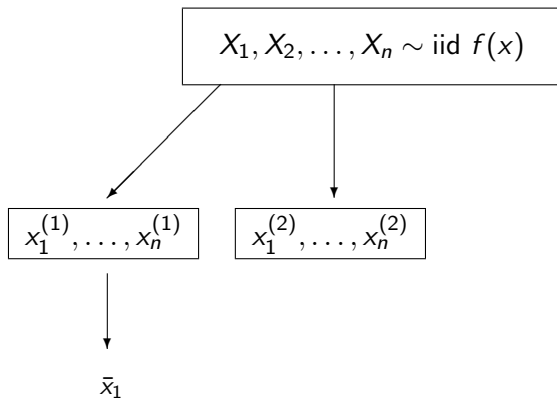
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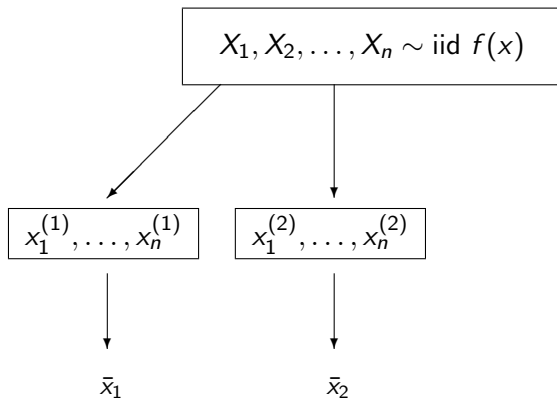
$$x_1^{(1)}, \dots, x_n^{(1)}$$

$$\bar{x}_1$$

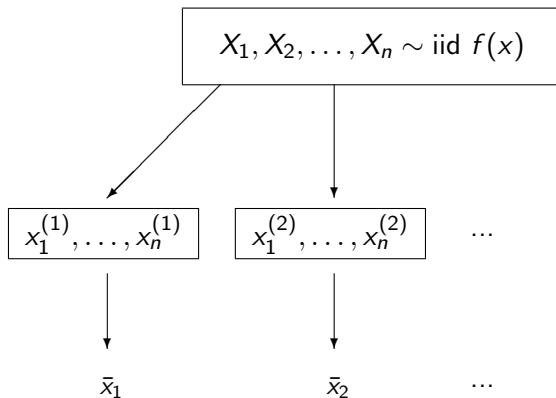
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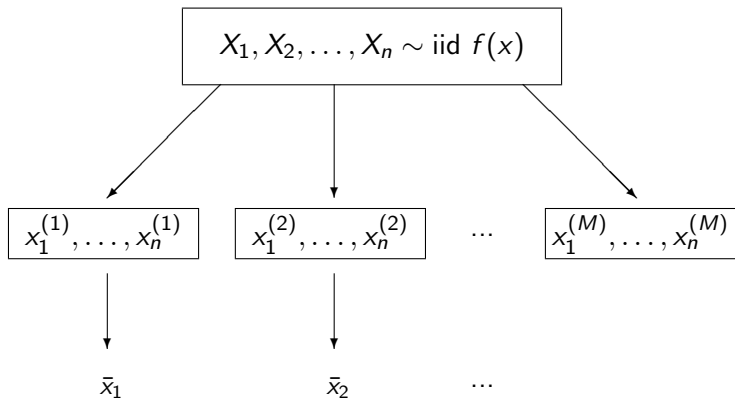
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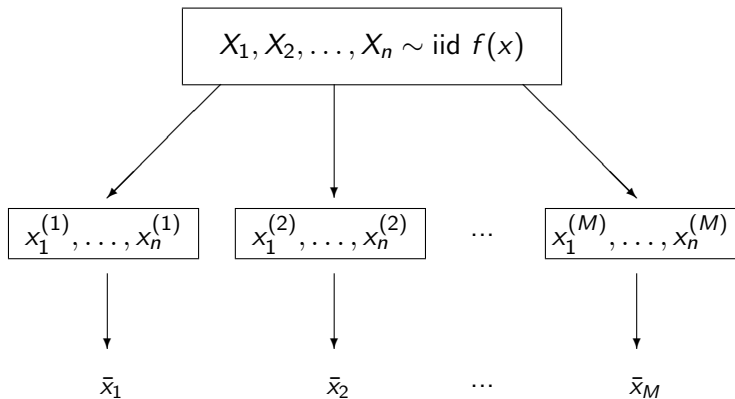
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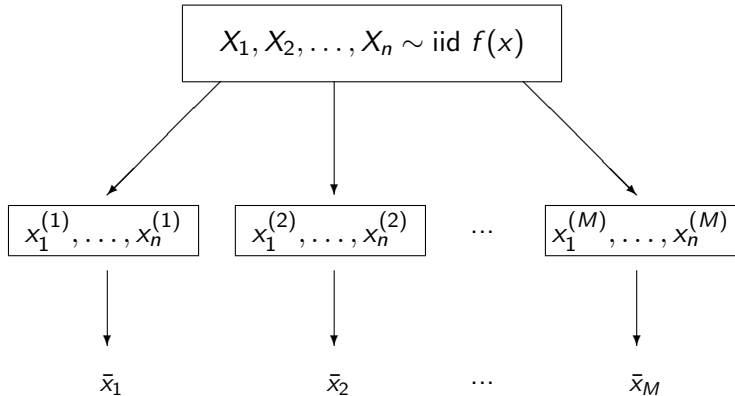
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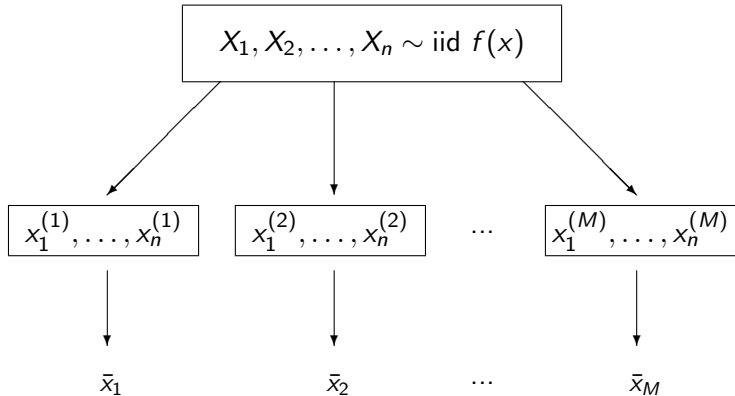


Random Sample of Size n



M Replications yield M different estimates

Random Sample of Size n



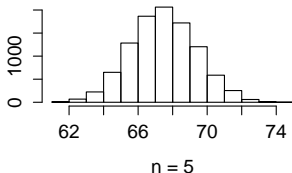
M Replications yield M different estimates

Sampling Distribution: Infinite Replications

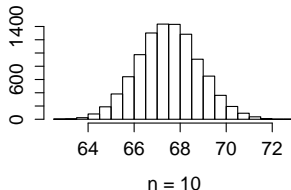
Histograms of sampling distribution of sample mean \bar{X}_n

Random Sampling With Replacement, 10000 Reps. Each

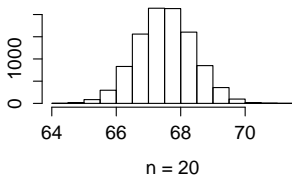
Mean = 67.6, Var = 3.6



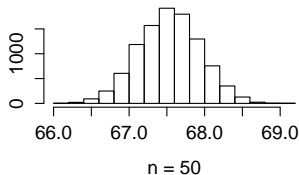
Mean = 67.5, Var = 1.8



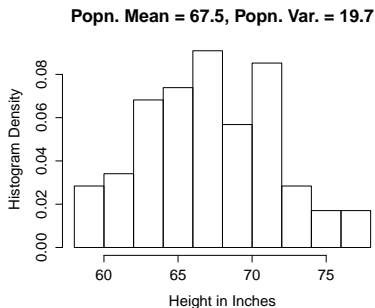
Mean = 67.5, Var = 0.8



Mean = 67.5, Var = 0.2



Population Distribution vs. Sampling Distribution of \bar{X}_n



Sampling Dist. of \bar{X}_n		
n	Mean	Variance
5	67.6	3.6
10	67.5	1.8
20	67.5	0.8
50	67.5	0.2

Two Things to Notice:

1. Sampling dist. “correct on average”
2. Sampling variability decreases with n

$X_1, \dots, X_9 \sim \text{iid}$ with $\mu = 5$, $\sigma^2 = 36$.



Calculate:

$$E(\bar{X}) = E \left[\frac{1}{9}(X_1 + X_2 + \dots + X_9) \right]$$

Mean of Sampling Distribution of \bar{X}_n

$X_1, \dots, X_n \sim \text{iid with mean } \mu$

$$E[\bar{X}_n] = E \left[\frac{1}{n} \sum_{i=1}^n X_i \right]$$

Mean of Sampling Distribution of \bar{X}_n

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Mean of Sampling Distribution of \bar{X}_n

$X_1, \dots, X_n \sim \text{iid}$ with mean μ

$$E[\bar{X}_n] = E\left[\frac{1}{n} \sum_{i=1}^n X_i\right] = \frac{1}{n} \sum_{i=1}^n E[X_i] = \frac{1}{n} \sum_{i=1}^n \mu = \frac{n\mu}{n} = \mu$$

Hence, sample mean is “correct on average.” The formal term for this is *unbiased*.

$X_1, \dots, X_9 \sim \text{iid}$ with $\mu = 5$, $\sigma^2 = 36$.



Calculate:

$$\text{Var}(\bar{X}) = \text{Var} \left[\frac{1}{9}(X_1 + X_2 + \dots + X_9) \right]$$

Variance of Sampling Distribution of \bar{X}_n

$X_1, \dots, X_n \sim \text{iid}$ with mean μ and variance σ^2

$$\text{Var}[\bar{X}_n] = \text{Var} \left[\frac{1}{n} \sum_{i=1}^n X_i \right]$$

Variance of Sampling Distribution of \bar{X}_n

$X_1, \dots, X_n \sim \text{iid}$ with mean μ and variance σ^2

$$\begin{aligned} \text{Var}[\bar{X}_n] &= \text{Var} \left[\frac{1}{n} \sum_{i=1}^n X_i \right] = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) \\ &= \end{aligned}$$

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Hence the variance of the sample mean *decreases linearly with sample size*.

$X_1, \dots, X_9 \sim \text{iid}$ with $\mu = 5$, $\sigma^2 = 36$.



Calculate:

$$SD(\bar{X}) = SD \left[\frac{1}{9}(X_1 + X_2 + \dots + X_9) \right]$$

Standard Error

Std. Dev. of estimator's sampling dist. is called **standard error**.

Standard Error of the Sample Mean

$$SE(\bar{X}_n) = \sqrt{\text{Var}(\bar{X}_n)} = \sqrt{\sigma^2/n} = \sigma/\sqrt{n}$$

Why $(n - 1)$ for sample variance?

Why $(n - 1)$ for sample variance?

We will show that having $n - 1$ in the denominator ensures:

$$E[S^2] = E \left[\frac{1}{n - 1} \sum_{i=1}^n (X_i - \bar{X})^2 \right] = \sigma^2$$

under random sampling.

Why $(n - 1)$ for sample variance?

Step # 1 – Tedious but straightforward algebra gives:

$$\sum_{i=1}^n (X_i - \bar{X})^2 = \left[\sum_{i=1}^n (X_i - \mu)^2 \right] - n(\bar{X} - \mu)^2$$

You are not responsible for proving Step #1 on an exam.

$$\begin{aligned}
\sum_{i=1}^n (X_i - \bar{X})^2 &= \sum_{i=1}^n (X_i - \mu + \mu - \bar{X})^2 = \sum_{i=1}^n [(X_i - \mu) - (\bar{X} - \mu)]^2 \\
&= \sum_{i=1}^n [(X_i - \mu)^2 - 2(X_i - \mu)(\bar{X} - \mu) + (\bar{X} - \mu)^2] \\
&= \sum_{i=1}^n (X_i - \mu)^2 - \sum_{i=1}^n 2(X_i - \mu)(\bar{X} - \mu) + \sum_{i=1}^n (\bar{X} - \mu)^2 \\
&= \left[\sum_{i=1}^n (X_i - \mu)^2 \right] - 2(\bar{X} - \mu) \sum_{i=1}^n (X_i - \mu) + n(\bar{X} - \mu)^2 \\
&= \left[\sum_{i=1}^n (X_i - \mu)^2 \right] - 2(\bar{X} - \mu) \left(\sum_{i=1}^n X_i - \sum_{i=1}^n \mu \right) + n(\bar{X} - \mu)^2 \\
&= \left[\sum_{i=1}^n (X_i - \mu)^2 \right] - 2(\bar{X} - \mu)(n\bar{X} - n\mu) + n(\bar{X} - \mu)^2 \\
&= \left[\sum_{i=1}^n (X_i - \mu)^2 \right] - 2n(\bar{X} - \mu)^2 + n(\bar{X} - \mu)^2 \\
&= \left[\sum_{i=1}^n (X_i - \mu)^2 \right] - n(\bar{X} - \mu)^2
\end{aligned}$$

Why $(n - 1)$ for sample variance?

Step # 2 – Take Expectations of Step # 1:

$$\begin{aligned} E \left[\sum_{i=1}^n (X_i - \bar{X})^2 \right] &= E \left[\left\{ \sum_{i=1}^n (X_i - \mu)^2 \right\} - n(\bar{X} - \mu)^2 \right] \\ &= \end{aligned}$$

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Why $(n - 1)$ for sample variance?

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Where we have used the linearity of expectation.

Why $(n - 1)$ for sample variance?

Step # 3 – Use assumption of random sampling:

$X_1, \dots, X_n \sim$ iid with mean μ and variance σ^2

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Why $(n - 1)$ for sample variance?

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Since we showed earlier today that $E[\bar{X}] = \mu$ and $\text{Var}(\bar{X}) = \sigma^2/n$ under this random sampling assumption.

Why $(n - 1)$ for sample variance?

Finally – Divide Step # 3 by $(n - 1)$:

$$E[S^2] = E\left[\frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2\right] = \frac{(n-1)\sigma^2}{n-1} = \sigma^2$$

Hence, having $(n - 1)$ in the denominator ensures that the sample variance is “correct on average,” that is *unbiased*.