Economics 103 – Statistics for Economists

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Lecture 15

Sampling Distributions and Estimation – Part II

Unbiased means "Right on Average"

Bias of an Estimator

Let $\widehat{\theta}_n$ be a sample estimator of a population parameter θ_0. The *bias* of $\widehat{\theta}_n$ is $E[\widehat{\theta}_n] - \theta_0$.

Unbiased Estimator

A sample estimator $\widehat{\theta}_n$ of a population parameter θ_0 is called *unbiased* if $E[\widehat{\theta}_n] = \theta_0$

A Different Estimator of the Population Variance

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$$

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$$E[\widehat{\sigma}^2] - \sigma^2 = \frac{(n-1)\sigma^2}{n} - \sigma^2 = \frac{(n-1)\sigma^2}{n} - \frac{n\sigma^2}{n} = -\sigma^2/n$$

Procedure versus Result of the Procedure

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Procedure? Long-Run Relative Freqencies?

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Procedure = Random Variable

Making a habit of playing the lottery. Expectation is negative.

Result of that Procedure = Constant

How much you win in a *particular* lottery. Could be greater than or less than cost of ticket in any *individual* instance.

Sampling Distribution of $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

Choose n Students from Class List with Replacement

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$$\begin{array}{c}\boxed{\text{Choose } n \text{ Students from Class List with Replacement}}\\ \swarrow \\ \boxed{\text{Sample 1}}\end{array}$$

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$$
\begin{tikzcd}
\boxed{\text{Choose } n \text{ Students from Class List with Replacement}} \arrow[d] \\
\boxed{\text{Sample 1}} \arrow[d] \\
\bar{x}_1
\end{tikzcd}
$$

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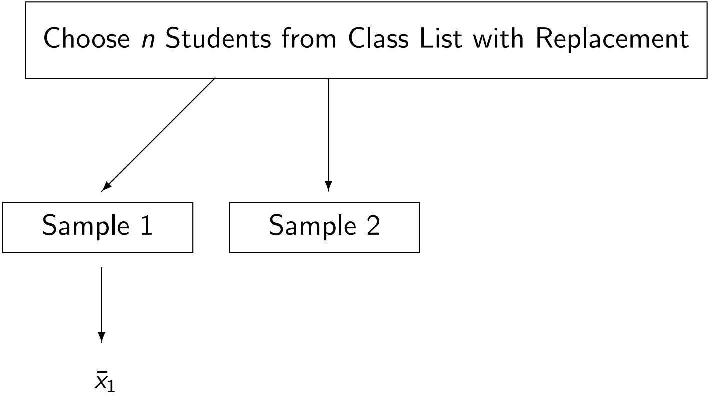

Sampling Distribution of $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

Choose n Students from Class List with Replacement

Sample 1 | Sample 2

$\bar{x}_1$ | $\bar{x}_2$

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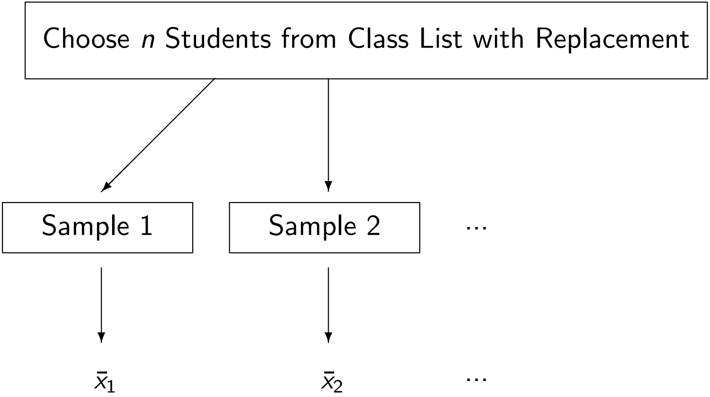

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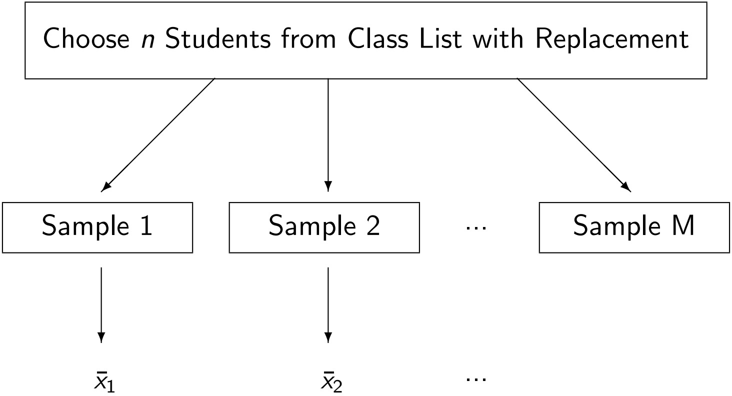

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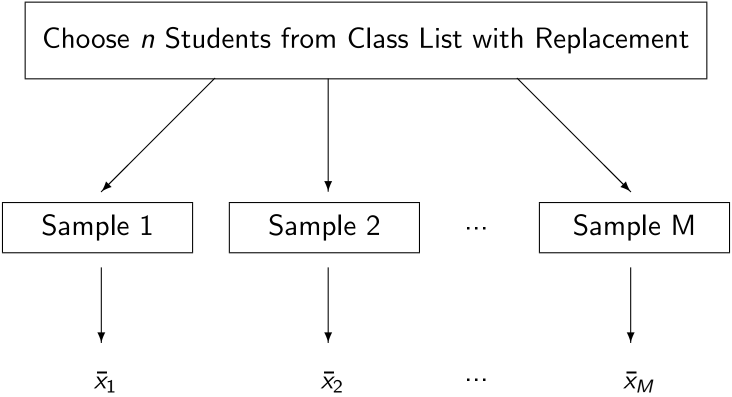

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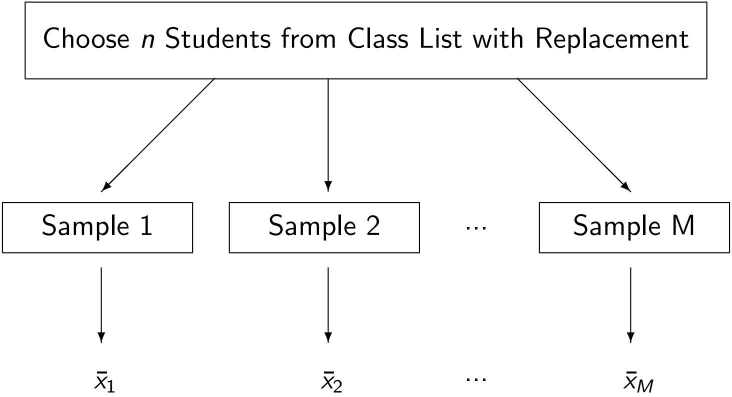

Repeat M times $\rightarrow$ get M different sample means

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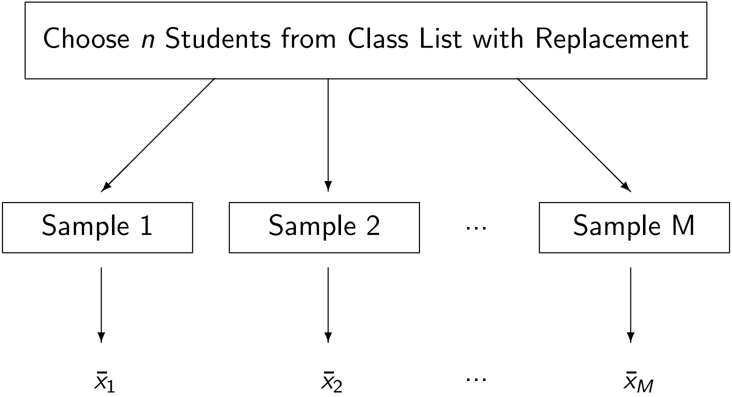

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Sampling Dist: long run relative frequencies of the $\bar{x}_i$

Height of Econ 103 Students

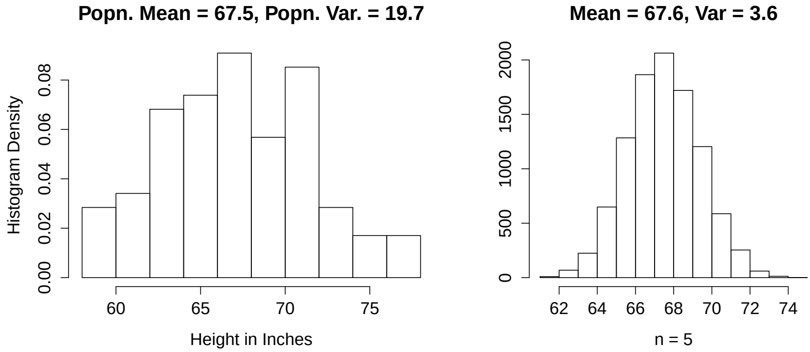

Figure : Left: Population, Right: Sampling distribution of $\bar{X}_5$

How Large is the Average Family?

How many brothers and sisters are in your family, including yourself?

The average number of children per family was about 2.0 twenty years ago.

What's Going On Here?

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- Zero children $\Rightarrow$ didn't send any to college

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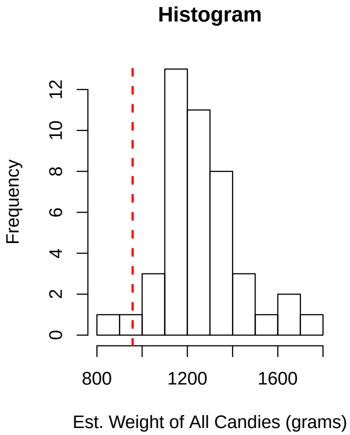
- Sampling by *children* so large families oversampled

Candy Weighing: 44 Estimates, Each With $n = 5$

$$\widehat{\theta} = 20 \times (X_1 + \ldots + X_5)$$

Summary of Sampling Dist.	
Overestimates	42
Exactly Correct	0
Underestimates	2
$E[\hat{\theta}]$	1269 grams
$\text{SD}(\widehat{\theta})$	189 grams

Actual Mass: θ_0 =958 grams



What was in the bag?

100 Candies Total:

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- 20 Fun Size Snickers Bars (large)

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So What Happened?

Not a random sample! The Snickers bars were *oversampled*.

Could we have avoided this? How?

Let $X_1, X_2, \ldots X_n \sim iid$ mean μ, variance σ^2 and define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. True or False:

$\bar{X}_n$ *is an unbiased estimator of* μ

(a) True

(b) False

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TRUE!

Let $X_1, X_2, \dots X_n \sim iid$ mean μ, variance σ^2. True or False:

X_1 is an unbiased estimator of μ

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(b) False

Let $X_1, X_2, \ldots X_n \sim iid$ mean μ, variance σ^2. True or False:

$$X_1 \textit{ is an unbiased estimator of } \mu$$

(a) True

(b) False

TRUE!

How to choose between two unbiased estimators?

Suppose $X_1, X_2, \ldots X_n \sim iid$ with mean μ and variance σ^2

$$E[\bar{X}_n] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \mu$$

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Var(\bar{X}_n) &= Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \sigma^2/n
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Var(X_1) &= \sigma^2
\end{aligned}$$

Efficiency - Compare Unbiased Estimators by Variance

Let $\widehat{\theta}_1$ and $\widehat{\theta}_2$ be unbiased estimators of θ_0. We say that $\widehat{\theta}_1$ is *more efficient* than $\widehat{\theta}_2$ if $Var(\widehat{\theta}_1) < Var(\widehat{\theta}_2)$.

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Mean Squared Error (MSE):

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Mean Squared Error (MSE): Squared Bias plus Variance

$$MSE(\widehat{\theta}) = \text{Bias}(\widehat{\theta})^2 + Var(\widehat{\theta})$$

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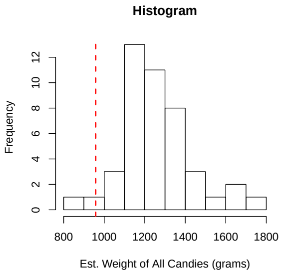
Mean Squared Error (MSE): Squared Bias plus Variance

$$\begin{aligned} MSE(\widehat{\theta}) &= \text{Bias}(\widehat{\theta})^2 + Var(\widehat{\theta}) \\ &= \left(E[\widehat{\theta}] - \theta_0\right)^2 + Var(\widehat{\theta}) \end{aligned}$$

Calculate MSE for Candy Experiment

$E[\hat{\theta}]$	1269 grams
θ_0	958 grams
$SD(\widehat{\theta})$	189 grams



$$\text{Bias} \quad =$$

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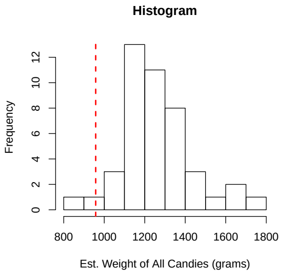

$$
\begin{aligned}
\text{Bias} &= 1269 \text{ grams} - 958 \text{ grams} \\
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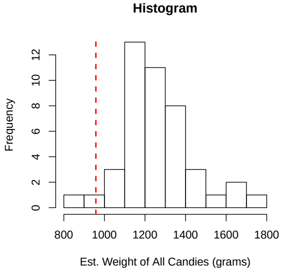

$$\begin{aligned} \text{Bias} &= 1269 \text{ grams} - 958 \text{ grams} \\ &= \color{red}{311 \text{ grams}} \\ \text{MSE} &= \end{aligned}$$

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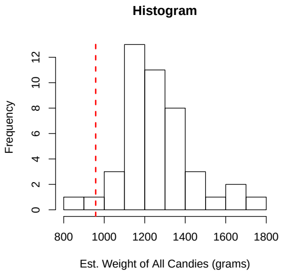

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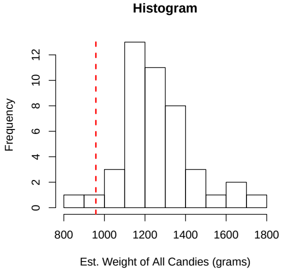

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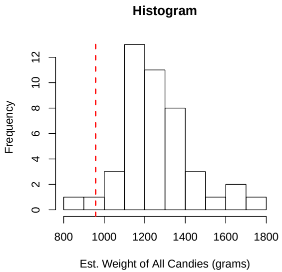

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&= (311^2 + 189^2) \text{ grams}^2 \\
&= 1.3244 \times 10^5 \text{ grams}^2
\end{aligned}$$

Calculate MSE for Candy Experiment

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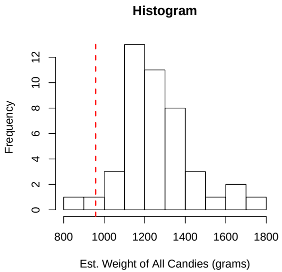

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\text{RMSE} &= \sqrt{\text{MSE}} = 364 \text{ grams}
\end{aligned}
$$

Finite Sample versus Asymptotic Properties of Estimators

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For *fixed sample size n* what are the properties of the sampling distribution of $\widehat{\theta}_n$? (E.g. bias and variance.)

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Asymptotic Properties

What happens to the sampling distribution of $\widehat{\theta}_n$ *as the sample size n gets larger and larger?* (That is, $n \to \infty$).

Why Asymptotics?

Law of Large Numbers

Make precise what we mean by "bigger samples are better."

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Law of Large Numbers

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Central Limit Theorem

As $n \to \infty$ *nearly all* sampling distributions behave behave like a normal random variable!

Consistency

Consistency

If an estimator $\widehat{\theta}_n$ (which is a RV) *converges* to θ_0 (a constant) as $n \to \infty$, we say that *$\widehat{\theta}_n$ is consistent for θ_0*.

What does it mean for a *RV* to converge to a *constant?*

For this course we'll use *MSE Consistency*:

$$\lim_{n \to \infty} \text{MSE}(\widehat{\theta}_n) = 0$$

This makes sense since $\text{MSE}(\widehat{\theta}_n)$ is a *constant*, so this is just an ordinary limit from calculus.

Law of Large Numbers (aka Law of Averages)

Let $X_1, X_2, \ldots X_n \sim iid$ mean μ, variance σ^2. Then the sample mean

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

is consistent for the population mean μ.

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Hence $\bar{X}_n$ is consistent for μ

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Bias of $\widehat{\sigma}^2$

$$E[\widehat{\sigma}^2] - \sigma^2 = \frac{(n-1)\sigma^2}{n} - \sigma^2 = -\sigma^2/n \to 0$$

Suppose $X_1, X_2, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$. What is the sampling distribution of $\bar{X}_n$?

(a) $\chi^2(n)$

(b) $t(n)$

(c) $F(n, n)$

(d) $N(\mu, \sigma^2/n)$

(e) Not enough information to determine.

But still, how can something random converge to something constant?

Sampling Distribution of $\bar{X}_n$ Collapses to μ

Look at an example where we can directly calculate not only the mean and variance of the sampling distribution of $\bar{X}_n$, but the *sampling distribution itself*:

$$X_1, X_2, \ldots, X_n \sim \text{iid } N(\mu, \sigma^2) \Rightarrow \bar{X}_n \sim N(\mu, \sigma^2/n)$$

Sampling Distribution of $\bar{X}_n$ Collapses to μ

$X_1, X_2, \ldots, X_n \sim$ iid $N(\mu, \sigma^2 \Rightarrow \bar{X}_n \sim N(\mu, \sigma^2/n)$.

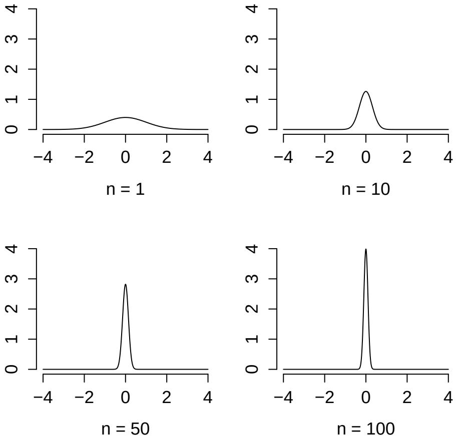

Figure : Sampling Distributions for $\bar{X}_n$ where $X_i \sim$ iid $N(0, 1)$

Another Visualization: Keep Adding Observations

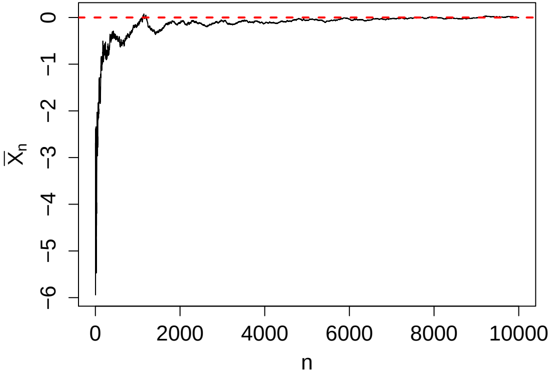

Figure : Running sample means: $X_i \sim$ iid $N(0, 100)$

n	$\bar{X}_n$
1	-2.69
2	-3.18
3	-5.94
4	-4.27
5	-2.62
10	-2.89
20	-5.33
50	-2.94
100	-1.58
500	-0.45
1000	-0.13
5000	-0.05
10000	0.00

Important!

Although I showed two examples involving normal RVs, the LLN holds IN GENERAL!