Economics 103 – Statistics for Economists

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Lecture # 12

Continuous Distributions – Part II

The Most Important RV of All

Normal Random Variable

Notation: $X \sim N(\mu, \sigma^2)$

Parameters: $\mu = E[X]$, $\sigma^2 = Var(X)$

Support: $(-\infty, +\infty)$

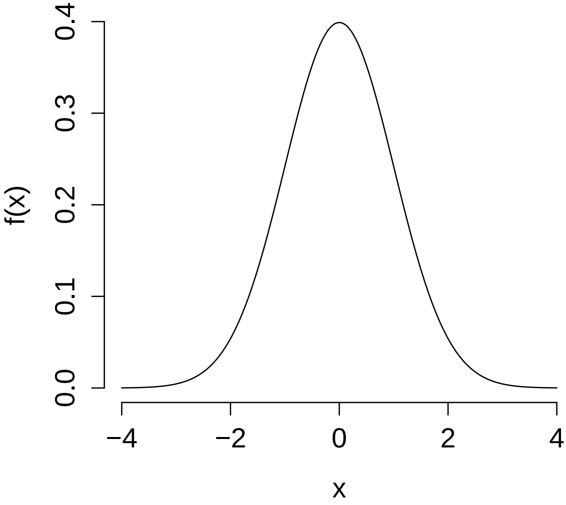
Probability Density Function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$$

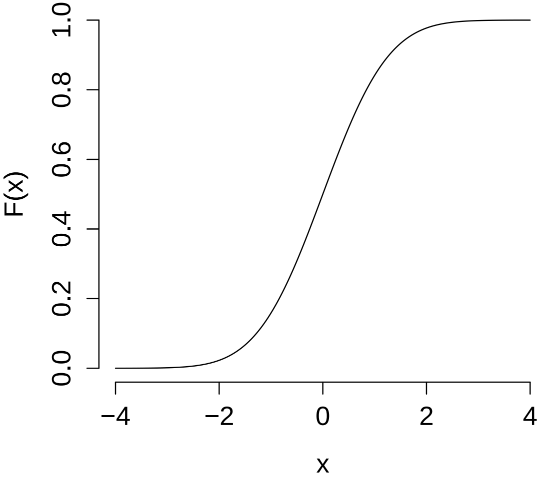
No Explicit Formula for CDF (use computer instead)

$$F(x_0) = \int_{-\infty}^{x_0} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \, dx$$

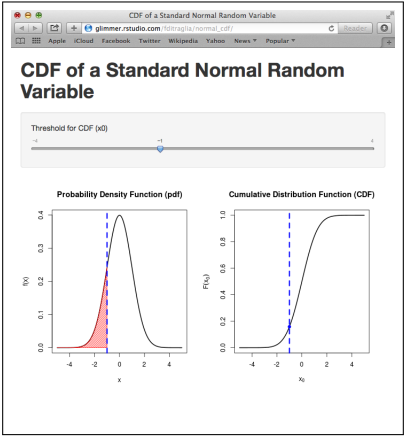
Normal PDF Centered at the Mean (Here $\mu = 0$, $\sigma^2 = 1$)



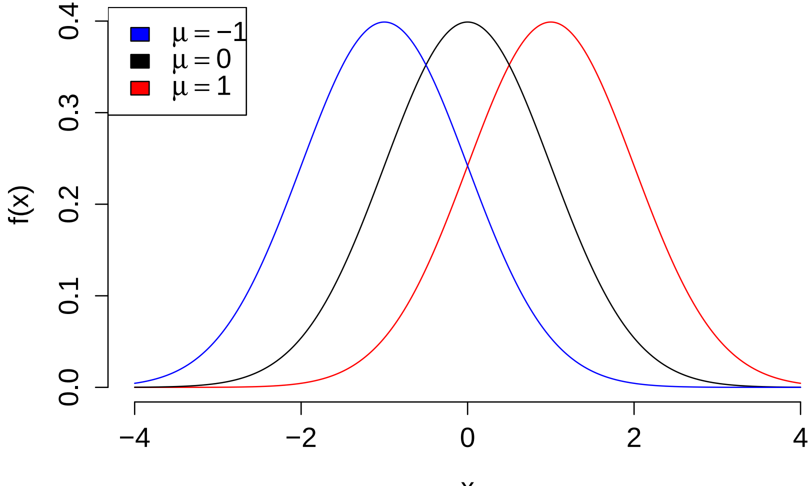
Normal CDF ($\mu = 0$, $\sigma^2 = 1$)



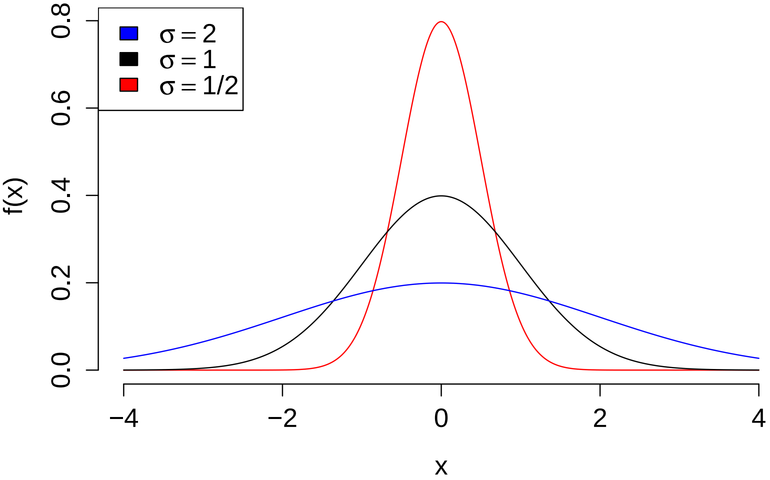
http://glimmer.rstudio.com/fditraglia/normal_cdf/



Different Means, Same Variance



Same Mean, Different Variances



Linear Function of Normal RV is a Normal RV

Suppose that $X \sim N(\mu, \sigma^2)$. Then if a and b constants,

$$\boxed{a + bX \sim N(a + b\mu, b^2\sigma^2)}$$

Important

- Using what know know about expectations of linear functions, no surprise what mean and variance are.
- Surprise is that the linear combination is *normal*
- Linear trans. does not preserve, e.g., Bernoulli or Binomial.

Example

Suppose $X \sim N(\mu, \sigma^2)$ and let $Z = (X - \mu)/\sigma$. What is the distribution of Z?

(a) $N(\mu, \sigma^2)$

(b) $N(\mu, \sigma)$

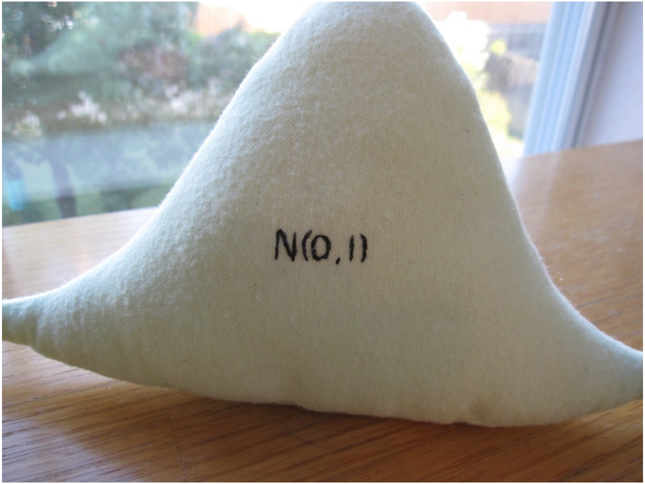
(c) $N(0, \sigma^2)$

(d) $N(0, \sigma)$

(e) $N(0, 1)$

Figure : Standard Normal Distribution (PDF)

Standard Normal Distribution: $N(0, 1)$



Standard Normal Distribution: $N(0,1)$

Mean = 0, Variance = Standard Deviation = 1

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Special symbol for Standard Normal CDF (no closed form):

$$\Phi(x_0) = \int_{-\infty}^{x_0} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

R Command: $\Phi(x_0) =$`pnorm()`

Where does the Empirical Rule come from?

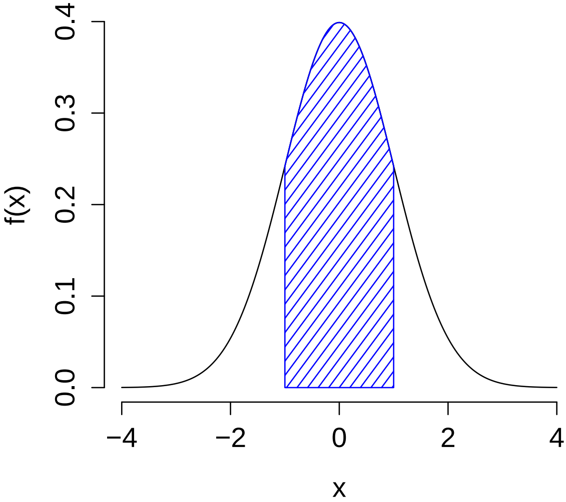
Empirical Rule

Approximately 68% of observations within $\mu \pm \sigma$

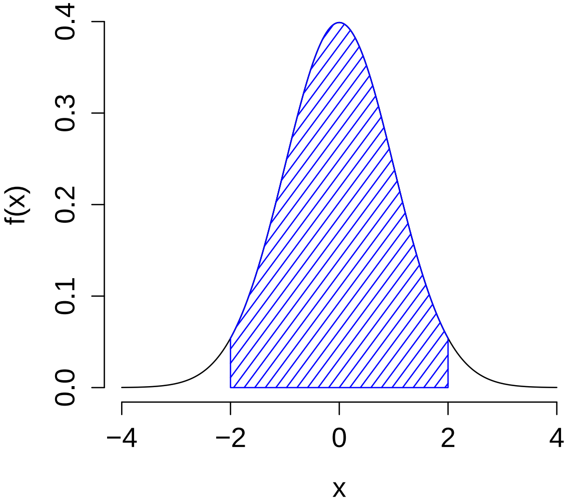
Approximately 95% of observations within $\mu \pm 2\sigma$

Nearly all observations within $\mu \pm 3\sigma$

Middle 68% of $N(0,1) \Rightarrow$ approx. $(-1, 1)$



Middle 95% of $N(0,1) \Rightarrow$ approx. $(-2, 2)$



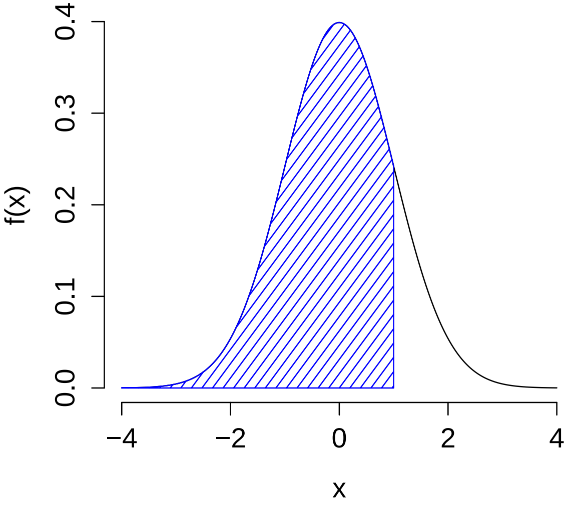
More Formally...

$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \approx 0.68$$

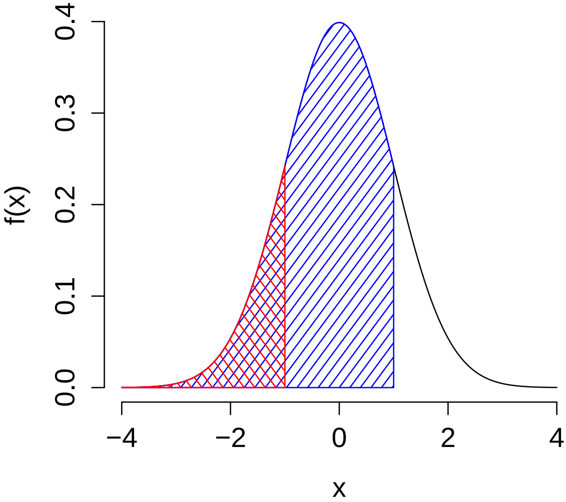
$$\int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \approx 0.95$$

But how do we know this?

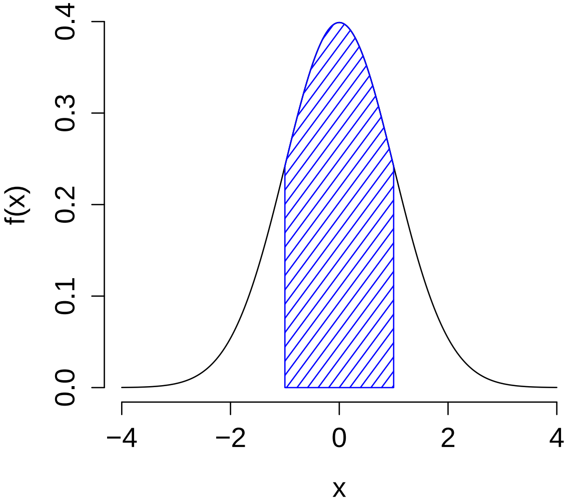
$\Phi(1) = \texttt{pnorm(1)} \approx 0.84$



$\Phi(1) - \Phi(-1) =$`pnorm(1) - pnorm(-1)`$\approx 0.84 - 0.16$



$\Phi(1) - \Phi(-1) =$`pnorm(1) - pnorm(-1)`≈ 0.68



Suppose $X \sim N(0,1)$

$$
\begin{aligned}
P(-2 \leq X \leq 2) &= \Phi(2) - \Phi(-2) \\
&= \texttt{pnorm(2) - pnorm(-2)} \\
&\approx 0.95
\end{aligned}
$$

$$
\begin{aligned}
P(-3 \leq X \leq 3) &= \Phi(3) - \Phi(-3) \\
&= \texttt{pnorm(3) - pnorm(-3)} \\
&\approx 1
\end{aligned}
$$

What if $X \sim N(\mu, \sigma^2)$?

$$
\begin{aligned}
P(X \leq a) &= P(X - \mu \leq a - \mu) \\
&= P\left(\frac{X - \mu}{\sigma} \leq \frac{a - \mu}{\sigma}\right) \\
&= P\left(Z \leq \frac{a - \mu}{\sigma}\right)
\end{aligned}
$$

Where Z is a standard normal random variable, i.e. $N(0, 1)$.

Which of these equals $P\left(Z \leq (a-\mu)/\sigma\right)$ if $Z \sim N(0,1)$?

(a) $\Phi(a)$

(b) $1 - \Phi(a)$

(c) $\Phi(a)/\sigma - \mu$

(d) $\Phi\left(\frac{a-\mu}{\sigma}\right)$

(e) None of the above.

What if $X \sim N(\mu, \sigma^2)$?

$$
\begin{aligned}
P(X \leq a) &= P(X - \mu \leq a - \mu) \\
&= P\left(\frac{X - \mu}{\sigma} \leq \frac{a - \mu}{\sigma}\right) \\
&= P\left(Z \leq \frac{a - \mu}{\sigma}\right) \\
&= \Phi\left(\frac{a - \mu}{\sigma}\right) \\
&= \texttt{pnorm}((a - \mu)/\sigma)
\end{aligned}
$$

Where Z is a standard normal random variable, i.e. $N(0, 1)$.

Suppose $X \sim N(\mu, \sigma^2)$

Which of these is $P(X \geq b)$?

(a) $\Phi(b)$

(b) $1 - \Phi\left(\frac{b-\mu}{\sigma}\right)$

(c) $1 - \Phi(b)$

(d) $1 - (\Phi(b)/\sigma - \mu)$

Suppose $X \sim N(\mu, \sigma^2)$

$$
\begin{aligned}
P(X \geq b) &= 1 - P(X \leq b) = 1 - P\left(\frac{X - \mu}{\sigma} \leq \frac{b - \mu}{\sigma}\right) \\
&= 1 - P\left(Z \leq \frac{b - \mu}{\sigma}\right) = 1 - \Phi\left(\frac{b - \mu}{\sigma}\right) \\
&= 1 - \texttt{pnorm}((b - \mu)/\sigma)
\end{aligned}
$$

Where Z is a standard normal random variable.

Suppose $X \sim N(\mu, \sigma^2)$

$$
\begin{aligned}
P(a \leq X \leq b) &= P\left(\frac{a-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}\right) \\
&= P\left(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right) \\
&= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \\
&= \texttt{pnorm}((b-\mu)/\sigma) - \texttt{pnorm}((a-\mu)/\sigma)
\end{aligned}
$$

Where Z is a standard normal random variable.

Suppose $X \sim N(\mu, \sigma^2)$

What is $P(\mu - \sigma \leq X \leq \mu + \sigma)$?

Suppose $X \sim N(\mu, \sigma^2)$

$$
\begin{aligned}
P(\mu - \sigma \leq X \leq \mu + \sigma) &= P\left(-1 \leq \frac{X - \mu}{\sigma} \leq 1\right) \\
&= P\left(-1 \leq Z \leq 1\right) \\
&= \Phi(1) - \Phi(-1) \\
&= \texttt{pnorm(1)} - \texttt{pnorm(-1)} \\
&\approx 0.68
\end{aligned}
$$

Suppose $X \sim N(\mu, \sigma^2)$

What is $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$?

Suppose $X \sim N(\mu, \sigma^2)$

$$
\begin{aligned}
P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) &= P\left(-2 \leq \frac{X - \mu}{\sigma} \leq 2\right) \\
&= P\left(-2 \leq Z \leq 2\right) \\
&= \Phi(2) - \Phi(-2) \\
&= \texttt{pnorm(2)} - \texttt{pnorm(-2)} \\
&\approx 0.95
\end{aligned}
$$

Percentiles/Quantiles for Continuous RVs

Quantile Function $Q(p)$ is the inverse of CDF $F(x_0)$

Plug in a probability p, get out the value of x_0 such that $F(x_0) = p$

$$Q(p) = F^{-1}(p)$$

In other words:

$$Q(p) = \text{the value of } x_0 \text{ such that } \int_{-\infty}^{x_0} f(x)\,dx = p$$

Inverse exists as long as $F(x_0)$ is *strictly increasing*.

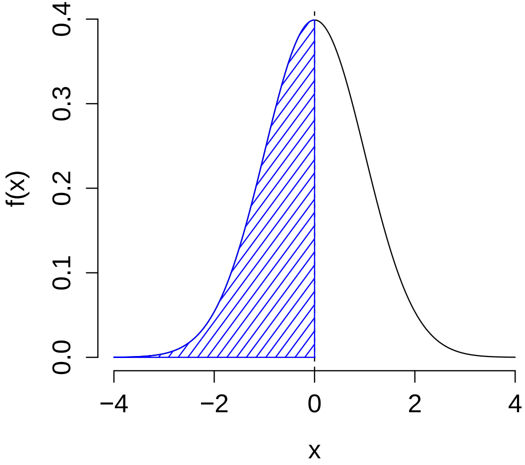
Example: Median

The median of a continuous random variable is $Q(0.5)$, i.e. the value of x_0 such that

$$\int_{-\infty}^{x_0} f(x)\,dx = 1/2$$

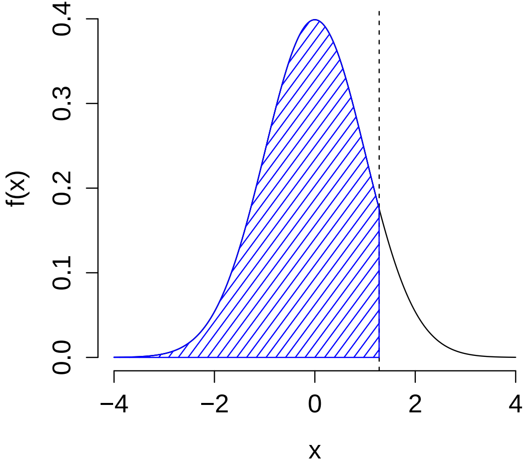
What is the median of a standard normal RV?

By symmetry, $Q(0.5) = 0$. R command: `qnorm()`



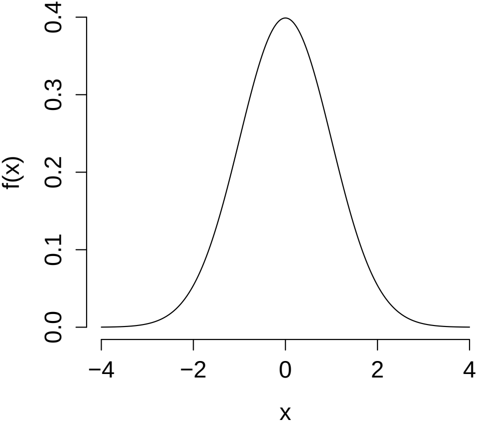
90th Percentile of a Standard Normal

`qnorm(0.9)` ≈ 1.28



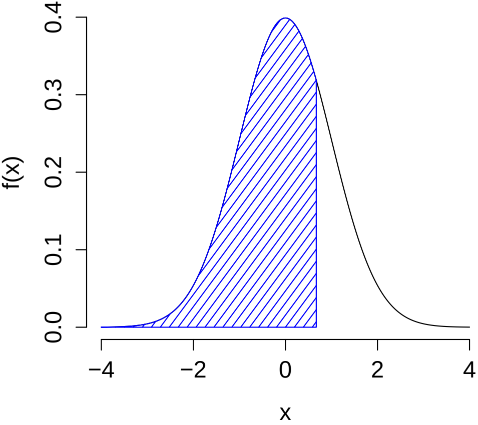
Using Quantile Function to find Symmetric Intervals

Suppose X is a standard normal RV. What is the value of c such that $P(-c \leq X \leq c) = 0.5$?



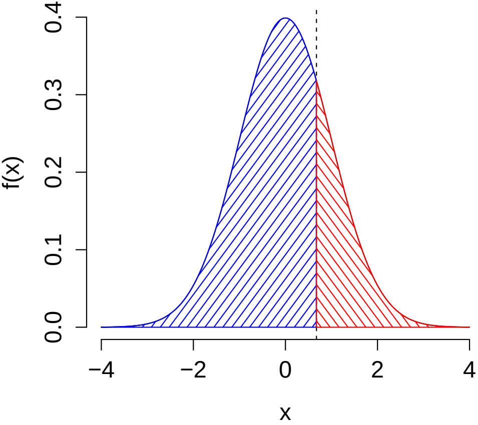
`qnorm(0.75)` ≈ 0.67

Suppose X is a standard normal RV. What is the value of c such that $P(-c \leq X \leq c) = 0.5$?



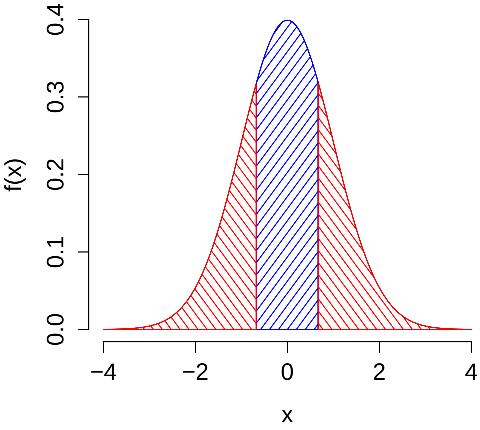
`qnorm(0.75)` ≈ 0.67

Suppose X is a standard normal RV. What is the value of c such that $P(-c \leq X \leq c) = 0.5$?



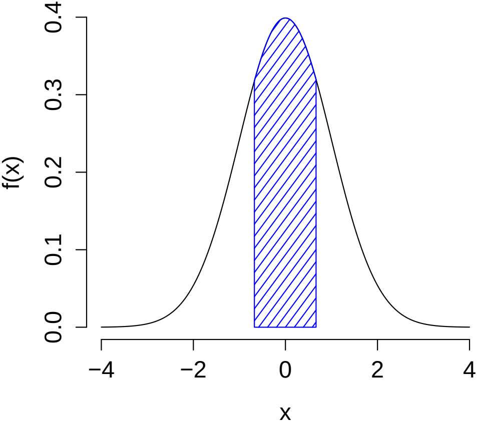
`pnorm(0.67)-pnorm(-0.67)`$\approx$?

Suppose X is a standard normal RV. What is the value of c such that $P(-c \leq X \leq c) = 0.5$?



pnorm(0.67)-pnorm(-0.67)$\approx$ 0.5

Suppose X is a standard normal RV. What is the value of c such that $P(-c \leq X \leq c) = 0.5$?



68% Central Interval for Standard Normal

Suppose X is a standard normal random variable. What value of c ensures that $P(-c \leq X \leq c) \approx 0.68$?

95% Central Interval for Standard Normal

Suppose X is a standard normal random variable. What value of c ensures that $P(-c \leq X \leq c) \approx 0.95$?

R Commands for *Arbitrary* Normal Distributions

Let $X \sim N(\mu, \sigma^2)$. Then we can use R to evaluate the CDF and Quantile function of X as follows:

CDF $F(x)$	`pnorm(x, mean =` μ`, sd =` σ`)`
Quantile Function $Q(p)$	`qnorm(p, mean =` μ`, sd =` σ`)`

Notice that this means you don't have to transform X to a standard normal in order to find areas under its pdf using R.

Example from Homework: $X \sim N(0, 16)$

One Way:

$$\begin{aligned} P(X \geq 10) &= 1 - P(X \leq 10) = 1 - P(X/4 \leq 10/4) \\ &= 1 - P(Z \leq 2.5) = 1 - \Phi(2.5) = 1 - \texttt{pnorm(2.5)} \\ &\approx 0.006 \end{aligned}$$

An Easier Way:

$$\begin{aligned} P(X \geq 10) &= 1 - P(X \leq 10) \\ &= 1 - \texttt{pnorm(10, mean = 0, sd = 4)} \\ &\approx 0.006 \end{aligned}$$

Suppose X has mean μ_x variance σ_x^2 and is independent of Y, which has mean μ_y variance σ_y^2. Let a, b be constants.

What is $E[aX + bY]$?

$E[aX + bY] = aE[X] + bE[Y] = a\mu_x + b\mu_y$

What is $Var(aX + bY)$?

$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) = a^2\sigma_x^2 + b^2\sigma_y^2$

By independence.

Now suppose $X \sim N(\mu_x, \sigma_x^2)$ independent of $Y \sim N(\mu_y, \sigma_y^2)$. Let a, b be constants.

What is $E[aX + bY]$?

$E[aX + bY] = aE[X] + bE[Y] = a\mu_x + b\mu_y$

What is $Var(aX + bY)$?

$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) = a^2\sigma_x^2 + b^2\sigma_y^2$

By independence.

Here's the Surprising Thing:

If X and Y are independent Normal Random Variables and a, b are constants, then $aX + bY$ is *also* a Normal Random Variable!

Linear Combinations of Independent Normals

Let $X \sim N(\mu_x, \sigma_x^2)$ independent of $Y \sim N(\mu_y, \sigma_y^2)$. Then if a, b, c are constants:

$$\boxed{aX + bY + c \sim N(a\mu_x + b\mu_y + c, a^2\sigma_x^2 + b^2\sigma_y^2)}$$

Important

- Result assumes independence
- Particular to Normal Distribution
- Extends to more than two Normal RVs

Suppose $X_1, X_2, \sim$ iid $N(\mu, \sigma^2)$

Let $\bar{X} = (X_1 + X_2)/2$. What is the distribution of $\bar{X}$?

(a) $N(\mu, \sigma^2/2)$

(b) $N(0, 1)$

(c) $N(\mu, \sigma^2)$

(d) $N(\mu, 2\sigma^2)$

(e) $N(2\mu, 2\sigma^2)$