Proof of the Law of Iterated Expectations

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Econ 103

This is not required for the course: only read it if you're interested. The proof I give below is for discrete RVs. The proof for continuous RVs is similar but uses integrals rather than sums. I've added subscripts to the expectations below to make it clear *which* pmf we're using to take the expectation.

$$\begin{aligned}
E_X\left[E_{Y|X}\left[Y|X\right]\right] &= E_X\left[\sum_y y\, p_{Y|X}(y|x)\right] \\
&= \sum_x \left(\sum_y y\, p_{Y|X}(y|x)\right) p_X(x) \\
&= \sum_x \sum_y y\, p_X(x)\, p_{Y|X}(y|x) \\
&= \sum_x \sum_y y\, p_{XY}(x,y) \\
&= \sum_y y \sum_x p_{XY}(x,y) \\
&= \sum_y y\, p_Y(y) \\
&= E[Y]
\end{aligned}$$

The second step is the tricky one. Here what we're doing is calculating the expected value of the function $g(X) = \sum_y p_{Y|X}(y|X)$ with respect to the *marginal* pmf of X. The other steps should be familiar from the derivations we did in class.