Economics 103 – Statistics for Economists

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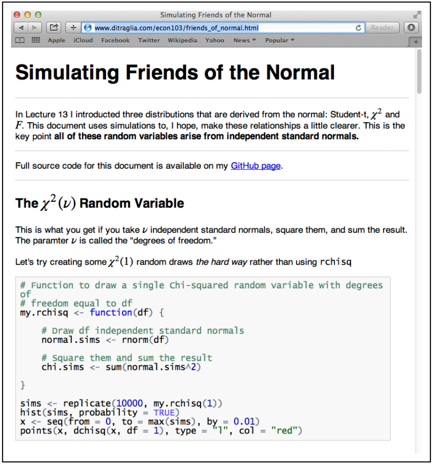
Lecture # 13

Continuous RVs – Part III

Figure: The Normal Distribution and Friends.

http://ditraglia.com/Econ103Public/Rtutorials/friends_of_normal.html

Source Code on my Github Page



Simulating Friends of the Normal

www.ditraglia.com/econ103/friends_of_normal.html

Simulating Friends of the Normal

In Lecture 13 I introducted three distributions that are derived from the normal: Student-t, χ^2 and F. This document uses simulations to, I hope, make these relationships a little clearer. This is the key point **all of these random variables arise from independent standard normals.**

Full source code for this document is available on my GitHub page.

The $\chi^2(\nu)$ Random Variable

This is what you get if you take ν independent standard normals, square them, and sum the result. The paramter ν is called the “degrees of freedom.”

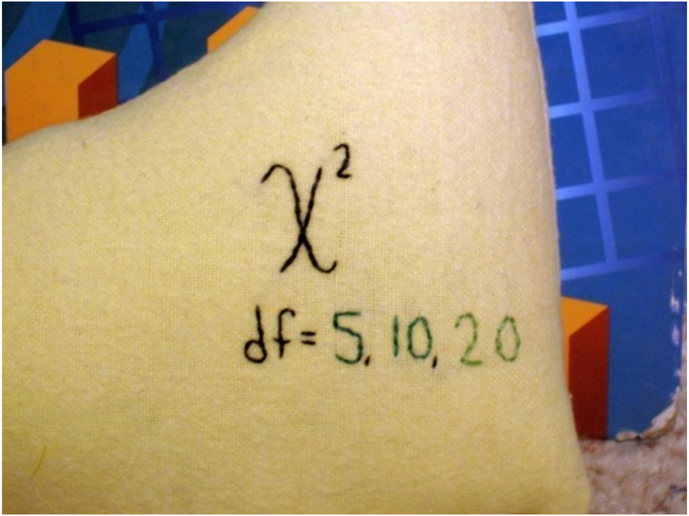
Let's try creating some $\chi^2(1)$ random draws *the hard way* rather than using `rchisq`

```r
# Function to draw a single Chi-squared random variable with degrees
of
# freedom equal to df
my.rchisq <- function(df) {

    # Draw df independent standard normals
    normal.sims <- rnorm(df)

    # Square them and sum the result
    chi.sims <- sum(normal.sims^2)

}

sims <- replicate(10000, my.rchisq(1))
hist(sims, probability = TRUE)
x <- seq(from = 0, to = max(sims), by = 0.01)
points(x, dchisq(x, df = 1), type = "l", col = "red")
```

Functions of Independent RVs are Independent

If X and Y are independent random variables and g and h are functions, then the random variables $g(X)$ and $h(Y)$ are also independent.

Figure: PDF for χ^2-Distribution


χ^2
df = 5, 10, 20

χ^2 Random Variable

Let $X_1, \ldots, X_\nu \sim$ iid $N(0,1)$. Then,

$$\left(X_1^2 + \ldots + X_\nu^2\right) \sim \chi^2(\nu)$$

where the parameter ν is the *degrees of freedom*

Support $= (0, \infty)$

χ^2 PDFs

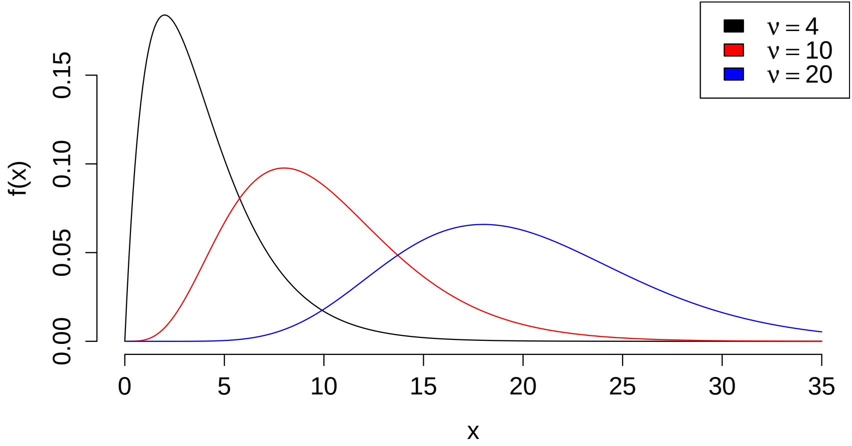
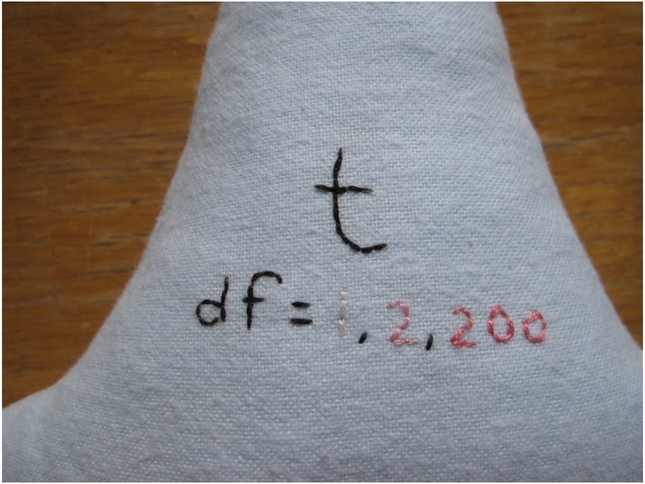


Figure: PDF for Student-t Distribution


t
df=1,2,200

Student-t Random Variable

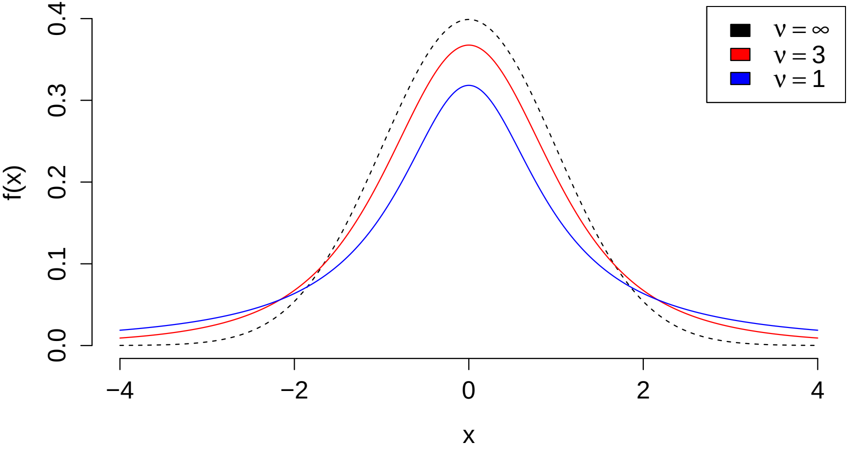
Let $X \sim N(0, 1)$ independent of $Y \sim \chi^2(\nu)$. Then,

$$\frac{X}{\sqrt{Y/\nu}} \sim t(\nu)$$

where the parameter ν is the degrees of freedom.

- Support $= (-\infty, \infty)$
- As $\nu \to \infty$, $t \to$ Standard Normal.
- Symmetric around zero, but mean and variance may not exist!
- Degrees of freedom ν control "thickness of tails"

Student-t PDFs



F Random Variable

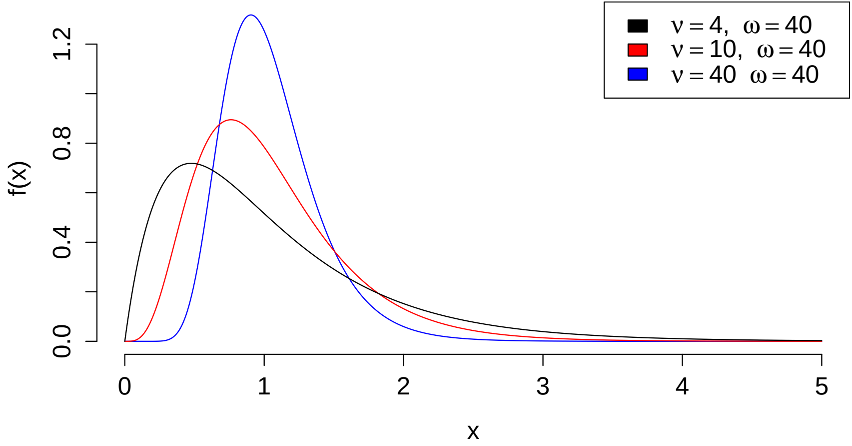
Suppose $X \sim \chi^2(\nu)$ independent of $Y \sim \chi^2(\omega)$. Then,

$$\frac{X/\nu}{Y/\omega} \sim F(\nu, \omega)$$

where ν is the numerator degrees of freedom and ω is the denominator degrees of freedom.

Support $= (0, \infty)$

F PDFs



R Commands – CDFs and Quantile Functions

$F(x) = P(X \leq x)$ is the CDF, $Q(p) = F^{-1}(p)$ the Quantile Function

	$F(x)$	$Q(p)$
$N(\mu, \sigma^2)$	`pnorm(x, mean =` μ`, sd =` σ`)`	`qnorm(p, mean =` μ`, sd =` σ`)`
$\chi^2(\nu)$	`pchisq(x, df =` ν`)`	`qchisq(p, df =` ν`)`
$t(\nu)$	`pt(x, df =` ν`)`	`qt(p, df =` ν`)`
$F(\nu, \omega)$	`pf(x, df1 =` ν`, df2 =` ω`)`	`qf(p, df1 =` ν`, df2 =` ω`)`

Mnemonic: "p" is for Probability, "q" is for Quantile.

R Commands – PDFs and Random Draws

	$f(x)$	Make n iid Random Draws
$N(\mu, \sigma^2)$	dnorm(x, mean = μ, sd = σ)	rnorm(n, mean = μ, sd = σ)
$\chi^2(\nu)$	dchisq(x, df = ν)	rchisq(n, df = ν)
$t(\nu)$	dt(x, df = ν)	rt(n, df = ν)
$F(\nu, \omega)$	df(x, df1 = ν, df2 = ω)	rf(n, df1 = ν, df2 = ω)

Mnemonic: "d" is for Density, "r" is for Random.

Example: $X_1, X_2, X_3 \sim$ iid $N(0,1)$

What is the distribution of $Y_1 = X_1^2 + X_2^2$?

Sum of squares of two indep. std. normals $\Rightarrow Y_1 \sim \chi^2(2)$

What is the distribution of $Y_2 = (Y_1/2)/(X_3^2)$?

$Y_1 \sim \chi^2(2)$ and $X_3^2 \sim \chi^2(1)$

Hence $Y_2 =$ ratio of two indep. χ^2 RVs, each divided by its degrees of freedom $\Rightarrow Y_2 \sim F(2,1)$

What is the distribution of $Z = X_3/\sqrt{Y_1/2}$?

Ratio of standard normal and square root of independent χ^2 RV divided by its degrees of freedom $\Rightarrow Z \sim t(2)$

Suppose $X_1, X_2, \sim$ iid $N(\mu, \sigma^2)$

Let $Y = (X_1 - \mu)^2 + (X_2 - \mu)^2$. What is the distribution of Y/σ^2?

(a) $F(2, 1)$

(b) $\chi^2(2)$

(c) $t(2)$

(d) $N(\mu, \sigma)$

(e) None of the above

$$Y_1 \sim \chi^2(2), \quad Y_2 \sim F(2,1), \quad Z \sim t(2)$$

What is the median of Y_1?

`qchisq(0.5, df = 2)` ≈ 1.4

What is $P(Y_2 \leq 5)$?

`pf(5, df1 = 2, df2 = 1)` ≈ 0.7

What value of c gives $P(-c \leq Z \leq c) = 0.5$?

Use Symmetry (like normal)

$c =$ `qt(0.75, df = 2)` ≈ 0.8

or equivalently $-c =$ `qt(0.25, df = 2)` ≈ -0.8