

# Economics 103 – Statistics for Economists

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Lecture # 6

# Basic Probability – Part II

# Classical Probability Examples Where Order Doesn't Matter

# Poker – Deal 5 Cards, Order Doesn't Matter

## Basic Outcomes

$\binom{52}{5}$  possible hands

How Many Hands have Four Aces?



48 (# of ways to choose the single card that is not an ace)

What is the Probability of Getting Four Aces?

$$48 / \binom{52}{5}$$

# Poker – Deal 5 Cards, Order Doesn't Matter



How Many Hands have Four of a Kind?

- ▶ 13 ways to choose *which* card we have four of
- ▶ 48 ways to choose the last card in the hand
- ▶  $13 \times 48 = 624$

What is the Probability of Being Dealt 4 of a Kind?

$$624 / \binom{52}{5}$$

Even if the basic outcomes are  
equally likely, the events of  
interest may not be...

## “Odd Question” # 4

To throw a total of 7 with a pair of dice, you have to get a 1 and a 6, or a 2 and a 5, or a 3 and a 4. To throw a total of 6 with a pair of dice, you have to get a 1 and a 5, or a 2 and a 4, or a 3 and another 3. With two fair dice, you would expect:

- (a) To throw 7 more frequently than 6.
- (b) To throw six more frequently than 7.
- (c) To throw 6 and 7 equally often.

## Basic Outcomes Equally Likely, Events of Interest Aren't

		Second Die					
		1	2	3	4	5	6
First Die	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

**Table :** There are 36 equally likely basic outcomes, of which 5 correspond to a sum of six and 6 correspond to a sum of seven.

$$P(7) = 6/36 = 1/6$$

$$P(6) = 5/36$$



# Derive Rules for Computing Probabilities from Axioms

## Recall: Axioms of Probability

Let  $S$  be the sample space. With each event  $A \subseteq S$  we associate a real number  $P(A)$  called the **probability of  $A$** , satisfying the following conditions:

**Axiom 1**  $0 \leq P(A) \leq 1$

**Axiom 2**  $P(S) = 1$

**Axiom 3** If  $A_1, A_2, A_3, \dots$  are mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \dots) = P(A_1) + P(A_2) + P(A_3) + \dots$$

## Key Point

The axioms of probability are our *starting assumptions* – they are a complete description of what we *mean* when we say “probability.” We use the axioms to derive various results for *computing* probabilities.

## The Complement Rule: $P(A^c) = 1 - P(A)$

Since  $A, A^c$  are mutually exclusive and collectively exhaustive:

$$P(A \cup A^c) = P(A) + P(A^c) = P(S) = 1$$

Rearranging:

$$P(A^c) = 1 - P(A)$$

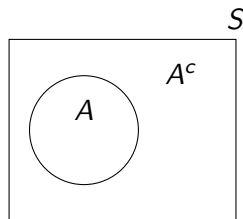


Figure :  $A \cap A^c = \emptyset$ ,  
 $A \cup A^c = S$

## Another Important Rule – Equivalent Events

If A and B are Logically Equivalent, then  $P(A) = P(B)$ .

In other words, if A and B contain exactly the same basic outcomes, then  $P(A) = P(B)$ .

Although this seems obvious it's important to keep in mind, especially later in the course...

# The Logical Consequence Rule

If  $B$  Logically Entails  $A$ , then  $P(B) \leq P(A)$

In other words,  $B \subseteq A \Rightarrow P(B) \leq P(A)$

Why is this so?

If  $B \subseteq A$ , then all the basic outcomes in  $B$  are also in  $A$ .

## Deriving The Logical Consequence Rule

Since  $B \subseteq A$ , we have  $B = A \cap B$  and  $A = B \cup (A \cap B^c)$ . Combining these,

$$A = (A \cap B) \cup (A \cap B^c)$$

Now since  $(A \cap B) \cap (A \cap B^c) = \emptyset$ ,

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^c) \\ &= P(B) + P(A \cap B^c) \\ &\geq P(B) \end{aligned}$$

because  $0 \leq P(A \cap B^c) \leq 1$ .

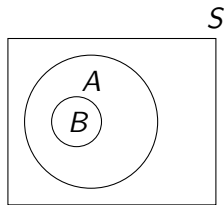


Figure :

$B = A \cap B$ , and  
 $A = B \cup (A \cap B^c)$

## “Odd Question” # 2

Pia is thirty-one years old, single, outspoken, and smart. She was a philosophy major. When a student, she was an ardent supporter of Native American rights, and she picketed a department store that had no facilities for nursing mothers. Rank the following statements in order from most probable to least probable.

- (a) Pia is an active feminist.
- (b) Pia is a bank teller.
- (c) Pia works in a small bookstore.
- (d) Pia is a bank teller and an active feminist.
- (e) Pia is a bank teller and an active feminist who takes yoga classes.
- (f) Pia works in a small bookstore and is an active feminist who takes yoga classes.



## “Odd Question” # 2 – Seven *Events*

Write events D, E, and F in terms of A, B, C, and Y.

A = Pia is an active feminist.

B = Pia is a bank teller.

C = Pia works in a small bookstore.

Y = Pia takes yoga classes.

D = Pia is a bank teller and an active feminist =  $A \cap B$

E = Pia is a bank teller and an active feminist who takes yoga classes =  $A \cap B \cap Y$

F = Pia works in a small bookstore and is an active feminist who takes yoga classes =  $A \cap C \cap Y$

## “Odd Question” # 2 – Which Events are Subsets?

A = Pia is an active feminist.

B = Pia is a bank teller.

C = Pia works in a small bookstore.

Y = Pia takes yoga classes.

$$D = A \cap B \Rightarrow D \subseteq A, D \subseteq B$$

$$E = A \cap B \cap Y \Rightarrow E \subseteq D$$

$$F = A \cap C \cap Y \Rightarrow F \subseteq A, F \subseteq C$$

## “Odd Question” # 2 – Apply Logical Consequence Rule

A = Pia is an active feminist.

B = Pia is a bank teller.

C = Pia works in a small bookstore.

Y = Pia takes yoga classes.

$D = A \cap B \Rightarrow D \subseteq A, D \subseteq B \Rightarrow P(D) \leq P(A), P(D) \leq P(B)$

$E = A \cap B \cap Y \Rightarrow E \subseteq D \Rightarrow P(E) \leq P(D)$

$F = A \cap C \cap Y \Rightarrow F \subseteq A, F \subseteq C \Rightarrow P(F) \leq P(A), P(F) \leq P(C)$

## “Odd Question” # 2 – Putting These Together...

- (a) Pia is an active feminist.
- (b) Pia is a bank teller.
- (c) Pia works in a small bookstore.
- (d) Pia is a bank teller and an active feminist.
- (e) Pia is a bank teller and an active feminist who takes yoga classes.
- (f) Pia works in a small bookstore and is an active feminist who takes yoga classes.

Any Correct Ranking Must Satisfy:

$$(a) \geq (d) \geq (e)$$

$$(b) \geq (d) \geq (e)$$

$$(a) \geq (f)$$

$$(c) \geq (f)$$

# Throw a Fair Die Once

$E$  = roll an even number

What are the basic outcomes?

$\{1, 2, 3, 4, 5, 6\}$

What is  $P(E)$ ?



$E = \{2, 4, 6\}$  and the basic outcomes are equally likely (and mutually exclusive), so

$$P(E) = 1/6 + 1/6 + 1/6 = 3/6 = 1/2$$

# Throw a Fair Die Once

$E$  = roll an even number

$M$  = roll a 1 or a prime number

What is  $P(E \cup M)$ ?



Key point:  $E$  and  $M$  are not mutually exclusive!

$$P(E \cup M) = P(\{1, 2, 3, 4, 5, 6\}) = 1$$

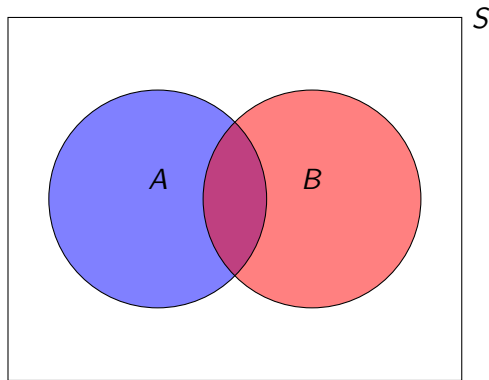
$$P(E) = P(\{2, 4, 6\}) = 1/2$$

$$P(M) = P(\{1, 2, 3, 5\}) = 4/6 = 2/3$$

$$P(E) + P(M) = 1/2 + 2/3 = 7/6 \neq P(E \cup M) = 1$$

## The Addition Rule – Don't Double-Count!

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Construct a formal proof as an optional homework problem.

Who's on the other side?



## Three Cards, Each with a Face on the Front and Back



1. Gaga/Gaga
2. Obama/Gaga
3. Obama/Obama

I draw a card at random and look at one side: it's Obama.  
What is the probability that the other side is also Obama?



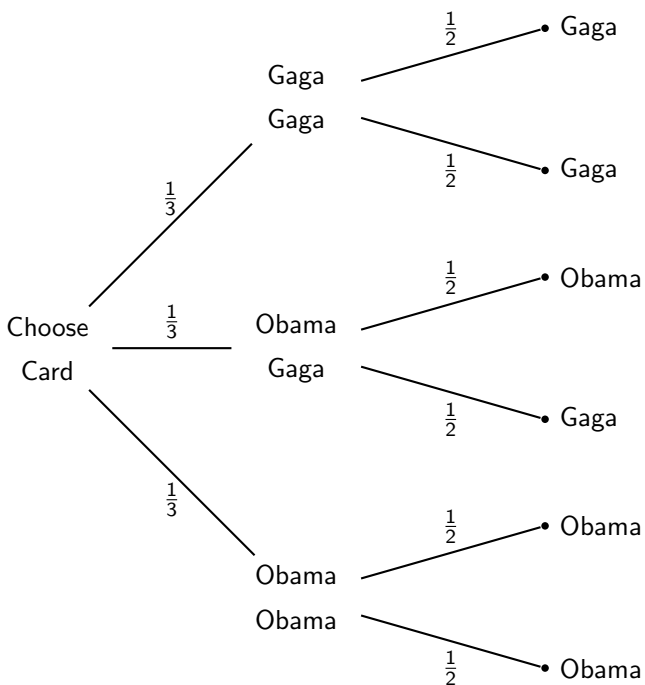
# Let's Try The Method of Monte Carlo...

When you don't know how to calculate, simulate.

## Procedure

1. Close your eyes and thoroughly shuffle your cards.
2. Keeping eyes closed, draw a card and place it on your desk.
3. Stand if Obama is face-up on your chosen card.
4. We'll count those standing and call the total  $N$
5. Of those standing, sit down if Obama is *not* on the back of your chosen card.
6. We'll count those *still* standing and call the total  $m$ .

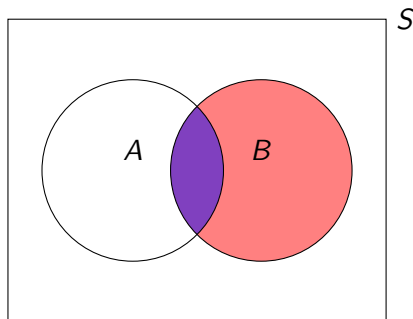
Monte Carlo Approximation of Desired Probability =  $\frac{m}{N}$



# Conditional Probability – Reduced Sample Space

Set of relevant outcomes restricted by condition

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) > 0$$



**Figure :**  $B$  becomes the “new sample space” so we need to re-scale by  $P(B)$  to keep probabilities between zero and one.

## Who's on the other side?

Let  $O_F$  be the event that Obama is on the front of the card of the card we draw and  $O_B$  be the event that he is on the back.

$$P(O_B|O_F) = \frac{P(O_B \cap O_F)}{P(O_F)} = \frac{1/3}{1/2} = 2/3$$