AG01

1. **Geo** Triangle ABC has side lengths $AB = 12$, $AC = 8$, and $BC = 15$. Let the altitude from A to BC intersect BC at D and let AE be the angle bisector of $\angle BAC$, where E is on BC. Compute the length of segment DE.

   Comment: This problem might be too trivial. Any ideas on how to add complexity, (e.g. interesting angle chase), would be appreciated.

   Nick: Problem has been fixed after team group test-solving by omitting the definition of the angle (a triangle is uniquely determined by its three side lengths).

   Answer: $\mathbf{\frac{7}{6}}$

   Solution: From the angle bisector theorem, $\frac{BE}{BA} = \frac{EC}{CA}$. Therefore, $BE = 9$ and $EC = 6$. Let $BD = x$, so $CD = 15 - x$ and by the Pythagorean Theorem, we have that $144 - x^2 = 64 - (15 - x)^2$. This has solution $x = \dfrac{61}{6}$, so therefore $DE = \dfrac{61}{6} - 9 = \boxed{\dfrac{7}{6}}$.

AJ01

2. **Calc** Compute

$$\int_0^6 \frac{x-3}{x^2-6x-7}\,dx$$

   Comment: Meant to be a number one. You can either use a substitution $u = x - 3$ or just compute the integral using elementary methods.

   Answer: 0

   Solution 1: Use the substitution $u = x - 3$ to get

$$\int_0^6 \frac{x-3}{x^2-6x-7}\,dx = \int_{-3}^3 \frac{u}{u^2-16} = 0$$

   since the integrand is odd.

   Solution 2: Another way of seeing this is using the substitution $u = x^2 - 6x - 7$ with $du = 2x - 6$ to see

$$\int_0^6 \frac{x-3}{x^2-6x-7}\,dx = \int_{-7}^{-7} \frac{2}{u}\,du = 0$$

   Solution 3: Finally, if we don't want to change the bounds of integration, the indefinite integral is $\frac{1}{2}\log(x^2 - 6x - 7)$. Plugging in the bounds gives us 0.

AJ02:cal

3. **Calc** Let f be a differentiable function such that $f'(0) = 4$ and $f(0) = 3$. Compute

$$\lim_{x\to\infty}\left(\frac{f(\frac{1}{x})}{f(0)}\right)^x.$$

   Comment: Should be around number 5.

   Answer: $\boldsymbol{e^{\frac{4}{3}}}$

   Solution 1: Denote the limit L. Let $h = \log \circ f$ so that

$$\log L = \lim_{x\to\infty} x\left(h(\frac{1}{x}) - h(0)\right).$$

   Now, let $y = \frac{1}{x}$. Then

$$\log L = \lim_{y\to 0^+} \frac{h(y) - h(0)}{y - 0} = h'(0).$$

   Now $h'(0) = \frac{f'(0)}{f(0)} = \frac{4}{3}$ so $L = e^{\frac{4}{3}}$.

AJ03:cal

4. **Calc** For what positive value k does the equation $\ln x = kx^2$ have exactly one solution?

Comment: Number 3ish

Answer: $\mathbf{\frac{1}{2e}}$

Solution: Let a be the x-coordinate of the single point of intersection. Both $y_1 = \ln x$ and $y_2 = kx^2$ will have the same tangent line and thus slope at $x = a$. Taking the derivative of both equations, we see

$$y_1'|_a = \frac{1}{a} = 2ka = y_2'|_a$$

Solving $\ln a = ka^2$ gives $\ln a = \frac{1}{2}$ so $a = e^{\frac{1}{2}}$. We also see

$$k = \frac{\ln a}{a^2} = \frac{1}{2e}$$

which is our solution. Thus, $k = \frac{1}{2e}$.

AJ04:cal

5. **Calc** If $f(x) = e^x g(x)$, where $g(2) = 1$ and $g'(2) = 2$, find $f'(2)$.

Comment: Meant to be a number one.

Answer: $\mathbf{3e^2}$

Solution: Using the product rule,

$$f'(x) = e^x g(x) + e^x g'(x)$$

so $f'(2) = e^2 + 2e^2 = 3e^2$.

AJ05

6. **Calc** Compute

$$\int_0^2 \sqrt{1 - \frac{x^2}{4}}\, dx$$

Comment: Tiebreaker

Answer: $\boldsymbol{\frac{\pi}{2}}$

Solution 1: We note that this area is just $\frac{1}{4}$ of the area of the ellipse

$$y^2 + \frac{x^2}{4} = 1,$$

which is 2π. Thus, the answer is $\frac{\pi}{2}$. **Solution 2:** Using the substitution $u = \frac{x}{2}$, we see $2\,du = dx$ and

$$\begin{aligned}\int_0^2 \sqrt{1 - \frac{x^2}{4}}\, dx &= 2\int_0^1 \sqrt{1 - u^2}\, du \\ &= 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}\end{aligned}$$

where the second to last equality comes from noticing that we are computing the area of a fourth of a circle of radius 1.

AJ06:cal

7. **Calc** Compute

$$\int_0^{\frac{\pi}{2}} \frac{e^x(\sin x + \cos x - 2)}{(\cos x - 2)^2}\, dx.$$

Answer: $\mathbf{1 - \frac{1}{2}e^{\pi/2}}$

Solution 1: We rewrite the integrand, noting that

$$\frac{e^x(\sin x + \cos x - 2)}{(\cos x - 2)^2} = \frac{e^x \sin x}{(\cos x - 2)^2} + \frac{e^x}{\cos x - 2}.$$

We first compute the indefinite integral of the first term, using integration by parts. We use the substitutions

$$u = e^x, \qquad dv = \frac{\sin x}{(\cos x - 2)^2} dx.$$

This gives us

$$du = e^x\, dx, \qquad v = \frac{1}{\cos x - 2}.$$

Completing by parts gives us

$$\int \frac{e^x \sin x}{(\cos x - 2)^2} dx = \frac{e^x}{\cos x - 2} - \int \frac{e^x}{\cos x - 2} dx.$$

Note the integrand of the second term cancels the term we split away earlier. Therefore,

$$\int_0^{\frac{\pi}{2}} \frac{e^x(\sin x + \cos x - 2)}{(\cos x - 2)^2}\, dx = \left.\frac{e^x}{\cos x - 2}\right|_0^{\frac{\pi}{2}} = 1 - \frac{1}{2}e^{\pi/2}.$$

AJ07

8. **Calc** Compute

$$\int_0^{\pi} \frac{1 - \cos(2016x)}{1 - \cos x}\, dx.$$

Comment: Number 10ish

Answer: $\boldsymbol{2016\pi}$

Solution 1: We do this in generality with $I_n = \int_0^{\pi} \frac{1-\cos(nx)}{1-\cos x}\, dx$. Then note that

$$\begin{aligned}
\frac{I_{n+1} + I_{n-1}}{2} &= \int_0^{\pi} \frac{2 - \cos(n+1)x - \cos(n-1)x}{2(1 - \cos x)}\, dx \\
&= \int_0^{\pi} \frac{1 - \cos nx \cos x}{1 - \cos x}\, dx \\
&= \int_0^{\pi} \frac{(1 - \cos nx) + (1 - \cos x)\cos nx}{1 - \cos x}\, dx \\
&= I_n + \int_0^{\pi} \cos nx\, dx = I_n
\end{aligned}$$

So, $I_n = \frac{I_{n+1} - I_{n-1}}{2}$ for $n \geq 1$ which means $\{I_k\}_{k\in\mathbb{N}}$ is an arithmetic sequence. So, since $I_0 = 0$ and $I_1 = \pi$, the result follows.

AJ08:cal

9. **Calc** Consider the curves with equations $x^n + y^n = 1$ for $n = 2, 4, 6, 8 \ldots$. Denote L_{2k} the length of the curve with $n = 2k$. Find $\lim_{k\to\infty} L_{2k}$.

Comment:

Answer: **8**

Solution 1: Note that geometrically, the limit tends to a square with vertices $(-1,-1), (-1,1), (1,1), (1,-1)$ so the quantity we are looking for is the perimeter of a square with side length 2, which is 8.

AJ09:cal 10. **Calc** Compute

$$\int_0^\pi \frac{1-\sin x}{1+\sin x}\,dx.$$

Comment:

Answer: $4-\pi$.

Solution 1: Let $x=\frac{\pi}{2}-2t$. Then the integral becomes

$$-2\int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} \frac{1-\cos(2t)}{1+\cos(2t)}\,dt = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1-\cos^2(t)}{\cos^2 t}\,dt = 2(\tan t - t)|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 4-\pi$$

AJ10:cal 11. **Calc** Compute

$$\int_0^\infty \frac{\ln\left(\frac{1+x^{11}}{1+x^3}\right)}{(1+x^2)\ln x}\,dx.$$

Comment:

Answer: 2π

Solution 1: Denote the integral I. Using the substitution $x=\frac{1}{u}$, we see

$$\begin{aligned}
I &= -\int_\infty^0 \frac{\ln\left(\frac{1+u^{-11}}{1+u^{-3}}\right)}{(1+u^{-2})\ln(u^{-1})}u^{-2}\,du \\
&= -\int_0^\infty \frac{\ln\left(\frac{1+u^{-11}}{1+u^{-3}}\right)}{(1+u^2)\ln u}\,du \\
&= -\int_0^\infty \frac{\ln\left(u^{-8}\frac{1+u^{11}}{1+u^3}\right)}{(1+u^2)\ln u}\,du \\
&= -\int_0^\infty \frac{\ln\left(\frac{1+u^{11}}{1+u^3}\right)}{(1+u^2)\ln u}\,du - \int_0^\infty \frac{8\ln u}{(1+u^2)\ln u}\,du \\
&= -I+4\pi
\end{aligned}$$

Thus, $2I=4\pi$ so $I=2\pi$.

AJ11 12. **Calc** Compute

$$\int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\cos^5 x+\sin^5 x}\,dx$$

Comment:

Answer: $\frac{\pi}{4}$.

Solution 1: Denote the integral I. Then, using the substitution $t=\frac{\pi}{2}-x$, we see

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\cos^5 x+\sin^5 x}\,dx \\
&= -\int_{\frac{\pi}{2}}^0 \frac{\sin^5 t}{\cos^5 t+\sin^5 t}\,dt \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin^5 t}{\cos^5 t+\sin^5 t}\,dt
\end{aligned}$$

Thus, $2I=\int_0^{\frac{\pi}{2}}\,dx=\frac{\pi}{2}$ so $I=\frac{\pi}{4}$.

AJ12 13. **Calc** A 13-ft ladder is leaning against a wall when its base starts to slide away from the wall. When the base is 12 ft from the wall, it is moving at a rate of 5ft/sec. At what rate is the area of the triangle formed by the wall, the ladder, and the ground changing when the base of the ladder is 12 ft from the wall?

Comment:

Answer: $-\frac{119}{2}\mathbf{ft^2/sec.}$

Solution 1: TO DO

AJ13 14. **Calc** Express

$$\sum_{n=1}^{24} \frac{1}{\sqrt{n+\sqrt{n^2-1}}}$$

as $a\sqrt{2}+b\sqrt{3}$ where a and b are integers.

Comment:

Answer: $\mathbf{2\sqrt{2}+2\sqrt{3}}$

Solution 1: Note that

$$\begin{aligned}
\frac{1}{\sqrt{n+\sqrt{n^2-1}}} &= \frac{1}{\sqrt{\left(\sqrt{\frac{n+1}{2}}+\sqrt{\frac{n-1}{2}}\right)^2}} \\
&= \frac{1}{\sqrt{\frac{n+1}{2}}+\sqrt{\frac{n-1}{2}}} \\
&= \frac{\sqrt{\frac{n+1}{2}}-\sqrt{\frac{n-1}{2}}}{\frac{n+1}{2}-\frac{n-1}{2}} \\
&= \sqrt{\frac{n+1}{2}}+\sqrt{\frac{n-1}{2}}
\end{aligned}$$

so the sum telescopes to $\sqrt{\frac{25}{2}}+\sqrt{\frac{24}{2}}-\sqrt{\frac{1}{2}}=2\sqrt{2}+2\sqrt{3}$.

AJ14:cal 15. **Calc** Using the fact that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

compute

$$\int_0^1 \ln x \ln(1-x)\, dx.$$

Comment:

Answer: $\mathbf{2-\frac{\pi^2}{6}}$

Solution 1: Consider the expansion $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ for $x \in (-1,1)$. Then the indefinite integral is:

$$\begin{aligned}
\int \ln(1-x)\ln x\,dx &= -\int sum_{n=1}^{\infty}\frac{x^n}{n}\ln x\,dx \\
&= -\sum_{n=1}^{\infty}\frac{1}{n}\int x^n \ln x\,dx \\
&= -\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x^{n+1}}{n+1}\ln x - \int\frac{x^n}{n+1}\,dx\right) + C \qquad \text{integration by parts} \\
&= -\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2}\right) + C.
\end{aligned}$$

Taking the definite integral over $[\epsilon, 1-\epsilon]$ and letting ϵ go to 0 gives our integral:

$$\int_0^1 \ln x\ln(1-x)\,dx = \sum_{n=1}^{\infty}\frac{1}{n(n+1)^2}.$$

The sum can be simplified as follows:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n(n+1)^2} &= \sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)^2}\right) \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \sum_{n=2}^{\infty}\frac{1}{n^2} \\
&= 1 - \left(\frac{\pi^2}{6} - 1\right) \\
&= 2 - \frac{\pi^2}{6}
\end{aligned}$$

AJ15:cal 16. **Calc** The radius r of a circle is increasing at a rate of 2 meters per minute. Find the rate of change of the area when r is 6 meters.

Comment:

Answer: 24π meters^2 per minute.

Solution 1: We have that the area A of a circle is $A = \pi r^2$. We want to find $\frac{dA}{dt}$ when $r = 6$ given that $\frac{dr}{dt} = 2$. Differentiating, we see

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}.$$

Plugging in values gives

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2\pi\cdot 6\cdot 2 = 24\pi$$

so that the rate of change of area is 24π meters^2 per minute.

AJ16:cal 17. **Calc** Let $f(x) = (x-1)^3$. Find $f'(0)$.

Comment:

Answer: 3

Solution 1: Taking the derivative, we see $f'(x) = 3(x-1)^2$ so $f'(0) = 3$.

AJ17 18. **Calc** Compute

$$\lim_{x\to 0}\frac{(1+x)^{-2}-1}{x}.$$

Comment:

Answer: -2

Solution 1:

$$\begin{aligned}\lim_{x\to 0}\frac{(1+x)^{-2}-1}{x} &= \lim_{x\to 0}\frac{1-(1+x)^2}{x(1+x)^2}\\ &= \lim_{x\to 0}\frac{-2x-x^2}{x(1+x)^2}\\ &= \lim_{x\to 0}\frac{-2-x}{(1+x)^2}\\ &= -2\end{aligned}$$

Solution 2: We can also apply L'hopital's rule:

$$\lim_{x\to 0}\frac{(1+x)^{-2}-1}{x} = \lim_{x\to 0}\frac{-2(1+x)^{-3}}{1} = -2$$

AJ18:cal 19. **Calc** Compute

$$\lim_{x\to 1}\frac{x^3-1}{x-1}.$$

Comment:

Answer: 3

Solution 1:

$$\begin{aligned}\lim_{x\to 1}\frac{x^3-1}{x-1} &= \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1}\\ &= \lim_{x\to 1} x^2+x+1 = 3\end{aligned}$$

Solution 2: We can also apply L'hopital's rule:

$$\lim_{x\to 1}\frac{x^3-1}{x-1} = \lim_{x\to 1}\frac{3x^2}{1} = 3$$

AL01:gut 20. **Team** A point P and a segment AB with length 20 are randomly drawn on a plane. Suppose that the probability that a randomly selected line passing through P intersects segment AB is $\frac{1}{2}$. Next randomly choose a point Q on segment AB. What is the probability with respect to choosing Q that a circle centered at Q passing through point P contains both A and B on the interior?

Comment: Cute but easy to guess.
Eric: Maybe we can change this problem to have a solution with non-zero probability? Heesu: Conceptually the problem is really awesome. This could be made harder (ie less than 1/2 prob), but I think it's so cute that I want all teams to get it ¿.¿

Answer: 0

Solution: Let M be the midpoint of AB. From the given probability, we can deduce that $\angle APB = 180 \cdot 12 = 90^\circ$. Since $\triangle APB$ is right, then $MP = MA = MB$. Thus, any point Q chosen between M and B exclusive will satisfy $QP < MP + MQ = AM + MQ = AQ$, so the circle will never contain point A. A similar statement can be said about points Q that are chosen between A and M. Thus, the probability is $\boxed{0}$.

`AL02:gut` 21. **Geo** An equilateral triangle $\triangle ABC$ with side length 3 has center O. A circle is drawn centered at O with radius 1. Find the area of the region contained inside both the triangle and circle.

Comment:

Answer: $\frac{\pi}{2} + \frac{3\sqrt{3}}{4}$

Solution: TODO(HS): add in diagram? Why is it commented out.

Label the points as shown. Notice that the intersection is composed of three 60° sectors and three equilateral triangles ($\triangle OED$, $\triangle OGF$, $\triangle OIH$). Since the radius is 1, the area of the three triangles $3 \cdot \frac{1^2 \cdot \sqrt{3}}{4} = 3\sqrt{3}/4$. The area of the three sectors is $3 \cdot 1^2 \cdot \pi/6 = \pi/2$. Thus, the area of the intersection is $\boxed{\frac{\pi}{2} + \frac{3\sqrt{3}}{4}}$.

`AL03` 22. **Geo** In a rectangular prism, points O and P are opposite vertices and A, B, and C are vertices that share an edge with O. Let E and F be the feet of O and P onto plane ABC, respectively. If $AB = 13$, $BC = 14$, and $CA = 15$, compute the length of EF.

Comment: Another interesting property: OP intersects ABC at its centroid.
Eric: Is the formula for computing the distance between the orthocenter and the circumcenter well known?

Answer: $\sqrt{265}/4$

Solution: Let R be the center of the rectangular prism and G be the foot of R onto plane ABC. The foot of O onto the plane is the orthocenter of triangle ABC. Since R is the center, we have that $RA = RB = RC = r$, so $GA = \sqrt{RA^2 - RG^2} = \sqrt{r^2 - RG^2} = GB = GC$. Thus, G is the circumcenter of triangle ABC. Since R is the midpoint of PQ, then G is the midpoint of EF, so F is the reflection of the orthocenter about the circumcenter of triangle ABC.

One can compute the distance between the orthocenter and the circumcenter in multiple ways, and arrive at the answer of $2 \cdot \frac{\sqrt{265}}{8} = \boxed{\frac{\sqrt{265}}{4}}$.

`AL04` 23. **Geo** Point C is chosen on the arc of a semicircle with diameter AB. The two circles with diameters of AC and BC intersect again at point D. If $DA = 20$ and $DB = 16$, compute the length of DC.

Comment:

Answer: $8\sqrt{5}$

Solution: Since AC and BC are diameters, $\angle ADC = \angle BDC = 90^\circ$, so A, B, and D are collinear. It turns out that D is the foot of C onto AB, so by similar triangles, one can arrive at the answer of $8\sqrt{5}$.

`AL05` 24. **Alg** Ash writes the positive integers from 1 to 2015 inclusive as a single positive integer $n = 1234567891011 \cdots 2015$. What is the result obtained by successively adding and subtracting the digits of n? (In other words, compute $1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 1 + 0 - 1 + \cdots$.)

Comment:

Answer: 3932

Solution: Breaking it down into the number of digits, we have

$$A = 1 - 2 + 3... + 9 = 5$$

$$B = (-1 + 0) + (-1 + 1) + ... + (-9 + 9) = 0$$

$$C = (-1 + 0 - 0) + ... + (-9 + 9 - 9) = 100 * (-1 - 2... - 9) = -4500$$

$$D = (1 - 0 + 0 - 0) + ... + (1 - 9 + 9 - 9) = 1000 - 4500 = -3500$$

$$E = (2 - 0 + 0 - 0) + ... + (2 - 0 + 1 - 5) = 16 * 2 + 5 + (1 - 0) + (1 - 1) + ... + (1 - 5) = 28$$

so $A + B + C + D + E = -7967$

AL06 25. **Geo** Point P is chosen inside square $ABCD$ such that $PA = 14$, $PB = \sqrt{158}$, and $PC = 10\sqrt{3}$. Compute the side length of the square.

Comment: Surprisingly difficult. May not be such a good problem given that you have to *notice* things.

Answer: TODO(AL): Find the answer.

Solution: Compute PD with British Flag Theorem. Rotate the square about point D by $90°$ to get square $DCB'C'$ with point P' inside. $\triangle PDP'$ is a $45 - 45 - 90$, so you have the side lengths of $\triangle PCP'$. Law of Cosines to get that $\angle CPP' = 30°$, so now you can find CD since $\angle CPD = 75°$.

AL07 26. **Geo** A circle of radius 1 is placed in each of the corners of a square of side length 5, such that each of the circles are tangent to two sides of the square. Compute the radius of the largest circle that lies entirely in the square and does not overlap with any of the other 4 circles.

Comment: Credits to Julia too. Homework problem.

Answer: $\frac{3\sqrt{2}-2}{2}$

Solution: To maximize the size, we want the circle to be tangent to the other circles. The tangent points lie on the diagonals of the square. Then the diameter of the circle is the distance from the centers of two of the unit circles diagonally opposite each other minus the two unit circles' radiuses $= 3\sqrt{2} - 2$. So the radius is $\frac{3\sqrt{2}-2}{2}$.

AL08:tea 27. **Team** Pooh has an unlimited supply of 1×1, 2×2, 3×3, and 4×4 squares. What is the minimum number of such squares he needs in order to fully cover a 5×5 square?

Comment: Interesting to look at general values of n maybe.

Answer: 8

Solution: TODO(AL): Write a solution

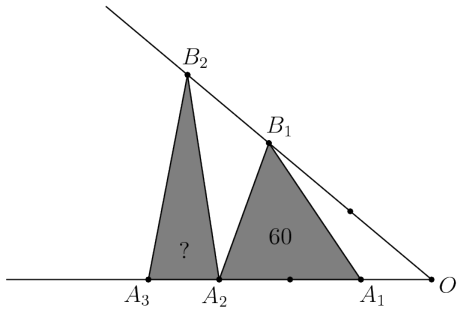
AL09:tea 28. **Geo**

In the figure below, O, A_1, A_2, A_3 are collinear with $2 \cdot OA_1 = A_1A_2 = 2 \cdot A_2A_3$. Also, O, B_1, B_2 are collinear with $OB_1 = 2 \cdot B_1B_2$. If the area of $\triangle A_1A_2B_1$ is 60, what is the area of $\triangle A_2A_3B_2$?

Comment: Asymptote is commented out

Answer: 45

Solution: TODO(AL): Write everything



AL10 29. **Geo** The midpoints of the sides of square $WXYZ$ form a square $ABCD$. A point P is chosen such that $PW^2+PX^2+PY^2+PZ^2 = 201$ and $WX = 6$. What is $PA^2+PB^2+PC^2+PD^2$?

Comment: General: $2(PW^2 + PX^2 + PY^2 + PZ^2) - 2s^2 = (PA^2 + PB^2 + PC^2 + PD^2)$. Julia: Not sure this answer works. Too lazy to truly test solve but feel like answer should be less than 201?

Answer: 330

Solution: Stewart's 4 times.

TODO(AL): Write a solution TODO(AL): Check that the inequalities/configuration is possible.

AL11 30. **Disc** Define/draw a picture of a X-pentagram. What is the minimum number of pentagrams needed to fully tile an 8×8 chessboard? The tiles may overlap and may overhang the edges of the chessboard.

Asymptote commented out.

Comment: Very similar to AL12.

Answer: 16

Solution: Break it up into 4×4 squares, or resolve how the corners must be tiled. TODO(AL): Finish up details

AL12 31. **Disc** Define/draw a picture of a X-pentagram. What is the maximum number of non-overlapping pentagrams that can be placed on an 8×8 chessboard? The tiles may not overhang the edges of the chessboard

Asymptote commented out.

Comment: Very similar to AL11.

Answer: Unclear

Solution: TODO(AL): Find answer and write solution.

AL13 32. **Disc** In the following diagram, the area of each of the smallest equilateral triangles is 1. What is the area of the shaded region?

Asymptote commented out.

Comment:

Answer:

Solution: TODO(AL): Write solution, get answer.

AL14 33. **Disc** Antony the Ant lives on Hextown. The streets of Hextown run in three different directions as shown below, each direction 60 degrees off from the other two street directions. Just as humans have four cardinal directions (north, south, east, west), ants use three cardinal directions. When ants navigate Hextown, they can go any number of blocks *dir1*, *dir2*, and *dir3*. The addresses of these street intersections are given as ordered triples of integers. To travel negative units in a cardinal direction is to walk in the opposite direction, just as walking −2 meters west would be equivalent to walking 2 meters east.

Anton is standing at the city hall at $(0, 0, 0)$. How many ways can he get to his home at $(-2, 3, 7)$ in exactly n steps?

Asymptote commented out. (Taken from AL13; very similar diagram.)

Comment:

Answer:

Solution: TODO(AL): This description is ass. Need another diagram to describe the dirs, choose n.

AL15 34. **Disc** A plane is tessellated with equilateral triangles as shown, in such a way that each equilateral triangle has one red, one yellow, and one green vertex. (I don't like using tessellated to describe this, because the triangles don't actually matter.) Define the coordinates the same way as in AL14, which is a complete pain in the ass to describe. If you, the ant, can only traverse edges in the order red, yellow, green, red, $\cdots$, how many ways can you get to [insert point]?

Asymptote commented out. (Taken from AL13; very similar diagram.)

Comment:

Answer:

Solution: TODO(AL): This description is ass. Need another diagram to describe the dirs. Problem statement incomplete.

AL16 35. **Team** At time $t = 0$, two beakers are on the desk. Beaker A contains 1 liter of apple juice, and beaker B 1 liter of orange juice. In each iteration n, you pour 1/3 of the contents of beaker A into beaker B, stir, pour the same volume of juice back from beaker B to A, and stir. Let a_n denote the fraction of apple juice in beaker A at the end of iteration n. (In this case, $a_0 = 1$.) For what value of k does $\sum_{i=0}^{\infty}(a_i - k)$ converge?

Assume that each beaker can hold at least 2 liters of fluids.

Comment:

Answer:

Solution: TODO(AL): I hate this problem.

AL17 36. **Alg** Randy is going to school. If she walks to school, it will take 16 minutes. If she waits for the bus, she may have to wait anywhere between 0 and 60 minutes for the bus to arrive. (uniform distributed) At any point, Randy may decide to stop waiting for the bus and walk instead, after which she cannot change her mind and continue waiting for the bus. However, if she catches the bus, she arrives at school in 2 minutes.

Randy wants to choose a strategy to minimize her regret. The bus will arrive at her stop uniformly at random in the next 60 minutes, and for all possible bus arrival times, she wishes to minimize the ratio of the actual time she spends in transit to transit time of her best option if she had known the bus's arrival time beforehand. On any given day, what is the maximum amount of time she should wait at the bus stop?

Comment:

Answer: 14

Solution: TODO(AL):

AL18 37. **Disc**

A *subword* of a string $\mathcal{S}$ of letters is any positive-length string using any number of the letters in $\mathcal{S}$ without repetition. How many substrings of $JULIA$ are there?

Comment:

Answer: 325

Solution: Using casework on the number of letters in the word, we have $5+5\cdot4+5\cdot4\cdot3+5\cdot4\cdot3\cdot2+5\cdot4\cdot3\cdot2\cdot1 = 325$.

AL19:tea 38. **Geo**

Rectangle $ABCD$ has $AB = 20$ and $BC = 15$. Two circles with diameters AB and BC intersect again at point E. What is the length of DE?

Comment:

Answer: $\sqrt{193}$

Solution: TODO(AL): Choose numbers, write solution

E lies on diagonal AC by two right angles, British flag theorem

AL20 39. **Disc**

Julia is an ant standing on a vertex of a regular dodecahedron. In how many ways can she walk on the edges of the dodecahedron so that she walks across each edge exactly once?

Comment: Maybe a viable problem is to ask for visit each vertex once.

Answer: 0

Solution: Each vertex has an odd degree.

AL21 40. **Geo**

A cube has a diagonal of length $\sqrt{27}$. What is its volume?

Comment:

Answer: 27

Solution: The side length is 3, so the volume is 27.

AL22 41. **Team**

Find the 10-digit positive integer X such that the first digit counts the number of digits equal to 0

Comment:

Answer: 6210001000

Solution: TODO(AL): Write soln

L23:alg:tea 42. **Alg**

$P(x)$ is a monic cubic polynomial. The lines $y = 0$ and $y = m$ intersect P at points A, C, E and B, D, F from left to right for a positive real number m. If $AB = \sqrt{7}$, $CD = \sqrt{15}$, and $EF = \sqrt{10}$, what is the value of m?

Comment:

Answer: $\sqrt{6}$

Solution: Let the corresponding x-coordinates of these intersection points be a, b, c, d, e, f. We know that $a+c+e = b+d+f$ or $(b-a)+(f-e) = (c-d)$ by Vieta's. Let $r = (b-a)$ and $s = (f-e)$. Then:

$$r^2 + m^2 = X^2$$
$$(r+s)^2 + m^2 = Y^2$$
$$s^2 + m^2 = Z^2$$

$$r^2 - s^2 = X^2 - Z^2$$
$$r^2 + 2rs = Y^2 - Z^2$$
$$s^2 + 2rs = Y^2 - X^2$$

TODO (AL): Choose numbers to make this computation work out nicely.

Closed form answer seems to be $h^2 = \frac{(a+b+c)-2\sqrt{a^2+b^2+c^2-(ab+bc+ca)}}{3}$, where a, b, c are X^2, Y^2, Z^2 (squared distances given). Negative root is taken because $h^2 < \min(a, b, c)$.

Choosing a=a, b=a+8, c=a+3 gives $h^2 = a - 1$.

`AL24:tea` 43. **Disc**

A $6 \times 14 \times 21$ rectangular prism is cut into unit cubes. Through or across how many unit cubes does a space diagonal of the rectangular prism touch?

Comment:

Answer: unclear

Solution: TODO (AL): solution

`AL25:tea` 44. **Alg**

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to a real number x. Compute

$$\sum_{n=1}^{\infty} \left\lfloor \frac{100}{n} \right\rfloor.$$

Comment:

Answer: unclear

Solution: TODO (AL): solution

`AL26:tea` 45. **Geo**

Circles ω_1 and ω_2 have radii $r_1 < r_2$ respectively and intersect at distinct points X and Y. The common external tangents intersect at point Z. One of the common tangents touches ω_1 and ω_2 at P and Q respectively. Line ZX intersects ω_1 and ω_2 again at points R and S and lines RP and SQ intersect again at point T. If $XT = 8$, $XZ = 15$, and $XY = 12$, then what is r_1/r_2?

Comment:

Answer: 9/25

Solution: TODO (AL): solution

AL27 46. **Disc** Ross the kangaroo is a funny kangaroo. He is standing at the origin of a coordinate plane and he wishes to hop to the $(20, 16)$ coordinate. Ross can move only by jumping exactly 5 length on each jump and Ross is only allowed to be at lattice points. How many ways can Ross reach $(20, 16)$ in n hops or less?

Comment: Taken from HS13; changed a number; different question

Answer: dunno

Solution: TODO(AL): finish.

AL28:tea 47. **Geo**

A cylinder has height 9 and volume 162π. What is the maximum distance between two points chosen on the surface of the cylinder?

Comment:

Answer: $\mathbf{3\sqrt{13}}$

Solution: The area of the base of the cylinder is $162\pi/9 = 18\pi$, so the radius of the base is $r = 3\sqrt{2}$.

The two points on the surface of the cylinder that have the maximum distance are points A and B on the circumferences of opposite bases such that the projection of A onto the opposite base forms a diameter with B.

By the Pythagorean Theorem, $AB = \sqrt{(2r)^2 + h^2} = \sqrt{4r^2 + h^2} = \sqrt{4 \cdot 18 + 9^2} = 3\sqrt{13}$.

AL29:tea 48. **Alg** The three roots of the quartic polynomial $f(x) = x^4 + ax^3 + bx + c$ are -1, 3, and 5. What is $a + b - c$?

Comment: Modified JH14.

Answer: 1

Solution: Notice that $f(-1) = 1 - a - b + c = 0$ since -1 is a root. Thus, $a + b - c = 1$.

AL30:tea 49. **Alg** A bar carries two brands of brandy, which have different alcohol content concentrations. Brandon mixes 4 shots of Brandy A and 9 shots of Brandy B to get a drink that is 28% alcohol. If Brandon instead mixed 4 shots of Brandy B with 9 shots of Brandy A, the drink would be 43% alcohol instead. What is the alcohol concentration of the brand of brandy with lower concentration?

Comment: should probably change flavor text

Answer: 16%

Solution: TODO(AL): soln

AZ01 50. **Calc** Find

$$\int_1^2 (x^4 - 5x^3 + 10x^2 - 10x + 5)\, dx.$$

Comment: Might be calc #3 or #4. -Albert

Answer: $\mathbf{\frac{47}{60}}$

Solution 1: Directly integrate:

$$\begin{aligned}\int_1^2 (x^4 - 5x^3 + 10x^2 - 10x + 5)\, dx &= \left(\frac{1}{5}x^5 - \frac{5}{4}x^4 + \frac{10}{3}x^3 - 5x^2 + 5x\right)\Bigg|_{x=1}^{2} \\ &= \frac{31}{5} - \frac{75}{4} + \frac{70}{3} - 15 + 5 \\ &= \frac{47}{60}.\end{aligned}$$

Solution 2: Note that since

$$(x-1)^5 = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1,$$

the integrand is equal to

$$\frac{(x-1)^5+1}{x} = \frac{t^5+1}{t+1} = t^4 - t^3 + t^2 - t + 1$$

for $t = x-1$. Thus, we compute

$$\begin{aligned}\int_1^2 (x^4 - 5x^3 + 10x^2 - 10x + 5)\,dx &= \int_0^1 (t^4 - t^3 + t^2 - t + 1)\,dt \\ &= \left(\frac{1}{5}t^5 - \frac{1}{4}t^4 + \frac{1}{3}t^3 - \frac{1}{2}t^2 + t\right)\Bigg|_{t=0}^1 \\ &= \frac{1}{5} - \frac{1}{4} + \frac{1}{3} - \frac{1}{2} + 1 \\ &= \frac{47}{60}.\end{aligned}$$

AZ02 51. **Calc** Consider the function $f(x) = \frac{1}{1+x+x^2}$. Find $f'(-1)$.

Comment: Might be calc #2 or #3. -Albert

Answer: 1

Solution 1: We compute $f'(x)$ directly using the quotient rule, then plug in -1:

$$f'(x) = -\frac{1+2x}{(1+x+x^2)^2},$$

so $f'(-1) = \boxed{1}$.

Solution 1: We can rewrite f in terms of $y = x+1$, as

$$f(y) = \frac{1}{1+(y-1)+(y-1)^2} = \frac{1}{1-y+y^2}.$$

We are now looking for $f'(0)$, since $y=0$ corresponds to $x=-1$. Now, expanding gives

$$\begin{aligned}f(y) &= 1 + (y-y^2) + (y-y^2)^2 + (y-y^2)^3 + \ldots \\ &= 1 + (y-y^2) + (y^2 - 2y^3 + y^4) + (y^3 - 3y^4 + 3y^5 - y^6) + \ldots\end{aligned}$$

No term past the first two contains a nonzero coefficient of y, so $f'(0)$, being the coefficient of y, is $\boxed{1}$.

AZ03 52. **Calc** Find

$$\lim_{x\to 0} \frac{\log(1+\sin(x)) - x}{e^{\cos(x)} - e}.$$

Answer: $\boldsymbol{\frac{1}{e}}$

Solution: First, we apply L'Hôpital's rule and obtain

$$\lim_{x\to 0} \frac{\log(1+\sin(x)) - x}{e^{\cos(x)} - e} = \lim_{x\to 0} \frac{(\log(1+\sin(x)) - x)'}{(e^{\cos(x)} - e)'} = \lim_{x\to 0} \left(-\frac{\frac{\cos(x)}{1+\sin(x)} - 1}{\sin(x)e^{\cos(x)}} \right).$$

We can factor out $\frac{1}{1+\sin x}$ since this has limit 1 as x approaches 0, so this limit is equal to

$$\lim_{x\to 0} \frac{1+\sin(x)-\cos(x)}{\sin(x)e^{\cos(x)}} = \lim_{x\to 0} \frac{(1+\sin(x)-\cos(x))'}{(\sin(x)e^{\cos(x)})'} = \lim_{x\to 0} \frac{\cos(x)+\sin(x)}{(\cos(x)-\sin^2(x))e^{\cos(x)}}.$$

Since both numerator and denominator are nonzero at $x = 0$, we find that the original limit is equal to

$$\frac{\cos(0)+\sin(0)}{(cos(0)-\sin^2(0))e^{\cos(0)}} = \boxed{\frac{1}{e}}.$$

AZ04:cal 53. **Calc** Find

$$\lim_{x\to 0} \frac{\sin(x)-x}{x\cos(x)-x}.$$

Comment: Easier version of AZ04.

Answer: $\mathbf{\frac{1}{3}}$

Solution: L'Hôpital's rule gives

$$\lim_{x\to 0} \frac{\sin(x)-x}{x\cos(x)-x} = \lim_{x\to 0} \frac{(\sin(x)-x)'}{(x\cos(x)-x)'} = \lim_{x\to 0} \frac{\cos(x)-1}{\cos(x)-x\sin(x)-1}.$$

Another application gives

$$\lim_{x\to 0} \frac{\cos(x)-1}{\cos(x)-x\sin(x)-1} = \lim_{x\to 0} \frac{(\cos(x)-1)'}{(\cos(x)-x\sin(x)-1)'} = \lim_{x\to 0} \frac{\sin(x)}{2\sin(x)+x\cos(x)},$$

and another application gives

$$\lim_{x\to 0} \frac{\sin(x)}{2\sin(x)+x\cos(x)} = \lim_{x\to 0} \frac{(\sin(x))'}{(2\sin(x)+x\cos(x))'} = \lim_{x\to 0} \frac{\cos(x)}{3\cos(x)-x\sin(x)} = \boxed{\frac{1}{3}}$$

BE01 54. **Calc** Let $\mathcal{F}$ be the family of curves on the plane given by $y - kx^2 = 0$, where each curve corresponds to a constant k. There exists a family of closed curves $\mathcal{F}^*$ also in the plane, such that any pair of curves $\gamma_1 \in \mathcal{F}, \gamma_2 \in \mathcal{F}^*$ are perpendicular whenever they intersect. Find the area enclosed by the curve $\gamma \in \mathcal{F}^*$ that passes through $(1, 2)$.

Answer: $\mathbf{\frac{9\sqrt{2}\pi}{2}}$.

Solution: First note that $\mathcal{F}$ mostly covers the plane without overlap – that is, for every point not on the y-axis, there is exactly one curve from $\mathcal{F}$ that passes through that point. Specifically, if $\gamma \in \mathcal{F}$ passes through (x_0, y_0), $k = \frac{y_0}{x_0^2}$. By taking the derivative of $y = kx^2$ at that point, we see that $\mathcal{F}$ produces a line field with slope $2kx_0 = \frac{2y_0}{x_0}$ at point (x_0, y_0) with $x_0 \neq 0$. If we want to find a family of curves perpendicular to $\mathcal{F}$, it must have slope $-\frac{x_0}{2y_0}$ at the point (x_0, y_0). This sets up the differential equation $\frac{dy}{dx} = -\frac{x}{2y}$. This separates into $2ydy = -xdx \Rightarrow y^2 = -x^2/2 + C$. Then the orthogonal family of curves is the family of ellipses $x^2 + 2y^2 = K$. The ellipse that passes through $(1, 2)$ is $x^2 + 2y^2 = 9$ which has major axis 3 and minor axis $3/\sqrt{2}$. Then the area is $\frac{9\pi}{\sqrt{2}} = \boxed{\frac{9\sqrt{2}\pi}{2}}$.

BE02 55. **Geo** A tetrahedron T has the following properties: the vertices of T have integer coordinates, and no edge of T is parallel to any of the vectors $(1, 1, 0), (1, 0, 1), (0, 1, 1)$. Find the smallest possible sidelength of T.

Answer: $\mathbf{10\sqrt{2}}$

Solution: An outline of the proof:

(a) Show that the sidelength of T is of the form $s = n\sqrt{2}$ for n an integer.

(b) If r is the radius of the tetrahedron, then $r = \frac{\sqrt{6}}{4}s$. Then $r^2 = \frac{3}{4}n^2 = x^2 + y^2 + z^2$ where x, y, z are quarter integers. Thus, we wish to find solutions in integers to $a^2 + b^2 + c^2 = 3k^2, 12k^2$ that minimize k^2.

(c) Here we have to be careful not to consider solutions which give us tetrahedra with edges parallel to one of the vectors.

BE03:alg 56. **Alg** Compute $f(1.13, 5.87)$ where $f(x, y) = \frac{x^2-y^2}{x-y}$.

Answer: 7

Solution: We simplify $f(x, y) = x + y$, so the answer is $1.13 + 5.87 = 7$.

BE04 57. **Calc** Find the limit

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}.$$

Answer: $\mathbf{\frac{1}{2}}$

Solution: TODO: Write a solution.

BE05 58. **Calc** Let

$$f(x) = \sum_{n=0}^{\infty} \frac{\sin(2^n x)}{2^n} \qquad g(x) = \sum_{n=0}^{\infty} \frac{\sin(3^n x)}{3^n}.$$

Both f and g are continuous and non-differentiable for all real values of x. Compute

$$\lim_{x\to 0} \frac{f(x)}{g(x)}$$

given that it exists.

Answer: $\mathbf{\log_2 3}$ **or** $\mathbf{\frac{\log 3}{\log 2}}$ **or** $\mathbf{\frac{\ln 3}{\ln 2}}$

Solution: TODO: Ask Boris for a solution, if it becomes needed.

BH01 59. **Alg** What is the maximum possible value of $(-5x^2 - 2y^2 - 2z^2) + (2xy + 4xz + 2yz) + (-18x - 4y + 14z)$?

Comment: Somewhat contrived.

HS: I agree that people will try to write this as a sum of squares, but a much more natural sum of squares doesn't use 3 terms in each squaring. EG, I think one that many people will try is $-((x - 2y)^2 + (y - 2z)^2 + (7z - 2x)^2 + 8xz)$ because seeing xy terms, you naturally try $(ax + by)^2$ terms and so on. This does yield a solution using calc (test $x < 0 < z$ and $z < 0 < x$, and take a derivative of first two terms and minimize in y. You get a sln for y in x,y and you can further take derivatives). So I think we give a very straightforward solution that can be used w/ calc for people who don't see a clever re-arrangement. I think the calc approach I outlined does take potentially more time (which I think is a good balancing feature). I guess I'm curious if anyone who has testsolved has gotten the solution as it is described here? If the official solution is too contrived that people all try bash it would be pointless.

XS: You can't actually give off a good expression without it being unsolvable apparently, so I made it as simple as possible. HS's comment is nice because it can be solved using sum of squares, but it can also just be calc-bashed and be a routine exercise in gaussian elimination, etc, which is why I think this is fine.

Answer: 26

Solution: The expression is equivalent to $26-(x-y+1)^2-(y-z+3)^2-(2x-z+4)^2$ as a sum of squares-type solution. Now setting the inner sums to 0, we can have that $x=0, y=1, z=4$ which attains a maximum value of 26.

BH02:gut 60. **Alg** Let $z \neq 0$ be a complex number satisfying $z^2 = 2016(z+i|z|)$. Find $z\overline{z}$, where $\overline{z} = a-bi$ is the complex conjugate of $z = a+bi$.

Comment: HS: We can easily bash this by letting $z = a+bi$ and comparing real and imaginary parts. Does it make it too easy? (for any testsolvers.)

Answer: 12192768

Solution: Let $w = \frac{z}{2016}$, so the equation becomes $w^2 = w + i|w| \leftrightarrow w^2 - w = i|w|$. This implies $|w-1| = 1$. In the complex plane, let $A = 0, B = 1, C = w$ and $D = w^2$; then, $AB = BC$ and $\triangle ABC \sim \triangle ACD$. Let the angle of AC be θ; then, the angle of BC is 2θ and the angle of CD is 3θ. But since $w^2 - w = i|w|$, the angle of CD must be $90°$, which implies $\theta \in \{30°, 150°, 260°\}$. However, the real part of C is positive, as ABC is isosceles and $C = w \neq 0$, so $\theta \in (-90°, 90°)$, so it must be that $\theta = 30°$. It follows that $w = \big(1+\cos(60°)\big) + \sin(60°)i = \frac{3}{2} + \frac{\sqrt{3}}{2}i$ and $z = 1008\cdot(3+\sqrt{3}i)$ and $z\overline{z} = 1008^2\cdot(9+3) = \boxed{12192768}$.

BH03 61. **Disc** A 5×5 grid G has each cell colored black or white. There are two operations: an α-operation consists of choosing a cell and flipping the colors of all its (2, 3, or 4) edge-adjacent cells, and a β-operationconsists of choosing a cell, flipping the colors of all its (2, 3, or 4) edge-adjacent cells, and flipping its own color. Call G α-solvable if, through a sequence of α-operations, G can be made all white; similarly, call G β-solvable if, through a sequence of β-operations, it can be made all white. How many distinct grids G are both α-solvable and β-solvable?

Answer: 262144

Solution: Let $\mathcal{S}$ denote the space spanned by 5×5 matrices over $\mathbb{F}_2$. We define the following operations on 5×5 matrices $A, B \in S$:

- $A\phi_\alpha B$ denotes the matrix derived from A where an α-operation is performed on every element marked 1 in B.
- $A\phi_\beta B$ denotes the matrix derived from A where an β-operation is performed on every element marked 1 in B.

Note that there is a one-to-one correspondence between grids and matrices $A \in \mathcal{S}$ (we take 0 to correspond to white and 1 to black). Furthermore, two α or β-operations performed on the same cell nullify one another, so we can assume that in any sequence of operations, an α or β-operation is performed on each cell either 0 or 1 times; then, there is a one-to-one correspondence between grids and sequences and binary operations $A\phi B$.

Now, note that $(A\phi_\alpha B)\phi_\alpha C = A\phi_\alpha(B+C)$ and $(A\phi_\alpha B) + (C\phi_\alpha D) = (A+C)\phi_\alpha(B+D)$; ϕ_β obeys similar properties.

Let $\mathcal{S}_\alpha$ and $\mathcal{S}_\beta$ denote the sets of matrices that are α-solvable and β-solvable, respectively. Note that $\mathcal{S}_\alpha = \mathbf{0}\phi_\alpha\mathcal{S}$, $\mathcal{S}_\beta = \mathbf{0}\phi_\beta\mathcal{S}$ and that $\mathcal{S}_\alpha\phi_\alpha\mathcal{S} \subset \mathcal{S}_\alpha$, $\mathcal{S}_\beta\phi_\beta\mathcal{S} \subset \mathcal{S}_\beta$. Furthermore, for $A = \mathbf{0}\phi_\alpha B \in \mathcal{S}_\alpha$, we have $A \in \mathcal{S}_\beta$ if and only if $B = A\phi_\beta B \in \mathcal{S}_\beta$. It follows that $\mathcal{S}_\alpha \cap \mathcal{S}_\beta = \mathbf{0}\phi_\alpha\mathcal{S}_\beta = \mathbf{0}\phi_\alpha(\mathbf{0}\phi_\beta\mathcal{S})$.

Now, let $K_\alpha(\mathcal{S}_\beta)$ be the set of $A \in \mathcal{S}_\beta$ such that $\mathbf{0}\phi_\alpha A = \mathbf{0}$; then, for $B, C \in \mathcal{S}_\beta$, $\mathbf{0}\phi_\alpha B = \mathbf{0}\phi_\alpha C$ if and only if $B - C \in K_\alpha(\mathcal{S}_\beta)$. It follows that $|\mathbf{0}\phi_\alpha\mathcal{S}_\beta| = \frac{|\mathcal{S}_\beta|}{|K_\alpha(\mathcal{S}_\beta)|}$. Likewise, $|\mathbf{0}\phi_\beta\mathcal{S}| = \frac{|\mathcal{S}|}{|K_\beta(\mathcal{S})|}$. Furthermore, for any $A \in K_\alpha(\mathcal{S})$, $\mathbf{0}\phi_\beta A = A$, so $A \in \mathcal{S}_\beta$; then, $K_\alpha(\mathcal{S}_\beta) = K_\alpha(\mathcal{S})$.

Combining, we have:

$$|\mathcal{S}_\alpha \cap \mathcal{S}_\beta| = \frac{|\mathcal{S}|}{|K_\alpha(\mathcal{S})| \cdot |K_\beta(\mathcal{S})|}$$

Now, note that for any $A \in \mathcal{S}$ satisfying $\mathbf{0}\phi_\alpha A = \mathbf{0}$, A is determined by its first row. This is because, after performing the α-operations indicated by the first k rows of A (call this matrix $A^{(k)}$), we must have $a_{k+1,i} = a^{(k)}_{k,i}$. Now, we have for the following first row configurations:

$$\begin{array}{c}\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\\ \begin{matrix}0&0&0&0&0\end{matrix}\end{array} \quad \begin{array}{c}\begin{pmatrix}0&1&0&0&0\\1&0&1&0&0\\0&1&0&1&0\\0&0&1&0&1\\0&0&0&1&0\end{pmatrix}\\ \begin{matrix}0&0&0&0&0\end{matrix}\end{array} \quad \begin{array}{c}\begin{pmatrix}0&0&1&0&0\\0&1&0&1&0\\1&0&1&0&1\\0&1&0&1&0\\0&0&1&0&0\end{pmatrix}\\ \begin{matrix}0&0&0&0&0\end{matrix}\end{array}$$

Any first row configurations produces a valid matrix A, so $|K_\alpha(S)| = 2^5$. Similarly, for matrices $B \in \mathcal{S}$ satisfying $\mathbf{0}\phi_\beta B = \mathbf{0}$:

$$\begin{array}{c}\begin{pmatrix}1&0&0&0&0\\1&1&0&0&0\\1&0&1&0&0\\0&1&1&1&0\\0&0&0&0&1\end{pmatrix}\\ \begin{matrix}0&1&1&0&1\end{matrix}\end{array} \quad \begin{array}{c}\begin{pmatrix}0&1&0&0&0\\1&1&1&0&0\\0&0&0&1&0\\1&1&0&1&1\\0&0&0&1&0\end{pmatrix}\\ \begin{matrix}1&1&1&0&0\end{matrix}\end{array} \quad \begin{array}{c}\begin{pmatrix}0&0&1&0&0\\0&1&1&1&0\\1&0&0&0&1\\1&0&1&0&1\\0&0&1&0&0\end{pmatrix}\\ \begin{matrix}1&1&0&1&1\end{matrix}\end{array}$$

Checking, the only first row configurations that produce a valid matrix B are $\begin{pmatrix}0&0&0&0&0\end{pmatrix}$, $\begin{pmatrix}1&1&0&1&1\end{pmatrix}$, $\begin{pmatrix}1&0&1&0&1\end{pmatrix}$, and $\begin{pmatrix}0&1&1&1&0\end{pmatrix}$, so $|K_\beta(S)| = 2^2$. Thus, $|\mathcal{S}_\alpha \cap \mathcal{S}_\beta| = \frac{2^{25}}{2^5 \cdot 2^2} = 2^{18} = \boxed{262144}$.

BH04 62. **Calc** Define the function:

$$f : (\mathbb{R}^+)^2 \longrightarrow \mathbb{R}^+ \mid f(x,y) = \sqrt{x + y\sqrt{x + y\sqrt{x + y\sqrt{\dots}}}}$$

Find the unique constants (a, b) such that:

$$\lim_{x\to\infty} x^a\big(f(x,2017) - 2f(x,2016) + f(x,2015)\big) = b$$

Answer: $\left(\frac{1}{2}, \frac{1}{4}\right)$

Solution: Note that $f(x,y)^2 = x + yf(x,y) \to f(x,y) = \frac{-y+\sqrt{y^2+4x}}{2}$. It follows that the series expansion at $x \to \infty$ is:

$$\lim_{x\to\infty} f(x,y) = \lim_{x\to\infty} x^{\frac{1}{2}} - \frac{y}{2}x^0 + \frac{y^2}{8}x^{-\frac{1}{2}} + O(x^{-\frac{3}{2}})$$

Then, the desired limit L is given by:

$$\begin{aligned} L &= \lim_{x\to\infty} x^a \left(\frac{2017^2 - 2\cdot 2016^2 + 2015^2}{8} x^{-\frac{1}{2}} + O(x^{-\frac{3}{2}})\right) \\ &= \lim_{x\to\infty} \frac{1}{4} x^{a-\frac{1}{2}} + O(x^{a-\frac{3}{2}}) \end{aligned}$$

Then, $a = \boxed{\frac{1}{2}}$ and $b = \boxed{\frac{1}{4}}$.

BH05 63. **Calc** Find the limit:

$$\lim_{x\to 0^+}\left(16\cos(2\sqrt{x})^{\frac{2\ln(2)}{x}}\right)^{-\frac{3}{x}}$$

Answer: 256

Solution: Let L be the desired limit and let $y=\sqrt{x}$, so:

$$\begin{aligned} L &= \lim_{y\to 0^+}\left(16\cos(2y)^{\frac{2\ln(2)}{y^2}}\right)^{-\frac{3}{y^2}} \\ &= \left(\lim_{y\to 0^+}\left(e^{2y^2}\cos(2y)\right)^{\frac{2}{y^4}}\right)^{-\frac{3}{\ln(2)}} \end{aligned}$$

Let $z=\left(e^{2y^2}\cos(2y)\right)^{\frac{2}{y^4}}$, so by many iterations of L'Hopital's rule:

$$\begin{aligned} \lim_{y\to 0^+}\ln(z) &= \lim_{y\to 0^+}\frac{4y^2+2\ln\big(\cos(2y)\big)}{y^4} = \frac{0+0}{0} \\ &= \lim_{y\to 0^+}\frac{8y-4\tan(2y)}{4y^3} = \lim_{y\to 0^+}\frac{2y-\tan(2y)}{y^2} = \frac{0-0}{0} \\ &= \lim_{y\to 0^+}\frac{2-2\sec^2(2y)}{3x^2} = \frac{2-2}{0} \\ &= \lim_{y\to 0^+}\frac{-8\tan(2y)\sec^2(2y)}{6y} = \lim_{y\to 0^+}\frac{-4\tan(2y)\sec^2(2y)}{3y} = \frac{0\cdot 0}{0} \\ &= \lim_{y\to 0^+}\frac{-8\big(2-\cos(4y)\big)\sec^4(2y)}{3} = -\frac{8}{3} \end{aligned}$$

Then, $L=\lim_{y\to 0^+} z^{-\frac{3}{\ln(2)}} = (e^{-\frac{8}{3}})^{-\frac{3}{\ln(2)}} = 2^8 = \boxed{256}$.

BH06 64. **Calc** Let $p(x)$ be a polynomial satisfying $p(x)-p''(x)=x^{2016}+2016$ and let N be the product of the roots of $p(x)$. Find the sum of all distinct (positive) prime numbers less than 2016 dividing N.

Answer: 13

Solution: Let $p(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$, so:

$$\begin{aligned} p''(x) &= n(n-1)a_nx^{n-2}+(n-1)(n-2)a_{n-1}x^{n-3}+\ldots+3\cdot 2a_3x+2\cdot 1a_2 \\ p(x)-p''(x) &= a_nx^n+a_{n-1}x^{n-1}+\Big(\big(a_{n-2}-n(n-1)a_n\big)x^{n-2}+\ldots+(a_0-2\cdot 1a_2)\Big) \end{aligned}$$

It follows that $n=2016$, $a_{2016}=1, a_{2014}=2016\cdot 2015,\ldots,a_2=2016\cdot\ldots\cdot 3, a_0=2016!+2016$, and $a_{2k+1}=0$. Then, by Vieta's formulas, $N=2016!+2016=2016\cdot(2015!+1)$. Since any prime less than 2016 divides 2015!, no prime less than 2016 divides $2015!+1$, so the only primes that divide N are those that divide 2016, which are 2, 3, and 7. Then, the desired sum is $2+3+7=\boxed{13}$.

BH07 65. **Calc** Let $p(x)$ be a polynomial of degree 2016 such that $p(x)\geq p'(x)$. Let n be the number of its roots that are real. How many possible values are there for n?

Answer: 1

Solution: Note that $\lim_{x\to\infty}p(x)=\infty$ and $\lim_{x\to-\infty}p(x)=\infty$ since $p(x)$ grows faster than $p'(x)$. Now, if $p(a)<0$ for some a, there exist $b<a<c$ such that $p(b)=p(c)=0$, and $p(x)<0$ for all $x\in(b,c)$. By the Mean Value Theorem there exists some $d\in(b,c)$ such that $f'(d) = 0 > f(d)$, a contradiction. Thus, $p(x) \geq 0$ for all x. Suppose $p(a) = 0$ for some a, so a is a root of multiplicity d. Then, $p'(a) = 0$ and a is a root of multiplicity $d-1$. This means that in the ϵ-neighborhood of a, $p'(a)$ grows faster than $p(a)$; furthermore, $p'(a) > 0$ on one side of a, so $p'(a) > p(a)$ on one side of a, a contradiction. Then, $p(x) > 0$ for all x. This implies that $p(x)$ cannot have any real roots, so the only possible value of n is 0.

BH08 66. **Calc** For a non-negative integer n, define:

$$F(n) = \int_{-\infty}^{\infty} \left(\frac{x}{2}\right)^n e^{-x^2} \mathrm{d}x$$

Let S be the set of the values $F(n)$ can take. What is the minimum positive element in S?

Answer: $\frac{105\sqrt{\pi}}{8192}$

Solution: Let $y = \frac{x}{2}$, so $F(n) = \frac{1}{2}\int_{-\infty}^{\infty} y^n e^{-4y^2}\mathrm{d}y$. Now, define:

$$G(t) = \int_{-\infty}^{\infty} e^{-ty^2}\mathrm{d}y = \frac{\sqrt{\pi}}{\sqrt{t}}$$

Then, $F(n) = (-1)^n G^{(n)}(4)$. Calculating:

$$\int_{-\infty}^{\infty} -y^2 e^{-ty^2}\mathrm{d}y = G^{(1)}(t) = -\frac{1}{2}\cdot\frac{\sqrt{\pi}}{t\sqrt{t}}$$
$$\int_{-\infty}^{\infty} (-y^2)^n e^{-ty^2}\mathrm{d}y = G^{(n)}(t) = \left(\prod_{k=1}^{n} -\frac{2k-1}{2}\right)\cdot\frac{\sqrt{\pi}}{t^n\sqrt{t}}$$

Which gives us:

$$G^{(n+1)}(t) = -\frac{2n+1}{2t}G^{(n)}(t)$$

Then, $F(n)$ is minimal when $2n-1 \leq 2t \leq 2n+1$, so $n = 4$. Thus:

$$F(4) = G^{(4)}(4) = \frac{1\cdot 3\cdot 5\cdot 7}{2^4}\frac{\sqrt{\pi}}{t^4\sqrt{t}} = \frac{105\sqrt{\pi}}{8192}$$

BH09 67. **Geo** $ABCD$ is a square of side length 1. Zhuoqun draws a semicircle ω_1 with side on AB of maximal size. Zhuoqun then draws a semicircle ω_2 with side on BC of maximal size such that it does not intersect ω_1. Zhuoqun then draws a semicircle ω_3 with side on CD of maximal size such that it does not intersect ω_2. Finally, Zhuoqun draws a semicircle ω_4 with side on DA of maximal size such taht it does not intersect ω_3 or ω_1. What is the radius if ω_4?

Comment: Just Pythagorean theorem over and over again.

Answer: $\frac{-28+16\sqrt{7}}{63}$

Solution: Let r_i denote the radius of ω_i; clearly, $r_1 = \frac{1}{2}$. Now, we solve for r_2, r_3, and r_4:

$$\begin{aligned} r_1^2 + (1-r_2)^2 &= (r_1+r_2)^2 \\ r_2 &= \frac{1}{2(1+r_1)} = \frac{1}{3} \\ r_2^2 + (1-r_3)^2 &= (r_2+r_3)^2 \\ r_3 &= \frac{1}{2(1+r_2)} = \frac{3}{8} \end{aligned}$$

And:

$$\begin{aligned}
\sqrt{(r_1+r_4)^2-r_1^2}+\sqrt{(r_3+r_4)^2-r_3^2} &= 1 \\
2\sqrt{\left((r_1+r_4)^2-r_1^2\right)\left((r_3+r_4)^2-r_3^2\right)} &= 1-2r^2-2r_1r_4-2r_3r_4 \\
4r_4^2(r_4+2r_1)(r_4+2r_3) &= 1-4r_4(r_1+r_3+r_4)+4r_4^2(r_1+r_3+r_4)^2 \\
r_4^2(8r_1r_3-4r_1^2-4r_3^2+4)+r_4(4r_1+4r_3)-1 &= 0
\end{aligned}$$

Which gives us:

$$r_4 = \frac{-(4r_1+4r_3)\pm\sqrt{(4r_1+4r_3)^2+4\big(4-4(r_1-r_3)^2\big)}}{2\big(4-4(r_1-r_3)^2\big)}$$

$$r_4 = \frac{-\frac{7}{2}\pm\sqrt{\frac{49}{4}+4\cdot\frac{63}{16}}}{2\cdot\frac{63}{16}} = \frac{-28+16\sqrt{7}}{63}$$

Then, the radius is $r_4 = \boxed{\frac{-28+16\sqrt{7}}{63}}$.

BH10:gut 68. **Team** Alex, Bill, and Charlie want to play a game of Dota. They each come online at a uniformly random time between 8:00 and 8:05 pm, and begin queuing for a game. However, if any of them see any other of them online while queuing, they immediately merge parties and restart the queue (For example, if a party of two of them are still in a queue when the third logs in, they merge all parties to have all three in one queue.) Given that queuing takes exactly 2 minutes, what is the probability that they will play as a party of 3?

Comment:

Answer: $\mathbf{\frac{72}{125}}$

Solution: We model the time players 1, 2, and 3 log on with coordinate points (x, y, z) where x is the number of minutes past 8pm when player 1 logs on, and similarly for y and player 2 and z and player 3. We assume player 1 gets on first, then 2, and then 3. We can then multiply the volume by 6 for the final answer for all the different combinations of orderings of players logging on.

Thus by assumption, we have $x \le y \le x+2$, $0 \le x, y, z \le 5$, and $y \le z \le y+2$. Split the base x-y plane into $y < 3$ and $y \ge 3$. Then the area of valid (x, y) on the base with $y < 3$ has area 4 and height $y+2-y = 2$, and has volume 8. For $y > 3$, we have the triangle along the line $x = y$ with height 2 and length $2\sqrt{2}$. This triangle is stretched between $y-2 \le x \le y$ and that parallelepiped has diagonal height $\sqrt{2}$, and so the volume with $y > 3$ is $\sqrt{2}\cdot 2\cdot 2\sqrt{2}/2 = 4$.

Thus total volume is $6(8+4) = 72$ and the overall probability space is 5^3. Therefore the probability is $\frac{72}{125}$.

BH11:gut 69. **Team** Let $S = \{0, 2016\}$, $T = \{1, 2, \ldots, 2015\}$, and $N = 0$. Brice randomly selects an element $t \in T$, removes it from T, and puts it in S. Then, he adds the following to N:

$$\big(t - \max(s \in S, s < t)\big)\cdot\big(\min(s \in S, s > t) - t\big)$$

Brice repeats this process until $T = \emptyset$. What is the expected value of N?

Answer: 2031120

Solution: It is easy to show through induction that using the numbers 0 to n in this manner, the final value of N will always be $1+2+\cdots+(n-1)$. We leave that solution to the reader and present another proof.

For each t chosen, consider the coordinate point (t,t) inside the box of $(0,0)$, $(2016,0)$, $(0,2016)$, and $(2016,2016)$. Now imagine coloring all lattice points (x,y) where $x \le t$ and $y \ge t$. All the lattice points that are colored that weren't before represent $\big(t - \max(s \in S, s < t)\big) \cdot \big(\min(s \in S, s > t) - t\big)$. Note that regardless of order of the t's chosen, the colored lattice points will be those within the box (including the edges of the box) and above or on the $y = x$ line. Thus the final value of N is invariant and equal to $1 + 2 + \ldots + 2015 == \frac{2015(2016)}{2} = 2031120$.

`BH13:geo` 70. **Geo** Triangle ABC has $AB = 4, BC = 6, CA = 5$. Let D, E be points on side BC such that $BD = 1, DE = 4, EC = 1$. Let F be a point on side AC and let BF intersect AD and AE at X and Y, respectively. Suppose $BF = 2XY$; find AF.

Answer: $\mathbf{7 - \sqrt{34}}$

Solution: Let $\frac{CF}{AF} = x$; by mass points, we get $\frac{XF}{BF} = \frac{5}{6+x}$ and $\frac{YF}{BF} = \frac{1}{6+6x}$. Since $BF = 2XY$, $\frac{5}{6+x} - \frac{1}{6+6x} = \frac{1}{2}$. Solving yields $x = \frac{4+\sqrt{34}}{3}$, so $AF = \frac{1}{1+x} \cdot 5 = 7 - \sqrt{34}$.

`BH14` 71. **Geo** Triangle ABC has $AB = 4, BC = 6, CA = 5$. Let M be the midpoint of BC and P the point on the circumcircle of ABC such that $\angle MPA = 90°$. Let D, E be the foot of the altitudes from B, C, respectively, and let PD, PE intersect BC at X, Y, respectively. Find XY.

Answer: **8**

Solution: Let H be the orthocenter of ABC; since the reflection H' of H over BC lies on (ABC), the reflection H'' of H over M lies on (ABC). Since AH'' is a diameter of (ABC), it follows that M, H, P are collinear. We then have $ADEHP$ concyclic. Let F be the foot of the altitude from A and M' the reflection of M over F. Angle chasing, $\angle CDX, BEY = \angle AEP, ADP = \angle AHP = \angle FHM = \angle M'HF$. Then, $\angle XDH, YEH = 90 - \angle M'HF = 180 - \angle HM'X, HM'Y$, so $DHM'X, EHM'Y$ are concyclic. By power of a point:

$$BM' \cdot BX = BH \cdot BD = BE \cdot BA \qquad CM' \cdot CY = CH \cdot CE = CD \cdot CA$$

By the Pythagorean theorem, we find that:

$$BE = \frac{27}{8}, BM' = \frac{3}{2} \qquad CD = \frac{9}{2}, CM' = \frac{9}{2}$$

This gives us $BX = 9, CY = 5$, and $XY = 9 + 5 - 6 = 8$.

`BP01` 72. **Calc** Consider the curve f obtained by drawing a line ℓ from the origin through a circle of radius $1/2$ centered at $(0, 1/2)$, then picking the point with the $y-$coordinate of the intersection of ℓ with the circle and the $x-$coordinate of the intersection of ℓ with the line $y = 1$. Compute $\int_{-\infty}^{\infty} f(x)dx$.

Answer: $\boldsymbol{\pi}$ **Comment:** The radius of the circle can be changed to make the answer less predictable **Solution:** Let the readius of the circle be a. The coordinates of the curve parametrized by the angle θ (between ℓ and the x-axis) are easily deduced from geometry:

$$x = 2a\cot(\theta)$$
$$y = a(1 - \cos(2t))$$

Eliminating the parameter is also easy,

$$y = \frac{8a^3}{x^2 + 4a^2}$$

This has integral:

$$\int_{-\infty}^{\infty} \frac{8a^3}{x^2+4a^2}dx = 4a^2\tan^{-1}\left(\frac{x}{2a}\right)\Big|_{-\infty}^{\infty} = 4\pi a^2$$

BP02 73. **Geo** Compute the area of the largest square such that its four vertices lie on the edges of a unit cube.

Answer: $\mathbf{\frac{9}{8}}$

Comment: I think it is reasonable for students to get the answer in a couple lines but it is unreasonable to expect a highly rigorous solution that is short. See Prince Rupert's Cube.

[04/16] (Sean) I think this problem might be too well-known, especially in China. I saw it in Chinese book '' (translated in Korean though) which I suppose to be quite famous there. Also it appeared in some national olympiads in recent years.

Eric: If we use this problem, we should also provide a complete proof.

Solution: It is clear from a sketch that each of the square vertices lie on a different cube edge. By symmetry, if one square vertex lies at a distance x from one cube vertex then each must. That is, each square vertex should divide the cube's edge uppon which it sits into two parts that are the same for each such edge. (In a very fluffy sense this comes from the fact that we need an "even" number of symmetries.) After drawing this, one can see that the side length of the square can be represented in two ways:

$$\sqrt{2x^2+1} = d = \sqrt{2(x-1)^2}$$

Thus, $x = 1/4$ so the area is $d^2 = \boxed{\frac{9}{8}}$.

BP03 74. **Team** For n even, let A be an $n \times n$ matrix where all entries are nonzero and B be an $n \times 1$ vector where the element in the ith row of B is b_i. Let $f(b_1, \ldots, b_n) = e^{B^TAB}$. After differentiating $f(b_1, \ldots, b_n)$ by each of b_1 through b_n once, compute the number of terms that are left when the expression is evaluated for $b_1 = \cdots = b_n = 0$.

Comment: Nick: This problem is awesome.

Eric: Is linear algebra okay to use for the tests? Also, this problem is not suitable for guts.

Heesu: Good problem. Aaron, take it since guts won't use calc material.

Answer: $\boldsymbol{\frac{n!}{(n/2)!}}$

Solution: $f = \exp\left[\sum_{\alpha,\beta} b_\alpha b_\beta a_{\alpha\beta}\right] = e^{(b_\alpha^2 a_{\alpha\alpha})}\prod_{\alpha\neq\beta} e^{b_\alpha b_\beta a_{\alpha\beta}}$. It is easy to see that the first part can be dropped immediately becasue after differentiation by one b_i that part will yeild a term with $2a_{ii}b_if$ which will vanish when we set $b_i = 0$. It is also easy to see that we can immediately set $b_i = 0$ after differentiation by b_i. Now,

$$\frac{df}{db_1} = \left(\sum_{\beta=2}^{n} b_\beta(a_{\beta 1}+a_{1\beta})\right) f$$

$$\frac{d^2f}{db_1db_2} = \left(a_{12}+a_{21}+\sum_{\beta=3}^{n} b_\beta(a_{\beta 1}+a_{1\beta})\right) f + \left(\sum_{\beta=3}^{n} b_\beta(a_{\beta 2}+a_{2\beta})\right)\left(\sum_{\beta=3}^{n} b_\beta(a_{\beta 1}+a_{1\beta})\right) f$$

In general,

BP04:alg 75. **Alg** Solve the recurrence as a function of a_1:

$$a_{n+1} = 4a_n^3 - 3a_n \qquad |a_1| \leq 1$$

Answer: $\cos\left(3^n \cos^{-1}(a_1)\right)$

Solution: Notice that the RHS looks like the cosine identity $\cos(3x) = 4\cos^3(x) - 3\cos(x)$, albeit hard to see. This suggests we make the substitution $a_n = \cos(b_n)$. Then $4\cos^3(b_n) - 3\cos(b_n) = \cos(3b_n)$, which gives us the recurrence:

$$\cos(b_{n+1}) = \cos(3b_n) \implies b_{n+1} = 3^n b_n$$

Plugging in $b_n = \cos^{-1}(a_n)$ gives us the desired recurrence $\boxed{a_n = \cos\left(3^n \cos^{-1}(a_1)\right)}$.

BP05 76. **Geo** Points A and B are fixed on a plane such that $AB = 2$. Cnsider the set of all points C such that there exists a point P inside of triangle ABC with $\angle PAB = \angle PBC = \angle PCA = 30°$. Given that the locus of point C traces out a circle, compute the square of its radius.

Comment: This problem is inspired by http://mathworld.wolfram.com/NeubergCircles.html. Anyone should feel free to write up the solution.

Answer: $4\left(1+\sqrt{3}\right)$

Solution: Rough solution outline (finish later). Someone else might be more coherent than me. (Aaron)

- From a construction standpoint, we can easily construct the line (ℓ_1) on which P lies given A and B. Given P, we can construct the line (ℓ_2) on which C lies. The trouble is how you locate the point on ℓ_2 that satisfies the $\angle PCA = \alpha$ part. - Starting with a constructed triangle that satisfies all three angle conditions, notice that the circumcircle of PAC will always be tangent to AB. Thus, given A, B, P, ℓ_1 and ℓ_2, the valid points C are the intersection point(s) of [the circle tangent to AB at A passing through P] (ω_1) and [ℓ_2]. Assume there are two such points: C_1 and C_2. - Rather than thinking about this construction as letting point P vary, we can also think about this as letting the center of ω_1, O, vary along the line through A perpendicular to AB. The intersection of ω_1 and ℓ_1 defines point P, and from that we can construct all points C. - By Power of a Point, $BC_1 \cdot BC_2 = BA^2 = 4$. This implies that all such C's must be on a circle. (Potential direction issues?) - Insert proof that all such points on this alleged circle are actually valid C's. I didn't actually check this. - Didn't bother to calculate the radius, but I suspect there's a solution with $Pow(\odot O, P) = OP^2 - r^2$ kind of solution.

Ben's solution (?) Let us calculate the equation for the circle that is traced out. To do this we will place the triangle in a planar coordinate system with A at $(0, 0)$ and B at $(2, 0)$. Then the third trangle vertex, C, has some coordinates (x, y) with $0 < x < 2$ and $y > 0$. Now because we know that the point C lies on a circle we can write two expressions involving the triangle sides $AC \equiv a$ and $BC \equiv b$. For any point in the locus those sides must satisfy:

$$x^2 + y^2 = a^2$$
$$(x-2)^2 + y^2 = b^2$$

becasue we can look at the circular locus from the perspecitve of A or B respectively. It may be instructive to think of those equations as representing lengths of the vector from A (resp. B) to the point on the circle, (x, y). Now we wish to compute the redius of the circle traced by (x, y) so we can write, $r^2 = (x - X)^2 + (y - X)^2$ in which X and Y are fixed. We wish to make the replacements for x and y however the expression is messy and it is not clear how to eliminate a and b yet.

To use the additional information about the angle we derive the Brocard identity, an identity relating the angle to the area of the triangle. The area can be calculated with Heron. The identiy is, $\cot\frac{\pi}{12} = \frac{a^2+b^2+2^2}{4\Delta}$ where Δ is the triangle area. We can show this by first showing the identity

$$\left(\cot\frac{\pi}{12} - \cot A\right)\left(\cot\frac{\pi}{12} - \cot C\right)\left(\cot\frac{\pi}{12} - \cot B\right) = \csc A \csc B \csc C$$

Then the cubic in $\cot\frac{\pi}{12}$ has the (only) root $\cot A + \cot B + \cot C$ which is equivalent to $\cot\frac{\pi}{12} = \frac{1+\cos A\cos B\cos C}{\sin A\sin B\sin C}$. Then the identity follows by dropping perpendiculars from each of the triangle vertices to the opposite sides and writing out the expression.

To finish we can write

$$\begin{aligned}
\cot\frac{\pi}{12} &= \frac{a^2+b^2+2^2}{\sqrt{4a^2b^2-(a^2+b^2-2^2)^2}} \\
2+\sqrt{3} &= \frac{x^2+y^2+(x-2)^2+y^2+2^2}{\sqrt{4(x^2+y^2)((x-2)^2+y^2)-(x^2+y^2+(x-2)^2+y^2-2^2)^2}}
\end{aligned}$$

which is an expression for the circle in which the cosine was evaluated with a half angle formula. Now solving for x reveals,

$$x = \frac{1}{2}\left(2 \pm \sqrt{\pm 8(2+\sqrt{3})y - 4y^2 - 12}\right)$$

Or,

$$(x-1)^2 + (y \pm 2 + \sqrt{3})^2 = (2+\sqrt{3})^2 - 3$$

at which point the result follows.

BP06 77. **Geo** Three identical circles are packed into a unit square. Each of the three circles are tangent to each other and tangent to at least one side of the square. What is the radius of the circles?

Comment: Eric: Reworded problem to eliminate the necessity of proof.

Answer: $\mathbf{\frac{2}{4+\sqrt{2}+\sqrt{6}}}$

Solution: After packing the circles into the square, the circles form an equilateral triangle of side length $2r$ after connecting the radii of the circles. With this shape, a right triangle can be constructed, and we may write the equation $\sin 75 = \frac{1-2r}{2r}$. Solving this equation yields the desired result. This can be achieved by writing $75 = 45 + 30$ to evaluate the sine and then the quadratic formula. (Include justification on why this is the largest radius.)

BP07 78. **Calc** Find the volume common to 3 cylinders of radius 1 that lie along the three coordinate axes.

Answer: $\mathbf{8(2-\sqrt{2})}$

Solution 1: Using multi-variable calculus we proceed as follows. Split the object into 16 congruent pieces. One of these sits in the first octant and is defined by, $\{(r,\theta,z) \mid 0 \le r \le 1,\ 0 \le \theta \le \pi/4,\ 0 \le z \le \sqrt{1-x^2}\}$ where $x = r\cos\theta$. Thus the entire volume is:

$$16\int_0^{\pi/4}\int_0^1\int_0^{\sqrt{1-r^2\cos^2\theta}} r\,dz\,dr\,d\theta = 16\int_0^{\pi/4}\int_0^1 \sqrt{1-r^2\cos^2\theta}\,r\,dr\,d\theta = 16 - 8\sqrt{2}$$

Solution 2: Using calculus of only one variable, we want to use the common technique that $V = \int A\,dh$ where A is the area of a cross section and h is the height at which that cross section sits. To do this, divide the region into 8 congruent pieces and consider the 1st quadrant part again. Now we will consider two regions divided by $z > 1/\sqrt{2}$ and $z < 1/\sqrt{2}$. For the first region, slicing the figure with a plane horizontally produces a square cross section. The sides of this square are of length $\sqrt{1-z^2}$. Thus the volume of this region is $\int_{1/\sqrt{2}}^{1}(1-z^2)dz = \frac{1}{12}\left(8-5\sqrt{2}\right)$.

For the second region we will obtain a cross section that consists of a circular arc and two triangles. This cross section is a subset of the unit disc in the first quadrant. For some point on the circle $\{x, \sqrt{1-x^2}\}$ the region of the cross section is equal to the intersection of the disc and a square of side length x with one corner at the origin and two sides parallel to the coordinate axes. The area of this region is $x\sqrt{1-x^2} + \pi/2 - \cos^{-1} x$. It is also easy to see that $x^2 + z^2 = 1 \implies dz = -\frac{xdx}{\sqrt{1-x^2}}$. So the volume of this region is,

$$\int_1^{\frac{1}{\sqrt{2}}} \left(\sqrt{1-x^2}x + \frac{1}{2}\left(\frac{\pi}{2} - 2\cos^{-1}(x)\right)\right) \frac{-x}{\sqrt{1-x^2}}\, dx = \frac{4}{3} - \frac{7}{6\sqrt{2}}$$

Summing these two answers gives $2-\sqrt{2}$, finally there are 8 regions, giving us an answer of $\boxed{8(2-\sqrt{2})}$.

BP08 79. **Calc** Let $y(x)$ be the solution to the differential equation $y'' = yx$. The following limit exists:

$$\lim_{x\to\infty} \frac{\ln y}{x^{3/2}}$$

- compute it.

Answer: $\mathbf{\frac{2}{3}}$

Solution: First of all this is Airy's equation so one can simply recall the asymptotic expansion for $Ai(x)$.

Because we know that the limit exists call it α. This implies that $y \approx e^{\alpha x^{3/2}}$ for $x \to \infty$. Using the differential equation we have

$$xe^{ax^{3/2}} = \frac{9}{4}a^2 x e^{ax^{3/2}} + \frac{3ae^{ax^{3/2}}}{4\sqrt{x}}$$

Thus $\alpha = \frac{3\sqrt{16x^3+1}-3}{18x^{3/2}}$ and in the limit $x \to \infty$, $\alpha = \boxed{\frac{2}{3}}$.

BP09 80. **Calc** Find f satisfying $4x(1-x)f'' + 2(1-2x)f' + f = 0$ and $f(1) = 1$, $f'(1) = 1/2$.

Answer: $\boldsymbol{\sqrt{x}}$

Solution: This is Mathieu's equation. Make the substitution $x = \cos^2 y$. Then the equation becomes, $f''(y) + f(y) = 0$. The solution to this satisfying the conditions is $\cos(y)$. The rest follows.

Finally suppose that we simply search for the taylor series of f at 1. Differentiating the equation once and evaluating at 1 gives $f''(1) = -1/4$. Then, using the differnetial equation we can write for $n \geq 2$:

$$\frac{d^n}{dx^n}(4x(1-x)f'' + 2(1-2x)f' + f) =$$
$$4x(1-x)f^{(n+2)} + [2(1-2x) + 4(1-2x)n]f^{(n+1)} + [1-4n-4n(n-1)]f^{(n)} = 0$$

So in general

$$f^{(n+1)}(1) = (1/2-n)f^{(n)}(1)$$

The solution to this is the obvious product which can be written as the generalized binomial coefficient: $f^{(n)}(1) = \prod_{m=0}^{n-1}\left(\frac{1}{2} - m\right) = n!\binom{\frac{1}{2}}{n}$. So the series is

$$\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(x-1)^n = \boxed{\sqrt{x}}$$

It should be easy to see how to sum this series from the standard decomposition, $(x+1)^n = \sum_{k=0}^{\infty}\binom{n}{k}x^k$.

BP10 81. **Calc** Consider the unit disc and a line passing through it such that the line is distance d away from the center of the disc. Compute the volume of the solid obtained by spinning the disc about the line.

Answer: $\boldsymbol{\frac{4\pi}{3}\left(1-d^2\right)^{3/2}}$

Solution: You can do this brute force with an integral of discs etc. or use Pappus' theorem which states that the volume of revolution has volume equal to the area rotated times the length that its centroid travels. It thus suffices to find the area of the larger part of the sliced disc and its centroid. The area is

$$2\int_{-1}^{d}\sqrt{1-x^2}dx = \sqrt{1-d^2}d + \sin^{-1}(d) + \frac{\pi}{2} = A$$

(It is not actually necessary to compute this area explicitly.) The centroid is clearly on the symmetry axis of the region. If we center the disc at the origin and the line at $x = d$ then the centroid has y coordinate 0 and x coordinate given by:

$$\bar{x} = \frac{\int_{-1}^{d}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x dy dx}{\int_{-1}^{d}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 dy dx} = \frac{-2}{3A}\left(1-d^2\right)^{3/2}$$

The distance traveled by the centroid is $2\pi|\bar{x}|$. Thus the volume is $A(2\pi|\bar{x}|) = \boxed{\frac{4\pi}{3}\left(1-d^2\right)^{3/2}}$.

BP11 82. **Calc** Compute

$$\int_0^{1/e}\frac{dx}{\sqrt{-\ln(x)-1}}$$

Answer: $\boldsymbol{\frac{\sqrt{\pi}}{e}}$

Solution: Set $u = \sqrt{\ln\left(\frac{1}{ex}\right)}$ then $e^{-u^2-1} = x$ and $-2ue^{-u^2-1}du = dx$ so the integral becomes,

$$\int_0^{1/e}\frac{dx}{\sqrt{-\ln(x)-1}} = \int_0^{1/e}\frac{dx}{\sqrt{\ln\left(\frac{1}{ex}\right)}} = \int_{\infty}^{0}\frac{-2ue^{-u^2-1}du}{u} = \int_{-\infty}^{\infty}e^{-u^2-1}du = \frac{\sqrt{\pi}}{e}$$

BP12 83. **Calc** Using the fact that $\sum_{n=1}^{\infty}\frac{1}{(nx)^2-1} = \frac{x-\pi\cot\left(\frac{\pi}{x}\right)}{2x}$, compute

$$\prod_{k=1}^{\infty}\left(1-\frac{1}{(2k)^2}\right).$$

Answer: $\boldsymbol{\frac{2}{\pi}}$

Solution: First of all, the interchanging of limits in this solution is justified but I won't show the proofs here.

To begin take a log and write the product as a sum:

$$\ln\left(\prod_{k=1}^{\infty}\left(1-\frac{1}{(2k)^2}\right)\right)=\sum_{k=1}^{\infty}\ln\left(1-\frac{1}{(2k)^2}\right)=\sum_{k=1}^{\infty}\ln((2k)^2-1)-\ln((2k)^2).$$

Now the plan will be to represent the summand as an integral of something like $\frac{1}{(nx)^2-1}$, interchange integration and summation, sum, and then finally integrate. This is a typical method in trying to sum series that are tricky but converge uniformly.

$$\begin{aligned}\sum_{k=1}^{\infty}\ln((2k)^2-1)-\ln((2k)^2)&=\sum_{k=1}^{\infty}\ln((2k)^2-x^2)\Big|_0^1\\&=\sum_{k=1}^{\infty}\int_0^1\frac{-2x}{(2k)^2-x^2}dx=\int_0^1\frac{-2}{x}\sum_{k=1}^{\infty}\frac{1}{\left(\frac{2k}{x}\right)^2-1}dx.\end{aligned}$$

using the sum formula we find,

$$\int_0^1-\frac{2-\pi x\cot\left(\frac{\pi x}{2}\right)}{2x}\,dx=\log\left(\sin\left(\frac{\pi x}{2}\right)\right)-\log(x)\Big|_0^1=\log\left(\frac{\sin\left(\frac{\pi x}{2}\right)}{x}\right)\Bigg|_0^1=0-\log\left(\frac{\pi}{2}\right)$$

where the last evaluation was done by taking limits at 0 and 1. Thus the answer is $\boxed{\frac{2}{\pi}}$.

BP13:tea 84. **Disc** Compute the least integer $n>1$ such that $\sum_{k=1}^{n}k^2$ is a perfect square.

Comment: In fact, 24 is the ONLY positive integer with this property. Actually it might be far too easy to bash this problem.

Schuyler: Well, except for 1...

Answer: 24

Solution: The sum in question is $\frac{n(n+1)(2n+1)}{6}$, divide the cases as follows,

BP14 85. **Alg** Solve the recurrence,

$$a_{n+1}=2a_n(1-a_n)\quad n=1,2,3,\ldots\qquad a_1=\frac{\pi}{2}.$$

That is, compute a_n.

Comment:

Answer: $\boldsymbol{\frac{1-(1-\pi)^{2^{n-1}}}{2}}$

Solution: The RHS begs for the following substitution, $a_n=\frac{b_n-1}{2}$ so that the recurrence becomes, $b_{n+1}=b_n^2$. But we can see the solution to this by inspection. At each step of the recurrence we square so by the $n+1$th step we have squared n times. $b_n=b_1^{2^{n-1}}$. So, $a_n=\frac{1-(1-\pi)^{2^{n-1}}}{2}$.

BP15 86. **Alg** Compute:

$$\sum_{k=1}^{n}\tan^2\left(\frac{\pi k}{2n+1}\right)$$

for $n=13$.

Comment:

Answer: 351

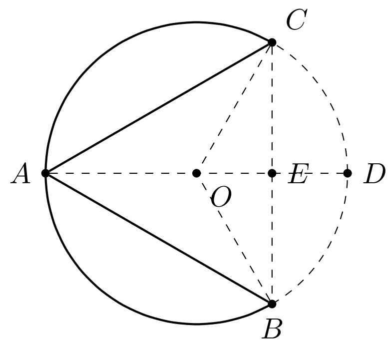
Solution: TODO(BP): Write a solution.

BP16 87. **Geo** Xavier is drilling out a cavity in a wooden sphere with a conical drill-bit. The sphere has radius R and the drill-bit is a right circular cone with opening angle $\pi/6$. The drill moves in a straight line along a sphere diameter removing all of the wood in the pierced region of the sphere, leaving a conical cavity. Xavier stops drilling when the drill is about to pierce the opposite side of the sphere. The drill bit is much longer than the sphere diameter so the cavity is conical. What is the surface area of the object?

Comment:

Answer: $\frac{9\pi}{2}R^2$

Solution:



This picture is a crossection of the object in a plane that contains the drilling axis. Because the angle of CAD is $\pi/6$, the angle COD is $\pi/3$. Therefore if the sphere has radius R then the length of CB is $\sqrt{3}R$. Thus the inner conical surface has area, $S = \pi(\sqrt{3}R)\left(\frac{\sqrt{3}R}{2}\right) = \frac{3\pi}{2}R^2$.

Turining to the outer spherical shell we enclose the sphere inside a cylinder of radius R and length $2R$ just touching at a great circle. Let the centers of the flat ends of the cylinder be at A and D. The projection of the sphere onto the lateral cylinder surface preserves area. The area of the entire sphere is same as lateral surface area of the cylinder which is $= (2\pi R)(2R) = 4\pi R^2$. The projection of the spherical shell on the cylinder has area $(2\pi R)|AE| = 2\pi R(R + R\sin(\pi/3)) = 3\pi R^2$.

Therefore the total surface area is $\frac{9\pi}{2}R^2$.

BP17:gut 88. **Team** Consider triangle $\triangle ABC$ with side lengths 4, 11, and 9 for AB, BC and CA respectively. The triangle is spun around a line that passes through B and the interior of the triangle (including the edges BC or BA). Of all possible lines with these constraints, what is the largest possible volume of the resulting solid?

Comment:

Answer: 96π

Solution: There are two factors that dtermine the area of the spun solid. The first is related to the projected area on the plane containing the spinning axis and the second is the "distance" of that area away from the spining axis. More precisely, by Pappus' theorem that the volume of a solid of revolution generated by the revolution of a lamina about an external axis is equal to 2π times the product of the area of the lamina and the distance traveled by the lamina's geometric centroid. Thus we should consider the maximization problem in terms of these two factors.

The distance travelled by the centroid is clearly maximized when the spinning axis is chosen along an edge BC, BA because all of the triangle mass is to one side of the axis and thus the centroid is farthest away. Coincidentally, the relevant area of the spun lamina is also maximized when the axis is along an edge! If there were area on both sides of the spinning axis (A_1, A_2) then when the triangle is spun the volume of the solid corresponds to the lamina made up of the intersection of A_1 with the reflection of A_2 across the axis. This area is strictly less than the triangle area. Thus the volume is maximized when the axis is either along BC or BA.

In both cases the area being spun is the same so the only difference is in the distance travelled by the centroid. We can now compute the volumes in the two cases. Notice that we *do not* need to compute the geometric centroid because the volumes that we need are made up of sums and differences of volumes of cones. In the case where BC is the axis, the volume is the sum of two cones with slants 4, and 9 and heights summing to 11. After applications of the pythagorean theorem we obtain equations for the two cone heights, x, y:

$$x + y = 11$$
$$y^2 - x^2 = 9^2 - 4^2$$

which have solution, $x = \frac{28}{11}, y = \frac{93}{11}$. The volume of a cone with slant l and height h is $\pi h \frac{l^2-h^2}{3}$. So the volume here is, $\frac{384\pi}{11}$. In the case BA the volume is the difference of two cones with slants 11, 9 and heights summing to 11. If the base has length b and the height is h we obtain the following two equations by the Pythagorean theorem:

$$h^2 + b^2 = 11^2$$
$$(h-4)^+ b^2 = 9^2$$

which have solution, $h = 7, b = 6\sqrt{2}$. The solid in this case has volume 96π which is larger.

BP18 89. **Alg** Find the maximum value of

$$\sum_{n=1}^{2015} \cos\left(\sqrt{2}t + n\right)$$

for $t \in \mathbb{R}$.

Answer: $\frac{\sin\left(\frac{2015}{2}\right)}{\sin\left(\frac{1}{2}\right)}$

Solution: The key in this problem is that all of the summands here have the same frequency, as functions of t they are just offset by different amounts. Now suppose that we have only two cosine functions, $\cos\left(\sqrt{2}t+n\right) + \cos\left(\sqrt{2}t+m\right)$. These can be combined with trigonometric addition formulas into, $\sqrt{2}\sqrt{\cos(m-n)+1}\cos\left(\sqrt{2}t + \frac{m+n}{2}\right)$. It follows by induction that the sum we have can be written, $A\cos(\sqrt{2}t+\delta)$ for appropriate A and δ. The maximum is thus A.

In the general case

$$\begin{aligned}\sum_{n=1}^{2015} \cos\left(\sqrt{2}t+n\right) &= \sum_{n=1}^{2015} \cos\left(\sqrt{2}t\right)\cos(n) - \sin\left(\sqrt{2}t\right)\sin(n) \\ &= \cos\left(\sqrt{2}t\right)\sum_{n=1}^{2015}\cos(n) - \sin\left(\sqrt{2}t\right)\sum_{n=1}^{2015}\sin(n) \\ &= A\cos(\sqrt{2}t+\delta)\end{aligned}$$

Where $A = \sqrt{\left(\sum_{n=1}^{2015}\cos(n)\right)^2 + \left(\sum_{n=1}^{2015}\sin(n)\right)^2}$. By the use of trigonometric formulas we can re-write this sum,

$$\begin{aligned} A &= \sqrt{2015 + \sum_{n=1,m<n}^{2015} 2\cos(n)\cos(m) + 2\sin(n)\sin(m)} \\ &= \sqrt{2015 + 2\sum_{n=1,m<n}^{2015} \cos(n-m)} \\ &= \sqrt{\sum_{n=1,m=1}^{2015} \cos(n-m)} = \boxed{\frac{\sin\left(\frac{2015}{2}\right)}{\sin\left(\frac{1}{2}\right)}} \end{aligned}$$

The evaluation of the double sum comes from applying Lagrange's Trigonometric Identity twice. This identity is,

$$\sum_{n=1}^{N} \cos n\theta = -\frac{1}{2} + \frac{\sin(N+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}$$

It can be easily derived by first expanding, $2\sin\frac{\theta}{2}\cos n\theta = \sin\frac{(2n+1)\theta}{2} - \sin\frac{(2n-1)\theta}{2}$ and then by telescoping we get,

$$\sum_{n=1}^{N} 2\sin\frac{\theta}{2}\cos n\theta = \sin\frac{(2n+1)\theta}{2} - \sin\frac{\theta}{2}$$

Finally divide by $2\sin\frac{\theta}{2}$.

BP19:gut 90. **Team** Consider the following functional equation for all $x > 1$:

$$a(x) = \sqrt{x}\cdot a(\sqrt{x}) + \ln(2)\cdot x$$

Given that that $a(e) = 0$, solve for $a(x)$. Solving the equation means giving a formula for $a(x)$ that does not reference the function a itself.

Comment:

Answer: $\boldsymbol{x\ln(\ln(x))}$

Solution: There is a basic strategy to solve functional equations whereby one writes the equation in a new form and re-defines the function in order to make it simpler. To do this we first divide the equation by x and then define $b(x) = \frac{a(x)}{x}$ to produce $b(x) = b(\sqrt{x}) + \ln(2)$. Next notice that this would be much nicer if there were no square roots. One potential way to approach this problem is to define $y = \sqrt{x}$ to get, $b(y^2) = b(y) + \ln(2)$ but then we still have the fundamental problem that an argument is squared which is a nonlinear operation. Instead, to make the arguments linearly related we should define $x = e^y$ and $c(y) = b(e^y)$ so that we obtain: $c(y) = c\left(\frac{y}{2}\right) + \ln(2)$ or $c(y) - c\left(\frac{y}{2}\right) = \ln(2)$. Finally exponentiate and define $e^{c(y)} = d(y)$ to produce: $\frac{d(y)}{d(y/2)} = 2$ whereupon it should be clear that $d(y) = Ky$ solves the equation for any K. Translating this solution back to the original function, $d(y) = Ky \implies c(y) = \ln(Ky) \implies \ln(Ky) = b(e^y) \implies \ln(K\ln(x)) = b(x) = \frac{a(x)}{x}$. So it is clear that the condition determines $K = 1$ and a unique solution, $x\ln(\ln(x))$.

Last we must ensure that inverting these substitutions were indeed valid. Note that the substitution $y = \sqrt{x}$ is allowable since $x > 1$ implies that $\sqrt{x}$ and x are bijective. Second, the substitution $x = e^y$ is invertible since e^y is a monotonically increasing function. Last, consider the substitution of $e^{c(y)}$ to $d(y)$. Supposing that $a(x) = x\ln(\ln(x))$, note that

$c(y) = \ln(y)$ and $d(y) = y$. Since $x = e^y$ where $x > 1$, we have also that $y > 0$. In order to be able to take the log of $e^{c(y)} = d(y)$ to receive $c(y) = \ln(d(y))$, it must be that $d(y) > 0$, which is true given $d(y) = y$ and $y > 0$. Therefore all the steps are indeed reversible.

BP20 91. **Team** Barnaby bought a large doughnut shaped cake; one thick wall of cake. He wants to share it with many people but only has time to make 22 cuts. What is the maximum number of pieces into which he can cut the cake? Assume that each cut slices the cylinder with an infinite plane dividing it along the intersection.

Comment: Eric: This solution needs to be elaborated on more.

TODO(HS): Re-write the problem http://www2.washjeff.edu/users/mwoltermann/Dorrie/67.pdf

Answer: 1794

Solution: The maximum number of pieces resulting from n cuts can be counted by thinking about the n planes along which he can cut. We obtain, $\binom{n+1}{3} + n + 1 = (n^3 + 5n + 6)/6$.

BP21 92. **Geo** A triangle with side lengths 3,4,5 sits with at least two vertices on the parabola x^2 in the $x-y$ plane. The maximum $y-$coordinate of the vertices is minimized, compute its value.

Comment:

Answer: TODO(BP): Find the answer.

Solution: Let us label that three triangle

TODO(BP): Finish the solution.

BP22:gut 93. **Disc** Ebeneezer is painting the edges of a cube. He wants to paint the edges so that the colored edges form a loop that does not intersect itself. For example, the loop should not look like a "figure eight" shape. If two colorings are considered equivalent if there is a rotation of the cubes so that the colored edges are the same, what is the number of possible edge colorings?

Comment:

Answer: 4

Solution: For notation's sake assume that the cube is centered at coordinates $\{(i, j, k) : i, j, k \in \{0, 1\}\}$ in $\mathbb{R}^3$. Imagine painting from any vertex. Note that for each direction (along the x, y, z-axes) that each direction must be traversed an even number of times in the positive and negative. Therefore the edges painted must be even, and since it completes a loop, must be inside $\{4, 6, 8, 10, 12\}$. Note 10 and 12 are too large since there will be overlap.

There is one loop of length 4 (the outline of a face). If the loop is of length 6, by rotation we may assume that we paint from vertex $(1, 0, 1)$ and has lines painted from that vertex to vertices $(0, 0, 1)$ and $(1, 0, 0)$. Painting a line from $(0, 0, 1)$ to $(0, 0, 0)$ results in the outline of two adjacent faces. Painting a line from $(0, 0, 1)$ to $(0, 1, 1)$ results in two possibilities to connect $(0, 1, 1)$ to $(1, 0, 0)$: another outline of two adjacent faces and a zig-zag pattern.

If the loop is of length 8, note that 4 edges are not painted and the edges come in pairs. Furthermore, if any vertex has 2 edges not painted, note that the third edge of that vertex has an end. Since each of the four non chosen edges has a unique vertex at its end, the 4 unpainted edges border all 8 edges. The only two arrangements of non-painted edges are exampled by $\{(0, 0, 0) - (1, 0, 0), (0, 0, 1) - (1, 0, 1), (0, 1, 0) - (1, 1, 0), (0, 1, 1) - (1, 1, 1)\}$ and $\{(0, 0, 0) - (1, 0, 0), (0, 1, 0) - (1, 1, 0), (0, 0, 1) - (0, 1, 1), (1, 0, 1) - (1, 1, 1)\}$. The first one has two disjoint painted loops and the latter is valid.

Therefore there are 1 of length 4, 2 of length 6, and 1 of length 8 for a total of 4.

BP23 94. **Disc** A family reunion with eight numbered tables is arranged so that at each table there is at least one person from each of the three generations that are present at the event. Each table seats six and 35 people come to the reunion including at least eight from each generation. Compute the maximum total possible number of seating arrangements ignoring placement at the tables.

Comment:

Answer: TODO(BP): Find the answer.

Solution: TODO(BP): Write a solution.

BP24 95. **Disc** Cedric is stuck in the Delayed-ville Airport for 3 hours. During this time he wants to visit as many of the 39 airport lounges as possible, but unfortunately even when sprinting he can only move at at speed 8ft/s. The airport terminal is shaped like a circle and the lounges are equispaced around the perimeter with a separation of 0.25 mi = 1320 ft, he begins at the location of one lounge. Compute the number of distinct lounges that he can visit if he spends only 2 minutes in each and must return to his starting point before three hours have elapsed.

Comment: Eric: Need to fill in answer and solution.

Answer: TODO(BP): Find the answer.

Solution: TODO(BP): Write a solution.

BP25 96. **Geo** A spider sits in a large semi-ellipsoidal bowl defined by $\{x^2+y^2/3+z^2/4=2, z<0\}$ at position $(x,y,z)=(1,1,2\sqrt{2/3})$. Suddenly the spider sees some food fall onto the surface at location $(1/2,\sqrt{3},\sqrt{3})$. The spider immediately walks to the food on the surface of the bowl along a path obtained by the vertical projection of the line connecting the spider's initial position and that of the food. Compute the lowest $z-$coordinate attained by the spider on this path.

Comment: Somebody should fiddle with numbers to make it nicer.

Answer: $\boldsymbol{\sqrt{\frac{2}{13}\left(1+2\sqrt{3}\right)}}$

Solution: The line connecting the spider and the food is $\vec{\ell}=(X,Y,Z)=[(1,1,2\sqrt{2/3})-(1/2,\sqrt{3},\sqrt{3})]t+(1,1,2\sqrt{2/3})(1-t)$. We can determine the projection onto the ellipsoid by evaluating the x,y coordinates of the line on the equation of the bowl:

$$z^2=4(2-x^2-y^2/3)=\left(-13t+8\sqrt{3}+4\right)t$$

We now need to obatin the absolute maximum of this curve on the interval $t\in[0,1]$. This can be computed by completing the square so that the lowest z-coordinate is, $\sqrt{\frac{2}{13}\left(1+2\sqrt{3}\right)}$.

BP26 97. **Disc** Two people play a game in which in turns they remove edges from a square grid of size 5x5 in which two opposite corners are marked with red dots. The objective is to take turns without preventing a path connection between the two dots along the remaining edges. The first person to prevent any such path looses. If number of turns in the game is equal to the number of edges removed including that of the loser, what is the expected number of turns in this game?

Comment: Eric: Need to fill in answer and solution.

Answer: TODO(BP): Find the answer.

Solution: TODO(BP): Write a solution.

BP27 98. **Alg** Compute

$$\tan\left(\frac{3\pi}{11}\right)+4\sin\left(\frac{2\pi}{11}\right)$$

in terms of radicals.

Comment:

Answer: $\sqrt{11}$

Solution: By repeated use of the angle sum identiy or a multiple angle formula it is easy to see that:

$$\sin(11\alpha)=\sin\alpha\left(11-220\sin^2\alpha+1232\sin^4\alpha-2816\sin^6\alpha+2816\sin^8\alpha-1024\sin^{10}\alpha\right)$$

Choosing $\alpha=\frac{\pi}{11}$ we find that $y=\sin^2\alpha$ is the solution to the polynomial equation:

$$\sin(\pi)=0=11-220y+1232y^2-2816y^3+2816y^4-1024y^5$$

Now, returning to the expression in the problem statement, let us consider it's square and use multiple angle identities to simplify the LHS:

$$\begin{aligned}\left(\tan\left(\frac{3\pi}{11}\right)+4\sin\left(\frac{2\pi}{11}\right)\right)^2&=\left(\frac{\tan^3(\alpha)-3\tan(\alpha)+24\sin^2(\alpha)\tan(\alpha)-8\sin(\alpha)\cos(\alpha)}{3\tan^2(\alpha)-1}\right)^2\\&=-\frac{y(4y(8y-11)+11)^2}{(1-4y)^2(y-1)}\end{aligned}$$

Here we have inserted $y=\sin^2\alpha, 1-y=\cos^2\alpha$, and $\frac{y}{1-y}=\tan^2\alpha$ and done heavy simplification. To proceed we manipulate the polynomial equation for y until we have the numerator of the quantity desired:

$$\begin{aligned}y(4y(8y-11)+11)^2&=y(4y(8y-11)+11)^2+11-220y+1232y^2-2816y^3+2816y^4-1024y^5\\&=-11(1-4y)^2(-1+y)\end{aligned}$$

Thus, $\tan\left(\frac{3\pi}{11}\right)+4\sin\left(\frac{2\pi}{11}\right)=\sqrt{11}$

BP28 99. **Geo** A line is positioned in such a way that it intersects the side of a square at an angle of $\frac{\pi}{6}$ and bisects the square into two unequal pieces such that the area of one is twice that of the other. If the square has sidelength $3^{1/4}$, compute the perimeter of the smaller bisected piece.

Comment:

Answer: $(3+\sqrt{3})\sqrt{2}$

Solution: Because the angle with which the line intersects a square edge is less than $\pi/4$, the second square edge intersected by the line must be an adjacent one. This implies that the smaller of the two bisected areas is a right 30-60-90 triangle whose hypotenuse is the line. If the shorter edge of the triangle has length x then it has area $\frac{\sqrt{3}}{2}x^2$. Given that the square area is $\sqrt{3}$, the bisection must occur such that the triangle has area $\sqrt{3}/3$. Thus, $x=\sqrt{2}$ and the perimeter of the triangle is $\sqrt{2}+\sqrt{3}\sqrt{2}+2\sqrt{2}=(3+\sqrt{3})\sqrt{2}$.

BP29 100. **Calc** Compute

$$\int_0^{2\pi}\frac{dx}{\sqrt{5}+\sin(x)}.$$

Comment: Note that the simplicity of this answer is not a result of the integral being independant of constants. The $\sqrt{5}$ has been carefully chosen.

Answer: π

Solution: Though not obvious from the problem, this integral can be decomposed into an arctangent integral. To make things smoother later we shall recall that

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right).$$

Notice that the sine function in the denominator is hard because it is not squared. If it were then we could make a substitution to get a rational function. In an effort to remedy this,

$$\int_0^{2\pi} \frac{1}{\sqrt{5}+\sin(x)} \cdot \frac{\sqrt{5}-\sin x}{\sqrt{5}-\sin x}\, dx = \int_0^{2\pi} \frac{\sqrt{5}-\sin x}{5-\sin^2(x)}\, dx = \int_0^{2\pi} \frac{\sqrt{5}-\sin x}{4+\cos^2(x)}\, dx.$$

Next observe that the integrand can now be split into two terms one of which is odd and one that it even. Over the integration domain the term containing $\frac{\sin x}{4+\cos^2(x)}$ vanishes. Thus we are left with the integral

$$\int_0^{2\pi} \frac{\sqrt{5}}{4+\cos^2(x)}\, dx.$$

Another way to see this manipulation is to note that $\sin(x) = -\sin(x+\pi)$ so we can split the integral as follows,

$$\int_0^{2\pi} \frac{dx}{\sqrt{5}+\sin(x)} = \int_0^{\pi} \frac{dx}{\sqrt{5}+\sin(x)} + \int_{\pi}^{2\pi} \frac{dx}{\sqrt{5}+\sin(x)} = \int_0^{\pi} \left(\frac{1}{\sqrt{5}+\sin(x)} + \frac{1}{\sqrt{5}-\sin(x)}\right) dx.$$

This integral also reduces to the one obtained above after rewriting the integration domain to be $[0, 2\pi]$.

The integral $\int_0^{2\pi} \frac{\sqrt{5}}{4+\cos^2(x)}\, dx$ is very similar to the first but critically only contains a squared trigonometric function. Thus we make substitution,

$$x = \arctan\left(\frac{u}{2}\right) \implies dx = \frac{2\,du}{4+u^2} \text{ and } \cos^2(x) = \frac{4}{4+u^2}.$$

To use this substitution we need to split the domain of integration into the subintervals

$$x \in [0, \pi/2] \cup [\pi/2, 3\pi/2] \cup [3\pi/2, 2\pi]$$

and realize that this substitution covers the domain $u \in (-\infty, \infty)$ twice. That is,

$$\left(\int_0^{\infty} + \int_{-\infty}^{\infty} + \int_{-\infty}^{0}\right) \frac{\sqrt{5}}{2(u^2+5)}\, du = 2\int_{-\infty}^{\infty} \frac{\sqrt{5}}{2(u^2+5)}\, du = \pi.$$

CA01:geo 101. **Geo** Let $ABCD$ be a unit square and let there be two unit circles, centered at C and D. Let P be the point of intersection of the two circles inside the square. Compute the measure of angle APB in degrees.

Comment: JH: rewording original problem: We have a square $ABCD$ with side length s. We draw arc AC such that ADC forms a quarter circle with center D and radius s. Similarly, we draw arc BD such that BCD forms a quarter circle with center C and radius s. Label the point at which the arcs BD and AC intersect as P. Find angle APB. A (potentially) harder variation of this problem is finding the area in the intersection of the two quarter circles. Still requires realizing that PDC is an equilateral triangle.

Answer: 150 degrees

Solution: Note that triangle DPC is an equilateral triangle because $PD = PC = 1$ as they are radii of the circles. Since $\angle PDC = 60°$, $\angle ADP = 30°$. We also know that triangle ADP is isosceles and $\angle ADP = 30°$, so $\angle DAP = \angle DPA = 75°$. We can then see that angle $PAB = 15°$, and by symmetry, $PBA = 15°$. Thus, we have angle APB is $180 - 15 - 15 = \boxed{150°}$.

CD01 102. **Gen** Compute the only integer solution of $x^4 - 8x^2 + x + 14 = 0$.

Comment: It is believed that this problem could work if numbers were changed to make rational root infeasible but the clever factorization more obvious.

Answer: 2

Solution: Note that $x^4 - 8x^2 + x + 14 = (x^4 - 8x^2 + 16) + (x - 2) = (x^2 - 4)^2 + (x - 2) = (x - 2)^2(x + 2)^2 + (x - 2)$ which clearly has a solution at $x = \boxed{2}$.

CD02 103. **Gen** You have just finished painting a fence that is 7 yards long and 2 yards high. You decide to convert to feet since larger numbers make you feel more productive. Compute the number of ft^2 you have painted.

Answer: 126

Solution: There are 9 ft^2 per yd^2, so our answer is $7 \cdot 2 \cdot 9 = \boxed{126}$.

CD03 104. **Gen** The swimming pool in Nick's backyard is 20 feet long, 10 feet wide, and 8 feet deep. He wants to fill it to a depth of 4 feet. Compute the number of cubic feet of water needed to fill the pool to the desired depth.

Answer: 800

Solution: $20 \cdot 10 \cdot 4 = \boxed{800}$.

DM01 105. **Geo** Find the maximum area of a triangle inscribed inside a unit circle.

Comment: Too easy?

Answer: $\frac{3\sqrt{3}}{4}$

Solution: The triangle inscribed in a unit circle is an equilateral triangle. The centroid of an inscribed equilateral triangle divides the median in a ratio of 2 : 1; the larger part of the median corresponds to the radius of the circle, 1. We know that the height of the triangle is $\frac{3}{2}$. Because the median creates two 30-60-90 triangles, the side of the triangle is $\sqrt{3}$, and the area is then $\frac{3\sqrt{3}}{4}$ (as the area of an equilateral triangle with length s is $\frac{s^2\sqrt{3}}{4}$).

DM02 106. **Disc** There are 8 dogs with the same starting weight w_0, with 7 dogs following the same growth function, $w_\alpha(t) = g_1 t + w_0$ and 1 dog following the growth function $w_\beta(t) = g_2 t + w_0$, where $g_1, g_2 \in \mathbb{N}$ and $g_2 > g_1$. Assume that t represents a time unit and increases by 1 time unit. Given an balance that allows you to weigh two groups of dogs to see which group is heavier, and that you are starting at $t = 0$, when is the earliest time unit you can find the dog with the faster growth rate. Assume that measuring with the balance is instantaneous (e.g. when you place the group of dogs on the scale at $t = 0$, you get the results back at $t = 0$), and that growth functions only apply to whole unit increments of t.

Comment: Can be simplified to the balance puzzle. Wording for expected answer might be confusing.

Answer: 2

Solution: Because at $t = 0$, all dogs will weigh te same, we can start measuring earliest at $t = 1$ for the differeing dog to display a heavier weight. At $t = 1$, we split the dogs into groups of 3, 3, and 2. At $t = 1$, we place the two groups of 3 on the balance scale. If the two groups (of 3) have the same weight, at $t = 2$, we weight the remaining group of 2 to determine the heaviest dog. If one of the two groups is heavier, we take the heavier group and choose two random dogs from that group and weigh them at $t = 2$: if the dogs on the balance have the same weight, the remaining dog of the chosen group is the heaviest.

DM03 107. **Disc** Eddy owns 5 different cats, and has 9 fish to distribute among the cats. Each cat gets at least 1 fish and at most 3 fish. Given that the fish are indistinguishable, how many ways can Eddy distribute the 9 fish among the 5 cats?

Comment: Comments?

Answer: 45

Solution 1: Since each cat must get at least 1 fish, this problem reduces to distributing 4 fish among 5 cats, with each cat getting at most 2 more fish. We proceed using casework:

(a) 2 cats get 2 more fish. There are $\binom{5}{2} = 10$ ways of choosing the 2 lucky cats in this case.

(b) 1 cat gets 2 more fish, and 1 cat gets 1 more fish. There are 5 ways of choosing the first cat and $\binom{4}{2} = 6$ ways of choosing the second cat, for a total of $5 * 6 = 30$ combinations.

(c) 4 cats get 1 more fish each. There are 5 ways of choosing the only cat which does not get more fish.

Hence, there are a total of $10 + 30 + 5 = \boxed{45}$ ways to distribute the 9 fish among the 5 cats.

Solution 2: Let c_i represent the number of fish that cat i has. The problem then reduces to finding the number of combinations of c_i, with $1 \leq c_i \leq 3$, such that $\sum_{i=1}^{5} c_i = 9$. The generating function for the problem is $(x + x^2 + x^3)^5 = x^5(1 + x + x^2)^5 = x^5 \left(\frac{1-x^3}{1-x}\right)^5$. Let $[x^r]$ represent the coefficient fo x^r. Our solution is thus given by

$$[x^r]\left(\frac{1-x^3}{1-x}\right) = \sum_{j=0}^{1}(-1)^j\binom{5}{j}\binom{4-3j+4}{4} = 45$$

DM04 108. **Calc** Given a right cone, and a cross section represented by $y = \sqrt{|x|}$ (the bounding function) and $y = x$, what is the volume of liquid the cone can hold to the brim. Assume that surface tension does not apply.

Comment: Problem wording might need to be more detailed?

Answer: $\boldsymbol{\frac{\pi}{5}}$

Solution: The intersection of $y = \sqrt{|x|}$ and $y = x$ are $(0, 0)$ and $(1, 1)$. We can determine the volume of the hollow space of the cone by rotating $y = \sqrt{x}$. The volume is $\pi \int_0^1 (y^2)^2 dy = \frac{\pi}{5}$. Or, $\frac{\pi}{3} - \pi \int_0^1 (y^2 - y^4) dy$ gives us the same result of $\frac{\pi}{5}$, where $\frac{\pi}{3}$ is the total volume of the cone with a height of 1.

DM05 109. **Geo** Given a hyperbola defined by the equation $y = \frac{1}{x}$, find the maximum area of the triangle given two points, $(1, 1)$ and $(4, \frac{1}{4})$, where the third point P of the triangle also lies on the hyperbola. Assume that P is in the first quadrant.

Comment: Comments?

Answer: $\boldsymbol{\frac{3}{8}}$

Solution: Let $(p, \frac{1}{p})$ be the third point of the triangle. To maximize the distance between the line AB determined by $(1, 1)$ and $(4, \frac{1}{4})$, the line tangent to $(p, \frac{1}{p})$ must be parallel to

AB. Because the slope of AB is $\frac{-1}{4}$ and the tangent lines to $y = \frac{1}{x}$ are $y = \frac{-1}{x^2}$, $x = \sqrt{ab}$ (as p lies in the first quadrant): $p = 2$.

$$Area of triangle = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 2 & \frac{1}{2} & 1 \\ 2 & \frac{1}{2} & 1 \end{vmatrix} = \frac{3}{8} \tag{1}$$

DM06:geo 110. **Geo** Given a pentagon $ABCDE$, where $AB = 12$, $BC = 20$, $CD = 7$, $DE = 24$, $EA = 9$, and $\angle EAB = \angle CDE = 90°$, compute the area of the pentagon.

Comment: Comments?

Answer: 288

Solution: The easiest solution is to see that the pentagon is formed of three Pythoagorean triple triangles, $\triangle EAB$, $\triangle BEC$, and $\triangle CDE$ are respectively, $9-12-15$, $15-20-25$, and $7-24-25$. So, the area of the pentagon is the sum of the area of the three triangles, $54 + 150 + 84 = 288$.

DM07 111. **Disc** Piglet and Hamlet plan to meet. Piglet is at coordinate $(0, 0)$ and Hamlet is at coordinate $(6, 5)$. However, most paths have little waves of water and will splash Piglet. The only paths that are free of little waves are paths through coordinates $(2, 4)$ and $(3, 3)$. Piglet will take each path with equal probability. What is the probability that Piglet will meet Hamlet without getting splashed?

Comment: Too easy?

Answer: $\mathbf{\frac{275}{462}}$

Solution: There are $\binom{6}{3}\binom{5}{2}$ paths through coordinate $(3, 3)$ and $\binom{6}{4}\binom{5}{1}$ paths through coordinate $(2, 4)$–275 paths. There are $\binom{11}{6} = 462$ paths in total. So, the probability of Piglet taking a path through coordinates $(2, 4)$ and $(3, 3)$ are $\frac{275}{462}$.

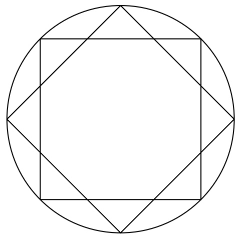
DM08 112. **Geo** You have a cup, a right conical frustrum, with milk tea. The cup's top radius is $\frac{1}{2}$ and the bottom radius is $\frac{1}{24}$. The slant height of the cup is $\frac{11}{12}$, and the tea fills the cup $\frac{5}{11}$ of the way up the cup's side. You tip the cup just to the point of spilling. What is the cosine of the angle formed by the tilted tea surface and the top cup circle?

Comment: TODO: Slightly concerned about wording (is it clear enough). If it's too easy, can make it all in terms of fractions to make the question more annoying. Solution isn't very clear–will include an image .

Answer: $\mathbf{\frac{5\sqrt{13}}{26}}$

Solution: Extend the conical frustrum to form a cone (and consequently, fill the new cone with more tea). Call the added side length x. Given the the angle formed by the slant height and bottom radius is $\frac{2\pi}{3}$, the cross section of the cone is an equilateral triangle; so, $x = \frac{1}{12}$. Let $q' = q + x$ be the length of the cup side where the tea is not filled to the cup top, where q is the slant height the tea reaches (after tilted). Let $p' = p + x$, where $p = \frac{5}{12}$. Let θ be the angle formed by the height of the cone and the slant height. Let $r = p' \sin\theta$ and $h = p' \cos\theta$. Because the volume is consistent, $V = \frac{\pi}{3}(p' \sin\theta)^2 p' \cos\theta = \frac{\pi}{3}p'(\sin\theta)(\frac{1}{2}q' \sin(2\theta))$; $q' = (p')^2$. Apply the law of cosines to the triangle formed the major axis (of the titled tea surface), a, diameter formed when the slant height is $\frac{1}{12}$, b, and the slant height from where the tea stopped on the other side to the brim of the cup, c. The angle formed by a and b and the angle formed by the titlted surface and the top cup circle (call this angle α. So, $\cos\alpha = \frac{c^2 - a^2 - b^2}{-2ab}$. $a^2 = 1 + (q')^2 - 2(q') \cos\frac{\pi}{3}$, where $q' = \frac{1}{4}$. So, $a =$, $b = \frac{1}{4}$, and $c = \frac{3}{4}$.

ED01 113. **Geo** Consider a circle with radius 1 and two squares, each inscribed within the circle such that one square is rotated 45 degrees from the other. Compute the area that is within the circle, but not within either of the squares.



Answer: $\boldsymbol{\pi - 8\frac{\sin(\pi/8)\sin(\pi/4)}{\sin(5\pi/8)}}$ **or** $\boldsymbol{\pi - 16\sin^2(\pi/8)}$

Solution: To do this problem we want to know the area of the triangle formed by the origin, a square vertex, and the point of intersection of two square edges (one of which emenates from the vertex). Let us label these points A, B, and C respectively. It is easy to see that $\angle CAB = \pi/8$ and that $\angle ABC = \pi/4$. It follows that $\angle ACB = 5\pi/8$. Because of the inscription in the circle, $|AB| = 1$ therefore by the law of sines $AC = \frac{\sin(\pi/4)}{\sin(5\pi/8)}$. Then the height of the triangle, measured along the perpendicular to AB that intersects C is $AC\sin(\pi/8)$. It follows that the area is: $\frac{\sin(\pi/8)\sin(\pi/4)}{2\sin(5\pi/8)}$. Now, there are 16 of these triangles making up the area of the union of the squares, and the circle has area π, so the result follows.

There is another way to write this trig ratio derived in this answer. It can be computed by writing the arguments of the sines in the form $n\frac{\pi}{8}$ and using trig identities.

ED02:gut 114. **Disc** Suppose we have 3 baskets and 4 indistinguishable balls. Each ball is placed into a randomly selected basket. Compute the probability that the basket with the most balls has at least 3 balls.

Comment:

Answer: $\boldsymbol{\frac{1}{3}}$

Solution: Suppose the baskets and balls are distinct (it does not matter as long as we are consistent within the problem). Let (a, b, c) represent the number of balls in each ith bucket for the ith coordinate. If a bucket has 4 balls, this implies the distribution is $(4, 0, 0)$ or any permutation for 3 cases.

Else if a bucket with the most balls has 3 balls, the cases look like a permutation of $(3, 1, 0)$. There are 6 ways to decide which bucket will have 3 balls, which has 1 ball, and which has 0. Then there is an additional 4 ways to choose which ball is alone in a basket. This gives 24 such cases.

Therefore there are 27 total cases with a basket having at least 3 balls and the total number of ways to distribute the balls is 3^4. Therefore the probability is $\frac{27}{3^4} = \frac{1}{3}$.

exampl 115. **Team** Compute $1 + 1$.

Comment: This is an optional comment.

Answer: 2

Solution: $1 + 1 = \boxed{2}$.

GO01 116. **Disc** Compute the number of ways one can cut a string of length n such that each piece has integral length.

Answer: $\boldsymbol{2^{n-1}}$

Solution: Notice that the way the string is cut can be represented by a string of length n made of 0's and 1's that starts with a 0. The m^{th} digit represents whether a cut is made at the m^{th} position (the first position being the beginning of the string which is why we always begin with a 0). It is now clear that a sequence of 0's and 1's that begins with 0 and has n digits can be written down in $\boxed{2^{n-1}}$ ways.

G002 117. **Geo** Suppose we having a shipping box that has dimensions $6 \times 10 \times 2$. What is the longest completely straight tube that will fit inside this box? (Note that given a rectangular box with dimensions $a \times b \times c$, the long diagonal has length $\sqrt{a^2+b^2+c^2}$ by the extension of the Pythagorean Theorem.)

Comment:

Answer: $\sqrt{\mathbf{140}}$**.**

Solution: The question is equivalent to asking what is the longest distance between any two points inside this box. This is clearly maximized by considering the distance between two opposite corners. To prove this, let the vertices of the box be $(0,0,0), (0,0,2), (0,10,0), (0,10,2), (6,0,0), (6,0,2$ Note that the distance between two points Then if the distance between (a,b,c) and (x,y,z) is the maximum, at least one the equations $a \neq x$, $b \neq y$, $c \neq z$ is true, since otherwise they are the same point. WLOG if $a < x$, if $a \neq 0$ or $x \neq 6$, we can set $a = 0$ and $x = 6$ and get a longer distance. Similarly we max out opposite coordinates and find that the maximum distance must be between opposite corners. This is length $\sqrt{6^2+10^2+2^2} = \sqrt{140}$.

HC01 118. **Calc** We have a triangle ABC with $AB = 2$, $BC = 3$, and $AC = 4$. Consider all lines XY such that X lies on AC, Y lies on BC, and triangle XYC has area half of that of ABC. What is the minimum possible length of XY?

Answer: $\frac{\sqrt{6}}{2}$

Solution: Let $x = CX$ and $y = CY$. Then since the area of XYC is half of that of ABC, we must have $xy = \frac{1}{2}AC \cdot BC = 6$. Now, let $\alpha = \angle ACB$. Then by the law of cosines, $XY = \sqrt{x^2+y^2-2xy\cos(\alpha)} = \sqrt{x^2+y^2-12\cos(\alpha)}$. Since $\cos(\alpha)$ is a constant, minimizing XY is equivalent to minimizing x^2+y^2.

Now, since $xy = 6$, we have $y = \frac{6}{x}$ and hence $x^2+y^2 = x^2+\frac{36}{x^2} = \frac{x^4+36}{x^2}$. The minimum is achieved when its derivative is zero. The numerator of the derivative is $4x^2-2(x^4+36)$. Setting the derivative to equal 0, we see that $x^4 = 36$ and hence $x = 36^{1/4} = \sqrt{6}$ which lies in the range $[2,4]$ and hence is a valid value of x. Therefore, $x = \sqrt{6}$ and $y = \frac{6}{x} = \sqrt{6}$.

We now proceed to compute $\cos(\alpha)$ so we may calculate the length of XY when $x = y = \sqrt{6}$. By the law of cosines an triangle ABC, $\cos(\alpha) = \frac{AC^2+BC^2-AB^2}{2AC\cdot BC} = \frac{9+16-4}{24} = \frac{7}{8}$. Therefore, the minimum value of XY is

$$XY = \sqrt{x^2+y^2-12\cos(\alpha)} = \sqrt{6+6-12\cdot\frac{7}{8}} = \boxed{\frac{\sqrt{6}}{2}}$$

HC02 119. **Calc** What is the least upper bound of the ratio of volume to surface area among all right cones with base radius 1?

Answer: $\frac{1}{3}$

Solution: Let the height of the cone be h. Then the volume is $V = \frac{1}{3}\pi h$. The surface area is $A = \pi + \pi s^2 \cdot \frac{2\pi}{2\pi s}$ where s is the slant length, so $s = \sqrt{1+h^2}$. Therefore $A = \pi(1+\sqrt{1+h^2})$ and hence $V/A = \frac{1}{3}\cdot\frac{h}{1+\sqrt{1+h^2}}$. This ratio is strictly increasing in h so sending $h \to \infty$, we see that the least upper bound is $\boxed{\frac{1}{3}}$.

HC03 120. **Disc** How many tuples $(a_1, a_2, \ldots, a_{2015})$ are there such that $0 \le a_i \le 2$ for each i and $a_1^2 + a_2^2 + \cdots + a_{2015}^2$ is divisible by 3?

Comment: In general, the number of solutions with n terms is the nth term in the sequence A101990 on OEIS.

Answer: $\mathbf{3^{2014}}$

Solution: Let b_n denote the number of tuples $(a_1, a_2, \ldots, a_n)$ such that $0 \le a_i \le 2$ for each i and $\sum_{i=1}^n a_i^2 \equiv 0 \pmod 3$. First, notice that $a_i^2 \equiv 0 \pmod 3$ if $a_i \equiv 0 \pmod 3$ and $a_i \equiv 1 \pmod 3$ if $a_i \equiv \pm 1 \pmod 3$. Since a_i^2 can only be equivalent to 0 or 1 modulo 3, the sum $\sum_{i=1}^n a_i^2$ is divisible by 3 only when the number of nonzero elements is a multiple of 3.

With this insight, we are then able to write down the formula $b_n = \sum_{i=0}^{\lfloor n/3 \rfloor} \binom{n}{3i} 2^i$. This does not give a nice closed form for b_n, but if we compute the first few values of b_n, we obtain $1, 1, 9, 33, 81, 225, 729$. From this, we may guess that alternating terms are powers of 9, or more precisely, $b_{2k+1} = 9^k = 3^{2k}$. If this conjecture holds true, then the answer is $b_{2015} = 3^{2014}$.

I don't know how to prove this conjecture is true. Someone needs to fill in a proof.

HC04 121. **Geo** According to the Constitution of the Kingdom of Nepal, the shape of the flag is constructed as follows:

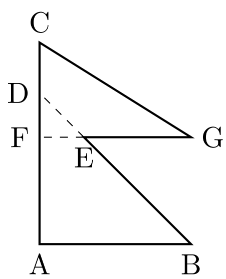
Draw a line AB of the required length from left to right. From A draw a line AC perpendicular to AB making AC equal to AB plus one third AB. From AC mark off D making line AD equal to line AB. Join BD. From BD mark off E making BE equal to AB. Touching E draw a line FG, starting from the point F on line AC, parallel to AB to the right hand-side. Mark off FG equal to AB. Join CG.

What is the ratio of the area of the flag to the length of AB?

Comment: I feel like I've seen this problem somewhere before.

Answer: $\mathbf{\frac{5+3\sqrt{2}}{12}}$

Solution: Here is a diagram of the construction:



Let $AB = 1$ so that the ratio is precisely the area of the flag. Then $AC = \frac{4}{3}$, $AD = 1$, $BE = 1$, and $FG = 1$. By Pythagoras, $BD = \sqrt{AB^2 + AD^2} = \sqrt{2}$. By similar triangles on DEF and ABD, we have that $\frac{AF}{AD} = \frac{BE}{BD}$ so $AF = \frac{AD \cdot BE}{BD} = \frac{1}{\sqrt{2}}$. Therefore $CF = \frac{4}{3} - \frac{1}{\sqrt{2}}$ and $DF = 1 - \frac{1}{\sqrt{2}}$. Again by similar triangles, $\frac{EF}{AB} = \frac{DF}{AD}$ so $EF = \frac{AB \cdot DF}{AD} = 1 - \frac{1}{\sqrt{2}}$.

Now, the area of the flag is the sum of the areas of triangle CFG and trapezoid $ABEF$. The area of CFG is $\frac{1}{2} \cdot CF \cdot FG = \frac{2}{3} - \frac{1}{2\sqrt{2}}$. The area of $ABEF$ is $\frac{1}{2} \cdot AF \cdot (AB + EF) = \frac{1}{\sqrt{2}} - \frac{1}{4}$.

Thus, the area of the flag is $\frac{2}{3} - \frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{4} = \boxed{\dfrac{5 + 3\sqrt{2}}{12}}$.

HC05:geo 122. **Geo** Let Γ_1 be a circle of radius 6 and let Γ_2 be a circle of radius 1. Let the circles be internally tangent at point P and let AP be a diameter of circle Γ_1. Let Y be the tangent point where line AY is tangent to Γ_2. Compute the length of PY.

Comment: HC: This was a problem from a set of 10 problems I wrote for my math club at high school. I'm not entirely sure that this is original and it looks like most of my problems were inspired by other problems I had seen. I suspect that I did not just rip this off another contest though.

Julia: Will modify to find length of PY (uses some law of cosines whee) (wording is a bit awk rn)

Original problem: Let Γ_1 be a circle of radius 6 and let Γ_2 be a circle of radius 1 that is internally tangent to circle Γ_1 at a point P. Let AP be a diameter of circle Γ_1. Let X be a point on the tangent line to Γ_1 going through P such that the line AX is tangent to Γ_2. Compute the length of PX. Original answer: $\frac{\sqrt{30}}{5}$ Original solution: We will use O_1 and O_2 to denote the respective centers of the circles.

First, note that since circle Γ_1 has radius 6 and AP is a diameter of Γ_1, we have $AP = 12$. Let AX be tangent to circle Γ_2 at Y. Now, since O_2P and O_2Y are both radii of circle Γ_2, we have $O_2P = O_2Y = 1$ and hence $AO_2 = AP - O_2P = 11$. Now, by Pythagoras, $AY = \sqrt{AO_2^2 - O_2P^2} = \sqrt{11^2 - 1^2} = \sqrt{120} = 2\sqrt{30}$. Next, notice that $\Delta APX \sim \Delta AO_2Y$ since $\angle APX = \angle AYO_2 = 90°$ and they both share angle $\angle PAX$. Therefore, we have that

$$\frac{PX}{O_2Y} = \frac{AP}{AY}$$
$$PX = \frac{AP \cdot O_2Y}{AY}$$
$$PX = \frac{12 \cdot 1}{2\sqrt{30}}$$
$$PX = \boxed{\frac{\sqrt{30}}{5}}$$

Answer: $\mathbf{\frac{2\sqrt{66}}{11}}$

Solution: Let O be the center of the smaller circle. Then, drawing $OY = 1$, we see that AOY is a right triangle, and thus $AY = \sqrt{120}$. Now, using law of cosines on triangle APY, we see that $YP^2{=}AY^2 + AP^2 - 2 \cdot AY \cdot AP \cdot \cos\angle YAP = 12^2 + 120 - 2 \cdot 12 \cdot \sqrt{120} \cdot \frac{AY}{AO} = 144 + 120 - 2 \cdot 12 \cdot \frac{120}{11} = \frac{24}{11}$. Then $PY = \frac{2\sqrt{66}}{11}$.

HC06 123. **Team** Consider a square with points $A = (1, 1)$, $B = (1, -1)$, $C = (-1, -1)$, and $D = (-1, 1)$. Find the minimum possible value of $f(X) = 2AX + BX + CX + DX$ where X is any point in the plane.

Comment: Heesu: err I think the result that X lies on $y = x$ could be done a lot faster? Unfortunately the calc isn't usable for guts, but seems good for Team.

Answer: $\mathbf{3\sqrt{2} + \sqrt{6}}$

Solution: First, notice that $X = (a, b)$ must lie on the line $y = x$. To see this, consider a change of coordinates such that $B = (1, 0)$ and $D = (-1, 0)$. Then we are asserting that X must lie on the y-axis in this coordinate system. For a fixed b, clearly, the distances to A and C in this coordinate system is minimized when $a = 0$. Now, the sum of the distances to B and D is given by $\sqrt{(a-1)^2 + b^2} + \sqrt{(a+1)^2 + b^2}$. Differentiating with respect to a and setting the derivative equal to 0, we get the equation

$$\begin{aligned}
\frac{2(a-1)}{\sqrt{(a-1)^2+b^2}} + \frac{2(a+1)}{\sqrt{(a+1)^2+b^2}} &= 0 \\
\frac{(a-1)}{\sqrt{(a-1)^2+b^2}} &= -\frac{(a+1)}{\sqrt{(a+1)^2+b^2}} \\
(a-1)\sqrt{(a+1)^2+b^2} &= -(a+1)\sqrt{(a-1)^2+b^2} \\
(a-1)^2(a+1)^2 + (a-1)^2b^2 &= (a+1)^2(a-1)^2 + (a+1)^2b^2 \\
(a-1)^2b^2 &= (a+1)^2b^2 \\
a^2-2a+1 &= a^2+2a+1 \\
a &= 0
\end{aligned}$$

Thus, the sum of the distances to B and D is minimized when $a = 0$, or when X lies on the y-axis in this coordinate system. Thus, in the original coordinate system, X must lie on the line $y = x$.

Now, if we denote $X = (a, a)$, then the value we are trying to minimize is

$$\begin{aligned}
&2\sqrt{(a-1)^2+(a-1)^2} + \sqrt{(a+1)^2+(a+1)^2} + \sqrt{(a-1)^2+(a+1)^2} + \sqrt{(a+1)^2+(a-1)^2} \\
=&2\sqrt{2}(1-a) + \sqrt{2}(1+a) + 2\sqrt{2}\sqrt{a^2+1}
\end{aligned}$$

We then solve for when the derivative is 0:

$$\begin{aligned}
-2\sqrt{2} + \sqrt{2} + 2\sqrt{2}\frac{a}{\sqrt{a^2+1}} &= 0 \\
2a &= \sqrt{a^2+1} \\
4a^2 &= a^2+1 \\
a &= \pm\frac{1}{\sqrt{3}}
\end{aligned}$$

Plugging both values of a in, we see that the minimum possible value is when $X = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and it is equal to $\boxed{3\sqrt{2}+\sqrt{6}}$.

HC07 124. **Geo** Consider a convex quadrilateral $ABCD$ with perimeter p and diagonal $AC = q$. Let E be the intersection of AC and BD. The quadrilateral formed by the incenters of triangles ABE, BCE, CDE, and ADE has perimeter r. Given that the incircle of triangle ABE is tangent to that of BCE and ADE, and that the incircle of triangle CDE is tangent to that of BCE and ADE, compute the area of $ABCD$ in terms of p, q, r.

Comment: Concrete numbers for p, q, r need to be chosen.

Aaron Lin: Final answer might be easy to guess given that someone can just assume it's a rectangle and make the numbers work out.

Answer: $\boldsymbol{\frac{r(p+4q)}{16}}$

Solution: It turns out that all the incircles have equal radius and the two diagonals of $ABCD$ are equal, and $ABCD$ must in fact be a rectangle.

HC08 125. **Team** Consider a rectangle $ABCD$ with side lengths $AB = 2$ and $BC = 1$. Let E be the midpoint of AB and let F be any point on CD. Let G be the intersection of EF and BD. What is the minimum possible sum of the areas of triangles BEG and DFG?

Comment: Heesu: consider for team or calc.

Answer: $\sqrt{2}-1$

Solution: Let x denote the length DF and let h denote the height of BEG. Then the height of DFG is $1-h$. Furthermore, note that triangles BEG and DFG are similar so

$$\begin{aligned}\frac{BE}{DF} &= \frac{h}{1-h}\\ \frac{1}{x} &= \frac{h}{1-h}\\ 1-h &= xh\\ h &= \frac{1}{1+x}\end{aligned}$$

Thus, the sum of the areas of triangles BEG and DFG is

$$\begin{aligned}\frac{1}{2}\cdot BE\cdot h+\frac{1}{2}\cdot DF\cdot(1-h) &= \frac{1}{2}\left(1\cdot\frac{1}{1+x}+x\cdot\frac{x}{1+x}\right)\\ &= \frac{1}{2}\cdot\frac{1+x^2}{1+x}\end{aligned}$$

To maximize the area, we take the derivative and set it to zero

$$\begin{aligned}\frac{1}{2}\cdot\frac{x^2+2x-1}{(1+x)^2} &= 0\\ x^2+2x-1 &= 0\\ x &= -1\pm\sqrt{2}\end{aligned}$$

Since $-1-\sqrt{2}<0$, it follows that the minimum is achieved when $x=\sqrt{2}-1$. In that case, the sum of the areas is equal to

$$\begin{aligned}\frac{1}{2}\cdot\frac{1+(\sqrt{2}-1)^2}{1+(\sqrt{2}-1)} &= \frac{1}{2}\cdot\frac{4-2\sqrt{2}}{\sqrt{2}}\\ &= \sqrt{2}-1\end{aligned}$$

HC09 126. **Team** Let $p(x)=ax^2+bx+c$ where the coefficients a,b,c are chosen uniformly and independently from the interval $[-1,1]$. What is the probability that $p(x)$ contains a positive root?

Answer: TODO: Find the answer.

Solution: First, the quadratic formula tells us that the roots of $p(x)$ are $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. First consider the case where $ac<0$, then $\sqrt{b^2-4ac}>|b|$ and hence there is necessarily a positive root. Now, consider the case where $ac>0$. Then $\sqrt{b^2-4ac}<|b|$ and the sign of the roots is equal to the sign of $-\frac{b}{2a}$, assuming the roots exist.

Thus, the probability that there exists a positive root is equal to the probability that $ac<0$ plus the probability that $ac>0$ and $-\frac{b}{2a}>0$ and $b^2-4ac>0$. The probability that $ac<0$ is $\frac{1}{2}$. To compute the probability that $ac>0$, $-\frac{b}{2a}>0$ and $b^2-4ac>0$, we consider 2 cases. Either $a>0,b<0,c>0$ or $a<0,b>0,c<0$. Either case happens with probability $\frac{1}{8}$ and the probability that $b^2-4ac>0$ in those cases is **I NEED TO CALCULATE THIS.**

TODO: Finish the solution.

HC10:tea 127. **Gen** A wizard is visiting numbers on the number line. He begins on 1 and he can either teleport 3 numbers to the right (e.g. $1 \to 4$) or he can teleport to the smallest prime number he has not visited yet. How many ways are there for him to visit all the integers from 1 to 31 inclusive?

Comment: Eric: Need to fill in a solution or change the problem so that it's more easily solved.

Answer: 256 I think

Solution: TODO: Write a solution.

HH01 128. **Geo** Let ABC be a triangle with $AB = 5$, $BC = 4$, and $AC = 7$. Let C_1 and C_2 be the two possible circles that are perpendicular to AB, AC, and BC when AC and BC are extended. Let C_1 have radius r_1 and C_2 have radius r_2, and suppose that $r_1 < r_2$. What is $\frac{r_1}{r_2}$?

Comment: Eric: The solution needs to be elaborated on.

Answer: $\boldsymbol{\frac{8}{3}}$

Solution: Use tangent lines.

TODO: Finish the solution.

HH02 129. **Disc** Imagine a 3-D representation of Pascal's triangle, where the nth level consists of an n square of numbers. The first level consists of a single 1. The second level consists of four 1's arranged in a square. In general, in any given level a number in a cell will be the sum of the at most four numbers in the previous level. In the 2015th level, what is the value of the element in the nth row and mth column?

Comment: Harrison: Is it important to include a diagram? This problem has the potential to ask for more interesting things.
Eric: The solution needs to be elaborated on.
Harrison: We need to find something to ask about this problem. This has the potential to be a relatively difficult AT problem.

Answer: $\boldsymbol{\binom{2014}{n-1}\binom{2014}{m-1}}$

Solution: Patterns.

TODO(HH): Finish the solution.

HH03 130. **Geo** Triangle ABC has side lengths $AB = 13$, $BC = 21$, and $AC = 20$. Let D be the orthocenter of ABC, the intersection of the three altitudes of ABC. Define the circumcenter of a triangle to be the center of the circle that is circumscribed about the triangle, and let E, F, and G be the circumcenters of BCD, ACD, and ABD, respectively. Compute the area of $\triangle EFG$.

Comment: I'm a little bit concerned that this problem is guessable, if students are able to draw an accurate diagram.

Alternate version of this problem that is less guessable, but slightly easier: given the same parameters, compute the distance between the circumcenter of ABC and the circumcenter of BCD.

Herman: This theorem may be too well known to use as a problem.

Answer: 126

Solution: Reflect D across BC to a point D'. By computing the area of $ABD'C$ in multiple ways, one can show that D' lies on the circumcircle of ABC. Hence, the circumcircle of ABC is the circumcircle of BCD', and the circumcircles of ABC and BCD have the same radius. Moreover, since BCD and BCD' are reflections across BC, the line connecting the circumcenters is perpendicular to BC. Similar results can be achieved with ACD and ABD.

Next, let H be the circumcenter of ABC. From the above results, $HE = AD$, as $HE = AD$ is the distance needed to translate the circumcircle of ABC to the circumcircle of BCD'. Similarly, we have that $HF = BD$ and $HG = CD$. Extending HE, HF, and HG reveal that H is actually the orthocenter of EFG, and that EFG is congruent to ABC. Hence, the area of EFG is equivalent to the area of ABC, which by using Heron's formula or the Pythagorean Theorem can be computed to be $\boxed{126}$.

HH04 131. **Team** A trapezoidal cone is formed by cutting off the tip of a cone with a plane horizontal to the cone's base, leaving a circle at the top of the trapezoidal cone. Suppose that a cylinder with radius 7 and height 10 has the same volume as a trapezoidal cone with the same height. Let x be the radius of the lower circle of the trapezoidal cone, and let y be the radius of the upper circle. If x and y are both integers, compute xy.

Comment: The selection of height doesn't matter. While based on geometry, it is more of an algebra problem.

TODO(HS): maybe. want to make the solution a little more detailed?

Answer: 22

Solution: One may derive the area of the trapezoidal cone to be $V = \frac{1}{3}\pi(x^2 + xy + y^2)h$, where h is the height. In addition, the volume of a cylinder is simply $V = \pi r^2 h$. It suffices to find the solutions to the equation

$$x^2 + xy + y^2 - 3r^2 = x^2 + xy + y^2 - 3 * 49 = 0$$

The determinant of this equation with respect to x is $y^2 - 4y^2 + 3 * 4 * 49 = 3(4 * 49 - y^2) = 3(14 - y)(14 + y)$. In order for x to take on an integral value, y must be chosen such that the determinant is a perfect square. The only choice of y that satisfies the above condition where $x > y$ is $y = 2$, resulting in $x = 11$. Hence, $xy = \boxed{22}$.

HH05:gut 132. **Team** Let ABC be a triangle such that $AB = 9$, $BC = 6$, and $AC = 10$. 2 points D_1, D_2 are labeled on BC such that BC is subdivided into 3 equal segments; 4 points $E_1, E_2, ..., E_4$ are labeled on AC such that AC is subdivided into 5 equal segments; and 8 points $F_1, F_2, ..., F_8$ are labeled on AB such that AB is subdivided into 9 equal segments. All possible cevians are drawn from A to each D_i; from B to each E_j; and from C to each F_k. At how many points in the interior of $\triangle ABC$ do at least two cevians intersect?

Comment: Side length does not matter.

Heesu: It's a really awesome problem, but it's entirely contigent on knowing Ceva's. Is that more allowable on team?

Answer: 48

Solution: There are $2 \cdot 4$ intersections involving some AD_i and BE_j; $4 \cdot 8$ intersections involving some BE_j and CF_k; and $8 \cdot 2$ intersections involving some CF_k and AD_i.

However, there are some points in the triangle where three cevians intersect, and they occur when $xyz = (3-x)(5-y)(9-z)$ by Ceva's Theorem, where $x = AF_i$, $y = BD_j$, and $z = CE_k$ for some i, j, k. Enumerating points reveals that there are four solutions to the equation: $(1, 1, 8)$, $(1, 4, 3)$, $(2, 1, 6)$, and $(2, 4, 1)$ (tricks such as noting that AF_i cannot be 5 since there could not be a 5 on the RHS help). Note that we counted each of these three intersections 3 times, for each pair of different types of cevians. Therefore we must subtract these twice.

Hence there are a total of $2 \cdot 4 + 4 \cdot 8 + 8 \cdot 2 - 4 \cdot 2 = 48$ intersections.

HH06:geo 133. **Geo** A four-pointed star is formed by placing four equilateral triangles of side length 4 in a coordinate grid. The triangles are placed such that their bases lie along one of the coordinate axes, with the midpoint of the bases lying at the origin, and such that the vertices opposite the bases lie at four distinct points. Compute the area contained within the star.

Comment: Needs a diagram?

Answer: $24\sqrt{3} - 24$

Solution: Through angle chasing, we can determine that the area contained within the overlapping of any two triangles is formed by eight 45-60-75 triangles. Since the equilateral triangles have side length 4, the side opposite the 75° angle has length 2. Using the Pythagorean Theorem shows that the height of the triangle, given that the base is length 2, is $3 - \sqrt{3}$, hence the total area of the overlap is $24 - 8\sqrt{3}$. Next, the area of an equilateral triangle with side length 4 is $\frac{4^2}{4}\sqrt{3} = 4\sqrt{3}$, hence the total area contained within the star is $4 \cdot 4\sqrt{3} - (24 - 8\sqrt{3}) = \boxed{24\sqrt{3} - 24}$.

HH07 134. **Disc** You throw three six-sided die. Given that the three die are indistinguishable, how many combinations of numbers can you receive?

Answer: 56

Solution: There are three possible cases. If the three die have three separate values, then there are $\binom{6}{3} = 20$ ways to select the combinations. If two of the three die have the same value, there are 6 ways to choose the first value, and 5 ways to choose the value of the last die, for a total of $6 \cdot 5 = 30$ ways. Finally, if all three die have the same value, then there are simply 6 ways to select the value. Hence, there are a total of $20 + 30 + 6 = \boxed{56}$ possible combinations.

HH08 135. **Alg** Eric takes a random four digit number and forms another four digit number by reversing the digits. He finds that the resulting sum of the two numbers is a palindrome (i.e., reading the digits forward is equivalent to reading the digits backward). How many such palindromes can Eric form in this way?

Answer: 82

Solution: Let $N = WXYZ$ be a four-digit number that Eric can choose. First, consider the four-digit numbers $ABBA$ that Eric can form. Since reversing N must also produce a four-digit number, the digit in the thousands place and ones place must be at least 1. In addition, the digit in the tens and hundreds place can be any digit. Hence, $2 \leq A \leq 9$, and $0 \leq B \leq 9$, so there are $8 \cdot 10 = 80$ possible four-digit numbers. Next, consider the five-digit numbers $ABCBA$ that Eric can form. The palindrome can only have five-digits if $W + Z = 11$. With this constraint, there are only two possible ways to choose X and Y: either $X + Y = 0$, or $X + Y = 11$. Hence, there are 2 possible five-digit numbers that Eric can form. In total, there are $\boxed{82}$ possible palindromes.

HH09 136. **Geo** Three unit circles are inscribed inside an equilateral triangle such that each circle is tangent to the other two circles and two sides of the triangle. Compute the area contained within the triangle and the unit circles.

Answer: $6 + 4\sqrt{3}$

Solution: Consider the triangle formed by the line segment connecting the center of one of the unit circles and the nearest vertex of the triangle and the line segment connecting the center of the unit circle and a nearest edge of the triangle. This forms a 30-60-90 triangle. Hence, the side length of a triangle is

$$\sqrt{3} + 1 + 1 + \sqrt{3} = 2 + 2\sqrt{3}$$

The area of the triangle is hence

$$\begin{aligned}\frac{(2+2\sqrt{3})^2}{4}\sqrt{3} &= \frac{4+8\sqrt{3}+12}{4}\sqrt{3}\\ &= (4+2\sqrt{3})\sqrt{3}\\ &= \boxed{6+4\sqrt{3}}\end{aligned}$$

HH10 137. **Alg** The Friday before ASMT 2016, the ASMT Problem Writing Committee works furiously to write the remaining problems for the tournament. Alone, a committee member can finish writing the tests in 96 hours. On that Friday, person A starts writing problems. Exactly two hours later, person B joins person A, and the two committee members write problems at twice the rate. This continues every two hours, with another person joining the current workers, until all of the problems have been written. Assuming that each committee member works an integral number of hours (they keep working until the end of the hour, even if the tests have already been completely written), how many hours will the committee need to finish writing all of the problems?

Comment: This is *definitely* not going to happen for this ASMT 2016, right? Right?

Might need some rewording for clarity.

Answer: 19

Solution: Let W denote the total amount of work for the tournament. Then the rate r of a single person can be written as $r = \frac{W}{96}$. Next, let n be the total number of workers needed for the Friday night, and let a be the number the number of hours the last member works for. Note that a can either be one or two. The second to last member will work for $a+2$ hours; the third to last member will work for $a+4$ hours; and so on, until the first member will work for $a+2(n-1)$ hours.

We must hence solve for the minimum n which satisfies

$$\begin{aligned}&W \leq \frac{W}{96}a + \frac{W}{96}(a+2) + ... + \frac{W}{96}(a+2(n-1))\\ &96 \leq n\frac{2a+2(n-1)}{2}\\ &96 \leq n(a+n-1)\end{aligned}$$

When $a = 1$, the minimum n is 10; when $a = 2$, the minimum n is also 10. Hence, there must be 10 committee members, with the last committee member working for 1 hour, for a total of $1 + 9 * 2 = 19$ hours.

HH11 138. **Disc** A candy bar is composed of three sticks, with the sticks having 3 pieces, 4 pieces, and 5 pieces of chocolate. Cynthia eats the candy bar as follows: after choosing a stick at random, she eats one piece of chocolate from the stick from one of the edges. She continues this process until the candy bar is completely eaten. Let x be the number of possible ways that Cynthia can eat the candy bar. Compute $\frac{x}{12!}$.

Answer: $\mathbf{\frac{4}{135}}$

Solution: Let A be a piece from the stick of length 3, B be a piece from the stick of length 4, and C be a piece from the stick of length 5. We can consider a similar problem of arranging 3 As, 4 Bs, and 5 Cs in a string of length 12. However, the first 2 bites from the stick of length 3 can be taken from either of the two sides; the first 3 bites from the stick of length 4 can be taken from either of the two sides; and the first 4 bites from the stick of length 5 can be taken from either of the two sides. Hence, we compute x to be

$$x = \frac{12!}{3!4!5!} * 2^2 2^3 2^4 = \frac{12! 2^2}{3 * 3 * 5 * 3} = 12! \frac{4}{135}$$

Thus $\frac{x}{12!} = \boxed{\frac{4}{135}}$.

HH12 139. **Geo** A square is inscribed inside a square with side length 3 such that the sides of the inner square form four 30-60-90 triangles with the outer square. This process is repeated indefinitely. Compute the total sum of the areas of the squares.

Comment: A geometry problem with some algebra.

Answer: $\mathbf{6\sqrt{3} - 9}$

Solution: Let s be the side length of a square. Note that for each of the 30-60-90 triangles formed by the square and its inscribed square, the sum of the legs is s, and the hypotenuse is the side length of the inscribed square. Hence, the shorter leg has length $\frac{s}{1+\sqrt{3}}$, the longer leg has length $\frac{s\sqrt{3}}{1+\sqrt{3}}$, and the hypotenuse has length $\frac{2s}{1+\sqrt{3}}$. Hence the area of the inscribed square is

$$\left(\frac{2s}{1+\sqrt{3}}\right)^2 = \frac{4s^2}{4+2\sqrt{3}} = s^2(4 - 2\sqrt{3})$$

The sequence of areas of consecutive squares thus forms a geometric sequence with initial term $3^2 = 9$ and common ratio $4 - 2\sqrt{3}$. Hence, the total sum of the areas of the squares is

$$\frac{9}{1 - (4 - 2\sqrt{3})} = \frac{9}{2\sqrt{3} - 3} = \boxed{6\sqrt{3} - 9}$$

HH13 140. **Geo** In regular pentagon $ABCDE$, $\angle CDE = \angle DEA = 90°$, and $AB = BC = 25$, $CD = 16$, $DE = 22$, and $EA = 12$. Circle O is 'incscribed' inside the pentagon such that O is tangent to AB, BC, and DE. Compute the radius of O.

Comment: Harrison: Alternate problem: draw the incircle of $\triangle BCD$, and compute the ratio of the areas. Although, I prefer the current problem, as it requires students to think about the side lengths of the hidden triangle.

Julia: I'll probably use this problem with different pentagon (ABCDE st CDE = DEA = 90 deg, and AB = BC = 25, CD = 16, DE = 22, EA = 12)

TODO(JH): Write solution.

Original problem: Regular hexagon $ABCDEF$ has side length 2. Line segment BD is drawn, and circle O is inscribed inside the pentagon $ABDEF$ such that O is tangent to AF, BD, and EF. Compute the radius of O. Original answer: $6\sqrt{3} - 9$ Original solution: Extend AF and BD to intersect at G, and extend FE and BD to intersect at H. Note O is inscribed within the triangle FGH. It suffices to compute the inradius of FGH.

Note that $\triangle ABG$ is a 30-60-90 triangle. Since $AB = 2$, $BG = 2\sqrt{3}$ and $AG = 4$. Similarly, $DH = 2\sqrt{3}$ and $EH = 4$. Now, we solve for the area of $\triangle FGH$ using two methods.

First, drop the altitude from F to GH, intersecting at a point I. Then $FI = 3$, $IG = 3\sqrt{3}$, and $GH = 6\sqrt{3}$. Hence, the area of $\triangle FGH$ is $\frac{3*6\sqrt{3}}{2} = 9\sqrt{3}$. Next, the area of $\triangle FGH$ can also be computed by taking half of the product of the inradius and the perimeter, which is $6 + 6 + 6\sqrt{3} = 12 + 6\sqrt{3}$. Solving for the radius r, we have

$$r * \frac{12 + 6\sqrt{3}}{2} = 9\sqrt{3}$$
$$r = \frac{9\sqrt{3}}{6 + 3\sqrt{3}}$$
$$r = \frac{9\sqrt{3}(2 - \sqrt{3})}{3(2 + \sqrt{3})(2 - \sqrt{3})}$$
$$r = \boxed{6\sqrt{3} - 9}$$

Answer: $\frac{45}{4}$

Solution:

HH14:geo 141. **Geo** Two concentric circles have differing radii such that a chord of the outer circle which is tangent to the inner circle has length 18. Compute the area of the outer circle which lies outside of the inner circle.

Answer: 81π

Solution: Let the radius of the inner circle be x, and the radius of the outer circle by y. Thus, we wish to compute $(y^2 - x^2)\pi$. By the Pythagorean Theorem, $9^2 + x^2 = y^2$, hence the area is $\boxed{81\pi}$.

HH15:dis 142. **Disc** $ABCD$ is a four digit number ($A \neq 0$) such that both ABC and BCD are divisible by 9 ($ABCD$ is not necessarily divisible by 9, and B, C, D may be 0). Compute the number of four digit numbers satisfying this property.

Answer: 112

Solution 1: Given the divisibility rule, we have $A + B + C \equiv B + C + D \equiv 0 \pmod 9$. Simplifying, we have $A \equiv D \pmod 9$. We proceed using casework digit by digit:

(a) Only $A \equiv D \equiv 0 \pmod 9$. There are 8 ways to choose B; each of these choices fixes C, since $A + B + C \equiv 0 \pmod 9$. Because there are 2 options for D (either 0 or 9), there are a total of $2 \cdot 8 = 16$ numbers.

(b) Only $B \equiv 0 \pmod 9$. There are 8 ways to choose C, and each of these choices fixes both A, D. Since there are 2 ways to choose B, there are $2 \cdot 8 = 16$ numbers.

(c) Only $C \equiv 0 \pmod 9$. There are 8 ways to choose B, and each of these choices fixes both A, D. Since there are 2 ways to choose C, there are $2 \cdot 8 = 16$ numbers again.

(d) All of the numbers are $0 \pmod 9$. There are 2 ways to choose each of B, C, D, for a total of $2^3 = 8$ numbers.

(e) None of the numbers are $0 \pmod 9$. There are 8 ways to choose B and 7 ways to choose C; each pair of B, C fixes A, D. Hence, there are $8 \cdot 7 = 56$ numbers.

Thus, there are a total of $16 + 16 + 16 + 8 + 56 = \boxed{112}$ numbers that satisfy the conditions.

Solution 2: We first select the digits BCD. Note that BCD can be any triple from 000 to 999 that is divisible by 9. In total, there are $\frac{999-0}{9} + 1 = 112$ such triples. Next, note that for each choice of B, C, D there is exactly one choice of A such that $A + B + C \equiv 0 \pmod 9$. We conclude that there are exactly $\boxed{112}$ four digit numbers satisfying the desired properties.

$\boxed{\texttt{HH16}}$ 143. **Team** Two random integers x, y are chosen with $1 \leq x, y \leq 10000$. What is the probability that xy has 7 digits? Your score will be (insert grading curve).

Comment: Detailed analysis necessary to see how hard/how easy it is.

TODO(HS): Write up a difficulty curve for grading. Easy estimation.

TODO(HS): Got a diff answer of $94404667/10^8$. Can someone else solve?

Answer: 0.27421532

Solution: We count the number of pairs (x, y) such that $x \cdot y < 10^6$ or . Imagine iterating over x. If $1 \leq x \leq 99$, note that $x \cdot y \leq 999900$ for any y. Otherwise if $100 \leq x \leq 10000$, consider $a = \lfloor \dfrac{999999}{x} \rfloor$. For any integer $1 \leq y \leq a$, note that $x \cdot a \leq 999999$ and $(a+1)x > 999999$ by definition. Therefore a is exactly the number of pairs (x, y) where x is fixed and $100 \leq x \leq 10000$ such that $x \cdot y < 1000000$.

Therefore the probability that $x \cdot y < 1000000$ is

$$10000 \cdot 99 + \sum_{i=100}^{10000} \lfloor \frac{999999}{x} \rfloor,$$

and the answer is the above divided by 10^8 subtracted from 1.

$\boxed{\texttt{HH17}}$ 144. **Disc** Anna has a magical compass which can point only in four directions: North, East, South, West. Initially, the compass points North. After each minute, the compass can either turn left, turn right, or stay at its current orientation, with each action occurring equally likely. The probability that the compass points South after 6 minutes can be expressed as $\frac{m}{n^p}$, where m, n, p are integers and m and n are relatively prime. Compute $m + n + p$.

Answer: 99

Solution: Note that the compass can point South either by rotating to the left 6 times, rotating to the left the equivalent of 2 times, rotating to the right the equivalent of 2 times, or rotating to the right 6 times. We analyze each of these cases:

(a) The compass moves 6 times to the left. There is clearly only 1 possible way that the compass can move this way.

(b) The compass moves 2 times to the left. The compass can either move 4 times to the left and 2 times to the right; 3 times to the left, 1 time to the right, and 2 null actions; or 2 times to the left and 4 null actions. Hence there are a total of $\frac{6!}{4!2!} + \frac{6!}{3!2!} + \frac{6!}{4!2!} = 15 + 60 + 15 = 90$ possible movements.

(c) The compass moves 2 times to the right. This case is equivalent to the one above, and there are again 90 possible movements.

(d) The compass moves 6 times to the right. This case is equivalent to the first case, and there is only 1 possible movement.

There are a total of 3^6 possible actions that the compass can take over the course of 6 minutes. Hence, the probability that the compass points south is $\frac{2*90+2*1}{3^6} = \frac{182}{3^6}$, and $m+n+p = \boxed{99}$.

$\boxed{\texttt{HH18}}$ 145. **Calc** On an infinite grid of unit squares, a unit square S is randomly placed with all edges parallel to those of the grid. Consider the largest contiguous section of S which is formed by edges of the squares on the grid. What is the expected value of the area of this contiguous section?

Answer: $\mathbf{\frac{9}{16}}$

Solution: Consider the center of S; it suffices to consider the area as the center is arranged within a single unit square. We can further split this area into four smaller subsquares of side length 0.5, with the side lengths of the contiguous section ranging from 0.5 to 1. We can obtain the expected value by taking the multivariate integral over the x and y-coordinates from 0.5 to 1, multiplying by 4, and dividing by the area of the unit square, which is simply 1. The expected value is thus

$$4 * \int_{0.5}^{1} \int_{0.5}^{1} xydydx = 4 * (\int_{0.5}^{1} xdx)(\int_{0.5}^{1} ydy) = 4 * \frac{1}{2} * \frac{3}{4} * \frac{1}{2} * \frac{3}{4} = \boxed{\frac{9}{16}}$$

HH19:dis 146. **Disc** Heesu, Xingyou, and Bill are in a class with 9 other children. The teacher randomly arranges the children in a circle for story time. However, Heesu, Xingyou, and Bill want to sit near each other. Compute the probability that all three children are seated within 5 seats of each other (in other words, that no two of the three children are separated by more than three other children).

Comment: Fun fact: if there are instead only 10 children, the probability is exactly $\frac{1}{2}$!

Answer: $\mathbf{\frac{18}{55}}$

Solution: We fix the position of the child sitting at the leftmost of the other two children. There are 3 ways to select the leftmost child out of Heesu, Xingyou, and Bill, and $4 \cdot 3 = 12$ ways to choose the positions of the other two children. After these 3 children have been given seats, there are 9! ways to arrange the remaining 9 children.

Next, we compute the total number of possible arrangements. After fixing any one of the 12 children, there are 11! total possible arrangements of the remaining children. Hence, the probability that the three children are sitting within 5 seats of each other is

$$\frac{3 \cdot 12 \cdot 9!}{11!} = \frac{36}{110} = \boxed{\frac{18}{55}}.$$

HH20 147. **Disc** Four mathematicians, four physicists, and four programmers gather in a classroom. The 12 people organize themselves into four teams, with each team having one mathematician, one physicist, and one programmer. How many possible arrangements of teams can exist?

Answer: 576

Solution: Consider each mathematician separately. To the first mathematician, we can assign one of four physicists and one of four programmers. To the second mathematician, we can assign one of three physicists and one of three programmers. We continue this process until the last mathematician, who can only be assigned one physicist and one programmer. This yields a total of $4 * 4 * 3 * 3 * 2 * 2 = \boxed{576}$ possible arrangements.

HH21 148. **Disc** 6 badminton teams $\{A, B, C, D, E, F\}$ play in a tournament. Each team plays every other team, with the outcome of each match being only a win or a less for either team. However, due to time constraints, teams A and D do not play each other, teams B and E do not play each other, and teams C and F do not play each other. Define the inferior set of a team t to be the recursive union of all of the teams which t has defeated and each of the inferior sets of the teams which t has defeated, after all teams have played each other. Next, define an ordered tournament to be one in which the inferior set for each team t does not include t. Compute the number of possible ordered tournaments.

Comment: Is there a better way to define a cycle?

The solution would benefit from a graph. Admittedly this problem is inspired by the 2016 ASMT Power Round, but it should be extremely difficult to apply the recurrence theorem from the Power Round, and ASMT and AMT are held in different countries.

Answer: 546

Solution: The problem statement is equivalent to finding the number of orientations of the graph which do not have a cycle. Denote a team that wins all its matches a source and a team that loses all its matches a sink. Suppose that team A is a source. Then there exists at most one other team, team D, which can also be a source. Similarly, if team A is a sink, then there exists at most one other team, team D, which can also be a sink. We can repeat this process for other team pairs.

Next, we show that the graph must have at least one sink and one source. Assume for the sake of contradiction that the graph has no sources. Then team t_1 must lose to some other team t_2, which must lose to some other team t_3. Team t_3 cannot lose to team t_1, as this would form a cycle. Hence, team t_3 must lose to some other team t_4; team t_4 must lose to some other team t_5; and team t_5 must lose to team t_6. But team t_6 must lose to one of the other five teams, forming a cycle, contradiction. Hence, the graph must have at least one sink and one source. (Note that we can extend this argument to any tournament with finitely many number of teams.)

We first deal with the case with one source and one sink, our most complicated case:

(a) The source and sink do not play each other. We have to ensure that there does not exist a cycle between the 4 other teams. Since there are 2 ways of achieving a cycle, there are $2^4 - 2 = 14$ ways of ordering the matches between the 4 other players. There are 6 ways of choosing the source and the sink, for a total of $14 * 6 = 84$ orientations.

(b) The source and sink do play each other. We have to ensure that there does not exist a cycle between the 4 other teams. Fixing one of the edges to either be a win or a loss, there are 3 ways of orienting each of the smaller triangles. There are $6 * 4 = 24$ ways of choosing the source and the sink, for a total of $24 * 2 * 3 * 3 = 432$ orientations.

Next, we deal with all remaining cases:

(a) There are two sources and one sink. By sketching out the tournament, we observe that choosing the sources and the sink fixes all of the possible matches. There are 3 ways to choose the sources and 4 ways to choose the sink, for a total of $3 * 4 = 12$ orientations.

(b) There are one source and two sinks. This case is equivalent to the case above, with sources swapped with sinks, for a total of 12 orientations.

(c) There are two sources and two sinks. Again, by sketching out the tournament, we observe that choosing the sources and the sinks fixes all of the possible matches. THere are 3 ways to choose the sources and 2 ways to choose the sinks, for a total of $3 * 2 = 6$ orientations.

Summing each of the cases, we conclude that there are $84 + 432 + 12 + 12 + 6 = \boxed{546}$ possible orientations.

HH22:gut 149. **Disc** Compute the number of nonnegative integer triples (x, y, z) which satisfy $4x+2y+z \leq 36$.

Answer: 1330

Solution 1: One way to approach this problem is thus to consider the sum $S = 4x + 2y + z$ in terms of increments of 4. When counting the number of triples, we increment x first, then y, then z, for simplicity. We use the notation (abc) to denote an increment of a, then an increment of b, and finally an increment of c. Thus, (211) would describe an increment of 2, followed by two increments of 1, equivalent to incrementing y by 1 and incrementing z by 2.

We observe that when x is incremented by 1, S increases by 4 (4); when y is incremented by 2, S increases by 4 (22); and when z is incremented by 4, S increases by 4 (1111). We can also increase y by 1 and z by 2 (211), y by 1 and z by 1 (21), or y by 1 with no increase in z (2), to result in an odd value of y. Note that we can only perform this type of increase once, as this reduces to the above case of incrementing y by 2 to get an increase of 4. Finally, we can increase z by 1 (1), 2 (11), or 3 (111) at most once at the end, by similar logic.

We break down these into cases:

(a) There only exist the increments (4), (22), (1111). Letting $a = x$, $b = 2y$, and $c = 4z$, this is equivalent to finding the number of nonnegative integer triples (a, b, c) which satisfies $a + b + c \leq 9$, or equivalently which satisfies $a + b + c + d = 9$, where d is also a nonnegative integer. The number of triples is equivalent to the number of ways of placing 3 dividers in a total of $9 + 3 = 12$ slots, for a total of $\binom{12}{3} = 220$ combinations.

(b) The increment (211) occurs once. This is equivalent to finding the number of nonnegative integer triples (a, b, c) which satisfies $a + b + c \leq 8$, since one increment of 4 must be reserved for (211). There are thus $\binom{8+3}{3} = \binom{11}{3} = 165$ combinations.

(c) One of the increments (1), (11), or (111) occurs once. This is equivalent to finding the number of nonnegative integer triples (a, b, c) which satisfies $a + b + c \leq 8$, since one increment of 4 must be reserved for (1), (11), or (111). There are thus $3 * \binom{8+3}{3} = 3 * \binom{11}{3} = 495$ combinations.

(d) The increment (211) and one of the increments (1), (11), or (111) occurs once. This is equivalent to finding the number of nonnegative integer triples (a, b, c) which satisfies $a + b + c \leq 7$, since two increments of 4 must be reserved. There are thus $3 * \binom{7+3}{3} = 3 * \binom{10}{3} = 360$ combinations.

(e) One of the increments (21) or (2) occurs once. This is equivalent to finding the number of nonnegative integer pairs (a, b) which satisfies $a + b \leq 8$ (note that we exclude c because (1111) cannot appear at all, as (21) or (2) must occur at the end of the increments since the total increase of either one is not 4). There are thus $2 * \binom{8+2}{2} = 2 * \binom{10}{2} = 90$ combinations.

Summing, there are a total of $220 + 165 + 495 + 360 + 90 = \boxed{1330}$ combinations.

Solution 2: Let us consider the inequality $z \leq 4a$. We know that there are $4a+1$ possibilities for z, each of the integers between 0 and $4a$ inclusive.

Now, consider the inequality $2y + z \leq 4a$. We know that there are $2a + 1$ possibilities for y, each of the integers between 0 and $2a$ inclusive, and for a given value $y = i$, we know that $z \leq 4a - 2i$, and there are $4a - 2i + 1$ possibilities for z. Summing this over all posible values of i, we see that the number of possible (y, z) pairs is $1 + 3 + 5 + ...(4a + 1) = (2a + 1)^2$ possibilities.

Now, consider the inequailty $4x + 2y + z \leq 4a$. We know that there are $a + 1$ possibilites for y, each of the integers between 0 and a inclusive, and for a given value $x = i$, we know that $2y + z \leq 4a - 4i$, and there are $(2(a - i) + 1)^2$ possibilities. Summing this over all possible values of i, we see that the number of possible (x, y, z) pairs is $1^2 + 3^2 + 5^2 + ...(2a + 1)^2$. For a larger a, you can derive the formula for the sum of the first n odd squares. But since we are confident in our abilities to add 3 digit numbers, we find that for $a = 9$, we have $1 + 9 + 25 + 49 + 81 + 121 + 169 + 225 + 289 + 361 = 1330$.

HH23:dis 150. **Disc** A container is filled with a total of 51 red and white balls and has at least 1 red ball and 1 white ball. The probability of picking up 3 red balls and 1 white ball, without replacement, is equivalent to the probability of picking up 1 red ball and two white balls, without replacement. Compute the original number of red balls in the container.

Answer: 29

Solution: Let m denote the number of red balls, and n denote the number of white balls. Note that $m+n=51$. We may write

$$\frac{\binom{m}{3}\binom{n}{1}}{\binom{51}{4}} = \frac{\binom{m}{1}\binom{n}{2}}{\binom{51}{3}}$$

$$\frac{m\cdot(m-1)\cdot(m-2)\cdot n\cdot 4\cdot 3\cdot 2}{3\cdot 2\cdot 51\cdot 50\cdot 49\cdot 48} = \frac{m\cdot n\cdot(n-1)\cdot 3\cdot 2}{2\cdot 51\cdot 50\cdot 49}$$

$$\frac{(m-1)\cdot(m-2)\cdot 4}{48} = (n-1)\cdot 3$$

$$(m-1)(m-2) = 36(n-1).$$

Looking at the product, we observe that the geometric mean of 36 and $n-1$ lies between $m-1$ and $m-2$. Hence, we make an initial guess that $m=30$ and $n=21$ and observe that

$$(30-1)(30-2) > 28^2 > 720 = 36\cdot 20.$$

Thus, our guess of $m=30$ is slightly too high. Indeed, we find that if we choose $m=29$ and $n=22$, then

$$(29-1)(29-2) = 27\cdot 27 = 36\cdot 21 = 36(22-1).$$

We conclude that there were originally $\boxed{29}$ red balls in the container.

HH24:tea 151. **Disc** Two arbitrary distinct factors of 144 are multipled. What is the probability that the product of the factors is greater than 144?

Answer: $\frac{7}{15}$

Solution: Let this probability be p. Then $2p+q=1$, where q is the probability that the product of the factors is equal to 144. This is achieved when we choose one of the 7 factor pairs $(1,144),(2,72),(3,48),(4,36),(6,24),(8,18),(9,16)$. Since $144=2^4 3^2$, there are a total of $(4+1)(2+1)=15$ factors of 144, for a total of $\frac{15*14}{2}=105$ factor pairs. Hence, the desired probability is

$$p = \frac{1-q}{2} = \frac{105-7}{2*105} = \boxed{\frac{7}{15}}$$

HH25 152. **Team** Compute the smallest positive integer x which satisfies $x^2-8x+1\equiv 0 \pmod{22}$ and $x^2-22x+1\equiv 0 \pmod{8}$.

Comment: Note to self: this problem is harder than it looks.

Answer: 35

Solution: Observing the first equivalence, we subtract $3*22=66$ from the left hand side to obtain

$$x^2-8x+1 \equiv x^2-8x-65 \equiv (x-13)(x+5) \equiv 0 \pmod{22}$$

Hence, $x \equiv 13 \pmod{22}$ or $x \equiv -5 \pmod{22}$. Looking at the second equivalence, we add $3 * 8x = 24x$ to the left hand side to obtain

$$x^2 - 22x + 1 \equiv x^2 + 2x + 1 \equiv (x+1)^2 \equiv 0 \pmod{8}$$

Hence, $x \equiv -1 \pmod{8}$ or $x \equiv 3 \pmod{8}$ (in the latter case, we have $(x+1)^2$ is divisible by 16 and hence is divisible by 8). It suffices to find the smallest x which satisfies the two equivalences through the Chinese Remainder Theorem.

Checking $x = 13$ and $x = 22 - 5 = 17$, we find that neither of these satisfies the second set of equivalences. Next, checking $x = 13 + 22 = 35$ and $x = 17 + 22 = 39$, we find that both of these satisfies the second set of equivalences. The smallest x is hence $x = \boxed{35}$.

HH26 153. **Disc** Let S be the set of positive integers less than or equal to 1000 such that for each element x in S, $6x$ is a perfect square. Compute the number of elements in S.

Answer: 11

Solution: All elements in S must take on the form $6n^2$, where n is a positive integer, as thus for each element we have that $6 * 6n^2 = (6n)^2$, a perfect square. Clearly, $6 = 6 * 1^2$ is an element in S; it suffices to find the largest value of n such that $6n^2 \leq 1000$, or $n^2 \leq 133.\bar{3}$. The largest n satisfying this equation is $n = 11$ with $n^2 = 121$. Since all $1 \leq n \leq 11$ produces a valid element of S, the number of elements is thus $11 - 1 + 1\boxed{11}$.

HH27 154. **Disc** Laurie throws a fair 6-sided die labelled with the integers from 1 through 6. She then throws a fair 4-sided die labelled with the integers from 1 through 4. What is the probability that the sum of the rolls is a multiple of 3?

Answer: $\frac{1}{3}$

Solution: By enumerating each of the possible combinations of the two die, we observe that for each face of the 4-sided die, there are exactly 2 faces of the 6-sided die such that the sum of the 2 faces is a multiple of 3. Hence, the probability is $\frac{2}{6} = \frac{1}{3}$ for each face of the 4-sided die, and the solution is simply $\boxed{\frac{1}{3}}$.

HH28:alg 155. **Alg** Let $f(x) = x^2 - 10x + 21$. Compute the distinct roots of $f(f(x) + 7)$.

Answer: 3, 5, 7

Solution: We observe that $f(x) = (x-3)(x-7)$, hence $f(x) = 0$ when $x = 3, 7$. Thus, the roots of $f(f(x))$ are the values of x which satisfy $f(x) + 7 = 3$ and $f(x) + 7 = 7$. In the first case, x must satisfy $x^2 - 10x + 21 = -4$, and $x^2 - 10x + 25 = (x-5)^2$. Hence, $x = 5$. In the other case, x must satisfy $x^2 - 10x + 21 = 0$, and we again have $x = 3, 7$. Hence, the distinct roots of $f(f(x) + 7)$ are $x = \boxed{3, 5, 7}$.

HH29:dis 156. **Disc** Julia adds up the numbers from 1 to 2016 in a calculator. However, every time she inputs a 2, the calculator malfunctions and inputs a 3 instead (for example, when Julia inputs 202, the calculator inputs 303 instead). How much larger is the total sum returned by the broken calculator? (No 2s are replaced by 3s in the output, and the calculator only malfunctions while Julia is inputting numbers.)

Comment: This problem is made easier by replacing 2016 with 2000.

Answer: 39202

Solution: For simplicity, we first consider the numbers from 1 to 1999, and then consider the numbers 2000 to 2016 separately. We classify the errors by place:

(a) Errors in the ones digit. There are two choices for the thousands digit, ten choices for the hundreds digit, and ten choices for the tens digit, for a total of $2 \cdot 10 \cdot 10 = 200$ incorrect inputs. Since each incorrect input contributes 1 to the total sum, the sum is increased by a total of $200 \cdot 1 = 200$.

(b) Errors in the tens digit. There are two choices for the thousands digit, ten choices for the hundreds digit, and ten choices for the ones digit, for a total of $2 \cdot 10 \cdot 10 = 200$ incorrect inputs. Since each incorrect input contributes 10 to the total sum, the sum is increased by a total of $200 \cdot 10 = 2000$.

(c) Errors in the hundreds digit. There are two choices for the thousands digit, ten choices for the tens digit, and ten choices for the ones digit, for a total of $2 \cdot 10 \cdot 10 = 200$ incorrect inputs. Since each incorrect input contributes 100 to the total sum, the sum is increased by a total of $200 \cdot 100 = 20000$.

For the numbers from 2000 to 2016, we observe that the thousands digit is incorrectly incremented 17 times, for a total increase of 17000. The ones digit is incremented only 2 times for the numbers 2002 and 2012, for a total increase of 2. Hence, the incorrect sum is off by a total of $200 + 2000 + 20000 + 17000 + 2 = \boxed{39202}$.

HH30:geo 157. **Geo** A circle with center O is drawn in the first quadrant of the 2D Cartesian plane (the quadrant with positive x and positive y values) such that it lies tangent to the x and y-axes. A line is drawn with slope m and passing through the origin. The line intersects the circle at two points A and B, with A closer to the origin than B. Suppose that ABO is an equilateral triangle. Compute m given that $m > 1$.

Answer: $\mathbf{4 + \sqrt{15}}$

Solution: We assume that the circle has radius 1. Let the midpoint of AB be C, and let the point of tangency of the circle on the x-axis be D. Then ACO is a 30-60-90 triangle, and $CO = \frac{\sqrt{3}}{2}$. Also, letting the origin be R, we observe that RDO is a 45-45-90 triangle, hence $RO = \sqrt{2}$. Finally, noting that RCO is a right triangle, we compute $RC = \frac{\sqrt{5}}{2}$ using the Pythagorean Theorem.

Using the tangent addition formula, we compute

$$\begin{aligned}
\tan\angle CRD &= \frac{\tan\angle CRO + \tan\angle ORD}{1 - \tan\angle CRO \tan\angle ORD} \\
m &= \frac{\frac{\sqrt{3}}{\sqrt{5}} + 1}{1 - \frac{\sqrt{3}}{\sqrt{5}}} \\
m &= \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}} \\
m &= \frac{5 + 3 + 2\sqrt{15}}{5 - 3} \\
m &= \boxed{4 + \sqrt{15}}
\end{aligned}$$

HH31:geo 158. **Geo** Let $ABCD$ be a rectangle with $AB = 9$ and $BC = 3$. Suppose that D is reflected across AC to a point E. Compute the area of trapezoid $AEBC$.

Answer: $\mathbf{\frac{243}{10}}$

Solution: Extend AB to a point F such that $\angle ACF = 90$, and extend CE to a point G such that $\angle CAG = 90$. Let H be the intersection of AB and CE. Observe that H is the midpoint of AF and the midpoint of CG, since $ACFG$ is also a rectangle. Next, since $\triangle ABC \sim \triangle CBF$, $BF = 1$ by using side similarity. Hence, $AH = 5$ and $BH = 4$. Noting that $\triangle HBE \sim HAC$, we have $\frac{BE}{AC} = \frac{4}{5}$ by the previous result. Finally, letting $A[\triangle ABC]$ denote the area of $\triangle ABC$, we observe that $\frac{A[\triangle ABE]}{A[\triangle ABC]} = \frac{4}{5}$, since their heights are equivalent. Since $A[\triangle ABC] = \frac{27}{2}$, we have $A[\triangle ABE] = \frac{27}{2} * \frac{4}{5} = \frac{108}{10}$. Finally, we compute the area of $AEBC$ to be $\frac{27}{2} + \frac{108}{10} = \frac{135+108}{10} = \boxed{\dfrac{243}{10}}$.

HH32:dis 159. **Disc** At the festival, Jing Jing plays a game where she must knock down ten targets with as few balls as possible. Every time Jing Jing knocks down a target, she can reuse the ball she just threw and does not have to pick up a new ball. Suppose that Jing Jing knocks down each target with a probability of $\frac{3}{4}$. Compute the expected number of balls that Jing Jing needs to knock down all ten targets.

Comment: Fun fact: if you choose her accuracy to be $\frac{2}{3}$, then she needs on average one additional ball for each target. Unfortunately that value as her accuracy makes this problem easy to guess.

Answer: $\mathbf{\frac{13}{3}}$

Solution 1: Jing Jing must use at least one ball to knock down any of targets. We thus consider the expected number of additional balls that Jing Jing needs to knock down any single target.

Jing Jing has a $\frac{3}{4}$ chance of hitting the target on her first try; in this case, she needs 0 additional balls. Jing Jing has a $\frac{1}{4} \cdot \frac{3}{4}$ chance of missing the target on her first try and hitting it on her second, and thus needs 1 additional ball. Next, Jing Jing has a $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4}$ chance of needing 2 additional balls, and so on. Her expected number of additional balls for a single target is thus

$$\begin{aligned}
1\left(\frac{1}{4}\cdot\frac{3}{4}\right) + 2\left(\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\right) + 3\left(\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\right) + \ldots &= \frac{3}{16} + 2\left(\frac{3}{64}\right) + 3\left(\frac{3}{256}\right) + \ldots \\
&= \sum_{i=2}^{\infty}\frac{3}{4^i} + \sum_{i=3}^{\infty}\frac{3}{4^i} + \sum_{i=4}^{\infty}\frac{3}{4^i} + \ldots \\
&= \frac{\frac{3}{16}}{1-\frac{1}{4}} + \frac{\frac{3}{64}}{1-\frac{1}{4}} + \frac{\frac{3}{256}}{1-\frac{1}{4}} + \ldots \\
&= \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots \\
&= \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}.
\end{aligned}$$

Since there are ten targets and Jing Jing must start with one ball, Jing Jing needs an expected value of $1 + 10\left(\frac{1}{3}\right) = \boxed{\dfrac{13}{3}}$ balls.

Solution 2: Since Jing Jing has a $\frac{3}{4}$ chance of hitting each target, she needs an expected value of $\frac{4}{3}$ balls for each of the 10 targets. However, since Jing Jing reuses the ball when she moves from one target to the next, we overcount by 9 balls. So, the expected number of balls is $10 \cdot \frac{4}{3} - 9 = \frac{40}{3} - \frac{27}{3} = \boxed{\dfrac{13}{3}}$.

HH33 160. **Team** Anne separates the 2-dimensional plane into a finite number of regions by drawing some number of distinct infinitely long lines, no three of which intersect at a single point.

For example, by drawing three lines, no two of which are parallel, Anne can separate the plane into 7 regions. Suppose that after drawing some number of lines, Anne separates the 2-dimensional plane into 2016 regions. Let a be the maximum number of lines that Anne has drawn, and let b be the minimum number of lines that Anne has drawn. Compute $a - b$.

Comment: Discrete/geo which requires algebra to solve. Easy-medium-ish problem for guts/team.

Answer: 1952

Solution: The maximal case is easy: Anne can form 2016 regions by simply drawing 2015 parallel lines.

The minimal case requires some more work. We consider the maximum number of additional regions that each line contributes. When no lines are drawn, there exists only 1 region: the entire plane. When 1 line is drawn, the plane is separated into 2 regions, and 1 more region is created. When 1 more line is drawn, 2 more regions are created, the maximal contribution. When 1 more line is drawn, 3 more regions are created, the maximal contribution. In general, when n lines are drawn, Anne can separate the 2-D plane into a maximum of $1 + \frac{n(n+1)}{2}$ different regions, when no two lines are parallel. Moreover, Anne can separate the plane into slightly fewer regions by choosing some lines to be parallel to each other. (Determining this selection of lines is left as an exercise to the reader.) Hence, we must find the smallest n such that $1 + \frac{n(n+1)}{2} \geq 2016$, or when $n(n+1) \geq 4030$.

Since $60^2 = 3600$ and $70^2 = 4900$, we guess that our choice of n lies between 60 and 70. Indeed, we find that when $n = 63$, $n(n+1) = 4032$. Hence, the minimal number of lines is 63.

Our answer is thus $2015 - 63 = \boxed{1952}$.

HH34:dis 161. **Disc** Define a *substring* of a string $\mathcal{S}$ of letters to be a positive-length string using any number of the letters in $\mathcal{S}$ in order. For example, a substring of $HARRISON$ is $ARRON$. Compute the number of substrings in $HARRISON$.

Comment: Inspired by AL18.

Answer: 191

Solution: There are two cases: either a substring has at most one R, or a substring has two Rs. In the first case, there are $2^7 - 1$ ways to choose letters from $HARISON$ to form a substring, since we can either select a character from $HARISON$ or not. Note that in this case, we have to subtract one for the case that no characters are selected, which gives an invalid substring. In the second case, there are 2^6 ways to choose letters from $HAISON$. Then, adding two Rs gives us new substrings of $HARRISON$ which are different from the ones in the first case. Note that in the second case, we do not need to subtract 1 from 2^6, since adding the two Rs guarantees a substring of positive length.

Hence the total number of substrings is $2^7 + 2^6 - 1 = 128 + 64 - 1 = \boxed{191}$.

HH35 162. **Disc** Heesu chooses an arbitrary 4 digit number $ABCD$. He then reverses the digits to form another 4 digit number $DCBA$. How many such numbers $ABCD$ have the property that $DCBA > ABCD$?

Comment: TODO: write checker.

Alternate problem: same problem, but with a 3 digit number.

Answer: 4005

Solution: There are two cases:

(a) $A < D$. There are 36 ways to choose pairs of A, D. For each of these pairs, the choice of B, C does not matter. Since B and C can take on 10 values each, there are $36 * 100 = 3600$ possible numbers.

(b) $A = D$. There are 9 ways to choose A, D. Next, there are 45 ways to choose B, C such that $C > B$. Hence, there are $9 * 45 = 405$ possible numbers.

We conclude that there are $3600 + 405 = \boxed{4005}$ total numbers which satisfy the conditions.

HH36 163. **Disc** Two arbitrary distinct lattice points are selected on the coordinate plane within the square marked by the points $(0, 0)$, $(3, 0)$, $(0, 3)$, and $(3, 3)$ (the lattice points may lie on a side or a corner of the square). What is the probability that the distance between the two points is at most $\sqrt{2}$?

Answer: $\mathbf{\frac{7}{20}}$

Solution: There are two cases: either the two points are a distance of 1 apart, or the two points are a distance of $\sqrt{2}$ apart. In the first case, there are 12 ways to draw horizontal line segments of length 1 and 12 ways to draw vertical line segments of length 1 within the square. In the second case, there are 9 1×1 unit squares within the 3×3 unit square, with each of these unit squares having two diagonals of length $\sqrt{2}$. Hence, there are $2 * 9 = 18$ such pairs of points that are a distance of $\sqrt{2}$ apart.

Finally, there are $\binom{16}{2} = \frac{16*15}{2} = 120$ total ways to select any two pairs of points in the figure. The desired probability is thus $\frac{18+24}{120} = \boxed{\frac{7}{20}}$.

HH37:alg 164. **Alg** A pet shop sells cats and two types of birds: ducks and parrots. In the shop, $\frac{1}{12}$ of animals are ducks, and $\frac{1}{4}$ of birds are ducks. Given that there are 56 cats in the pet shop, how many ducks are there in the pet shop?

Answer: 7

Solution: Let x be the number of ducks in the pet shop. Then there are $4x$ birds and $12x$ total animals. Thus, there are $12x - 4x = 8x = 56$ cats. Solving for x, we have $x = \boxed{7}$, which is precisely the number of ducks in the shop.

HH38 165. **Disc** Seven children sit down in a circle. However, two of the children, Bill and Heesu, want to sit next to each other. Given that arrangements achieved by rotation are considered identical, how many ways are there to arrange the seven children?

Answer: 240

Solution: We can either have Bill sit to the left of Heesu, or Bill sit to the right of Heesu. Now, suppose that the seating of Bill and Heesu are fixed. Then, there are $5! = 120$ ways to seat the remaining five children. Thus, there are $2 * 120 = \boxed{240}$ ways to arrange the seven children.

HH39:dis 166. **Disc** Moor owns three shirts, one of each black, red, and green. Moor also owns three pairs of pants, with one of each white, red, and green. Being stylish, he decides to wear an outfit consisting of one shirt and one pair of pants that are different colors. How many combinations of shirts and pants can Moor choose?

Answer: 7

Solution: There are three choices of shirts and three choices of pants. However, he cannot wear an all red outfit or an all green outfit, which eliminates two possible choices. Thus, he can wear a total of $3 \cdot 3 - 2 = \boxed{7}$ outfits.

HH40:dis 167. **Disc** The largest factor of n not equal to n is 35. Compute the largest possible value of n.

Answer: 175

Solution: We must have $n = 35x$ for some positive integer x. We claim that x is at most 5.

First, suppose that x is composite. Then, we may write $x = ab$ for some integers $a, b > 1$, hence $n = 35ab$. Then $35a$ is also a factor of n, which is impossible, showing that x cannot be composite.

Next, suppose that x is prime and greater than 5. Then $7x > 35$, and $7x$ is thus a larger factor of n not equal to n. Thus, x cannot be greater than 5.

Hence, we claim that x is at most 5. Indeed, we observe that the largest factor of $5 \cdot 35 = 175$ which is not equal to 175 is 35. Thus, the largest possible value of n is $\boxed{175}$.

HS01 168. **Disc** Moor has 2016 white rabbit candies! He and his n friends split them equally amongst themselves, and find that they all have an integer number of candies. Given that n is a positive integer (Moor has at least one friend!), how many possible n exist?

Comment: Easy/very low medium NT problem.

Answer: 35

Solution: We can factor $2016 = 2^5 \cdot 3^2 \cdot 7$. The condition given in the problem is that $\frac{2016}{n+1}$ is an integer. Therefore as usual, any factor of 2016 is of the form $2^a \cdot 3^b \cdot 7^c$ where $0 \le a \le 5$, $0 \le b \le 2$, and $0 \le c \le 1$ for integers a, b, c. This gives $6 \cdot 3 \cdot 2 = 36$ divisors of 2016. Each of these factors is a valid solution except for $n = 0$ since n must be positive. Therefore the answer is $36 - 1$.

HS02 169. **Alg** Bill has taken 3 tests, each having an integer score from 0 to 100. Let the scores be $0 \le a \le b \le c \le 100$. Bill takes a 4th test and gets a score of d where $b \le d \le c$. Let x be the increase in mean from the set of 3 tests to 4 tests, and let y be the increase in median from the set of 3 tests to 4 tests. What is the maximum possible difference of $x - y$?

Comment: Median level of algebra? Easy to bash out algebra. Might require a little thinking to reduce number of variables.

Answer: $\mathbf{\frac{25}{3}}$

Solution: We may explicitly write the means $\frac{a+b+c}{3}$ and $\frac{a+b+c+d}{4}$, and medians b and $\frac{b+d}{2}$. Therefore the value we are asked to maximize is

$$\left(\frac{a+b+c+d}{4} - \frac{a+b+c}{3}\right) - \left(\frac{b+d}{2} - b\right),$$

which simplifies to

$$\frac{-a+5b-c-3d}{12}.$$

We can maximize this by making a, c, and d as small as possible. Thus we let $a = 0$ and $d = b$. And since $b = d \le c$, we may further let $c = b$. This reduces the expression to $\frac{b}{100}$, which is maximized when $b = 100$. Therefore the answer is $\frac{100}{12}$. An example of this possible solution is $(a, b, c, d) = (0, 100, 100, 100)$.

HS03 170. **Geo** Princeton is building a new open lawn which will be sectioned off for different activities. The space will have circular fence on the outside, a square fence circumscribed inside that outer fence, and one last circular fence circumscribed in the square. If the outer circle has a radius of 40 meters and fencing costs $\$\frac{8}{\pi}$/m, what is the difference in cost of the outer circular fence than the inner circular fence?

Comment: Easy geo. Just crunching numbers.

Answer: $\mathbf{640 - 320\sqrt{2}}$

Solution: Given radius r, the circumference of a circle is $2\pi r$. The radius of the inner circle is the distance from the center of the fences to the edge of a square which we are told has its vertex 40m from the center. Thus the radius of the smaller circle is $20\sqrt{2}$. Thus the length of the outer circumference is 80π and the length of the inner is $40\sqrt{2}\pi$. The cost difference is therefore

$$\frac{8}{\pi}(80\pi - 40\sqrt{2}\pi) = 640 - 320\sqrt{2}.$$

HS04:cal 171. **Calc** Suppose a and b are two variables that satisfy $\int_0^2 (-ax^2 + b)\,dx = 0$. What is $\frac{a}{b}$?.

Comment: Easy calc. Hopefully just algebra bashing.

Answer: $\mathbf{\frac{3}{4}}$

Solution: We may simply expand the integrals:

$$\int_0^2 -ax^2 + bdx = 0$$
$$\left[\frac{-ax^3}{3} + bx\right]_0^2 = 0$$
$$-\frac{8}{3}a + 2b = 0,$$

so $\frac{a}{b} = \frac{3}{4}$.

HS05 172. **Calc** Eddy has an ice cream cone that is shaped like a regular cone with a radius of 5cm and a height of 15cm. He fills it with soft serve ice cream such that at any point, the ice cream fills the cone from the tip and the top of the volume of ice cream is flat. The height of the ice cream increases as a constant function, and it takes 10 seconds to fill the cone completely. Let $f(t)$ be the function of rate of ice cream being poured into the cone for any time $0 \le t \le 10$. Given that $f(t)$ polynomial, what is the product of the degree of the leading term of f and its coefficient?

Comment: Easy calc. ie 2 or 3?

Answer: $\boldsymbol{\frac{\pi}{4}}$

Solution: Note that at any time $0 \le t \le 10$, the ice cream is in the shape of a cone that is a small dilation of the entire cone, and the ratio of the ice cream volume to the entire cone is $\frac{t}{10}$ since it takes 10 seconds to fill. Therefore the volume of the ice cream at time t is $\dfrac{\pi \cdot r^2 \cdot h}{3} \cdot \left(\frac{t}{10}\right)^3 = \dfrac{\pi t^3}{8}$, where $r = 5$ and $h = 15$ are the radius and height. Therefore $f(t)$ is simply the derivative of this function, since f is exactly the rate at which the volume of the ice cream is increasing.

Therefore $f(t) = \dfrac{d}{dt}\dfrac{\pi t^3}{8} = \dfrac{3\pi t^2}{8}$, and our answer is $\frac{3\pi}{8} \cdot 2 = \frac{3\pi}{4}$.

HS06 173. **Calc** Ben is playing catch with a dog and he accidentally throws a ball into a hot-tub that is shaped like a circle of radius 200cm. The dog rushes to get the ball anyway! Because the hot-tub is on and creates a current, the dog swims at a speed of $r/second$, where r is the distance in centimeters that the dog is away from the center of the hot-tub. Given that the dog can grab the ball from $5cm$ away and that the ball is floating in the center of the pool, how long would it take the dog to reach the ball from the edge of the hot-tub in seconds?

Comment: Med calc? somewhere in 4-6 maybe.

Answer: $\mathbf{\ln 39}$

Solution: The problem asks if speed is r for $5 \leq r \leq 200$ where r is also the distance from a point, how long it would take to move from 5cm to 200cm away from that point. Note at any distance r that to move δr further, the change in time is $\Delta t = \dfrac{\Delta r}{r}$. Thus to sum over distances r from 5 to 200, the answer is the summation over Δt, and therefore we have

$$\int_5^{195} \frac{1}{r} dr = [\ln r]_5^{195} = \ln 39.$$

HS07:gut 174. **Disc** Let a be the least positive integer with 20 positive divisors and b be the least positive integer with 16 positive divisors. What is $a+b$? (Note that for any integer n, both 1 and n are considered divisors of n.)

Comment: HH: Many people may not know the trick behind these kinds of problems. Would recommend putting on team or guts.

Answer: 456

Solution: Suppose $n = \prod_{i=1}^{k} p_i^{e_i}$ is a prime factorization of any number n. Then the number of divisors of n is $\prod_{i=1}^{k} (e_i + 1)$ since for each divisor d of n, the exponent of p_i is an integer d_i satisfying $0 \leq d_i \leq e_i$.

Therefore for an integer with 20 divisors, we must have $\prod_{i=1}^{k} (e_i + 1) = 10$ for some k and e_i where e_i is the exponent of the prime divisors of it. We may express 20 as 20, $10 \cdot 2$, $5 \cdot 4$, or $5 \cdot 2 \cdot 2$.

- The smallest integer satisfying the first case is 2^{19}.
- The smallest integer in the second case is $2^9 \cdot 3$.
- The smallest integer in the third case is $2^4 \cdot 3^3$.
- The smallest integer in the fourth case is $2^4 \cdot 3 \cdot 5$.

Therefore the smallest integer with 20 divisors is $2^4 \cdot 3 \cdot 5 = 240$.

Similarly the smallest integer with 16 divisors is $2^3 \cdot 3^3 = 216$. Thus $a = 240$, $b = 216$ implies $a + b = 456$.

HS08:dis 175. **Disc** Consider all fractions $\frac{a}{b}$ where $1 \leq b \leq 100$ and $0 \leq a \leq b$. Of these fractions, let $\frac{m}{n}$ be the smallest fraction such that $\frac{m}{n} > \frac{2}{7}$. What is $\frac{m}{n}$?

Comment: Med NT? If someone know the technique it is fairly simple.

Answer: 27/94

Solution: In general, suppose that an ordered list of such fractions are made where $b \leq n$, and suppose that $\frac{a}{b}$ and $\frac{c}{d}$ are two consecutive fractions in the list. Now consider the case when we add additional fractions such that the denominator is at most n', where $n' > n$. We claim that the next fraction that will be inserted between $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{a+c}{b+d}$. As an example, here is the ordered list when the denominators are bounded by 5:

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}.$$

If we increase the maximum possible denominator to 9, the next fraction to be inserted between $\frac{1}{5}$ and $\frac{1}{4}$ is $\frac{2}{9}$.

Following this logic, we compile a list of all fractions where the denominator is less than or equal to 7, and observe that $\frac{2}{7}$ and $\frac{1}{3}$ are consecutive fractions. Therefore, imagining inserting many fractions, the fraction just above $\frac{2}{7}$ will be $\frac{1+a\cdot 2}{3+7\cdot a}$, where $3+7\cdot a \le 100$. Thus, $a = 13$ and the desired fraction is $\frac{27}{94}$.

As justification for our claim, we may proceed using induction on n. After writing out all fractions with denominator bounded by some n, we observe that the difference between 2 consecutive fractions $\frac{a}{b}$ and $\frac{c}{d}$ is

$$\frac{c}{d} - \frac{a}{b} = \frac{bc-ad}{bd} = \frac{1}{bd},$$

hence $bc - ad = 1$. Next, consider the parallelogram formed by the coordinates $(0,0)$, (b,a), (d,c), $(b+d, a+c)$ and note that the area of the parallelogram is 1. By Pick's Theorem, the parallelogram also contains no interior points. Finally, observe that the fractions $\frac{a}{b}$ and $\frac{c}{d}$ correspond exactly to the slopes of the lines from the origin to (b,a) and (d,c), respectively. Hence, there is no other lattice point with slope between $\frac{a}{b}$ and $\frac{c}{d}$, showing that $\frac{a+c}{b+d}$ is the next fraction to add.

HS09 176. **Team** Let a, b, c be two independently random integers $0 \le a, b, c < 2016$. Let n be the number of pairs of (a, b, c) such that

$$\lfloor \frac{a+b+c}{2016} \rfloor = \lfloor \frac{a}{2016} \rfloor + \lfloor \frac{b}{2016} \rfloor + \lfloor \frac{c}{2016} \rfloor.$$

What is the sum of the digits of n?

Comment:

Answer: 42

Solution: By integer division, note that $a = q \cdot 2016 + r$ where $0 \le r < 2016$. Let $\left\{\frac{a}{2016}\right\} := \frac{r}{2016}$ denote the remainder so that $\frac{a^2}{2016} = \lfloor \frac{a^2}{2016} \rfloor + \left\{\frac{a^2}{2016}\right\}$. Therefore with this notation, $\lfloor \frac{a^2+b^2}{2016} \rfloor = \lfloor \frac{a^2}{2016} \rfloor + \lfloor \frac{b^2}{2016} \rfloor$ only when $\left\{\frac{a^2}{2016}\right\} + \left\{\frac{b^2}{2016}\right\} < 1$.

A rough solution is to note that by the relative randomness of squaring in a modulus, $\{x^2 \pmod{2016} : 0 \le x < 2016\}$ is relatively evenly split between those less than $\frac{2016}{2}$ and greater. Thus $\left\{\frac{a^2}{2016}\right\} + \left\{\frac{b^2}{2016}\right\}$ should be expected to be about 1. Therefore there should be roughly half pairs of (a, b) such that $\left\{\frac{a^2}{2016}\right\} + \left\{\frac{b^2}{2016}\right\} < 1$. Thus the naive answer is $\frac{2016^2}{2} = 2032128$.

HS10 177. **Team** A Gaussian integer is a number of the form $a+bi$ for integers $a, b \in \mathbb{Z}$. We define the norm as $N(a+bi) = a^2+b^2$. A Gaussian integer α is a Gaussian prime if the only divisors of α are units (± 1 or $\pm i$) and α times units. (0, ± 1, and $\pm i$ are not primes.) Compute the number of Gaussian primes α such that $N(\alpha) < 2016$.

Comment: TODO(HS): make a grading curve for this...

Answer: 1258

Solution: In general let $\alpha \in \mathbb{Z}[i]$ be a Gaussian integer. Suppose that $\alpha = \beta \cdot \gamma$. We may show that $N(\alpha) = N(\beta) \cdot N(\gamma)$.

If either a or b is 0, WLOG suppose $b = 0$. Then $N(\alpha) = a^2$. If a is not prime itself in $\mathbb{Z}$, it is still not prime in $\mathbb{Z}[i]$. If a is prime in $\mathbb{Z}$, it is only not prime in $\mathbb{Z}[i]$ if there exists some $m+ni$ such that $N(m+ni) = a \implies m^2+n^2 = a$. This happens iff $a = 1, 2 \pmod 4$. (It is easy to see that primes are either 2 or 1 or 2 mod 4. If $a = 2$ then $2 = (1+i)\cdot(1-i)$. If $a = 3 \pmod 4$, note that the sum of two squares is either $0, 1, 2 \pmod 4$. Lastly if $a = 1 \pmod 4$ is prime, by Fermat's Two Square Theorem, there do exist $m, n \in \mathbb{Z}$ such that $a = m^2 + n^2$. Thus $a = (m + ni)(m - ni)$ is not a Gaussian prime.)

Otherwise if both $a, b \neq 0$, we again have $N(\alpha) \in \mathbb{Z}$. if $N(\alpha)$ is prime, then α must be a Gaussian prime. If $N(\alpha)$ is composite, $N(\alpha) = m \cdot n$ where $m, n \in \mathbb{Z}$ and $1 < m < N(\alpha)$, note also that $N(\alpha) = a^2 + b^2 = (a + bi)\cdot(a - bi) = m \cdot n$. Since there is unique factorization in $\mathbb{Z}[i]$, we may not have $m \cdot n | a - bi$ since then $(a + bi) \cdot \frac{(a - bi)}{m \cdot n} = 1$, which cannot happen as $N(a + bi) > N((a + bi) \cdot \frac{(a - bi)}{m \cdot n}) > N(1)$. Therefore $N(\alpha)$ composite implies α is not prime. Similarly $N(\alpha)$ prime implies that α is prime.

Therefore if WLOG $b = 0$, we may count the numbers $a \in \mathbb{Z}$ such that a is prime and $a = 3 \pmod 4$ and a and ai are Gaussian primes. Else, we count $a + bi$ where $a, b \neq 0$, $a^2 + b^2 < 2016$, and $a^2 + b^2$ is prime.

By computation, this gives 1258 Gaussian primes.

HS11 178. **Alg** How many integer solutions are there to $y^2 + y = x^3 - 4$?

Comment:

Answer: 0

Solution: Consider the substitution $y = \frac{z-1}{2}$. Thus we have

$$\begin{aligned}\frac{1}{4}(z^2 - 2z + 1) + \frac{1}{2}(z - 1) &= x^3 - 2\\ \frac{1}{4}z^2 &= x^3 - \frac{15}{4}\\ z^2 &= 4x^3 - 15.\end{aligned}$$

However, consider this equation $\mod 4$, which gives $z^2 = -3 \pmod 4$. Note that squares modulo 4 are $0, 1, 2$. Therefore there are no integer solutions (x, z) to the latter equation. Note that if (x, y) were an integer solution to the original that $z = 2y - 1$ shows that (x, z) would also be an integer solution to the latter equation. Therefore there are no solutions (x, y).

HS12:gut 179. **Disc** There are n integers a such that $0 \leq a < 91$ and a is a solution to $x^3 + 8x^2 - x + 83 \equiv 0 \pmod{91}$. What is n?

Comment:

Answer: 6

Solution: Note that $8 \equiv -83 \pmod{91}$, and therefore the polynomial equals $x^3 - x + 8x^2 - 8$ and factors as $x(x^2 - 1) + 8(x - 1) = (x + 1)(x - 1)(x + 8) \pmod{91}$. Since 91 factors as $91 = 7 \cdot 13$, we find that $x \equiv \pm 1 \pmod 7$ and $x \equiv \pm 1, 5 \pmod{13}$ and by the Chinese Remainder Theorem each pair of combinations gives a unique solution. Therefore there are $3 \cdot 2 = 6$ total solutions.

HS13:tea 180. **Disc** Ross the kangaroo is a funny kangaroo. He is standing at the origin of a coordinate plane and he wishes to hop to the $(20, 16)$ coordinate. Ross can move only by jumping exactly $\sqrt{10}$ length on each jump and Ross is only allowed to be at lattice points. What is the fewest number of jumps Ross has to take to jump to $(20, 16)$?

Comment:

Answer: 10

Solution: Note by the Pythagorean theorem that jumping $\sqrt{10} = \sqrt{3^2+1}$ units is equivalent to moving 3 lattice points in one direction and 1 in the other. In other words, if Ross stands at point (a, b), he may only move to $(a + \pm 3, b + \pm 1)$ or $(a + \pm 1, b + \pm 3)$ in one jump.

We claim the answer is 10, and one such way to get to $(20, 16)$ in 10 jumps is by taking 3 $(3, 1)$ jumps, 5 $(1, 3)$ jumps, and 2 $(3, -1)$ jumps.

To see why there cannot be 9 or fewer jumps to $(20, 16)$, Note that only using positive jumps (ie, $(3, 1)$ or $(1, 3)$ jumps), there is no integer solution to $a(3, 1)+b(1, 3) = (20, 16)$. Therefore Ross must each at least one $(3, -1)$, $(1, -3)$, $(-1, 3)$, $(-3, 1)$ jump.

TODO(HS): finish.

`HS15:gut` 181. **Alg** What is

$$\sum_{n=1996}^{2016} \lfloor \sqrt{n} \rfloor ?$$

Comment:

Answer: 924

Solution: Knowing that $1600 = 40^2$ and $2500 = 50^2$, it is natural to guess and check around there to find that $44^2 = 1936$ and $45^2 = 2025$. Therefore for all $1996 \le i \le 2016$, $\lfloor \sqrt{i} \rfloor = 44$. Note that the given sum includes 21 terms from 1996 inclusive to 2016. Therefore the answer is $21 \cdot 44 = 924$.

`HS16:dis` 182. **Disc** Find the 2016th smallest positive integer that is a solution to $x^x \equiv x \pmod 5$.

Answer: 3360

Solution: First, note that it for primes p, the units of $\mathbb{Z}/p\mathbb{Z}$ are cyclic. For $p = 5$ we simply compute $2^2 \equiv -1 \pmod 5$ to see that 2 is a generator of the unit group of $\mathbb{Z}/5\mathbb{Z}$. Therefore every number not 0 is equal to 2^a for some $0 \le a < 5$. We show this relationship in the following table:

a	1	2	3	4
$2^a \pmod 5$	2	4	3	1

There are two cases:

(a) Suppose that we can write $x \equiv 2^a \pmod 5$ for some a. Then note that we have

$$(2^a)^x = 2^{ax} \equiv 2^a \pmod 5 \implies ax \equiv a \pmod 4,$$

since the order of $2 \pmod 5$ is 4.

If a is coprime with 4, we have $x \equiv 1 \pmod 4$, and hence either $2^a \equiv 2^1 \equiv 2 \pmod 5$ or $2^a \equiv 2^3 \equiv 3 \pmod 5$. Otherwise, if a is not coprime with 4, we have the following two cases:

$a = 0:$ Then $x \equiv 2^0 \equiv 1 \pmod 5$, and $x^x \equiv 1^x \equiv 1 \pmod 5$, and no other conditions are necessary.

$a = 2:$ Then $x \equiv 2^2 \equiv 4 \pmod 5$, and $2x \equiv 2 \pmod 4 \Leftrightarrow x \equiv 1 \pmod 2$.

(b) Suppose that we cannot write $x \equiv 2^a \pmod 5$ for some a. Then x is not a unit, and equivalently $x \equiv 0 \pmod 5$. This is sufficient as $x^x \equiv 0 \equiv x \pmod 5$.

Therefore we have the following 5 modular conditions: $x \equiv 1 \pmod 4$ and $x \equiv 2 \pmod 5$; $x \equiv 1 \pmod 4$ and $x \equiv 3 \pmod 5$; $x \equiv 1 \pmod 2$ and $x \equiv 4 \pmod 5$; $x \equiv 1 \pmod 5$; and finally, $x \equiv 0 \pmod 5$.

By the Chinese Remainder Theorem, there are $1+1+2+4+4 = 12$ unique solutions modulo $5 \cdot 4 = 20$. Note that $2016 = 12 \cdot 168$. Thus, we have to find the 168th positive integer which is $0 \pmod{20}$ to obtain the 2016th solution overall. Hence, our answer is $168 * 20 = \boxed{3360}$.

HS17 183. **Team** Let $a(n)$ be the number of paths from $(0,0)$ to $(2n,0)$ using steps of the form (k,k) and $(k,-k)$ where k is any positive integer such that the path does not go below the x-axis. What is $a(10)$?

Comment: HS: Really credits to Bill.

TODO(HS): make a grading curve/answer format.

Answer: 29324405

Solution: First let $f(a,b)$ be the number of paths from $(0,0)$ to (a,b) again only using paths of the form $(k,\pm k)$ for $k \in \mathbb{Z}^+$. Then examine the set of $A = \{(a-i, b+i) : i \in \mathbb{Z}^+, a-i \geq b+i\}$ and $B = \{(a-i, b-i) : i \in \mathbb{Z}^+, b-i \geq 0\}$. Then note that given $0 \leq b \leq a$,

$$f(a,b) = \sum_{a\in A} f(a) + \sum_{b \in B} f(b).$$

For example, $f(4,2) = f(3,3) + f(3,1) + f(2,0)$. Therefore we may create a recursive algorithm for $a(n) = f(n,n)$.

Claim: for any $i \in \mathbb{Z}^+$, $f(i,i) = 2^{i-1}$.

Proof. For base cases note that $f(0,0) = 1$ by definition (in the same way that combinatorially $\binom{n}{0} = 1$) and that $f(1,1) = 1$ since the only path from $(0,0)$ to $(1,1)$ is using the step $(1,1)$.

By induction suppose for all $i < k$ for some k suppose $f(i,i) = 2^{i-1}$. Then note that

$$f(k,k) = \sum_{i=0}^{k-1} f(i,i) = 1 + \sum_{i=1}^{k-1} f(i,i) = 1 + \sum_{i=0}^{k-2} 2^i = 1 + (2^{k-1} - 1) = 2^{k-1},$$

and by induction we are done. $\square$

Given this it is clear how we may algorithmically construct any $f(a,b)$. Example $f(a,b)$ are given below:

⋮								
6							32	
5						16		
4					8		48	
3				4		20		
2			2		8		45	
1		1		3		17		
0	1		1		5		29	
	0	1	2	3	4	5	6	...

In this was, we may compute $f(20,0) = a(10)$ is 90 additions to be 29324405.

HS18:tea 184. **Team** Note that $\phi(n)$ is defined to be the number of positive integers i such that $1 \leq i \leq n$ and $\gcd(i,n) = 1$. For example, $\phi(6) = 2$ and $\phi(7) = 6$. Find the number of positive integers x such that $\phi(x) = 2016$.

Comment: Can be estimation or actual prob (hard end for that b/c of computation).

Answer: 53

Solution: TODO(HS): write a sln. is easy

HS19:gut 185. **Alg** Eddy is traveling to England and needs to exchange USD to GBP (US dollars to British pounds). The current exchange rate is 1 GBP equals 1.3 USD. He exchanges x USD to GBP and while in England, uses $x/2$ GBP. When he returns, the value of the British pound has changed so that 1 GBP equals α USD. After exchanging all the GBP, he notes that he has $x/2$ USD left. What is α?

Comment:

Answer: $\mathbf{\frac{13}{7}}$.

Solution: Note that x USD $= \frac{x}{1.3}$GBP. Therefore at the end of her trip she has $\frac{x}{1.3} - \frac{x}{2} = \frac{.7}{2.6}x$ GBP. After exchanging she has $\frac{.7x}{2.6} \cdot \alpha = \frac{x}{2}$USD. Therefore we see that $\alpha = \frac{2.6}{1.4} = \frac{13}{7}$.

HS20:gut 186. **Alg** Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence. The partial sums $\{s_n\}_{n=1}^{\infty}$ are defined by $s_n = \sum_{i=1}^{n} a_i$. The Cesàro sums are then defined as $\{A_n\}_{n=1}^{\infty}$ where $A_n = \frac{1}{n} \cdot \sum_{i=1}^{n} s_i$. What is the limit of the Cesàro sums of the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = (-1)^{n+1}$ as n goes to infinity?

Comment:

Answer: $\mathbf{\frac{1}{2}}$.

Solution: Given that $a_n = (-1)^{n+1}$, the series is $1, -1, 1, -1, 1, -1, \ldots$. Clearly $s_n = 1$ if n is odd, and $s_n = 0$ if n is even. Therefore for even n, $A_n = \frac{1}{2}$ and for odd n, where $n = 2k+1$, $A_n = \frac{k}{2k+1}$. Clearly we have $\lim\limits_{n\to\infty} A_n = \frac{1}{2}$.

HS21 187. **Geo** A shape $X \subset \mathbb{R}^3$ (the 3-D coordinate plane) is convex if for any two points $a, b \in X$, for all $t \in [0,1]$, $ta + (1-t)b \in X$. The convex hull of a shape $X \subset \mathbb{R}^3$ is the smallest convex set S such that $X \subset S$. Consider the equilateral triangle A lying in the plane $z = 1$, centered at $(0,0,1)$, with one vertex at $(1,0,1)$, and the equilateral triangle B lying in the plane $z = -1$, centered at $(0,0,-1)$, with one vertex at $(0,1,-1)$. Let X be the convex hull of $\mathcal{A} \cup \mathcal{B}$. What is the volume of X?

Comment: Original question: A shape $X \subset \mathbb{R}^3$ (the 3-D coordinate plane) is convex if for any two points $a, b \in X$, for all $t \in [0,1]$, $ta + (1-t)b \in X$. The convex hull of a shape $X \subset \mathbb{R}^3$ is the smallest convex set S such that $X \subset S$. Consider the cube $\mathcal{A}$ with the set of vertices $\{(x,y,z) : x,y,z \in \{0,20\}\}$ and cube $\mathcal{A}$ with the set of vertices of $\{(x,y,z) : x,y,z \in \{0,-16\}\}$. Let X be the convex hull of $\mathcal{A} \cup \mathcal{B}$. What is the volume of X?

Original Answer: 23616

Original Solution: Note that the $\mathcal{A}$ is a dilation of the $\mathcal{B}$ with center $P = (-80,-80,-80)$, so the convex hull $\mathcal{X}$ consists of the space covered in the dilation $\mathcal{B} \to \mathcal{A}$.

Then, $\mathcal{X}$ consists of 3 truncated pyramids with vertex P and bases the sides of $\mathcal{A}$ and $\mathcal{B}$, as well as $\mathcal{B}$. This gives us a volume of:

$$V = 3 \cdot \frac{1}{3} \cdot (80 \cdot 20^2 - 64 \cdot 16^2) + 20^3 = 23616.$$

Answer: TODO(JH): find real answer

Solution: TODO(JH): find real solution

HS22:gut 188. **Geo** Bill is buying cans of soup. Cans come in two containers. Can A is a rectangular prism shaped can with dimensions $20 \times 16 \times 10$, and can B is a cylinder shaped can with radius 10 and height 10. Which can holds more volume, A or B?

Comment:

Answer: A

Solution: The volume of a rectangular box is simply the product of the dimensions. The volume of a cylinder is the area of the circles times the height. Thus A has volume $20 \cdot 16 \cdot 10 = 3200$, and B has volume $\pi \cdot 10^2 \cdot 10 = 1000\pi$. Since $\pi \sim 3.14$, note that $1000\pi \sim 3140$ and so can A has a greater volume.

HS23:gut 189. **Alg** What is the number of real solutions to the equation $x^2 + 20x + 16 = 0$?

Comment:

Answer: **2**

Solution: By the quadratic formula, the solutions to this equation are $\dfrac{-20 \pm \sqrt{20^2 - 4 \cdot 16}}{2}$. Since $20^2 - 4 \cdot 16 = 400 - 64 = 336$ is a positive number, the square root of the above exists, and there are two solutions.

HS24:gut 190. **Team** A lattice point is a coordinate pair (a, b) where both a, b are integers. What is the number of lattice points (x, y) that satisfy $\frac{x^2}{2016} + \frac{2y^2}{2016} < 1$ and $y \equiv 2x \pmod 7$?

Let C be the actual answer, A be the answer you submit, and $D = |A - C|$. Your score will be rounded up from $\max\left(0, 21 - e^{\frac{D}{100}}\right)$.

Comment:

Answer: **641**

Solution: We first consider the set of all regular lattice points inside the ellipse created by the graph, and then consider the subset of points that can further satisfy the modular condition.

Note that the ellipse has semi-radii $\sqrt{2016}$ and $\sqrt{1008}$. The area of an ellipse with radii a, b is $ab\pi$. Therefore this ellipse has area $\sqrt{2016} \cdot \sqrt{1008}\pi = 1008\pi\sqrt{2}$. Note that this is approximately $1008 \cdot 3.14 \cdot 1.41 = 4462.8192$.

Note that there is approximately 1 lattice point per unit square inside the ellipse (for example, take the correspondence between each unit square and the bottom left lattice point). Furthermore, our area approximation is slightly smaller than area since $3.14 < \pi$ and $1.41 < \sqrt{2}$. Lastly, since this estimation of 1 lattice per unit square over uses the amount of partial unit squares on the edges of the ellipse, the actual area should be more than the number of lattice points. Therefore this 4462.8192 should be a good approximation of the number of all regular lattice points inside the ellipse.

Last, the number of lattice points (x, y) that satisfy $y \equiv 2x \pmod 7$ is randomly, uniformly distributed among all lattice points, in as intuitive of the sense is necessary. Given any random $x \in \mathbb{Z}$, there is exactly $\frac{1}{7}$th chance that another random $y \in \mathbb{Z}$ satisfies $y \equiv 2x \pmod 7$. Therefore our answer should be $4463/7 \sim 638$.

The actual answer is 641.

HS25:gut 191. **Team** The Euclidean Algorithm on inputs a and b is a way to find the greatest common divisor $\gcd(a, b)$. Suppose WLOG that $a > b$. On each step of the Euclidean Algorithm, we do the division $a = bq + r$ for integers q, r such that $0 \leq r < b$, and repeat on b and r. Thus $\gcd(a, b) = \gcd(b, r)$, and we repeat. If $r = 0$, we are done. For example, $\gcd(100, 15) = \gcd(15, 10) = \gcd(10, 5) = 5$, because $100 = 15 \cdot 6 + 10$, $15 = 10 \cdot 1 + 5$, and $10 = 5 \cdot 2 + 0$. Thus the Euclidean Algorithm here takes 3 steps. What is the largest number of steps that the Euclidean Algorithm can take on some integer inputs a, b where $0 < a, b < 10^{2016}$?

Let C be the actual answer, A be the answer you submit, and $D = |A - C|$. if $\frac{D}{C} > \frac{1}{2}$, then your score will be 0. Else, your score will be rounded up from $20 - (D/10)^{\frac{1}{2.2}}$.

Comment:

Answer: 9644

Solution: We can show that the given such a bound, the solutions a and b that maximize the number of Euclidean Algorithm steps are when a and b are consecutive terms of the Fibonnaci sequence. This happens because we wish to maximize the number of successive divisions we must make. For example, if $a = bq + r$ is a step of the division, it is clearly optimal to make $q = 1$ to ensure that $r = a - bq$ is as large as possible. Therefore we see in each step of the Euclidean Algorithm that a, b are numbers such that $a = b \cdot 1 + r_1$ for some $0 \leq r_1 < b$. On the second step, we should have $b = r_1 \cdot 1 + r_2$, and $r_1 = r_2 \cdot 1 + r_3$, and so forth. Thus the remainder terms, as well as b and a, satisfy the recurrence relation of the Fibonacci sequence. Clearly it is optimal for the initial values of r_k, where r_k is the last non-zero remainder of the Euclidean Algorithm, is 1.

Note that for $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, that the Euclidean Algorithm on (F_n, F_{n-1}) takes $n - 2$ steps. Thus it suffices to find the largest n such that $F_n < 10^{2016}$, and our answer is $n - 2$.

Binet's formula is a closed formula for the Fibonacci sequence and is $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$. For large n, note that the latter term tends to 0. Let $\phi = \frac{1+\sqrt{5}}{2}$. Therefore we must solve

$$\begin{aligned}\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n &< 10^{2016}\\ \log_\phi(\phi^n) &< \log_\phi(\sqrt{5} \cdot 10^{2016})\\ n &< 2016 \log_\phi(10 \cdot 5^{1/4032})\\ &\sim 2016 \cdot \log_\phi(10)\end{aligned}$$

Thus we need to estimate $\log_\phi(10)$. One possible method is to approximate $\phi \sim 1.61$, and to proceed from there. Another method is to note that 987 is a Fibonnaci number close to 10^3 and use log manipulations.

The exact value of $2016 \cdot \log_\phi(10)$ is closer to 9646.5 and gives us the exact answer of 9644.

HS26:gut 192. **Team** Find the best rational approximation to $\sqrt[3]{2}$. That is, find $\frac{a}{b}$ where a, b are coprime integers that is as close to $\sqrt[3]{2}$ as possible.

Let C be the actual answer, A be the answer you submit, and $D = |A - C|$. We have a special constant α. Your score will be rounded up from $\frac{20}{1+e^{\frac{D-\alpha}{0.001}}}$. ($\alpha$ is a baseline approximation.)

Comment:

Answer: 1.2599210498948732

Solution: One estimation process that works is simply guess and check using decimals. Note that $1.2^3 = 1.728$ and $1.3^3 = 2.197$. Therefore a very rough guess may be that $\sqrt[3]{2} \sim 1.25 = \frac{5}{4}$. Note that $\frac{125}{64} < 2$, and therefore this approximation is slightly low. Therefore we can create further refinements with more guess and check to find more base 10 digits. As it is, $\sqrt[3]{2} \sim 1.2599210....$

There exist other estimation methods, such as continued fractions. Note that since $\frac{5}{4}$ is such an easy guess with minimal work, we let $\alpha = 1.25$ in our grading curve above.

JH01:geo 193. **Geo** Compute the radius of the sphere inscribed in the tetrahedron with coordinates $(2, 0, 0)$, $(4, 0, 0)$, $(0, 1, 0)$, $(0, 0, 3)$.

Comment: whee it's really ugly if you hero's bash it?

Answer: $\mathbf{\frac{3}{7}}$

Solution: The inradius can be found in a similar way to the 2D version. We know that the surface area $\cdot$ the radius $\cdot \frac{1}{3}$ = the volume.

The volume can be found by taking the tetrahedrons with base $(0, 0, 0)$, $(0, 1, 0)$, $(0, 0, 3)$, with heights 2 and 4, and subtracting the smaller one from the larger one. Then the volume is 2.

To find the surface area, the area of two triangles lying on the xy and xz plane can be found using the base along the x axis and the height as 1 and 3. To find the area of the remaining triangles with coordinates $(0, 0, a)$, $(0, b, 0)$, $(a+b, 0, 0)$, we can draw congruent triangle in 2D with coordinates $(0, 0)$, $(a + b, a)$, $(b, a + b)$. To calculate area, draw square with coordinates of a diagonal being $(0, 0)$ and $(a+b, a+b)$. Find area of square and subtract area of triangles. $\frac{a^2+ab+b^2}{2}$, with $a = 1, b = 2$ or $a = 1, b = 3$. Then the area is 14. (The area of these two triangles can also be found by taking half of the cross product between the two vectors).

Using the formula from above, you can find that the inradius is $\frac{3}{7}$.

JH02 194. **Geo** Let O be the center of a regular hexagon ABCDEF with side length 4. What is the volume of the square pyramid formed by cutting off quadrilateral ABCO and connecting sides AO and CO of the remaining hexagon AOCDEF?

Comment: Just getting my math juices flowing again. Probably won't use this problem. TODO come up with a better version of this problem, write the solution

Answer: $\mathbf{\frac{32\sqrt{2}}{3}}$

Solution: TODO

JH03 195. **Geo** Let circle O with radius 3 and cirlce P with radius 1 be externally tangent to each other at point A. Circles O and P are also both tangent to line l at points B and C respectively. Define circle Q as the cirlce passing through A, B, and C, and define circle R as the circle internally tangent to Q but externally tangent to circles O and P. What is the radius of circle R?

Comment: TODO(JH): Find nice numbers, solve, and write the solution.

Answer: $\boldsymbol{x}$

Solution: $1 + 1 = \boxed{2}$.

JH04 196. **Geo** Triangle ABC has side lengths AB=6, BC=8, and AC=10. Let triangle $A'B'C'$ be congruent to ABC but 180 degrees and oriented such that the midpoints of the hypotenuses of each triangle lies on the right angle of the other triangle. Let square $ACDE$ (centered at O) lie outside the triangles such that it shares a side with triangle ABC. What is the area of triangle $A'OC'$?

Comment: TODO(JH): Include a picture. Write solution better.

Answer: **49**

Solution: We know the area of triangle $A'B'C'$ is $\frac{1}{2} \cdot 6 \cdot 8$. If we draw OB', we form two triangles with base OB' and heights that add up to AC. So the sum of the area of the two triangles is $\frac{1}{2} \cdot 10 \cdot 5$. So, the total area $= \frac{1}{2} \cdot 6 \cdot 8 + \frac{1}{2} \cdot 5 \cdot 10 = 49$.

Another solution - shoelace (-3,3), (4,-4), (7,7) for 49.

JH05 197. **Geo** I am filling my water bottle, which is a cylinder of radius 2 and height 9, at a water fountain by tilting it so that the water coming out of the fountain is falling into the water bottle. If the maximum height the arch of the water ever reaches is 6 (measuring from the base of the water fountain on which my water bottle is resting), what is the maximum height I can fill my water bottle to when I orient my water bottle straight up?

Comment: Should be a fairly easy similar triangles problem?

TODO(JH): Include a picture. Word problem better. Write solution better.

Answer: $\mathbf{9 - \sqrt{5}}$

Solution: Drawing the side view, the cylinder becomes a rectangle slanted to catch the water. Let us label the rectangle $ABCD$, with B as the bottom most point of the cylinder. We know that $AB = 4$ and $BC = 9$. If we draw a horizontal line through C, we get the maximum amount of water we can put in it - let C hit AD at E. If we draw a horizontal line through A that hits BC at F, then we see that AFB is congruent to CED, both of which are similar to CBG, where G is the point when we drop C to the base of the water fountain. Then, we can see that $BF = ED = 2\sqrt{5}$. By symmetry, we see that the height when the water bottle is straight up is $9 - \frac{1}{2} \cdot ED = 9 - \sqrt{5}$.

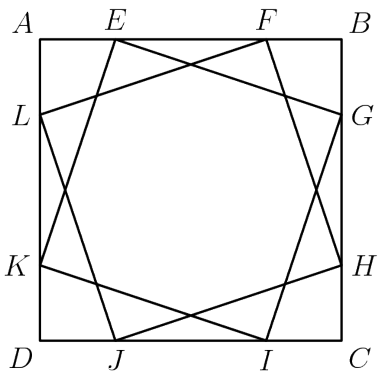
JH06 198. **Geo** In regular hexagon ABCDEF, if we draw AC, BD, CE, DF, EA, and FB, we have a smaller regular hexagon. What is the ratio of the areas of the smaller hexagon to the larger hexagon ABCDEF? Express your answer as a common fraction.

Comment: Should be a fairly easy similar triangles problem? For harder version, use a pentagon instead with ans $\frac{7-3\sqrt{5}}{2}$. JK, apparently was on ASMT team round RIP.

Answer: $\mathbf{\frac{1}{3}}$

Solution: Original solution: We see that the triangles formed by a vertice on the larger hexagon and two vertices of the smaller hexagon closest to it are equilateral. Then each of the drawn diagonals are split into 3 congruent parts by the other diagonals. We can find the length of the diagonal using triangle ABC and splitting it into two 30-60-90 triangles. Doing so, we find that the side length ratios is $\frac{\sqrt{3}}{3} : 1$ and thus the ratios of the areas is $\frac{1}{3} : 1 = \frac{1}{3}$

199. **Geo** In the diagram to the right, square $ABCD$ has side length of 4. Two congruent squares are drawn such that $AE = FB = BG = HC = CI = JD = DK = LA = 1$ and $EF = GH = IJ = KL = 2$. Compute the area of the region that lies in both square $EGIK$ and $FHJL$.



JH07:geo

Comment: Medium to hard difficulty

Answer: $\mathbf{\frac{25}{3}}$

Solution: Let us label the remaining points as follows: M is the intersection of LJ and KI, N is the intersection of KI and JH, and O is the midpoint of CD. The total area of $ABCD$ is 16. We can see that the area of the desired region is $[ABCD] - 8[DMI] + 4[JIN]$. The area of $[DMI] = \frac{1}{2} \cdot 3 \cdot \frac{3}{4} = \frac{9}{8}$, where the height $\frac{3}{4}$ can be found by finding intersection of $y = x$ and $y = -\frac{1}{3}x + 1$. The area of $[EFN]$ is $\frac{1}{2} \cdot 2 \cdot \frac{1}{3} = \frac{1}{3}$. We can find the height by noting triangles INO and IKD are similar in a 1:3 ratio. Then, the area of the desired region is $16 - 9 + \frac{4}{3} = \frac{25}{3}$.

JH08 200. **Geo** An ant is at point A of square $ABCD$ of side length $2\sqrt{2}$ trying to get to C. However, there is a circle of radius 1 sharing a center with $ABCD$ that the ant cannot step in. What is the minimum he has to travel to get to C?

Comment: TODO(JH) eh wording struggz, reword

Answer: $\mathbf{2\sqrt{3}+\frac{\pi}{3}}$

Solution: Letting the center be O, we draw the tangent from A to the circle such that the tangent point is E. We can find the length of $AE=\sqrt{3}$ because AOE is a 30-60-90 triangle. We can see that the measure of the arc the ant walks over is 60 degrees. The remaining part the ant has to walk is also $\sqrt{3}$, so the total is $\sqrt{3}+\frac{\pi}{3}+\sqrt{3}$.

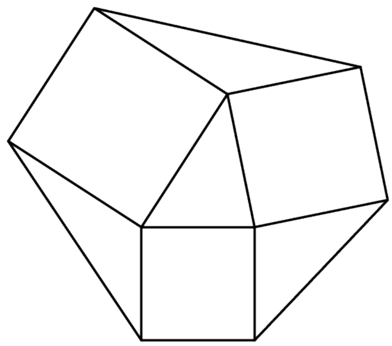
JH09 201. **Geo** In a right triangle, the product of the two legs is 108. The square of the sum of the two legs is 441. What is the length of the hypotenuse?

Comment:

Answer: 15

Solution: Given $ab=108$ and $(a+b)^2=441$, we can see that $c^2=a^2+b^2=441-2*108=225$, so $c=15$. Also, a 9-12-15 triangle.

JH10:gut 202. **Geo** In the following diagram, squares are drawn on the sides of the triangle with side lengths 5, 6, and 7 as shown below. The corners of adjacent squares are then connected. What is the area of the resulting hexagon?



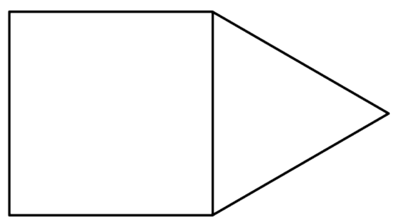
Comment:

Answer: $\mathbf{110+24\sqrt{6}}$

Solution: We may use Heron's formula or the law of cosines to find area of 5-6-7 triangle to be $6\sqrt{6}$. For all of the other triangles, we use the law of sines to calculate their areas using the angle adjacent to the inner most triangle. For example, suppose the angle of the inner triangle in-between the sides of lengths 5 and 6 is α. Then consider the triangle between the squares of lengths 5 and 6. It has angle $180-\alpha$ between the squares. Since $\sin(\alpha)=\sin(180-\alpha)$, the law of sines shows that these triangles have the same area (The area of a triangle is $\frac{1}{2}ab\sin C$ where a and $b$are two side lengths and C is the angle in-between them.). This is true of all the triangles, and therefore they all have the same area of $6\sqrt{6}$.

The area of squares is $25+36+49=110$, and therefore the total area is $110+4\cdot 6\sqrt{3}=110+24\sqrt{3}$.

203. **Geo** An equilateral triangle shares a side with a square. If the resulting pentagon has a perimeter of 20, what is the area of the pentagon?



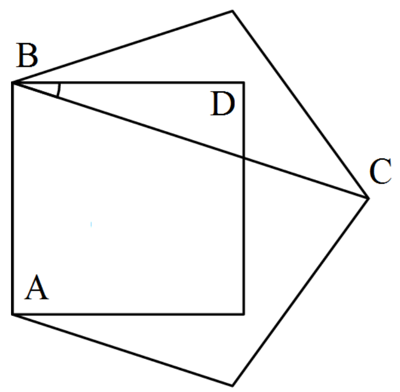
JH11

Comment:

Answer: $\mathbf{16 + 4\sqrt{3}}$

Solution: Side length of everything is 4. The area of the square is 16. The area of the triangle is $\frac{s^2\sqrt{3}}{4} = 4\sqrt{3}$. So the total area is $16 + 4\sqrt{3}$.

JH12:gut 204. **Geo** In the following diagram, the square and pentagon are both regular and share the side as designated. What is the measure of $\angle CBD$?



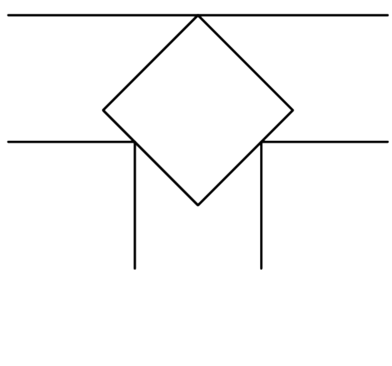
Comment:

Answer: **18**

Solution: For any regular polygon of n sides, the angle of any vertex is $180 \cdot \frac{n-2}{n}$. To see this, note that the sum of all of the angles of a triangle is 180. For any regular n-gon, we may pick a specific vertex and draw diagonals to all the other vertices to split the n-gon into $n-2$ triangles. Thus the total sum of all the angles of the n-gon is $180(n-2)$. Therefore the angle of any of the n-vertices of the n-gon is exactly $180 \cdot \frac{n-2}{n}$.

Now in the isosceles triangle created inside the pentagon, note that the large angle is therefore $180 \cdot \frac{3}{5} = 108$, and so the smaller angles are $\frac{180-108}{2} = 36$. Thus $\angle ABC = 108 - 36 = 72$. Lastly, $\angle CBD = 90 - \angle ABC = 90 - 72 = 18$.

JH13 205. **Geo** A cone with a square base is placed on the intersection of two equally spaced lines as shown below. What is the ratio of the side length of the cone to the width of the lines?



Comment:

Answer: $\mathbf{\frac{3\sqrt{2}}{4}}$

Solution: If we let the width of the line be 1, we can see that the diagonal of the cone's base is $1 + \frac{1}{2}$. Then, the side length of the square is $\frac{3}{2\sqrt{2}}$, which simplifies to $\frac{3\sqrt{2}}{4}$.

JH14 206. **Alg** The three roots of the polynomial $f(x) = x^4 + ax^3 + bx + c$ are -1, 3, and 5. What is the sum of $a + b + c$?

Comment: Inspired / quite similar to HMMT 1998 Alg 7.

A funny alternate to this problem is finding $a+b-c$, where you plug in $f(-1) = 1-a-b+c = 1-(a+b-c) = 0$. It is 0 because -1 is a root, and so the answer is 1.

Answer: 31

Solution: We see that the coefficient of the x^2 term is 0. Then we see that if the fourth root (a double root) is x, we can express this term as $-1\cdot3+-1\cdot5+3\cdot5+-x+3x+5x = 7+7x = 0$. So $x = -1$. To find $a + b + c$, we can plug in 1 into the polynomial: $f(1) = (x+1)^2(x-3)(x-5) = 2^2 \cdot -2 \cdot -4 = 32 = 1 + a + b + c$. Thus, the sum $a + b + c = 32 - 1 = 31$.

JH15:geo 207. **Geo** Compute the sine of the smaller angle between the diagonals AC and BD of the cyclic quadrilateral $ABCD$ with side lengths $AB = 14$, $BC = 19$, $CD = 26$, $DA = 29$.

Comment:

Answer: $\mathbf{\frac{60}{61}}$

Solution:

The sine is actually the same for all 4 angles formed by the two diagonals. Then, letting the 4 segments of the diagonals defined by the intersection of the diagonals and each of the vertices of the quadrilateral have lengths w, x, y, z (in that order), we see that the total area of the quadrilateral is $\frac{1}{2}\cdot\sin(\theta)(wx+xy+yz+wz) = \frac{1}{2}\cdot\sin(\theta)(w+y)(x+z) = \frac{1}{2}\cdot\sin(\theta)\cdot D_1\cdot D_2$, where D_1 and D_2 are the diagonals of the cyclic quadrilateral. We can use Ptolemy's to find this product $D_1 \cdot D_2 = AB\cdot CD + BC\cdot DA = 14\cdot26 + 19\cdot29 = 915$. We can also find the area of the quadrilateral using Brahmagupta to be $\sqrt{(s-14)(s-19)(s-26)(s-29)} = \sqrt{30\cdot25\cdot18\cdot15} = 450$, where s is the semiperimeter. Then the sine is $\frac{60}{61}$.

JH16:geo 208. **Geo** Let O be a circle with chord AB. Let C be a randomly chosen point inside the circle, and G be the centroid of triangle ABC. What is the ratio of the area formed by all the possible positions for G to the area of circle O?

Comment: Better way to explain locus would be good.

Answer: $\mathbf{\frac{1}{9}}$

Solution: Let M be the midpoint of AB. Then we know that the ratio of MG to MC is $\frac{1}{3}$ because G is a centroid. Since C can be anywhere on the circle O, then we know that the possible positions G can be is the circle formed by shrinking O down in a $\frac{1}{3}$ ratio around M. So the ratio of the areas is $\frac{1}{9}$.

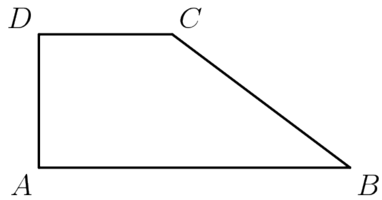
JH17:geo 209. **Geo** The side lengths of triangle ABC are 13, 14, and 15. Let I be the incenter of the circle. Compute the product of $AI \cdot BI \cdot CI$.

Comment: Also nice round numbers for side lengths 51, 52, 53, with answer as 27030.

Answer: 520

Solution: We can find the radius of the incircle - it's the area of the triangle divided by the semiperimeter. The area can be found using Hero's formula to be 84. The semiperimeter is $\frac{1}{2}\cdot(13+14+15) = 21$. Then the inradius is $\frac{84}{21} = 4$. We find that the lengths from the points of tangency to the vertices are 6, 7, 8 by noting that the two segments from a vertex are congruent in length. We can thenfind AI, BI, CI using pythagorean theorem, and their product is $\sqrt{65}\cdot\sqrt{52}\cdot\sqrt{80} = 520$.

210. **Geo** Compute the area of the trapezoid $ABCD$ with right angles BAD and ADC and side lengths of $AB = 7$, $BC = 5$, and $CD = 3$.



JH18:geo

Answer: 15

Solution: Drawing the perpendicular from C to AB with the foot as E, we see that $BE = 4$, and thus, $CE = AD = 3$. Then the area of the trapezoid is $\frac{1}{2} \cdot 3 \cdot (3 + 7) = 15$.

JL01 211. **Disc** How many ways are there to assign 5 jobs to 3 employees so that each employee is assigned at least one job?

Comment: .

Answer: 150

Solution: We have, by inclusion-exclusion, $3^5 - 3 \cdot 2^5 + 3 \cdot 1^5 = 150$ ways.

LJ01 212. **Gen** Arthur has a 30 cm paintbrush, which he grips by the very end of its handle at a point 24 cm away from a large piece of paper. Keeping his hand and the end of the paintbrush fixed, he sweeps the brush around and it makes paint marks in all possible places it can touch the paper. What is the area in cm^2 of the figure that Arthur paints?

Answer: 324π

Solution: The set of all points a fixed distance away from a given point that lie on given a plane is a circle, with radius that is one leg of a right triangle with hypotenuse being the distance away from the point in space, and other leg the distance from point to plane.

The hypotenuse here is 30 cm and one leg is 24 cm, thus the other leg is $\sqrt{30^2 - 24^2} = 18$ cm. Then the area of the circle is $A = \pi r^2 = \pi 18^2 = \boxed{324\pi}$ cm^2.

LJ02 213. **Team** Piotr has a very thin metal pole that is 2 meters long. Its density varies along its length according to the function $f(x) = (1 + \sqrt{16 - 4x^2})$ kg/m, if we consider $x = 0$ at one end and $x = 2$ at the other. Compute the total weight (in kg) of Piotr's metal pole.

Comment: This is a "calculus" problem that involves no calculus. I'm not sure if it belongs on general or elsewhere. Maybe team?

Nick: I protest this problem. I don't believe that it is well-known whether the area of an ellipse is πab without calculus.

Lennart: oh well, oops. I thought πab was relatively trivial given the intuition that an ellipse is a "stretched" circle.

Answer: $2\pi + 2$

Solution: The total weight is $\int_0^2 f(x)\mathrm{d}x$, which can be computed by finding the area under the curve using geometry. The area is the sum of a 1×2 rectangle and a quarter of an ellipse with semi-major axis 4 and semi-minor axis 2 (this can be seen easily from a plot of the function). The rectangle area is 2, the full ellipse has area $\pi ab = 4 \cdot 2\pi = 8\pi$, so one quarter of the ellipse is 2π, making the total $\boxed{2\pi + 2}$ kg.

LJ03 214. **Gen** How many ways are there to draw a path between all dots in a 3×3 grid exactly once, where a path from dot to dot may only go horizontally or vertically?

Comment: Idea from Herman Chau, after staring at a Shinteki puzzle

Nick: Has anyone raised concerns about clarity on this question? In particular, I feel like there's an unwritten "the path cannot cross itself" here.

Answer: 20

Solution: There are three basic shapes for such paths, one that looks like an S, one that looks like a 6, and one that looks like a G (with a spur). The S pattern has 4 symmetries (counting rotation and reflection), and the 6 and G patterns have 8 symmetries. The answer is $4 + 8 + 8 = \boxed{20}$.

LJ04 215. **Gen** Find the sum of all the integers from 3 to 10 inclusive.

Comment: Suitable for General 1.

Answer: 52

Solution: Just add them: $3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \boxed{52}$.

Alternatively, you may have known that $1 + 2 + \cdots + 9 + 10 = 55$. $3 + 4 + \cdots + 9 + 10 = 55 - (1 + 2) = \boxed{52}$.

LJ05 216. **Gen** There are 3 teaspoons in a tablespoon, there are 8 fluid ounces in a cup, and there are 0.5 fluid ounces in a tablespoon. How many teaspoons are there in a cup?

Comment: Suitable for General 3 or 4.

Answer: 48

Solution:

$$\frac{3 \text{ teaspoons}}{1 \text{ tablespoon}} \cdot \frac{1 \text{ tablespoon}}{0.5 \text{ fluid ounces}} \cdot \frac{8 \text{ fluid ounces}}{1 \text{ cup}} = \frac{\boxed{48} \text{ teaspoons}}{1 \text{ cup}}.$$

MA01:gut 217. **Team** If $(\underline{ABC})^2 = \underline{CCAAC}$, where A, B, and C are distinct nonzero digits, then find the three digit number $\underline{ABC}$. We use the notation $\underline{ABC}$ to mean the number comprised of the digits A, B, and C with $0 \leq A, B, C \leq 9$. Thus $\underline{ABC} = 100A + 10B + C$ and $\underline{CCAAC} = 10000C + 1000C + 100A + 10A + C$.

Answer: 235

Solution: Since $\underline{ABC}$ and $(\underline{ABC})^2$ end in the same digit C, we must have $C \in \{0, 1, 5, 6\}$. But C is nonzero, and if $C = 1$, that would imply $(\underline{AB1})^2 = \underline{11AA1} < 200^2$ or $A = 1$, which is not allowed since A and C must be distinct. Thus $C = 5$ or $C = 6$. In both cases, we have $200^2 < \underline{CCAAC} < 300^2$, so $A = 2$. Therefore, either $(\underline{2B5})^2 = 55225$ or $(\underline{2B6})^2 = 66226$.

If $(\underline{2B6})^2 = 66226$, then since $250^2 < 66226 < 260^2$, we have $250 < \underline{2B6} < 260$, or $B = 5$. But $256^2 = 65536 \neq 66226$, which is a contradiction.

Thus $(\underline{2B5})^2 = 55225$, and since $230^2 < 55225 < 240^2$, we have $230 < \underline{2B5} < 240$, or $B = 3$. Indeed, $235^2 = 55225$, so $\underline{ABC} = \boxed{235}$.

MB01 218. **Gen** A polygon has exactly 90 diagonals. How many sides does the polygon have?

Answer: 15

Solution: The number of diagonals of a polygon with n sides is $\frac{n(n-3)}{2}$, since there are n choices for the first vertex and $n - 3$ choices for the second vertex, which cannot be adjacent to the first. Note that we must divide by 2, since the order of the vertices does not matter. Solving $\frac{n(n-3)}{2} = 90$ for n, we have that $n = \boxed{15}$.

NB01 219. **Disc** In computer science, a "queue" is a data type that can be thought of as an ordered set of elements, and at any point, an element can be enqueued (added to the end of the queue as the last element), or dequeued (removed from the beginning of the queue). For example, we can start out with the following queue of integers: $\{1, 2\}$. Enqueueing 3 into this queue would produce the queue $\{1, 2, 3\}$, and dequeueing this queue would produce the queue $\{2, 3\}$. A "stack" is a data type that can be thought of as an ordered set of elements, and at any point, an element can be pushed (added to the end of the stack as the last element), or popped (removed from the end of the stack). For example, we can start out with the following stack of integers: $\{1, 2\}$. Pushing 3 into this stack would produce the stack $\{1, 2, 3\}$, and popping this stack would produce the stack $\{1, 2\}$.

Let A be a queue of integers initialized as $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ in that order, and let B and C be empty stacks. Suppose we have a computer program that randomly does one of the following two things: a) if A is not empty, it dequeues an integer from A and pushes that integer into B, or b) if B is not empty, it pops an integer from B and pushes that integer into C. Our objective is to run this program successively until it transfers all the elements of A to C. Given that at time T, 9 is in C, after time T, in how many possible ways can integers be popped into C by running the program successively until we reach our objective?

Answer: 1536

Solution: Since 9 has already been popped into C, 9 must have been popped into B, meaning $1, 2, ..., 8$ have all been popped into B by time T. 10 may or may not have been popped into B by time T. Hence, at time T, B contains some subset of $\{1, 2, ..., 8\}$ in increasing order, and may contain 10 at the end. Given that B contains n integers out of $\{1, 2, ..., 8\}$, there are $n + 1$ possible intervals in which 10 could be popped into C after time T, or 10 might have already been popped into C by time T. Thus this gives $n + 2$ possible ways 10 could be popped into C after time T. Hence the total number of ways integers could be popped from B to C after time T is:

$$\sum_{n=0}^{8} \binom{8}{k} (n + 2) = \boxed{1536}$$

NW01 220. **Gen** You have 17 apples and 7 friends, and you want to distribute apples to your friends. The only requirement is that Steven, one of your friends, does not receive more than half of the apples. Given that apples are indistinguishable and friends are distinguishable, compute the number of ways the apples can be distributed.

Comment: This problem was NW30 in the 2013 repo.

Answer: 97944.

Solution: Note that without the restriction that Steven cannot receive more than half of the apples, there are $\binom{23}{6}$ ways to distribute apples. There are $\binom{23-9}{6} = \binom{14}{6}$ ways to distribute apples given that Steven does receive more than half of the apples. Therefore, there are $\binom{23}{6} - \binom{14}{6} = \boxed{97944}$ ways to distribute apples such that Steven does not receive more than half of the apples.

NW02:gut 221. **Disc** A 2016-digit number is *dominant* if the first 1008 digits are in non-decreasing order, the last 1008 digits are in non-increasing order, and there are two consecutive 9's somewhere in the number. Compute how many 2016-digit numbers are dominant.

Comment: Nick: This answer is somewhat unwieldy. I'm not sure there's an easier way to express this though. If we don't like this answer, we can probably ask for some sort of signature of the answer.

Answer: $\mathbf{2 \cdot \binom{1016}{1008} + \binom{1016}{1007}^2}$

Solution: There are three cases.

(a) The last 9 is the 1008th digit of the number, counting from the left.

(b) The first 9 is the 1009th digit of the number, counting from the left.

(c) Both the 1008th and 1009th digits are 9.

The first two cases are equivalent - we can reverse the number to obtain a bijection between the two cases. We will now count the number of dominant numbers in both of these cases.

Let $f(n,d)$ be the number of d-digit numbers which are strictly decreasing and start with n, possibly with leading zeroes. We claim that $f(n,d) = \binom{n+d-1}{d-1}$. Note that $f(n,d) = \binom{n+d-1}{n}$ because we can convert any such number into a sequence of nonnegative numbers that sums to at most d by taking deltas between digits - the number of such sequences is $\binom{n+d-1}{n}$ by stars and bars. As a result, the number of dominant numbers that satisfy the first case is $\sum_{i=0}^{8} \binom{1007+i}{1007} = \binom{1016}{1008}$. This is the same for the second case by our bijection.

In the third case, the number of dominant numbers is $\left(\sum_{i=0}^{9} \binom{i+1006}{i}\right)^2 = \binom{1016}{1007}^2$.

This gives us an answer of $\boxed{2 \cdot \binom{1016}{1008} + \binom{1016}{1007}^2}$.

NW03 222. **Disc** Cathy owns a fair 2016-sided die. Each side contains a unique integer from 1 to 2016. Cathy rolls the die twice. Compute the probability that the sum of the two die rolls is even.

Answer: $\mathbf{\frac{1}{2}}$

Solution: The probability that the sum of the two die rolls is even is equal to the probability that both die rolls is odd, which is $\frac{1}{4}$, added to the probability that both die rolls is even, which is $\frac{1}{4}$. The answer is therefore $\boxed{\frac{1}{2}}$.

NW04 223. **Geo** A right triangle with area 1 is inscribed inside a circle with radius 2016. Compute the length of the hypotenuse of the right triangle.

Answer: 4032

Solution: Any right triangle inscribed in a circle has a hypotenuse which is also the diameter of the circle, so the length is $2 \cdot 2016 = \boxed{4032}$.

NW05 224. **Team** We say that a number is *ascending* if its digits are, from left-to-right, in nondecreasing order. We say that a number is *descending* if the digits are, from left-to-right, in nonincreasing order. Let a_n be the number of n-digit positive integers which are ascending, and b_n be the number of n-digit positive integers which are descending. Compute the ordered pair (x,y) such that $\lim_{n\to\infty} \frac{b_n}{a_n} - xn - y = 0$.

Answer: $\left(\mathbf{\frac{1}{9}}, \mathbf{1}\right)$

Solution: We claim that $b_n = \binom{n+9}{n} - 1$ and $a_n = \binom{n+8}{n}$. The answer follows.

NW06:gut 225. **Disc** A number n is *almost prime* if any of $n-2$, $n-1$, n, $n+1$, or $n+2$ is prime. Compute the smallest positive integer that is not *almost prime*.

Answer: 26

Solution: We need to find 5 consecutive numbers that are not prime. The smallest such 5 are 24 through 28, making 26 the answer.

NW07 226. **Disc** 2016 is divisible by 9. Compute the number of four-digit positive integers divisible by 9.

Answer: 1000

Solution 1: There are 1111 multiples of 9 less than or equal to 9999. There are 111 multiples of 9 less than or equal to 999. This leaves $\boxed{1000}$ four-digit positive integers divisible by 9.

Solution 2: There are 10 ways to pick each of the last three digits of the number, or 1000 different combinations. For each such combination, there is exactly one digit from 1 to 9 that can force the sum of all four digits to be divisible by 9. Therefore, there are $\boxed{1000}$ four-digit positive integers divisible by 9.

NW08:alg 227. **Alg** Given positive numbers x and y where $x + y = \frac{1}{x} + \frac{1}{y} = 5$, compute $x^2 + y^2$.

Answer: 23

Solution: Note that $\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy}$, so therefore $xy = 1$. Note that $x^2+y^2 = (x+y)^2-2xy = 5^2 - 2 = \boxed{23}$.

NW09 228. **Alg** Alice and Bob are painting a house. Alice can paint a house in 20 hours by herself. Bob can paint a house in 40 hours by himself. If both people paint at a constant rate and work together to paint one house, what fraction of the house was painted by Alice?

Answer: $\mathbf{\frac{2}{3}}$

Solution: Alice paints twice as quickly as Bob. Therefore, if Alice paints x of the house, Bob paints $\frac{x}{2}$ of the house. We have that $x + \frac{x}{2} = 1$, so $\boxed{x = \dfrac{2}{3}}$.

RC01 229. **Geo** Let there by a circle with R and 3 circles with centers I, C, E that intersect R orthogonally. The radius of circle I, $r_I = 7$. Given that $IR = 25$ $CR = 26$, and $ER = 30$, find $r_I \cdot r_C \cdot r_E$, the product of the radii of circles I, C, and E.

Comment: Aaron Lin: Meh

Answer: 1260

Solution: Solution: Let X be one of the points of intersection between circles I and R. Because these circles are orthogonal, the triangle formed from the center of the two circles and X is a right triangle. Therefore, the radius of circle R equals $\sqrt{25^2 - 7^2} = 24$. Using the same argument, we can find the radii of the other two circles, $r_C = \sqrt{26^2 - 24^2} = 10$ and $r_E = \sqrt{30^2 - 24^2} = 18$, and hence $r_I \cdot r_C \cdot r_E = \boxed{1260}$.

RC02 230. **Calc** Let $L(f(x))$ and $A(f(x))$ denote the arc-length and area, respectively, of $f(x)$ from $x = a$ to $x = b$. Find $\frac{L(f(x))}{A(f(x))}$ for the hyperbolic cosine function $f(x) = \frac{e^x+e^{-x}}{2}$.

Answer: 1

Solution: Since $f(x) > 0$ for all x,

$$\begin{aligned} A(f(x)) &= \int_a^b \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2}\, dx \\ &= \int_a^b \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2}\, dx \\ &= \int_a^b \frac{e^x - e^{-x}}{2}\, dx \\ &= L(f(x)). \end{aligned}$$

Hence $\frac{L(f(x))}{A(f(x))} = \boxed{1}$.

RC03:dis 231. **Disc** Define $\varphi_n(x)$ to be the number of integers y less than or equal to n such that $\gcd(x,y) = 1$. Also, define $m = \text{lcm}(2016, 6102)$. Compute

$$\frac{\varphi_{m^m}(2016)}{\varphi_{m^m}(6102)}.$$

Comment: RC: As is, this is pretty easy to guess. Could make it harder by changing 4102 to a number with a repeated prime factor, or by changing to two numbers that don't have the same number of distinct prime factors. Also, I think the limit will converge if we just take it in n without multiplying by the lcm, but I'm not sure. At the very least, I think we need to pick a smaller number than 4012.

Eric: This problem changes quite a bit when updated to 2016. Does anyone want to update it?

Harrison: Updated to 2016. It's actually a bit cleaner now! Yay!

Answer: $\mathbf{\frac{339}{392}}$

Solution: We first need to determine some properties of $\varphi_n(x)$. For simplicity, we assume that n is a multiple of x, which we can safely assume for this problem since m is a common multiple of 2016 and 6102. Observe that for x prime, we have $\varphi_n(x) = n - \frac{n}{x} = n(1 - \frac{1}{x})$, since there are exactly $\frac{n}{x}$ values y such that $\gcd(x,y) = x \neq 1$. Next, observe that for any integer $k \geq 1$, we have

$$\varphi_n(x) = \varphi_n\left(x^k\right) = n\left(1 - \frac{1}{x}\right),$$

since there similarly are exactly $\frac{n}{x}$ values y such that $\gcd(x,y) \neq 1$.

Finally, suppose that $x = ab$ where a and b are distinct primes. Then $\frac{1}{a}$ of the n numbers will be divisible by a, $\frac{1}{b}$ of the n numbers will be divisible by b, and $\frac{1}{ab}$ of the n numbers will be divisible by ab. By inclusion-exclusion, the number of positive integers $y \leq n$ such that $\gcd(x,y) = 1$ is

$$n - \frac{n}{a} - \frac{n}{b} + \frac{n}{ab} = n\left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right).$$

This intuitively means that of the $n(1 - \frac{1}{a})$ integers y such that $\gcd(a,y) = 1$, only $1 - \frac{1}{b}$ of these integers m will also satisfy $\gcd(b,y) = 1$. Through some work, we can generalize that if

$$x = p_1^{e_1} p_2^{e_2} ... p_q^{e_q}$$

for some distinct primes $p_1, ..., p_q$, positive integers $e_1, ..., e_q$, and n that is a multiple of x, then

$$\varphi_n(x) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_q}\right).$$

(Proving this is left as an exercise to the reader. Induction will be useful.)

We now apply this result to our equation. First, we note that $2016 = 2^5 \cdot 3^2 \cdot 7$ and $6102 = 2 \cdot 3^3 \cdot 113$. With this in mind, we expand the fraction and find

$$\begin{aligned}
\frac{\varphi_{m^m}(2016)}{\varphi_{m^m}(6102)} &= \frac{m^m\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{7}\right)}{m^m\left(1 - \frac{1}{2}\right)\left(\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{113}\right)\right.} \\
&= \frac{m^m}{m^m} \cdot \frac{6/7}{112/113} \\
&= \frac{3 \cdot 113}{7 \cdot 56} = \boxed{\frac{339}{392}}.
\end{aligned}$$

RC04 232. **Calc** Compute $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

Answer: $\boldsymbol{\pi/4}$

Solution: Noting that the series equals $1 - \frac{1}{3} + \frac{1}{5}... = x - \frac{x^3}{3} + \frac{x^5}{5}...$ when $x = 1$, we differentiate the right hand side to obtain the known series $1 - x^2 + x^4... = \frac{1}{1+x^2}$. Integrating, $x - \frac{x^3}{3} + \frac{x^5}{5}... = \arctan(x) + C$ and setting $x = 0$ shows that $C = 0$. Setting $x = 1$ yields the answer $1 - \frac{1}{3} + \frac{1}{5}... = \arctan(1) = \boxed{\pi/4}$.

RC05 233. **Gen** At $t = 0$ Tal starts walking in a line from the origin. He continues to walk forever at a rate of $1\frac{m}{s}$ and lives happily ever after. But Michael (who is Tal's biggest fan) can't bear to say goodbye. At $t = 10s$ he starts running on Tal's path at a rate of n such that $n > 1\frac{m}{s}$. Michael runs to Tal, gives him a high-five, runs back to the origin, and repeats the process forever. Assuming that the high-fives occur at time $t_0, t_1, t_2...$, compute the limiting value of $\frac{t_z}{t_{z-1}}$ as $z \to \infty$.

Comment: Eric: I'm concerned about putting limits in the general test. Maybe move to an easy calculus problem?

Answer: $\boldsymbol{\frac{n+1}{n-1}}$

Solution: Suppose the two meet at $t_j = x$ a distance x from the origin. Michael would need to run to the origin and back to the same spot, which would take him $2x/n$ time, by which point Tal would move further away by this amount of distance. To reach Tal's new location would require Michael $2x/n^2$ time, and the process repeats until they meet. Thus, it takes Michael a total of $\frac{2x}{n} + \frac{2x}{n^2} + ... = \frac{2x}{n-1}$ time to reach Tal again. Hence $t_{j+1} = x + \frac{2x}{n-1} = \frac{n+1}{n-1}x$ and $\frac{t_z}{t_{z-1}} = \boxed{\dfrac{n+1}{n-1}}$ is constant for all z.

RC06 234. **Calc** Given an ellipse with major axis a and minor axis b, construct the rhombus that completely encompasses the ellipse with the minimum possible perimeter. Find the area outside of the ellipse but within the rhombus.

Answer: $\boldsymbol{2(a+b)\sqrt{ab} - \pi ab}$

Solution: We write the ellipse as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Using symmetry, we can restrict ourselves to the first quadrant and finding the line segment of minimum length such that it is tangent to the ellipse and has endpoints along the coordinate axes. The two points of the rhombus along the axes are written as $(0, t)$ and $(s, 0)$ and the tangent point will be (x, y). Using the normal vector $(\frac{2x}{a^2}, \frac{2y}{b^2})$ and the tangent vector from $(s, 0)$ to (x, y), $(x - s, y) \cdot (\frac{x}{a^2}, \frac{y}{b^2}) = 0$. Solving for s (and then interchanging $a \leftrightarrow b$ and $x \leftrightarrow y$ to get t) yields $s = \frac{a^2}{x}$, $t = \frac{b^2}{y}$. Thus, we are trying to minimize $L^2 = s^2 + t^2 = \frac{a^4}{x^2} + \frac{b^4}{y^2}$. Using the equation of the ellipse as the constraint (keeping in mind that $x \le a$), we minimize the function and find that $(x, y) = \left(\sqrt{\frac{a^3}{a+b}}\ ,\ \sqrt{\frac{b^3}{a+b}}\right)$, $s = \sqrt{a}\sqrt{a+b}$, and $t = \sqrt{b}\sqrt{a+b}$. Therefore, the area of rhombus equals $A = \frac{2s \cdot 2t}{2} = 2(a+b)\sqrt{ab}$. We subtract the area of the ellipse (πab) to get the final answer, $\boxed{2(a+b)\sqrt{ab} - \pi ab}$.

RC07 235. **Calc** Given $P(n) = \frac{N!}{n!(N-n)!}p^n q^{N-n}$ and $p + q = 1$, find the mean value $\mu = \frac{\sum_{n=0}^{N} nP(n)}{\sum_{n=0}^{N} P(n)}$.

Answer: $\boldsymbol{Np}$

Solution: Using the binomial theorem, $\sum_{n=0}^{N} P(n) = (p+q)^N$ and thus $\sum_{n=0}^{N} nP(n) = \sum_{n=0}^{N} n\binom{N}{n}p^n q^{N-n} = p\frac{\partial}{\partial p}[\sum_{n=0}^{N}\binom{N}{n}p^n q^{N-n}] = p\frac{\partial}{\partial p}[(p+q)^N] = pN(p+q)^{N-1}$. Since $p+q = 1$ we obtain $\mu = \boxed{Np}$.

RC08 236. **Geo** What is the distance between the center of mass of a unit sphere of uniform density with one octant missing and the center of the sphere.

Comment: Aaron Lin: I don't think the solution reasoning for this is valid (how the volume of the two parts have to be the same. I didn't verify though. Also, this problem strongly favors anyone who knows calculus.

Answer: $\mathbf{\sqrt{3}\sqrt[3]{\pi/2}/3}$

Solution: We can look at the volume in question as the sum of a unit sphere of uniform density and an octant with negative mass. Thus, the problem reduces to calculating the center of mass of a single octant. Because of the symmetry, the center of mass must lie on the line that is equidistant from the three edges of the octant. The plane perpendicular to this line through the center of mass forms a pyramid which must have mass (or volume since the density is uniform) equal to half the total mass of the octant. Since the pyramid has three right angles, its volume is $1/6l^3$, where l is the length of one of the edges. $1/6l^3 = 1/2 * 1/8 * 4/3\pi$, so $l = \sqrt[3]{\pi/2}$.

Now we need to find the center of mass's distance along the line. If we recalculate the area of the pyramid with the equilateral triangle as the base, the height of the pyramid will be the desired distance. Each edge of the equilateral triangle is the hypotenuse of a 45-45-90 right triangle with legs of length l, so the equilateral triangle has side lengths $s = \sqrt{2}l$ and area $b = 1/4s^2 * \sqrt{3} = 1/2l^2\sqrt{3}$.

By setting the two equations for the volume of the pyramid equal to each other, we get $1/6l^3 = 1/3bh$. $h = \frac{l^3}{l^2\sqrt{3}} = \frac{\sqrt[3]{\pi/2}}{\sqrt{3}} = \boxed{\sqrt{3}\sqrt[3]{\pi/2}/3}$

RJ01 237. **Team** For a positive integer k, let $f(k)$ be the largest positive root of the polynomial

$$x^k - (kx^{k-1} + k^2x^{k-2} + \cdots + k^{k-1}x + k^k).$$

Find the ordered pair (c, L) such that the limit

$$\lim_{k\to\infty} \frac{f(k)}{k^c}$$

exists, is nonzero, and is equal to L.

Comment: Specifying "largest" makes the analysis easy. We might be able to also say "unique," but I'm too lazy at the moment to think - Robin

I think this is a good problem, but we had too much Calc on Team already in 2014. Could be useful in 2015 - Robin.

Answer: $\mathbf{(1, 2)}$

Solution: Let $P_k(x)$ be the polynomial described in the problem, let α be the largest positive root of P_k, and let $a = \frac{\alpha}{k}$ We then have

$$\alpha^k = a^k k^k = k^k + k^k a + \cdots + k^k a^{k-1} = \frac{a^k - 1}{a - 1} k^k.$$

Now, we have $a^k(a-1) = a^k - 1$, or $a^{k+1} = 2a^k - 1$. First, note that $2^{k+1} > 2 \cdot 2^k - 1$ for all k. This inequality holds if we substitute a for any number greater than 2, so we must have $\alpha < 2k$. Furthermore, for all $\varepsilon > 0$, there exists an N such that $k > N \implies (2-\varepsilon)^{k+1} < 2 \cdot (2-\varepsilon)^k - 1$. In particular, we can choose k such that $(2-\varepsilon)^k > \frac{1}{\varepsilon}$, so

$$(2-\varepsilon)^k \cdot 2 - (2-\varepsilon)^{k+1} = (2-\varepsilon)^k \cdot \varepsilon > 1.$$

Hence, as $k \to \infty$, a must approach 2.

RJ02 238. **Disc** Let $a_1, a_2, a_3, \ldots, a_{2^n}$ be a permutation of the integers $\{1, 2, 3, \ldots, 2^n\}$ for some positive integer n. Call this permutation *balanced* if and only if for every positive integer $k < n$ and any integer i with $0 \le i < 2^{n-k-1}$,

$$\min(a_{1+2i\cdot 2^k}, a_{2+2i\cdot 2^k}, a_{3+2i\cdot 2^k}, \ldots, a_{2^k+2i\cdot 2^k}) +$$
$$\min(a_{1+(1+2i)2^k}, a_{2+(1+2i)2^k}, a_{3+(1+2i)2^k}, \ldots a_{2^k+(1+2i)2^k})$$

is a value independent of i (but it may depend on k). Find the number of *balanced* permutations of $\{1, 2, 3, \ldots, 2^{10}\}$.

Comment: I fear that this question does not do complete justice to the elegant underlying structure at work here: the NCAA March Madness bracket. Each regional bracket is a balanced element of S_{16}. It's a very pretty structure, with lots of nice symmetry. There's also a group-theoretic perspective: once a particular balanced permutation has been selected, the set of permutations that compose with it to get another balanced permutation form a group. Moreover, this group is in fact isomorphic to the symmetry group of a binary search tree with 2^n leaf nodes, i.e. the group generated by all the operations that swap two children of the same parent. If you draw out the complete March Madness-style bracket and pick chalk all the way (which is modeled in the problem by all the min expressions), you'll find that you have constructed a min-segment tree. Only these isometries will preserve the desired properties. - Robin

This is not a good problem, but I still think that something about a tournament bracket would make for a good problem - Robin

Answer: 2^{45}

Solution: Let $f(n)$ denote the number of balanced permutations of $\{1, 2, \ldots, 2^n\}$. We claim that $f(n) = 2^{\frac{n(n-1)}{2}}$, so $f(10) = 2^{45}$.

First, $f(1) = 2$ because both permutations of $\{1, 2\}$ are balanced.

Now, we induct. Setting $k = 1$, we see that $a_1 + a_2 = a_3 + a_4 = \cdots = a_{2^n-1} + a_{2^n}$. Since these 2^{n-1} pairs together sum to $1 + 2 + 3 + \cdots + 2^n = 2^{n-1}(2^n + 1)$, each pair sums to $2^n + 1$. Therefore, by pigeonhole, the smaller element of the pair is $\le 2^{n-1}$. Let the sequence $b_1, b_2, \ldots, b_{2^{n-1}}$ be defined by $b_i = \min(a_{2i-1}, a_{2i})$. The b_i are 2^{n-1} distinct integers $\le 2^{n-1}$, and hence are a permutation of $\{1, 2, 3, \ldots, 2^{n-1}\}$. Moreover, since the (a_i) defined a balanced permutation, we can see that the (b_i) are also a balanced permutation: the properties about minimums still hold.

Now, fix the sequence $b_1, b_2, b_3, \ldots, b_{2^{n-1}}$ to be some balanced permutation. For each $i = 1, 2, 3, \ldots, 2^{n-1}$, there are exactly 2 ways to achieve b_i: either $a_{2i-1} = b_i$ and $a_{2i} = 2^n + 1 - b_i$, or $a_{2i-1} = 2^n + 1 - b_i$ and $a_{2i} = b_i$. Since these choices are independent for each i, there are exactly 2^{n-1} to achieve the entire sequence $b_1, b_2, b_3, \ldots, b_{2^{n-1}}$. Since we can do this for each balanced permutation of $\{1, 2, 3, \ldots, 2^{n-1}\}$, we obtain the recurrence $f(n) = 2^{n-1} f(n-1)$. By the inductive hypothesis, $f(n-1) = 2^{\frac{(n-1)(n-2)}{2}}$, so $f(n) = 2^{\frac{n(n-1)}{2}}$ as claimed. Therefore, $f(10) = \boxed{2^{45}}$.

RJ03 239. **Calc** Two concentric circles have radius 1 and 2, respectively. Point A is chosen uniformly at random from the boundary of the larger circle, and point B is chosen independently and uniformly at random from the boundary of the smaller circle. Find the expected value of AB^2.

Comment: This general idea can be riffed upon to make harder questions, but I thought we'd start here and make it harder if desired - Robin

Answer: 5

Solution: Let O be the center of the circle, and θ be the angle between OA and OB. Clearly, θ is uniformly distributed within $[0, \pi]$. By Law of Cosines,

$$AB^2 = 1^2 + 2^2 - 1 \cdot 2 \cdot \cos\theta.$$

Since $\int_0^\pi \cos\theta \, d\theta = 0$, the expected value of AB^2 is simply $1^2 + 2^2 = \boxed{5}$.

RK01:dis 240. **Disc** You own two cats, Chocolate and Tea. Chocolate and Tea sleep for C and T hours a day, respectively, where C and T are chosen independently and uniformly at random from the interval $[5, 10]$. In a given day, what is the probability that Chocolate and Tea will together sleep for a total of at least 14 hours?

Comment: Pretty basic continuous probability problem. Not very original.

HH: Could use an actual picture.

Answer: $\mathbf{\frac{17}{25}}$

Solution: We can solve this problem by drawing a picture. In the 2-D coordinate plane, we can represent the x-axis as the time that Chocolate spends sleeping and represent the y-axis as the time that Tea spends sleeping. Thus, the set of all possible sleeping times for the two cats lies within a 5×5 box. Moreover, the set of all possible times where the two cats sleep for less than 14 hours lies within a triangle with base 4 and height 4.

Thus, our desired probability is $\frac{5*5-4*4/2}{5*5} = \boxed{\frac{17}{25}}$.

RK02 241. **Geo** Given a circle with center O and radius 4, select two distinct points on the circle, P and Q, uniformly at random. Let A be the area of the smaller sector formed from OP, OQ. What is the probability that $A \geq 2\pi$?

Comment:

Answer: $\mathbf{\frac{3}{4}}$

Solution: From the formula for the area of a sector, this requires $\theta \geq \frac{\pi}{4}$. Since each $\theta \in [0, \pi]$ is equally likely, the probability is $\frac{\pi - \frac{\pi}{4}}{\pi} = \frac{3}{4}$.

RM01 242. **Geo** Circle A has a radius 5 and is centered at the origin, and circle B has radius $5\sqrt{3}$ and is centered at (10,0). What is the area of the intersection of A and B?

Answer: $\mathbf{\frac{125\pi}{6} - 25\sqrt{3}}$

Solution: Drawing radii to the intersection points, one gets 30°-60°-90° right triangles. The area of the circular segment subtended in A is

$$\left(\frac{1}{3}\right)(25\pi) = \frac{25\pi}{3}$$

and the area of the sector in B is

$$\left(\frac{1}{6}\right)(75\pi) = \frac{75\pi}{6}$$

so that the area of the intersection, D, has the following relations:

$$\frac{75\pi}{6} - D = X$$
$$\frac{25\pi}{3} - D = Y$$

And

$$X + Y + D = 25\sqrt{3}, \text{ the area of the kite}$$

So that:

$$D = \frac{125\pi}{6} - 25\sqrt{3}$$

RM02 243. **Team** Two points are a distance 2 apart, and each one has a line attached to it, with the two lines rotating in the same direction at the same constant, nonzero angular velocity. When one line is vertical, the angle from the vertical to the other line is δ. Determine the area enclosed by the path traced out by the point of intersection of the two lines.

Answer: $\pi \csc^2 \delta$

Solution: Let the points be $(-1, 0)$ and $(1, 0)$, with the lines making angles of θ and $\theta + \delta$ with the positive y axis, respectively. Their equations are $y = \tan\theta(x + 1)$ and $y = \tan(\theta + \delta)(x - 1)$, respectively. The intersection has x coordinate $x = \frac{\tan(\theta+\delta)+\tan\theta}{\tan(\theta+\delta)-\tan\theta}$. $\tan(\theta + \delta) = \frac{\tan\theta+\tan\delta}{1-\tan\theta\tan\delta}$, so $x = \frac{\tan\theta+\tan\delta+\tan\theta-\tan^2\theta\tan\delta}{\tan\theta+\tan\delta-\tan\theta+\tan^2\theta\tan\delta} = \frac{2\tan\theta+(1-\tan^2\theta)\tan\delta}{(1+\tan^2\theta)\tan\delta} = \frac{2\tan\theta+(1-\tan^2\theta)\tan\delta}{\sec^2\theta\tan\delta} = 2\sin\theta\cos\theta\cot\delta + (\cos^2\theta - \sin^2\theta) = \sin(2\theta)\cot\delta + \cos(2\theta)$. Then $y = \tan\theta(\sin(2\theta)\cot\delta + \cos(2\theta) + 1) = 2\sin^2\theta\cot\delta + 2\sin\theta\cos\theta = (1 - \cos(2\theta))\cot\delta + \sin(2\theta)$. $x^2 + (y - \cot\delta)^2 = (\sin^2(2\theta) + \cos^2(2\theta))(1 + \cot^2\delta) = 1 + \cot^2\delta = \csc^2\delta$, so the path is a circle with center $(0, \cot\delta)$ and radius $|\csc\delta|$, and hence its area is $\pi\csc^2\delta$.

RM03 244. **Gen** In a classroom, there are 47 students in 6 rows and 8 columns. Every student's position is expressed by (i, j). After moving, the position changes to (m, n). Define the change of every student as $(i - m) + (j - n)$. Find the maximum of the sum of changes of all students.

Answer: 12

Solution: Notice that

$$(i - m) + (j - n) = (i + j) - (m + n)$$

Thus:

$$\begin{aligned} \sum [(i - m) + (j - n)] &= \sum [(i + j) - (m + n)] \\ &= \sum (i + j) - \sum (m + n) \end{aligned}$$

This is the sum of all the student's initial coordinates minus the sum of their final coordinates. The maximum is achieved when the first empty position is the lowest, $(1, 1)$ and the final empty position is the greatest, $(6, 8)$. Since all the other positions remain filled, the total change is:

$$(6 + 8) - (1 + 1) = 12$$

RM04 245. **Team** Let C_1 be the circle in the complex plane with radius 1 centered at 0. Let C_2 be the circle in the complex plane with radius 2 centered at $4 - 2i$. Let C_3 be the circle in the complex plane with radius 4 centered at $3 + 8i$.

Let S be the set of points which are of the form $\frac{k_1+k_2+k_3}{3}$ where $k_1 \in C_1, k_2 \in C_2, k_3 \in C_3$. What is the area of S? (Note: a circle or radius r only contains the points at distance r from the center and does not include the points inside the circle)

Comment:

Answer: $16\pi/3$

Solution: First note that the average of the centers of C_1, C_2, C_3 is $7/3 + 2i$. Also, any point in C_1 can be written as m_1, where $|m_1| = 1$, any point in C_2 can be written as $m_2 + (4 - 2i)$, where $|m_2| = 2$, and any point in C_3 can be written as $m_3 + (3 + 8i)$, where $|m_3| = 4$.

Then, any point in S can be written as $\frac{m_1+m_2+m_3}{3}+7/3+2i$. We can translate S by $-7/3-2i$ to make $S' = \frac{m_1+m_2+m_3}{3}$ which has the same area as S and is radially symmetric.

Since $\max\left|\frac{m_1+m_2+m_3}{3}\right| = \frac{1+2+4}{3} = 7/3$ and

$$\min\left|\frac{m_1+m_2+m_3}{3}\right| = \min\left|\frac{m_3-m_2-m_1}{3}\right| \geq \min\frac{|m_3|-|m_2|-|m_1|}{3} = 1/3,$$

this minimum occurs when $m_1 = -1, m_2 = -2, m_3 = 4$, so $\min\left|\frac{m_1+m_2+m_3}{3}\right| = 1/3$. Thus, $\left|\frac{m_1+m_2+m_3}{3}\right|$ can be any value between 1/3 and 7/3 so S' is a disk with radius 7/3 with a hole of radius 1/3. Its area is $\pi((7/3)^2-(1/3)^2) = \pi * 48/9 = \boxed{16\pi/3}$

RM05 246. **Calc** Define $f : \mathbb{R}^2 \to \mathbb{Z}$ where $f(x,y)$ is the number of lines passing through (x,y) that are tangent to $y = \sin(x)$ at a point $(x^\star, y^\star)$ such that $-\pi \leq x^\star \leq \pi$.

Find $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\,\mathrm{d}x\mathrm{d}y$.

Comment: This one needs verifying.

Answer: $\mathbf{11/2\pi^2 - 8}$

Solution: The easiest solution is to draw tangents and observe the graph above. What

follows below is a more rigorous solution.

... justify graph ...

Let $A = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\,\mathrm{d}x\mathrm{d}y$. Thus, we can break the integral into 3 regions based on the value of f. Since $f \geq 1$ on the entire region,

$$A = (2\pi)^2 + 1 * (\text{area where } f = 2) + 2 * (\text{area where } f = 3).$$

The region where $f = 2$ is simply two congruent triangles. The one in the first quadrant has a base length of $\pi/2$ in the x direction and a height of π in the y direction. Thus

$$A = (2\pi)^2 + 1 * \left(2 * \left(\frac{1}{2}\frac{\pi}{2}\pi\right)\right) + 2 * (\text{area where } f = 3).$$

The region where $f = 3$ consists of 4 symmetric pieces. Consider the one with x coordinates between 0 and $\pi/2$. It is the region where $\sin(x) \leq y \leq x$. It's area is $\int_0^{\pi/2} x - \sin x\,\mathrm{d}x = (\pi/2)^2/2 + \cos(\pi/2) - \cos(0) = \pi^2/8 - 1$. Since there are 4 of these regions, the total area where $f = 3$ is $\pi^2/2 - 4$.

$$A = (2\pi)^2 + 1 * \left(2 * \left(\frac{1}{2}\frac{\pi}{2}\pi\right)\right) + 2 * (\pi^2/2 - 4)$$

$$A = 4\pi^2 + \frac{\pi^2}{2} + \pi^2 - 8 = \boxed{11\pi^2/2 - 8}$$

SH01 247. **Geo** Let V be a polyhedron formed by 24 vertices of $(\pm 2, \pm 1, 0)$, $(\pm 2, 0, \pm 1)$, $(\pm 1, \pm 2, 0)$, $(\pm 1, 0, \pm 2)$, $(0, \pm 2, \pm 1)$, $(0, \pm 1, \pm 2)$. (The $\pm$ signs can be of any combination: $(2,1,0)$, $(2,-1,0)$, $(-2,1,0)$, $(-2,-1,0)$ are all vertices of V.) Find the volume of V.

Comment: [04/09/2014 19:40] (Sean) Updated. I admit that current formulation (of problem and solution) is quite crappy, so feel free to improve.

Answer: 32

Solution: Faces of V consist of 8 hexagons in each octant (eight regions divided by xy, yz, zx planes) which are reflections of hexagon with vertices $(2,0,1)$, $(2,1,0)$, $(1,2,0)$, $(0,2,1)$,

$(0,1,2)$, $(1,0,2)$, and six squares such as ones like $(2,1,0)$, $(2,0,1)$, $(2,-1,0)$, $(2,0,-1)$. Observe that extension of six hexagonal faces gives a regular octahedron X with six vertices $(3,0,0)$, $(-3,0,0)$, $(0,3,0)$, $(0,-3,0)$, $(0,0,3)$, $(0,0,-3)$. Our figure V is obtained by cutting out a square section out of all six vertices of X. (Draw the picture to check!)

Volume of X can be obtained by dividing it along xy-plane into two square pyramids having side length of base $3\sqrt{2}$ and height 3, so that

$$\mathrm{vol}(X) = 2 \cdot (\frac{1}{3} \cdot (3\sqrt{2})^2 \cdot 3) = 36.$$

Meanwhile the six parts cut out from X are all square pyramids having side length of base $\sqrt{2}$ and height 1, so their volumes are all $((\sqrt{2})^2 \cdot 1)/3 = 2/3$. Thus we have

$$\mathrm{vol}(V) = 36 - 6 \cdot \frac{2}{3} = 32$$

as volume of V.

SH02 248. **Geo** A tetrahedron $ABCD$ and a point X satisfies $XA = \sqrt{2}$, $XB = XC = \sqrt{3}$ and $XD = \sqrt{5}$. If we denote V the volume of $ABCD$, find the maximum value of V^2.

Comment: The answer $\frac{5}{2}$ is wrong. Confirmed by a checker and the following counterexample: $X = (0,0,0)$, $A = (\sqrt{2},0,0)$, $B = (-\sqrt{3},0,0)$, $C = (0,\sqrt{3},0)$, $D = (0,0,\sqrt{5})$. The height is $\sqrt{5}$ and the base is a triangle with area $\frac{1}{2} \cdot (\sqrt{2}+\sqrt{3}) \cdot \sqrt{3}$, so the volume square is $\frac{15}{36} \cdot (\sqrt{2}+\sqrt{3})^2 \approx 4.12$. – Herman

Comment: [04/12/2014 15:00] (Sean) Updated.

Answer: 5/2

Solution: Let $XA = \vec{a}$, $XB = \vec{b}$, etc. If we fix X, B, C, D and move A around, the volume is maximized when A is farthest from plane BCD. Thus when $ABCD$ has maximal volume, XA should be orthogonal to plane BCD. This translates in language of vector as follows:

$$\begin{aligned} &XA \cdot BC = \vec{a} \cdot (\vec{b} - \vec{c}) = 0, \quad XA \cdot CD = \vec{a} \cdot (\vec{c} - \vec{d}) = 0 \\ &\Rightarrow \vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}. \end{aligned}$$

Similarly we have $\vec{b} \cdot \vec{a} = \vec{b} \cdot \vec{c} = \vec{b} \cdot \vec{d}$ etc., so all inner products between $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are same. Denote its value as $-p < 0$.

Now it is time to solve for p. As four vectors in 3-dimensional space are linearly dependent, there exists x, y, z, w with $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = 0$. We take inner product of this with $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ to get

$$(x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d}) \cdot \vec{a} = x(\vec{a} \cdot \vec{a}) - (y+z+w)p = (|XA|^2 + p)x - (x+y+z+w)p = 0$$

so

$$\begin{aligned} \frac{x}{x+y+z+w} &= \frac{p}{|XA|^2+p}, \quad & \frac{y}{x+y+z+w} &= \frac{p}{|XB|^2+p}, \\ \frac{z}{x+y+z+w} &= \frac{p}{|XC|^2+p}, \quad & \frac{w}{x+y+z+w} &= \frac{p}{|XD|^2+p}. \end{aligned}$$

Adding all those give

$$\frac{p}{p+2} + \frac{p}{p+3} + \frac{p}{p+3} + \frac{p}{p+5} = 1$$

and solving (or guessing!) gives $p = 1$.

By applying the law of cosines $BC = \sqrt{XB^2 + XC^2 - 2XB \cdot XC}$ etc, we have $BC = 2$ and $BD = CD = \sqrt{6}$. Thus $[BCD] = \sqrt{5}$, as height from D is $\sqrt{5}$. Meanwhile let XP be the distance of X to plane BCD. As XP is perpendicular to BCD, we have $XP \cdot XA = XP \cdot XA + PB \cdot XA = XB \cdot XA = -p$. So length of XP is $p/|XA| = 1/\sqrt{2}$. Therefore we have

$$[ABCD] = \frac{1}{3}|AP| \cdot [BCD] = \frac{1}{3} \cdot (\sqrt{2} + \frac{1}{\sqrt{2}})\sqrt{5} = \sqrt{\frac{5}{2}}.$$

SH03:gut 249. **Disc** Suppose we have a 2×5 grid and we wish to write 0's and 1's inside so that for any 2×2 sub-block, the *determinant* is 0. The determinant of a 2×2 block $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$. For example, the following is a valid configuration:

0	1	0	1	0
0	1	0	0	0

However, the following is not valid because the last 2×2 sub-block has determinant 1.

0	1	0	1	0
0	1	0	0	1

How many such valid 2×5 configurations are there?

Comment:

Answer: 208

Solution: We use a technique called dynamic programming. Suppose we have a valid configuration for a $2 \times n$ length block. Then we may build a valid $2 \times (n+1)$ block by considering only what the last column of the $2 \times n$ block was, as we only need to add one more column so that the determinant of the last two columns of the new $2 \times (n+1)$ is 0. More importantly, this idea is invertible: given a valid $2 \times (n+1)$ configuration, chopping off the last column gives us a valid $2 \times n$ configuration.

Denote $A(n, 00)$ to denote the number of valid $2 \times n$ blocks whose last column is $\substack{0\\0}$. Similarly for $A(n, 01)$ denoting $\substack{0\\1}$, and so on for $A(n, 10)$ and $A(n, 11)$. Then we may make these recursive equations: $A(1, 00) = A(1, 01) = A(1, 10) = A(1, 11) = 1$, and for $n \geq 2$, $A(n, 00) = A(n-1, 00) + A(n-1, 01) + A(n-1, 10) + A(n-1, 11)$, $A(n, 01) = A(n-1, 00) + A(n-1, 01)$, $A(n, 10) = A(n-1, 00) + A(n-1, 10)$, and $A(n, 11) = A(n-1, 00) + A(n-1, 11)$. Thus we may simply compute these recursive formulas. (If the reader is suspicious of letting $A(1, 01)$ mean anything, note we may simply start with the $n = 2$ case.)

Let a_n denote $a_n = (A(n, 00), A(n, 01), A(n, 10), A(n, 11))$. Thus we have $a_1 = (1, 1, 1, 1)$, $a_2 = (4, 2, 2, 2)$, $a_3 = (10, 6, 6, 6)$, $a_4 = (28, 16, 16, 16)$, and $a_5 = (76, 44, 44, 44)$. Therefore the total number of valid configurations of size 2×5 is $76 + 44 + 44 + 44 = 208$.

SH04 250. **Disc** Let n_a be the number of solution $x^2 + y^2 \equiv a (\text{mod } 59)$. Find $n_1 - n_{-1}$.

Comment: [05/12/2014] (Sean) Updated. The number 59 can be any $4k+3$ prime. As well as some of my other problems I wanted this to be solvable by bash but taking lots of time.

Answer: 0

Solution: As $p = 59$ is $4k + 3$ form prime, there does not exist solution for $x^2 + y^2 \equiv 0 (\text{mod } p)$. That means, the set $S = \{k \text{ mod } p : k \equiv x^2 \text{ for some } x\}$ and $T = \{l \text{ mod } p : l \equiv -y^2 \text{ for some } x\}$ are disjoint except at zero. (incomplete yet)

SH05:alg 251. **Alg** A polynomial of degree 5 $f(x)$ satisfies $f(0) = f(1) = 1$, $f(2) = 2$, $f(3) = 6$, $f(4) = 24$, and $f(5) = 120$. Find $f(7)$.

Comment: [05/11/2014] (Sean) Updated. Pretty standard polynomial interpolation problem I like, but I haven't seen this case before. For the number choice it can be generalized, but my intent was to trick some people into brute-force calculation (of actually finding f).

Nick: The given solution is wrong, the answer is 1331, verified by Herman and WolframAlpha. Sean, your solution doesn't work because $f(x+1) - (x+1)f(x)$ is a 6th-degree polynomial and therefore has 6 zeroes.

Harrison: Added a somewhat legit solution. The problem is somewhat lengthy, but the substitution is very nice and still contributes to the solution.

Eric: Potentially guessable using Lagrange interpolation.

Answer: 1331

Solution: Define $g(x) = f(x+1) - (x+1)f(x)$. Then $g(x)$ has $x = 0, 1, 2, 3, 4$ as roots, with one extra root b. Thus there exists some a such that

$$g(x) = ax(x-1)(x-2)(x-3)(x-4)(x-b)$$

Through various means (*), we can determine that the coefficient of the leading term in $f(x)$ is $\frac{11}{30}$. Since $g(x) = f(x+1) - (x+1)f(x)$, the coefficient of the leading term of $g(x)$ is the opposite of that of $f(x)$, since $(x+1)f(x)$ contributes the term of the largest power in $g(x)$. Hence $a = -\frac{11}{30}$, the opposite of the value above. Substituting $x = -1$ gives $g(-1) = f(0) = 1 = 44(b+1)$, so $b = -\frac{45}{44}$. Now, substituting $x = 5$ gives $f(6) - 6f(5) = -265$, hence $f(6) = 455$. Finally, substituting $x = 6$ gives $f(7) - 7f(6) = -1854$, hence $f(7) = \boxed{1331}$.

Various means: consider a polynomial $P(x)$ of degree n with coefficient of leading term a. When $n = 1$ (the polynomial is linear), computing $P(x+1) - P(x)$ for any x yields a. When $n = 2$ (the polynomial is quadratic), computing $P(x+1) - P(x)$ for any x yields $2a$. In general, computing $P(x+1) - P(x)$ for any x and $n \geq 1$ yields $an!$. (Try to prove this with induction!) Using this rule repeatedly on the given values allows us to compute that the coefficient of the leading term of $f(x)$ in this problem is $\frac{11}{30}$.

SS01 252. **Calc** In tennis, players have two chances to hit a serve in. If the first serve is in, the point is played to completion (until either player wins the point). If the first serve is out, the player hits a second serve. If the second serve is in, the point is played to completion; otherwise, the server automatically loses the point. Andy can precisely control the velocity v of his serve up to 100mph. The faster his serve, the higher the probability of him winning the point if the serve goes in, but the higher the probability that the serve goes out. For a given v, the probability that Andy's serve is in is $p(v) = \frac{150-v}{150}$, and the probability that he wins the point after his serve goes in is $q(v) = \frac{v}{100}$. Assuming that he chooses optimal velocities for his first and second serves, compute the probability that Andy wins the point.

Comment: This is basically an AT problem with a little bit of calculus thrown in, but I think it's suitable as a diversity problem in the middle of the calc test.

Eric: This problem is very verbose. Is there a way to shorten the problem somehow?

Answer: $\mathbf{\frac{75}{128}}$

Solution: We have to find the optimal second-serve velocity first. The probability of winning the point after missing the first serve, as a function of v the speed of the second serve, will be

$$p(v) \cdot q(v) = \frac{v(150-v)}{100 \cdot 150}$$

The derivative of this is $\frac{150-2v}{15000}$, which has only one zero at $v = 75$, so this is the maximum. At this v, Andy's probability of winning the point is $\frac{75}{150} \cdot \frac{75}{100} = \frac{3}{8}$.

Now, given this, the probability of winning the first point as a function of v, the speed of the first serve, will be

$$p(v) \cdot q(v) + (1 - p(v)) \cdot \frac{3}{8} = \frac{v(150 - v)}{100 \cdot 150} + \frac{3v}{8 \cdot 150}$$

The derivative of this is $\frac{150-2v}{100 \cdot 150} + \frac{3}{8 \cdot 150}$, setting this equal to 0 and solving, we find $\frac{150-2v}{100} = \frac{-3}{8} \Rightarrow 8(150 - 2v) = -3 \cdot 100 \Rightarrow 1500 = 16v \Rightarrow v = \frac{375}{4}$.

Finally, this means the probability Andy wins the point is

$$p(v) \cdot q(v) + (1 - p(v)) \cdot \frac{3}{8}\bigg|_{v=\frac{375}{4}} = \frac{3}{8} \cdot \frac{15}{16} + \frac{5}{8} \cdot \frac{3}{8} = \boxed{\frac{75}{128}}$$

Note that this means Andy will always attempt his first serve at 93.75mph, and his second serve at a more conservative 75mph. In fact, real tennis players also hit their second serves significantly slower than their first serves.

SS02 253. **Calc** Your retina can be modeled as the inner surface of a hemisphere with radius $1cm$. While the exact distribution isn't nearly this simple, the density of rod cells (the type of photoreceptor used in your peripheral and night vision) in your retina can be approximated by $f(\theta) = 50 - \frac{100\theta}{\pi}$ million cells per square centimeter, where $\theta \in [0, \frac{\pi}{2}]$ is the angle away from the center of the hemisphere (so the number of rods per square centimeter ranges linearly from 50 million at the center of your retina, to zero at the very edge). Using this approximation, compute the total number of rod cells in your retina, in millions.

Comment: We may as well make this semi-accurate. According to wikipedia people have 75-150M rods, and the distribution tapers approximately linearly as in the problem except for our blind spot (which this problem ignores).

Harrison: This problem should avoid the science part, as it adds an additional (and unnecessary) layer of complexity to the problem.

Answer: $\mathbf{100(\pi - 2)}$

Solution: TODO: Write a solution.

TJ01 254. **Team** An ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is tangent to each of the circles $(x - 1)^2 + y^2 = 1$ and $(x + 1)^2 + y^2 = 1$ at two points. Find the ordered pair (a, b) that minimizes the area of the ellipse.

Comment: I have not been able to find a non-calculus solution, but if you have one please tell me.

BP: It seems that if you choose $a = 2$, any value of b will satisfy the tangency condition. In that case there is no good answer to this. Is it supposed to mean tangent at two points AND non-intersecting?

TJ: I said tangent to EACH circle at two points. $a = 2$ only produces one point of tangency per circle.

MX: Since this is only mildly calculus, I consider this mistagged. I am retagging as team, where mildly calculus problems should be placed.

Answer: $\left(\frac{3\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\right)$

Solution: Since the two circles are symmetric, we can just pick one of them to work on. So we have the system

$$\begin{cases} (x-1)^2 + y^2 = 1 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \end{cases}$$

Solving the first equation gives us $y^2 = 1 - (x-1)^2$, and clearing denominators gives us $b^2x^2 + a^2y^2 = a^2b^2$ in the second. Substituting for y^2 and simplifying, we obtain

$$(b^2 - a^2)x^2 + 2a^2x - a^2b^2 = 0,$$

which is a quadratic in x.

The tangency condition means that there is a double root, so the discriminant must be zero:

$$(2a^2)^2 - 4(b^2 - a^2)(-a^2b^2) = 0$$
$$a^2b^4 - a^4b^2 + a^4 = 0.$$

Applying the quadratic formula to b^2 gives us

$$b^2 = \frac{a^2 \pm \sqrt{a^4 - 4a^2}}{2}.$$

Since $b < a$ (otherwise there would only be one point of tangency per circle),

$$b^2 = \frac{a^2 - \sqrt{a^4 - 4a^2}}{2}.$$

The quantity we wish to minimize is the area of the ellipse, given by the formula πab. Since a and b are positive, this is minimized if and only if a^2b^2 is minimized. Substituting in b^2 gives us

$$a^2b^2 = \frac{a^4 - a^3\sqrt{a^2 - 4}}{2}.$$

Differentiating this expression produces

$$\frac{4a^3 - \frac{a^4}{\sqrt{a^2-4}} - 3a^2\sqrt{a^2-4}}{2},$$

which we set equal to zero. Multiplying by $2\sqrt{a^2-4}$, we solve:

$$4a^3\sqrt{a^2-4} - a^4 - 3a^2(a^2-4) = 0$$
$$a^3\sqrt{a^2-4} = a^4 - 3a^2$$
$$a^6(a^2-4) = a^8 - 6a^6 + 9a^4$$
$$2a^6 - 9a^4 = 0$$
$$a^4(2a^2 - 9) = 0,$$

so $a^2 = \frac{9}{2}$. Plugging this into the expression for b^2 gives us $b^2 = \frac{3}{2}$. So $(a, b) = \boxed{\left(\frac{3\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\right)}$.

VD01 255. **Geo** In quadrilateral $ABCD$, E and F are on BC and AD respectively such that each of the areas of triangle AED and triangle BFC are $4/7$ of the area of $ABCD$. R is the intersection of AC and BD, and we have $AR/RC = 3/5$ and $BR/RD = 5/6$. N is the intersection of AD and BC. Find the ratio of the areas of $ABCD$ and NDC.

Comment: This problem was VD02 in the 2013 repo.

Answer: TODO: Find the answer.

Solution: TODO: Write a solution.

WV01 256. **Disc** You and your friend play a game where each turn, you gain 3 points, while your friend flips a fair coin and gains 5 points if it lands heads, and 0 points otherwise. On the turn that your point total first becomes greater than 5, what is the expected difference between your and your friend's total scores?

Answer: 1

Solution: Since you gain 3 points at a time, your point total will first become greater than 5 on turn 2, with 6 points on turn 2. So your friend will flip the coin 2 times.

With probability $\frac{1}{4}$, he will flip two heads, getting 10 points.

With probability $\frac{1}{2}$, he will flip one head, getting 5 points.

With probability $\frac{1}{4}$, he will flip no heads, getting 0 points.

The expected difference between your scores is therefore $\frac{1}{4}(6-10)+\frac{1}{2}(6-5)+\frac{1}{4}(6-0)=$ $\boxed{1}$.

WW01 257. **Disc** Andrew needs to take 9 large dogs and 7 small dogs out for walks. Since he can take a maximum of 5 large dogs and 5 small dogs on a single walk, he is planning on making two trips: one in the morning, and one in the evening. How many different ways can he accomplish walking each dog exactly once?

Comment:

Answer: 28224

Solution: $(\sum_{k=2}^{5}\binom{7}{k})\cdot(\sum_{k=4}^{5}\binom{9}{k})=\boxed{28224}$

WW02 258. **Team** Two dogs start at the origin and independently walk 1 meter in a straight line in a random direction. What is the expected area in square meters of the triangle formed by the origin and locations of the two dogs?

Comment:

Answer: $\boldsymbol{\frac{1}{\pi}}$

Solution: If we rotate ourselves such that one dog always walks in the positive x direction, we can model the (smaller) angle between the two dogs as a uniform distribution from 0 to π. Then the expression for the expected value is:

$$\int_0^\pi \frac{1}{\pi}\frac{1\cdot 1\cdot \sin\theta}{2}d\theta=\boxed{\frac{1}{\pi}}$$

WW03 259. **Geo** A cylinder with radius r and height h is sliced by a plane passing tangent to one of the cylinder's circles and through a diameter of the other. What is the volume of the smaller piece?

Comment: this may not have a solution not involving calculus... maybe better as a team problem

Answer:

$$\boldsymbol{\frac{2}{3}r^2h}$$

Solution: integrate with respect to height

WW04 260. **Calc** Compute the following:

$$\sum_{\substack{i+j+k=n \\ i,j,k\geq 0}} \frac{1}{i!j!k!}$$

Comment:

Answer:

$$\boldsymbol{\frac{3^n}{n!}}$$

Solution: This is just looking for the n^{th} coefficient in the Taylor expansion of e^{3x}.

XS01:dis 261. **Disc** Let $\mathcal{S}$ be the set of all possible 9-digit numbers that use $1, 2, 3, ..., 9$ each exactly once as a digit. What is the probability that a randomly selected number n from $\mathcal{S}$ is divisible by 27?

Comment: Deceptively hard discrete problem.

Answer: 27/80

Solution: Let $A, B, C, D, E, F, G, H, I$ be the digits of an arbitrary 9-digit number satisfying the properties.

By the problem statement, we must have $ABCDEFGHI \equiv 0 \mod 27$. Expanding and taking the left side modulo 27, we have that

$$\begin{aligned} 10^8 \cdot A + 10^7 \cdot B + ... + 10H + I &\equiv 0 \mod 27 \\ 19A + 10B + C + 19D + 10E + F + 19G + 10H + I &\equiv 0 \mod 27. \end{aligned}$$

We observe a pattern of $19, 10, 1, 19, 10, 1, \ldots$. Note that

$$A + B + ... + I = 45 \equiv 18 \mod 27,$$

so by subtracting $A + B + ... + I$ from both sides, we have

$$\begin{aligned} 18A + 9B + 18D + 9E + 18G + 9H &\equiv 9 \mod 27 \\ 2A + B + 2D + E + 2G + H &\equiv 1 \mod 3 \\ -A + B - D + E - G + H &\equiv 1 \mod 3. \end{aligned}$$

Now, we recall that there are 9! numbers which use each of the digits from 1 to 9 exactly once. Let x be the number of combinations of digits $A, B, ..., I$ that satisfy the above equation. We observe that by taking the negative of the above equation,

$$A - B + D - E + G - H \equiv -1 \equiv 2 \mod 3,$$

there are still x combinations of the digits that satisfy this equation. Moreover, we can rearrange digits to obtain

$$-A + B - D + E - G + H \equiv 2 \mod 3,$$

giving an equation which still has x solutions. Finally, let y be the number of combinations of digits $A, B, ..., I$ that satisfy

$$-A + B - D + E - G + H \equiv 0 \mod 3.$$

Note that all 9! combinations of digits must satisfy one of the three equations listed above. Hence, $2x + y = 9!$. Thus, we can solve for x by first solving for y and computing $x = \frac{9!-y}{2}$, and our desired probability is $\frac{x}{9!}$.

For simplicity, we consider each of the digits modulo 3, and solve the problem

$$A + D + G \equiv B + E + H \mod 3.$$

There are three cases:

(a) $A + D + G \equiv B + E + H \equiv 0 \mod 3$. We must either have ADG, BEH, CFI be some permutation of the digits $000, 111, 222$ or have ADG, BEH, CFI be some permutation of the digits $012, 012, 012$. In the first case, there are $3! = 6$ ways of arranging the combinations, and in the second case, there are $3! \cdot 3! \cdot 3!$ ways of arranging the digits within each of the combinations.

(b) $A + D + G \equiv B + E + H \equiv 1 \mod 3$. Here, we must have ADG, BEH, CFI be some permutation of permutations of the digits $001, 112, 022$. There are $3 \cdot 3 \cdot 3$ ways of arranging the different digit in each of the sets of three digits and 3! ways of arranging the permutations in general, for a total of $3 \cdot 3 \cdot 3 \cdot 3!$ combinations.

(c) $A + D + G \equiv B + E + H \equiv 2 \mod 3$. Here, we must have ADG, BEH, CFI be some permutation of permutations of the digits $011, 002, 122$. Again, there are $3 \cdot 3 \cdot 3$ ways of arranging the different digit in each of the sets of three digits and 3! ways of arranging the permutations in general, for a total of $3 \cdot 3 \cdot 3 \cdot 3!$ combinations.

Summing, there are a total of

$$6 + (3! \cdot 3! \cdot 3!) + 2(3 \cdot 3 \cdot 3 \cdot 3!) = 546$$

solutions with the simplification. To remove the simplification that we only consider the digits modulo 3, we have to multiply by 3! for each set of three digits. Hence, $y = 546 \cdot 3^3 \cdot 2^3$.

Finally, we compute our desired probability:

$$\begin{aligned}
\frac{x}{9!} &= \frac{9! - y}{2 \cdot 9!} \\
&= \frac{(9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2) - (546 \cdot 3^3 \cdot 2^3)}{2 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} \\
&= \frac{(2^7 \cdot 3^4 \cdot 5 \cdot 7) - (2 \cdot 3 \cdot 7 \cdot 13 \cdot 3^3 \cdot 2^3)}{2^8 \cdot 3^4 \cdot 5 \cdot 7} \\
&= \frac{(2^7 \cdot 3^4 \cdot 5 \cdot 7) - (2^4 \cdot 3^4 \cdot 7 \cdot 13)}{2^8 \cdot 3^4 \cdot 5 \cdot 7} \\
&= \frac{(2^3 \cdot 5) - 13}{2^4 \cdot 5} = \frac{40 - 13}{16 \cdot 5} = \boxed{\frac{27}{80}}.
\end{aligned}$$

XS02:alg 262. **Alg** Solve, in positive reals for a, b, c, d, the system of equations:

$$a + 4b + 8c + 4d = 53$$

$$3a^2 + 4b^2 + 12c^2 + 2d^2 = 159$$
$$9a^3 + 4b^3 + 18c^3 + d^3 = 477$$

Comment:

Answer: $\boldsymbol{a = 1, b = 3, c = 2, d = 6}$

Solution: Using Cauchy Schwarz,

$$(a + 4b + 8c + 4d)(9a^3 + 4b^3 + 18c^3 + d^3) \geq (3a^2 + 4b^2 + 12c^2 + 2d^2)^2$$

But since the conditions give $53 \cdot 477 = 159^2$, we have equality. Using the equality condition on Cauchy, we get that

$$9a^2 = b^2 = \frac{18}{8}c^2 = \frac{1}{4}d^2$$

Plugging this in the second equation, we get the desired solutions.

XS03 263. **Alg** Suppose we have the set $S = \{1, 2, ..., n\}$, and $f : T \subseteq S \to \mathbb{R}$ be a function that satisfies

$$f(A) + f(B) = f(A \cap B) + f(A \cup B)$$

for all $A, B \subseteq S$. Assume that for the nullset $\{\}$, $f(\{\}) = 0$. Now suppose that $f(\{k\}) = 2^k \ \ \forall 1 \leq k \leq n$. Compute

$$\sum_{C \subseteq S} f(C)$$

Comment:

Answer: $\boldsymbol{2^n(2^{n+1} - 2)}$

Solution: We can prove that (due to submodularity), f admits a linear basis (in the set definition), i.e. $f(\{x_1, ..., x_m\}) = f(\{x_1\}) + ... + f(\{x_m\})$ for all sets. Then it follows that the sum is equivalent to

$$\sum_{1 \leq k \leq n} 2^{n-1} f(\{k\}) = 2^{n-1} \sum_{1 \leq k \leq n} 2^k = 2^n(2^{n+1} - 2)$$

The proof of the linearity comes from Wikipedia - Sumbodular Set Function; the equivalent condition will be that $f(X \cup \{x\}) - f(X) = f(Y \cup \{x\}) - f(Y)$ for $X \subseteq Y$. Then it follows that plugging in any set Y and a corresponding singleton set X shows the linearity.

XS04 264. **Alg** Suppose in a graph G, we have 2^4 vertices (call this set as V), all of which correspond to all the 4-length binary vectors in $\{0, 1\}^4$. Suppose that $\Gamma = \{0011, 0110, 1000\}$. Connect two vertices u, v in G if their corresponding binary vectors, $b(u), b(v)$, if $b(u) - b(v) \in \Gamma$. (Here, subtraction is taken mod 2 for each location, so for example, $0001 - 1000 = 1001$). For a set $S \subseteq V$, let $E(S, V - S)$ be the set of edges with one end in S and the other end in $V - S$. What is

$$\max_{S \subseteq V} E(S, V - S)$$

Comment: This is somewhat hard conceptually, but not so much mechanically.

Answer: $\boldsymbol{r = 1010,\ E(S, V - S) = 4 \cdot 2^4 - \frac{1}{4} \cdot 2^4(-3) = 64 + 12 = 76}$

Solution:

Suppose for notation, each cut is represented as a function $f_c : \{0,1\}^n \to \{-1,1\}$ for cut c.

Now if $G = Cayley(\Gamma, T)$, then

$$2 \cdot E(S, V-S) = \frac{2^n|T|}{2} - \frac{1}{2} \sum_{x\in\{0,1\}^n} \sum_{t\in T} f(x)f(x+t)$$

Then it's clear we need to minimize

$$M = \sum_{x\in\{0,1\}^n} \sum_{t\in T} f(x)f(x+t)$$

Now consider all cuts of the form $f(x) = \chi_r(x) = (-1)^{\langle r,x\rangle}$ for some $r \in \{0,1\}^n$ we will find later. Plugging into the above and cancelling the double $\langle r, x\rangle$ term,

$$M_r = \sum_{x\in F_2^n} \sum_{t\in T} \chi_r(x)^2(-1)^{\langle r,t\rangle} = 2^n \sum_{t\in T} (-1)^{\langle r,t\rangle}$$

Now just find r that minimizes the above.

Now to prove that this will produce the MAX-CUT, note that using the above again, if we had any cut function f_c, then it can be represented as a Fourier sum of the basis functions, which are the characters of the group Γ. The characters are of the form $\chi_r(x)$, where $r \in \{0,1\}^n$. (This was proven in the notes).

Then

$$f_c(x) = \sum_{\chi} \hat{f}(\chi) \cdot \chi(x)$$

Now for maxcut, f_c needs to minimize

$$M_c = \sum_{x\in\{0,1\}^n} \sum_{t\in T} f_c(x)f_c(x+t)$$

Plugging in the Fourier expansion directly and taking the product, we will have terms of the form (with their coefficients hidden)

$$\sum_{x\in\{0,1\}^n} \sum_{t\in T} \chi_{r_1}(x)\chi_{r_2}(x+t) = \sum_{x\in\{0,1\}^n} \sum_{t\in T} (-1)^{\langle r_1+r_2,x\rangle + \langle r_2,t\rangle}$$

If $r_1 \neq r_2$, then $r_1 + r_2 \neq 0^n$, the n-length 0 bit, and therefore applying symmetry, we will always have

$$\sum_{x\in\{0,1\}^n} (-1)^{\langle r_1+r_2,x\rangle + \langle r_2,t\rangle} = 0$$

Then the only terms remain are when $r_1 = r_2$, which implies

$$M_c = \sum_{x\in\{0,1\}^n} \sum_{t\in T} \sum_{\chi} \hat{f}(\chi)^2 \chi(x)\chi(x+t) = 2^n \sum_{\chi} \left(\hat{f}(\chi)^2 \sum_{t\in T} (-1)^{r_\chi t} \right)$$

with, from Parseval's identity,

$$\sum_{\chi} \hat{f}(\chi)^2 = 1$$

This implies that every cut f_c is supposed to minimize a weighted sum of the cuts formed by the characters. Then obviously $M_c \geq M_{r^*}$, the minimum value, and the MAXCUT can only be formed by using the character with r^*.

XS05:alg 265. **Alg** Suppose we have a continuous function $f : \mathbb{R} \to \mathbb{R}$, where

$$f(2x) = 2f(x)^2 - 1$$

and $f(\frac{1}{2}) = 0.3$. Find the number of solutions to $f(x) = 0.5$.

Comment:

Answer: ∞

Solution: Note that $f(x) = \cos(kx)$ where $\cos(\frac{k}{2}) = 0.3$ and $k \neq 0$. But since the cos function is periodic, we have infinite solutions when it hits 0.5.

XS06 266. **Alg** Consider all n - degree polynomials $f(x) = a_n x^n + ... + a_0$, that satisfy

$$|f(x)| \leq 1 \ \ \forall x \in [-1, 1]$$

What is the maximal possible value of a_n?

Comment:

Answer: $\mathbf{2^{n-1}}$

Solution: Consider the Cheybyshev polynomial $T_n(x)$ such that $T_n(\cos x) = \cos(nx)$. Note that it is well known that the leading coefficient of T_n is 2^{n-1}. Then let $f(x) = T_n(x)$. To see it is the best possible, suppose that $f(x) = T_n(x) + g(x)$. Then the condition implies that when we plug in $x = k\pi/n$ where $k = 0, 1, ..., n+1$, $g(x)$ alternates negative and positive $n+1$ times, which implies that g cannot be at most degree n, contradiction.

XS07 267. **Alg** Real numbers x, y, z are in arithmetic progression, and

$$x + y + z = 5$$

$$xy + yz + xz = 6$$

What is the absolute value of their common difference?

Comment: Easy

Answer: $\sqrt{\frac{7}{3}}$

Solution: $x = a - d, y = a, z = a + d \implies 3a = 5$ from first equation, and

$$6 = xy + yz + xz = (a-d)a + a(a+d) + (a-d)(a+d) = 3a^2 - d^2 = \frac{25}{3} - d^2$$

$$d^2 = \frac{7}{3} \implies |d| = \sqrt{\frac{7}{3}}$$

XS08 268. **Alg** What is $\sin(4^\circ) + \sin(8^\circ) + ... + + \sin(172^\circ) + \sin(176^\circ)$?

Comment: medium.

Answer: $\frac{\mathbf{1}}{\mathbf{\tan(2^\circ)}}$

Solution: Let S be the value we want. Multiplying by $\sin(2^\circ)$ will then give us by the sum product formula,

$$S\sin(2^\circ) = \frac{1}{2}(\cos(2^\circ) - \cos(6^\circ) + \cos(6^\circ) - \cos(8^\circ) + ... - \cos(178^\circ)) = \cos(2^\circ)$$

Thus $S = 1/\tan(2^\circ)$.

XS09 269. **Alg** For a positive integer n, let $f(n) = \lfloor \sqrt{n} \rfloor$. Find the value of

$$\sum_{k=1}^{1000} f(k)$$

Comment: medium but bashy

Answer: 20615

Solution: If $f(n) = m$, then $m^2 \le n < (m+1)^2$. Thus there are $(m+1)^2 - m^2 = 2m+1$ integers n for which $f(n) = m$. We need to end this when $m = 31 \implies m^2 = 961$ and there are only $1 + 1000 - 961 = 40$ solutions to $f(n) = 31$ when $n \le 1000$. Thus our total sum is

$$40 \cdot 31 + \sum_{m=1}^{30} (2m+1)m = 20615$$

XS10:gut 270. **Alg** Find the number of real solutions x to

$$\sin(x) = \frac{x}{1000}.$$

Here, the inputs for sin are in radians.

Comment: Somewhat numeric, when you do $500/\pi$.

Answer: 317

Solution: Note that by symmetry, $x = 0$ is a solution, and then we only need to look at solutions $x \in (0, 1000]$ Every 2π cycle, we will have two solutions as the line $y = \frac{x}{1000}$ and the curve $y = \sin(x)$ will hit, except for at the first cycle, where we've already counted $x = 0$. Thus, we will have $\lfloor \frac{1000}{2\pi} \rfloor - 1 = 158$ solutions for $x \in (0, 1000]$. Thus, in total, we will have $2 \cdot 158 + 1 = 317$ solutions in total.

XS11:alg 271. **Alg** Let $f(x) = x^3 - 13x^2 + 40x - 27$. It is known that this function has a unique inverse x_0 for every y_0 in its image; i.e. $f(x_0) = y_0$. Now let $f^{-1}(x)$ be its inverse. Find all such x that satisfy $f(x) = f^{-1}(x)$.

Comment:

Answer: $\boldsymbol{x = 1, 3, 9}$

Solution: Since the inverse is the reflection about the line $y = x$, then it is obvious the only times when the curve and its inverse intersect is at the line, i.e. $f(x) = x$. This comes out to $(x-1)(x-3)(x-9) = 0$, fromm which we get our solutions.

XS12:alg 272. **Alg** Let x, y be real numbers satisfying $\frac{x^2}{9} + \frac{y^2}{16} = 1$. What is the maximum possible value xy?

Comment:

Answer: 6

Solution: By parametrization, we may let $x = 3\cos\theta, y = 4\sin\theta$ and therefore $xy = 12\cos\theta\sin\theta = 6\sin(2\theta) \le 6$, achieved when $\theta = 45$ degrees.

As an alternative solution for those unfamiliar with trig parametrization, note by the AM-GM inequality that

$$\frac{xy}{12} = \sqrt{\frac{x^2y^2}{9 \cdot 16}} \le \frac{x^2/9 + y^2/16}{2}.$$

Therefore $xy \le 6$. To check this can be achieved, note that equality comes when $x^2/9 = y^2/16 \implies \frac{x^2}{9} = 1/2 \implies x = \frac{3}{\sqrt{2}}$ and $y = \frac{4}{\sqrt{2}}$, which gives the maximum as desired.

XS13 273. **Alg** Let $\mathcal{G} = \{1, 2, ..., n\}$ and let $f : S \subseteq \mathcal{G} \to \mathbb{R}$ be a function from the subsets of $\mathcal{G}$ to the reals. f is called *submodular* if for all $S \subseteq T \subseteq \mathcal{G}$ and all base elements ρ,

$$f(S \cup \{\rho\}) - f(S) \geq f(T \cup \{\rho\}) - f(T)$$

Suppose that we wanted to find the set M_k^* out of all k-element subsets $M_k \subseteq \mathcal{G}$ that maximizes $f(M_k)$. Instead of calculating exactly, we construct a possible solution M_k° greedily, i.e. recursively at each time $t \leq k$ choosing $\rho_t \in \mathcal{G} - M_t^\circ$ such that $f(M_{t-1}^\circ \cup \rho_t)$ is maximized over all possible choices ρ_t.

When $n \to \infty$ and $k \to \infty$, what is the least c such that $f(M_k^\circ) \geq c \cdot f(M_k^*)$?

Comment: Haven't constructed example yet.

Answer: $\mathbf{(1 - 1/e)}$

Solution: TODO(XS): Write a solution.

XS14:alg 274. **Alg** What interval of $\mathbb{R}$ contains all such t such that $5x^2y^2 - 4yx - 2x + t = 0$ has a solution in reals x, y?

Comment: Easy-Medium.

Answer: $t \in \mathbb{R}$ **or** $t \in (-\infty, \infty)$

Solution: When looking at it in terms of x, the discriminant will need to be non-negative, i.e.

$$(-4y - 2)^2 - 20y^2t \geq 0 \iff (4 - 5t)y^2 + 4y + 1 \geq 0$$

There is a guarenteed solution when $4 - 5t \geq 0$, and when $4 - 5t < 0$, checking the vertex, i.e. at $y = \frac{-2}{4-5t} \implies \frac{4}{4-5t} - \frac{8}{4-5t} + 1 \geq 0 \implies 1 \geq \frac{4}{4-5t}$ which is always true because $4 - 5t$ is negative.

ZY01:alg 275. **Alg** Let a_n be a sequence of real numbers satisfying

$$8a_{n+2} + (a_na_{n+1} + a_{n+1}a_{n+2} + a_{n+2}a_n) - a_na_{n+1}a_{n+2} = 10(a_{n+1} + a_n - 1).$$

Given that $a_1 = 1 + \sqrt{3}$ and $a_2 = 4$, find a_{2016}.

Answer: $\mathbf{7 - 3\sqrt{3}}$

Solution: We begin by moving all of the terms to one side of the equation. Reversing the sign gives us

$$a_na_{n+1}a_{n+2} - (a_na_{n+1} + a_{n+1}a_{n+2} + a_{n+2}a_n) - 8a_{n+2} + 10a_{n+1} + 10a_n - 10 = 0.$$

The presence of the $a_na_{n+1}a_{n+2}$ and $a_na_{n+1}, a_{n+1}a_{n+2}, a_{n+2}a_n$ terms suggest that we first partially factorize the recursion using a product of the form $(a_n - 1)(a_{n+1} - 1)(a_{n+2} - 1)$. Doing so gives us the recurrence

$$(a_n - 1)(a_{n+1} - 1)(a_{n+2} - 1) - 9a_{n+2} + 9a_{n+1} + 9a_n - 9 = 0.$$

We'd like to make this recurrence look nicer, and the equation suggests we make a substitution of the form $a_n = kb_n + 1$, where k is some integer, to remove the -1 from the products. Due to the multiple occurrences of 9's outside of the product, we try $k = 3$ and substitute to get

$$b_nb_{n+1}b_{n+2} + b_n + b_{n+1} - b_{n+2} = 0.$$

This looks much nicer. Isolating b_{n+2} gives us

$$b_{n+2} = \frac{b_n + b_{n+1}}{1 - b_nb_{n+1}}$$

which looks suspiciously like the tangent angle sum formula. Indeed, allowing $c_n = \tan^{-1}(b_n)$ makes our recurrence become

$$\tan(c_{n+2}) = \frac{\tan(c_n) + \tan(c_{n+1})}{1 - \tan(c_n)\tan(c_{n+1})} = \tan(c_n + c_{n+1}).$$

This looks like the Fibonacci recurrence $c_{n+2} = c_n + c_{n+1}$, but if we note that tan is periodic with period π (i.e. $\tan(x + \pi) = \tan(x)$), then we actually have the recurrence

$$c_{n+2} \equiv c_n + c_{n+1} \pmod{\pi}.$$

Armed with this information, we begin solving for a_{2016}. The initial condition $a_1 = 1 + \sqrt{3}$ gives us $b_1 = \frac{\sqrt{3}}{3}$, and so $c_1 = \frac{\pi}{6}$, while the initial condition $a_2 = 4$ gives us $b_2 = 1$, and so $c_1 = \frac{\pi}{4}$. We make the final substitution $d_n = \frac{12}{\pi}c_n$ to make our life easier while computing the recurrence. The problem then reduces down to computing d_{2016} with initial conditions $d_1 = 2$, $d_2 = 3$, and the recurrence

$$d_{n+2} \equiv d_n + d_{n+1} \pmod{12}.$$

Computing the first few terms (and knowing that the recurrence will have a period of less than 144), we quickly find that $d_{24} = 1$, $d_{25} = 2$, and $d_{26} = 3$. Hence, the recurrence has a period of 24, so $d_{2016} = d_0 = 1$. This gives us $c_{2016} = \frac{\pi}{12}$, $b_{2016} = 2 - \sqrt{3}$ (which can be computed using half angle formulas), and finally $a_{2016} = \boxed{7 - 3\sqrt{3}}$.

ZY02 276. **Alg** Consider the sequence of real numbers a_n satisfying the recurrence

$$a_n a_{n+2} - a_{n+1}^2 - (n+1)a_n a_{n+1} = 0.$$

Given that $a_1 = 1$ and $a_2 = 2016$, compute

$$\frac{a_{2017} \cdot a_{2015}}{a_{2016}^2}.$$

Comment: TODO: Give some justification on why we should divide by $a_n a_{n+1}$.

Answer: $\mathbf{\frac{1009}{1007}}$

Solution: Dividing both sides by $a_n a_{n+1}$ and rearranging gives us

$$\frac{a_{n+2}}{a_{n+1}} = \frac{a_{n+1}}{a_n} + (n+1).$$

If we make the substitution $b_n = \frac{a_{n+1}}{a_n}$, then our recurrence becomes

$$b_{n+1} = b_n + (n+1).$$

Solving this recurrence gives us the general formula

$$b_n = b_1 + (2 + 3 + \cdots + (n-1)) = 2015 + \frac{n(n-1)}{2}.$$

The answer is therefore

$$\frac{a_{2017} \cdot a_{2015}}{a_{2016}^2} = \frac{b_{2016}}{b_{2015}} = \frac{2015 + \frac{2016 \cdot 2015}{2}}{2015 + \frac{2015 \cdot 2014}{2}} = \frac{\frac{2018 \cdot 2015}{2}}{\frac{2016 \cdot 2015}{2}} = \boxed{\frac{1009}{1007}}.$$

ZY03 277. **Alg** Given that x and y are real numbers, compute the minimum value of

$$x^2 + 4x^3 + 8x^2 + 4xy + 6x + 4y^2 + 10.$$

Answer: 8

Solution: The coefficients of x^4 and $4x^3$ suggest that we should try to separate the quartic $(x+1)^4$ from the rest of our expression. In doing so, we are left with

$$(x+1)^4 + 2x^2 + 4xy + 2x + 4y^2 + 9.$$

Note that the remaining terms can be grouped as follows:

$$(x+1)^4 + x^2 + 2x + 1 + x^2 + 4xy + 4y^2 + 8.$$

Completing the squares gives us

$$(x+1)^4 + (x+1)^2 + (x+2y)^2 + 8.$$

Because squares are always non-negative, their minimum is achieved when they are equal to 0. Hence, the minimum is $\boxed{8}$, and we can easily check that the solution $x = -1, y = \frac{1}{2}$ achieves this minimum.

General: 12
Algebra: 44
Geometry: 71
Discrete: 62
Calculus: 54
Team: 34
Unsorted: 0