BMT 2016 — **DISCRETE ROUND** — 12 March 2016

DISCRETE ROUND

1. A bag is filled with quarters and nickels. The average value when pulling out a coin is 10 cents. What is the least number of nickels in the bag possible?

2. How many integers from 1 to 2016 are divisible by 3 or 7, but not 21?

3. How many five-card hands from a standard deck of 52 cards are full houses? A full house consists of 3 cards of one rank and 2 cards of another rank.

4. Three 3-legged (distinguishable) Stanfurdians take off their socks and trade them with each other. How many ways is this possible if everyone ends up with exactly 3 socks and nobody gets any of their own socks? All socks originating from the Stanfurdians are distinguishable from each other. All Stanfurdian feet are indistinguishable from other feet of the same Stanfurdian.

5. What are the last two digits of $9^{8^{\cdot^{\cdot^{\cdot^{2}}}}}$?

6. Bob plays a game on the whiteboard. Initially, the numbers $\{1, 2, ..., n\}$ are shown. On each turn, Bob takes two numbers from the board x, y, erases them both, and writes down $2x + y$ onto the board. In terms of n, what is the maximum possible value that Bob can end up with?

7. Consider the graph on 1000 vertices $v_1, v_2, ...v_{1000}$ such that for all $1 \le i < j \le 1000$, v_i is connected to v_j if and only if i divides j. Determine the minimum number of colors that must be used to color the vertices of this graph such that no two vertices sharing an edge are the same color.

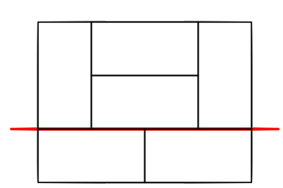
8. Let $(v_1, \cdots, v_{2^n})$ be the vertices of an n-dimensional hypercube. Label each vertex v_i with a real number x_i. Label each edge of the hypercube with the product of labels of the two vertices it connects. Let S be the sum of the labels of all the edges. Over all possible labelings, find the minimum possible value of $\dfrac{S}{x_1^2 + x_2^2 + \cdots + x_{2^n}^2}$ in terms of n.

   Note: an n dimensional hypercube is a graph on 2^n vertices labeled labeled with the binary strings of length n, where two vertices have an edge between them if and only if their labels differ in exactly one place. For instance, the vertices 100 and 101 on the 3 dimensional hypercube are connected, but the vertices 100 and 111 are not.

9. $(\sqrt{6} + \sqrt{7})^{1000}$ in base ten has a tens digit of a and a ones digit of b. Determine $10a + b$.

10. An $m \times n$ rectangle is tiled with $\dfrac{mn}{2}$ 1×2 dominoes. The tiling is such that whenever the rectangle is partitioned into two smaller rectangles, there exists a domino that is part of the interior of both rectangles. Given $mn > 2$, what is the minimum possible value of mn?

    For instance, the following tiling of a 4×3 rectangle doesn't work because we can partition along the line shown, but that doesn't necessarily mean other 4×3 tilings don't work.



1. A $2 \times 4 \times 8$ rectangular prism and a cube have the same volume. What is the difference between their surface areas?

2. Cyclic quadrilateral $ABCD$ has side lengths $AB = 6, BC = 7, CD = 7, DA = 6$. What is the area of $ABCD$?

3. Let S be the set of all non-degenerate triangles with integer sidelengths, such that two of the sides are 20 and 16. Suppose we pick a triangle, at random, from this set. What is the probability that it is acute?

4. ABC is an equilateral triangle, and $ADEF$ is a square. If D lies on side AB and E lies on side BC, what is the ratio of the area of the equilateral triangle to the area of the square?

5. Convex pentagon $ABCDE$ has the property that $\angle ADB = 20^\circ$, $\angle BEC = 16^\circ$, $\angle CAD = 3^\circ$, and $\angle DBE = 12^\circ$. What is the measure of $\angle ECA$?

6. Triangle ABC has sidelengths $AB = 13$, $AC = 14$, and $BC = 15$ and centroid G. What is the area of the triangle with sidelengths AG, BG, and CG?

7. Let ABC be a right triangle with $AB = BC = 2$. Construct point D such that $\angle DAC = 30^\circ$ and $\angle DCA = 60^\circ$, and $\angle BCD > 90^\circ$. Compute the area of triangle BCD.

8. A regular unit 7-simplex is a polytope in 7-dimensional space with 8 vertices that are all exactly a distance of 1 apart. (It is the 7-dimensional analogue to the triangle and the tetrahedron.) In this 7-dimensional space, there exists a point that is equidistant from all 8 vertices, at a distance d. Determine d.

9. Given right triangle ABC with right angle at C, construct three external squares $ABDE$, $BCFG$, and $ACHI$. If $DG = 19$ and $EI = 22$, compute the length of FH.

10. Triangle ABC has side lengths $AB = 5$, $BC = 9$, and $AC = 6$. Define the incircle of ABC to be C_1. Then, define C_i for $i > 1$ to be externally tangent to C_{i-1} and tangent to AB and BC. Compute the sum of the areas of all circles C_n.

You will be given 15 minutes to solve 3 problems. You may submit multiple answers for a problem, and the time will be recorded for each submission. Only your last submission for a given problem will be graded. Competitors will be ranked by the number of correct (graded) submissions, and ties will be broken by earlier time for the last correct (graded) submission.

1. Find all triples of real numbers (a, b, c) such that $a^2 - 4ab + 5b^2 - 2c - 4bc + 5c^2 + 1 = 0$.

2. Suppose that $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$, $f_0 = 0$, $f_1 = 1$. Define $s_n = F_0 + F_1 + \ldots F_n$ for $n \geq 0$. If $s_{n+3} = as_{n+2} + bs_{n+1} + cs_n$ for all $n \geq 0$, then find (a, b, c).

3. For what value of $b > 1$ do the graphs of $y = b^x$ and $y = \log_b(x)$ in the xy-plane have one point of intersection?

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1. Let $Z(n)$ denote the number of zeros at the end of the decimal representation of $n!$. What is the least real number r such that, for all natural numbers n,

$$\frac{Z(n)}{n} > r$$

2. Suppose we have an infinite geometric sequence $\{a_n\}$, starting at $a_1 = 1$. Now suppose that the sum of the areas of circles with radii a_i is equal to the sum of the volumes of spheres with radii a_i, where $i \geq 1$. What is a_2?

3. Suppose that the vertices of triangle ABC lie on $y = x^2 + x + 2$. If the slopes of AB, BC, CA are 4, 5, 6 What is the centroid of ABC? Give your answer as an ordered pair (u, v).

1. Define a_n such that $a_1 = \sqrt{3}$ and for all integers i, $a_{i+1} = a_i^2 - 2$. What is a_{2016}?

2. Jennifer wants to do origami, and she has a square of side length 1. However, she would prefer to use a regular octagon for her origami, so she decides to cut the four corners of the square to get a regular octagon. Once she does so, what will be the side length of the octagon Jennifer obtains?

3. A little boy takes a 12 in long strip of paper and makes a Mobius strip out of it by tapping the ends together after adding a half twist. He then takes a 1 inch long train model and runs it along the center of the strip at a speed of 12 inches per minute. How long does it take the train model to make two full complete loops around the Mobius strip? A complete loop is one that results in the train returning to its starting point.

4. How many graphs are there on 6 vertices with degrees 1,1,2,3,4,5?

5. Let ABC be a right triangle with $AB = BC = 2$. Let ACD be a right triangle with angle $DAC = 30$ degrees and angle $DCA = 60$ degrees. Given that ABC and ACD do not overlap, what is the area of triangle BCD?

6. How many integers less than 400 have exactly 3 factors that are perfect squares?

7. Suppose $f(x, y)$ is a function that takes in two integers and outputs a real number, such that it satisfies
$$f(x, y) = \frac{f(x, y+1) + f(x, y-1)}{2}$$
$$f(x, y) = \frac{f(x+1, y) + f(x-1, y)}{2}$$
What is the minimum number of pairs (x, y) we need to evaluate to be able to uniquely determine f?

8. How many ways are there to divide 10 candies between 3 Berkeley students and 4 Stanford students, if each Berkeley student must get at least one candy? All students are distinguishable from each other; all candies are indistinguishable.

9. How many subsets (including the empty-set) of $\{1, 2..., 6\}$ do not have three consecutive integers?

10. What is the smallest possible perimeter of a triangle with integer coordinate vertices, area $\frac{1}{2}$, and no side parallel to an axis?

11. Circles C_1 and C_2 intersect at points X and Y. Point A is a point on C_1 such that the tangent line with respect to C_1 passing through A intersects C_2 at B and C, with A closer to B than C, such that $2016 \cdot AB = BC$. Line XY intersects line AC at D. If circles C_1 and C_2 have radii of 20 and 16, respectively, find the ratio of $\sqrt{1 + BC/BD}$.

12. Consider a solid hemisphere of radius 1. Find the distance from its center of mass to the base.

13. Consider an urn containing 51 white and 50 black balls. Every turn, we randomly pick a ball, record the color of the ball, and then we put the ball back into the urn. We stop picking when we have recorded n black balls, where n is an integer randomly chosen from $\{1, 2, \ldots, 100\}$ What is the expected number of turns?

14. Consider the set of axis-aligned boxes in $\mathbb{R}^d$, $B(a, b) = \{x \in \mathbb{R}^d : \forall i, a_i \leq x_i \leq b_i\}$ where $a, b \in \mathbb{R}^d$. In terms of d, what is the maximum number n, such that there exists a set of n points $S = \{x_1, ..., x_n\}$ such that no matter how one partition $S = P \cup Q$ with P, Q disjoint and P, Q can possibly be empty, there exists a box B such that all the points in P are contained in B, and all the points in Q are outside B?

15. Let s_1, s_2, s_3 be the three roots of $x^3 + x^2 + \frac{9}{2}x + 9$.

$$\prod_{i=1}^{3}(4s_i^4 + 81)$$

can be written as $2^a 3^b 5^c$. Find $a + b + c$.