Math Basics

1.1 Summation and Product Notation

Exercise 1.1.1

For the set $\{x_1 = 4,\ x_2 = 3,\ x_3 = 2,\ x_4 = -1,\ x_5 = 10,\ x_6 = 0,\ x_7 = -11\}$ calculate:

a) $\sum_{i=1}^{5} x_i$
b) $\sum_{i=2}^{3} x_i$
c) $\sum_{i=4}^{7} x_i$
d) $\sum_{i=1}^{7} x_i$
e) $\prod_{i=2}^{4} x_i$
f) $\prod_{i=1}^{3} x_i$
g) $\prod_{i=1}^{7} x_i$
h) $\prod_{i=4}^{5} x_i$

Exercise 1.1.2

For the set $\{x_1 = 4,\ x_2 = 3,\ x_3 = 2,\ x_4 = -1,\ x_5 = 10,\ x_6 = 0,\ x_7 = -11\}$ calculate:

a) $\sum_{i=1}^{5} x_i$
b) $\sum_{i=2}^{3} x_i$
c) $\sum_{i=4}^{7} x_i$
d) $\sum_{i=1}^{7} x_i$
e) $\prod_{i=2}^{4} x_i$
f) $\prod_{i=1}^{3} x_i$
g) $\prod_{i=1}^{7} x_i$
h) $\prod_{i=4}^{5} x_i$

1.2 Exponentiation and Logarithms

Exercise 1.2.1

Simplify:

a) $x^3 \cdot x^7$
b) $x^{-3} \cdot x^5$
c) $x^{-5} \cdot x^4$
d) $x^3 : x^4$
e) $x^3 : x^{-3}$
f) $a^{-7} : a^{-7}$
g) $\left(x^3\right)^5$
h) $\left[\left(c^3\right)^3\right]^3$
i) $\left[\left(a^2\right)^b\right]^2$

Exercise 1.2.2

Simplify:

a) $\log_2 16$
b) $\log_{\frac{1}{2}} 16$
c) $\log_2 64$
d) $\log_{\frac{1}{2}} 64$
e) $\log_3 \mathrm{x}^{\frac{1}{2}}$
f) $\ln(\mathrm{e})$
g) $\ln\left(\mathrm{e}^8\right)$
h) $\ln(2) + \ln(10)$
i) $\ln(9) - \ln(3)$
j) $\ln(27)$
k) $\ln(81)$
l) $\ln(2) - \ln(1)$
m) $\ln(2) + \ln(15) - \ln(3)$

1.3 Important Functions and Plots

Exercise 1.3.1

Draw and find X and Y-intercept of the following functions:

a) $y = x$
b) $y = -x$
c) $y = 2x + 3$
d) $y = 3x - 2$
e) $y = -2x + 5$
f) $y = -2x - 1$

Exercise 1.3.2

Find zeros, coordinates of the vertex and draw the following functions:

a) $y = 3x^2 + x + 5$
b) $y = 16x^2 + 8x + 1$
c) $y = -2x^2 + 3x + 7$
d) $y = -2x^2 + 6x + 5$
e) $y = -2x^2 - 8x + 10$
f) $y = 3x^2 + 2\text{x}$
g) $y = -0,5x^2 - 4x - 8$
h) $y = x^2 + 2x\_6$

Exercise 1.3.3

Draw the following hyperbolic functions:

a) $y = \frac{1}{x}$
b) $y = \frac{1}{x+1} + 3$
c) $y = \frac{1}{x^2} - 2$
d) $y = \frac{2}{(1+x)^2} - 1$
e) $y = \left|\frac{1}{x}\right|$
f) $y = \left|\frac{1}{x^2}\right|$
g) $y = -\left|\frac{1}{x}\right| - 2$

exponential functions:

a) $y = 2^x$
b) $y = (\frac{1}{2})^x + 3$
c) $y = 3 \cdot 2^x$
d) $y = 3 \cdot (\frac{1}{2})^x$
e) $y = 2^{2x}$
f) $y = (\frac{1}{2})^{2x}$
g) $y = 2 \cdot 3^{x-1} - 1$
h) $y = 2 \cdot (\frac{1}{4})^{2x-2}$

logarithmic functions:

a) $y = \log_2 x$
b) $y = \log_{\frac{1}{2}} \text{x}$
c) $y = |\log_2 \text{x}|$
d) $y = \left|\log_{\frac{1}{2}} x\right|$
e) $y = \log_2(x - 1) + 2$
f) $y = \log_{\frac{1}{2}}(x + 1) + 3$

trigonometric functions:

a) $y = \sin(x)$
b) $y = 2\sin(x)$
c) $y = \sin(2x)$
d) $y = \cos(x)$
e) $y = 2\cos(x)$
f) $y = \text{tg}(x)$
g) $y = \text{tg}(2x)$
h) $y = \text{ctg}(x)$

Exercise 1.3.4

Compare:

a) $y = 2^x$ and $y = \log_2 x$
b) $y = (\frac{1}{2})^x$ and $y = \log_{(\frac{1}{2})} x$

1.4 Logic

Exercise 1.4.1

Check whether the following are tautologies:

a) $\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$

b) $[(p \vee q) \wedge \neg p] \rightarrow q$

c) $[(\neg p) \rightarrow q] \leftrightarrow [(\neg q) \rightarrow p]$

d) $[p \rightarrow (\neg p)] \leftrightarrow \neg p$

e) $(p \rightarrow q) \leftrightarrow [(\neg q) \rightarrow \neg p]$

f) $p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$

g) $p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$

h) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

1.5 Set Thoery

Exercise 1.5.1

Given the following: $\Omega = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, \mathrm{A} = \{1, 2, 3, 4, 5\}$,
$\mathrm{B} = \{0, 2, 4, 6, 8, 10\}, \mathrm{C} = \{2, 3, 5, 7\}$ find:

a) $\mathrm{A} \cup \mathrm{B}$

b) $\mathrm{A}' \cup \mathrm{B}'$

c) $\mathrm{A}' \cap \mathrm{B}$

d) $\mathrm{A} \cap (\mathrm{B} \cup \mathrm{C})$

e) $\mathrm{A} \backslash \mathrm{B}$

f) $\mathrm{A} \backslash (\mathrm{B} \backslash \mathrm{C})$

g) $\mathrm{A} \backslash \mathrm{B} \cap \mathrm{C} \backslash \mathrm{B}$

h) $\mathrm{A} \cap \mathrm{C}$

i) $\mathrm{A} \cap \mathrm{B} \cap \mathrm{C}$

j) $\mathrm{A} \cup (\mathrm{B} \cap \mathrm{C})$

k) $(\mathrm{A} \cup \mathrm{B})'$

l) $\mathrm{B} \backslash \mathrm{A}$

m) $\mathrm{C} \backslash (\mathrm{B} \backslash \mathrm{A})$

n) $(\mathrm{A} \backslash \mathrm{B}) \cap (\mathrm{A} \backslash \mathrm{C})$

Chapter 2

Linear Algebra

2.1 Eigenvalues and Eigenvectors

Exercise 2.1.1

Express each of the following quadratic forms as a matrix product involving symmetric coefficient matrix:

a) $q = 4x_1^2 - 4x_1x_2 + 9x_2^2$

b) $q = x_1^2 + 7x_1x_2 + 3x_2^2$

c) $q = 8x_1x_2 - x_1^2 + 5x_2^2$

d) $q = 6x_!x_2 + 5x_2^2 - 2x_1^2$

e) $q = 3x_1^2 - 2x_1x_2 + 4x_1x_3 + 5x_2^2 + 4x_3^2 - 2x_2x_3$

Exercise 2.1.2

Find eigenvalues of the following matrices:

a) $A = \begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix}$ b) $A = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$

c) $A = \begin{bmatrix} 5 & 3 \\ 3 & 0 \end{bmatrix}$ d) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Exercise 2.1.3

Determine the eigenvalues, eigenvectors and trace of the matrices below. In each case, check whether the sum of all eigenvalues is equal to the trace.

a) $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ c) $A = \begin{bmatrix} 5 & 1 \\ 15 & 3 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ e) $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ f) $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$

g) $A = \begin{bmatrix} -5 & 0 \\ -7 & 4 \end{bmatrix}$ h) $A = \begin{bmatrix} -5 & 2 \\ 0 & 6 \end{bmatrix}$ i) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

j) $A = \begin{bmatrix} 1 & 4 & 4 \\ 0 & 2 & 0 \\ 0 & -2 & -1 \end{bmatrix}$ k) $A = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ l) $A = \begin{bmatrix} -1 & 3 & 0 \\ 0 & -2 & 0 \\ 4 & -6 & 1 \end{bmatrix}$

m) $A = \begin{bmatrix} 1 & -2 & -2 \\ -4 & -11 & -8 \\ 4 & 13 & 10 \end{bmatrix}$ n) $A = \begin{bmatrix} 2 & 6 & 6 \\ 0 & 3 & 0 \\ 0 & -3 & -1 \end{bmatrix}$ o) $A = \begin{bmatrix} 3 & 10 & 10 \\ -1 & 0 & -2 \\ 1 & -2 & 0 \end{bmatrix}$

Exercise 2.1.4

Determine whether the matrices from Exercise 2.1.1 are definite or indefinite. If they are definite, determine whether they are positive, semi-positive, negative or semi-negative definite.

Exercise 2.1.5

Find the cosine of the angle between vectors a and b:

a) $\mathbf{a} = [1, 0]$ and $\mathbf{b} = [0, 1]$
b) $\mathbf{a} = [3, 4]$ and $\mathbf{b} = [-3, -4]$
c) $\mathbf{a} = [3, 4]$ and $\mathbf{b} = [4, 3]$
d) $\mathbf{a} = [3, 4]$ and $\mathbf{b} = [1, 0]$
e) $\mathbf{a} = [1, 2, 3]$ and $\mathbf{b} = [-1, 2, 4]$
f) $\mathbf{a} = [1, 2, 1]$ and $\mathbf{b} = [-1, 2, 4]$

Exercise 2.1.6

Verify whether the following vectors are linearly independent:

a) $\mathbf{a} = [1, 0]$ and $\mathbf{b} = [0, 1]$
b) $\mathbf{a} = [3, 4]$ and $\mathbf{b} = [-3, -4]$
c) $\mathbf{a} = [3, 4]$ and $\mathbf{b} = [4, 3]$
d) $\mathbf{a} = [3, 4]$ and $\mathbf{b} = [1, 0]$
e) $\mathbf{a} = [1, 0, 0]$, $\mathbf{b} = [0, 2, 0]$ and $\mathbf{c} = [0, 0, 8]$
f) $\mathbf{a} = [1, 2, 1]$, $\mathbf{b} = [0, 2, 0]$ and $\mathbf{c} = [-1, 2, 4]$

2.2 MATHEMATICS EXCERCISES–SET 2 - before eigenvalues

Exercise 2.2.1

Given:

$$\mathrm{A} = \begin{bmatrix} 7 & -1 \\ 6 & 9 \end{bmatrix}, \mathrm{B} = \begin{bmatrix} 0 & 4 \\ 3 & -2 \end{bmatrix}, \mathrm{C} = \begin{bmatrix} 8 & 3 \\ 6 & 1 \end{bmatrix}, \text{ find:}$$

a) $\mathrm{A} + \mathrm{B}$
b) $\mathrm{C} - \mathrm{A}$
c) $\mathrm{A} - \mathrm{C} + \mathrm{B}$
d) $3\ \mathrm{A}$
e) $2.5\ \mathrm{C}$
f) $2\ \mathrm{A} - 3\ \mathrm{B}$
g) $4\ \mathrm{B} + 2\ \mathrm{C}$
h) $\mathrm{A} + 2\ \mathrm{B} - 3\ \mathrm{C}$
i) $2\ \mathrm{C} - 3\ \mathrm{A} + 2\ \mathrm{B}$

Exercise 2.2.2

Give:

$$\mathrm{A} = \begin{bmatrix} 2 & 1 & 3 \\ 5 & 0 & 9 \\ 17 & 4 & 5 \end{bmatrix}, \mathrm{B} = \begin{bmatrix} 3 & 12 & 1 \\ 2 & 4 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \mathrm{C} = \begin{bmatrix} 1 & 8 & 12 \\ 1 & 9 & 20 \\ 1 & 7 & 6 \end{bmatrix}, \mathrm{D} = \begin{bmatrix} 5 & 22 & 4 \\ -5 & 9 & -1 \\ 3 & 8 & -7 \end{bmatrix}, \text{ find:}$$

a) A + B
b) A − B
c) B − A
d) C + A
e) C − A
f) 4 C − 3 D
g) 2 B − C
h) D + C
i) 2 C − 3 A + 4 D
j) D − 2 A
k) 2 C + D − 4 A
l) A + B − C − D
m) D − A + C − B

Exercise 2.2.3

Find:

$$\begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \end{bmatrix} =?$$

Exercise 2.2.4

Given:

$$\mathrm{A} = \begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix}, \mathrm{B} = \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix}, \mathrm{C} = \begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}, \text{ find:}$$

a) Is AB defined? Calculate AB. Can you calculate BA? Why?

b) Is BC defined? Calculate BC. Is CB defined? If, so calculate CB. Is it true that BC = CB.

Exercise 2.2.5

Find the product matrices in the following (in each case, append beneath every matrix a dimension indicator):

a) $\begin{bmatrix} 7 & -1 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 3 & -2 \end{bmatrix}$

b) $\begin{bmatrix} 8 & 3 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 7 & -1 \\ 6 & 9 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 4 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 1 \\ 3 & 5 \end{bmatrix}$

d) $\begin{bmatrix} 3 & 5 & 0 \\ 4 & 2 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

e) $\begin{bmatrix} 6 & 5 & -1 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 5 & 2 \\ 0 & 1 \end{bmatrix}$

f) $\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 2 \\ 1 & 4 \end{bmatrix}$

g) $\begin{bmatrix} 0 & 2 & 4 \\ 3 & 0 & 4 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 9 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 5 & 0 \end{bmatrix}$

h) $\begin{bmatrix} 1 & 2 & 9 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix} \begin{bmatrix} 9 & 0 & 10 \\ 3 & 0 & 11 \\ 7 & 1 & 0 \end{bmatrix}$

i) $\begin{bmatrix} 5 & 1 & 2 \\ -7 & 2 & 1 \\ 2 & 5 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 8 \\ 1 & 8 & 11 \\ 3 & 1 & 0 \end{bmatrix}$

j) $\begin{bmatrix} -1 & 5 & 1 \\ 2 & 5 & 1 \\ 4 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \\ 3 & 3 & 1 \\ 3 & 1 & 0 \end{bmatrix}$

k) $\begin{bmatrix} 2 & 1 & 3 \\ 5 & 0 & 9 \\ 17 & 4 & 5 \end{bmatrix} \begin{bmatrix} 3 & 12 & 1 \\ 2 & 4 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

l) $\begin{bmatrix} 1 & 8 & 12 \\ 1 & 9 & 20 \\ 1 & 7 & 6 \end{bmatrix} \begin{bmatrix} 5 & 22 & 4 \\ -5 & 9 & -1 \\ 3 & 8 & -7 \end{bmatrix}$

m) $\begin{bmatrix} 3 & 12 & 1 \\ 2 & 4 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 22 & 4 \\ -5 & 9 & -1 \\ 3 & 8 & -7 \end{bmatrix}$

n) $\begin{bmatrix} 1 & 8 & 12 \\ 1 & 9 & 20 \\ 1 & 7 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 5 & 0 & 9 \\ 17 & 4 & 5 \end{bmatrix}$

o) $\begin{bmatrix} 3 & 12 & 1 \\ 2 & 4 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 22 & 4 \\ -5 & 9 & -1 \\ 3 & 8 & -7 \end{bmatrix}$

p) $\begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 4 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 5 & 0 & 9 \\ 17 & 4 & 5 \end{bmatrix}$

q) $\begin{bmatrix} 5 & 22 & 4 \\ -5 & 9 & -1 \\ 3 & 8 & -7 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 4 \\ 2 & 3 & 0 \end{bmatrix}$

r) $\begin{bmatrix} 1 & 0 & 2 & 9 \\ 0 & 2 & 1 & 5 \\ 2 & 4 & 1 & 2 \\ 1 & 6 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 & 0 \\ 4 & 0 & 5 & 3 \\ 7 & 1 & 0 & 4 \\ 3 & 2 & 1 & 1 \end{bmatrix}$

s) $\begin{bmatrix} 2 & 1 & 1 & 9 \\ 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 2 \\ 7 & 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 9 & 4 & 7 & 0 \\ 4 & 0 & 1 & 0 \\ 6 & 1 & 0 & 4 \\ 3 & 2 & 1 & 0 \end{bmatrix}$

t) $\begin{bmatrix} 8 & 1 & 2 & 0 \\ 3 & 2 & 0 & 0 \\ 2 & 4 & 4 & 1 \\ 5 & 3 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 6 & 2 & 1 \\ 4 & 0 & 0 & 3 \\ 5 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 \end{bmatrix}$

Exercise 2.2.6

Expand the following summation expressions:

a) $\sum_{i=1}^{5} x_i$

b) $\sum_{i=5}^{8} a_i x_i$

c) $\sum_{i=1}^{4} b x_i$

d) $\sum_{i=1}^{n} a_i x^{i-1}$

e) $\sum_{i=0}^{3} (x-i)^2$

Exercise 2.2.7

Rewrite the following in Σ notation:

a) $x_1 (x_1 - 1) + 2x_2 (x_2 - 1) + 3x_3 (x_3 - 1)$

b) $a_2 (x_3 - 2) + a_3 (x_4 - 3) + a_4 (x_5 - 4)$

c) $\frac{1}{x} + \frac{1}{x^2} + \ldots + \frac{1}{x^n}$

d) $1 + \frac{1}{x} + \frac{1}{x^2} + \ldots + \frac{1}{x^n}$

Exercise 2.2.8

Show that the following are true:

a) $(\sum_{i=0}^{n} x_i) + x_{n+1} = \sum_{i=0}^{n+1} x_i$ b) $\sum_{j=1}^{n} ab_j y_j = a \sum_{j=1}^{n} b_j y_j$

c) $\sum_{j=1}^{n} (x_j + y_i) = \sum_{j=1}^{n} x_j + \sum_{j=1}^{n} y_j$

Exercise 2.2.9

Given $\mathrm{u}' = \begin{bmatrix} 3 & 4 \end{bmatrix}$ and $\mathrm{v} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$ find:

a) u′v b) uv c) vu d) vu′

Exercise 2.2.10

Given $\mathrm{u}' = \begin{bmatrix} 5 & 1 & 3 \end{bmatrix}, \mathrm{v}' = \begin{bmatrix} 3 & 1 & -1 \end{bmatrix}, \mathrm{w}' = \begin{bmatrix} 7 & 5 & 8 \end{bmatrix}$, and $\mathrm{x}' = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$, write out the column vectors, u, v, w, and, x and find:

a) uv′ c) xx′ e) u′v g) u′u

b) uw′ d) v′u f) w′x h) x′x

Exercise 2.2.11

Given $\mathrm{u} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $\mathrm{v} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, find the following graphically:

a) 2 v c) u − v e) 2 u + 3 v

b) u + v d) v − u f) 4 u − 2 v

Exercise 2.2.12

Given $\mathrm{A} = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}, \mathrm{B} = \begin{bmatrix} -1 & 7 \\ 8 & 4 \end{bmatrix}$, and $\mathrm{C} = \begin{bmatrix} 3 & 4 \\ 1 & 9 \end{bmatrix}$, verify that:

a) (A + B) + C = A + (B + C) b) (A + B) − C = A + (B − C)

Exercise 2.2.13

Given $\mathrm{A} = \begin{bmatrix} 0 & 2 \\ 1 & -4 \end{bmatrix}, \mathrm{B} = \begin{bmatrix} 7 & 1 \\ -2 & 5 \end{bmatrix}$, and $\mathrm{C} = \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix}$, verify that:

a) (A + B) + C = A + (B + C) b) (A + B) − C = A + (B − C)

Exercise 2.2.14

Test the associative law of multiplication with the following matrices:

$$\mathrm{A} = \begin{bmatrix} 5 & 3 \\ 0 & 5 \end{bmatrix}, \mathrm{B} = \begin{bmatrix} -8 & 0 & 7 \\ 1 & 3 & 2 \end{bmatrix}, \text{ and } \mathrm{C} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 7 & 1 \end{bmatrix}$$

Exercise 2.2.15

Test the associative law of multiplication with the following matrices:

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 4 \end{bmatrix}, B = \begin{bmatrix} 7 & 0 & 8 \\ 1- & 1 & 0 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 2 & 4 \\ 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Exercise 2.2.16

Given $A = \begin{bmatrix} -1 & 5 & 7 \\ 0 & -2 & 4 \end{bmatrix}, b = \begin{bmatrix} 9 \\ 6 \\ 0 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, calculate:

a) AI b) IA c) Ix d) bI e) x′I

Indicate dimension of the identity matrix used in each case.

Exercise 2.2.17

Given $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -4 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, calculate:

a) AI b) IA c) Ix d) bI e) x′I

Indicate dimension of the identity matrix used in each case.

Exercise 2.2.18

Given $A = \begin{bmatrix} 0 & 4 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & -8 \\ 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 & 9 \\ 6 & 1 & 1 \end{bmatrix}$, find A', B' and C'.

Exercise 2.2.19

Given $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 7 \\ 0 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 6 \end{bmatrix}$, find A', B' and C'

Exercise 2.2.20

Use the matrices given in Exercise 2.2.12, 2.2.13 2.2.14 and 2.2.15 to verify that:

a) $(A + B)' = A' + B'$ b) $(AC)' = C'A'$

2.3 MATHEMATICS EXCERCISES–SET 3

Exercise 2.3.1

Find A′ if:

a) $A = \begin{bmatrix} 5 & 2 \\ 0 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} -1 & 0 \\ 9 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 3 & 7 \\ 3 & -1 \end{bmatrix}$

d) $A = \begin{bmatrix} 5 & 0 \\ 8 & 1 \end{bmatrix}$

e) $A = \begin{bmatrix} 6 & 3 \\ 8 & 4 \end{bmatrix}$

f) $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$

g) $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

h) $A = \begin{bmatrix} -7 & 0 & -3 \\ 9 & 1 & 4 \\ 0 & 6 & 5 \end{bmatrix}$

i) $A = \begin{bmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ 7 & 8 & 9 \end{bmatrix}$

j) $A = \begin{bmatrix} 8 & 1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 3 \end{bmatrix}$

k) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -7 & 5 \\ 3 & 6 & 9 \end{bmatrix}$

l) $A = \begin{bmatrix} 4 & 0 & 2 \\ 6 & 0 & 3 \\ 8 & 2 & 3 \end{bmatrix}$

m) $A = \begin{bmatrix} 1 & 1 & 2 \\ 8 & -11 & 3 \\ 0 & 4 & 3 \end{bmatrix}$

n) $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

o) $A = \begin{bmatrix} x & 5 & 0 \\ 3 & y & 2 \\ 9 & -1 & 8 \end{bmatrix}$

Exercise 2.3.2

Given: $A = \begin{bmatrix} 0 & 4 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & -8 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 9 \\ 6 & 1 & 1 \end{bmatrix}$, verify:

a) $(A + B)' = A' + B'$

b) $(AC)' = C'A'$

Exercise 2.3.3

Use simplified formula and Laplace expansion to find values of determinants of following matrices:

a) $A = \begin{bmatrix} 5 & 2 \\ 0 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} -1 & 0 \\ 9 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 3 & 7 \\ 3 & -1 \end{bmatrix}$

d) $A = \begin{bmatrix} 4 & 2 \\ 8 & 0 \end{bmatrix}$

e) $A = \begin{bmatrix} -3 & 0 \\ 2 & 1 \end{bmatrix}$

f) $A = \begin{bmatrix} 2 & 4 \\ 9 & -1 \end{bmatrix}$

g) $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

h) $A = \begin{bmatrix} -7 & 0 & 3 \\ 9 & 1 & 4 \\ 0 & 6 & 5 \end{bmatrix}$

i) $A = \begin{bmatrix} -2 & 1 & 3 \\ -6 & 3 & 9 \\ 7 & 8 & 9 \end{bmatrix}$

j) $A = \begin{bmatrix} 8 & -1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 3 \end{bmatrix}$

k) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \\ 3 & 6 & 9 \end{bmatrix}$

l) $A = \begin{bmatrix} 4 & 0 & 2 \\ 6 & 0 & -3 \\ 8 & 2 & 3 \end{bmatrix}$

m) $A = \begin{bmatrix} 1 & 1 & 2 \\ 8 & 11 & 3 \\ 0 & 4 & 3 \end{bmatrix}$

n) $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

o) $A = \begin{bmatrix} x & 5 & 0 \\ 3 & y & 2 \\ 9 & -1 & 8 \end{bmatrix}$

Exercise 2.3.4

Evaluate determinants of the following matrices:

a) $\begin{bmatrix} 1 & 2 & 0 & 9 \\ 2 & 3 & 4 & 6 \\ 1 & 6 & 0 & -1 \\ 0 & -5 & 0 & 8 \end{bmatrix}$

b) $\begin{bmatrix} 2 & 7 & 0 & 1 \\ 5 & 6 & 4 & 8 \\ 0 & 0 & 9 & 0 \\ 1 & -3 & 1 & 4 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 3 & 0 & 3 \\ 2 & 1 & 2 & 7 \\ 5 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 8 & 0 & 0 & 5 \\ 3 & 0 & 0 & 1 \\ 7 & 1 & 9 & 7 \\ 5 & 0 & 1 & 8 \end{bmatrix}$

e) $\begin{bmatrix} 7 & 0 & 1 & 0 \\ 6 & -9 & 8 & 0 \\ 3 & 8 & 2 & 0 \\ 6 & 3 & 8 & 1 \end{bmatrix}$

Exercise 2.3.5

Use the determinant $\begin{vmatrix} 4 & 0 & -1 \\ 2 & 1 & -7 \\ 3 & 3 & 9 \end{vmatrix}$ to verify following properties of determinants:

a) $|A| = |A'|$

b) Change of two rows/columns will alter the sign of determinant numerical value

c) The multiplication of one row/column by scalar k will change the value of determinant k-fold.

Exercise 2.3.6

Which properties of determinants enable us to write the following?

a) $\begin{vmatrix} 9 & 18 \\ 27 & 56 \end{vmatrix} = \begin{vmatrix} 9 & 18 \\ 0 & 2 \end{vmatrix}$

b) $\begin{vmatrix} 9 & 27 \\ 4 & 2 \end{vmatrix} = 18 \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$

Exercise 2.3.7

Find the inverse of each of the following matrices:

a) $\begin{bmatrix} 5 & 2 \\ 0 & 1 \end{bmatrix}$

b) $\begin{bmatrix} -1 & 0 \\ 9 & 2 \end{bmatrix}$

c) $\begin{bmatrix} 3 & 7 \\ 3 & -1 \end{bmatrix}$

d) $\begin{bmatrix} 4 & -2 & 1 \\ 7 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 3 \\ 4 & 0 & 2 \end{bmatrix}$

f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

g) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Exercise 2.3.8

Solve the system Ax = d by matrix inversion:

a) $\begin{cases} 4x + 3y = 28 \\ 2x + 5y = 42 \end{cases}$

b) $\begin{cases} 4x_1 + x_2 + -5x_3 = 8 \\ -2x_1 + 3x_2 + x_3 = 12 \\ 3\mathrm{x}_1 - \mathrm{x}_2 + 4\mathrm{x}_3 = 5 \end{cases}$

Exercise 2.3.9

Use Cramer's rule and matrix inversion to solve the following equation systems:

a) $\begin{cases} 3x_1 - 2x_2 = 6 \\ 2x_1 + x_2 = 11 \end{cases}$

b) $\begin{cases} -x_1 + 3x_2 = -3 \\ 4x_1 - x_2 = 12 \end{cases}$

c) $\begin{cases} 8x_1 - 7x_2 = 9 \\ x_1 + x_2 = 3 \end{cases}$

d) $\begin{cases} 5x_1 + 9x_2 = 14 \\ 7x_1 - 3x_2 = 4 \end{cases}$

e) $\begin{cases} 8x_1 - x_2 = 16 \\ 2x_2 + 5x_3 = 5 \\ 2x_1 + 3x_3 = 7 \end{cases}$

f) $\begin{cases} -x_1 + 3x_2 + 2x_3 = 24 \\ x_1 + x_3 = 6 \\ 5x_2 - x_3 = 8 \end{cases}$

g) $\begin{cases} 4x + 3y - 2z = 1 \\ x + 2y = 6 \\ 3x + z = 4 \end{cases}$

h) $\begin{cases} -x + y + z = a \\ x - y + z = b \\ x + y - z = c \end{cases}$

i) $\begin{cases} x + y + z = 0 \\ 2x - y - z = -3 \\ 4x - 5y - 3z = -7 \end{cases}$

j) $\begin{cases} x - y + 2z = -3 \\ -x + y + z = 0 \\ 2x - y + 2z = -3 \end{cases}$

k) $\begin{cases} 2x + y + z = 0 \\ 4x - 3y + z = 1 \\ 6x + 2z = -2 \end{cases}$

l) $\begin{cases} x + y + z + u = 0 \\ -x + 2y - 2z + 3u = 0 \\ 2x + 3y + 3z + u = 0 \\ 3y - z + 4u = 1 \end{cases}$

m) $\begin{cases} x + y + z + t = -2 \\ -x + y - z - t = 0 \\ x - y - z - t = 1 \\ 2x - y - z - 3t = -1 \end{cases}$

One Variable Calculus

3.1 Limits

Exercise 3.1.1

Calculate the following limits:

a) $\lim_{x\to 1}(x+2)(x-3)$
b) $\lim_{x\to 3}(x+2)(x-3)$
c) $\lim_{x\to 5}(x+2)(x-3)$
d) $\lim_{x\to 0}\frac{3x+5}{x+2}$
e) $\lim_{x\to 5}\frac{3x+5}{x+2}$
f) $\lim_{x\to -1}\frac{3x+5}{x+2}$
g) $\lim_{x\to 0} 7-9x-x^2$
h) $\lim_{x\to 3} 7-9x-x^2$
i) $\lim_{x\to -1} 7-9x-x^2$
j) $\lim_{x\to 1}\frac{x^2+x-56}{x-7}$
k) $\lim_{x\to 7}\frac{x^2+x-56}{x-7}$
l) $\lim_{x\to 2}\frac{(x+2)^3-8}{x}$
m) $\lim_{x\to 0}\frac{(x+2)^3-8}{x}$
n) $\lim_{x\to 2}\frac{x^2-5x+6}{x^2-x-2}$
o) $\lim_{x\to\infty}(5-\frac{1}{x})$
p) $\lim_{x\to-\infty}(5-\frac{1}{x})$
q) $\lim_{x\to\infty}\frac{100}{x}$
r) $\lim_{x\to\infty}\frac{1}{x^2}$
s) $\lim_{x\to\infty}\frac{5}{\sqrt{x}}$
t) $\lim_{x\to\infty}\frac{7}{9^x}$
u) $\lim_{x\to\infty}\frac{5\cdot 8^x+11}{3\cdot 8^x-1}$
v) $\lim_{x\to\infty}\frac{2\cdot 4^x+5\cdot 3^x-13}{7\cdot 4^x+3}$
w) $\lim_{x\to\infty}\sqrt{x+2}-\sqrt{x}$
x) $\lim_{x\to\infty} x-\sqrt{x^2+7x-1}$

Exercise 3.1.2

Determine the limits of the function at the borders of the function domain. Then draw a graph of the function.

a) $f(x)=2x-3$
b) $f(x)=-x+2$
c) $f(x)=(x-2)^2-3$
d) $f(x)=-(x+3)^2+2$
e) $f(x)=x^3+2$
f) $f(x)=-(x-1)^3-2$
g) $f(x)=\frac{1}{x}$
h) $f(x)=\frac{1}{x-3}+2$
i) $f(x)=\frac{1}{x^2}-3$
j) $f(x)=-\frac{1}{x^2+2x+1}-3$
k) $f(x)=e^x$
l) $f(x)=e^{x-3}+2$
m) $f(x)=2^{2x-2}+2$
n) $f(x)=(\frac{1}{3})^x+2$
o) $f(x)=(\frac{1}{2})^x-1$
p) $f(x)=\ln x$
q) $f(x)=-\ln x$
r) $f(x)=\ln(x-2)+1$
s) $f(x)=\log_2 x+1$
t) $f(x)=\log_{\frac{1}{3}} x$
u) $f(x)=\log_2(x-1)+2$

3.2 Definition of a derivative

Exercise 3.2.1

Find difference quotient, derivative (using the concept of limit) and value of derivative at $x = 0$, $x = 1$ and $x = 3$ for the following functions:

a) $f(x) = x^2 - 5$
b) $f(x) = -2x^2 + 7$
c) $f(x) = 3x^2 - 2x$
d) $f(x) = -x^2 + x - 12$
e) $f(x) = 2x^2 - 3x + 4$
f) $f(x) = x^2 - 4x + 3$
g) $f(x) = 3x^2 - 2x + 3$
h) $f(x) = -x^2 + 3x + 2$
i) $f(x) = 4x^2 + 5x + 2$
j) $f(x) = x^3 - 17$
k) $f(x) = -x^3 + 3x^2 + 3x + 6$

3.3 Formulas for derivatives

$f(x)$	$f'(x)$	Assumptions
a	0	$a \in \mathbb{R}$
x^n	nx^{n-1}	$n \in \mathbb{R}$
e^x	e^x	
a^x	$a^x \ln a$	$a > 0$
$\ln x$	$\frac{1}{x}$	$x > 0$
$\log_a x$	$\frac{1}{x \ln a}$	$a > 0, a \neq 1, x > 0$
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$a \cdot g(x)$	$a \cdot g'(x)$	$a \in \mathbb{R}$
$g(x) + h(x)$	$g'(x) + h'(x)$	
$g(x)h(x)$	$g'(x)h(x) + g(x)h'(x)$	
$\frac{g(x)}{h(x)}$	$\frac{g'(x)h(x)-g(x)h'(x)}{h^2(x)}$	
$h(g(x))$	$h'(g(x))g'(x)$	

Exercise 3.3.1

Find derivatives of the following functions:

a) $f(x) = 3x + 1$
b) $f(x) = x^{12} - 2$
c) $f(x) = 63$
d) $f(x) = 7x^5 + 30$
e) $f(u) = 3u^{-1} + 4$
f) $f(u) = -4u^{\frac{1}{2}} - 10$
g) $f(u) = 4u^{\frac{1}{4}} - 1$
h) $f(x) = -x^4 + e$
i) $f(x) = 9x^{\frac{1}{3}}$
j) $f(w) = 5w^4 + 17$
k) $f(x) = cx^2 - 80$
l) $f(x) = ax^b + \pi$
m) $f(x) = ax^{-b} + d$

Exercise 3.3.2

Find $f'(1)$ and $f'(2)$:

a) $f(x) = 18x + 1$
b) $f(x) = cx^3 - c$
c) $f(x) = -5x^{-2}$
d) $f(x) = \frac{3}{4}x^{\frac{4}{3}} + 17$
e) $f(u) = 6x^{\frac{1}{6}}$
f) $f(u) = -3w^{-\frac{1}{6}}$

Exercise 3.3.3

Find derivatives of the following functions:

a) $f(x) = 12x^{\frac{7}{4}} + 5x^{-\frac{4}{3}} + 5x^3 + 4x^2 - 3x + 0.67$

b) $f(x) = 6x^{\frac{2}{3}} - 2x^{-\frac{1}{3}} + 2x^4 + 4x^5 + 81x + 112.5$

c) $f(x) = \frac{3}{\sqrt[3]{x^4}} - \frac{2}{\sqrt[5]{x^2}} + \sqrt[-2]{x^7} + \sqrt[7]{x^8} + 5x^3 + 2x - 12.3$

d) $f(x) = \frac{5}{\sqrt[2]{x^5}} - \frac{7}{\sqrt[4]{x^3}} + \sqrt[3]{x^7} + \sqrt[5]{x^3} + 6x^7 + 13x - 11.11$

e) $f(x) = \frac{7}{\sqrt[3]{x^2}} - \frac{3}{\sqrt[7]{x^8}} + \sqrt[-3]{x^4} + \sqrt[3]{x^3} + 3x^5 - 9x + 8.7$

f) $f(x) = 4x^3 - \frac{3}{\sqrt[4]{x^5}} + \sqrt[-4]{x^9} + \sqrt[7]{x^8} + 12x^2 - 4x - 5.5$

g) $f(x) = \frac{-1}{\sqrt[3]{x^5}} - \frac{2}{\sqrt[4]{x^3}} + \sqrt[-7]{x^6} + \sqrt[2]{x^5} + 7x^7 + 3x - 1.32$

Exercise 3.3.4

Differentiate using the product rule and check the result by first simplifying the formula and then differentiating:

a) $f(x) = x(x-2)$
b) $f(x) = x^3(2-x)$
c) $f(x) = (9x^2-2)(3x+1)$
d) $f(x) = (2x-1)(5x-2)$
e) $f(x) = (3x+10)(6x^2-7x)$
f) $f(x) = (1+x^3)(2x^2-3)$
g) $f(x) = (x^3+3)(2x^2-2x)$
h) $f(x) = (2x^4+3x)(x^3-5x^2)$
i) $f(x) = (2x-x^4)(2x^3-x^2)$
j) $f(x) = x^2(4x+6)$
k) $f(x) = (ax-b)(cx^2)$
l) $f(x) = x^3(3-x^2)(x-2)$
m) $f(x) = (2-3x)(1+x)(x+2)$
n) $(2x-2)(3x-4)(x2-1)(3+x)$
o) $(5-x^2)x^5$
p) $(x^2+3)x^{-1}$
q) $(x^3+1)x^{-7}$
r) $(2x+x^4)x^{-3}$

Exercise 3.3.5

Find the derivatives using quotient rule:

a) $\frac{(x^2+3)}{x}$
b) $\frac{(x+9)}{x}$
c) $\frac{6x}{x+5}$
d) $\frac{3+x}{1-x}$
e) $\frac{9x^2-2}{3x+1}$
f) $\frac{2x-1}{5x-2}$
g) $\frac{3x+10}{6x^2-7x}$
h) $\frac{x^3+3}{2x^2-2x}$
i) $\frac{2x^4+3x}{x^3-5x^2}$
j) $\frac{2x-x^4}{2x^3-x^2}$
k) $\frac{1+x^3}{2x^2-3}$
l) $\frac{5-x^2}{x^5}$
m) $\frac{x^3+1}{x^7}$
n) $\frac{2x+x^4}{x^{-3}}$
o) $\left(ax^2+b\right)\left(cx+d\right)$

Exercise 3.3.6

Draw the following functions using first and second derivatives as your guide:

a) $f(x) = x^2 - 3x + 3$
b) $f(x) = -x^2 + 2x - 4$
c) $f(x) = x^3 - 3x + 9$
d) $f(x) = -x^3 + 2x + 8$
e) $f(x) = x^3 - 3x^2$
f) $f(x) = 3x^5 - 5x^4$
g) $f(x) = x^4 - 10x^2 + 9$
h) $f(x) = x^5 + 2x^2$
i) $f(x) = x^3 - 3x^2$

Exercise 3.3.7

Given $y = u^3$, where $u = 5 - x^2$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by the chain rule.

Exercise 3.3.8

Given $w = ay^2$, where $y = bx^2 + cx$, find $\frac{\mathrm{d}w}{\mathrm{d}x}$ by the chain rule.

Exercise 3.3.9

Given $z = y^2$, where $y = 2x + 5$, find $\frac{\mathrm{d}z}{\mathrm{d}x}$ by the chain rule.

Exercise 3.3.10

Given $z = (1/2)y^2 - 3$, where $y = 3x^2$, find $\frac{\mathrm{d}z}{\mathrm{d}x}$ by the chain rule.

Exercise 3.3.11

Given $y = 2u^4 + 3$, where $u = 5 - 3x^3$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by the chain rule.

Exercise 3.3.12

Given $w = 5y + 4$, where $y = ax^2 - cx$, find $\frac{\mathrm{d}w}{\mathrm{d}x}$ by the chain rule.

Exercise 3.3.13

Given $z = y^2$, where $y = 2x + 5$, find $\frac{\mathrm{d}z}{\mathrm{d}x}$ by the chain rule.

Exercise 3.3.14

Given $z = y^3$, where $y = x^3$, find $\frac{\mathrm{d}z}{\mathrm{d}x}$ by the chain rule.

Exercise 3.3.15

Use the chain rule to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the following:

a) $y = \left(3x^2 - 13\right)^3$
b) $y = \left(7x^3 - 5\right)^9$
c) $y = (ax + b)^5$
d) $y = \left(2x^3 + 3x^2 - 7x + 9\right)^{70}$
e) $y = \left(3x^4 + 2x^7 - 2x + 3\right)^{11}$
f) $y = \left(5x^4 + x^5 + 3x + 4\right)^{31}$
g) $y = \left[(-2x + 3)^7 + 4\right]^{12}$
h) $y = \left[(x + 1)^{-3} + 6\right]^{10}$

Exercise 3.3.16

Use product or quotient and the chain rule to calculate derivatives of the following:

a) $(x-1)^{10}(x+2)^{12}$

b) $\frac{(x-1)^{10}}{(x+2)^{12}}$

c) $\left(x^2-3\right)^{100}\left(4x^3+2x\right)^{10}$

d) $\frac{\left(x^2-3\right)^{100}}{(4x^3+2x)^{10}}$

e) $(ax+b)^c(d-x)^{10}$

f) $\frac{(ax+b)^c}{(d-x)^{10}}$

g) $\left(7x^3-5\right)^9\left(3x^2-13\right)^3$

h) $\frac{\left(3x^2-13\right)^3}{(3x^2-13)^3}$

Exercise 3.3.17

For $y = 6x + 36$, find inverse function and find $\frac{\mathrm{d}x}{\mathrm{d}y}$.

Exercise 3.3.18

Given:

a) $y = x^5 + x$, find $\frac{\mathrm{d}x}{\mathrm{d}y}$

b) $y = x^6 + x^2 + 1$, find $\frac{\mathrm{d}x}{\mathrm{d}y}$

c) $y = 7x^3 + 2x + 1$, find $\frac{\mathrm{d}x}{\mathrm{d}y}$

d) $y = -4x^6 + 4x^5 - 4x^2 + 2x - 14$, find $\frac{\mathrm{d}x}{\mathrm{d}y}$

e) $y = 5x^5 - 3x^4 - 2x^3 - 7x - 14$, find $\frac{\mathrm{d}x}{\mathrm{d}y}$

Exercise 3.3.19

Given $y = 7x + 21$, find its inverse function. Then find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}x}{\mathrm{d}y}$, and verify the inverse function rule. Also verify that the graphs of the two functions bear a mirror-image relationship to each other.

Exercise 3.3.20

Given $y = 0.5x + 5$, find its inverse function. Then find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}x}{\mathrm{d}y}$, and verify the inverse function rule. Also verify that the graphs of the two functions bear a mirror-image relationship to each other.

Exercise 3.3.21

Find derivatives of:

a) $y = 4e^x$

b) $y = 5^x$

c) $y = e^{2t+4}$

d) $y = e^{1-9t}$

e) $y = 3^x$

f) $y = 5^{3x\hat{}2-e}$

g) $y = 2^*2^x$

h) $y = 13^{2x+3}$

i) $y = e^{\left(t^2 2\right)+1}$

j) $y = 5e^{2-\left(t\hat{}2\right)}$

k) $y = e^{a\left(x\hat{}2\right)+bx+c}$

l) $y = 7e^{2x\hat{}3+2x-89}$

m) $y = 2e^{7x\hat{}4-8x-125}$

n) $y = xe^x$

o) $y = x^2e^{2x}$

p) $y = axe^{bx+c}$

q) $y = 3x2^{x+1}$

r) $y = 7x(\frac{1}{2})^{x^{12+2x-17}}$

s) $y = x^3(\frac{5}{7})^{2x^{3+5x-24}}$

t) $y = e^{x-3}(\frac{1}{3})^{x^{5-7x-34}}$

Exercise 3.3.22

Find derivatives of:

a) $y = \ln\left(7x^5\right)$
b) $y = \ln\left(ax^c\right)$
c) $y = \ln(x + 19)$
d) $y = \ln\left(7x^3 + 2x + 1\right)$
e) $y = 5\ln(t + 1)^2$
f) $y = \ln\left[\left(3x^2 - 13\right)^3\right]$
g) $y = \ln(x) - \ln(1 + x)$
h) $y = \ln\left[x(1 - x)^8\right]$
i) $y = \ln\left[\frac{2x}{1+x}\right]$
j) $y = 5x^4 \ln\left(x^2\right)$
k) $y = 9x^3 \ln\left(2x^3 + 7x^2 - 3x + 12\right)$
l) $y = 3x^4 \ln\left(3x^2 + 2x + 66\right)$

Exercise 3.3.23

Find derivatives of:

a) $y = \log_2\left(3x^2 - 2x + 3\right)$
b) $y = \log_2(x + 1)$
c) $y = \log_7\left(7x^2 + 17x\right)$
d) $y = \log_2\left(8x^2 + 3x - 12\right)$
e) $y = \log_2\left(-6x^3 - x + 4\right)$
f) $y = 9x^3 \ln\left(7x^{3,5} - 5x^2 - 5x + 8\right)$
g) $y = x2\log 3(x + 1)$

Exercise 3.3.24

Find derivatives of the following functions:

a) $y = \frac{e^x}{(x+4)}$
b) $y = \frac{2e^{3x}}{\ln(2-x)}$
c) $y = \frac{x^3-2x^2+3x}{\ln(2-x)}$
d) $y = (2x - 3)e^{3x^3}$
e) $y = \left(x^2 + 3\right)e^{x^2+1}$
f) $y = x^a e^{kx-c}$
g) $y = dx^{-a}e^{kx^b-c}$
h) $y = \frac{7e^{3x^3+2x-7}}{\ln(3x-2x^2)}$
i) $y = \frac{ae^{bx^c+dx}}{\ln(k-x^2)}$

Exercise 3.3.25

Find $\frac{\partial y}{\partial x_1}$ and $\frac{\partial y}{\partial x_2}$ for each of the following functions:

a) $y = 2x_1^3 - 11x_1^2x_2 + 3x_2^2$
b) $y = 7x_1^3 + 6x_1x_2^2 - 9x_2^3$
c) $y = (2x_1 + 3)(x_2 - 2)$
d) $y = \frac{5x_1+3}{x_2-2}$

Exercise 3.3.26

Find f_x and f_y from the following:

a) $f(x, y) = x^2 + 5xy - y^3$
b) $f(x, y) = \left(x^2 - 3y\right)(x - 2)$
c) $f(x, y) = \frac{2x-3y}{x+y}$
d) $f(x, y) = \frac{x^2-1}{xy}$

Exercise 3.3.27

Draw the following functions using first and second derivatives as your guide:

a) $y = x^2 - 3x + 3$
b) $y = -x^2 + 2x - 4$
c) $y = x^3 - 3x + 9$
d) $y = -x^3 + 2x + 8$
e) $y = x^3 - 3x + 9$
f) $y = x^3 - 3x^2$
g) $y = 3x^5 - 5x^4$
h) $y = x^4 - 10x^2 + 9$
i) $y = x^5 + 2x^2$
j) $y = x^3 - 3x^2$
k) $y = 5x^5 - 2x^3 - 3x^2$
l) $y = 2x^5 + 1,5x^3 - 5x^2$
m) $y = \frac{x-1}{x-2}$
n) $y = \frac{x^2}{x-1}$
o) $y = \frac{x+2}{(x-1)^2}$
p) $y = e^{-\frac{1}{2}x^2}$
q) $y = \ln\left(x^2 - 1\right)$
r) $y = xe^{-x}$
s) $y = x \ln x$

Exercise 3.3.28

Find the differential dy for:

a) $y = x^3 - 3x^2$
b) $y = e^{2x} - \ln\left(x^2 + 2\right)$
c) $y = 4x^5 - 3x^4y + 4x^3 - 3x^2$

3.4 Function optimisation

Exercise 3.4.1

Let q denote the level of production in your company. Moreover, let p, r and c be functions of q which yields the price of the product, total revenue and total costs of the company, respectively. Find optimal level of production in the following cases:

a) $\begin{cases} p(q) = 520 - 3q \\ c(q) = \frac{1}{2}q^2 + 100q + 4000 \end{cases}$

b) $\begin{cases} r(q) = -\frac{1}{3}q^2 + 4000q \\ c(q) = 2000q + 1000000 \end{cases}$

c) $\begin{cases} p(q) = 90 - 8q \\ c(q) = 10q + 150 \end{cases}$

d) $\begin{cases} p(q) = -\frac{1}{2}q + 12 \\ c(q) = \frac{1}{4}q^2 + 3q \end{cases}$

e) $\begin{cases} p(q) = -300q + 9000 \\ c(q) = 200q^2 + 1000q + 30000 \end{cases}$

f) $\begin{cases} p(q) = -\frac{2}{100}q + 500 \\ c(q) = \frac{3}{100}q^2 + 150q + 12500 \end{cases}$

Exercise 3.4.2

Find fix cost function, average cost function and marginal cost function, when total cost function is given by:

a) $\mathrm{C} = \mathrm{Q}^3 - 4\mathrm{Q}^2 + 10\mathrm{Q} + 75$
b) $\mathrm{C} = \mathrm{Q}^3 - 5\mathrm{Q}^2 + 12\mathrm{Q} + 100$
c) $C = Q^2 - 4Q + 174$
d) $C = Q^3 - 3Q^2 + 15Q + 200$

Exercise 3.4.3

Find the marginal and average functions for the following total functions and graph the results:

a) $\mathrm{C} = 3\mathrm{Q}^2 + 7\mathrm{Q} + 12$

b) $R = 10Q - Q^2$

c) $Q = aL + bL^2 - c^3(a, b, c > 0)$

Exercise 3.4.4

Given the production function $Q = 96K^{0.3}L^{0.7}$, find the MPP_K and MPP_L functions. Is MMP_K a function of K alone, or both K and L ? What about MP_L ?

Multiple Variables Calculus

4.1 Optimisation of multivariate functions

Exercise 4.1.1

Find local minimums and maximums of the following functions:

a) $f(x,y) = x^2 + xy + 2y^3 + 3$ b) $f(x,y) = -x^2 + y^2 + 6x + 2y$

c) $f(x,y) = x + 2ey - e^x - e^{2y}$ d) $f(x,y) = e^{2x} - 2x + 2y^2 + 3$

e) $f(x,y,z) = x^2 + 3y^2 - 3xy + 4yz + 6z^2$ f) $f(x,y,z) = 29 - (x^2 + y^2 + z^2)$

g) $f(x,y,z) = xz + x^2 - y + yz + y^2 + 3z^2$ h) $f(x,y,z) = e^{2x} + e^{-y} + e^{z^2} - (2x + 2e^z - y)$

Exercise 4.1.2

Let x and y denote the levels of production for two goods X and Y. Moreover, let π denote the function which yields profit based on production levels. Find an optimal level of production in the following cases:

a) $\pi(x,y) = -2x^2 - y^3 + 6x + 12y$ b) $\pi(x,y) = 64x - 2x^2 + 4xy - 4y^2 + 32y - 14$

c) $\pi(x,y) = \frac{-x^2y + xy^2 - 1}{3y^3 + 9y}$ d) $\pi(x,y) = 8x + 10y - \frac{1}{1000}(x^2 + xy^2 + y^2) - 10000$

Exercise 4.1.3

Let x and y denote the levels of production for two goods X and Y that your company is selling. Moreover, let $p_x = 12$ and $p_y = 18$ be the prices of the products X and Y respectively. Finally, let $c(x,y) = 2x^2 + xy + 2y^2 + 24$ denote the function which yields total costs based on production levels. Find an optimal level of production.

Exercise 4.1.4

Let x and y denote the levels of production for two goods X and Y that your company is selling. Moreover, let $p_x = p_y = 150$ be the prices of the products X and Y respectively. Finally, let $c_x(x) = \frac{2}{1000}x^2 + 4x + 456000$ and $c_y(y) = \frac{5}{1000}y^2 + 4y + 274000$ denote functions which yield total costs based on production levels for products X and Y, respectively. Find an optimal level of production.

Exercise 4.1.5

Let x and y denote the levels of production for two goods X and Y that your company is selling. Moreover, let $r(x,y) = -\frac{5}{1000}x^2 - \frac{3}{1000}y^2 - \frac{2}{1000}xy + 20x + 17y$ and $c(x,y) = 6x + 3y + 1000$ denote functions of production levels which yield total revenue and total costs, respectively. Find an optimal level of production.

Exercise 4.1.6

Find extremums of function f with respect to given restrictions:

a) $\begin{cases} f(x,y) = xy + 2x \\ 4x + 2y = 60 \end{cases}$ b) $\begin{cases} f(x,y) = xy \\ 4x + 2y = 40 \end{cases}$

Exercise 4.1.7

A company CoalMeMaybe (CMM) has access to two power plants A and B. If we denote by x and y tons of coal used in power plant A and B, respectively, then production cost functions can be expressed as $c_A(x) = 2(x-1)$ and $c_B(y) = (y-3)^2$ in each power plant respectively. In the first power plant out of one ton of coal it is possible to produce 5MWh and in the second power plant 2MWh. CMM is supposed to product 100 MWh. What division of resources between two power plants will be optimal?

Exercise 4.1.8

Denote by a, b and t the cost of the hour labour, the cost of unit capital and the total budget for the project, respectively. Moreover, denore by q the function which yiedls level of production based on level of labor l and capital k. Find level of labor and capital which maximizes level of production when:

a) $\begin{cases} a = 3 \\ b = 5 \\ t = 1200 \\ q(l,k) = 6l + 10k + lk \end{cases}$ b) $\begin{cases} a = 6 \\ b = 3 \\ t = 69 \\ q(l,k) = 5l + 2k + kl \end{cases}$

a) $\begin{cases} a = 2 \\ b = 1 \\ t = 99 \\ q(l,k) = 2\sqrt{l} + 3\sqrt{k} \end{cases}$ b) $\begin{cases} a = 2 \\ b = 3 \\ t = 1200 \\ q(l,k) = 50lk \end{cases}$

Integrals

5.1 Indefinite Integrals

5.1.1 Simple Integration

Exercise 5.1.1

Find the following:

a) $\int x dx$
b) $\int x^4 dx$
c) $\int \frac{1}{x^2} dx \quad x \neq 0$
d) $\int \sqrt[3]{x} dx$
e) $\int \frac{3}{\sqrt{x}} dx \quad x \neq 0$
f) $\int \frac{1}{\sqrt[4]{x^5}} dx \quad x \neq 0$
g) $\int 3e^x dx$
h) $\int \frac{1}{5x} dx \quad x \neq 0$
i) $\int \frac{x^3-x+1}{x^2} dx$
j) $\int \frac{7}{x^8} dx \quad x \neq 0$
k) $\int \frac{x^2-1}{x+1} dx$
l) $\int \sqrt[5]{x} dx$
m) $\int 16x^{-3} dx \quad x \neq 0$
n) $\int 9x^8 dx$
o) $\int \left(\pi - x + \sqrt[3]{\frac{2}{x}} + 100x^2\right) dx$
p) $\int (5^x - 3x)\, dx$
q) $\int \left(7^x - 7x^7\right) dx$
r) $\int (\pi^x + x^e + e^x)\, dx$
s) $\int 2e^{-2x} dx$
t) $\int 2e^{-4x+2} dx$
u) $\int \frac{3dx}{x}$
v) $\int \frac{dx}{x-2}$
w) $\int \frac{4x}{x^2+1} dx$
x) $\int \frac{16x^3+8x}{x^4+x^2} dx$
y) $\int \frac{x^4+x^2}{3x^2+5} dx$

5.1.2 Integration by substitution

Exercise 5.1.2

Find the following:

a) $\int 2x\left(x^2+1\right) dx$
b) $\int 6x^2\left(x^3+1\right)^{99} dx$
c) $\int 8e^{2x+3} dx$
d) $\int (2ax+b)\left(ax^2+bx\right) dx$
e) $\int 3e^{-(2x+7)} dx$
f) $\int 4ex^{x^2+3} dx$
g) $\int e^{-10x} dx$
h) $\int 2^{1-x} dx$
i) $\int \sqrt{4x-1} dx$
j) $\int (3x+1)^{10} dx$
k) $\int 10^{2x+3} dx$
l) $\int (4x+17+\pi)^{199} dx$

5.1.3 Integration by parts

Exercise 5.1.3

Find the following, in some examples(*) you need to use the rule twice:

a) $\int x(x+1)^{1/2}dx$
b) $\int \ln(x)dx \quad x>0$
c) $\int xe^x dx$
d) $\int (x+3)(x+1)^{1/2}dx$
e) $\int x\ln(x)dx$
f) $\int x^2e^x dx$ (*)
g) $\int (\ln(x))^2 dx \quad x>0$ (*)
h) $\int xe^{3x}dx$
i) $\int x^2e^{4x}dx$ (*)

5.2 Definite Integrals

5.2.1 Simple Integration

Exercise 5.2.1

Find the following:

a) $\int_1^5 3x^2dx$
b) $\int_a^b ke^x dx$
c) $\int_0^4 \left(\frac{1}{1+x}+2x\right)dx$
d) $\int_1^2 \left(2x^3-1\right)^2\left(6x^2\right)dx$
e) $\int_1^3 \frac{1}{2}x^2dx$
f) $\int_0^1 x\left(x^2+6\right)dx$
g) $\int_1^3 3\sqrt{x}dx$
h) $\int_2^4 \left(x^3-6x^2\right)dx$
i) $\int_{-1}^1 \left(ax^2+bx+c\right)dx$
j) $\int_4^2 x^2\left(\frac{1}{3}x^3+1\right)dx$
k) $\int_1^2 e^{-2x}dx$
l) $\int_{-1}^{e-2} \frac{dx}{x+2}$
m) $\int_2^3 \left(e^{2x}+e^x\right)dx$
n) $\int_4^2 \left(\frac{1}{3}x^3+1\right)dx$
o) $\int_e^6 \left(\frac{1}{x}+\frac{1}{1+x}\right)dx$

5.2.2 Integration of the areas under functions

Exercise 5.2.2

Find the following areas under functions and present them on the graph:

a) $\int_0^1 xdx, \int_{-1}^0 xdx, \int_3^4 xdx$
b) $\int_0^1 2xdx, \int_{-1}^0 2xdx, \int_6^7 2xdx$
c) $\int_0^1 x^2dx, \int_{-1}^0 x^2dx, \int_3^4 x^2dx$
d) $\int_0^1 x^3dx, \int_{-1}^0 x^3dx, \int_2^3 x^3dx$
e) $\int_0^1 \sqrt{x}dx, \int_{-1}^0 \sqrt{x}dx, \int_4^5 \sqrt{x}dx$
f) $\int_1^2 \frac{1}{x}dx, \int_{-2}^{-1} \frac{1}{x}dx, \int_3^4 \frac{1}{x}dx$
g) $\int_1^2 \frac{1}{x^2}dx, \int_{-2}^{-1} \frac{1}{x^2}dx, \int_3^4 \frac{1}{x^2}dx$
h) $\int_0^1 e^xdx, \int_{-1}^0 e^xdx, \int_3^4 e^xdx$
i) $\int_{-2}^3 \left(1-x^2\right)dx$
j) $\int_{-2}^3 \left(3-x^2\right)dx$

5.2.3 Area between functions

Exercise 5.2.3

Find areas in between functions and present them on the graph:

a) $y = f(x) = x; y = g(x) = x^2$ in interval $(0,1)$
b) $y = f(x) = x; y = g(x) = x^2$ in interval $(-1,0)$
c) $y = f(x) = x; y = g(x) = x^2$ in interval $(-2,1)$
d) $y = f(x) = x; y = g(x) = x^3$ in interval $(0,1)$
e) $y = f(x) = x; y = g(x) = x^3$ in interval $(-1,0)$
f) $y = f(x) = x; y = g(x) = x^3$ in interval $(-1,1)$
g) y = f(x) = x; y = g(x) = $\sqrt{x}$ in interval $(0,1)$
h) y = f(x) = x^2; y = g(x) = $\sqrt{x}$ in interval $(0,1)$

5.3 Proper and Improper Integrals

5.3.1 Area between functions

Exercise 5.3.1

Find the following areas whenever possible and present your answer on the graph:

a) $\int_1^\infty \frac{1}{x}dx$
b) $\int_1^\infty \frac{1}{x^2}dx$
c) $\int_{-1}^{-\infty} \frac{1}{x}dx$
d) $\int_{-1}^{-\infty} \frac{1}{x^2}dx$
e) $\int_0^1 \frac{1}{x}dx$
f) $\int_0^1 \frac{1}{x^2}dx$
g) $\int_{-1}^0 \frac{1}{x}dx$
h) $\int_{-1}^0 \frac{1}{x^2}dx$
i) $\int_{-1}^1 \frac{1}{x}dx$
j) $\int_{-1}^1 \frac{1}{x^2}dx$
k) $\int_0^9 x^{-\frac{1}{2}}dx$

5.3.2 Proper or Improper

Exercise 5.3.2

Which of the following integrals are improper, and why? Evaluate them.:

a) $\int_0^\infty e^{-rt}dt$
b) $\int_2^3 x^4dx$
c) $\int_0^1 x^{-2/3}dx$
d) $\int_{-\infty}^0 e^{-rt}dt$
e) $\int_1^5 \frac{dx}{x-2}$
f) $\int_{-3}^4 6xdx$

5.4 Application of Integrals to Economic Problems

Exercise 5.4.1

If the marginal cost (MC) of a firm is the following function of output, $C'(Q) = 2e^{0,2Q}$, and if the fixed cost is $C_F = 90$, find total cost function $C(Q)$.

Exercise 5.4.2

If the marginal propensity to save (MPS) is the following function of income, $S'(Y) = 0.3 - 0.1Y^{-1/2}$, and if the aggregate savings S is zero when income Y is 81 , find saving function S(Y).

Exercise 5.4.3

Given the following marginal revenue functions:

a) $R'(Q) = 28Q - e^{0,3Q}$

b) $R('Q) = 10(1+Q)^{-2}$ Find in each case total revenue function $R(Q)$. What initial condition can you introduce do definitive the constant of integration?

Exercise 5.4.4

Given:

a) the marginal propensity to import $M'(Y) = 0,1$ and information that $M = 20$ when $Y = 0$, find the import function $M(Y)$.

b) the marginal propensity to consume $C'(Y) = 0.8 + 0.1Y^{-1/2}$ and the information that C = Y when Y = 100, find the consumption function C(Y).

Exercise 5.4.5

If net investment is a constant flow at I(t) = 1000 (dollars per year), what will be the total net investment (capital accumulation) during t = 0 to t = 1, and what during t = 1 and t = 2.

Exercise 5.4.6

If I(t) = 3t − 1/2 (thousands of dollars per year) − a nonconstant flow − what will be the capital formation during the interval[1,4], that is, during the second, third, and fourth years?

Exercise 5.4.7

Assume that the rate of investment is described by the function $I(t) = 12t^{1/2}$ and that $K(0) = 25$:

a) Find the time path of capital stock K.

b) Find the amount of capital accumulation during intervals $[0, 1]$ and $[1, 3]$, respectively.

Exercise 5.4.8

What is the present value of a continuous revenue flow lasting for y years at the constant rate of D dollars per year and discounted at the rate of r per year.

Exercise 5.4.9

WINE STORAGE PROBLEM REVISITED: You buy case of wine for C dollars. If cost of storing wine is s dollars per year and annual interest rate is r, find optimal time to sell the wine.

Exercise 5.4.10

Given a continuous income stream at the constant rate of $1000 per year:

a) What will be the present value Π if the income stream lasts for 2 years and the continuous discount rate is 0.05 per year?

b) What will be the present value Π if the income stream terminates after exactly 3 years and the discount rate is 0.04 ?

Exercise 5.4.11

What is the present value of a perpetual cash flow of:

a) \$1450 per year, discounted at $r = 5\%$?

b) \$2460 per year, discounted at $r = 8\%$?

Complex numbers

6.1 Algebraic form and conjugation of a complex number

Exercise 6.1.1

Simplify:

a) $(-2+3i)+(7-8i)$ b) $(4i-3)-(1+10i)$

c) $(\sqrt{2}+i)\cdot(3-\sqrt{3}i)$ d) $\frac{2-3i}{5+4i}$

Exercise 6.1.2

Find the real numbers x, y satisfying the given equations:

a) $x(2+3i)+y(4-5i)=6-2i$ b) $(x-i)\cdot(2-yi)=11-23i$

c) $\frac{x}{2-3i}+\frac{y}{3+2i}=1$

Exercise 6.1.3

Solve the given equations in the set of complex numbers:

a) $z^2+3\bar{z}=0$ b) $2z+(1+i)\bar{z}=1-3i$

c) $z^2-z+1=0$ d) $\frac{z+1}{\bar{z}-1}=-1$

e) $(z+\bar{z})+i(z-\bar{z})=2i-6$ f) $(i-3)z=5+i-z$

g) $\frac{1-3i}{3z+2i}=\frac{2i-3}{5-2iz}$

Exercise 6.1.4

For which values of the real parameters a, b the equation $3\bar{z}-2z=a+bi$ has a solution?

Exercise 6.1.5

On the complex plane, draw sets of numbers z that satisfy the given conditions:

a) $\mathrm{Im}[(1+2i)z-3i]<0$ b) $\mathrm{Re}(z-i)^2\geq 0$

c) $z^2=2\,\mathrm{Re}(iz)$ d) $\mathrm{Re}\left(z^3\right)\geq \mathrm{Im}\left(z^3\right)$

Exercise 6.1.6

Sketch the set of all complex numbers z for which the number $\omega = \frac{z}{z+i}$ is:

a) real b) purely imaginary

Exercise 6.1.7

Points $z_1 = -1 + 2i$, $z_2 = i$, and $z_4 = 2 + 4i$ are the vertices of the parallelogram. Find the position of vertex z_3 of this parallelogram.

6.2 Modulus and argument of a complex number. The trigonometric form of a complex number.

Exercise 6.2.1

Calculate the modules of the given complex numbers:

a) $4i$ b) $12i - 5$

c) $\sqrt{7} + \sqrt{29}i$ d) $(\sqrt{5} - \sqrt{3}) + (\sqrt{5} + \sqrt{3})i$

e) $\sin\alpha + i\cos\alpha$, and $\alpha \in \mathbb{R}$

Exercise 6.2.2

Give a geometric interpretation of the modulus of difference of complex numbers. Using this interpretation, draw sets of complex numbers a satisfying the given conditions:

a) $|z + 1 - 2i| = 3$ b) $2 \leq |z + i| < 4$

c) $|(1 + i)z - 2| \geq 4$ d) $\left|\frac{z+3}{z-2i}\right| \geq 1$

e) $\mathrm{Re}(z + 1) < 0$ and $|i - z| \leq 3$ f) $\left|z^2 + 4\right| \leq |z - 2i|$