Learning Objectives

> This review session has one objective, and that is to *work in groups to identify learning objectives that you are not comfortable with.*.

1. The following sentences have errors or are missing something. Correct them or add the missing details.

   (a) **(Subspaces & Bases; Span; Linear Independence)**
   A basis for the subspace V is a set S such that $\dim S = \dim V$ and $\operatorname{span}(S) = V$.

   (b) **(Vectors)**
   $\vec{w}$ is a convex combination of u and v if:
   $$\vec{w} = \left\{ \vec{w} : \vec{w} = a\vec{u} + b\vec{v} \text{ for all } a, b \in [0,1] \right\}$$

   (c) **(Determinants; Dot products)**
   If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are pairwise orthogonal vectors in $\mathbb{R}^3$, then $\det\left[\vec{a} \mid \vec{b} \mid \vec{c}\right] = 1$.

   (d) **(Eigenvectors & Diagonalization)**
   Every matrix has a basis of eigenvalues.

2. **(Linear Transformations)**
   A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called *almost linear* if for all $x, y \in \mathbb{R}^n$, and $\alpha \geq 0$, we have
   $$\|T(x+y)\| \leq \|T(x) + T(y)\|$$
   $$\|T(\alpha x)\| \leq \alpha \|T(x)\|$$

   (a) Write down a clear and concise explanation of what one must do in order to show that a transformation is almost linear.

   (b) Show that the transformation $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not almost linear.

   (c) Show that any linear transformation in almost linear.

3. **(Inverses)**
   Mohammed has partially row-reduced the matrix A using the following steps. Use his work to write A as a product of elementary matrices. Use this to compute $\det A$.
   $$A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 3 & 0 \end{bmatrix} \xrightarrow{r_4 \to r_4 - r_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \end{bmatrix} \xrightarrow{r_4 \leftrightarrow r_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

4. **(Determinants and Inverses; Subspaces)**
   We define:
   $$A = \begin{bmatrix} 0 & 3 & 2 \\ a & 0 & 0 \\ 8 & 2 & 0 \end{bmatrix}$$

(a) For which values of a is the matrix A invertible?

(b) What is the rank of A when it is not invertible?

(c) In the above case, find a basis for the range of the transformation $T_A(x) = Ax$.

5. **(Eigenvalues & Diagonalization; Similar Matrices; Projections)**
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation which projects vectors onto the subspace $V = \text{span}\{\vec{v}\}$, where $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   (a) Write T in the standard basis, and in one other basis. Which one do you prefer?

   (b) Is T one-to-one? Is T onto?

   (c) What are the eigenvectors of T? Is T diagonalizable?

   (d) Is T invertible?

6. **(Computational Objectives; Representations of Lines; SLE)**
Let $\vec{a} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Define:

$$V = \left\{ \vec{x} \in \mathbb{R}^2 : \left(\vec{x} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \perp \vec{a} \text{ and } \left(\vec{x} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \perp \vec{b} \right\}$$

   (a) Write down a system of linear equations where V is the compelte solution set

   (b) Write down the corresponding augmented matrix.

   (c) Express V in vector form.

   (d) Express V as the span or translated span of vectors.

   (e) Sketch V.

1. Here are the things students should identify for each question:

   (a) Dimension does not make sense in this context, and the proper definition includes linear independence

   (b) $=$ should be $\in$, and the 'for all' makes no sense there. Additionally, the condition is wrong. Students should also be able to make this answer shorter.

   (c) Students should replace 1 with $\pm\|a\|\|b\|\|c\|$, and do this by understanding that the unit cube is sent to another cube.

   (d) The word 'eigenvalues' should be replaced with 'eigenvectors,' and even so, it does not make sense for a matrix to have a basis. This question is quite open-ended, and there are many directions the students can take this.

2. (a) To show that a transformation T is almost linear, one needs to show that for any $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ such that $\lambda \geq 0$:
   - $\|T(x+y)\| \leq \|T(x)+T(y)\|$
   - $\|T(\lambda x)\| \leq \lambda\|T(x)\|$

   (b) Take $x = 2$ and $\lambda = 2$. Then $\|f(\lambda x)\| = 16$, but $\lambda\|T(x)\| = 2\|4\| = 8$.

   (c) In a linear transformation, the first condition holds and is an equality. Now, let $\lambda \geq 0$ be given, and let $x \in \mathbb{R}^n$. Then by linearity:

$$\|T(\lambda x)\| = \|\lambda T(x)\| = |\lambda|\|T(x)\| = \lambda\|T(x)\|$$

   Showing almost linearity.

3.

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

   and the determinant of A is 6.

4. (a) When $a = 0$, the columns of A are linearly dependent, so A is not invertible. When $a \neq 0$, they are linearly independent, so the determinant is nonzero.

   (b) When A is not invertible, $a = 0$, in which case the columns of A span a 2-dimensional subspace of $\mathbb{R}^3$, meaning that the rank of A is 2.

   (c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

5. (a) $[T]_{\mathcal{E}} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and if $\mathcal{B} = \left\{\begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} -1, 1 \end{bmatrix}\right\}$, then $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

   (b) T is neither one-to-one nor onto. It is not one-to-one because $\vec{0}$ and $\begin{bmatrix} -1, 1 \end{bmatrix}$ are both sent to the same point. It is not onto because $\begin{bmatrix} 0, 1 \end{bmatrix}$ is not in the image.

   (c) The eigenvalues of T are 1 and 0, and T is diagonalizable.

   (d) Since T is not one to one, it is not invertible.

6. (a)

$$\begin{aligned} -x_1 + x_3 &= 1 \\ x_1 &= 0 \end{aligned}$$

(b) TODO

(c) TODO

(d) TODO

Session Objectives

We are pretty open about the fact that this session is for students to find their weak spots. We are happy if every student at least attempts every problem.

What to Do

Unlike tutorials during the year, in this tutorial, it is ok if students do not finish a question before moving on to the next one. Walk around the room helping students for a fixed time for each question. Encourage multiple-group discussions, and allow groups to exchange ideas, like a mini-tutorial. After the allotted time for a question, there will be a short wrap-up. Rinse and repeat.

Some students will want to skip ahead to do things they either know how to do, or think they should practice. Try not to let students do this, and instead offer them to discuss with other groups.

Notes

- Question 1 has math notation mistakes and math content mistakes. Try to get each group to notice at least one of each.
- Students will struggle with question 2, and it's important that they are able to do parts a) and c) for this question.
- Question 3 can be done by inspection, but students should do it by writing A as a product of elementary matrices
- Question 4 can be done by inspection, but if students insist on it being done this way, they should have a very good explanation to go along.
- In Question 5 a), students might try to do some complicated calculus to find the minimal distance. This will work, but the shorter approach of using plane geometry is preferred.