8. e^x-TRA-ORDINARY DIFFERENTIAL EQUATIONS

Example 2. Consider the **system** of linear differential equations[2]:

$$\begin{aligned} f_1' &= a_{1,1}f_1 + \cdots + a_{1,n}f_n \\ f_2' &= a_{2,1}f_1 + \cdots + a_{2,n}f_n \\ \vdots \quad & \quad \vdots \qquad \ddots \qquad \vdots \\ f_n' &= a_{n,1}f_1 + \cdots + a_{n,n}f_n \end{aligned}$$

AKA $\frac{d}{dx}\vec{f} = A\vec{f}$, where $A = (a_{ij})_{ij}$ is a matrix of constants and $\vec{f} = (f_1, \ldots, f_n)$ is a vector of functions of a single variable.
Let $\vec{f}(0) = v \in \mathbb{R}^n$ be the initial condition. Then:

$$\frac{d}{dx}e^{Ax}v = \left(\frac{d}{dx}e^{Ax}\right)v + e^{Ax}\frac{d}{dx}v = Ae^{Ax}v$$

Meaning that $\vec{f} = e^{Ax}v$ solves the system, and meets the initial condition $\vec{f}(0) = e^0 v = I_n v = v$.

Remark. This also works when replacing a matrix with any linear operator. This is how physicists solve Schr😺dinger's equation.

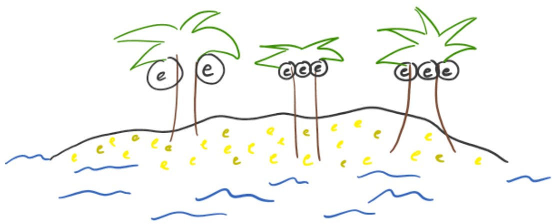
[2]Whoa what's this? We only learned what e was a few pages ago and now we're doing differential equations?

Have no fear, and let me be your guide. I will teach you about

NUMBER

1. How can our e's be real if our numbers aren't real?

If real numbers were an ocean, e would be the tropical island where there's nothing to eat but coconuts.



Definition. For any $\text{✂} \in \mathbb{R}$, we define:

$$e^{\text{✂}} = \sum_{\text{🚲}=0}^{\infty} \frac{\text{✂}^{\text{🚲}}}{\text{🚲}!}$$

This definition sucks, let's try again:

Definition. We define:

$$e = \lim_{\text{🚲}\to\infty} \left(1 + \frac{1}{\text{🚲}}\right)^{\text{🚲}}$$

and its side-kick:

$$e^{\text{✂}} = \lim_{\text{🚲}\to\infty} \left(1 + \frac{\text{✂}}{\text{🚲}}\right)^{\text{🚲}}$$

Shit, we accidentally defined the same thing.

Proof. Expand for finite 🚲. □

Nothing[1] prevents x from being another kind of number. Another number? Different numbers? Other numbers? Complex numbers.

Theorem. *Let $a + bi = z \in \mathbb{C}$. Then*

$$e^z = e^a e^{bi} = e^a(\cos(b) + i\sin(b))$$

Proof. To prove $e^z = e^a e^{bi}$, we have to use multinomial theorem on the first definition, like in the real case. To prove $e^{bi} = \cos(b) + i\sin(b)$, exponentiate and and use Taylor's formulas for sin and cos. □

Example 1. We can take the exponent of a square matrix to get another matrix.
It turns out that if $A \in \mathrm{Mat}(n)$, then $e^A e^{-A} = I_n$, so e^A is always invertible.

However, some of the exponent rules don't work. If $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, then:

$$e^A e^B \neq e^B e^A$$

[1] well, some things, like being able to take powers, sums, division over $\mathbb{Q}$, convergence...

"$\log(x)$, being the inverse function of e^x, has always been controversial. Some say that log makes them feel ”

Captain's log day 1: e-log-ical musings

```
>How does a number theorist drown?
>logloglogloglog
>What do you get when you integrate
 one over cabin with respect to cabin?
>log cabin
>No, it's a houseboat, you forgot to
 add the c!
>No, it's a priceless yacht, you
 disregarded the absolute value
```

Figure 1. log for scale. Scale pictured on previous page

2. Properties of e

Ok, so we have this number, but what is it good for? Well, it's a good approximation for 2.7, which is a good approximation for 3, which is a good approximation for π for sufficiently bad approximations of good. Succinctly:

$$e \approx 3 \approx \pi$$

Theorem. *e is irrational*

Proof. If $e = \frac{a}{b}$, then:

$$x := b!\left(\frac{a}{b} - \sum_{n=0}^{b} \frac{1}{n!}\right) = a(b-1)! - \sum_{n=0}^{b} \frac{b!}{n!} \in \mathbb{Z}$$

But then by definition of $e = \sum_n \frac{1}{n!}$:

$$x = b!\left(\sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{b} \frac{1}{n!}\right) = \sum_{n=b+1}^{\infty} \frac{b!}{n!} > 0$$

Next, note that if $n > b$, then $\frac{b!}{n!} < \frac{1}{(b+1)^{n-b}}$, so:

$$0 < x < \sum_{n=b+1}^{\infty} \frac{1}{(b+1)^{n-b}} = \sum_{k=1}^{\infty} \frac{1}{(b+1)^k} = \frac{1}{b} \leq 1$$

so $0 < x < 1$, a contradiction to $x \in \mathbb{Z}$. $\square$

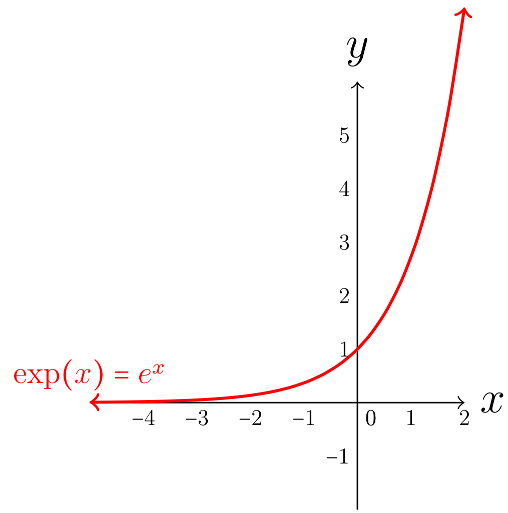
3. A baker's story

A baker wants to bake the perfect cookie: a cookie of 100% chocolate. She bakes it, and sets her cookie to cool. In the ☾, the ♕ of fairies sends a mischevious fairy upon the cookie, to ⚡ it with her 🪄⋆⭑☆ that turns chocolate into cookie. The next ☼, the baker finds that her perfect cookie is now $1 - 1 =$ 🪫 chocolate! She decides to outsmart the ♕ by splitting her 🔋 chocolate cookie into two pieces. The ♕ decides to send two fairies to ⚡ the halves with their 🪄⋆⭑☆. However, being simple creatures of ⋆⭑☆, neither fairy knows which half the other ⚡ and each ⚡ the halves randomly. The next day, the baker wakes up to find that her 🔋 cookie is now $\left(1 - \frac{1}{2}\right)^2 =$ 🪫 chocolate! She decides outsmart the ♕ by splitting her 🔋 chocolate cookie into three pieces...

4. The function of the function e^x

Yes, we defined $e^{⚖}$ as a series, or as some limit of a sequence, but that's ok. It's still a function, with some nice properties.

(1) e^x is a monotonically increasing positive function from $\mathbb{R}$ to $\mathbb{R}$.
(2) $e^{x+y} = e^x e^y$, $e^0 = 1$ (homomorphism!)
(3) If you made a slide in the shape of the plot of $y = e^x$, you will never stop slidin' leftward.



But most importantly:

Theorem. *$\frac{d}{dx}e^x = e^x$ ($\Rightarrow \frac{d}{dx}e^{cx} = ce^{cx}$ by 🔗 rule).*

Proof. Use the first definition, and differentiate term by term. □