## Homework 2 Solutions

## CS 511 Formal Methods

October 7, 2024

**Exercise 1: Lecture Slides 06, Page 9.** For all strings  $s, t \in A^*$ , it holds that

$$reverse(s \cdot t) = reverse(t) \cdot reverse(s)$$

*Proof.* Suppose  $s \in A^*$  is an arbitrary string. We prove by structural induction on  $t \in A^*$ . First assume that  $t = \varepsilon$ , that is, t is the empty string. Then

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\begin{aligned} \mathbf{reverse}(s \cdot t) &= \mathbf{reverse}(s \cdot \varepsilon) \\ &= \mathbf{reverse}(s) \\ &= \varepsilon \cdot \mathbf{reverse}(s) \\ &= \mathbf{reverse}(\varepsilon) \cdot \mathbf{reverse}(s) \\ &= \mathbf{reverse}(t) \cdot \mathbf{reverse}(s) \end{aligned}
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Next, assume  $t = t' \cdot x$  where  $t', s \in A^*$ ,  $x \in A$ , and  $\mathbf{reverse}(s \cdot t') = \mathbf{reverse}(t') \cdot \mathbf{reverse}(s)$ . Then

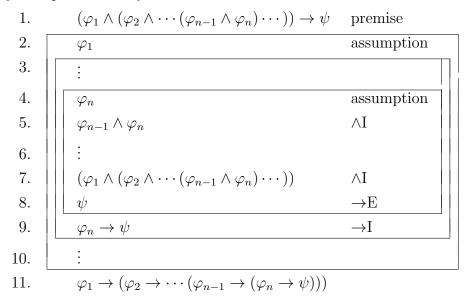
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\begin{aligned} \mathbf{reverse}(s \cdot t) &= \mathbf{reverse}(s \cdot (t' \cdot x)) \\ &= \mathbf{reverse}((s \cdot t') \cdot x) \\ &= x \cdot \mathbf{reverse}(s \cdot t') \\ &= x \cdot \mathbf{reverse}(t') \cdot \mathbf{reverse}(s) \\ &= \mathbf{reverse}(t' \cdot x) \cdot \mathbf{reverse}(s) \\ &= \mathbf{reverse}(t) \cdot \mathbf{reverse}(s) \end{aligned}
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By the principle of structural induction, we have that  $\mathbf{reverse}(s \cdot t) = \mathbf{reverse}(t) \cdot \mathbf{reverse}(s)$  for all  $s, t \in A^*$ .

**Exercise 2: LCS Exercise 1.4.15.** Use mathematical induction on n to prove the theorem

$$(\varphi_1 \land (\varphi_2 \land \cdots (\varphi_{n-1} \land \varphi_n) \cdots)) \rightarrow \psi \vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow \cdots (\varphi_{n-1} \rightarrow (\varphi_n \rightarrow \psi)))$$

*Proof.* We prove directly instead.



**Problem 1.** Show that any of the three rules  $\{(LEM), (PBC), (\neg \neg E)\}$  are interderivable.

*Proof.* First, we assume  $(\neg \neg E)$  and wish to show (PBC).

1. 
$$\neg \varphi \rightarrow \bot$$
 premise  
2.  $\neg \varphi$  assumption  
3.  $\bot \rightarrow E 1,2$   
4.  $\neg \neg \varphi$   $\neg I 2-3$   
5.  $\varphi$   $\neg \neg E 4$ 

We prove one of De Morgan's laws as a lemma before continuing. Specifically we show:

$$\neg(p \lor q) \vdash \neg p \land \neg q$$

1.
 
$$\neg (p \lor q)$$
 premise

 2.
  $p$ 
 assumption

 3.
  $p \lor q$ 
 $\lor I_1 2$ 

 4.
  $\bot$ 
 $\neg E 1,3$ 

 5.
  $\neg p$ 
 $\neg I 2-4$ 

 6.
  $q$ 
 assumption

 7.
  $p \lor q$ 
 $\lor I_2 6$ 

 8.
  $\bot$ 
 $\neg E 1,7$ 

 9.
  $\neg q$ 
 $\neg I 6-8$ 

 10.
  $\neg p \land \neg q$ 
 $\land I 5,9$ 

Next, we assume (PBC) and show (LEM).

1. 
$$\neg(\varphi \lor \neg \varphi) \quad \text{assumption}$$
2. 
$$\neg \varphi \land \neg \neg \varphi \quad \text{De Morgan}$$
3. 
$$\neg \varphi \quad \land E_1 \ 2$$
4. 
$$\neg \neg \varphi \quad \land E_2 \ 2$$
5. 
$$\bot \quad \neg E \ 3,4$$
6. 
$$\varphi \lor \neg \varphi \quad \text{PBC}$$

Finally, we assume (LEM) and show  $(\neg \neg E)$ .

1.	$\neg \neg \varphi$	premise
2.	$\varphi \vee \neg \varphi$	LEM
3.	φ	assumption
4.	$\neg \varphi$	assumption
5.		$\neg E 1,4$
6.	$\varphi$	$\perp E 5$
7.	$\varphi$	$\vee E 2,3,4-6$