Homework 2 Solutions

CS 511 Formal Methods

October 7, 2024

Exercise 1: Lecture Slides 06, Page 9. For all strings $s, t \in A^*$, it holds that

$$\textbf{reverse}(s \cdot t) = \textbf{reverse}(t) \cdot \textbf{reverse}(s)$$

Proof. Suppose $s \in A^*$ is an arbitrary string. We prove by structural induction on $t \in A^*$. First assume that $t = \varepsilon$, that is, t is the empty string. Then

$$\begin{aligned}
\textbf{reverse}(s \cdot t) &= \textbf{reverse}(s \cdot \varepsilon) \\
&= \textbf{reverse}(s) \\
&= \varepsilon \cdot \textbf{reverse}(s) \\
&= \textbf{reverse}(\varepsilon) \cdot \textbf{reverse}(s) \\
&= \textbf{reverse}(t) \cdot \textbf{reverse}(s)
\end{aligned}$$

Next, assume $t = t' \cdot x$ where $t', s \in A^*$, $x \in A$, and $\textbf{reverse}(s \cdot t') = \textbf{reverse}(t') \cdot \textbf{reverse}(s)$. Then

$$\begin{aligned}
\textbf{reverse}(s \cdot t) &= \textbf{reverse}(s \cdot (t' \cdot x)) \\
&= \textbf{reverse}((s \cdot t') \cdot x) \\
&= x \cdot \textbf{reverse}(s \cdot t') \\
&= x \cdot \textbf{reverse}(t') \cdot \textbf{reverse}(s) \\
&= \textbf{reverse}(t' \cdot x) \cdot \textbf{reverse}(s) \\
&= \textbf{reverse}(t) \cdot \textbf{reverse}(s)
\end{aligned}$$

By the principle of structural induction, we have that $\textbf{reverse}(s \cdot t) = \textbf{reverse}(t) \cdot \textbf{reverse}(s)$ for all $s, t \in A^*$. $\square$

Exercise 2: LCS Exercise 1.4.15. Use mathematical induction on n to prove the theorem

$$(\varphi_1 \wedge (\varphi_2 \wedge \cdots (\varphi_{n-1} \wedge \varphi_n) \cdots)) \to \psi \vdash \varphi_1 \to (\varphi_2 \to \cdots (\varphi_{n-1} \to (\varphi_n \to \psi)))$$

Proof. We prove directly instead.

1.	$(\varphi_1 \wedge (\varphi_2 \wedge \cdots (\varphi_{n-1} \wedge \varphi_n) \cdots)) \to \psi$	premise
2.	φ_1	assumption
3.	$\vdots$	
4.	φ_n	assumption
5.	$\varphi_{n-1} \wedge \varphi_n$	$\wedge$I
6.	$\vdots$	
7.	$(\varphi_1 \wedge (\varphi_2 \wedge \cdots (\varphi_{n-1} \wedge \varphi_n) \cdots))$	$\wedge$I
8.	ψ	$\to$E
9.	$\varphi_n \to \psi$	$\to$I
10.	$\vdots$	
11.	$\varphi_1 \to (\varphi_2 \to \cdots (\varphi_{n-1} \to (\varphi_n \to \psi)))$	

$\square$

Problem 1. Show that any of the three rules $\{$(LEM), (PBC), ($\neg\neg$E)$\}$ are interderivable.

Proof. First, we assume ($\neg\neg$E) and wish to show (PBC).

1.	$\neg\varphi \to \bot$	premise
2.	$\neg\varphi$	assumption
3.	$\bot$	$\to$E 1,2
4.	$\neg\neg\varphi$	$\neg$I 2–3
5.	φ	$\neg\neg$E 4

We prove one of De Morgan's laws as a lemma before continuing. Specifically we show:

$$\neg(p \vee q) \vdash \neg p \wedge \neg q$$

1.	$\neg(p \lor q)$	premise
2.	p	assumption
3.	$p \lor q$	$\lor\mathrm{I}_1$ 2
4.	$\bot$	$\neg$E 1,3
5.	$\neg p$	$\neg$I 2–4
6.	q	assumption
7.	$p \lor q$	$\lor\mathrm{I}_2$ 6
8.	$\bot$	$\neg$E 1,7
9.	$\neg q$	$\neg$I 6–8
10.	$\neg p \land \neg q$	$\land$I 5,9

Next, we assume (PBC) and show (LEM).

1.	$\neg(\varphi \lor \neg\varphi)$	assumption
2.	$\neg\varphi \land \neg\neg\varphi$	De Morgan
3.	$\neg\varphi$	$\land\mathrm{E}_1$ 2
4.	$\neg\neg\varphi$	$\land\mathrm{E}_2$ 2
5.	$\bot$	$\neg$E 3,4
6.	$\varphi \lor \neg\varphi$	PBC

Finally, we assume (LEM) and show ($\neg\neg$E).

1.	$\neg\neg\varphi$	premise
2.	$\varphi \lor \neg\varphi$	LEM
3.	φ	assumption
4.	$\neg\varphi$	assumption
5.	$\bot$	$\neg$E 1,4
6.	φ	$\bot$E 5
7.	φ	$\lor$E 2,3,4–6

□