Homework 1 Solutions

CS 511 Formal Methods

October 6, 2024

Exercise 1.2.1

(h) $p \vdash (p \to q) \to q$

1.	p	premise
2.	$p \to q$	assumption
3.	q	$\to$E 1,2
4.	$(p \to q) \to q$	$\to$I 2–3

(i) $(p \to r) \land (q \to r) \vdash p \land q \to r$

1.	$(p \to r) \land (q \to r)$	premise
2.	$p \to r$	$\land$E$_1$ 1
3.	$p \land q$	assumption
4.	p	$\land$E$_1$ 3
5.	r	$\to$E 2,4
6.	$p \land q \to r$	$\to$I 3–5

(j) $q \to r \vdash (p \to q) \to (p \to r)$

1.	$q \to r$	premise
2.	$p \to q$	assumption
3.	p	assumption
4.	q	$\to$E 2,3
5.	r	$\to$E 1,4
6.	$p \to r$	$\to$I 3–5
7.	$(p \to q) \to (p \to r)$	$\to$I 2–6

Exercise 1.4.2

(g)

p	q	$p \to q$	$(p \to q) \to p$	$((p \to q) \to p) \to p$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

(h)

p	q	r	$p \lor q$	$p \to r$	$q \to r$	$(p \lor q) \to r$	$(p \to r) \lor (q \to r)$	$((p \lor q) \to r) \to ((p \to r) \lor (q \to r))$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	T	T	T	T	T	T
T	F	F	T	F	T	F	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T	T
F	F	T	F	T	T	T	T	T
F	F	F	F	T	T	T	T	T

(i)

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \to \neg q$	$(p \to q) \to (\neg p \to \neg q)$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Exercise 1.5.3

(b)

Show that, if $C \subseteq \{\neg, \wedge, \vee, \rightarrow, \perp\}$ is adequate for propositional logic, then $\neg \in C$ or $\perp \in C$. (Hint: suppose C contains neither $\neg$ nor $\perp$ and consider the truth value of a formula ϕ, formed by using only the connectives in C, for a valuation in which every atom is assigned $\top$.)

Proof. We prove the contrapositive of the original statement. Assume that neither $\neg$ nor $\perp$ are in C. I claim that $C \subseteq \{\wedge, \vee, \rightarrow\}$ is not adequate for propositional logic because it can make neither $\neg$ nor $\perp$ from $\wedge$, $\vee$, and $\rightarrow$. To see this, observe the following truth table:

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \vee q$	$p \wedge q$	$\neg p$	$\perp$
T	T	T	T	T	T	F	F
T	F	F	T	T	F	F	F
F	T	T	F	T	F	T	F
F	F	T	T	F	F	T	F

Note that, in the case where p is true, none of the connectives in $C \subseteq \{\wedge, \vee, \rightarrow\}$ with an arbitrary Boolean variable q can turn the truth value of the statement to false. For both $\neg$ and $\perp$ in the case where p is true, they change the truth value to false. We can look at the partial truth table to emphasize this:

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \vee q$	$p \wedge q$	$\neg p$	$\perp$
T	T	T	T	T	T	F	F

This means that no combination of $\rightarrow$, $\vee$, and $\wedge$ can create $\neg$ or $\perp$, which implies that $C \subseteq \{\wedge, \vee, \rightarrow\}$ is not adequate for propositional logic. (This is an informal description of structural induction, but I wanted to go easy on its use here since we have not formally covered the technique in class, yet.)

□

(c)

Is $\{\leftrightarrow, \neg\}$ adequate? Prove your answer.

Proof. $\{\leftrightarrow, \neg\}$ is not adequate for propositional logic. This can be seen easiest from looking at a truth table:

p	q	$\neg p$	$p \to q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

p	q	$p \leftrightarrow q$	$p \to q$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	T	T

The connectives $\neg$ and $\leftrightarrow$ have an even number of true values and $\to$ has an odd number of true values. This means that any combination of $\neg$ and $\leftrightarrow$ cannot form $\to$ as functions of variables p and q (do you see why?).

$\square$