# Inner Product Spaces

## Inner Product

Definition of an Inner Product:

- In a Real Vector Space $V$
- An Inner Product $f : V \times V \to \mathbb{R}$ s.t. $\forall u, v, w \in V, k \in \mathbb{R}$
  $\textbf{Notation:}$ $\langle u, v \rangle$ means the **Inner Product** of vectors $u$ and $v$

### The 4 Axioms of an Inner Product:

1. **Symmetry:** $\langle u, v \rangle = \langle v, u \rangle$
2. **Linearity (Additivity):** $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
3. **Homogeneity:** $\langle ku, v \rangle = k\langle u, v \rangle$
4. **Positivity:** $\langle v, v \rangle \geqslant 0$, and $\langle v, v \rangle = 0 \text{ iff } v = 0$.
   - Inner product of itself is $\geqslant 0$, and only zero, when using the zero vector

## Proof that the dot product is an Inner Product:

- Proving that the standard **Dot Product** on $\mathbb{R}^n$ satisfies the axioms of an Inner Product.

#### 1. Symmetry Proof:

Prove that: $\langle u, v \rangle = \langle v, u \rangle$

$$\begin{aligned} &\text{Let } u, v \in \mathbb{R}^n \\ &\quad \langle u, v \rangle = u_1v_1 + u_2v_2 + \cdots + u_nv_n \qquad &&\text{(Dot Product Definition)} \\ &\quad \phantom{\langle u, v \rangle} = v_1u_1 + v_2u_2 + \cdots + v_nu_n \qquad &&\text{(Commutativity: } ab = ba\text{)} \\ &\quad \langle u, v \rangle = \langle v, u \rangle \quad \checkmark \end{aligned}$$

#### 2. Linearity (Additivity) Proof:

Prove that: $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

$$\begin{aligned} &\text{(Start with the Dot Product definition)} \\ \langle u + v, w \rangle &= (u_1 + v_1)w_1 + (u_2 + v_2)w_2 + \cdots + (u_n + v_n)w_n \\ &= \sum_{i=1}^{n} (u_i + v_i)w_i \\ &= \sum_{i=1}^{n} (u_iw_i + v_iw_i) \\ &= \sum_{i=1}^{n} u_iw_i + \sum_{i=1}^{n} v_iw_i \\ \langle u + v, w \rangle &= \langle u, w \rangle + \langle v, w \rangle \quad \checkmark \end{aligned}$$

#### 3. Homogeneity Proof:

Prove that: $\langle ku, v\rangle = k\langle u, v\rangle$

$$\begin{aligned}
&\text{(Start with the Dot Product definition)}\\
\langle ku, v\rangle &= k(u_1v_1) + k(u_2v_2) + \cdots + k(u_nv_n)\\
&= k(u_1v_1 + u_2v_2 + \cdots + u_nv_n)\\
\langle ku, v\rangle &= k\langle u, v\rangle \quad \checkmark
\end{aligned}$$

**4. Positivity Proof:**

Prove that: $\langle v, v\rangle \geqslant 0$, and $\langle v, v\rangle = 0 \iff v = 0$.

- Inner product of itself is $\geqslant 0$, and only zero when $v$ is the zero vector

$$\begin{aligned}
&\text{Part A: Show } \langle v, v\rangle \geqslant 0\\
&\quad \langle v, v\rangle = v_1v_1 + v_2v_2 + \cdots + v_nv_n\\
&\qquad = v_1^2 + v_2^2 + \cdots + v_n^2 && \text{(Sum of Squares)}\\
&\qquad \text{Since } v_i \in \mathbb{R},\ \text{then } v_i^2 \geqslant 0 && \text{(Real squares are non-negative)}\\
&\qquad \therefore \sum_{i=1}^{n} v_i^2 \geqslant 0 \implies \langle v, v\rangle \geqslant 0 \quad \checkmark
\end{aligned}$$

$$\begin{aligned}
&\text{Part B: Show } \langle v, v\rangle = 0 \iff v = \mathbf{0}\\
&\quad \langle v, v\rangle = 0\\
&\quad \sum_{i=1}^{n} v_i^2 = 0\\
&\implies v_i^2 = 0 \text{ for all } i && \text{(Sum is 0 only if all terms are 0)}\\
&\implies v_i = 0 \text{ for all } i\\
&\quad \therefore v = \mathbf{0} \quad \checkmark
\end{aligned}$$

$$\text{Extra: If } v = \mathbf{0} \implies \langle v, v\rangle = 0^2 + \cdots + 0^2 = 0. \quad \checkmark \qquad \text{(Proof of the other direction of iff)}$$