

Converging or Diverging Series Tests

Choosing the Right Test

Step 1: Divergence Test

- Take the Limit of the term (a_k) as $k \rightarrow \infty$
- Limit $\neq 0 \rightarrow$ Divergent
- Limit $= 0 \rightarrow$ Test is Inconclusive (Move onto the next step)

Example:

$$\sum_{k=0}^{\infty} \frac{k}{k+1} \rightarrow \lim_{k \rightarrow \infty} \frac{k}{k+1} = 1 \neq 0 \therefore \text{The Series Diverges}$$

Step 2: The “Standard Series” Check

- Check if the series fits a “memorised” pattern
- **Geometric Series:** $(\sum ar^n)$
 - Converges if $|r| < 1$
 - Diverges if $|r| > 1$
- **P-Series:** $(\sum \frac{1}{n^p})$
 - Converges if $|p| > 1$
 - Diverges if $|p| \leq 1$

Step 3: Deciding which Test to Use

- If the series isn't a Standard Form, look at the algebra of term a_n , to pick the right test

If you see this in a_n	Use this Test	Reasoning
Factorials $k!$ or Exponentials (3^k)	Ratio Test	Ratio Test cancels out factorials perfectly (e.g. $\frac{(n+1)!}{n!} = n + 1$)
Entire term raised to the n^{th} power $(\dots)^k$	Root Test	The n^{th} root $\sqrt[n]{a_n}$ removes the power
Rational Functions (Polynomial over Polynomial) e.g. $\frac{n}{n^3+1}$	Limit Comparison (Quotient Test)	The Ratio Test usually fails here (results in 1) So we compare it to a 'similar' P-Series, based of the highest powers (e.g. $\frac{k}{k^3} = \frac{1}{k^2}$)
Logarithms $(\ln(k))$ or simple decreasing functions	Integral Test	Use only if we can easily integrate the function

If you see this in a_n	Use this Test	Reasoning
Oscillating Term ($\sin k, \cos k$)	Direct Comparison Test	Limits don't exist for Oscillating terms So we replace them with their min/max values (e.g. $\cos k \leq 1$) to build an inequality
Alternating Signs (-1^n)	Alternating Sign Test	This test is designed for series with negative terms

Tests:

1. The Ratio Test:

- Best for Factorials ($k!$) and Exponentials (3^k)

Formula:

- Calculate the limits of the “next term divided by the current term”

$$\rho = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|$$

Results:

- $\rho < 1$: The series **converges** (Absolutely)
- $\rho > 1$: The series **diverges**
- $\rho = 1$: **Test Inconclusive** (Try a different test)

2. The n^{th} Root Test:

- Best for Entire terms raised to the (k^{th}) power e.g. $(\dots)^k$

Formula:

- Calculate the k^{th} root of the absolute term (get rid of the power)

$$\rho = \lim_{k \rightarrow \infty} |a_k|^{\frac{1}{k}}$$

Results:

- $\rho < 1$: The series **converges** (Absolutely)
- $\rho > 1$: The series **diverges**
- $\rho = 1$: **Test Inconclusive** (Try a different test)

3. The Limit Comparison Test (Quotient Test):

- Best for Rational Functions (polynomials over polynomials)

Method:

1. Create a simplified comparison series by keeping only the highest powers of k
2. Calculate the limit ratio

$$\rho = \lim_{k \rightarrow \infty} \frac{a_k}{b_k}$$

Rules:

- If $0 < \rho < \infty$ (Finite and non-zero): Both series behave the same
 - If $\sum b_k$ converges $\rightarrow \sum a_k$ converges
 - If $\sum b_k$ diverges $\rightarrow \sum a_k$ diverges
- If $\rho = 0$:
 - If $\sum b_k$ converges $\rightarrow \sum a_k$ converges
- If $\rho = \infty$:
 - If $\sum b_k$ diverges $\rightarrow \sum a_k$ diverges

4. Direct Comparison Test:

- Best for Oscillating terms ($\sin k, \cos k$)

Method:

- Find a simple series $\sum b_k$ that is strictly larger or smaller than the original series $\sum a_k$

Rules:

- The ceiling Rule (**Convergence**):
 - If $0 \leq a_k \leq b_k$, and $\sum b_k$ converges $\rightarrow \sum a_k$ converges
- The Floor Rule (**Divergence**):
 - If $0 \leq b_k \leq a_k$, and $\sum b_k$ diverges $\rightarrow \sum a_k$ diverges

Example:

For $\frac{2 + \cos k}{k^2}$, use the fact that $\cos k \leq 1$

$$\therefore \frac{2 + \cos k}{k^2} \leq \frac{3}{k^2}$$

5. Integral Test:

- Best for Logarithms, or easy to integrate functions

Formula:

- Convert the sequence a_k into a continuous function $f(x)$ and integrate from 1 to ∞

$$I = \lim_{N \rightarrow \infty} \int_1^N f(x) dx$$

Condition:

- The function $f(x)$ must be positive and decreasing for $x \geq 1$

Rules:

- Integral is **Finite** → Series **Converges**
- Integral is **Infinite** → Series **Diverges**

6. Alternating Sign Test:

- Best for series with $(-1)^k$

Method:

- Ignore the negative sign (look at $|a_k|$) and check two conditions

Conditions:

1. **Limit:** $\lim_{k \rightarrow \infty} |a_k| = 0$
2. **Decreasing:** $|\lim_{k \rightarrow \infty} |a_k|| < |a_k|$, the terms eventually get smaller (in magnitude)

Rules:

If **both** conditions are met → The series **Converges**

Note: Does not check for absolute convergence

Related Pages:

- [Double Summation](#)
- [Methods to Find Limits](#)
- [Taylor Series](#)