Resolution

Resolution

Resolution Rule:

$$\frac{p \vee q \quad \neg p \vee r}{q \vee r}$$

Concept:

- If you have a long clause containing x, and another long clause containing $\neg x$, you can remove the $x \text{ and } \neg x$, and join all the remaining literals with ORs $(\vee)$

Clause:

- A clause (C_i) is a disjunction of literals (a sequence of literals joined by ORs, $\vee$) of literals
  For conjunctive normal form $(C_1 \wedge C_2 \wedge \cdots \wedge C_m)$ each C is a collection of literals joined with ORs $\vee$.

Soundness & Completeness

Resolution is both sound and complete

- **Soundness:** If Resolution announces that ϕ is a theorem, then ϕ is a tautology
- **Completeness:** If ϕ is a tautology, then Resolution announces that ϕ is a theorem

Note: In the worst case, Resolution involves an exponential number of applications

Satisfiability & Tautologies

- **Sat-Solvers:** Checks if a given formula of propositional logic is satisfiable
- **Proof Systems:** Systems like Resolution aim to prove theorems
- **Link:** A formula ϕ is satisfiable if, and only if, its negation $\neg\phi$ is not a tautology
  - If ϕ is satisfiable, there exists a truth assignment making ϕ true
  - This means there exists a truth assignment making $\neg\phi$ false, meaning that $\neg\phi$ is not a tautology

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The Resolution Algorithm

- Natural Deduction tries to prove theorems from scratch, but Resolution takes a different approach by taking a given formula and working with it to decide if it is a theorem

Method:

1. Start with a given formula ϕ
2. Negate it (because Resolution works be showing the negation is unsatisfiable)
3. Convert the negated formula to C.N.F $(C_1 \wedge C_2 \wedge \cdots \wedge C_m)$
4. Extract the individual Clauses $C_1, C_2, \ldots, C_m$
5. Repeatedly resolve each clause

Halting Conditions:

- **Proving a Theorem:** If we ever infer the empty clause $\emptyset$, then we halt and output that ϕ is a theorem
- **Disproving a Theorem:** If we get to the point where we have not inferred the empty clause and we cannot infer any new clauses, then we halt and output that ϕ is not a theorem

Extra Rule:

- **Deleting Literals:** When resolving, we are also allowed to delete repeated literals in any clause

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Example:

- **Question:** Let ϕ be the formula $\neg((p \vee q) \wedge (\neg p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q))$. Is ϕ a theorem?

Negate the Formula: $(p \vee q) \wedge (\neg p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$
- The Formula is already in Conjunctive Normal Form

List the Clauses:

Clause 1: $p \vee q$
Clause 2: $\neg p \vee q$
Clause 3: $p \vee \neg q$
Clause 4: $\neg p \vee \neg q$

Resolution Rule: Combine the clauses with clashing literals

Clause 1 and 2: (The p and $\neg p$ clash)

$$(p \vee q) \quad (\neg p \vee q) \longrightarrow q \vee q \longrightarrow q$$

Clause 3 and 4: (The p and $\neg p$ clash)

$$(p \vee \neg q) \quad (\neg p \vee \neg q) \longrightarrow \neg q \vee \neg q \longrightarrow \neg q$$

Now the Clauses p and $\neg p$ clash, leaving nothing left $(\emptyset)$

Since we have inferred the Empty Clause, it means $\neg\phi$ is a contradition
$\therefore \phi$ is a Theorem

Diagram:

```
(p ∨ q)      (¬p ∨ q)        (p ∨ ¬q)     (¬p ∨ ¬q)
     \           /               \            /
      \         /                 \          /
       \       /                   \        /
        ---(q)---                   -(¬q)--
            \                          /
             \                        /
              \                      /
               \                    /
                ---- Empty Clause ----
                         (∅)
```

Shortcut when converting to c.n.f. $(\neg(A \Rightarrow B) \equiv A \wedge \neg B)$

- Used when Implications are involved $(\Rightarrow)$

The negation of $A \Rightarrow B$ is equivalent to $A \wedge \neg B$

Proof:
Start with the Negation:

$\neg(A \Rightarrow B)$	Implication Law
$\neg(\neg A \vee B)$	De Morgan's
$(A \wedge \neg B)$	

Truth Table:

```
 A | B || A ⇒ B | ¬(A ⇒ B) || ¬B | A ∧ ¬B
---|---||-------|----------||----|--------
 T | T ||   T   |    F     || F  |   F
 T | F ||   F   |    T     || T  |   T
 F | T ||   T   |    F     || F  |   F
 F | F ||   T   |    F     || T  |   F
```

Related Pages:

- Logic
- Sat-Solvers
- Logic Practical Week 15