

# Converging or Diverging Series Tests

## Choosing the Right Test

### Step 1: Divergence Test

- Take the Limit of the term ( $a_k$ ) as  $k \rightarrow \infty$
- Limit  $\neq 0 \rightarrow$  **Divergent**
- Limit = 0  $\rightarrow$  Test is **Inconclusive** (Move onto the next step)

**Example:**

$$\sum_{k=0}^{\infty} \frac{k}{k+1} \rightarrow \lim_{k \rightarrow \infty} \frac{k}{k+1} = 1 \neq 0 \therefore \text{The Series Diverges}$$

### Step 2: The “Standard Series” Check

- Check if the series fits a “memorised” pattern
- **Geometric Series:** ( $\sum ar^n$ )
  - **Converges** if  $|r| < 1$
  - **Diverges** if  $|r| > 1$
- **P-Series:** ( $\sum \frac{1}{n^p}$ )
  - **Converges** if  $|p| > 1$
  - **Diverges** if  $|p| \leq 1$

### Step 3: Deciding which Test to Use

- If the series isn't a Standard Form, look at the algebra of term  $a_n$ , to pick the right test

If you see this in $a_n$	Use this Test	Reasoning
Factorials $k!$ or Exponentials ( $3^k$ )	Ratio Test	Ratio Test cancels out factorials perfectly (e.g. $\frac{(n+1)!}{n!} = n+1$ )
Entire term raised to the $n^{th}$ power $(\dots)^k$	Root Test	The $n^{th}$ root $\sqrt[n]{a_n}$ removes the power
Rational Functions (Polynomial over Polynomial)  e.g. $\frac{n}{n^3+1}$	Limit Comparison (Quotient Test)	The Ratio Test usually fails here (results in 1) So we compare it to a 'similar' P-Series, based on the highest powers (e.g. $\frac{k}{k^3} = \frac{1}{k^2}$ )
Logarithms ( $\ln(k)$ ) or simple decreasing functions	Integral Test	Use <b>only</b> if we can easily integrate the function

If you see this in $a_n$	Use this Test	Reasoning
Oscillating Term ( $\sin k, \cos k$ )	Direct Comparison Test	Limits don't exist for Oscillating terms So we replace them with their mix/max values (e.g. $\cos k \leq 1$ ) to build an inequality
Alternating Signs ( $-1^n$ )	Alternating Sign Test	This test is designed for series with negative terms

## Tests:

### 1. The Ratio Test:

- Best for Factorials ( $k!$ ) and Exponentials ( $3^k$ )

Formula:

- Calculate the limits of the “next term divided by the current term”

$$\rho = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|$$

Results:

- $\rho < 1$  : The series **converges** (Absolutely)
- $\rho > 1$  : The series **diverges**
- $\rho = 1$  : **Test Inconclusive** (Try a different test)

### 2. The $n^{th}$ Root Test:

- Best for Entire terms raised to the ( $k^{th}$ ) power e.g.  $(\dots)^k$

Formula:

- Calculate the  $k^{th}$  root of the absolute term (get rid of the power)

$$\rho = \lim_{k \rightarrow \infty} |a_k|^{\frac{1}{k}}$$

Results:

- $\rho < 1$  : The series **converges** (Absolutely)
- $\rho > 1$  : The series **diverges**
- $\rho = 1$  : **Test Inconclusive** (Try a different test)

### 3. The Limit Comparison Test (Quotient Test):

- Best for Rational Functions (polynomials over polynomials)

Method:

1. Create a simplified comparison series by keeping only the highest powers of  $k$
2. Calculate the limit ratio

$$\rho = \lim_{k \rightarrow \infty} \frac{a_k}{b_k}$$

Rules:

- If  $0 < \rho < \infty$  (Finite and non-zero): Both series behave the same
  - If  $\sum b_k$  converges  $\rightarrow \sum a_k$  converges
  - If  $\sum b_k$  converges  $\rightarrow \sum a_k$  converges
- If  $\rho = 0$  :
  - If  $\sum b_k$  converges  $\rightarrow \sum a_k$  converges
- If  $\rho = \infty$  :
  - If  $\sum b_k$  converges  $\rightarrow \sum a_k$  converges

## 4. Direct Comparison Test:

- Best for Oscillating terms ( $\sin k, \cos k$ )

Method:

- Find a simple series  $\sum b_k$  that is strictly larger or smaller than the original series  $\sum a_k$

Rules:

- The ceiling Rule (**Convergence**):
  - If  $0 \leq a_k \leq b_k$ , and  $\sum b_k$  converges  $\rightarrow \sum a_k$  **converges**
- The Floor Rule (**Divergence**):
  - If  $0 \leq b_k \leq a_k$ , and  $\sum b_k$  diverges  $\rightarrow \sum a_k$  **diverges**

Example:

For  $\frac{2 + \cos k}{k^2}$ , use the fact that  $\cos k \leq 1$

$$\therefore \frac{2 + \cos k}{k^2} \leq \frac{3}{k^2}$$

## 5. Integral Test:

- Best for Logarithms, or easy to integrate functions

Formula:

- Convert the sequence  $a_k$  into a continuous function  $f(x)$  and integrate from 1 to  $\infty$

$$I = \lim_{N \rightarrow \infty} \int_1^N f(x) dx$$

Condition:

- The function  $f(x)$  must be positive and decreasing for  $x \geq 1$

Rules:

- Integral is **Finite**  $\rightarrow$  Series **Converges**
- Integral is **Infinite**  $\rightarrow$  Series **Diverges**

## 6. Alternating Sign Test:

- Best for series with  $(-1)^k$

Method:

- Ignore the negative sign (look at  $|a_k|$ ) and check two conditions

Conditions:

1. **Limit:**  $\lim_{k \rightarrow \infty} |a_k| = 0$
2. **Decreasing:**  $|\lim_{k \rightarrow \infty} |a_k|| < |a_k|$ , the terms eventually get smaller (in magnitude)

Rules:

If **both** conditions are met  $\rightarrow$  The series **Converges**

**Note:** Does not check for absolute convergence

**Related Pages:**

- [Double Summation](#)
- [Methods to Find Limits](#)
- [Taylor Series](#)