# Logic

## Intro to Logic

- Logic provides a formal language to avoid the ambiguity of natural languages (e.g. English)

**3 Components of Logic:**

- **Syntax:** The grammatical rules of a language (so we know if a statement is 'valid')
- **Semantics:** The meaning of the syntax (to determine the truth or falsity of a statement)
- **Proof System:** A rigorous method for manipulating formulae to construct proofs

**Desirable properties of a logical system:**

- **Completeness:** Every statement that is true (according to semantics) can be proven (using the proof system)
- **Soundness:** The proof system only derives true conclusions from statements that are both valid and true

**Propositions:**

- A declarative 'statement of fact' that can either be true or false, but not both (Questions, Commands and gibberish are *not* propositions)
- We can use **propositional variables** to represent propositions, and they can take the **truth values** $T$ (True) or $F$ (False)

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## Truth Tables & Semantics

| p | q | $\neg p$ (Not) | $p \land q$ (And) | $p \lor q$ (Or) | $p \implies q$ (Implies) | $p \iff q$ (Iff) |
|---|---|---|---|---|---|---|
| **T** | **T** | F | **T** | **T** | **T** | **T** |
| **T** | **F** | F | F | **T** | **F** | F |
| **F** | **T** | **T** | F | **T** | **T** | F |
| **F** | **F** | **T** | F | F | **T** | **T** |

**Syntax:**

- $\Rightarrow$ (Implies): $p \Rightarrow q$ if only False when $p$ **is True and** $q$ **is False** (broken promise)
- $\Leftrightarrow$ **(Iff):** True only when $p$ and $q$ are the **same**
- $\boldsymbol{\phi}$ **(phi)**, $\boldsymbol{\psi}$ **(psi):** Represent **any** complex formula (e.g. $\phi$ could be $(p \land q) \Rightarrow r$)

- letters (like $p, q, r$) are used to represent **atomic** statements

**Tautology:**

- A statement $(\phi)$ that is True for **every** truth assignment (True for every possible input)

**Contradiction:**

- A statement $(\phi)$ that is False for **every** truth assignment (False for every possible input)

**Satisfiable:**

- A statement $(\phi)$ that can be satisfied (There exists an input that results in $\phi$ being True)

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## Logical Equivalence & Algebra

### Implication Law:

- This is used to convert an implication $(\Rightarrow)$ into standard Or/Not Logic

$$p \Rightarrow q \equiv \neg p \vee q$$

**English Example:**

- Think about the promise "If it rains, I will bring an umbrella." The only time I break my promise is if "It rains" ($p = T$) **AND** "No umbrella" ($q = F$). So, as long as "It doesn't rain" ($\neg p$) **OR** "I have the umbrella" ($q$), I am safe.

### De Morgan's Laws:

- **Negating AND:** $\neg(p \wedge q) \equiv \neg p \vee \neg q$.
  - *Translation:* "Not (A and B)" is the same as "(Not A) or (Not B)"
- **Negating OR:** $\neg(p \vee q) \equiv \neg p \wedge \neg q$.
  - *Translation:* "Not (A or B)" is the same as "(Not A) and (Not B)"

### Distribution Laws:

- **Distributing OR over AND:**

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

  - *Note:* The outside operator $(\vee)$ goes inside the new brackets. The inside operator $(\wedge)$ moves to the middle.
- **Distributing AND over OR:**

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

## Normal Forms & Completeness

### Functional Completeness:

- Set of logical operators is **Functionally Complete** if *any* possible propositional formula can be written using *only* the operators in that set

#### Examples of Functionally Complete Sets:

- $\{\wedge, \vee, \neg\}$
- Even $\{\wedge, \neg\}$ since $A \vee B \equiv \neg(\neg A \wedge \neg B)$
- $\{\Rightarrow, \neg\}$
- $\text{NAND}$: False only if both are True
- $\text{NOR}$: True only if both are False

## Disjunctive Normal Form: (DNF)

#### Definition:

- A formula is in **DNF** if it is a **Disjunction of Conjunctions** (an OR of ANDs)
- **Structure:** $(A \wedge B) \vee (\neg A \wedge C) \vee \dots$
- Think of it as: "A list of ways to win" (A collection of different ways to make the formula True)

#### How to build it (Truth Table Method):

1. Look at the Truth Table
2. Identify every row where the result is **True**
3. Write a conjunction for that row $(\text{e.g. if } p = T, q = F, \text{write } (p \wedge \neg q))$
4. Join each row with $\vee$

## Conjunctive Normal Form: (CNF)

#### Definition:

- A formula is in **CNF** if it is a **Conjunction of Disjunctions** (an AND of ORs)
- **Structure:** $(A \vee B) \wedge (\neg A \vee C) \wedge \dots$
- Think of it as: "A list of requirements, all of which must be met"

#### How to build it (Truth Table Method):

1. Identify the **False** rows
2. Negate the variables in the "False" row
3. Join then with **ORs** $(\vee)$ to make a clause
    - *Example:* If the false row is $p = T, q = F, r = T$, write $(\neg p \vee q \vee \neg r)$

4. Then join the clauses (False rows) with **ANDs** ($\wedge$)

**How to build it (Syntactic Method):**

1. Eliminate the arrows ($\Rightarrow, \Leftrightarrow$)
2. Push any negations inside brackets (De Morgan's)
3. **Distribute** $\vee$ over $\wedge$ until you have the correct shape $(p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r))$

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# Natural Deduction

**The Sequent:**

$$\phi_1, \phi_2 \vdash \psi$$

- **Left side** $(\phi_1, \phi_2)$: The **Premises** (What you are given/allowed to assume is true)
- **Symbol** $(\vdash)$: Pronounced "turnstile". It means "provably entails"
- **Right side ($\psi$):** The **Conclusion** (What you are trying to reach)
- **English:** "Given that $\phi_1$ and $\phi_2$ are True, prove that $\psi$ is also True"

**Rules:**

1. Lines $1 \text{ to } n$ are the **Premises** (state what we have been given at the start)
2. Every line after that must be justified by a **Rule** applied to previous line(s)
3. The final line must be the **Conclusion** $(\psi)$
   ***Important:*** *Note that every line evaluates to True*

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### Conjunction $(\wedge)$ and Double Negation $(\neg\neg)$:

1. **And-Introduction** $(\wedge i)$
   - **Effect:** If you have two lines $x(A)$ and $y(B)$, you can write $A \wedge B$
   - **Notation:** $\wedge i \; x, y$
   - Where $x$ and $y$ are the line numbers and $A$ and $B$ are previously stated propositions
   - **Example:**

$$\frac{\frac{\phi_1}{\phi_2}}{\phi_1 \wedge \phi_2}(\wedge i \; 1, 2)$$

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2. **And-Elimination** $(\wedge e)$
   - **Rule:** If you have $A \wedge B$, you can extract just $A$ (or just $B$)

- **Notation:** $\wedge e\ 1\ (or \wedge e\ 2 \text{ for B})$
- **Example:**

$$\frac{\phi \wedge \psi}{\phi}\ (\wedge e1) \quad \text{or} \quad \frac{\phi \wedge \psi}{\psi}\ (\wedge e2)$$

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3. **Double Negation Elimination $(\neg\neg e)$
   - **Rule:** $\neg\neg A$ is the same as $A$, so you can remove the double negative
   - **Notation:** $\neg\neg e\ x$

**Example:**

- Prove $p, \neg\neg(q \wedge r) \vdash p \wedge r$

$$\begin{array}{ll}
1.\ p & \text{Premise} \\
2.\ \neg\neg(q \wedge r) & \text{Premise} \\
3.\ q \wedge r & \neg\neg e\ 2 \\
4.\ r & \wedge e_2\ \ 3 \longleftarrow \text{(Take the 2}^{\text{nd}}\text{ part of the 3}^{\text{rd}}\text{ line)} \\
5.\ p \wedge r\ \ \checkmark & \wedge i\ 1, 4 \longleftarrow \text{(AND the 1}^{\text{st}}\text{ line and the 4}^{\text{th}}\text{ line)}
\end{array}$$

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## Implication & The Boxes

- To prove an "If... then..." statement $(\Rightarrow)$, you have to build a **simulation**

### 1. Implication Introduction $(\Rightarrow i)$

- To prove $A \Rightarrow B$, you start a sub-proof where you **assume** $A$ to be True, then use that to reach $B$

**Boxes (Sub-Proof):**

1. **Top of Box:** You **Assume** the left side $(A)$ to be True
2. **Bottom of Box:** You end with $B$ (proven from $A$)
3. **Exit:** Close the box and write $A \Rightarrow B$ so you can use it from now on

***Important:*** *Note that you* ***cannot*** *use* ***any*** *statements from the sub-proof outside of the box*

**Example:**

$$
\begin{array}{ll}
1. \ \ldots & \text{(some previous work)} \\
\boxed{\begin{array}{ll} 2.\ A & \text{assumption} \\ \vdots & \\ 5.\ B & \text{(proven from A)} \end{array}} & \\
6.\ A \Rightarrow B & \Rightarrow i\ 2\text{--}5
\end{array}
$$

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**2. Implication Elimination** $(\Rightarrow e)$

*Also known as Modus Ponens*

**Rule:**

- If you know **"If A then B"** $(A \Rightarrow B)$ is true...
- And you know **"A"** is true...
- Then you can smash them together to get **"B"**

**Example:**

- Prove $A, A \Rightarrow B \vdash B$

$$
\begin{array}{lll}
1. & A & \text{Premise} \\
2. & A \Rightarrow B & \text{Premise} \\
3. & B & \Rightarrow e\ 1, 2
\end{array}
$$

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## Disjunction

### 1. Disjunction Introduction $(\vee i)$

- If a statement is True, then you can **OR** any other statement to it, and the whole thing remains True

**Example:**

$$
\frac{\phi}{\phi \vee \psi}\ (\vee i_1) \quad \text{or} \quad \frac{\psi}{\phi \vee \psi}\ (\vee i_2)
$$

- **Syntax:** Use $\vee i_1$ (if adding to the right) or $\vee i_2$ (if adding to the left) + line number

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### 2. Disjunction Elimination $(\vee e)$

- If you know $A \vee B$ is True, but you don't know *which* one is True. So, to prove a conclusion $C$, you must prove it works in **both scenarios**

### The Structure (Two Boxes):

1. **Box 1:** Assume $A$ is true $\rightarrow$ Prove $C$
2. **Box 2:** Assume $B$ is true $\rightarrow$ Prove $C$
3. **Conclusion:** Since you get $C$ in *either* case, $C$ is definitely True

### Structure: (Proof by Cases)

$$
\begin{array}{ll}
1.\ A \vee B & \text{Premise} \\
\boxed{\begin{array}{lll} 2.\ A & \text{Assumption} & \leftarrow \text{(Case 1)} \\ \vdots & & \\ 4.\ C & \text{(Result)} & \end{array}} & \\
\boxed{\begin{array}{lll} 5.\ B & \text{Assumption} & \leftarrow \text{(Case 2)} \\ \vdots & & \\ 7.\ C & \text{(Result MUST be same)} & \end{array}} & \\
8.\ C & \vee e\ 1, 2{-}4, 5{-}7
\end{array}
$$

- **Syntax:** Use $\vee e$ + Line of OR + Range of Box 1 + Range of Box 2

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### Disjunction Example: $(p \vee q \vdash q \vee p)$

- Proof that $\vee$ is Commutative
- **Plan:**
  1. **Start:** You have $p \vee q$. You need to use $\vee e$ (Proof by Cases)
  2. **Case 1 (Box 1):** Assume $p$. Use $\vee i$ to turn it into $q \vee p$
  3. **Case 2 (Box 2):** Assume $q$. Use $\vee i$ to turn it into $q \vee p$
  4. **Finish:** Cite the rule `ve` to conclude $q \vee p$
     **Sequent:** $p \vee q \vdash q \vee p$

$$
\begin{array}{ll}
1.\ p \vee q & \text{Premise} \\
\boxed{\begin{array}{ll} 2.\ p & \text{Assume} \\ 3.\ p \vee q & \vee i_2\ 2 \end{array}} & \\
\boxed{\begin{array}{ll} 4.\ q & \text{Assume} \\ 5.\ p \vee q & \vee i_1\ 4 \end{array}} & \\
6.\ q \vee p & \vee e\ 1, 2{-}3, 4{-}5
\end{array}
$$

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## Negation $(\neg)$ & Contradiction $(\bot)$

### Notation: $(\bot)$

- The $\perp$ symbol (called "Bottom") represents a **Contradiction** (False)
- Its means that we have hit a logical dead end. We have proved that something is both True AND False at the same time (e.g. $P \land \neg P$)

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### 1. Negation Elimination $(\neg e)$ / Bottom Introduction $(\perp i)$

- **Rule:** If you have a statement $(A)$ on one line, and its exact opposite $(\neg A)$ on another, you can smash them together to create a **Contradiction** $(\perp)$

**Example:**

$$\frac{\phi \quad \neg\phi}{\perp}\ (\neg e \text{ or } \perp i)$$

- **Notation:** $\neg e$ (or $\perp i$) + the line number of the combined contradicting statements

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### 2. Bottom Elimination

*Ex Falso Quodlibet ("From falsehood, anything follows")*

- Once you have a Contradiction $(\perp)$, you can conclude **ANYTHING** you want
- Because if the rules are broken, logic goes out the window

**Example:**

$$\frac{\perp}{\phi}\ (\perp e)$$

- **Notation:** $\perp e$ + the line number of the contradiction

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### 3. Negation Introduction $(\neg i)$

*Also known as "Proof by Contradiction" (for negations)*

**Steps to prove that something is False $(\neg A)$:**

1. **Assume** the statement is True $A$
2. Show that this assumption leads to a **Contradiction** $(\perp)$
3. **Conclusion:** Since assuming $A$ to be True, broke the logic, $A$ must be False
4. $\therefore \neg A$ is True

**Structure:**

$$
\begin{array}{lll}
1. \ \ldots & & \\
\boxed{\begin{array}{lll} 2.\ A & \text{Assumption} & \\ \vdots & & \\ 5.\ \bot & \neg e\ \ldots & \leftarrow \text{(Crash found)} \end{array}} & & \\
6.\ \neg A & & \neg i\ 2{-}5
\end{array}
$$

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### Negation Example - Proof of Modus Tollens (MT)

$x \Rightarrow y, \neg y \vdash \neg x$

$$
\begin{array}{ll}
1.\ x \Rightarrow y & \text{Premise} \\
2.\ \neg y & \text{Premise} \\
\boxed{\begin{array}{ll} 3.\ x & \text{Assume} \\ 4.\ y & e \Rightarrow 3, 1 \\ 5.\ \bot & \neg e\ 4, 2 \end{array}} & \\
6.\ \neg x & \neg i\ 3{-}5
\end{array}
$$

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## Derived Rules (Theorems)

### 1. Modus Tollens (MT)

*(As proved above)*

- **Rule:** If $A \Rightarrow B$ is True, and $B$ is False $(\neg B)$, then $A$ must be False $(\neg A)$
- **Notation:** MT $x, y$

**Example:**

$$\frac{A \Rightarrow B \quad \neg B}{\neg A}\ (MT)$$

### 2. Reductio Ad Absurdum (RAA)

*("Proof by Contradiction" for **positive** conclusions)*

- **Method:** To prove $A$, assume $\neg A$, find a contradiction $(\bot)$ then conclude $A$
- **Difference from Negation Introduction** $(\neg i)$**:**
  - $\neg i$: Assume $A \rightarrow$ Contradiction $\rightarrow$ Conclude $\neg A$
  - $RAA$: Assume $\neg A \rightarrow$ Contradiction $\rightarrow$ Conclude $A$
- **Notation:** RAA (applied to a box assuming $\neg A$ and ending in $\bot$)

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### 3. Law of Excluded Middle (LEM)

- **Concept:** Everything is either True or False ($A \lor \neg A$). There is no third option
- **Usage:** You can write $A \lor \neg A$ on any line, (without any requirements) usually used to set up a **Proof by Cases** $(\lor e)$
- **Notation:** LEM (nothing else needed)

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### Proof of De Morgan's Law

- **Sequent:** $\neg(p \lor q) \vdash \neg p \land \neg q$

$$
\begin{array}{ll}
1.\ \neg(p \lor q) & \text{Premise} \\
\quad \text{(Plan: Get } \neg p\text{)} & \\
\boxed{\begin{array}{ll} 2.\ p & \text{Assume} \\ 3.\ p \lor q & \lor i_1\ 2 \\ 4.\ \bot & \bot i\ 3,1 \end{array}} & \\
5.\ \neg p & \neg i\ 2\text{–}4 \\
\quad \text{(Plan: Get } \neg q\text{)} & \\
\boxed{\begin{array}{ll} 6.\ q & \text{Assume} \\ 7.\ p \lor q & \lor i_2\ 6 \\ 8.\ \bot & \bot i\ 7,1 \end{array}} & \\
9.\ \neg q & \neg i\ 6\text{–}8 \\
10.\ \neg p \land \neg q\ \checkmark & \land i\ 5,9
\end{array}
$$

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