Taylor Series

Standard Maclaurin Expansions

Maclaurin Series = A Taylor series expansion of a function $f(x)$ about the point $a = 0$. It is given by the power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

- Simple Explanation: A way of turning functions into an infinite sum of simple powers of $x(1, x, x^2, \ldots)$ all centred at zero.

Standard Maclaurin Expansions to Memorise:

Geometric Series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ \ldots \ = \sum_{n=0}^{\infty} x^n \text{ for } |x| < 1$$

Geometric Series (General Form):

$$\text{Sum } = \frac{a}{1-r} = a + ar + ar^2 + \ldots = \sum_{n=0}^{\infty} ar^n$$

$$a = \text{ first term, } r = \text{ common ratio (The common ratio normally includes an } x)$$

Exponential Function:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ \ldots \text{ for all } x$$

Sine (Odd powers, alternating signs):

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ \ldots \text{ for all } x$$

Cosine (Even powers, alternating signs):

$$cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ \ldots \text{ for all } x$$

Logarithms:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ \ldots \text{ for } |x| < 1$$

Substitution:

For functions that are similar to standard forms you can use substitution e.g. $\left(e^{-3x}\right)$
Example: Finding the series for e^{-3x}

- **Standard Series:** $e^u = 1 + u + \frac{u^2}{2!} + \dots$
- **Substitute:** Let $u = -3x$:

$$e^{-3x} = 1 + (-3x) + \frac{(-3x)^2}{2!} + \frac{(-3x)^3}{3!} + \dots$$

- **Calculate Powers:**

$$e^{-3x} = 1 - 3x + \frac{9x^2}{2} - \frac{27x^3}{6} + \dots$$

  Note on Limits:
- If the original series converges for $|x| < 1$, the new series converges for $|u| < 1$
- *Example:* $\frac{1}{1+w^2}$ converges when $|-w^2| < 1$, which simplifies to $|w| < 1$

Algebraic Operations:

For questions like $\left(\frac{sin(x)}{1-3x}\right)$ that combine multiple series:

- Treat the original function as a product of multiple functions $f(x)g(x)$
  - Find the series expansion for each sub-function individually and multiply them together

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General Taylor Series

- To be used when the question asks for an expansion "about $x = a$" (where $a \neq 0$)
  Complex Definition:
- Let f be indefinitely differentiable at a.
- The Taylor Series of f about a is the power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$
  Simple Explanation:
- A method to approximate a curve near a specific point a (the "base point") rather than zero.
- Instead of powers of x, we use powers of the distance from the base point $(x - a)$

Formula:

The Taylor Series of f about point a:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

Example:

For $f(x) = \ln(x)$ about $a = 1$ (degree 2):

- $f(x) = \ln(x) \to f(1) = 0$
- $f'(x) = 1/x \to f'(1) = 1$
- $f''(x) = -1/x^2 \to f''(1) = -1$
  Result: $P_2(x) = 0 + 1(x-1) + \frac{-1}{2}(x-1)^2$
  Note that $P_2(x)$ means the polynomial (degree 2) with respect to x

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Langrage Remainder (Error Analysis):

- The Taylor polynomial $P_N(x)$ is just an approximation
- The "Remainder" (R_{N+1}) is the exact error difference between the approximation and the real function

The Theorem:

$$f(x) = P_{N,a}(x) + R_{N+1,a}(x)$$

$f(x) =$ (Taylor Polynomial of Degree N centred at a) + (Remainder Term)

The Lagrange Form Formula:

$R_{N+1}(x)$ is the Nth **remainder term**

$$R_{N+1}(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!}(x-a)^{N+1} \text{ for some } \xi \in (x, a)$$

for some value ξ in the interval between a and x

- The error in the approximation (R) is determined by the **next derivative** $f^{(N+1)}$ at some unknown point (ξ), multiplied by the distance from the centre $(x-a)$, and divided by a huge factorial number
- We cannot calculate the exact value of ξ, instead we assume the **worst-case scenario** (the maximum possible value the derivative could have between a and x)

Bounding the Error ('Worst-Case Scenario'):

Since we cannot calculate the exact value of ξ, we find the **maximum possible error** (an upper bound).

1. **Write the Formula:** Write down $R_{N+1}(x)$, leaving ξ inside the derivative

2. **Maximise the Derivative:** After differentiating f^{N+1}, look at the term $|f^{(N+1)}(\xi)|$. Find the maximum value (M) this term could possibly take for any number ξ in the integral (a, x).
   - *Tip:* For $\sin(\xi)$ or $\cos(\xi)$, the max is always 1.
   - *Tip:* For increasing functions like e^{ξ}, the max is at the larger end of the interval
3. **Substitute and Solve:** Replace $f^{(N+1)}(\xi)$ with M to form the inequality:

$$|R_{N+1}(x)| \leq \frac{M}{(N+1)!}|x-a|^{N+1}$$