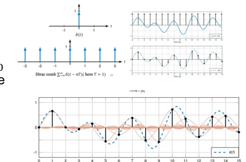
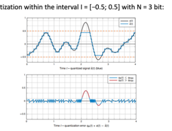
Signals: flow of information, measured quantity that varies with time (or position); Types: Continuous (voltage, current, temperature, speed) analogue infinite values over time, Discrete (daily max temperature, sampled continuous signals) sampled at discrete intervals; Signal transformation for: amplify or filter out embedded info, detect patterns, prevent interference, undo distortions; Digital Signal Processing: adv: noise easy to control, flexibility, insensitive to component tolerances; disadv: aliasing, more power, cause interference, application; applications: communication system, consumer electronics, music, medical diagnostics, transportation, security; Periodic signals: signal with base period $T_0 > 0$ if for all n ∈ N holds s(t) = s(t + nT_0) where T_0 is the smallest value for the definition, $f_0 = 1/T_0$ denotes frequency in Hz, $\omega_0 = 2\pi f_0$ is the angular frequency in rad/s where 1 rad = 1/2π Hz, the period of a resulting signal is the LCM of the periods of the individual signals, to get period from sin(xt) where x = 2π /T and T is the period; Rectangle impulse: rect(t) = A for –T/2 < t < T/2, A/2 for |t| = T/2, 0 otherwise; Gaussian impulse: gauss(t) = exp(-π(2t/T)) (smooth edges); Sampling: discretization in the time domain; Quantization: discretization in value domain; Dirac δ: properties: identity: $\int_{-\infty}^{\infty}\delta(t)dt = 1$, sampling: $\int_{-\infty}^{\infty}\delta(t)s(t)dt = s(0)$, displacement: $\int_{-\infty}^{\infty}\delta(t-t')s(t)dt = s(t')$, Dirac comb: $\int_{-\infty}^{\infty}s(t)\sum_{-\infty}^{\infty}\delta(t-nt_s)dt = \sum_{-\infty}^{\infty}\int_{-\infty}^{\infty}s(t)\,\delta(t-nt_s)dt$ expresses the sampling of a continuous signal s(t), with the n-th time discrete sample denoted as s[n] for all n ∈ Z and sampling period t_s

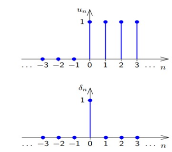


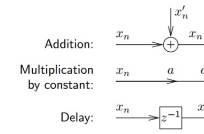
Energy (Joules): E_s of a time-continuous signal is the squared magnitude of the signal $E_s = \int_{-\infty}^{\infty}|s(t)|^2dt$ and for a time-discrete signal $E_s = \sum_{n=-\infty}^{\infty}|s[n]|^2$ with finite energy, is called an energy signal Power (Watts): P_s of a time-continuous signal is the time average of the squared magnitude of the signal $P_s = \lim_{T\to\infty} 1/2T\int_{-T}^{T}|s(t)|^2dt < \infty$, P_s of $\sin(x) = \frac{1}{2}$; Reconstruction: by means of samples s[n] we can reconstruct original signal s(t), each sample used as weight for a time-shifted basis function, we use the sinc impulse $sinc(t) = \sin(\pi t)/\pi t$, sinc reaches max at t=0 and has value of 0 at all integers of t, we use a series of sinc impulses shifted by T_S (sampling interval) and weighted by the series of the time-discrete samples s[n]: $s(t) \approx \sum_{n=-\infty}^{\infty}s[n]\cdot sinc(\frac{t-nT_S}{T_S})$, and the sincs don't effect other sincs, since they have a value of 0 at integer values of t; Quantization maps the value-continuous samples s[n] to set defined signal levels, for each level we define a binary code word, $M = 2^N$ signal stages we need words of N bits, optimal if $I_Q = [a,b]$ occur with equal probability, distance between levels is $\Delta = \frac{b-a}{M} = \frac{b-a}{2^N}$ with max quantization error $q_{max} = \frac{\Delta}{2}$ inside I_Q, and outside I_Q we map to lowest or highest value in I_Q so q_{max} is unbounded; SNR: ratio comparing the level of desired signal to level of background noise = P_S/P_n, for a uniform distribution variance is $(b-a)^2/12$ so the $P_n = \Delta^2/12$; $SNR_{db} = 10\log_{10}(SNR)$

• Linear quantization within the interval I = [−0.5; 0.5] with N = 3 bit:

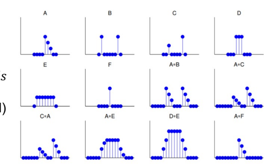


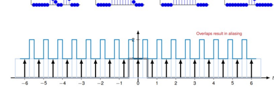
Discrete sequence: $\{x_n\}_{n=-\infty}^{\infty}$ is a sequence of numbers $.., x_{-1}, x_0, x_1, ..$ where x_n denotes the n-th number in the sequence, discrete sequence samples a continuous function $x(t)$ as $x_n = x(t_s\cdot n)$ where t_s is the sampling frequency; Unit step sequence $u_n = 0\ if\ n<0, = 1\ if\ n \geq 0$, Impulse sequence: $\delta_n = 1, if\ n = 0,\ = 0\ if\ n \neq 0$, $\delta_n = u_n - u_{n-1}$; Sinusoidal sequences: $x(t) = Acos(2\pi ft + \varphi)$ where the part inside the cosine is the phase, and sampling at rate f_s results in the discrete sequence $x_n = Acos\left(\frac{2\pi fn}{f_s} + \varphi\right) = Acos(\dot{\omega}n + \varphi)$ where $\dot{\omega} = 2\pi f/f_s$ is frequency in radians per sample; Properties of sequences: absolutely summable, square summable, a continuous function with period t_p is still periodic after sampling iff $\frac{t_p}{t_s} \in Q$; Why take square and abs when computing energy: to get a real number even if the components where imaginary $(a+bi)(a-bi)$; DC: direct current (constant), DC component = $mean\ voltage = \frac{1}{t_2-t_1}\int_{t_1}^{t_2}f(\tau)d\tau$, AC: alternating current (zero mean), AC component = f(t) – DC component, strength of AC signal using RMS = $\sqrt{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}|f(t)|^2dt}$ is the strength of a DC signal of equal average power, rms of sin wave = $A/\sqrt{2}$; Sensation limit (SL, ϕ_0) = lowest intensity stimulus that can still be perceived, Difference limit (DL, $\Delta\phi$) = smallest perceivable stimulus difference at a given intensity level; Weber's law: Difference limit $\Delta\phi$ is proportional to the intensity ϕ of the stimulus (except for a small correction constant a, to describe deviation of experimental results near SL): $\Delta\phi = c(\phi + a)$; Fechner's scale: Define a perception intensity scale ψ using the sensation limit ϕ_0 as the origin and the respective difference limit $\Delta\phi = c\phi$ as a unit step. $\psi = \log_c \phi/\phi_0$; A sound that is 20 DL over SL is perceived as more than twice as loud as one that is 10 DL over SL, $\psi = k(\phi - \phi_0)^a$; Why logarithmic units: quantities vary over many orders of magnitude, ratio of quantities more interesting, perception is logarithmic; Units: Neper (Np), Bel (B); Decibel: where F is a field quantity (ex voltage) and F_0 the corresponding reference level: 10 dB · $\log_{10}$(P/ P_0) = 20 dB · $\log_{10}$(F/ F_0), If you want absolute signal strength, common reference values are indicated with a suffix after "dB": 0 dBW = 1 W, 0 dBm = 1 mW = –30 dBW, 0 dBµV = 1 µV, 0 dBu = 0.775 V = √(600Ω × 1 mW), 0 dBSPL = 20 µPa (sound pressure level), 0 dB = perception threshold (sensation limit), 0 dBFS = full scale (clipping limit of analog/digital converter) Decibel (cont.) • Remember 3 dB = 2× power, 6 dB = 2× voltage, 10 dB = 10× power, 20 dB = 10× voltage; Types of discrete systems: causal: $y_n = f(x_n, x_{n-1}, ...)$, memory-less: $y_n = f(x_n)$, time invariant: $\{y_n\} = T\{x_n\} \Leftrightarrow \{y_{n-d}\} = T\{x_{n-d}\}$, linear: $T\{a\cdot x_n + b\cdot x'_n\} = a\cdot T\{x_n\} + b\cdot T\{x'_n\}$, Examples: M-point moving average system: $y_n = \frac{1}{M}\sum_{k=0}^{M-1}x_{n-k}$ C,TI,L,M; Exponential averaging system: $y_n = \alpha\cdot x_n + (1-\alpha)\cdot y_{n-1}$ C,TI,L,M; Accumulator system: $y_n = \sum_{k=-\infty}^{n}x_k$ C,TI,L,M; Backward difference system: $y_n = x_n - x_{n-1}$ C,TI,L,M; $y_n = x_n{}^2$ C,TI,NL,ML; $y_n = \log_2 x_n$ C,TI,NL,ML; $y_n = \max\{\min\{\lfloor 256x_n\rfloor, 255\}, 0\}$ C,TI,NL,ML; $y_n = x_n\cdot u_n$ C,TV,L,ML; $y_n = \frac{1}{2}(x_{n-1} + x_{n+1})$ NC,TI,L,M; $y_n = x_{\lfloor n/4\rfloor}$ C,TV,L,M





Convolution: for an LTI system $y_n = \sum_{-\infty}^{\infty}a_k\cdot x_{n-k}$ the operation over sequences is called convolution and is defined as $\{p_n\} * \{q_n\} = \{r_n\} \Leftrightarrow r_n = \sum_{k=-\infty}^{\infty}p_k\cdot q_{n-k}$; Impulse response: If $\{y_n\} = \{a_n\} * \{x_n\}$ is a representation of an LTI system T, with $\{y_n\} = T\{x_n\}$, then we call the sequence $\{a_n\}$ the impulse response of T, because $\{a_n\} = T\{\delta_n\}$ as $\{a_n\} * \{\delta_n\} = \{a_n\}, \sum_k a_k\cdot\delta_{n-k} = a_n$; Properties: associative, commutative, linear, impulse sequence neutral under convolution, sequence shifting is equivalent to convolving with shifted impulse; Continuous convolution: $(f*g)(t) = \int_{-\infty}^{\infty}f(s)g(t-s)ds$ Convolving two-unit rectangular impulses: $0\ for\ t < -1, increasing\ for\ -1 \leq t \leq 0, \max at\ t = 0\ with\ value\ of\ 1, decreasing\ 0 \leq t \leq 1, 0\ for\ t > 1$; Shifting: convolution $f(t) * \delta(t-t')$ results in a displacement of f(t) by t', given a Dirac comb, the convolution $\sum_{n=-\infty}^{\infty}f(t)*\delta(t-nT)$ leads to a periodic repetition of f(t) in intervals of T, if displacement t – t; is smaller than the timely extension (period) of f(t) the copies of f(t) start to overlap; Optics example: if a projective lens is out of focus, the blurred image is equal to the original image convolved with the aperture shape (filled circle), $point\ spread\ function\ h\left(disk, r = \frac{as}{2f}\right)$: $h(x,y) = \begin{cases}\frac{1}{r^2\pi}, & x^2+y^2 \leq r^2\\ 0, & x^2+y^2 > r^2\end{cases}$, $original\ image\ I, blurred\ image\ B = I * h$: $B(x,y) = \iint I(x-x', y-y')\cdot h(x',y')\cdot dx'dy'$; Electronics: any passive network (resistor, capacitor, inductor) convolves its input voltage U_{in} with an impulse response function h, leading to $U_{out} = U_{in} * h$, that is $U_{out}(t) = \int_{-\infty}^{\infty}U_{in}(t-\tau)\cdot h(\tau)\cdot d\tau$, in example: $\frac{U_{in}-U_{out}}{R} = C\cdot\frac{dU_{out}}{dt}$, $h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}\ if\ t \geq 0, 0\ otherwise$, Images: convolving an image with a kernel is a powerful tool for image processing, to blur or implement Sobel edge detection





Fourier Series: representing periodic continuous signals as a superposition of sine and cosine oscillations of different frequencies as $s(t) = a_0 + \sum_{k=1}^{\infty}(a_k\cos(k\omega t) + b_k\sin(k\omega t))$, the sum component with index k is called the k harmonic, and $\omega = 2\pi/T$, the weights a_k and b_k can be calculated as: $a_k = \frac{2}{T}\int_0^T s(t)\cos(k\omega t)\,dt$ and $b_k = \frac{2}{T}\int_0^T s(t)\sin(k\omega t)\,dt$ and $a_0 = \frac{2}{T}\int_0^T s(t)dt$; Gibbs phenomenon: around jump discontinuities of a continuously differentiable periodic function there, any partial Fourier Series exhibits an oscillatory behavior, these overshoots converge to approximately 8.95% of signal's amplitude, however the infinite series converges anywhere except for the points of discontinuities where it converges to the midpoint of the jump; If signal has zero average then $a_0 = 0$; Sawtooth function: $a_0 = 1, b_k = -\frac{1}{\pi k}$; Even functions: $f(-x) = f(x), then\ b_k = 0$, Odd functions: $f(-x) = -f(x), then\ a_k = 0$

Why are sine waves useful: Sine-wave sequences form a family of discrete sequences that is closed under convolution with arbitrary, Sine waves are orthogonal to each other: in the context of an (infinitely dimensional) vector space, where the "vectors" are functions of the form f: ℝ→ℝ (or f: ℝ→ℂ) and the scalar product is defined as $f\cdot g = \int f(t)\cdot g(t)dt$, Over integer periods or half periods: $m, n \in \mathbb{N}, m \neq n \Rightarrow \int_0^{\pi}\sin(nt)\sin(mt)dt = 0$; and $m, n \in \mathbb{N} \Rightarrow \int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt = 0$, they can be used to form an orthogonal function basis for a transform sequences; Why are exponential functions useful: closed under convolution with arbitrary sequences, they give us all n solutions of equations involving polynomials up to degree n, they give us great unifying theory: $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})\ and\ \sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$ or $\cos(\dot{\omega}n + \varphi) = \Re\left(e^{j(\dot{\omega}n+\varphi)}\right)\ and\ \sin(\dot{\omega}n + \varphi) = \Im\left(e^{j(\dot{\omega}n+\varphi)}\right)$, complex multiplication allows us to modify the amplitude and phase of a complex rotating vector using a single operation and value; Fourier Transform: $\mathcal{F}\{g(t)\}(f) = G(f) = \int_{-\infty}^{\infty}g(t)\cdot e^{-2\pi jft}dt\ and\ \mathcal{F}^{-1}\{G(f)\}(t) = g(t) = \int_{-\infty}^{\infty}G(f)\cdot e^{2\pi jft}df$, $could\ also\ use\ \omega\ as\ \mathcal{F}\{g(t)\}(f) = G(f) = \alpha\int_{-\infty}^{\infty}g(t)\cdot e^{\pm j\omega t}dt$ and $\mathcal{F}^{-1}\{G(f)\}(t) = g(t) = \beta\int_{-\infty}^{\infty}G(\omega)\cdot e^{\pm j\omega t}d\omega$ where $\alpha\beta = \frac{1}{2\pi}\ and\ \alpha = 1\ and\ \beta = \frac{1}{2\pi}\ or\ both\ equaling\ the\ fraction$; Properties: linearity: $ax(t) + by(t) \to aX(f) + bY(f)$, time scaling: $x(at) \to \frac{1}{|a|}X(\frac{f}{a})$, frequency scaling: $\frac{1}{|a|}x\left(\frac{t}{a}\right) \to X(af)$, time shifting: $x(t-\Delta t) \to X(f)\cdot e^{2\pi j\Delta ft} \to X(f-\Delta f)$, Parseval's theorm (total energy): $\int_{-\infty}^{\infty}|x(t)|^2dt = \int_{-\infty}^{\infty}|X(f)|^2df$; rect and sinc: $\mathcal{F}\{rect(t)\}(f) = \int_{-1/2}^{1/2}e^{-2\pi jft}dt = \frac{\sin(\pi f)}{\pi f} = sinc(f)$ and vice versa, some properties: $\int_{-\infty}^{\infty}sinc(t)dt = 1 = \int_{-\infty}^{\infty}rect(t)dt$, proof: $Arect\left(\frac{t}{T}\right) \to \int_{-\infty}^{\infty}Arect\left(\frac{t}{T}\right)e^{-j2\pi ft}dt = \frac{A}{2\pi f}([\sin(2\pi ft)]_{-T/2}^{T/2} + j[\cos(2\pi ft)]_{-T/2}^{T/2}) = \frac{A\sin(\pi fT)}{\pi f} = ATsinc(fT)$; Convolution theorm: convolution in the time domain is equivalent to scalar multiplication in the frequency domain $\mathcal{F}\{(f*g)(t)\} = \mathcal{F}\{f(t)\}\cdot\mathcal{F}\{g(t)\}$, and vice versa (modulation theorem), also works on the inverse fourier transform; Dirac delta function: continuous equivalent of the impulse response $\{\delta_n\}$, $\delta(x) = x\ if\ x \neq 0, and\ \infty\ if\ x = 0\ and\ the\ integral\ is\ 1$, Properties: $\int_{-\infty}^{\infty}f(x)\delta(x-a)dx = f(a)$ and $\int_{-\infty}^{\infty}e^{\pm 2\pi jixa}dx = \delta(a)\ (x\ is\ time\ and\ a\ is\ frequency) \to \sum_{-\infty}^{\infty}e^{\pm 2\pi jixa} = \frac{1}{|a|}\sum_{-\infty}^{\infty}\delta(x-ia) \to \delta(ax) = \frac{1}{|a|}\delta(x)$, and relating to the FT: $\mathcal{F}\{\delta(t)\}(f) = \int_{-\infty}^{\infty}\delta(t)e^{-2\pi jft}dt = e^0 = 1$ and the same for the inverse FT of 1

Linking Dirac delta with FT: Dirac delta's ability to sample inside an integral: $g(t) = \mathcal{F}^{-1}(\mathcal{F}(g))(t) = \int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}g(s)e^{-2\pi jfs}ds)e^{2\pi jft}df = \int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}e^{-2\pi jf(t-s)}df)g(s)ds = \int_{-\infty}^{\infty}\delta(t-s)g(s)ds$ then $\int_{-\infty}^{\infty}e^{-2\pi jf(t-s)}df = \delta(t-s)$, we can also approximate $\delta(t) \approx \sum_{i=1}^{\infty}\cos(2\pi f_i t)\ since\ the\ \cos = \frac{1}{2}(e^{2\pi jf_i t} + e^{-2\pi jf_i t})$ and we can add discrete different frequencies that are multiples of the base frequency to repeat the Dirac delta function; Sine and cosine in frequency domain: $\cos(2\pi f_0 t) = \frac{1}{2}e^{2\pi jf_0 t} + \frac{1}{2}e^{-2\pi jf_0 t}\ and\ \sin(2\pi f_0 t) = \frac{1}{2j}e^{2\pi jf_0 t} - \frac{1}{2j}e^{-2\pi jf_0 t}\ so\ \mathcal{F}\{\cos(2\pi f_0 t)\}(f) = \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$ and $\mathcal{F}\{\sin(2\pi f_0 t)\}(f) = -\frac{j}{2}\delta(f-f_0) + \frac{j}{2}\delta(f+f_0)$; FT symmetries: x(t) is real ⇔ X(–f) = [X(f)]*, x(t) is imaginary ⇔ X(–f) = –[X(f)]*, x(t) is even ⇔ X(f) is even, x(t) is odd ⇔ X(f) is odd, x(t) is real and even ⇔ X(f) is real and even, x(t) is real and odd ⇔ X(f) is imaginary and odd, x(t) is imaginary and even ⇔ X(f) is imaginary and even, x(t) is imaginary and odd ⇔ X(f) is real and odd, example of FT: amplitude modulation: comm. Channels usually permit use within a frequency interval, so modulating with a carrier frequency shifts the spectrum to the desired band: $y(t) = Acos(2\pi tf_c)x(t) \to X(f) * f_c = Y(f)$; Sampling using a Dirac comb: The loss of information in the sampling process that converts a continuous function x(t) into a discrete sequence $\{x_n\}$ defined by $x_n = x(t_s n) = x(\frac{n}{f_s})$ can be modelled through multiplying x(t) by a comb of Dirac impulses $s(t) = t_s\sum_{n=-\infty}^{\infty}\delta(t-t_s n)$ to obtain the sampled function $\hat{x}(t) = x(t)s(t)$ The function $\hat{x}(t)$ now contains exactly the same information as the discrete sequence $\{x_n\}$, but is still in a form that can be analyzed using the Fourier transform on continuous functions; The FT of a Dirac comb $s(t) = t_s\sum_{n=-\infty}^{\infty}\delta(t-t_s n) = \sum_{n=-\infty}^{\infty}e^{2\pi jnt/t_s}$ is another Dirac comb $S(f) = ts\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(t-t_s n)e^{-2\pi jft}dt = \sum_{n=-\infty}^{\infty}\delta(f-\frac{n}{t_s})$; Discrete-time Fourier Transform (DTFT): commonly expressed using the normalized frequency $\dot{\omega} = 2\pi f/f_s$ where $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty}x_n e^{-j\omega n}\ and\ the\ inverse\ x_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega n}d\dot{\omega}$; Properties of DTFT: periodic $\hat{X}(f) = \hat{X}(f+kf_s)\ or\ X(e^{j\omega}) = X(e^{j\omega+2\pi k})$, DTFT is just the FT applied to a discrete sequence, and inherits the properties of the continuous FT (linearity, symmetry, convolution and modulation theorem: $\{x_n\} * \{y_n\} = \{z_n\} \Leftrightarrow X(e^{j\omega})\cdot Y(e^{j\omega}) = Z(e^{j\omega})$ and $x_n\cdot y_n = z_n \Leftrightarrow \int_{-\pi}^{\pi}X\left(e^{j\omega'}\right)\cdot Y\left(e^{j(\omega-\omega')}\right)d\omega' = Z(e^{j\omega})$, Examples: (1) $x(n) = u(n) \to \sum_{-\infty}^{\infty}u(n)e^{-j\omega n} = \sum_{-\infty}^{\infty}e^{-j\omega n} \Rightarrow X(e^{j\omega}) = \frac{1}{1-e^{-j\omega}}$, (2) $x(n) = u(n-k) \to X(e^{j\omega}) = \sum_{n=k}^{\infty}e^{-j\omega n} = e^{-j\omega k} + e^{-j\omega(k+1)} + ... = \frac{e^{-j\omega k}}{1-e^{-j\omega}}$, (3) $x(n) = \delta(n-k) \to X(e^{-j\omega}) = \sum_{n=-\infty}^{\infty}\delta(n-k)e^{-j\omega n} = [e^{-j\omega n}]\ for = k \to = e^{-j\omega k}$, (4) $x(n) = \{1,3,-2.5,2\} \to X(e^{-j\omega}) = x(0) + x(1)e^{-j\omega} + x(2)e^{-j2\omega} + \cdots = 1 + 3e^{-j\omega} - 2e^{-j2\omega} + 5e^{-j3\omega} + 2e^{-j4\omega}$

Let's Prove it from Fourier Series

$x(t) = \sum_{n=-\infty}^{\infty}\delta(t-n)$

$\sum_{n=-\infty}^{\infty}e^{\pm 2\pi jnt} = \sum_{n=-\infty}^{\infty}\delta(t-n)$

• This is a **periodic impulse train** with period $T_0 = 1$

$\omega_0 = \frac{2\pi}{1} = 2\pi$

$x(t) = \sum_{k=-\infty}^{\infty}c_k e^{jk\omega_0 t}$

• $c_k = \frac{1}{1}\int_0^1\sum_{n=-\infty}^{\infty}\delta(t-n)\cdot e^{-jk2\pi t}dt$

$c_k = \frac{1}{T_0}\int_{\langle T_0\rangle}x(t)e^{-jk\omega_0 t}dt$

Only one Dirac delta survives in the interval period [0,1) the one at t=0

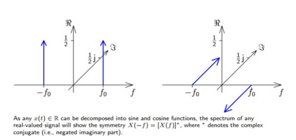
$\omega_0 = \frac{2\pi}{T_0}$

$c_k = \int_0^1\delta(t)\cdot e^{-jk2\pi t}dt = e^0 = 1$ for all k

• Plug Back into the Series

$x(t) = \sum_{n=-\infty}^{\infty}\delta(t-n) = \sum_{k=-\infty}^{\infty}c_k e^{jk2\pi t} = \sum_{k=-\infty}^{\infty}e^{jk2\pi t}$

Multiplication of s(t) with a Dirac comb in the time domain corresponds to convolutions in the frequency domain, This convolution with Dirac pulses corresponds to periodic repetitions of S(f) along the frequency axis. Consequently, the sampling of the signal s(t) at intervals ts corresponds to the periodic repetition of its spectrum S(f) at intervals $fs = 1/t_s$, Shannon and Nyquist theorem: A signal $s(t)$ band-limited to $|f| \leq B$ is fully described by equidistant samples $s[n]$, provided they are no farther apart than $t_s \leq 1/2B$. The sampling frequency which allows a complete signal reconstruction, is consequently bounded below by $fs \geq$ 2B, If one chooses fs < 2B, the periodic repetitions of the spectrum overlap this is known as aliasing and lossless construction no longer possible; If the bandwidth of a signal to be sampled is larger than the sampling frequency fs , the images of the signal that emerge during sampling may overlap with the original spectrum, Such overlap will hinder the reconstruction of the original continuous signal by removing the aliasing frequencies with a reconstruction filter, Therefore, it is advisable to limit the bandwidth of the input signal to the sampling frequency fs before sampling, using an anti-aliasing filter, In the common case of a real-valued base-band signal (with frequency content down to 0 Hz), all frequencies that occur in the signal with non-zero power should be limited to the interval $-fs/2 < f < fs/2$, The upper limit $fs/2$ for the single-sided bandwidth of a baseband signal is known as the "Nyquist limit", Wiki: An anti-aliasing filter is a filter that restricts the bandwidth of a signal to satisfy the Nyquist–Shannon sampling theorem over the band of interest; The ideal anti-aliasing filter for eliminating any frequency content above $fs/2$ before sampling with a frequency of fs has the Fourier transform $H(f) = \begin{cases}1\ if\ |f| < f_s/2\\ 0\ if\ |f| > f_s/2\end{cases} = rext(t_s f)$, this leads, after an inverse Fourier transform, to the impulse response $h(t) = f_s\cdot\sin\left(\frac{\pi tf_s}{\pi tf_s}\right) = \frac{1}{t_s}sinc(\frac{t}{t_s})$, the original band limited signal can be reconstructed by convolving this with the sampled signal $\hat{x}(t)$ which eliminates the periodicity of the frequency domain introduced by the sampling process $x(t) = h(t) * \hat{x}(t)$; Reconstruction filters: ideal filter: $x(t) = \sum_{n=-\infty}^{\infty}x_n\cdot\sin(\pi(t/t_s - n)/\pi(t/t_s - n)$; DFT matrix: $\begin{matrix}\omega^0 & \omega^0\\ \omega^0 & \omega^1\end{matrix}$, every row corresponds to sample points from the unit circle at distinct integer analyzing frequencies, DFT matrix is an orthogonal matrix; Cont. vs discrete FT: sampling a continuous signal makes its spectrum periodic, a periodic signal has a sampled spectrum; Discrete Fourier Transform: $X_k = \sum_{i=0}^{n-1}x_i\cdot e^{-2\pi j\frac{ik}{n}}\ and\ its\ inverse\ x_k = \frac{1}{n}\sum_{i=0}^{n-1}X_i\cdot e^{2\pi j\frac{ik}{n}}$



As any $x(t) \in \mathbb{R}$ can be decomposed into sine and cosine functions, the spectrum of any real-valued signal will show the symmetry $X(-f) = [X(f)]^*$, where * denotes the complex conjugate (i.e., negated imaginary part).

• If before being sampled $x_n = x(t/f_s)$ with the signal $x(t)$ satisfied the Nyquist limit

$$\mathcal{F}\{x(t)\}(f) = \int_{-\infty}^{\infty}x(t)\cdot e^{-2\pi jft}\,dt = 0 \quad \text{for all } |f| \geq \frac{f_s}{2}$$

then it can be reconstructed by interpolation with $h(t) = \frac{1}{t_s}\operatorname{sinc}\left(\frac{t}{t_s}\right)$

$$\begin{aligned}x(t) &= \int_{-\infty}^{\infty}h(s)\cdot\hat{x}(t-s)\cdot ds\\ &= \int_{-\infty}^{\infty}\frac{1}{t_s}\operatorname{sinc}\left(\frac{s}{t_s}\right)\cdot t_s\sum_{n=-\infty}^{\infty}x_n\cdot\delta(t-s-t_s\cdot n)\cdot ds\\ &= \sum_{n=-\infty}^{\infty}x_n\cdot\int_{-\infty}^{\infty}\operatorname{sinc}\left(\frac{s}{t_s}\right)\cdot\delta(t-s-t_s\cdot n)\cdot ds\\ &= \sum_{n=-\infty}^{\infty}x_n\cdot\operatorname{sinc}\left(\frac{t-t_s\cdot n}{t_s}\right) = \sum_{n=-\infty}^{\infty}x_n\cdot\operatorname{sinc}(t/t_s - n)\\ &= \sum_{n=-\infty}^{\infty}x_n\cdot\frac{\sin\pi(t/t_s-n)}{\pi(t/t_s-n)}\end{aligned}$$

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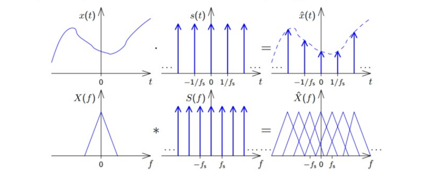
Frequency-domain view of sampling



Sampling a signal in the time domain corresponds in the frequency domain to convolving its spectrum with a Dirac comb. The resulting copies of the original signal spectrum in the spectrum of the sampled signal are called "images"

The Fourier transform of a Dirac comb is another Dirac comb. Why?

• We'll compute the Fourier transform of $x(t) = \sum_{n=-\infty}^{\infty}\delta(t-nt_s)$ using the Fourier transform definition $X(f) = \int_{-\infty}^{\infty}x(t)\cdot e^{-j2\pi ft}dt$

• We will get

$$X(f) = \int_{-\infty}^{\infty}[\sum_{n=-\infty}^{\infty}\delta(t-nt_s)]e^{-j2\pi ft}dt$$

• Remember that $\int_{-\infty}^{\infty}f(x)\delta(x-a)dx = f(a)$

• Thus,

$$X(f) = \sum_{n=-\infty}^{\infty}e^{-j2\pi fnt_s}$$

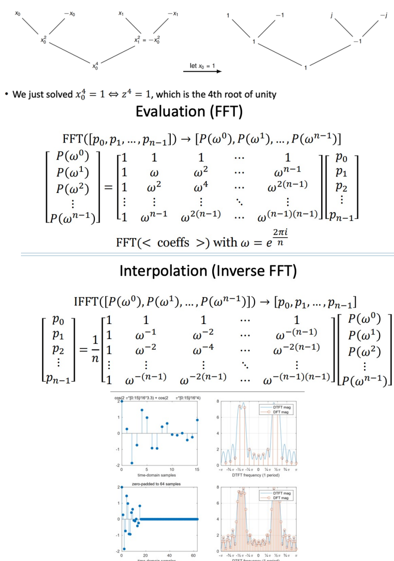
This is a Fourier series of a periodic impulse train

• Thus,

$$X(f) = \sum_{n=-\infty}^{\infty}e^{-j2\pi fnt_s} = \frac{1}{t_s}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{t_s}\right)$$

• The Fourier transform of a Dirac comb is another Dirac comb

Essentially, the FFT is just an algorithm to efficiently calculate the DFT if the input vector is a power of 2. In that case, complexity (for a 1D input) is reduced from $\mathcal{O}\ N2$ to $\mathcal{O}(N\log N)$, Consider the multiplication of two polynomials of degree $N-1$ $a(x)\cdot b(x) = aN{-}1x\ N{-}1 + aN{-}2x\ N{-}2 + \cdots + a1x + a0\ bN{-}1x\ N{-}1 + bN{-}2xN{-}2 + \cdots + b1x + b0 = c2N{-}1x\ 2N{-}2 + c2N{-}2x\ 2N{-}3 + \cdots\ c1x + c0 = c(x)$, The resulting polynomial c(x) is of degree $2N-2$ and the multiplication takes $\mathcal{O}\ N2$ operations because we need to multiply each term in a(x) with each term in b(x); To represent polynomials in a compact form, we can choose from two alternatives 1. Coefficient representation: we write the coefficients as vector, i.e., $a(x) = [a0, a1, \ldots, aN{-}1]$ 2. Value representation: a polynomial of degree N − 1 is uniquely determined by N evaluations at different points, i.e., a set of points like $\{(x_0, a(x_0)), \{(x_1, a(x_1)), \ldots, \{(x_{N-1}, a(x_{N-1}))\}$; Thus instead of evaluating a(x)·b(x) directly, we can also evaluate both polynomials at 2N points and multiply the value representations pairwise. That would reduce the complexity of multiplying both polynomials to $\mathcal{O}(N)$; Steps: split polynomial into even and odd parts, even (x^0, x^2, x^4) and odd (x^1, x^3, x^5) and derive $p_e\ and\ p_o\ such\ that\ p(x_n) = p_e(x_n^2) + x\cdot p_o(x_n^2)$, In general, we can split any polynomial of degree N − 1 into even and odd parts, resulting in two polynomials of degree N/2 − 1. Since we can exploit the symmetry, we get away by evaluating N/2 instead of N points in order to do the polynomial multiplication in the value domain, If we can do this recursively it would result in an O(N log N) algorithm, but there's a problem: The two resulting polynomials $p_e(x_n^2) + x\cdot p_o(x_n^2)$ and are functions in x^2. Thus, we do not have any more pairs like $\pm\ x_n^2$ as all x_n^2 are positive, so we choose $j^2 = -1$, to generalize: $x_0 = 1, x_1 = j, so\ x_0^2 = 1\ and\ x_1^2 = -1\ as\ intended, remember\ that\ e^{j\theta} = \cos(\theta) + \mathrm{j}\sin(\theta)\ so\ if\ we\ choose\ x_2 = e^{-\frac{j2\pi3}{8}}, we\ get\ x_2^2 = e^{-\frac{j\pi}{2}} = j$ and at the same time $-x_2 = -e^{-\frac{j\pi3}{8}}$, Solutions to $x^n = 1\ are\ x_i = e^{\frac{2\pi i}{n}}, where\ i = 1, \ldots, n\ and\ for\ a\ polynomial\ of\ degree n\ d\ we\ need\ n \geq (d+1)\ points, which\ are\ the\ n^{th}\ roots\ of\ unity$, Why nth roots of unity: let $\omega = e^{\frac{2\pi i}{n}}\ then\ \omega^{j+n/2} = -\omega^j \to \left(\omega^j, \omega^{j+\frac{n}{2}}\right) are \pm paired\ wich\ are\ on\ the\ diagonal\ of\ the\ circle$, so squaring the n roots of unity results in n/2 roots of unity; Now we have transformed from Coeff -> Value, to transform back the Coeff we can interpolate using the inverse DFT matric; FFT-based convolution: calculating the convolution of two sequences xi and yi of lengths m and n $z_i = \sum_{j=\max(0,i-(n-1))}^{\min(m-1,j)} x_j \cdot y_{i-j}, 0 \leq i < m+n-1$ takes mn multiplications, we can apply FFT in just O(mlogm + nlogn) multiplications as $\{z_i\} = \mathcal{F}^{-1}(\mathcal{F}\{x_i\}\cdot\mathcal{F}\{y_i\})$ but xi and yi have to be periodic with equal period lengths. Deconvolution: A signal u(t) was distorted by convolution with a known impulse response h(t), how do we bring back original image or estimate u(t), the convolution theorm turns the problem into one of the multiplication $s(t) = \int u(t-\tau)\cdot h(\tau)\cdot d\tau, s = u * h \to \mathcal{F}\{s\} = \mathcal{F}\{u\}\cdot\mathcal{F}\{h\} \to u = \mathcal{F}^{-1}\{\frac{\mathcal{F}\{s\}}{\mathcal{F}\{h\}}\}$ in practice we can also record some noise n(t), but at frequincies f where F{h}(f) approaches zero, the noise will be amplified during deconvolution, workaround: modify the FT of the impulse response, st < ∈, sor some threshold, Wiener filter: $W(f) = \frac{|\mathcal{F}\{s\}(f)|^2}{|\mathcal{F}\{s\}(f)|^2 + |\mathcal{F}\{n\}(f)|^2}$ and before deconvolution: $\breve{u} = \mathcal{F}^{-1}\{W\cdot\frac{\mathcal{F}\{c\}}{\mathcal{F}\{h\}}\}$; Spectral estimation: artifacts: The reason for the leakage and scalloping losses is easy to visualize with the help of the convolution theorem: The operation of cutting a sequence of the size of the DFT input vector out of a longer original signal (the one whose continuous Fourier spectrum we try to estimate) is equivalent to multiplying this signal with a rectangular function. This destroys all information and continuity outside the "window" that is fed into the DFT, Multiplication with a rectangular window of length T in the time domain is equivalent to convolution with $\sin(\pi fT)/(\pi fT)$ in the frequency domain; Effect of Hann function: Zeros between two peaks, DFT is taken assuming the 16 samples as one period of a periodic signal, DTFT is taken while assuming the 16 samples and 0 everywhere else, DFT is a sampling of DTFT; Hann window goes to zero quickly while reducing the artifacts; Triangular window (Barlett window): $w_i = 1 - |1 - \frac{i}{n/2}|$; Hann window: $w_i = 0.5 - 0.5{\times}\cos(2\pi\frac{i}{n-1})$, Hamming window: $w_i = 0.54 - 0.46{\times}\cos(2\pi\frac{i}{n-1})$; Frequency resolution: the ability to perfectly distinguish one frequency from another, frequency resolution of DFT is $\Delta F = f_s/N$ where N is the length of the input signal; Zero paddings increase DFT resolution; Notes: DFT is taken while assuming all samples as one period of a periodic signal, DTFT is taken while assuming all samples and zeros elsewhere, Zero padding did not change the DTFT, as the zeros are already considered, The block size is increased for DFT by considering the zero padding, DFT is sampling at a higher frequency resolution, If we have more real values instead of zeros, the window size is increased, and the two frequency peaks are very accurate and clear



Filter: suppresses (removes, attenuates) unwanted signal components, low-pass filter: suppress all frequencies above a cut-off frequency, high-pass filter: suppress all frequencies below a cut-off frequency, including DC (direct current = 0 Hz), band-pass filter: suppress signals outside a frequency interval (=passband), band-stop filter (aka: band-reject filter): suppress signals inside a single frequency interval (= stopband), notch filter: narrow band-stop filter, ideally suppressing only a single frequency; For digital filters, we also distinguish, finite impulse response (FIR) filters (have finite memory), infinite impulse response (IIR) filters depending on how far their memory reaches back in time; Recall the low-pass filter from before H(f) and impulse response h(t), the problems are (1) impulse response is intently long, (2) filter is not causal; to solve this: (1) make the impulse response finite by multiplying the sampled h(t) with a windowing function, (2) make the impulse response causal by adding a delay of half the window size, so now $h_i = \frac{2f_c}{f_s}\cdot\frac{\sin\left[\frac{2\pi(i-n/2)f_c}{f_s}\right]}{\frac{2\pi(i-n/2)f_c}{f_s}}\cdot w_i$ where {wi} is a windowing sequence such as the Hamming window with wi = 0; Filter performance: An ideal filter has a gain of 1 in the pass-band and a gain of 0 in the stop band, and nothing in between. This idea filter requires an infinite impulse response, which requires infinite memory, A practical filter will have: (1) frequency-dependent gain near 1 in the passband (2) frequency-dependent gain below a threshold in the stopband (3) a transition band between the pass and stop bands, We truncate the ideal, infinitely long impulse response by multiplication with a window sequence, In the frequency domain, this will convolve the rectangular frequency response of the ideal low-pass filter with the frequency characteristic of the window, The width of the main lobe determines the width of the transition band, and the side lobes cause ripples in the passband and stopband; How to obtain band-pass filter from a low-pass filter: To obtain a band-pass filter that attenuates all frequencies f outside the range$fl < f < fh$, we first design a low-pass filter with a cut-off frequency $fh - fl$ /2. We then multiply its impulse response with a sine wave of frequency $fh + fl$ /2, effectively amplitude modulating it, to shift its center frequency. Finally, we apply a window function: $h_i = (f_h - f_l)/f_s\cdot\frac{\sin[\pi(i-n/2)(f_h-f_l)/f_s]}{\pi(i-n/2)(f_h-f_l)/f_s}\cdot\cos(\pi i(f_h+f_l)/f_s]\cdot w_i$; Low-pass to high-pass filter conversion through frequency inversion: In order to turn the spectrum $X(f)$ of a real-valued signal xi sampled at fs into an inverted spectrum $X'(f) = X(fs/2 - f)* = X(f \pm fs/2)$, we merely have to shift the periodic spectrum by fs/2, This can be accomplished by multiplying the sampled sequence xi with $yi = \cos\pi fs\ t = \cos\pi i$ = e jπi, which is nothing but multiplication with the sequence … , 1, −1,1, −1,1, −1,1, −1, …, So in order to design a discrete high-pass filter that attenuates all frequencies f outside the range $fc < |f| < fs/2$, we merely have to design a low-pass filter that attenuates all frequencies outside the range $-fc < f < fc$, and then multiply every second value of its impulse response with -1; Z-transform: $X(z) = \sum_{n=-\infty}^{\infty} x_n z^{-n}$, X(z) is exactly the factor with which an exponential sequence {zn} is multiplied, if it is convolved with {xn}: $\{z^n\} * \{x_n\} = \{y_n\} \Rightarrow y_n = \sum_{-\infty}^{\infty} z^{n-k}x_k = z^n\sum_{k=-\infty}^{\infty} z^{-k}x_k = z^n\cdot X(z)$ convolving exp sequence with a discrete seq results in exponential sequence multiplied by a scalar, The z-transform defines for each sequence a continuous complex-valued surface over the complex plane ℂ, For finite sequences, its value is defined across the entire complex plane (except possibly at z = 0 or |z| = ∞), The z-transform is a generalization of the discrete-time Fourier transform, which it contains on the complex unit circle (|z| = 1) : $t_s^{-1}\cdot\mathcal{F}\{\hat{x}(t)\}(f) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_n e^{-j\omega n}\ where\ \omega\ is\ the\ sampling\ rate$, here z is a complex var z= re^jw, It gives a way to analyze discrete signals in the complex frequency domain • It captures both magnitude and phase behavior, including system stability and causality, The z-transform is a more general tool, because it allows any complex value of z, not just those on the unit circle;
Properties: linearity $\{ax_n + by_n\} \leftrightarrow$ aX(z) + bY(z), convolution $\{x_n\} * \{y_n\} \leftrightarrow$ X(z) · Y(z), time shift $\{x_{n+k}\} \leftrightarrow z^k$X(z), time reversal $\{x_{-n}\} \leftrightarrow \mathrm{X}(z^{-1})$, multiplication with exponential $\{a^{-n}x_n\} \leftrightarrow \mathrm{X}(az)$, complex conjugate $\{x_n *\} \leftrightarrow$ X ∗ (z ∗), real/imaginary parts $\{\Re\{x_n\}\} \leftrightarrow \frac{1}{2}(X(z) + X^*(z^*))\ and\ \{\Im\{x_n\}\} \leftrightarrow \frac{1}{2j}(X(z) - X^*(z^*))$, initial value $x_0 = \lim_{z\to\infty} X(z)\ if\ x_n = 0$