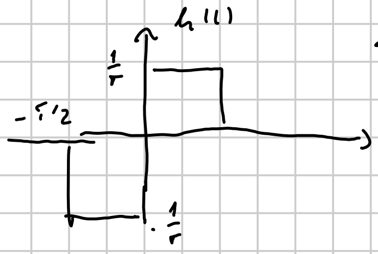
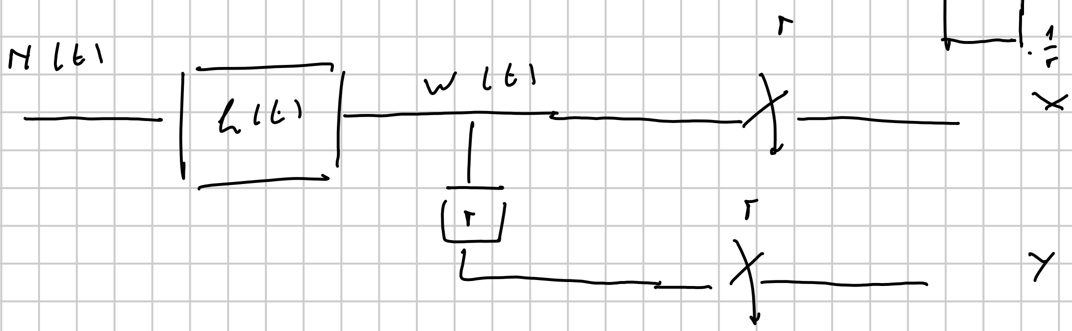
ESERCIZIO 2



$S_n(f) = \frac{N_0}{2}$



$f_{xy}(x,y)$?

$X = w(T)$ $\qquad \eta_x \,;\, \eta_y \,;\, \sigma_x^2 \,;\, \sigma_y^2 \,;\, cov(x,y)$

$Y = w(0)$

$$E\{w(t)\} = E\{n(t)\}\, H(0) = 0 \quad \Rightarrow \quad \eta_x = \eta_y = 0$$

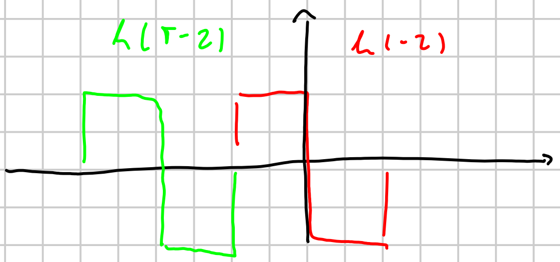
$$E\{x^2\} = E\{y^2\} = \int_{-\infty}^{+\infty} S_n(f)\, |H(f)|^2 \, df = \frac{N_0}{2} \int_{-\infty}^{+\infty} |H(f)|^2 \, df =$$

$$= \frac{N_0}{2} \int_{-\infty}^{+\infty} h^2(t)\, dt = \frac{N_0}{2} \frac{1}{T^2} T = \frac{N_0}{2T} = \sigma_x^2 + \sigma_y^2 = \sigma^2$$

$$cov(X,Y) = E\{XY\} = E\{w(0)\, w(T)\} = R_w(T) =$$

$$= \left. \underbrace{\frac{N_0}{2}\, \delta(\tau)}_{S_n(f)} \otimes h(-\tau) \otimes h(\tau) \right|_{\tau = T} =$$

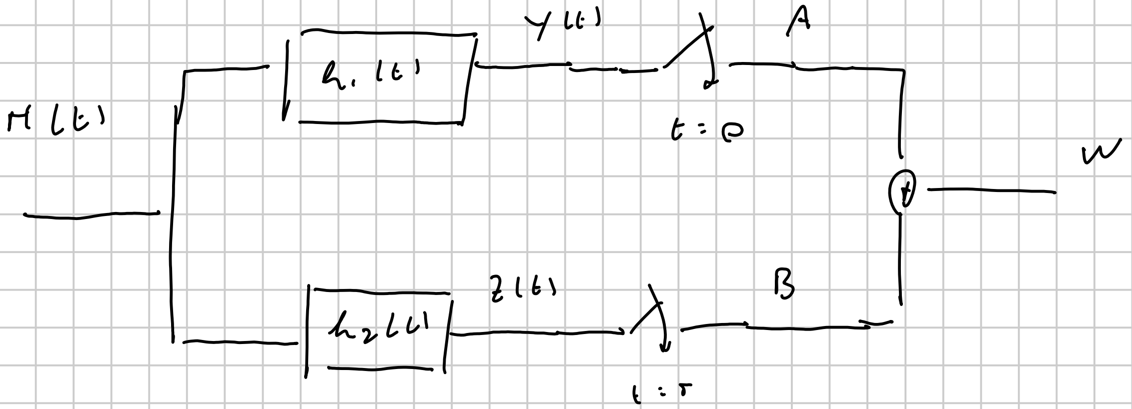
$$= \left. \frac{N_0}{2}\, h(-\tau) \otimes h(\tau) \right|_{\tau = T} = \frac{N_0}{2} \int_{-\infty}^{+\infty} h(-\tau)\, h(T-\tau)\, d\tau = 0$$



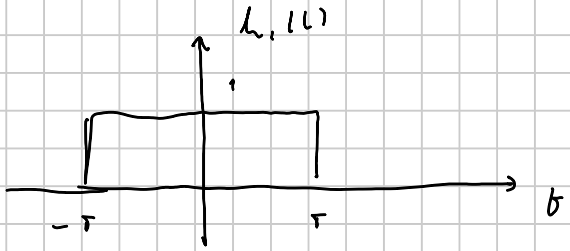
$$cov(x,y) = 0$$

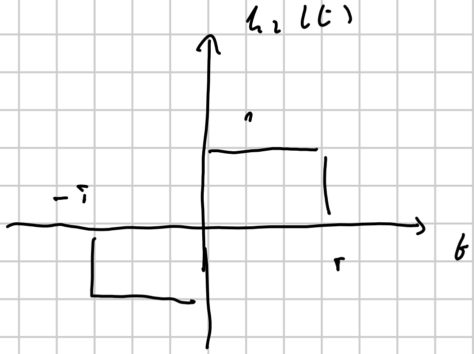
$$f_{xy}(x,y) = \frac{1}{2\pi\sigma^2}\, e^{-\frac{x^2+y^2}{2\sigma^2}}$$

ESERCIZIO 3



$f_W(w)$?





$n(t)$ GAUSSIANO BIANCO

$E\{n(t)\} = 0$ $\qquad R_n(\tau) = \frac{N_0}{2}\,\delta(\tau)$ $\qquad S_n(f) = \frac{N_0}{2}$

$Y(t)$ e $Z(t)$ SONO PROCESSI ALEATORI GAUSSIANI S.S.L.

A e B SONO V.A. CONG. GAUSSIANE

$$A = \int_{-\infty}^{+\infty} n(\tau)\, h_1(t-\tau)\, d\tau \Big|_{t=0}$$

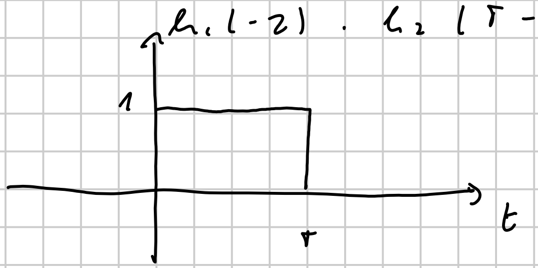
$$E\{A\} = \int_{-\infty}^{+\infty} E\{n(\tau)\}\, h_1(-\tau) = 0$$

$$B = \int_{-\infty}^{+\infty} n(\tau)\, h_2(t-\tau)\, d\tau \Big|_{t=T}$$

$$E\{B\} = 0$$

$$E\{AB\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underbrace{E\{n(z_1)\, n(z_2)\}}_{R_n(z_1 - z_2) = \frac{N_0}{2}\delta(z_1 - z_2)} h_1(-z_1)\, h_2(T - z_2)\, dz_1\, dz_2$$

$$= \int_{-\infty}^{+\infty} \frac{N_0}{2}\, h_1(-z)\, h_2(T - z)\, dz$$



$$E\{AB\} = \frac{N_0}{2} T \neq \eta_A\, \eta_B$$

A e B SONO CORRELATE

$$A \in \mathcal{N}(0, \sigma_A^2)$$

$$B \in \mathcal{N}(0, \sigma_B^2)$$

$$\mathrm{cov}\{AB\} = \frac{N_0}{2} T$$

$$E\{A^2\} = E\{Y^2(0)\} = \int_{-\infty}^{+\infty} \frac{N_0}{2} |H_1(f)|^2\, df =$$

$$= \frac{N_0}{2} \int_{-\infty}^{+\infty} h_1^2(t)\, dt = \frac{N_0}{2} \cdot 2T = N_0 T = \sigma_A^2$$

$$E\{B^2\} = \frac{N_0}{2} \int_{-\infty}^{+\infty} h_2^2(t)\, dt = N_0 T$$

$$\rho = \frac{\mathrm{cov}(A, B)}{\sigma_A\, \sigma_B} = \frac{N_0 T/2}{N_0 T} = \frac{1}{2}$$

$$w \in \mathcal{N}(\eta_w, \sigma^2_w)$$

$$w = A + B$$

$$\eta_w = E\{A + B\} = E\{A\} + E\{B\} = 0$$

$$\sigma^2_w = E\{(w - \eta_w)^2\} = E\{w^2\} =$$

$$= E\{(A+B)^2\} = E\{A^2\} + E\{B^2\} + 2E\{AB\} =$$

$$= N_0\sigma + N_0\sigma + 2\frac{N_0\sigma}{2} = 3N_0\sigma$$

$$f_w(w) = \frac{1}{\sqrt{2\pi\, 3N_0\sigma}}\, e^{-\frac{w^2}{6N_0\sigma}}$$

ESERCIZIO 4

X e Y v.a. i.i.d. $\in \mathcal{N}(0,1)$

$$\begin{cases} V = 5X + 3Y \\ Z = -3X + 5Y \end{cases}$$

Z e V SONO INCORRELATE $\Rightarrow$ INDIPENDENTI

$$C_{ZV} \triangleq R_{XZ} - \eta_Z \eta_V = R_{XZ}$$

$$\eta_Z = E\{-3X + 5Y\} = -3\eta_X + 5\eta_Y = 0$$

$$\eta_V = 5\eta_X + 3\eta_Y = 0$$

$$C_{ZV} = E\{ZV\} = E\{(5X+3Y)(-3X+5Y)\} =$$

$$= E\{-15X^2 + 25XY - 9XY + 15Y^2\} =$$

$$= -15\,E\{X^2\} + 16\,E\{XY\} + 15\,E\{Y^2\}$$

$$E\{X^2\} = \sigma_X^2 + \eta_X^2 = 1$$

$$E\{Y^2\} = \sigma_Y^2 + \eta_Y^2 = 1$$

$$E\{XY\} = R_{XY} = C_{XY} = 0$$

$\uparrow$

X e Y INDIPENDENTI

$C_{ZV} = -15 + 15 = 0 \Rightarrow$ INCORRELATE e QUINDI INDIPENDENTI

$$f_{zv}(z,v) = f_z(z)\, f_v(v)$$

$$\begin{cases} v = 5x + 3y \\ z = -3x + 5y \end{cases} \qquad \bar{x} = \frac{5v - 3z}{34} \qquad \bar{y} = \frac{3v + 5z}{34}$$

$$\underline{\underline{J}} = \begin{bmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} = \begin{bmatrix} 5 & +3 \\ -3 & 5 \end{bmatrix}$$

$$|\underline{\underline{J}}| = 34$$

$$f_{zv}(v,z) = \frac{1}{34} f_{xy}(x,y)\Big|_{\substack{x=\bar{x} \\ y=\bar{y}}} =$$

$$= \underbrace{\frac{1}{34}}\ \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}_{f_x(x)}\ \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}}_{f_y(y)}\Bigg|_{\substack{x=\bar{x} \\ y=\bar{y}}} =$$

$$f_{xy}(x,y) = f_x(x)\, f_y(y)$$

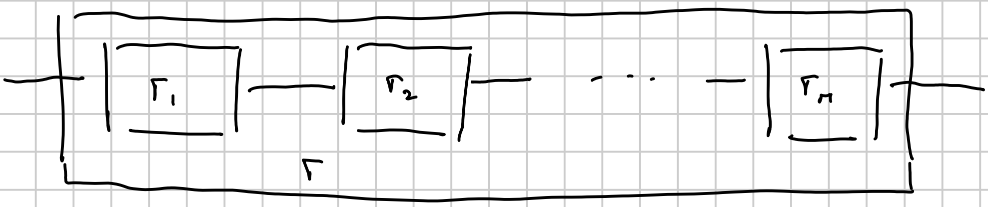
$$= \frac{1}{34\, 2\pi} e^{-\frac{1}{34^2}\frac{1}{2}\left(25v^2 + 9z^2 - \cancel{30vz} + 9v^2 + 25z^2 + \cancel{30vz}\right)} =$$

$$= \frac{1}{2\pi\, 34} e^{-\frac{1}{2}\left(\frac{\cancel{34}v^2}{34^2} + \frac{\cancel{34}z^2}{34^2}\right)} =$$

$$= \frac{1}{\sqrt{2\pi\, 34}} e^{-\frac{1}{2}\frac{v^2}{34}} \quad \frac{1}{\sqrt{2\pi\, 34}} e^{-\frac{1}{2}\frac{z^2}{34}}$$

v e z $\Rightarrow$ INDIPENDENTI.

ESERCIZIO 5



T_i = TEMPO DI VITA DELL i-esima LAMPADINA

$E\{T\}$ $\qquad T = \min\{T_1, T_2, \ldots, T_M\}$ $\qquad E\{T\}$

$$f_{T_i}(t_i) = \frac{1}{\eta} e^{-t_i/\eta} \qquad t_i \geq 0$$

$$E\{T_i\} = \eta$$

$$P_r\{\text{i-esima sia accesa al tempo } t\} =$$

$$= P_r\{T_i > t\} = \int_t^{+\infty} \frac{1}{\eta} e^{-t_i/\eta}\, dt_i = \left[-e^{-t_i/\eta}\right]_t^{+\infty} =$$

$$= e^{-t/\eta}$$

$$P_r\{T > t\} = P_r\left\{\bigcap_{i=1}^{M} T_i > t\right\} = \prod_{i=1}^{M} P_r\{T_i > t\} =$$

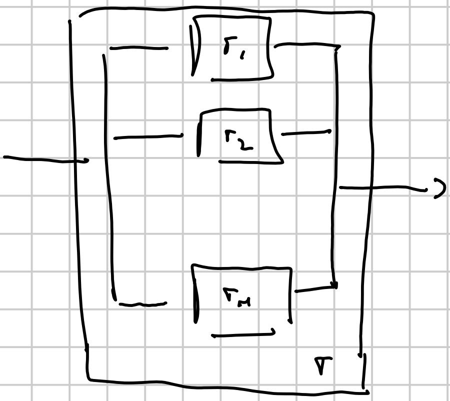
$$= e^{-Mt/\eta}$$

$$F_T(t) = P_r\{T \leq t\} = 1 - P_r\{T > t\} =$$

$$= 1 - e^{-Mt/\eta}$$

$$f_T(t) = \frac{M}{\eta} e^{-Mt/\eta} \qquad E\{T\} = \frac{\eta}{M} < \eta$$

PARALLELO



$$T = \max\{ T_1, \dots, T_M \}$$

$$E\{T\}$$

$$P_r\{ T_i \leq t \} = 1 - e^{-t/\eta}$$

$$P_r\{ T \leq t \} = P_r\left\{ \bigcap_{i=1}^{M} T_i \leq t_i \right\} = \prod_{i=1}^{M} P_r\{ T_i \leq t \} =$$

$$= \left(1 - e^{-t/\eta}\right)^M = F_T(t)$$

$$f_T(t) = \frac{\partial}{\partial t} F_T(t) = \frac{M}{\eta}\left(1 - e^{-t/\eta}\right)^{M-1} e^{-t/\eta} \quad \text{per } t > 0$$

$$E\{T\} = \int_{-\infty}^{+\infty} t \, f_T(t) \, dt$$

$$M = 2$$

$$f_T(t) = \frac{2}{\eta}\left(e^{-t/\eta} - e^{-2t/\eta} \right) = 2 \cdot \frac{1}{\eta} e^{-t/\eta} - \frac{1}{\eta/2} e^{-t/\eta/2}$$

$$E\{T\} = 2\eta - \frac{\eta}{2} = \frac{3}{2}\eta > \eta$$

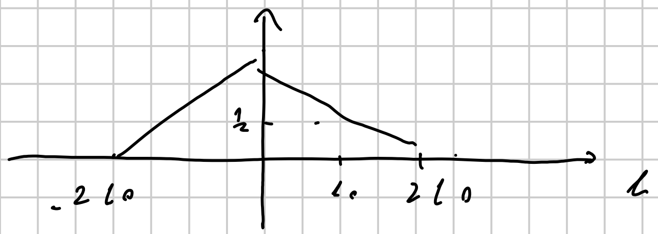
ESERCIZIO 6 | Processi parametrici

$$X(t) = A\cos(2\pi f_0 t) + B\sin(2\pi f_0 t)$$

$A, B \in \mathcal{N}(0, \sigma^2)$ $\qquad \sigma^2 = 4$

$X(t) \longrightarrow \boxed{H(f)} \longrightarrow Y(t)$

$f_Y(y;t)$



$$x_n(t) \Rightarrow X_n(f) = \frac{A}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right) + \frac{B}{2j}\left(\delta(f-f_0) - \delta(f+f_0)\right)$$

$$Y_n(f) = X_n(f)\,H(f) = \frac{1}{2}X_n(f) \Rightarrow$$

$$y_n(t) = \frac{1}{2}x_n(t)$$

$$Y(t) = \frac{1}{2}X(t)$$

$$Y = Y(\bar{t}) = \frac{A}{2}\cos(2\pi f_0\bar{t}) + \frac{B}{2}\sin(2\pi f_0\bar{t}) =$$

$$= K_1\frac{A}{2} + K_2\frac{B}{2} = \qquad K_1 = \cos(2\pi f_0\bar{t})$$

$$K_2 = \sin(2\pi f_0\bar{t})$$

$$= Y_A + Y_B$$

Y è V.A. GAUSSIANA

A e B IND. $\Rightarrow$ Y_A e Y_B INDIPENDENTI $\Rightarrow$ INCORRELATE

$$\eta_{Y_A} = E\left\{ K_1 \frac{A}{2} \right\} = 0$$

$$\eta_{Y_B} = E\left\{ K_2 \frac{B}{2} \right\} = 0$$

$$\sigma^2_A = E\left\{ \left(K_1 \frac{A}{2} \right)^2 \right\} = \frac{K_1^2}{4} \sigma^2$$

$$\sigma^2_B = E\left\{ \left(K_2 \frac{B}{2} \right)^2 \right\} = \frac{K_2^2}{4} \sigma^2$$

$$\eta_Y = 0$$

$$\sigma^2_Y = E\left\{ (Y_A + Y_B)^2 \right\} = E\left\{ Y_A^2 \right\} + E\left\{ Y_B^2 \right\} + \underbrace{2E\left\{ Y_A Y_B \right\}}_{0} =$$

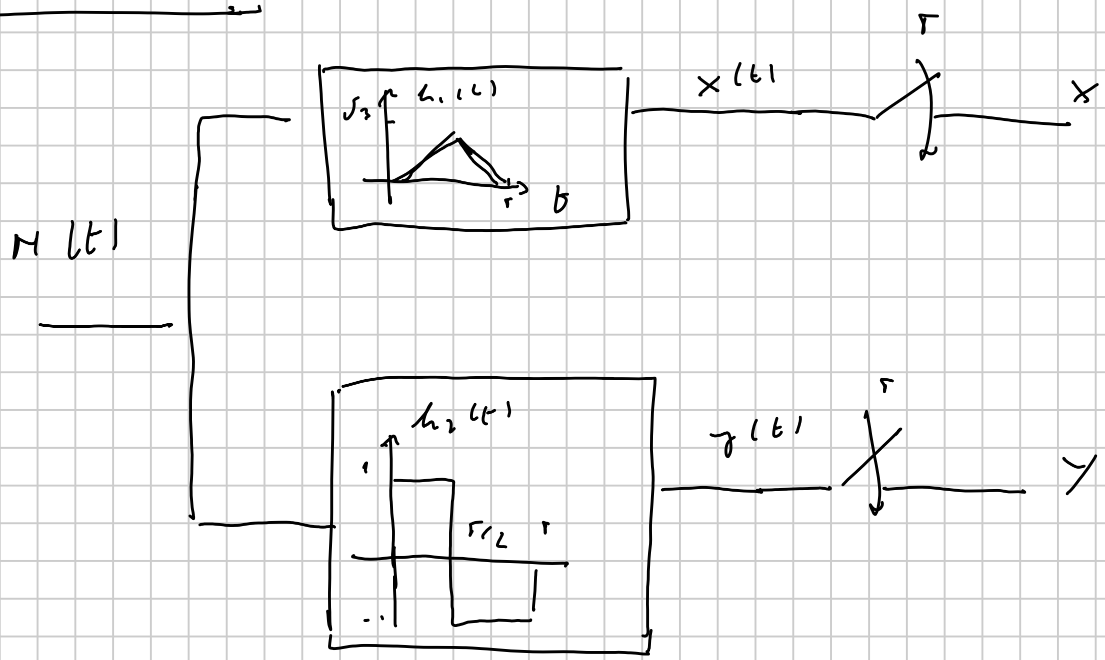
$$= \frac{1}{4} \underbrace{\left(K_1^2 + K_2^2 \right)}_{1} \sigma^2 =$$

$$= \frac{1}{4} \left(\cos^2(2\pi f_0 \bar{t}) + \sin^2(2\pi f_0 \bar{t}) \right) \sigma^2 = 1$$

$$Y \in N(0, 1)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

ESERCIZIO 7



$n(t)$ GAUSSIANO BIANCO $\quad S_n(f) = \frac{N_0}{2}$

- $P_r\{X > Y\}$
- $E\{XY\}$

$$x(t) = \int_{-\infty}^{+\infty} n(\tau)\, h_1(t-\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{+\infty} n(\tau)\, h_2(t-\tau)\, d\tau$$

$X = x(T)$

$Y = y(T)$

$\eta_n(t) = 0$

$\eta_x(t) = \eta_n(t) \otimes h_1(t) = 0$

$\eta_Y(t) = 0$

$$S_x(f) = S_n(f)\,|H_1(f)|^2$$

$$P_x = \int_{-\infty}^{+\infty} S_n(f)\,|H_1(f)|^2\,df = \frac{N_0}{2}\int_{-\infty}^{+\infty} h_1^2(t)\,dt = \frac{N_0}{2}\cdot 3\,\frac{2}{3}\,\frac{T}{2} = \frac{N_0 T}{2}$$

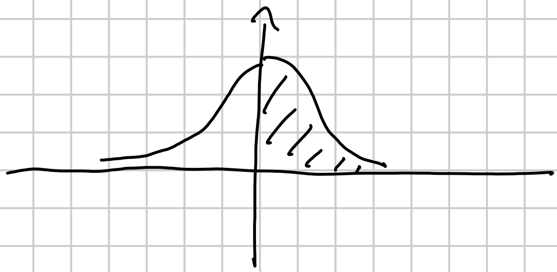
$$= \sigma_x^2$$

$$P_Y = \sigma_Y^2 = \frac{N_0}{2}\int_{-\infty}^{+\infty} |h_2(t)|^2\,dt = N_0\,\frac{T}{2}$$

X e $Y \in \mathcal{N}\left(0, \frac{N_0 T}{2}\right)$

$$P_r\{X > Y\} = P_r\{X - Y > 0\} = \frac{1}{2}$$

$$Z = X - Y \in \mathcal{N}(0, \sigma_z^2)$$



$$E\{X\,Y\} = E\Big\{ \underbrace{\int_{-\infty}^{+\infty} n(z)\,h_1(T-z)\,dz}_{X} \; \underbrace{\int_{-\infty}^{+\infty} n(\alpha)\,h_2(T-\alpha)\,d\alpha}_{Y} \Big\}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underbrace{E\{n(z)\,n(\alpha)\}}_{} \; h_1(T-z)\; h_2(T-\alpha)\; dz\,d\alpha =$$

$\frac{N_0}{2}\,\delta(z-\alpha) \neq 0$ se $z = \alpha$

$$= \frac{N_0}{2} \int_{-\infty}^{+\infty} h_1(\tau - \lambda)\, h_2(\tau - \lambda)\, d\lambda = 0$$