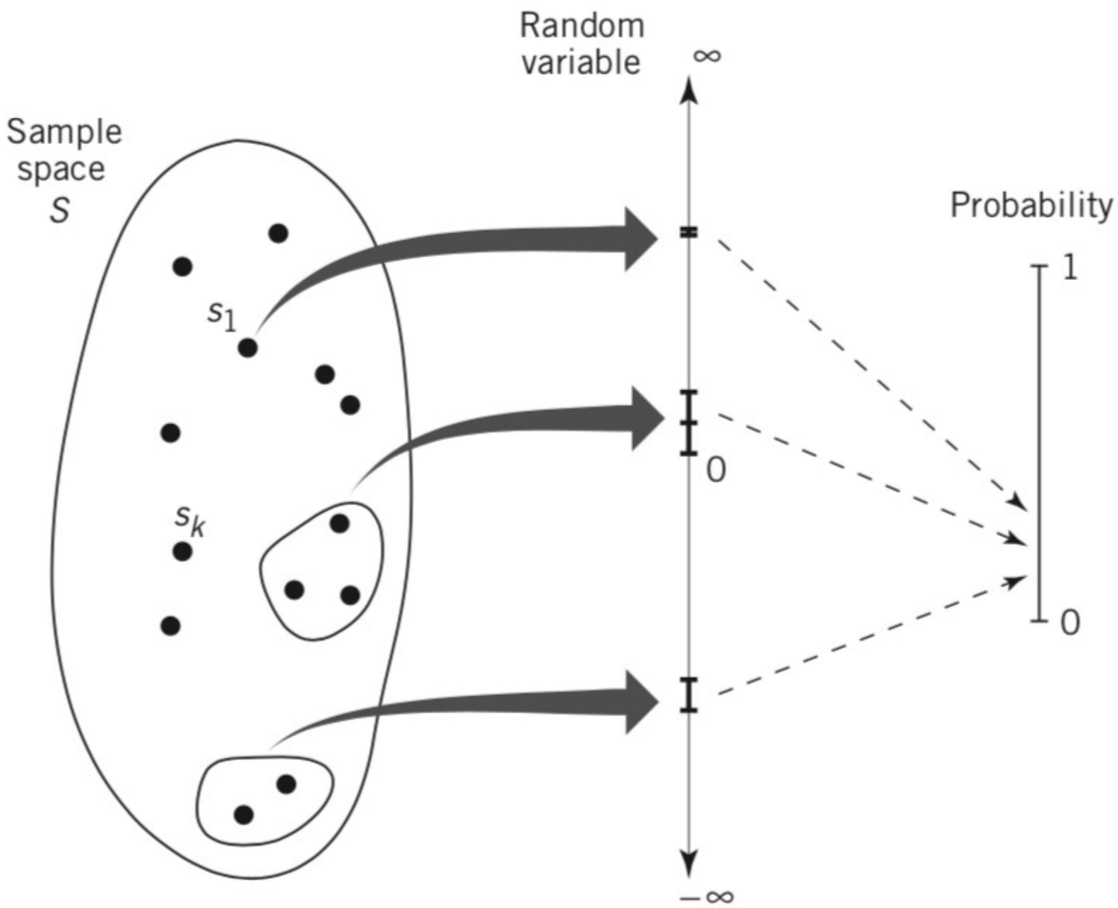
Random Variables

Random variable: function whose domain is a sample space and whose range is some set of real numbers.



Distribution Functions

Consider the random variable X and the probability of the event $X \leq x$. Denote this probability by $P[X \leq x]$. To simplify notation:

$$F_X(x) = \mathbb{P}[X \leq x] \qquad \text{for all } x$$

The function $F_X(x)$ is called the distribution function of the random variable X.

- Note that $F_X(x)$ is a function of x, not of the random variable X.

Properties:

- Boundedness: It lies between zero and one;
- Monotonicity: The distribution function is a monotone nondecreasing function of x

Probability density function (pdf)

The random variable X is said to be continuous if the distribution function $F_X(x)$ is differentiable:

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) \qquad \text{for all } x$$

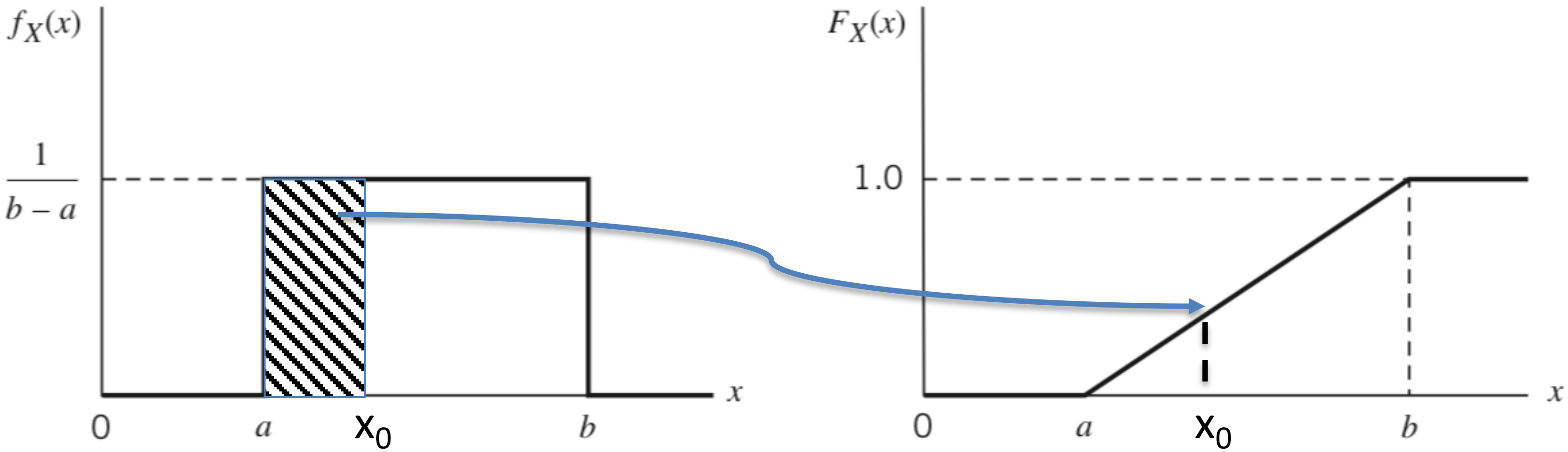
The function $f_X(x)$ is called the probability density function (pdf):

$$\begin{aligned} \mathbb{P}[x_1 < X \le x_2] &= \mathbb{P}[X \le x_2] - \mathbb{P}[X \le x_1] \\ &= F_X(x_2) - F_X(x_1) \\ &= \int_{x_1}^{x_2} f_X(x)\,\mathrm{d}x \end{aligned}$$

Properties:

- Nonnegativity;
- Normalization: The total area of the pdf is equal to unity.

Example – Uniform Distribution



Probability Mass Function

Consider next the case of a discrete random variable, X, that can take a finite or countably infinite number of values.

- The distribution function $F_X(x)$ also applies to discrete random variables.
- …however, it is not differentiable;
- To get around, define the probability mass function $p_X(x)$ as

$$p_X(x) = \mathbb{P}[X = x]$$

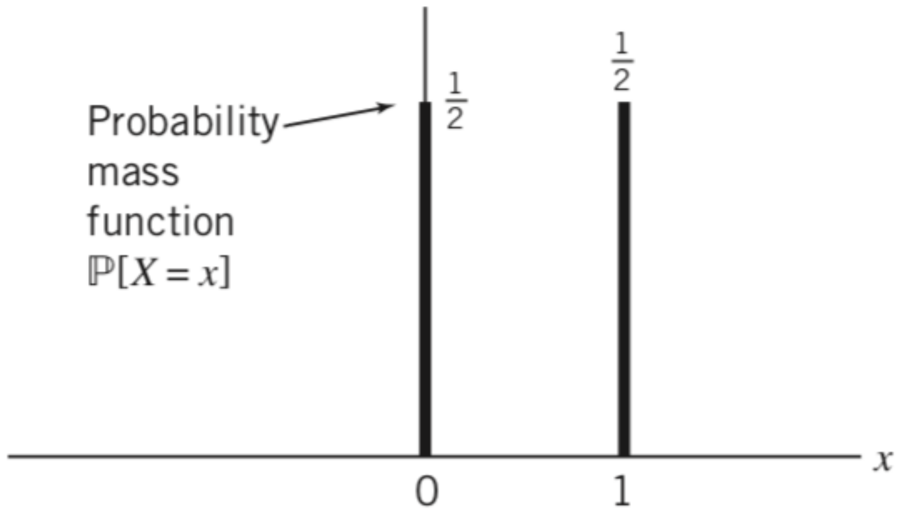
Defined as the probability of the event X = x, which consists of all possible outcomes of an experiment that lead to a value of X equal to x.

Example – Bernoulli Random Variable

Consider a probabilistic experiment that takes one of two possible values:

- the value 1 with probability p;
- the value 0 with probability 1 – p.

Such a random variable is called the *Bernoulli random variable:*



Multiple Random Variables

Consider two random variables X and Y

$$F_{X,Y}(x,y) = \mathbb{P}[X \leq x, Y \leq y]$$

The joint distribution function $F_{X,Y}(x,y)$ is the probability that X is less than or equal to a specified value x, and that Y is less than or equal to another specified value y.

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

The joint probability density function $f_{X,Y}(x,y)$ contains all is needed for the probability analysis of joint random variables.

Conditional Probability Density Function

Suppose that X and Y are two continuous random variables with $f_{X,Y}(x,y)$.

- The conditional probability density function of Y, such that X = x, is defined by

$$f_Y(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

Suppose that knowledge of X can, in no way, affect the distribution of Y

1. Then, $f_Y(y|x)$ reduces to the marginal density $f_Y(y)$ and…
2. The joint pdf becomes $f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$

If the joint probability density function of the random variables X and Y equals the product of their marginal densities, then X and Y are statistically independent.

Sum of Independent Random Variables

Let X and Y be two statistically independent continuous random variables with probability density functions are denoted by $f_X(x)$ and $f_Y(y)$. Define

$$Z = X + Y$$

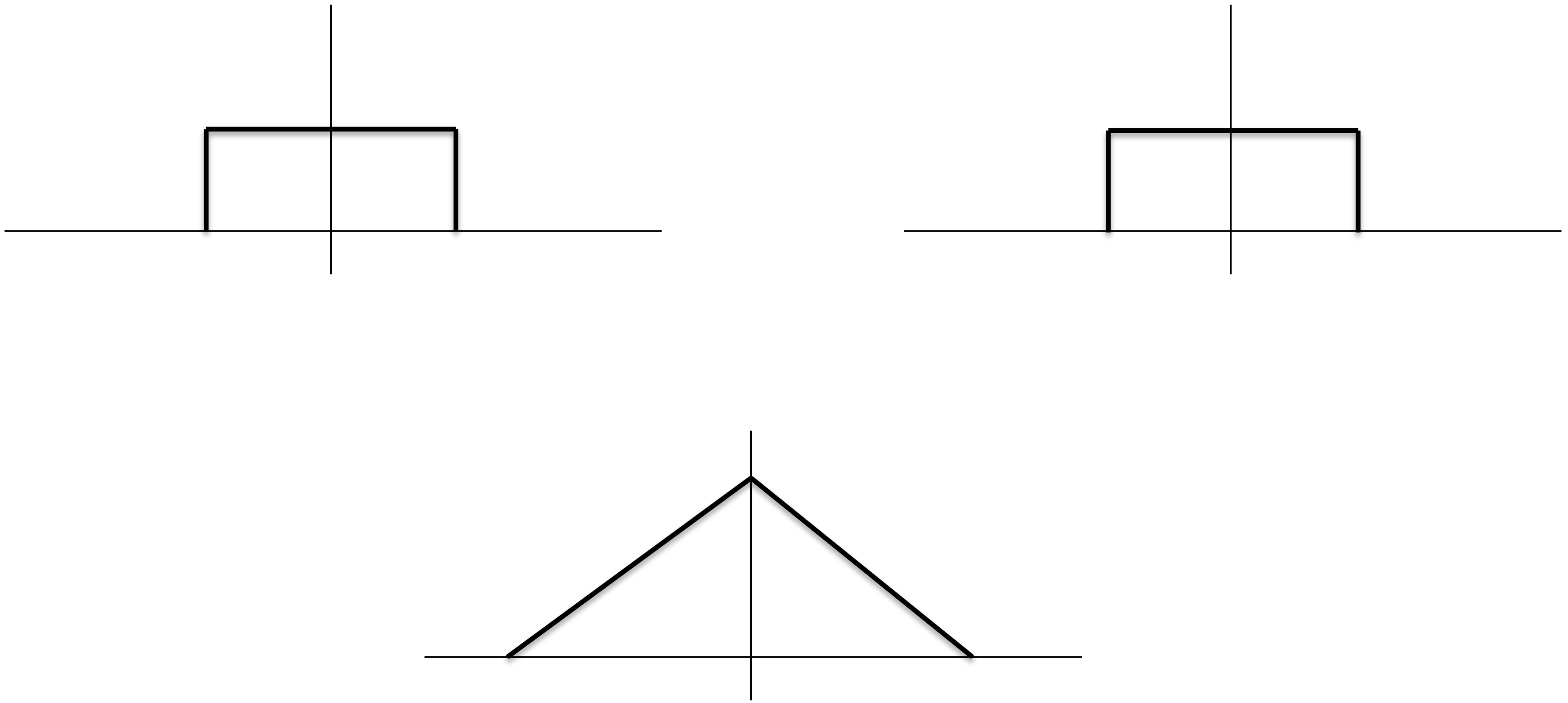
The pdf $f_Z(z)$ is:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx$$

The summation of two independent continuous random variables leads to the convolution of their respective probability density functions.

Sum of Independent Random Variables

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,\mathrm{d}x$$



The Mean Value of Random Variables

The expected value or mean of a continuous random variable X is formally defined by

$$\mu_X = \mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x \qquad\qquad \mathbb{E}[X] = \sum_x x p_X(x)$$

The mean locates the center of gravity of the area under the probability density curve of the random variable X.

Properties

1. Linearity: E[Z] = E[X + Y] = E[X] + E[Y]
2. Statistical independence: E[Z] = E[XY] = E[X] E[Y] if X and Y are independent.

Variance

The variance σ^2_X of a random variable X is defined as

$$\mathrm{var}[X] = \mathbb{E}(X-\mu_x)^2 = \int_{-\infty}^{\infty}(x-\mu_X)^2 f_X(x)\,\mathrm{d}x$$

$$\sigma_X^2 = \mathbb{E}[X^2 - 2\mu_X X + \mu_X^2] = \mathbb{E}[X^2] - 2\mu_X\mathbb{E}[X] + \mu_X^2 = \mathbb{E}[X^2] - \mu_X^2$$

In a sense, the variance of X is a measure of the variable's "randomness"

$$\mathbb{P}[|X-\mu_X| \geq \varepsilon] \leq \frac{\sigma_X^2}{\varepsilon^2}$$

Covariance

Let X and Y be two random variables. The covariance is defined as

$$\begin{aligned}\mathrm{cov}[XY] &= \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] \\ &= \mathbb{E}[XY]-\mu_X\mu_Y\end{aligned}$$

The *correlation coefficient* of X and Y is (measure of similarity between X and Y)

$$\rho(X,Y) = \frac{\mathrm{cov}[XY]}{\sigma_X\sigma_Y}$$

Two random variables X and Y are said to be

1. Uncorrelated *if cov[XY] = 0;*
2. Orthogonal *if E[XY] = 0.*

The Gaussian Distribution

Among the many distributions, the Gaussian distribution stands out, by far, as the most commonly used distribution in the statistical analysis of communications systems

- The variable X is said to be Gaussian distributed if its pdf has the general form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

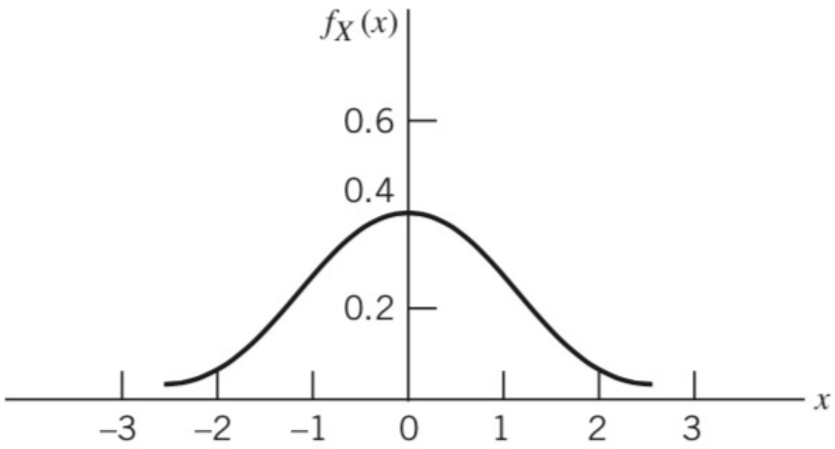
Properties

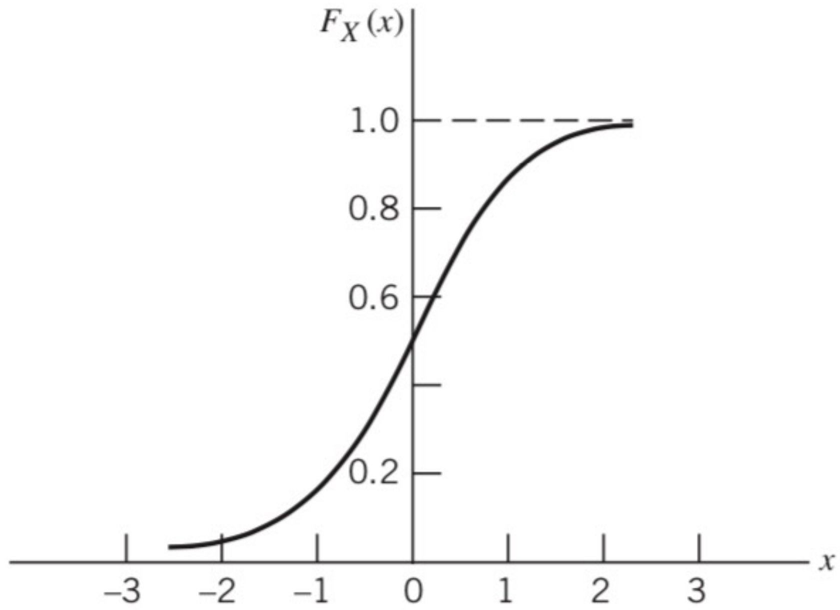
1. Uniquely defined by its mean and variance
2. Gaussianity is preserved by a linear transformation.
3. The sum Z = X + Y of independent Gaussian random variables is also a Gaussian random variable, with E[Z] = E[X] + E[Y] and var[Z] = var[X] + var[Y]

The Standard Gaussian Distribution

When E[X] = 0 and var[X] = 1, then (standard form)

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \qquad F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t$$

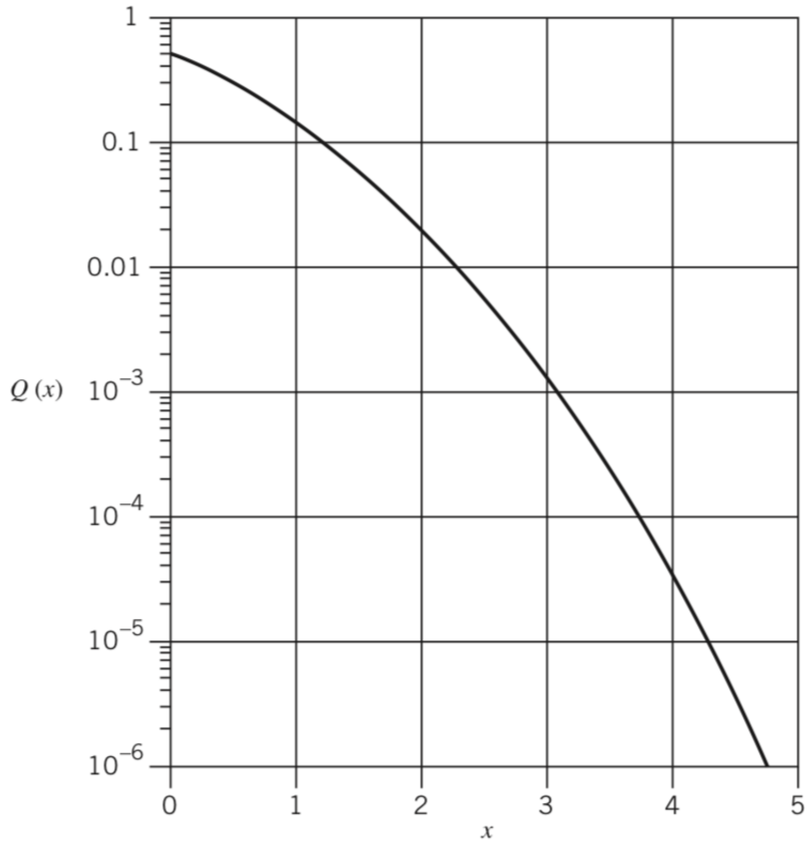




The Standard Gaussian Distribution

The function commonly used in communication systems is the Q-function:

$$
\begin{aligned}
Q(x) &= 1 - F_X(x) \\
&= \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t
\end{aligned}
$$



The Central Limit Theorem

Let X_1, X_2, ..., X_n denote a sequence of independently and identically distributed (iid) random variables with mean $\boldsymbol{\mu}$ and variance $\boldsymbol{\sigma}^2$. Define:

$$Y_n = \frac{1}{\sigma\sqrt{n}}\left(\sum_{i=1}^{n} X_i - n\mu\right)$$

As the number of random variables n approaches infinity, Yn converges to the standard Gaussian random variable:

$$F_Y(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp^{-\frac{x^2}{2}} dx$$

Mathematical justification for using the Gaussian distribution as a model for an observed random variable result of a large number of random events

Sum of Uniformly Distributed Random Variables

Consider the random variable

$$Y_n = X_1 + X_2 + \ldots + X_n$$

where X_i are independent and uniformly distributed random variables on the interval from -1 to $+1$. Let's compute the pdf by using Matlab.