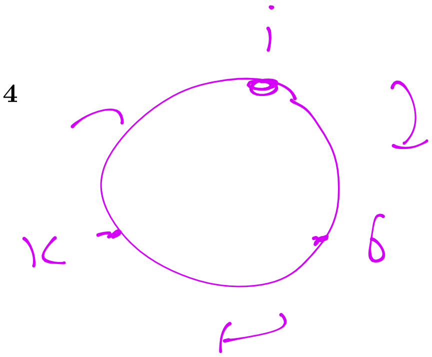
$\hat{a} \times \hat{a} = 0$

591AA 21/22 – ELENCO DEI PROBLEMI 4



Problema 1. Calcolare

(a) $(1,1,1) \times (1,2,3)$;
(b) $(1,1,1) \times (2,3,4)$.

(a) $(i+j+k)\times(i+2j+3k)$

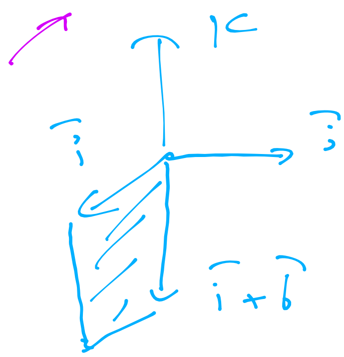
$= i\times i + (i\times 2j) + i\times 3k$
$+ j\times i + j\times 2j + j\times 3k$
$+ k\times i + k\times 2j + k\times 3k$

$= 2k - 3j - k + 3i + j - 2i$

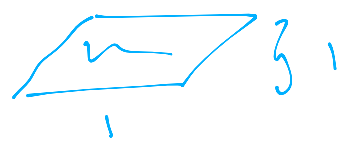
$i\times j = k$
$j\times k = i$
$k\times i = j$

$u\times v = -v\times u$

Problema 2. Calcola $\mathbf{i} \times (\mathbf{i}+\mathbf{j})$ usando la definizione geometrica del prodotto vettoriale.



$i\times(i+j) = ?$



$i\times(i+j)$
$= i\times i + i\times j = k$

Problema 3. Quali dei seguenti vettori sono perpendicolari

No (a) $(1,1,1)$ e $(1,2,3)$.
Si (b) $(1,1,-1)$ e $(1,2,3)$.
Si (c) $(3,4,5)$ e $(3,4,-5)$

$$1 \quad -1 \quad 2 \quad +3$$

$$(a)\quad ((1,1,1),(1,2,3)) = (1)(1)+(1)(2)+(1)(3) = 6$$

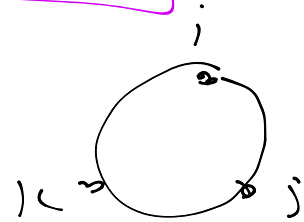
$$(b)\quad ((1,1,-1),(1,2,3)) = (1)(1)+(1)(2)+(-1)(3) = 0$$

$$(c)\quad ((3,4,5),(3,4,-5)) = 3^2+4^2-5^2 = 0$$

Problema 4. Trova l'equazione del piano che passa per i punti

$$\boxed{x+y-z=1}$$ l'equazione del piano

$$P = (1,0,0), \qquad Q = (0,1,0), \qquad R = (0,0,-1)$$



①
$$u = Q-P = (-1,1,0)$$
$$v = R-P = (-1,0,-1)$$

②
$$N = u\times v = (-i+j)\times(-i-k)$$
$$= (-i)\times(-i) + (-i)\times(-k) + (j)\times(-i) + (j)\times(-k)$$
$$= 0 + i\times k - j\times i - j\times k$$
$$= -j + i\times j + k\times j = -j+k-i = (-1,-1,1)$$

$$\det\begin{pmatrix} i & j & k \\ -1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix} = i\det\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - j\det\begin{pmatrix}-1 & 0\\ -1 & -1\end{pmatrix} + k\det\begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix} = (-1,-1,1)$$

$$i\times j = k,\quad j\times k = i,\quad k\times i = j$$

③ l'Equazione $s=(x,y,z)$, $P=(1,0,0)$

$$w = s-P = (x-1,y,z)$$
$$\Rightarrow (w,N) = ((x-1,y,z),(-1,-1,1))$$
$$= 1-x-y+z = 0 \iff \boxed{x+y-z=1}$$

Ricontrolla la risposta ③

$(x,y,z) = (1,0,0) \Rightarrow x+y-z \overset{?}{=} 1$
$(x,y,z) = (0,1,0) \Rightarrow x+y-z = 1$
$(x,y,z) = (0,0,-1) \Rightarrow x+y-z = 1$

Ricontrolla la risposta ②

$((-1,1,0),(-1,-1,1)) = 1-1+0 = 0$
$((-1,0,-1),(-1,-1,1)) = 1+0-1 = 0$

$u\times v \perp u$, $u\times v \perp v$

Problema 5. Trova l'intersezione dei piani: $x + y - z = 1$ and $x - 2y + z = 0$.

$$\{Ax + By + Cz = D\} \longleftrightarrow \{x + y - z = 1\} \Rightarrow (A, B, C) = (1, 1, -1)$$

$$\{A'x + B'y + C'z = D'\} \longleftrightarrow \{x - 2y + z = 0\} \Rightarrow (A', B', C') = (1, -2, 1)$$

Ricontrolla

$$((1,1,-1),(-1,-2,-3)) = -1 - 2 + 3 = 0$$

$$((1,-2,1),(-1,-2,-3)) = -1 + 4 - 3 = 0$$

(1)

$$(A,B,C)\times(A',B',C') = (\vec{i} + \vec{j} - \vec{k})\times(\vec{i} - 2\vec{j} + \vec{k}) = (\vec{i}\times\vec{i}) - 2(\vec{i}\times\vec{j}) + (\vec{i}\times\vec{k})$$
$$+ (\vec{j}\times\vec{i}) - 2(\vec{j}\times\vec{j}) + (\vec{j}\times\vec{k})$$
$$- (\vec{k}\times\vec{i}) + 2(\vec{k}\times\vec{j}) + (\vec{k}\times\vec{k})$$
$$= (-1, -2, -3)$$

(2)

$\underline{x = 1}$: $\quad 1 + y - z = 1, \quad 1 - 2y + z = 0$

$$\Rightarrow y - z = 0, \; -2y + z = -1 \Rightarrow z = 1$$

$y = z$

$$\Rightarrow (1,1,1) \in \{x + y - z = 1\} \cap \{x - 2y + z = 0\}$$

(3) $L(t) = (1,1,1) + t(-1,-2,-3)$

Ricontrolla $\quad x(t) = 1 - t, \; y(t) = 1 - 2t, \; z(t) = 1 - 3t$

$$x(t) + y(t) - z(t) = (1 - t) + (1 - 2t) - (1 - 3t) = 1$$

$$x(t) - 2y(t) + z(t) = (1 - t) - 2(1 - 2t) + (1 - 3t)$$
$$= (1 - 2 + 1) - t + 4t - 3t = 0$$

Problema 6. Trova l'angolo tra i vettori

$$u = (1, 0, 1), \qquad v = (1, 2, 2)$$

$$\cos\theta = \frac{(u,v)}{\sqrt{(u,u)}\sqrt{(v,v)}}$$

$$(u,v) = ((1,0,1),(1,2,2)) = (1)(1) + (0)(2) + (1)(2) = 3$$

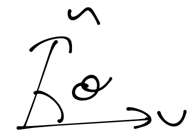
$$(u,u) = 1^2 + 0^2 + 1^2 = 2, \quad (v,v) = 1^2 + 2^2 + 2^2 = 9$$

$$\cos\theta = \frac{3}{\sqrt{2}\sqrt{9}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \theta = \pi/4$$

Problema 7. Trova l'angolo tra i piani

$$x+y+z=1, \qquad x+y-2z=0$$

$$Ax+By+Cz=D \qquad A'x+B'y+C'z=D'$$

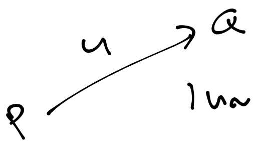


$$u=(A,B,C)=(1,1,1), \quad v=(A',B',C')=(1,1,-2).$$

$$\cos\theta = \frac{(u,v)}{\sqrt{(u,u)}\sqrt{(v,v)}} = \frac{(1)(1)+(1)(1)+(1)(-2)}{\sqrt{(u,u)}\sqrt{(v,v)}} = 0 \Rightarrow \theta = \pi/2$$

Problema 8. Trova la distanza tra i vettori

$$\underbrace{(4,6,15)}_{P}, \qquad \underbrace{(1,2,3)}_{Q}$$



lunghezza u = distanza P a Q

$$u=(-3,-4,-12)$$

$$(u,u)=3^2+4^2+12^2=9+16+144$$

$$=25+144=169$$

$$\sqrt{(u,u)}=\sqrt{169}=13$$

Problema 9 Trova il vettori unitario nella direzione:

$$v = (1, 2, 2)$$

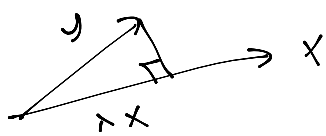
u unitario $\iff |u| = 1 \qquad |u| = \sqrt{(u,u)}$

u ha la stessa direzione di v

$$u = \frac{v}{|v|} = \frac{(1,2,2)}{\sqrt{1^2+2^2+2^2}} = \frac{(1,2,2)}{3}$$

Ricontrollo: $\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{1+4+4}{9} = 1$

Problema 10 Trova la proiezione di $y = (1, 1, 1)$ sul direzione di $x = (0, 1, 2)$.



$$\lambda x = \frac{(y,x)}{(x,x)} x \quad (\text{proiezione di } y \text{ sul } x)$$

$$\frac{(y,x)}{(x,x)} x = \frac{((1,1,1),(0,1,2))}{0^2+1^2+2^2} (0,1,2) = \frac{3}{5}(0,1,2)$$

Ricontrollo $y - \lambda x \perp x$

$y - \lambda x = (1,1,1) - \frac{3}{5}(0,1,2) = (1, \frac{2}{5}, -\frac{1}{5})$

$(1, \frac{2}{5}, -\frac{1}{5}) \cdot (0,1,2) = 0 + \frac{2}{5} - \frac{2}{5} = 0$