TEOREMA SCHWARZ

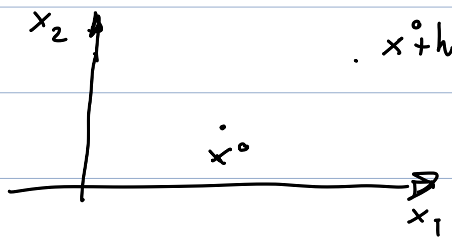
$f : A \to \mathbb{R}$ $\quad$ A aperto $\subseteq \mathbb{R}^n$ $\quad$ $n \geq 2$

$\dfrac{\partial^2 f}{\partial x_i \partial x_j}$, $\dfrac{\partial^2 f}{\partial x_j \partial x_i}$ esistono $B(x^0, R)$ $i \neq j$, sono continue in x^0

$$\Rightarrow \frac{\partial^2 f}{\partial x_i \partial x_j}(x^0) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x^0)$$

$\boxed{Hf(x^0) \text{ è simmetrica}}$

Dim



$x^0 + h = (x_1^0 + h_1, x_2^0 + h_2)$

$n = 2$, ma non è riduttivo

$h \neq 0$ $\quad$ $(h_1, h_2) \neq (0,0)$

$$\omega(h) = \frac{f(x_1^0 + h_1, x_2^0 + h_2) - f(x_1^0, x_2^0 + h_2) - f(x_1^0 + h_1, x_2^0) + f(x^0)}{h_1 h_2}$$

$\varphi(x_1) := f(x_1, x_2^0 + h_2) - f(x_1, x_2^0)$ $\qquad$ $\psi(x_2) := f(x_1^0 + h_1, x_2) - f(x_1^0, x_2)$

$$\omega(h) = \frac{\psi(x_2^0 + h_2) - \psi(x_2^0)}{h_1 h_2} = \frac{\psi'(\xi_2)}{h_1} = \frac{\partial_2 f(x_1^0 + h_1, \xi_2) - \partial_2 f(x_1^0, \xi_2)}{h_1}$$

$$= \partial_1 \partial_2 f(\xi_1, \xi_2)$$

$\downarrow h \to 0$

$\partial_1 \partial_2 f(x_1^0, x_2^0)$

$$\omega(h) = \frac{\varphi(x_1^0 + h_1) - \varphi(x_1^0)}{h_1 h_2} =$$

$$= \frac{\partial_1 \varphi(\eta_1)}{h_2} = \frac{\partial_1 f(\eta_1, x_2^0 + h_2) - \partial_1 f(\eta_1, x_2^0)}{h_2} = \partial_2 \partial_1 f(\eta_1, \eta_2)$$

$\downarrow h$

$\partial_2 \partial_1 f(x_1^0, x_2^0)$

OSS $\quad \dfrac{\partial^3 f}{\partial x_1^2 \partial x_2} = \dfrac{\partial^3 f}{\partial x_1 \partial x_2 \partial x_1} = \dfrac{\partial^3 f}{\partial x_2 \partial x_1^2}$ $\qquad f \in C^3$

$\alpha = (\alpha_1, \dots, \alpha_n)$ $\qquad \alpha_i \in \mathbb{N} \cup \{0\}$ $\qquad$ multi-indice

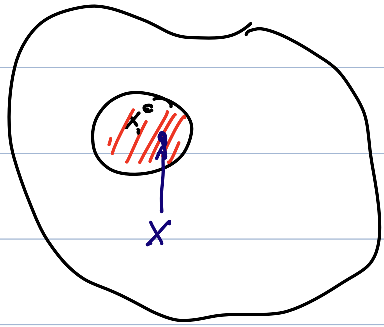
$\alpha \quad \partial^{\alpha_1 + \dots + \alpha_n} f \quad \partial^{|\alpha|} f$

$$D^{\alpha} f = \frac{\partial f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} = \frac{\partial f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

$f : A \to \mathbb{R}$ $\quad$ A aperto $\subseteq \mathbb{R}^n$ $\quad$ x^0 c'è max/min locale

$f(x) \geq f(x^0)$ $\quad \forall x \in B(x^0, \varepsilon)$ $\quad \varepsilon > 0$ $\quad$ x^0 è pt di min locale

$m = f(x^0)$ è il minimo locale



$\nabla f(x^0)$ ESISTE $\xrightarrow{\quad}$ $\nabla f(x^0) = 0$?

$(\partial_1 f(x^0), \dots, \partial_n f(x^0)) \neq (0, \dots, 0)$

$$F(t) = f(tx + (1-t)x^0)$$

$F(0) = f(x^0)$ $\qquad$ $F(1) \geq F(0)$ $\qquad t \in [0,1]$

$F(1) = f(x)$

$$F(t) = F(0) + F'(0)t + \frac{F''(\xi)t^2}{2}$$ $\quad n = 1$ Resto Lagrange

$t = 1$ $\quad F(1) = F(0) + F'(0) + \frac{F''(\xi)}{2}$

$$f(x) = f(x^0) + (\nabla f(x^0), x - x^0) +$$

$$\frac{d}{dt} F(t) = \frac{d}{dt} f(tx + (1-t)x^0)$$

x^0 è pt di minimo $\quad t = 0$

$$0 = (\nabla f(tx + (1-t)x^0), x - x^0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(tx + (1-t)x^0) \cdot \frac{d}{dt}(tx_i + (1-t)x_i^0)$$

$$\frac{d^2}{dt^2} F(t) = \frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(tx + (1-t)x^0)(x_i - x_i^0)\right)$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(tx + (1-t)x^0)(x_i - x_i^0)(x_j - x_j^0)$$

$$= (x-x^0)\ \frac{\partial^2 f}{\partial x_i \partial x_j}(tx+(1-t)x^0)\ (x-x^0)$$

$$= (x-x^0)^T\ Hf(\ )(x-x^0) = (Hf(\ )(x-x^0), x-x^0)$$

$$F(1) = F(0) + F'(0) + \frac{F''(\xi)}{2} \qquad \text{TAYLOR IN } t \quad t_0=0 \quad n=1 \text{ Resto Lag}$$

$$f(x) = f(x^0) + 0 + (x-x^0)^T \frac{Hf(\xi x + (1-\xi)x^0)}{2}(x-x^0)$$

$$0 \le f(x) - f(x^0) = (x-x^0)^T \frac{Hf(\xi x + (1-\xi)x^0)}{2}(x-x^0) \ge 0$$

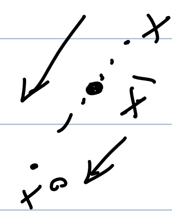
$$= (x-x^0)^T \frac{Hf(x^0)}{2}(x-x^0) + (x-x^0)^T \frac{Hf(\xi x+(1-\xi)x^0) - Hf(x^0)}{2}(x-x^0)$$

$$\parallel$$

$$(x-x^0)^T \cdot \frac{H(\bar{x}) - H(x^0)}{2}(x-x^0)$$

$$f \in C^2 \Rightarrow Hf \text{ simmetrica}$$

$$(x-x^0)^T\ \frac{H(\bar{x}) - H(x^0)}{2}(x-x^0) \xrightarrow[x \to x^0]{} 0$$



$$\left| (x-x^0)^T \left(\frac{H(\bar{x}) - H(x^0)}{2}\right)(x-x^0) \right| \le \|x-x^0\|^2 \left\| \frac{H(\bar{x}) - H(x^0)}{2} \right\| = o(\|x-x^0\|^2)$$

$$f(x) - f(x^0) = (x-x^0)^T \frac{Hf(x^0)}{2}(x-x^0) + o(\|x-x^0\|^2)$$

SE $\quad (x-x^0)^T \frac{Hf(x^0)}{2}(x-x^0) \ge \alpha \|x-x_0\|^2 \qquad \alpha > 0$

$\Rightarrow$ VICINO A x^0 (TEO PERMANENZA SEGNO)

$$f(x) - f(x^0) \ge \frac{\alpha}{2}\|x-x^0\|^2 \ge 0$$

$\boxed{n=2}$ $\quad \frac{1}{2}(x-x^0, y-y^0)\ Hf(x^0, y^0)\ (x-x^0, x-x^0)$

$$Hf(x^0, y^0) = \begin{pmatrix} a & b \\ b & d \end{pmatrix} = A \qquad Q^{-1} A Q = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

λ_i autovalori di A $\quad \lambda_i \in \mathbb{R} \quad A = A^T$

$Q^{-1} = Q^T$ $\quad Q$ è ortogonale $\quad Q = (v_1 | v_2) = \left(\begin{matrix} v_1^1 \\ v_1^2 \end{matrix} \middle| \begin{matrix} v_2^1 \\ v_2^2 \end{matrix} \right)$

$A v_1 = \lambda_1 v_1 \qquad \|v_1\| = \|v_2\|$
$A v_2 = \lambda_2 v_2$

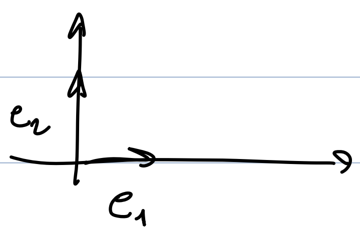
$$Q^T Q = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \big(v_1 | \dots | v_n \big) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

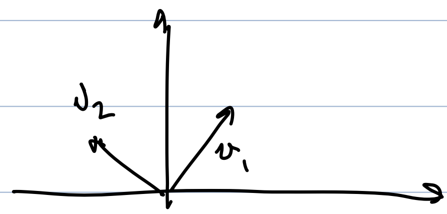
$\det A = \det(Q^T A Q) = \lambda_1 \lambda_2$
$\mathrm{tr}\, A = \mathrm{tr}(Q^T A Q) = \lambda_1 + \lambda_2$

$$\langle Ax, x \rangle = x^T A x \geq \alpha \|x\|^2$$

$\lambda_1, \lambda_2 > 0$

$$(x_1, x_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_1 x_1^2 + \lambda_2 x_2^2$$
$$\geq \min\{\lambda_1, \lambda_2\} (x_1^2 + x_2^2) = \min\{\lambda_1, \lambda_2\} \|x\|^2$$





$A v_1 = \lambda_1 v_1$
$A v_2 = \lambda_2 v_2$

$A \in M(2 \times 2, \mathbb{R}) \qquad A = A^T \qquad \underset{>0}{\det A} \qquad \underset{>0}{\mathrm{tr}\, A} \Rightarrow \lambda_1, \lambda_2 > 0$

$\Rightarrow \quad \langle Ax, x \rangle = \langle x^T, Ax \rangle \geq \alpha \|x\|^2$

$\langle Ax, x \rangle > 0 \qquad \forall\, x \neq 0$
$\langle Ax, x \rangle = 0 \iff x = 0$

$f(x,y) = x^2 + 2y^2 - x^3 \qquad \mathbb{R}^2$

$\nabla f(x,y) = (2x - 3x^2, 4y) \qquad \nabla f(0,0) = (0,0)$

$$Hf(0,0) = \begin{pmatrix} 2-6x & 0 \\ 0 & 4 \end{pmatrix}\Bigg|_{(x,y)=(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

$(0,0)$ è pt di min locale

$g(x,y) = x^2 - y^3$

$\nabla g(x,y) = (2x, -3y^2)$

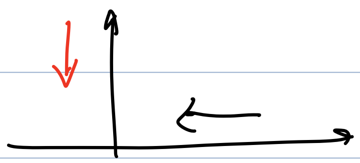
$\nabla g(0,0) = 0$

$$Hg(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -6y \end{pmatrix}\Bigg|_{(x,y)=(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$x^T Hg(0,0)\, x \geq 0$

$x^T Hg(0,0)\, x = 2x_1^2 \geq 0 \qquad x^T Hg(0,0)\, x = 0 \qquad x = (0, x_2)$

$Hg(0,0)$ è SEMI-DEFINITA POSITIVA $\overset{?}{\Rightarrow}$ x^0 è di minimo locale



$(x,0) \qquad g(x,0) = x^2 \qquad x=0$ è minimo

$(0,y) = g(0,y) = -y^3 \qquad y=0$ non è di minimo

$h(x,y) = x^2 + y^4 \qquad h(0,0) = 0$ e $h(x,y) > 0 \quad \forall (x,y) \neq (0,0)$

$\nabla h = (2x, 4y^3)$

$Hh = \begin{pmatrix} 2 & 0 \\ 0 & 12y^2 \end{pmatrix} \qquad Hh(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ SEMI DEF POSITIVA

$\varphi(x,y) = x^2 - y^2 \qquad H\varphi = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ PT DI SELLA

$\nabla \varphi = (2x, -2y) \qquad (x,y)\, H\varphi \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 - 2y^2$

$$f(x,y,z) = 3x^2 + 2y^2 + z^2 - 2xz + 2x + 2y + 1 \qquad (x,y,z) \in \mathbb{R}^3$$

$$\nabla f = \left(6x - 2z + 2,\ 4y + 2,\ 2z - 2x\right)$$

$$\nabla f(x,y,z) = (0,0,0) \qquad \begin{cases} 6x - 2z + 2 = 0 \\ 4y + 2 = 0 \\ 2z - 2x = 0 \end{cases} \qquad \begin{cases} 3x - z = -1 \\ y = -1/2 \\ x = z \end{cases} \quad 2x = -1$$

$$\begin{cases} x = -1/2 \\ y = -1/2 \\ z = -1/2 \end{cases} \qquad \begin{cases} \lambda_1 = 4 > 0 \\ \lambda_2 = 4 + 2\sqrt{2} > 0 \\ \lambda_3 = 4 - 2\sqrt{2} > 0 \end{cases}$$

$$S^{n-1} = \{ x \in \mathbb{R}^n : \|x\| = 1 \} \qquad Hf(x^0) = H$$

$$f \in C^2(\mathbb{R}^n) \qquad x^0 \in \mathbb{R}^n$$

S^{n-1} è chiuso e limitato $\Rightarrow$ compatto

$\min_{S^{n-1}} \varphi$ $\qquad \max_{S^{n-1}} \varphi$ ESISTONO SE φ è continua

$$\varphi(x) = x^T H x = \langle Hx, x \rangle = \sum_{i,j=1}^{n} H_{ij} x_i x_j$$

$$m = \min_{S^{n-1}} \varphi \qquad M = \max_{S^{n-1}} \varphi$$

$$\varphi(x) = \langle Hx, x \rangle = \left\langle H \frac{x}{\|x\|}, \frac{x}{\|x\|} \right\rangle \|x\|^2 \qquad \forall\ x \neq 0$$

$$\varphi(x) = \varphi\left(\frac{x}{\|x\|}\right) \|x\|^2 \qquad \forall\ x \in \mathbb{R}^n \setminus \{0\} \text{ è aperto}$$

$$m \leq \varphi\left(\frac{x}{\|x\|}\right) \leq M \qquad \left\| \frac{x}{\|x\|} \right\| = \frac{\|x\|}{\|x\|} = 1 \quad \forall x \neq 0$$

$$m\|x\|^2 \leq \varphi(x) \leq M\|x\|^2 \qquad m\|x\|^2 \leq \langle Hx, x \rangle \leq M\|x\|^2$$

$\Rightarrow$ SE $m > 0$ $\qquad m = \min_{S} \varphi(x)$ $\Rightarrow$ $\langle Hx, x \rangle$ è DEF. POSITIVA

$$\frac{\varphi(x)}{\|x\|^2} = \varphi\left(\frac{x}{\|x\|}\right) \qquad \forall x \neq 0 \qquad m = \min_{S^{n-1}} \varphi(x) = \varphi(x^0)$$

$\|x^0\| = 1$

x^0 é pt. di minimo di $\varphi\left(\frac{x}{\|x\|}\right) \Rightarrow x^0$ é pt di minimo $\frac{\varphi(x)}{\|x\|^2}$

FERMAT $\quad \nabla\left(\frac{\varphi(x)}{\|x\|^2}\right)\Big|_{x^0} = 0$

$$\frac{\partial}{\partial x_i}\left(\frac{\sum_{p,q=1}^n H_{pq} x_p x_q}{\sum_{k=1}^n x_k^2}\right) = \frac{(H_{pq}\delta_{ip}x_q + H_{pq}x_p\delta_{iq})\|x\|^2 - H_{pq}x_p x_q\, 2x_k\delta_{ik}}{\|x\|^4}$$

$$(H_{iq}x_q + H_{qi}x_q)\|x\|^2 - 2x_i(H_{pq}x_p x_q)$$

$H_{iq} = H_{qi}$

$$(2H_{iq}x_q^0)\underbrace{\|x^0\|^2}_{=1} - 2x_i^0\underbrace{(H_{pq}x_p^0 x_q^0)}_{=m} = 0 \qquad i = 1, \dots n$$

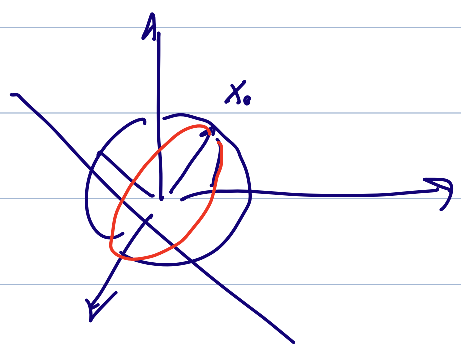
$$2H_{iq}x_q^0 = 2m x_i^0$$

$\|x^0\| = 1$

$$Hx^0 = mx^0$$

x^0 é autovettore relativo all autovalore $\lambda = m$

$$\mathbb{R}^n = \{x^0\} \oplus (x^0)^\perp$$



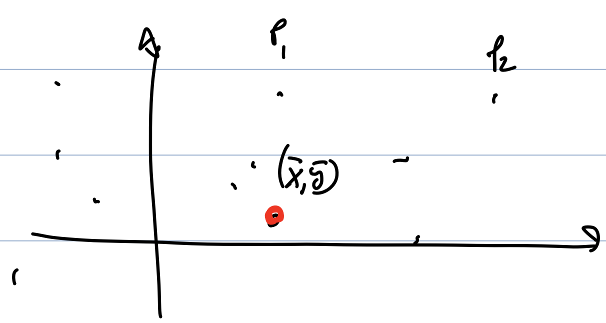
$\varphi(x) \quad \min \left\{x : \langle x^0, x\rangle = 0\right\}$
$\|x\| = 1$

$p_i \in \mathbb{R}^2 \qquad i = 1, \dots, n$

$\| $

(x_i, y_i)

$\bar{p} = (\bar{x}, \bar{y})$



$$\boxed{f(\bar{p}) = \sum_{i=1}^{n} (\bar{x}-x_i)^2 + (\bar{y}-y_i)^2}$$

$\min f(\bar{p})$ $\bar{p}$ baricentro

$\{p_i\} \subseteq \mathbb{R}^2$