Routh-Hurwitz Criterion

In [ ]:
```
#| default_exp routh_hurwitz_criterion
```

In [ ]:
```
#| include: false
%load_ext autoreload
%autoreload 2
```

In [ ]:
```
#| export
import numpy as np
import matplotlib.pyplot as plt
import control
```

Background

- Given a system with transfer function $G(s) = \frac{N(s)}{D(s)}$: in order for a system to be stable, all the roots of its characteristic equation $D(s) = 0$ must have $Re < 0$
- The roots of the characteristic equation $D(s) = 0$ are the poles of the transfer function $G(s)$.

For example:

$$G(s) = \frac{1}{s+a}$$

whose inverse Laplace transform to get the time domain representation is:

$$\mathcal{L}^{-1}(G(s)) = e^{-at} = e^{st}$$

- if $a > 0$, the system is stable: signals go to zero as time goes to infinity.
- if $a < 0$, the system is unstable: the response of the system goes to infinity.

We can have more complex transfer functions:

$$G(s) = \frac{1}{s+a}\frac{1}{s+b}\frac{1}{s+c}\dots$$

and we know that we can always simplify a transfer function using partial fraction expansion:

$$G(s) = \frac{A}{s+a} + \frac{B}{s+b} + \frac{C}{s+c}\dots$$

and

$$\mathcal{L}^{-1}(G(s)) = Ae^{-at} + Be^{-bt} + Ce^{-ct}\dots$$

If there is one root that is unstable, the system is unstable.

- We know that we can determine the stability of the system calculating the roots of the characteristic equation

- Calculating the roots of a polynomial for $n > 2$ is time consuming, and possibly even impossible in closed form

$$12s^5 + 14s^4 + 3s^3 + s^2 + 16s + 11 = 0$$

- We would like to determine stability (hence the roots of $D(s) = 0$) without solving for the roots directly

- This is where the Routh-Hurwitz criterion can help us

Routh–Hurwitz stability criterion

> *All the roots of a polinomial have $Re < 0$ if and only if a certain set of algebraic combinations (i.e. fill out the RH array) of its coefficients have the same signs*

- The **Routh–Hurwitz stability criterion** is a mathematical test that is a necessary and sufficient condition for the stability of a linear time invariant (LTI) control system
- Determine whether all the roots of the characteristic polynomial of a linear system have negative real parts
- The importance of the criterion is that the roots p of the characteristic equation of a linear system with negative real parts represent solutions e^{pt} of the system that are stable (bounded).
- The criterion provides a way to determine if the equations of motion of a linear system have only stable solutions, without solving the system directly

Given:

$$G(s) = \frac{N(s)}{D(s)}$$

- If all the signs of the coefficients are *NOT* the same, then the system is **unstable**
  - e.g. $s^5 + 3s^3 - 4s^2 + s + 1 \rightarrow$ *unstable*
- If all the signs are the same, then the system can be stable or unstable

For example:

$$G(s) = \frac{1}{(s^2 - s + 4)(s+2)(s+1)}$$

the roots are $0.5 \pm j1.9365$ (unstable roots), $-2, -1 \Rightarrow$ we have two roots with $Re > 0$.

If we write the characteristic equation however:

$$s^4 + 2s^3 + 3s^2 + 10s + 8 = 0$$

- All coefficients have the same sign. We need to use the RHC and populate the Routh array

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Routh-Hurwitz array

A tabular method can be used to determine the stability when the roots of a higher order characteristic polynomial are difficult to obtain.

For an nth-degree polynomial

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

the table has $n+1$ rows and the following structure:

s^n	a_n	a_{n-2}	a_{n-4}	$\dots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\dots$
$\dots$	b_1	b_2	b_3	$\dots$
$\dots$	c_1	c_2	c_3	$\dots$
s^1	$\dots$	$\dots$	$\dots$	$\dots$
s^0	$\dots$	$\dots$	$\dots$	$\dots$

where the elements b_i and c_i can be computed as follows:

$$b_i = \frac{a_{n-1} \times a_{n-2i} - a_n \times a_{n-(2i+1)}}{a_{n-1}}.$$

$$c_i = \frac{b_1 \times a_{n-(2i+1)} - a_{n-1} \times b_{i+1}}{b_1}.$$

- When completed, the number of sign changes in the first column will be the number of non-negative roots.

Examples

$$G(s) = \frac{1}{(s^2 - s + 4)(s+2)(s+1)} = \frac{1}{s^4 + 2s^3 + 3s^2 + 10s + 8}$$

s^4	1	3	8
s^3	2	10	
s^2	$\frac{2\cdot3-1\cdot10}{2} = -2$	$\frac{2\cdot8-1\cdot0}{2} = 8$	
s^1	$\frac{-2\cdot10-2\cdot8}{-2} = 18$	0	
s^0	$\frac{18\cdot8-2\cdot0}{2} = 8$		

- Determine the number of roots in RHP by counting the number of sign changes in the first column: **the are two sign changes, hence there are two roots with $Re > 0$**
- The system is unstable.

$$G(s) = \frac{1}{s^4 + 2s^3 + 3s^2 + 4s + 5}$$

s^4	1	3	5
s^3	2	4	0
s^2	$\frac{2\cdot3-1\cdot4}{2} = 1$	$\frac{2\cdot5-1\cdot0}{2} = 5$	
s^1	$\frac{1\cdot4-2\cdot5}{1} = -6$	0	
s^0	$\frac{-6\cdot5-1\cdot0}{-6} = 5$		

- We have two roots with $Re > 0$
- Roots: $-1.28 \pm j0.858, 0.28 \pm j1.416$

Routh-Hurwitz criterion: special cases

Special case 1)

- A zero in a row with at least one non-zero appearing later in the row
- The system is always unstable
- We can still fill out the table to know how many are unstable

- Example 1:

$$G(s) = \frac{1}{1s^4 + 2s^3 + 0s^2 + 3s + 4}$$

s^4	1	0	4
s^3	2	3	0
s^2	$\frac{2 \cdot 0 - 1 \cdot 3}{2} = \frac{-3}{2}$		
s^1			
s^0			

We can already say that the system is unstable

- Example 2:

$$G(s) = \frac{1}{1s^4 + 2s^3 + 2s^2 + 4s + 5}$$

s^4	1	2	5
s^3	2	4	0
s^2	$0 \to \epsilon$	5	
s^1	$\frac{\epsilon \cdot 4 - 2 \cdot 5}{\epsilon}$	0	
s^0	5		

- To calculate the coefficients we need to calculate the values for $\lim\limits_{\epsilon \to 0}$

s^4	1	2	5
s^3	2	4	0
s^2	0^+	5	
s^1	$4 - \frac{10}{\epsilon} = -\infty$	0	
s^0	5		

- We have two roots with $Re > 0$

Special case 2)

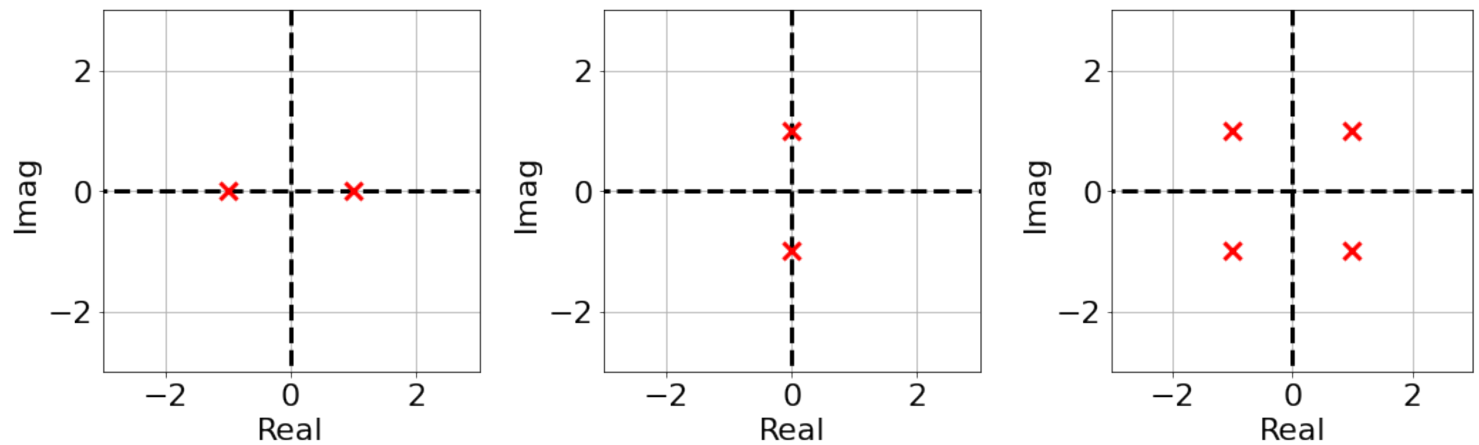
Special case 2)

- An entire row is zeros

$$G(s) = \frac{1}{1s^5 + 2s^4 + 6s^3 + 10s^2 + 8s + 12}$$

s^5	1	6	8
s^4	2	10	12
s^3	1	2	0
s^2	6	12	0
s^1	0	0	0
s^0	$?$		

- There are only 3 possible conditions that can lead to a Routh array with all zeros in a row:
  - two real roots, equal and opposite in sign $\Rightarrow$ **unstable**
  - two imaginary roots, that are complex conjugate of each other $\Rightarrow$ **marginally stable - response is oscillatory**
  - four roots that are all equal distance from the origin $\Rightarrow$ **unstable**



- To determine the system stability:

- Here is the original table: | | | | | | ------ | ------ | - | - | | s^5 | 1 | 6 | 8 | | s^4 | 2 | 10 | 12 | | s^3 | 1 | 2 | 0 | | s^2 | 6 | 12 | 0 | | s^1 | 0 | 0 | 0 | | s^0 | $?$ | | |

  - We build the **auxiliary polinomial** using the row right above the one that is zero:
    - Those are the coefficients of the auxiliary polinomial
    - $p(s) = 6s^2 + 12s^0 = 0 \rightarrow p(s) = s^2 + 2$
    - Note that we are skipping every other power
  - Take the derivative of $p(s)$: $\frac{d}{ds}p(s) = 2s$ and replace the all zero row with the coefficient of $\frac{d}{ds}p(s)$
  - Complete the table as we would normally

s^5	1	6	8
s^4	2	10	12
s^3	1	2	0
s^2	6	12	0
s^1	2	0	0
s^0	12		

- No sign changes in the first column, hence no roots with $Re > 0$.
- This means that the system must have two imaginary roots and is marginally stable.

Additional comments:

- The auxiliary polinomial $p(s)$ exists if and only if there is an all zero row in the routh array, and it is a factor of the original polinomial $q(s)$ (it divides the original polinomial with no reminder):

$$p(s)r(s) = q(s)$$

- This makes it possible to calculate how many roots have $Re < 0$, how many $Re = 0$ and how many $Re > 0$
- We can determine $r(s) = \frac{q(s)}{p(s)}$ (polinomial division)

In our case:

$$(s^5 + 2s^4 + 6s^3 + 10s^2 + 8s + 12) \quad : \quad (s^2 + 2)$$

$$\Downarrow$$

$$r(s) = s^3 + 2s^2 + 4s + 6$$

and this means:

$$q(s) = (s^2 + 2)(s^3 + 2s^2 + 4s + 6)$$

- Only true if we have a row of all zeros, or we will have some non zero reminder when we do the division

If we now re-write our table here:

s^5	1	6	8
s^4	2	10	12
s^3	1	2	0
s^2	6	12	0
s^1	2	0	0
s^0	12		

- Any part of the table above the auxiliary polinomial $p(s) = 6s^2 + 12 = s^2 + 2$ is due to the factor $r(s) = s^3 + 2s^2 + 4s + 6$, and since there are not sign changes we can say that $r(s)$ is stable.
- The other part of the table is due to the auxiliary polinomial $p(s)$: the number of sign changes after $p(s)$ in the table predicts the number of $Re > 0$ roots for $p(s)$
- Given that we have $p(s) = s^2 + 2 \Rightarrow s = \sqrt{-2} = \pm j\sqrt{2} \Rightarrow$ Two complex conjugate roots, the system is marginally stable (what we expected)

Practical uses: beyond stability

- How can we use the Routh Criterion for more than just assessing stability

Suppose you have an open loop system:

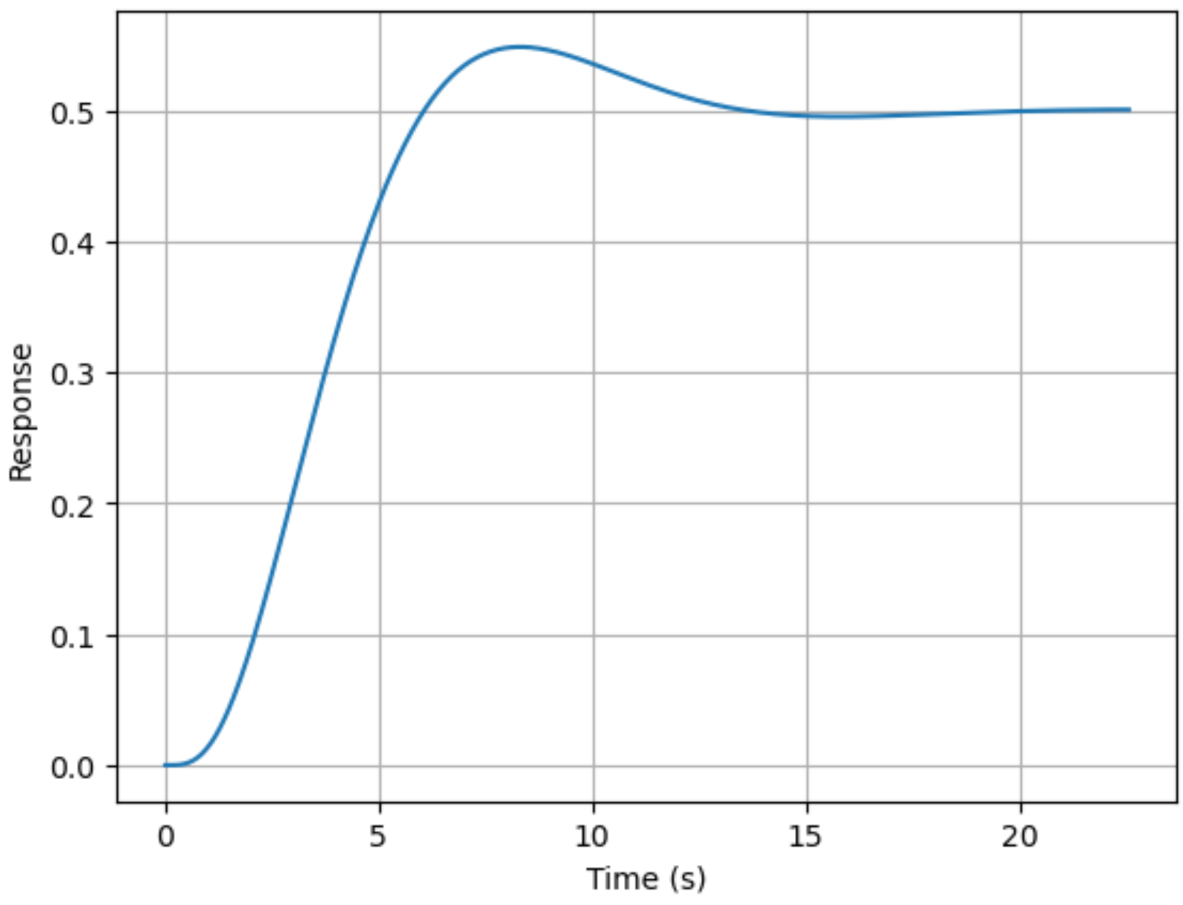
$$G(s) = \frac{1}{s^4 + 6s^3 + 11s^2 + 6s + 2}$$

And we can verify its step response:

In [ ]:
```
# Import relevant libraries (we have imported them already so we comment these out)
# import control
# import matplotlib.pylab as plt
```

In [ ]:
```
sys = control.tf([1], [1, 6, 11, 6, 2])
T, yout = control.step_response(sys)
```

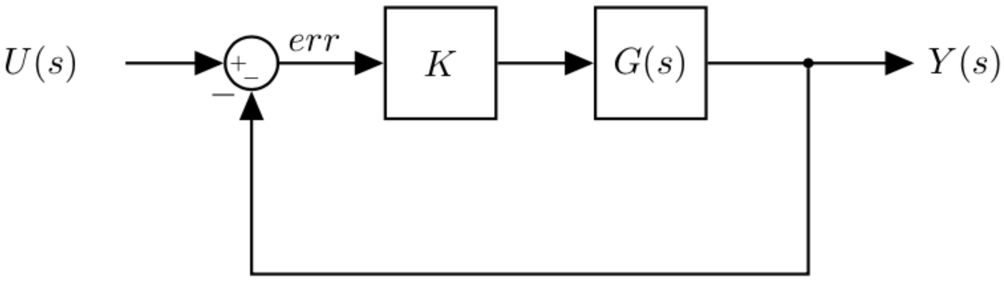
In [ ]:
```
fig = plt.figure()

plt.plot(T, yout)
plt.grid()
plt.xlabel('Time (s)')
plt.ylabel('Response');
```



Design requirement:

- improve system response time

We can add feedback:



- $U(s)$ is the reference signal (e.g. desired angle, etc.)
- Note that the feedback path is unitary: perfect sensor (it always knows the output signal (e.g. system angle) with no delay or error)
- We want to tune the gain K to get the response that we desire
  - Adjusting the gain increases the amplitude of the signal (err)
  - In an open loop linear system we can increase the amplitude "as much as we want" (it does not affect stability)
  - The feedback loop however changes the dynamics
- Typical requirement: **increase the gain large enough to meet our requirements (e.g. fastest response) while keeping the system stable**

- Let's assess the closed loop stability using RHC

- 1. Simplify the block diagram:

$$\hat{G}(s) = \frac{\frac{K}{s^4+6s^3+11s^2+6s+2}}{1+\frac{K}{s^4+6s^3+11s^2+6s+2}} = \frac{K}{s^4+6s^3+11s^2+6s+2+K}$$

  The new characteristic equation for the closed loop system is:

$$s^4+6s^3+11s^2+6s+2+K=0$$

- 1. Build the Routh array

s^4	1	11	$24+K$
s^3	6	6	0
s^2	10	$K+2$	
s^1	$48-6K$		
s^0	$K+2$		

- To have asymptotic stability:
  - $48-6K>0 \to K<8$
  - $K+2>0 \to K>-2$
  - Stability range is $-2<K<8$
- When $K=-2$, we have a pole at the origin

- When $K=8$, we have a pair of imaginary poles and the system is marginally stable

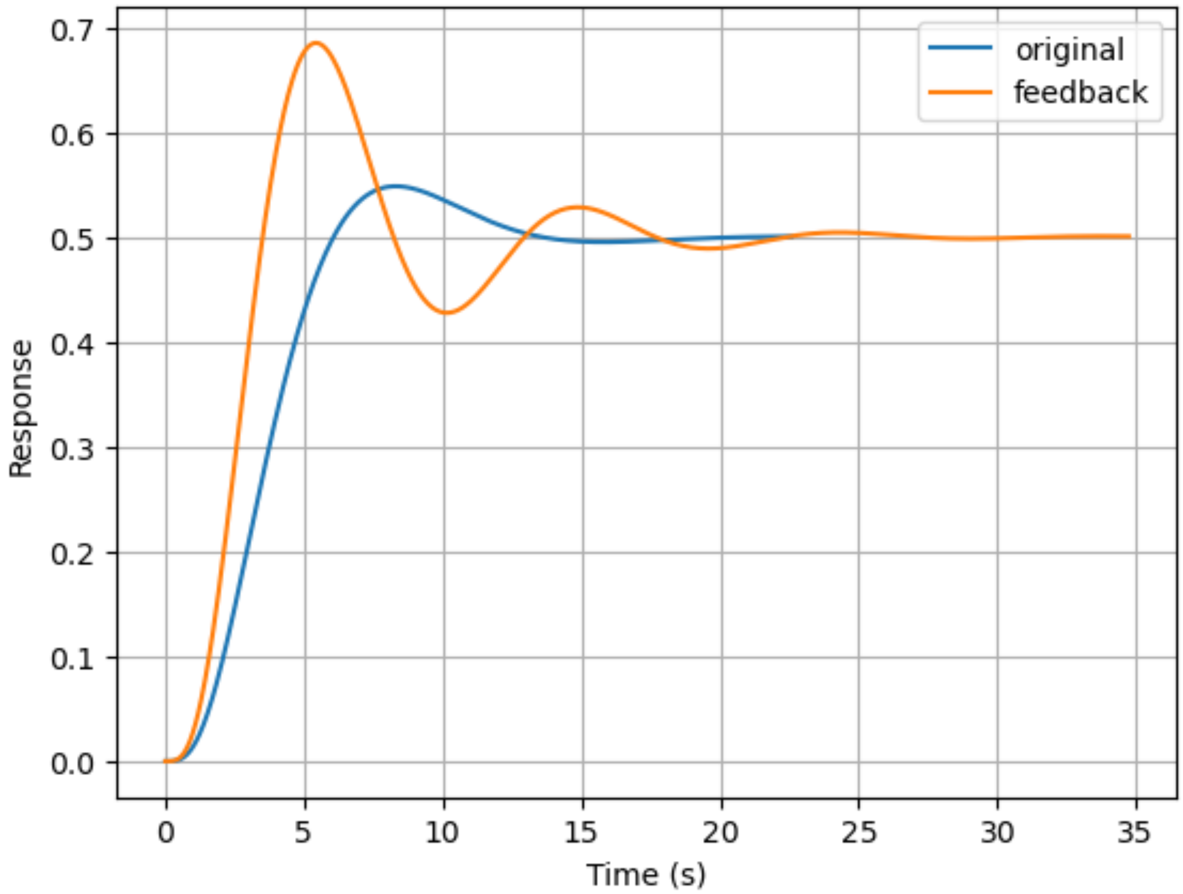
- We know the gain margin that we have before the system becomes unstable

- We do not know where the poles are exactly (we only know when they cross the imaginary axis)

In [ ]:
```
K = 2 #8
```

In [ ]:
```
sys = control.tf([K], [1, 6, 11, 6, 2+K])
T_k, yout_k = control.step_response(sys)
```

In [ ]:
```
fig = plt.figure()

plt.plot(T, yout, label='original')
plt.plot(T_k, yout_k, label='feedback')
plt.grid()
plt.legend()
plt.xlabel('Time (s)')
plt.ylabel('Response');
```



Example 2

Suppose you have an open loop system:

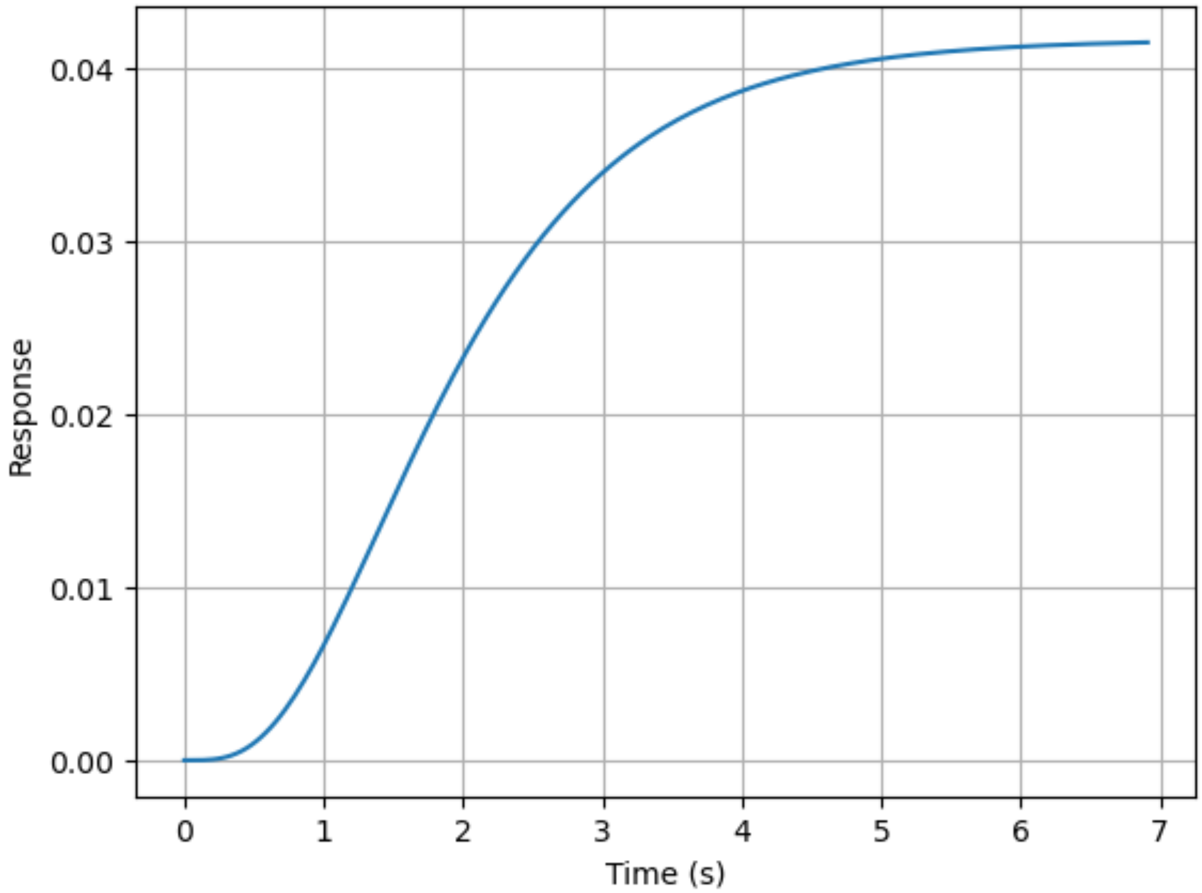
$$G(s) = \frac{1}{s^4 + 10s^3 + 35s^2 + 50s + 24} = \frac{1}{(s+1)(s+2)(s+3)(s+4)}$$

And we can plot its step response:

In [ ]:
```
sys = control.tf([1], [1, 10, 35, 50, 24])
T, yout = control.step_response(sys)
```

In [ ]:
```
fig = plt.figure()

plt.plot(T, yout, label='original')
plt.grid()
plt.xlabel('Time (s)')
plt.ylabel('Response')
```

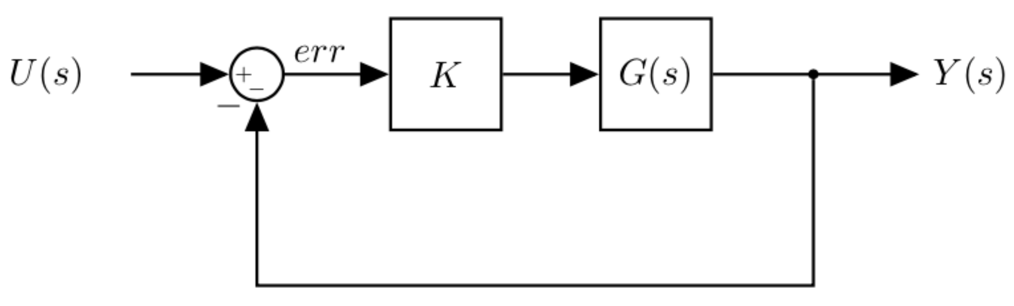
Out[ ]:
```
Text(0, 0.5, 'Response')
```



Design requirement:

- improve system response time

We can add feedback:



- $U(s)$ is the reference signal (e.g. desired angle, etc.)
- Note that the feedback path is unitary: perfect sensor (it always knows the output signal (e.g. system angle) with no delay or error)
- We want to tune the gain K to get the response that we desire
  - Adjusting the gain increases the amplitude of the signal (err)
  - In an open loop linear system we can increase the amplitude as much as we want
  - The feedback loop however changes the dynamics
- Typical requirement: increase the gain large enough to meet our requirements (e.g. fastest response) while keeping the system stable

- Let's assess the closed loop stability using RHC

- 1. Simplify the block diagram:

$$\hat{G}(s) = \frac{\frac{K}{s^4+10s^3+35s^2+50s+24}}{1+\frac{K}{s^4+10s^3+35s^2+50s+24}} = \frac{K}{s^4+10s^3+35s^2+50s+24+K}$$

  The new characteristic equation for the closed loop system is:

$$s^4+10s^3+35s^2+50s+24+K=0$$

- 1. Build the Routh array

s^4	1	35	$24+K$
s^3	10	50	0
s^2	$\frac{10\cdot 35-50}{10}=30$	$24+K$	
s^1	$42-\frac{K}{3}$		
s^0	$24+K$		

- How many sign changes?
    - $42 - K/3 > 0 \rightarrow K < 126$
    - $K > -24$
    - and solving the system of equations: $K < 126$
- When $K = -24$ we have a pole at the origin
- When $K = 126$ the poles would be going from $Re < 0$ to $Re > 0$ (imaginary poles) and the system would be marginally stable

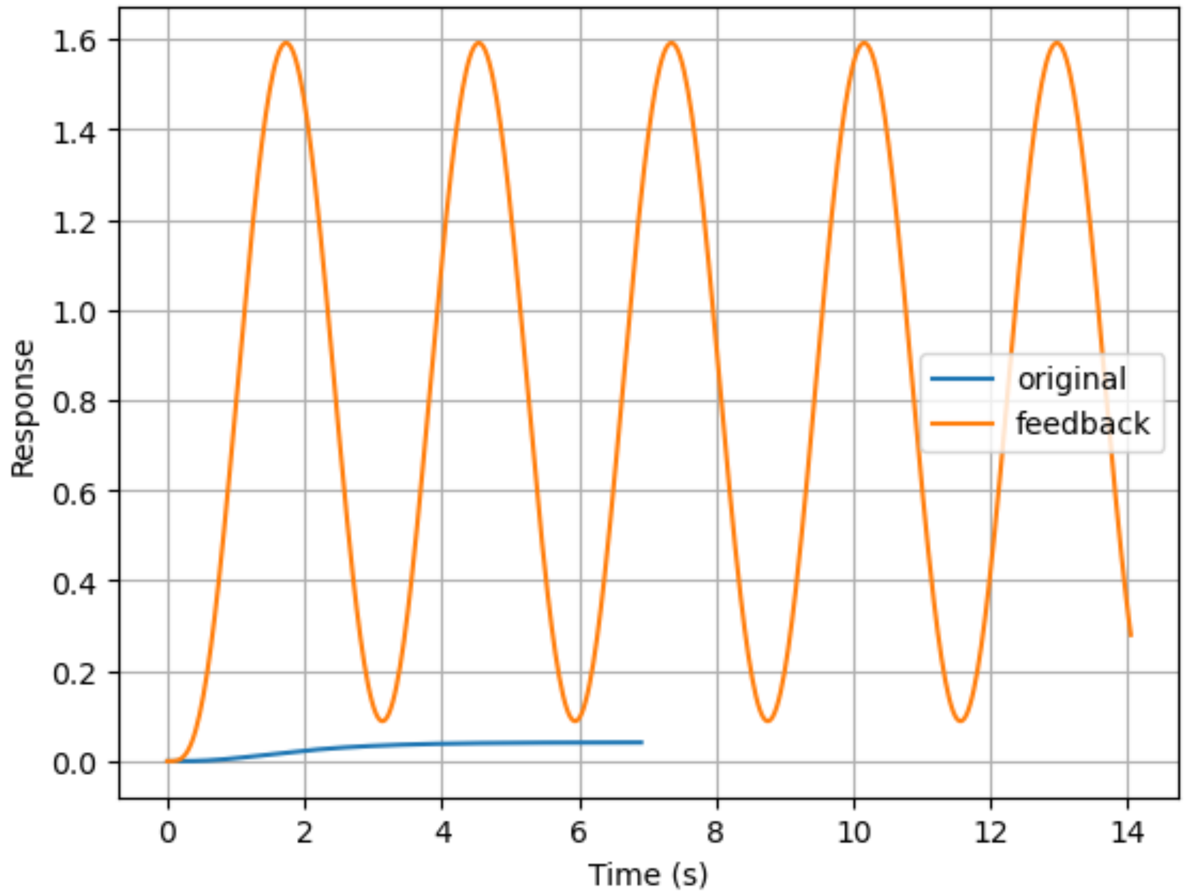
Again,

- We know the gain margin that we have
- We do not know where the poles are exactly (we only know when they cross the imaginary axis)

In [ ]:
```
K = 126
```

In [ ]:
```
sys = control.tf([K], [1, 10, 35, 50, 24+K])
T_k, yout_k = control.step_response(sys)
```

In [ ]:
```
fig = plt.figure()

plt.plot(T, yout, label='original')
plt.plot(T_k, yout_k, label='feedback')
plt.grid()
plt.legend()
plt.xlabel('Time (s)')
plt.ylabel('Response');
```



Having additional requirements: time to half

- Speed of the response depends on the position of the poles on the real axis
  - The further from the imaginary axis, the faster the response (true for both stable and unstable poles)
  - When $s = j\omega$ (imaginary poles), the response is a sinusoid ($e^{j\omega}$ is a sinusoid)
  - the closer the poles are to the real axis, the slower the frequency of the response (i.e., slower oscillation)

- The time to half is the time for a signal to half (or double, if unstable) the initial signal in magnitude
- Typical requirement, especially when dealing with systems that interact with humans: the system should not be too fast or too slow to respond

- A time to half requirements is a requirement on the real part of the poles

- We would like the poles to be to the left of a line called the *z-line*

- Set $s = z + \sigma_{des}$

  - When $z = 0$, $s = +\sigma_{des}$, and the poles have the desired real part.
- Given a characteristic equation $D(s) = 0$

$$s^3 + 5s^2 + 25s + 30 = 0$$

we replace $s = z + \sigma_{des}$ (e.g σ_{des}=-1)

$$(z-1)^3 + 5(z-1)^2 + 25(z-1) + 30 = 0$$

$$\Downarrow$$

$$z^3 + 2z^2 + 18s + 9 = 0$$

- We can now use the Routh array to verify if the system has any roots to the right of the *z-line*

s^3	1	18
s^2	2	9
s^1	$\frac{27}{2}$	
s^0	9	

- No sign changes in the first column: all roots are to the left side of $s = -1$.

RHC Final Comments

- RHC provides necessary and sufficient conditions to analyse the stability of a system
- The stability analysis can be done with respect to any parameters of the system (including the gain K)
- Building the routh array:
  - when we have an all zero row, the roots of the auxiliary polynomial are a subset of the roots of the original polinomial (i.e., of the characteristic equation of the system)
  - the roots of the auxiliary polynomial are symmetric with respect to the origin (i.e., wrt real and imaginary axis).

In [ ]: