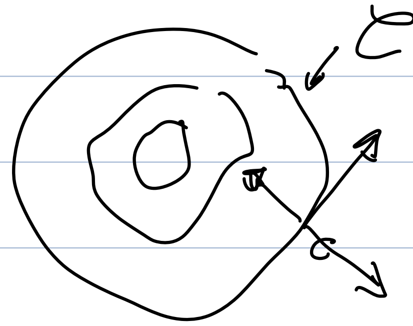
$\{ f(x,y) = c \} = L_c$



$\gamma : [0,1] \to \mathbb{R}^2 \qquad (\gamma_1(t), \gamma_2(t))$

$f(\gamma_1(t), \gamma_2(t)) \equiv c$

$$\frac{d}{dt} f(\gamma_1(t), \gamma_2(t)) = 0 = \partial_x f(\gamma_1(t), \gamma_2(t))\, \dot{\gamma}_1(t) + \partial_y f(\gamma_1(t), \gamma_2(t))\, \dot{\gamma}_2(t)$$

$$= \left(\nabla f(\gamma_1(t), \gamma_2(t)), (\dot{\gamma}_1, \dot{\gamma}_2) \right) = 0$$

$f : [a,b] \to \mathbb{R}$ $\quad f$ DERIVABILE $\quad$ pt. di max, min $\underset{a,b}{f'} = 0$

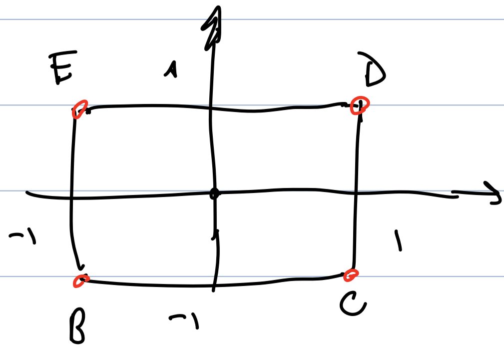
$f : \bar{A} \to \mathbb{R}$ $\quad f \in C^1(\bar{A})$ $\quad A$ limitato

$\bar{A} \subset \mathbb{R}^2$

$x^0 \in \mathring{A}$ $\quad x^0$ é pt di max o min $\quad \nabla f(x^0) = 0$

$0 \leq f(x,y) = x^2 + y^2$ $\quad A = [-1,1] \times [-1,1]$

$\nabla f(x,y) = (2x, 2y)$ $\quad \nabla f = 0 \iff (x,y) = (0,0)$



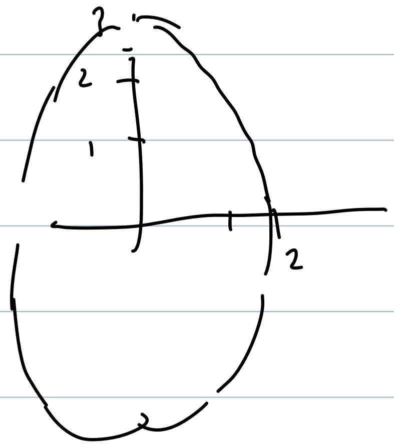
$\partial A = \overline{BC} \cup \overline{CD} \cup \overline{DE} \cup \overline{EB}$

$\overline{BC} = \{(x,y) \in \mathbb{R} : |x| \leq 1 \ \ y = -1\}$

$f(x,y)\big|_{\overline{BC}} = f(x,-1) = x^2+1$ $\quad x \in [-1,1]$

$\max_{\overline{BC}} f(x,y) = 2$

$f(x,y) = x^2+y^2$ $\qquad A = \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{2^2} + \frac{y^2}{3^2} \leq 1\}$



$\gamma(t) = (2\cos(t), 3\sin(t))$

$$\frac{2^2\cos^2(t)}{2^2} + \frac{3^2\sin^2(t)}{3^2} = \cos^2(t) + \sin^2(t) = 1$$

$f(\gamma_1(t), \gamma_2(t))$ $\qquad t \in [0, 2\pi]$

$f|_{\partial A} = 2^2\cos^2(t) + 3^2\sin^2(t)$

$F(x,y) = \frac{x^2}{2^2} + \frac{y^2}{3^2} - 1$

$\partial A = \{(x,y) \in \mathbb{R}^2 : F(x,y) = 0\}$

$\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1 \qquad \frac{y^2}{3^2} = 1 - \frac{x^2}{4} \qquad y = \pm 3\sqrt{1 - \frac{x^2}{4}}$

$f :]a,b[\to \mathbb{R}$ $\quad x_0 \in]a,b[$

$f'(x_0) = 0 \quad f''(x_0) > 0 \quad \Rightarrow \quad x_0$ è pt minimo locale

DERIVATE SUCCESSIVE

$f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}^2$ $\quad \partial_x f(x^0)$, $\partial_y f(x^0)$ $\quad x^0 \in \mathring{A}$

$\partial_x f : A \to \mathbb{R}^n$

$\partial_y f : A \to \mathbb{R}^n$

$$\partial_x (\partial_x f)(x^0) = \lim_{h \to 0} \frac{\partial_x f(x^0 + he_1) - \partial_x f(x^0)}{h}$$

$$\partial_{11} f(x^0) = \partial_{xx} f(x^0) = f_{xx}(x^0) = \frac{\partial^2 f}{\partial x^2}(x^0)$$

$(x,y) = (x_1, x_2)$

$$\partial_y(\partial_x f(x^0)) = \lim_{k \to 0} \frac{\partial_x f(x^0 + ke_2) - \partial_x f(x^0)}{k}$$

$(\partial_{xx} f, \partial_{xy} f, \partial_{yx} f, \partial_{yy} f)$

$$= \partial_{yx} f(x^0) = f_{yx}(x^0) = \frac{\partial^2 f}{\partial y \partial x}(x^0)$$

$f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$ $\quad \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x^0) \right)$ $\quad i,j = 1, \dots n$

$$f(x,y) = \sin(xy)$$

$$\partial_x f(x,y) = \cos(xy)\cdot y \qquad \partial_y f(x,y) = \cos(xy)\, x$$

$$\partial_{xx} f = \partial_x \left(\cos(xy)\cdot y\right) = -\sin(xy)\, y^2$$

$$\partial_{yx} f = \partial_y \cos(xy)\, y = -\sin(xy)\, xy + \cos(xy)$$

$$\partial_{xy} f = \partial_x \left(\cos(xy)\, x\right) = -\sin(xy)\, yx + \cos(xy)$$

$$\partial_{yy} f = \partial_y \left(\cos(xy)\, x\right) = -\sin(xy)\, x^2$$

$$f(x,y) = x + |y| \qquad \partial_x f = 1 \qquad \partial_y \partial_x f = 0$$

$$\partial_y f(0,0) = N.E. \qquad \partial_x \partial_y f(0,0) = N.E.$$

$f_{xy}(x^0) \neq f_{yx}(x^0)$ in generale, anche se esistono entrambe.

$$f(x,y) = \begin{cases} xy \dfrac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \qquad f \in C(\mathbb{R}^2 \setminus \{0\})$$

$$f(x,y) = \rho^2 \cos\theta \sin\theta \, \frac{\rho^2\cos^2\theta - \rho^2 \sin^2\theta}{\rho^2} = \rho^2 \cos\theta \sin\theta \left(\cos^2\theta - \sin^2\theta\right) \longrightarrow 0 \quad \rho \to 0$$

f è continua in $(0,0)$

$$\partial_x f(x,y) = y \frac{x^2-y^2}{x^2+y^2} + xy \frac{2x(x^2+y^2) - 2x(x^2-y^2)}{(x^2+y^2)^2} \qquad (x,y) \neq (0,0)$$

$$= \frac{y(x^2+y^2)(x^2-y^2) + xy\, 4y^2}{(x^2+y^2)^2} = \frac{\rho^5}{\rho^4} \varphi(\theta) \xrightarrow{\rho \to 0} 0$$

$\partial_{xy} f(0,0) \neq \partial_{yx} f(0,0)$

$$f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2} \\ 0 \end{cases}$$

$$\partial_x \partial_y f(0,0) = \lim_{h\to 0} \frac{\partial_y f(h,0) - \partial_y f(0,0)}{h}$$

$$\partial_y f(x,0) = \lim_{k\to 0} \frac{f(x,k) - f(x,0)}{k} = \lim_{k\to 0} \frac{x\cancel{k}\left(\frac{x^2-k^2}{x^2+k^2}\right)^{\to 1} - 0}{\cancel{k}} = x$$

$$\partial_{xy} f(x,0) = \lim_{h\to 0} \frac{h-0}{h} = 1$$

$$\partial_y \partial_x f(0,0) = \lim_{k\to 0} \frac{\partial_x f(0,k) - \partial_x f(0,0)}{k}$$

$$\partial_x f(0,y) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\cancel{h}y}{\cancel{h}} \left(\frac{h^2-y^2}{h^2+y^2}\right)^{\to -1} = -y$$

$$\partial_y \partial_x f(0,0) = -1$$

TEOREMA SCHWARZ

$f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$ $\qquad x^0 \in \mathring{A}$ $\qquad \partial_{x_i} f \quad \partial_{x_j} f \qquad i \neq j$

ESISTONO IN $B(x^0, r)$ $\quad r > 0$

$\partial_{x_i} \partial_{x_j} f \quad \partial_{x_j} \partial_{x_i} f$ sono continue in x^0

$$\Rightarrow \quad \partial_{x_i} \partial_{x_j} f(x^0) = \partial_{x_j} \partial_{x_i} f(x^0)$$

$Hf(x^0) =$ HESSIANO DI f in x^0 $\qquad \mathcal{M}(n \times n, \mathbb{R})$

$$(Hf(x^0))_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(x^0) \qquad i,j = 1, \dots, n$$

$f \in C^2 \quad \Rightarrow \quad H$ è SIMMETRICA

$$f(x) = \sum_{i=1}^{n} a_i x_i + b \quad \in C^\infty(\mathbb{R}^n) \qquad a = (a_1, \dots, a_n) \in \mathbb{R}^n$$

$$\frac{\partial f}{\partial x_j}(x) = \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{n} a_i x_i + b\right) = \sum_{i=1}^{n} \frac{\partial}{\partial x_j}\left((a_i x_i) + b\right)$$

$$= \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_j} x_i$$

$$\frac{\partial}{\partial x_j} x_i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = \delta_{ij} \quad \delta \text{ di KRONECKER}$$

$$(I)_{ij} = \delta_{ij} \to \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$\frac{\partial f}{\partial x_j} = \sum_{i=1}^{n} a_i \delta_{ij} = a_j \qquad \nabla f(x) = (a_1, \dots, a_n)$$

$$b = 0 \qquad f(x) = \sum_i a_i x_i = (a, x) \qquad \nabla f = a = df$$

LINEARE

$$f(x) = x^T A x + bx + c \qquad A = A_{ij} \text{ simmetrica}$$

$$x \in \mathbb{R}^n \quad b \in \mathbb{R}^n \quad c \in \mathbb{R}$$

$$= \sum_{p,q=1}^{n} A_{pq} x_p x_q + \sum_{p=1}^{n} b_p x_p + c \qquad \in C^\infty(\mathbb{R}^n)$$

$$\frac{\partial f}{\partial x_i}(x) = \frac{\partial}{\partial x_i} \sum A_{pq} x_p x_q + \frac{\partial}{\partial x_i} \sum b_p x_p$$

$$= \sum A_{pq} \delta_{pi} x_q + \sum A_{pq} x_p \delta_{qi} + \sum b_p \delta_{ip}$$

$$= \sum A_{iq} x_q + \sum A_{pi} x_p + b_i$$

$$= \sum A_{ip} x_p + \sum A_{pi} x_p + b_i$$

$$= 2 \sum A_{ip} x_p + b_i = 2(Ax)_i + b_i$$

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(x) = \frac{\partial}{\partial x_j}\left(2 \sum A_{ip} x_p\right) = 2 \sum A_{ip} \frac{\partial}{\partial x_j} x_p = 2 \sum A_{ip} \delta_{pj}$$

$$= 2 A_{ij}$$

$$Hf = 2A$$

$$f : \mathbb{R}^n \to \mathbb{R}^m \qquad \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} = \begin{pmatrix} f_1(x_1 \dots x_n) \\ \vdots \\ f_m(x_1 \dots x_n) \end{pmatrix}$$

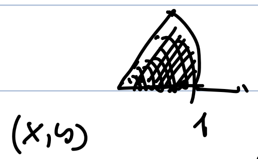
$$\frac{\partial f}{\partial x_i} = \begin{pmatrix} \partial_i f_1(x) \\ \vdots \\ \partial_i f_m(x) \end{pmatrix}$$

$$\nabla f = \begin{pmatrix} \partial_1 f_1 & \dots & , \partial_n f_1 \\ \vdots & & \end{pmatrix} = J$$

matrice di JACOBI

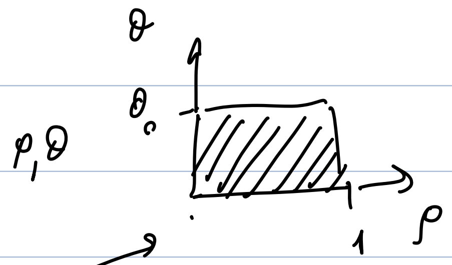
matrice JACOBIANA

$(\partial_1 f_m \quad \dots \quad \partial_n f_m)$



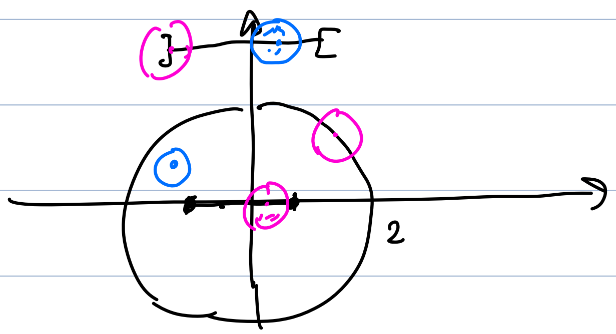
$0 < \rho < 1$

$0 < \theta < \theta_0$



$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$B = \left(B(0,2) \setminus [-1,1] \times \{0\}\right) \cup \left(]-1,1[\times \{3\}\right) \subseteq \mathbb{R}^2$$



NO CONNESSO

NO CONVESSO

LIMITATO $B \subset B(0,10)$

$$\mathring{B} = \{x \in B : \exists\, r > 0 \;\; B(x,r) \subset B\} = \left(B(0,2) \setminus \left([-1,1] \times \{0\}\right)\right)$$

$$\overline{B} = \overline{B(0,2)} \cup \left([-1,1] \times \{3\}\right)$$

$$\partial B = \partial B(0,2) \cup \left([-1,1] \times \{3\}\right) \cup \left([-1,1] \times \{0\}\right)$$

$x \in \mathbb{R}^2 \quad x \in \partial B$

$\forall \; B(x,r) \cap B \neq \emptyset$

$B(x,r) \cap (\mathbb{R}^2 \setminus B) \neq \emptyset$