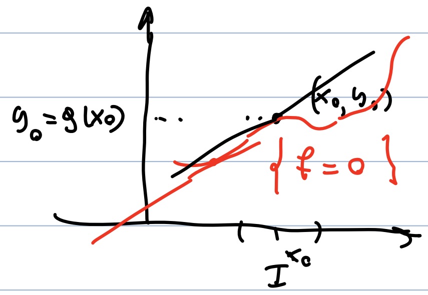
$A \subseteq \mathbb{R}^2$

$f \in C^1(A) \quad (x_0, y_0) \in A \qquad f(x_0,y_0)=0 \qquad \frac{\partial f}{\partial y}(x_0,y_0) \neq 0$

$\Rightarrow \quad \exists \; I \ni x_0 \qquad g \in C^1(I) \qquad f(x, g(x)) = 0 \qquad \forall x \in I$



$$g'(x) = -\frac{\partial_x f(x, g(x))}{\partial_y f(x, g(x))} \qquad g'(x_0) = -\frac{\partial_x f(x_0, g(x_0))}{\partial_y f(x_0, g(x_0))}$$

$f(x_0,y_0)=0 \qquad \frac{\partial f}{\partial x}(x_0,y_0) \neq 0$

$h \in C^1 \qquad f(h(y), y) = 0$

$$h'(y) = -\frac{\partial_y f(h(y), y)}{\partial_x f(h(y), y)}$$

$y - y_0 = m(x - x_0)$

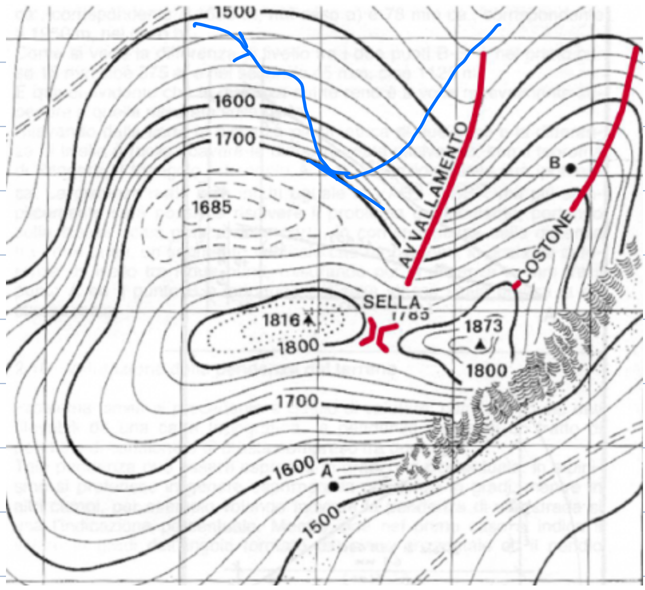
$y - y_0 = g'(x_0)(x - x_0)$

$$y - y_0 = -\frac{\partial_x f(x_0,y_0)}{\partial_y f(x_0,y_0)}(x - x_0) \iff \boxed{\partial_y f(x_0,y_0)(y-y_0) + \partial_x f(x_0,y_0)(x-x_0) = 0}$$

$\nabla f(x_0, y_0) \perp (x - x_0, y - y_0)$

$\langle \nabla f(x_0,y_0), (x-x_0, y-y_0) \rangle = 0$

f : $\nabla f(x,y) \neq (0,0)$ retta tg $f(x,y) = c$



$z = f(x,y)$ altezza sul livello del mare

$G(x,y) = 0$

$(\bar{x},\bar{y})$

$$f(\bar{x},\bar{y}) = \max_{G(x,y)=0} f(x,y)$$

massimo vincolato

$$\nabla f(\bar{x},\bar{y}) \parallel \nabla G(\bar{x},\bar{y})$$

METODO DEI MOLTIPLICATORI DI LAGRANGE

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda G(x,y)$$

i massimi e minimi di f ristretti a $\{G(x,y)=0\}$

si trovano (sotto ipotesi) $\nabla \mathcal{L} = 0$

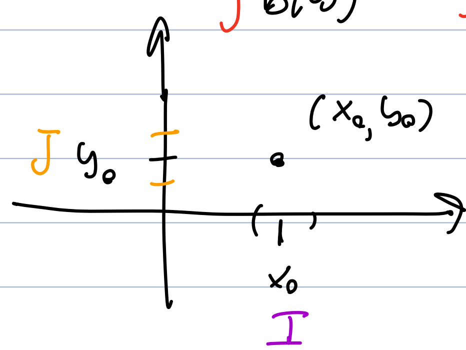
$$\begin{cases} \partial_x \mathcal{L} = 0 \\ \partial_y \mathcal{L} = 0 \\ \partial_\lambda \mathcal{L} = 0 \end{cases} \qquad \begin{cases} f_x - \lambda G_x = 0 \\ f_y - \lambda G_y = 0 \\ G = 0 \end{cases}$$

$$\begin{cases} y'(x) = a(x)\, b(y) \\ y(x_0) = y_0 \end{cases} \qquad \int \frac{dy}{b(y)} = \int dx\, a(x)$$

a, b continue

$a \in C(I) \qquad x_0 \in I$

$b \in C(J) \qquad y_0 \in J$



$b(y_0) \neq 0 \quad \Rightarrow \quad b(y) \neq 0$ in J per un segno

$A'(x) = a(x)$ $\quad$ A è primitiva di a

$B'(y) = -\dfrac{1}{b(y)}$ $\quad$ $\dfrac{1}{b} \in C(J)$ $\quad$ $b \neq 0$ in J

(1) $F(x,y) = A(x) - A(x_0) + B(y) - B(y_0) \quad \in C^1$

(2) $F(x_0,y_0) = A(x_0) - A(x_0) + B(y_0) - B(y_0) = 0$

(3) $\dfrac{\partial F}{\partial y}(x_0,y_0) = B'(y_0) = -\dfrac{1}{b(y_0)} \neq 0$

$$\Rightarrow \quad \exists\ y \in C^1 : \quad y'(x) = -\frac{\partial_x F(x,y(x))}{\partial_y F(x,y(x))}$$

TEO F. IMPLICITA

$$\begin{cases} y'(x) = -\dfrac{a(x)}{-\dfrac{1}{b(y)}} = a(x)\,b(y(x)) \\ y(x_0) = y_0 \end{cases}$$

$f(x,y) = e^{xy} + x - y - 1$ $\qquad$ $e^{xy} + x - y - 1 = 0$

$(x_0,y_0) = (0,0)$ $\qquad$ $y(x) = ?$

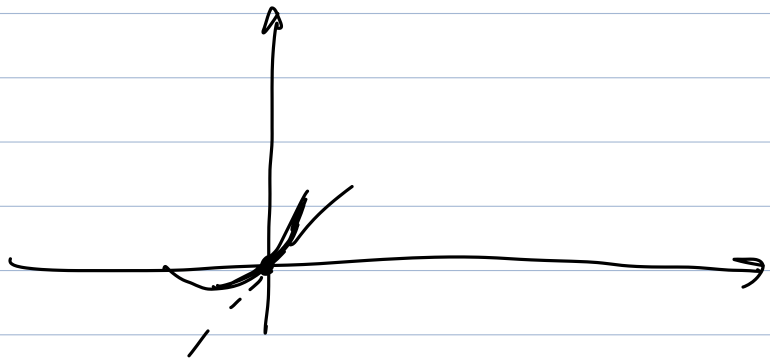
$\partial_x f(x,y) = e^{xy}\, y + 1$ $\qquad$ $\partial_x f(0,0) = 1 \neq 0$

$\partial_y f(x,y) = e^{xy}\, x - 1$ $\qquad$ $\partial_y f(0,0) = -1 \neq 0$

$$g'(x) = -\frac{\partial_x f(x,g(x))}{\partial_y f(x,g(x))} \qquad x \in I \qquad I =]-\delta,\delta[$$

$g \in C^1(I)$

$$g'(0) = -\frac{\partial_x f(0,g(0))}{\partial_y f(0,g(0))} = -\frac{\partial_x f(0,0)}{\partial_y f(0,0)} = -\frac{1}{-1} = 1$$



$$g''(x) = \frac{d}{dx} g'(x) = \frac{d}{dx}\left(-\frac{\partial_x f(x,g(x))}{\partial_y f(x,g(x))}\right) = -$$

$$= -\frac{\left(\partial_{xx} f(x,g(x)) + \partial_{xy} f(x,g(x))\, g'(x)\right) \partial_y f(x,g(x)) - f_x(x,g(x)) \cdot \left(\partial_{yx} f(x,g(x)) + \partial_{yy} f(x,g(x))\, g'(x)\right)}{\left[\partial_y f(x,g(x))\right]^2}$$

$$g''(x_0) = -\frac{\left(f_{xx}(x_0,y_0) + f_{xy}(x_0,y_0)\, g'(0)\right) f_y(0,0) - f_x(0,0)\left(f_{xy}(0,0) + f_{yy}(0,0)\, g'(0)\right)}{\left[f_y(0,0)\right]^2}$$

$$g''(0) = 2$$

$$g(x) = g(0) + g'(0)x + g''(0)\frac{x^2}{2} + \cdots$$

$$= 0 + x + x^2 + \cdots$$

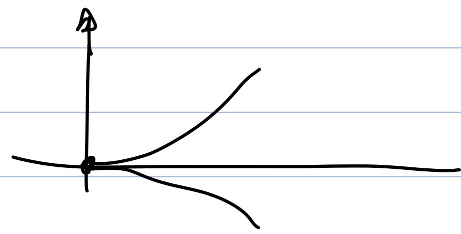
$f(x,y) = y^2 + \lambda x^2 - x^3$ $\qquad \lambda \geq 0$ $\qquad y^2 = x^3 - \lambda x^2$

$y^2 = x^2(x - \lambda)$

$\{f(x,y)=0\}$ al variare di $\lambda \geq 0$ $\qquad y = \pm x\sqrt{x-\lambda}$

$\lambda = 0$

$f(x,y) = y^2 - x^3$



$f(0,0) = 0 \quad \forall \lambda$

$\nabla f(0,0) = (2\lambda x - 3x^2, 2y)\big|_{(x,y)=(0,0)} = (0,0)$