f semplice $\mathbb{R}^2 \to \mathbb{R}$ $\qquad f(x) = \sum_{i=1}^{K} c_i \chi_{(Q_i)}$

$Q_i = [a_i, b_i[\times [c_i, d_i[$

f semplice $\mathbb{R}^n \to \mathbb{R}$ $\qquad f(x) = \sum_{i=1} c_i \chi_{(Q_i)}$

$Q_i = [a_1^i, b_1^i[\times [a_2^i, b_2^i[\times \dots \times [a_n^i, b_n^i[$ $\qquad$ ("rettangolo" e n-dimensioni)

$m(Q_i) = \prod_{k=1}^{n} (b_k^i - a_k^i)$

$$\int_{\mathbb{R}^n} f(x)\,dx = \sum_{i=1}^{k} c_i\, m(Q_i)$$

$f: \mathbb{R}^n \to \mathbb{R}$ limitata e (non nulla solo in un insieme limitato) $\quad \operatorname{supp} f \subseteq B(0,R)$

$$\sup_{f_1 \in S_1} \int_{\mathbb{R}^n} f_1\,dx = \inf_{f_2 \in S_2} \int f_2\,dx \qquad f_1 \in S_1 = \{f \text{ semplici } f_1 \le f\} \quad f_2 \in S_2 = \{f \text{ semplici } f_2 \ge f\}$$

$$\int_{\mathbb{R}^n} f(x)\,dx \overset{\text{def}}{=} \underbrace{\int_{\mathbb{R}} \dots \int_{\mathbb{R}}}_{n \text{ volte}} f(x_1, x_2, \dots, x_n)\,dx_1\,dx_2 \dots dx_n$$

$$\int_A f(x)\,dx \overset{\text{def}}{=} \int_{\mathbb{R}^n} f(x)\,\chi_A(x)\,dx \qquad A \text{ misurabile}$$

$A = A_1 \cup A_2 \qquad A_1 \cap A_2 = \emptyset$

$$\int_A f\,dx = \int_{A_1} f\,dx + \int_{A_2} f\,dx$$

DEF.

A misurabile $\quad \forall \varepsilon > 0 \quad \exists\, R_1, R_2 \quad R_1 \subseteq A \subseteq R_2$

R_1, R_2 PLURIRETTANGOLI $\qquad m(R_2 \setminus R_1) < \varepsilon$

(UNIONE FINITA DI "RETTANGOLI" APERTI A DESTRA NON INTERSECANTESI)

$f, g \in \mathcal{R}(\mathbb{R}^n)$
$\alpha \in \mathbb{R}$

$$\int_{\mathbb{R}^n} f + g \, dx = \int_{\mathbb{R}^n} f \, dx + \int_{\mathbb{R}^n} g \, dx$$

$$\int_{\mathbb{R}^n} \alpha f \, dx = \alpha \int_{\mathbb{R}^n} f \, dx$$

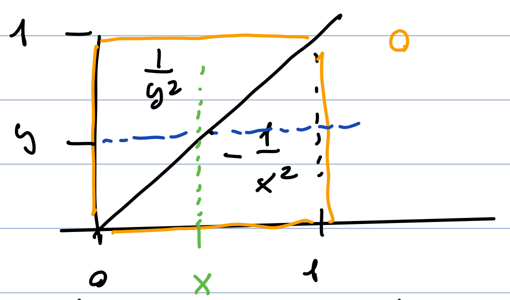
$$\left| \int_{\mathbb{R}^n} f(x) dx \right| \leq \int_{\mathbb{R}^n} |f(x)| dx \qquad f \geq 0 \Rightarrow \int_{\mathbb{R}^n} f \geq 0$$

FUBINI $\int_{\mathbb{R}^2} f(x,y) \, dx dy$ esiste $f \in \mathcal{R}(\mathbb{R}^2)$

$x \longrightarrow \int_{\mathbb{R}} f(x,y) dy$ é integrabile in dx

$$\int_{\mathbb{R}} dx \left(\int_{\mathbb{R}} f(x,y) dy \right) = \int_{\mathbb{R}^2} f(x,y) \, dx dy$$

$f \in \mathcal{R}(\mathbb{R}^n)$ $\quad \int_{\mathbb{R}^n} f \, dx = \int dx_1 \Big(\qquad \Big(\int \Big(\int f(x_1, \dots x_n) dx_n \Big) dx_{n-1} \Big)$



$$f = \begin{cases} 0 & x = 0,1 \text{ oppure } y = 0,1 \\ -\frac{1}{x^2} & 0 < y < x \\ +\frac{1}{y^2} & x < y < 1 \end{cases}$$

$$\int_0^1 f(x,y) dx = \int_0^y \frac{1}{y^2} dx + \int_y^1 -\frac{1}{x^2} dx = \frac{1}{y^2} \int_0^y 1 \, dx \quad \int_y^1 -\frac{1}{x^2} dx$$

$$= \frac{1}{y^2} (y - 0) + \frac{1}{x}\Big|_y^1 = 1$$

$$\int_0^1 \left(\int_0^1 f(x,y) dx \right) dy = 1$$

$$f \in \mathcal{R}([0,1] \times [0,1])$$

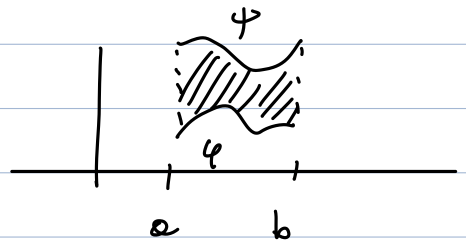
$$\int_0^1 f(x,y) \, dy = -1 \quad \neq$$

$$\int_0 dx \left(\int_0 f(x,y)\,dy\right) = -1$$

INSIEME NORMALE RISPETTO A Y DEL PIANO

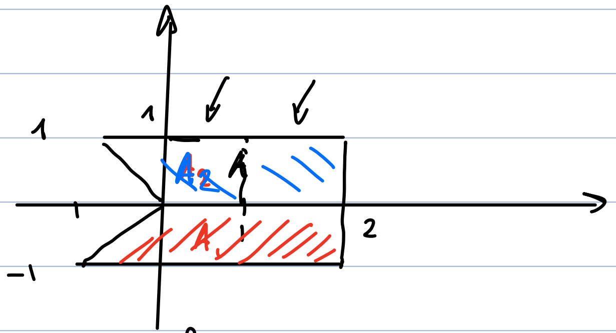
$$A = \{(x,y)\in\mathbb{R}^2 : \quad a\le x\le b, \quad \varphi(x)\le y\le\psi(x)\}$$

$\varphi(x)\le\psi(x)$ $\quad \varphi,\psi\in C[a,b]$



$$\int_A f(x,y)\,dx\,dy = \int_a^b dx \int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy$$

$f\in C(A)$



$$\int_A x\,dx\,dy$$

$A = \{(x,y)\in A \quad -1\le x\le 2 ,$

$A = A_1\cup A_2$

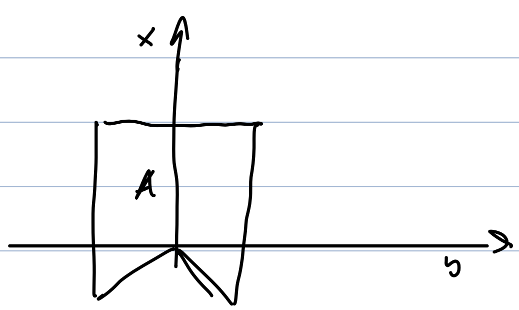
$A_1 = \{(x,y)\in\mathbb{R}^2 : \quad -1\le x\le 2 \quad -1\le y\le\varphi(x)\}$

$\varphi(x) = \begin{cases} x & -1\le x\le 0 \\ 1 & 0\le x\le 2\end{cases}$

$$\int_A f(x,y)\,dx\,dy = \int_{A_1} f + \int_{A_2} f$$

$A_2 = \{ \quad -1\le x\le 2 \quad \psi(x)\le y\le 2\}$

$\psi(x) = \begin{cases} -x & x\le 0 \\ 0 & x\ge 0\end{cases}$



$A = \{(x,y)\in\mathbb{R}^2 \quad -1\le y\le 1 \quad -|y|\le x\le 2\}$

NORMALE RISPETTO A X

$A = \{(x,y)\in\mathbb{R}^2 \quad a\le y\le b \quad \varphi(y)\le x\le\psi(y)\}$

$$\int_A f\,dx\,dy = \int^b dy \int^{\psi(y)} f(x,y)\,dx$$

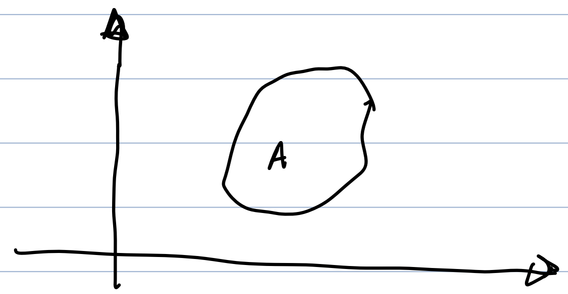
$A \qquad \varphi(y)$

$\parallel$

$$\int_{-1}^{1} dy \int_{-|y|}^{2} x\,dx = \int_{-1}^{1} dy \left(\frac{x^2}{2}\Big|_{-|y|}^{2} \right) = \int_{-1}^{1} dy \;\; \frac{4}{2} - \left(\frac{|y|^2}{2}\right)$$

$$= \int_{-1}^{1} 2 - \frac{y^2}{2}\,dy = 2y - \frac{y^3}{6}\Big|_{-1}^{1} = 4 - \frac{2}{6} = \frac{22}{6} = \frac{11}{3}$$

$$m(A) = \int_A 1\,dx\,dy = 5$$

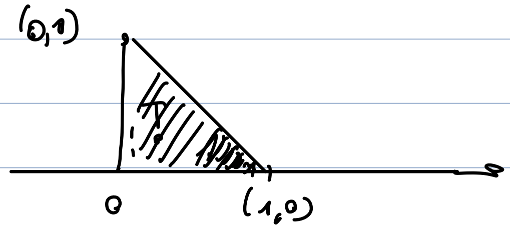


ρ densità $\quad \rho \geq 0$

$$M = \int_A \rho(x)\,dx$$

BARICENTRO

$$\bar{x} = \frac{1}{M}\int_A x\,\rho(x,y)\,dx\,dy \qquad \bar{y} = \frac{1}{M}\int_A y\,\rho(x,y)\,dx\,dy$$



$\rho = 1 \qquad M = \int_T 1\,dx\,dy = \frac{1}{2}$

$$\bar{x} = \frac{1}{\frac{1}{2}}\int_T x\,dx\,dy \qquad T = \left\{ (x,y) \in \mathbb{R} \quad \begin{array}{l} 0 \leq x \leq 1 \\ 0 \leq y \leq 1-x \end{array} \right\}$$

$$= 2\int_0^1 dx \int_0^{1-x} x\,dy = 2\int_0^1 dx\; x \int_0^{1-x} dy$$

$$= 2\int_0^1 dx\; x\;(1-x) = 2\int_0^1 x - x^2 = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 \right) = \frac{1}{3}$$

a) $\boxed{\bar{x} = \bar{y}}$

$\rho = K(1+x)$

b) $\bar{x} \qquad \bar{y}$