Esercizio – Somma di sinusoidi

$$x(t) = A\cos(2\pi f_0 t + \varphi) + B\sin(4\pi f_0 t)$$

·) Calcolare lo spettro X_n

·) Disegnare spettro di ampiezza e di fase

Svolgimento:

$$x(t) = y(t) + z(t)$$

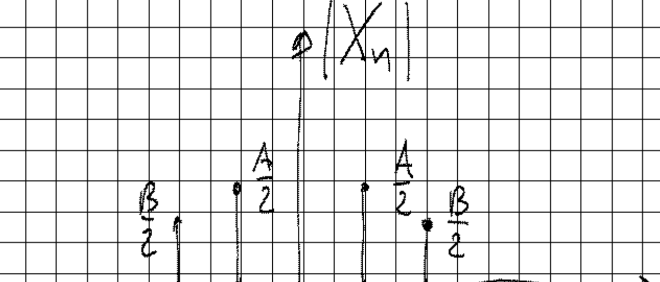
$$y(t) = A\cos(2\pi f_0 t + \varphi), \quad z(t) = B\sin(4\pi f_0 t)$$

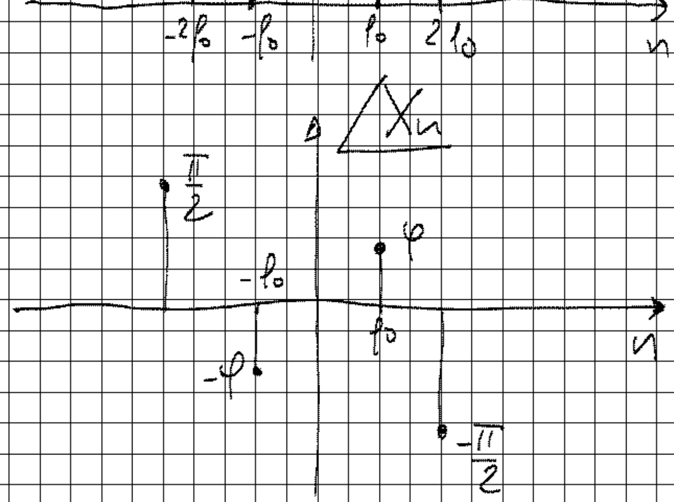
$$X_n = Y_n + Z_n$$

$$Y_n = \begin{cases} \frac{A}{2} e^{j\varphi} & n = 1 \\ \frac{A}{2} e^{-j\varphi} & n = -1 \\ 0 & n \neq \pm 1 \end{cases}$$

$$z(t) = B\cos\left(4\pi f_0 t - \frac{\pi}{2}\right)$$

$$Z_n = \begin{cases} \frac{B}{2} e^{-j\frac{\pi}{2}} & n = +2 \\ \frac{B}{2} e^{j\frac{\pi}{2}} & n = -2 \\ 0 & n \neq \pm 2 \end{cases}$$





ESERCIZIO - PRODOTTO DI SINUSOIDI

$$z(t) = A\cos(2\pi f_0 t + \varphi)\cos(4\pi f_0 t)$$

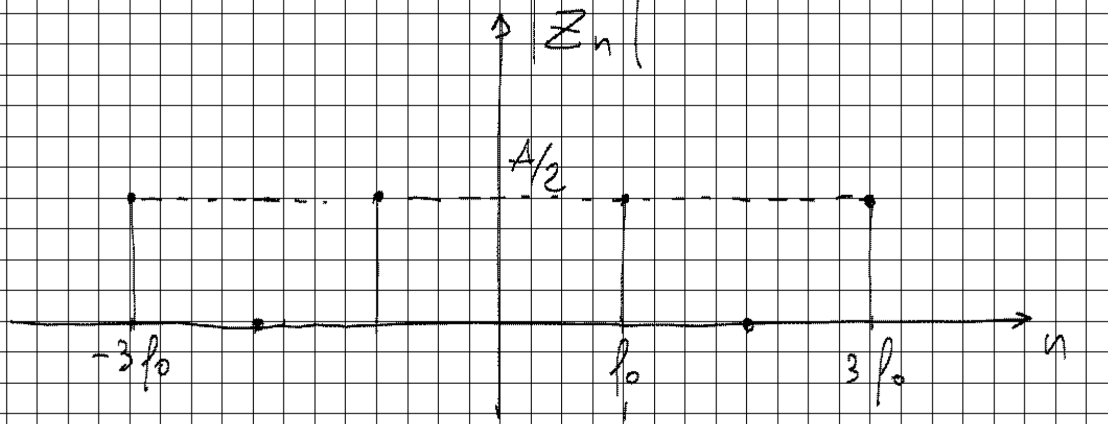
$$z(t) = \frac{A}{2}\left[\cos(6\pi f_0 t + \varphi) + \cos(2\pi f_0 t - \varphi)\right]$$

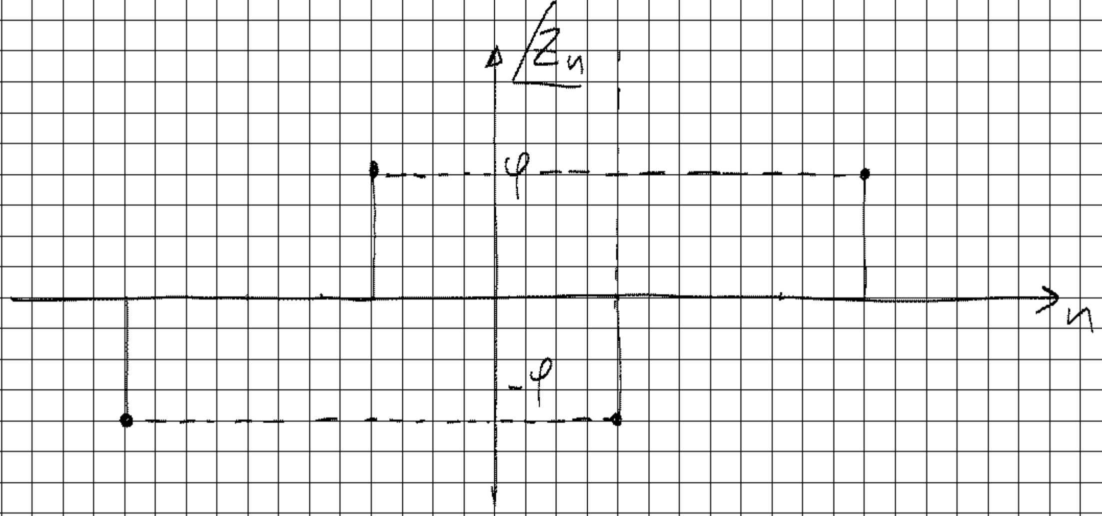
$$Z_n = X_n + Y_n$$

$$X_n \overset{TSF}{\Longleftrightarrow} x(t) = \frac{A}{2}\cos(6\pi f_0 t + \varphi)$$

$$Y_n \overset{TSF}{\Longleftrightarrow} y(t) = \frac{A}{2}\cos(2\pi f_0 t - \varphi)$$

$$X_n = \begin{cases} X_3 = \frac{A}{2}e^{j\varphi} \\ X_{-3} = \frac{A}{2}e^{-j\varphi} \\ 0 & n \neq \pm 3 \end{cases}, \quad Y_n = \begin{cases} Y_1 = \frac{A}{2}e^{-j\varphi} \\ Y_{-1} = \frac{A}{2}e^{j\varphi} \\ 0 & n \neq \pm 1 \end{cases}$$

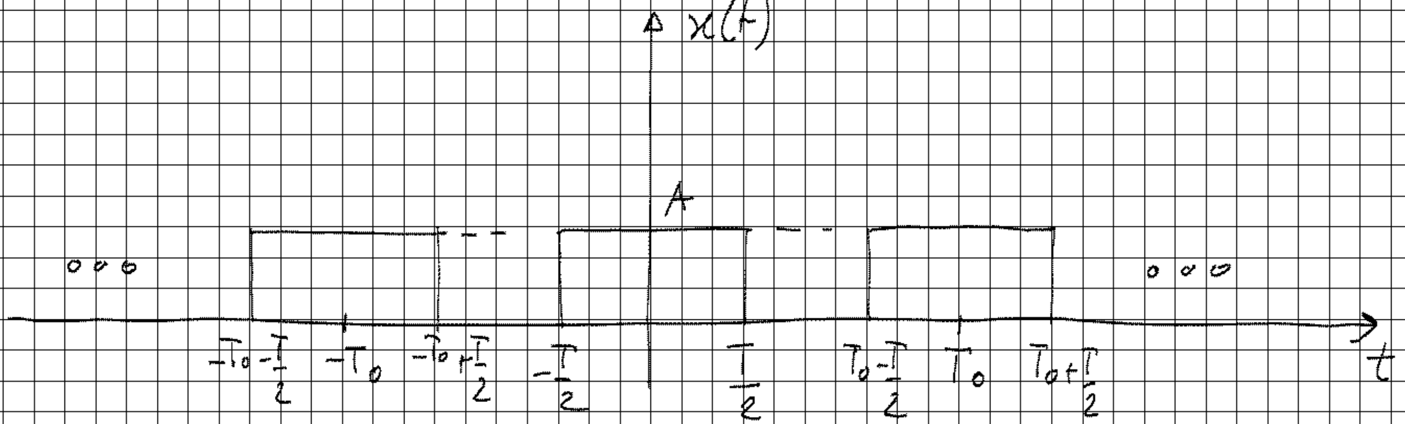




ESERCIZIO – "Treno di impulsi rettangolari"

$$x_0(t) = A\,\mathrm{rect}\left(\frac{t}{T}\right)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} x_0(t - nT_0) = A \sum_{k=-\infty}^{+\infty} \mathrm{rect}\left(\frac{t - kT_0}{T}\right)$$



Calcolare $X_n \overset{TSF}{\Longleftrightarrow} x(t)$

$$X_n \triangleq \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\,dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A\,\mathrm{rect}\left(\frac{t}{T}\right) e^{-j2\pi n f_0 t}\,dt$$

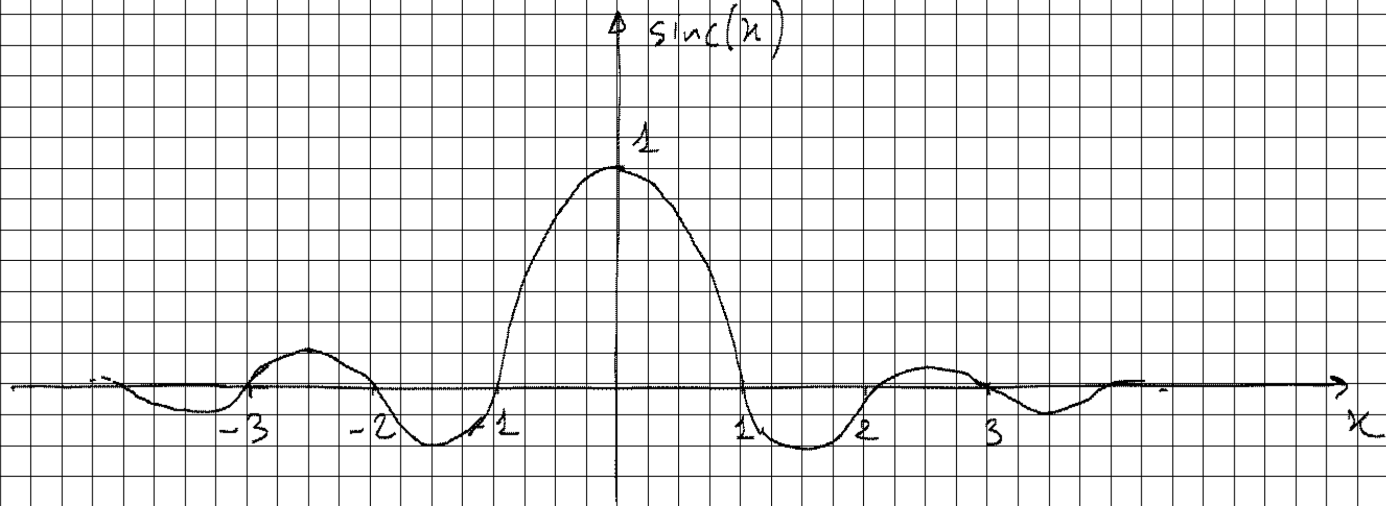
$$= \frac{A}{T_0}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j2\pi n f_0 t}\,dt = \frac{A}{T_0}\,\frac{-1}{j2\pi f_0}\, e^{-j2\pi n f_0 t}\Big|_{-\frac{T}{2}}^{\frac{T}{2}} =$$

$$= -\frac{A}{j2\pi n f_0 T_0}\left[e^{-j2\pi n f_0 \frac{T}{2}} - e^{j2\pi n f_0 \frac{T}{2}}\right] =$$

$$= \frac{A}{\pi n}\,\frac{e^{+j\pi n f_0 T} - e^{-j\pi n f_0 T}}{2j} = \frac{A}{\pi n}\sin(\pi n f_0 T) =$$

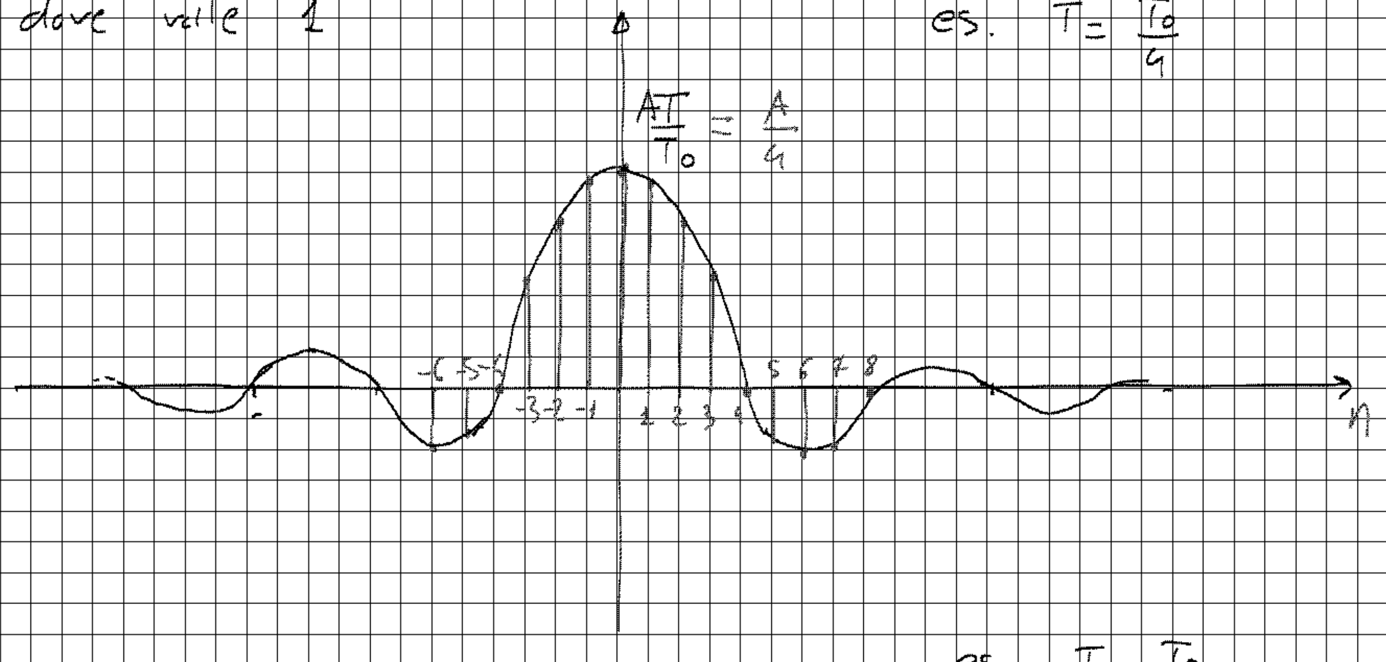
$$= A f_0 T\,\frac{\sin(\pi n f_0 T)}{\pi n f_0 T} = A f_0 T\,\mathrm{sinc}(n f_0 T) = \frac{AT}{T_0}\,\mathrm{sinc}\left(n\frac{T}{T_0}\right)$$

Funzione $\mathrm{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$

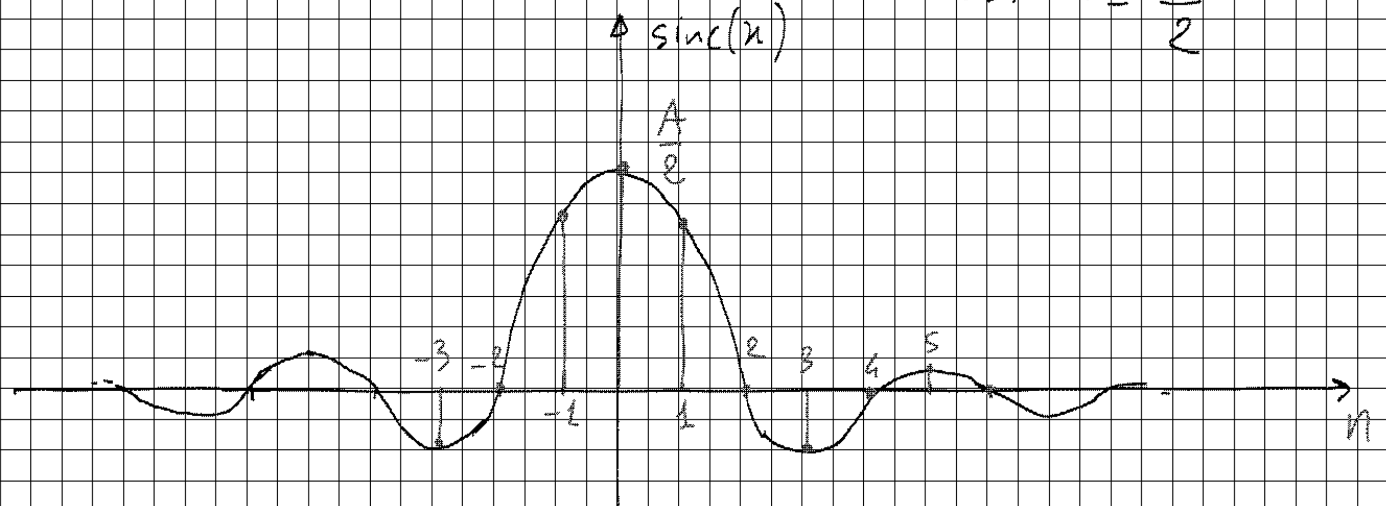


⇒ Si annulla per valori interi dell'argomento, eccetto $x=0$ dove vale 1

es. $T = \dfrac{T_0}{4}$

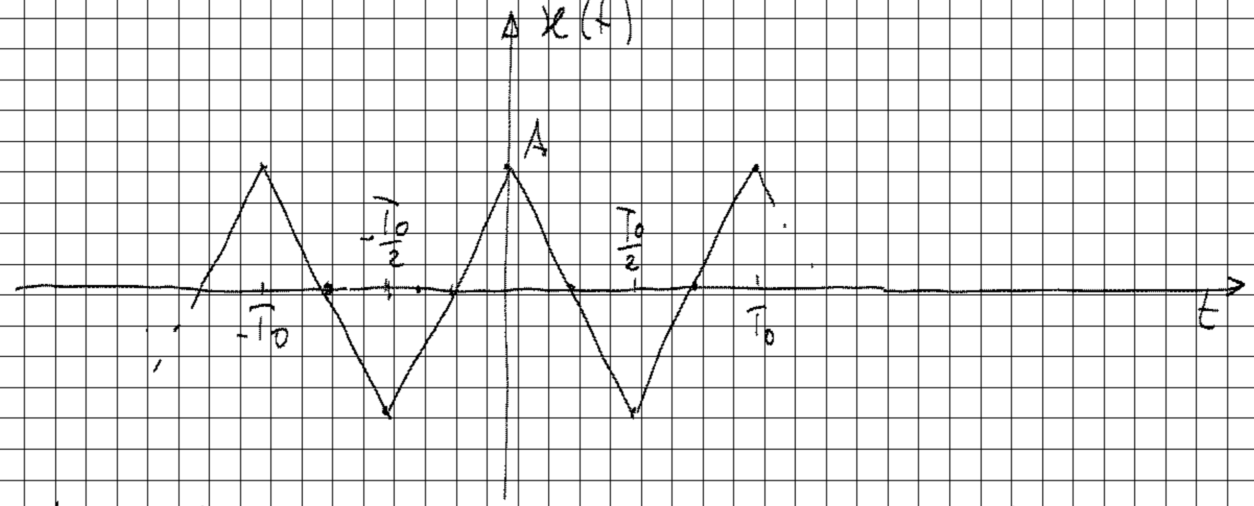


es. $T = \dfrac{T_0}{2}$



⇒ Il treno di rettangoli è reale e pari x cui la sua TSF è reale e pari.

SPETTRO DI UN ONDA TRIANGOLARE



Calcolare lo spettro X_n e disegnare gli spettri di ampiezza e di fase per le componenti $n=0, \pm 1, \pm 2$

•) Segnale reale, pari, alternativo

$$X_{-n} = X_n, \quad \mathrm{Im}\{X_n\} = 0, \quad X_n = 0 \quad n \text{ pari}$$

$$X_n = \frac{2}{T_0}\int_0^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt = \frac{2}{T_0}\int_0^{\frac{T_0}{2}} \left(A - \frac{4At}{T_0}\right) e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{2A}{T_0}\int_0^{\frac{T_0}{2}} e^{-j2\pi n f_0 t}\, dt - \frac{8A}{T_0^2}\int_0^{\frac{T_0}{2}} t\, e^{-j2\pi n f_0 t}\, dt =$$

$$= \frac{2A}{T_0}\left(-\frac{1}{j2\pi n f_0}\right) e^{-j2\pi n f_0 t}\Big|_0^{\frac{T_0}{2}} +$$

$$- \frac{8A}{T_0^2}\left[\frac{t}{-j2\pi n f_0} e^{-j2\pi n f_0 t}\Big|_0^{\frac{T_0}{2}} - \int_0^{\frac{T_0}{2}} \frac{1}{-j2\pi n f_0} e^{-j2\pi n f_0 t}\, dt\right] =$$

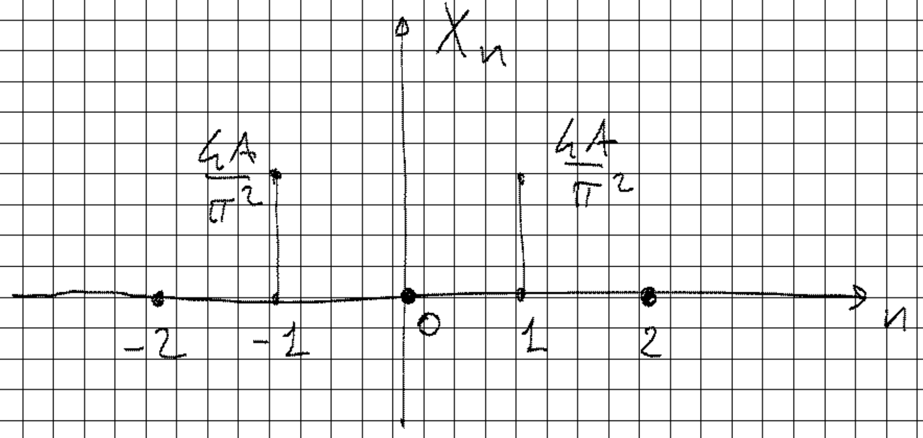
$$= -\frac{2A}{j2\pi n}\left[e^{-j\pi n} - 1\right] + \frac{8A}{T_0^2}\frac{T_0}{j4\pi n f_0}\left(e^{-j\pi n} - 0\right) +$$

$$- \frac{8A}{T_0^2}\cdot\frac{1}{4\pi^2 n^2 f_0^2}\, e^{-j2\pi n f_0 t}\Big|_0^{\frac{T_0}{2}} =$$

$$= \frac{2A}{j\pi n} - \frac{2A}{j\pi n} - \frac{2A}{\pi^2 n^2}\left(e^{-j\pi n} - 1\right) = \frac{4A}{\pi^2 n^2}$$

$X_0 = 0$ $\qquad$ $X_{\pm 1} = \frac{4A}{\pi^2}$

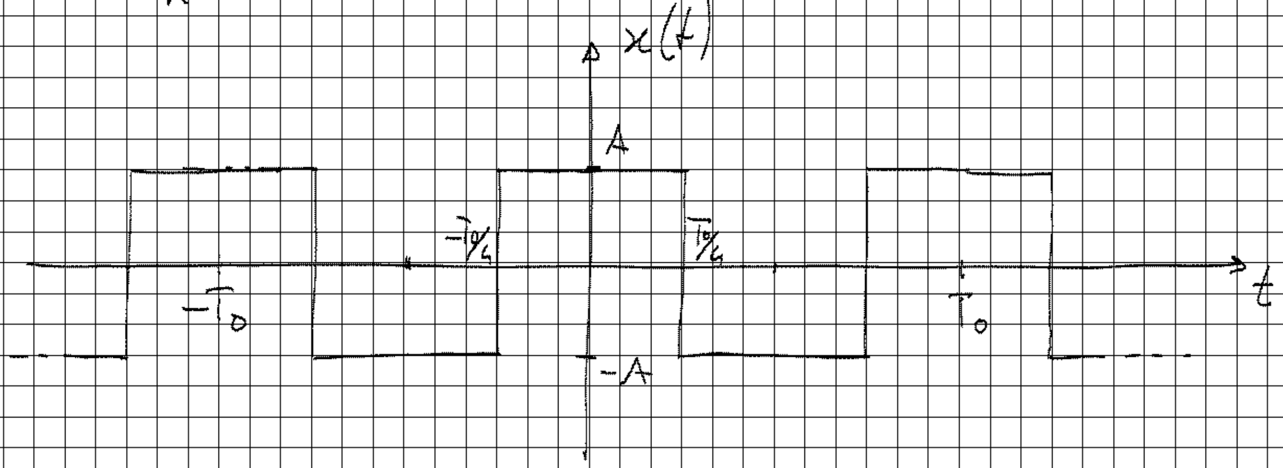
$X_{\pm 2} = 0$



ESERCIZIO – "Treno di impulsi rettangolari" (VARIANTE)

$$x_0(t) = 2A\,\mathrm{rect}\left(\frac{t}{T_0/2}\right)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} x_0(t-kT_0) - A$$



$x(t)$ è reale e pari $\Rightarrow$ X_n è reale e pari

$x(t)$ è un segnale alternativo $\Rightarrow$ $X_n = 0$ n pari

$$X_n = \frac{2}{T_0}\int_0^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt \qquad n \text{ dispari}$$

$$= \frac{2A}{T_0}\int_0^{\frac{T_0}{4}} e^{-j2\pi n f_0 t}\, dt - \frac{2A}{T_0}\int_{\frac{T_0}{4}}^{\frac{T_0}{2}} e^{-j2\pi n f_0 t}\, dt =$$

$$= \frac{2A}{T_0}\,\frac{1}{-j2\pi n f_0}\, e^{-j2\pi n f_0 t}\Big|_0^{\frac{T_0}{4}} - \frac{2A}{T_0}\,\frac{1}{-j2\pi n f_0}\, e^{-j2\pi n f_0 t}\Big|_{\frac{T_0}{4}}^{\frac{T_0}{2}} =$$

$$= -\frac{2A}{j2\pi n}\left[e^{-j2\pi n f_0 \frac{T_0}{4}} - 1\right] + \frac{2A}{j2\pi n}\left[e^{-j2\pi n f_0 \frac{T_0}{2}} - e^{-j2\pi n f_0 \frac{T_0}{4}}\right] =$$

$$= -\frac{2A}{j2\pi n}\left[e^{-j\frac{\pi n}{2}} - 1\right] + \frac{2A}{j2\pi n}\left[e^{-j\pi n} - e^{-j\frac{\pi n}{2}}\right] =$$

$\Rightarrow$ (n dispari)

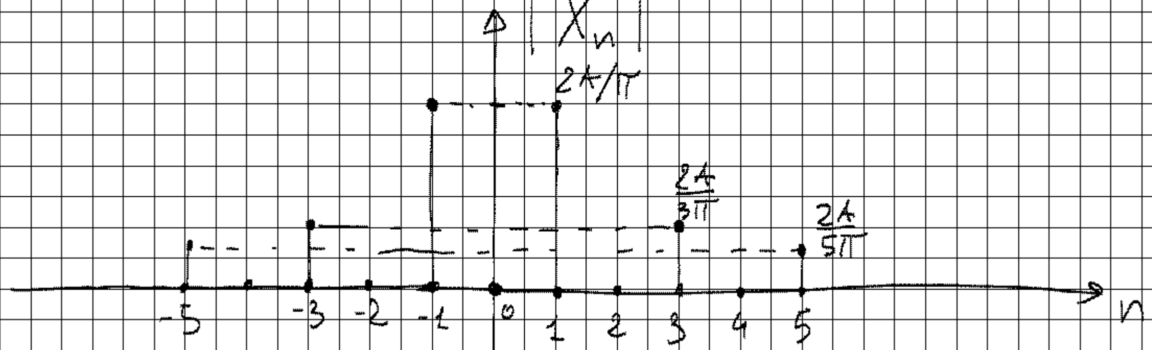
$$= -\frac{2A}{j2\pi n}\left[(-j)^n - 1\right] + \frac{2A}{j2\pi n}\left[-1-(-j)^n\right] =$$

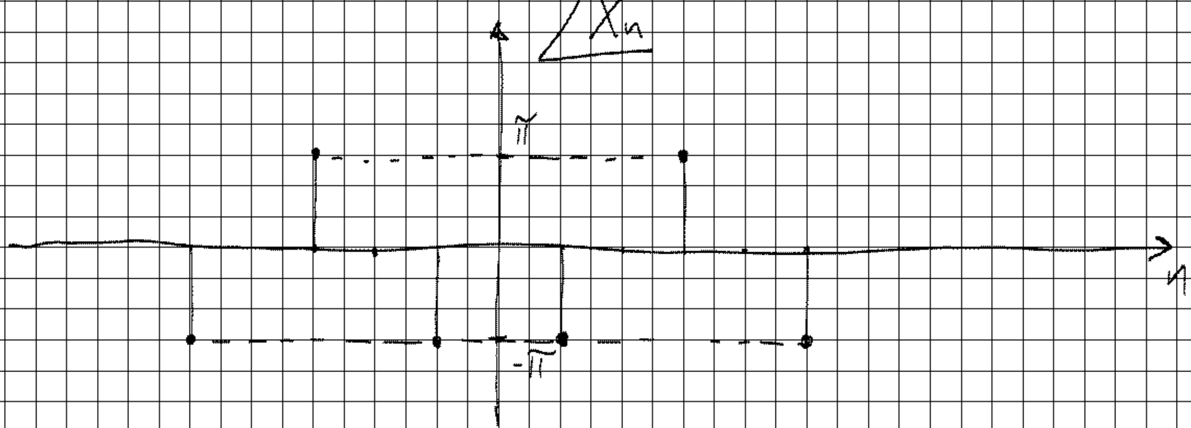
$$= +\frac{2A}{j2\pi n} - \frac{2A}{j2\pi n} - \frac{2A}{2\pi n}(-1)^{\frac{n-1}{2}} - \frac{2A}{2\pi n}(-1)^{\frac{n-1}{2}} =$$

$$= -\frac{2A}{\pi n}(-1)^{\frac{n-1}{2}}$$

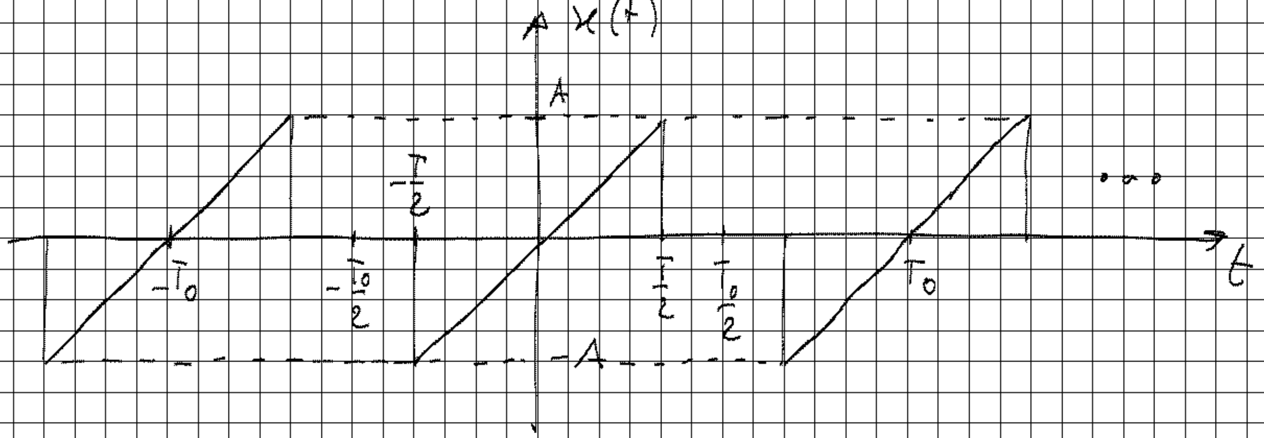
$X_1 = -\frac{2A}{\pi}$ $\qquad X_3 = \frac{2A}{3\pi}$ $\qquad X_5 = -\frac{2A}{5\pi}$

$X_{-1} = -\frac{2A}{\pi}$ $\qquad X_{-3} = \frac{2A}{3\pi}$ $\qquad X_{-5} = -\frac{2A}{5\pi}$





ESERCIZIO - SPETTRO DI UNA RAMPA PERIODICA



Calcolare X_n.

Svolgimento:

.) Determinare l'espressione analitica di $x(t)$ in un periodo

$$x_0(t) = \frac{2A}{T} t \; \mathrm{rect}\left(\frac{t}{T}\right)$$

$$X_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \frac{2A}{T} t \; \mathrm{rect}\left(\frac{t}{T}\right) e^{-j2\pi n f_0 t} \, dt =$$

$$= \frac{2A}{T T_0}\int_{-\frac{T}{2}}^{\frac{T}{2}} t \, e^{-j2\pi n f_0 t} \, dt =$$

$$= \frac{2A}{T T_0}\left[\frac{-t}{j2\pi n f_0} e^{-j2\pi n f_0 t}\Bigg|_{-\frac{T}{2}}^{\frac{T}{2}} - \int_{-\frac{T}{2}}^{\frac{T}{2}} -\frac{1}{j2\pi n f_0} e^{-j2\pi n f_0 t} \, dt\right]$$

$$= \frac{2A}{T T_0}\left[-\frac{T}{j4\pi n f_0} e^{-j\pi n f_0 T} - \frac{T}{j4\pi n f_0} e^{j\pi n f_0 T} - \left(\frac{1}{j2\pi n f_0}\right)^2 e^{-j2\pi n f_0 t}\Bigg|_{-\frac{T}{2}}^{\frac{T}{2}}\right]$$

$$= \frac{2A}{T T_0}\left[\frac{jT}{2\pi n f_0}\cos(\pi n f_0 T) + \frac{1}{4\pi^2 n^2 f_0^2}\left(e^{-j\pi n f_0 T} - e^{+j\pi n f_0 T}\right)\right]$$

$$= \frac{2A}{T T_0}\left[\frac{jT}{2\pi n f_0}\cos(\pi n f_0 T) - \frac{j}{2\pi^2 n^2 f_0^2}\sin(\pi n f_0 T)\right]$$

$$= \frac{2jA}{T T_0}\left[\frac{T}{2\pi n f_0}\cos(\pi n f_0 T) - \frac{T}{2\pi n f_0}\,\mathrm{sinc}(n f_0 T)\right]$$

$$= \frac{jA}{\pi n}\left[\cos(\pi n f_0 T) - \mathrm{sinc}(n f_0 T)\right]$$