RELAZIONE TRA TCF e TSF

PERIODOCIZZAZIONE

$$y(t) = \sum_{n=-\infty}^{+\infty} x(t - nT_0) \qquad T_0 = \text{periodo di } y(t)$$

$$y(t) = y(t - kT_0) \qquad \text{periodico}$$

Dimostrazione

$$y(t - kT_0) = \sum_{n=-\infty}^{+\infty} x(t - kT_0 - nT_0) = \sum_{n=-\infty}^{+\infty} x[t - (k+n)T_0]$$

$$= (k+n = n') = \sum_{n'=-\infty}^{+\infty} x(t - n'T_0) = y(t)$$

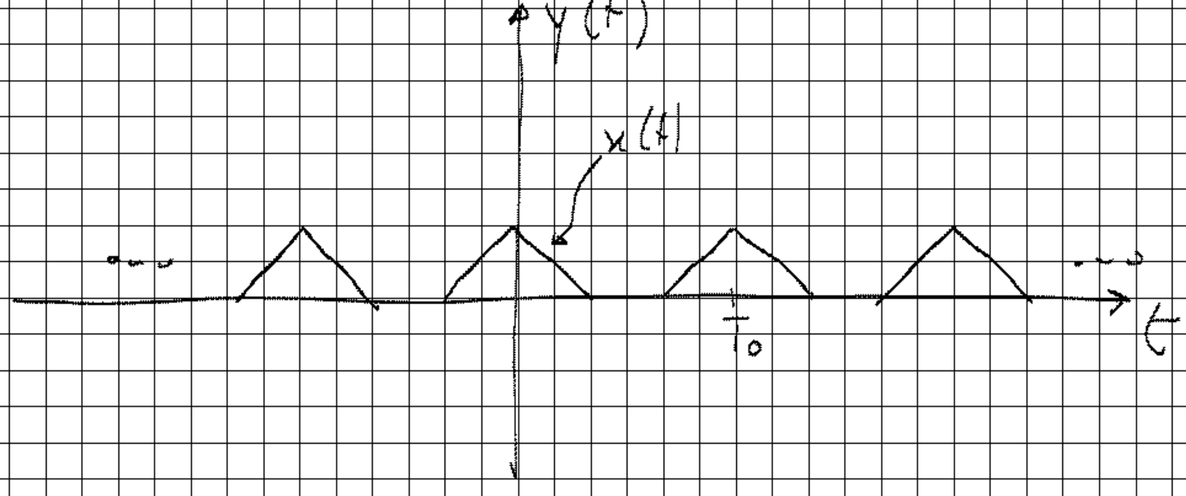
$$\boxed{y(t) \overset{TSF}{\Longleftrightarrow} Y_n = \frac{1}{T_0} X\left(\frac{n}{T_0}\right)}$$

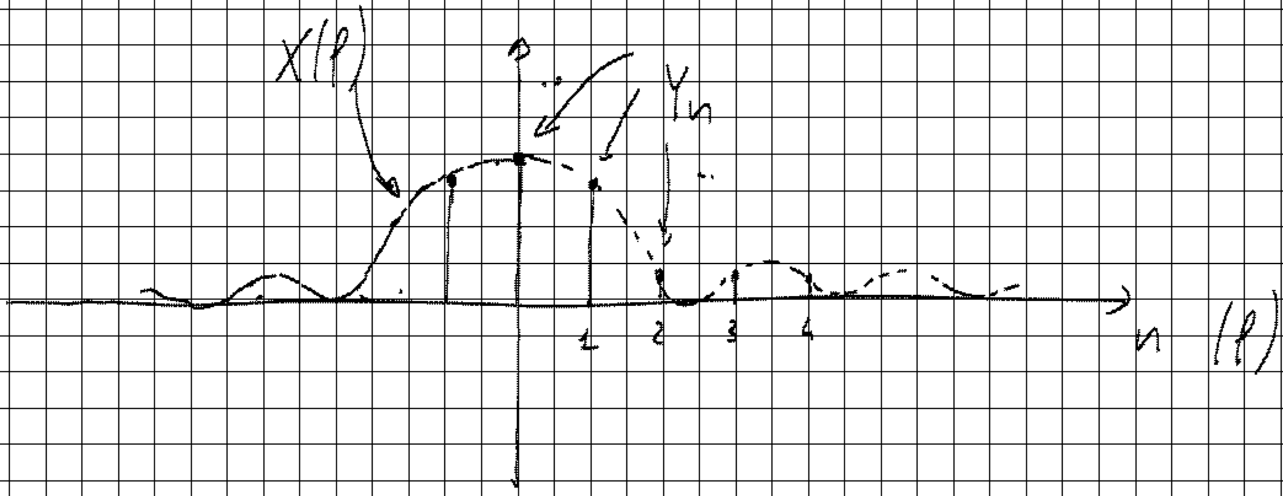
$$Y_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sum_{k=-\infty}^{+\infty} x(t - kT_0)\, e^{-j2\pi n f_0 t}\, dt =$$

$$= \frac{1}{T_0} \sum_{k=-\infty}^{+\infty} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t - kT_0)\, e^{-j2\pi n f_0 t}\, dt = (t - kT_0 = \alpha)$$

$$= \frac{1}{T_0} \sum_{k=-\infty}^{+\infty} \int_{-\frac{T_0}{2} - kT_0}^{\frac{T_0}{2} - kT_0} x(\alpha)\, e^{-j2\pi n f_0 \alpha}\, d\alpha \, \underbrace{e^{-j2\pi n f_0 k T_0}}_{1}$$

$$= \frac{1}{T_0} \int_{-\infty}^{+\infty} x(\alpha)\, e^{-j2\pi n f_0 \alpha}\, d\alpha = \frac{1}{T_0} X\left(\frac{n}{T_0}\right)$$





La Y_n è un campionamento della $X(f)$ pesato con un fattore $\frac{1}{T_0}$.

I^a FORMULA DI POISSON

$$\sum_{k=-\infty}^{+\infty} x(t-kT_0) = \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} X\left(\frac{k}{T_0}\right) e^{j2\pi k\frac{t}{T_0}}$$

Dimostrazione

$$y(t) = \sum_{k=-\infty}^{+\infty} x(t-kT_0) = \sum_{k=-\infty}^{+\infty} Y_k\, e^{j2\pi k\frac{t}{T_0}} =$$

$$= \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} X\left(\frac{k}{T_0}\right) e^{j2\pi\frac{kt}{T_0}}$$

II^a FORMULA DI POISSON

$$\sum_{k=-\infty}^{+\infty} x(kT_0)\, e^{-j2\pi f k T_0} = \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} X\left(f-\frac{k}{T_0}\right)$$

Dimostrazione (sarà chiarita in seguito)

$$Y(f) = \sum_{k=-\infty}^{+\infty} X\left(f-\frac{k}{T_0}\right) = \sum_{k=-\infty}^{+\infty} y[k]\, e^{-j2\pi n f T_0} =$$

$$\Rightarrow y[kT_0] = T_0\, x(kT_0)$$

$$= T_0 \sum_{k=-\infty}^{+\infty} x(kT_0)\, e^{-j2\pi f k T_0}$$

FORMULE DI POISSON APPLICATE ALLA $\delta(t)$

$$\sum_{k=-\infty}^{+\infty} \delta(t-kT_0) = \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} e^{j2\pi k \frac{t}{T_0}} \qquad I^a$$

$$\sum_{k=-\infty}^{+\infty} e^{-j2\pi f k T_0} = \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} \delta\left(f-\frac{k}{T_0}\right) \qquad II^a$$

PROPRIETÀ - TRENO DI DELTA DI DIRAC

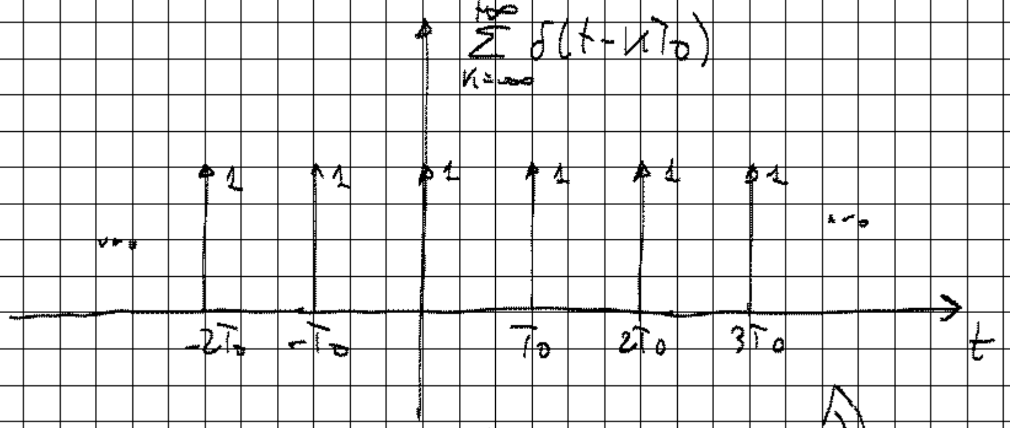
$$\sum_{k=-\infty}^{+\infty} \delta(t-kT_0) \overset{TCF}{\Longleftrightarrow} \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} \delta\left(f-\frac{k}{T_0}\right)$$

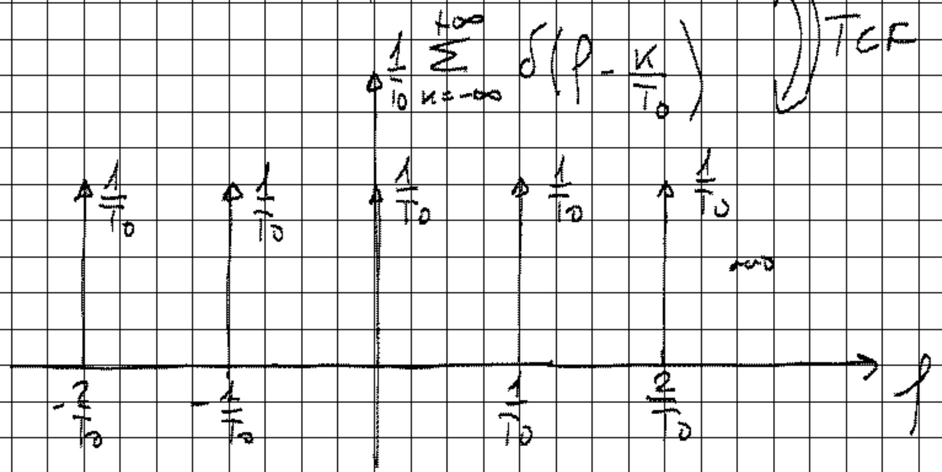
Dimostrazione

$$\int_{-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \delta(t-kT_0)\, e^{-j2\pi f t}\, dt =$$

$$= \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(t-kT_0)\, e^{-j2\pi f t}\, dt =$$

$$= \sum_{k=-\infty}^{+\infty} e^{-j2\pi f k T_0} = \frac{1}{T_0} \sum_{k=-\infty}^{+\infty} \delta\left(f-\frac{k}{T_0}\right)$$





TCF DI SEGNALI PERIODOCIZZATI

$$y(t) = \sum_{k=-\infty}^{+\infty} x(t-kT_0) \iff Y(f) = \frac{1}{T_0} \sum_{k=-\infty}^{+\infty} X\left(\frac{k}{T_0}\right) \delta\left(f-\frac{k}{T_0}\right)$$

Dimostrazione:

$$Y(f) = \int_{-\infty}^{+\infty} y(t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} x(t-kT_0)\, e^{-j2\pi f t}\, dt$$

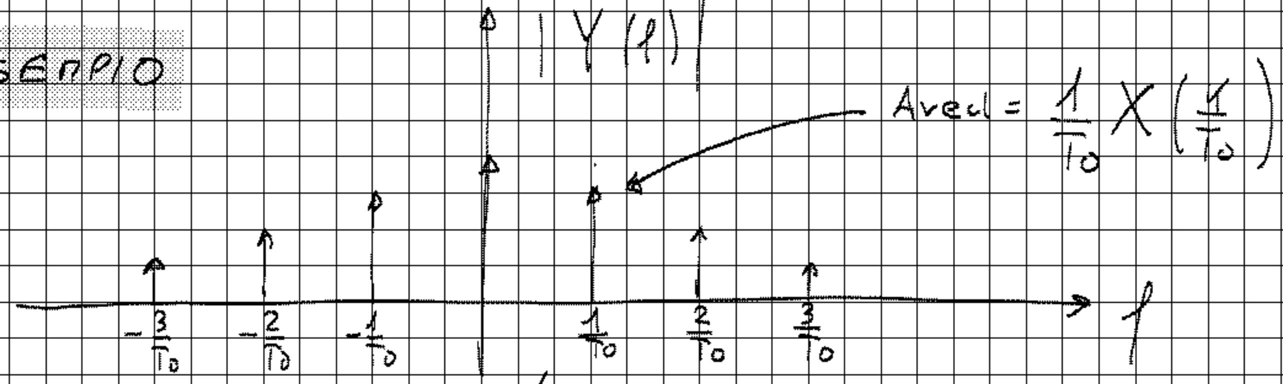
$$= \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(t-kT_0)\, e^{-j2\pi f t}\, dt = \quad (\alpha = t-kT_0)$$

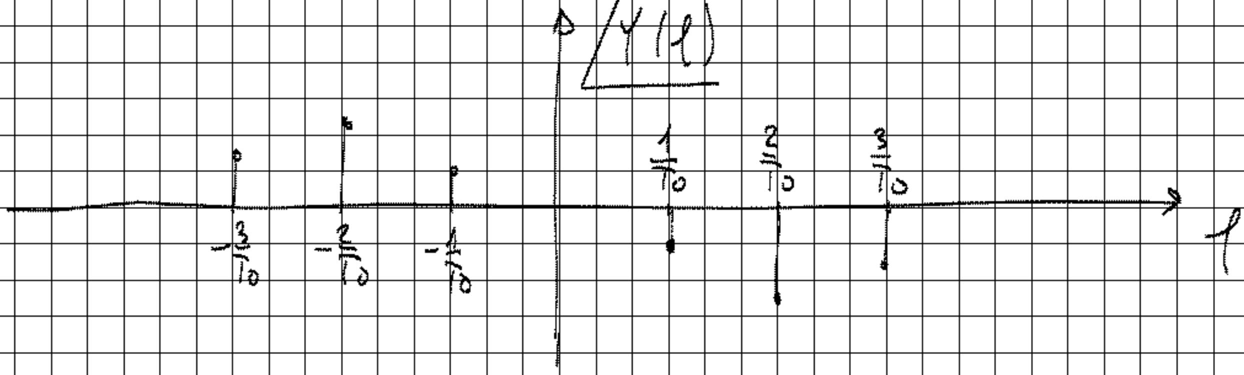
$$= \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(\alpha)\, e^{-j2\pi f\alpha}\, dt\; e^{-j2\pi f n T_0} =$$

$$= X(f) \sum_{n=-\infty}^{+\infty} e^{-j2\pi f n T_0} = \frac{1}{T_0} X(f) \sum_{k=-\infty}^{+\infty} \delta\left(f - \frac{k}{T_0}\right) =$$

$$= \frac{1}{T_0} \sum_{k=-\infty}^{+\infty} X\left(\frac{k}{T_0}\right) \delta\left(f - \frac{k}{T_0}\right) = \sum_{k=-\infty}^{+\infty} Y_k\, \delta\left(f - \frac{k}{T_0}\right)$$

ESEMPIO





TEOREMA DI PARSEVAL PER SEGNALI PERIODICI (CON TSF)

$$\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, y^*(t)\, dt = \sum_{n=-\infty}^{+\infty} X_n Y_n^*$$

Dimostrazione

$$\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, y^*(t)\, dt = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sum_{k=-\infty}^{+\infty} X_k\, e^{j2\pi k f_0 t}\, y^*(t)\, dt$$

$$= \sum_{k=-\infty}^{+\infty} X_k \cdot \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} y^*(t)\, e^{j2\pi k f_0 t}\, dt =$$

$$= \sum_{k=-\infty}^{+\infty} X_k \left[\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} y(t)\, e^{-j2\pi k f_0 t}\, dt \right]^* = \sum_{k=-\infty}^{+\infty} X_k Y_k^*$$

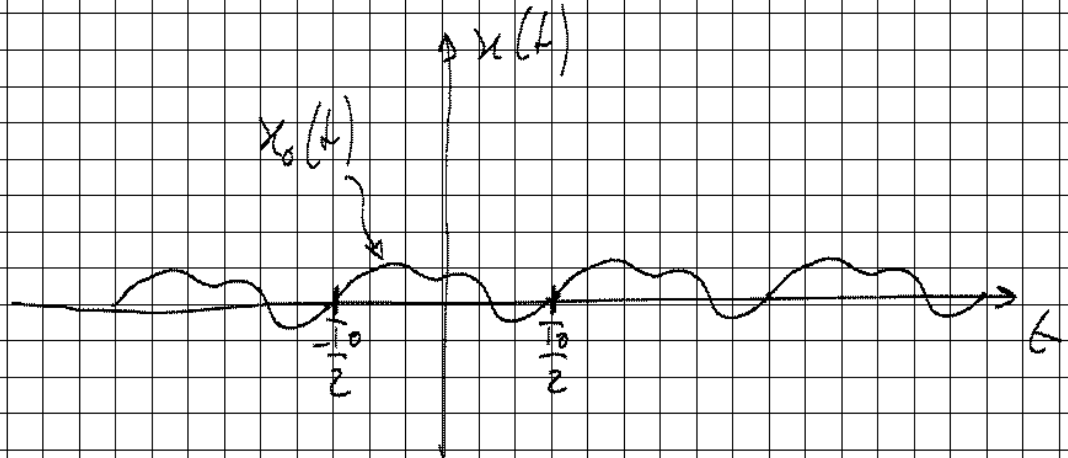
TEOREMA DI PARSEVAL PER SEGNALI PERIODICI (CON TCF)

$$\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, y^*(t)\, dt = \frac{1}{T_0^2} \sum_{k=-\infty}^{+\infty} X_0\left(\frac{k}{T_0}\right) Y_0^*\left(\frac{k}{T_0}\right)$$

$$X_k = \frac{1}{T_0} X_0\left(\frac{k}{T_0}\right) \quad , \quad Y_k = \frac{1}{T_0} Y_0\left(\frac{k}{T_0}\right)$$

$$X_0(f) \overset{TCF}{\Leftrightarrow} x_0(t) \quad , \quad x_0(t) = x(t)\, rect\left(\frac{t}{T_0}\right)$$

$$Y_0(f) \overset{TCF}{\Leftrightarrow} y_0(t) \quad , \quad y_0(t) = y(t)\, rect\left(\frac{t}{T_0}\right)$$

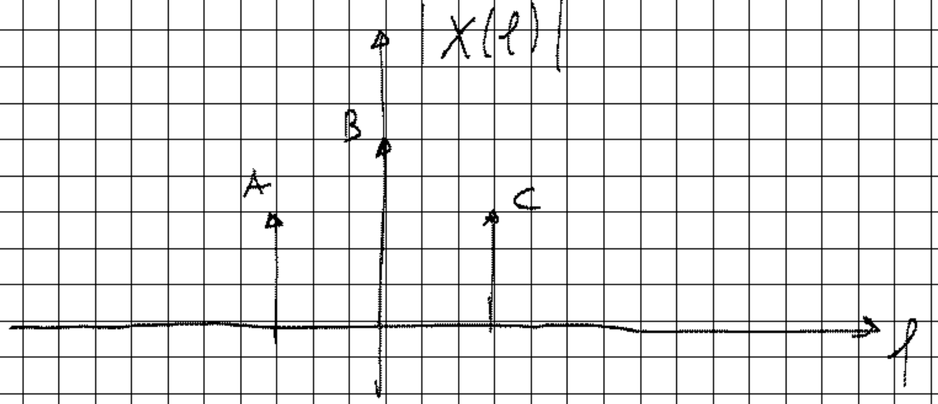


POTENZA DI UN SEGNALE PERIODICO

$$P_x = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2\, dt = \sum_{n=-\infty}^{+\infty} |X_n|^2 = \underbrace{\frac{1}{T_0^2}\sum_{n=-\infty}^{+\infty}\left|X_0\left(\frac{k}{T_0}\right)\right|^2}$$

Somma delle aree delle delta al quadrato

ESEMPIO



$$P_x = A^2 + B^2 + C^2$$

DENSITÀ SPETTRALE DI POTENZA PER SEGNALI PERIODICI

$$P_x \triangleq \int_{-\infty}^{+\infty} S_x(f)\, df$$

$$S_x(f) = \sum_{n=-\infty}^{+\infty} \frac{1}{T_0^2}\left|X\left(\frac{k}{T_0}\right)\right|^2 \delta\left(f - \frac{k}{T_0}\right)$$

Dimostrazione

$$P_x = \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \frac{1}{T_0^2}\left|X\left(\frac{k}{T_0}\right)\right|^2 \delta\left(f-\frac{k}{T_0}\right) df =$$

$$= \sum_{k=-\infty}^{+\infty} \frac{1}{T_0^2} \left| X\left(\frac{k}{T_0}\right) \right|^2 \underbrace{\int_{-\infty}^{+\infty} \delta\left(f - \frac{k}{T_0}\right) df}_{=1} =$$

$$= \frac{1}{T_0^2} \sum_{k=-\infty}^{+\infty} \left| X\left(\frac{k}{T_0}\right) \right|^2 = P_x$$

DENSITÀ SPETTRALE DI POTENZA PER SEGNALI APERIODICI

$$S_x(f) = \lim_{T \to \infty} \frac{\left| X_T(f) \right|^2}{T}$$

$$X_T(f) \overset{TCF}{\Longleftrightarrow} x_T(t) = x(t)\, \mathrm{rect}\left(\frac{t}{T}\right)$$