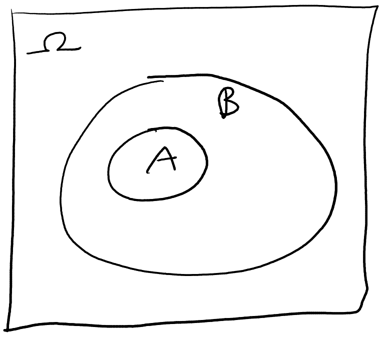
1) Eventi A, B

Se $A \subset B$, $P(A) \leq P(B)$



$$B = \underbrace{A \cup (A^c \cap B)}_{\text{eventi disgiunti}}$$

eventi disgiunti → ASSIOMA DI ADDITIVITÀ

$$P(B) = P(A) + \underbrace{P(A^c \cap B)}_{\geq 0} \Longrightarrow P(B) \geq P(A)$$

≥ 0 → ASSIOMA DI NONNEGATIVITÀ

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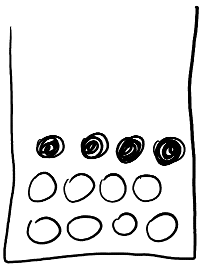
2)



$$P(2\ \text{nere}) = \frac{4}{12} \cdot \frac{3}{11} = \frac{1}{11}$$

metodo alternativo →

$$= \frac{\binom{4}{2}}{\binom{12}{2}} = \frac{\frac{4!}{2!\,2!}}{\frac{12!}{2!\,10!}} = \frac{4 \cdot 3}{2} \cdot \frac{2}{12 \cdot 11} = \frac{1}{11}$$

$$P(2\ \text{bianche}) = \frac{8}{12} \cdot \frac{7}{11} = \frac{14}{33}$$

metodo alternativo →

$$= \frac{\binom{8}{2}}{\binom{12}{2}} = \frac{\frac{8 \cdot 7}{2}}{\frac{12 \cdot 11}{2}} = \frac{14}{33}$$

3) $\mathbb{P}\left\{\begin{matrix}\text{tiro nel}\\ \text{bersaglio}\end{matrix}\right\} = p = \frac{1}{3}$ $\qquad P\left\{\begin{matrix}\text{tiro fuori}\\ \text{bersaglio}\end{matrix}\right\} = 1-p = \frac{2}{3}$

$A = \{\text{Almeno un bersaglio in } N \text{ tiri}\} \longrightarrow N$ tale che $P(A) > 0{,}8$?

$A^c = \{\text{Nessun bersaglio in } N \text{ tiri}\}$

$$P(A) = 1 - P(A^c)$$

$$P(A^c) = (1-p)^N$$

$$P(A) = 1 - (1-p)^N$$

$$\Rightarrow 1 - \left(\frac{2}{3}\right)^N > 0{,}8 \Longrightarrow \left(\frac{2}{3}\right)^N < 0{,}2$$

$$N \cdot \ln\left(\frac{2}{3}\right) < \ln\left(\frac{1}{5}\right) \qquad N \cdot \ln\left(\frac{3}{2}\right) > \ln 5 \qquad N > \frac{\ln 5}{\ln 1.5} \cong 3.97$$

$$\underline{N = 4}$$

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4) a. estrazione con reimmissione $\rightarrow$ Composizione di esperimenti indipendenti

$$P(3_{\text{assi c.r.}}) = \frac{4}{40} \cdot \frac{4}{40} \cdot \frac{4}{40} = \frac{1}{1000}$$

b. estrazione senza reimmissione

$$P(3_{\text{assi s.r.}}) = \frac{4}{40} \cdot \frac{3}{39} \cdot \frac{2}{38} = \frac{1}{10} \cdot \frac{1}{13} \cdot \frac{1}{19}$$

$$\stackrel{!}{=} \binom{4}{3} \Big/ \binom{40}{3} = 4 \cdot \frac{3 \cdot 2}{40 \cdot 39 \cdot 38} = \frac{1}{2470}$$

5) $$P(\text{tris di assi}, 7b, 4d) = \frac{\binom{4}{3}\cdot 1\cdot 1}{\binom{40}{5}} = \cancel{4}\cdot\frac{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}}{\cancel{40}\cdot 39\cdot 38\cdot 37\cdot \cancel{36}_{3}}$$

$$= \frac{1}{39\cdot 38\cdot 37\cdot 3} \simeq 6{,}08\cdot 10^{-6}$$

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6) $P\{C\} = 75\% = p$

FORMULA DI BERNOULLI

$$P_{N,K} = \binom{N}{K} p^{K} (1-p)^{N-K}$$

$N = 6$ studenti estratti

$K =$ studenti con calcolatrice

$$P\left\{\begin{matrix}\text{non più di 2 studenti}\\ \text{con la calcolatrice}\end{matrix}\right\} = P\{0 \text{ stud}\} + P\{1 \text{ stud}\} + P\{2 \text{ stud}\}$$

$$P\{0 \text{ stud}\} = \binom{6}{0} p^0 (1-p)^6 = (1-p)^6 = \frac{1}{4^6} = \frac{1}{2^{12}}$$

$$P\{1 \text{ stud}\} = \binom{6}{1} p (1-p)^5 = 6\cdot\frac{3}{4}\cdot\frac{1}{4^5} = \frac{18}{2^{12}}$$

$$P\{2 \text{ stud}\} = \binom{6}{2} p^2 (1-p)^4 = \frac{6\cdot 5}{2}\,\frac{3^2}{4^2}\cdot\frac{1}{4^4} = \frac{3^3\cdot 5}{4^6} = \frac{135}{2^{12}}$$

$$= \frac{1 + 18 + 135}{2^{12}} = \frac{154}{4096} \simeq 3{,}76\%$$

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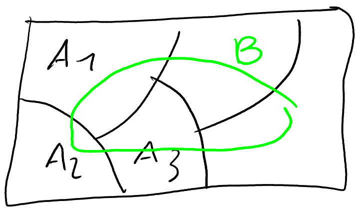
7) 

$\{A_1, A_2, \dots, A_N\}$ partizione di Ω

$$A_i \cap A_k = \emptyset \;, \; i \neq k \;; \quad \bigcup_{i=1}^{N} A_i = \Omega$$

$$P(B) = P(B\cap\Omega) = P\left[B \cap \bigcup_{i=1}^{N} A_i\right] = P\left[\bigcup_{i=1}^{N} B\cap A_i\right] = \sum_{i=1}^{N} P(B\cap A_i) = P(B)$$

8) $P(C) = 0{,}01$ Sensibilità $= P(+|C) = 0{,}8$

Specificità $= P(-|\bar{C}) = 0{,}8$

a) $P(C|+) = ?$

Teorema di Bayes : $P(C|+) = \dfrac{P(+|C)\cdot P(C)}{P(+)}$

Teorema della probabilità totale : $P(+) = P(+|C)\cdot P(C) + P(+|\bar{C})\cdot P(\bar{C})$

$P(+|\bar{C}) + P(-|\bar{C}) = 1 \longrightarrow P(+|\bar{C}) = 1 - P(-|\bar{C}) = 0{,}2$

$P(\bar{C}) = 1 - P(C) = 0{,}99$

$P(+) = 0{,}8\cdot 0{,}01 + 0{,}2\cdot 0{,}99 = 0{,}206$

$P(C|+) = \dfrac{0{,}8\cdot 0{,}01}{0{,}206} \cong 3{,}88\%$

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b) $P(+|C) = 0{,}8$

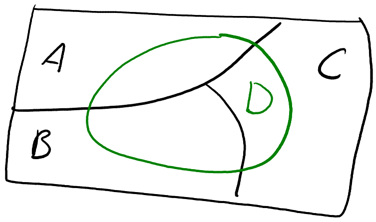
$P(-|\bar{C}) = 0{,}999 \longrightarrow P(+|\bar{C}) = 0{,}001$

$P(+) = 0{,}8\cdot 0{,}01 + 0{,}001\cdot 0{,}99 = 0{,}00899$

$P(C|+) = \dfrac{0{,}8\cdot 0{,}01}{0{,}00899} \cong 88.99\%$

9)

	A	B	C	
	50%	30%	20%	PROD.
	2%	3%	4%	pezzi difettosi



$$P(D) = P(D|A)\cdot P(A) + P(D|B)\cdot P(B) + P(D|C)\cdot P(C)$$

$$= 0{,}02 \cdot 0{,}5 + 0{,}03 \cdot 0{,}3 + 0{,}04 \cdot 0{,}2 = 2{,}7\%$$

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10) $P_1(G_1) = 0{,}2 = P_2(G_2)$

$$\Omega_1 = \{\xi_1 = F_1 \,;\, \xi_2 = G_1\}$$

$$\Omega_2 = \{\lambda_1 = F_2 \,;\, \lambda_2 = G_2\}$$

$$\Omega = \Omega_1 \times \Omega_2 = \{(F_1, F_2); (F_1, G_2); (G_1, F_2); (G_1, G_2)\}$$

$$P\left(\begin{matrix}\text{lampadario almeno}\\ \text{parzialmente acceso}\end{matrix}\right) = 1 - P\{(G_1, G_2)\} = 1 - P_1(G_1)\cdot P_2(G_2)$$

$$= 1 - 0{,}2 \cdot 0{,}2 = \underline{0{,}96}$$

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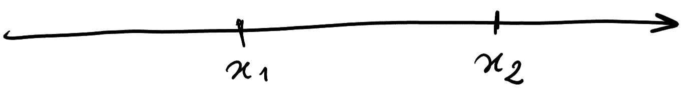
11)



$F_X(x)$ è nota: $P\{x_1 < X \leq x_2\} = ?$

$$\{X \leq x_2\} = \{X \leq x_1\} \cup \{x_1 < X \leq x_2\}$$

EVENTI DISGIUNTI

$$P\{X \leq x_2\} = P\{X \leq x_1\} + P\{x_1 < X \leq x_2\}$$

$$P\{x_1 < X < x_2\} = P\{X \leq x_2\} - P\{X \leq x_1\} = F_X(x_2) - F_X(x_1)$$

12) $f_X(x)\,dx = P\{x < X \leq x + dx\}$

Supponiamo di ripetere l'esperimento N volte e contare il numero di occorrenze $\Delta n(x)$ per cui $x < X \leq x + \Delta x$

$$\Rightarrow f_X(x)\cdot\Delta x \simeq P(x < X \leqslant x + \Delta x) \simeq \frac{\Delta n(x)}{N}$$

frequenza relativa

$$\Rightarrow f_X(x) \simeq \frac{\Delta n(x)}{N\cdot\Delta x}$$

[Formalmente $f_X(x) = \lim\limits_{\substack{N\to\infty\\ \Delta x\to 0}} \dfrac{\Delta n(x)}{N\cdot\Delta x}$]

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13) $f_X(x) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}} u(x)$ $\qquad x_0 \mid P\{X > x_0\} = 0{,}05 \quad x_0 = ?$

$\lambda = 10$ minuti

$$F_X(x) = \int_{-\infty}^{x} f_X(\alpha)\,d\alpha = \begin{cases} 0, & x \leqslant 0 \\ 1 - e^{-\frac{x}{\lambda}}, & x > 0 \end{cases}$$

$$P\{X > x\} = 1 - P\{X \leqslant x\} = 1 - F_X(x) = 1 - \left(1 - e^{-\frac{x}{\lambda}}\right) = e^{-\frac{x}{\lambda}}$$

$$\Rightarrow P\{X > x_0\} = 0{,}05 = e^{-\frac{x_0}{\lambda}} \longrightarrow \ln(0{,}05) = -\frac{x_0}{\lambda}$$

$$x_0 = -\lambda\cdot\ln(0{,}05) = \lambda\cdot\ln 20 \cong \underline{30 \text{ minuti}}$$

14) $X \in \mathcal{N}(\eta, \sigma^2)$ $\quad \eta = 502\,g$

$\sigma = 5\,g$

a) $P\{X < 495\} = 1 - P\{X \geq \underset{x_0}{\underset{\shortparallel}{495}}\} = 1 - Q\left(\frac{x_0 - \eta}{\sigma}\right)$

$= 1 - Q\left(\frac{495 - 502}{5}\right) = 1 - Q\left(-\frac{7}{5}\right) = Q\left(\frac{7}{5}\right) = Q(1,4) \cong 0,0808$

b) $N = 20$; $p = 0,0808$

$P\left(\begin{smallmatrix}\text{Almeno } 1\\ \text{pacco} < 495\end{smallmatrix}\right) = 1 - P\left(\begin{smallmatrix}\text{Nessun pacco}\\ < 495\end{smallmatrix}\right)$

$P\left(\begin{smallmatrix}\text{Nessun pacco}\\ < 495\end{smallmatrix}\right) = \binom{N}{0} p^0 (1-p)^{N-0} = (1-p)^N \cong 0,185$

$P\left(\begin{smallmatrix}\text{Almeno } 1\\ \text{pacco} < 495\end{smallmatrix}\right) = 1 - 0,185 = 0,815$

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15) X, Y v.a. discrete $\quad p_X(x_i) = P(\{X = x_i\}) \quad i = 1, 2, 3, \ldots$

$Y = g(X) \quad p_X(y_k) = P(\{Y = y_k\}) \quad K = 1, 2, 3, \ldots$

$$\eta_Y = \sum_K y_k \cdot P(\{Y = y_k\}) = \underbrace{y_1 \cdot P(\{Y = y_1\})} + y_2 \cdot P(\{Y = y_2\}) + \ldots$$

- $y_1 \cdot P(\{Y = y_1\}) = y_1 \cdot \sum_{G(y_1)} P(\{X = x_i\}) = \sum_{G(y_1)} g(x_i) \cdot P(\{X = x_i\})$

Somma estesa a tutti gli x_i tali che $g(x_i) = y_1$

FORMALMENTE : $G(y_1) \triangleq \{x_i : g(x_i) = y_1\}$

$\Downarrow$

- $y_2 \cdot P(\{Y = y_2\}) = y_2 \cdot \sum_{G(y_2)} g(x_i) \cdot P(\{X = x_i\})$ $\quad \Rightarrow \eta_Y = \sum_i g(x_i) P(\{X = x_i\})$

$= E\{g(X)\}$

16) $f_X(x) = Kx\, e^{-\frac{x^2}{2}} u(x)$, $K \in \mathbb{R}$

a) K tale che $f_X(x)$ è una ddp.

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = 1 = \int_0^{+\infty} Kx\, e^{-\frac{x^2}{2}} dx = (-K)\int_0^{+\infty} (-x)\, e^{-\frac{x^2}{2}} dx = (-K)\left[e^{-\frac{x^2}{2}}\right]\Bigg|_0^{+\infty}$$

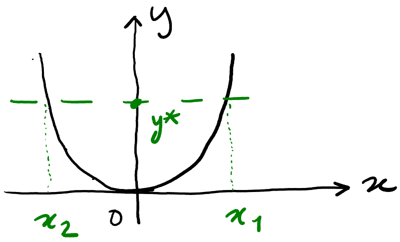
$$= (-K)\cdot(0-1) = K \implies K = 1$$

b) $Y = X^2 \rightarrow$ Teorema fondamentale per la trasformazione di v.a.

$$f_Y(y) = \sum_{i=1}^{K} \frac{f_X(x_i)}{|g'(x_i)|}\Bigg|_{x_i = g^{-1}(y)}$$

$g(x) = x^2$

$g'(x) = 2x$



- $y < 0$: non ci sono soluz. a $y = g(x) \rightarrow f_Y(y) = 0$
- $y \geq 0$: $f_Y(y) = \dfrac{f_X(x_1)}{|g'(x_1)|}\Bigg|_{x_1=\sqrt{y}} + \dfrac{f_X(x_2)}{|g'(x_2)|}\Bigg|_{x_2=-\sqrt{y}}$

$f_X(x_2) = 0 \ \forall x_2 < 0$ data la ddp di X

$$\implies f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}} = \frac{1}{2\sqrt{y}}\cdot\sqrt{y}\, e^{-\frac{y}{2}} = \frac{1}{2} e^{-y/2}$$

Riassumendo: $f_Y(y) = \frac{1}{2} e^{-\frac{y}{2}} u(y)$; $Y \in Exp(\lambda = 2)$

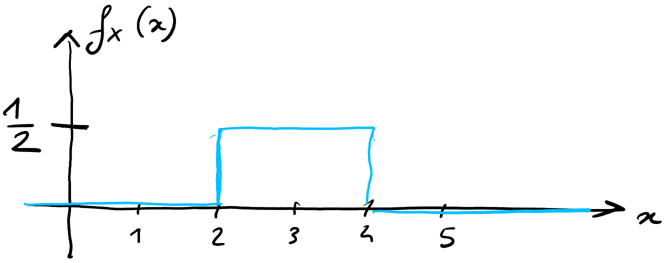
$\boxed{E\{Y\} = 2}$ Potevo calcolarlo anche $E\{Y\} = E\{X^2\} = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx = \int_0^{+\infty} x^3 e^{-\frac{x^2}{2}} dx$

$$= -\int_0^{+\infty} x^2 (-x) e^{-\frac{x^2}{2}} dx = -\left[x^2 \cdot e^{-\frac{x^2}{2}}\right]\Bigg|_0^{+\infty} + \int_0^{+\infty} 2x \cdot e^{-\frac{x^2}{2}} dx = [0-0] - 2\int_0^{+\infty}(-x)\, e^{-\frac{x^2}{2}} dx =$$

$$= (-2)\cdot\left[e^{-\frac{x^2}{2}}\right]_0^{+\infty} = (-2)(0-1) = 2 = E\{Y\}$$

17) $X \in U(2,4)$

a) $f_X(x) = \begin{cases} 1/2, & 2 \leq x \leq 4 \\ 0, & \text{altrimenti} \end{cases}$

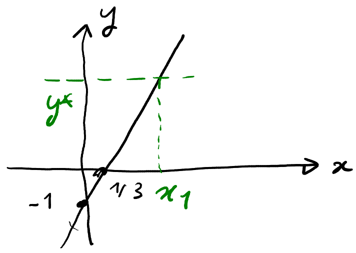


Osservo che la ddp è simmetrica rispetto a $x = 3 \rightarrow \underline{\eta_x = 3}$

$$E\{X^2\} = \int x^2 f_X(x)\,dx = \int_2^4 x^2 \cdot \frac{1}{2}\,dx = \frac{1}{2} \cdot \frac{1}{3} \int_2^4 3x^2\,dx = \frac{1}{6}\left[x^3\right]\Big|_2^4 = \frac{1}{6}(64-8) = \frac{56}{6} = \frac{28}{3}$$

$$\sigma_X^2 = E\{X^2\} - \eta_X^2 = \frac{28}{3} - 9 = \frac{28-27}{3} = \underline{\frac{1}{3} = \sigma_X^2}$$

b) $Y = 3X - 1$



$g(x) = 3x - 1$

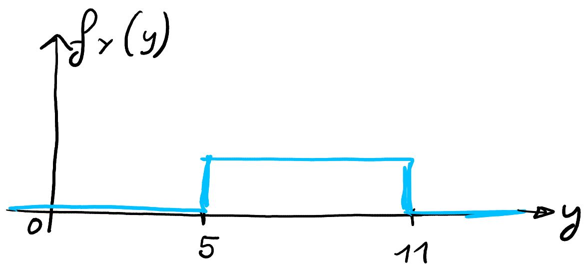
$g'(x) = 3$

$$f_Y(y) = \left.\frac{f_X(x_1)}{|g'(x_1)|}\right|_{x_1 = \frac{y+1}{3}}$$

↑ unico termine perché g è biunivoca

$$f_Y(y) = \frac{f_X\left(\frac{y+1}{3}\right)}{3} = \begin{cases} \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}, & \text{se } 2 \leq \frac{y+1}{3} \leq 4 \\ & \Updownarrow \\ & \text{cioè se } 5 \leq y \leq 11 \\ 0, & \text{altrimenti} \end{cases}$$

$Y \in U(5, 11)$



18) $n = 100$ telefonate in una giornata lavorativa

Durata telefonata $X \in Exp(\lambda = 3 \text{ minuti})$ $\begin{cases} \eta_X = \lambda = 3 \\ \sigma_X^2 = \lambda^2 \Rightarrow \sigma_X = \lambda = 3 \end{cases}$

$$Y_n = \sum_{i=1}^{n} X_i$$

Durata della giornata lavorativa: Y_{100}

$$\mathbb{E}\{Y_{100}\} = \eta_{100} = 100 \cdot \eta_X = 300$$

$$\mathbb{E}\{(Y_{100} - \eta_{100})^2\} = \sigma_{100}^2 = 100 \cdot \sigma_X^2 = 900 = \sigma_{100}^2 \rightarrow \sigma_{100} = \sqrt{900} = 30$$

$$S_n = \frac{Y_n - \eta_n}{\sigma_n} \approx Z$$

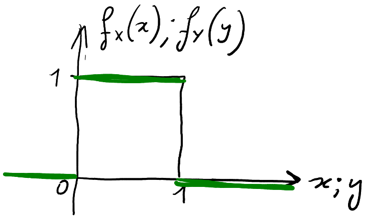
$$Y_n \approx \sigma_n \cdot Z + \eta_n \rightarrow Y_n \in \mathcal{N}(\eta_n, \sigma_n^2)$$

$$Y_{100} \in \mathcal{N}(\eta_{100}, \sigma_{100}^2)$$

$$P\left(\substack{\text{giornata lavorativa} \\ > 6 \text{ ore}}\right) = P(Y_{100} > 6 \text{ ore}) = P(Y_{100} > \underset{y_0}{360} \text{ minuti})$$

$$= Q\left(\frac{y_0 - \eta_{100}}{\sigma_{100}}\right) = Q\left(\frac{360 - 300}{30}\right) = Q(2) = 0{,}0228$$

19)



X, Y indipendenti

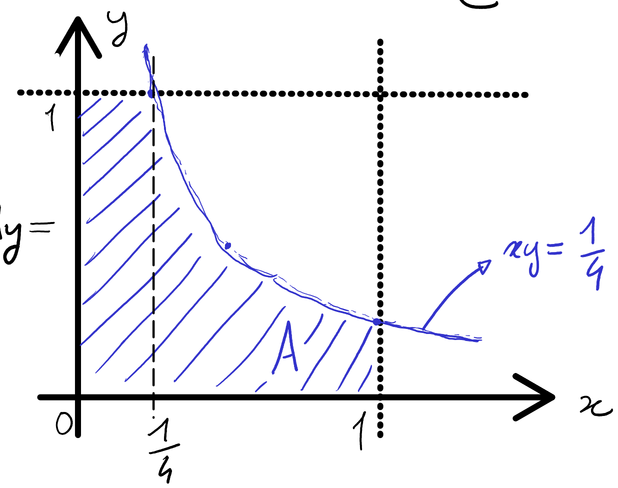
$\Downarrow$

a) $f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) = \begin{cases} 1, & 0 \le x \le 1 \text{ e } 0 \le y \le 1 \\ 0, & \text{altrimenti} \end{cases}$

b) $P\left(\frac{XY}{2} < \frac{1}{8}\right) = ?$

$$\| $$

$$P\left(XY < \frac{1}{4}\right) = \int_A f_{XY}(x,y)\,dx\,dy =$$



$$= 1\cdot\frac{1}{4} + \int_{\frac{1}{4}}^{1} \frac{1}{4x}\,dx$$

$$= \frac{1}{4} + \frac{1}{4}\cdot \ln x \Big|_{\frac{1}{4}}^{1} = \frac{1}{4} + \frac{1}{4}\left(\ln 1 - \ln \frac{1}{4}\right) = \frac{1}{4}\left(1 + \ln 4\right) \simeq 0{,}596$$

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20) X $\quad \eta_x$, σ_X^2 $\qquad Y = 2X - 3$ $\quad$ a) $\eta_Y = ?$ $\quad \sigma_Y^2 = ?$

$$\eta_Y = \mathbb{E}\{2X-3\} = 2\cdot\mathbb{E}\{X\} - 3 = \underline{2\cdot\eta_x - 3 = \eta_x}$$

LINEARITÀ DELL'OPERATORE ASPETTAZIONE

$$\sigma_Y^2 = \mathbb{E}\{Y^2\} - \eta_Y^2$$

$$\mathbb{E}\{Y^2\} = \mathbb{E}\{(2X-3)^2\} = \mathbb{E}\{4X^2 - 12X + 9\} = 4\mathbb{E}\{X^2\} - 12\eta_x + 9$$

$$\sigma_Y^2 = 4\mathbb{E}\{X^2\} - 12\eta_x + 9 - 4\eta_x^2 + 12\eta_x - 9 = 4\left(\mathbb{E}\{X^2\} - \eta_x^2\right) = \underline{4\sigma_X^2 = \sigma_Y^2}$$

b) $r_{XY} = \mathbb{E}\{XY\} = \mathbb{E}\{X(2X-3)\} = 2\mathbb{E}\{X^2\} - 3\eta_x = \underline{2\left(\sigma_X^2 + \eta_x^2\right) - 3\eta_x = r_{XY}}$

$$c_{XY} = \mathbb{E}\{(X-\eta_x)(Y-\eta_y)\} = r_{XY} - \eta_x\cdot\eta_y = 2\sigma_X^2 + 2\eta_x^2 - 3\eta_x - \eta_x(2\eta_x - 3)$$

$$= 2\sigma_X^2 + 2\eta_x^2 - 3\eta_x - 2\eta_x^2 + 3\eta_x = \underline{2\sigma_X^2 = c_{XY}}$$

$$\rho_{XY} = \frac{c_{XY}}{\sigma_X\cdot\sigma_Y} = \frac{2\sigma_X^2}{\sigma_X\cdot 2\sigma_X} = \underline{1 = \rho_{XY}}$$