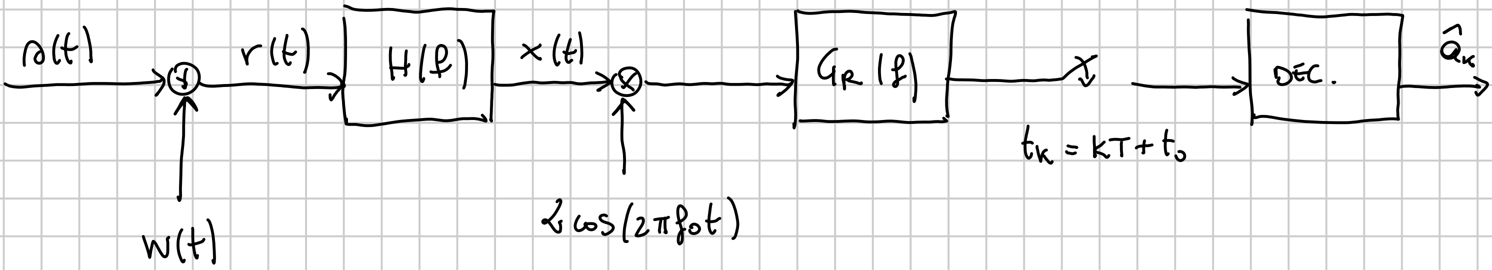
Esercizio 3.1



$w(t)$ AWGN

$$s(t) = \sum_i a_i \, p(t - iT) \cos(2\pi f_0 t)$$

$a_i \in \{\pm 1\}$ indipendenti ed equiprobabili

$$P(f) = \begin{cases} \sqrt{T} \cos(\pi f T/2) & |fT| \leq 1 \\ 0 & \text{altrove} \end{cases}$$

$$H(f) = \begin{cases} e^{-j[2\pi(f-f_0)\tau + \pi/6]} & |f-f_0| \leq \frac{1}{T} \\ e^{-j[2\pi(f+f_0)\tau - \pi/6]} & |f+f_0| \leq \frac{1}{T} \\ 0 & \text{altrove} \end{cases}$$

$$r(t) = s(t) + w(t)$$

$$x(t) = \sum_i a_i \left[p(t-iT) \cos(2\pi f_0 t) \right] \otimes h(t) + \underbrace{w(t) \otimes h(t)}_{n(t)}$$

$$g(t) = \left(p(t-iT)\cos(2\pi f_0 t)\right) \otimes h(t)$$

$$G(f) = \left[P(f)\, e^{-j2\pi f iT} \otimes \left(\frac{\delta(f-f_0) + \delta(f+f_0)}{2} \right) \right] \cdot H(f) =$$

$$\left[\frac{1}{2} P(f-f_0)\, e^{-j2\pi (f-f_0)\cdot iT} + \frac{1}{2} P(f+f_0)\, e^{-j2\pi (f+f_0)\cdot iT} \right] \cdot H(f) =$$

$$= \frac{1}{2} P(f-f_0)\, e^{-j2\pi (f-f_0) iT} \cdot e^{-j2\pi (f-f_0)\tau}\, e^{-j\pi/6} +$$

$$\frac{1}{2} P(f+f_0)\, e^{-j2\pi (f+f_0) iT}\, e^{-j2\pi (f+f_0)\tau}\, e^{+j\pi/6} =$$

$$= \frac{1}{2} P(f-f_0)\, e^{-j2\pi (f-f_0)(iT+\tau)}\, e^{-j\pi/6} + \frac{1}{2} P(f+f_0)\, e^{-j2\pi (f+f_0)(iT+\tau)}\, e^{+j\pi/6}$$

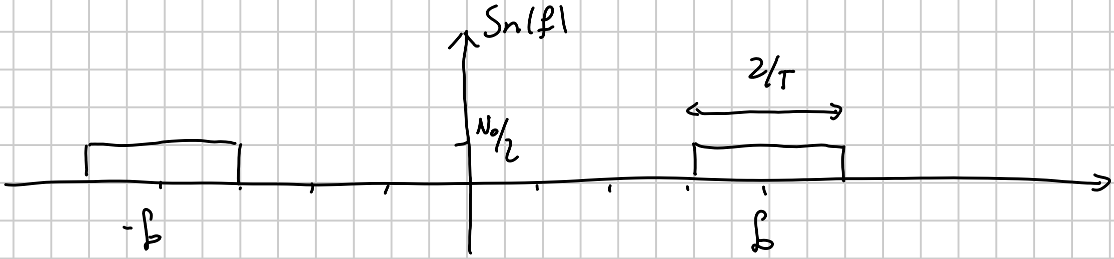
Trasformata Continua di Fourier Inversa

$$g(t) = p(t-iT-\tau)\cos(2\pi f_0 t + \pi/6) =$$

$$= p(t-iT-\tau)\left[\cos(2\pi f_0 t)\cos\frac{\pi}{6} - \sin(2\pi f_0 t)\sin\frac{\pi}{6}\right]$$

$$x(t) = \sum_i a_i\, p(t - iT - \tau)\left[\cos(2\pi f_0 t)\cos \pi/6 - \sin(2\pi f_0 t)\sin(\pi/6)\right] + n(t)$$

$n(t)$ è un processo passa banda, a valor medio nullo, SSL, Gaussiano



$$n(t) = n_c(t)\cos(2\pi f_0 t) - n_s(t)\sin(2\pi f_0 t)$$

$$\Rightarrow z(t) = x(t)\cdot 2\cos(2\pi f_0 t) = z_s(t) + z_n(t)$$

$$z_s(t) = 2\sum_i a_i\, p(t-iT-\tau)\left[\cos^2(2\pi f_0 t)\cos(\pi/6) - \sin(2\pi f_0 t)\cos(2\pi f_0 t)\cdot\sin(\pi/6)\right] =$$

$$= \sum_i a_i\, p(t-iT-\tau)\left[\left(1+\cos(4\pi f_0 t)\right)\cos(\pi/6) - \sin(4\pi f_0 t)\sin(\pi/6)\right]$$

$$z_n(t) = 2n(t)\cdot\cos(2\pi f_0 t) = \left[n_c(t)\cdot\cos(2\pi f_0 t) - n_s(t)\sin(2\pi f_0 t)\right]\cdot 2\cos(2\pi f_0 t) =$$

$$= n_c(t)\cdot\left(1+\cos(4\pi f_0 t)\right) - n_s(t)\sin(4\pi f_0 t)$$

$G_R(f)$ realizza il filtro adattato

$$G_R(f) = P(f) = \begin{cases} \sqrt{T} \cos(\pi f T/2) & |fT| \leq 1 \\ 0 & \text{altrove} \end{cases}$$

$$y(t) = z(t) \otimes g_R(t) = y_s(t) + y_n(t)$$

$$y_s(t) = z_s(t) \otimes g_R(t)$$

$$y_n(t) = z_n(t) \otimes g_R(t)$$

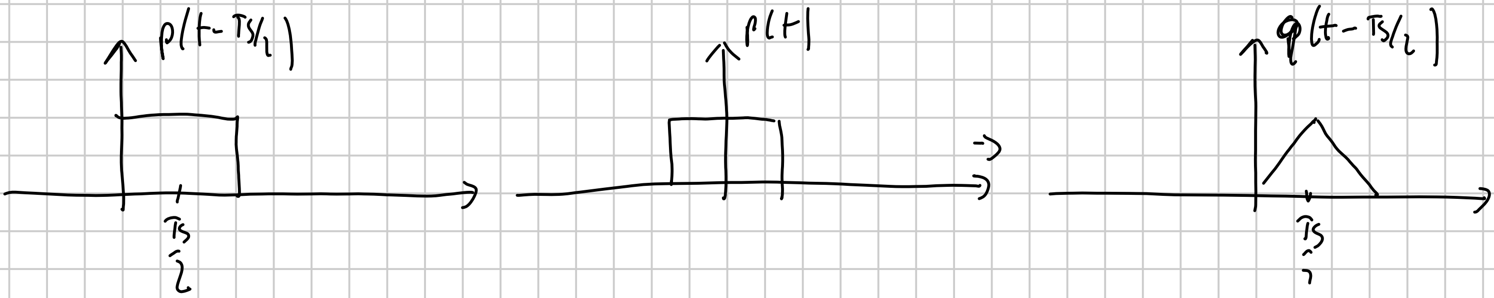
$$y_s(t) = \sum_i a_i \cos(\pi/6)\, p(t - iT - \tau) \otimes \underset{\overset{\|}{g_R(t)}}{p(t)}$$

$$y_n(t) = z_n(t) \otimes p(t) = n_c(t) \otimes p(t)$$

$$y(t) = \sum_i a_i \frac{\sqrt{3}}{2}\, p(t - iT - \tau) \otimes p(t) + n_c(t) \otimes p(t)$$

$$q(t) = p(t) \otimes p(t)$$

$$Q(f) = P^2(f) = \begin{cases} T \cos^2(\pi f T/2) & |fT| \leq 1 \\ 0 & \text{altrove} \end{cases} \qquad \cos^2\alpha = \frac{1}{2} + \frac{1}{2}\cos 2\alpha$$



$$Q(f) = \begin{cases} \frac{T}{2}\left(1 + \cos(\pi f T)\right) & |fT| \leq 1 \\ 0 & \text{altrove} \end{cases}$$

Funzione Raised Cosine

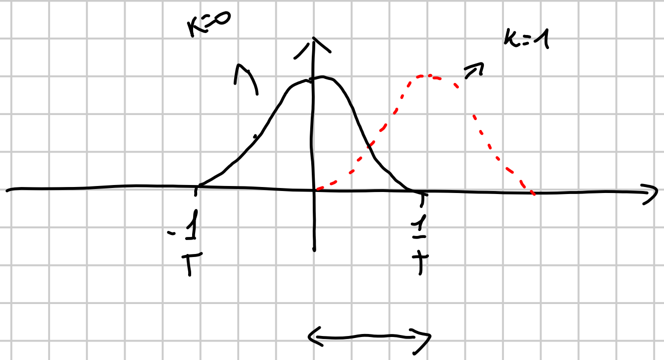
$\alpha = 1$

$$q(kT) = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases}$$

$$y(t) = \sum_i a_i \frac{\sqrt{3}}{2} q(t - iT - \tau) + n_c(t) \otimes p(t)$$

$t_k = kT + \tau \Rightarrow t_0 = \tau$ istante di campionamento ottimo

$$\sum_k Q\left(f - \frac{k}{T}\right) = T$$



$$\frac{T}{2}\left(1 + \cos(\pi f T)\right) + \frac{T}{2}\left(1 + \cos\left(\pi\left(f - \frac{1}{T}\right)T\right)\right) =$$

$$= \frac{T}{2} + \frac{T}{2}\cos(\pi f T) + \frac{T}{2} + \frac{T}{2}\cos(\pi f T - \pi) \rightarrow -\frac{T}{2}\cos(\pi f T) \quad = T$$

Se campiono all'istante ottimo

$$y[k] = a_k \cdot \frac{\sqrt{3}}{2} + n_{cp}[k] \qquad n_{cp}(t) = n_c(t) \otimes p(t)$$

DOBBIAMO RICAVARE LA DSP DI $n_{cp}(t)$

Dato un processo $x(t)$ con $\tilde{x}(t)$, il suo inviluppo complesso, SSL e a valor medio nullo, si dimostra che

1) $$x_c(t) \triangleq \mathrm{Re}\{\tilde{x}(t)\}$$

$$x_s(t) \triangleq \mathrm{Im}\{\tilde{x}(t)\}$$

sono processi SSL a media nulla

2) $$R_{x_c}(\tau) = R_{x_s}(\tau) \Rightarrow S_{x_c}(f) = S_{x_s}(f) \text{ pari e reale}$$

$$R_{x_s x_c}(\tau) = -R_{x_s x_c}(-\tau) \Rightarrow j\,S_{x_s x_c}(f) = -j\,S_{x_s x_c}(f) \text{ immaginario e dispari}$$

3) $$x(t) \triangleq \mathrm{Re}\left\{\tilde{x}(t)\, e^{j2\pi f t}\right\} \quad \text{SSL e a media nulla}$$

$$R_{\tilde{x}}(t) = R_{x_c}(\tau) + j\,R_{x_s x_c}(\tau) \qquad (\text{eq. } 1)$$

$$R_x(t) = R_{x_c}(\tau)\cos(2\pi f \tau) - R_{x_s x_c}(\tau)\sin(2\pi f \tau) \triangleq \mathrm{Re}\left\{R_{\tilde{x}}(\tau)\, e^{j2\pi f t}\right\}$$

La conoscenza di $R_{\tilde{x}}(\tau)$ permette di risalire a $R_{x_c}(\tau)$ e $R_{x_s}(\tau)$

N.B. poiché $R_{x_s x_c}(\tau) = -R_{x_s x_c}(-\tau)$

Allora per $\tau = 0$ deve essere

$$R_{x_s x_c}(0) \triangleq E\{x_c(t_1)\, x_s(t_1)\} = 0$$

le variabili aleatorie $x_c(t_1)$ e $x_s(t_1)$ sono incorrelate

Antitrasformando (eq. 1)

$$S_{\tilde{x}}(f) = S_{x_c}(f) + j\, S_{x_s x_c}(f)$$

$$\begin{cases} S_{x_c}(f) = S_{x_s}(f) = \dfrac{S_{\tilde{x}}(f) + S_{\tilde{x}}(-f)}{2} \\ \\ j\, S_{x_s x_c}(f) = -j\, S_{x_s x_c}(f) = \dfrac{S_{\tilde{x}}(f) - S_{\tilde{x}}(-f)}{2} \end{cases}$$

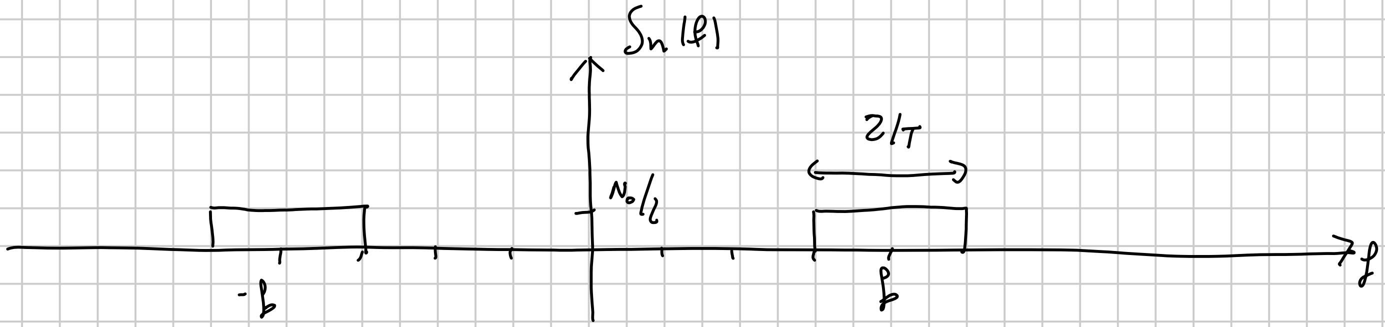
Se il processo è passa banda

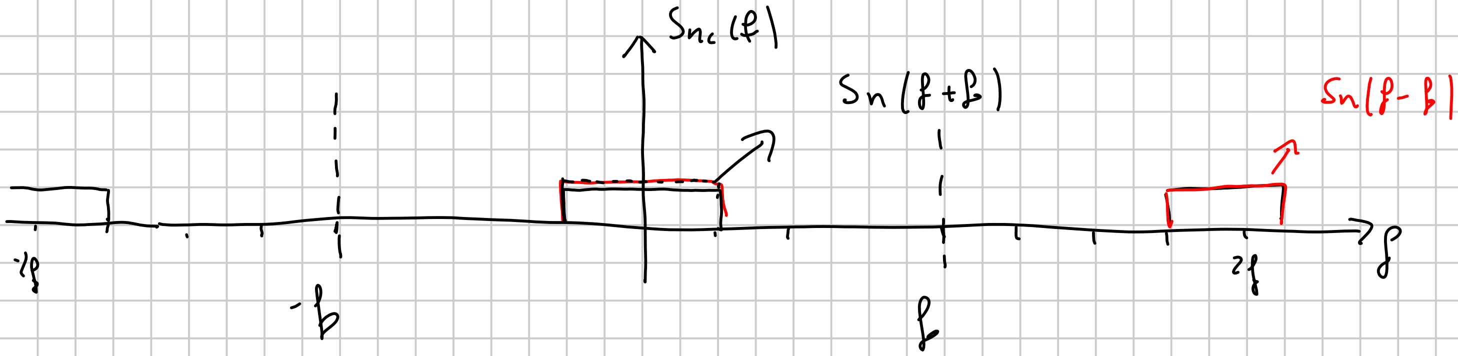
$$S_x(f) = 0 \qquad \text{per} \qquad |f| \geq 2f_0$$

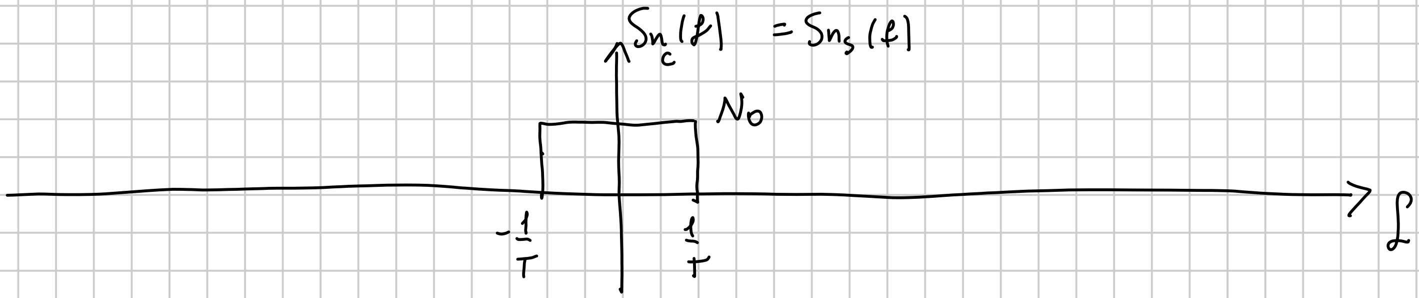
$$S_{\tilde{x}}(f) = \begin{cases} 2\, S_x(f + f_0) & |f| \leq f_0 \\ 0 & \text{altrove} \end{cases}$$

$$S_{x_c}(f) = S_{x_s}(f) = \begin{cases} S_x(f+f_0) + S_x(f-f_0) & |f| \le f_0 \\ 0 & \text{altrove} \end{cases}$$

$$j\, S_{x_s x_c}(f) = \begin{cases} S_x(f+f_0) - S_x(f-f_0) & |f| \le f_0 \\ 0 & \text{altrove} \end{cases}$$







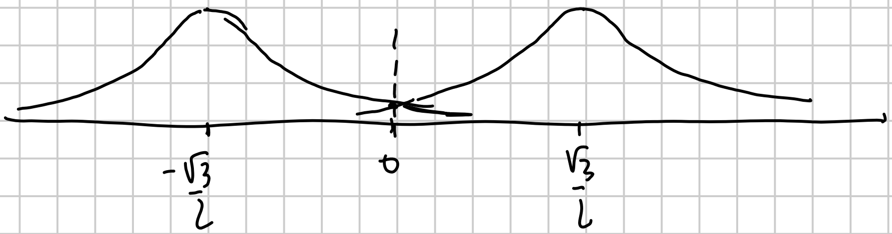
$$S_{x_s x_c}(f) = 0 \qquad \forall f$$

All'uscita del campionatore

$$y[k] = a_k \frac{\sqrt{3}}{2} + n_{cp}[k] \qquad n_{cp}(t) = n_c(t) \otimes p(t)$$

$$S_{n_{cp}}(f) = S_{n_c}(f) \cdot |P(f)|^2$$

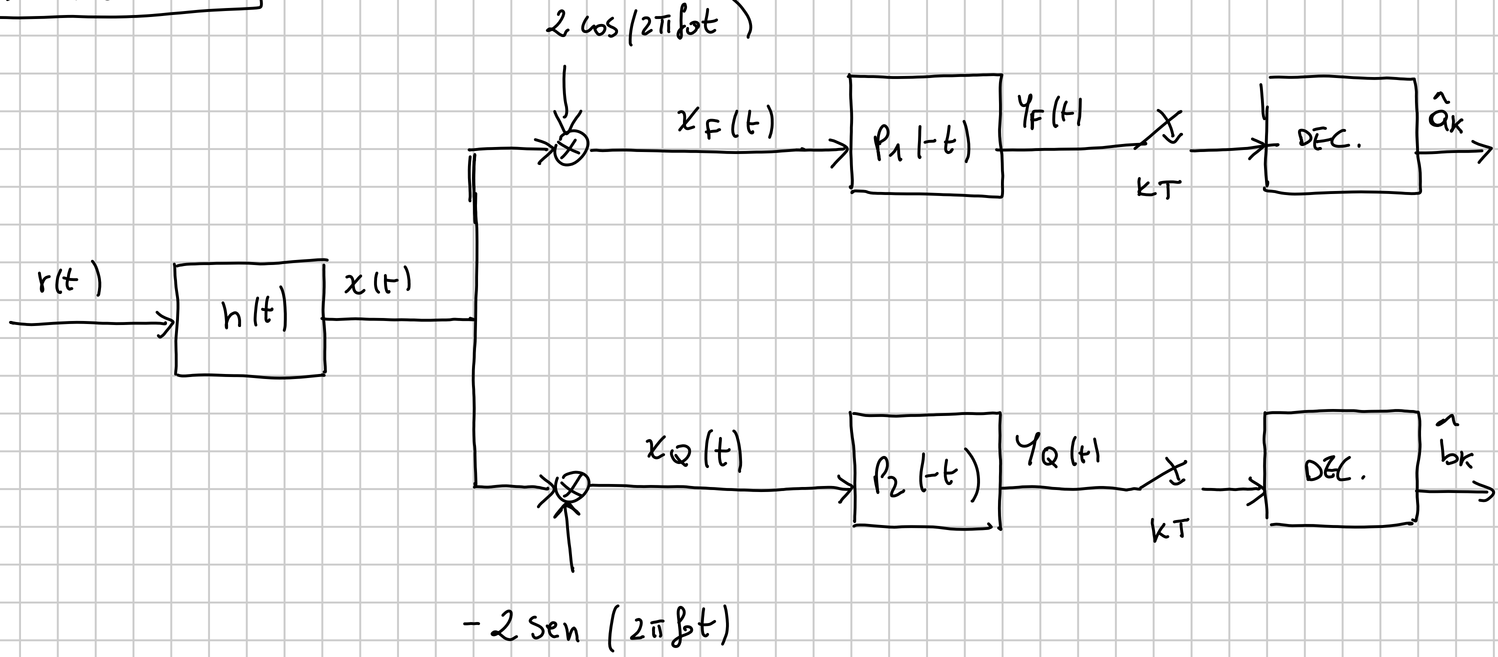
$$\sigma^2_{n_{cp}} = \int_{-\infty}^{+\infty} S_{n_{cp}}(f)\, df = N_0 \int_{-\frac{1}{T}}^{1/T} |P(f)|^2\, df = N_0$$



$$P(e) = \frac{1}{2} P(e \mid a_k = 1) + \frac{1}{2} P(e \mid a_k = -1) = P(e \mid a_k = 1) =$$

$$= Q\left(\frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{N_0}}\right) = Q\left(\sqrt{\frac{3}{4 N_0}}\right)$$

Esercizio #2



$$r(t) = \sum_i a_i \, p_1(t - iT)\cos(2\pi f_0 t) - \sum_i b_i \, p_2(t-iT)\,\text{sen}(2\pi f_0 t) + w(t)$$

a_i e b_i sono indipendenti $\quad a_i \in \{\pm 2\} \quad b_i \in \{\pm 1\}$

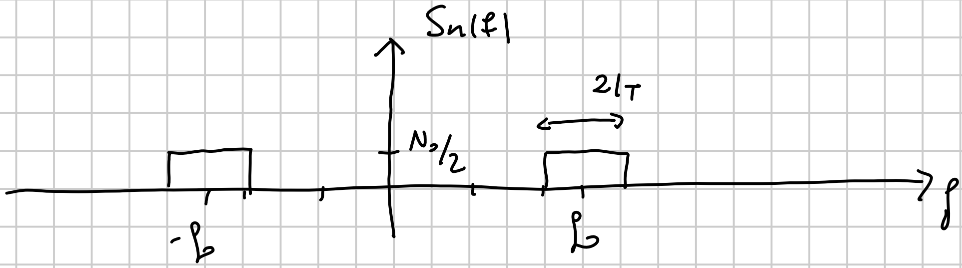
$H(f)$ è passa banda ideale di banda $\frac{2}{T}$ alle frequenze $\pm f_0$

$$P_1(f) = \begin{cases} T\sqrt{|f|} & |fT| \leq 1 \\ 0 & \text{altrove} \end{cases}$$

$$P_2(f) = \begin{cases} \sqrt{T}\ \text{sen}(\pi |f| T/2) & |fT| \leq 1 \\ 0 & \text{altrove} \end{cases}$$

$$x(t) = \sum_i a_i\, p_1(t-iT)\cos(2\pi f_0 t) - \sum_i b_i\, p_2(t-iT)\,\text{sen}(2\pi f_0 t) + n(t)$$

$$n(t) = w(t) \otimes h(t)$$



$$x_F(t) = x(t)\cdot 2\cos(2\pi f_0 t) =$$

$$= \sum_i a_i\, p_1(t-iT)\left(1+\cos(4\pi f_0 t)\right) - \sum_i b_i\, p_2(t-iT)\,\text{sen}(4\pi f_0 t) +$$

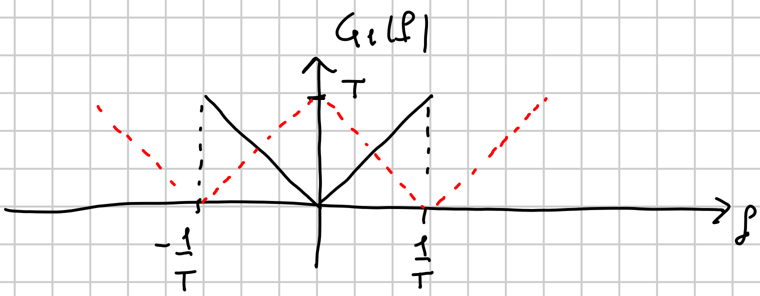
$$+ n(t)\cdot 2\cos(2\pi f_0 t)$$

$$n(t) = n_c(t)\cos(2\pi f_0 t) - n_s(t)\,\text{sen}(2\pi f_0 t)$$

$$y_F(t) = \sum_i a_i\, p_1(t-iT)\otimes p_1(-t) + n_c(t)\otimes p_1(-t)$$

$$g_1(t) = p_1(t)\otimes p_1(-t)$$

$$G_1(f) = P_1(f)\cdot P_1^*(f) = \begin{cases} T^2\,|f| & |fT|\le 1 \\ 0 & \text{altrove} \end{cases}$$



$$\sum_k G_1\left(f-\frac{k}{T}\right) = T \qquad \text{ok} \qquad g_1(kT) = \begin{cases} 1 & k=0 \\ 0 & k\neq 0 \end{cases}$$

All'uscita del campionatore

$$y_F[k] = a_k + n_{cp_1}[k] \qquad n_{cp_1}(t) = n_c(t) \otimes p_1(-t)$$

RAMO IN QUADRATURA

$$x_Q(t) = x(t)\left(-2\,\text{sen}(2\pi f_0 t)\right) =$$

$$\sum_i a_i\, p_1(t-iT)\left(-\,\text{sen}(4\pi f_0 t)\right) + \sum_i b_i\, p_2(t-iT)\, 2\,\text{sen}^2(2\pi f_0 t) +$$

$$+\, n(t)\left(-2\,\text{sen}(2\pi f_0 t)\right)$$

$$\text{sen}^2(2\pi f_0 t) = \frac{1}{2} - \frac{1}{2}\cos(4\pi f_0 t)$$

$$n(t) = n_c(t)\cos(2\pi f_0 t) - n_s(t)\,\text{sen}(2\pi f_0 t)$$

$$x_Q(t) = -\sum_i a_i\, p_1(t-iT)\,\text{sen}(4\pi f_0 t) + \sum_i b_i\, p_2(t-iT)\left(1-\cos(4\pi f_0 t)\right) +$$

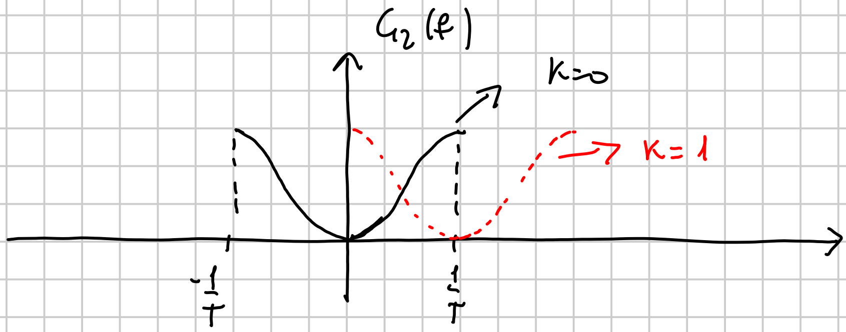
$$n(t)\left(-2\,\text{sen}(2\pi f_0 t)\right)$$

$$y_Q(t) = \sum_i b_i\, p_2(t - iT) \otimes p_2(-t) + n_s(t)$$

$$g_2(t) = p_2(t) \otimes p_2(-t)$$

$$G_2(f) = P_2(f) \cdot P_2^*(f) = \begin{cases} T\, \text{sen}^2\left(\pi |f| T/2\right) & |fT| \le 1 \\ 0 & \text{altrove} \end{cases}$$

$$G_2(f) = \begin{cases} \frac{T}{2}\left(1 - \cos(\pi f T)\right) & |fT| \le 1 \\ 0 & \text{altrove} \end{cases}$$



$$\sum_k G_2\left(f - \frac{k}{T}\right) = T$$

$$\frac{T}{2}\left(1 - \cos\left(\pi f T\right)\right) + \frac{T}{2}\left(1 - \cos\left(\pi\left(f - \frac{1}{T}\right)T\right)\right) =$$

$$\frac{T}{2} - \frac{T}{2}\cos(\pi f T) + \frac{T}{2} - \frac{T}{2}\cos\left(\pi f T - \pi\right)$$

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

$$= \frac{T}{2} - \cancel{\frac{T}{2}\cos(\pi f T)} + \frac{T}{2} + \cancel{\frac{T}{2}\cos(\pi f T)} = T$$

$$g_2(kT) = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases}$$

$$y_Q[k] = b_k + n_{sp_2}[k] \qquad n_{sp}(t) = n_s(t) \otimes p_2(-t)$$

All'uscita del campionatore, sui due rami, si ha che

$$y_F[k] = a_k + n_{cp_1}[k] = a_k + \nu_c$$

$$y_Q[k] = b_k + n_{sp_2}[k] = b_k + \nu_s$$

$$S_{\nu_c}(f) = S_{n_c}(f) \cdot |P_1(f)|^2$$

$$S_{\nu_s}(f) = S_{n_s}(f) \cdot |P_2(f)|^2$$

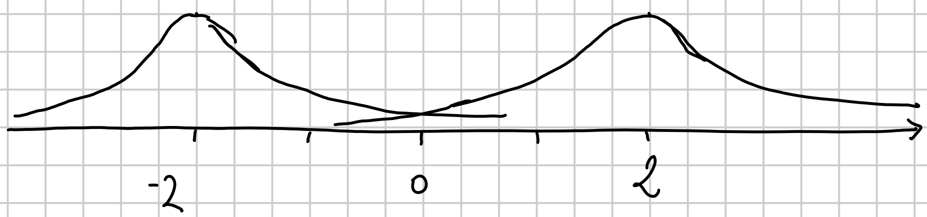
$$S_{n_c}(f) = S_{n_s}(f) = \begin{cases} N_0 & |fT| \leq 1 \\ 0 & \text{oltrove} \end{cases}$$

$$\sigma^2_{v_c} = N_0 \int_{-\infty}^{+\infty} |P_1(f)|^2 \, df = N_0$$

$$\int_{-\infty}^{+\infty} |P_1(f)|^2 \, df = \int_{-\infty}^{+\infty} G_1(f) \, df = g_1(0) = 1$$

$$\sigma^2_{v_s} = N_0 \int_{-\infty}^{+\infty} |P_2(f)|^2 \, df = N_0$$

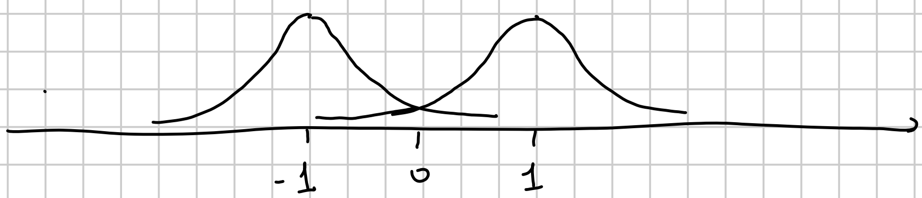
<u>Ramo in fase</u> $\quad y_F[k]\big|_{a_k} \in \mathcal{N}\left(a_k, \sigma^2_{v_c}\right)$



$$P_r\left\{ a_k + v_c < 0 \mid a_k = 2 \right\} = P_r\left\{ a_k + v_c > 0 \mid a_k = -2 \right\}$$

$$P_r(e) = P_r\left\{ a_k + v_c < 0 \mid a_k = 2 \right\} = Q\left(\frac{2}{\sqrt{N_0}}\right)$$

RAMO IN QUADRATURA $\quad y_Q[k]\big|_{b_k} \in \mathcal{N}\left(b_k, \sigma^2_{v_s}\right)$



$$P_r(e) = P_r\{b_k + \nu_s < 0 \mid b_k = 1\} = Q\left(\frac{1}{\sqrt{N_0}}\right)$$

Calcoliamo l'energia media per intervallo di segnalazione del segnale trasmesso.

$$s(t) = \sum_i a_i\, p_1(t - iT)\cos(2\pi f_0 t) - \sum_i b_i\, p_2(t - iT)\, \text{sen}(2\pi f_0 t)$$

$$E_s = \int_0^T E\{s^2(t)\}\, dt$$

$$E\{a_i \cdot a_{i+k}\} = \begin{cases} E\{a_i^2\} & k = 0 \\ E\{a_i\} \cdot E\{a_{i+k}\} & k = 0 \end{cases}$$ perché indipendenti

$$E\{a_i\} = (-2) \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = 0$$

$$E\{a_i^2\} = \frac{4}{2} + \frac{4}{2} = 4$$

$$R_a[k] = 4\,\delta[k]$$

$$E\{b_i\, b_{i+k}\} = \begin{cases} E\{b_i^2\} = 1 & k = 0 \\ E\{b_i\} \cdot E\{b_{i+k}\} = 0 & k \neq 0 \end{cases}$$

$$R_b[k] = \delta[k]$$

$$E\{a_i b_{i+k}\} = E\{a_i\} \cdot E\{b_{i+k}\} = 0 \quad \forall k$$

$$E\left\{ \int_0^T \left(\sum_i a_i p_1(t-iT)\cos(2\pi f_0 t) - \sum_i b_i p_2(t-iT)\,\mathrm{sen}(2\pi f_0 t) \right) \cdot \right.$$

$$\left(\sum_k a_k p_1(t-kT)\cos(2\pi f_0 t) - \sum_k b_k p_2(t-kT)\,\mathrm{sen}(2\pi f_0 t) \right) dt$$

$$= \int_0^T \sum_i \sum_k E\{a_i a_k\} p_1(t-iT) p_1(t-kT)\cos^2(2\pi f_0 t) \; -$$

$$- \int_0^T \sum_i \sum_k \cancel{E\{a_i b_k\} p_1(t-iT) p_2(t-kT)\cos(2\pi f_0 t)\cdot \mathrm{sen}(2\pi f_0 t)}$$

$$- \int_0^T \sum_i \sum_k \cancel{E\{b_i a_k\} p_2(t-iT) p_1(t-kT)\,\mathrm{sen}(2\pi f_0 t)\cdot\cos(2\pi f_0 t)}$$

$$+ \int_0^T \sum_i \sum_k E\{b_i b_k\} p_2(t-iT) p_2(t-kT)\,\mathrm{sen}^2(2\pi f_0 t)$$

$$E_S = \int_0^T \sum_i 4\, p_1^2(t-iT)\cos^2(2\pi f_0 t)\,dt + \int_0^T \sum_i 1\, p_2^2(t-iT)\,\mathrm{sen}^2(2\pi f_0 t)\,dt =$$

$$= 4\int_{-\infty}^{+\infty} p_1^2(t)\cos^2(2\pi f_0 t)\,dt + \int_{-\infty}^{+\infty} p_2^2(t)\,\mathrm{sen}^2(2\pi f_0 t)\,dt$$

$$= 2\int_{-\infty}^{+\infty} p_1^2(t)\left(1+\cos\left(4\pi f_0 t\right)\right)dt + \int_{-\infty}^{+\infty} p_2^2(t)\left(\frac{1}{2}-\frac{1}{2}\,\mathrm{sen}(4\pi f_0 t)\right)dt =$$

$$= 2\int_{-\infty}^{+\infty} p_1^2(t)\,dt + \frac{1}{2}\int_{-\infty}^{+\infty} p_2^2(t)\,dt = 2+\frac{1}{2}=\frac{5}{2}$$

$$||$$

$$2\int_{-\infty}^{+\infty} |P_1(f)|^2\,df + \frac{1}{2}\int_{-\infty}^{+\infty} |P_2(f)|^2\,df = \frac{5}{2}$$

$$\int_{-\infty}^{+\infty} \underbrace{p_1^2(t)\cdot\cos(4\pi f_0 t)}_{q(t)}\,df = Q(0) \qquad \text{se } f_0 \gg \frac{2}{T}$$