Esercizio # 1

Sia $X(t) = A\cos(2\pi f_0 t + \Theta)$

A Variabile aleatoria esponenziale con valor medio η

Θ $\mathcal{U}(-\pi, \pi)$

A e Θ sono indipendenti.

$$f_A(a) = \frac{1}{\eta} e^{-a/\eta} u(a)$$

$$f_\Theta(\theta) = \frac{1}{2\pi} \operatorname{rect}\left(\frac{\theta}{2\pi}\right)$$

Calcolare $\eta_x(t)$, $P_x(t)$ e determinare se è un processo SSL.

1) $\eta_x(t) = E\{X(t)\} = E\{A\cos(2\pi f_0 t + \Theta)\} =$

perché A e Θ sono indipendenti

$$= E\{A\} \cdot E\{\cos(2\pi f_0 t + \Theta)\} =$$

$$= \eta \cdot E\{\cos(2\pi f_0 t + \Theta)\} =$$

$$= \eta \cdot \int_{-\infty}^{+\infty} g(\theta) f_\Theta(\theta)\, d\theta = \eta \cdot \int_{-\infty}^{+\infty} \cos(2\pi f_0 t + \theta) \cdot \frac{1}{2\pi} \operatorname{rect}\left(\frac{\theta}{2\pi}\right) d\theta =$$

$$= \frac{\eta}{2\pi} \cdot \int_{-\pi}^{\pi} \cos(2\pi f_0 t + \theta)\, d\theta = \frac{\eta}{2\pi} \left[\sin(2\pi f_0 t + \theta)\right]_{-\pi}^{\pi} =$$

$$= \frac{\eta}{2\pi} \left[\sin(2\pi f_0 t + \pi) - \sin(2\pi f_0 t - \pi)\right] =$$

$$= \frac{\eta}{2\pi}\left[-\sin(2\pi f_0 t) - \left(\sin(2\pi f_0 t)\cdot(-1)\right)\right] = 0$$

$$\eta_x(t) = \eta_x = 0$$

$$2)\quad P_x(t) = E\{x^2(t)\} = E\{A^2 \cdot \cos^2(2\pi f_0 t + \Theta)\} \; = $$

$$\cos^2(\alpha) = \frac{1}{2} + \frac{1}{2}\cos(2\alpha)$$

$$= E\left\{A^2\left(\frac{1}{2} + \frac{1}{2}\cos(4\pi f_0 t + 2\Theta)\right)\right\} =$$

$$= E\left\{\frac{A^2}{2} + \frac{A^2}{2}\cos(4\pi f_0 t + 2\Theta)\right\} =$$

$$= E\left\{\frac{A^2}{2}\right\} + E\left\{\frac{A^2}{2}\cos(4\pi f_0 t + 2\Theta)\right\} =$$

$$E\{A^2\} = 2\eta^2$$

$$= \eta^2 + E\left\{\frac{A^2}{2}\right\}\cdot E\{\cos(4\pi f_0 t + 2\Theta)\} =$$

$$= \eta^2 + \eta^2 \cdot E\{\cos(4\pi f_0 t + 2\Theta)\} =$$

$$= \eta^2 + \eta^2 \int_{-\pi}^{\pi} \cos(4\pi f_0 t + 2\theta)\, d\theta =$$

$$= \eta^2 + \eta^2 \cdot \frac{1}{2}\left[\sin(4\pi f_0 t + 2\theta)\right]_{-\pi}^{\pi} =$$

$$= \eta^2 + \frac{\eta^2}{2}\left[\sin(4\pi f_0 t + 2\pi) - \sin(4\pi f_0 t - 2\pi)\right] =$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cdot\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$= \eta^2 + \frac{\eta^2}{2}\left[\sin(4\pi f_0 t) - \sin(4\pi f_0 t)\right] = \eta^2$$

$$P_x(t) = P_x = \eta^2$$

$X(t)$ è SSL se $\eta_x(t) = \eta_x$ e $R_x(t_1, t_2) = R_x(\tau)$ dove $\tau = t_2 - t_1$.

$$\begin{cases} X(t_1) = A\cos(2\pi f_0 t_1 + \Theta) \\ X(t_2) = A\cos(2\pi f_0 t_2 + \Theta) \end{cases}$$

2 variabili Aleatorie

$$R_x(t_1, t_2) \stackrel{\Delta}{=} E\{X(t_1)\, X(t_2)\} =$$

$$E\{A\cos(2\pi f_0 t_1 + \Theta)\cdot A\cos(2\pi f_0 t_2 + \Theta)\} =$$

$$= E\{A^2\cdot \cos(2\pi f_0 t_1 + \Theta)\cos(2\pi f_0 t_2 + \Theta)\} =$$

$$= E\{A^2\}\cdot E\{\cos(2\pi f_0 t_1 + \Theta)\cos(2\pi f_0 t_2 + \Theta)\} =$$

$$\cos(\alpha)\cdot\cos(\beta) = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta)$$

$$= 2\eta^2 \cdot E\left\{\frac{1}{2}\cos\left(2\pi f_0 (t_1 - t_2)\right) + \frac{1}{2}\cos\left(2\pi f_0 (t_1+t_2) + 2\Theta\right)\right\} =$$

$$= 2\eta^2 E\left\{\frac{1}{2}\cos\left(2\pi f_0 (t_1-t_2)\right)\right\} + \underbrace{2\eta^2 E\left\{\frac{1}{2}\cos\left(2\pi f_0 (t_1+t_2)+2\Theta\right)\right\}}_{=\,0} =$$

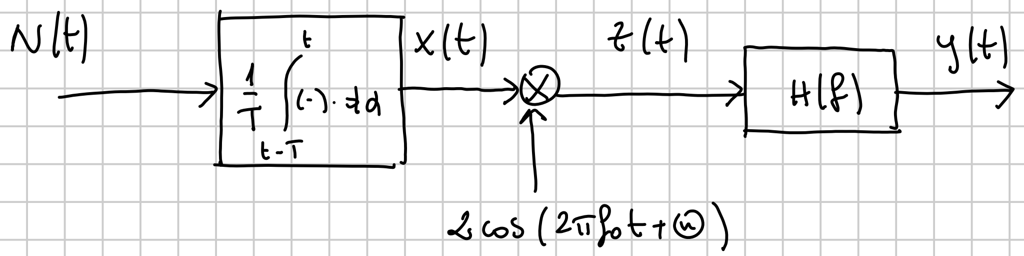
$$= \frac{2\eta^2}{2} E\left\{\cos\left(2\pi f_0 (t_1-t_2)\right)\right\} = \eta^2\cdot\cos\left(2\pi f_0 (t_1-t_2)\right) = \eta^2\cos(2\pi f_0 \tau)$$

Una proprietà delle funzione di autorelazione di processi stazionari è:

$\lim_{\tau\to+\infty} R_x(\tau) = \eta_x^2$ SE $R_x(\tau)$ NON CONTIENE COMPONENTI PERIODICHE.

In questo caso, essendo $R_x(\tau)$ periodica questa proprietà non è infatti verificata.

Esercizio #2



$N(t)$ è un processo SSL con $S_N(f) = \xi$ (processo Gaussiano Bianco)

Θ è una Variabile Aleatoria indipendente da $N(t)$ e $U(-\pi,\pi)$

Calcolare la $S_y(f)$.

- Poiché l'integratore è un sistema Lineare e Stazionario

$X(t)$ sarà un processo Gaussiano e SSL.

$$\eta_X(t) = \eta_N(t) \cdot H_1(0) = 0$$

$$R_X(\tau) = R_N(\tau) \otimes h_1(\tau) \otimes h_1(-\tau)$$

$$S_X(f) = S_N(f) \cdot |H_1(f)|^2$$

$$h_1(t) = \frac{1}{T} \operatorname{rect}\left(\frac{t - T/2}{T}\right)$$

$$H_1(f) = \operatorname{sinc}(fT)\, e^{-j2\pi f \frac{T}{2}} \Rightarrow |H_1(f)|^2 = \operatorname{sinc}^2(fT)$$

$$R_X(\tau) = \xi\, \delta(\tau) \otimes \underbrace{h_1(\tau) \otimes h_1(-\tau)}_{R_h(\tau)} = \xi\, \delta(\tau) \otimes R_h(\tau) =$$

$$= \xi\, R_h(\tau) = \xi\, \frac{1}{T}\left(1 - \frac{|\tau|}{T}\right) \operatorname{rect}\left(\frac{\tau}{2T}\right)$$

$$S(t) = 2\cos(2\pi f_0 t + \Theta) \qquad \Theta \in U(-\pi, \pi)$$

$$\left.\begin{aligned} &\eta_S = 0 \\ &R_S(\tau) = 2\cos(2\pi f_0 \tau) \end{aligned}\right\} \text{Ricavate nell'esercizio 1}$$

$$z(t) = x(t) \cdot s(t)$$

$$\eta_z(t) = E\{z(t)\} = 2\,E\{x(t) \cdot \cos(2\pi f_0 t + \Theta)\} =$$

$$= 2\,E\{x(t)\} \cdot E\{\cos(2\pi f_0 t + \Theta)\} = 0$$

$$R_z(t_1, t_2) = R_z(t, t-\tau) = E\{z(t)\, z(t-\tau)\} =$$

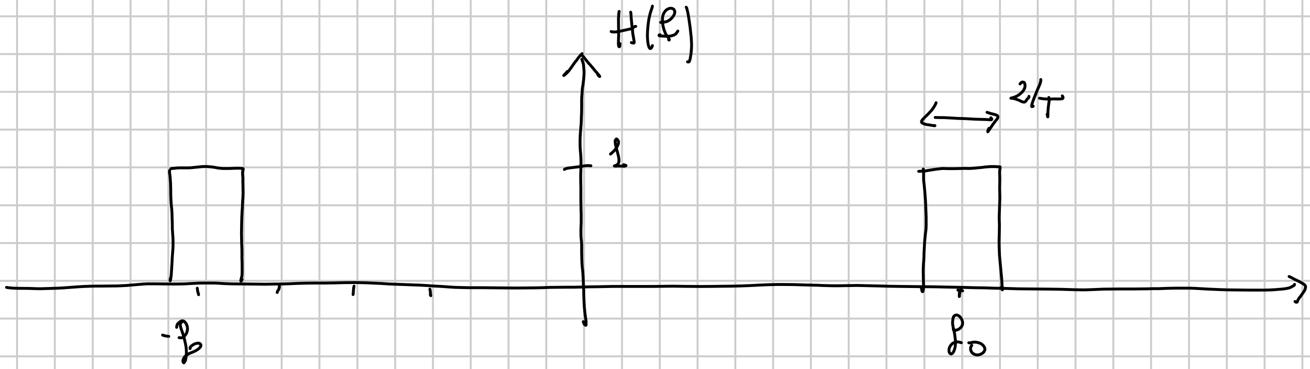
$$= 4\,E\{x(t)\, x(t-\tau) \cos(2\pi f_0 t + \Theta) \cdot \cos(2\pi f_0 (t-\tau) + \Theta)\} =$$

Θ e $x(t)$ INDIPENDENTI

$$= 4\,E\{x(t)\, x(t-\tau)\} \cdot E\{\cos(2\pi f_0 t + \Theta) \cdot \cos(2\pi f_0 (t-\tau) + \Theta)\} =$$

$$= R_x(\tau) \cdot R_s(\tau) = 2\, R_x(\tau) \cdot \cos(2\pi f_0 \tau)$$

$$R_z(\tau) = 2\, R_x(\tau) \cdot \cos(2\pi f_0 \tau)$$

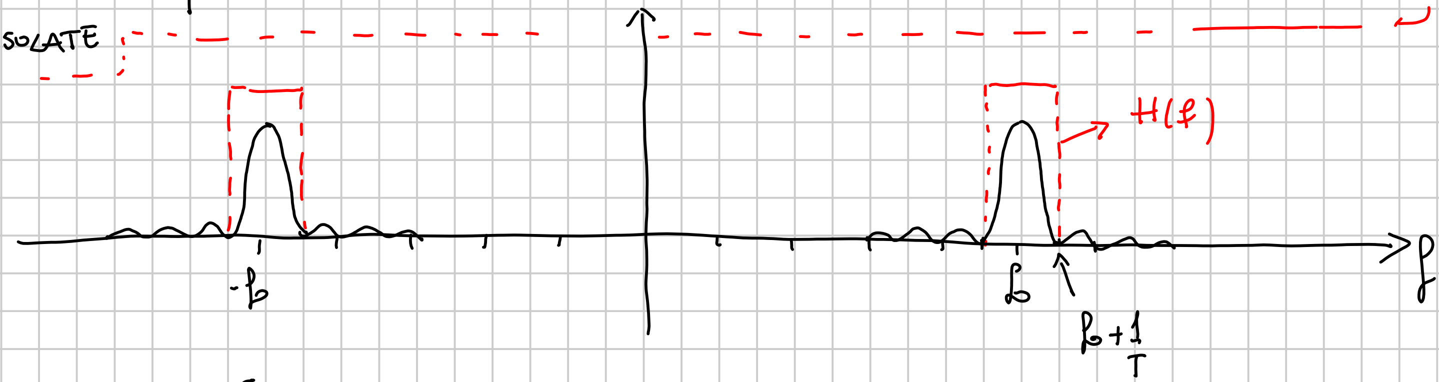


$$S_y(f) = S_z(f) \cdot |H(f)|^2$$

$$R_z(\tau) = 2\, R_x(\tau) \cdot \cos(2\pi f_0 \tau) \qquad R_x(\tau) = \frac{\varepsilon}{T}\left(1 - \frac{|\tau|}{T}\right) \mathrm{rect}\left(\frac{\tau}{2T}\right)$$

$$S_z(f) = \xi \left\{ \mathrm{sinc}^2\left((f+f_0)T\right) + \mathrm{sinc}^2\left((f-f_0)T\right) \right\}$$

Se $f_0 \gg \frac{1}{T}$ le due repliche di $S_X(f)$ centrate a $\pm f_0$ possono considerarsi ISOLATE



$$S_y(f) = \xi \left[\mathrm{sinc}^2\left((f+f_0)T\right) \mathrm{rect}\left(\frac{(f+f_0)T}{2}\right) + \mathrm{sinc}^2\left((f-f_0)T\right) \mathrm{rect}\left(\frac{(f-f_0)T}{2}\right) \right]$$

È possibile ricavare la ddp di 1° ordine di $y(t)$, $f_y(y;t)$?

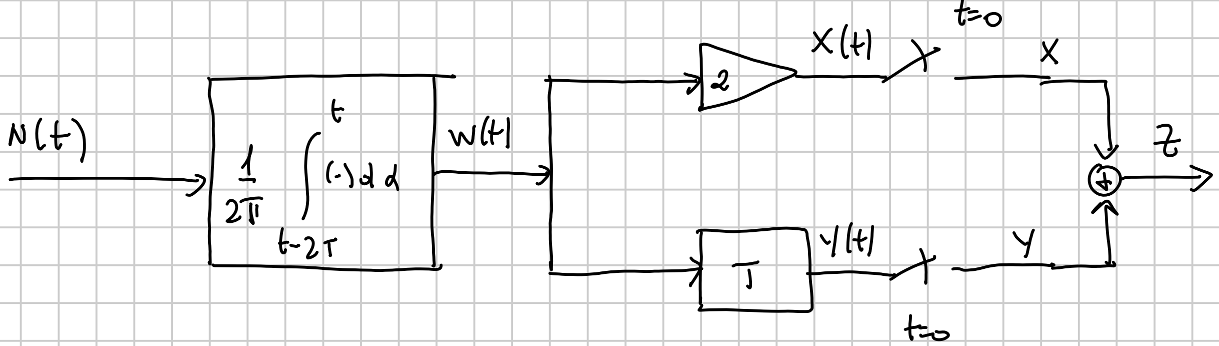
OSSERVAZIONE: Condizionatamente ad un valore di Θ, il processo $Y(t)$ è Gaussiano quindi $f_{Y|\Theta}(y|\theta;t)$ è la ddp di una Gaussiana.
Quindi la ddp congiunta tra le variabili $Y(t)$ e Θ è

$$f_{Y\Theta}(y,\theta;t) = f_{Y|\Theta}(y|\theta;t) \cdot f_\Theta(\theta)$$

È possibile quindi ricavare adesso la ddp marginale dalla congiunta.

$$f_Y(y;t) = \int_{-\infty}^{+\infty} f_{Y\Theta}(y,\theta;t)\, d\theta$$

Esercizio 3



Trovare la ddp di Z

$N(t)$ é Gaussiono Bionco $S_N(f) = 2T$

$$h(t) = \frac{1}{2T} \operatorname{rect}\left(\frac{t-T}{2T}\right)$$

$$X(t) = 2W(t)$$

$$y(t) = W(t-T)$$

Quondo campiono a $t=0$

$$X = 2W(0)$$

$$Y = W(-T)$$

Poiché $W(t)$ é Gaussiono allora X e Y sono Variabili Aleatorie Congiuntamente Gaussione.

$$Z = X+Y = 2W(0) + W(-T)$$

La combinazione Lineare di Variabili Aleatorie Congiuntamente Gaussione é una Variabile Aleatoria Gaussiana.

$$\eta_W(t) = \eta_N(t) \cdot H(0) = \eta_N \cdot H(0) = 0$$

$$R_W(\tau) = \left(1 - \frac{|\tau|}{2T}\right) \operatorname{rect}\left(\frac{\tau}{4T}\right)$$

$$E\{z\} = E\{x+y\} = E\{x\} + E\{y\} = 2E\{w(0)\} + E\{w(-T)\} = 0$$

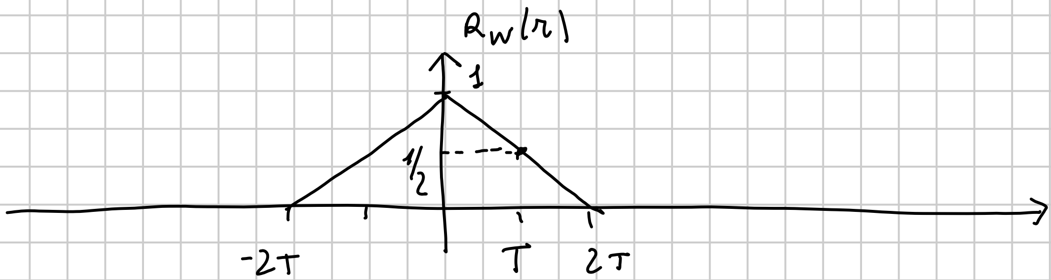
$$\sigma_z^2 = E\{(z-\eta_z)^2\} = E\{z^2\} =$$

$$= E\{[2w(0) + w(-T)]^2\} =$$

$$= E\{4w^2(0) + w^2(-T) + 4w(0)w(-T)\} =$$

$$= E\{4w^2(0)\} + E\{w^2(-T)\} + 4E\{w(0)w(-T)\} =$$

$$= 4R_w(0) + R_w(0) + 4R_w(T)$$



$R_w(0) = 1$

$R_w(T) = \frac{1}{2}$

$$\sigma_z^2 = 4 + 1 + 4\cdot\frac{1}{2} = 5 + 2 = 7$$

$$f_z(z;t) = \frac{1}{\sqrt{14\pi}}\, e^{-z^2/14}$$