Segnali Periodici

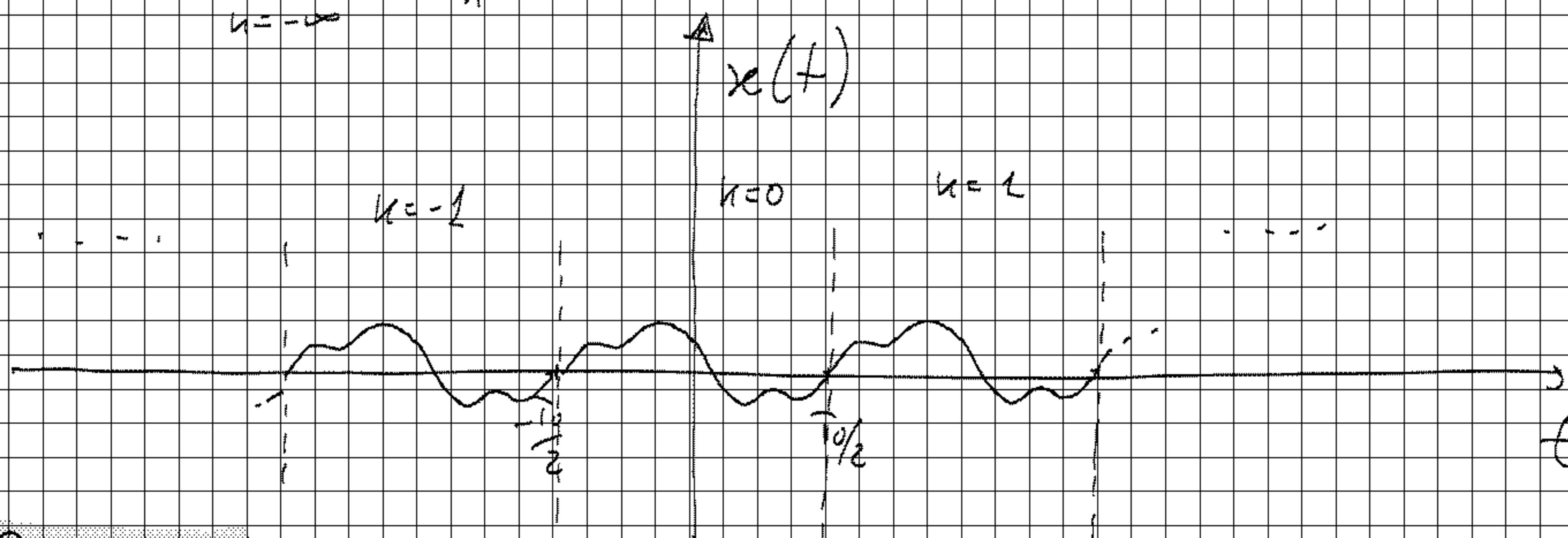
$$x(t) = x(t - kT_0)$$

T_0 = periodo , $f_0 \triangleq \dfrac{1}{T_0}$ frequenza

ENERGIA

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \sum_{k=-\infty}^{+\infty} \int_{-\frac{T_0}{2}+kT_0}^{\frac{T_0}{2}+kT_0} |x(t)|^2 \, dt =$$

$$= \sum_{k=-\infty}^{+\infty} E_{x_k} = \infty$$



POTENZA

$$P_x = \lim_{T \to \infty} \frac{E_{Tx}}{T} = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \, dt$$

$\Rightarrow$ discretizziamo la variabile $T = kT_0$

$$P_x = \lim_{k \to \infty} \frac{1}{kT_0} \int_{-\frac{kT_0}{2}}^{\frac{kT_0}{2}} |x(t)|^2 \, dt = \lim_{k \to \infty} \frac{1}{kT_0} \, k \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 \, dt$$

$$= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 \, dt$$

VALOR MEDIO

$$x_m \triangleq \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \, dt = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \, dt$$

ANALISI DI FOURIER

Ogni segnale periodico di periodo T_0 che soddisfa la condizione di Dirichelet può essere scritto come la somma di infinite sinusoidi a frequenze multiple di $1/T_0$:

$$x(t) = \sum_{n=0}^{\infty} A_n \cos(2\pi n f_0 t + \varphi_n) \quad , \quad f_0 = \frac{1}{T_0} \quad , \quad T_0 = \text{periodo}$$

$$= A_0 + \sum_{n=1}^{\infty} A_n \cos(2\pi n f_0 t + \varphi_n)$$

SVILUPPO IN SERIE DI FOURIER (FORMA POLARE)

$A_0 \Rightarrow$ componente continua

$$A_0 = x_m = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, dt = \underbrace{\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A_0\, dt}_{= A_0} +$$

$$+ \underbrace{\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} A_1 \cos(2\pi f_0 t + \varphi_1)\, dt}_{= 0} + \underbrace{\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} A_2 \cos(4\pi f_0 t + \varphi_2)\, dt + \dots}_{= 0}$$

$A_1 \cos(2\pi f_0 t + \varphi_1) \Rightarrow$ prima armonica

$A_2 \cos(4\pi f_0 t + \varphi_2) \Rightarrow$ seconda armonica

$\vdots$

Criterio di Dirichelet

1) $x(t)$ è assolutamente integrabile nel periodo T_0

$$\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|\, dt < \infty$$

2) $x(t)$ è continua o presenta un numero finito di discont. di prima specie nel periodo T_0

3) $x(t)$ è derivabile su tutto il periodo escluso al più un numero finito di punti dove esistono finite le derivate dx e sx.

SVILUPPO IN SERIE DI FOURIER IN FORMA COMPLESSA

$$\boxed{x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{j2\pi n f_0 t}}$$

Equazione di sintesi della TRASFORMATA SERIE DI FOURIER (ANTITRASFORMATA)

$$\boxed{X_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt}$$

Equazione di analisi della TRASFORMATA SERIE DI FOURIER (TRASFORMATA)

$$x(t) \overset{TSF}{\Longleftrightarrow} X_n$$

$$\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sum_{m=-\infty}^{+\infty} X_m e^{j2\pi m f_0 t}\, e^{-j2\pi n f_0 t}\, dt =$$

$$= \frac{1}{T_0}\sum_{m=-\infty}^{+\infty} X_m \underbrace{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi (m-n) f_0 t}\, dt}_{= \begin{cases} T_0 & m=n \\ 0 & m\neq n \end{cases}} =$$

$$= \frac{1}{T_0}\sum_{m=-\infty}^{+\infty} X_m \left[\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos\left[2\pi(m-n) f_0 t\right] dt + \underbrace{j\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sin\left[2\pi(m-n) f_0 t\right] dt}_{=0 \quad \forall m,n}\right]$$

$$= \frac{1}{T_0} X_n T_0 = X_n$$

) La TSF è biunivoca : $\forall x(t)\ \exists!\ X_n$

$$x(t) \underset{}{\overset{TSF}{\Longleftrightarrow}} X_n$$

segnale analogico periodico ⟺ sequenza complessa

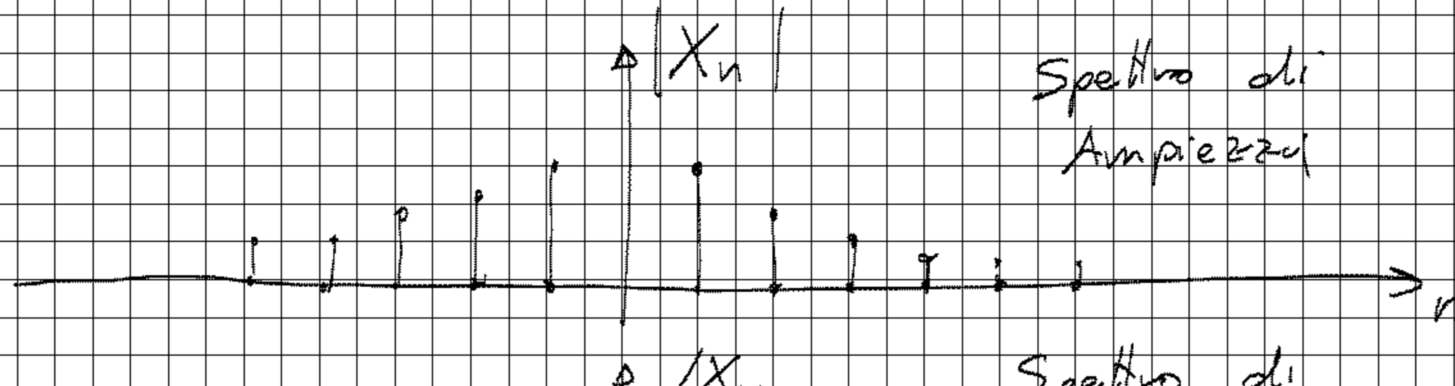
SPETTRO DI UN SEGNALE PERIODICO

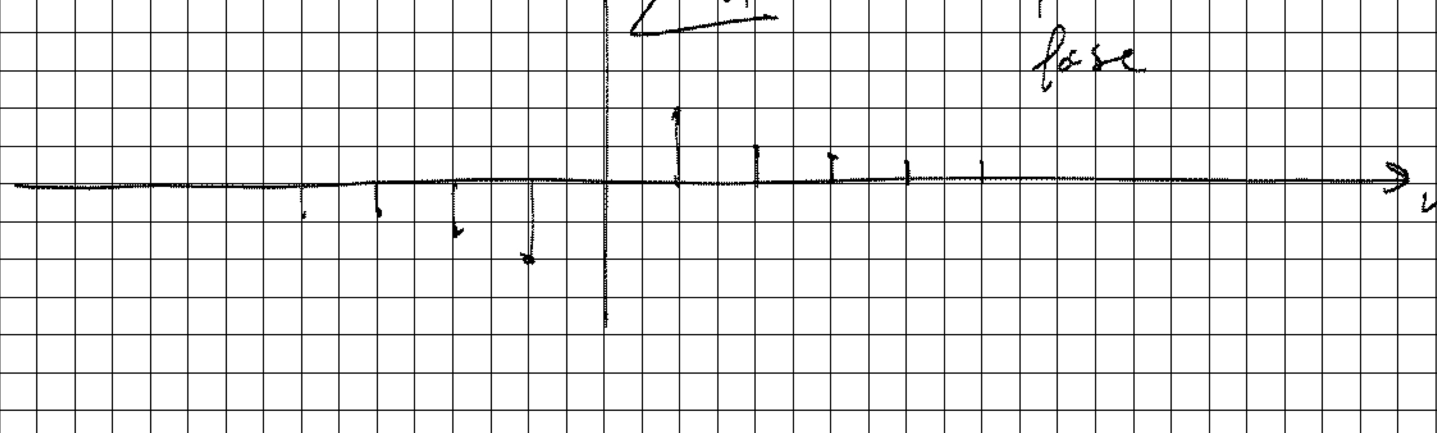
) Rappresentazione del contenuto frequenziale di un segnale periodico.

) Lo spettro di un segnale periodico è una sequenza complessa

⇓ **spettro di ampiezza** $|X_n|$ ⟹ **spettro di fase** $\angle X_n$

$$X_n = C_n e^{j\varphi_n} \Rightarrow \begin{cases} |X_n| = C_n \\ \angle X_n = \varphi_n \end{cases}$$





Esempio "Spettro di un coseno"

$$x(t) = A\cos(2\pi f_0 t)$$

$$X_0 = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A\cos(2\pi f_0 t)\, e^{-j2\pi 0\cdot f_0 t}\, dt =$$

$$= \frac{A}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(2\pi f_0 t)\, dt = 0$$

$$X_1 = \frac{A}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(2\pi f_0 t)\, e^{-j2\pi f_0 t}\, dt =$$

$$= \frac{A}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \left[\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right] e^{-j2\pi f_0 t}\, dt =$$

$$= \frac{A}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} 1\, dt + \frac{A}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{-j4\pi f_0 t}\, dt = \frac{A}{2}$$

$$X_{-1} = \frac{A}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(2\pi f_0 t)\, e^{+j2\pi f_0 t}\, dt =$$

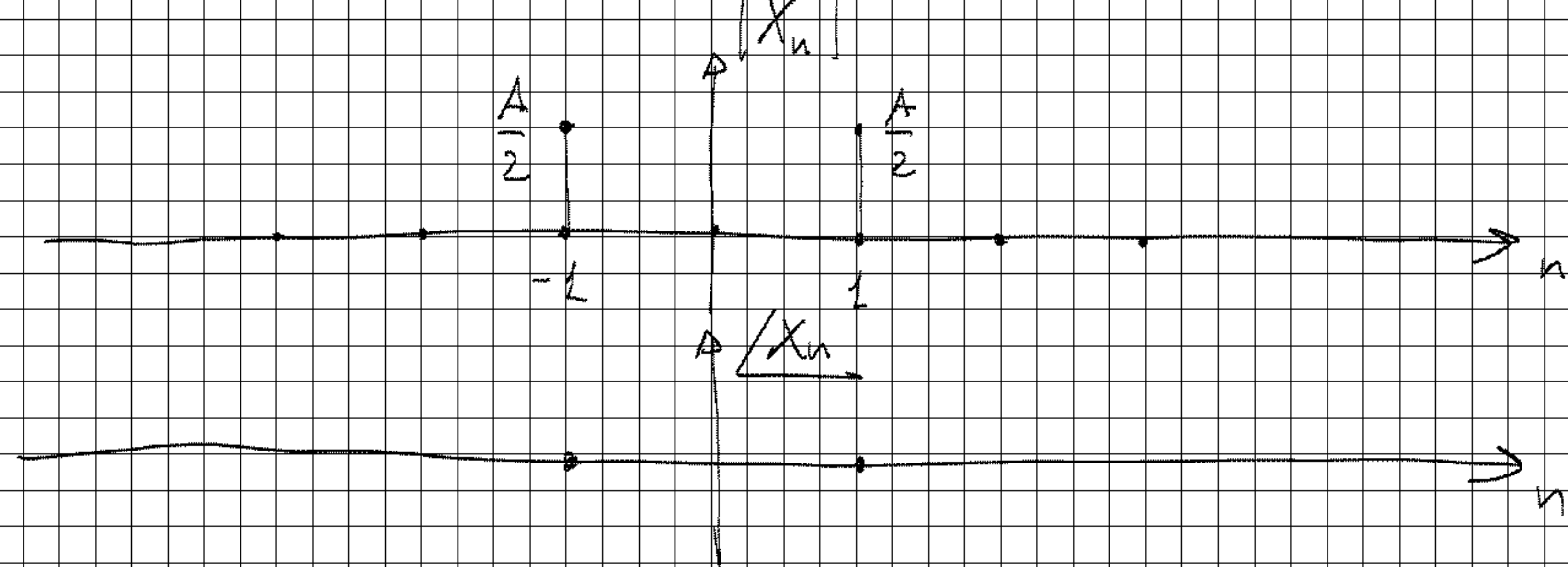
$$= \frac{A}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j4\pi f_0 t}\, dt + \frac{A}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} 1\, dt = \frac{A}{2}$$

$$X_n = \frac{A}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(2\pi f_0 t)\, e^{-j2\pi n f_0 t}\, dt =$$

$$= \frac{A}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi f_0 t}\, e^{-j2\pi n f_0 t}\, dt + \frac{A}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{-j2\pi f_0 t}\, e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{A}{2T_0}\underbrace{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{-j2\pi (n-1) f_0 t}\, dt}_{= \begin{cases} T_0 & n=1 \\ 0 & \text{altrove} \end{cases}} + \frac{A}{2T_0}\underbrace{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{-j2\pi (n+1) f_0 t}\, dt}_{= \begin{cases} T_0 & n=-1 \\ 0 & \text{altrove} \end{cases}}$$

$X_n \neq 0$ solo per $n = \pm 1$



Si poteva anche notare l'equazione di sintesi

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n\, e^{+j2\pi n f_0 t}$$

$$x(t) = A\cos(2\pi f_0 t) = \frac{A}{2}\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t} \right]$$

sono presenti solo le armoniche $n = \pm 1$

Esempio "Spettro di un seno"

$$x(t) = A \sin(2\pi f_0 t)$$

$$X_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A \sin(2\pi f_0 t)\, e^{-j2\pi n f_0 t}\, dt =$$

$$= \frac{A}{j2T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi f_0 t}\, e^{-j2\pi n f_0 t} - \frac{A}{j2T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{-j2\pi f_0 t}\, e^{-j2\pi n f_0 t}\, dt$$

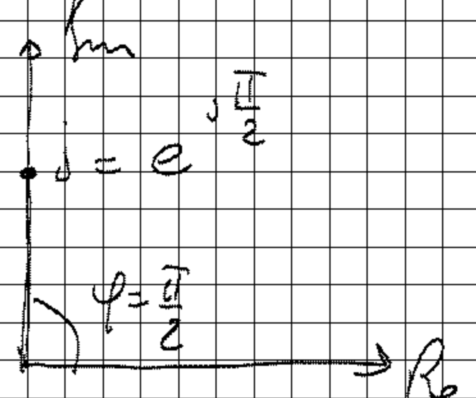
$$= \frac{A}{j2T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{-j2\pi (n-1) f_0 t}\, dt - \frac{A}{j2T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{-j2\pi (n+1) f_0 t}\, dt$$

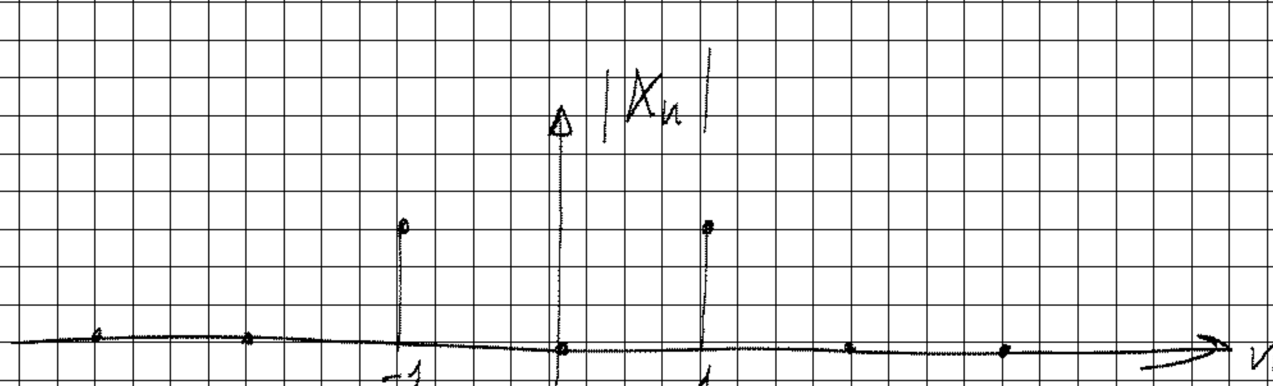
$$X_1 = \frac{A}{j2} = \frac{A}{2} e^{-j\frac{\pi}{2}} \qquad X_n = 0 \quad n \neq \pm 1$$

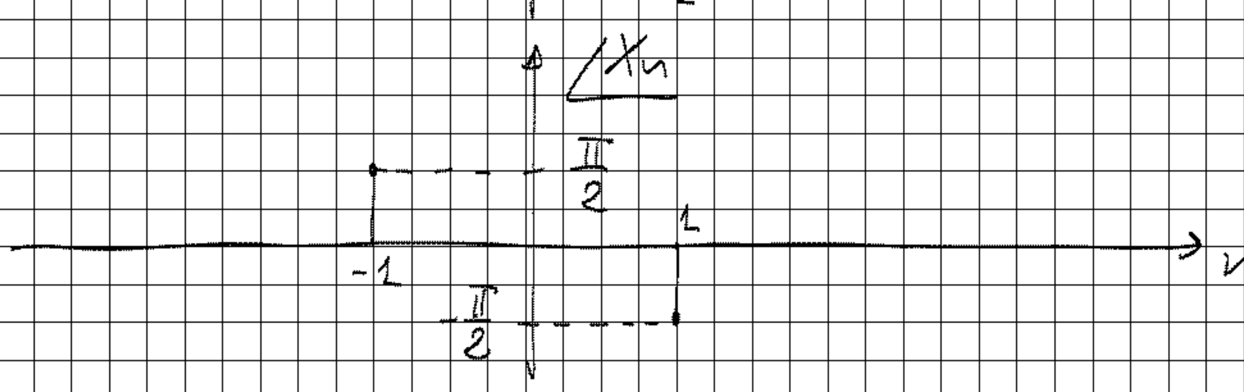
$$X_{-1} = -\frac{A}{j2} = \frac{A}{2} e^{j\frac{\pi}{2}}$$

$$j = e^{j\frac{\pi}{2}}$$

$$-j = e^{-j\frac{\pi}{2}}$$







Linearità della TSF

$$z(t) = x(t) + y(t) \iff Z_n = X_n + Y_n$$

$$x(t) \iff X_n \quad , \quad y(t) = Y_n$$

Dimostrazione:

$$Z_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} z(t)\, e^{-j2\pi n f_0 t}\, dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} [x(t)+y(t)]\, e^{-j2\pi n f_0 t}\, dt$$

$$= \underbrace{\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt}_{X_n} + \underbrace{\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} y(t)\, e^{-j2\pi n f_0 t}\, dt}_{Y_n}$$

Segnali reali e periodici - Simmetria Hermitiana

$$x(t) = x(t - kT_0) \quad , \quad k \in \mathbb{Z} \;,\; T_0 \in \mathbb{R}^+ \;,\; x \in \mathbb{R}$$

$$X_{-n} = X_n^*$$

Dimostrazione:

$$X_{-n} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi(-n)f_0 t}\, dt =$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{j2\pi n f_0 t}\, dt = \frac{1}{T_0}\left[\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x^*(t)\, e^{-j2\pi n f_0 t}\, dt\right]^* =$$

$$= \left[\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt\right]^* = X_n^*$$

Segnali periodici reali e pari

$$x(t) = x(-t)$$

$$X_{-n} = X_n$$

$$X_{-n} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi(-n)f_0 t}\, dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{j2\pi n f_0 t}\, dt =$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(-t')\, e^{-j2\pi n f_0 t'}\, dt' = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t')\, e^{-j2\pi n f_0 t'}\, dt' = X_n$$

Notare che essendo reale vale anche

$$X_{-n} = X_n^* \quad \Rightarrow \quad X_n = X_n^*$$

$\Rightarrow$ Segnali periodici reali e pari hanno TSF reale e pari

Segnali periodici e dispari

$$x(t) = -x(-t)$$

$$X_{-n} = -X_n$$

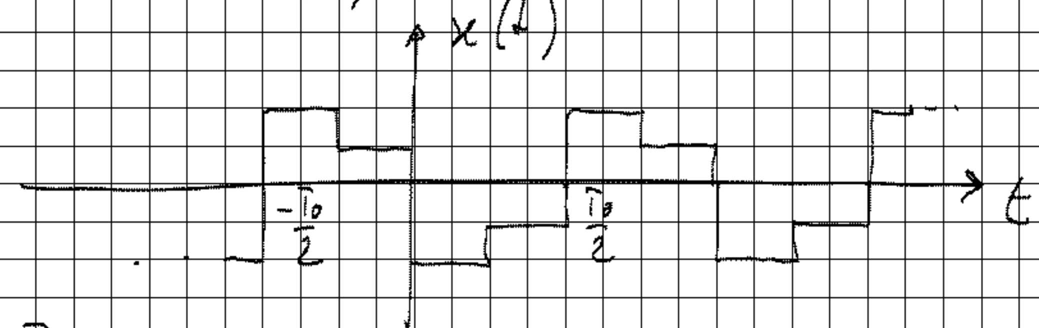
$$X_{-n} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{+j2\pi n f_0 t}\, dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(-t')\, e^{-j2\pi n f_0 t'}\, dt' =$$

$$= -\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t')\, e^{-j2\pi n f_0 t'}\, dt' = -X_n$$

$X_{-n} = X_n^* = -X_n \quad \Rightarrow \quad X_n$ è una sequenza immaginaria dispari

Segnali periodici e alternativi

$$x(t) = -x\left(t - \frac{T_0}{2}\right)$$



$$X_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt =$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{0} x(t)\, e^{-j2\pi n f_0 t}\, dt + \frac{1}{T_0}\int_{0}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt =$$

$$\Rightarrow \quad t' = t + \frac{T_0}{2}$$

$$= \frac{1}{T_0}\int_{0}^{\frac{T_0}{2}} x\left(t' - \frac{T_0}{2}\right) e^{-j2\pi n f_0 \left(t' - \frac{T_0}{2}\right)}\, dt' + \frac{1}{T_0}\int_{0}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{0}^{\frac{T_0}{2}} \left[-x(t)\, e^{+j2\pi n f_0 \frac{T_0}{2}} + x(t)\right] e^{-j2\pi n f_0 t}\, dt =$$

$$= \frac{1}{T_0}\int_{0}^{\frac{T_0}{2}} x(t)\underbrace{\left[1 - e^{j\pi n}\right]}_{= \begin{cases} 0 & n \text{ pari} \\ 2 & n \text{ dispari}\end{cases}} e^{-j2\pi n f_0 t}\, dt =$$

$$= \begin{cases} \frac{2}{T_0}\int_{0}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt & n \text{ dispari} \\ 0 & n \text{ pari} \end{cases}$$

Definizione di segnali periodici a partire da segnali aperiodici

$x_0(t)$ = segnale aperiodico

$$x(t) = \sum_{n=-\infty}^{+\infty} x_0(t - nT_0)$$

Prova:

$$x(t) = x(t - kT_0)$$

$$x(t - kT_0) = \sum_{n=-\infty}^{+\infty} x_0(t - kT_0 - nT_0) = \sum_{n=-\infty}^{+\infty} x_0[t - (n+k)T_0]$$

$$= \sum_{n'=-\infty}^{+\infty} x_0(t - n'T_0) = x(t) \qquad , n' = n + k$$