TEOREMA (MOLTIPLICATORI DI LAGRANGE)

DEF. $(\bar{x},\bar{y})$ è ESTREMO VINCOLATO PER f CON VINCOLO G se $f(\bar{x},\bar{y}) = \max\limits_{G(x,y)=0} f(x,y)$ oppure $f(\bar{x},\bar{y}) = \min\limits_{G(x,y)=0} f(x,y)$

$f \in C^1(\mathbb{R}^2)$ $\quad G \in C^1(\mathbb{R}^2)$ $\quad (\bar{x},\bar{y})$ SIA ESTREMO VINCOLATO

$\nabla G(\bar{x},\bar{y}) \neq 0$ $\quad \left(\partial_x G(\bar{x},\bar{y}) \neq 0 \overset{\text{oppure}}{\vee} \partial_y G(\bar{x},\bar{y}) \neq 0 \right)$

$\Longrightarrow \exists\, \bar{\lambda} \in \mathbb{R} \quad \nabla f(\bar{x},\bar{y}) = \lambda \nabla G(\bar{x},\bar{y})$

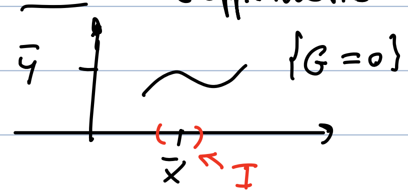
<u>OSS</u> massimi e minimi vincolati vanno cercati dove

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda G(x,y)$$

è tale che $\nabla \mathcal{L}(x,y,\lambda) = 0 \iff \begin{cases} \partial_x \mathcal{L}(x,y,\lambda) = 0 = \partial_x f - \lambda \partial_x G \\ \partial_y \mathcal{L}(x,y,\lambda) = 0 = \partial_y f - \lambda \partial_y G \\ \partial_\lambda \mathcal{L}(x,y,\lambda) = 0 = G \end{cases}$

<u>Dim</u> supponiamo $G \in C^1$ $\quad G(\bar{x},\bar{y}) = 0$ $\quad \partial_y G(\bar{x},\bar{y}) \neq 0$ $\Rightarrow \exists\, I \ni \bar{x}$ $\quad \exists\, g : I \to \mathbb{R}$

$g(\bar{x}) = \bar{y}$ $\quad G(x,g(x)) = 0$ $\quad \forall x \in I$



$$g'(x) = -\frac{\partial_x G(x,g(x))}{\partial_y G(x,g(x))} \qquad x \in I$$

$\Phi(x) = f(x,g(x)) \geq \Phi(\bar{x})$ $\qquad \phi \in C^1(I)$

$\Rightarrow$ FERMAT $\Phi'(\bar{x}) = 0$

$$\frac{d}{dx}\Phi(x)\Big|_{x=\bar{x}} = \frac{d}{dx} f(x,g(x))\Big|_{x=\bar{x}} = \frac{\partial}{\partial x} f(x,g(x)) + \frac{\partial f}{\partial y}(x,g(x))\, g'(x)\Big|_{x=\bar{x}} = 0$$

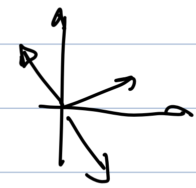
$$f_x(\bar{x},g(\bar{x})) + f_y(\bar{x},g(\bar{x})) \left[-\frac{\partial_x G(\bar{x},g(\bar{x}))}{\partial_y G(\bar{x},g(\bar{x}))} \right] = 0$$

$$f_x(\bar{x},\bar{y}) + f_y(\bar{x},\bar{y}) \left[-\frac{\partial_x G(\bar{x},\bar{y})}{\partial_y G(\bar{x},\bar{y})} \right] = 0$$

$$\partial_y G(\bar x,\bar y)\,\partial_x f(\bar x,\bar y) - \partial_x G(\bar x,\bar y)\,\partial_y f(\bar x,\bar y) = 0$$

$$(\partial_x f, \partial_y f)(\bar x,\bar y) \perp (\partial_y G, -\partial_x G)(\bar x,\bar y) \iff \langle(\partial_x f,\partial_y f),(\partial_y G,-\partial_x G)\rangle = 0$$

$(v_1,v_2)\in\mathbb{R}^2$ $\qquad \langle (v_1,v_2),(-v_2,v_1)\rangle = 0$



$(-v_2,v_1)\perp(v_1,v_2)$

$(\partial_y G, -\partial_x G)\perp(\partial_x G,\partial_y G)$

$\Rightarrow (\partial_x f,\partial_y f) \parallel (\partial_x G,\partial_y G) \iff \exists\lambda,\quad \partial_x f=\lambda\partial_x G, \quad \partial_y f = \lambda \partial_y G$

SE $\partial_x G(\bar x,\bar y)\neq 0 \qquad \exists J\ni\bar y \qquad \exists h\in C^1(J)$

$G(h(y),y)=0 \qquad h'(y) = -\dfrac{\partial_y G(h(y),y)}{\partial_x G(h(y),y)}$

$\varphi(y) = f(h(y),y) \qquad \dfrac{d}{dy}\varphi'(y)\Big|_{y=\bar y}=0$

$$\frac{d}{dy} f(h(y),y) = f_x(h(y),y)\,h'(y)+f_y(h(y),y)\Big|_{y=\bar y}=0$$

$$= f_x(\bar x,\bar y)\left[-\frac{\partial_y G(\bar x,\bar y)}{\partial_x G(\bar x,\bar y)}\right]+f_y(\bar x,\bar y)=0$$

max e min locali

$f(x,y)=(x-y)(x^2+y^2-1) \qquad (x,y)\in\mathbb{R}^2$

$\boxed{x_n^2+y_n^2\to+\infty}$

$\max\limits_{\mathbb{R}^2} f = \text{N.E.} \qquad \min\limits_{\mathbb{R}^2} f = \text{N.E.} \qquad (x_n,y_n)\to\infty \qquad f(x_n,y_n)\to+\infty$

$(x_n',y_n')\to\infty \qquad f(x_n',y_n')\to-\infty$

$\partial_x f = x^2+y^2-1+(x-y)2x = 3x^2-2xy+y^2-1=0 \qquad \nabla f=0$

$\partial_y f = -(x^2+y^2-1)+(x-y)2y = -3y^2+2xy-x^2+1=0$

I+II $\quad 3x^2-x^2+y^2-3y^2=0 \qquad y=\pm x$

$2(x^2 - y^2) = 0$

$\partial_x f = 3x^2 - 2x^2 + x^2 - 1 = 0 \qquad 2x^2 = 1 \qquad x = \pm \frac{1}{\sqrt{2}}$

$\partial_y f = 0$

P_1: $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$ $\qquad$ P_2: $x = -\frac{1}{\sqrt{2}}$, $y = -\frac{1}{\sqrt{2}}$ $\qquad$ P_3: $x = \frac{1}{\sqrt{6}}$, $y = -\frac{1}{\sqrt{6}}$ $\qquad$ P_4: $x = -\frac{1}{\sqrt{6}}$, $y = \frac{1}{\sqrt{6}}$

uniche soluzioni del sistema $\nabla f = (0,0)$

$$Hf(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6x - 2y & -2x + 2y \\ -2x + 2y & -6y + 2x \end{pmatrix}$$

$$Hf\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \begin{pmatrix} \frac{6}{\sqrt{2}} - \frac{2}{\sqrt{2}} & 0 \\ 0 & -\frac{6}{\sqrt{2}} + \frac{2}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{2}} & 0 \\ 0 & -\frac{4}{\sqrt{2}} \end{pmatrix}$$

$$Hf(P_2) = \begin{pmatrix} -\frac{4}{\sqrt{2}} & 0 \\ 0, & \frac{4}{\sqrt{2}} \end{pmatrix}$$

$$Hf(P_3) = \frac{4}{\sqrt{6}} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

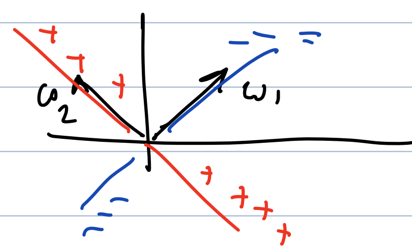
$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 4 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0 \qquad \lambda = \frac{2 \pm \sqrt{4-3}}{1} = 2 \pm 1$$

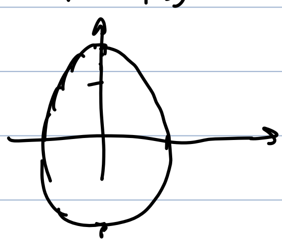
$Hf = A \qquad \lambda_1 < 0 \qquad \lambda_2 > 0$

$\exists\, w_1, w_2 \in \mathbb{R}^2 \qquad A w_1 = \lambda_1 w_1 \qquad w_1 \perp w_2 \Leftarrow A = A^T$

$\|w_1\| = \|w_2\| = 1 \qquad A w_2 = \lambda_2 w_2$



$f(x,y) = x^2 + 3y \qquad$ max e min $\qquad \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 = G$



$\max\limits_{G=0} f \qquad \{G = 0\}$ limitato e chiuso

$$\{G = 0\} = \left\{ \begin{pmatrix} 2\cos(t) \\ 3\sin(t) \end{pmatrix} \quad t \in [0, 2\pi] \right\}$$

$$\phi(t) = f|_G = [2\cos(t)]^2 + 3\cdot 3\sin(t) = 4\cos^2(t) + 8\sin(t) \qquad t\in[0,2\pi]$$

$$\phi'(t) = 8\cos(t)(-\sin(t)) + 8\cos(t) = \cos(t)[8 - 8\sin(t)]$$

$$\phi'(t)=0 \iff \cos(t)=0 \qquad t = \frac{\pi}{2}, \frac{3\pi}{2}$$

$t=0 \quad t=2\pi \qquad t=\frac{\pi}{2} \quad t=\frac{3\pi}{2}$

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda G(x,y)$$
$$= x^2+3y - \lambda\left(\frac{x^2}{4}+\frac{y^2}{8}-1\right)$$

$$\nabla G = \left(\frac{2x}{2}, \frac{2y}{8}\right) = \left(x, \frac{2y}{8}\right) = 0$$

$\iff (x,y)=(0,0)$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x} = 2x - 2\lambda\dfrac{x}{4} = 2x\left(1-\dfrac{\lambda}{4}\right) = 0 \\ \dfrac{\partial \mathcal{L}}{\partial y} = 3 - 2\lambda\dfrac{y}{8} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} = \dfrac{x^2}{4}+\dfrac{y^2}{8}-1 = 0 \end{cases}$$

$(\bar{x},\bar{y},\bar{\lambda})$ t.c. $\nabla\mathcal{L}(\bar{x},\bar{y},\bar{\lambda})=0$

$\Downarrow$

$(\bar{x},\bar{y})$ sarà stazionario di f.

$\boxed{x=0} \qquad \dfrac{y^2}{8}=1 \implies y = \pm 3$

$3-\dfrac{2}{8}\lambda y=0 \qquad 3-\dfrac{2}{8}\lambda(\pm 3)=0$

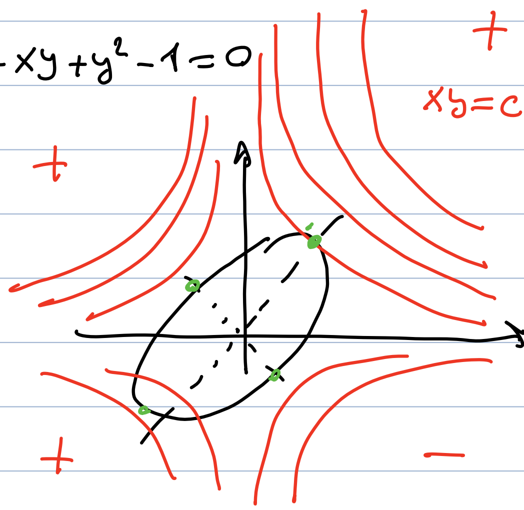
$3-\dfrac{2\lambda}{8}y=0$

VERIFICARE CHE NON CI SONO ALTRE SOLUZIONI

$f(x,y)=xy \qquad G = x^2-xy+y^2-1=0$

$$\mathcal{L}(x,y,\lambda) = xy - \lambda(x^2-xy+y^2-1)$$
$$f - \lambda G$$

$\nabla\mathcal{L}=0$



$$\begin{cases} \partial_x\mathcal{L} = y - \lambda(2x-y) \\ \partial_y\mathcal{L} = x - \lambda(-x+2y) \\ \partial_\lambda\mathcal{L} = x^2-xy+y^2-1 \end{cases}$$

$$\begin{cases} \partial_x \mathcal{L} = y - 2\lambda x + \lambda y = 0 \\ \partial_y \mathcal{L} = x + \lambda x - 2\lambda y = 0 \\ \partial_\lambda \mathcal{L} = x^2 - xy + y^2 - 1 = 0 \end{cases} \qquad \begin{cases} (1+\lambda) y = 2\lambda x \\ (1+\lambda) x = 2\lambda y \\ x^2 - xy + y^2 - 1 = 0 \end{cases} \qquad y = \frac{2\lambda}{1+\lambda} x \quad x = \frac{2\lambda}{1+\lambda} y$$

$$y = \frac{2\lambda}{1+\lambda} x = \frac{2\lambda}{1+\lambda} \frac{2\lambda}{1+\lambda} y = \left(\frac{2\lambda}{1+\lambda}\right)^2 y$$

$$\frac{2\lambda}{1+\lambda} = \pm 1 \quad \nearrow \; 2\lambda = 1+\lambda \quad \lambda = 1$$
$$\searrow \; 2\lambda = -1-\lambda \quad \lambda = -1/3$$

$\lambda = 1$ $\quad \frac{2\lambda}{1+\lambda} = 1 \Rightarrow y = x \quad y = x \quad x^2 - \cancel{x^2} + \cancel{x^2} - 1 = 0 \quad x^2 = 1$

$\lambda = -1/3$ $\quad \frac{2\lambda}{1+\lambda} = -1 \Rightarrow y = -x \quad x^2 - x(-x) + (-x)^2 - 1 = 0 \quad 3x^2 = 1$

$\lambda = 1 \quad x = \pm 1 \quad y = x \quad P_1 = (1, 1, 1)$

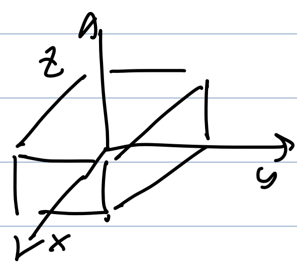
$P_2 = (-1, -1, 1)$

$\lambda = -1/3 \quad x = \pm \frac{1}{\sqrt{3}} \quad y = -x \quad P_3 = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{3}\right)$

$P_4 = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, -\frac{1}{3}\right)$

$f(x,y) = xy$

$$\begin{array}{cccc} f(P_1) & f(P_2) & f(P_3) & f(P_4) \\ \parallel & \parallel & \parallel & \parallel \\ 1 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ \uparrow & & \uparrow & \end{array}$$



$V = xyz \qquad x, y, z \geq 0$

$G(x,y,z) = xy + xz + yz - A = 0$

$f - \lambda G$

$\mathcal{L}(x,y,z,\lambda) = xyz - \lambda(xy + xz + yz - A)$

$$\nabla \mathcal{L} = \begin{cases} \partial_x \mathcal{L} = yz - \lambda(y+z) \\ \partial_y \mathcal{L} = xz - \lambda(x+z) \\ \partial_z \mathcal{L} = xy - \lambda(y+x) \\ \partial_\lambda \mathcal{L} = xy + xz + yz - A \end{cases}$$

$x \quad yz - \lambda(y+z)x = 0$

$y \quad xz - \lambda(x+z)y = 0$

$-\lambda(y+z)x + \lambda(x+z)y = 0$

$\cancel{\lambda}(x+z)y = \cancel{\lambda}(y+z)x$

$\lambda = 0$ ma scartiamo

$$\cancel{xy} + yz = \cancel{xy} + xz$$

$$\Downarrow$$

$$y = x$$

$$x = y = z$$

$$xx + xx + xx = A$$

$$3x^2 = A \qquad x^2 = \frac{A}{3} \qquad x = \sqrt{A/3}$$

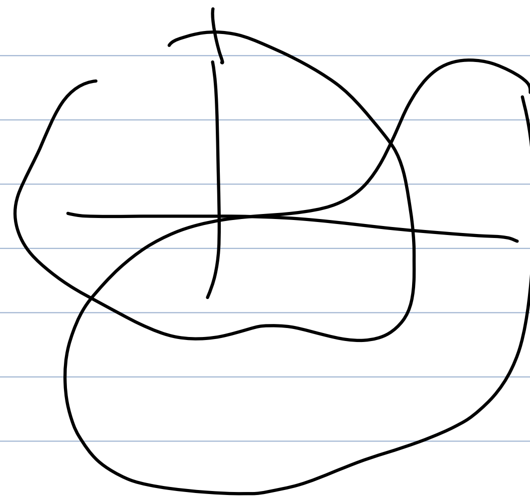
$$G(x,y,z) = 0 \qquad \nabla G \neq 0 \qquad \partial_z G \neq 0$$

$$\Rightarrow \quad z = g(x,y) \qquad G(x,y,g(x,y)) = 0$$

$$\partial_x \big(G(x,y,g(x,y)) \big) = 0 \qquad \partial_x G(x,y,g(x,y)) + \partial_y \cancel{G(\,)} \frac{\partial y}{\partial x} + \partial_z G \frac{\partial g}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial g(x,y)}{\partial x} = - \frac{\partial_x G(x,y,g(x,y))}{\partial_z G(x,y,g(x,y))}$$

$$\frac{\partial g}{\partial y} = - \frac{\partial_y G}{\partial_z G}$$



$$x^2 - xy + y^2 = 1$$

$$(x \; y) \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1$$

$$\frac{1}{2} \qquad \frac{3}{2}$$