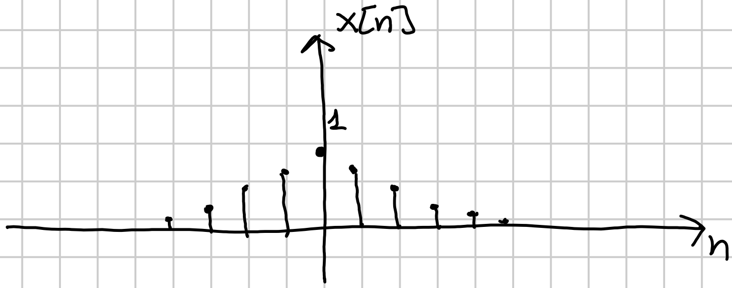
Esercitazione 21 - 03 - 2012 RICEVIMENTO VENERDI 9:30 11:30

Esercizio #1

Calcolare la TFS

$x[n] = a^{|n|} \qquad 0 \leq a < 1$



$x[n] = a^{n} u[n] + a^{-n} u[-n] - x[0]\,\delta[n]$

$x[n] = y[n] + y[-n] - x[0]\,\delta[n]$

TFS di $y[n]$

$$y[n] = a^n u[n]$$

$$\bar{Y}(f) = \sum_{n=0}^{+\infty} a^n e^{-i2\pi f nT} = \sum_{n=0}^{+\infty} \left(a e^{-i2\pi f T}\right)^n$$

$$\boxed{\sum_{n=0}^{+\infty} q^n = \frac{1}{1-q} \qquad |q| < 1}$$

$$\bar{Y}(f) = \frac{1}{1 - a e^{-i2\pi f T}}$$

$$x[n] = y[n] + y[-n] - x[0]\,\delta[n]$$

$$\bar{X}(f) = \bar{Y}(f) + \bar{Y}(-f) - x[0]$$

$y[n]$ è una sequenza reale $\Rightarrow \bar{Y}(f)$ è a simmetria Hermitiana

$$\bar{Y}(-f) = \bar{Y}^*(f)$$

$$\bar{X}(f) = \bar{Y}(f) + \bar{Y}^*(f) - x[0] = 2\,\mathrm{Re}\left\{\bar{Y}(f)\right\} - x[0]$$

$$\bar{X}(f) = \frac{1}{1-\alpha e^{-j2\pi fT}} + \frac{1}{1-\alpha e^{+j2\pi fT}} - 1 =$$

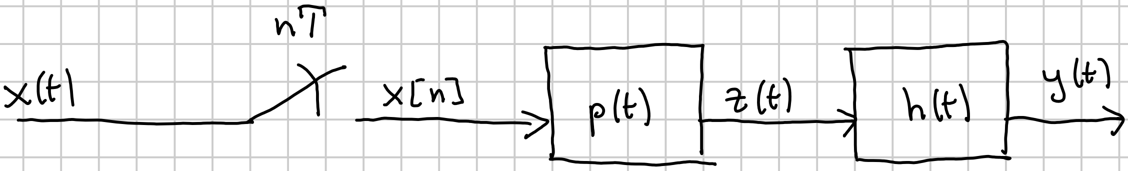
$$= \frac{1-\alpha e^{+j2\pi fT} + 1-\alpha e^{-j2\pi fT} - \left(1-\alpha e^{-j2\pi fT}\right)\left(1-\alpha e^{+j2\pi fT}\right)}{\left(1-\alpha e^{-j2\pi fT}\right)\left(1-\alpha e^{+j2\pi fT}\right)}$$

$$= \frac{1-\cancel{\alpha e^{+j2\pi fT}} + 1-\cancel{\alpha e^{-j2\pi fT}} - 1-\alpha^2 + \cancel{2\alpha\cos(2\pi fT)}}{1+\alpha^2-2\alpha\cos(2\pi fT)} =$$

$$= \frac{1-\alpha^2}{1+\alpha^2-2\alpha\cos(2\pi fT)}$$

<u>Esercizio #2</u>

Ricostruzione di un segnale analogico a partire da una sequenza



Ricavare $y(t)$

$$x(t) = \operatorname{sinc}\left(\frac{t}{T}\right)$$

$$p(t) = \operatorname{rect}\left(\frac{t-T/2}{T}\right) \qquad \text{INTERPOLATORE A MANTENIMENTO}$$

$$H(f) = \frac{\pi fT}{\sin(\pi fT)}\operatorname{rect}(fT)$$

Svolgimento

- $x[n] = \mathrm{sinc}\left(\frac{nT}{T}\right) = \mathrm{sinc}(n) = \begin{cases} 1 & n=0 \\ 0 & n\neq 0 \end{cases}$

$$x[n] = \delta[n]$$

- $z(t) = \sum_{n=-\infty}^{+\infty} x[n]\, p(t-nT) = p(t)$

- $y(t) = z(t) \otimes h(t)$

$$Y(f) = Z(f)\, H(f) = P(f)\, H(f)$$

- $p(t) = \mathrm{rect}\left(\frac{t - T/2}{T}\right)$

$$P(f) = T\, \mathrm{sinc}(fT)\, e^{-j 2\pi f \frac{T}{2}} = T\, \mathrm{sinc}(fT)\, e^{-j\pi f T}$$

- $Y(f) = T\, \mathrm{sinc}(fT)\, e^{-j\pi f T} \cdot \frac{\pi f T}{\sin(\pi f T)}\, \mathrm{rect}(fT) =$

$$= T\, \frac{\cancel{\sin(\pi f T)}}{\cancel{\pi f T}}\, e^{-j\pi f T}\, \frac{\cancel{\pi f T}}{\cancel{\sin(\pi f T)}}\, \mathrm{rect}(fT)$$

$$Y(f) = T\, \mathrm{rect}(fT)\, e^{-j\pi f T}$$

- $y(t) = \mathrm{sinc}\left(\frac{t - T/2}{T}\right)$ replica non distorta del segnale $x(t)$

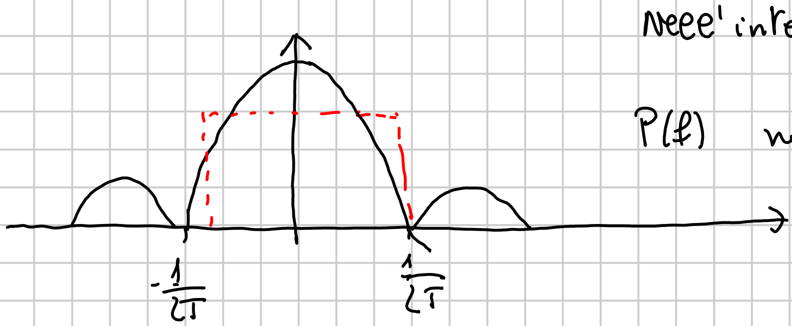
Sembra che l'interpolatore a mantenimento non introduce distorsioni!

Se però in ingresso ho un segnale $x(t) = \text{sinc}^2(t/T)$ allora l'interpolatore non è in grado di ricostruire il segnale di partenza.

DISTORSIONI INTRODOTTE DALL'INTERPOLATORE A MANTENIMENTO

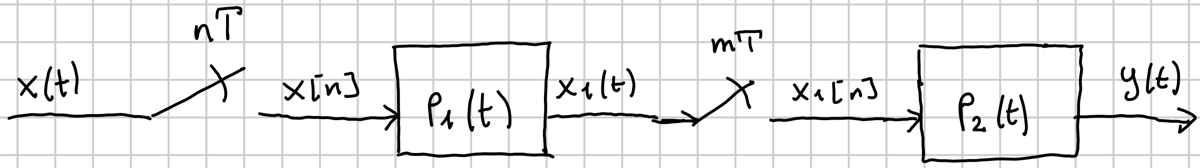
Le discontinuità nel dominio del tempo introducono delle componenti frequenziali che non sono presenti nel segnale analogico, denominate IMMAGINI.

$$p(t) = \text{rect}\left(\frac{t - T/2}{T}\right) \Rightarrow P(f) = T\,\text{sinc}(fT)\,e^{-j2\pi f T/2}$$



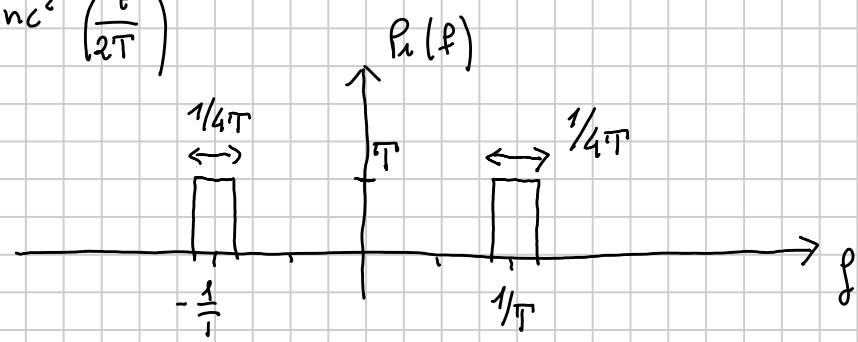
Nell'intervallo base $\left[-\frac{1}{2T} ; \frac{1}{2T}\right]$ $P(f)$ non è costante.

Esercizio #3



$$x(t) = \text{sinc}^2\left(\frac{t}{2T}\right)$$

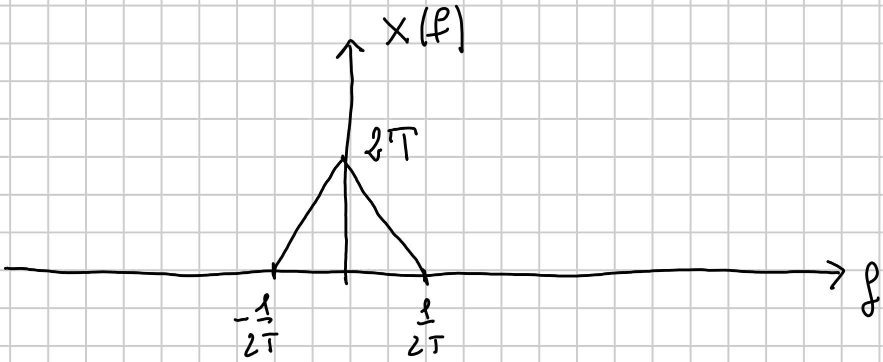
$P_1(f)$



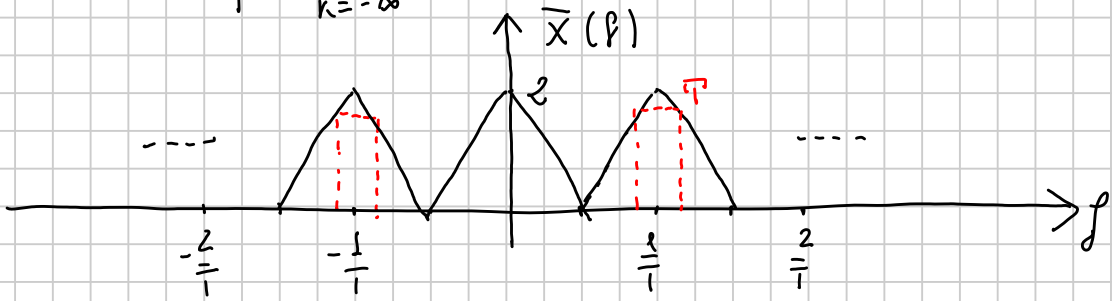
$$p_2(t) = \text{sinc}\left(\frac{t}{T}\right)$$

Trovare l'espressione di $y(t)$

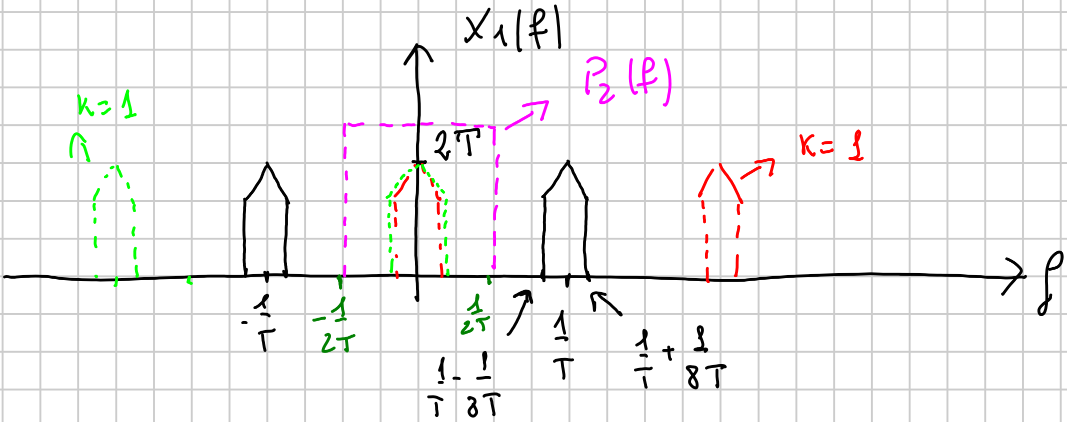
- $X(f) = 2T\left(1 - |f|\,2T\right)\operatorname{rect}(fT)$ $\qquad B = \frac{1}{T}$



- $\overline{X}(f) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X\left(f - \frac{k}{T}\right)$



$X_1(f) = P_1(f)\cdot\overline{X}(f)$



$\overline{X}_1(f) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_1\left(f - \frac{k}{T}\right)$

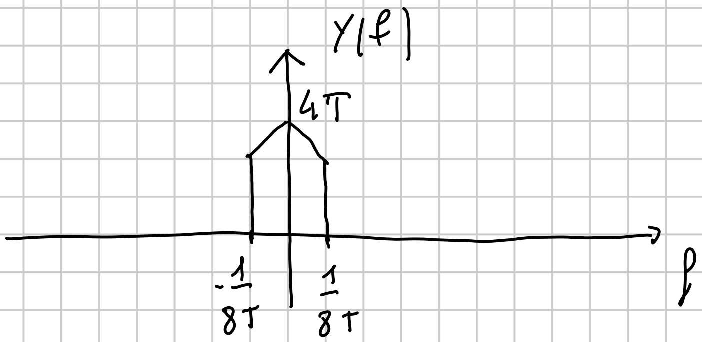
$$\bar{X}_1(f) = 2\,X(f)\,\text{rect}\,(f\,4T) \qquad -\frac{1}{2T} \le f \le \frac{1}{2T}$$

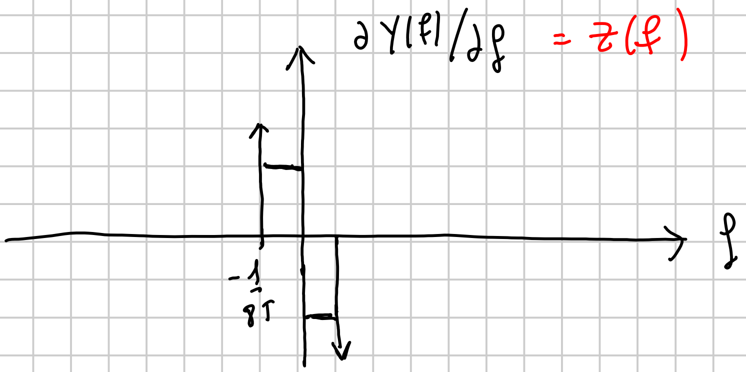
$$p_2(t) = \text{sinc}\left(\frac{t}{T}\right)$$

$$P_2(f) = T\,\text{rect}\,(f\,T)$$

$$Y(f) = P_2(f)\cdot X_1(f) = 2\,X(f)\,\text{rect}\,(f\,4T)\,T\,\text{rect}\,(f\,T)$$

$$Y(f) = 2T\,X(f)\,\text{rect}\,(f\,4T)$$





$$Z(f) = \frac{\partial Y(f)}{\partial f} \quad \Rightarrow \quad z(t) = -j2\pi t\,y(t)$$

$$\text{se } z(t)\Big|_{t=0} = 0$$

$$y(t) = \frac{z(t)}{-j2\pi t}$$

$$z(t)\Big|_{t=0} = 0 \quad \Rightarrow \quad \int_{-\infty}^{+\infty} Z(f) = 0$$

$$Y(f) = 4T \left(1 - \frac{|f|}{1/2T}\right) \text{rect}\left(\frac{f}{1/4T}\right)$$

$$f = \pm \frac{1}{8T} \qquad Y(f)\Big|_{f=\pm\frac{1}{8T}} = 3T$$

$a = 8T^2$ pendenza delle rette che compongono $Y(f)$

$$Z(f) = \frac{\partial Y(f)}{\partial f} = 8T^2 \,\text{rect}\left(\frac{f + \frac{1}{16T}}{\frac{1}{8T}}\right) - 8T^2 \,\text{rect}\left(\frac{f - \frac{1}{16T}}{\frac{1}{8T}}\right)$$

$$+ 3T\,\delta\left(f + \frac{1}{8T}\right) - 3T\,\delta\left(f - \frac{1}{8T}\right)$$

$$z(t) = \cancel{8}T^{\cancel{2}} \left(\frac{1}{\cancel{8T}} \text{sinc}\left(\frac{t}{8T}\right) e^{j2\pi t \frac{1}{16T}} - \frac{1}{8T} \text{sinc}\left(\frac{t}{8T}\right) e^{-j2\pi t \frac{1}{16T}}\right)$$

$$+ 3T \frac{2j}{2j} \left(\delta\left(f + \frac{1}{8T}\right) - \delta\left(f - \frac{1}{8T}\right)\right)$$

$$z(t) = T \frac{2j}{2j} \text{sinc}\left(\frac{t}{8T}\right) \left[e^{j2\pi t \frac{1}{16T}} - e^{-j2\pi t \frac{1}{16T}}\right] - 3T \cdot 2j \cdot \sin\left(\pi \frac{t}{8T}\right)$$

$$z(t) = 2j\,T\,\mathrm{sinc}\left(\frac{t}{8T}\right) \cdot \sin\left(\frac{\pi t}{8T}\right) - 6Tj\,\sin\left(\frac{\pi t}{8T}\right)$$

$$y(t) = \frac{z(t)}{-j2\pi t}$$

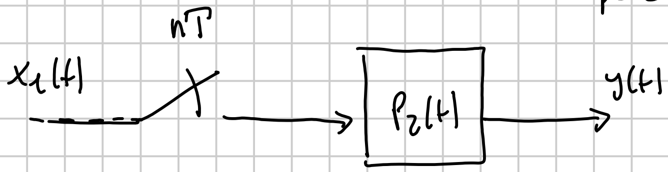
$$y(t) = \frac{2Tj}{-2j\pi t}\,\mathrm{sinc}\left(\frac{t}{8T}\right)\sin\left(\frac{\pi t}{8T}\right) - \frac{6Tj}{-2j\pi t}\,\sin\left(\frac{\pi t}{8T}\right) =$$

$$= -\frac{T}{\pi t}\,\mathrm{sinc}\left(\frac{t}{8T}\right)\sin\left(\frac{\pi t}{8T}\right) + \frac{3T}{\pi t}\,\sin\left(\frac{\pi t}{8T}\right) =$$

$$= -\frac{1}{8} \cdot \frac{1}{\pi t \cdot \frac{1}{8T}}\,\sin\left(\frac{\pi t}{8T}\right)\mathrm{sinc}\left(\frac{t}{8T}\right) + \frac{3}{8}\,\frac{1}{\pi t \frac{1}{8T}}\,\sin\left(\frac{\pi t}{8T}\right) =$$

$$= -\frac{1}{8}\,\mathrm{sinc}^2\left(\frac{t}{8T}\right) + \frac{3}{8}\,\mathrm{sinc}\left(\frac{t}{8T}\right)$$

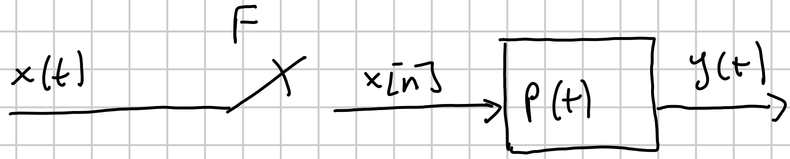
OSSERVAZIONE: $x_1(t)$ e' un segnale modulato. Il sistema quindi a valle di $x_1(t)$ opera come un demodulatore.



$$x_1(t) = x_0(t)\cos(2\pi f_0 t) \qquad f_0 = \frac{k}{T}$$

$$y(t) = a\,x_0(t - t_0)$$

Esercizio #4



- $p(t)$ interpolatore a mantenimento $p(t) = \mathrm{rect}\left(\frac{t - T/2}{T}\right)$

$$x(t) = \sum_{n=-\infty}^{+\infty} x[n]\, p(t - nT) \qquad x(nT) \doteq x[n]$$

$$F = \frac{1}{T} = \frac{10}{\alpha}$$

$$x(t) = e^{-t/\alpha}\, u(t)$$

Calcolare E_x ed E_y

$$E_x = \int_{-\infty}^{+\infty} e^{-2t/\alpha}\, u(t)\, dt = \int_{0}^{+\infty} e^{-2t/\alpha}\, dt = -\frac{\alpha}{2} e^{-2t/\alpha}\Big|_{0}^{+\infty} = \frac{\alpha}{2}$$

$$y(t) = \sum_{n=-\infty}^{+\infty} x[n]\, \mathrm{rect}\left(\frac{t - nT - T/2}{T}\right)$$

$$E_y = \int_{-\infty}^{+\infty} |y(t)|^2\, dt$$

$$E_y = \int_{-\infty}^{+\infty} \sum_n x^2[n] \operatorname{rect}\left(\frac{t - nT - T/2}{T}\right) dt$$

$$|y(t)|^2 = \sum_n x[n] \operatorname{rect}\left(\frac{t - nT - T/2}{T}\right) \sum_m x^*[m] \operatorname{rect}\left(\frac{t - mT - T/2}{T}\right) =$$

$$= \sum_n x^2[n] \operatorname{rect}\left(\frac{t - nT - T/2}{T}\right)$$

$$E_y = \sum_n x^2[n] \int_{-\infty}^{+\infty} \operatorname{rect}\left(\frac{t - nT - T/2}{T}\right) dt = T \sum_{n=0}^{+\infty} \left(e^{-2T/\alpha}\right)^n$$

$$E_y = T \cdot \frac{1}{1 - e^{-2T/\alpha}} = \frac{\frac{\alpha}{10}}{1 - e^{-1/5}} = \frac{\alpha}{2} \frac{1}{5\left(1 - e^{-1/5}\right)} =$$

$$= E_x \frac{1}{5\left(1 - e^{-1/5}\right)} \cong 1,1\ E_x$$

È giusto che $E_y > E_x$ perché: