TEOREMI SULLA TCF

Linearità

$$x(t) = a x_1(t) + b x_2(t)$$

$$\Updownarrow \text{TCF}$$

$$X(f) = a X_1(f) + b X_2(f)$$

dove $x_1(t) \Leftrightarrow X_1(f)$, $x_2(t) \Leftrightarrow X_2(f)$

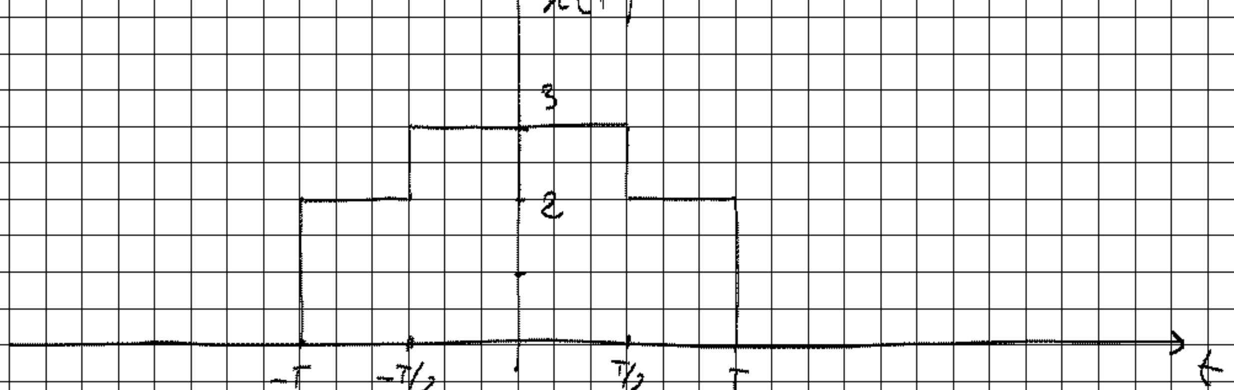
Dimostrazione:

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t} = \int_{-\infty}^{+\infty} \left[a x_1(t) + b x_2(t)\right] e^{-j2\pi f t} dt$$

$$= a \underbrace{\int_{-\infty}^{+\infty} x_1(t) e^{-j2\pi f t} dt}_{X_1(f)} + b \underbrace{\int_{-\infty}^{+\infty} x_2(t) e^{-j2\pi f t} dt}_{X_2(f)} =$$

$$= a X_1(f) + b X_2(f)$$

Esempio: Calcolare $X(f)$ del segnale



Soluzione:

$$x_1(t) = 2\,\mathrm{rect}\left(\frac{t}{2T}\right), \quad x_2(t) = \mathrm{rect}\left(\frac{t}{T}\right)$$

$$x(t) = 2\,\mathrm{rect}\left(\frac{t}{2T}\right) + \mathrm{rect}\left(\frac{t}{T}\right)$$

Calcolo di $X_1(f)$

$$\mathrm{rect}\left(\frac{t}{T}\right) \Leftrightarrow T\,\mathrm{sinc}(Tf)$$

$$T' = 2T$$

$$x_1(t) = 2\,\mathrm{rect}\left(\frac{t}{T'}\right) \Leftrightarrow$$

$$X_1(f) = 2T'\,\mathrm{sinc}\,(T'f) = 4T\,\mathrm{sinc}\,(2Tf)$$

$$X_2(f) = T\,\mathrm{sinc}(Tf)$$

$$X(f) = T\left[4\,\mathrm{sinc}(2Tf) + \mathrm{sinc}\,(Tf)\right]$$

Dualità

se $x(t) \overset{TCF}{\Longleftrightarrow} X(f)$

allora $X(t) \overset{TCF}{\Longleftrightarrow} x(-f)$

Dimostrazione:

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

scambio t ed f

$$X(t) = \int_{-\infty}^{+\infty} x(f)\, e^{-j2\pi f t}\, df = \qquad (f' = -f)$$

$$= \int_{-\infty}^{+\infty} x(-f')\, e^{j2\pi f' t}\, df$$

$$X(t) \overset{TCF}{\Longleftrightarrow} x(-f)$$

Esempio generico

$$x(t) = \mathrm{rect}\left(\frac{t}{T}\right) \overset{TCF}{\Longleftrightarrow} X(f) = T\,\mathrm{sinc}(Tf)$$

$$X(t) = T\,\mathrm{sinc}(Tt) \Longleftrightarrow x(f) = \mathrm{rect}\left(\frac{-f}{T}\right) = \mathrm{rect}\left(\frac{f}{T}\right)$$

Esempio

$$x(t) = A\,\mathrm{sinc}(Ft)$$

$$x(t) = \frac{A}{F}\, F\,\mathrm{sinc}(Ft)$$

$$X(f) = \frac{A}{F}\,\mathrm{rect}\left(\frac{f}{F}\right)$$

RITARDO

$$y(t) = x(t-t_0) \Longleftrightarrow Y(f) = X(f)\, e^{-j2\pi f t_0}$$

$$Y(f) = \int_{-\infty}^{+\infty} x(t-t_0)\, e^{-j2\pi f t}\, dt = \qquad (t' = t - t_0)$$

$$= \int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi f (t'+t_0)}\, dt' =$$

$$= \int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi f t'}\, dt' \cdot e^{-j2\pi f t_0} =$$

$$= X(f)\, e^{-j2\pi f t_0}$$

$$\Rightarrow \left| Y(f) \right| = \left| X(f) \right|$$

$$\Rightarrow \angle Y(f) = \angle X(f) - 2\pi f t_0$$

Esempio:

$$y(t) = \mathrm{rect}\left(\frac{t-t_0}{T}\right)$$

Soluzione

1) $$Y(f) = \int_{-\infty}^{+\infty} \mathrm{rect}\left(\frac{t-t_0}{T}\right) e^{-j2\pi f t}\, dt$$

2) $$x(t) = \mathrm{rect}\left(\frac{t}{T}\right) \Rightarrow y(t) = x(t-t_0)$$

$$Y(f) = X(f)\, e^{-j2\pi f t_0} = T \operatorname{sinc}(Tf)\, e^{-j2\pi f t_0}$$

CAMBIAMENTO DI SCALA

$$y(t) = x(\alpha t) \qquad \alpha \neq 0$$

$$\Updownarrow \text{TCF}$$

$$Y(f) = \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right)$$

Dimostrazione

$\alpha > 0$

$$Y(f) = \int_{-\infty}^{+\infty} x(\alpha t)\, e^{-j2\pi f t}\, dt = \qquad (\alpha t = t')$$

$$= \int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi f \frac{t'}{\alpha}}\, \frac{dt'}{\alpha} =$$

$$= \frac{1}{\alpha} \int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi \frac{f}{\alpha} t'}\, dt' = \frac{1}{\alpha} X\left(\frac{f}{\alpha}\right)$$

$\alpha < 0$

$$Y(f) = \int_{-\infty}^{+\infty} x(\alpha t)\, e^{-j2\pi f t}\, dt = \qquad (\alpha t = t')$$

$$= \int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi f \frac{t'}{\alpha}}\, \frac{dt'}{-\alpha} =$$

NB: il segno "-" tiene conto dell'inversione degli estremi di integrazione

$$= \frac{1}{-\alpha} \int_{-\infty}^{+\infty} x(t') \, e^{-j 2\pi \frac{f}{\alpha} t'} \, dt' = -\frac{1}{\alpha} X\left(\frac{f}{\alpha}\right)$$

Quindi unendo i due casi $(\alpha \neq 0)$

$$x(\alpha t) = \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right)$$

N.B. $|\alpha| > 1 \Rightarrow$ compressione scala dei tempi

$|\alpha| < 1 \Rightarrow$ dilatazione scala dei tempi

$\alpha < 0 \Rightarrow$ inversione scala dei tempi

Esempio:

$$y(t) = \text{rect}\left(\frac{t}{2T}\right)$$

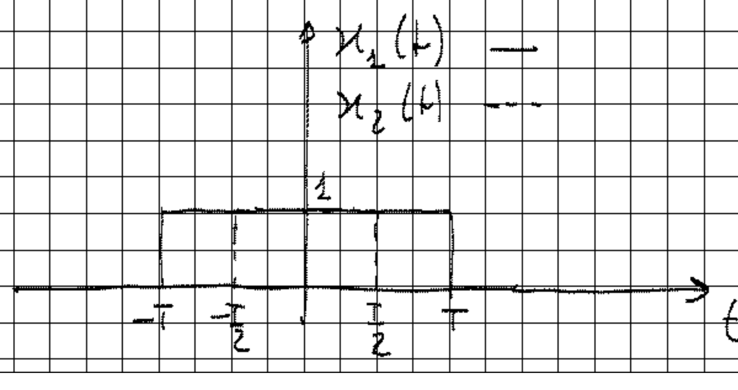
Calcolare $Y(f)$

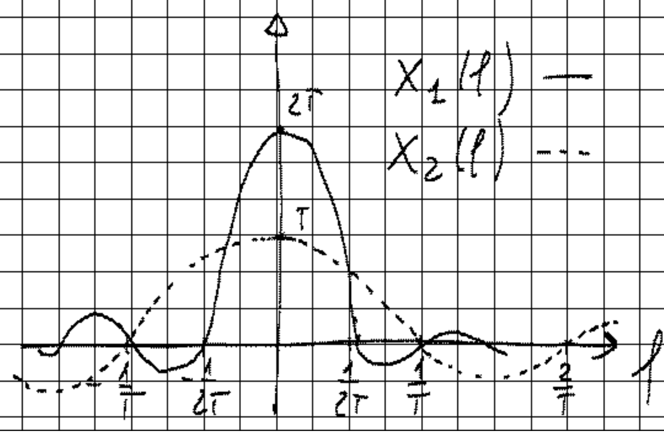
Soluzione:

$$x(t) = \text{rect}\left(\frac{t}{T}\right) \overset{TCF}{\Longleftrightarrow} T \, \text{sinc}(Tf) = X(f)$$

$$y(t) = x(\alpha t), \quad \alpha = \frac{1}{2}$$

$$Y(f) = \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right) = 2T \, \text{sinc}(2Tf)$$





TEOREMA DELLA MODULAZIONE

$$y(t) = x(t)\cos(2\pi f_0 t)$$

$$\Updownarrow$$

$$Y(f) = \frac{X(f-f_0) + X(f+f_0)}{2}$$

Dimostrazione

$$Y(f) = \int_{-\infty}^{+\infty} x(t)\cos(2\pi f_0 t)\, e^{-j2\pi f t}\, dt =$$

$$= \int_{-\infty}^{+\infty} x(t)\left[\frac{1}{2} e^{j2\pi f_0 t} + \frac{1}{2} e^{-j2\pi f_0 t}\right] e^{-j2\pi f t}\, dt =$$

$$= \frac{1}{2}\underbrace{\int x(t)\, e^{-j2\pi(f-f_0)t}\, dt}_{X(f-f_0)} + \frac{1}{2}\underbrace{\int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi(f+f_0)t}\, dt}_{X(f+f_0)}$$

$$= \frac{1}{2}\left[X(f-f_0) + X(f+f_0)\right]$$

Lo spettro di un segnale modulato è una versione traslata dello spettro del segnale di partenza (segnale modulante)

$$\underbrace{y(t)}_{\text{SEGNALE MODULATO}} = \underbrace{x(t)}_{\text{SEGNALE MODULANTE}} \cdot \underbrace{\cos(2\pi f_0 t)}_{\text{OSCILLAZIONE}}$$

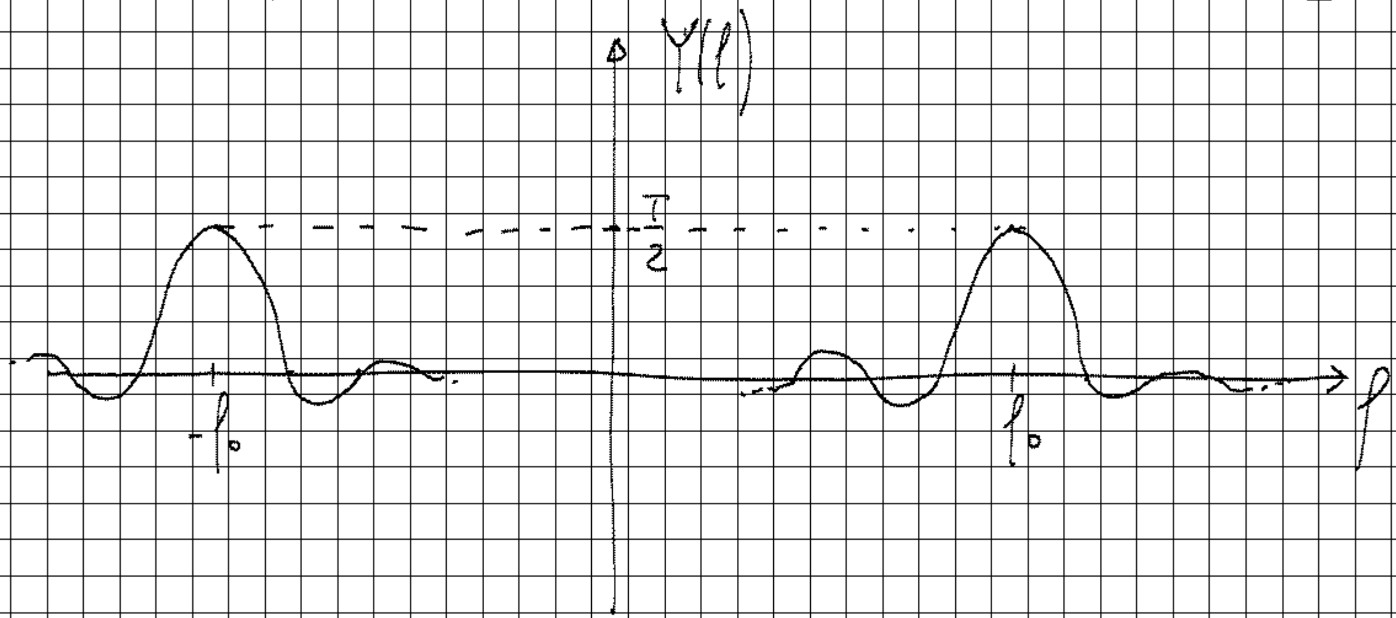
Esempio

$$y(t) = \text{rect}\left(\frac{t}{T}\right)\cos(2\pi f_0 t)$$

Soluzione:

$$x(t) = \text{rect}\left(\frac{t}{T}\right) \Leftrightarrow X(f) = T\,\text{sinc}(Tf)$$

$$Y(f) = \frac{1}{2}T\,\text{sinc}\left[T(f-f_0)\right] + \frac{1}{2}T\,\text{sinc}\left[T(f+f_0)\right]$$



MODULAZIONE CON SENO

$$y(t) = x(t)\sin(2\pi f_0 t)$$

$$\Updownarrow \text{TCF}$$

$$Y(f) = \frac{X(f-f_0) - X(f+f_0)}{2j}$$

Dimostrazione

$$Y(f) = \int_{-\infty}^{+\infty} x(t) \sin(2\pi f_0 t)\, e^{-j2\pi f t}\, dt =$$

$$= \frac{1}{2j} \int_{-\infty}^{+\infty} x(t)\, e^{j2\pi f_0 t} e^{-j2\pi f t}\, dt - \frac{1}{2j} \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f_0 t} e^{-j2\pi f t}\, dt$$

$$= \frac{1}{2j} \left[\int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi (f-f_0) t}\, dt - \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi (f+f_0) t}\, dt \right] =$$

$$= \frac{1}{2j} \left[X(f-f_0) - X(f+f_0) \right]$$

MODULAZIONE CON FASE GENERICA

$$y(t) = x(t) \cos(2\pi f_0 t + \varphi)$$

$$\Updownarrow \text{ TCF}$$

$$Y(f) = \frac{e^{j\varphi}}{2} X(f-f_0) + \frac{e^{-j\varphi}}{2} X(f+f_0)$$

Dimostrazione

$$Y(f) = \int_{-\infty}^{+\infty} x(t) \cos(2\pi f_0 t + \varphi)\, e^{-j2\pi f t}\, dt =$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} x(t)\, e^{j(2\pi f_0 t + \varphi)} e^{-j2\pi f t}\, dt + \frac{1}{2} \int_{-\infty}^{+\infty} x(t)\, e^{-(j2\pi f_0 t + \varphi)} e^{-j2\pi f t}\, dt$$

$$= \frac{e^{j\varphi}}{2}\int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi(f-f_0)t}\, dt + \frac{e^{-j\varphi}}{2}\int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi(f+f_0)t}\, dt$$

$$= \frac{e^{j\varphi}}{2} X(f-f_0) + \frac{e^{-j\varphi}}{2} X(f+f_0)$$

MODULAZIONE CON ESPONENZIALE COMPLESSO

$$y(t) = x(t)\, e^{j2\pi f_0 t}$$

$$\Updownarrow \text{ TCF}$$

$$Y(f) = X(f-f_0)$$

Dimostrazione

$$Y(f) = \int_{-\infty}^{+\infty} x(t)\, e^{j2\pi f_0 t}\, e^{-j2\pi f t}\, dt =$$

$$= \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi(f-f_0)t}\, dt = X(f-f_0)$$

NB. Può essere letto come il teorema del ritardo in frequenza

$$X(f-f_0) \Leftrightarrow x(t)\, e^{j2\pi f_0 t}$$

in analogia con:

$$x(t-t_0) \Leftrightarrow X(f)\, e^{-j2\pi f t_0}$$

NB. attenzione solo al segno!!

TEOREMA DELLA DERIVAZIONE

$$y(t) = \frac{d}{dt} x(t)$$

$$\Updownarrow \text{ TCF}$$

$$Y(f) = j2\pi f \, X(f)$$

Dimostrazione:

$$y(t) = \frac{d}{dt} x(t) = \frac{d}{dt} \int_{-\infty}^{+\infty} X(f) e^{j2\pi f t} \, df =$$

$$= \int_{-\infty}^{+\infty} X(f) \left[\frac{d}{dt} e^{j2\pi f t} \right] df = \int_{-\infty}^{+\infty} j2\pi f \, X(f) e^{j2\pi f t} \, df$$

quindi $Y(f) = j2\pi f \, X(f)$

TEOREMA DELL' INTEGRAZIONE

$$y(t) = \int_{-\infty}^{t} x(\alpha) \, d\alpha \qquad \text{con} \int_{-\infty}^{+\infty} x(t) \, dt = 0$$

$$\Updownarrow \text{ TCF} \qquad\qquad \| $$

$$X(0) = 0$$

$$Y(f) = \frac{X(f)}{j2\pi f}$$

Dimostrazione

$$x(t) = \frac{d}{dt} y(t) = \frac{d}{dt} \int_{-\infty}^{t} x(\alpha)\, d\alpha = x(t)$$

$$\Updownarrow \text{TCF}$$

$$X(f) = j2\pi f\, Y(f)$$

$$Y(f) = \frac{X(f)}{j2\pi f}$$

N.B. La condizione $X(0)$ e' necessaria per

$$f \to 0 \Rightarrow \lim_{f \to 0} Y(f) < \infty$$

TEOREMA DELLA DERIVAZIONE IN FREQUENZA

$$Y(f) = \frac{d}{df} X(f)$$

$$\Updownarrow \text{TCF}$$

$$y(t) = -j2\pi t\, x(t)$$

Dimostrazione

$$Y(f) = \frac{d}{df} \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt =$$

$$= \int_{-\infty}^{+\infty} x(t) \left[\frac{d}{df} e^{-j2\pi f t}\right] dt = -j2\pi t \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt = -j2\pi t\, x(t)$$

TEOREMA DELLA INTEGRAZIONE IN FREQUENZA

$$Y(f) = \int_{-\infty}^{f} X(\alpha)\, d\alpha$$

$$\Updownarrow \text{TCF}$$

$$y(t) = -\frac{x(t)}{j2\pi t}$$

Dimostrazione:

$$X(f) = \frac{d}{df} Y(f)$$

$$x(t) = -j2\pi t\, y(t)$$

$$y(t) = -\frac{x(t)}{j2\pi t}$$

TEOREMA DELLA CONVOLUZIONE

Definizione di prodotto di convoluzione

$$z(t) = x(t) \otimes y(t) = \int_{-\infty}^{+\infty} x(\alpha)\, y(t-\alpha)\, d\alpha$$

$$z(t) = x(t) \otimes y(t)$$

$$\Updownarrow \text{TCF} \qquad X(f) \overset{\text{TCF}}{\Leftrightarrow} x(t) \quad , \quad Y(f) \overset{\text{TCF}}{\Leftrightarrow} y(t)$$

$$Z(f) = X(f)\, Y(f)$$

Dimostrazione

$$Z(f) = \int_{-\infty}^{+\infty} \left[x(t) \otimes y(t)\right] e^{-j2\pi f t}\, dt =$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(\alpha)\, y(t-\alpha)\, d\alpha\, e^{-j2\pi f t}\, dt =$$

$$= \int_{-\infty}^{+\infty} x(\alpha) \int_{-\infty}^{+\infty} y(t-\alpha)\, e^{-j2\pi f t}\, dt\, d\alpha =$$

$$= \int_{-\infty}^{+\infty} x(\alpha)\, Y(f)\, e^{-j2\pi f \alpha}\, d\alpha = Y(f) \int_{-\infty}^{+\infty} x(\alpha)\, e^{-j2\pi f \alpha}\, d\alpha = X(f)\, Y(f)$$

PROPRIETÀ DELLA CONVOLUZIONE

Proprietà commutativa

$$x(t) \otimes y(t) = y(t) \otimes x(t)$$

Dimostrazione:

$$y(t) \otimes x(t) =$$

$$\int_{-\infty}^{+\infty} y(\alpha)\, x(t-\alpha)\, d\alpha = \left(t - \alpha = \alpha'\right) = \int_{-\infty}^{+\infty} y(t-\alpha')\, x(\alpha')\, d\alpha'$$

$$= \int_{-\infty}^{+\infty} x(\alpha')\, y(t-\alpha')\, d\alpha' = x(t) \otimes y(t)$$

Proprietà distributiva

$$x(t) \otimes \left[y(t) + z(t) \right] = x(t) \otimes y(t) + x(t) \otimes z(t)$$

Dimostrazione

$$\int_{-\infty}^{+\infty} x(\alpha) \left[y(t-\alpha) + z(t-\alpha) \right] d\alpha =$$

$$= \int_{-\infty}^{+\infty} x(\alpha)\, y(t-\alpha)\, d\alpha + \int_{-\infty}^{+\infty} x(\alpha)\, z(t-\alpha)\, d\alpha =$$

$$= x(t) \otimes y(t) + x(t) \otimes z(t)$$

Proprietà associativa

$$x(t) \otimes \left[y(t) \otimes z(t) \right] = \left[x(t) \otimes y(t) \right] \otimes z(t)$$

Dimostrazione

$$FT\left\{ x(t) \otimes \left[y(t) \otimes z(t) \right] \right\} = X(f) \left[Y(f) \cdot Z(f) \right] =$$

$$= \left[X(f) \cdot Y(f) \right] Z(f)$$

$$FT^{-1}\left\{ \left[X(f) \cdot Y(f) \right] Z(f) \right\} = \left[x(t) \otimes y(t) \right] \otimes z(t)$$

TEOREMA DEL PRODOTTO

$$z(t) = x(t) \cdot y(t) \qquad x(t) \overset{TCF}{\Longleftrightarrow} X(f)$$

$$\Updownarrow TCF$$

$$Z(f) = X(f) \otimes Y(f) \qquad y(t) \overset{TCF}{\Longleftrightarrow} Y(f)$$

Dimostrazione

$$Z(f) = \int_{-\infty}^{+\infty} x(t)\, y(t)\, e^{-j2\pi f t}\, dt =$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} X(\nu)\, e^{j2\pi\nu t}\, d\nu\; y(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{+\infty} X(\nu) \int_{-\infty}^{+\infty} y(t)\, e^{-j2\pi(f-\nu)t}\, dt\, d\nu$$

$$= \int_{-\infty}^{+\infty} X(\nu)\, Y(f-\nu)\, d\nu = X(f) \otimes Y(f)$$