Equazioni del Secondo Ordine: risposta forzata

Caso Poli reali coincidenti

$$T^2 \triangleq \frac{1}{p^2} \triangleq \frac{a_2}{a_0}$$

Risposta al gradino

$$G(s)\cdot U(s) = \frac{b_0/a_2}{(s+1/T)^2}\cdot\frac{1}{s} = \frac{A}{s} + \frac{B}{s+1/T} + \frac{C}{(s+1/T)^2}$$

$$A = \lim_{s\to 0}\frac{b_0/a_2}{(s+1/T)^2} = \frac{b_0\cdot T^2}{a_2} = \frac{b_0}{a_0} = G(0)$$

$$C = \lim_{s\to -1/T}\frac{b_0}{a_2}\cdot\frac{1}{s} = -\frac{b_0\cdot T}{a_2} = -\frac{b_0}{a_0}\cdot\frac{a_0}{a_2}\cdot T = -G(0)\cdot\frac{T}{T^2} = -\frac{G(0)}{T}$$

$$A + B = 0 \implies B = -A = -G(0)$$

Equazioni del Secondo Ordine: risposta forzata

Caso Poli reali coincidenti

$$T^2 \triangleq \frac{1}{p^2} \triangleq \frac{a_2}{a_0}$$

$$Y(s) = \frac{G(0)}{s} - \frac{G(0)/T}{(s+1/T)^2} - \frac{G(0)}{s+1/T}$$

$$\downarrow$$

$$F(s-a) = \mathcal{L}\{e^{at} \cdot f(t)\}$$

$$\downarrow$$

$$\frac{1}{s^2} \longrightarrow t \cdot 1(t) \longrightarrow \mathcal{L}^{-1}\left\{\frac{G(0)}{T} \cdot \frac{1}{(s+1/T)^2}\right\} = \frac{G(0)}{T} \cdot \mathcal{L}^{-1}\left\{\frac{1}{(s+1/T)^2}\right\} = \frac{G(0)}{T} \cdot e^{-t/T} \cdot t \cdot 1(t)$$

$$y(t) = G(0)\left(1 - e^{-t/T}\left(1 + \frac{t}{T}\right)\right) \cdot 1(t)$$

Equazioni del Secondo Ordine: risposta forzata

Caso Poli Complessi e Coniugati

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{a_2 s^2 + a_1 s + a_0}$$

$$\alpha = -\frac{a_1}{2a_2} \qquad \beta = \sqrt{\frac{a_0}{a_2} - \left(\frac{a_1}{2a_2}\right)^2} = \sqrt{\frac{a_0}{a_2}\left(1 - \frac{a_1^2}{4a_2a_0}\right)}$$

$$p_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4 \cdot a_0 \cdot a_2}}{2 \cdot a_2}$$

$$p_{1,2} = \alpha \pm i\beta$$

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Caso $\alpha = 0 \xrightarrow{a_1 = 0} a_2 s^2 + a_0 = 0 \longrightarrow s^2 = -\frac{a_0}{a_2}$

$$\omega_0 = \sqrt{\frac{a_0}{a_2}}$$ Pulsazione di risonanza

$$\xi \triangleq \frac{a_1}{2\sqrt{a_0 \cdot a_2}}$$ Smorzamento

$a_1 \neq 0$

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$$\omega_0 = \sqrt{\frac{a_0}{a_2}}$$

Pulsazione di risonanza

Ed infine:

$$\alpha = -\xi \cdot \omega_0$$

$$\beta = \omega_0 \sqrt{1 - \xi^2}$$

$$\xi \triangleq \frac{a_1}{2\sqrt{a_0 \cdot a_2}}$$

Smorzamento

$a_1 \neq 0$

Equazioni del Secondo Ordine: risposta forzata

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Pulsazione di risonanza

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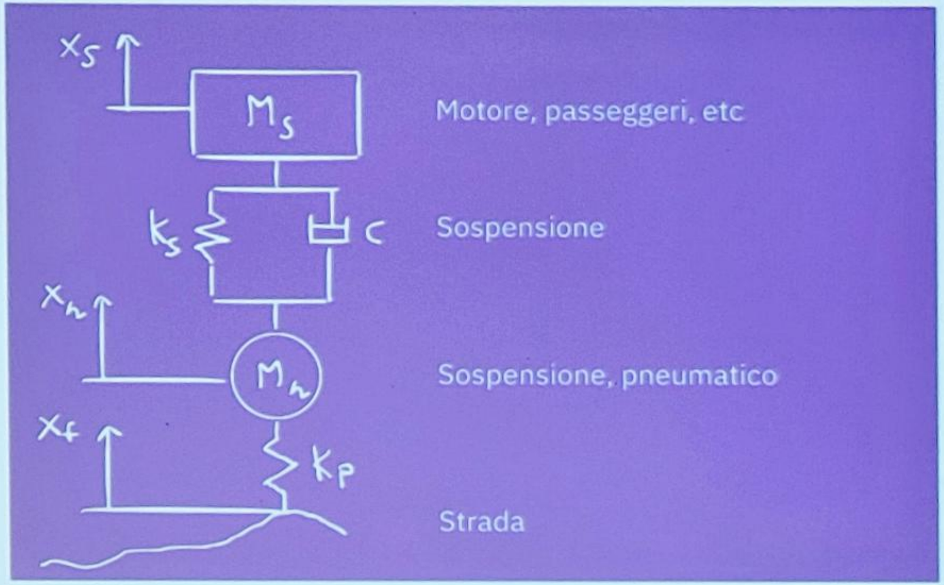
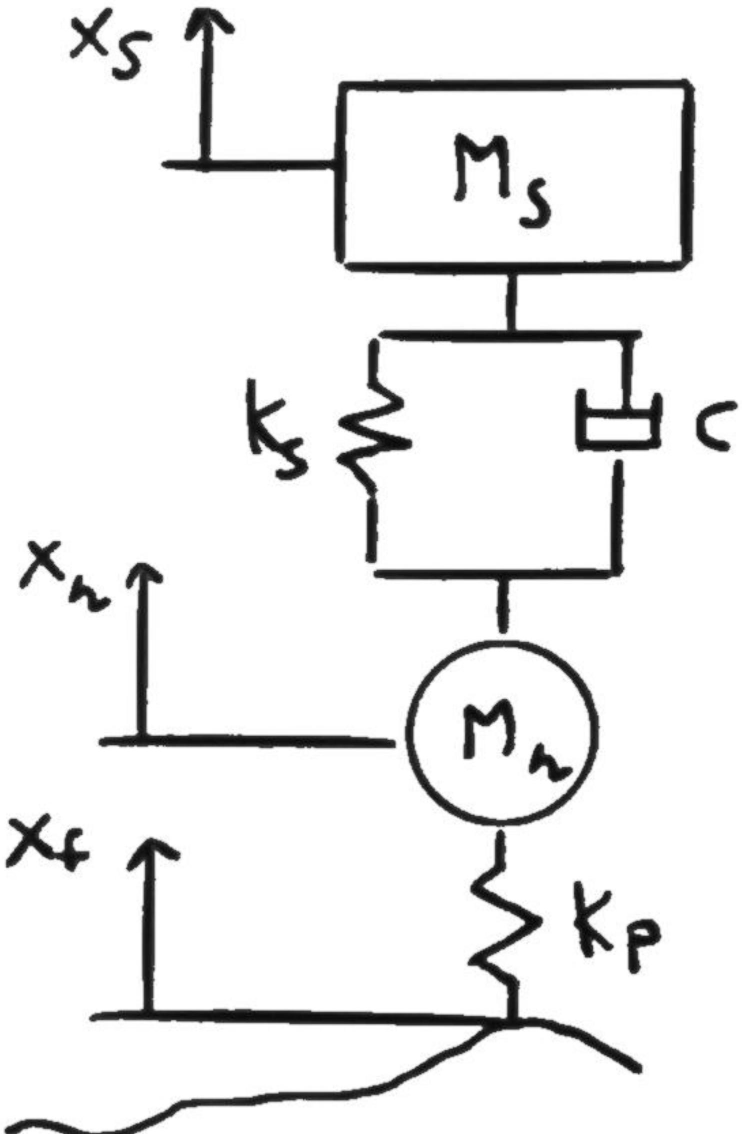
Smorzamento

Forma di Bode

$$G(s) = \frac{G(0)}{1 + 2 \cdot \xi \cdot \frac{s}{\omega_0} + \frac{s^2}{\omega_0^2}}$$

Equazioni del Secondo Ordine: Esempio





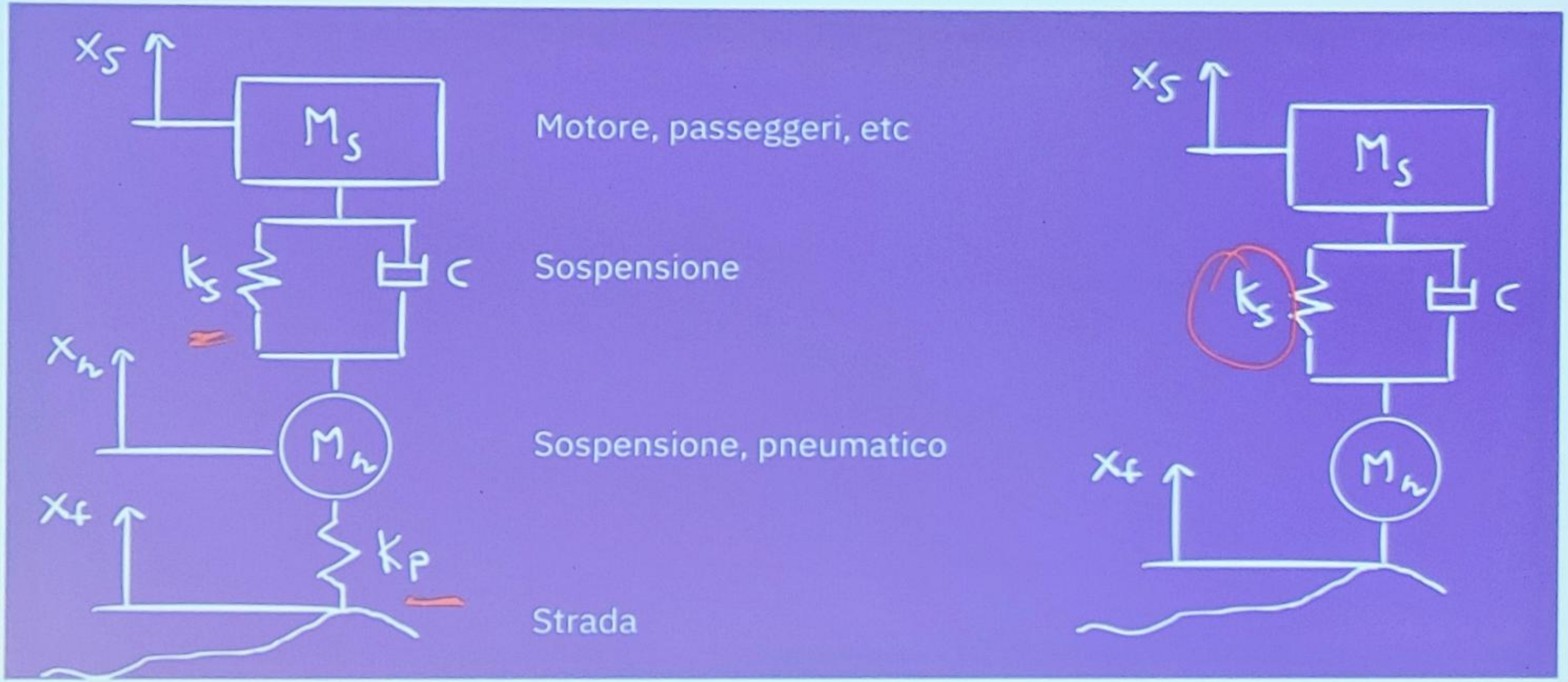
Motore, passeggeri, etc

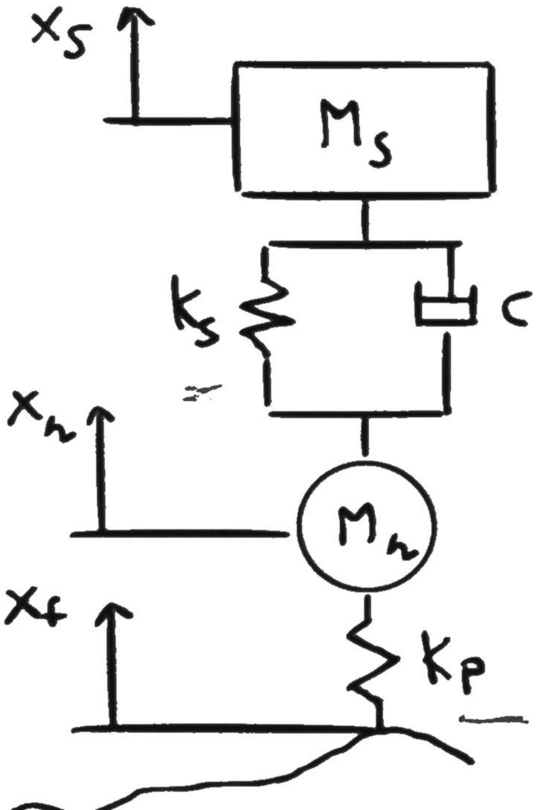
Sospensione

Sospensione, pneumatico

Strada

Equazioni del Secondo Ordine: Esempio





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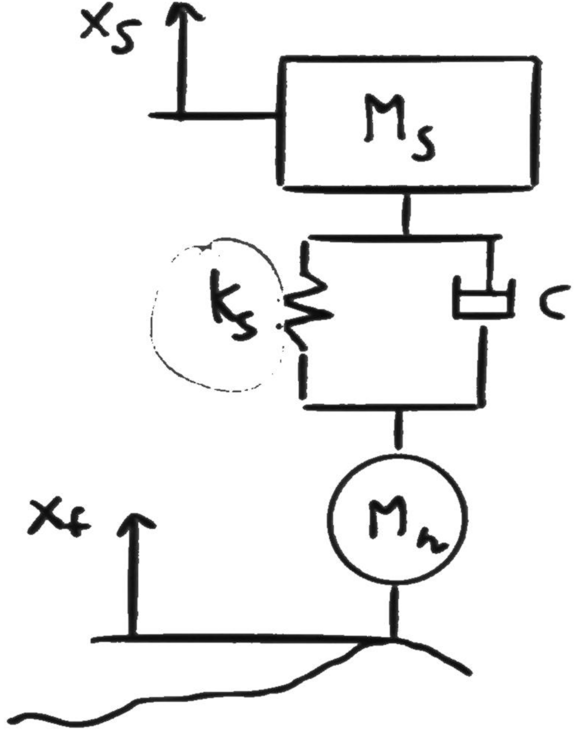
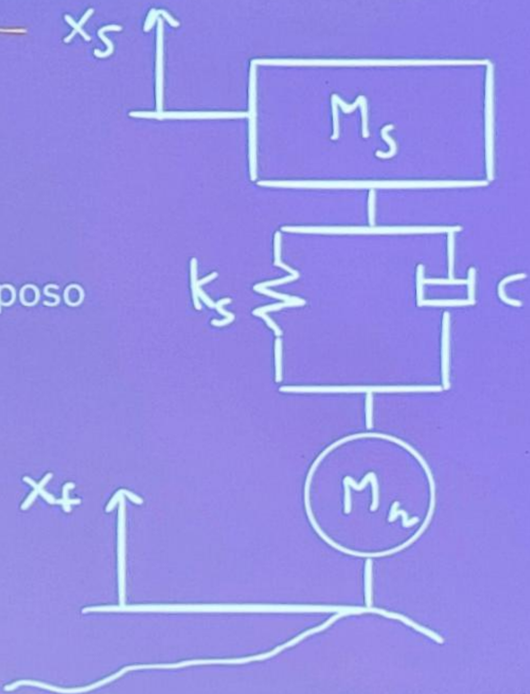


Equazioni del Secondo Ordine: Esempio

Uscita

L_R Lunghezza molla a riposo

Ingresso



Il sistema funziona bene se la massa sospesa Ms rimane ferma (comfort)

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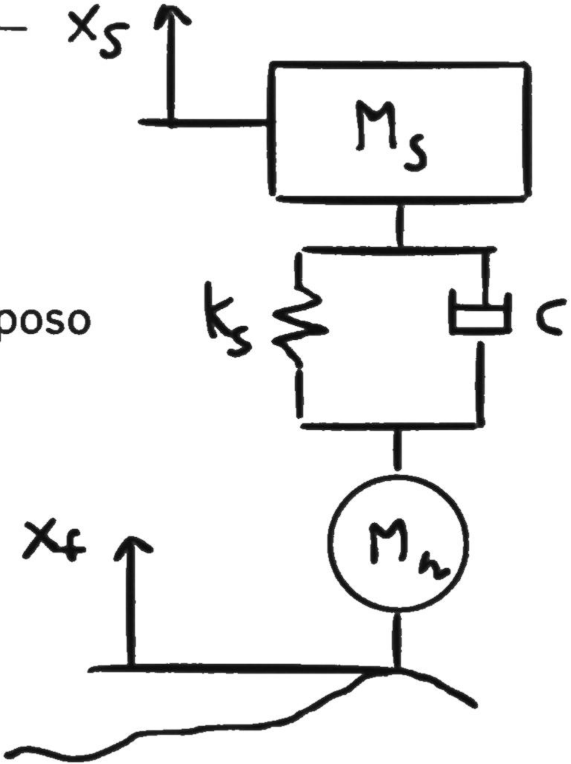

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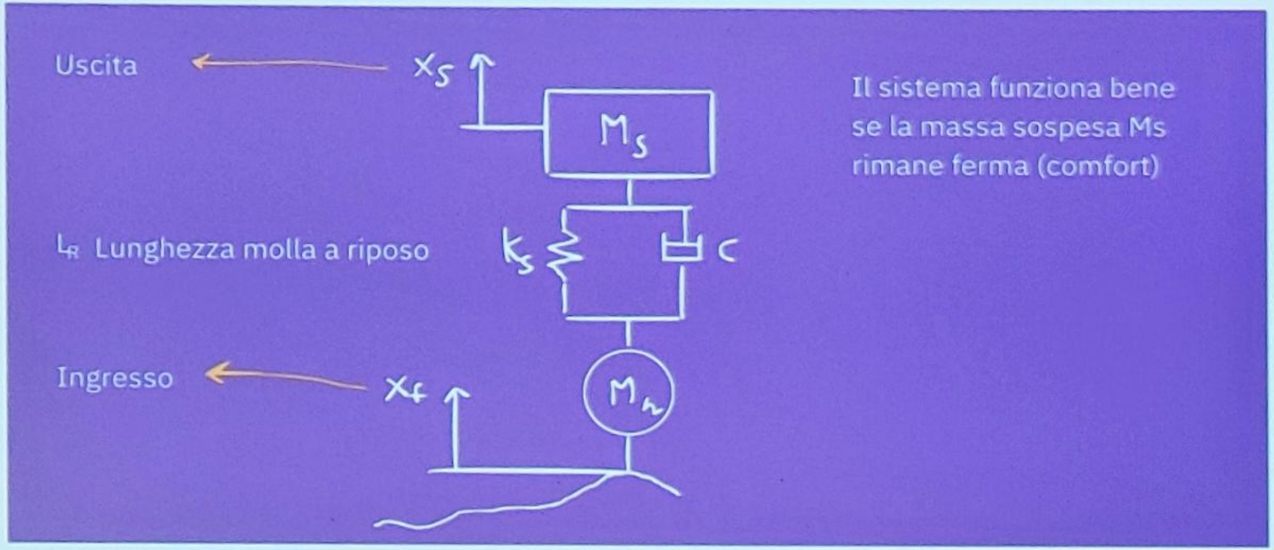
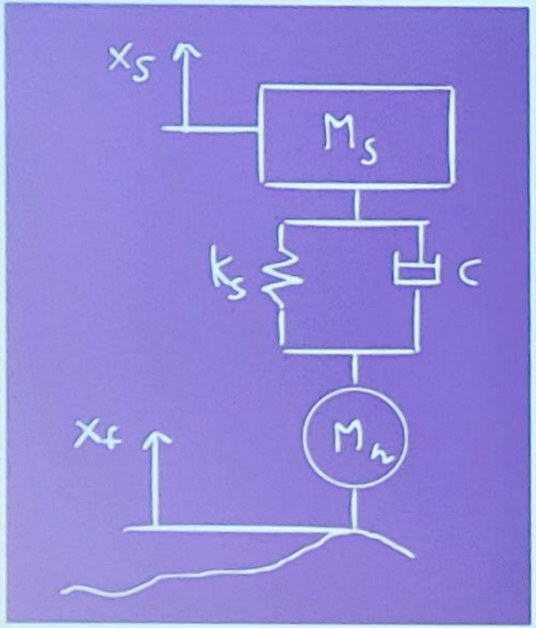
Equazioni del Secondo Ordine: Esempio



$$M_s \frac{d^2 x_s}{dt^2} = c \frac{d(x_f - x_s)}{dt} + k_s(x_f - x_s) + k_s L_r - M_s g$$

Equazioni del Secondo Ordine: Esempio



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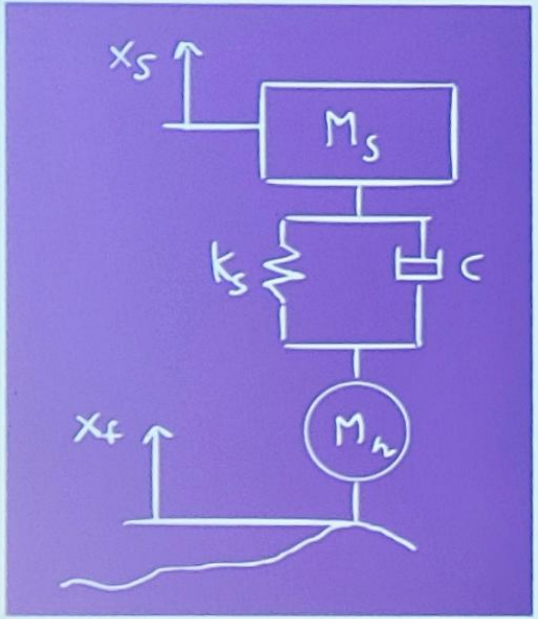
$$x_s = \Delta x_s + \bar{x}_s \qquad \bar{x}_s = L_r - \frac{M_s g}{k_s}$$

Condizione di equilibrio

Equazioni del Secondo Ordine: Esempio



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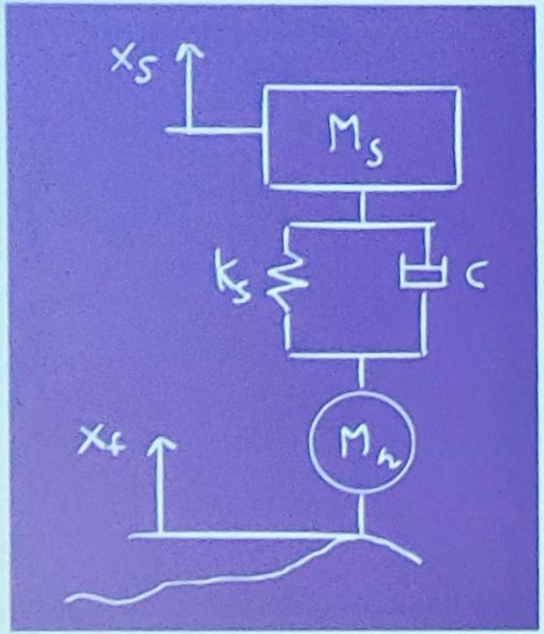
Condizione di equilibrio

$$\Delta x_s = x_s - \bar{x}_s = x_s - L_r + \frac{M_s g}{k_s}$$

$$\frac{d^2 x_s}{dt^2} = \frac{d^2 \Delta x_s}{dt^2} \qquad \frac{dx_s}{dt} = \frac{d\Delta x_s}{dt}$$

$$M_s \frac{d^2 \Delta x_s}{dt^2} = c\frac{d(x_f - \Delta x_s)}{dt} + k_s(x_f - \Delta x_s)$$

Equazioni del Secondo Ordine: Esempio



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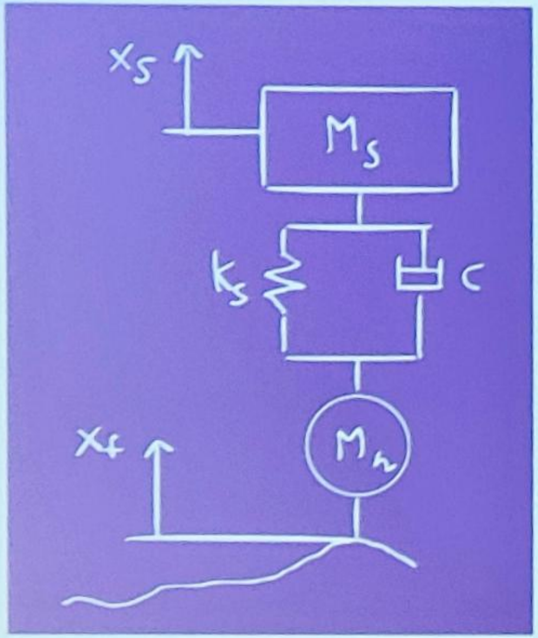
$$M_s \frac{d^2 \Delta x_s}{dt^2} + c \frac{d \Delta x_s}{dt} + k_s \Delta x_s = c \frac{dx_f}{dt} + k_s x_f$$

$$(s^2 M_s + cs + k_s) \Delta x_s = (cs + k_s) x_f$$

$$G(s) = \frac{\Delta x_s}{x_f} = \frac{cs + k_s}{s^2 M_s + cs + k_s}$$

Equazioni del Secondo Ordine: Esempio



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$$\omega_0 = \sqrt{\frac{a_0}{a_2}} = \sqrt{\frac{k_s}{M_s}} \qquad \xi = \frac{a_1}{2\sqrt{a_0 \cdot a_2}} = \frac{c}{2\sqrt{k_s M_s}}$$