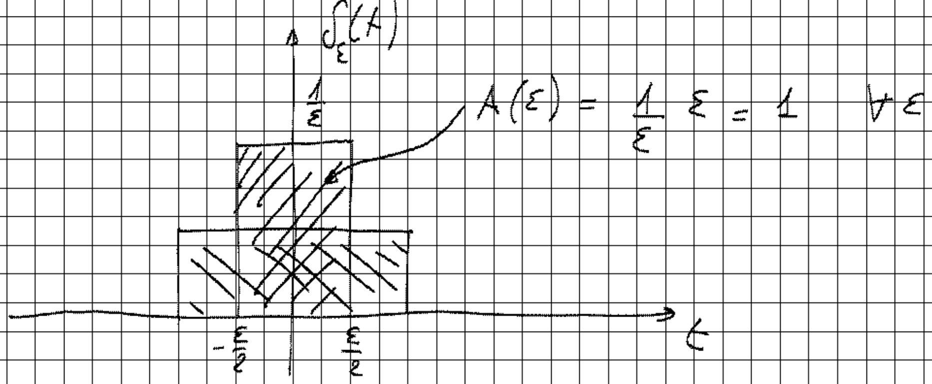
IMPULSO (DELTA) DI DIRAC

$$\delta_\varepsilon(t) = \frac{1}{\varepsilon}\,\mathrm{rect}\left(\frac{t}{\varepsilon}\right)$$



$$\delta(t) \triangleq \lim_{\varepsilon \to 0} \delta_\varepsilon(t)$$

N.B. $\delta(t)$ non è una funzione in senso canonico

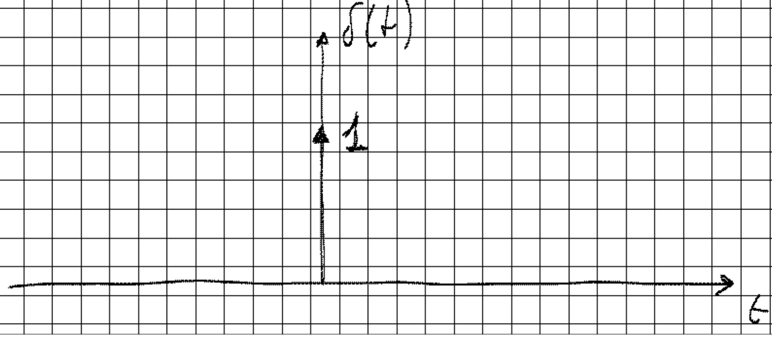
PROPRIETÀ

1) $\displaystyle\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$ AREA DELLA $\delta(t)$

Dimostrazione

$$\int_{-\infty}^{+\infty} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\,\mathrm{rect}\left(\frac{t}{\varepsilon}\right) dt = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \underbrace{\int_{-\infty}^{+\infty} \mathrm{rect}\left(\frac{t}{\varepsilon}\right) dt}_{\varepsilon} = 1$$

Rappresentazione su un grafico della $\delta(t)$

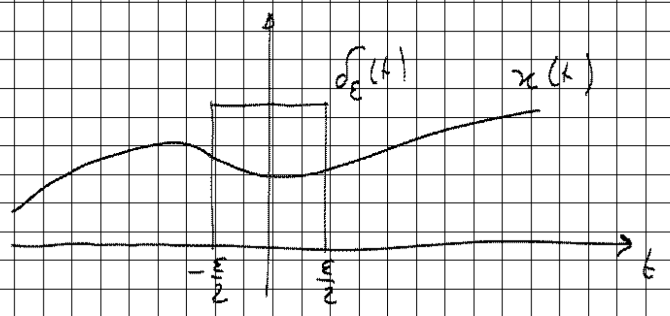


2) $\int_{-\infty}^{+\infty} x(t)\,\delta(t)\,dt = x(0)$ PROPRIETÀ CAMPIONATRICE

$x(t)$ continua in $t=0$

Dimostrazione

$$\int_{-\infty}^{+\infty} x(t) \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \operatorname{rect}\left(\frac{t}{\varepsilon}\right) dt = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} x(t) \operatorname{rect}\left(\frac{t}{\varepsilon}\right) dt =$$



$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} x(t)\,dt = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \cdot \varepsilon\, x(\tilde{t}) \quad , -\frac{\varepsilon}{2} \le \tilde{t} \le \frac{\varepsilon}{2}$$

TEOREMA DELLA MEDIA

$$= \lim_{\varepsilon\to 0} x(\tilde{t}) = x(0)$$

NB. le seguenti proprietà saranno dedotte dalla proprietà campionatrice

3) $\delta(t) = \delta(-t)$ PARITÀ

$$\int_{-\infty}^{+\infty} \delta(-t)\,x(t)\,dt = (t' = -t) = \int_{-\infty}^{+\infty} \delta(t')\,x(-t')\,dt' = x(0)$$

4) $\int_{-\infty}^{+\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$ TRASLAZIONE

Dimostrazione

$$\int_{-\infty}^{+\infty} x(t)\,\delta(t-t_0) = (t' = t-t_0) = \int_{-\infty}^{+\infty} x(t'+t_0)\,\delta(t')\,dt' =$$
$$= x(t_0)$$

4 bis) $x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$

Dimostrazione

$$\int_{-\infty}^{+\infty} x(t)\,\delta(t-t_0) = x(t_0)\int_{-\infty}^{+\infty}\delta(t-t_0)\,dt =$$
$$= \int_{-\infty}^{+\infty} x(t_0)\,\delta(t-t_0)\,dt$$

5) $x(t) \otimes \delta(t) = x(t)$ CONVOLUZIONE

Dimostrazione

$$\int_{-\infty}^{+\infty} x(\alpha)\,\delta(t-\alpha)\,d\alpha = x(t)$$ per la proprietà campionatrice

$$x(t) \otimes \delta(t-t_0) = x(t-t_0)$$

Dimostrazione

$$\int_{-\infty}^{+\infty} x(\alpha)\,\delta[(t-t_0)-\alpha]\,d\alpha = \int_{-\infty}^{+\infty} x(\alpha)\,\delta[\alpha-(t-t_0)]\,d\alpha = x(t-t_0)$$

6) $\int_{-\infty}^{+\infty} x(t)\,\delta(at)\,dt = \dfrac{x(0)}{|a|}$ $(a \neq 0)$

Dimostrazione

$a > 0$

$$\int_{-\infty}^{+\infty} x(t)\,\delta(at)\,dt = \left(at = \alpha \right) = \int_{-\infty}^{+\infty} x\left(\frac{\alpha}{a}\right)\delta(\alpha)\,\frac{d\alpha}{a} = \frac{x(0)}{a}$$

$a < 0$

$$\int_{-\infty}^{+\infty} x(t)\,\delta(at)\,dt = \left(at = \alpha \right) = -\int_{-\infty}^{+\infty} x\left(\frac{\alpha}{a}\right)\delta(\alpha)\,\frac{d\alpha}{a} = -\frac{x(0)}{a}$$

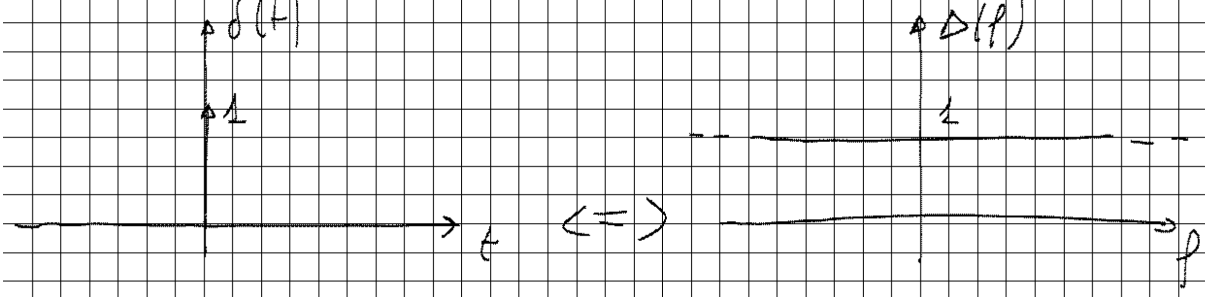
$$a \neq 0 \quad \Rightarrow \quad \int_{-\infty}^{+\infty} x(t)\,\delta(at)\,dt = \frac{x(0)}{|a|}$$

TRASFORMATA CONTINUA DI FOURIER GENERALIZZATA

$$\delta(t) \overset{TCF}{\Longleftrightarrow} 1$$

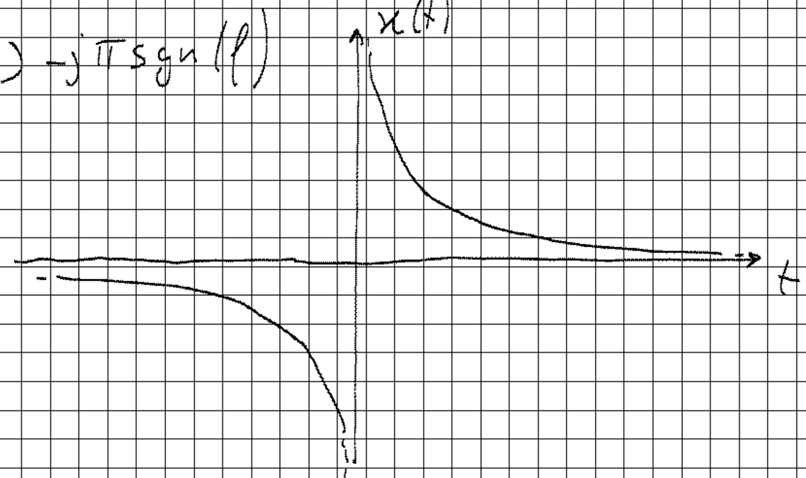
Dimostrazione

$$\int_{-\infty}^{+\infty} \delta(t)\, e^{-j2\pi f t}\, dt = 1$$ per la proprietà campionatrice



TRASFORMATA DI $\frac{1}{t}$

$$x(t) = \frac{1}{t} \Leftrightarrow -j\pi\,\mathrm{sgn}(f)$$



$$X(f) = \int_{-\infty}^{+\infty} \frac{1}{t} e^{-j2\pi f t} \, dt =$$

$$= \int_{-\infty}^{+\infty} \frac{1}{t} \left[\cos(2\pi f t) - j \sin(2\pi f t) \right] dt =$$

$$= \int_{-\infty}^{+\infty} \underbrace{\frac{1}{t} \cos(2\pi f t)}_{C(t)} \, dt - j \int_{-\infty}^{+\infty} \frac{1}{t} \sin(2\pi f t) \, dt$$

$C(t)$ è una funzione dispari integrata tra $-\infty$ e $+\infty$

$$\int_{-\infty}^{+\infty} C(t) \, dt = 0$$

A rigore devo tener presente la discontinuità in $t=0$

$$\lim_{T \to 0} \left[\int_{-\infty}^{-T} C(t) \, dt + \int_{T}^{+\infty} C(t) \, dt \right] = 0$$

Valore Principale Cauchy

$$X(f) = -j \int_{-\infty}^{+\infty} \frac{\sin(2\pi f t)}{t}\, dt = -j 2\pi f \int_{-\infty}^{+\infty} \mathrm{sinc}(2 f t)\, dt$$

N.B. $\int_{-\infty}^{+\infty} y(t)\, dt = Y(0)$

ricordando la trasformata

$$\mathrm{sinc}(2Bt) \Leftrightarrow \frac{1}{2B} \mathrm{rect}\left(\frac{f}{2B}\right)$$

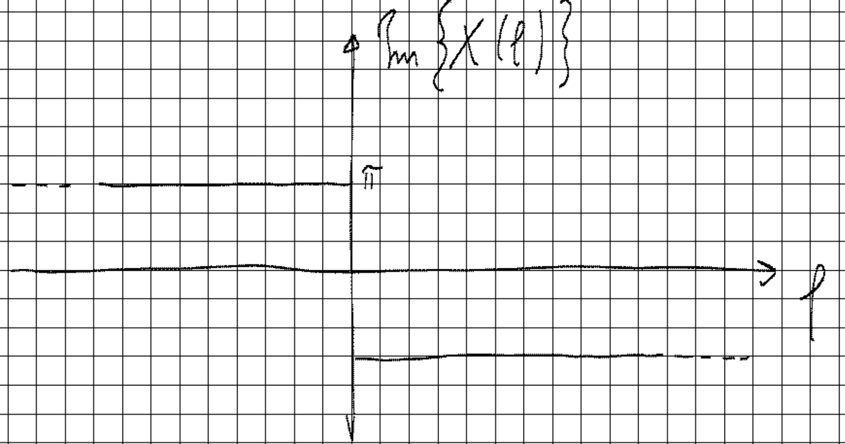
$$\int_{-\infty}^{+\infty} \mathrm{sinc}(2Bt)\, dt = \frac{1}{2B} \mathrm{rect}\left(\frac{0}{2B}\right) = \frac{1}{2B}$$

Se $B<0$

$$\int_{-\infty}^{+\infty} \mathrm{sinc}(2Bt)\, dt = \int_{-\infty}^{+\infty} \mathrm{sinc}(-2|B|t)\, dt =$$

$$= \int_{-\infty}^{+\infty} \mathrm{sinc}(2|B|t)\, dt = \frac{1}{2|B|}$$ Vale anche quando $B>0$

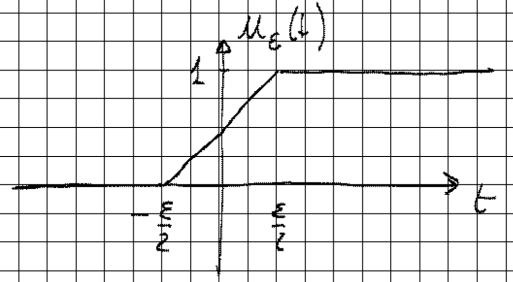
$$X(f) = -\frac{j 2\pi f}{2|f|} = -j\pi\, \mathrm{sgn}(f)$$



FUNZIONI $u(t)$ e $\delta(t)$

$$u(t) = \int_{-\infty}^{t} \delta(\alpha)\, d\alpha$$

$$\delta(t) = \frac{d}{dt} u(t)$$



$$\delta_\varepsilon(t) = \frac{d}{dt} u_\varepsilon(t)$$

$$\lim_{\varepsilon \to 0} \delta_\varepsilon(t) = \lim_{\varepsilon \to 0} \frac{d}{dt} u_\varepsilon(t)$$

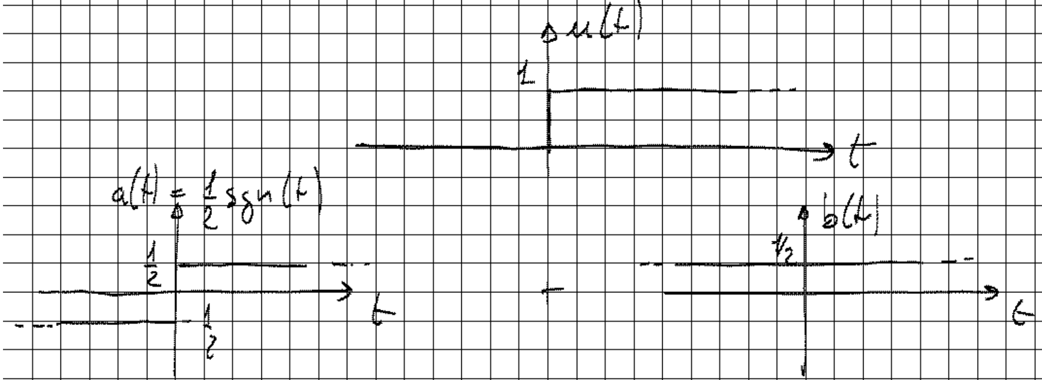
$$= \frac{d}{dt} \lim_{\varepsilon \to 0} u_\varepsilon(t) = \frac{d}{dt} u(t)$$

TRASFORMATA DI $u(t)$

$$u(t) \overset{TCF}{\Leftrightarrow} U(f) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$

Dimostrazione

$$u(t) = \frac{1}{2}\left[1 + \mathrm{sgn}(t)\right]$$



$$U(f) = A(f) + B(f) \quad , \quad a(t) \Leftrightarrow A(f) \ , \ b(t) \Leftrightarrow B(f)$$

Ricordando che

$$\frac{1}{t} \Leftrightarrow -j\pi \,\mathrm{sgn}(f) \quad \overset{\text{DUALITÀ}}{\Longrightarrow} \quad -j\pi \,\mathrm{sgn}(t) \Leftrightarrow -\frac{1}{f}$$

$$\mathrm{sgn}(t) = \frac{1}{j\pi f} \Rightarrow a(t) = \frac{1}{2}\mathrm{sgn}(t) \Leftrightarrow A(f) = \frac{1}{j2\pi f}$$

$$b(t) = \frac{1}{2} \Leftrightarrow B(f) = \frac{1}{2}\delta(f)$$

$$U(f) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$

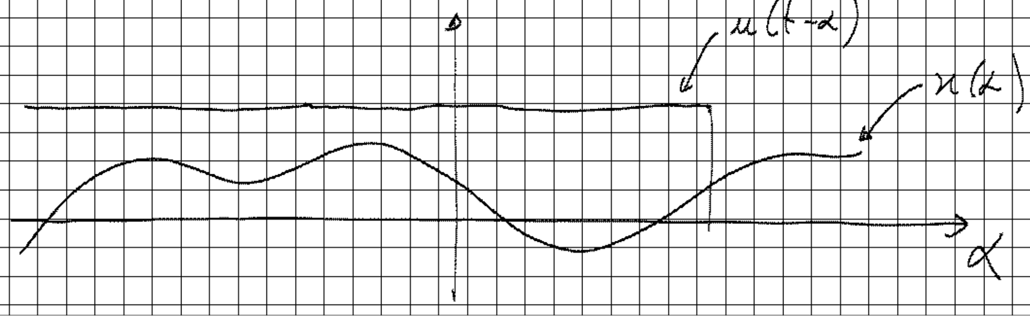
TEOREMA DELL'INTEGRAZIONE COMPLETO

$$y(t) = \int_{-\infty}^{t} x(\alpha)\, d\alpha \Leftrightarrow Y(f) = \frac{X(0)}{2}\delta(f) + \frac{X(f)}{j2\pi f}$$

Dimostrazione

$$y(t) = x(t) \otimes u(t) = \int_{-\infty}^{+\infty} x(\alpha)\, u(t-\alpha)\, d\alpha =$$

$$= \int_{-\infty}^{t} x(\alpha)\, d\alpha$$



$$Y(f) = X(f)\,U(f) = X(f)\left[\frac{\delta(f)}{2} + \frac{1}{j2\pi f}\right] =$$

$$= \frac{X(0)}{2} + \frac{X(f)}{j2\pi f}$$

TCF della $\delta(t-t_0)$

$$\delta(t-t_0) \Leftrightarrow e^{-j2\pi f t_0}$$

Dimostrazione

$$\int_{-\infty}^{+\infty} \delta(t-t_0)\, e^{-j2\pi f t}\, dt = e^{-j2\pi f t_0}$$ proprietà campionatrice

$$e^{j2\pi f_0 t} \Leftrightarrow \delta(f-f_0)$$ x dualità

TCF DI SINUSOIDI

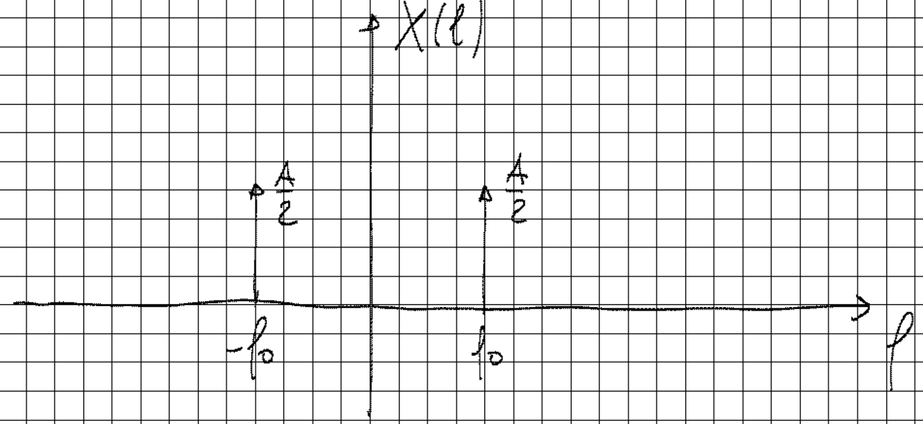
$$x(t) = A\cos(2\pi f_0 t)$$ TCF DI UN COSENO

$$X(f) = \int_{-\infty}^{+\infty} A\cos(2\pi f_0 t)\, e^{-j2\pi f t}\, dt =$$

$$= \frac{A}{2}\int_{-\infty}^{+\infty} e^{j2\pi f_0 t}\, e^{-j2\pi f t}\, dt + \frac{A}{2}\int_{-\infty}^{+\infty} e^{-j2\pi f_0 t}\, e^{-j2\pi f t}\, dt =$$

$$= \frac{A}{2}\int_{-\infty}^{+\infty} e^{-j2\pi (f-f_0)t}\, dt + \frac{A}{2}\int_{-\infty}^{+\infty} e^{-j2\pi (f+f_0)t}\, dt =$$

$$= \frac{A}{2}\,\delta(f-f_0) + \frac{A}{2}\,\delta(f+f_0) = \frac{A}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right]$$



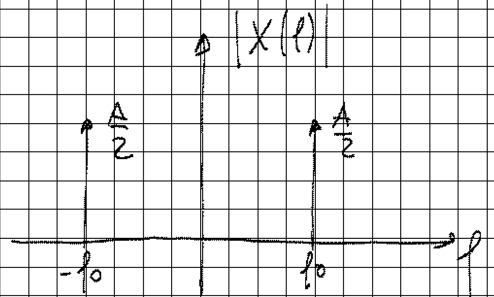
TCF DI UN SENO

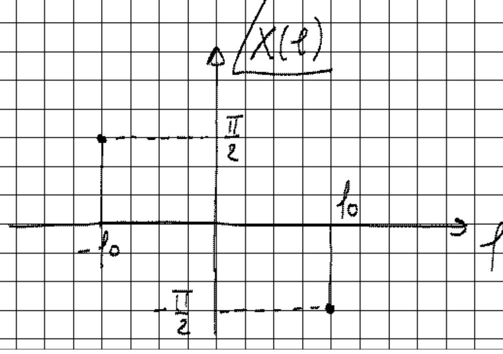
$$x(t) = A \sin(2\pi f_0 t)$$

$$X(f) = \int_{-\infty}^{+\infty} A \sin(2\pi f_0 t)\, e^{-j2\pi f t}\, dt =$$

$$= \frac{A}{2j}\int_{-\infty}^{+\infty} e^{-j2\pi(f-f_0)t}\, dt - \frac{A}{2j}\int_{-\infty}^{+\infty} e^{-j2\pi(f+f_0)t}\, dt =$$

$$= \frac{A}{2j}\,\delta(f-f_0) - \frac{A}{2j}\,\delta(f+f_0) = \frac{A}{2}\left[e^{-j\frac{\pi}{2}}\delta(f-f_0) + e^{j\frac{\pi}{2}}\delta(f+f_0)\right]$$





BIUNIVOCITÀ DELLA TCF

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df$$

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

Dimostrazione

$$\int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df =$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(\alpha)\, e^{-j2\pi f \alpha}\, d\alpha \; e^{j2\pi f t}\, df =$$

$$= \int_{-\infty}^{+\infty} x(\alpha) \int_{-\infty}^{+\infty} e^{-j2\pi f(\alpha - t)}\, df \; d\alpha =$$

$$= \int_{-\infty}^{+\infty} x(\alpha)\, \delta(\alpha - t)\, d\alpha = x(t)$$