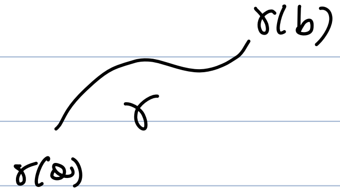
$$F = F_1\, i + F_2\, j + F_3\, k \qquad F: \mathbb{R}^3 \to \mathbb{R}^3$$

$$dL = F \cdot dx$$

$$\int_\gamma F\cdot dx \overset{def}{=} \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt$$



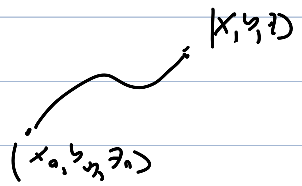
INTEGRALE CURVILINEO DI

SECONDA SPECIE

CIRCUITAZIONE $\gamma(a)=\gamma(b)$

$$F = \nabla U \quad \text{POTENZIALE } U \iff \oint_\gamma F\cdot dx = 0$$

$$U(x,y,z) = \int_\gamma F\cdot dx \qquad \begin{aligned}\gamma(a) &= (x_0,y_0,z_0)\\ \gamma(b) &= (x,y,z)\end{aligned}$$



$$F = \nabla U \iff F_i(x) = \frac{\partial}{\partial x_i} U(x) \qquad U \in C^2(\Omega)$$

$$\frac{\partial}{\partial x_j} F_i(x) = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} U = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} U = \frac{\partial}{\partial x_i} F_j$$

$$\partial_j F_i = \partial_i F_j \qquad \forall\, i,j = 1,2,3$$

<u>TEOREMA</u> $F = \nabla U$ (F DERIVA DAL POTENZIALE U)

$$\Rightarrow \quad \mathrm{rot}\, F = 0$$

$$\mathbb{R}^3 \ni \mathrm{rot}\, F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

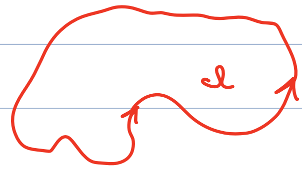
$$= \left(\frac{\partial}{\partial x_2} F_3 - \frac{\partial}{\partial x_3} F_2\right) i - \left(\frac{\partial}{\partial x_1} F_3 - \frac{\partial}{\partial x_3} F_1\right) j + \left(\frac{\partial}{\partial x_1} F_2 - \frac{\partial}{\partial x_2} F_1\right) k$$

CONDIZIONE NECESSARIA F CON POTENZIALE È CHE $\mathrm{rot}\, F = 0$

$$F_1\, i + F_2\, j$$

$$\boxed{\Omega = \mathbb{R}^2 \setminus \{0,0\}} \qquad F(x,y) = -\frac{y}{x^2+y^2}\, i + \frac{x}{x^2+y^2}\, j$$

$\gamma \qquad x^2+y^2=1$



$\mathbb{R}^2 \setminus \Omega$

$$\gamma_1(t) = \cos(t)$$
$$\gamma_2(t) = \sin(t)$$
$$t \in [0, 2\pi]$$

$$\oint_\gamma F \cdot dx = \int_0^{2\pi} -\frac{\sin(t)}{\sin^2(t)+\cos^2(t)}(-\sin(t)) + \frac{\cos(t)}{\sin^2(t)+\cos^2(t)}\cos(t)\, dt$$

$$= \int_0^{2\pi} 1\, dt = \boxed{2\pi}$$

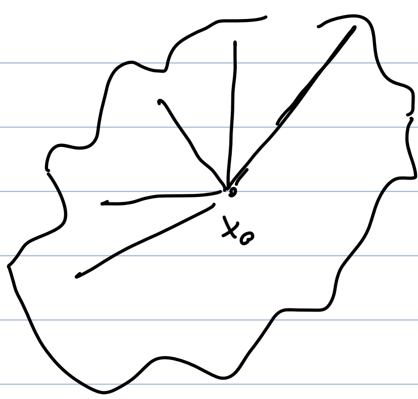
$$\partial_2 F_1 = \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) \quad = \quad \partial_1 F_2 = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

$$= -\frac{x^2+y^2 - y\,2y}{(x^2+y^2)^2} = -\frac{x^2-y^2}{(x^2+y^2)^2} \qquad = \frac{x^2+y^2 - x\,2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

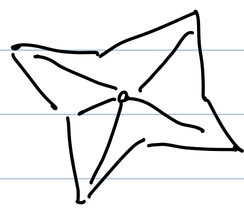
$$F_1 = \partial_1 U \qquad F_2 = \partial_2 U \quad \Rightarrow \quad \boxed{\partial_2 F_1 = \partial_1 F_2}$$
$$\Leftarrow$$

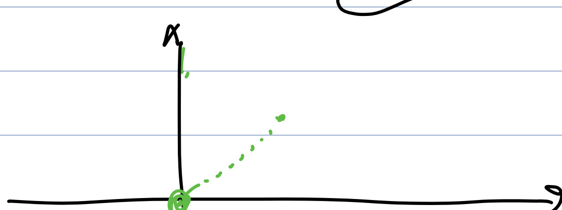
TEOREMA $\quad \mathrm{rot}\, F = 0 \qquad F \in C^1(\Omega)$

$\Rightarrow \quad F = \nabla U \quad$ SE $\quad \Omega$ è STELLATO RISPETTO A UN PUNTO



DEF $\exists\, x_0$ t.c.
$\forall x \in \Omega \quad [x_0, x] \subseteq \Omega$





$$\gamma(t) = x^0 + t(x-x^0) \qquad t\in[0,1]$$

$$U(x) = \int_\gamma F\cdot dx = \int_0^1 F(\gamma(t))\cdot\gamma'(t)\,dt = \int_0^1 \langle F(\gamma(t)), x-x^0\rangle\,dt$$

$$= \int_0^1 \langle F(x^0+t(x-x^0)), x-x^0\rangle\,dt$$

$$\frac{\partial}{\partial x_j}U(x) \overset{?}{=} F_j(x)$$

$$\boxed{\frac{\partial}{\partial x_j}U(x)} = \frac{\partial}{\partial x_j}\int_0^1 \sum_{i=1}^3 F_i(x^0+t(x-x^0))(x_i-x_i^0)\,dt$$

$$= \sum_{i=1}^3 \frac{\partial}{\partial x_j}\int_0^1 F_i(x^0+t(x-x^0))(x_i-x_i^0)\,dt$$

$$\frac{\partial}{\partial x_j}x_k = \begin{cases}1 & j=k\\ 0 & j\neq k\end{cases} = \delta_{jk}$$

$$= \sum_{i=1}^3 \int_0^1 \frac{\partial}{\partial x_j}\left[F_i(x^0+t(x-x^0))(x_i-x_i^0)\right]dt$$

$$= \sum_{i=1}^3 \int_0^1 \left[\sum_k \frac{\partial F_i}{\partial x_k}(x^0+t(x-x^0))\frac{\partial}{\partial x_j}[x^0+t(x-x^0)]_k (x_i-x_i^0) + F_i(\ )\frac{\partial}{\partial x_j}(x_i-x_i^0)\right]dt$$

$$= \int_0^1\left[\sum_{i=1}^3 t\,\frac{\partial F_i}{\partial x_j}(x^0+t(x-x^0))(x_i-x_i^0) + F_j(x^0+t(x-x^0))\right]dt$$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \qquad \text{PERCHÉ } \operatorname{rot} F = 0$$

$$= \int_0^1 t\,\boxed{\sum_i \frac{\partial F_j}{\partial x_i}(x^0+t(x-x^0))(x_i-x_i^0)} + F_j(\ )\Big]\,dt$$

$$\int_0^1 t\,\boxed{\frac{d}{dt}F_j(x^0+t(x-x^0))} + F_j(x^0+t(x-x^0))\,dt$$

$$\boxed{\int_0^1 g f'\,dt = gf\Big|_0^1 - \int_0^1 g'f} \qquad g(t)=t \qquad f' = \frac{d}{dt}F_j(\ )$$

$$= t\, F_j(x^0 + t(x-x^0))\Big|_0^1 - \int_0^1 1\, \cancel{F_j(x^0+t(x-x^0))} + \int_0^1 \cancel{F_j(x^0 + t(x-x^0))}\, dt$$

$$= 1\, F_j(x^0 + 1(x-x^0)) - 0\, \cancel{F_j(\,)}$$

$$= F_j(x)$$

ES $\quad F = -y\,i + x\,j \qquad$ CONSERVATIVA $\quad (F = \nabla U)$

$$\partial_2 F_1 \neq \partial_1 F_2 \qquad \frac{\partial}{\partial y}(-y) = \frac{\partial}{\partial x} x$$

$$-1 = 1 \qquad \text{NO}$$

$$\oint_\gamma F \cdot dx \neq 0$$

ES $\quad F(x,y) = \dfrac{x}{x^2+y^2}\, i + \dfrac{y}{x^2+y^2}\, j \qquad$ CONSERVATIVO $\quad F = \nabla U$

$$F : \mathbb{R}^2 \setminus \{(0,0)\}$$

$$\partial_2 F_1 = \frac{\partial}{\partial y} \frac{x}{x^2+y^2} \qquad \frac{\partial}{\partial x} \frac{y}{x^2+y^2} = \partial_1 F_2$$

$$= \frac{-x\, 2y}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2} = \frac{\partial}{\partial x} \frac{y}{x^2+y^2} = \frac{-y\, 2x}{(x^2+y^2)^2}$$

$$\nabla \times F = 0 \qquad \partial_1 F_2 = \partial_2 F_1$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = i\left(\partial_y 0 - \frac{\partial}{\partial z} F_2\right) - j\left(- \frac{\partial}{\partial z} F_1\right)$$

$$+ k\left(\partial_x F_2 - \partial_y F_1\right)$$

$$U \text{ t.c. } \quad \partial_x U = \frac{x}{x^2+y^2} \qquad \partial_y U = \frac{y}{x^2+y^2}$$

$$\frac{x}{x^2+y^2} = \frac{1}{2}\frac{2x}{x^2+y^2} = \frac{1}{2}\frac{\partial}{\partial x}\ln(x^2+y^2)$$

$$U(x,y) = \frac{1}{2}\ln(x^2+y^2) + \varphi(y)$$

$$\frac{\partial}{\partial x}U(x,y) = \frac{x}{x^2+y^2} + 0$$

$$\frac{\partial}{\partial y}U(x,y) = \frac{\partial}{\partial y}\left(\frac{1}{2}\ln(x^2+y^2) + \varphi(y)\right) = \frac{\cancel{2}y}{\cancel{2}(x^2+y^2)} + \varphi'(y) \Rightarrow \varphi'(y)=0$$

$$\varphi = c$$

$$\boxed{U(x,y) = \frac{1}{2}\ln(x^2+y^2) + c}$$

$$\boxed{F = F_1 i + F_2 j + F_3 k} \qquad L = \int_\gamma F \cdot dx$$

FORME DIFFERENZIALI $\qquad \omega(x,y,z) = a(x,y,z)\,dx + b(x,y,z)\,dy + c(x,y,z)\,dz$

$a,b,c : \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^3$

LAVORO ELEMENTARE

$dL = F_1\,dx + F_2\,dy + F_3\,dz$

FORMA DIFFERENZIALE

$\omega = F_1\,dx + F_2\,dy + F_3\,dz$

LAVORO FATTO DA F SU γ

$$\int_\gamma F \cdot dx = \int_a^b F(\gamma(t))\gamma'(t)\,dt$$

INTEGRALE DI ω SU CAMMINO γ

$$\int_\gamma \omega = \int_a^b F(\gamma(t))\gamma'(t)\,dt$$

F CONSERVATIVA

$F = \nabla U$

ω È ESATTA

$\omega = df = f_x\,dx + f_y\,dy + f_z\,dz$

F IRROTAZIONALE

$\nabla \times F = 0$

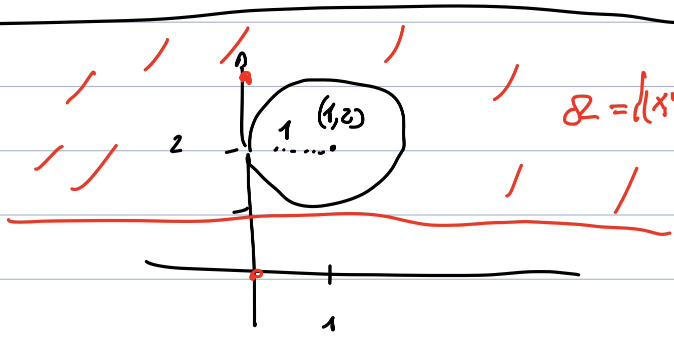
ω È CHIUSA

$\mathrm{rot}\, f = 0$

$\Pi_1(x_1,x_2,x_3) = x_1$ È LINEARE

$d\Pi_1 = \Pi_1$

$$F = -\frac{y}{x^2+y^2}\, i + \frac{x}{x^2+y^2}\, j$$



$\Omega = \{(x,y) : y > \frac{1}{2}\}$

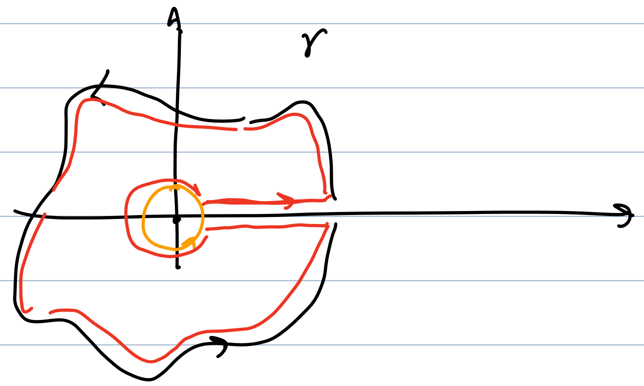
$$0 = \int_\gamma F \cdot dx \qquad \gamma$$

$(x-1)^2 + (y-2)^2 = 1$

$\text{rot} F = 0 \qquad \Omega = \mathbb{R}^2 \setminus \{(0,0)\}$

$\Omega = \{ (x,y) : y > \frac{1}{2} \}$ È SEMI SPAZIO È STELLATO

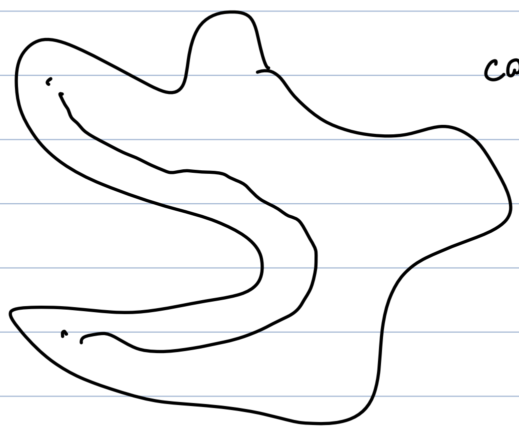
$U(x,y) = \text{arctg}\left(\frac{y}{x}\right) \qquad y > \frac{1}{2}$



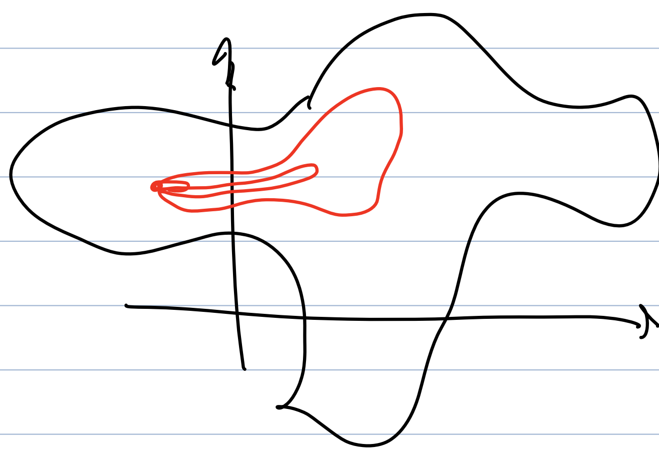
$$\int_\gamma F\,dx = 0$$

DOMINIO SEMPLICEMENTE CONNESSO (CONNESSO)

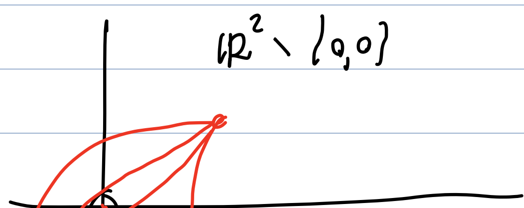
$\nabla \times F = 0 \quad \Rightarrow \quad F = \nabla U$



CONNESSO



$\mathbb{R}^2 \setminus \{0,0\}$

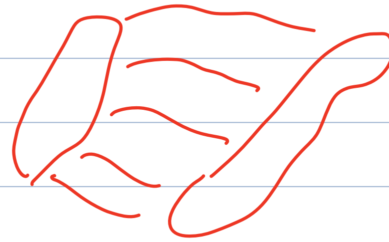


C^1

$F: [0,1]\times[0,1] \to \mathbb{R}^2$

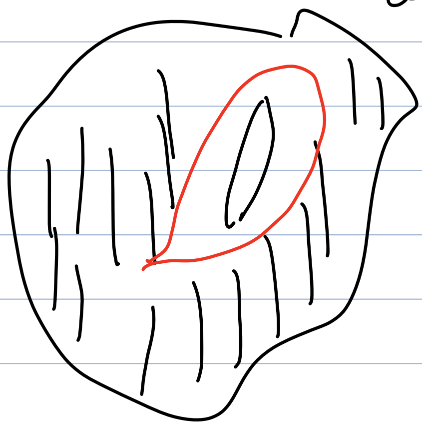
$F(0,t) = \gamma_0(t)$

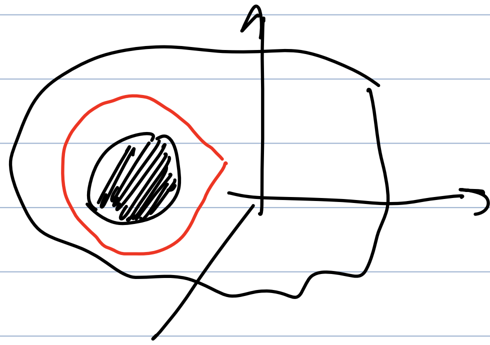
$F(1,t) = \gamma_1(t)$

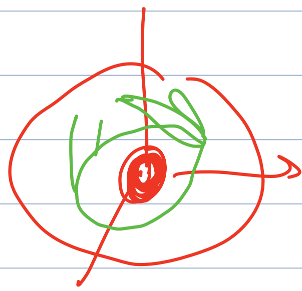


$$\int_{\gamma_0} F\cdot dx = \int_{\gamma_1} F\cdot dx$$

NON SEMPLICEMENTE CONNESSO

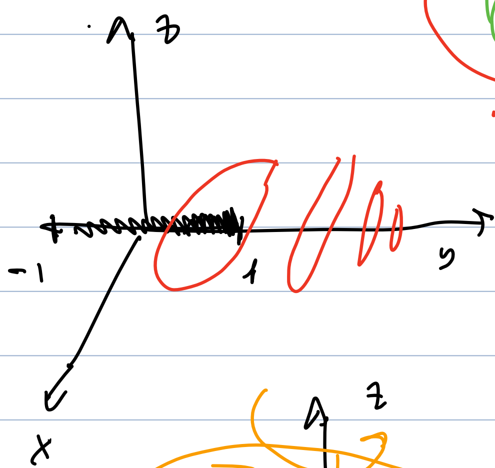




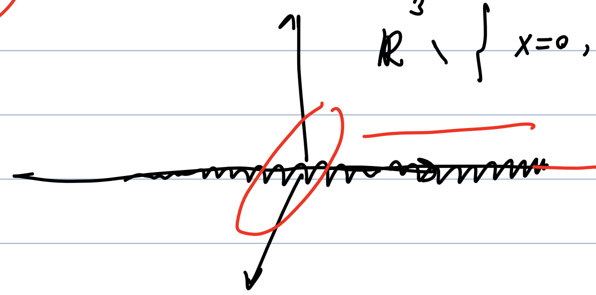


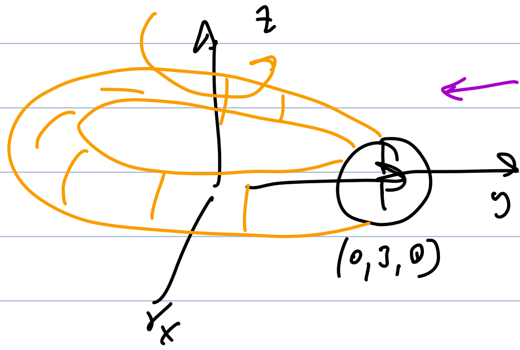
$\{1 < \rho < 2\}$

$\mathbb{R}^3$



$\mathbb{R}^3 \setminus \{x=0, y=0\}$





TORO

SEMPLICEMENTE CONNESSO ??