$f :]-\pi,\pi] \to \mathbb{R}$ PROLUNGATA PERIODIC.

$$f(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx}$$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx \xrightarrow{} = \hat{f}_k \qquad f(x) = \sum_{k\in\mathbb{Z}} \hat{f}_k e^{ikx}$$

ANALISI DI FOURIER SINTESI DI FOURIER

f C^1 A TRATTI $\quad \sum_{k=-n}^{n} \hat{f}_k e^{ikx} \to f(x) \quad \forall x \in]-\pi,\pi]$

$f \in L^2(-\pi,\pi)$ $\quad \int_{-\pi}^{\pi} \left| \sum_{k=-n}^{n} \hat{f}_k e^{ikx} - f(x) \right|^2 dx \to 0$

$\int_{-\pi}^{\pi} f^2 < +\infty$

$\left\| \sum_{k=-n}^{n} \hat{f}_k e^{ikx} - f \right\|_{L^2} \to 0$

$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\cdot \overline{g(x)}\,dx$

$$\sum_{k\in\mathbb{Z}} |c_k|^2 = \sum_{k\in\mathbb{Z}} |\hat{f}_k|^2 \le c\int_{-\pi}^{\pi} |f|^2 dx \Rightarrow c_k \xrightarrow[k\to+\infty]{} 0$$

$$(f(x))' = \left(\sum_{k\in\mathbb{Z}} c_k e^{ikx}\right)' = \sum_{k\in\mathbb{Z}} ik\, c_k e^{ikx}$$

IL COMPORTAMENTO È "PEGGIORE"

$c_k \to 0$ $\quad k c_k \to ??$

$k \to +\infty$

$f \in C^1(\mathbb{R})$ 2π PERIODICA

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underset{f}{f(x)} \underset{g'}{e^{-ikx}}\,dx = \frac{1}{2\pi} \cancel{\underset{f}{f(x)} \underset{g}{\frac{e^{-ikx}}{-ik}}\Big|_{-\pi}^{\pi}} - \frac{1}{2\pi}\int_{-\pi}^{\pi} \underset{f'}{f'(x)} \underset{g}{\frac{e^{-ikx}}{-ik}}\,dx$$

$$= \frac{1}{2\pi\, ik}\int_{-\pi}^{\pi} f'(x) e^{-ikx} = \frac{i}{k}\, \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x) e^{-ikx}$$

$i \quad c_k'$

$f' \in C([-\pi,\pi])$

$\Rightarrow \sum_{k\in\mathbb{Z}} |c_k'|^2 < +\infty$

$\bar{k}$ $\quad \|$

$-\frac{1}{k} \quad \frac{i}{k} \quad \int_{-\pi}^{\pi} f'' e^{-ikx}$

$$S_n(x) = \sum_{k=-n}^{n} \hat{f}_k e^{ikx} \qquad f(x) = \sum_{k=-n}^{n} \hat{f}_k e^{ikx} + R_n$$

$$|S_n(x) - f(x)| < \; ? \qquad R_n = \sum_{|k|>n} \hat{f}_k e^{ikx}$$

modulo di numero complesso

$$|R_n(x)| = \left|\sum_{|k|>n} \hat{f}_k e^{ikx}\right| \le \sum_{|k|>n} |\hat{f}_k e^{ikx}|$$

$$\le \sum_{|k|>n} |\hat{f}_k|\,|e^{ikx}| = \sum_{|k|>n} |\hat{f}_k| \qquad \forall x \in [-\pi,\pi]$$

$f \in C^2(-\pi,\pi)$ INTEGRO 2 VOLTE PER PARTI

$$\hat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx} = \frac{1}{ik}\frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x) e^{-ikx} = \frac{1}{(ik)^2}\frac{1}{2\pi}\int_{-\pi}^{\pi} f'' e^{-ikx}\,dx$$

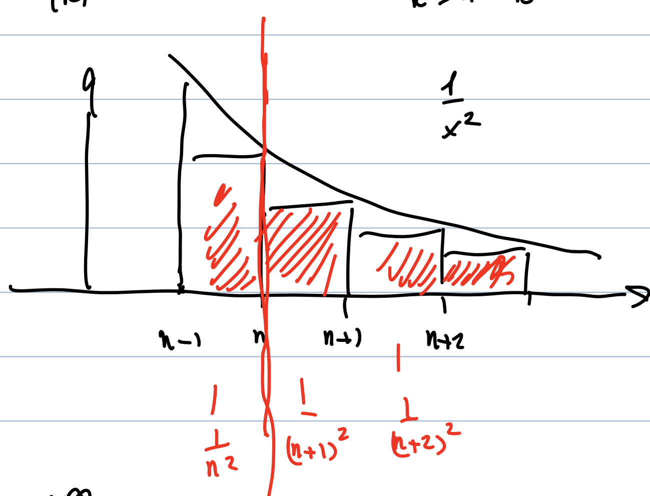
$$|\hat{f}_k| \le \frac{1}{|k|^2}\frac{1}{2\pi}\int_{-\pi}^{\pi} \max|f''|\,\underbrace{|e^{-ikx}|}_{=1}\,dx$$

$$\le \max|f''|\frac{1}{|k|^2}\frac{1}{\cancel{2\pi}}\cancel{2\pi} \le \max|f''|\frac{1}{|k|^2}$$

$$|R_n(x)| \le \max|f''| \sum_{|k|>n}\frac{1}{|k|^2} = 2\max|f''|\sum_{k>n}\frac{1}{k^2}$$

$$\sum_{k>n}\frac{1}{k^2} \xrightarrow{n\to+\infty} 0$$

$\|$



$$\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots \le \int_n^{+\infty} \frac{1}{x^2}\,dx$$

$$= \frac{1}{1-2} x^{1-2} \Big|_n^{+\infty}$$

$$= \frac{1}{n}$$

$$|R_n(x)| \leq \frac{2}{n} \max |f''| < \varepsilon \qquad n > \frac{2 \max |f''|}{\varepsilon}$$