$\sin(x)$ periodiche di periodo 2π
$\cos(x)$
$\sin(2x)$ $\sin(x) + \sin(3x) + \cos(5x)$ periodico di periodo 2π
$\sin(3x)$

$$P_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx)$$

polinomio trigonometrico periodico di periodo 2π

$f(x) = \cos(x) + \cos(\sqrt{2}x)$ $f(0) = 2$

$f(\bar{x}) = 2$ $\cos(\bar{x}) = 1$ $\cos(\sqrt{2}\bar{x}) = 1$

$\bar{x} = 2k\pi$ $\sqrt{2}\bar{x} = 2m\pi$ $\bar{x} = \sqrt{2}m\pi$

$2k\cancel{\pi} = \sqrt{2}m\cancel{\pi}$ $k, m \in \mathbb{N}$

$\sqrt{2} = \frac{m}{k}$ ASSURDO

SUPPONIAMO CHE ESISTA SOMMA DELLA SERIE

$$\lim_{n\to+\infty} P_n(x) \overset{!}{=} \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) = f(x)$$

- $\int_{-\pi}^{\pi} \sin(kx)\sin(mx)\,dx = \begin{cases} 0 & k \neq m \\ \pi & k = m \end{cases}$ $k, m \geq 1$

$$\int_{-\pi}^{\pi} \frac{e^{ikx} - e^{-ikx}}{2i} \cdot \frac{e^{imx} - e^{-imx}}{2i} = 0 \quad k \neq m$$

- $\int_{-\pi}^{\pi} \sin(mx)\cos(kx)\,dx = 0$

$$\int_{-\pi}^{\pi} \left(\frac{e^{ikx} - e^{-ikx}}{2i}\right)^2 dx = \int_{-\pi}^{\pi} \frac{e^{i2ikx} - 2 + e^{-i2ikx}}{-4}\,dx = \frac{1}{2} 2\cdot\pi = \pi$$

- $\int_{-\pi}^{\pi} \cos(kx)\cos(mx)\,dx = \begin{cases} 0 & k \neq m \\ \pi & k = m \end{cases}$ $k, m \geq 1$

$$\int_{-\pi}^{\pi} \cos(0x)\cdot\cos(0x) = \int_{-\pi}^{\pi} 1 = 2\pi \quad k = m = 0$$

$$\int_{-\pi}^{\pi} \sin(mx)\left(a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)\right) = \int_{-\pi}^{\pi} f(x) \cdot \sin(mx)\,dx$$

$$\int^{\pi} b_m \sin(mx)\sin(mx) = \int^{\pi} f(x) \sin(mx)\,dx$$

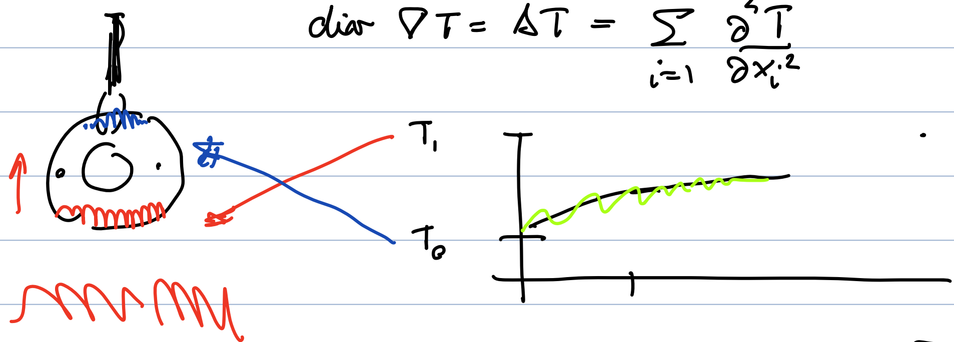
$$\pi b_m = \int_{-\pi}^{\pi} f(x)\sin(mx)\,dx$$

$$\boxed{\begin{array}{ll} b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx & a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx \quad m \geq 1 \\ a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx & \end{array}}$$

FOURIER STUDIO CALORE

$$\frac{d}{dt}\int_V T = \int_{\partial V} k \nabla T \hat{n}\, dS \qquad \partial_t T = -k\Delta T \quad \text{in } (x,t) \in (0,T)\times V$$

$$\text{div}\, \nabla T = \Delta T = \sum_{i=1}^{3} \frac{\partial^2 T}{\partial x_i^2}$$



$$r(t) = (\cos(t), \sin(t))$$

$$\boxed{\partial_t T = k\,\partial_{xx} T}$$

$$T(t,x) = a(t)\,b(x)$$

$$\partial_t T(t,x) = \partial_t (a(t)\,b(x)) = a'(t)\,b(x)$$

$$\partial_{xx} T(t,x) = \partial_{xx}(a(t)\,b(x)) = a(t)\,b''(x) \qquad k=1$$

$$a'(t)\,b(x) = \cancel{k}\, a(t)\,b''(x) \qquad k=1$$

$$-m^2 = \frac{a'(t)}{a(t)} = \frac{b''(x)}{b(x)} = -m^2 \qquad m \in \mathbb{R}$$

$$a'(t) = -m^2 a(t) \qquad a(t) = c\, e^{-m^2 t}$$ DECADIMENTO RADIOATTIVO

$$b''(x) + m^2 b(x) = 0$$ OSCILLATORE ARMONICO

$$\lambda^2 + m = 0 \qquad \lambda = \pm\sqrt{m}$$

$$b(x) = A\sin(mx) + B\cos(mx)$$

$$T(t,x) = e^{-m^2 t}\left(A_m \sin(mx) + B_m \cos(mx)\right) +$$

$$e^{-k^2 t}\, A_k \sin(kx) + B_k \cos(kx)$$

$$= \sum_{m=0}^{\infty} e^{-m^2 t}\left(A_m \sin(mx) + B_m \cos(mx)\right)$$

$$T(t,x)\Big|_{t=0} = T_0(x) = \sum_{m=0}^{\infty} A_m \sin(mx) + B_m \cos(mx)$$

$u_j \in V \qquad \langle u_j, u_k \rangle = \delta_{jk} = \begin{cases} 1 & j=k \\ 0 & j\neq k \end{cases} \qquad (V, \langle\,,\rangle)$

u_j ORTONORMALI

TEOREMA DI PITAGORA $\quad \langle u_j, u_k\rangle = 0 \quad j \neq k$

$$\Big\| \sum_{j=1}^{n} u_j \Big\|^2 = \sum_{j=1}^{n} \|u_j\|^2$$

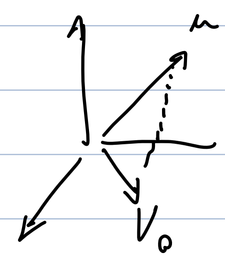
DIM $\quad \left\langle \sum_{j=1}^{n} u_j , \sum_{k=1}^{n} u_k \right\rangle = \sum_{j=1}^{n}\sum_{k=1}^{n} \langle u_j, u_k\rangle = \sum_{j=1}^{n} \langle u_j, u_j\rangle = \sum_{j=1}^{n} \|u_j\|^2$

TEOREMA DELLE PROIEZIONI

$V_0 \subseteq V \qquad V_0$ sottospazio di V vettoriale di dimensione finita $\qquad (V, \langle\,,\rangle)$

e_j base ortonormale di $V_0 \quad j = 1, \dots n$

$$P_{V_0} u = \sum_{j=1}^{n} \langle u, e_j\rangle e_j$$



(1) $\quad u - P_{V_0} u \perp v \qquad \forall v \in V_0$

(2) $\quad \|P_{V_0} u\| \leq \|u\|$

(3) $\quad \sum_{j=1}^{n} |\langle u, e_j\rangle|^2 \leq \|u\|^2$

(4) $\quad \|u - P_{V_0} u\| \leq \|u - v\| \qquad \forall v \in V_0$

$\| u - P_{V_0} u \| = \min_{v \in V_0} \| u - v \|$

$$\boxed{V = \{ C(0,2\pi) : f(0) = f(2\pi) \quad , \quad \langle u, v \rangle = \int_{-\pi}^{\pi} u(x) v(x) \, dx \}}$$

$\sin(kx) \qquad \cos(kx) \qquad k = 0, \dots m$

Dim

(1) $u - \sum_{j=1}^{n} \langle u, e_j \rangle e_j \perp e_k \qquad \forall \, k = 1, \dots n$

$$\langle u - \sum_{j=1}^{n} \langle u, e_j \rangle e_j \, , e_k \rangle = \langle u, e_k \rangle - \sum_{j=1}^{n} \langle u, e_j \rangle \underbrace{\langle e_j, e_k \rangle}_{\delta_{jk}}$$

$$= \langle u, e_k \rangle - \langle u, e_k \rangle \underbrace{\langle e_k, e_k \rangle}_{1} = 0$$

(2) $u = \underbrace{u - P_{V_0} u}_{\in V_0^\perp} + \underbrace{P_{V_0} u}_{\in V_0}$

$$\| u \|^2 = \underbrace{\| u - P_{V_0} u \|^2}_{\geq 0} + \| P_{V_0} u \|^2 \geq \| P_{V_0} u \|^2$$

(3) $\| P_{V_0} u \|^2 = \sum | \langle u, e_j \rangle |^2$

(4) $u - v = \underbrace{u - P_{V_0} u}_{\in V_0^\perp} + \underbrace{P_{V_0} u - v}_{\in V_0} \qquad v \in V_0$

$$\| u - v \|^2 = \| u - P_{V_0} u \|^2 + \underbrace{\| P_{V_0} u - v \|^2}_{\geq 0} \geq \| u - P_{V_0} u \|^2 \qquad \forall v \in V_0$$

$V_0 = \text{span} \{ 1, \cos(kx), \sin(kx) \}_{k = 1, \dots n} \qquad f \in C_{per}[0, 2\pi]$

$P_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx) \qquad \langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) \, dx$

$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) \, dx \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) \, dx$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$\|f-g\|^2 = \langle f-g, f-g\rangle = \int_{-\pi}^{\pi} |f-g|^2\,dx$$

$$\int_{-\pi}^{\pi} |f(x) - P_n(x)|^2\,dx \le \int_{-\pi}^{\pi} |f(x) - Q_n(x)|^2\,dx \qquad Q_n \in V_0$$

$$\boxed{\sum_{0}^{n} (\underset{\geq 0}{a_k^2} + \underset{\geq 0}{b_k^2}) \le c\int_{-\pi}^{\pi} |f|^2\,dx} \qquad \forall n \in \mathbb{N}$$

$$\sup_n \left(\sum_0^n a_k^2 + b_k^2\right) < +\infty$$

$$\sum_{k=0}^{+\infty} a_k^2 + b_k^2 < +\infty \implies \boxed{\begin{array}{l} k \to +\infty \\ a_k \to 0 \\ b_k \to 0 \end{array}}$$

Lemma (Cauchy-Riemann)

$V_0 = \text{span}\{1, \cos(kx), \sin(kx)\}_{k=1,\dots,n}$ $\qquad f \in C_{per}(0,2\pi)$

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx) = P_{V_0} f$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$$

f È PERIODICA DI PERIODO 2π $\qquad \int_{-\pi}^{\pi} f(x)\,dx = \int_{a}^{a+2\pi} f(x)\,dx$

$$S_n(x) = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt}_{\frac{a_0}{2}} + \sum_{k=1}^{n} \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)}_{a_k} \cos(kx) + \sum \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)}_{b_k} \sin(kx)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left\{\frac{1}{2} + \sum_{k=1}^{n} \cos(kt)\cos(kx) + \sin(kt)\sin(kx)\right\}dt$$

FORMULA SOTTRAZIONE COSENO

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \left\{ \frac{1}{2} + \sum_{k=1}^{n} \cos[k(t-x)] \right\} dt$$

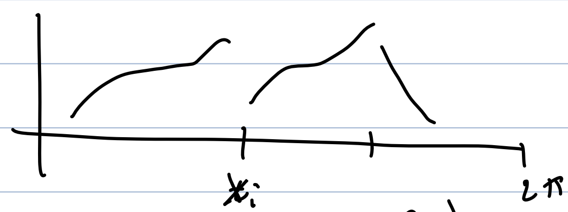
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \frac{\sin\left(\frac{2n+1}{2}(t-x)\right)}{2\sin\frac{(t-x)}{2}} dt$$

TEOREMA $f \in C^1[0,2\pi]$ e periodica di periodo 2π

$$\lim_{n\to+\infty} S_n(x) = f(x) \quad \forall\, x \in [0,2\pi]$$

f è C^1 a tratti e periodica

$\exists\; t_1 \dots t_m \in [0,2\pi] : f_{|]t_i, t_{i+1}[} \in C^1$ $\quad \lim_{t\to t_i^{\pm}} f'(t)$ ESISTE FINITO



$$\lim_{n\to+\infty} S_n(x) = \begin{cases} f(x) & \text{DOVE } f \text{ È DERIVABILE} \\ \frac{f(x_i^+)+f(x_i^-)}{2} & \text{DOVE CI SONO I SALTI} \end{cases}$$

$$\frac{1}{2} + \sum_{k=1}^{n} \cos(ky) = \frac{1}{2} + \sum_{k=1}^{n} \frac{e^{iky}+e^{-iky}}{2}$$

$$= \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{n} e^{iky} + \frac{1}{2}\sum_{k=1}^{n} e^{-iky} \qquad \sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$$

$$= \frac{1}{2}\left\{ 1 + \sum_{k=1}^{n} (e^{iy})^k + \sum_{k=1}^{n} (e^{-iy})^k \right\}$$

$$= \frac{1}{2}\left\{ 1 + \sum_{k=0}^{n} (e^{iy})^k + \sum_{k=0}^{n} (e^{-iy})^k - 2 \right\} \qquad \sin y = \frac{e^{iy}-e^{-iy}}{2i}$$

$$\frac{1}{2}\left\{ \frac{1-e^{+i(n+1)y}}{1-e^{iy}} + \frac{1-e^{-i(n+1)y}}{1-e^{-iy}} - 1 \right\}$$

$$= \frac{1}{2}\left\{ \frac{e^{-iy/2}}{e^{-iy/2}} \frac{1-e^{i(n+1)y}}{1-e^{+iy}} + \frac{e^{iy/2}}{e^{iy/2}} \frac{1-e^{-i(n+1)y}}{1-e^{-iy}} - 1 \right\}$$

$$= \frac{1}{2}\left\{ \frac{e^{-iy/2}-e^{i(n+\frac{1}{2})y}}{e^{-iy/2}+e^{iy/2}} + \frac{e^{iy/2}-e^{-i(n+\frac{1}{2})y}}{e^{iy/2}-e^{-iy/2}} \right.$$

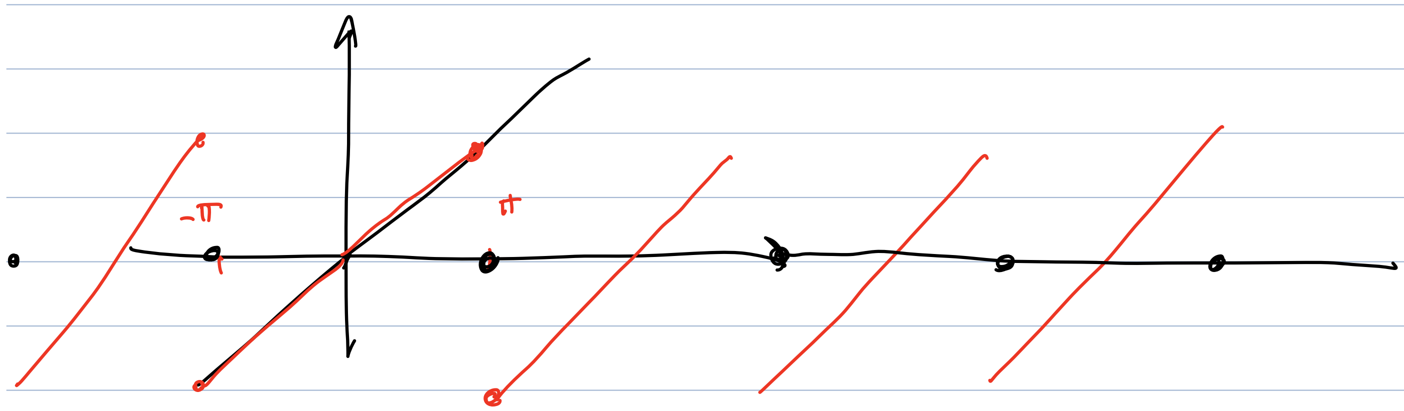
$$= \frac{1}{2}\left\{ \frac{\frac{e^{-iy/2}-e^{i(n+\frac{1}{2})y}}{2i}}{\frac{e^{-iy/2}+e^{iy/2}}{2i}} \right.$$

$$= \frac{1}{2} \frac{\frac{e^{i(n+\frac{1}{2})y}-e^{-iy/2}}{2i}}{\sin y/2} \qquad = \frac{1}{2}\frac{\sin(n+\frac{1}{2})y}{\sin(\frac{y}{2})}$$

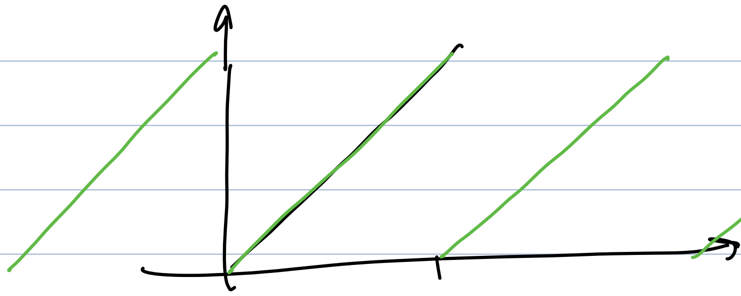
- $f(x) = x$ PERIODICO DI PERIODO 2π

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx)$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(kx)\,dx \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)\,dx$$



$f(x) = x \quad x \in [\pi,\pi]$ e POI PROLUNGATA PER PERIODICITÀ

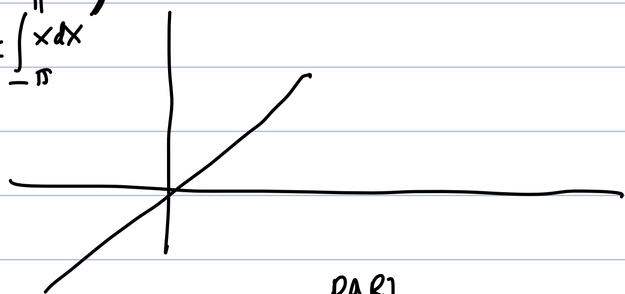


$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{f}{x}\ \underset{g'}{\cos(kx)}\,dx = \frac{1}{\pi}\ x\ \frac{\sin(kx)}{k}\Bigg|_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} 1\ \frac{\sin(kx)}{k}\,dx$$

$$= \frac{1}{\pi}\ \pi \underbrace{\left(\frac{\sin(k\pi)}{k} - \frac{\sin(-k\pi)}{k}\right)}_{=0} + \frac{1}{\pi}\ \frac{\cos(kx)}{k^2}\Bigg|_{-\pi}^{\pi} = 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0$$

DISPARI



DISPARI

PARI

$$x = \frac{a_0}{2} + \sum_{k=1}^{\infty} \underbrace{a_k}_{=0}\cos(kx) + b_k \sin(kx)$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin(kx)\,dx \neq 0$$