Control Systems
Cheat Sheet

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Introduction

This document serves as a comprehensive cheat sheet for control systems, covering essential topics such as root locus plots, Nyquist diagrams, and lead compensator design. It includes theoretical foundations, mathematical properties, and step-by-step procedures to aid in the analysis and design of control systems.

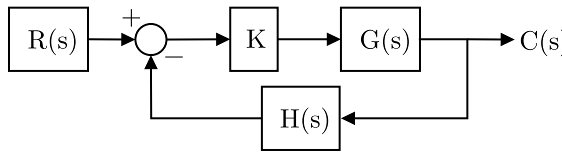
Note

This cheat sheet is written by a student and may contain errors or inaccuracies. It is intended as a study aid and should not be used as a primary source of information. Please consult textbooks, lecture notes, or other reliable sources for critical information.

Rules for Making Root Locus Plots

The closed loop transfer function of the system shown is:

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$



So the characteristic equation (c.e.) is

$$1 + KG(s)H(s) = 1 + K\frac{N(s)}{D(s)} = 0, \text{ or } D(s) + KN(s) = 0$$

As K changes, so do locations of closed loop poles (i.e., zeros of c.e.). The table below gives rules for sketching the location of these poles for $K \in [0, +\infty[$ (i.e., $K \geq 0$).

Rule Name	**Description**
Definitions	• The loop gain is $KG(s)H(s)$ or $K\frac{N(s)}{D(s)}$ • $N(s)$, the numerator, is an m^{th} order polynomial; $D(s)$ is n^{th} order • $N(s)$ has zeros at z_i $(i = 1..m)$; $D(s)$ has them at p_i $(i = 1..n)$ • The difference between n and m is q, so $q = n - m$ $(q \geq 0)$
Symmetry	The locus is symmetric about real axis (i.e., complex poles appear as conjugate pairs)
Number of Branches	There are n branches of the locus, one for each closed loop pole
Starting and Ending Points	The locus starts $(K = 0)$ at poles of loop gain, and ends $(K \to +\infty)$ at zeros. Note: this means that there will be q roots that will go to infinity as $K \to +\infty$
Locus on Real Axis*	For any point s on the real axis to be a point on the root locus, the total number of poles and zeros of $GH(s)$ to the right of that point should be odd
Asymptotes as $\|s\| \to \infty$*	If $q > 0$ there are asymptotes of the root locus that intersect the real axis at $\sigma = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{q}$ and radiate out with angles $\theta = \pm r\frac{180^\circ}{q}$, where $r = 1, 3, 5 \ldots$
Break-Away/-In Points	Break-away or break-in points of the locus exist where $N(s)D'(s) - N'(s)D(s) = 0$

Rule Name	Description
Angle of Departure*	Angle of departure from pole, p_j is $$\theta_{depart,p_j} = 180^\circ + \sum_{i=1}^{m} \angle(p_j - z_i) - \sum_{i=1, i\neq j}^{n} \angle(p_j - p_i)$$
Angle of Arrival*	Angle of arrival at zero, z_j, is $$\theta_{arrive,z_j} = 180^\circ - \sum_{i=1, i\neq j}^{m} \angle(z_j - z_i) + \sum_{i=1}^{n} \angle(z_j - p_i)$$
Imaginary Axis Crossing	Use Routh-Hurwitz to determine where the locus crosses the imaginary axis
Finding Poles for Given K	Rewrite c.e. as $D(s) + KN(s) = 0$. Put value of K into equation, and find roots of c.e.. (This may require a computer)
Finding K for Given Pole	Rewrite c.e. as $K = -\frac{D(s)}{N(s)}$, replace "s" by desired pole location and solve for K. Note: if "s" is not exactly on locus, K may be complex (small imaginary part). Use real part of K

*These rules change to draw complementary root locus ($K \leq 0$). See below for details.

Complementary Root Locus (or *inverse root locus*)

To sketch complementary root locus ($K \leq 0$), most of the rules are unchanged except for those in table below.

Rule Name	Description
Locus on Real Axis	For any point s on the real axis to be a point on the root locus, the total number of poles and zeros of $GH(s)$ to the left of that point should be odd
Asymptotes	If $q > 0$ there are asymptotes of the root locus that intersect the real axis at $$\sigma = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{q}$$ and radiate out with angles $$\theta = \pm p\frac{180^\circ}{q}, \text{ where } p = 0, 2, 4 \ldots$$
Angle of Departure	Angle of departure from pole, p_j is $$\theta_{depart,p_j} = \sum_{i=1}^{m} \angle(p_j - z_i) - \sum_{i=1, i\neq j}^{n} \angle(p_j - p_i)$$

Rule Name	Description
Angle of Arrival	Angle of arrival at zero, z_j, is $$\theta_{arrive,z_j} = \sum_{i=1,i\neq j}^{m} \angle(z_j - z_i) - \sum_{i=1}^{n} \angle(z_j - p_i)$$

1 The Nyquist Diagram: Theoretical Foundations

1.1 Complex Analysis Foundations

The Nyquist diagram is rooted in complex analysis. For a transfer function G(s), we consider its behavior along a contour in the complex plane. The fundamental theorem underlying the Nyquist plot is Cauchy's argument principle:

$$Z = N + P$$

where:

- Z = number of zeros enclosed by the contour
- N = number of encirclements of a point (clockwise for N > 0, counter-clockwise for N < 0)
- P = number of poles enclosed by the contour

1.2 Phase and Gain Margins

From the Nyquist plot:

- Phase Margin: $\phi_m = \angle G(j\omega_{gc}) + 180°$
- Gain Margin: $G_m = \frac{1}{|G(j\omega_{pc})|}$

where:

- ω_{gc} is the gain crossover frequency where $|G(j\omega_{gc})| = 1$
- ω_{pc} is the phase crossover frequency where $\angle G(j\omega_{pc}) = -180°$

1.3 Transfer Function Analysis

Given a rational transfer function:

$$G(s) = K\frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{j=1}^{n}(s - p_j)}$$

where:

- K is the gain
- z_i are zeros
- p_j are poles

The frequency response is obtained by evaluating G(s) along $s = j\omega$:

$$G(j\omega) = |G(j\omega)|e^{j\angle G(j\omega)}$$

1.4 Mathematical Properties

1.4.1 Magnitude Calculation

For any frequency ω:

$$|G(j\omega)| = K\frac{|(j\omega + z_1)||(j\omega + z_2)| \cdots |(j\omega + z_k)|}{|s^n||(j\omega + p_1)||(j\omega + p_2)| \cdots |(j\omega + p_n)|}\Bigg|_{s \to j\omega}$$

1.4.2 Phase Calculation

The phase is given by:

$$\begin{aligned}\angle G(j\omega) &= \angle K + \angle(j\omega + z_1) + \angle(j\omega + z_2) + \cdots + \angle(j\omega + z_k) \\ &- n\angle j\omega - \angle(j\omega + p_1) - \angle(j\omega + p_2) - \cdots - \angle(j\omega + p_n) \\ &= \angle K - n \cdot 90^\circ + \tan^{-1}\left(\frac{\omega}{z_1}\right) + \tan^{-1}\left(\frac{\omega}{z_2}\right) + \cdots + \tan^{-1}\left(\frac{\omega}{z_k}\right) \\ &\quad - \tan^{-1}\left(\frac{\omega}{p_1}\right) - \tan^{-1}\left(\frac{\omega}{p_2}\right) - \cdots - \tan^{-1}\left(\frac{\omega}{p_n}\right)\end{aligned}$$

1.5 Strictly Proper Functions

For strictly proper functions (n > m):

$$\lim_{\omega\to\infty} G(j\omega) = 0$$

$$\lim_{\omega\to\infty} |G(j\omega)| = \lim_{\omega\to\infty} K\frac{\omega^m}{\omega^n} = 0$$

2 Nyquist Plot Construction

The construction of the Nyquist plot is divided into three main sections, each corresponding to different portions of the Nyquist contour.

2.1 General Procedure Overview

For any transfer function $G(s)$, the Nyquist plot is constructed by following these steps in order:

Step 0. Compute $G(j\omega)$ and $\angle G(j\omega)$, don't simplify numerator/denominator because it will be used to find the sign of the real and imaginary parts of $G(j\omega)$ in step 2.

Step 1. First Arc: $\omega \to 0^+, \theta \in [0°, 90°]$

Let $s = j\omega = \varepsilon e^{j\theta}$, where $\omega \to 0$, then $\varepsilon \to 0$, compute the limit:

$$\lim_{\varepsilon \to 0} G(s) = \lim_{\varepsilon \to 0} \frac{K}{\varepsilon} e^{-nj\theta}$$

where n is the order of the pole in the origin.

- Move along arc from $\theta = 0°$ to $\theta = 90°$
- Magnitude stays at $\varepsilon \to 0$ ($s = j\omega \to 0$)
- Substitute θ values into $\lim_{\varepsilon \to 0} G(s)$ to find $\angle G(j\omega)$

	$G(j\omega)$	$\angle G(j\omega)$
$\theta_{\mathrm{i}} = 0°$	$\lvert G(0^+)\rvert$	$\angle G(0°)$
$\theta_{\mathrm{f}} = 90°$	$\lvert G(0^+)\rvert$	$\angle G(90°)$

Step 2. Imaginary Axis Traversal: $\omega \in]0^+, +\infty[$, $\theta = 90°$

- Start at end of first arc
- Move along imaginary axis with $s = j\omega$
- Phase maintained at 90° throughout

	$G(j\omega)$	$\mathfrak{R}\{G(j\omega)\}$	$\mathfrak{I}\{G(j\omega)\}$	$\angle G(j\omega)$
$\omega_{\mathrm{i}} = 0^+$	$G(j\omega_{\mathrm{i}})$	$\mathfrak{R}\{G(j\omega_{\mathrm{i}})\}$	$\mathfrak{I}\{G(j\omega_{\mathrm{i}})\}$	$\angle G(j\omega_{\mathrm{i}})$
$\omega_{\mathrm{f}} = +\infty$	$G(j\omega_{\mathrm{f}})$	$\mathfrak{R}\{G(j\omega_{\mathrm{f}})\}$	$\mathfrak{I}\{G(j\omega_{\mathrm{f}})\}$	$\angle G(j\omega_{\mathrm{f}})$

- Note 1: $\mathfrak{R}\{G(j\omega)\}$ and $\mathfrak{I}\{G(j\omega)\}$ are respectively real and imaginary parts of $G(j\omega)$
- Note 2: $\angle G(j\omega)$ is the phase of $G(j\omega)$

Step 3. Infinite Radius Arc: $\omega \to +\infty$, $\theta \in [90°, 0°]$

Let $s = \varepsilon e^{j\theta}$, where $\varepsilon \to +\infty$, compute the limit:

$$\lim_{\varepsilon \to +\infty} G(s) = \lim_{\varepsilon \to +\infty} \frac{K}{\varepsilon} e^{-nj\theta}$$

where n is the order of the pole in the origin.

- Move along arc from $\theta = 90°$ to $\theta = 0°$
- Magnitude stays at $\varepsilon \to +\infty$ ($s = j\omega \to +\infty$)
- Substitute θ values into $\lim_{\varepsilon \to +\infty} G(s)$ to find $\angle G(j\omega)$

	$G(j\omega)$	$\angle G(j\omega)$
$\theta_{\mathrm{i}} = 90°$	$\lvert G(+\infty)\rvert$	$\angle G(90°)$
$\theta_{\mathrm{f}} = 0°$	$\lvert G(+\infty)\rvert$	$\angle G(0°)$

Draw the Nyquist plot by connecting the points obtained in each step.
Considering the simmetry of the Nyquist plot, once the plot is completed for $\omega \in]0^+, +\infty[$, the plot must be mirrored respect to the real axis to obtain the plot for $\omega \in]-\infty, 0^-[$.

3 Special Cases

3.1 Systems Without Poles on $j\omega$-axis

For systems without poles on the imaginary axis, the following steps can be simplified:

- Step 1 can be omitted
- Step 3 should be done just to check if the angle of point 2 matches with the starting angle of point 3.

4 Worked Example

Consider the transfer function:

$$G(s) = \frac{K}{s(s+2)}$$

Let's construct the Nyquist plot following our procedure:

Step 0: Initial Analysis

Substitute $s = j\omega$:

$$G(j\omega) = \frac{K}{j\omega(j\omega+2)} = \frac{K}{-\omega^2 + 2j\omega}$$

Multiply numerator and denominator by complex conjugate:

$$G(j\omega) = \frac{K(-\omega^2 - 2j\omega)}{(-\omega^2)^2 + (2\omega)^2} = \frac{-K\omega^2}{4\omega^2+\omega^4} + j\frac{-2K\omega}{4\omega^2+\omega^4}$$

Therefore:

$$\Re\{G(j\omega)\} = \frac{-K\omega^2}{4\omega^2+\omega^4}$$
$$\Im\{G(j\omega)\} = \frac{-2K\omega}{4\omega^2+\omega^4}$$

Phase:

$$\angle G(j\omega) = -90° - \tan^{-1}\left(\frac{\omega}{2}\right)$$

Step 1: First Arc $(\omega \to 0^+, \theta \in [0°, 90°])$

For $s = \varepsilon e^{j\theta}$:

$$G(s) = \frac{K}{\varepsilon e^{j\theta}(\varepsilon e^{j\theta}+2)} = \frac{K}{\varepsilon^2 e^{2j\theta} + 2\varepsilon e^{j\theta}}$$

As $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} G(s) = \frac{K}{2\varepsilon} e^{-j\theta} \to \infty$$

Then we have:

	$G(j\omega)$	$\angle G(j\omega)$
$\theta_{\mathrm{i}} = 0°$	$+\infty$	$0°$
$\theta_{\mathrm{f}} = 90°$	$+\infty$	$-90°$

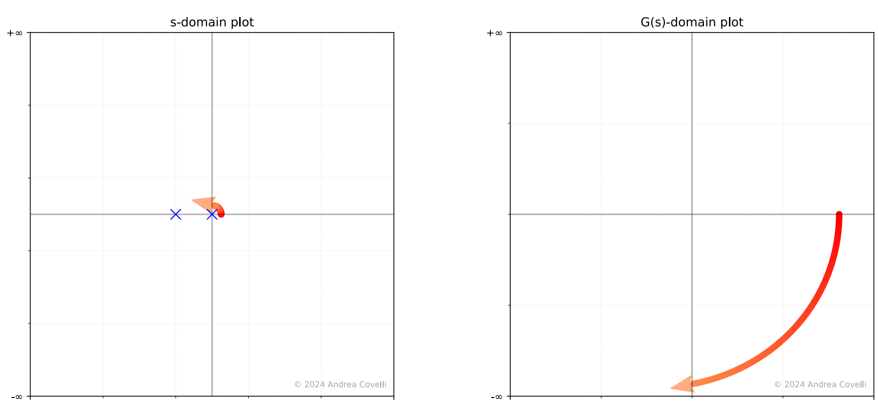


Figure 1: The image is a numerical approximation of the Nyquist plot, direction of the arc may vary slightly

Step 2: Imaginary Axis Traversal ($\omega \in (0, +\infty)$)

Key points along the imaginary axis:

ω	$\Re\{G(j\omega)\}$	$\Im\{G(j\omega)\}$	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0^+	$-\infty$	$-\infty$	$+\infty$	$-90°$
$+\infty$	0^-	0^-	0^+	$-180°$

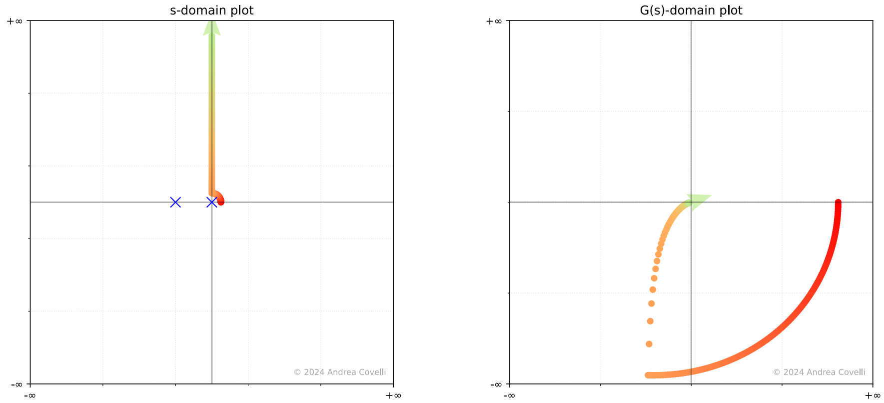


Figure 2: The image is a numerical approximation of the Nyquist plot, direction of the arc may vary slightly

Step 3: Infinite Radius Arc ($\omega \to \infty, \theta \in [90°, 0°]$)

For $s = \varepsilon e^{j\theta}$ as $\varepsilon \to +\infty$:

$$\lim_{\omega \to +\infty} G(\varepsilon e^{j\theta}) = \lim_{\omega \to +\infty} \frac{K}{\varepsilon^2 e^{2j\theta}} = \frac{K}{\varepsilon^2} e^{-2j\theta}$$

Then we have:

	$G(j\omega)$	$\angle G(j\omega)$
$\theta_i = 90°$	0^+	$-180°$
$\theta_f = 0°$	0^+	$0°$

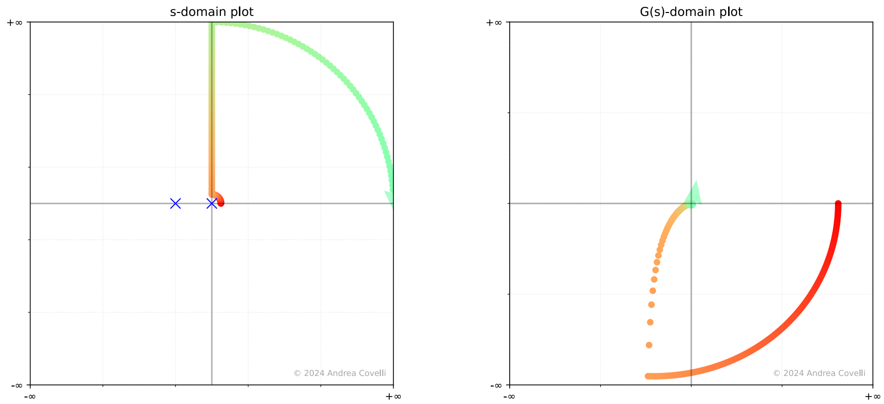


Figure 3: The image is a numerical approximation of the Nyquist plot, direction of the arc may vary slightly

Final Plot

The Nyquist plot is obtained by connecting the points obtained in each step. The plot is then mirrored with respect to the real axis to obtain the plot for $\omega \in]-\infty, 0^-[$.

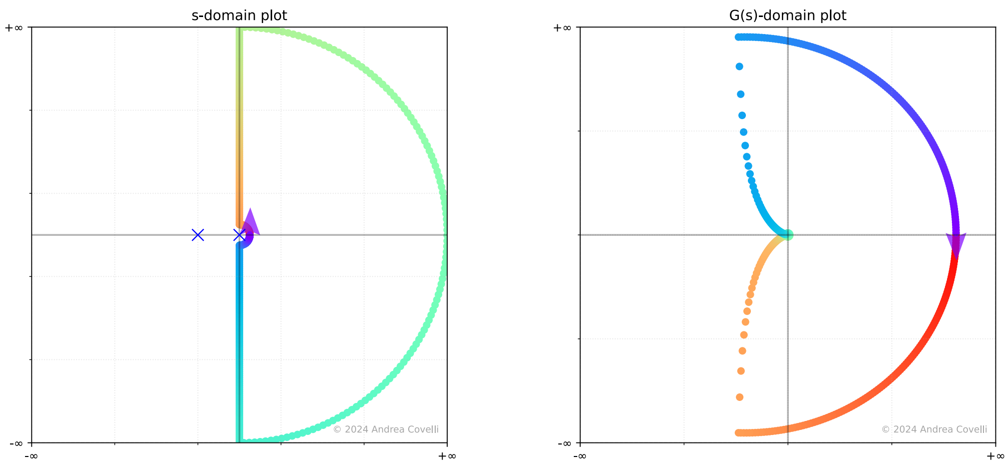


Figure 4: The image is a numerical approximation of the Nyquist plot, direction of the arc may vary slightly

Considering Nyquist Criterion, the system has $P = 0$, $N = 0 \implies Z = N + P = 0$.
From Cauchy's argument principle, the closed-loop system has no poles in the right half-plane: the closed-loop system is stable.

5 Common Mathematical Pitfalls

- Sign errors in phase calculations:

$$\angle(a + jb) = \arctan\left(\frac{b}{a}\right) + \begin{cases} 0^\circ & \text{if } a > 0 \\ 180^\circ & \text{if } a < 0 \end{cases}$$

- Rotation direction in Nyquist diagram:

$$\text{If } \Delta\theta = \theta_f - \theta_i, \text{ then } \Delta\theta = \begin{cases} > 0 & \text{counter-clockwise rotation in G(s) plot} \\ < 0 & \text{clockwise rotation in G(s) plot} \end{cases}. \tag{1}$$

6 Build a Lead Compensator

6.1 Introduction

Lead compensators are used to improve the transient response of a system. They are designed to increase the phase margin of a system, which in turn increases the stability of the system.

The transfer function of a lead compensator is given by:

$$D(s) = K\frac{T_D s + 1}{\alpha T_d s + 1}, \quad \alpha < 1$$

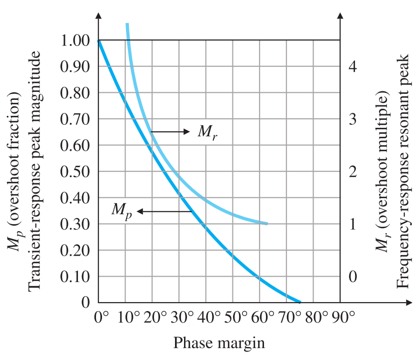
where:

- K is the gain of the compensator
- $T_d = \dfrac{1}{\omega_{gc}}$ where ω_{gc} is the gain crossover frequency
- α is the zero-pole ratio, $\alpha = \dfrac{|z|}{|p|}$

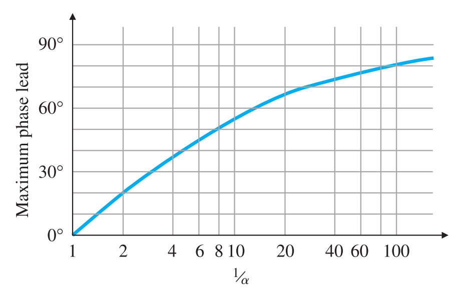
From the transfer function, the zero and pole of the compensator are given by:

- Zero: $s = -\dfrac{1}{T_d}$
- Pole: $s = -\dfrac{1}{\alpha T_d}$

Transient-response overshoot (M_p) and frequency-response resonant peak (M_r) versus PM for $T(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$



Maximum phase increase for lead compensation



6.2 Design Procedure

To design a lead compensator, follow these steps:

1. Compute $T_d = \dfrac{1}{\omega_{gc}}$, $\quad \omega_{gc} \rightarrow$ gain crossover frequency
2. Place zero at $s = -\dfrac{1}{T_d}$
3. Choose $\alpha < 1$ $\left(\dfrac{1}{a} > 1\right)$ from the image above on the right to increase the phase margin
4. Place pole at $s = -\dfrac{1}{\alpha T_d}$
5. Compute K to ensure the DC gain is preserved
6. Verify the phase margin has increased and the criterias are met