HW 7

Problema 1 Sia $v=(x_1, x_2, x_3, x_4)$.

Dobbiamo risolvere il systema di equazioni lineari

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ -1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 \end{array}\right) \xrightarrow[R_2 \to R_2+R_1]{} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 1 & -1 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3-R_1]{} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3+R_2]{} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \end{array}\right)$$

$$\xrightarrow[\substack{R_2 \to R_2/2 \\ R_3 \to R_3/2}]{} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

$$\Rightarrow x_3 + x_4 = 0 \Rightarrow x_3 = -x_4.$$

$$\Rightarrow x_2 + x_3 = 0 \Rightarrow x_2 = -x_3 = x_4$$

$$\Rightarrow x_1 + x_2 + x_3 - x_4 = 0 \Rightarrow x_1 - x_4 = 0 \Rightarrow x_1 = x_4$$

$$\Rightarrow v = c(1,1,-1,1), \quad c \neq 0.$$

Problema 2 $f = x^2 + x$, $g = x^3+1$, $[a,b]=[1,0]$

$$|f|^2 = \int_0^1 f^2(x)\,dx = \int_0^1 (x^4 + 2x^3 + x^2)\,dx$$

$$= \frac{x^5}{5} + \frac{2x^4}{4} + \frac{x^3}{3}\Big|_0^1 = \frac{1}{5} + \frac{1}{2} + \frac{1}{3}$$

$$= \frac{6}{30} + \frac{15}{30} + \frac{10}{30} = \frac{31}{30}$$

$$|g|^2 = \int_0^1 (x^3+1)^2\,dx = \int_0^1 (x^6 + 2x^3 + 1)\,dx$$

$$= \frac{x^7}{7} + \frac{2x^4}{4} + x\Big|_0^1 = \frac{1}{7} + \frac{1}{2} + 1$$

$$= \frac{1}{7} + \frac{3}{2} = \frac{2}{14} + \frac{21}{14} = \frac{23}{14}$$

$$\langle f,g\rangle = \int_0^1 f(x)g(x)\,dx$$

$$= \int_0^1 (x^2+x)(x^3+1)\,dx$$

$$= \int_0^1 (x^5+x^4+x^2+x)\,dx$$

$$= \frac{x^6}{6} + \frac{x^5}{5} + \frac{x^3}{2} + \frac{x^2}{2}\Big|_0^1$$

$$= \frac{1}{6} + \frac{1}{5} + \frac{1}{3} + \frac{1}{2} = 1 + 1/5 = 6/5$$

(a) $\therefore \langle f,g\rangle = |f|\,|g|\cos\theta$

$$\Rightarrow \cos\theta = \frac{\langle f,g\rangle}{|f|\,|g|} = \frac{(6/5)}{\sqrt{\frac{31}{30}}\sqrt{\frac{23}{14}}}$$

(b) $d(f,g) = |f-g|$, $|f-g|^2 = \int_0^1 (f-g)^2\,dx$

$$|f-g|^2 = \int_0^1 (x^2+x-x^3-1)^2\,dx$$

$$= \frac{x^7}{7} - \frac{x^6}{3} - \frac{x^5}{5} + x^4 - x^2 + x\Big|_0^1$$

$$= \frac{29}{105}$$

$$\Rightarrow d(f,g) = \sqrt{\frac{29}{105}}$$

(c) Dobbiamo verificare

$$(f,g)^2 \le |f|^2\,|g|^2$$

$$(6/5)^2 \le \left(\frac{31}{30}\right)\left(\frac{23}{14}\right) \quad ?$$

$$\frac{A}{B} \le \frac{C}{D} \iff AD \le BC \quad (B,D>0)$$

$$\underbrace{(36)(30)(14)}_{15120} \le \underbrace{(25)(31)(23)}_{17825}$$

(d) Dobbiamo verificare che

$$|f+g| < |f| + |g|$$

$$|f+g|^2 = \int_0^1 (x^2+x+x^3+1)^2 dx$$

$$= \frac{x^7}{7} + \frac{x^6}{3} + \frac{3x^5}{5} + x^4 + x^3 + x^2 + x \Big|_0^1$$

$$= \frac{533}{105}$$

$$\underbrace{\sqrt{\frac{533}{105}}}_{2,253\ldots} \leq \underbrace{\sqrt{\frac{31}{30}}}_{1,016} + \underbrace{\sqrt{\frac{23}{14}}}_{1,281\ldots}$$

$$2,26 < 1.01 + 1,28$$

Problema 3 $\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1 y_1 + 2x_2 y_2 + 3 x_3 y_3.$

mi dispiace, ho dimenticato $+ 4x_4 y_4$. Il metodo è la stessa

Dobbiamo verificare che

(A1) $\langle a+b, c \rangle = \langle a, c \rangle + \langle b, c \rangle$

$\langle \lambda a, c \rangle = \lambda \langle a, c \rangle$

(A2) $\langle a, b \rangle = \langle b, a \rangle$

(A3) $\langle a, a \rangle \geq 0$, $\langle a, a \rangle = 0 \iff a = 0$

(A1) $\langle \underbrace{(a_1, a_2, a_3)}_{a} + \underbrace{(b_1, b_2, b_3)}_{b}, \underbrace{(c_1, c_2, c_3)}_{c} \rangle$

$$= \langle (a_1+b_1, a_2+b_2, a_3+b_3), (c_1, c_2, c_3) \rangle$$

$$= (a_1+b_1)c_1 + 2(a_2+b_2)c_2 + 3(a_3+b_3)c_3$$

$$= a_1 c_1 + b_1 c_1 + 2a_2 c_2 + 2 b_2 c_2 + 3 a_3 c_3 + 3 b_3 c_3$$

$$= (a_1 c_1 + 2a_2 c_2 + 3a_3 c_3) + (b_1 c_1 + 2b_2 c_2 + 3 b_3 c_3)$$

$$= \langle (a_1, a_2, a_3), (c_1, c_2, c_3) \rangle + \langle (b_1, b_2, b_3), (c_1, c_2, c_3) \rangle$$

$$= \langle a, c \rangle + \langle b, c \rangle$$

$$\langle \lambda \underbrace{(a_1, a_2, a_3)}_{a}, \underbrace{(c_1, c_2, c_3)}_{c} \rangle = \lambda a_1 c_1 + 2\lambda a_2 c_2 + 3 \lambda a_3 c_3$$

$$= \lambda \langle (a_1, a_2, a_3), (c_1, c_2, c_3) \rangle$$

$$= \lambda \langle a, c \rangle$$

(A2) $\langle (a_1,a_2,a_3),(b_1,b_2,b_3)\rangle$

$$\left.\begin{array}{l} = a_1b_1 + 2a_2b_2 + 3a_3b_3 \\ \langle (b_1,b_2,b_3),(a_1,a_2,a_3)\rangle \\ = b_1a_1 + 2b_2a_2 + 3b_3a_3 \end{array}\right] =$$

(A3) $\langle (a_1,a_2,a_3),(a_1,a_2,a_3)\rangle$

$$= a_1^2 + 2a_2^2 + 3a_3^2 \geq 0$$

$$a_1^2 + 2a_2^2 + 3a_3^2 = 0 \Rightarrow a_1 = 0,\ a_2 = 0,\ a_3 = 0$$

<u>Problema 4</u> $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$$\langle A,B\rangle = \sum_{i=1}^{2}\sum_{j=1}^{3} A_{ij} B_{ij}$$

$$= (1)(1) + (2)(0) + (0)(0) + (2)(0) + (0)(1) + (4)(0)$$

$$= 1$$

$$|A|^2 = \langle A,A\rangle = 1^2 + 2^2 + 0^2 + 2^2 + 0^2 + 4^2$$
$$= 9 + 16 = 25$$

$$\Rightarrow |A| = 5$$

$$|B|^2 = 1^2 + 0^2 + 0^2 + 0^2 + 1^2 + 0^2 = 2 \Rightarrow |B| = \sqrt{2}$$

(a) $$\cos\theta = \frac{\langle A,B\rangle}{|A|\,|B|} = \frac{1}{5\sqrt{2}}$$

(b) Dobbiamo verificare che

$$\langle A,B\rangle^2 \leq |A|^2\,|B|^2$$
$$1^2 \leq (25)(2) \checkmark$$

(c) Dobbiamo verificare che

$$|A+B| \leq |A| + |B|$$

$$A + B = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 4 \end{pmatrix}$$

$$|A+B|^2 = 2^2 + 2^2 + 0^2 + 2^2 + 1^2 + 4^2$$
$$= 4 + 4 + 0 + 4 + 1 + 16$$
$$= 29$$

$|A+B| = \sqrt{29} = 5.385\ldots < 5.39$

$|A| = 5$, $|B| = \sqrt{2} = 1.41\ldots$

$\therefore |A+B| < |A| + |B|$

<u>Problema 5</u>

(a) $tr(A) = \sum_{k=1}^{N} a_{kk}$, $A = (a_{ij})$ $1 \leq i,j \leq N$

(i) $tr(A+B) = \sum_{k=1}^{N} a_{kk} + b_{kk}$

$= \left(\sum_{k=1}^{N} a_{kk}\right) + \left(\sum_{k=1}^{N} b_{kk}\right)$

$= tr(A) + tr(B)$

$A = (a_{ij})$
$B = (b_{ij})$

(ii) $tr(\lambda A) = \sum_{k=1}^{N} \lambda a_{kk} = \lambda \sum_{k=1}^{N} a_{kk} = \lambda Tr(A)$

$(A = (a_{ij})\ , \lambda \in \mathbb{R})$

$\Longrightarrow$ $tr : Mat_{N \times N} \to \mathbb{R}$ è una mappa lineare.

(b) $A = (a_{ij}) \Longrightarrow A^t = (\alpha_{ij})$ $\alpha_{ij} = a_{ji}$

$B = (b_{ij}) \Longrightarrow B^t = (\beta_{ij})$ $\beta_{ij} = b_{ji}$

$(A+B)^t = C^t$, $C = A + B = (c_{ij})$, $c_{ij} = a_{ij} + b_{ij}$

$\shortparallel$

(γ_{ij}) , $\gamma_{ij} = c_{ji} = a_{ji} + b_{ji} = \alpha_{ij} + \beta_{ij}$

$\Longrightarrow (A+B)^t = A^t + B^t$

(c) $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \rightarrow B^t = \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix}$

$$AB^t = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{11}b_{21} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{12} & a_{21}b_{21} + a_{22}b_{22} \end{pmatrix}$$

$$\Rightarrow tr(AB^t) = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$$

$$= \langle A, B \rangle$$

Problema 6

$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ $a > 0$ ✓ $d > 0$ ✓ $ad - b^2 > 0$ ✗
$(1)(1) - 3^2$

$\Longrightarrow$ A non è una matrice definita positiva

$B = \begin{pmatrix} 1 & -2 \\ -2 & -3 \end{pmatrix}$ $a > 0$ ✓ $d > 0$ ✗ B non è una matrice definita positiva

$C = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$ $a > 0$, $d > 0$
$ad - b^2 = (1)(5) - 2^2 = 1 > 0$

$\therefore$ C è una matrice definita positiva