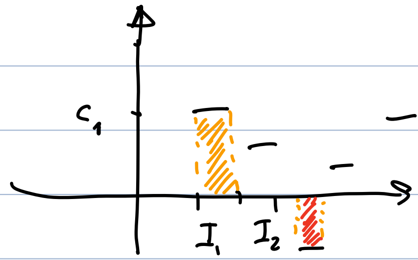


funzione semplice $\qquad I_i$ INTERVALLO

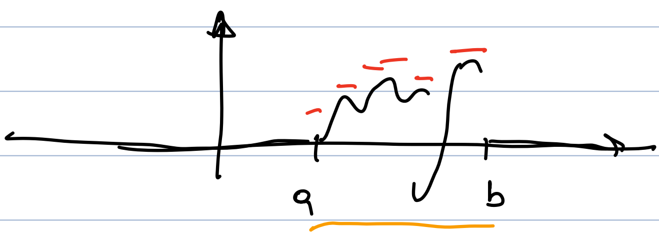
$$f = \begin{cases} c_1 & [a_1,b_1] \\ c_2 & [a_2,b_2] \\ \vdots & \\ c_n & [a_n,b_n] \end{cases} \iff f(x) = \sum_{i=1}^{n} c_i \, \chi_{I_i}(x)$$

FUNZIONE INDICATRICE $\qquad \chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \sum_{i=1}^{n} c_i (b_i - a_i) = \sum_{i=1}^{n} c_i \, |I_i| = \sum c_i \, m(I_i)$$

$|I_i| = b_i - a_i = m(I_i)$ la misura di I_i

f è limitata e nulla fuori da $[a,b]$



$\mathcal{S}_1 = \{ f_1 : \mathbb{R} \to \mathbb{R}$ semplici $f_1(x) \leq f(x) \ \forall x \}$

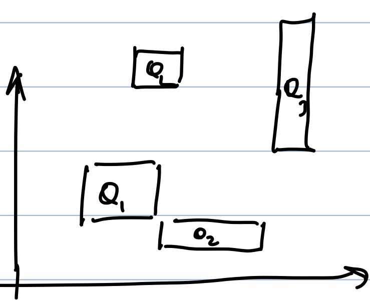
$\mathcal{S}_2 = \{ f_2 : \mathbb{R} \to \mathbb{R}$ semplici $f_2(x) \geq f(x) \ \forall x \in \mathbb{R} \}$

$$\sup_{f_1 \in \mathcal{S}_1} \int_{\mathbb{R}} f_1(x)\,dx = \inf_{f_2 \in \mathcal{S}_2} \int_{\mathbb{R}} f_2(x)\,dx \overset{\text{def}}{=} \int_a^b f(x)\,dx$$

$f \in \mathcal{R}$ integrabile (secondo Riemann)

$f : \mathbb{R}^2 \to \mathbb{R}$

f semplice



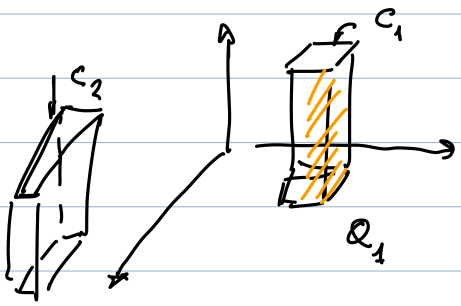
supporto $f = \overline{\{ x : f(x) \neq 0 \}}$

$f_i \in \mathbb{R}$

$$f = \sum_{i=1}^{n} f_i \, \chi_{Q_i}(x)$$

$Q_i = [a_i, b_i[\times [c_i, d_i[$

$a_i < b_i \qquad c_i < d_i$



$$\int_{\mathbb{R}^2} f(x)\,dx \overset{\text{def}}{=} \sum_{i=1}^{n} f_i \, \boxed{m(Q_i)}$$

$$= \sum^{n} f_i \cdot \boxed{(b_i - a_i) \cdot (d_i - c_i)}$$

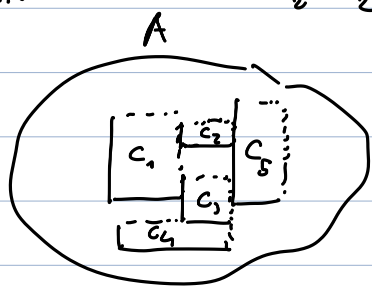
$\sum_{i=1}$

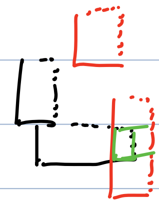
$f : \mathbb{R}^2 \to \mathbb{R}$ limitata nulla fuori da $B(0,R)$ $\quad R > 0$

$\mathcal{S}_1 = \{ f_1 : \mathbb{R}^2 \to \mathbb{R}$ semplici $f_1(x) \leq f(x) \quad \forall x \in \mathbb{R}^2 \}$

$\mathcal{S}_2 = \{ f_2 : \mathbb{R}^2 \to \mathbb{R}$ semplici $f_2(x) \geq f(x) \quad \forall x \in \mathbb{R}^2 \}$

$$\sup_{f_1 \in \mathcal{S}_1} \int_{\mathbb{R}^2} f_1(x)\,dx = \inf_{f_2 \in \mathcal{S}_2} \int_{\mathbb{R}^2} f_2(x)\,dx \overset{\text{def}}{=} \int_{\mathbb{R}^2} f(x)\,dx$$





$f_2 = \sum C_k \chi_{Q_k}$

$f_1 = \sum D_k \chi_{Q_k}$

$f \in \mathcal{R}(\mathbb{R}^2)$ se $\forall \varepsilon > 0$ $\exists f_1, f_2$ $\quad f_1 \in \mathcal{S}_1 \quad f_2 \in \mathcal{S}_2 \quad \int_{\mathbb{R}^2} f_2 - f_1 \, dx < \varepsilon$

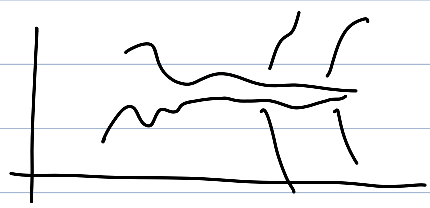
$\boxed{f \in \mathcal{R}(\mathbb{R}^2)}$ TEOREMA DELLA MEDIA INTEGRALE

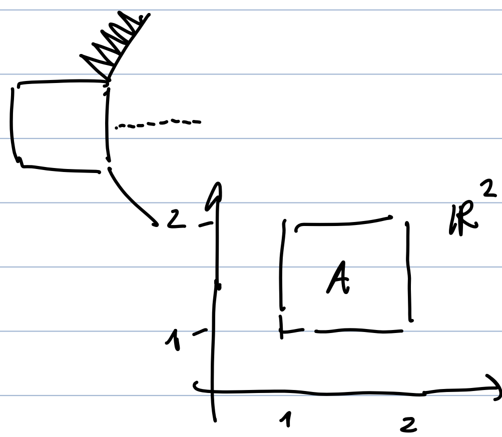
$$\inf f \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq \sup f \qquad f : [a,b] \to \mathbb{R}$$

$$\int_A f(x)\,dx \qquad A \subseteq \mathbb{R}^2 \qquad f : A \to \mathbb{R}$$

$$\shortparallel$$

$$\int_{\mathbb{R}^2} f(x)\,\chi_A(x)\,dx$$

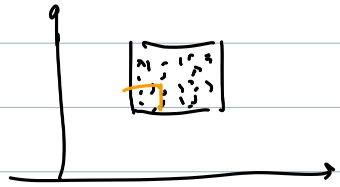




$f \in C([a,b]) \Rightarrow f$ è integrabile

$$f = \begin{cases} 1 & \text{su } A \\ 0 & \text{su } \mathbb{R}^2 \setminus A \end{cases}$$

$f(x)=1 \qquad \forall x \in D \qquad D = \{ x \in [1,2] \times [1,2] \quad (x_1,x_2) \in \mathbb{Q}^2 \}$



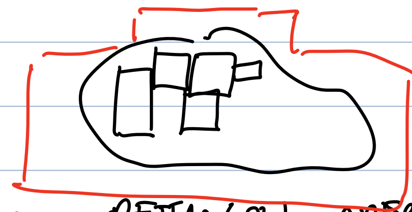
$(\sqrt{2},\sqrt{2}) \notin D \qquad (\sqrt{2},\frac{3}{2}) \notin D$

$(\frac{3}{2},\frac{3}{2}) \in D$

$\int_D f = \text{NE}$

$$\int_A f(x)\,dx = \int_{\mathbb{R}^2} f(x)\chi_A(x)\,dx$$

A è misurabile — DEFINIZIONE



- SI PUÒ APPROSSIMARE BENE CON PLURIRETTANGOLO $\bigcup_{i=1}^{N} Q_i$ — Q_i sono RETTANGOLI APERTI A DESTRA

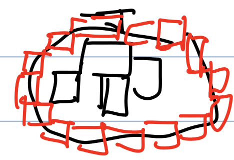
  $Q_i = [a_i, b_i[$

- $\forall \varepsilon > 0 \quad \exists \; R_1, R_2$ PLURIRETTANGOLI — $R_1 = \bigcup_{i=1}^{N} Q_i$, $R_2 = \bigcup_{i=1}^{n} Q_i'$

$$R_1 \subseteq A \subseteq R_2$$

$$m(R_1) = \sum_{i=1}^{n} m(Q_i) \qquad m(R_2) = \sum_{i=1}^{n'} m(Q_i')$$

$$m(R_2 \setminus R_1) < \varepsilon$$



TEOREMA: A è misurabile $\Leftrightarrow m(\partial A) = 0$

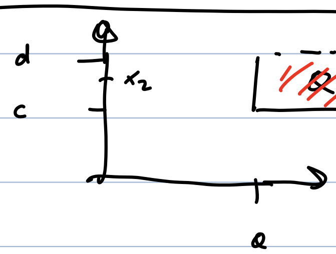
(1) TEOREMA MEDIA INTEGRALE $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}^2$ — A misurabile, f è INTEGRABILE

$$\inf_A f \le \frac{\int_A f\,dx}{m(A)} \le \sup_A f \qquad \inf_{[a,b]} f \le \frac{\int_a^b f}{b-a} \le \sup_{[a,b]} f$$

(2) $f \in C(A)$ TEOREMA

$\exists \xi \in A$ t.c. $\frac{\int_A f(x)\,dx}{\quad} = f(\xi) \qquad \frac{\int_a^b f(x)\,dx}{b-a} = f(\xi)$

$m(A)$



$[a,b[\times [c,d[\qquad \alpha$

$f(x) = \alpha \chi_Q$

$$\int_{\mathbb{R}^2} f(x)dx = \alpha \; m(Q) = \boxed{\alpha (b-a)(d-c)}$$

$f(x) = f(x_1, x_2)$ $\qquad \int_{\mathbb{R}} f(x_1,x_2)\,dx_1 = \int_{-\infty}^{+\infty} f(x_1,x_2)\,dx_1$

$$\int_{-\infty}^{+\infty} f(x_1,x_2)dx_1 = \begin{cases} 0 & x_2 < c \\ \alpha (b-a) & c \le x_2 < d \\ 0 & x_2 \ge d \end{cases}$$

$$\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x_1,x_2)\,dx_1 \right) dx_2 = \boxed{\alpha (b-a)(d-c)}$$

$$\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x_1,x_2)\,dx_2 \right) dx_1 \quad /\!/$$

$\Rightarrow \quad f \in \mathcal{R}(\mathbb{R}^2)$ $\qquad \int_{\mathbb{R}^2} f(x)dx = \int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} f(x_1,x_2)dx_2$

TEOREMA (FUBINI)

$$= \int_{\mathbb{R}} dx_2 \int_{\mathbb{R}} f(x_1,x_2)\,dx_1$$

$A = [0,1]^2 = [0,1] \times [0,1]$

$$\int_{[0,1]^2} x^2 y \, dx dy$$

$$= \int_0^1 dx \left(\int_0^1 dy \; x^2 y \right) = \int_0^1 dx \; x^2 \left(\int_0^1 y\, dy \right)$$

$$= \int_0^1 dx \; x^2 \left(\frac{y^2}{2} \Big|_0^1 \right) = \int_0^1 dx \; x^2 \left(\frac{1}{2} - 0 \right) =$$

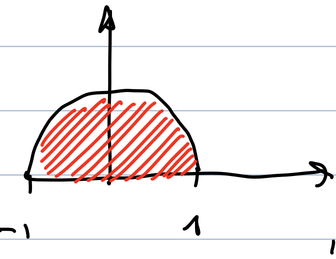
$$= \int_0^1 \frac{x^2}{2} dx = \frac{x^3}{6} \Big|_0^1 = \frac{1}{6}$$

$$\int x^2 y \, dx dy \qquad A = \left\{ (x,y) \in \mathbb{R}^2 \quad \begin{array}{l} -1 \le x \le 1 \\ 0 \le y \le \sqrt{1-x^2} \end{array} \right\}$$

$0 \le y \le \sqrt{1-x^2}$

A ||

$$\int_{-1}^{1} dx \int_{0}^{\sqrt{1-x^2}} x^2 y \, dy$$



$$= \int_{-1}^{1} dx \; x^2 \left(\frac{y^2}{2} \Big|_{0}^{\sqrt{1-x^2}} \right)$$

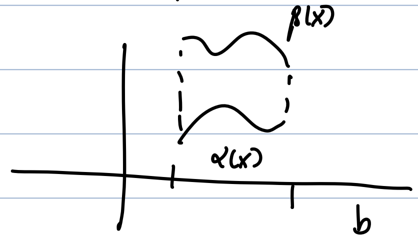
"A È INSIEME NORMALE"

$$= \int_{-1}^{1} dx \; x^2 \left(\frac{1-x^2}{2} - 0 \right) = \int_{-1}^{1} \frac{x^2}{2} - \frac{x^4}{2} = \frac{x^3}{6} - \frac{x^5}{10} \Big|_{-1}^{1} = \frac{2}{6} - \frac{2}{10}$$

A È INSIEME NORMALE

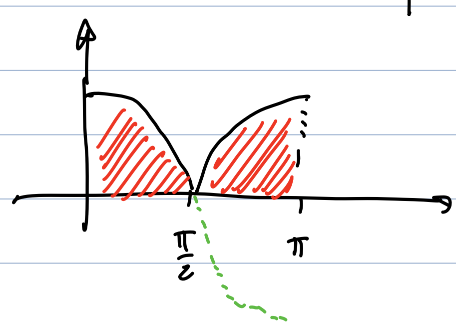
$$A = \{ (x,y) \in \mathbb{R}^2 : \quad a \le x \le b \qquad \alpha(x) \le y \le \beta(x) \}$$

α, β continue $\qquad \alpha(x) \le \beta(x) \quad \forall x \in [a,b]$



$$\int_A f(x,y)\, dx\, dy = \int_a^b dx \int_{\alpha(x)}^{\beta(x)} f(x,y)\, dy$$

$$\int_A xy \sin(x) \, dx\, dy \qquad A = \{ (x,y) \in \mathbb{R}^2 : \; 0 \le x \le \pi, \; 0 \le y \le |\cos(x)| \}$$



$$\int_0^{\pi} dx \int_0^{|\cos(x)|} xy \sin(x)\, dy = \int_0^{\pi} dx \; x \sin(x) \int_0^{|\cos(x)|} y \, dy$$

$$= \int_0^{\pi} dx \; x \sin(x) \left(\frac{y^2}{2} \Big|_0^{|\cos(x)|} \right) = \int_0^{\pi} dx \; \frac{x}{2} \sin(x) \cos^2(x)$$

$f \qquad g'$

$\pi \qquad \pi$

$$= -\frac{x}{2}\,\frac{\cos^3(x)}{3}\Big|_0^{\pi} + \int_0^{\pi} \frac{1}{2}\,\frac{\cos^3(x)}{3}\,dx$$

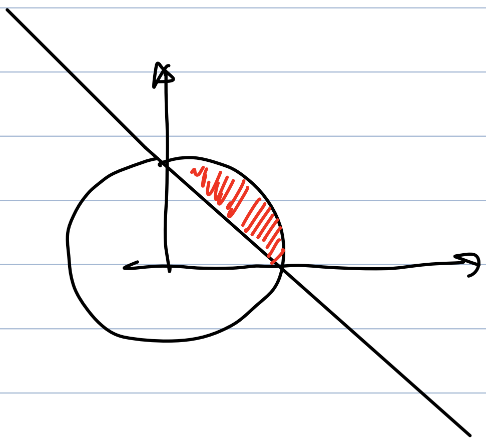
$$= -\frac{\pi}{6}\cos^3(\pi) + \underbrace{\boxed{\frac{1}{6}\int_0^{\pi} \cos^3(x)\,dx}}_{=0}$$

$$e^{ix} = \cos(x) + i\sin(x)$$

$$e^{i3x} = \cos(3x) + i\sin(3x) = \left(\cos(x) + i\sin(x)\right)^3 = \cos^3(x) + 3\cos^2(x)\, i\sin(x)$$
$$+ 3\cos(x)\left(i\sin(x)\right)^2 + \left(i\sin(x)\right)^3$$

$$\cos(3x) = \cos^3(x) - 3\cos(x)\sin^2(x)$$
$$= \cos^3(x) - 3\cos(x)\left(1-\cos^2(x)\right)$$
$$= \cos^3(x) - 3\cos(x) + 3\cos^3(x) = 4\cos^3(x) - 3\cos(x)$$

$$\cos^3(x) = \frac{\cos(3x)}{4} + \frac{3}{4}\cos(x)$$



$A =$ REGIONE DI PIANO LIMITATA TRA $y = 1-x$ e LA CIRCONFERENZA UNITARIA CENTRATA NELL'ORIGINE NEL SEMIPIANO DESTRO

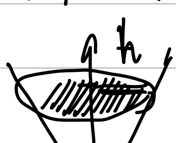
$$m(A) = \int_A 1\,dx\,dy = \int_{\mathbb{R}^2} \chi_A(x)\,dx$$

$$A = \left\{ (x,y) \in \mathbb{R}^2 \quad 0 \le x \le 1 \quad 1-x \le y \le \sqrt{1-x^2} \right\}$$

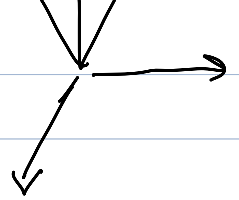
$$\int_A 1\,dx\,dy = \int_0^1 dx \int_{1-x}^{\sqrt{1-x^2}} 1\,dy = \int_0^1 dx \left(\sqrt{1-x^2} - (1-x)\right) dx = \dots$$

$$\int \sqrt{1-x^2}\,dx = \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2}$$

$$\int_A f(x,y,z)\,dx\,dy\,dz \qquad A = \left\{ z \ge \sqrt{x^2+y^2} \qquad 0 \le z \le h \right\}$$



$$\int 1\,dx\,dy\,dz = Vol(A)$$



$\boxed{x^2+y^2=z^2}$

A

$$\int_0^h dz \boxed{\int_{S(z)} 1\, dx\,dy} = \int_0^h dz\, \pi z^2 = \pi \frac{z^3}{3}\Big|_0^h = \pi \frac{h^3}{3}$$

$\parallel$

CIRCONFERENZA DI RAGGIO z

AREA πz^2

- $\displaystyle\int_{S^+} 1\, dx\,dy\,dz$

$S^+ = \{ 0 < z \le \sqrt{1-x^2-y^2} \}$