$f \in C^2(A)$ $\quad A \subseteq \mathbb{R}^n$ aperto $\quad x^0, x^0+h \in A$ $\quad S[x^0; x^0+h] \subseteq A$

TEOREMA $\quad \exists\, \xi = \xi(x^0, h) \quad 0 < |\xi| < 1$

$$f(x^0+h) = f(x^0) + \sum_{i=1}^{n} \partial_i f(x^0) h_i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f(x^0+\xi h)}{\partial x_i \partial x_j} h_i h_j$$

TAYLOR 1 RESTO LAGRANGE

$$f(x^0+h) = f(x^0) + (\nabla f(x^0), h) + \frac{1}{2} h^T Hf(x^0+\xi h)\, h$$

$$f(x^0+h) = f(x^0) + \sum \partial_i f(x^0) h_i + \frac{1}{2} \sum \partial_i \partial_j f(x^0) h_i h_j + \omega(h)$$

TAYLOR ORDINE 2 RESTO PEANO $\quad \exists\, \omega : \mathbb{R}^n \to \mathbb{R}$

$\lim_{h \to 0} \frac{\omega(h)}{\|h\|^2} = 0 \qquad \omega(h) = o(\|h\|^2)$

$$f(x^0+h) = f(x^0) + (\nabla f(x^0), h) + \frac{1}{2} h^T Hf(x^0) h + \omega(h)$$

TEOREMA (STESSE IPOTESI) $\quad x^0$ interno

x^0 è pt di minimo (locale) $\Rightarrow \nabla f(x^0) = 0 \quad (Hf(x^0) \geq 0)$

$Hf(x^0)$ È SEMI DEFINITA POSITIVA $\quad v^T Hf(x^0) v \geq 0 \quad \forall v \in \mathbb{R}^n$

SE $\nabla f(x^0) = 0$ e SE $(Hf(x^0) > 0)$

$Hf(x^0)$ È DEFINITA POSITIVA $\quad v^T Hf(x^0) v \geqslant 0$

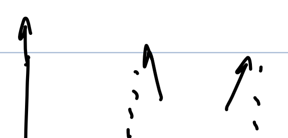
$v^T Hf(x^0) v > 0 \quad \forall v \neq 0$

$\exists\, \alpha > 0 \quad v^T Hf(x^0) v \geqslant \alpha \|v\|^2$

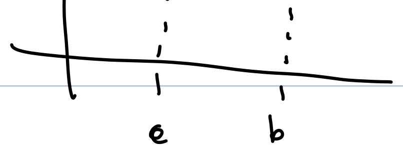
$\alpha = \min \lambda_j \; : \; Hf(x_0) v = \lambda_j v \quad \forall v \in \mathbb{R}^n \setminus \{0\}$

$m = \min_{\|x\|=1} x^T Hf(x_0) x \Rightarrow \exists\, \bar{x} \quad Hf(x_0) \bar{x} = m \bar{x}$

$\det(Hf(x_0) - \lambda I) = 0 \qquad 0 = p(\lambda) = \sum_{k=0}^{n} a_k \lambda^k$ polinomio



$f \to +\infty \qquad f \in C^0(]a,b[)$



$x \to a^+$

$x \to b^-$

$\Longrightarrow \exists\ m = \min_{]0,b[} f$

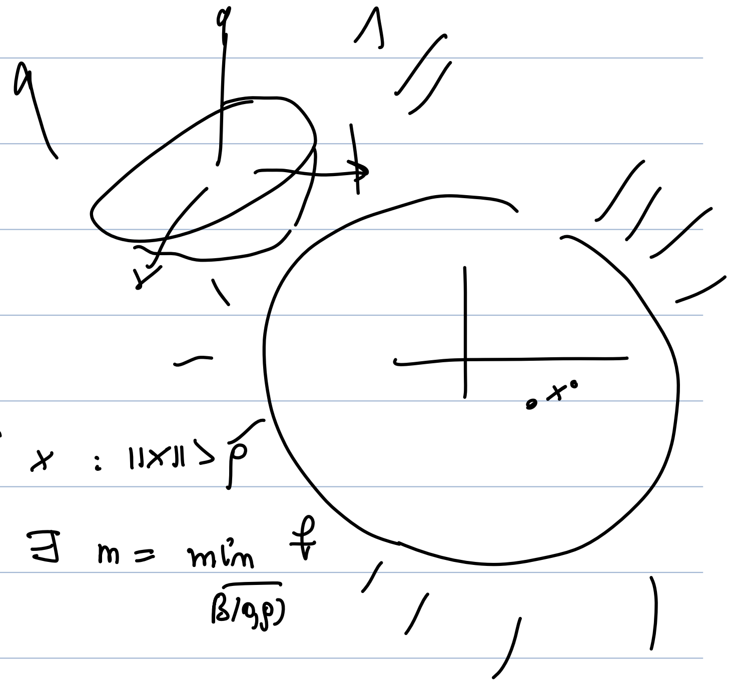
• $\lim_{x \to \pm\infty} f = +\infty \qquad f \in C^0(\mathbb{R})$

$\Rightarrow \quad \exists\ m = \min_{\mathbb{R}} f$

$f \in C(\mathbb{R}^n) \qquad \lim_{x \to \infty} f = +\infty \qquad \lim_{\|x\| \to +\infty} f(x) = +\infty$

$\forall M \in \mathbb{R} \quad \exists\ \rho > 0 \ : \ \forall x \in \mathbb{R}^n \quad \|x\| > \rho \qquad f(x) > M$

$\Rightarrow \quad \exists\ m = \min_{\mathbb{R}^n} f$



$x^0 \in \mathbb{R}^n \qquad f(x^0)$

$M = f(x^0) + 10 > f(x^0)$

$f(x) > f(x^0) + 10 \qquad \forall\ x \ : \ \|x\| > \rho$

$f|_{\overline{B(0,\rho)}} \qquad$ WEIERSTRASS $\qquad \exists\ m = \min_{\overline{B(0,\rho)}} f$

$\overline{B(0,\rho)}$ CHIUSO E LIMITATO

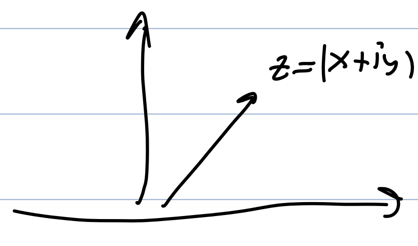
$m = \min_{\overline{B(0,\rho)}} f \leq f(x^0)$

$\Rightarrow \quad m = \min_{\mathbb{R}^n} f \qquad$ perchè $\quad f(x) > f(x^0) + 10 \quad \forall x : \|x\| > \rho$

$$p(z) = \sum_{k=0}^{n} a_k z^k \qquad z \in \mathbb{C} \qquad a_k \in \mathbb{C}$$

$z = x + iy$

$$p(z) = f(x,y) = \sum_{k=0}^{n} a_k (x+iy)^k$$



$f : \mathbb{R}^2 \to \mathbb{C}$

?? $\lim_{z \to \infty} p(z) = +\infty$

?? $\lim_{|z| \to +\infty} p(z) = +\infty$

$|z| = \sqrt{x^2 + y^2}$

$$\lim_{|z|\to+\infty} |p(z)| = +\infty \qquad |p(z)| \in \mathbb{R}^+$$

$$p(z) = \sum_{k=0}^{n} a_k z^k \qquad a_n \neq 0 \qquad n > 1$$

POLINOMIO NON COSTANTE

$$= z^n \left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \dots + a_n \right)$$

$$|z| \to +\infty \qquad \frac{a_0}{z^n} \to 0 \qquad \text{perché } |z^n| = |z|^n \to +\infty$$

$$|p(z)| = |a_n z^n| = |a_n|\,|z|^n \to +\infty$$

TEOREMA FONDAMENTALE DELL'ALGEBRA

$$p(z) = \sum_{k=0}^{n} a_k z^k \qquad a_n \neq 0 \qquad \text{non costante}$$

$$\Rightarrow \exists\, \bar{z} : \quad p(\bar{z}) = 0$$

$$f(z) = |p(z)| \qquad f : \mathbb{C} \to \mathbb{R}^+ \qquad \lim_{|z|\to+\infty} |p(z)| = +\infty$$

$$\Rightarrow \exists \min f(z) \qquad \exists\, z_* : \quad f(z_*) = |p(z_*)| \leq f(z) \quad \forall z \in \mathbb{C}$$

1) $f(z_*) = 0$ FINITO $|p(z_*)| = 0 \Rightarrow p(z_*) = 0$

2) $f(z_*) > 0$ (ASSURDO) $|p(z_*)| > 0 \Rightarrow p(z_*) \neq 0$

LEMMA SE $|p(z)| > 0 \Rightarrow \exists\, w \in \mathbb{C} \quad |p(w)| < |p(z)|$

$$q(w) = \frac{p(z_* + w)}{p(z_*)} = \sum_{k=0}^{n} \frac{a_k (z_* + w)^k}{p(z_*)}$$

POLINOMIO DI GRADO n

$$q(0) = \frac{p(z_*)}{p(z_*)} = 1$$

$$q(w) = 1 + \beta_1 w + \beta_2 w^2 + \dots + \beta_n w^n \qquad \beta_n \neq 0$$

β_i potrebbero essere nulli $1 < i < n$

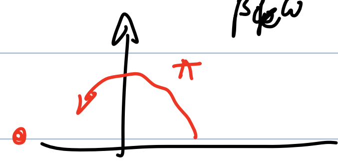
$q(\omega) = 1 + \dots^0 + \beta_k \omega^k + \beta_{k+1}\omega^{k+1} + \dots + \beta_n \omega^n$ $\quad k \in \mathbb{N}$ t.c. β_k è il primo diverso da zero

$q(\omega) = 1 + \beta_k \omega^k + \omega^{k+1}\left[\beta_{k+1} + \beta_{k+2}\omega + \dots \beta_n \omega^{n-(k+1)}\right]$

$= 1 + \beta_k \omega^k + \omega^{k+1} \tilde{q}(\omega)$ $\quad \tilde{q}$ POLINOMIO

SCELGO $\omega \in \mathbb{C}$ $\quad \beta_k \omega^k \in \mathbb{R}_-$ $\quad \beta_k \omega^k \in \mathbb{R}$, $\beta_k \omega^k < 0$

$\beta_k \neq 0$



$\arg(\beta_k \omega^k) = \pi$

$\arg(\beta_k) + \arg(\omega^k)$

$= \arg(\beta_k) + k \arg(\omega) = \pi$ $\quad \omega = \rho e^{i\theta_0}$

$\arg(\omega) = \dfrac{\pi - \arg(\beta_k)}{k} = \theta_0$

$|1 + \beta_k \omega^k| = 1 - |\beta_k||\omega|^k$

$= 1 - |\beta_k| \rho^k$

$\lim_{\omega \to 0} \omega^{k+1} \tilde{q}(\omega) = 0$

$|q(\omega)| = |1 + \beta_k \omega^k + \omega^{k+1} \tilde{q}(\omega)|$

$= |1 + \omega^k [\beta_k + \omega \tilde{q}(\omega)]|$ $\quad 0 < \rho < \rho_0$

$\omega \tilde{q}(\omega) \to 0$ $\quad |\omega \tilde{q}(\omega)| < \dfrac{|\beta_k|}{2}$

< 1

$0 < \rho < \rho_0$ $\quad$ TEO PERM. SEGNO

$f : \mathbb{C} \to \mathbb{C}$ $\qquad z = x + iy$

$f : \mathbb{R}^2 \to \mathbb{R}^2$

$f(z) = u(x,y) + i\, v(x,y)$

$F : \mathbb{R} \to \mathbb{C}$

$F(x) = F_1(x) + i F_2(x)$

$F' = F_1'(x) + i F_2'(x)$

- $$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$z, z_0 \in \mathbb{C} \qquad z \neq z_0$

$z = x + iy \qquad z_0 = x_0 + iy_0$

$$\frac{(u(x,y) + i\, v(x,y)) - (u(x_0,y_0) + i\, v(x_0,y_0))}{(x - x_0) + i(y - y_0)} \xrightarrow[(x,y) \to (x_0,y_0)]{} f'(z_0)$$

SE IL LIMITE ESISTE

$y = y_0$
$$\frac{u(x,y_0) + i\, v(x,y_0) - (u(x_0,y_0) + i\, v(x_0,y_0))}{x - x_0}$$

$$\frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + i \left(\frac{v(x,y_0) - v(x_0,y_0)}{x - x_0}\right) \xrightarrow[x \to x_0]{} \partial_x u(x_0,y_0) + i\, \partial_x v(x_0,y_0)$$

$x = x_0$
$$\frac{u(x_0,y) + i\, v(x_0,y) - (u(x_0,y_0) + i\, v(x_0,y_0))}{i(y - y_0)}$$

$$\frac{1}{i} \frac{u(x_0,y) - u(x_0,y_0)}{y - y_0} + \frac{\cancel{i}}{\cancel{i}} \frac{v(x_0,y) - v(x_0,y_0)}{(y - y_0)} \xrightarrow[y \to y_0]{} \frac{1}{i} \partial_y u(x_0,y_0) + \partial_y v(x_0,y_0)$$

$$- i\, \partial_y u(x_0,y_0) + \partial_y v(x_0,y_0)$$

$$\boxed{\begin{aligned} \partial_x u(x_0,y_0) &= \partial_y v(x_0,y_0) \\ \partial_x v(x_0,y_0) &= -\partial_y u(x_0,y_0) \end{aligned}}$$

CAUCHY - RIEMANN

OLOMORFA

$C\text{-}R \implies f \in C^\infty \qquad f(z) = \sum_{k=0}^{+\infty} a_k z^k$

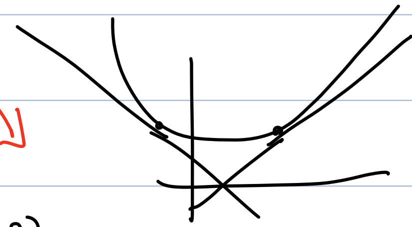
$+\infty \qquad +\infty$

$$f(z) = e^z = \sum_{k=0} \frac{z^k}{k!} \qquad \int_0 \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$$

- $f = \sqrt{1-\|x\|^2} \qquad \|x\| \leq 1 \qquad x \in \mathbb{R}^n$

$\|x^0\| < 1$

$f''(z) \geq 0$

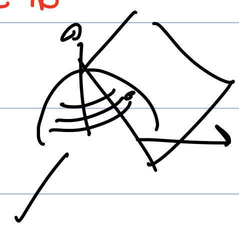


f convesse $\qquad f(x) \geq f(x_0) + f'(x_0)(x-x_0)$

$\forall x, x^0 \in A \qquad f(x) \geq f(x^0) + (\nabla f(x^0), x - x^0)$

A convesso $\qquad Hf(y)$ SEMI DEFINITA POSITIVA

$v^T Hf(y) v \geq 0 \qquad \forall v \in \mathbb{R}^n \qquad \forall y \in A$



$$f(x,y) = \sqrt{1-x^2-y^2}$$

$$f(x) = \sqrt{1-\|x\|^2} \qquad x^0$$

$\varphi(x) = f(x^0) + (\nabla f(x^0), x - x^0)$ PIANO TG. IN $(x^0, f(x^0))$

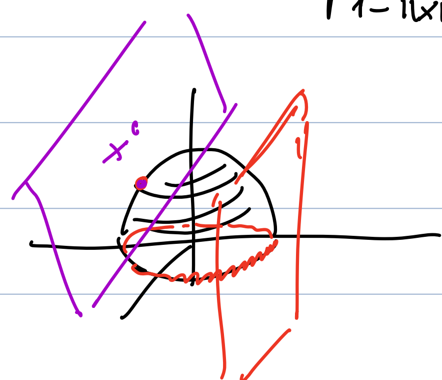
$1 - (x_1^2 + x_2^2 + \dots + x_n^2)$

$$\partial_i f(x) = \frac{1}{2}(1-\|x\|^2)^{-1/2} \cdot \frac{\partial}{\partial x_i}(1-\|x\|^2) = \frac{1}{2}\frac{-2x_i}{\sqrt{1-\|x\|^2}} = -\frac{x_i}{\sqrt{1-\|x\|^2}}$$

$$\varphi(x^0) = \sqrt{1-\|x^0\|^2} + \sum_i \left(-\frac{x_i^0}{\sqrt{1-\|x^0\|^2}}, x_i - x_i^0\right)$$

$$= \sqrt{1-\|x^0\|^2} + \frac{1}{\sqrt{1-\|x^0\|^2}}\left(-(x,x^0) + \|x_i^0\|^2\right) = \frac{1 - \cancel{\|x^0\|^2} - (x,x^0) + \cancel{\|x^0\|^2}}{\sqrt{1-\|x^0\|^2}}$$

$$= \frac{1-(x,x^0)}{\sqrt{1-\|x^0\|^2}}$$



$$f(x,y) = \sqrt{1-x^2-y^2}$$

$$\begin{pmatrix} \partial_{xx} f & \partial_{xy} f \\ \partial_{yx} f & \partial_{yy} f \end{pmatrix}$$ SEMI DEFINITA NEGATIVA

$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ $\det(A - \lambda I) = 0$ $\lambda_1, \lambda_2 \in \mathbb{R}$ $\lambda_1, \lambda_2 < 0$

$a + d = \lambda_1 + \lambda_2$

$\det A = \lambda_1 \lambda_2$

se $\lambda_1, \lambda_2 < 0 \Rightarrow$ $\det A > 0$, $\operatorname{tr} A < 0$