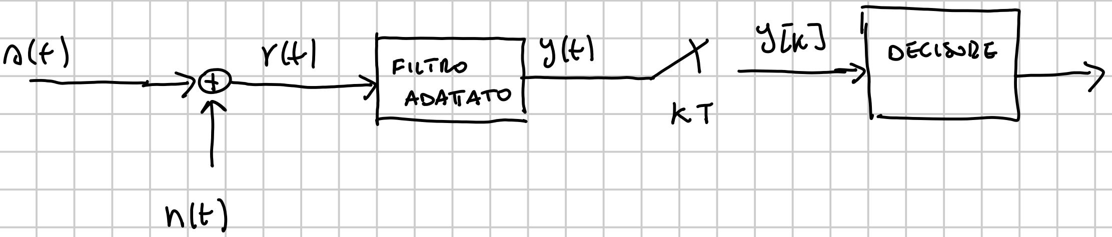
Esercizio #1

PAM binario



$$s(t) = \sum_{i=-\infty}^{+\infty} x[i]\, p(t - iT)$$

$a_i \in \{\pm 1\}$ equiprobabili e indipendenti

$$Pr\{x[i] = a_1\} = Pr\{x[i] = a_{-1}\} = \frac{1}{2}$$

$n(t)$ è AWGN

$\lambda = 0$

$$\hat{x}[k] = \begin{cases} 1 & \text{se } y[k] \geq \lambda \\ -1 & \text{se } y[k] < \lambda \end{cases}$$ DECISORE

$$h(t) = p(t) \otimes h_{FA}(t)$$

$$r(t) = s(t) + n(t)$$

$$y(t) = s_u(t) + n_u(t)$$

$$y[k] = x[k]\, h(0) + n_u[k]$$

$$\sigma^2_{nu} = \int_{-\infty}^{+\infty} S_{nu}(f)\, df = \int_{-\infty}^{+\infty} S_n(f)\, |H_{FA}(f)|^2\, df = \frac{N_0}{2} \int_{-\infty}^{+\infty} |H_{FA}(f)|^2\, df =$$

$$= (\text{per la proprietà del filtro adattato}) = \frac{N_0}{2} \int_{-\infty}^{+\infty} H(f)\, df = \frac{N_0}{2} h(0)$$

$$SNR = \frac{h^2(0)}{\frac{N_0}{2} h(0)} = \frac{2h(0)}{N_0}$$

$$E\{x^2[n]\, h^2(0)\} = h^2(0)\, E\{x^2[n]\} = h^2(0) \left\{ \frac{1}{2} \cdot (1)^2 + \frac{1}{2} \cdot (-1)^2 \right\} = h^2(0)$$

$$P_E[b] = Pr\{x[n] = +1\}\, Pr\{\hat{x} = -1 \mid x = +1\} +$$

$$Pr\{x = -1\}\, Pr\{\hat{x} = 1 \mid x = -1\} =$$

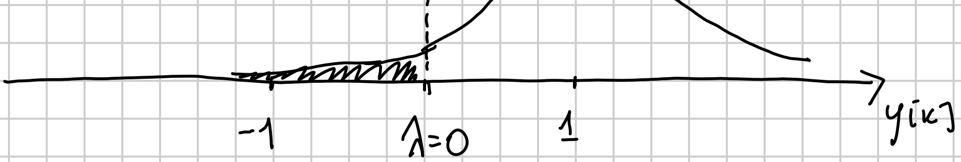
$$= \frac{1}{2} Pr\{\hat{x} = -1 \mid x = 1\} + \frac{1}{2} Pr\{\hat{x} = 1 \mid x = -1\}$$

$$Pr\{\hat{x} \,|{-1}|\, x = 1\}$$

$$y[k] = x[k]\, h(0) + n_u(k)$$

$$y[k]\big|_{x=1} = +h(0) + n_u(k) \in \mathcal{N}\left(+h(0), \sigma^2_{nu}\right)$$

$$Pr\{\hat{x} - 1 \mid x = 1\} = \int_{-\infty}^{\lambda} f_y(y \mid x = 1)\, dy = \int_{-\infty}^{0} f_y(y \mid x = 1)\, dy$$



$$f_y(y \mid x = 1) = \frac{1}{\sqrt{2\pi \sigma^2_{nu}}}\, e^{-\frac{(y - h(0))^2}{2\sigma_n^2}}$$

$$P\{\hat{x} = -1 \mid x = 1\} = 1 - Q\left(\frac{0 - h(0)}{\sigma_{nu}}\right) = \qquad \sigma_{nu} = \sqrt{\frac{N_0}{2}\, h(0)}$$

$$= Q\left(\sqrt{\frac{2h(0)}{N_0}}\right) = Q\left(\sqrt{SNR}\right) = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right)$$

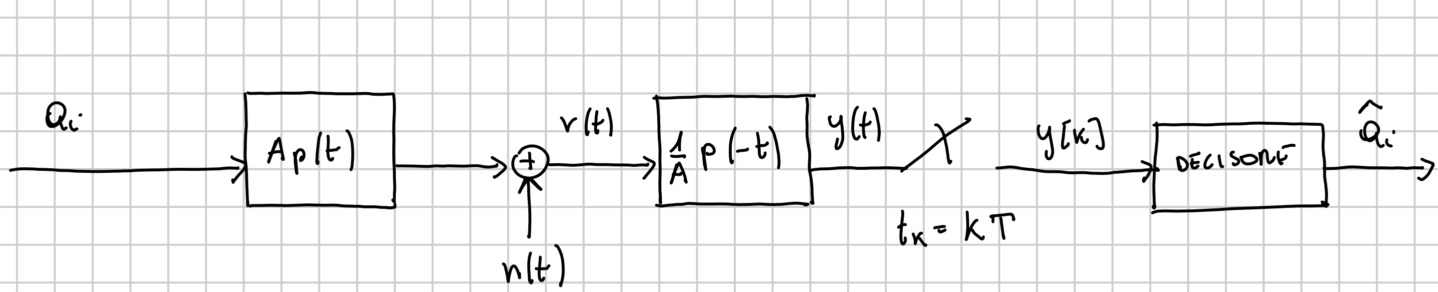
Si dimostra per simmetria che

$$P_r\{\hat{x} = 1 \mid x = -1\} = P_r\{\hat{x} = -1 \mid x = 1\} = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right)$$

$$P_E(b) = \frac{1}{2}\cdot\frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right) + \frac{1}{2}\,\frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right) = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right)$$

Esercizio #2

sistema di comunicazione PAM a 4 livelli



$$r(t) = \sum_{i=-\infty}^{+\infty} a_i\, A\, p(t - iT) + n(t)$$

$a_i \in \{\pm 3;\ \pm 1\}$ indipendenti ed equiprobabili.

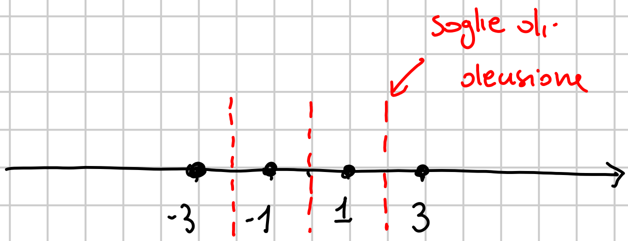
$n(t)$ AWGN $\quad S_n(f) = \frac{N_0}{2} \quad \forall f$

$p(t)$ ha trasformata di Fourier

$$P(f) = \begin{cases} \sqrt{T}\,\sqrt{1-|fT|} & |fT| \leq 1 \\ 0 & \text{altrove} \end{cases}$$

La strategia di decisione è:

$$\hat{a}_k = \begin{cases} 3 & \text{se} \quad y[k] \geq 2 \\ 1 & \text{se} \quad 0 \leq y[k] < 2 \\ -1 & \text{se} \quad -2 \leq y[k] < 0 \\ -3 & \text{se} \quad y[k] < -2 \end{cases}$$



Si determini:

1) Il SNR $= \frac{E_s}{N_0}$ dove E_s è l'energia media per intervallo di segnalazione del segnale trasmesso.

2) La $P_E(M)$ = probabilità di errore in funzione di E_s/N_0

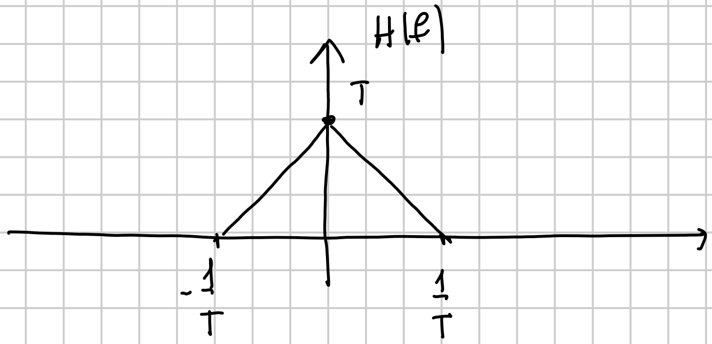
Svolgimento:

Il segnale all'uscita del FA

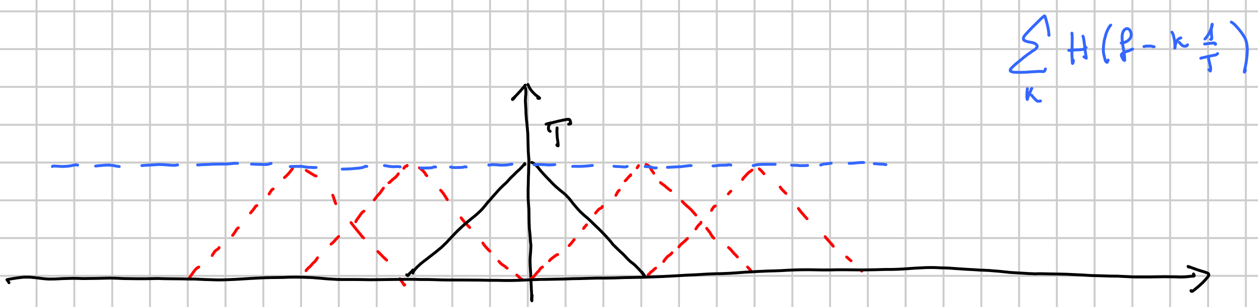
$$y(t) = \sum_{i=-\infty}^{+\infty} a_i\, h(t-iT) + n(t)$$

$h(t) = p(t) \otimes h_{FA}(t) = p(t) \otimes p(-t)$

$$H(f) = P(f)\, P^*(f) = P(f)\, P(f) = P^2(f) = \begin{cases} T\,(1-|fT|) & |fT| \le 1 \\ 0 & \text{altrove} \end{cases}$$



Soddisfa la condizione di Nyquist!



$$\sum_k H\left(f - \frac{k}{T}\right) = T \quad \Rightarrow \quad h(0) = 1$$

$$n_u(t) = n(t) \otimes h_R(t) = \frac{1}{A}\, n(t) \otimes p(-t)$$

$$\sigma^2_{n_u} = \frac{N_0}{2} \int_{-\infty}^{+\infty} |H_R(f)|^2\, df = \frac{N_0}{2A^2} \underbrace{\int_{-\infty}^{+\infty} |P(f)|^2\, df}_{h(0)} = \frac{N_0}{2A^2}\, h(0) = \frac{N_0}{2A^2}$$

$$E_S = E\left\{\int_0^T s^2(t)\,dt\right\} = A^2 E\left\{\int_0^T \sum_i \sum_n a_i a_n\, p(t-iT)\, p(t-nT)\right\} =$$

$$= A^2 \sum_i \sum_n E\{a_i a_n\} \int_0^T p(t-iT)\, p(t-nT)\, dt$$

Osserviamo che

$$E\{a_i\} = \frac{1}{4}\cdot(-3) + \frac{1}{4}(-1) + \frac{1}{4}(1) + \frac{1}{4}(3) = 0$$

$$E\{a_i^2\} = \frac{1}{4}(3)^2 + \frac{1}{4}(-3)^2 + \frac{1}{4}(-1)^2 + \frac{1}{4}(1)^2 = 5$$

Quindi:

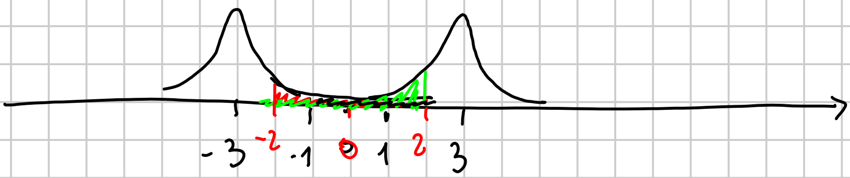
$$E\{a_i a_n\} = \begin{cases} E\{a_i^2\} & \text{se } i=n \\ E\{a_i\}\cdot E\{a_n\} & \text{se } i\neq n \end{cases} = \begin{cases} 5 & \text{se } i=n \\ 0 & \text{se } i\neq n \end{cases}$$

$$E_S = A^2\, 5 \sum_i \int_0^T p^2(t-iT) = 5A^2 \int_{-\infty}^{+\infty} p^2(t)\,dt = 5A^2 \int_{-\infty}^{+\infty} P^2(f)\,df = 5A^2$$

Calcoliamo le probabilità

$$P_E = \frac{1}{4}\Big\{ \Pr\{e \mid a_k=-3\} + \Pr\{e \mid a_k=-1\} + \Pr\{e \mid a_k=1\} +$$
$$+ \Pr\{e \mid a_k=3\}\Big\}$$

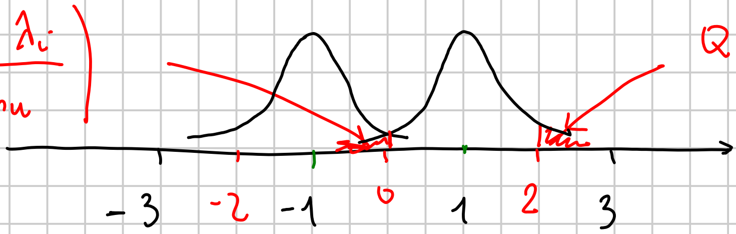
$$y(k) \mid a_k = 3 \quad \in \mathcal{N}(a_k, \sigma^2_{nu})$$



$$Pr\{e \mid a_k=3\} = Pr\{e \mid a_k=-3\} = Pr\{-3+n_k > -2\} = Q\left(\frac{-2+3}{\sigma_{nu}}\right) = Q\left(\frac{1}{\sigma_{nu}}\right)$$

$$Pr\{e \mid a_k=1\} = Pr\{e \mid a_k=-1\} = Pr\{-1+n_k > 0\ ,\ -1+n_k \leq -2\}$$

$$Q\left(\frac{\eta_i-\lambda_i}{\sigma_{nu}}\right)$$



$$Q\left(\frac{\lambda_i-\eta_i}{\sigma_{nu}}\right)$$

$\eta_i =$ valor medio delle Gaussiane

$\lambda_i =$ soglia di decisione

$$Pr\{e \mid a_k=1\} = Pr\{-1+n_k > 0\} + Pr\{-1+n_k \leq -2\} =$$

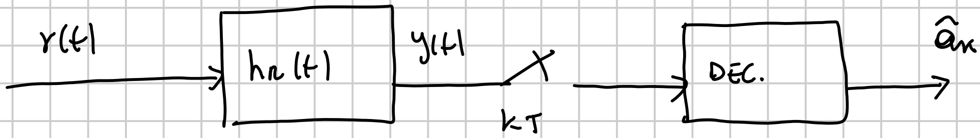
$$= Q\left(\frac{0+1}{\sigma_{nu}}\right) + Q\left(\frac{+2-1}{\sigma_{nu}}\right) = 2Q\left(\frac{1}{\sigma_{nu}}\right)$$

$$P_e = \frac{3}{2}\, Q\left(\frac{1}{\sigma_{nu}}\right)$$

$$\sigma^2_{nu} = \frac{N_0}{2A^2} = \frac{5}{5}\,\frac{N_0}{2A^2} = \frac{5}{2}\,\frac{N_0}{E_s}$$

$$P_e = \frac{3}{2}\, Q\left(\sqrt{\frac{2}{5}\,\frac{E_s}{N_0}}\right)$$

Esercizio #3



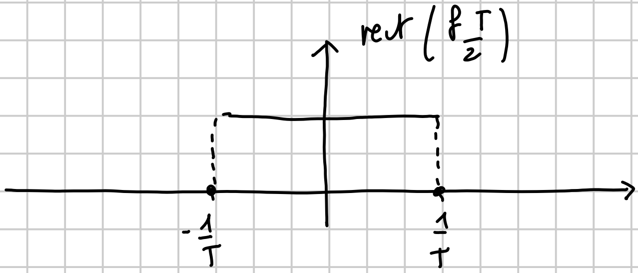
$a_i \in \{-e \,;\, 2e\}$ indipendenti ed equiprobabili.

$$r(t) = \underbrace{\sum_i a_i \, p(t - iT)}_{s(t)} + n(t)$$

$S_n(f) = \frac{N_0}{2}$ AWGN

$$P(f) = T \, |f| \, e^{-|f|} \, \mathrm{rect}\left(\frac{fT}{2}\right)$$

Si assume che la $\mathrm{rect}(x) = 0$ per $x = \pm \frac{1}{2}$



$$H_R(f) = e^{|f|} \, \mathrm{rect}\left(\frac{fT}{2}\right)$$

$$\hat{a}_k = \begin{cases} -e & x_k \le \lambda \\ 2e & x_k > \lambda \end{cases}$$

$\lambda = \frac{3e}{2}$

1) E_s del segnale trasmesso

2) $P_r\{e\}$ nel caso in cui $\lambda = \frac{3e}{2}$

3) Il valore ottimo di λ che minimizza la $Pr\{e\}$

① E_s si può calcolare in due modi:

1) $$E_s = E\{a_i^2\} \int_{-\infty}^{+\infty} |P(f)|^2 \, df$$

2) $E_s = P_T \cdot T$ dove $P_T = \int_{-\infty}^{+\infty} S_s(f) \, df$ e

$$S_s(f) = \frac{1}{T} S_a(f) \, |P(f)|^2 \qquad S_a(f) \triangleq \sum_m R_a(m) \, e^{-j2\pi f m T}$$

$$R_a(m) = E\{a_{i+m} \, a_i\} = \begin{cases} E\{a_i^2\} & m=0 \\ E\{a_{i+m}\} \cdot E\{a_i\} & m \neq 0 \end{cases}$$

$\boxed{\text{funzione di autocorrelazione dei simboli trasmessi } a_i}$

$$E\{a_i\} = \frac{1}{2}(-e) + \frac{1}{2}(2e) = \frac{e}{2}$$

$$E\{a_i^2\} = \frac{1}{2} e^2 + \frac{1}{2} 4e^2 = \frac{5}{2} e^2$$

$$R_a(m) = \begin{cases} \frac{5}{2} e^2 & m=0 \\ \frac{e}{2} & m \neq 0 \end{cases} \Rightarrow \boxed{R_a(m) = \frac{5}{2} e^2 \delta(m) + \frac{1}{2} e}$$

$$S_a(f) = \frac{5}{2} e^2 + \frac{e}{2T} \sum_k \delta\left(f - \frac{k}{T}\right)$$

$$S_s(f) = \frac{1}{T} |P(f)|^2 \left[\frac{5}{2} e^2 + \frac{e}{2T} \sum_k \delta\left(f - \frac{k}{T}\right)\right]$$

$$E_S = \int_{-\infty}^{+\infty} |P(f)|^2 \cdot \left[\frac{5}{2} e^2 + \frac{e}{2T} \sum_k \delta\left(f - \frac{k}{T}\right) \right] =$$

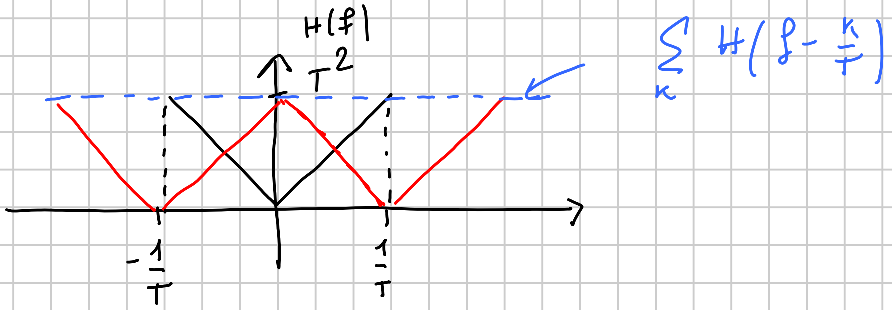
$$= \frac{5}{2} e^2 \int_{-\infty}^{+\infty} |P(f)|^2 df + \frac{e}{2T} \sum_k \int_{-\infty}^{+\infty} |P(f)|^2 \delta\left(f - \frac{k}{T}\right) df =$$

$$\int_{-\infty}^{+\infty} |P(f)|^2 df = \frac{1}{4} \left[1 - e^{-2/T} \left(1 + \frac{2}{T} + \frac{2}{T^2} \right) \right]$$ ✱ SVOLTO ALLA FINE

$$\int_{-\infty}^{+\infty} |P(f)|^2 \delta\left(f - \frac{k}{T}\right) df = \left| P\left(\frac{k}{T}\right) \right|^2 \Rightarrow \sum_k \left| P\left(\frac{k}{T}\right) \right|^2 = |P(0)|^2 = 0$$

$$E_S = \frac{5}{2} e^2 \cdot E_P \qquad E_P = \frac{1}{4} \left[1 - e^{-2/T} \left(1 + \frac{2}{T} + \frac{2}{T^2} \right) \right]$$

(2) $H(f) = P(f) \, H_R(f) = T^2 |f| \, \text{rect}\left(\frac{fT}{2}\right)$



$$\sum_k H\left(f - \frac{k}{T}\right) = T^2 \Rightarrow \boxed{h(0) = T}$$

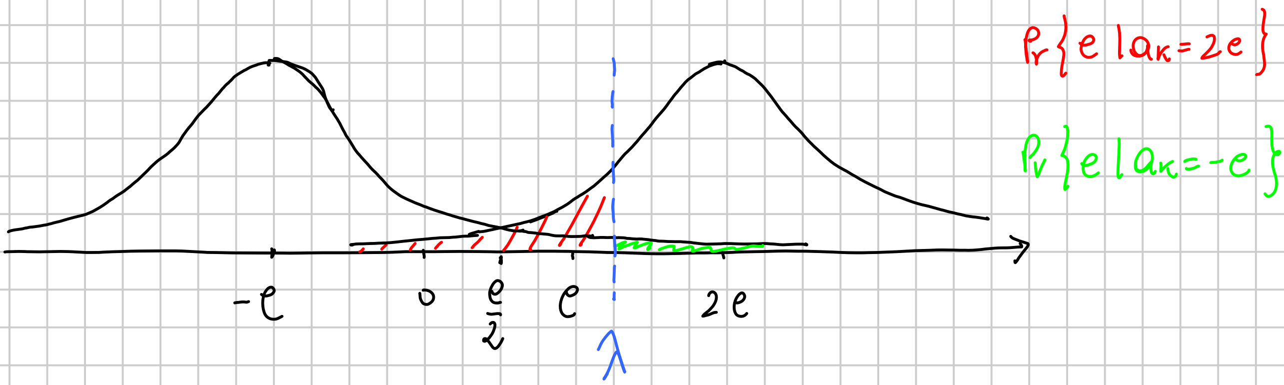
$$y_k = h(0)\, a_k + n_k = a_k + n_k$$

$n_u(t) = n(t) \otimes h_R(t)$

$S_{n_u}(f) = \frac{N_0}{2} |H_R(f)|^2$

$\sigma_{n_u}^2 = \frac{N_0}{2} \int_{-\infty}^{+\infty} |H_R(f)|^2 df = \frac{N_0}{2} 2 \int_0^{+\infty} e^{2f} df = \frac{N_0}{2} \left(e^{2/T} - 1 \right)$

$P(e) = \frac{1}{2} Pr\{e | a_K = -e\} + \frac{1}{2} Pr\{e | a_K = 2e\}$



(3) $Pr\{e | a_K = 2e\} = Q\left(\frac{-\lambda + 2e}{\sigma_{n_u}}\right)$

$Pr\{e | a_K = -e\} = Q\left(\frac{\lambda + e}{\sigma_{n_u}}\right)$

$P_E = \frac{1}{2} Q\left(\frac{-\lambda + 2e}{\sigma_{n_u}}\right) + \frac{1}{2} Q\left(\frac{\lambda + e}{\sigma_{n_u}}\right)$

Si può calcolare la P_E nel caso in cui $\lambda = \frac{3e}{2}$

Per trovare la soglia che minimizza la probabilità di errore si impone che.

$$\frac{\partial P_E}{\partial \lambda} = 0$$

$$Q'(x) \triangleq \frac{\partial Q}{\partial x} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\frac{\partial P_E}{\partial \lambda} = -\cancel{\frac{1}{2}}\left(+\cancel{\frac{1}{\sqrt{2\pi}}}\right) e^{-\frac{(-\lambda+2e)^2}{2\sigma_n^2}} + \cancel{\frac{1}{2}}\left(\cancel{\frac{1}{\sqrt{2\pi}}}\right) e^{-\frac{(\lambda+e)^2}{2\sigma_{nu}^2}}$$

$$e^{-\frac{(-\lambda+2e)^2}{2\sigma_{nu}^2}} = e^{-\frac{(\lambda+e)^2}{2\sigma_{nu}^2}}$$

$$\frac{(-\lambda+2e)^2}{\cancel{2\sigma_{nu}^2}} = \frac{(\lambda+e)^2}{\cancel{2\sigma_{nu}^2}}$$

$$\cancel{\lambda^2} + 4e^2 - 4\lambda e = \cancel{\lambda^2} + e^2 + 2\lambda e$$

$$\lambda(-4e - 2e) = -4e^2 + e^2$$

$$-6e\lambda = -3e^2$$

$$\lambda = \frac{3e^2}{6e} = \frac{e}{2}$$

Come ci si doveva aspettare λ soglia ottima nel caso di simboli equiprobabili è equidistante dai simboli.

(d) l'integrale si risolve per parti

$$\int_0^{1/T} f^2 e^{-2T}\, df = f^2\left[-\frac{1}{2} e^{-2f}\right]_0^{1/T} - \int_0^{1/T} 2f\left[-\frac{1}{2} e^{-2f}\right] df =$$

$$= -\frac{f^2}{2} e^{-2f}\Big|_0^{1/T} + \int_0^{1/T} f e^{-2f}\, df =$$

$$= -\frac{f^2}{2} e^{-2f}\Big|_0^{1/T} + \left[-\frac{f}{2} e^{-2f}\right]_0^{1/T} + \int_0^{1/T} \frac{1}{2} e^{-2f}\, df =$$

$$= \left[-\frac{f^2}{2} e^{-2f}\right]_0^{1/T} + \left[-\frac{f}{2} e^{-2f}\right]_0^{1/T} - \frac{1}{4} e^{-2f}\Big|_0^{1/T} =$$

$$= -\frac{1}{2T^2} e^{-2/T} - \frac{1}{2T} e^{-2/T} - \frac{1}{4} e^{-2/T} + \frac{1}{4}$$