Probability Theory

Introduction

Two classes of mathematical models:

1. Deterministic
2. Probabilistic

A model is said to be deterministic if there is no uncertainty about its time-dependent behavior at any instant of time

- linear time-invariant systems are examples of a deterministic model.

In real-world problems, the use of a deterministic model is inappropriate:

- the underlying physical phenomenon involves too many unknown factors;
- a probabilistic model accounts for uncertainty in mathematical terms.

Introduction

Probabilistic models are needed for the design of systems that are

- reliable in performance in the face of uncertainty,
- efficient in computational terms, and
- cost effective in building them.

Consider a digital communication system over a wireless channel: subject to uncertainties, the sources of which include

1. noise, internally generated in electronic devices at the front-end of the TX/RX;
2. fading of the channel, due to the multipath phenomenon-an inherent characteristic of wireless channels;
3. interference from other transmitters.

To account for these uncertainties, we need a probabilistic model for channel.

Introduction

The objective of probability theory is twofold:

1. the mathematical description of probabilistic models and
2. the development of probabilistic reasoning procedures for handling uncertainty.

We begin the study of probability theory with a review of set theory:

- This is because probabilistic models assign probabilities to the collections (sets) of possible outcomes of random experiments.

Set Theory

The objects constituting a set are called the elements of the set. Let A be a set and x be an element of the set A. To describe this statement:

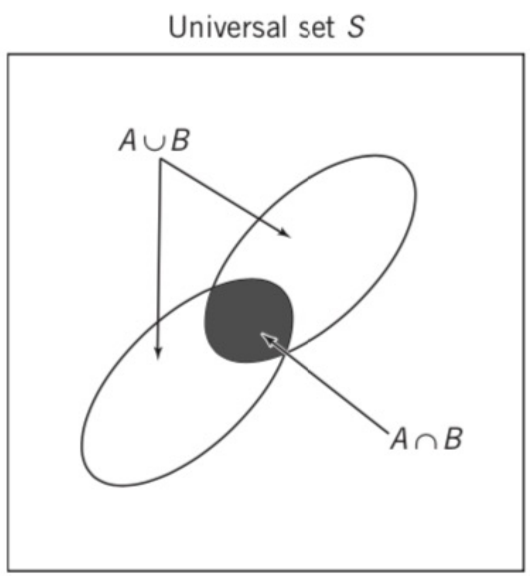
- We write $x \in A$; otherwise, we write $x \notin A$
- If the set A is empty, we denote it by $\emptyset$
- If $x_1, x_2, \ldots, x_N$ are all elements of the set A, then

$$A = \{x_1, x_2, \ldots, x_N\}$$

Boolean operation on sets

Unions and Intersections

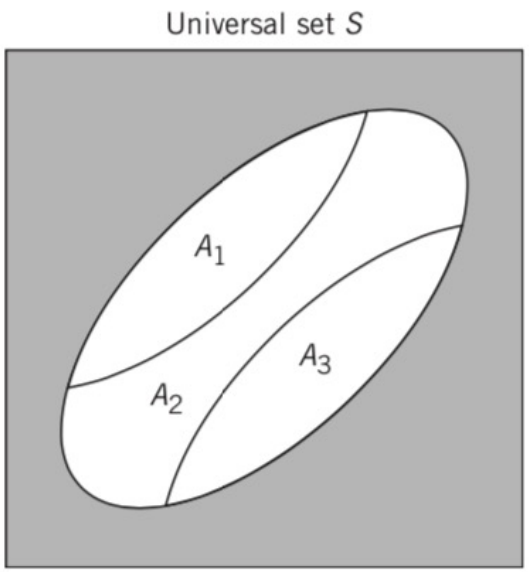
- The union $A \cup B$ of two sets A and B is defined by the set of elements that belong to A or B, or to both.
- The intersection $A \cap B$ of two sets A and B is defined by the particular set of elements that belong to both A and B.



Boolean operation on sets

Disjoint and Partition Sets

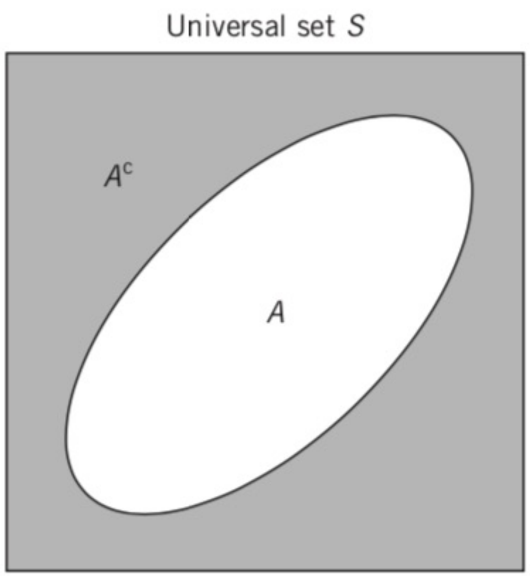
- Two sets A and B are said to be disjoint if their intersection is empty; that is, they have no common elements.
- The partition of a set $A = A_1 \cup A_2 \cup A_3$ refers to a collection of disjoint sets.



Boolean operation on sets

Complements

- The set A^c is said to be the complement of the set A, with respect to the universal set S, if it is made up of all the elements of S that do not belong to A



Algebra of Sets

1. *Idempotence property*

   $(A^c)^c = A$

2. *Commutative property*

   $$A \cup B = B \cup A$$
   $$A \cap B = B \cap A$$

3. *Associative property*

   $$A \cup (B \cup C) = (A \cup B) \cup C$$
   $$A \cap (B \cap C) = (A \cap B) \cap C$$

4. *Distributive property*

   $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
   $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

   Note that the commutative and associative properties apply to both the union and intersection, whereas the distributive property applies only to the intersection.

Probabilistic Models

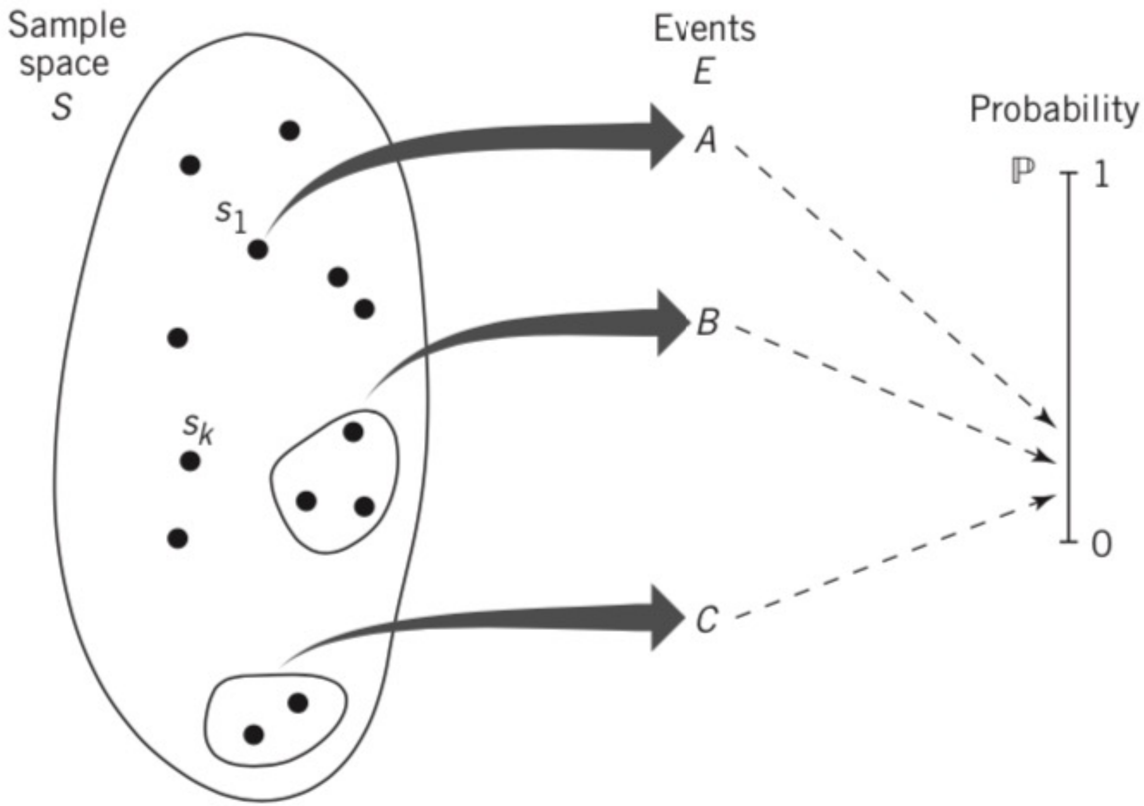
The mathematical description of an experiment with uncertain outcomes is called a probabilistic model, the formulation of which rests on three fundamental ingredients:

1. Sample space or universal set S, which is the set of all conceivable outcomes of a random experiment under study.
2. A class of events that are subsets of S.
3. Probability law, according to which a nonnegative measure P[A] is assigned to A.

The measure P[A] is called the probability of event A.

- Encodes our belief in the likelihood of event A occurring when the experiment is conducted.

Probabilistic Models



- An event may involve a single outcome or a subset of possible outcomes in S.

Axioms of Probability

1. **Axiom 1 – Non negativity** The probability of event A is a nonnegative number bounded by unity

$$O \leq \mathbb{P}[A] \leq 1 \qquad \text{for any event } A$$

2. **Axiom 2 – Additivity** The second axiom states that if A and B are two disjoint events, then the probability of their union satisfies the equality

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B]$$

3. **Axiom 3 Normalization** The third and final axiom states that the probability of the entire sample space S is equal to unity.

Axioms of Probability

These three axioms can be to develop some other basic properties of probability

- The probability of an impossible event is zero.
- $P[A^c] = 1 - P[A]$
- If event A lies within the subspace of another event B, then $P[A] \leq P[B]$
- If two events A and B are not disjoint, then $P[A \cup B] = P[A] + P[B] - P[A \cap B]$
- …

Axioms of Probability

- $P[A^c] = 1 - P[A]$

- If event A lies within the subspace of another event B, then $P[A] \leq P[B]$

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Conditional probability

Suppose we perform an experiment that involves a pair of events A and B.

- Let P[A|B] denote the probability of event A given that event B has occurred.
- The probability P[A|B] is called the conditional probability of A given B.

Assuming that B has nonzero probability, this is given by

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

Bayes' rule

Suppose

- the conditional probability P[A|B] and the individual probabilities P[A] and P[B] are all easily determined directly, but …
- the conditional probability P[B|A] is desired.

$$\mathbb{P}[B|A] = \frac{\mathbb{P}[A|B]\mathbb{P}[B]}{\mathbb{P}[A]}$$

Independence

Suppose

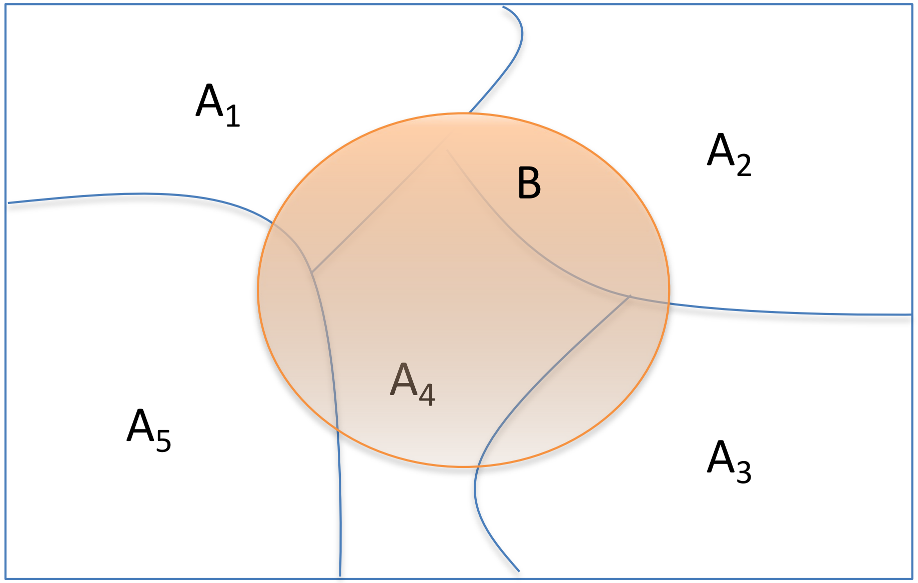
- the occurrence of event A provides no information whatsoever about event B; that is P[B|A] = P[B]
- From the Bayes's rule, we have that P[A|B] = P[A]
- The two events are independent and such that

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$$

If an event A is independent of another event B, then B is independent of A, and A and B are therefore independent events.

Law of total probability

Suppose $\{A_n; n=1,\ldots,N\}$ is a set of disjoint events



$$P[B] = \sum_{n=1}^{N} P[B \cap A_n]$$

Radar Detection Problem

In the radar detection problem, there are three probabilities of particular interest:

- $\mathbb{P}[A]$ probability that a target is present in the area; this probability is called the *prior probability*.
- $\mathbb{P}[B|A]$ probability that the radar receiver detects a target, given that a target is actually present in the area; this second probability is called the *probability of detection*.
- $\mathbb{P}[B|A^c]$ probability that the radar receiver detects a target in the area, given that there is no target in the surveillance area; this third probability is called the *probability of false alarm*.

Suppose these three probabilities have the following values:

$$\mathbb{P}[A] = 0.02$$
$$\mathbb{P}[B|A] = 0.99$$
$$\mathbb{P}[B|A^c] = 0.01$$

The problem is to calculate the conditional probability $\mathbb{P}[A|B]$ which defines the probability that a target is present in the surveillance area given that the radar receiver has made a target detection.

Communication Problem (Homework)

In a communication system,:

- 1 is transmitted with probability $p = 0.3$ and 0 with $1 - p = 0.7$.
- The error probability is $P_e = 0.01$

Assume that 0 has been received:

what is the probability that 0 has been effectively transmitted?

- Define A = { 0 has been transmitted}, A^c = { 1 has been transmitted},
- Define B = { 0 has been received}, B^c = {1 has been received}
- Compute $P[A|B] = 0.996$.