Stochastic Processes

Introduction

In simple terms:

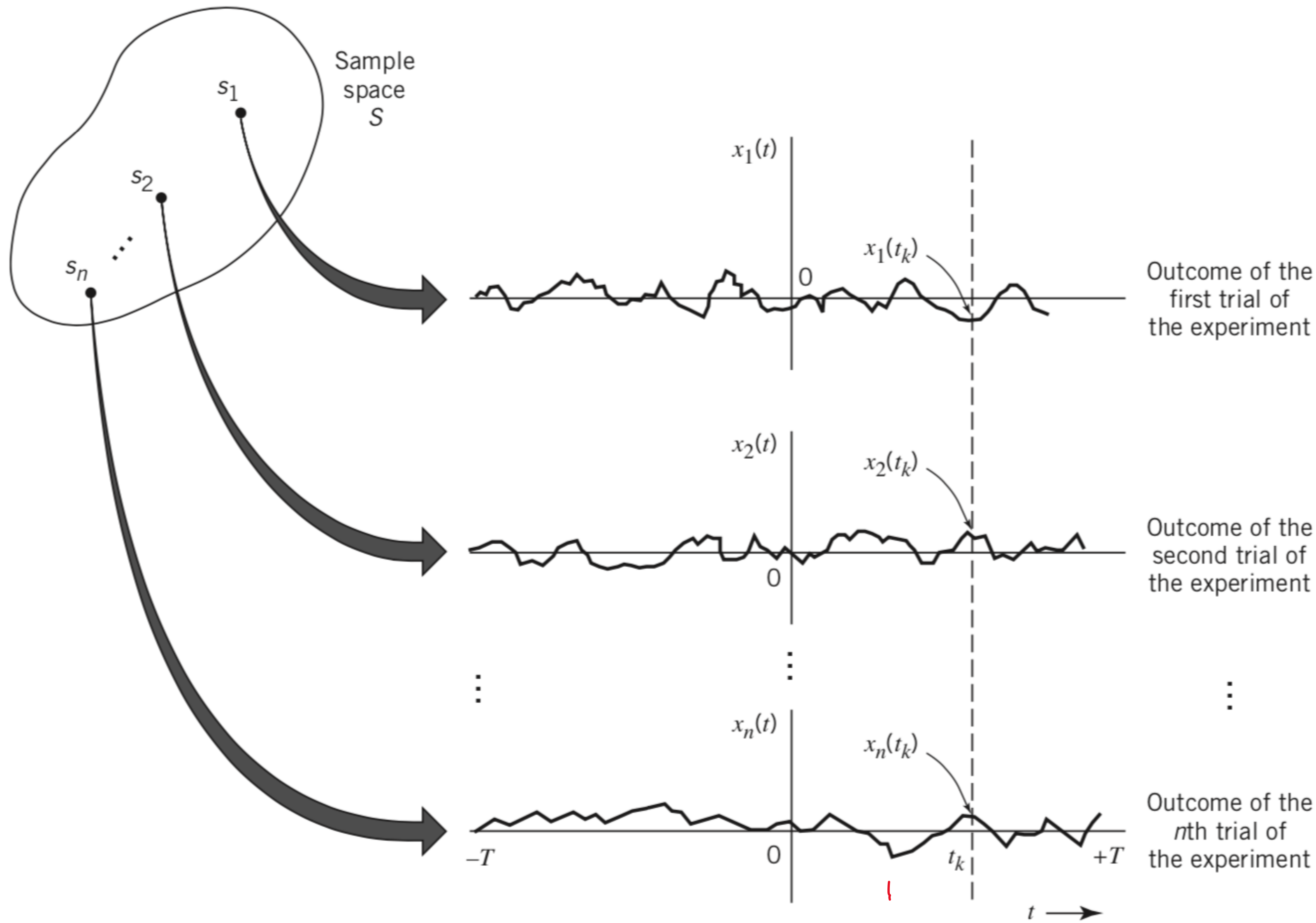
A stochastic process is a set of random variables indexed in time.

Properties:

- they are functions of time;
- they are random in the sense that, before conducting an experiment, it is not possible to define the waveforms that will be observed in the future exactly

- Although not being possible to predict a signal drawn from a stochastic process, it is possible to characterize the process in terms of statistical parameters such as average power, correlation functions, and power spectra.

Introduction

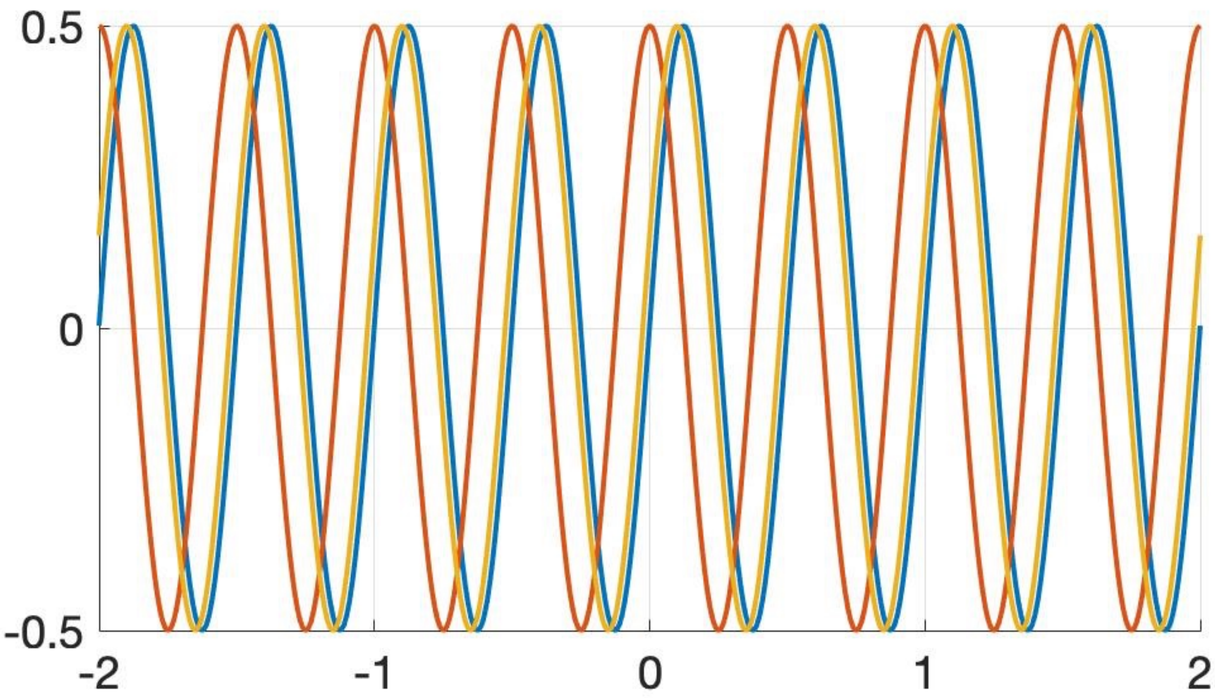


Example

Consider a sinusoidal signal with random phase

$$X(t) = A\cos(2\pi f_c t + \Theta) \qquad f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi \le \theta \le \pi \\ 0, & \text{elsewhere} \end{cases}$$

where A and f_c are constants and is a random variable that is uniformly distributed



Mean and Correlation

Consider a real-valued stochastic process X(t).

- The mean of X(t) is the expectation of the random variable obtained by sampling the process at some time t

$$\mu_X(t) = \mathbb{E}[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)\,\mathrm{d}x$$

- The autocorrelation function of X(t) is the expectation of the product of two random variables, $X(t_1)$ and $X(t_2)$

$$M_{XX}(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2$$

Weakly Stationary Processes

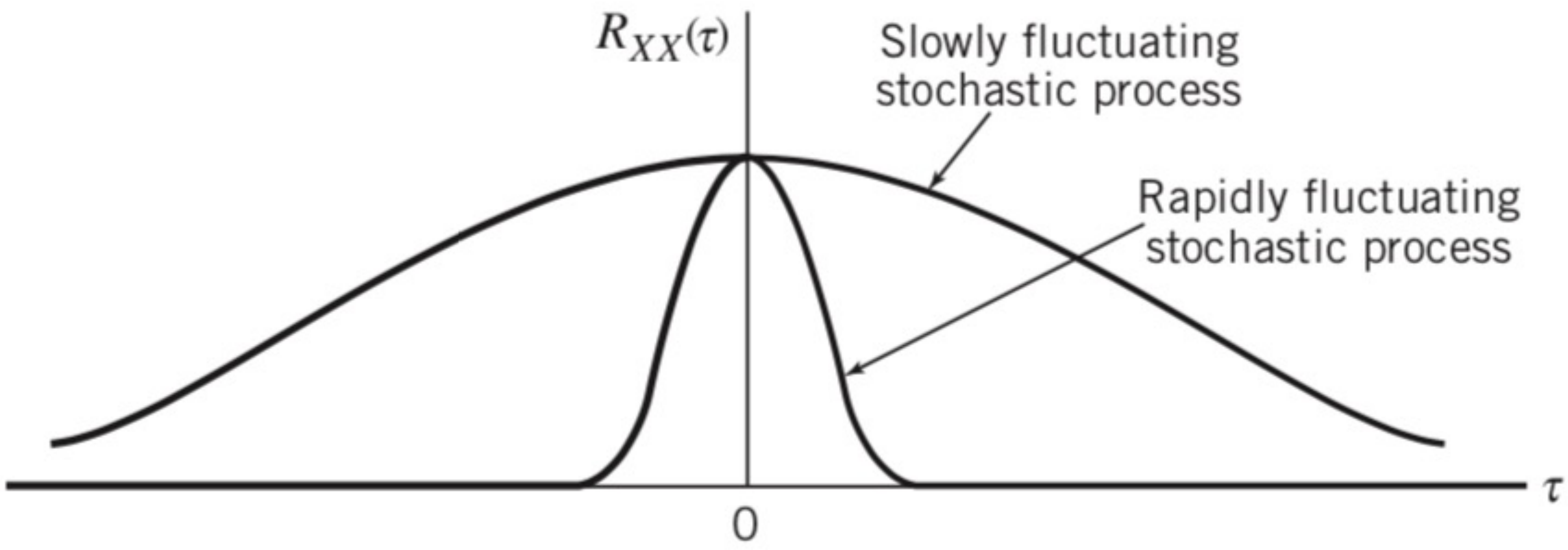
A stochastic process $X(t)$ is said to be weakly stationary if its second-order moments satisfy the following two conditions:

1. The mean of the process $X(t)$ is constant for all time t.

2. The autocorrelation function of the process $X(t)$ depends solely on the difference between any two times at which the process is sampled.

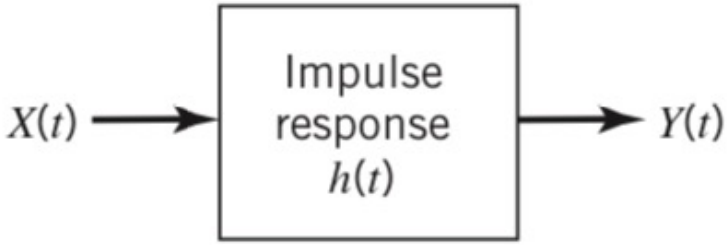
Physical Significance of Autocorrelation Function

- The autocorrelation function is significant because it provides a means of describing the interdependence of two random variables obtained by sampling the stochastic process X(t) at seconds apart.
- The more rapidly the stochastic process X(t) changes with time, the more rapidly will the autocorrelation function decrease from its maximum $R_{XX}(0)$



Transmission of a Weakly Stationary Process through a Linear Time-invariant Filter

- Suppose that a stochastic process X(t) is applied as input to a linear time-invariant filter of impulse response h(t), producing a new stochastic process Y(t)



- The transmission of a process through a linear time-invariant filter is governed by the convolution integral:

$$Y(t) = \int_{-\infty}^{\infty} h(\tau_1) X(t - \tau_1)\, d\tau_1$$

Mean and Correlation

- The mean of the stochastic process Y(t)

$$\mu_Y = \mu_X H(0)$$

- The autocorrelation function of the output stochastic process is

$$R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_{XX}(\tau + \tau_1 - \tau_2)\, d\tau_1\, d\tau_2$$

Power Spectral Density

- Suppose you want to characterize the output Y(t) by using frequency-domain ideas

$$\mathbb{E}[Y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 S_{XX}(f)\,\mathrm{d}f$$

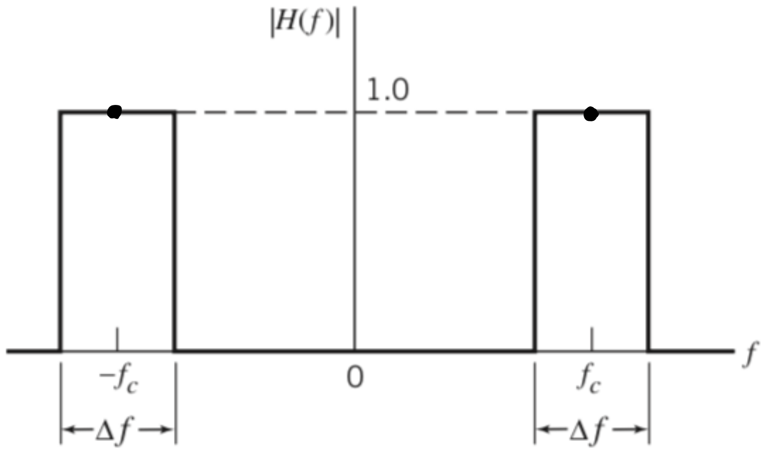
where $S_{XX}(f)$ is the power spectral density

$$S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau)\,\exp(-j2\pi f\tau)\,\mathrm{d}\tau$$

Why is it called power spectral density ?

Physical Significance of Power Spectral Density

- To investigate the physical significance of the power spectral density, suppose that the weakly stationary process X(t) is passed through an ideal narrowband filter



$$\mathbb{E}[Y^2(t)] \approx (2\Delta f)S_{XX}(f) \qquad \text{for all } f$$

$S_{XX}(f)$ represents the density of the average power of X(t), evaluated at frequency f.

The power spectral density is therefore measured in watts per hertz (W/Hz).

Relationship between the Power Spectral Densities of Input and Output

- Let $S_{YY}(f)$ denote the power spectral density of the output process Y(t)

$$S_{YY}(f) = |H(f)|^2 S_{XX}(f)$$

The power spectral density of Y(t) equals the power spectral density of the input process X(t), multiplied by the squared magnitude response of the filter.

The Gaussian process

- A process Y(t) is said to be a Gaussian process if the random variable Y is a Gaussian-distributed random variable.

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

- The stochastic processes produced by physical phenomena are often such that a Gaussian model is appropriate.
- The use of a Gaussian model to describe physical phenomena is often confirmed by experiments.
- Last, but by no means least, the central limit theorem provides mathematical justification for the Gaussian distribution.

White Noise

- This source of noise is idealized, in that its power spectral density is assumed to be constant and, therefore, independent of the operating frequency.

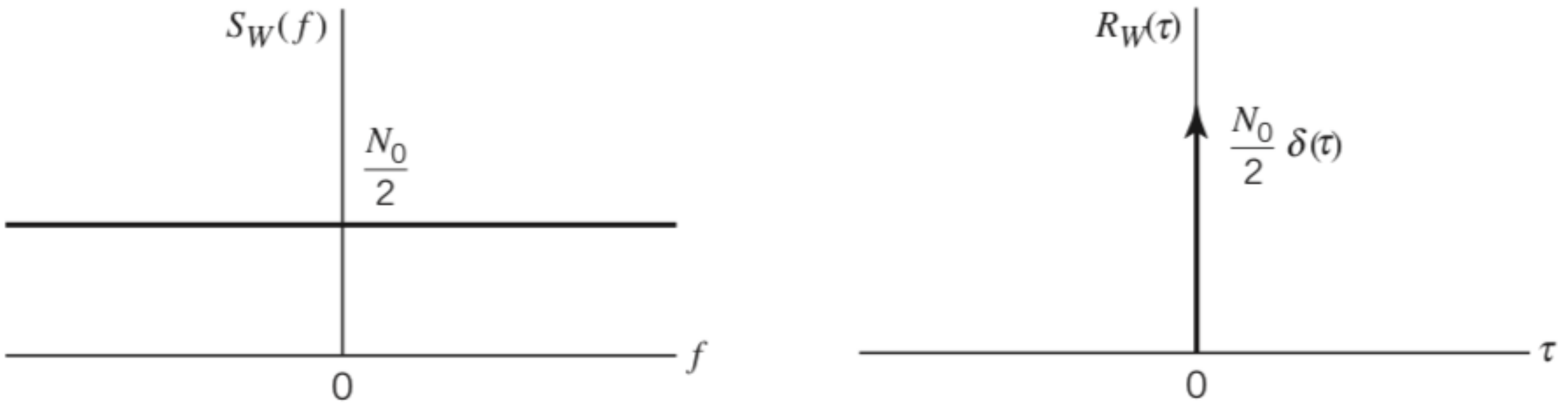
- The adjective "white" is used in the sense that white light contains equal amounts of all frequencies within the visible band of electromagnetic radiation.

- We may thus make the statement:

White noise, denoted by W(t), is a stationary process whose power spectral density $S_W(f)$ has a constant value across the entire frequency interval

$$S_{WW}(f) = \frac{N_0}{2} \qquad \text{for all } f$$

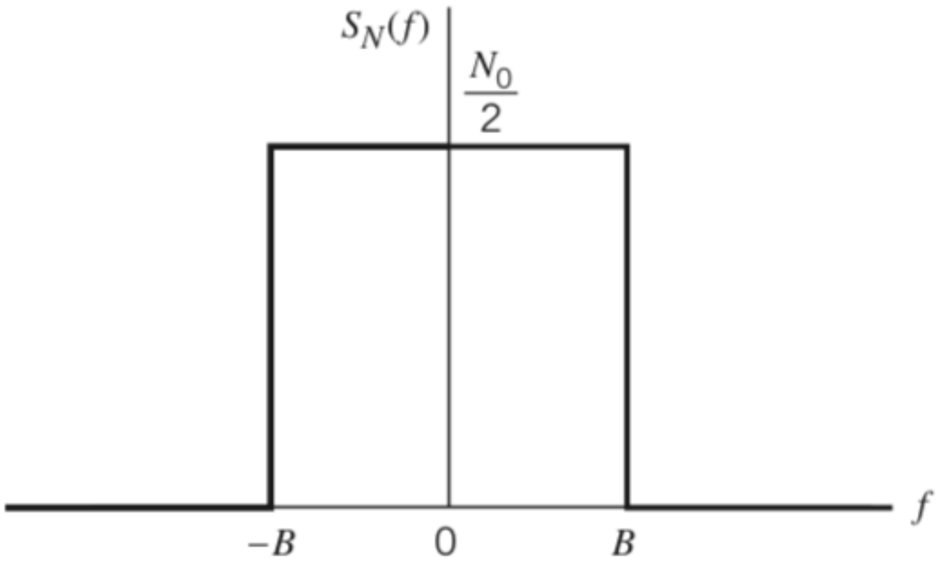
White Noise



- Since $R_W(\tau) = 0$ zero for $\tau \neq 0$, it follows that any two different samples of white noise are uncorrelated no matter how closely together in time.
- If the white noise is Gaussian, then the two samples are statistically independent
- As long as the bandwidth of a noise process at the input of a system is appreciably larger than the bandwidth of the system itself, then we may model the noise process as white noise.

Ideal Low-pass Filtered White Noise

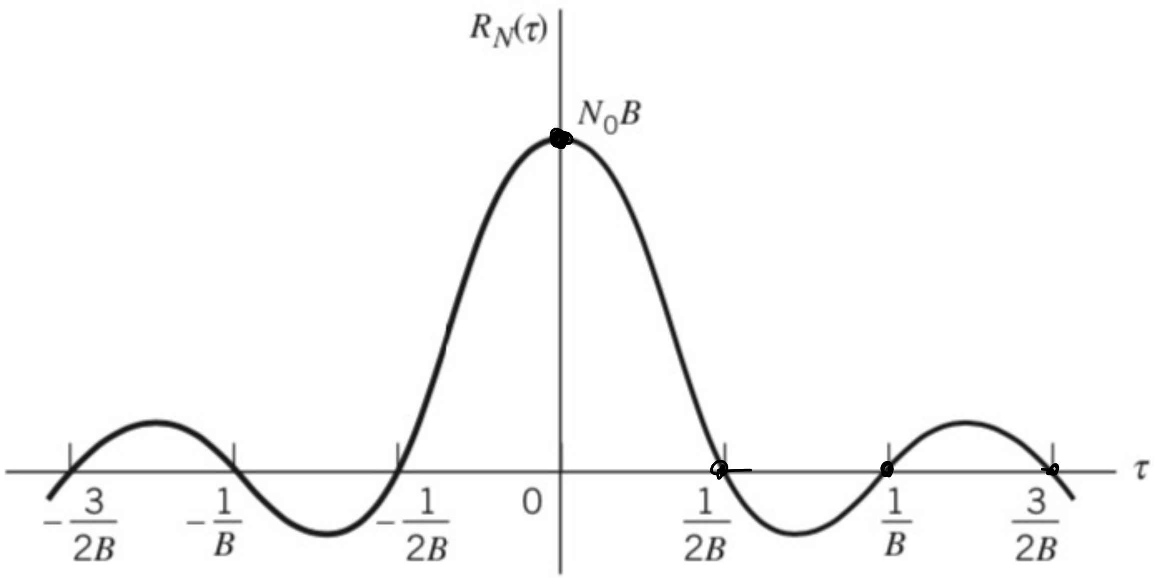
- Suppose that a white Gaussian noise of zero mean and power spectral density N0/2 is applied to an ideal low-pass filter of bandwidth B.
- The power spectral density of the ouput noise N(t) is:



Since the input noise W(t) is Gaussian (by hypothesis), it follows that the band-limited noise N(t) at the filter output is also Gaussian.

Ideal Low-pass Filtered White Noise

- Suppose, then, that N(t) is sampled at the rate of 2B times per second.
- The resulting noise samples are uncorrelated and, being Gaussian, they are statistically independent.



- Note that each such noise sample has a mean of zero and variance of N_0B.