591AA 21/22 – ELENCO DEI PROBLEMI 8

Problema 1. Quali dei seguenti sono sistemi lineari:

(a) $p \in \mathbb{R}[x]$ tali che $p(1) = 1$, $p(2) = 2$, $p(3) = 3$.
(b) $p \in \mathbb{R}[x]$ tale che $dp/dx + p(x) = x$.
(c) $p \in \mathbb{R}[x]$ tale che $(dp/dx)^2 + p(x) = x$

$$\frac{d}{dx} : \mathbb{R}[x] \to \mathbb{R}[x]$$

$$\frac{d}{dx}(p+q) + (p+q) = \left(\frac{dp}{dx} + p\right) + \left(\frac{dq}{dx} + q\right)$$

$$\frac{d}{dx}(cp) + (cp)(x) = c\frac{dp}{dx} + c\,p(x) = c\left(\frac{dp}{dx} + p\right)$$

$$\left(\frac{d}{dx}(p+q)\right)^2 = \left(\frac{dp}{dx} + \frac{dq}{dx}\right)^2 = \left(\frac{dp}{dx}\right)^2 + 2\frac{dp}{dx}\frac{dq}{dx} + \left(\frac{dq}{dx}\right)^2$$

Problema 2.

(a) Trova polinomi tali che $p(-1) = 1$, $p(0) = 0$, $p(1) = 1$.
(b) Trova polinomi tali che $p(x) + p(-x) = 1$.

(a) $S = \{-1, 0, 1\}$

$$x^2 + \tilde{p}(x)$$
$$x^2 + (x-(-1))(x-0)(x-1)\,q(x) = x^2 + (x^3 - x)\,q(x)$$

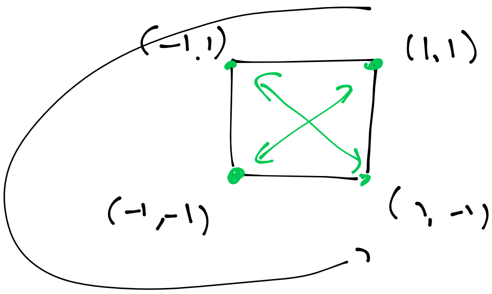
$$p_{-1}(x) = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{x(x-1)}{2}$$

$p_{-1}(-1) = 1$
$p_{-1}(0) = 0$
$p_{-1}(1) = 0$

$$p_1(x) = \frac{(x-(-1))(x-0)}{(1-(-1))(1-0)} = \frac{(x+1)x}{2}$$

$$p(x) = \frac{x(x-1)}{2} + \frac{x(x+1)}{2} = x^2$$

Problema 3. Sia $T = \{(1,1),(1,-1),(-1,1),(-1,-1)\}$. Trova tutte le funzioni da T a $\mathbb{R}$ tali che $f(x,y)+f(-x,-y) = xy$. [Qui stiamo usando il fatto che $(x,y) \in T \implies (-x,-y) \in T$].



$$z_1 = f(1,1)$$
$$z_2 = f(-1,1)$$
$$z_3 = f(-1,-1)$$
$$z_4 = f(1,-1)$$

$$\begin{cases} z_1 + z_3 = 1 \\ z_2 + z_4 = -1 \end{cases} \implies \begin{cases} z_1 = 1 - z_3 \\ z_2 = -1 - z_4 \end{cases}$$

$$\underset{f(1,1)}{\overset{\shortparallel}{z_1}} + \underset{f(-1,-1)}{\overset{\shortparallel}{z_3}} = (1)(1) = 1 \qquad \underset{f(-1,1)}{\overset{\shortparallel}{z_2}} + \underset{f(1,-1)}{\overset{\shortparallel}{z_4}} = (-1)(1)$$

Problema 4. Moltiplica le seguenti matrici e vettori.

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 1 \\ 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} (2)(1) + (1)(-1) \\ (5)(1) + (3)(-1) \end{pmatrix} = \begin{pmatrix} 2-1 \\ 5-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-2+1 \\ 1-4+9 \end{pmatrix}$$

Problema 5. Trasposizione di una matrice.

$$A = (a_{ij}) \implies A^t = (\alpha_{ij}), \quad \alpha_{ij} = a_{ji}$$

Per esempio

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}^t = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Trova tutte le matrici 2x2 tali che

$$A + A^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^t = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

$$A + A^t = \begin{pmatrix} a+d & b+c \\ c+b & a+d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\implies \begin{matrix} c+b = 0 \\ a+d = 1 \end{matrix} \implies \begin{matrix} a = 1-d \\ c = -b \end{matrix}$$

$$\left\{ \begin{pmatrix} 1-d & b \\ -b & d \end{pmatrix} \,\middle|\, b, d \in \mathbb{R} \right\}$$

Problema 6. Scrivi i seguenti sistemi lineari in forma matriciale.

(a) $x + y + z = 1$, $x - 2y + 3z = 2$.
(b) $x + 2y + 3z = 6$, $3x + y + 2z = 6$, $x + y + z = 3$.

(a)

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\left(\begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{matrix} \,\middle|\, \begin{matrix} 1 \\ 2 \end{matrix} \right)$$

(b)

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 3 \end{pmatrix}$$

$$\left(\begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & 1 & 1 \end{matrix} \,\middle|\, \begin{matrix} 6 \\ 3 \\ 3 \end{matrix} \right)$$

Problema 7. Scrivi le matrici aumentate per i sistemi lineari in Problema 6.

Problema 8. Trova i pivot delle seguenti matrici.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 3 & 4 \end{pmatrix}$$

Problema 9. Risolvi i seguenti sistemi linear

(a)

$$\begin{pmatrix} 1 & -3 & -2 & \mid 6 \\ 0 & 2 & 1 & \mid -4 \\ 0 & 0 & 7 & \mid 14 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \mid 0 \\ 0 & 0 & 1 & 1 & \mid 0 \end{pmatrix}$$