HW #6

Problema 1

(a)

$$(0,1,0) = (1,1,0) - (1,0,0)$$
$$(0,0,1) = (1,1,1) - (1,1,0)$$
$$\therefore\ (1,0,1) = (1,0,0) + \{(1,1,1) - (1,1,0)\}$$
$$\Longrightarrow L(1,0,1) = L(1,0,0) + L(1,1,1) - L(1,1,0)$$
$$2 = 1 + (3-2) = 2 \checkmark$$

$$\therefore\ L(1,0,0) = 1 \leftarrow L(1,1,0) - L(1,0,0)$$
$$L(0,1,0) = 2-1 = 1$$
$$L(0,0,1) = 3-2 = 1 \leftarrow L(1,1,1) - L(1,1,0)$$

$$\Longrightarrow L(x,y,z) = x+y+z$$

(b) ~~[illegible]~~

$$(1,1,1) - (0,1,1) = (1,0,0)$$
$$\Longrightarrow L(1,0,0) = L(1,1,1) - L(0,1,1) = 4-3 = 1$$
$$(1,1,1) - (1,0,1) = (0,1,0)$$
$$\Longrightarrow L(0,1,0) = L(1,1,1) - L(1,0,1) = 4-2 = 2$$
$$(0,0,1) = (1,1,1) - (1,1,0)$$
$$\Longrightarrow L(0,0,1) = L(1,1,1) - L(1,1,0) = 4-1 = 3$$
$$\Longrightarrow L(x,y,z) = x + 2y + 3z$$

Osserva che $L(1,1,1) = 1 + 2(1) + 3(1) = 6$

$\Longrightarrow$ Quindi non esiste una tale mappa lineare

(a) + (b): Metodo sistematico che utilizza l'eliminazione gaussiana

(a)
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 0 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(b)
$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right) \leftarrow \text{No soluzione}$$

Problema 3 $\quad 3x+4y+12z=0 \iff ((x,y,z),\underbrace{(3,4,12)}_{N})=0$

Abbiamo bisogno di un vettore unitario invece di N

$$u=\frac{N}{|N|}, \qquad |N|^2=3^2+4^2+12^2=25+144=169=13^2$$

Formula: $\pi(x,y,z) = (x,y,z)-((x,y,z),u)u$

$$= (x,y,z)-\left((x,y,z),\frac{N}{|N|}\right)\frac{N}{|N|}$$

$$= (x,y,z)-\frac{1}{|N|^2}((x,y,z),N)\,N$$

$$= (x,y,z)-\frac{1}{169}((x,y,z),(3,4,12))(3,4,12)$$

$$= (x,y,z)-\frac{1}{169}(3x+4y+12z)(3,4,12)$$

$$= (L_1(x),L_2(x),L_3(x))$$

Problema 4: Ricorda (lezione 6), la composizione di due mappe lineari $U\xrightarrow{L}V$, $V\xrightarrow{L'}W$ è una mappa lineare.

Quindi: Calcolare $(p_2\circ p_1)(1,0)$, $(p_2\circ p_1)(0,1)$

$$p_1(1,0)=(1,0)-2((1,0),(\cos\theta,\sin\theta))(\cos\theta,\sin\theta)$$

(La formula $|u|=1$, $p_u(v)=v-2(v,u)u$
$u=(\cos\theta,\sin\theta)\Rightarrow|u|=1$)

$$= (1,0)-2\cos\theta(\cos\theta,\sin\theta)$$

$$= (1-2\cos^2\theta,-2\sin\theta\cos\theta)$$

$$= (\cos^2\theta+\sin^2\theta-2\cos^2\theta,-2\sin\theta\cos\theta)$$

$$= (\sin^2\theta-\cos^2\theta,-2\sin\theta\cos\theta)$$

$$= (-\cos 2\theta,-\sin 2\theta)$$

$$\rho_1(0,1) = (0,1) - 2((0,1), (\cos\theta, \sin\theta))(\cos\theta, \sin\theta)$$
$$= (0,1) - 2\sin\theta(\cos\theta, \sin\theta)$$
$$= (-2\sin\theta\cos\theta,\ 1 - 2\sin^2\theta)$$
$$= (-\sin 2\theta,\ \cos^2\theta + \sin^2\theta - 2\sin^2\theta)$$
$$= (-\sin 2\theta,\ \cos^2\theta - \sin^2\theta) = (-\sin 2\theta,\ \cos 2\theta)$$

$$(\rho_2 \circ \rho_1)(1,0) = \rho_2(-\cos 2\theta, -\sin 2\theta)$$
$$= (-\cos 2\theta, -\sin 2\theta)$$
$$- 2((-\cos 2\theta, -\sin 2\theta), (1,0))(1,0)$$

(la formula $\rho_2(v) = v - 2(v,u)u$, $u = (1,0)$)

$$= (-\cos 2\theta, -\sin 2\theta) + 2\cos 2\theta\,(1,0)$$
$$= (\cos 2\theta, -\sin 2\theta)$$

$$(\rho_2 \circ \rho_1)(0,1) = \rho_2(-\sin 2\theta, \cos 2\theta)$$
$$= (-\sin 2\theta, \cos 2\theta) - 2((-\sin 2\theta, \cos 2\theta), (1,0))(1,0)$$
$$= (-\sin 2\theta, \cos 2\theta) + 2\sin 2\theta\,(1,0)$$
$$= (\sin 2\theta, \cos 2\theta)$$

$$\therefore \left.\begin{array}{l}(\rho_2\circ\rho_1)(1,0) = (\cos 2\theta, -\sin 2\theta) \\ (\rho_2\circ\rho_1)(0,1) = (\sin 2\theta, \cos 2\theta)\end{array}\right\} = L_{(-2\theta)} \quad !$$

$$L_\varphi = \left\{\begin{array}{l} L_\varphi(1,0) = (\cos\varphi, \sin\varphi) \\ L_\varphi(0,1) = (-\sin\varphi, \cos\varphi)\end{array}\right\}$$

per lezione 6

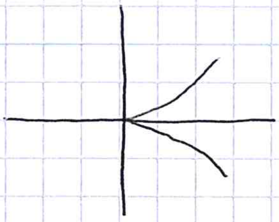
Problema 5

(a) $\frac{d}{dx}(x^3 - x) = 3x^2 - 1$

$\therefore \frac{d}{dx}(x^3 - x) = 0 \iff 3x^2 - 1 = 0 \iff x = \pm \frac{1}{\sqrt{3}}$

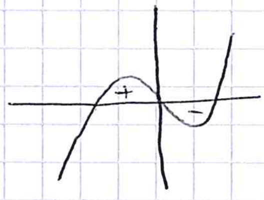
(b) $\frac{dr}{dt} = \frac{d}{dt}(t^2, t^3, 0) = (2t, 3t^2, 0)$

$\frac{dr}{dt}(0) = 0 \implies L(t) = r(0) + t\left(\frac{dr}{dt}\right)(0)$
$= 0 + t(0) = 0$



La curva non è derivabile a $t=0$

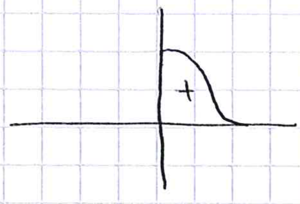
(c) $\int_{-1}^{1} (x^3 - x)\,dx = \frac{x^4}{4} - \frac{x^2}{2} \Big|_{-1}^{1} = 0$



$\int_{-1}^{1} (x^3 - x)\,dx = 0$ per simmetria

(d) $\int_0^1 (x^4 - 2x^2 + 1)\,dx = \frac{x^5}{5} - \frac{2}{3}x^3 + x \Big|_0^1$
$= \frac{1}{5} - \frac{2}{3} + 1 > 0$

$x^4 - 2x^2 + 1 = (x^2 - 1)^2 \geq 0$



$\int_0^1 (x^2-1)^2\,dx > 0.$

Problema 6

$U, V =$ spazi vettoriali

$L(U, V) = \{ L : U \to V \mid L$ è una mappa lineare$\}$

- $(L_1 + L_2)(u) = L_1(u) + L_2(u)$
- $(cL)(u) = cL(u)$

Passo 1: Verificare che due operazione mappino da $L(U,V)$ a $L(U,V)$

$$(L_1 + L_2)(u+v) = \underbrace{L_1(u+v) + L_2(u+v)}_{\text{definizione di } L_1+L_2}$$

$$= L_1(u) + L_1(v) + L_2(u) + L_2(v)$$

$$= (L_1+L_2)(u) + (L_1+L_2)(v)$$

$\therefore \Lambda = L_1 + L_2 \Rightarrow \Lambda(u+v) = \Lambda(u) + \Lambda(v)$

$$(cL)(u) = cL(u) = L(cu)$$

$\therefore \Lambda = L \Rightarrow c\Lambda(u) = \Lambda(cu)$

Passo 2: Verificare $L(U,V)$ soddisfi il resto degli assiomi

(1) $$\left.\begin{aligned}(L_1+L_2)(u) &= L_1(u) + L_2(u) \\ &= L_2(u) + L_1(u) = (L_2+L_1)(u)\end{aligned}\right\} =$$

(2) $$\left.\begin{aligned}((L_1+L_2)+L_3)(u) &= (L_1+L_2)(u) + L_3(u) \\ &= L_1(u) + L_2(u) + L_3(u) \\ (L_1+(L_2+L_3))(u) &= L_1(u) + (L_2+L_3)(u) \\ &= L_1(u) + L_2(u) + L_3(u)\end{aligned}\right\} =$$

(3) $0(u) := 0$ (definizione di 0)

$$\left.\begin{aligned}(0+L)(u) &= 0(u) + L(u) = L(u) \\ (L+0)(u) &= L(u) + 0(u) = L(u)\end{aligned}\right\} =$$

(4) $(-L)(u) := -L(u)$ (definizione di $-L$)

$$(-L+L)(u) = -L(u) + L(u) = 0 \Rightarrow -L + L = 0$$

⑤ $((rs)L)(u) = rsL(u)$

$(r(sL))(u) = r((sL)(u)) = rsL(u)$

⑥ $((r+s)L)(u) = (r+s)L(u) = rL(u) + sL(u)$

$(rL)(u) + (sL)(u) = rL(u) + sL(u)$

⑦ $(r(L_1+L_2))(u) = r(L_1+L_2)(u) = rL_1(u) + rL_2(u)$

$(rL_1)(u) + (rL_2)(u) = rL_1(u) + rL_2(u)$

⑧ $(1 \cdot L)(u) = 1L(u) = L(u)$