TRASFORMAZIONE DI UN VETTORE ALEATORIO

$\underline{X}$

$\underline{Y} = g(\underline{X}) \qquad \underline{x}_i = g^{-1}(\underline{y})$

$f_{\underline{Y}}(\underline{y})$?

$$f_{\underline{Y}}(\underline{y}) = \sum_i \frac{f_{\underline{X}}(\underline{x}_i)}{|\det(J(\underline{x}_i))|}$$

$$\underline{J}(\underline{x}_i) = \begin{bmatrix} \frac{\partial g_1(\underline{x})}{\partial x_1} & \frac{\partial g_1(\underline{x})}{\partial x_2} & \cdots & \frac{\partial g_1(\underline{x})}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_n(\underline{x})}{\partial x_1} & \frac{\partial g_2(\underline{x})}{\partial x_2} & \cdots & \frac{\partial g_n(\underline{x})}{\partial x_n} \end{bmatrix}$$

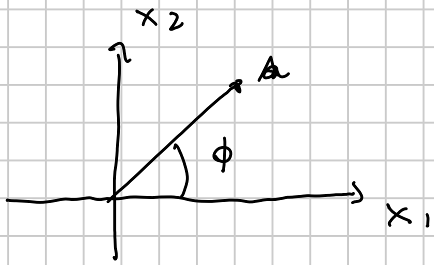
TRASF. DA 2 V.A. a 2 V.A.

$$f_{Y_1,Y_2}(y_1, y_2) = \sum_i \frac{f_{X_1,X_2}(x_{1i}, x_{2i})}{\left| \begin{vmatrix} \frac{\partial g_1(\underline{x})}{\partial x_1} & \frac{\partial g_1(\underline{x})}{\partial x_2} \\ \frac{\partial g_2(\underline{x})}{\partial x_2} & \frac{\partial g_2(\underline{x})}{\partial x_2} \end{vmatrix} \right|_{\underline{x} = \underline{x}_i}}$$

$\{ x_{1,i} \quad x_{2,i} \}$ soluzioni:

$\underline{x} = g^{-1}(\underline{y})$

ESEMPIO



$X_1 \, , \, X_2$ indipendenti

$X_1 \, , \, X_2 \in \mathcal{N}(0, \sigma^2)$

$$\begin{cases} A = \sqrt{x_1^2 + x_2^2} \\ \phi = tg^{-1}\left(\dfrac{x_2}{x_1}\right) \end{cases} \qquad \begin{cases} X_1 = A \cos \phi \\ X_2 = A \sin \phi \end{cases}$$

$$f_{A\phi}(a, \varphi) = \sum_i \frac{f_{x_1 x_2}(x_{1i}, x_{2i})}{\left| \det\left(J(x_{1i}, x_{2i}) \right) \right|}$$

$$\underline{\underline{J}} = \begin{bmatrix} \dfrac{\partial g_1(\underline{x})}{\partial x_1} & \dfrac{\partial g_1(\underline{x})}{\partial x_2} \\[2ex] \dfrac{\partial g_2(\underline{x})}{\partial x_1} & \dfrac{\partial g_2(\underline{x})}{\partial x_2} \end{bmatrix}$$

$$\frac{\partial g_1(\underline{x})}{\partial x_1} = \frac{\partial}{\partial x_1}\left(x_1^2 + x_2^2 \right)^{\frac{1}{2}} = \frac{1}{2}\left(x_1^2 + x_2^2 \right)^{-\frac{1}{2}} 2x_1 =$$

$$= \frac{x_1}{\sqrt{x_1^2 + x_2^2}}$$

$$\frac{\partial g_1(\underline{x})}{\partial x_2} = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}$$

$$\frac{\partial g_2(\underline{x})}{\partial x_1} = \frac{1}{1 + \frac{x_2^2}{x_1^2}} \cdot x_2 \left(-x_1 \right)^{-2}$$

$$\frac{\partial g_2(\underline{x})}{\partial x_2} = \frac{1}{1 + \frac{x_2^2}{x_1^2}} \cdot \frac{1}{x_1} = \frac{x_1}{x_1^2 + x_2^2}$$

$$\frac{\partial g_1(x_1,x_2)}{\partial x_1} = \frac{a \cos\varphi}{a} = \cos\varphi$$

$$\frac{\partial g_1(x_1,x_2)}{\partial x_2} = \frac{a \sin\varphi}{a} = \sin\varphi$$

$$\frac{\partial g_2(x_1,x_2)}{\partial x_1} = -\frac{a \sin\varphi}{a^2} = -\frac{\sin\varphi}{a}$$

$$\frac{\partial g_2(x_1,x_2)}{\partial x_2} = \frac{\cos\varphi}{a}$$

$$\underline{\underline{J}} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\frac{\sin\varphi}{a} & \frac{\cos\varphi}{a} \end{bmatrix}$$

$$\left| \det(\underline{\underline{J}}) \right| = \left| \frac{\cos^2\varphi}{a} + \frac{\sin^2\varphi}{a} \right| = \left| \frac{1}{a} \right| = \frac{1}{a}$$

$$f_{x_1 x_2}(x_1, x_2) = \frac{1}{\sqrt{(2\pi)^2 \sigma^2 \sigma^2}} e^{-\frac{x_1^2 + x_2^2}{2\sigma^2}}$$

$$f_{A\Phi} = \frac{a}{2\pi\sigma^2} e^{-\frac{a^2}{2\sigma^2}} u(a)$$

$$f_A(a) = \int_{-\infty}^{+\infty} f_{A\Phi}(a,\varphi)\, d\varphi = \int_{-\pi}^{\pi} \frac{a}{2\pi\sigma^2} e^{-\frac{a^2}{2\sigma^2}} u(a)\, d\varphi =$$

$$= \frac{a}{\sigma^2} e^{-\frac{a^2}{2\sigma^2}} u(a)$$

ddp. di RAYLEIGH

$$f_\phi(u) = \int_0^{+\infty} \frac{a}{2\pi\sigma^2} e^{-\frac{a^2}{2\sigma^2}} \, da =$$

$$a^2 = b$$

$$= \int_0^{+\infty} \frac{1}{4\pi\sigma^2} e^{-\frac{b}{2\sigma^2}} \, db = \qquad db = 2a\, da$$

$$= \frac{1}{4\pi\sigma^2} \left[-2\sigma^2 e^{-\frac{b}{2\sigma^2}} \right]_0^{+\infty} = -\frac{1}{2\pi} e^{-\frac{b}{2\sigma^2}} \Big|_0^{+\infty} =$$

$$= \frac{1}{2\pi} \operatorname{rect}\left(\frac{u}{2\pi}\right)$$

ESERCIZIO 1

$$X \in \mathcal{U}[0,1]$$

$$Y \in \mathcal{U}[0,1]$$

$$\begin{cases} Z = X - Y \\ V = X + Y \end{cases}$$

$$A = \{V \leq 1\}$$

$$f_{Z|A}(z|A) \quad ?$$

$$f_{Z|A}(z|A) \triangleq \frac{d}{dz} F_{Z|A}(z|A)$$

$$F_{Z|A}(z|A) \triangleq \frac{\Pr\{Z \leq z, A\}}{\Pr\{A\}}$$

$$f_{ZV}(z,v) = \sum_i \frac{f_{XY}(x_i, y_i)}{|\det(J(x_i, y_i))|}$$

x_i, y_i soluzioni del sistema inverso

$$\begin{cases} X = \frac{Z+V}{2} \\ Y = \frac{V-Z}{2} \end{cases}$$

$$\underline{\underline{S}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow |\det(S)| = 2$$

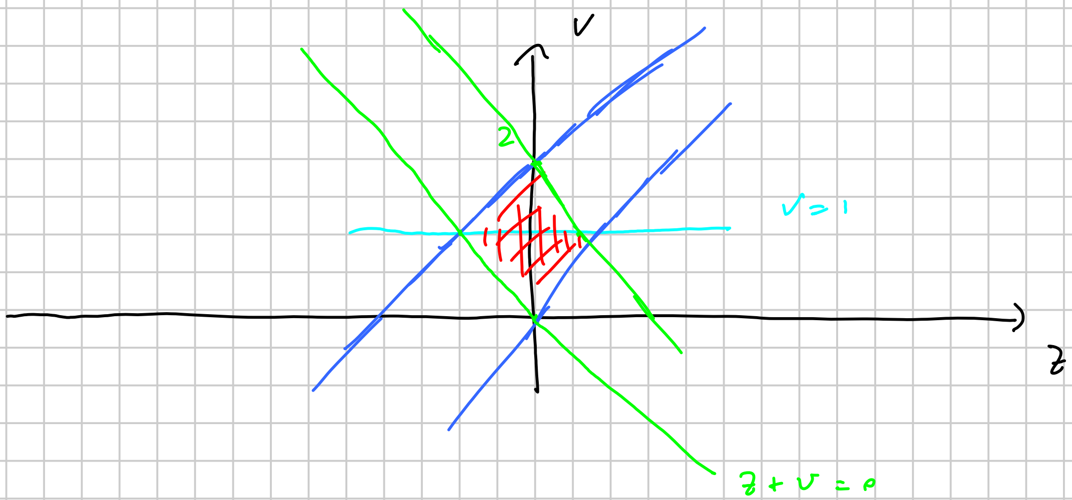
$$f_{XY}(x,y) \overset{\text{IND.}}{=} f_X(x) f_Y(y) = \text{rect}\left(\frac{x-\frac{1}{2}}{1}\right) \text{rect}\left(\frac{y-\frac{1}{2}}{1}\right)$$

$$f_{ZV}(z,v) = \frac{1}{2} \text{rect}\left(\frac{\frac{z+v}{2}-\frac{1}{2}}{1}\right) \text{rect}\left(\frac{\frac{v-z}{2}-\frac{1}{2}}{1}\right) =$$

$$= \frac{1}{2} \text{rect}\left(z + \frac{v}{2} - 1\right) \text{rect}\left(v - \frac{z}{2} - 1\right)$$

$$-1 \leq z + v - 1 \leq 1 \Rightarrow 0 \leq z + v \quad 2$$

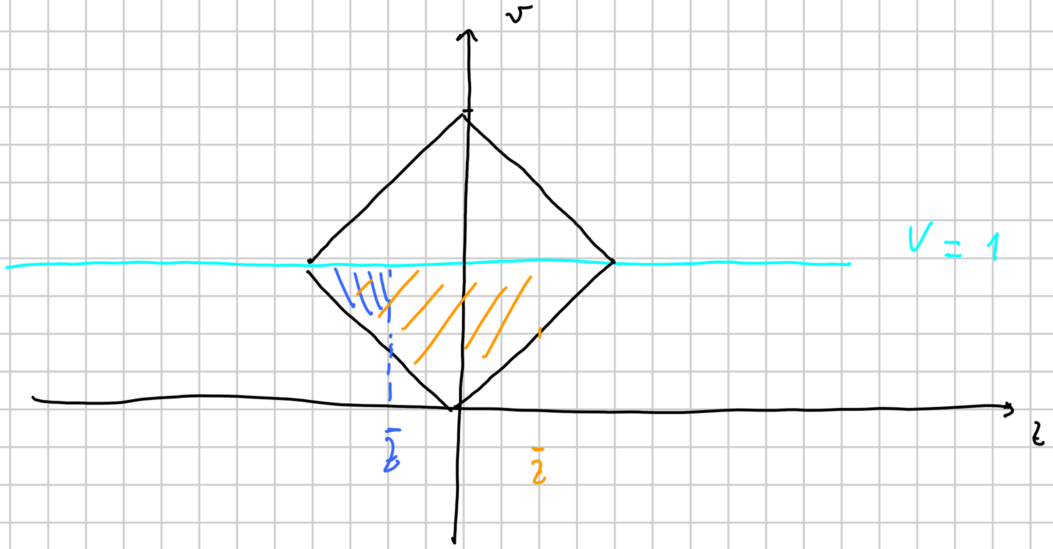
$$-1 \leq v - z - 1 \leq 1 \Rightarrow 0 \leq v - z \leq 2$$



$$P_r\{A\} = P_r\{V \leq 1\} = \int_{z=-\infty}^{+\infty} \int_{v=-\infty}^{1} f_{ZV}(z,v) = F_V(1) = \frac{1}{2}$$

$$P_r\{Z \leq z, A\} = \int_{-\infty}^{z} \int_{-\infty}^{1} f_{ZV}(z,v)\, dz\, dv =$$

$$= \int_{-\infty}^{z} \int_{-\infty}^{1} \frac{1}{2} \text{rect}\left(z + \frac{v}{2} - 1\right) \text{rect}\left(v - \frac{z}{2} - 1\right) dz\, dv =$$



$-1 \le z \le 0$

$$\Pr\{Z \le z, V \le 1\} = \left(\frac{1+z}{2}\right)^2 \cdot \frac{1}{2}$$

$0 \le z \le 1$

$$\Pr\{Z \le z, V \le 1\} =$$

$$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2} - \left(\frac{(1-z)^2}{2}\right)\right) = \frac{1}{2}\left[1 - \frac{(1-z)^2}{2}\right]$$

$$F_{Z|A}(z|A) = \begin{cases} \frac{(1+z)^2}{2} & -1 \le z \le 0 \\ 1 - \frac{(1-z)^2}{2} & 0 \le z \le 1 \end{cases}$$

$$f_{Z|A}(z|A) = \frac{d}{dz} F_{Z|A}(z|A) = \begin{cases} 1+z & -1 \le z \le 0 \\ 1-z & 0 \le z \le 1 \end{cases}$$

$$f_{Z|A}(z|A) = 1 - |z| \qquad -1 \le z \le 1$$

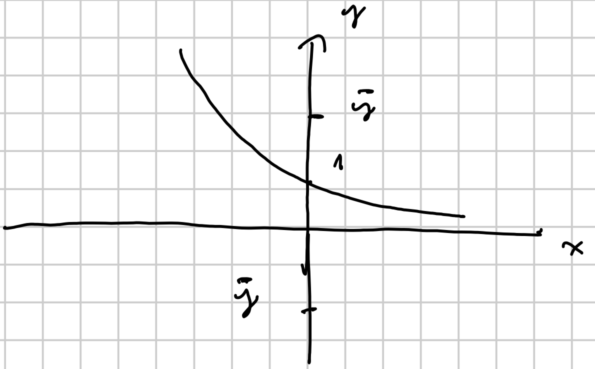
ESERCIZIO 1

$X \in \mathcal{N}(0,1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$Y = e^{-X}$ $\qquad f_Y(y)$? $\qquad \eta_Y$?

$$f_Y(y) = \sum_i \frac{f_X(x_i)}{|g'(x_i)|}$$



$\bar{y} \le 0 \Rightarrow f_Y(y) = 0$

$\bar{y} > 0$ $\exists$ 1 soluzione

$x_1 = -\ln(\bar{y})$

$$f_X(x_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\ln^2 y}{2}}$$

$$g'(x_1) = -e^{-x_1} = -y$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi y^2}} \exp\left\{-\frac{\ln^2 y}{2}\right\} u(y)$$

$$\eta_Y = E\{Y\} = \int_{-\infty}^{+\infty} y \, f_Y(y) \, dy =$$

$$= \int_{-\infty}^{+\infty} g(x) \, f_X(x) \, dx = \int_{-\infty}^{+\infty} e^{-x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx =$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \, e^{-\left(\frac{x^2}{2}+x\right)} \, dx =$$

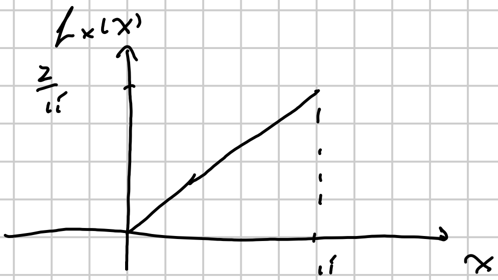
$$\frac{x^2}{2} + x = \frac{1}{2}\left((x+1)^2 - 1\right)$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \, e^{\frac{1}{2}} \, e^{-\frac{1}{2}(x+1)^2} \, dx =$$

$$= e^{+\frac{1}{2}} \underbrace{\underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}(x+1)^2} \, dx}_{\mathcal{N}(-1,\,1)}}_{1} = e^{+\frac{1}{2}}$$

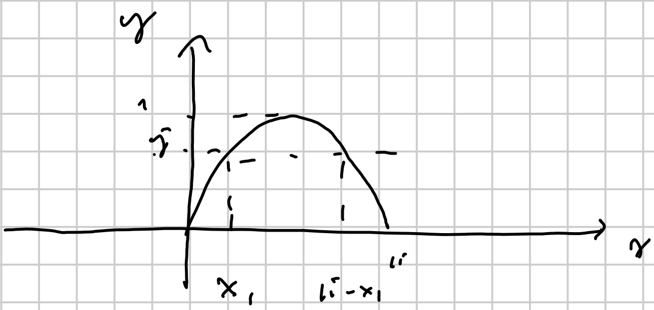
ESERCIZIO 2

X v.a. t.c. $\quad f_X(x) = \frac{2x}{\pi^2} \, \mathrm{rect}\left(\frac{x - \pi/2}{\pi}\right)$



$Y = \sin(X)$

$f_Y(y)$? $\quad \eta_Y$?



Se $\bar{y} > 1$ e $\bar{y} < 0$ no soluzioni, 5c $\quad 0 \leq y \leq 1$

$$X_1 = \operatorname{asin}(y)$$

$$x_2 = \pi - x_1$$

$$g(x) = \sin(x)$$

$$g'(x) = \cos(x)$$

$$g'(x_1) = \cos(x_1)$$

$$g'(x_2) = \cos(x_2) = \cos(\pi - x_1) = -\cos(x_1)$$

$$f_X(x_1) = \frac{2x_1}{\pi^2} \qquad f_X(x_2) = 2\frac{(\pi - x_1)}{\pi^2}$$

$$f_Y(y) = \frac{2x_1}{\pi^2} \cdot \frac{1}{\cos(x_1)} + \frac{2(\pi - x_1)}{\pi^2 \cos(x_1)} = \frac{2}{\pi \cos(x_1)} =$$

$$= \frac{2}{\pi \cos(\operatorname{asin}(y))} \cdot \operatorname{rect}\left(\frac{y - \frac{1}{2}}{1}\right)$$

$$\eta_Y = \int_{-\infty}^{+\infty} y\, f_Y(y)\, dy = \int_{-\infty}^{+\infty} g(x)\, f_X(x)\, dx =$$

$$= \int_0^1 y \cdot \frac{2}{\pi \cos(\operatorname{asin}(y))}\, dy =$$

$$z = \operatorname{asin}(y)$$

$$y = \sin(z)$$

$$dy = \cos z\, dz$$

$$y = 0 \Rightarrow z = 0$$

$$y = 1 \Rightarrow z = \frac{\pi}{2}$$

$$= \int_0^{\pi/2} \sin(z) \frac{2}{\pi \cos(z)} \cos z \, dz =$$

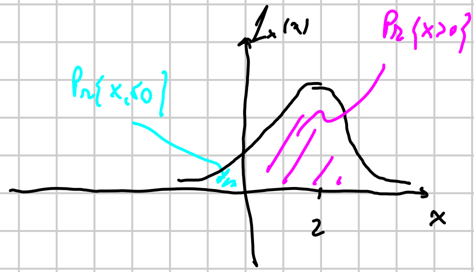
$$= \frac{2}{\pi} \int_0^{\pi/2} \sin(z) \, dz = -\frac{2}{\pi} \int_0^{\pi/2} -\sin z \, dz = -\frac{2}{\pi} \cos(z) \Big|_0^{\pi/2} =$$

$$= \frac{2}{\pi}$$

ESERCIZIO

$$X \sim \mathcal{N}(2, 2)$$

$$Y = \begin{cases} 1 & \text{se } X > 0 \\ -1 & \text{se } X \leq 0 \end{cases} \qquad \eta_Y \qquad \sigma^2_Y$$



$$\eta_Y = 1 \, P_1 - 1 \, P_{-1}$$

$$P_1 = Pr\{Y = 1\} \qquad P_{-1} = Pr\{Y = -1\}$$

$$P_1 = \int_0^{+\infty} f_X(x) \, dx = Pr\{X > 0\}$$

$$\eta_Y = \int_{-\infty}^{+\infty} g(x) \, f_X(x) \, dx$$

$$\hookrightarrow \quad g(x) = \operatorname{sgn}(x)$$

$$= -\int_{-\infty}^{0} f_X(x) \, dx + \int_0^{+\infty} f_X(x) \, dx$$

$$\int_{-\infty}^{0} f_X(x)\,dx = \phi\left(\frac{0-2}{\sqrt{2}}\right) = \phi\left(-\frac{2}{\sqrt{2}}\right)$$

$$\int_{0}^{+\infty} f_X(x)\,dx = 1 - \phi\left(-\frac{2}{\sqrt{2}}\right)$$

$$\eta_Y = -\phi\left(-\frac{2}{\sqrt{2}}\right) + 1 - \phi\left(-\frac{2}{\sqrt{2}}\right) = 1 - 2\phi\left(-\frac{2}{\sqrt{2}}\right)$$

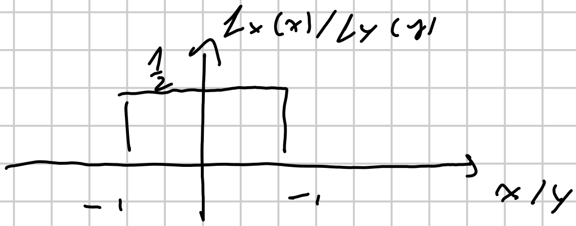
$$\sigma_Y^2 = E\{(y-\eta_Y)^2\} = \int_{-\infty}^{+\infty} (y-\eta_Y)^2 f_Y(y)\,dy =$$

$$= (1-\eta_Y)^2 P_1 + (-1-\eta_Y)^2 P_{-1}$$

ESERCIZIO

$X \in \mathcal{U}(-1, 1)$

$Y \in \mathcal{U}(-1, 1)$



$\alpha^2 + 2X\alpha + Y = 0$ abbia radici reali

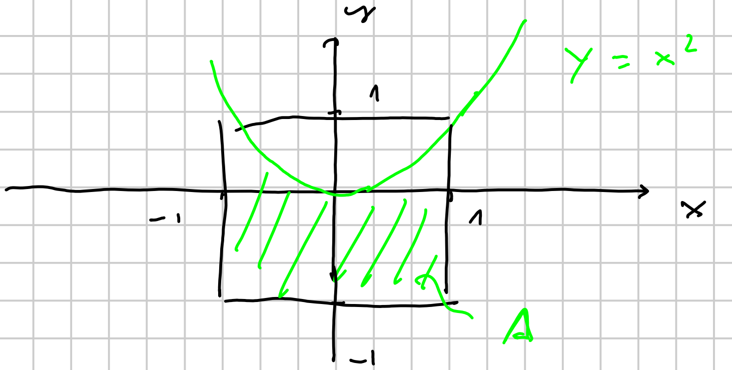
$\Delta = b^2 - 4ac = \underline{4X^2 - 4Y} \geq 0$

$X^2 - Y \geq 0$

$Y \leq X^2$

$Pr\{Y \leq X^2\}$?

$$f_{XY}(x,y) \overset{IND.}{=} f_X(x) \cdot f_Y(y)$$



$$f_{XY}(x,y) = \frac{1}{4}\, \mathrm{rect}\left(\frac{x}{2}\right) \mathrm{rect}\left(\frac{y}{2}\right)$$

$$P_r\{Y \le x^2\} = \iint_A f_{XY}(x,y)\,dx\,dy =$$

$$= \frac{1}{4} \iint_A dx\,dy = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{x^2} dx\,dy = \frac{1}{4} \int_{-1}^{1} y \Big|_{-1}^{x^2} dx =$$

$$= \frac{1}{4} \int_{-1}^{1} (x^2+1)\,dx = \frac{1}{2} \int_{0}^{1} (x^2+1)\,dx =$$

$$= \frac{1}{2} \left(\frac{X^3}{3} + x\right)\Big|_0^1 = \frac{1}{2} + \left(\frac{1}{3} + 1\right) = \frac{2}{3}$$

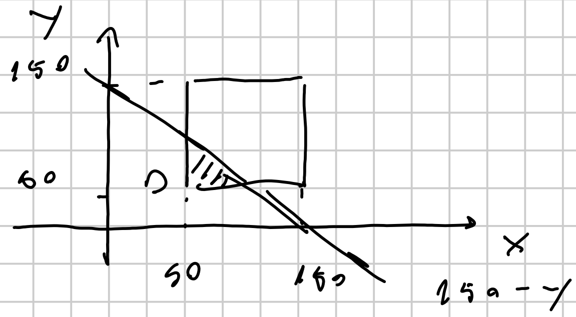
ESERCIZIO

$X, Y \in \mathcal{U}[50 : 150]$ km/h

P_r { CHE I 2 TRENI SI INCONTRINO DOPO 6 ORE }

$$900 \geq 6X + 6Y \Rightarrow X + Y \le 150$$

$$X \le 150 - Y$$



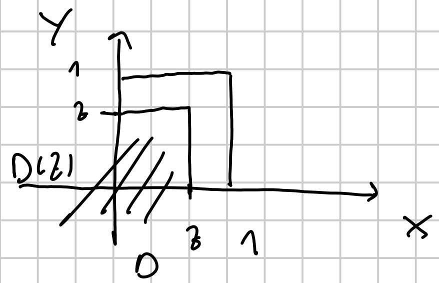
$$Pr\{X \le 150 - Y\} = \iint_D f_{XY}(x,y)\,dx\,dy =$$

$$= \frac{1}{100^2}\iint_D dx\,dy = \frac{1}{100^2}\,\frac{50^2}{2} = 0{,}125 = 12{,}5\%$$

ESERCIZIO 1

$X, Y \in \mathcal{U}(0,1)$

$Z = \max(x,y)$ $\qquad f_Z(z)$

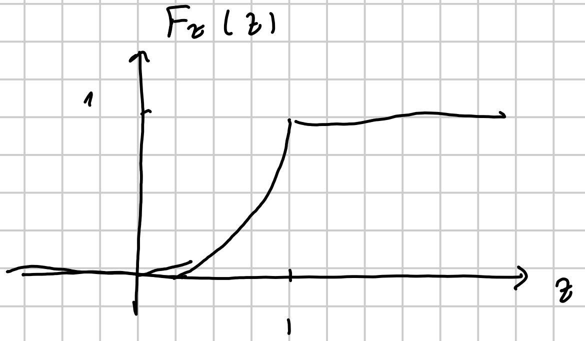


$$F_Z(z) = Pr\{Z \le z\} =$$

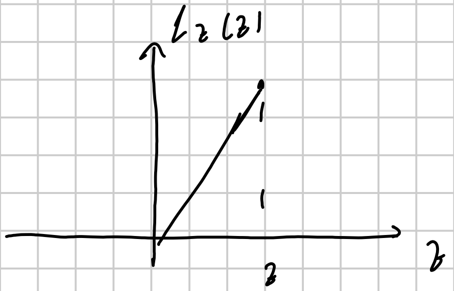
$$= Pr\{\max(x,y) \le z\} =$$

$$= \iint_{D(z)} f_{XY}(x,y)\,dx\,dy =$$

$$= \begin{cases} 0 & \text{se } z < 0 \\ z^2 & \text{se } 0 \le z < 1 \\ 1 & \text{se } z \ge 1 \end{cases}$$



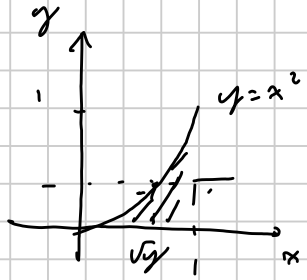
$$f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} 2z & se\ 0 \le z \le 1 \\ 0 & altrimenti \end{cases}$$



ESERCIZIO

X, Y v.a

$$f_{XY}(x,y) = \begin{cases} A\,x & (x,y) \in D \\ 0 & altrove \end{cases}$$



A ? $\quad f_X(x), \; f_Y(y)$?

$f_{X|Y}(x|y)$?

X e Y INDIPENDENTI ?

. A)

$$\iint_D f_{XY}(x,y)\, dx\, dy = 1 \qquad D = \{(x,y) : 0 \le x \le 1, \; 0 \le y \le x^2\}$$

$$\int_0^1 \int_0^{x^2} A x \, dx \, dy = 1$$

$$A \int_0^1 x x^2 \, dx = A \int_0^1 x^3 \, dx = A \left. \frac{x^4}{4} \right|_0^1 = \frac{A}{4} \Rightarrow \boxed{A = 4}$$

$$f_x(x) = \int_{-\infty}^{+\infty} f_{xy}(x,y) \, dy = \int_0^{x^2} 4x \, dy = 4x^3$$

$$0 < x < 1$$

$$= 4x^3 \operatorname{rect}\left(\frac{x - \frac{1}{2}}{1} \right)$$

$$f_y(y) = \int_{-\infty}^{+\infty} f_{xy}(x,y) \, dx \int_{\sqrt{y}}^1 4x \, dx = \cancel{4}^{2} \frac{x^2}{\cancel{2}} \Bigg|_{\sqrt{y}}^1 = 2 - 2y$$

NON SONO IND. PERCHE' $\boxed{f_{xy}(x,y) \neq f_x(x) f_y(y)}$

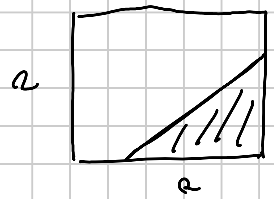
$$f_{x|y}(x|y) = \frac{f_{xy}(x,y)}{f_y(y)} = \frac{4x}{2(1-y)} \operatorname{rect}\left(\frac{x - \frac{1}{2}}{1} \right)$$

SE VALE IND. $f_{x|y}(x|y) = f_x(x)$

$$P_r\{ X < 0,5 \mid Y = 0,5 \} = P_r\{B\}$$

$$f_{x|y}(x|y=0,5) = 4x \qquad P_r\{B\} = \int_0^{\frac{1}{2}} 4x \, dx = 4 \frac{x^2}{2} \Bigg|_0^{\frac{1}{2}} = \frac{1}{2}$$

ESERCIZIO 1



$$A_T < \frac{1}{8} A_Q$$

$$A_Q = a^2 \Rightarrow Pr\left\{ A_T < \frac{a^2}{8} \right\}$$

X, Y ind $\in \mathcal{U}[0, a]$

$X = a X_1$ $\qquad X_1, Y_1 \in \mathcal{U}[0,1]$

$Y = a Y_1$

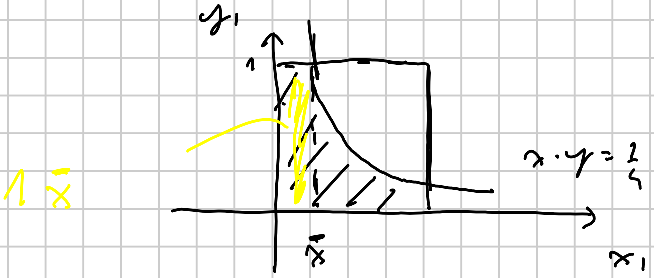
$$A_T = \frac{XY}{2} = a^2 \frac{X_1 Y_1}{2}$$

$$Pr\left\{ A_T < \frac{1}{8} A_Q \right\} = Pr\left\{ \cancel{a^2} \frac{X_1 Y_1}{2} < \frac{1}{8} \cancel{a^2} \right\} =$$

$$= Pr\left\{ \frac{X_1 Y_1}{2} < \frac{1}{8} \right\} = Pr\left\{ \underbrace{X_1 Y_1}_{Z} < \frac{1}{4} \right\}$$

$$Pr\left\{ Z < \frac{1}{4} \right\} = F_Z\left(\frac{1}{4}\right)$$

$$f_{X_1 Y_1}(x_1, y_1) = \mathrm{rect}\left(\frac{x_1 - \frac{1}{2}}{1}\right) \mathrm{rect}\left(\frac{y_1 - \frac{1}{2}}{1}\right)$$



$$\bar{y} = 1 \Rightarrow \bar{x} = \frac{1}{4}$$

$$A = 1\,\bar{x} + \int_{\bar{x}}^{1} \frac{1}{4x}\,dx = \bar{x} + \frac{1}{4} \ln(x) \Big|_{\bar{x}}^{1} =$$

$$= \bar{x} + \frac{1}{4}\left[\ln(1) - \ln(\bar{x}) \right]$$

$$P_r\left\{ Z \leq \frac{1}{4} \right\} = P_r\left\{ A_r < \frac{1}{8} A_e \right\} = \frac{1}{4}\left[1 - \ln\left(\frac{1}{4}\right) \right]$$