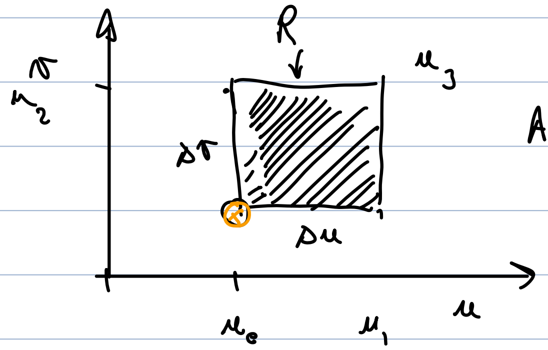
$$\phi : \Omega' \longrightarrow \Omega$$
$$\Omega' \subset \mathbb{R}^2_{u,v} \qquad \Omega \subset \mathbb{R}^2_{x,y}$$
$$\mathbb{R}^n \qquad \mathbb{R}^n$$

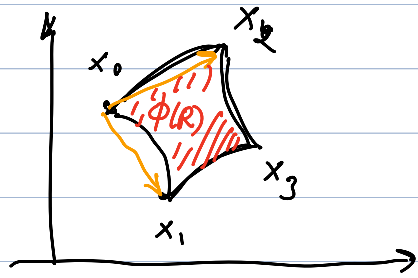
$$\int_{\Omega} f(x,y)\,dx\,dy = \int_{\Omega'} f(\varphi(u,v),\psi(u,v))\,|\det J\phi|\,du\,dv$$

$$x = \varphi(u,v) \qquad \Phi(u,v) = \begin{pmatrix} x \\ y \end{pmatrix}$$
$$y = \psi(u,v)$$




$$A = \Delta u\,\Delta v = \|(u_1-u_0)\overset{\times}{\wedge}(u_2-u_0)\|$$

$$\Phi$$




$$x_i = \phi(u_i)$$

$$x_1 - x_0 = \phi(u_1) - \phi(u_0) = \frac{\partial \phi}{\partial u}(u_0,v_0)(u_1-u_0) + o(\ )$$

$$x_2 - x_0 = \phi(u_2) - \phi(u_0) = \frac{\partial \phi}{\partial v}(u_0,v_0)(u_2-u_0) + o(\ )$$

$$A(\phi(R)) \approx \|(x_1-x_0)\wedge(x_2-x_0)\| = \left\| \frac{\partial \phi}{\partial u}\Delta u \wedge \frac{\partial \phi}{\partial v}\Delta v \right\|$$

$$= \left\| \frac{\partial \phi}{\partial u} \wedge \frac{\partial \phi}{\partial v} \right\| \Delta u\,\Delta v$$

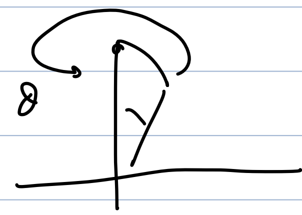
$$u = (u_1, u_2, 0) \qquad u \wedge v = \det \begin{vmatrix} i & j & k \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{vmatrix} = k\,(u_1 v_2 - v_1 u_2)$$
$$v = (v_1, v_2, 0)$$

$$\frac{\partial \phi}{\partial u} \wedge \frac{\partial \phi}{\partial v} \qquad \begin{pmatrix} \frac{\partial \phi_1}{\partial u} & \frac{\partial \phi_2}{\partial u} \\ \frac{\partial \phi_1}{\partial v} & \frac{\partial \phi_2}{\partial v} \end{pmatrix} \qquad \frac{\partial \phi_1}{\partial u}\frac{\partial \phi_2}{\partial v} - \frac{\partial \phi_1}{\partial v}\frac{\partial \phi_2}{\partial u}$$

$$\begin{pmatrix} \frac{\partial \phi_1}{\partial u} & \frac{\partial \phi_2}{\partial u} \\ \frac{\partial \phi_1}{\partial v} & \frac{\partial \phi_2}{\partial v} \end{pmatrix}$$

$m\,(B(0,R))$ $\qquad$ $B(0,R) = \{ x \in \mathbb{R}^3 : \|x\| \leq R \}$

$$\begin{cases} x = \rho \sin(\varphi) \cos(\theta) \\ y = \rho \sin(\varphi) \sin(\theta) \\ z = \rho \cos(\varphi) \end{cases} \qquad \begin{array}{l} \varphi \in [0,\pi] \\ \theta \in [0, 2\pi] \\ \rho \in [0,R] \end{array}$$



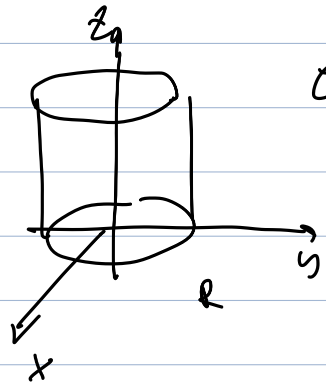

$$\det J = \rho^2 \sin(\varphi)$$

$$\mathrm{Vol}\,(B(0,R)) = \int_{B(0,R)} 1\, dx\,dy\,dz = \int_0^{\pi} d\varphi \int_0^{2\pi} d\theta \int_0^R \rho^2 \sin(\varphi)\, d\rho$$

$$= 2\pi \left( \int_0^{\pi} \sin \varphi \right) \left( \int_0^R \rho^2 d\rho \right)$$

$$= 2\pi \left( -\cos(\varphi) \Big|_0^{\pi} \right) \left( \frac{\rho^3}{3} \Big|_0^R \right)$$

$$= 2\pi\,(1+1)\,\frac{R^3}{3} = \frac{4\pi R^3}{3}$$




$$\mathcal{C} = \{ 0 \leq x^2+y^2 \leq R^2 \quad z \in [0,h] \} \subseteq \mathbb{R}^3$$

$$\begin{cases} x = \rho \cos \theta \\ y = \rho \sin \theta \\ z = t \end{cases} \qquad \begin{array}{l} \rho \in [0,R] \\ \theta \in [0,2\pi] \\ t \in [0,h] \end{array}$$

$$m(\mathcal{C}) = \mathrm{Vol}(\mathcal{C}) = \int_{\mathcal{C}} 1\, dx\,dy\,dz = \int_0^h dt \int_0^{2\pi} d\theta \int_0^R d\rho\, \rho$$

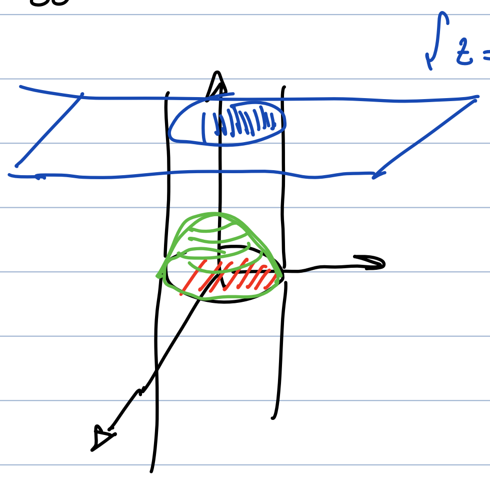
$$J = \begin{pmatrix} \cos\theta & -\rho \sin \theta & 0 \\ \sin \theta & \rho \cos \theta & 0 \end{pmatrix} = \qquad \det J = \rho$$

$0 \quad 0 \quad \boxed{1}$

$$h \, 2\pi \left( \frac{\rho^2}{2} \Big|_0^R \right) = h \pi \cdot \cancel{2} \frac{R^2}{\cancel{2}}$$

$$= \pi R^2 h$$

$$\int_\Omega (x^2+y^2)\,dx\,dy\,dz \qquad \Omega \subseteq \mathbb{R}^3$$




a) $\Omega \subseteq \{x^2+y^2 \le 1\} \subset \mathbb{R}^3$

cilindro infinito retto di base $0 < x^2+y^2 \le 1 \quad z=0$

b) $\Omega$ sta sotto a $\{z=3\}$

c) $\Omega$ sta sopra $x^2+y^2+z=1$

$z = 1-(x^2+y^2) = 1-\rho^2$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = t \end{cases}$$

$$\boxed{\begin{array}{l} \rho \in [0,1] \\ \theta \in [0,2\pi] \\ t \in [1-\rho^2, 3] \end{array}}$$

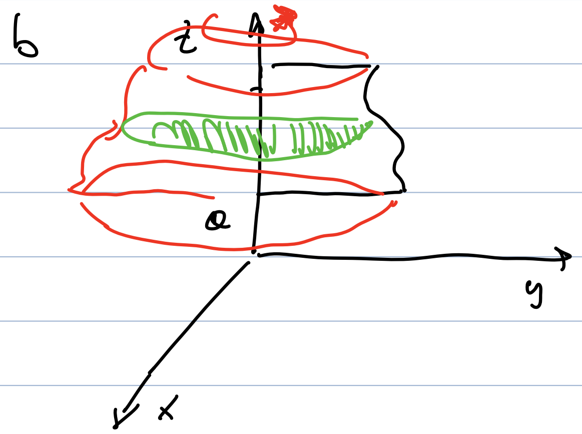
$$\int_0^{2\pi} d\theta \int_0^1 d\rho \int_{1-\rho^2}^{3} \underbrace{\rho^2}_{f} \underbrace{\rho}_{\det J}\, dt = 2\pi \int_0^1 d\rho \int_{1-\rho^2}^{3} \rho^3\, dt$$

$$= 2\pi \int_0^1 d\rho\, \rho^3 \left( t \Big|_{1-\rho^2}^{3} \right) = 2\pi \int_0^1 d\rho\, \rho^3 \left[ 3-(1-\rho^2) \right]$$

$$= 2\pi \int_0^1 d\rho\, \rho^3 (2+\rho^2) = 2\pi \int_0^1 d\rho\, (2\rho^3+\rho^5) = 2\pi \left( \frac{\cancel{2}}{\cancel{4}}\rho^4 + \frac{\rho^6}{6} \Big|_0^1 \right)$$

$$= 2\pi \left( \frac{1}{2} + \frac{1}{6} \right)$$

# VOLUME DEI SOLIDI DI ROTAZIONE




$$y = f(z) \qquad z \in [a,b]$$

$$\int_{\Omega} 1 \, dx\,dy\,dz = \int_a^b dz \int_{A_z} dx\,dy$$

$$= \int_a^b dz \quad m(A_z)$$

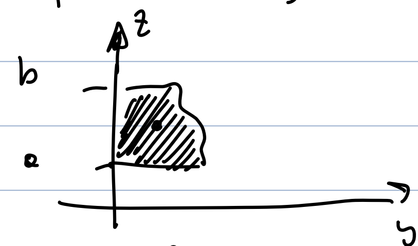
$$= \int_a^b dz \quad \pi \, (f(z))^2 = \pi \int_a^b (f(z))^2 \, dz$$

TEOREMA (PAPPO - GULDINO)

VOLUME DI SOLIDO DI ROTAZIONE È UGUALE AL PRODOTTO TRA AREA SEZIONE MERIDIANA PER LA LUNGHEZZA DELLA CIRCONFERENZA DESCRITTA DAL BARICENTRO ATTORNO ALL' ASSE z.

$$\bar{y} = \frac{\int_T y \, dy\,dz}{\int_T 1 \, dy\,dz} \qquad T = \{ \; y \in [0, f(z)] \quad z \in [a,b] \}$$

$$= \frac{\int_T y \, dy\,dz}{m(T)}$$




$$= \frac{\int_a^b dz \int_0^{f(z)} y \, dy}{m(T)} = \frac{\int_a^b dz \left. \frac{y^2}{2} \right|_0^{f(z)}}{m(T)} = \frac{\int_a^b \frac{(f(z))^2}{2} \, dz}{m(T)}$$

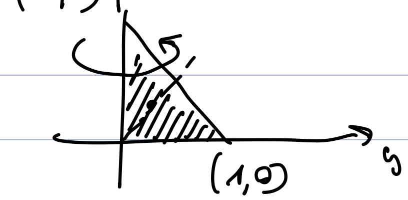
$$Vol(\Omega) = \pi \int_a^b (f(z))^2 = 2\pi \, m(T) \; \overbrace{\frac{\int_a^b (f(z))^2 \, dz}{2\, m(T)}}^{= \bar{y}}$$

$$= 2\pi \bar{y} \; m(T)$$

LUNGHEZZA CIRCONFERENZA DESCRITTA DA BARICENTRO

AREA SEZIONE MERIDIANA

$(0,1)$ q z

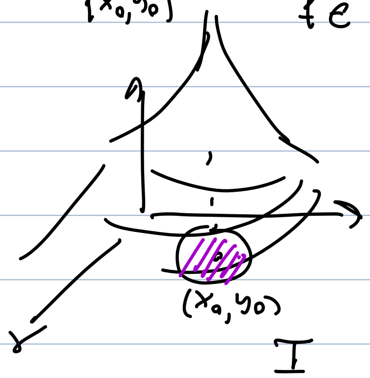



$\bar{x} = \bar{y} = \frac{1}{3}$

---

$f : \mathbb{R}^2 \setminus \{x_0, y_0\} \to \mathbb{R}$ supporto compatto $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = +\infty$

$f \in C(\mathbb{R}^2 \setminus \{x_0, y_0\})$




$\int_{\mathbb{R}^2} f(x,y)\,dx\,dy \qquad f \geq 0$

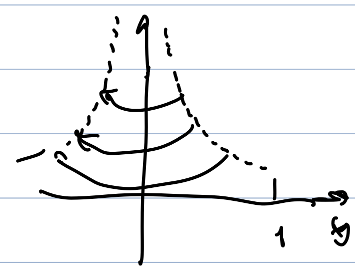
$$I_\varepsilon = \int_{\mathbb{R}^2 \setminus B(x_0, y_0, \varepsilon)} f(x,y)\,dx\,dy \xrightarrow{\varepsilon \to 0} I \in \mathbb{R}$$

$$B(x_0, y_0, \varepsilon) = \{(x,y) : (x-x_0)^2 + (y-y_0)^2 < \varepsilon^2\}$$

$$I \overset{\text{def}}{=} \int_{\mathbb{R}^2} f(x,y)\,dx\,dy$$

$f \geq 0 \implies I_\varepsilon = \int_{\mathbb{R}^2 \setminus B(\,,\varepsilon)} f \downarrow$ ammette limite essendo monotone

$$f = \begin{cases} \frac{1}{\|x\|^\beta} & 0 < \|x\| \leq 1 \\ 0 & \|x\| > 1 \end{cases}$$




$\beta > 0 \qquad \int_{\mathbb{R}^2} f(x,y)\,dx\,dy = \int_{B(0,1)} f(x,y)\,dx\,dy$ NON ESISTE

$f$ non è limitata intorno $0$ $\quad x = 0$

$$\lim_{\varepsilon \to 0^+} \left( \int_{B(0,1) \setminus B(0,\varepsilon)} f(x,y)\,dx\,dy \right) \qquad \begin{matrix} \theta \\ \rho \end{matrix}$$

$2\pi \quad 1 \quad 1 \quad \left( \quad 2-\beta \quad 1 \right)$

$$\int_0 d\vartheta \int_\varepsilon d\rho \ \frac{1}{\rho^\beta}\rho = 2\pi \int_\varepsilon \rho^{1-\beta} d\rho = 2\pi \left( \frac{1}{2-\beta} \rho^{2-\beta} \Big|_\varepsilon \right)$$

$$= \frac{2\pi}{2-\beta} \left(1 - \varepsilon^{2-\beta}\right)$$

$\mathbb{R}^2$

$$\lim_{\varepsilon \to 0^+} \frac{2\pi}{2-\beta}\left(1-\varepsilon^{2-\beta}\right) \quad \& \quad \beta < 2 \quad \lim_{\varepsilon\to 0} (\ ) = \frac{2\pi}{2-\beta}$$

$$\beta \geq 2 \quad \lim_{\varepsilon \to 0} (\ ) = +\infty$$

$$\beta = 2 \qquad 2\pi \int_\varepsilon^1 \frac{1}{\rho} d\rho = 2\pi \left( \ln \rho \Big|_\varepsilon^1 \right) = 2\pi (\ln 1 - \ln \varepsilon) \to +\infty$$

$\beta \geq 0$

$$\int_{B(0,1)} \frac{1}{\|x\|^\beta} dx_1 \dots dx_n < +\infty \iff 0 \leq \beta < n$$

$$n = 1 \qquad \int_0^1 \frac{1}{x^\beta} dx < +\infty \qquad \beta < 1 \qquad \int_1^\infty \frac{1}{x^\alpha} dx < +\infty \qquad \alpha > 1$$

$$n = 2 \qquad \int_{B(0,1)} \frac{1}{\|x\|^\beta} dx_1 dx_2 < +\infty \qquad \beta < 2 \qquad \int_{\mathbb{R}^2 \setminus B(0,1)} \frac{1}{\|x\|^\alpha} dx_1 dx_2 < +\infty \qquad \alpha > 2$$

$$\int_{\mathbb{R}^2 \setminus B(0,1)} \frac{|\sin(xy)|}{\sqrt{(x^2+y^2)^3}} dx dy$$

$$\leq f = \frac{|\sin(xy)|}{\sqrt{(x^2+y^2)^3}} \leq \frac{1}{\|x\|^{2\cdot 3}}$$

$$\text{Volume} \quad \left\{ 0 \leq \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\} = E \qquad \int_E 1 \, dx dy dz = \mathrm{Vol}(E)$$

$$z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

$$\mathrm{Vol}(E) = 2c \int_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dx\, dy$$

$$\int_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dx\, dy$$

$x \rightarrow \frac{x}{a}$

$y \rightarrow \frac{y}{b}$

$z \rightarrow \frac{z}{c}$

$(\rho, \theta) \longrightarrow \Phi$

$$\begin{cases} x = a\rho\cos\theta \\ y = b\rho\sin\theta \end{cases}$$

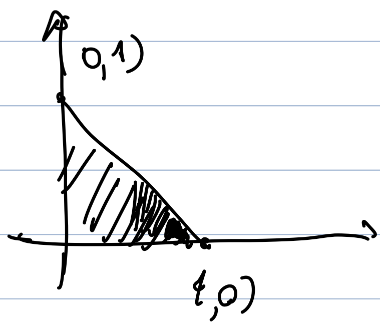
$\frac{x}{a} = \rho\cos\theta$

$\frac{y}{b} = \rho\sin\theta$

$$\int_0^{2\pi} d\theta \int_0^1 \sqrt{1-\rho^2}\, ab\rho\, d\rho$$

$\rightarrow \boxed{\frac{4\pi}{3} abc}$ VOLUME ELLISSOIDE

$\boxed{\frac{4\pi}{3} R^3}$ VOLUME SFERA




$k > 0$

$$\rho(x,y) = k(1+x) \qquad T = \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1-x \end{cases}$$

$$m(T) = \frac{1}{2}$$

$$\bar{x} = \frac{\int_T x\rho(x,y)\,dx\,dy}{\int_T \rho(x,y)\,dx\,dy} = \frac{\int_0^1 dx \int_0^{1-x} x\rho(x,y)\,dy}{\int_0^1 dx \int_0^{1-x} \rho(x,y)\,dy} = \frac{\int_0^1 dx\; x\,k(1+x)\int_0^{1-x} dy}{\int_0^1 dx\; k(1+x)\int_0^{1-x} dy}$$

massa totale

$$= \frac{\int_0^1 dx\; x\,k(1+x)(1-x)}{\int_0^1 dx\; k(1+x)(1-x)}$$

$$\bar{y} = \frac{\int_0^1 dx \int_0^{1-x} y\,\rho(x,y)\,dy}{(\quad)}$$

$$= \frac{k\int_0^1 (1-x^2)\cdot x}{k\int_0^1 k(1-x^2)} =$$

$$M\bar{x} = \frac{1}{4}$$

$$M\bar{y} = \frac{1}{6}$$