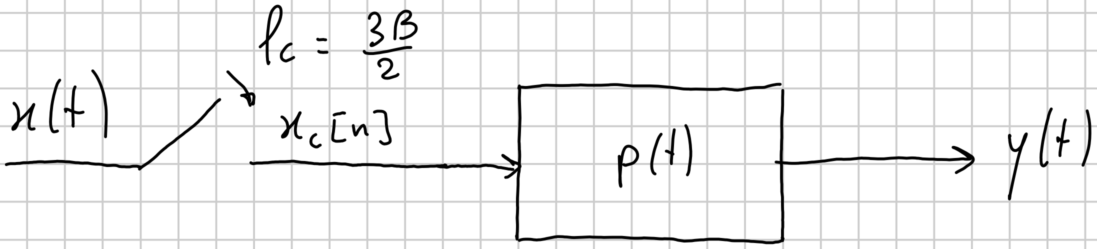
ES. 1



$p(t)$ è un interp. cardinale di banda B

$x(t) = B \, \mathrm{sinc}^2(Bt)$

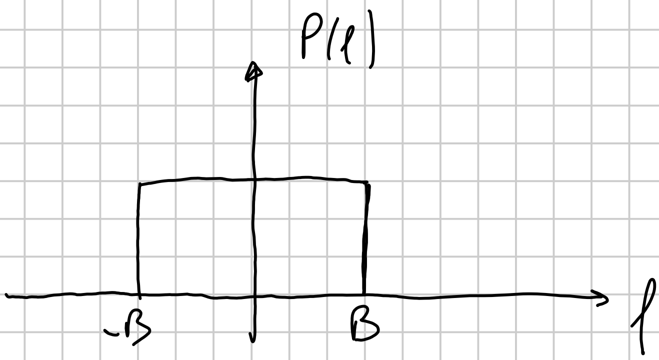
a) $y(t) = ?$

b) $E_y, P_y = ?$

c) $x_c[n]$, $\bar{X}_c(f)$ quando $f_c = B$

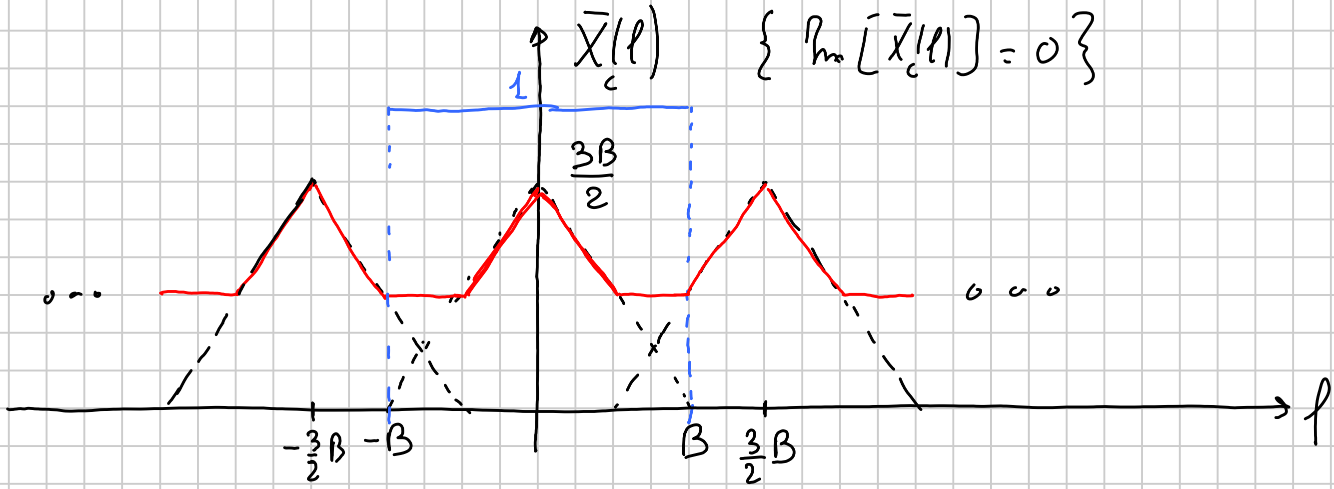
SOLUZIONE

$$P(f) = \mathrm{rect}\left(\frac{f}{2B}\right)$$

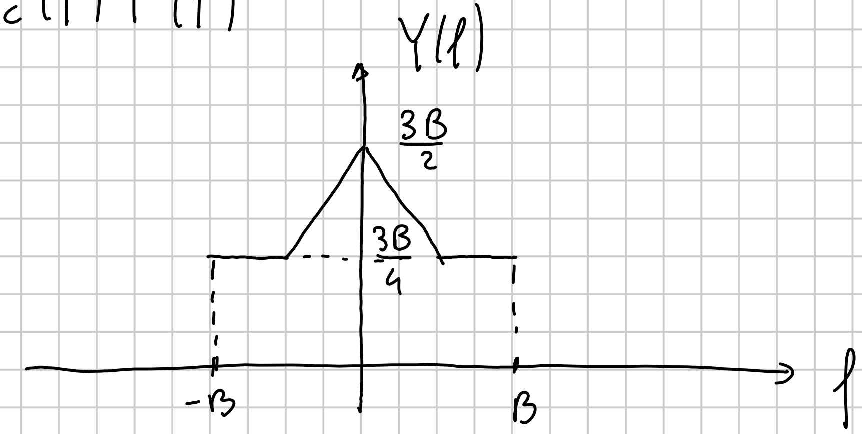


$$X(f) = \left(1 - \frac{|f|}{B}\right) \mathrm{rect}\left(\frac{f}{2B}\right)$$

$$\bar{X}_c(f) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} X\left(f - \frac{n}{T}\right) \quad , \quad T = \frac{1}{f_c} = \frac{2}{3B}$$



$$Y(f) = \bar{X}_c(f)\,P(f)$$



$$y(t) = TCF^{-1}\left[Y(f)\right]$$

$$Y(f) = Y_1(f) + Y_2(f)$$

$$Y_1(f) = \frac{3B}{4}\,\text{rect}\left(\frac{f}{2B}\right) \quad , \quad Y_2(f) = \frac{3B}{4}\left(1 - \frac{|f|}{B/2}\right)\text{rect}\left(\frac{f}{B}\right)$$

$$y_1(t) = \frac{3B}{4}\cdot 2B\,\text{sinc}(2Bt) = \frac{3B^2}{2}\,\text{sinc}(2Bt)$$

$$y_2(t) = \frac{3B}{4}\cdot\frac{B}{2}\,\text{sinc}^2\left(\frac{B}{2}t\right) = \frac{3B^2}{8}\,\text{sinc}^2\left(\frac{B}{2}t\right)$$

$$y(t) = y_1(t) + y_2(t)$$

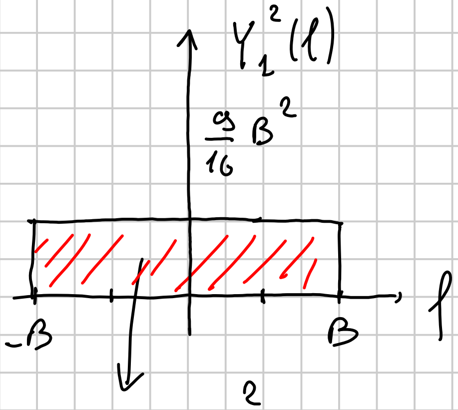
b) $E_y = ?$, $P_y = ?$

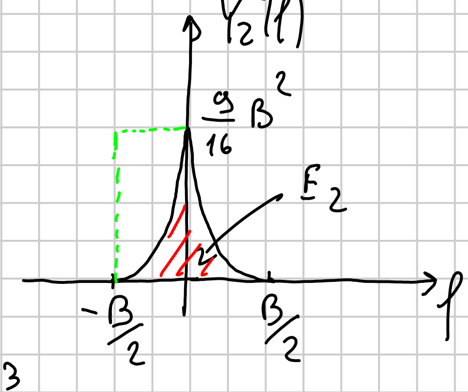
$$E_y = \int_{-\infty}^{+\infty} |y(t)|^2\,dt = \int_{-\infty}^{+\infty} |Y(f)|^2\,df$$

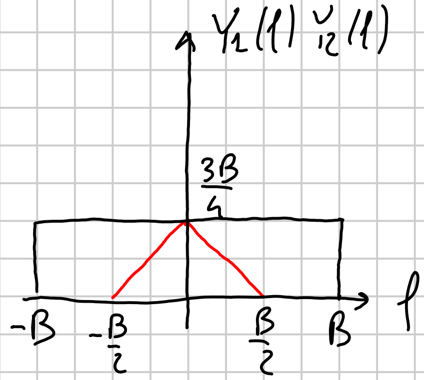
$$Y(f) = Y_1(f) + Y_2(f)$$

$$|Y(f)|^2 = Y^2(f) = Y_1^2(f) + Y_2^2(f) + 2\,Y_1(f)\,Y_2(f)$$

$$E_y = \int_{-\infty}^{+\infty} Y_1^2(f)\,df + \int_{-\infty}^{+\infty} Y_2^2(f)\,df + 2\int_{-\infty}^{+\infty} Y_1(f)\,Y_2(f)\,df$$

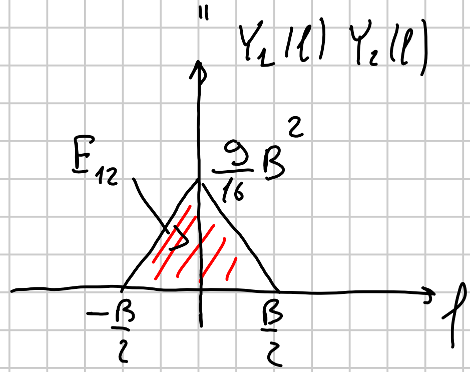






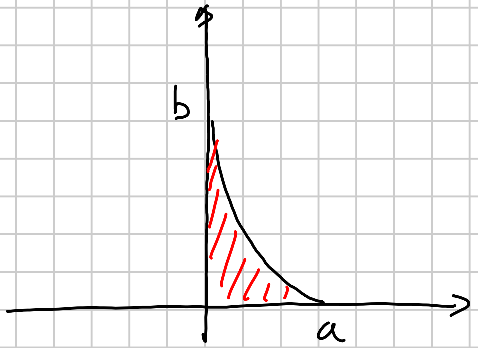
$$E_1 = \frac{9}{16}B^2 \cdot 2B = \frac{9}{8}B^3$$

$$E_2 = \frac{2}{3}\,\frac{9}{16}B^2 \cdot \frac{B}{2} = \frac{3}{16}B^3$$

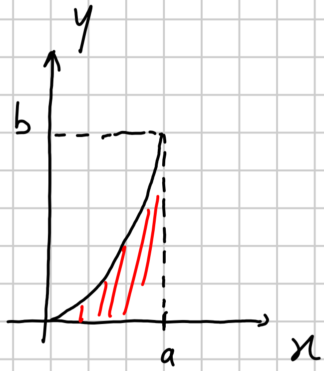


$$E_{12} = \frac{9}{16}B^2 \cdot \frac{B}{2} = \frac{9}{32}B^3$$

$$E = E_1 + E_2 + 2E_{12} = \left(\frac{9}{8} + \frac{3}{16} + \frac{9}{16}\right)B^3 = \frac{15}{8}B^3$$



$\Rightarrow$



$\Downarrow$

$$y = a_1 x^2 + bx + c \qquad\qquad b = c = 0$$

$$y = a_1 x^2$$

$$\int_0^a y(x)\,dx = \int_0^a a_1 x^2\,dx = \int_0^a \frac{b}{a^2}x^2\,dx = \frac{b}{a^2}\frac{x^3}{3}\Big|_0^a =$$

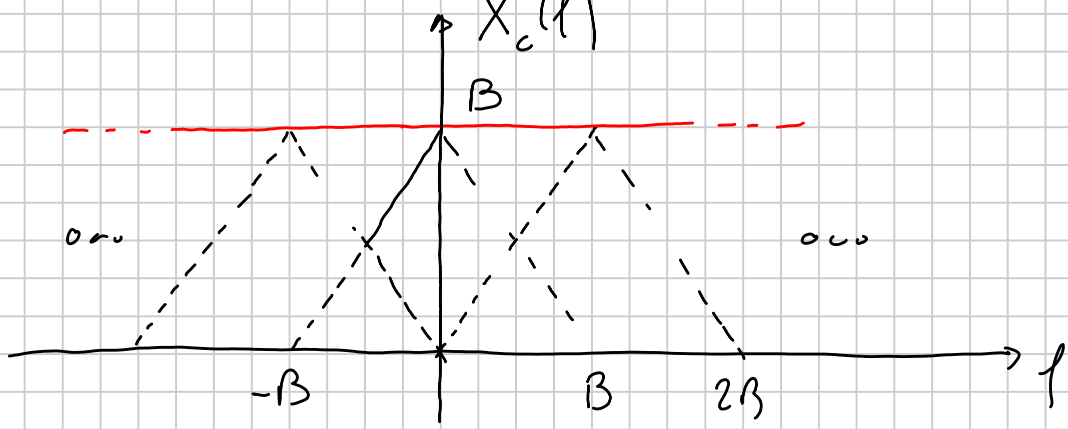
$$= \frac{b}{a^2}\frac{a^3}{3} = \boxed{\frac{ab}{3}}$$

$$a_1 = ? \qquad x = a \Rightarrow y = b$$

$$b = a_1 a^2 \Rightarrow a_1 = \frac{b}{a^2}$$

$$E_y < +\infty \Rightarrow P_y = 0$$

c)



$$\bar{X}_c(f) = B \quad \Rightarrow \quad x_c[n] = B\,\delta[n]$$

$$\bar{X}_c(f) = \sum_{n=-\infty}^{+\infty} x_c[n]\, e^{-j2\pi f n T} =$$

$$= \sum_{n=-\infty}^{+\infty} B\,\delta[n]\, e^{-j2\pi f n T} = B$$

ESERCIZIO ②

$X \in \mathcal{U}[0,1]$, $Y \in \mathcal{U}[0,1]$ e indipendenti

$f_{Z|A}(z|A) = ?$

$$\begin{cases} Z = X - Y \\ V = X + Y \end{cases} , \quad A \equiv \{V \leq 1\}$$

SOLUZIONE

$$f_{Z|A}(z|A) \triangleq \frac{d}{dz} F_{Z|A}(z|A)$$

$$F_{Z|A}(z|A) \triangleq \frac{P\{Z \leq z, A\}}{P\{A\}}$$

$$f_{ZV}(z,v) = \sum_{n=1}^{N} \frac{f_{XY}(x_i, y_i)}{\left|\det \underline{\underline{J}}(x_i, y_i)\right|} \quad , \quad (x_i, y_i) \text{ sol. inv.}$$

SOL. DEL PROB. INV.

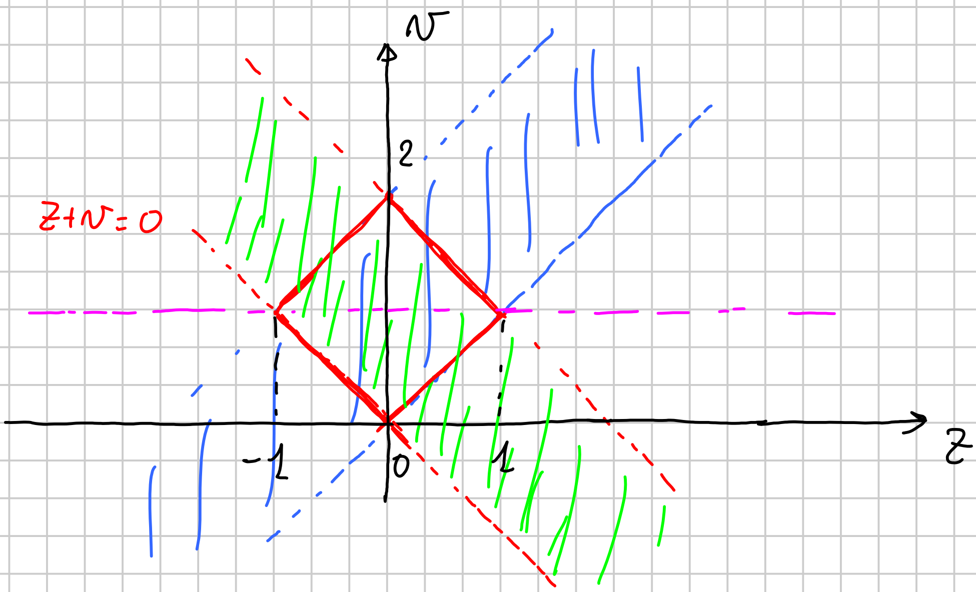
$$\begin{cases} x = \dfrac{z+v}{2} \\[2ex] y = \dfrac{v-z}{2} \end{cases}$$

$$\underline{\underline{J}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow \left|\det\{\underline{\underline{J}}\}\right| = 2$$

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) = \text{rect}\left(\frac{x - 1/2}{1}\right) \text{rect}\left(\frac{y - \frac{1}{2}}{1}\right)$$

$$f_{ZV}(z,v) = \frac{1}{2} \cdot \text{rect}\left(\frac{\frac{z+v}{2} - \frac{1}{2}}{1}\right) \text{rect}\left(\frac{\frac{v-z}{2} - \frac{1}{2}}{1}\right)$$

$$= \frac{1}{2}\, \text{rect}\left(\frac{z+v-1}{2}\right) \text{rect}\left(\frac{v-z-1}{2}\right)$$



$$-1 \leq z+v-1 \leq +1 \quad \Rightarrow \quad 0 \leq z+v \leq 2$$

$$-1 \leq v-z-1 \leq 1 \quad \Rightarrow \quad 0 \leq v-z \leq 2$$

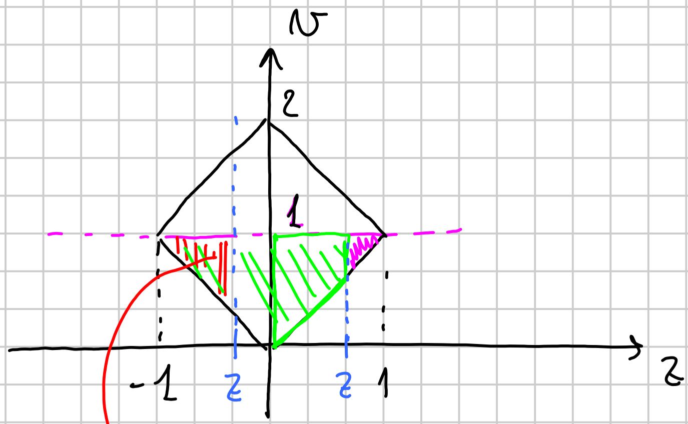
$$P\{A\} = P\{V \leq 1\} = \frac{1}{2} = \int_{-\infty\,(z)}^{+\infty} \int_{-\infty\,(v)}^{1} f_{ZV}(z,v)\,dz\,dv = F_V(1)$$

$$\int_{-\infty}^{+\infty} f_{ZV}(z,v)\,dz = f_V(v)$$

$$P\{V \leq 1\} = \int_{-\infty}^{1} f_V(v)\,dv = F_V(1)$$

$$P\{Z \leq z, A\} = \int_{-\infty}^{z} \int_{-\infty}^{1} f_{ZV}(z,v)\,dz\,dv =$$

$$= \int_{-\infty}^{z} \int_{-\infty}^{1} \frac{1}{2}\,\mathrm{rect}\left(\frac{z+v-1}{2}\right)\mathrm{rect}\left(\frac{v-z-1}{2}\right)dz\,dv$$



$-1 \leq z \leq 0$

$$P\{Z \leq z, V \leq 1\} = \frac{(1+z)^2}{2} \cdot \frac{1}{2}$$

$0 \leq z \leq 1$

$$P\{Z \leq z, V \leq 1\} = \frac{1}{2}\left[\frac{1}{2} + \frac{1}{2} - \frac{(1-z)^2}{2}\right] = \left[1 - \frac{(1-z)^2}{2}\right]\frac{1}{2}$$

$$F_{Z|A}(z|A) = \frac{1}{1/2} \cdot \begin{cases} \dfrac{(1+z)^2}{4} & -1 \leq z \leq 0 \\ \dfrac{1}{2}\left(1 - \dfrac{(1-z)^2}{2}\right) & 0 \leq z \leq 1 \end{cases}$$

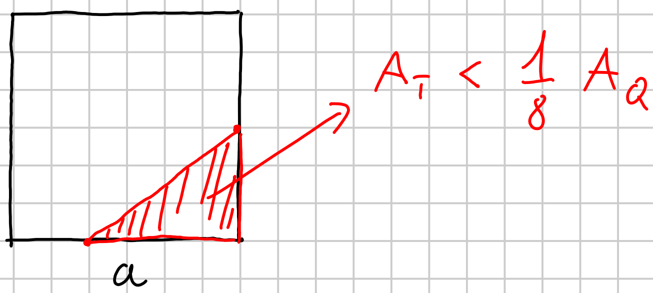
$$F_{Z|A}(z|A) = \begin{cases} \dfrac{(1+z)^2}{2} & -1 \leq z \leq 0 \\ 1 - \dfrac{(1-z)^2}{2} & 0 \leq z \leq 1 \end{cases}$$

$$f_{Z|A}(z|A) = \frac{d}{dz} F_{Z|A}(z|A) =$$

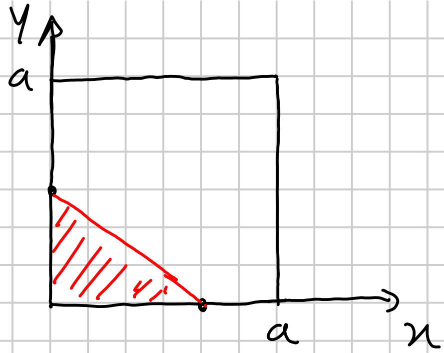
$$= \begin{cases} 1+z & -1 \leq z \leq 0 \\ 1-z & 0 \leq z \leq 1 \end{cases}$$

$$f_{Z|A}(z|A) = \begin{cases} 1 - |z| & -1 \leq z \leq 1 \\ 0 & \text{———} \end{cases}$$

ESERCIZIO 3



$$A_Q = a^2 \quad \Rightarrow \quad P\left\{A_T < \frac{a^2}{8}\right\}$$



X, Y ind. $\in \mathcal{U}[0, a]$

$\Rightarrow a = 1$

$X = a \cdot X_1 \quad , \quad X_1 \in \mathcal{U}[0, 1]$

$Y = a \cdot Y_1 \quad , \quad Y_1 \in \mathcal{U}[0, 1]$

$$A_T = \frac{XY}{2} = \frac{aX_1 \cdot aY_1}{2} = a^2 \frac{X_1 Y_1}{2}$$

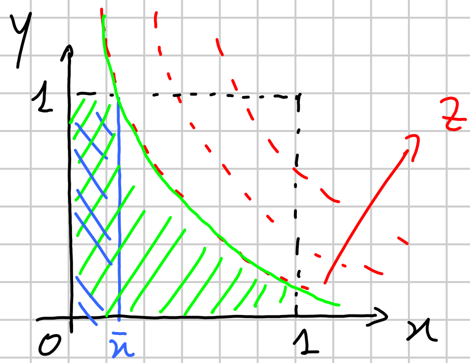
$$P\left\{ A_T < \frac{1}{8} A_Q \right\} = P\left\{ a^2 \frac{X_1 Y_1}{2} < \frac{1}{8} a^2 \right\} =$$

$$= P\left\{ \frac{X_1 Y_1}{2} < \frac{1}{8} \right\}$$

$$= P\left\{ X_1 Y_1 < \frac{1}{4} \right\}$$

$Z = X_1 Y_1$

$$P\left\{ Z < \frac{1}{4} \right\} = F_Z\left(\frac{1}{4}\right)$$



$P\{ Z \leq z \} =$

$= A_{verde} \cdot 1$

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) = \mathrm{rect}\left(\frac{x-1/2}{1}\right) \mathrm{rect}\left(\frac{y-1/2}{1}\right)$$

$$x \cdot y = \frac{1}{4} \Rightarrow y = \frac{1}{4x}$$

$$\bar{x} \Rightarrow 1 = \frac{1}{4\bar{x}} \Rightarrow \bar{x} = \frac{1}{4}$$

$$A_{verde} = \bar{x} \cdot 1 + \int_{\bar{x}}^{1} \frac{1}{4x}\, dx =$$

$$= \bar{x} + \frac{1}{4} \ln x \Big|_{\bar{x}}^{1} = \bar{x} + \frac{1}{4}\left[\ln(1) - \ln(\bar{x})\right]$$

$$P\left(Z \leq \frac{1}{4}\right) = P\left\{A_T < \frac{1}{8} A_Q\right\} = \frac{1}{4}\left[1 - \ln(\bar{x})\right]$$