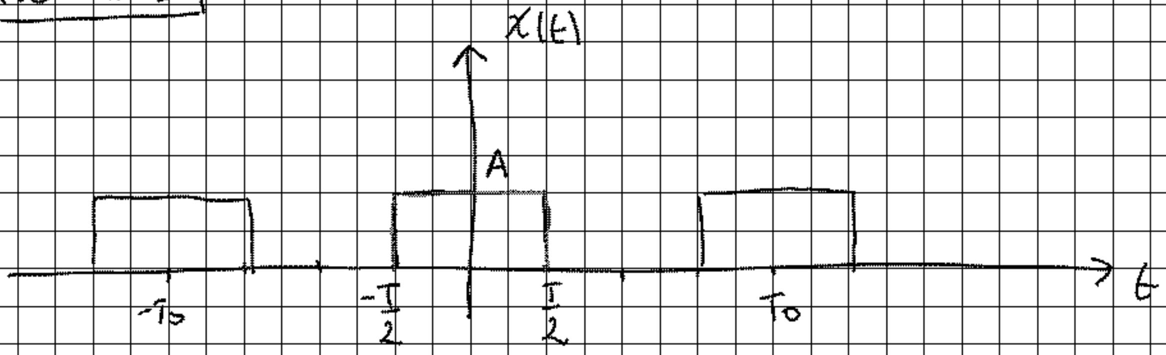
Esercitazione SEGNALI PERIODICIZZATI

Esercizio # 1



- $X(f)$
- X_n

$$X(f) = \frac{1}{T_0} \sum_{k=-\infty}^{+\infty} X_0\left(\frac{k}{T_0}\right) \delta\left(f - \frac{k}{T_0}\right)$$

$$X_k = \frac{1}{T_0} X_0\left(\frac{k}{T_0}\right)$$

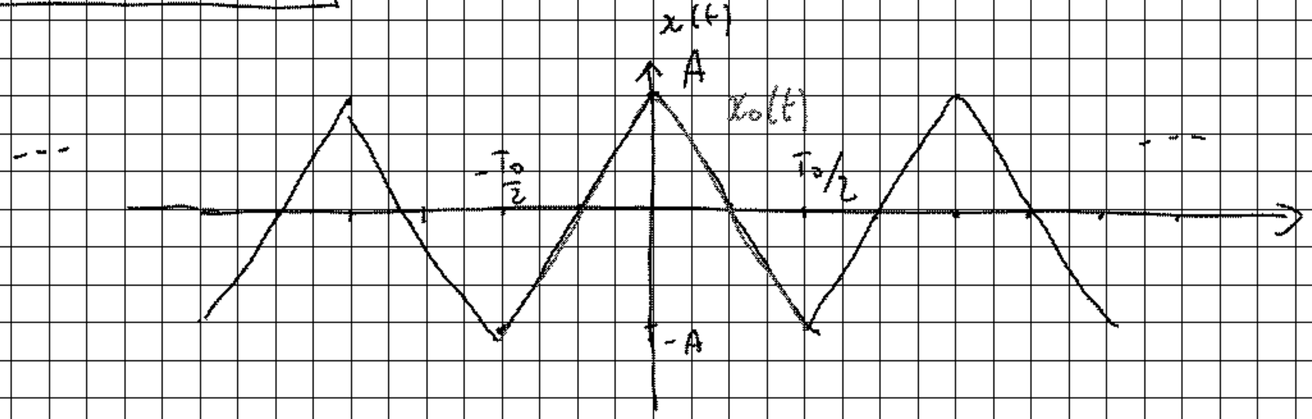
$$x_0(t) = A \operatorname{rect}\left(\frac{t}{T}\right)$$

$$X_0(f) = AT \operatorname{sinc}(fT)$$

$$X(f) = \frac{1}{T_0} \sum_{k=-\infty}^{+\infty} AT \operatorname{sinc}\left(\frac{k}{T_0}T\right) \delta\left(f - \frac{k}{T_0}\right)$$

$$X_k = \frac{1}{T_0} AT \operatorname{sinc}\left(\frac{k}{T_0}T\right)$$

Esercizio # 2 ONDA TRIANGOLARE



Determinare $X(f)$ e X_k

$$x_0(t) = 2A\left(1 - \frac{|t|}{T_0/2}\right)\text{rect}\left(\frac{t}{T_0}\right) - A\,\text{rect}\left(\frac{t}{T_0}\right)$$

$$X_0(f) = 2A\frac{T_0}{2}\,\text{sinc}^2\left(f\frac{T_0}{2}\right) - AT_0\,\text{sinc}(fT_0)$$

$$\boxed{X_k = \frac{1}{T_0}X_0\left(\frac{k}{T_0}\right)}$$

$$X_k = A\,\text{sinc}^2\left(\frac{\cancel{T_0}}{2}\cdot\frac{k}{\cancel{T_0}}\right) - A\,\text{sinc}\left(\cancel{T_0}\frac{k}{\cancel{T_0}}\right) =$$

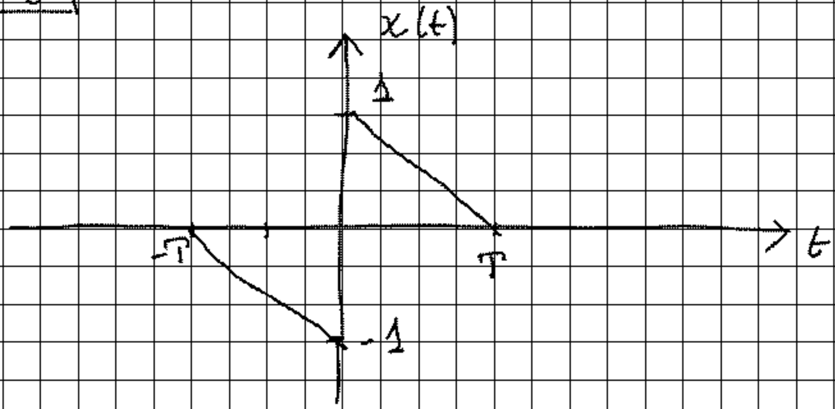
$$= A\,\text{sinc}^2\left(\frac{k}{2}\right) - A\,\text{sinc}(k) = \begin{cases} 0 & \text{per } k \text{ pari} \\ \frac{4}{k^2\pi^2} & \text{per } k \text{ dispari} \end{cases}$$

$\text{sinc}(k) = 0 \quad \forall\ k = \pm 1, \pm 2, \pm 3, \dots$

$$\text{sinc}^2\left(\frac{k}{2}\right) = \frac{\sin^2\left(\frac{k\pi}{2}\right)}{\left(\frac{k\pi}{2}\right)^2} = \begin{cases} 0 & \text{per } k \text{ pari} \\ \frac{4}{k^2\pi^2} & \text{per } k \text{ dispari} \end{cases}$$

$$X(f) = \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} X_0\left(\frac{k}{T_0}\right)\delta\left(f - \frac{k}{T_0}\right)$$

Esercizio #3

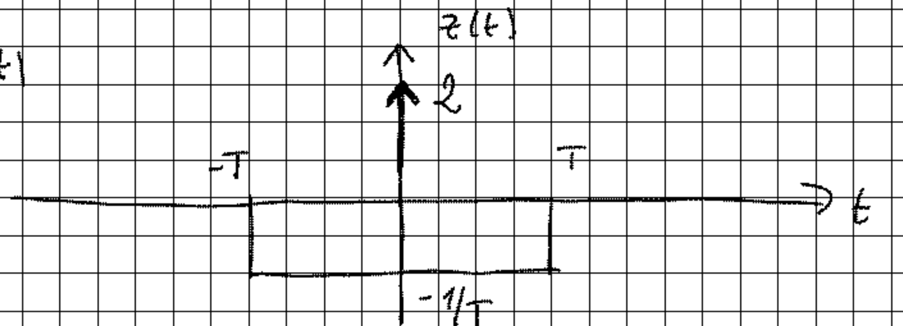


$$y(t) = \sum_{n=-\infty}^{+\infty} x(t - 4Tn) \qquad T_0 = 4T$$

Determinare $Y(f)$ e Y_k

- teorema di derivazione

$$z(t) = \frac{\partial}{\partial t} x(t)$$



$$x(t) = \int_{-\infty}^{t} z(t)\, dt$$

la funzione integrale di z(t) deve essere x(t)

$$z(t) = -\frac{1}{T} \text{rect}\left(\frac{t}{2T}\right) + 2\delta(t)$$

$$Z(f) = -\frac{1}{\cancel{T}} 2\cancel{T} \text{ sinc}(f2T) + 2 = -2 \text{ sinc}(f2T) + 2$$

$\boxed{Z(0) = 0} \Rightarrow$ è valida l'ipotesi di applicabilità del teorema di derivazione incompleto.

$$Z(f) = j2\pi f X(f)$$

$$X(f) = \frac{Z(f)}{j2\pi f} = \frac{2\left[1 - \mathrm{sinc}(f\,2T)\right]}{j2\pi f} \qquad T_0 = 4T$$

$$Y_k = \frac{1}{T_0} X\left(\frac{k}{T_0}\right) = \frac{1}{4T} X\left(\frac{k}{4T}\right) =$$

$$= \frac{\left[1 - \mathrm{sinc}\left(\frac{k}{4T} 2T\right)\right]}{j\pi \frac{k}{\cancel{4T}}} \cdot \frac{1}{\cancel{4T}} =$$

$$= \frac{1 - \mathrm{sinc}\left(\frac{k}{2}\right)}{j\pi k}$$

$$Y(f) = \sum_{k=-\infty}^{+\infty} \frac{1 - \mathrm{sinc}(k/2)}{j\pi k} \cdot \delta\left(f - \frac{k}{4T}\right)$$

Esercizio 4

$$s(t) = \mathrm{sinc}\left(2Bt - 1/2\right) + \mathrm{sinc}\left(2Bt + 1/2\right)$$

Si calcolino Energia e Potenza dei segnali

$$x(t) = s(t) \sin(2\pi B t)$$

$$y(t) = \sum_{k=-\infty}^{+\infty} s(t - 2k/B)$$

$$s(t) = \text{sinc}\left(2B\left(t - \overset{t_0}{\frac{1}{4B}}\right)\right) + \text{sinc}\left(2B\left(t + \overset{t_0}{\frac{1}{4B}}\right)\right)$$

$$S(f) = \frac{1}{2B}\,\text{rect}\left(\frac{f}{2B}\right) e^{-i2\pi f\frac{1}{4B}} + \frac{1}{2B}\,\text{rect}\left(\frac{f}{2B}\right) e^{i2\pi f\frac{1}{4B}} =$$

$$= \frac{1}{B}\,\text{rect}\left(\frac{f}{2B}\right) \cdot \cos\left(2\pi f\frac{1}{4B}\right) = \frac{1}{B}\,\text{rect}\left(\frac{f}{2B}\right)\cos\left(\frac{\pi f}{2B}\right)$$

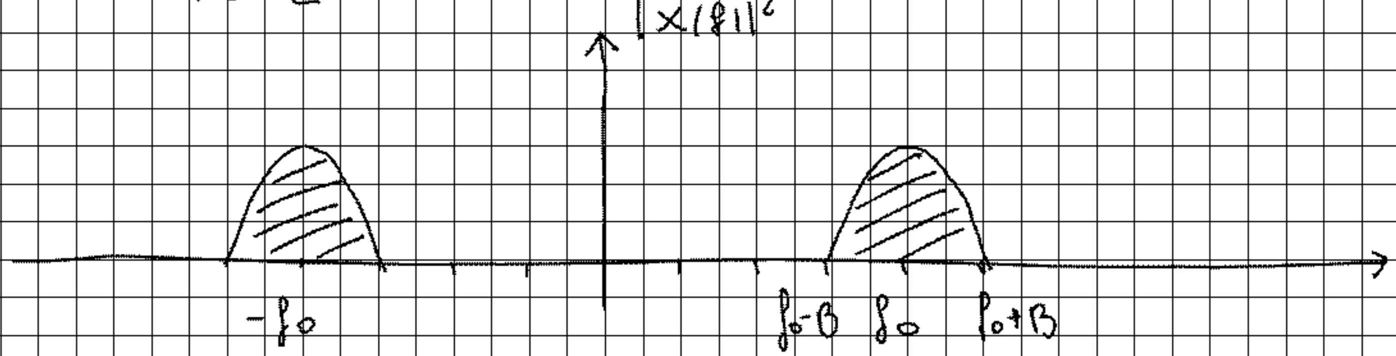
$$x(t) = s(t)\sin(2\pi B t)$$

$$X(f) = \frac{1}{2i}\left[S(f-f) - S(f+f_0)\right] =$$

$$= \frac{1}{2iB}\left[\text{rect}\left(\frac{f-f_0}{2B}\right)\cos\left(\frac{\pi(f-f_0)}{2B}\right) - \text{rect}\left(\frac{f+f_0}{2B}\right)\cos\left(\frac{\pi(f+f_0)}{2B}\right)\right]$$

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2\,dt = \int_{-\infty}^{+\infty} |X(f)|^2\,df$$

$$|X(f)|^2 = \frac{1}{4B^2}\left[\text{rect}\left(\frac{f-f_0}{2B}\right)\cos^2\left(\frac{\pi(f-f_0)}{2B}\right) + \text{rect}\left(\frac{f+f_0}{2B}\right)\cos^2\left(\frac{\pi(f+f_0)}{2B}\right)\right]$$

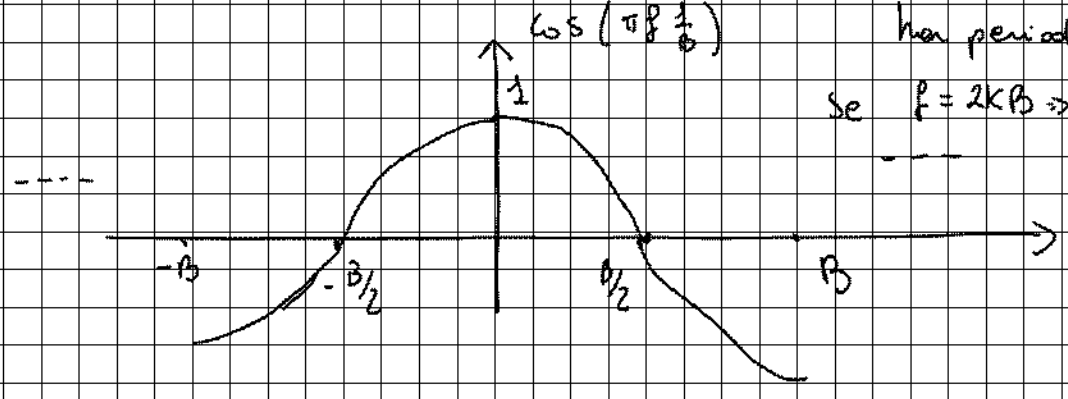


$$E_x = \frac{2}{4B^2}\int_{f_0-B}^{f_0+B} \cos^2\left(\frac{\pi(f-f_0)}{2B}\right) df \qquad f-f_0=f'$$

$$E_x = \frac{2}{4B^2}\int_{-B}^{B}\cos^2\left(\frac{\pi f'}{2B}\right)df' \qquad \cos^2\alpha = \frac{1}{2}+\frac{1}{2}\cos 2\alpha$$

$$E_x=\frac{1}{2B^2}\int_{-B}^{B}\left[\frac{1}{2}+\frac{1}{2}\cos\left(\pi f'\frac{1}{B}\right)\right]df'$$

$$E_x=\frac{1}{2B^2}\int_{-B}^{B}\frac{1}{2}\, df' = \frac{1}{2B^2}\,\frac{f}{2}\Big|_{-B}^{B} = \frac{1}{4B^2}\,2B=\frac{1}{2B}$$



ha periodo 2B

se $f = 2kB \Rightarrow \cos(2k\pi)$

$$E_x=\frac{1}{2B} \qquad P_x = 0$$

$$y(t) = \sum_k x\left(t-\frac{2k}{B}\right)$$

$$E_y = \infty$$

$$P_y = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{T_0/2} |y(t)|^2 = \sum_{k=-\infty}^{+\infty} |Y_k|^2$$

$$Y_k = \frac{1}{T_0} S\left(\frac{k}{T_0}\right) = \frac{B}{2} S\left(\frac{kB}{2}\right) =$$

$$= \frac{B}{2} \frac{1}{B} \text{rect}\left(\frac{k\,B/2}{2B}\right) \cos\left(\frac{\pi k\,B/2}{2B}\right) = \frac{1}{2} \text{rect}\left(\frac{k}{4}\right) \cos\left(\frac{\pi}{4} k\right)$$

$$Y_k = \begin{cases} 1/2 & k = 0 \\ \frac{\sqrt{2}}{4} & k = \pm 1 \\ 0 & \text{altrove} \end{cases}$$

$$P_y = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

Esercizi su SISTEMI

Esercizio #1

Sia data la trasformazione ingresso-uscita di un sistema

$$y(t) = \int_{+T}^{t} x(\alpha)\, d\alpha$$

Dire se il sistema gode delle seguenti proprietà:

1) Linearità
2) stazionarietà
3) stabilità (BIBO)
4) istantaneità

Svolgimento

1) $x(t) = a\, x_1(t) + b\, x_2(t)$

$$y(t) = T\left[x(t)\right] = a\, T\left[x_1(t)\right] + b\, T\left[x_2(t)\right]$$

$$y(t) = \int_{T}^{t} \left[a\, x_1(t) + b\, x_2(t)\right] dt =$$

per la proprietà di linearità dell'integrale

$$= \int_{T}^{t} a\, x_1(t)\, dt + \int_{T}^{t} b\, x_2(t)\, dt =$$

$$= a\, T\left[x_1(t)\right] + b\, T\left[x_2(t)\right]$$

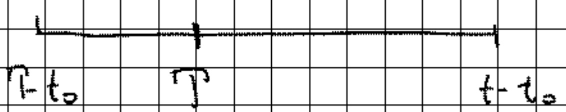
LINEARE

2) STAZIONARIETÀ

$$y(t-t_0) = T\left[x(t-t_0)\right]$$

$$\int_{T}^{t} x(\alpha - t_0)\, d\alpha = \qquad \boxed{\alpha - t_0 = \beta}$$

$$= \int_{T-t_0}^{t-t_0} x(\beta)\, d\beta = \int_{T-t_0}^{T} x(\beta)\, d\beta + \int_{T}^{t-t_0} x(\beta)\, d\beta =$$



$$= \int_{T-t_0}^{T} x(\beta)\, d\beta + y(t-t_0)$$

$\boxed{\text{NON STAZIONARIO}}$

3) STABILITÀ

$$|x(t)| \leq M \qquad |y(t)| \leq K$$

Dimostriamo l'INSTABILITÀ del sistema

$$x(\alpha) = M$$

$$\int_{T}^{t} x(\alpha)\,d\alpha = \int_{T}^{t} M\,d\alpha = M\,(t-T) = y(t)$$

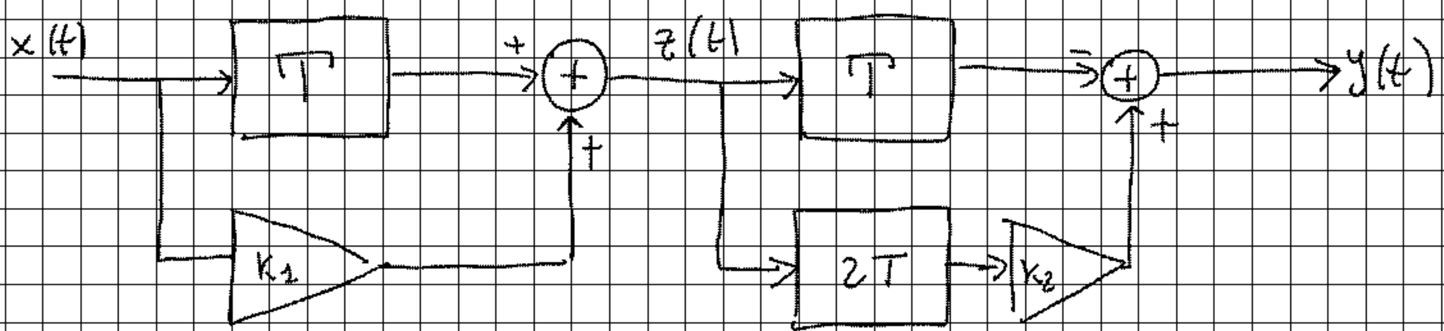
$\boxed{\text{NON È STABILE}}$

4) ISTANTANEITÀ

$$y(t) = \int_{T}^{t} x(\alpha)\,d\alpha$$

$\boxed{\text{CON MEMORIA}}$

Esercizio #2 STUDIO DI SISTEMA



Determinare l'espressione di $y(t)$ e dire se il sistema è:

- stazionario
- lineare
- senza memoria
- causale
- stabile

$$z(t) = x(t-T) + k_1\,x(t)$$

$$y(t) = -z(t-T) + k_2\,z(t-2T)$$

$$y(t) = -\Big(x(t-2T) + k_1\,x(t-T)\Big) + k_2\Big(x(t-3T) + k_1\,x(t-2T)\Big)$$

$$y(t) = -k_1\, x(t-T) + x(t-2T)\left(-1 + k_1 k_2\right) + k_2\, x(t-3T)$$

1) STAZIONARIETÀ

$$x(t-t_0)$$

$$-k_1\, x(t-t_0-T) + x(t-t_0-2T)\left(k_1k_2-1\right) + k_2\, x(t-t_0-3T) =$$

$$= y(t-t_0)$$

STAZIONARIO

2) LINEARITÀ

$$x(t) = a\, x_1(t) + b\, x_2(t)$$

$$y(t) = -k_1\, x(t-T) + x(t-2T)(k_1k_2-1) + k_2\, x(t-3T)$$

$$-k_1\left(a\, x_1(t-T) + b\, x_2(t-T)\right) + \left(a\, x_1(t-2T) + b\, x_2(t-2T)\right)\left(k_1k_2-1\right)$$

$$+ k_2\left(a\, x_1(t-3T) + b\, x_2(t-3T)\right) =$$

$a\, y_1(t)$

$$= \left(-k_1\, a\, x_1(t-T) + a\, x_1(t-2T)(k_1k_2-1) + a\, k_2\, x_1(t-3T)\right) +$$

$$\left(-k_1\, b\, x_2(t-T) + a\, x_2(t-2T)(k_1k_2-1) + a\, k_2\, x_2(t-3T)\right) =$$

$b\, y_2(t)$

$$= a\, y_1(t) + b\, y_2(t) = a\,T\left[x_1(t)\right] + b\,T\left[x_2(t)\right]$$

LINEARE

3) MEMORIA

Il sistema é con memoria perché l'uscita all'istante t $(y(t))$ dipende da istanti precedenti a t $(x(t-T), x(t-2T), x(t-3T))$

4) CAUSALITÀ

Il sistema é causale perché $\underbrace{\text{l'uscita}}_{}$ all'istante t non dipende da istanti temporali successivi a t.

5) STABILITÀ

$$|x(t)| \leq M \quad \forall t$$

$$|y(t)| = \left| -k_1 x(t-T) + (k_2 k_1 - 1) x(t-2T) + k_2 x(t-3T) \right| \leq$$

$$\left| -k_1 x(t-T) \right| + \left| (k_2 k_1 - 1) x(t-2T) \right| + \left| k_2 x(t-3T) \right| \leq$$

$$|k_1| M + |k_2 k_1 - 1| M + |k_2| M < +\infty$$

SISTEMA STABILE

- Calcolare la risposta impulsiva e in frequenza del sistema

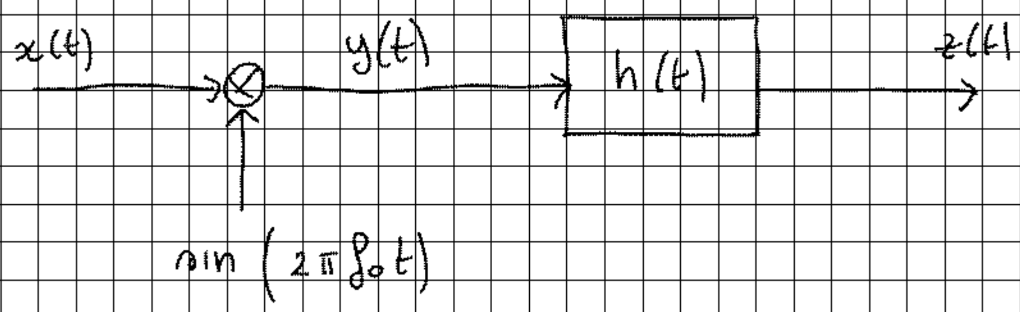
Il sistema é SLS quindi ha senso calcolarne la risposta impulsiva

$x(t) = \delta(t)$

$$y(t) = -k_1\,\delta(t-T) + (k_2 k_1 - 1)\,\delta(t-2T) + k_2\,\delta(t-3T)$$

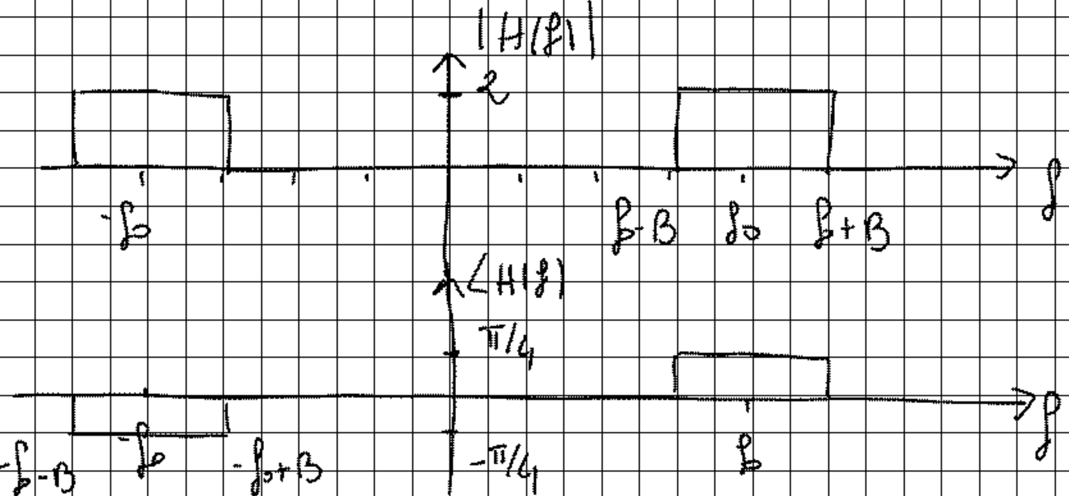
$$Y(f) = -k_1 \cdot 1 \cdot e^{-j2\pi f T} + (k_2 k_1 - 1)\, e^{-j 2\pi f 2T} + k_2\, e^{-j2\pi f 3T}$$

Esercizio #3



$$X(f) = A\left(1 - \frac{|f|}{B}\right) \mathrm{rect}\left(\frac{f}{2B}\right)$$

H(f)



- Calcolare espressione di $z(t)$
- Calcolare l'energia di $z(t)$

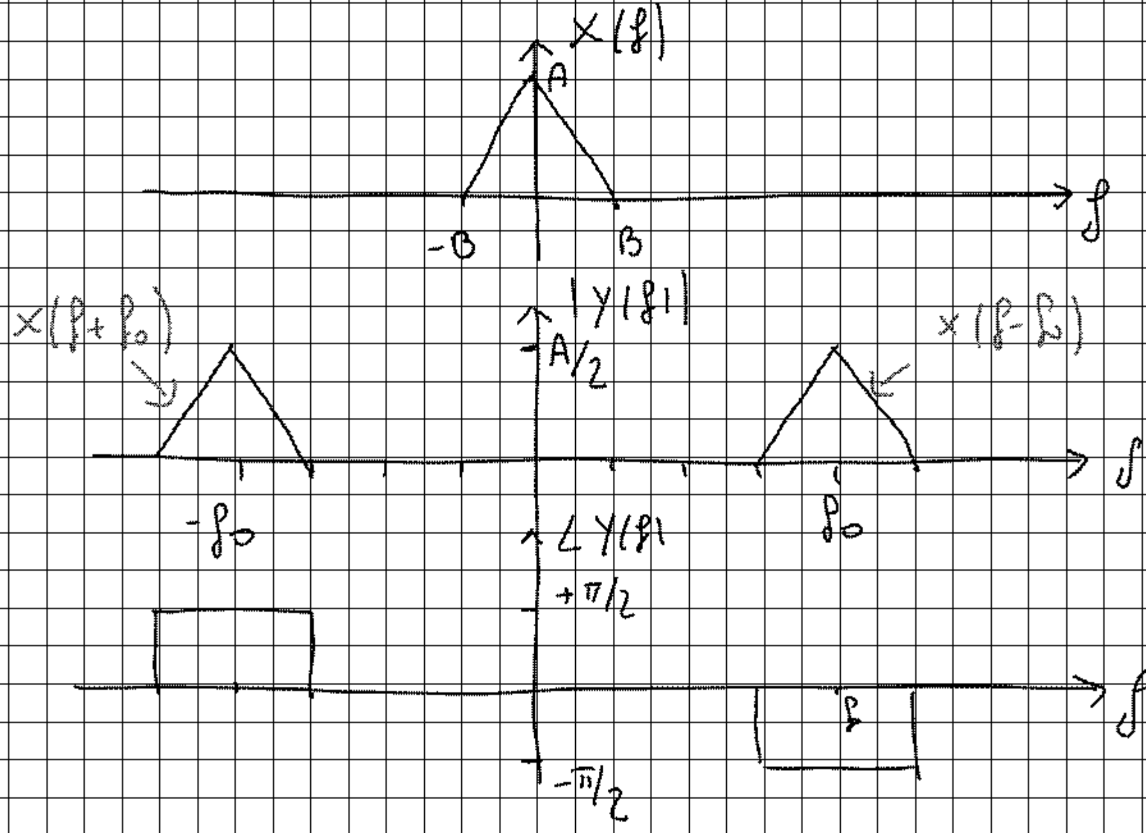
Svolgimento

$$y(t) = x(t) \cdot \sin(2\pi f_0 t)$$

$$Y(f) = X(f) \otimes \frac{1}{2j} \left(\delta(f - f_0) - \delta(f + f_0) \right) =$$

$$= \frac{1}{2j} \left(X(f - f_0) - X(f + f_0) \right) =$$

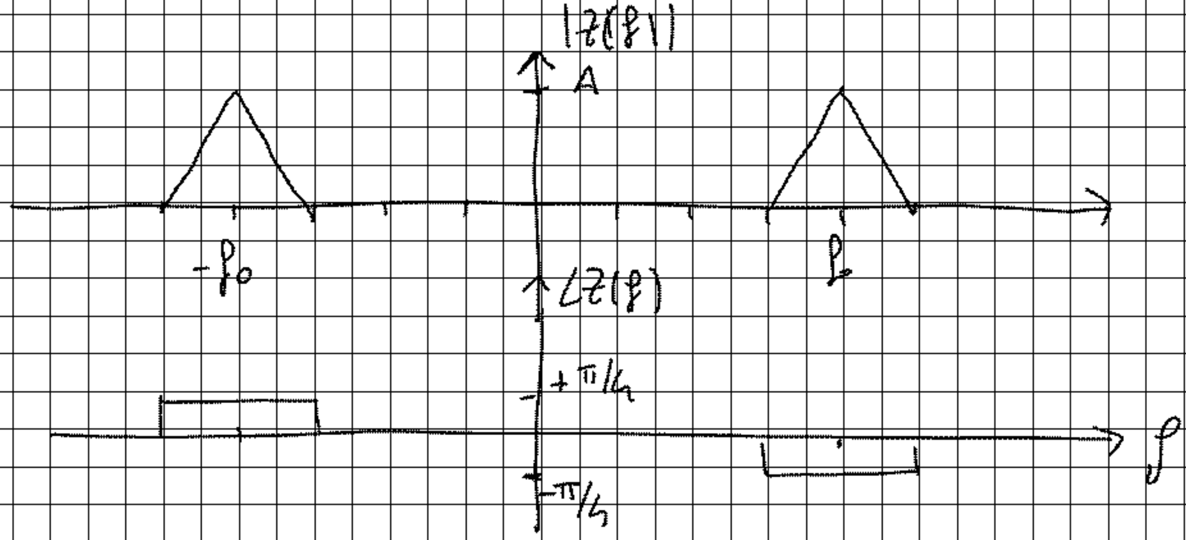
$$= \frac{e^{-j\pi/2}}{2} X(f - f_0) + \frac{e^{j\pi/2}}{2} X(f + f_0)$$



$$Z(f) = Y(f) H(f)$$

$$|Z(f)| = |Y(f)| \, |H(f)|$$

$$\angle Z(f) = \angle Y(f) + \angle H(f)$$



$$Z(f) = A\left(1-\frac{|f-f_0|}{B}\right)\mathrm{rect}\left(\frac{f-f_0}{2B}\right)e^{-i\pi/4} + A\left(1-\frac{|f+f_0|}{B}\right)\mathrm{rect}\left(\frac{f+f_0}{2B}\right)e^{i\pi/4}$$

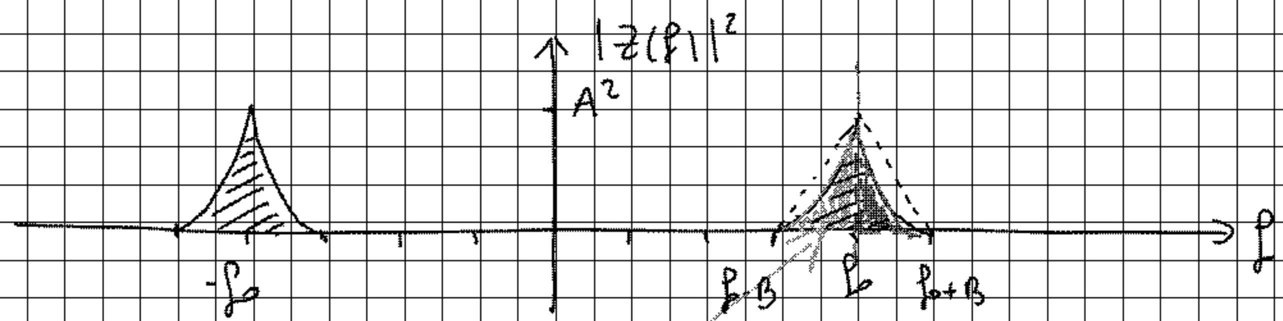
$$Z(f) = A\left(1-\frac{|f|}{B}\right)\mathrm{rect}\left(\frac{f}{2B}\right) \otimes \left[\delta(f-f_0)\,e^{-i\pi/4} + \delta(f+f_0)\,e^{i\pi/4}\right]$$

$$z(t) = AB\,\mathrm{sinc}^2(fB)\cdot\left[e^{+i2\pi f_0 t}\,e^{-i\pi/4} + e^{-i2\pi f_0 t}\,e^{i\pi/4}\right] =$$

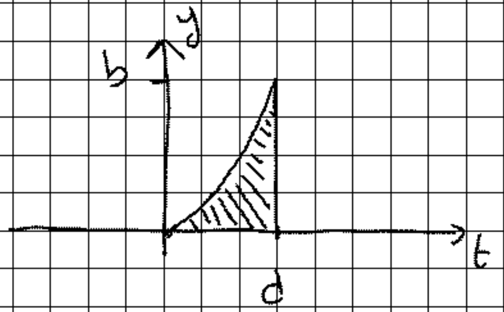
$$= 2AB\,\mathrm{sinc}^2(fB)\cdot\cos(2\pi f_0 t - \pi/4)$$

$$- E_z = \int_{-\infty}^{+\infty} |z(t)|^2\,dt = \int_{-\infty}^{+\infty} |Z(f)|^2\,df$$

$$Z(f) = A\left(1-\frac{|f+f_0|}{B}\right)\mathrm{rect}\left(\frac{f+f_0}{2B}\right)e^{i\pi/4} + A\left(1-\frac{|f-f_0|}{B}\right)\mathrm{rect}\left(\frac{f-f_0}{2B}\right)e^{-i\pi/4}$$



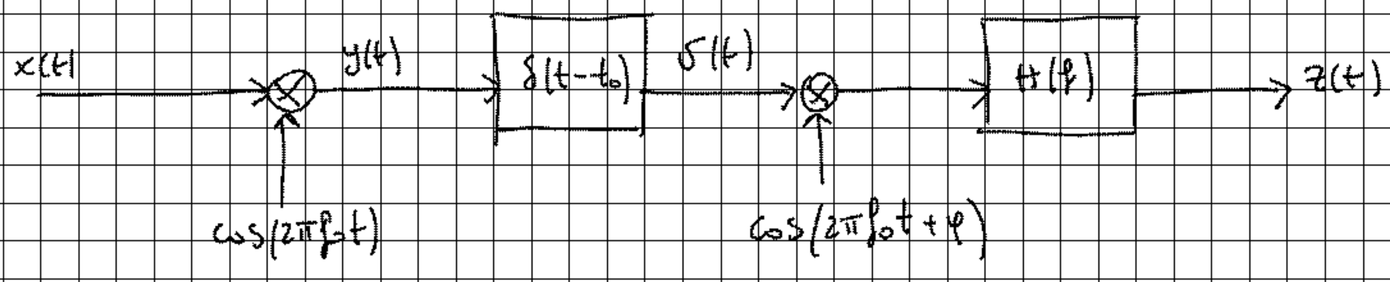
$$E_z = 4\int_{f_0-B}^{f_0} |Z(f)|^2 \, df$$



$y = \frac{b}{a^2} t^2$ $\qquad \int_0^a \frac{b}{a^2} t^2 \, dt = \frac{ab}{3}$

$$E_z = 4\,\frac{A^2 B}{3}$$

Esercizio #4 | SISTEMA WIRELESS



Il sistema rappresenta un sistema di comunicazione di tipo wireless, nel quale il segnale trasmesso $y(t)$ si propaga in spazio libero.
Il blocco $\delta(t-t_0)$ schematizza il canale di propagazione.
L'onda e.m. ricevuta, $s(t)$, è una copia ritardata ed attenuata di $y(t)$. t_0 è il tempo che l'onda e.m. impiega a percorrere

la distanza tra il trasmettitore e il ricevitore.

$$X(f) = \left[1 + \cos\left(\frac{\pi f}{B}\right)\right] \mathrm{rect}\left(\frac{f}{2B}\right)$$

$H(f)$ è un filtro passa basso ideale di banda B

Calcolare:

1) $Y(f)$

2) $W(f)$

3) $Z(f)$

4) Determinare il valore di φ affinché $z(t)$ abbia energia massima

Svolgimento

1) $y(t) = x(t)\cos(2\pi f_0 t)$ teorema della modulazione

$$Y(f) = \frac{X(f-f_0) + X(f+f_0)}{2}$$

$$Y(f) = \frac{1}{2}\left[1 + \cos\left(\frac{\pi(f-f_0)}{B}\right)\right] \mathrm{rect}\left(\frac{f-f_0}{2B}\right) +$$

$$\frac{1}{2}\left[1 + \cos\left(\frac{(f+f_0)\pi}{B}\right)\right] \mathrm{rect}\left(\frac{f+f_0}{2B}\right)$$

2) $v(t) = y(t) \otimes \delta(t - t_0)$

$$V(f) = Y(f) \cdot e^{-j2\pi f t_0}$$

teorema della modulazione

teorema del ritardo

$$w(t) = v(t) \cdot \cos(2\pi f_0 t + \varphi)$$

$$W(f) = V(f) \otimes \left[\frac{1}{2}\delta(f - f_0) e^{j\varphi} + \frac{1}{2}\delta(f + f_0) e^{-j\varphi}\right] =$$

$$= \frac{1}{2} V(f - f_0) e^{j\varphi} + \frac{1}{2} V(f + f_0) e^{-j\varphi}$$

$$W(f) = \frac{1}{2} Y(f - f_0) e^{-j2\pi(f - f_0)t_0} e^{j\varphi} + \frac{1}{2} Y(f + f_0) e^{-j2\pi(f + f_0)t_0} e^{-j\varphi} =$$

$$= \frac{1}{2} e^{-j2\pi f t_0} \left[Y(f - f_0) e^{+j2\pi f_0 t_0 + j\varphi} + Y(f + f_0) e^{-j2\pi f_0 t_0 - j\varphi} \right] =$$

$$= \frac{1}{2} e^{-j2\pi f t_0} \left\{ \left[\left(\frac{1}{2} + \frac{1}{2} \cos\left(\frac{\pi(f - 2f_0)}{B} \right) \right) \mathrm{rect}\left(\frac{f - 2f_0}{2B} \right) + \right. \right.$$

$$+ \left(\frac{1}{2} + \frac{1}{2} \cos\left(\frac{\pi f}{B} \right) \right) \mathrm{rect}\left(\frac{f}{2B} \right) \Big] e^{j2\pi f_0 t_0 + j\varphi} +$$

$$+\left[\left(\frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi f}{B}\right)\right)\mathrm{rect}\left(\frac{f}{2B}\right)+\left(\frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi(f+2f_0)}{2B}\right)\right)\mathrm{rect}\left(\frac{f+2f_0}{2B}\right)\right]\cdot e^{-j2\pi f_0 t_0-j\varphi}\Bigg\}$$

Definisco

$$W_{2f_0}(f)=\frac{1}{4}\,e^{-j2\pi f t_0}\,e^{j2\pi f_0 t_0+j\varphi}\left(1+\cos\left(\frac{\pi(f-2f_0)}{B}\right)\right)\mathrm{rect}\left(\frac{f-2f_0}{2B}\right)$$

$$W_{-2f_0}(f)=\frac{1}{4}\,e^{-j2\pi f t_0}\,e^{-j2\pi f_0 t_0-j\varphi}\left(1+\cos\left(\frac{\pi(f+2f_0)}{B}\right)\right)\mathrm{rect}\left(\frac{f+2f_0}{2B}\right)$$

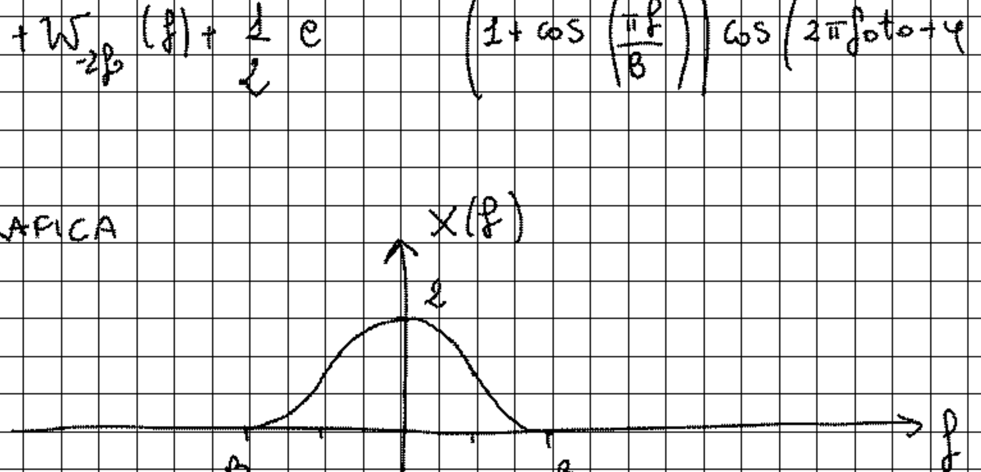
L'espressione di $W(f)$ si può riscrivere quindi come segue:

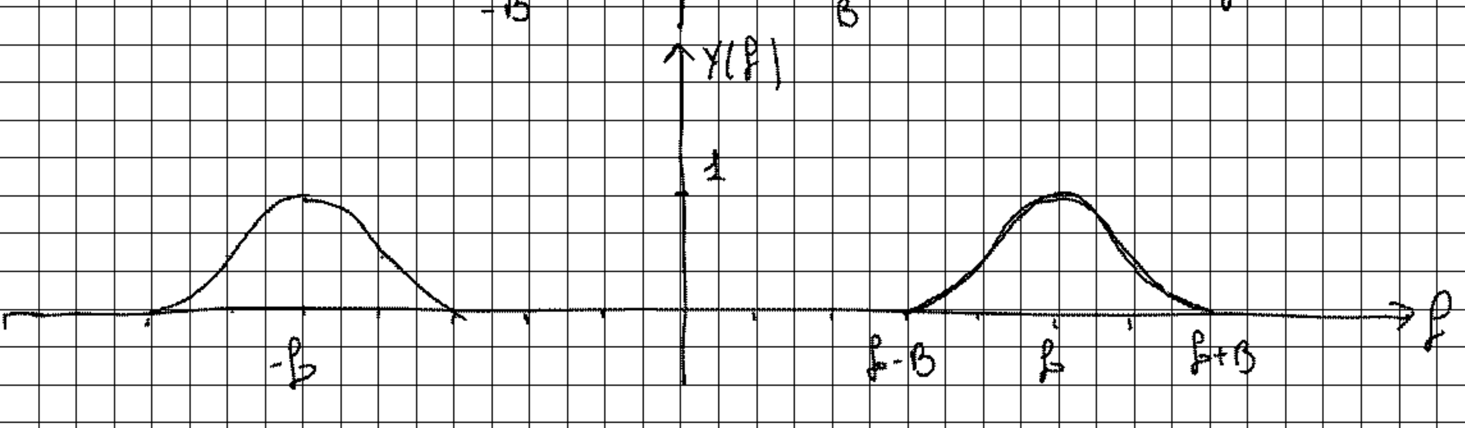
$$W(f)=$$

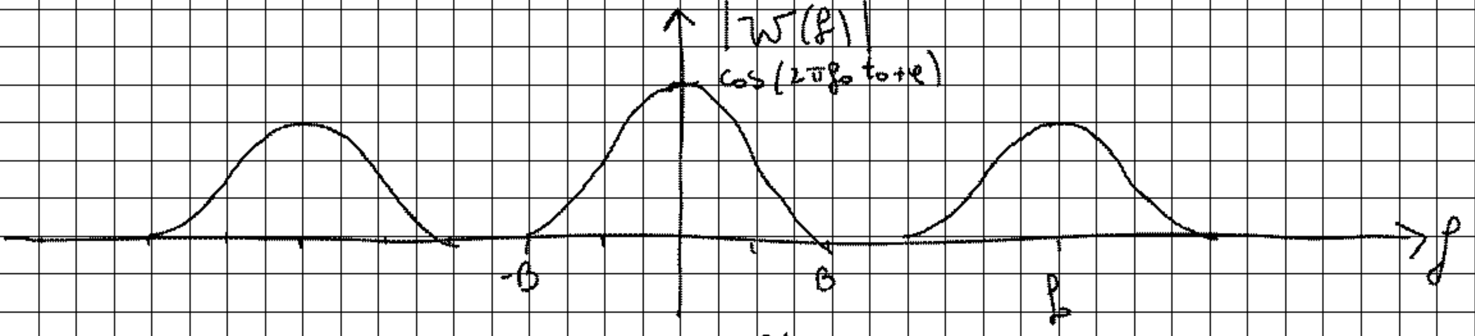
$$=W_{2f_0}(f)+W_{-2f_0}(f)+\frac{1}{2}\,e^{-j2\pi f t_0}\left(1+\cos\left(\frac{\pi f}{B}\right)\right)\cos\left(2\pi f_0 t_0+\varphi\right)\mathrm{rect}\left(\frac{f}{2B}\right)$$

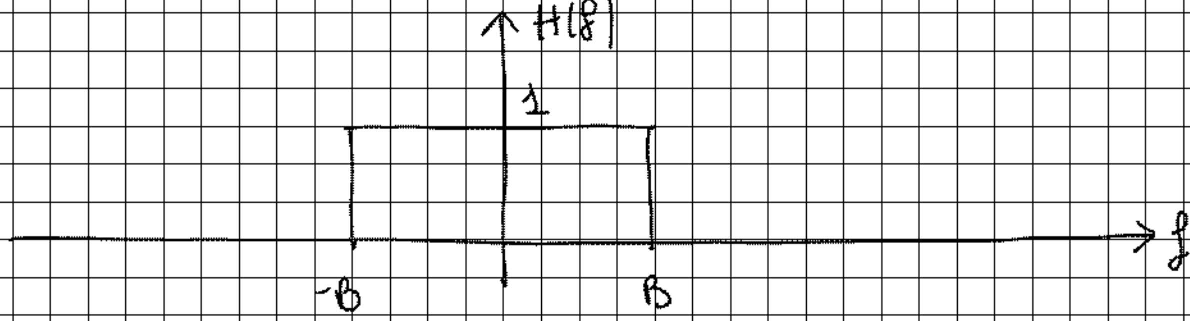
3)

ANALISI GRAFICA









h(t) è un filtro passa basso ideale quindi:

$$Z(f) = \frac{1}{2} e^{-j 2\pi f t_0} \cos(2\pi f_0 t_0 + \varphi) \left(1 + \cos\left(\frac{\pi f}{B}\right)\right) \mathrm{rect}\left(\frac{f}{2B}\right)$$

$$Z(f) = \frac{1}{2} \cos(2\pi f_0 t_0 + \varphi) \cdot X(f)\, e^{-j 2\pi f t_0}$$

teorema del ritardo

$$z(t) = \frac{1}{2} \cos(2\pi f_0 t + \varphi) \cdot x(t - t_0)$$

z(t) è una replica ritardata e attenuata del segnale trasmesso

$$X(f) = \mathrm{rect}\left(\frac{f}{2B}\right) + \mathrm{rect}\left(\frac{f}{2B}\right) \cos\left(\frac{\pi f}{B}\right) \qquad t_0 = \frac{1}{2B}$$

$$x(t) = 2B\, \mathrm{sinc}(t\, 2B) + B\, \mathrm{sinc}(t\, 2B) \otimes \left[\delta(t - t_0) + \delta(t + t_0)\right] =$$

$= 2B \operatorname{sinc}(t2B) + B \operatorname{sinc}(2B(t-t_0)) + B \operatorname{sinc}(2B(t+t_0))$

4) $$E_z = \int_{-\infty}^{+\infty} |z(t)|^2 dt = \int_{-\infty}^{+\infty} |Z(f)|^2 df$$

$$z(t) = \frac{1}{2} \cos(2\pi f_0 t_0 + \varphi)\, x(t-t_0)$$

$$E_z = \frac{1}{4} \cos^2(2\pi f_0 t_0 + \varphi) \int_{-\infty}^{+\infty} |x(t-t_0)|^2 dt = \frac{1}{4} \cos^2(2\pi f_0 t_0 + \varphi)\, E_x$$

dove E_x è l'energia di $x(t)$. Allora E_z è massima quando $\cos^2(2\pi f_0 t_0 + \varphi) = 1$

$$E_{z,max} = \frac{1}{4} E_x$$

$$\cos^2(2\pi f_0 t_0 + \varphi) = 1 \Rightarrow 2\pi f_0 t_0 + \varphi = k\pi$$

$$\boxed{\varphi = -2\pi f_0 t_0 + k\pi}$$