$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx)$ polinomio trigonometrico

COEFFICIENTI DI FOURIER

$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\overset{t}{x}) \cos(k\overset{t}{x})\, d\overset{t}{x}$ $\qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\overset{t}{x}) \sin(k\overset{t}{x})\, d\overset{t}{x}$

$P : f \to S_n(x) \in \text{span}\{1, \cos(kx), \sin(kx)\}_{k=1,\dots,n} = V_n$

$\| f - S_n \| \leq \| f - g \| \qquad g \in V_n$

$\| f \| = \left(\int_{-\pi}^{\pi} |f|^2 \right)^{1/2}$

$\frac{a_0^2}{2} + \sum_{k=1}^{n} a_k^2 + b_k^2 \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx \qquad \forall n \in \mathbb{N}$

DISEG. DI BESSEL

$\lim_{k \to +\infty} a_k = \lim_{k \to +\infty} b_k = 0$

PROBLEMA $\quad S_n(x) \to f(x) \qquad f \in C(0, 2\pi)$

- TEOREMA $\quad \int_{-\pi}^{\pi} |S_n(x) - f(x)|^2 dx \xrightarrow[n \to +\infty]{} 0 \qquad \left(\text{ma basta } \int_{-\pi}^{\pi} f^2 < +\infty \right)$

$\shortparallel$

$\| S_n(x) - f(x) \|^2 \to 0$

- TEOREMA $\quad f \in C^1(\mathbb{R}) \quad f$ periodica di periodo 2π

$\Rightarrow \quad S_n(x) \to f(x) \quad \forall x \in \mathbb{R}$

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left\{ \frac{1}{2} + \sum_{k=1}^{n} \cos(k(t-x))\, dt \right\} dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \frac{\sin\left(\frac{2n+1}{2} (t-x) \right)}{2 \sin\left(\frac{t-x}{2} \right)} dt \qquad t - x = z$$

$$= \frac{1}{\pi} \int^{\pi} f(x+z) \sin\left(\frac{2n+1}{2} z \right) dz$$

$$\pi\int_{-\pi} \qquad \overline{\quad 2\sin\left(\frac{z}{2}\right)}$$

$$S_n(x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+z)\,\frac{\sin\left(\frac{2n+1}{2}z\right)}{2\sin\left(\frac{z}{2}\right)} - f(x)\,\overset{\overset{1}{\parallel}}{\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin\frac{2n+1}{2}z}{2\sin\frac{z}{2}}\,dz}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[f(x+z)-f(x)\right]\frac{\sin\left(\frac{2n+1}{2}z\right)}{2\sin\left(\frac{z}{2}\right)}\,dz$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\boxed{\underbrace{\overbrace{\frac{f(x+z)-f(x)}{z}}^{\text{REGOLARE}}}_{\text{ok}}\;\underbrace{\overbrace{\frac{z}{2\sin\left(\frac{z}{2}\right)}}^{\text{LIMITATA}}}_{\text{LIMITATO}}}\;\underset{\parallel}{\sin\left(\frac{2n+1}{2}z\right)}\,dz$$

SE $f\in C^1$ $\qquad \lim_{z\to0}\frac{\sin z}{z}=1$

$$\underset{\parallel}{\sin\left(nz+\frac{z}{2}\right)}$$

funzione integrabile

$g(x+z)$

$$\sin(nz)\cos\left(\frac{z}{2}\right)+\cos(nz)\sin\left(\frac{z}{2}\right)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\left[g(x+z)\cos\left(\frac{z}{2}\right)\right]\sin(nz)}_{\beta_n \xrightarrow[n\to+\infty]{} 0} + \underbrace{\left[g(x+z)\sin\left(\frac{z}{2}\right)\right]\cos(nz)}_{\alpha_n \xrightarrow[n\to+\infty]{} 0}$$

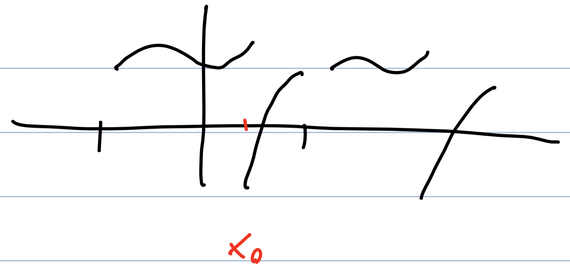
TEOREMA $f\in C(\mathbb{R})$

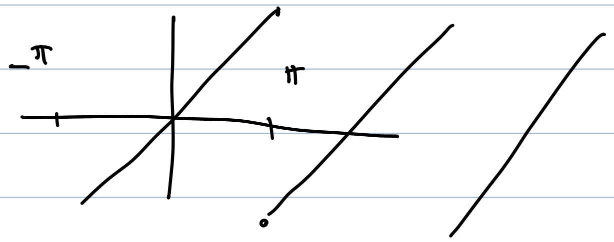
$f'_+(x_0)$, $f'_-(x_0)$ finite

$\Rightarrow$ $S_n(x_0)\to f(x_0)$

TEOREMA f è C^1 a tratti

$$S_n(x)\xrightarrow[n\to+\infty]{}\frac{f(x^+)+f(x^-)}{2}$$





PROPOSIZIONE f è pari $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin(kx)\,dx = 0$

f è dispari $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx = 0$

f è pari $f(x)\sin(kx)$ è dispari $\Rightarrow \int_{-\pi}^{\pi} f \sin(kx) = 0$

LEMMA f pari $f \in C^1(\mathbb{R})$ e periodica di periodo 2π

dispari

$$f(x) = \lim_{n\to+\infty} \frac{a_0}{2} + \sum_{1}^{n} a_k\cos(kx)$$

$$f(x) = \lim_{n\to+\infty} \sum_{1}^{n} b_k \sin(kx)$$

$$\boxed{S_n(x)} = \frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos(kx) + b_k\sin(kx) \xrightarrow{n\to+\infty} f(x)$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \frac{e^{ikx}+e^{-ikx}}{2} + b_k \frac{e^{ikx}-e^{-ikx}}{2i}$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} \left(\frac{a_k}{2} + \frac{b_k}{2i}\right) e^{ikx} + \left(\frac{a_k}{2} - \frac{b_k}{2i}\right) e^{-ikx}$$

$$= \frac{a_0}{2} + \frac{1}{2}\sum_{k=1}^{n} (a_k - ib_k)\, e^{ikx} + (a_k + ib_k)\, e^{-ikx}$$

$c_0 = \frac{a_0}{2}$ $c_k = \frac{a_k - ib_k}{2}$ $k>0$ $c_k = \frac{a_k + ib_k}{2}$ $k<0$

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} \xrightarrow{n\to+\infty} f(x)$$

$$\sum_{k\in\mathbb{Z}} c_k e^{ikx} = f(x)$$ SERIE BILATERA

$$\sum_{k\in\mathbb{Z}} c_k e^{ikx} = \lim_{n\to+\infty} \sum_{k=-n}^{n} c_k e^{ikx} = f(x)$$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-ikx}\,dx$$

● ESERCIZIO $f:\mathbb{R}\to\mathbb{R}$ $c_{-k} = \overline{c_k}$

$A = A^T$ $A \in M(n\times n;\mathbb{R})$ $\exists\ v_1 \,..\, v_n$ autovettori

$\lambda_1 \,..\, \lambda_n$

$$A \rightsquigarrow \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$Q^T A Q = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$ $A v_k = \lambda_k v_k$

$$A = -\frac{d^2}{dx^2}$$ $$\frac{d^2}{dx^2}\sin(kx) = \frac{d}{dx}\, k\cos(kx) = -k^2 \sin(kx)$$

$$\frac{d^2}{dx^2}\cos(kx) = -k^2\cos(kx)$$

$$-\frac{d^3}{dx^2}\sin(kx) = k^2 \sin(kx)$$ $$\mathcal{L} = -\frac{d^2}{dx^2}$$

$$-\frac{d^2}{dx^3}\cos(kx) = k^2\cos(kx)$$ $\sin(kx)$, $\cos(kx)$ AUTOFUNZIONI

k^2 $k \in \mathbb{N}$ AUTOVALORI

$f(x) = x$ $x \in [-\pi, \pi]$ E POI PROLUNGATA PER PERIODICITÀ

DISPARI

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx = 0$$

$$b_k = \frac{1}{}\int^{\pi} x \sin(kx)\,dx = \frac{1}{}\left\{ x\left[-\cos(kx)\right]\Big|^{\pi} + \overset{\overset{0}{\parallel}}{\int^{\pi} 1\cos(kx)\,dx}\right]$$

$k \quad \pi \int_{-\pi} \quad f \quad g' \qquad \pi \Big[\Big(\frac{}{k} \Big) \Big|_{-\pi} \quad \int_{-\pi} \quad \frac{}{k} \Big]$

$$= -\frac{1}{\pi k} \quad x \cos(kx) \Big|_{-\pi}^{\pi}$$

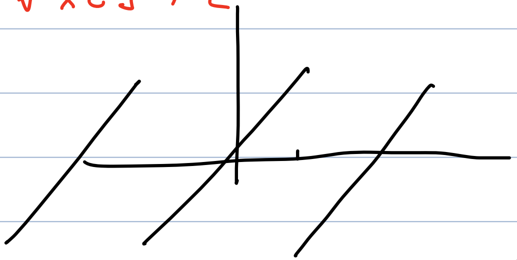
$$= -\frac{1}{k\pi} \left[\pi \cos(k\pi) - (-\pi) \cos(-k\pi) \right]$$

$$= -\frac{1}{k\pi} \left[\pi \cos(k\pi) + \pi \cos(k\pi) \right]$$

$$= -\frac{2}{k} \cos(k\pi) = -\frac{2}{k} (-1)^k$$

$$x = \sum_{k=1}^{\infty} -\frac{2}{k} (-1)^k \sin(kx) = \sum_{k=1}^{\infty} b_k \sin(kx)$$

$\forall\, x \in \,]-\pi, \pi[$



$$\sum_{k=1}^{\infty} -\frac{2}{k} (-1)^k \sin(k\pi) = 0$$

$x = \frac{\pi}{2}$

$$\frac{\pi}{2} = \sum_{k=1}^{\infty} \frac{2}{k} (-1)^{k+1} \sin\left(\frac{k\pi}{2}\right)$$

$$\sin\left(k\frac{\pi}{2}\right) = \begin{array}{|ccccc} 0 & 1 & 0 & -1 & 0 \\ k=0 & k=1 & k=2 & k=3 & k=4 \end{array}$$

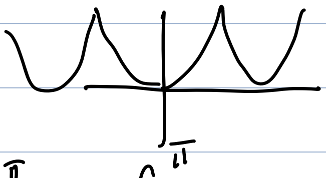
- $\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{1}{k} (-1)^{k+1} \sin\left(\frac{k\pi}{2}\right) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$

$\text{arctg}\, x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \dots \qquad x = 1$

$|x| < 1$

$x^2 \qquad x \in [\pi, \pi] \qquad$ PARI $\qquad \int_{-\pi}^{\pi} x^2 \sin(kx) = 0$

$$x^2 = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx)$$



$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(kx) dx = \frac{1}{} \Big[x^2 \sin(\cancel{kx}) \Big|^{\pi} - \int^{\pi} 2x \sin(kx) dx \Big]$

$$\pi \int_{-\pi} \quad f \quad g' \qquad\qquad \pi \Big[\quad \cancel{k} \quad \Big|_{-\pi} \quad \int_{-\pi} \quad f' \quad g \quad k \Big]$$
$$\qquad\qquad\qquad\qquad f \quad g$$

$k \geq 1$

$$= -\frac{1}{2k}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} x \sin(kx)\,dx\right]$$

$$= -\frac{1}{k}\;\frac{2}{k}(-1)^{k+1} = \frac{4}{k^2}(-1)^{k+2} = \frac{4}{k^2}(-1)^{k}$$

$$b_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\,\frac{x^3}{3}\Big|_{-\pi}^{\pi} = \frac{1}{\pi}\,\frac{2}{3}\pi^3 = \frac{2}{3}\pi^2$$

$$x^2 = \frac{1}{3}\pi^2 + \sum_{k=1}^{\infty}\frac{4}{k^2}(-1)^k \cos(kx) \qquad = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx)$$

$x = \pi$

$$\pi^2 = \frac{1}{3}\pi^2 + \sum_{k=1}^{\infty}\frac{4}{k^2}(-1)^k\cos(k\pi) = \frac{1}{3}\pi^2 + 4\sum_{k=1}^{\infty}\frac{1}{k^2}(-1)^k\overset{\overset{1}{\shortparallel}}{(-1)^k}$$

$$\frac{2}{3}\pi^2 = \left(1-\frac{1}{3}\right)\pi^2 = 4\sum_{k=1}^{\infty}\frac{1}{k^2}\cdot 1$$

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{2\pi^2}{4\cdot 3} = \frac{\pi^2}{6} \qquad\qquad \sum_{k=1}^{\infty}\frac{1}{k^3} = ?$$

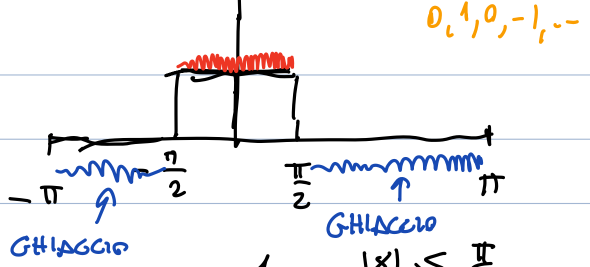
$$x^4 \qquad\qquad \sum\frac{1}{k^4} \overset{?}{=} \frac{\pi^4}{96}$$

$$u_t = u_{xx} \qquad\qquad u(-\pi) = u(\pi)$$

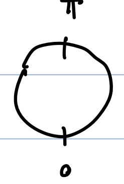
$$u(t,x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k e^{-k^2 t}\cos(kx) + b_k e^{-k^2 t}\sin(kx)$$

$$u(0,x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(kx) + \underset{\overset{\shortparallel}{0}}{b_k}\sin(kx)$$

a



$0, 1, 0, -1, \ldots$



$$u(0,x) = \begin{cases} 1 & |x| \leq \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{cases}$$

PARI

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} u(0,x) dx = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \, dx = \frac{1}{\pi} \pi = 1$$

$$a_k = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} 1 \cdot \cos(kx) dx = \frac{1}{\pi} \frac{\sin(kx)}{k} \Big|_{-\frac{\pi}{2}}^{\pi/2}$$

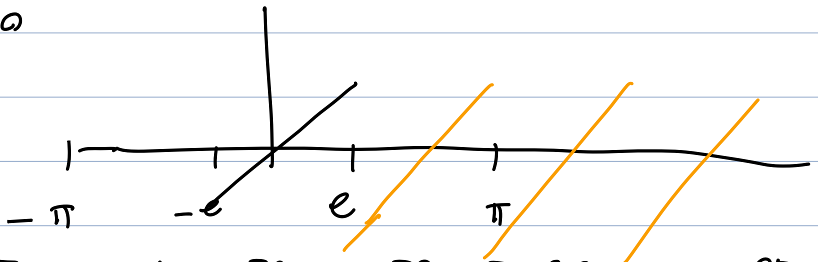
$$= \frac{1}{\pi k} \left[\sin\left(k\frac{\pi}{2}\right) - \sin\left(-k\frac{\pi}{2}\right) \right]$$

$$= \frac{1}{\pi k} 2 \sin\left(\frac{k\pi}{2}\right)$$

$f : [-\pi, \pi] \to \mathbb{R}$

$f : [-\ell, \ell] \to \mathbb{R}$ PROLUNGATA PERIODICAMENTE

$\ell > 0$



SERIE DI FOURIER PER FUNZIONI DI PERIODO 2ℓ $\ell > 0$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi}{\ell} x\right) + b_k \sin\left(\frac{k\pi}{\ell} x\right)$$

$kx \longmapsto \frac{k\pi}{\ell} x$

$\pi \searrow$

$$a_k = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos\left(\frac{k\pi}{\ell} x\right) dx \qquad b_k = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin\left(\frac{k\pi}{\ell} x\right)$$

$\pi \nearrow$ $\leftarrow -\pi$

$$a_0 = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) dx$$

X VARIABILE FISICA

K NUMERI DI ONDA (FREQUENZE)