Semester Two Examination, 2022

Question/Answer booklet

MATHEMATICS METHODS UNITS 1&2

Section Two: Calculator-assumed

SOLUTIONS

Student Name ______________________________

Teacher Name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Number of additional answer booklets used (if applicable): ☐

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators, which can include scientific, graphic and Computer Algebra System (CAS) calculators, are permitted in this ATAR course examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	12	12	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **twelve** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

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Question 8 **(8 marks)**

In an industrial process to create hydrogen, an engineer observed that the higher the power density P W/cm^2, the shorter the cathode lifetime T in hours, so that when $P = 0.8, T = 180$ and when $P = 3.8, T = 45$.

The relationship between the variables is of the form $T = \dfrac{a}{bP + 0.1}$, where a and b are constants.

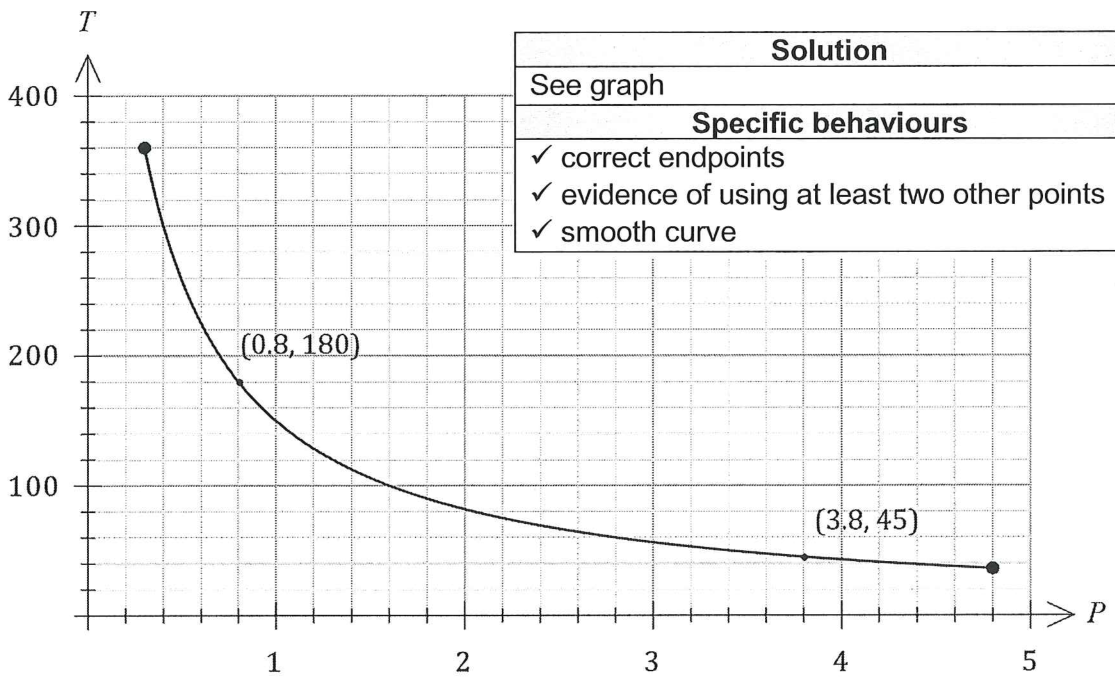
(a) Determine the value of a and the value of b. (3 marks)

Solution
$180 = \dfrac{a}{0.8b + 0.1}, \qquad 45 = \dfrac{a}{3.8b + 0.1}$ Solving simultaneously with CAS gives $a = 90$ and $b = 0.5$.
Specific behaviours
✓ uses points to form two equations   ✓ correct value of a ✓ correct value of b

(b) If the process is only possible for $0.3 \leq P \leq 4.8$, state the corresponding range of T. (2 marks)

Solution
$T(0.3) = 360, \qquad T(4.8) = 36$ Hence range of T is $36 \leq T \leq 360$.
Specific behaviours
✓ obtains lower or upper bound   ✓ correct domain as inequality

(c) Graph the relationship on the axes below over the domain $0.3 \leq P \leq 4.8$. (3 marks)



Solution
See graph
Specific behaviours
✓ correct endpoints   ✓ evidence of using at least two other points   ✓ smooth curve

Question 9 **(8 marks)**

An online store offers 32 different art prints, of which 9 are abstract, 12 are modern and the remainder impressionist.

(a) Determine the number of different selections that can be made

(i) when 5 different prints are bought from the store. (2 marks)

Solution
$\binom{32}{5} = 201\,376$
Specific behaviours
✓ indicates correct use of combination notation ✓ correct number of selections

(ii) when 4 different impressionist prints are bought from the store. (2 marks)

Solution
$\binom{32-9-12}{4} = \binom{11}{4} = 330$
Specific behaviours
✓ indicates correct use of combination notation ✓ correct number of selections

(b) A random selection of 3 different prints sold by the store is made. Determine the probability that

(i) none of the prints in the selection are abstract. (3 marks)

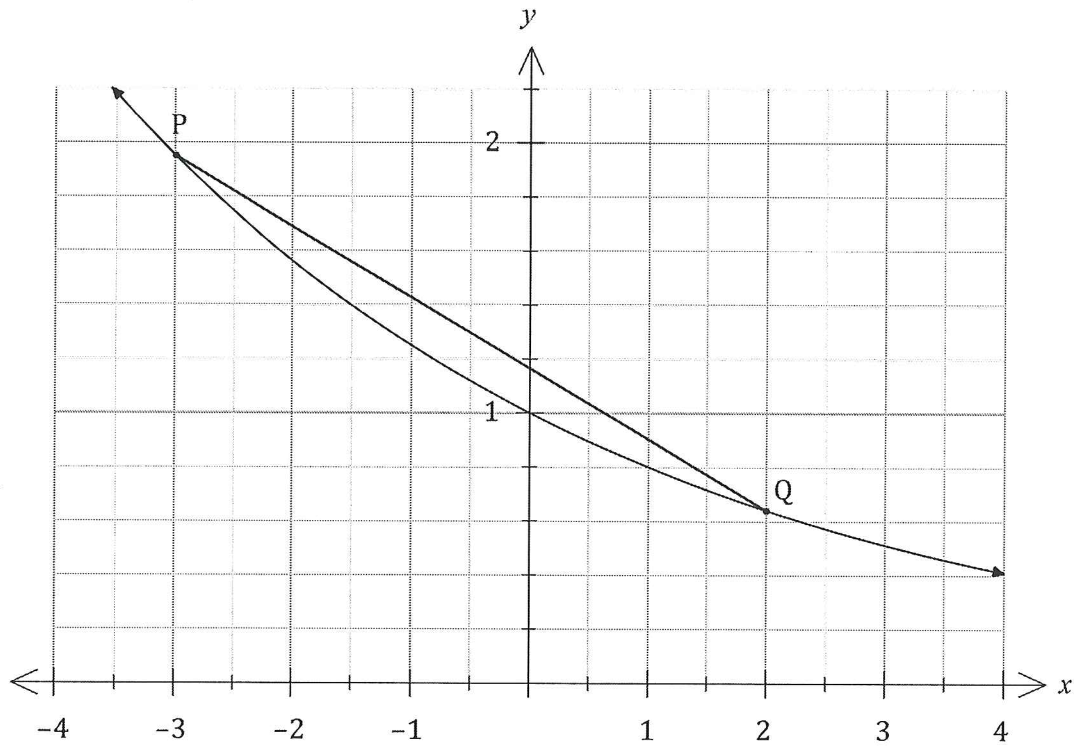
Solution
$P = \dfrac{\binom{32-9}{3}}{\binom{32}{3}} = \dfrac{\binom{23}{3}}{\binom{32}{3}} = \dfrac{1771}{4960} \approx 0.3571$
Specific behaviours
✓ indicates correct method for numerator ✓ indicates correct method for denominator ✓ correct probability

(ii) at least one of the prints in the selection is abstract. (1 mark)

Solution
$P = 1 - \dfrac{1771}{4960} = \dfrac{3189}{4960} \approx 0.6429$
Specific behaviours
✓ correct probability

Question 10 **(8 marks)**

A function is defined by $f(x) = 0.8^x$ and the graph of $y = f(x)$ is shown below.



(a) Draw the chord between the points P and Q on the curve $y = f(x)$ that have x-coordinates -3 and 2 respectively and determine the slope of this chord as an exact value. (3 marks)

Solution
$$\frac{f(2)-f(-3)}{2-(-3)} = \frac{\frac{16}{25}-\frac{125}{64}}{5} = -\frac{2101}{8000}$$ or $$\frac{f(2)-f(-3)}{2-(-3)} = \frac{0.64-1.953}{5} = -0.2625$$
Specific behaviours
✓ draws chord with ruler ✓ calculates y-coordinates of points ✓ calculates gradient

The point R with x-coordinate $-3+h$ lies on the curve between P and Q, where $h>0$.

(b) Use $\frac{\delta y}{\delta x}=\frac{f(x+h)-f(x)}{h}$ to calculate the slope of chord PR when

(i) $h=0.5$. (2 marks)

Solution
$\frac{\delta y}{\delta x}=\frac{f(-2.5)-f(-3)}{0.5}=\frac{1.747-1.953}{0.5}=-0.412$
Specific behaviours
✓ uses correct values of $x+h$ and x
✓ correct slope, to at least 2 dp

(ii) $h=0.1$. (1 mark)

Solution
$\frac{\delta y}{\delta x}=\frac{f(-2.9)-f(-3)}{0.1}=\frac{1.910-1.953}{0.1}=-0.431$
Specific behaviours
✓ correct slope, to at least 2 dp

(c) Show use of $\frac{\delta y}{\delta x}=\frac{f(x+h)-f(x)}{h}$ to determine the slope of tangent to the curve at P that is correct to 3 decimal places. (2 marks)

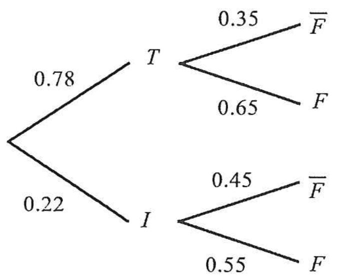
Solution
$h=0.0001\Rightarrow\frac{\delta y}{\delta x}=\frac{f(-2.9999)-f(-3)}{0.0001}=-0.436$
Specific behaviours
✓ chooses h where $0<h<0.0068$
✓ correct slope

Question 11 **(9 marks)**

Clinical records for sports injuries classified the type of sport involved as either individual or team. The records show that 78% of injuries were the result of team sport, and that after initial treatment, 35% of team sport and 45% of individual sport related injuries required no further treatment.

Determine the probability that a randomly chosen sports injury

(a) was sustained through individual sport and required no further treatment. (2 marks)



Solution
$P(I \cap \overline{F}) = (1 - 0.78) \times 0.45 = 0.22 \times 0.45 = 0.099$
Specific behaviours
✓ shows correct use of probability rule ✓ correct probability

(b) required further treatment. (3 marks)

Solution
$P(F) = P\big((T \cap F) \cup (I \cap F)\big) = 0.78 \times (1 - 0.35) + 0.22 \times (1 - 0.45)$ $= 0.78 \times 0.65 + 0.22 \times 0.55$ $= 0.507 + 0.121$ $= 0.628 = \dfrac{157}{250}$
Specific behaviours
✓ indicates correct probability for $T \cap F$ ✓ indicates correct probability for $I \cap F$ ✓ correct probability

(c) was sustained through individual sport or required no further treatment. (2 marks)

Solution
$P\big(I \cup (T \cap \overline{F})\big) = 0.22 + 0.78 \times 0.35$ $= 0.22 + 0.273$ $= 0.493$
Specific behaviours
✓ shows correct use of probability rules ✓ correct probability

(d) was sustained through team sport, given that the patient was known to have required no further treatment. (2 marks)

Solution
$P(T\|\overline{F}) = \dfrac{P(T \cap \overline{F})}{P(\overline{F})}$ $= \dfrac{0.273}{1 - 0.628} = \dfrac{0.273}{0.372} = \dfrac{91}{124} \approx 0.734$
Specific behaviours
✓ shows correct use of conditional probability rule ✓ correct probability

Question 12 **(8 marks)**

(a) A large outdoor amphitheatre has 15 seats in the first row, 19 in the second row and so on, where each row has four more seats than the previous row. There are 34 rows of seats in the amphitheatre.

(i) Determine the number of seats in the last row of the amphitheatre. (2 marks)

Solution
$T_n = 15 + 4(n-1)$ $T_{34} = 15 + 4(33) = 147$ seats.
Specific behaviours
✓ indicates first term and common difference of sequence ✓ correct number of seats

(ii) Determine the total number of seats in the last 16 rows of the amphitheatre. (3 marks)

Solution
$S_{34} = \frac{34}{2}\big(2(15) + 4(34-1)\big) = 2754$ $S_{34-16} = S_{18} = \frac{18}{2}\big(2(15) + 4(18-1)\big) = 882$ Hence in last 16 rows there are $2754 - 882 = 1872$ seats.
Specific behaviours
✓ sum of all rows ✓ sum of first 18 rows ✓ correct number of seats in last 16 rows

(b) The sum of the first and second terms of a geometric series is 18, and the sum of the second and third terms of the series is 63. Determine the sum of the first three terms of the series. (3 marks)

Solution
Let a = first term and r = common ratio so that $a + ar = a(1+r) = 18, \qquad ar + ar^2 = ar(1+r) = 63$ $\frac{ar(1+r)}{a(1+r)} = r = \frac{63}{18} = 3.5, \qquad a(1+3.5) = 18 \rightarrow a = 4$ Hence sum of first three terms is $4 + 14 + 49 = 67$.
Specific behaviours
✓ expressions using a and r for both sums ✓ obtains correct ratio ✓ correct sum of first three terms

Question 13 **(9 marks)**

Let S_n be the sum of the first n terms of an arithmetic sequence with first term a and common difference d.

The sum of the first 16 terms of the sequence is 432.

(a) Show that $2a + 15d = 54$. (2 marks)

Solution
Using sum of AP formula: $$\frac{16}{2}\big(2a + d(16-1)\big) = 432$$ $$8(2a + 15d) = 432$$ $$2a + 15d = 54$$
Specific behaviours
✓ substitutes correctly into sum of AP formula ✓ shows at least one correct simplification step

The sum of the first 76 terms of the sequence is 684.

(b) Determine the sum of the first 53 terms of the sequence. (3 marks)

Solution
Equation using second sum, as in (a): $$\frac{76}{2}(2a + 75d) = 684$$ $$38(2a + 75d) = 684$$ $$2a + 75d = 18$$ Solving simultaneously $2a + 15d = 54$ and $2a + 75d = 18$ (CAS) gives $a = 31.5$ and $d = -0.6$. Hence $S_{53} = 842.7$.
Specific behaviours
✓ forms second equation ✓ solves equations for a, d ✓ calculates correct sum

The sum S_n can be expressed in the form $S_n = pn^2 + qn$, where p and q are constants.

(c) Determine the value of p, the value of q and then use a method involving differentiation to show that S_{53} is the maximum sum of the sequence. (4 marks)

Solution
$$S_n = \frac{n}{2}\left(2(31.5) - 0.6(n-1)\right) = -\frac{3n^2}{10} + \frac{159n}{5} = -0.3n^2 + 31.8n$$ $$p = -0.3, \qquad q = 31.8$$ $$\frac{dS}{dn} = -0.6n + 31.8$$ $$\frac{dS}{dn} = 0 \Rightarrow n = 53$$ S_n is a quadratic with a negative n^2 coefficient, so it has a maximum at $n = 53$.
Specific behaviours
✓ values of p and q ✓ correctly differentiates   ✓ correctly solves derivative equal to 0 for n ✓ makes reasonable case for maximum

Question 14 **(9 marks)**

(a) Express $108°$ as an exact radian measure and hence determine the length of an arc of a circle that subtends an angle of $108°$ at the centre of a circle of radius 15 cm. (2 marks)

Solution
$108° = \dfrac{108\pi}{180} = \dfrac{3\pi}{5}$ $l = r\theta = 15 \times \dfrac{3\pi}{5} = 9\pi \approx 28.3 \text{ cm}$
Specific behaviours
✓ angle in radians ✓ correct arc length

(b) Determine the area of a segment of a circle that subtends an angle of $54°$ at the centre of a circle of radius 46 cm. (2 marks)

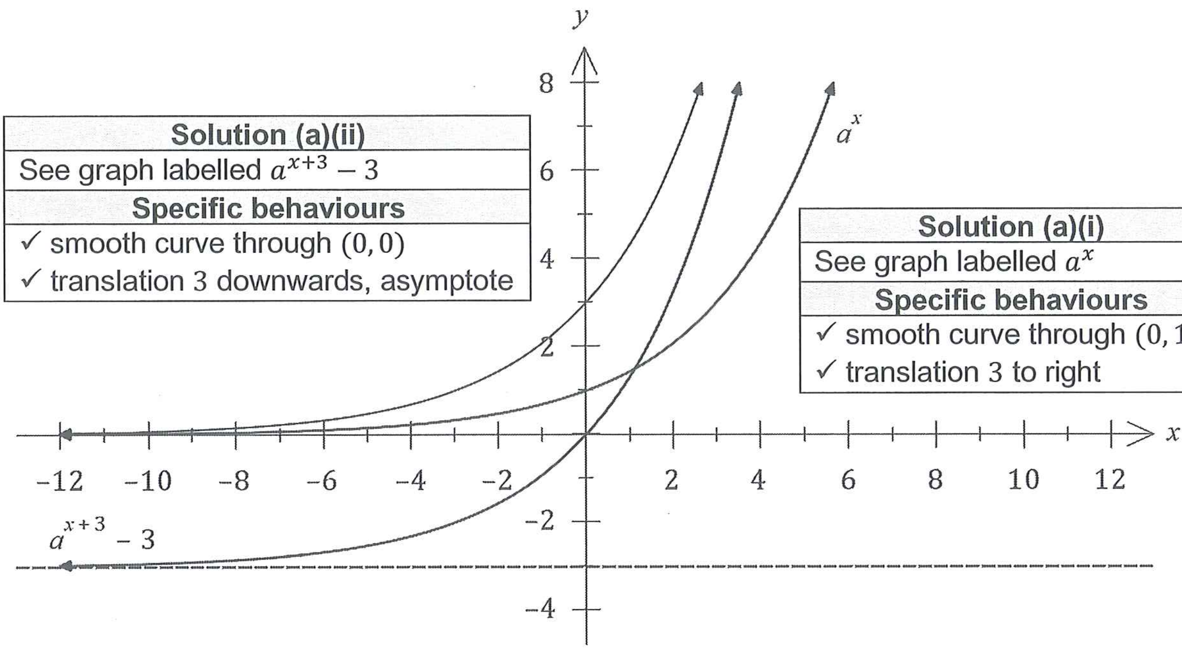
Solution
$A = \dfrac{46^2}{2}\left(\dfrac{3\pi}{10} - \sin\dfrac{3\pi}{10}\right)$ $= 141.2 \text{ cm}^2$
Specific behaviours
✓ substitutes correctly ✓ correct area of segment

(c) The lengths of the sides of a triangle with an area of 190 cm^2 are in the ratio $4:5:8$. Determine the length of the shortest side of the triangle. (5 marks)

Solution
Let angle opposite longest side be θ so that $\cos\theta = \dfrac{4^2 + 5^2 - 8^2}{2(4)(5)}$ $\theta = 125.1°$ (Other possible angles are $24.1°$ or $30.8°$) Then using area formula with sides of $4x, 5x$ and $8x$: $190 = \dfrac{1}{2}(4x)(5x)\sin 125.1°$ $x = 4.819$ Hence shortest side is $4 \times 4.819 = 19.3 \text{ cm}$.
Specific behaviours
✓ shows use of cosine rule to obtain angle ✓ one correct angle in triangle ✓ shows use of area formula with variable ✓ solves for variable ✓ correct length of required side

Question 15 **(7 marks)**

The graph of $y = a^{x+3}$ is shown below, where a is a positive constant.



Solution (a)(ii)
See graph labelled $a^{x+3} - 3$
Specific behaviours
✓ smooth curve through $(0, 0)$
✓ translation 3 downwards, asymptote

Solution (a)(i)
See graph labelled a^x
Specific behaviours
✓ smooth curve through $(0, 1)$
✓ translation 3 to right

(a) On the axes above sketch and label the graphs of

(i) $y = a^x$. (2 marks)

(ii) $y = a^{x+3} - 3$. (2 marks)

(b) Given that the curve $y = a^{x+3}$ passes through the point $(0, 3)$, solve $a^x = a^{x+3} - 3$ for x. (3 marks)

Solution
$3 = a^3 \Rightarrow a = \sqrt[3]{3} = 1.442$ $\sqrt[3]{3}^{\,x} = \sqrt[3]{3}^{\,x+3} - 3$ $x = 1.107$
Specific behaviours
✓ forms equation for a ✓ correct value of a ✓ correct value of x

Question 16 **(9 marks)**

(a) Determine the global minimum and maximum values of the function $f(x) = 10 - 4x - 2x^2$ when the domain of f is restricted to $-3 \leq x \leq 2$. (2 marks)

Solution
Using CAS, f has a local maximum at $(-1, 12)$, $f(-3) = 4$ and $f(2) = -6$. Hence global minimum value of function is -6 and global maximum is 12.
Specific behaviours
✓ correct global minimum ✓ correct global maximum

(b) Determine the discriminant of $g(x) = 6x - 9 - x^2$ and use it to explain how many roots the function g has. (2 marks)

Solution
$\Delta = (6)^2 - 4(-1)(-9) = 36 - 36 = 0$ Hence g has 1 root as the discriminant is equal to 0.
Specific behaviours
✓ correct value of Δ ✓ number of roots, with reasoning

(c) The graph of $y = ax^2 + bx + c$ is symmetrical about the line $x = 1$ and passes through the points $(0, -21)$ and $(-2, 11)$. Determine the roots of the graph. (5 marks)

Solution
When $x = 0, y = -21 \Rightarrow c = -21$. Using line of symmetry $$-\frac{b}{2a} = 1 \Rightarrow b = -2a$$ Using $(-2, 11)$: $(-2)^2a + (-2)b - 21 = 4a - 2(-2a) - 21 = 11$ $8a = 32$ $a = 4, \quad b = -2(4) = -8$ Hence $$y = 4x^2 - 8x - 21$$ Using CAS, $4x^2 - 8x - 21 = 0$ when $x = -1.5, x = 3.5$.
Specific behaviours
✓ obtains value of c ✓ obtains equation for a, b using line of symmetry ✓ obtains equation for a, b using point ✓ solves for a, b ✓ correct roots

Question 17 **(8 marks)**

The length of a rectangular prism is three times its width, and the sum of its height, width and length is 156 cm. Let the width of the rectangular prism be x cm.

(a) Show that the volume of the rectangular prism is $468x^2 - 12x^3$ cm^3. (2 marks)

Solution
$w = x, \quad l = 3x, \quad h = 156 - 4x$ $V = lwh$ $= x(3x)(156 - 4x)$ $= 468x^2 - 12x^3$
Specific behaviours
✓ correct expression for height ✓ correct factored expression for volume

(b) Use a method involving differentiation to determine the height of the rectangular prism that maximises its volume. (3 marks)

Solution
$\frac{dV}{dx} = 936x - 36x^2$ $= 36x(26 - x)$ Hence $\frac{dV}{dx} = 0$ when $x = 26$. Height for maximum volume is $h = 156 - 4(26) = 52$ cm.
Specific behaviours
✓ differentiates correctly ✓ equates derivative to 0 and solves for x ✓ states correct height

(c) Determine the maximum possible total surface area of the rectangular prism. (3 marks)

Solution
$A = 2(lw + wh + hl)$ $= 2\left(3x^2 + x(156 - 4x) + 3x(156 - 4x)\right)$ $= 1248x - 26x^2$ $\frac{dA}{dx} = 1248 - 52x, \qquad \frac{dA}{dx} = 0 \Rightarrow x = 24$ $A_{MAX} = 1248(24) - 26(24)^2 = 14\,976 \text{ cm}^2$.
Specific behaviours
✓ expression for TSA ✓ determines optimum value of x ✓ states correct maximum TSA

Question 18 **(8 marks)**

For two events A and B, $P(A) = 0.45, P(B) = b$ and $P(\bar{A} \cap B) = 0.22$. Determine the value of the constant b in each of the following cases:

(a) A and B are mutually exclusive. (1 mark)

Solution
$P(B) = P(\bar{A} \cap B) \rightarrow b = 0.22$
Specific behaviours
✓ correct value of b

(b) $P(A \mid B) = 0.6$. (2 marks)

Solution
$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} \rightarrow \dfrac{b - 0.22}{b} = 0.6 \rightarrow b = 0.55$
Specific behaviours
✓ formulates correct equation for b ✓ correct value of b

(c) $P(\bar{B} \mid A \cup B) = 0.3$. (2 marks)

Solution
$P(\bar{B} \mid A \cup B) = \dfrac{P(A \cap \bar{B})}{P(A \cup B)} \rightarrow \dfrac{0.45 + 0.22 - b}{0.45 + 0.22} = 0.3$ $\dfrac{0.67 - b}{0.67} = 0.3 \rightarrow b = 0.469$
Specific behaviours
✓ formulates correct equation for b ✓ correct value of b

(d) $P(B \mid A) = P(B)$. (3 marks)

Solution
Since $P(B \mid A) = P(B)$ then A and B are independent. $P(A \cap B) = P(A)P(B)$ $b - 0.22 = 0.45b$ $b = 0.4$
Specific behaviours
✓ indicates independence ✓ formulates correct equation for b ✓ correct value of b

Question 19 **(7 marks)**

On the first day of the year 1912 an industrialist established a fund with a deposit of \$8000 so that 110 years later it could be used to award research grants. The funds were to be invested at an interest rate of 4% per annum compounded yearly, with interest added to the fund on the last day of each year. Let A_n be the balance of the fund in dollars after n years, so that $A_0 = 8000$.

(a) Show that $A_0, A_1, A_2, \ldots$ form a geometric sequence and hence determine A_{110}, the balance of the fund after 110 years to the nearest cent. (3 marks)

Solution
$A_0 = 8000, \quad A_1 = 8000 \times 1.04, \quad A_2 = 8000 \times 1.04^2$ Hence the terms form a geometric sequence with $a = 8000$ and $r = 1.04$. $A_{110} = 8000 \times 1.04^{110}$ $= 598\,077.29$
Specific behaviours
✓ shows expressions for first three terms   ✓ states first term and common ratio   ✓ calculates balance (to nearest cent – penalise once in question)

Suppose instead that \$8000 was deposited into the fund on the first day of each year, starting in 1912, and let B_n be the balance of this modified fund in dollars after n years so that $B_0 = 8000$.

(b) Show that the balance of the modified fund after one year is \$16 320. (1 mark)

Solution
$B_1 = 8000 + 8000 \times 1.04 = 8000 + 8320 = 16\,320.$
Specific behaviours
✓ shows interest calculation and adds next deposit

(c) Express B_n as the sum of the terms of a geometric sequence and hence determine B_{110}, the balance of the modified fund after 110 years to the nearest cent. (3 marks)

Solution
$B_n = 8000 + 8000 \times 1.04 + 8000 \times 1.04^2 + \cdots + 8000 \times 1.04^n$ Note that B_n has $n + 1$ terms and so sum of first 111 terms of series is $S_{111} = \dfrac{8000(1.04^{111} - 1)}{(1.04 - 1)}$ $= 15\,350\,009.47$ $\therefore B_{110} = 15\,350\,009.47$
Specific behaviours
✓ expresses B_n as a series   ✓ correctly uses sum of series formula for first 111 terms   ✓ calculates balance to nearest cent

End of questions

Supplementary page

Question number: _________