12

SIMULTANEOUS LINEAR EQUATIONS AND THEIR APPLICATIONS, PIECEWISE LINEAR AND STEP GRAPHS

APPLICATIONS OF SIMULTANEOUS EQUATIONS, PIECEWISE AND STEP GRAPHS

TERMINOLOGY

elimination method
extrapolating
general form of a linear equation
gradient
interpolating
linear rule
linear graph
parallel lines
piecewise linear graph
point of intersection
simultaneous linear equations
step graph
substitution method
x-intercept
y-intercept

Prior learning

SIMULTANEOUS LINEAR EQUATIONS AND THEIR APPLICATIONS

- solve a pair of simultaneous linear equations, using technology when appropriate (ACMGM044)
- solve practical problems that involve finding the point of intersection of two straight-line graphs; for example, determining the break-even point where cost and revenue are represented by linear equations. (ACMGM045)

PIECEWISE LINEAR GRAPHS AND STEP GRAPHS

- sketch piecewise linear graphs and step graphs, using technology when appropriate (ACMGM046)
- interpret piecewise linear and step graphs used to model practical situations; for example, the tax paid as income increases, the change in the level of water in a tank over time when water is drawn off at different intervals and for different periods of time, the charging scheme for sending parcels of different weights through the post. (ACMGM047) AC

12.01 REVIEW OF LINEAR GRAPHS

A **linear equation** connecting y and x is one that results in a straight line when you graph it.

Equations such as $y = 2x + 1$; $y = \frac{1}{2}x$; $3x - 2y = 12$; $y = 8 - x$ are examples of linear equations.

The **general form of a linear equation** is $y = a + bx$, where a represents the y-intercept and b represents the **gradient** of the straight line.

IMPORTANT

Note that the value representing the gradient is the coefficient of the term containing the pronumeral x. In some instances you may need to rearrange the equation into the general form, by making y the subject, before identifying the gradient and the y-intercept.

Remember that the gradient of a line can be calculated using $\frac{\text{rise}}{\text{run}}$ or $\frac{y_2 - y_1}{x_2 - x_1}$.

Example 1

State the gradient and y-intercept of the lines below.

a $y = 4x - 3$ **b** $y - 2x = 6$ **c** $4x + 3y = 9$

Solution

a Write the equation in the form $y = a + bx$.

$y = -3 + 4x$

State the values of the gradient and y-intercept.

$b = 4$, $a = -3$

The gradient is 4 and the y-intercept is −3.

b Write the equation in the form $y = a + bx$.

$y - 2x = 6$

$y = 6 + 2x$

State the values of the gradient and y-intercept.

$b = 2$, $a = 6$

The gradient is 2 and the y-intercept is 6.

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c Write the equation in the form $y = a + bx$.

$$4x + 3y = 9$$
$$3y = 9 - 4x$$
$$y = 3 - \frac{4}{3}x$$

State the values of the gradient and y-intercept.

$b = -\frac{4}{3}$, $a = 3$

The gradient is $-\frac{4}{3}$ and the y-intercept is 3.

The gradient and y-intercept can be used to sketch a linear graph.

- Plot the y-intercept on the y-axis.
- Use the gradient to locate another point on the line.
- Rule a line through both marked points.
- Label the graph with its equation.

Remember that:

- a positive gradient slopes up from left to right
- a negative gradient slopes down from left to right.

IMPORTANT

To sketch the graph of a straight line, only two points are required.

Example 2

Sketch the graphs for each of the following linear functions using the gradient and y-intercept method.

a $y = 2x + 1$ **b** $2x + 3y = 12$

Solution

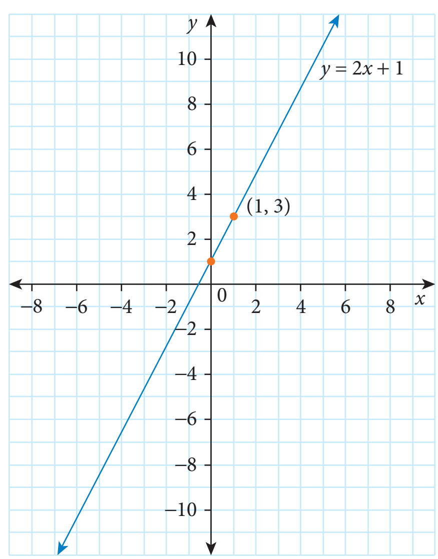
a Write the equation.

$y = 2x + 1$

Identify the gradient and the y-intercept from the equation.

Gradient: $b = 2$
y-intercept: $a = 1$

Plot the coordinates of the y-intercept: (0, 1).
Move 1 unit across to the right followed by 2 units up, plotting a point at (1, 3).
Join these points to sketch the line.
Label the line with its equation.



b Write the equation. $2x + 3y = 12$

Rearrange the equation to make y the subject. Subtract $2x$ from both sides. $3y = 12 - 2x$

Divide both sides by 3.

$$y = \frac{12}{3} - \frac{2}{3}x$$

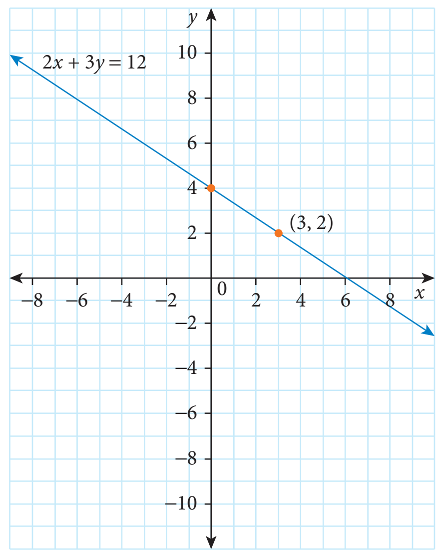
$$= 4 - \frac{2}{3}x$$

Identify the gradient and the y-intercept.

Gradient: $b = -\frac{2}{3}$

y-intercept: $a = 4$

Plot the coordinates of the y-intercept: (0, 4).
Move 2 units down and 3 units across, plotting a point at (3, 2).
Join these points to sketch the line.
Label the line with its equation.



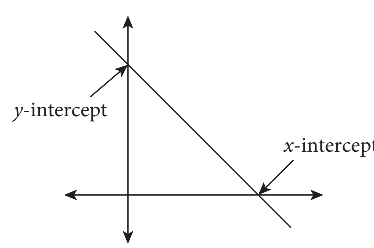
Two points that can be used to sketch the graph of a straight line are the ***x*-intercept** and the ***y*-intercept**.

All the points along the x-axis have coordinates with $y = 0$.

To find the x-intercept, let $y = 0$.

All the points along the y-axis have coordinates with $x = 0$.

To find the y-intercept, let $x = 0$.



IMPORTANT

Lines with equations in the form $y = a$, where a is a constant, are horizontal lines (or lines parallel to the x-axis). These lines pass through the number represented by the constant.

Lines with equations in the form $x = a$, where a is a constant, are vertical lines (or lines parallel to the y-axis). These lines pass through the number represented by the constant.

Example 3

Find the x- and y-intercepts for each of the following linear functions, and hence sketch their graphs.

a $y = 3 + 2x$ **b** $2x - 3y = 18$ **c** $y = 3$ **d** $x = -2$

Solution

a Write the equation. $y = 3 + 2x$

Find the x-intercept by letting $y = 0$. $0 = 3 + 2x$

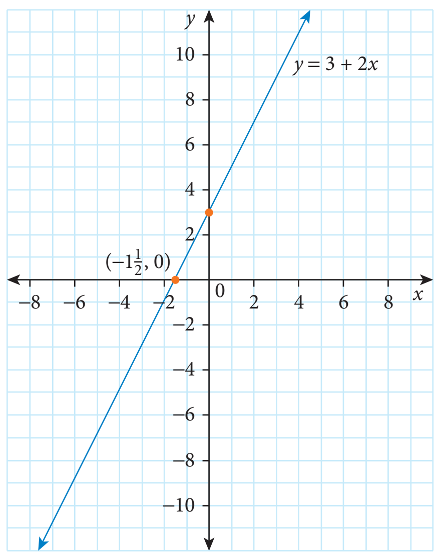
Solve the equation.

$$2x = -3$$
$$x = -\frac{3}{2}$$
$$= -1\frac{1}{2}$$

Find the y-intercept by letting $x = 0$.

$$y = 3 + 2 \times 0$$
$$y = 3$$

Plot the intercepts.
Join these points to sketch the line. Extend the line.
Remember to write the equation of the line next to its graph.



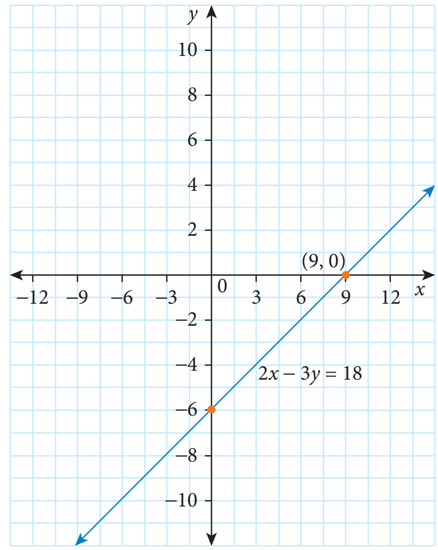
b Write the equation.
Find the x-intercept by letting $y = 0$.
Solve the equation for x.

$$2x - 3y = 18$$
$$2x - 3 \times 0 = 18$$
$$2x = 18$$
$$x = 9$$

Find the y-intercept by letting $x = 0$.
Solve the equation.

$$2 \times 0 - 3y = 18$$
$$-3y = 18$$
$$y = -6$$

Plot the intercepts.
Join these points to sketch the line. Extend the line.
Remember to write the equation of the line next to its graph.

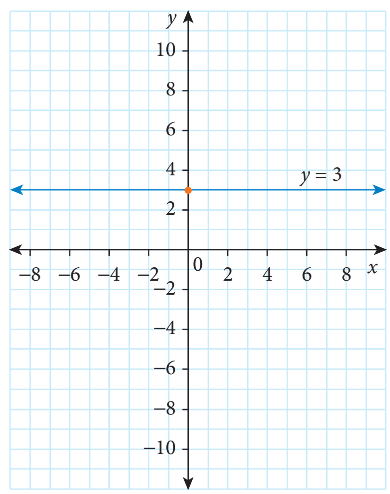


c Write the equation.

The equation is in the form $y = a$, hence the graph is a line parallel to the x-axis. The y-intercept is 3.

$y = 3$

Draw a horizontal line passing through the point (0, 3).
Remember to write the equation of the line next to its graph.



d Write the equation.

$x = -2$

The equation is in the form $x = a$. There is no y value present in the equation; hence the graph is a line parallel to the y-axis. The x-intercept is −2.

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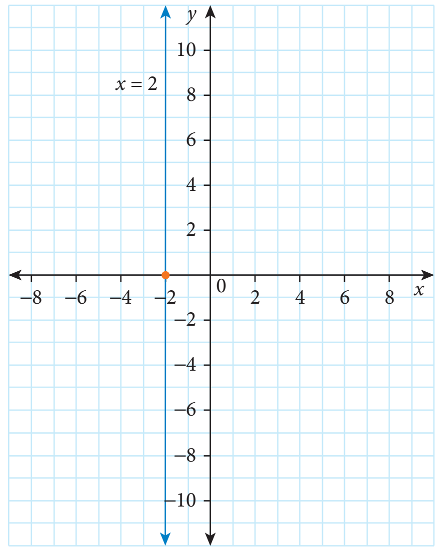
Draw a vertical line passing through the point (–2, 0). Remember to write the equation of the line next to its graph.



EXERCISE 12.01 Review of linear graphs

WS Substitution

Concepts and techniques

1 Example 1 For each of the linear functions given below, find:

i the gradient
ii the y-intercept

a $y = x - 2$
b $y = 6 + 3x$
c $y = 7 - \frac{1}{4}x$
d $x + 2y = 6$
e $8x - 4y = -24$
f $3y - 2x = 8$
g $3x + 5y - 20 = 0$
h $3x - 4y - 6 = 0$

2 Find the equation of each linear function.

a gradient = 1, y-intercept = –2
b gradient = –2, y-intercept = 2
c gradient = 5, y-intercept = 3
d gradient = –10, y-intercept = 0
e gradient = $\frac{1}{2}$, y-intercept = 1
f gradient = $-\frac{3}{4}$, y-intercept = –5
g gradient = $-\frac{3}{2}$, y-intercept = 3
h gradient = $\frac{2}{3}$, y-intercept = 0

3 Example 2 Sketch the following graphs using the gradient and y-intercept.

a $y = 3 + x$
b $y = -2 - x$
c $y = \frac{1}{3}x + 1$
d $4x + y = 8$
e $y = 3x$
f $4x + 6y = 0$
g $10y - 6x + 5 = 0$
h $5x + 4y - 100 = 0$

4 a The y-intercept of the line on the right is:

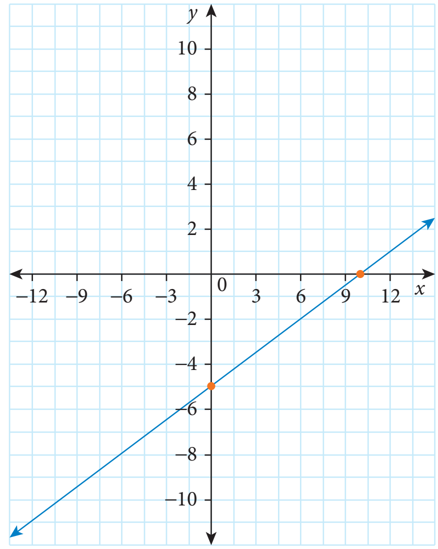
A -5 B -4 C $-\frac{1}{2}$

D $\frac{1}{2}$ E 2

b The gradient of the line on the right is:

A -5 B -2 C $-\frac{1}{2}$

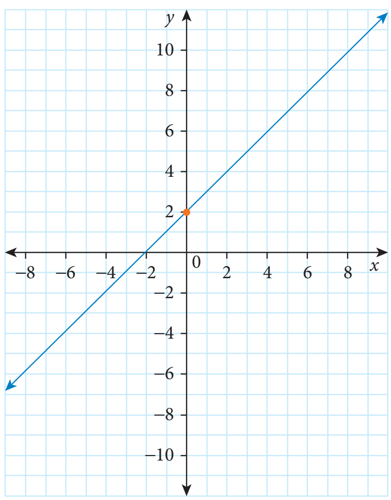
D $\frac{1}{2}$ E 2



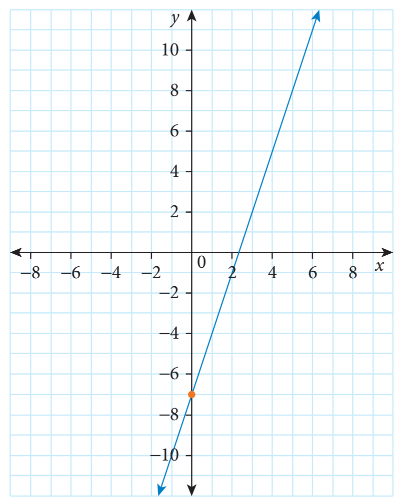
5 For each of the following graphs find:

i the y-intercept ii the gradient iii the equation of the line.

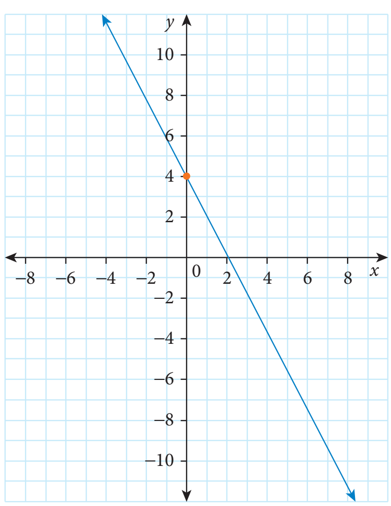
a



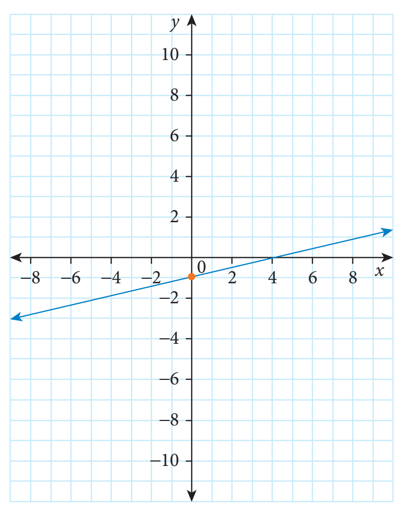
b



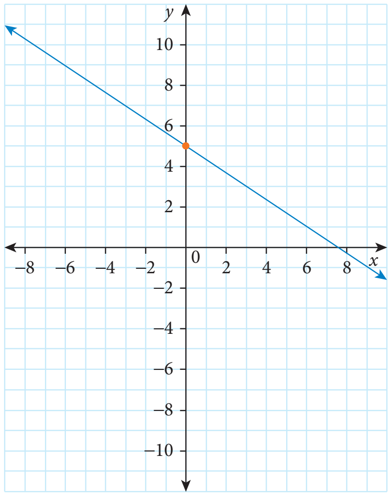
c



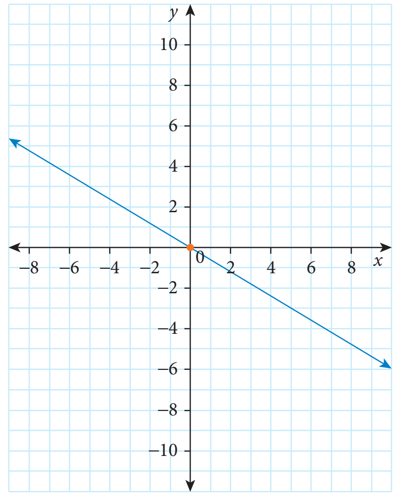
d



e



f



6 Example 3 The coordinates of the x- and y-intercepts respectively for the graph below are:

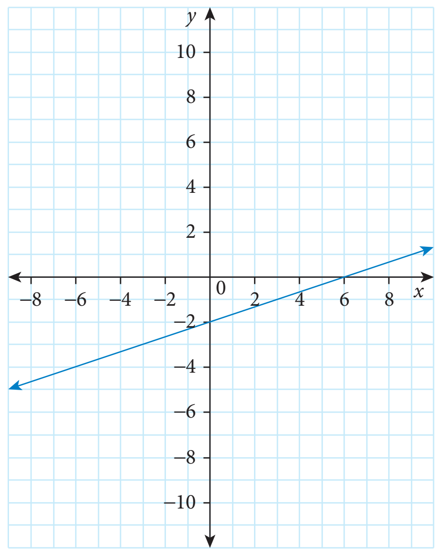
A (6, 0) and (0, –2)

B (–2, 6)

C (–2, 0) and (0, 6)

D (–2, 0) and (6, 0)

E (6, –2)



7 Calculate the x- and y-intercepts for each linear equation. Use the intercepts to sketch their graphs.

a $y = 2x - 2$

b $y = 8 - 2x$

c $4x + 3y = 12$

d $2x + y = 7$

e $y = \frac{2}{3}x + 5$

f $5y - 2x + 15 = 0$

g $x = -5$

h $y = 6$

8 Sketch the following graphs.

a $y = -5x$

b $y = \frac{1}{4}x$

c $y = 3$

d $x = 7$

e $x = -9$

f $y = -5.5$

g $y = 0$

h $x = 0$

Reasoning and communication

9 Write a sentence explaining what is special about the lines $x = 0$ and $y = 0$. (Hint: use your answers from question **8**.)

WS Finding the gradient between two points on a line

10 Find the equation (in the form $y = a + bx$) of the straight line passing through the points:

a $(-1, 3)$ and $(0, 4)$ b $(5, 1)$ and $(0, 5)$ c $(-2, 4)$ and $(0, -6)$

d $(3, 6)$ and $(1, 0)$ e $(-4, 2)$ and $(0, 0)$ f $(-3, 7)$ and $(3, 3)$.

11 Write down the gradient and C-intercept for the equation $C = 125 + 42n$.

12 A taxi fare costs \$5 flag fall plus \$2 per km.

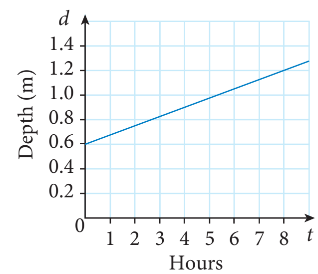
a Write an equation for the cost (C) of the taxi when travelling x km.

b Sketch the graph of this equation on the number plane.

c Find the gradient of this graph. What does the gradient represent?

d What is the vertical axis intercept of this graph? What does this point represent?

13 A swimming pool is being filled and the depth of water is recorded as it rises. The graph shows the changes in depth.



a What is the depth of water after 6.5 hours?

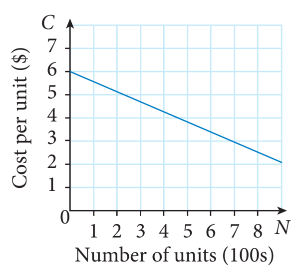
b When is the pool 1.2 m deep?

c Where does the graph intersect the vertical axis and what does this point represent?

d Find the gradient of the line. What does the gradient tell us about the depth of the water in the pool?

e Find the equation of the depth (d) of water over time (t).

14 The cost per unit of an electrical part decreases with more orders. The graph shows the cost.



In part **i** you found an answer within the given values or range of the graph. This is called **interpolating**.

In part **ii** you found an answer that fell outside the range of the graph. This is called **extrapolating**.

a Find the equation for the cost per unit of n electrical parts.

b Use this equation to find:

i the cost per unit of 300 parts

ii the number of parts that would each cost \$1.50.

15 The table below shows the quarterly cost of water usage for different volumes of water.

Volume, v (kL)	16	25	48	60	81	99
Cost, C (\$)	168.24	183	220.72	240.40	274.84	304.36

a What is the independent variable and what does it represent?

b Find a linear function for C in terms of v.

c Sketch a graph of this linear function.

d What does the gradient represent?

e What does the vertical axis intercept represent?

f What is the cost of using 54 kL of water?

g Find the volume of water usage that costs \$265.

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12.02 SOLVING SIMULTANEOUS EQUATIONS GRAPHICALLY

A pair of linear equations, such as $y = x + 2$ and $y = 2x - 5$, can be solved together to find the values of x and y that satisfy *both* equations. As they are solved at the same time, they are called **simultaneous linear equations**.

If simultaneous linear equations are graphed on the same Cartesian plane, their solution is the coordinates of the point where their graphs intersect.

The point where two or more graphs intersect is called the **point of intersection**.

You can check that your solution is correct by substituting the x and y values back into the original equations.

Sometimes the equations represent parallel lines that never intersect and therefore the simultaneous equations have no solution.

Example 4

Graph the equations $y = x + 2$ and $y = 2x - 3$ on the same set of axes and find their point of intersection.

Solution

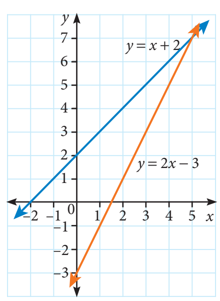
Both equations are written in the form $y = a + bx$ so it is easiest to use the gradient and y-intercept method.

$y = x + 2$ $\quad$ $y = 2x - 3$

Find the gradient and y-intercept for each equation.

gradient = 1 $\quad$ gradient = 2
y-intercept = 2 $\quad$ y-intercept = −3

Sketch both lines on the same set of axes.
Remember to write the equation of each line next to its graph.



Use the graph to read the coordinates of the point of intersection.

Point of intersection = (5, 7)

Check the solutions by substituting (5, 7) into the original equations.

$y = x + 2$
$7 = 5 + 2$ ✓
$y = 2x - 3$
$7 = 2 \times 5 - 3$ ✓

Write the answer.

The point of intersection is (5, 7).

It may be easier to use the x- and y-intercept method to sketch the graphs of the pair of simultaneous equations.

Example 5

Solve the simultaneous equations $2x + y = 7$ and $x - y = -4$ graphically.

Solution

Write down the equation of the first line. $2x + y = 7$

Find the x-intercept by letting $y = 0$. $2x + 0 = 7$

Solve the equation.
$$2x = 7$$
$$x = 3\tfrac{1}{2}$$

State the coordinates of the x-intercept. The coordinates of the x-intercept are $\left(3\tfrac{1}{2}, 0\right)$.

Find the y-intercept by letting $x = 0$. $2 \times 0 + y = 7$

Solve the equation. $y = 7$

State the coordinates of the y-intercept. The coordinates of the y-intercept are (0, 7).

Write down the equation of the second line. $x - y = -4$

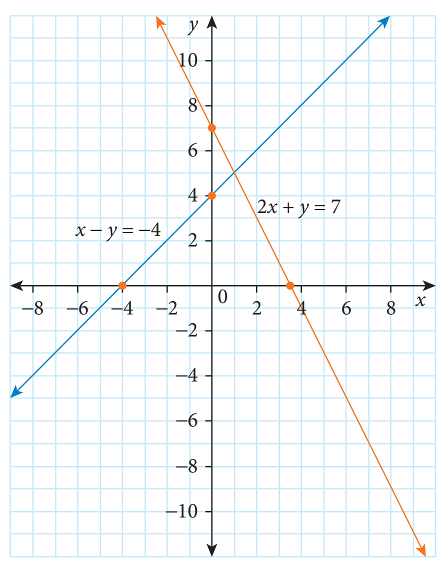
Find the x-intercept by letting $y = 0$ and solve.
$$x - 0 = -4$$
$$x = -4$$

State the coordinates of the x-intercept. The coordinates of the x-intercept are (−4, 0).

Find the y-intercept by letting $x = 0$ and solve.
$$0 - y = -4$$
$$y = 4$$

State the coordinates of the y-intercept. The coordinates of the y-intercept are (0, 4).

Sketch both lines on the same set of axes. Remember to write the equation of each line next to its graph.



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Use the graph to read the coordinates of the point of intersection.	Point of intersection = (1, 5)
Check the solutions by substituting (1, 5) into the original equations.	$2x + y = 7$ $2 + 5 = 7$ ✓ $x - y = -4$ $1 - 5 = -4$ ✓
Write the answer.	The solution to the simultaneous equations is $x = 1, y = 5$.

Example 6

Use a CAS calculator to graph and solve the equations $y = 3x - 2$ and $x + 2y = 10$.

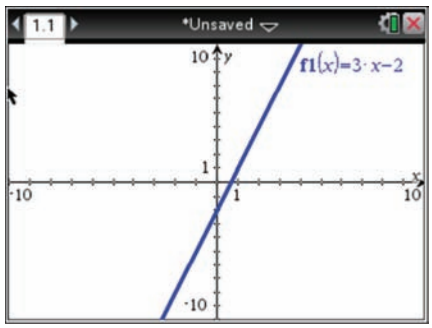
Solution

Write the equation.

$y = 3x - 2$

TI-Nspire CAS

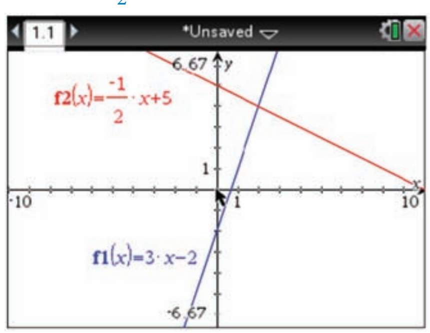
Open a New Document with a Graphs page.
In the entry line with the prompt $f1(x)$ =, type $3x - 2$.
Press enter.



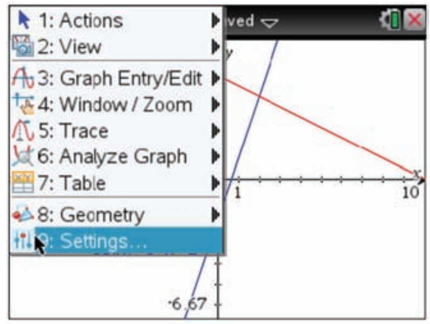
To type in the second equation we need to first write the equation in the form $y = a + bx$.

$$x + 2y = 10$$
$$2y = -x + 10$$
$$y = -\frac{1}{2}x + 5$$

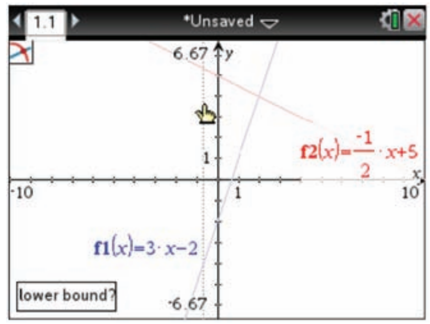
Press ctrl G to view another entry line.
Type in the second equation.
Press enter.
(You can use the arrow ▲▼ keys to move from one entry line to another.)



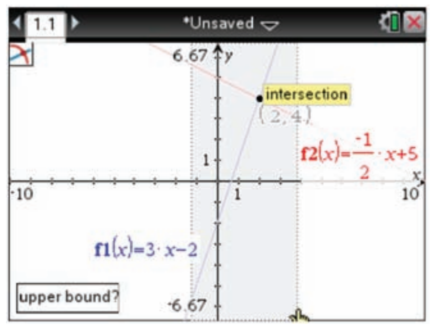
To find the point of intersection:
Press menu.
Select: 6: Analyze Graph then
Select: 4: Intersection
Press enter.



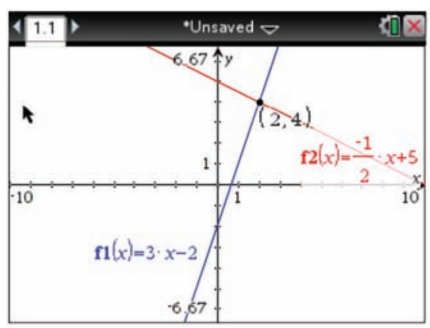
Use the ◀ key to move the lower bound line to the left of the point of intersection and press enter.



Use the ▶ key to move the upper bound line to the right of the point of intersection.



Press enter.



Use the graph to read the coordinates of the point of intersection.

Point of intersection = (2, 4)

Remember you can check that the solution is correct by substituting back into the original equations.

$y = 3x - 2$
$4 = 6 - 2$ ✓
$x + 2y = 10$
$2 + 8 = 10$ ✓

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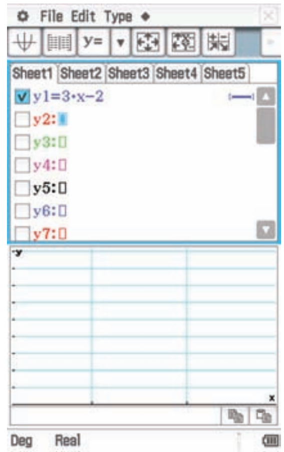
Write the answer.

The solution to the simultaneous equations is $x = 2$, $y = 4$.

ClassPad

Tap Menu then the **Graph** & Table application.

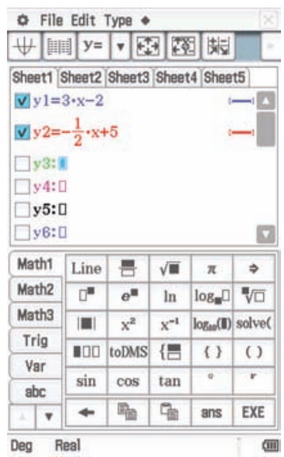
In the entry line next to $y1=$ type 3 $\boxed{x}$ $\boxed{-}$ 2 then press $\boxed{\text{EXE}}$.



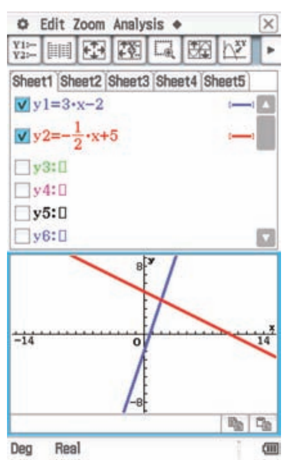
To type in the second equation, we need to first write the equation in the form $y = a + bx$.

$$x + 2y = 10$$
$$2y = -x + 10$$
$$y = -\frac{1}{2}x + 5$$

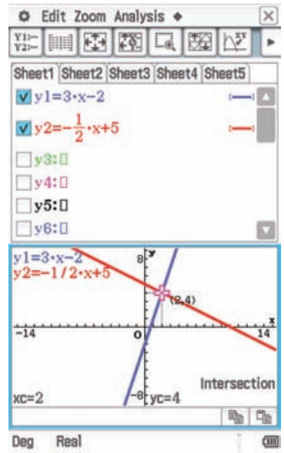
In the entry line next to $y2=$ type $-\frac{1}{2}x + 5$ then press $\boxed{\text{EXE}}$.



Tap the graph icon. To ensure that you can see the lines, you may need to tap **Zoom** then **Zoom Out,** or tap the view window icon to change the maximum and minimum of the x or y values.



Tap the graph screen then tap **Analysis** then **G-Solve** then **Intersection**.



The coordinates of the point of intersection are shown at the bottom of the graph screen.

Point of intersection = (2, 4)

Remember that you can check that the solution is correct by substituting back into the original equations.

$y = 3x - 2$
$4 = 6 - 2$ ✓
$x + 2y = 10$
$2 + 8 = 10$ ✓

Write the answer.

The solution to the simultaneous equations is $x = 2$, $y = 4$.

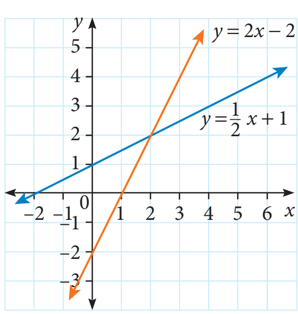
EXERCISE 12.02 Solving simultaneous equations graphically

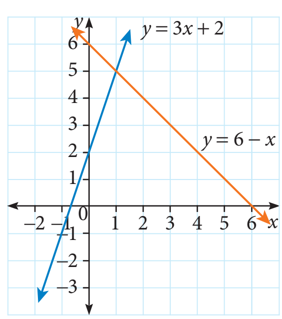
Concepts and techniques

1 Example 4 Find the point of intersection of each pair of simultaneous equations given their graphs below.

a $y = \frac{1}{2}x + 1$
$y = 2x - 2$

b $y = 3x + 2$
$y = 6 - x$





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2 Example 5 a Graph the equations $y = 2x + 2$ and $y = x + 5$ on the same set of axes.

b Use the graphs constructed in part **a** to solve the simultaneous equations $y = 2x + 2$ and $y = x + 5$.

WS Sketching simultaneous equations

3 Sketch each pair of linear functions to find the point of intersection.

a $y = 2x$
$y = x + 3$

b $y = x + 4$
$y = -x - 6$

c $y = 3x + 4$
$y = x + 1$

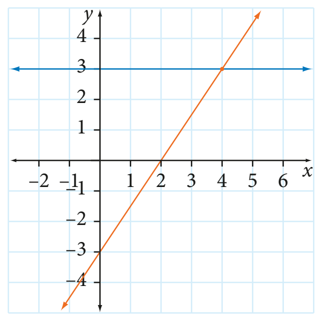
d $y = 2x + 1$
$y = 7 - x$

e $y = -4$
$y = 7x - 4$

f $y = -4x - 1$
$y = 3x + 6$

Remember to check that the solution is correct by substituting back into the original equations.

4 The graph shows the intersection of two linear functions.



a The equations of the lines are:

A $y = \frac{2}{3}x - 3$ and $x = 3$

B $y = \frac{3}{2}x - 3$ and $y = 3$

C $y = -\frac{2}{3}x - 3$ and $y = 3$

D $y = \frac{3}{2}x - 3$ and $x = 3$

E $y = \frac{2}{3}x - 3$ and $y = 3$

b The coordinates of their point of intersection are:

A (0, 3) B (2, −3) C (4, 3) D (2, 0) E (3, 4)

5 Examples 5, 6 Solve the following pairs of simultaneous equations graphically and check the solutions using a CAS.

a $y = x + 3$
$2x - y = -4$

b $x + y = 2$
$x - y = -6$

c $2x + y = 3$
$4x - y = 3$

d $3x - y = 4$
$2x - y = 2$

e $2x - y = -1$
$4x + y + 5 = 0$

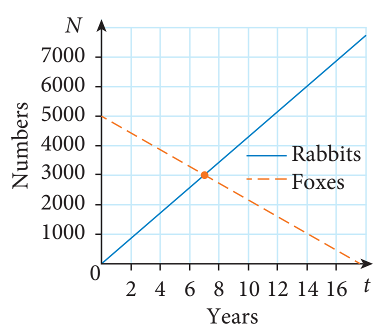
f $5y - 2x = 20$
$2y - 2x = 5$

Reasoning and communication

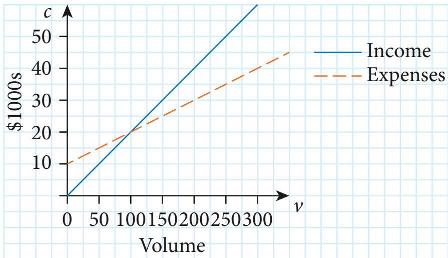
6 The number of rabbits in Blackwood Forest is increasing steadily, while the number of foxes is falling. The graph shows this information.

a When will the numbers of foxes and rabbits be equal?

b How many of each will there be at that time?



7 The graph below shows the income and expenses for the Gosford Petroleum Company.



a What is the point of intersection?

b Write a sentence discussing the value of the expenses and income at this point. What does the point of intersection represent?

8 a Sketch the line: $y = \frac{1}{2}x - 4$.

b On the same set of axes, sketch the line $4y = 2x + 12$.

c Comment on the type of lines that have been drawn.

d Will these lines have a point of intersection? Give reasons for your answer.

e How can you determine if two lines will intersect or not intersect without sketching their graphs? (*Hint*: What does the gradient determine when sketching a line?)

9 a Sketch the lines $3x - 2y = 5$ and $6y = 9x - 15$ on the same set of axes.

b What do you notice about the graphs?

c These lines are called **coincident lines**. Using your answers to parts **a** and **b**, write a definition of coincident lines.

12.03 SOLVING SIMULTANEOUS EQUATIONS BY SUBSTITUTION

You have seen how to graphically find solutions for pairs of simultaneous linear equations. This does not always give accurate results and can be time consuming. Algebraic methods can be more accurate and may also provide a solution more quickly.

The first algebraic method is called the **substitution method**.

IMPORTANT

Steps for the substitution method

1 Use one equation to construct a more complex equation that can be used to solve for one variable.

2 Substitute the formula for the variable into the second equation.

3 Solve the second equation.

4 Substitute the solution into one of the original equations.

Check the values in *both* of the original equations.

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Example 7

Solve the following simultaneous equations by substitution.

$y = 3 - 2x$

$-5x + 2y = -12$

Solution

Write and label the equations.	$y = 3 - 2x$ $-5x + 2y = -12$	[1] [2]
Substitute equation [1] into equation [2].	$-5x + 2(3 - 2x) = -12$	y from [1] into [2]
Expand the brackets and simplify the terms on the left-hand side.	$-5x + 6 - 4x = -12$ $-9x + 6 = -12$ $-9x = -18$	
Solve for x.	$x = 2$	
Substitute $x = 2$ into equation [1].	$y = 3 - 2 \times 2$	$x = 2$ into [1]
Simplify to solve for y.	$= -1$	
Remember that you can check if the solution is correct by substituting back into the original equations. Substitute $(2, -1)$ into equations [1] and [2].	[1] $-1 = 3 - 2 \times 2$ ✓ [2] $-5x + 2y = -5 \times 2 + 2 \times (-1)$ $= -10 - 2$ $= -12$ ✓	
Write the answer.	The solution to the simultaneous equations is $x = 2$ and $y = -1$.	

In some cases it may be necessary to transpose one equation to write one unknown in terms of the other before using the substitution method.

Example 8

Solve the following simultaneous equations by substitution:

$6x + 3y = 12$

$3x + 2y = -3$

Solution

Write and label the equations.	$6x + 3y = 12$ $3x + 2y = -3$	[1] [2]
Rearrange equation [1] to make y the subject. Label this equation [3].	$3y = 12 - 6x$ $y = 4 - 2x$	Rearrange [1] [3]
Substitute equation [3] into equation [2].	$3x + 2(4 - 2x) = -3$	y from [3] into [2]
Expand the brackets. Simplify the terms on the left-hand side. Solve for x.	$3x + 8 - 4x = -3$ $-x + 8 = -3$ $-x = -11$ $x = 11$	

Substitute $x = 11$ into equation [3].	$y = 4 - 2 \times 11$	$x = 11$ into [3]
Simplify to solve for y.	$y = -18$	
Remember that you can check if the solution is correct by substituting back into the original equations. Substitute $(11, -18)$ into equations [1] and [2].	[1] $6 \times 11 + 3 \times (-18) = 12$ $66 - 54 = 12$ ✓ [2] $3 \times 11 + 2 \times (-18) = -3$ $33 - 36 = -3$ ✓	
Write the answer.	The solution to the simultaneous equations is $x = 11$, $y = -18$.	

EXERCISE 12.03 Solving simultaneous equations by substitution

Concepts and techniques

WS Solving simultaneous equations

1 Example 7 Find the point of intersection for the lines $y = 3x - 1$ and $5x - 2y = 9$.

2 Solve the following simultaneous equations by substitution.

a $y = 11 - 2x$
$3x - 2y = 6$

b $y = -3 + 2x$
$3x + 2y = 22$

c $y = 2x - 1$
$2x - 3y = 11$

d $y = -4 - x$
$2x - 3y = 7$

e $y = -3x$
$2x + 3y = 14$

f $y = 2x - 12$
$3x + 2y = 4$

g $x = 4 - 5y$
$2y - 3x = 22$

h $x = 17 - 2y$
$3x - y = 2$

3 The equations $y = 3x - 6$ and $y = x + 4$ are solved simultaneously using the substitution method. The solution is:

A $x = -1, y = -9$
B $x = 5, y = 9$
C $x = -1, y = 3$
D $x = 10, y = 14$
E $x = -5, y = -1$

4 Example 8

$$y - x = 2 \quad [1]$$
$$3y - 4x = 12 \quad [2]$$

a Make y the subject in equation [1].
b Use your answer to part **a** to solve the equations using the substitution method.

5 Solve the following simultaneous equations.

a $y - x = 3$
$4x - 3y = 3$

b $2x - y = -10$
$7x - 2y = 1$

c $3x + y = 7$
$5x - 2y = -3$

d $3y - 12x = 9$
$4x - 3y = -1$

e $x + 3y = 23$
$3x - 2y = 3$

f $21x + 7y = 84$
$2x - 5y = -9$

g $2y = 16 - 3x$
$x - 8y = -129$

h $5x - 2y = -3$
$7x + 4y = 40$

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Reasoning and communication

6 Using the substitution method, solve the following equations.

$3t - 2d = -12$

$7t - 3d = 48$

7 The following equations represent the cost (C) of running two different dance parties depending on the number of people (n) attending.

$C = 12n$ and $C = 5.8n + 155$

a Solve the equations simultaneously using the substitution method to find the number of people attending when the cost of the parties are the same.

b What is the cost of each party when this occurs?

8 Joseph's solution to the simultaneous equations $y = x + 2$ and $5x - 2y = 5$ is shown below.

$y = x + 2$ [1]

$5x - 2y = 5$ [2]

Substitute [1] into [2].

$5x - 2(x + 2) = 5$

$5x - 2x + 4 = 5$

$3x = 1$

$x = \frac{1}{3}$

Substitute $x = \frac{1}{3}$ into [2].

$y = \frac{1}{3} + 2$

$y = 2\frac{1}{3}$

The solution is $x = \frac{1}{3}$, $y = 2\frac{1}{3}$.

a Identify the step in the working shown where the error was made.

b Solve the equations correctly.

9 The point where the equations $y = 11 - 2x$ and $y = 6$ intersect is:

A $(2\frac{1}{2}, 6)$ B $(-5, 6)$ C $(6, -1)$ D $(-1, 6)$ E $(-1, 2\frac{1}{2})$

10 Can you find another line apart from $y = 11 - 2x$, which will give the same point of intersection as your answer to question **9** when passing through the line $y = 6$?

11 a Solve the following simultaneous equations.

$y = 11 - 2x$ and

$2y = 22 - 4x$

b Did you find a solution to the simultaneous equations above? If not, why do you think a solution does not exist for this pair of equations?

12.04 SOLVING SIMULTANEOUS EQUATIONS BY ELIMINATION

The next algebraic method of finding simultaneous solutions is called the **elimination method**. In this method we add suitable multiples of the equations to *eliminate* one variable at a time.

IMPORTANT

Steps for the elimination method

1. Choose which variable to eliminate.
2. Make the coefficients of that variable the same size, but opposite in sign, by multiplying by suitable numbers.
3. Add the equations to eliminate the variable.
4. Solve the new equation.
5. Substitute the solution into either of the original equations and solve for the other variable *or* repeat the elimination method for the other variable.
6. Check the values in *both* of the original equations.

Example 9

Solve the following simultaneous equations by elimination.

$2x - y = 0$

$3x + y = 10$

Write down and label the equations.	$2x - y = 0$ $3x + y = 10$	[1] [2]
We will eliminate y first. The coefficients of y are the same and the signs are different. Eliminate y by adding equations [1] and [2].	$5x = 10$	[1] + [2]
Solve for x.	$x = 2$	
Substitute $x = 2$ into equation [1].	$2 \times 2 - y = 0$	$x = 2$ into [1]
Simplify. Subtract 4 from both sides. Solve for y.	$4 - y = 0$ $-y = -4$ $y = 4$	
Remember to check that the solution is correct by substituting back into the original equations. Substitute (2, 4) into equations [1] and [2].	[1] $2 \times 2 - 4 = 0$ $4 - 4 = 0$ ✓ [2] $3 \times 2 + 4 = 10$ $6 + 4 = 10$ ✓	
Write the answer.	The solution to the simultaneous equations is $x = 2$, $y = 4$.	

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Sometimes the coefficient of one of the variables needs to be changed to the same size and opposite sign before we can eliminate the variable and then solve the simultaneous equations. This can be achieved by performing an operation on one of the equations as shown in Example **10a**. In some instances you may need to change both equations as shown in Example **10b**. In some instances you may need to subtract equation 2 from equation 1 to eliminate the variable as shown in Example **10c**. You must always examine the equations carefully and select the correct method to use.

Example 10

Solve the following simultaneous equations by elimination.

a $2x + y = 12$ $3x - 4y = 7$	**b** $4x + 3y = -12$ $6x + 2y = 8$	**c** $2x + y = 9$ $2x + 3y = 15$

a	Write and label the equations.	$2x + y = 12$ $3x - 4y = 7$	[1] [2]
	We will eliminate y first. Multiply equation [1] by 4 and label the new equation [3].	$4 \times [1]$ $8x + 4y = 48$	 [3]
	Eliminate y by adding equations [2] and [3].	$11x = 55$	[2] + [3]
	Solve for x.	$x = 5$	
	Substitute for $x = 5$ into equation [2].	$3 \times 5 - 4y = 7$	$x = 5$ into [2]
	Simplify. Subtract 15 from both sides. Solve for y.	$15 - 4y = 7$ $-4y = -8$ $y = 2$	
	Remember to check that the solution is correct by substituting back into the original equations. Substitute (5, 2) into equations [1] and [2].	[1] $2 \times 5 + 2 = 12$ $10 + 2 = 12$ ✓ [2] $3 \times 5 - 4 \times 2 = 7$ $15 - 8 = 7$ ✓	
	Write the answer.	The solution to the simultaneous equations is $x = 5$, $y = 2$.	
b	Write and label the equations.	$4x + 3y = -12$ $6x + 2y = 8$	[1] [2]
	We will eliminate y first. Multiply equation [1] by 2 and label the new equation [3].	$2 \times [1]$ $8x + 6y = -24$	 [3]
	Multiply equation [2] by −3 and label the new equation [4].	$-3 \times [2]$ $-18x - 6y = -24$	 [4]
	Eliminate y by adding equations [3] and [4].	$-10x = -48$	[3] + [4]
	Solve for x.	$\frac{-10}{-10}x = \frac{-48}{-10}$ $x = 4\frac{8}{10}$ or $4\frac{4}{5} = 4.8$	

Substitute for $x = 4.8$ into equation [1].
Simplify and solve for y.

$$\begin{aligned} 4x + 3y &= -12 \qquad x = 4.8 \text{ into [1]} \\ 4(4.8) + 3y &= -12 \\ 19.2 + 3y &= -12 \\ 3y &= -12 - 19.2 \\ 3y &= -31.2 \\ \frac{3y}{3} &= \frac{-31.2}{3} \\ y &= -10.4 \end{aligned}$$

Remember you can check that the solution is correct by substituting back into the original equations.
Substitute (4.8, −10.4) into equations [1] and [2].

[1]
$$\begin{aligned} 4x + 3y &= -12 \\ 4(4.8) + 3(-10.4) &= -12 \\ 19.2 + (-31.2) &= -12 \\ -12 &= -12 \quad ✓ \end{aligned}$$

[2]
$$\begin{aligned} 6x + 2y &= 8 \\ 6(4.8) + 2(-10.4) &= 8 \\ 28.8 + (-20.8) &= 8 \\ 8 &= 8 \quad ✓ \end{aligned}$$

Write the answer.

The solution to the simultaneous equations is $x = 4.8$, $y = -10.4$.

c Write and label the equations.

$$\begin{aligned} 2x + y &= 9 \qquad [1] \\ 2x + 3y &= 15 \qquad [2] \end{aligned}$$

We will eliminate x first.
Subtract equation [2] from equation [1].

$$-2y = -6 \qquad [1] - [2]$$

Solve for y.

$$\begin{aligned} \frac{-2y}{-2} &= \frac{-6}{-2} \\ y &= 3 \end{aligned}$$

Substitute $y = 3$ into equation [2]. Simplify and solve for x.

$$\begin{aligned} 2x + 3y &= 15 \qquad y = 3 \text{ into [2]} \\ 2x + 3(3) &= 15 \\ 2x + 9 &= 15 \\ 2x &= 15 - 9 \\ 2x &= 6 \\ \frac{2x}{2} &= \frac{6}{2} \\ x &= 3 \end{aligned}$$

Remember that you can check if the solution is correct by substituting back into the original equations.
Substitute (3, 3) into equations [1] and [2].

[1]
$$\begin{aligned} 2x + y &= 9 \\ 2(3) + 3 &= 9 \\ 6 + 3 &= 9 \\ 9 &= 9 \quad ✓ \end{aligned}$$

[2]
$$\begin{aligned} 2x + 3y &= 15 \\ 2(3) + 3(3) &= 15 \\ 6 + 9 &= 15 \\ 15 &= 15 \quad ✓ \end{aligned}$$

Write the answer.

The solution to the simultaneous equations is $x = 3$, $y = 3$.

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Example 11

Solve the equations

$y = 2x + 1$

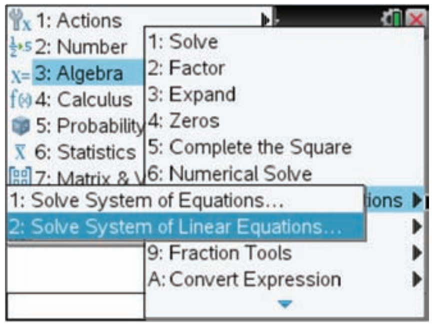
$4x + y + 5 = 0$

using a CAS calculator.

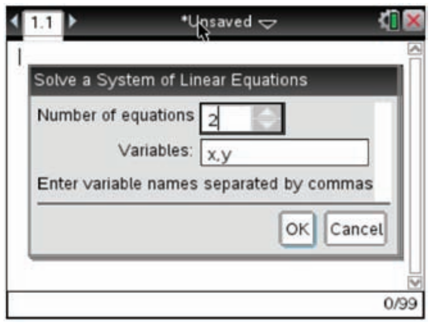
Solution

TI-Nspire CAS

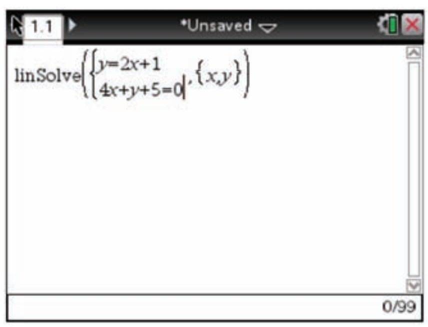
Open a New Document with a Calculator page.
Press menu.
Select 3: Algebra.
Select 7: Solve System of Equations.
Select 2: Solve System of Linear Equations.



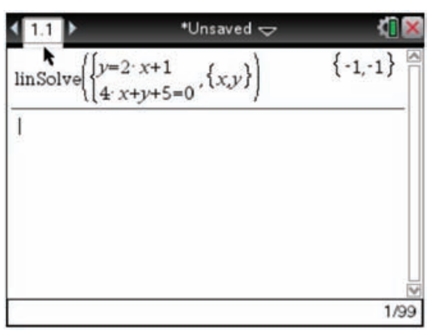
Complete the windows shown on the screen.
When OK is highlighted, press enter.

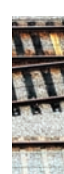


Enter the equations into the boxes as shown.
You can move between boxes by using the tab key.



Press enter to show the solution:
$x = -1, y = -1$.





Remember that you can check if the solution is correct by substituting back into the original equations.
Substitute $(-1, -1)$ into equations [1] and [2].

[1] $-1 = 2 \times (-1) + 1$ ✓

[2] $4 \times (-1) - 1 + 5 = 0$

$-4 + 4 = 0$ ✓

Write the answer.

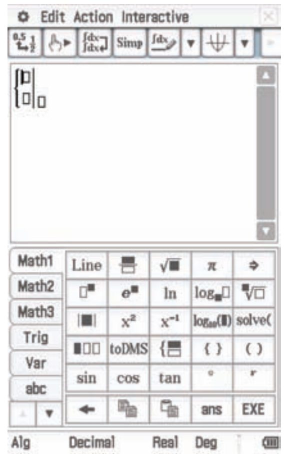
The solution to the simultaneous equations is $x = -1$, $y = -1$.

ClassPad

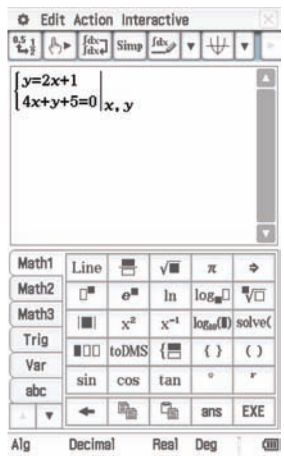
Use the Main $\sqrt{\alpha}$ application.

Press Keyboard

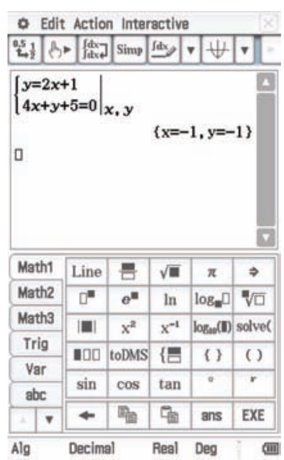
then tap {■.



In the top line type y = 2 x + 1.
In the bottom line type 4 x + y + 5 = 0.
After the vertical line, in the box provided by the template, type x , y.



Press EXE.



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Remember that you can check if the solution is correct by substituting back into the original equations. Substitute $(-1, -1)$ into equations [1] and [2].	[1] $-1 = 2 \times (-1) + 1$ ✓ [2] $4 \times (-1) - 1 + 5 = 0$ $-4 + 4 = 0$ ✓
Write the answer.	The solution to the simultaneous equations is $x = -1, y = -1$.

EXERCISE 12.04 Solving simultaneous equations by elimination

Concepts and techniques

Solving simultaneous equations

1 Example 9 Solve the simultaneous equations below using the elimination method.

$2x + y = -5$

$5x - y = 19$

2 The equations $7x - y = -8$ and $-5x - y = 16$ are to be solved simultaneously using the elimination method.

a The equation that results when one variable is eliminated is:

A $2x = 8$ B $12x = -24$ C $12x = 8$

D $2x = -24$ E $2x = -8$

b The point of intersection between the two lines is:

A $(4, 36)$ B $(-12, -44)$ C $(-2, 11)$ D $(-4, -36)$ E $(-2, -6)$

3 Solve the following simultaneous equations by elimination.

a $x + y = 3$
$3x - y = -15$

b $3x + y = 3$
$2x - y = 7$

c $3x + 2y = 12$
$4x - 2y = 2$

d $-5x + 5y = 25$
$3x - 5y = -21$

e $-6x - 2y = 22$
$3x + 2y = -16$

f $-3x + 2y = -6$
$3x + y = 18$

g $-5x + 2y = 16$
$5x + 7y = 11$

h $3x - 4y = -13$
$-3x - 4y = -19$

4 Example 10 For parts **a** and **b** use the simultaneous equations below.

$$-3x + 2y = -3 \quad [1]$$
$$x + 3y = 23 \quad [2]$$

a The equations are to be solved using the elimination method. To eliminate the y variable by adding the equations we must first:

A multiply equation [1] by -3 and multiply equation [2] by -3

B multiply equation [1] by 3 and multiply equation [2] by 2

C multiply equation [1] by 2 and multiply equation [2] by 2

D multiply equation [1] by -3 and multiply equation [2] by 2

E multiply equation [1] by 1 and multiply equation [2] by 3 b The point of intersection is:

A $(5, -6)$ B $(-6, -5)$ C $(5, 6)$ D $\left(5\frac{1}{5}, 6\frac{11}{13}\right)$ E $(4, 11)$

5 Solve the following simultaneous equations by elimination.

a $3x + 4y = -5$
$x - y = 3$

b $3x - 4y = 8$
$2x + y = 9$

c $2x + 3y = 5$
$3x + 5y = 9$

d $2x + 3y = -6$
$3x + y = 5$

e $x + 2y = 3$
$2x + y = 9$

f $2x + 3y = 10$
$4x - 2y = -4$

g $2x - 9y = -7$
$3x + 2y = 36$

h $4x - 3y = 11$
$3x - 4y = 17$

6 Example 11 Solve the following simultaneous equations.

a $3x + 2y = 14$
$4x - 2y = 7$

b $-2x + 5y = 20$
$-2x + 2y = 5$

c $3x - 2y = 10$
$-3x - 2y = -14$

d $2x + 3y = 11$
$5x - 2y = -1$

e $4x + 7y = 10$
$2x - 5y = -46$

f $2x - 3y = 13$
$3y - x = -8$

Reasoning and communication

7 a Solve the following cost equations simultaneously by using the elimination method.
$C = 60n$
$C = 20n - 256$

b Is the elimination method the most appropriate method to use when solving these simultaneous equations? Give reasons for your answer.

8 Jack and Jenny solved the simultaneous equations below by elimination, but obtained two different answers.
$4x - 3y = -4$ and
$6x - 4y = 9$
Jack's answer was $x = 21\frac{1}{2}$, $y = 30$ whilst Jenny insisted that $x = 5\frac{1}{2}$, $y = 8\frac{2}{3}$
Solve the equations to find who was correct.

9 Solve the following simultaneous equations.

a $5(2x - y) + 4y = 3(3x + 1)$
$5(2x - 1) = 5x - 2y + 38$

b $\frac{x}{2} + y = 5$
$\frac{3x}{4} + \frac{y}{3} = 4$

c $\frac{2x}{3} = 13 + y$
$3x - \frac{3y}{2} = 7$

12.05 PRACTICAL APPLICATIONS OF SIMULTANEOUS EQUATIONS

Simultaneous equations can be used to solve problems involving linear modelling. You may be required to draw the graphs for two equations on the same set of axes and use the graph to solve the problem.

Example 12

The income and expenses equations for the Superman Products Company are given below.

Expenses: $V = 15x$

Income: $V = 2000 + \dfrac{5x}{3}$

where V is measured in thousands of dollars and x is the number of units sold.

a Graph both equations on the same set of axes. (Use values of x from 0 to 300).

b Find the break-even point for the company.

The break-even point is the point where the expenses equal the income.

Solution

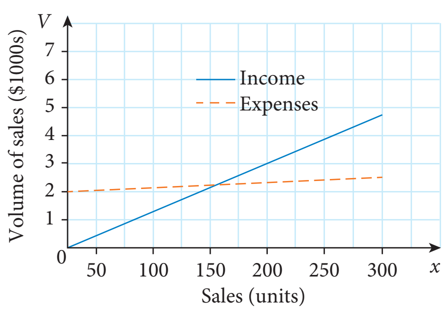
a Write the first equation.
Identify its gradient and vertical intercept.

$V = 15x$
Gradient = 15, vertical intercept = 0

Write the second equation.
Identify its gradient and vertical intercept.

$V = 2000 + \dfrac{5x}{3}$
Gradient = $\dfrac{5}{3}$, vertical intercept = 2000

Graph both lines on the same set of axes. Remember to use an appropriate scale on the horizontal and vertical axis.



Alternatively, the graphs could also be drawn by finding two points on each line.
For expenses:
if $x = 0$, $V = 15 \times 0 = 0$, so (0, 0)
if $x = 300$, $V = 15 \times 300 = 4500$, so (300, 4.5)
For Income:
if $x = 0$, $V = 2000 + \dfrac{5}{3} \times 0 = 2000$, so (0, 2)

if $x = 300$, $V = 2000 + \dfrac{5}{3} \times 300 = 2500$, so (300, 2.5)

b The break-even point is where the lines intersect.
Use the graph to identify the point of intersection.

Point of intersection = (150, 2250)

Write the answer.

The break-even point is at sales of 150 units, where both the income and expenses are \$2250.

Sometimes it is necessary to construct simultaneous linear equations from a worded problem.

Steps for solving worded problems

1. Read the question carefully.
2. Identify the unknowns and assign a pronumeral for the variables.
3. Set up the equations by converting the given information into mathematical sentences.
4. Solve the simultaneous equations using the most appropriate method.
5. Check the values in *both* of the original equations.
6. Answer the original question in sentence form.

Example 13

The length of a rectangle is 5 cm longer than its width. If the perimeter of the rectangle is 38 cm, find its dimensions.

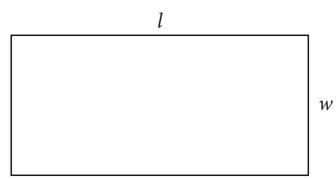
Solution

Define the variables.

Let l = length and w = width.

You can dra

Remember that the formula for the perimeter of a rectangle is: $P = 2l + 2w$.



Use the given information to write two linear equations. Label the equations [1] and [2].

$l = w + 5$ [1]
$38 = 2l + 2w$ [2]

Equation [1] is written in the form $y = a + bx$, so the substitution method is the most appropriate method to use. Substitute equation [1] into equation [2].

$38 = 2(w + 5) + 2w$
l from [1] into [2]

Expand the brackets and solve for w.

$$38 = 2w + 10 + 2w$$
$$38 = 4w + 10$$
$$28 = 4w$$
$$7 = w$$
$$w = 7$$

Substitute $w = 7$ into equation [1] to find the value of l.

$l = 7 + 5$ $w = 7$ into [1]
$l = 12$

Alternatively, solve using a CAS calculator.

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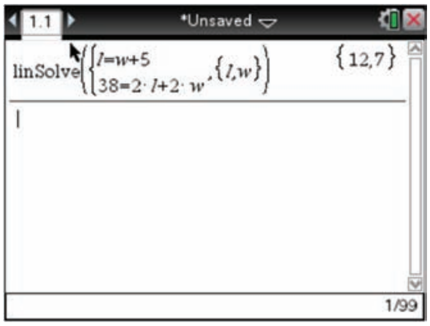
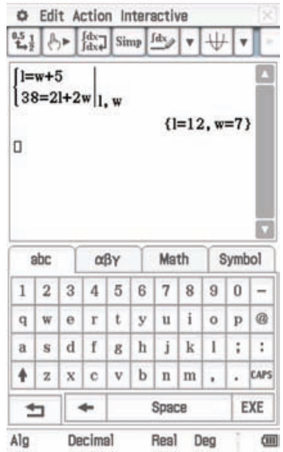
TI-Nspire CAS



ClassPad



Remember to check that the solution is correct by substituting back into the original equations. Substitute $l = 12$ and $w = 7$ into equations [1] and [2].

[1] $12 = 7 + 5$ ✓

[2] $38 = 2 \times 12 + 2 \times 7$

$38 = 24 + 14$ ✓

Write the answer.

The length of the rectangle is 12 cm and the width of the rectangle is 7 cm.

Cost equations are common examples of practical applications of simultaneous equations.

Always check that variables are given in the same units. Convert units where necessary.

Example 14

Toni spent \$16.40 on pens and pencils. She purchased a total of 8 items. If the pens cost \$2.80 each and the pencils cost 80 cents each, how many pens and pencils did Toni purchase?

Solution

Define the variables.

Let p = number of pens
q = number of pencils

Use the given information to write two linear equations. Label the equations [1] and [2]. Remember to convert the 80 cents to \$0.80.

$p + q = 8$ [1]
$2.8p + 0.8q = 16.4$ [2]

The coefficients of the variables are decimals, so it is easier to use the elimination method.
We will eliminate q first.
Multiply equation [1] by −0.8 and label the new equation [3].

$-0.8 \times$ [1]
$-0.8p - 0.8q = -6.4$ [3]

Eliminate q by adding equations [2] and [3].

$2p = 10$ [2] + [3]

Solve for p.

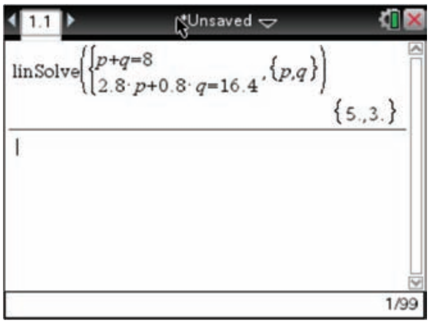
$p = 5$

Substitute $p = 5$ into equation [1]. $5 + q = 8$ $p = 5$ into [1]

Solve for q. $q = 3$

Alternatively, a CAS calculator can be used to solve the equations.

TI-Nspire CAS



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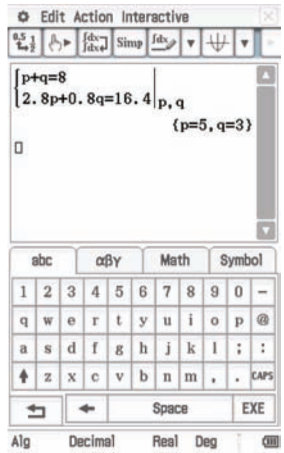

Remember to check that the solution is correct by substituting back into the original equations. Substitute (5, 3) into equations [1] and [2].

[1] $5 + 3 = 8$ ✓

[2] $2.8 \times 5 + 0.8 \times 3 = 16.4$

$14 + 2.4 = 16.4$ ✓

Write the answer. Toni purchased 5 pens and 3 pencils.

EXERCISE 12.05 Practical applications of simultaneous equations

Concepts and techniques

1 Example 12 David jogged to the park while Huong rode her bike. David was already 3 km ahead of Huong when she started from home, and jogged at 10.8 km/h, while Huong rode at a speed of 18 km/h. The travel graph below shows their journeys, with the distances from Huong's home shown.

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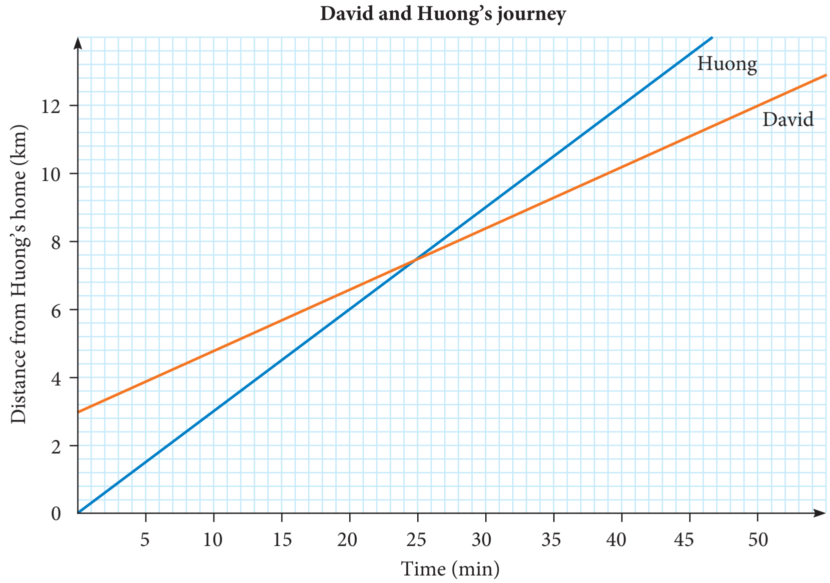
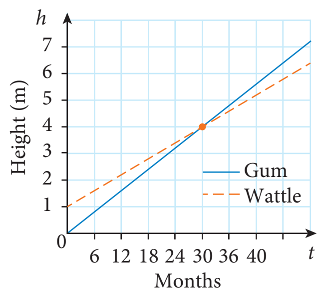


a Did Huong catch up to David? If so, when?
b If the park is 12 km from Huong's home, who got there first and by how many minutes?

2 The graph shows the growth of 2 different types of trees, measured from the time they were seedlings.



a Which tree was taller as a seedling and what was the difference in height?
b After how many months do the trees reach the same height?
c What is that height?
d Find an equation for the height of the:
   i gum tree ii wattle tree.
e Are these equations good models for the height of the trees after 10 years? Why or why not?

Use the following information to answer questions **3** and **4**.

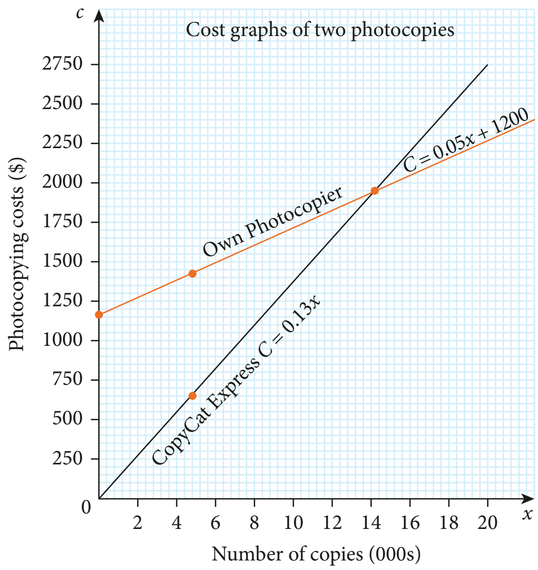
At CopyCat Express, the charge for photocopying is 13 cents per sheet. Klaus is investigating the purchase of his own photocopier for \$1200, which brings the cost down to 5 cents per sheet. The two cost functions are:

CopyCat Express: $C = 0.13x$

Own photocopier: $C = 0.05x + 1200$

where C is in dollars and x represents the number of copies made.

The graph of the functions is shown below.



3 Which option is cheaper for making 5000 copies and by what amount?

A Own photocopier by \$1450 **B** CopyCat Express by \$650

C Own photocopier by \$800 **D** CopyCat Express by \$800

E None of the above

4 What are the coordinates of the break-even point?

A (16 000, 2080) **B** (15 000, 1950) **C** (0, 1200)

D (21 000, 2730) **E** (0, 0)

5 Example 13 The length of a rectangle is three times as long as its width. If the perimeter of the rectangle is 52 cm, find the dimensions of the rectangle.

6 The sum of two consecutive numbers is 93. Find the numbers.

7 The sum of two numbers is 39. The larger number is 5 more than the smaller number. What are the numbers?

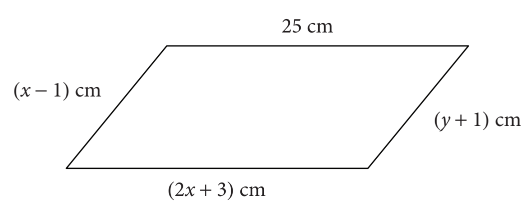
8 The average of two numbers is 18. When 2 times the first number is added to 5 times the second number, the result is 120. Find the two numbers.

9 Petra is four times as old as her daughter Philippa. In 8 years time she will be five times as old as Philippa. How old are Philippa and Petra?

10 **a** Find the values of x and y.

b State the dimensions of the parallelogram.

c Find the perimeter of the parallelogram.



11 Example 14 Jan saves money regularly, while Tony spends his money. The amount they have in the bank is given by the simultaneous equations below, where a = the amount of money (in \$) and m = number of months.

Jan: $2a - 5m = 20$

Tony: $a + 5m = 40$

Solve the equations simultaneously to find the number of months it will take for them to have the same amount of money in the bank. What is this amount?

12 Beachside Ice-creams sells ice-cream in single cones and double cones. The price of a single cone is \$3.80, whereas the price of a double cone is \$5.20. On a particular day, they sold a total of 450 cones and their takings for the day were \$1990. How many of each type of cone did Beachside Ice-creams sell on the day?

13 Jenni is comparing two Internet cafes. QuickByte has a monthly membership fee of \$8 and charges 60 cents per hour of Internet use. Game Hunters charges \$1 per hour of Internet use but has no monthly access fee. The two costs are represented by the formulas:

QuickByte: $C = 0.6t + 8$

Game Hunters: $C = t$

where C is in dollars and t is the number of hours of Internet use.

Alamy/David R.Frazier Photolibrary, Inc.

a If Jenni uses an Internet cafe for 24 hours per month, which cafe is better for her? Why?

b For what number of hours is the cost the same for both plans? What is this cost?

14 Geoff wants to hire a car over the three days of a long weekend. There are two rental companies he can choose from.

Shifty: \$45 per day plus \$1.85 per kilometre

Megahertz: \$80 per day plus \$1.55 per kilometre

a If d is the distance travelled in kilometres, and C is the cost in dollars, write a formula representing the cost of hiring a car for three days from:

i Shifty ii Megahertz

b After how many kilometres will the cost for hiring a car be the same for both companies? What is the cost at this time?

c If Geoff plans to drive an average of 100 km each day on the long weekend, which plan would be better for him and what would be his cost? Give reasons for your answer.

Reasoning and communication

15 The cost, \$$C$, of making n yo-yos is given by the formula $C = 3n + 90$, while the revenue, \$$R$, from selling n yo-yos is given by the formula $R = 5n$.

a Graph both formulas on the same axes, for values of n from 0 to 50.

b What is the cost of making 20 yo-yos?

c What is the revenue from selling 20 yo-yos?

d What is the loss from making 20 yo-yos?

e How much does one yo-yo sell for?

f What is the break-even point? (Hint: For what value of n does the revenue equal the cost?)

Revenue means income or money received.

16 Megan and Jane both sell kitchenware and earn commissions. Megan's commission is \$240 plus 5% of her sales, while Jane's commission is 20% of her sales (with no retainer). These rates can be expressed by the formulas below.

Megan's commission: $C = 0.05x + 240$

Jane's commission: $C = 0.2x$

where \$$x$ is the total value of sales and \$$C$ is the commission.

a Graph both commission formulas on the same axes, for values of x from \$0 to \$2400.

b If each person sells \$1200 worth of kitchenware, who earns more?

c For what value of sales does Megan earn the same amount as Jane?

d What is the value of their commissions at this point?

e For what value of sales does Megan earn more than Jane?

17 Dave claims that a quick 'rule-of-thumb' for converting Celsius temperatures to Fahrenheit is 'double it and add 30'.

a Graph Dave's rule and the actual conversion formula for values of C from 0 to 100.

Dave's rule: $F = 2C + 30$

Actual conversion formula: $F = \frac{9}{5}C + 32$

where C represents the temperature in °C and F represents the temperature in °F.

b For what value of C is Dave's rule exactly equal to the actual conversion formula?

c For what values of C does Dave's rule give answers that are too high?

d For what values of C is the difference between the two formulas not more than 5°F?

18 Mathew purchases 4 CDs and 2 DVDs and pays \$86. Dion buys 3 CDs and 5 DVDs and pays \$131.

a Write an equation which represents Mathew's purchases.

b Write an equation which represents Dion's purchases.

c Solve the equations above simultaneously to find the price of each CD and each DVD.

INVESTIGATION Taxi charges

Burntrubber Taxis charges \$4.60 flag fall plus 90 cents per kilometre. Their rival company, Whiteknuckle Cabs, charges \$2.50 flag fall plus \$1.20 per kilometre.

a If C represents the dollars charged and k represents the kilometres travelled, write an equation that represents the trip charges for each company.

b Which variable is the dependent variable and which one is the independent variable?

c Graph both formulas for values of k from 0 to 11 on the same set of axes.

d State the vertical intercept for both lines.

e How much will each company charge to travel the following distances?

i 2 km ii 5 km iii 12 km

f There is no value for $k = 12$ on the horizontal axis. How can you determine the cost of travelling 12 kilometres using the graph?

g Use the graph to find their point of intersection.

h What does the point of intersection represent?

i Solve the equations simultaneously using one of the algebraic methods (substitution or elimination). Does your solution agree with your answer to part **g**? Give reasons for your answer.

j When is each company cheaper to use?

k Jake travels to and from work each day by taxi, a distance of 11 km each way. Determine which is the better taxi for him to use, and calculate the saving he would make using it over 5 days.

12.06 DRAWING PIECEWISE GRAPHS

A **piecewise graph** is a combination of two or more straight lines. It is made up of separate line segments ('pieces') with different gradients.

For example, we can draw the graph of $y = -3x$ for $-1 \le x < 3$ and then draw the graph of $y = 2x - 1$ for $3 \le x \le 6$. The second line segment starts from where the first line segment finishes. The two line segments have different gradients (or slopes) and form the piecewise linear graph.

The equation of the piecewise linear function above is written as:

$$y = \begin{cases} -3x, & -1 \le x < 3 \\ 2x - 1, & 3 \le x \le 6 \end{cases}$$

Remember:

$<$ means less than

$>$ means greater than

$\le$ means less than or equal to

$\ge$ means greater than or equal to

IMPORTANT

$-1 \le x < 3$ means all the x values between -1 and 3. The -1 is included but 3 is not.

That is, x is greater than or equal to -1 but is less than 3.

Example 15

A piecewise linear equation is given below.

$$y = \begin{cases} 3x, & -1 \le x < 3 \\ 12 - x, & 3 \le x \le 6 \end{cases}$$

a Complete a table of values for this equation.

b Use the table of values to draw its graph.

Solution

a There are two parts to the piecewise linear function. Hence, we will need to draw two separate line segments.

Identify the first line segment.

$y = 3x$

Complete a table of values for this equation. We only require the x values between −1 and 3.

x	−1	0	1	2	3
y	−3	0	3	6	9

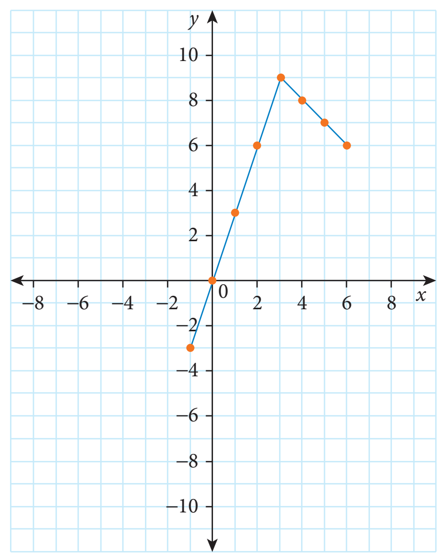
Identify the second line segment.

$y = 12 - x$

Complete a table of values for this equation. We only require the x values between 3 and 6.

x	3	4	5	6
y	9	8	7	6

b Plot the points for $y = 3x$ from the first table of values.
On the same set of axes, plot the points for $y = 12 - x$ from the second table of values.
This line segment begins where the first line segment ended.



Using a table of values to plot a piecewise linear graph can be very time-consuming. You can use your knowledge of sketching linear functions to draw the graph of each line segment in the piecewise linear function. The critical points are the end points of each line segment and therefore we find the coordinates of these points and join them to draw the line segment.

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Example 16

a Sketch the graph of $y = 3x - 2$ for $-3 \le x < 2$.
b On the same set of axes, sketch the graph of $y = 4$ for $2 \le x \le 5$.
c Comment on the graph formed by sketching these two lines on the same set of axes.

Solution

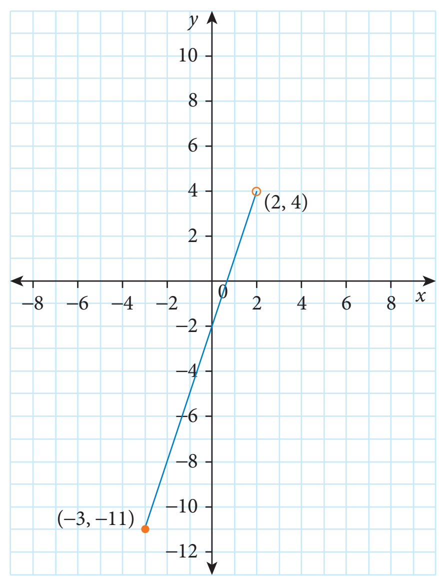
a Write the equation.

$y = 3x - 2$

The graph of $y = 3x - 2$ is only required for x values between -3 and 2.
Find the value of y at each of the end points, that is, where $x = -3$ and $x = 2$.

When $x = -3$,
$y = -9 - 2$
$= -11$
When $x = 2$,
$y = 6 - 2$
$= 4$
The coordinates of the end points are $(-3, -11)$ and $(2, 4)$.

Plot the coordinates of the end points: $(-3, -11)$ and $(2, 4)$.
Join these points to sketch the line.



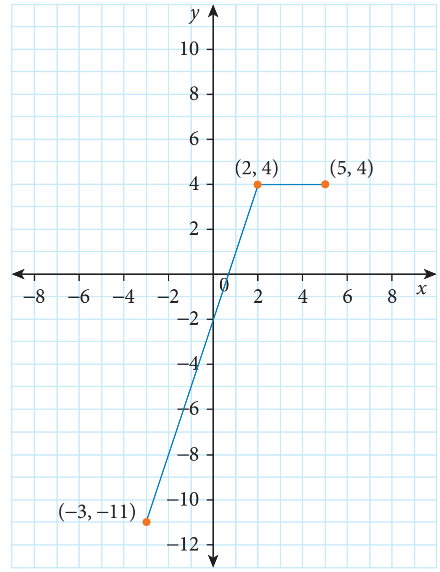
b Write the equation.

$y = 4$

The graph of $y = 4$ is only required for x values between 2 and 5.
State the coordinates of the points at $x = 2$ and $x = 5$.

Coordinates of the end points are: $(2, 4)$ and $(5, 4)$.

The graph of $y = 4$ starts where the graph of $y = 3x - 2$ finishes.
Draw a horizontal line from (2, 4) to (5, 4).



c Examine the graphs and write a comment.

The graph is made up of two different line segments.
The first line segment has a positive gradient, hence the graph is increasing.
The coordinates of its end points are (−3, −11) and (2, 4).
The second line segment starts from the point where the first line segment ends, that is, at (2, 4) and finishes at (5, 4). It is a horizontal line.

Example 17

For the piecewise equation

$$y = \begin{cases} 2 - x, & -6 \le x < 1 \\ x, & 1 \le x \le 5 \end{cases}$$

a sketch a graph

b check your answer to part **a** with your CAS calculator.

Solution

a Write the equation of the first line.

$y = 2 - x$

The graph of $y = 2 - x$ is only required for x values between −6 and 1.
Find the value of y at the end points, that is where $x = -6$ and $x = 1$.

When $x = -6$,
$y = 2 + 6$
$= 8$
When $x = 1$,
$y = 2 - 1$
$= 1$
The coordinates of the end points are (−6, 8) and (1, 1).

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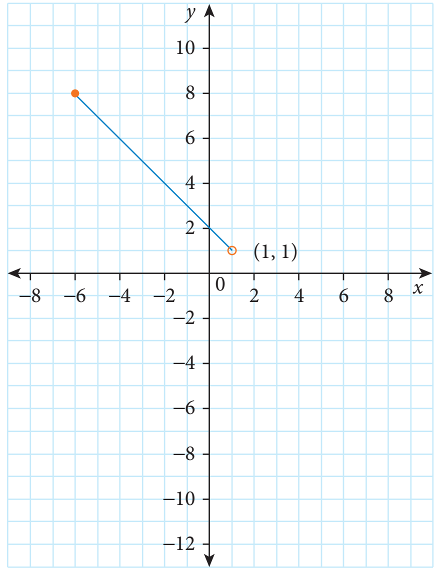
Use the points above to sketch the first line segment.



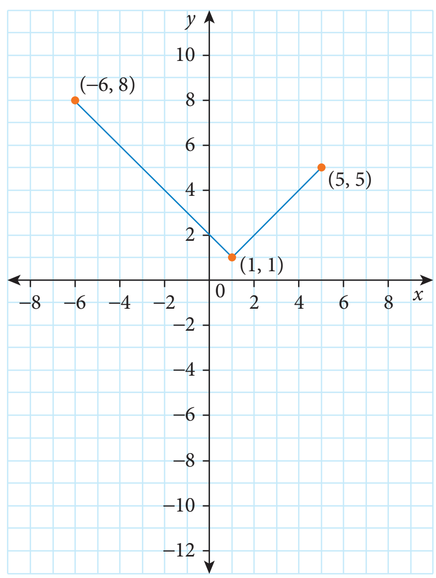
Write the equation of the second line segment.

$y = x$

The graph of $y = x$ is only required for x values between 1 and 5.
Find the value of y at the end points, that is, where $x = 1$ and $x = 5$.

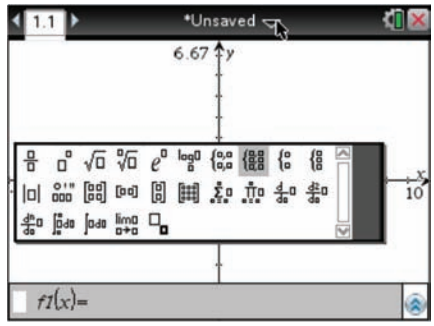
When $x = 1$, $y = 1$
When $x = 5$, $y = 5$
The coordinates of the end points are (1, 1) and (5, 5).

Use the points above to sketch the second line segment on the same set of axes.

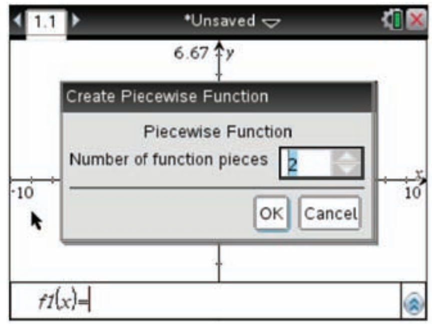


TI-Nspire CAS

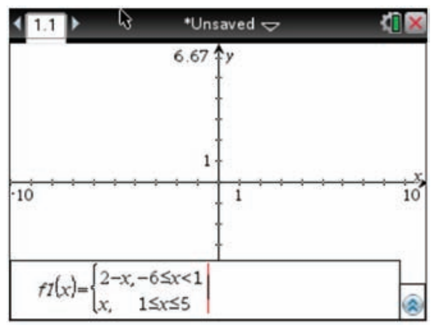
Open a New Document with a Graphs page.
Press [templates key], then select the piecewise equation template.
Press enter.



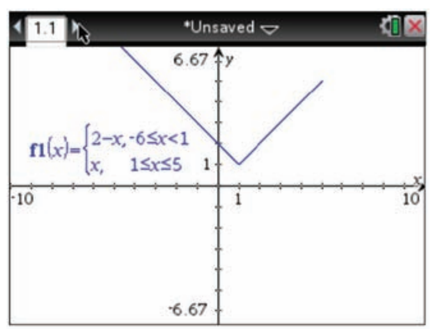
In the pop up screen set the number of function pieces to 2.
Press tab so OK is highlighted, then press enter.



Type the piecewise equation into the template. Inequality signs < and ≤ can be found by pressing ctrl =.



Press enter.



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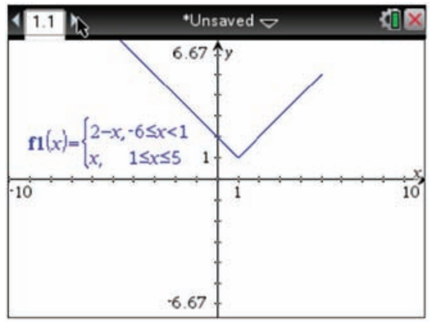
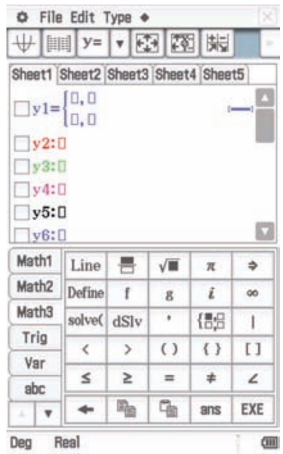
Press [menu].
4: Window/Zoom
4: Zoom-Out
Move the cursor (Center?) over the origin and press [enter].



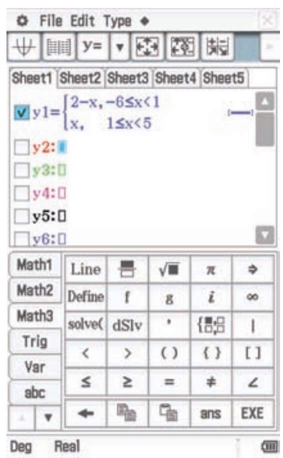
ClassPad

Use the **Graph & Table** application.
With the cursor in the entry line next to $y1=$,
press [Keyboard].
Tap [Math3] then the piece-wise equation template [piecewise template].

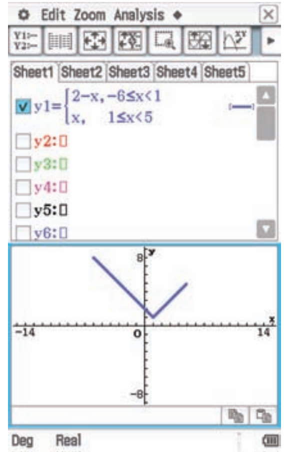
For a piecewise equation with three parts, tap [piecewise template] twice to get the correct template.



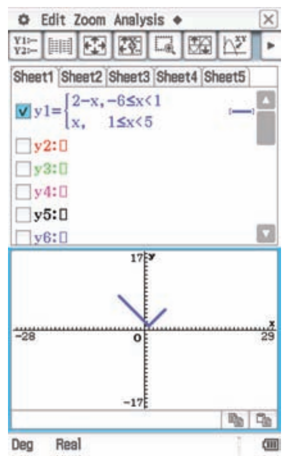
Type the piecewise equation into the template.
Inequality signs [<] and [≤] can be found on the screen given by [Math3].
Press [EXE] when you have finished entering the equation.



Tap [icon].



Tap **Zoom** then **Zoom Out**.



EXERCISE 12.06 Drawing piecewise graphs

Concepts and techniques

1 Example 15 A piecewise linear equation is given below.

$$y=\begin{cases} x+3, & -5\le x<0 \\ 3-2x, & 0\le x\le 4 \end{cases}$$

a Complete a table of values for the first line segment.

b Complete a table of values for the second line segment.

c Use the tables of values to create a graph.

2 A piecewise graph consists of 3 line segments given by the rule below.

$$y = \begin{cases} -x, & -7 \le x < -3 \\ 3, & -3 \le x < 2 \\ x+1, & 2 \le x \le 7 \end{cases}$$

a Complete a table of values for the rule.

b Create a graph using the points in the table of values.

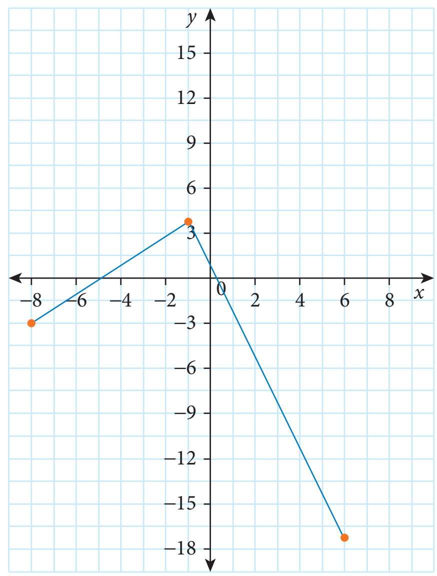
3 Example 16 a Sketch the graph of $y = \frac{1}{2}x$ for $-8 \le x < 2$.

b On the same set of axes, sketch the graph of $y = 1$ for $2 \le x \le 8$.

c Comment on the graph formed by sketching these two lines on the same set of axes.

For questions **4** and **5**, use the graph below.

$$y = \begin{cases} x+5, & -8 \le x < -1 \\ 1-3x, & -1 \le x \le 6 \end{cases}$$



4 The first line segment will be drawn for all x values that fall between –8 and –1 but ...

A does not include the points where $x = -8$ and $x = -1$

B does not include the point where $x = -8$, but includes the point where $x = -1$

C includes the points where $x = -8$ and $x = -1$

D includes the points where $y = -8$ and $y = -1$

E includes the point where $x = -8$, but does not include the point where $x = -1$.

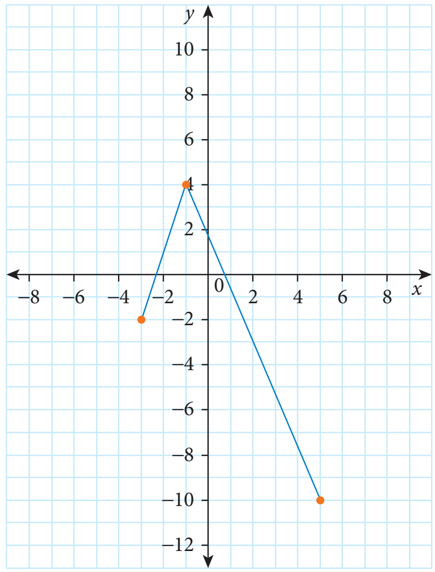
5 The point where the two line segments meet is:

A $(-8, -3)$ B $(4, -1)$ C $(6, -15)$ D $(-1, 4)$ E $(-1, 10)$

6 Which of the following graphs best represents the piecewise linear equation below?

$$y = \begin{cases} 3x + 7, & -3 \le x < -1 \\ x + 5, & -1 \le x \le 5 \end{cases}$$

A



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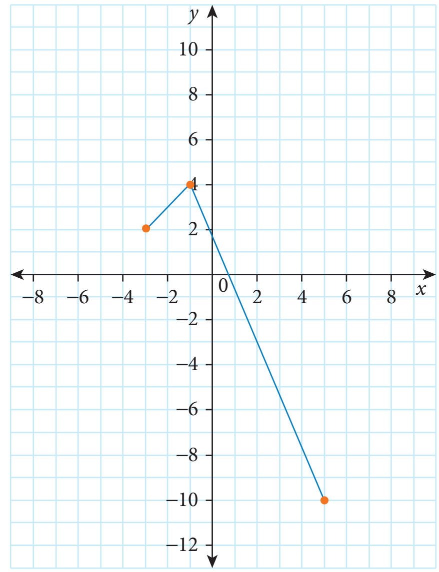

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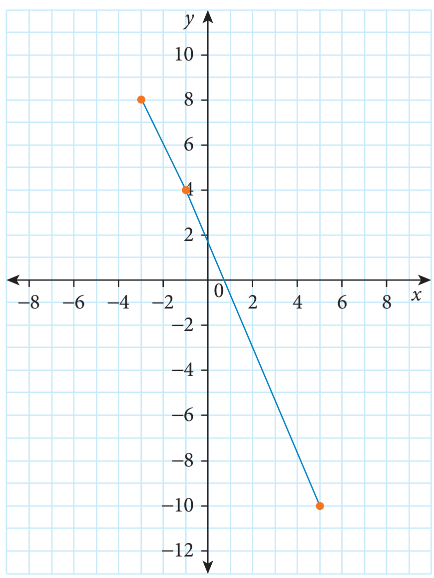

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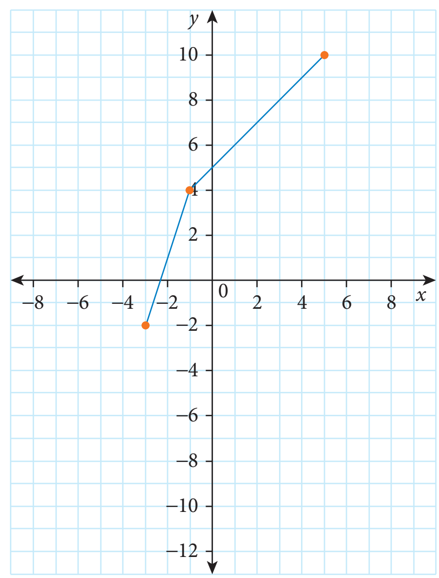

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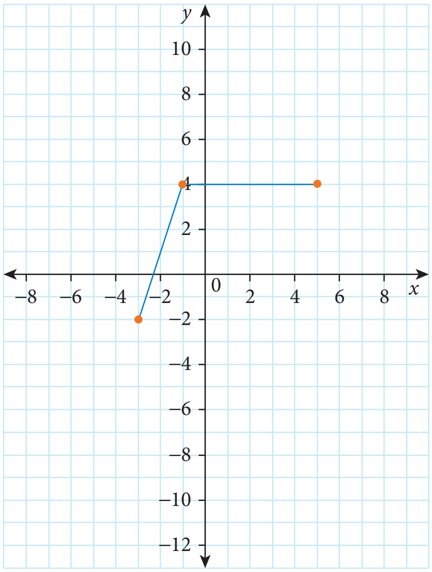
E



7 Example 17 Sketch a graph for each of the following rules.

a $y=\begin{cases}2x-5, & -1\le x<4\\ 3, & 4\le x\le 10\end{cases}$

b $y=\begin{cases}-6, & -6\le x<2\\ -3x, & 2\le x\le 3\end{cases}$

c $y=\begin{cases}-x+5, & -4\le x<2\\ 9-3x, & 2\le x\le 5\end{cases}$

d $y=\begin{cases}7-2x, & -5\le x<1\\ 4x+1, & 1\le x\le 4\end{cases}$

e $y=\begin{cases}x-3, & -7\le x<-1\\ -4, & -1\le x<5\\ 1-x, & 5\le x\le 11\end{cases}$

f $y=\begin{cases}8, & -8\le x<-2\\ -4x, & -2\le x<3\\ 2x-18, & 3\le x<10\end{cases}$

8 Check the accuracy of your graphs in question **7** by using your CAS calculator to draw each piecewise linear graph. Check to see if your answers match.

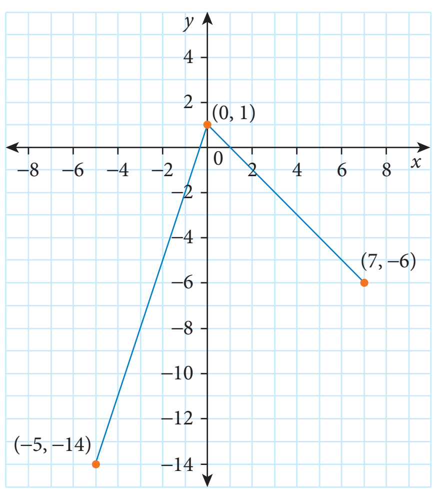
Reasoning and communication

9 a The piecewise graph on the right consists of two line segments. Between what values of x does each line segment lie?

b Find the equation of the first line segment.

c Find the equation of the second line segment.

d Using your answers above, write an equation for the piecewise linear graph.



10 The equation below is for a piecewise linear graph

$$y = \begin{cases} 2x, & x < 1 \\ 2, & x \geq 1 \end{cases}$$

a State the equation of the first part of the linear piecewise graph.

b For what values of x do you need to draw the line segment? How does this differ from the way the previous piecewise linear equations were written?

c How can you show this when drawing the graph?

d State the equation of the second part of the linear piecewise graph.

e For what values of x do you need to draw this line segment?

f Use your answer from parts **a** to **e** to draw the graph of this equation.

Remember: A line segment has a starting point and an end. A ray has a starting point but no end point.

11 Sketch the piecewise linear graph given by the equation

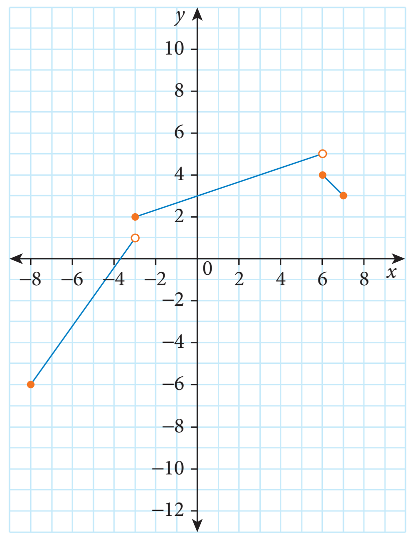
$$y = \begin{cases} 2 - x, & x \leq -1 \\ 3, & -1 < x \leq 2 \\ 2x - 1, & x \geq 2 \end{cases}$$

12 a Examine the graph on the right. How does this graph differ from the previous graphs?

b How many line segments form this piecewise linear graph?

c State the values of x used for each line segment.

d Write a sentence explaining why some of the end points have open circles whilst others have closed circles.



12.07 APPLICATIONS INVOLVING PIECEWISE GRAPHS

Piecewise linear graphs are useful for displaying information that involves different rates of change. Examples of situations where piecewise linear rules and graphs are useful include:

- taxation rates
- commissions
- mobile phone rates, and
- utility rates.

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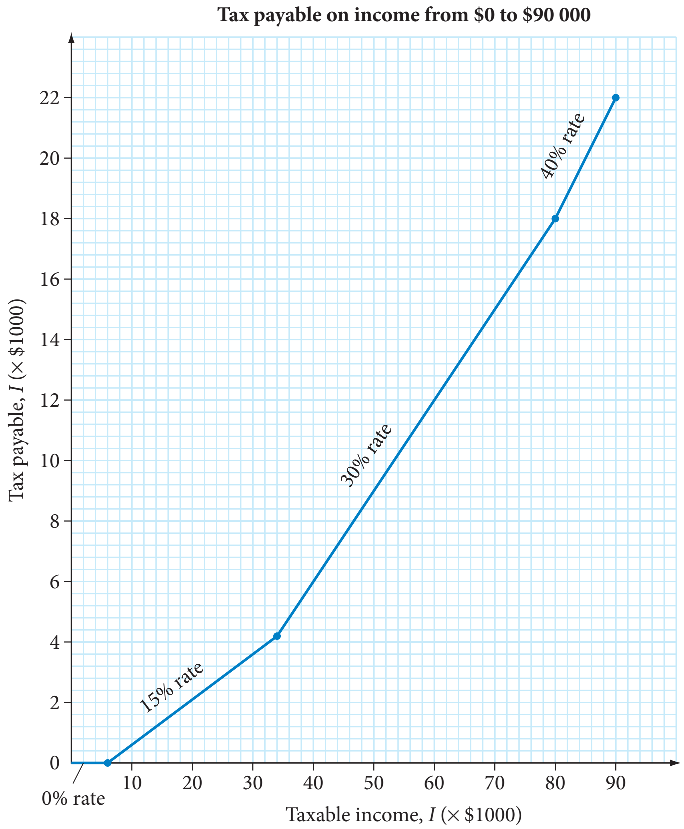
Example 18

This piecewise linear graph illustrates the income tax payable for incomes up to \$90 000.



a Describe the graph and what it represents.

b Use the graph to find the tax payable on an income of:

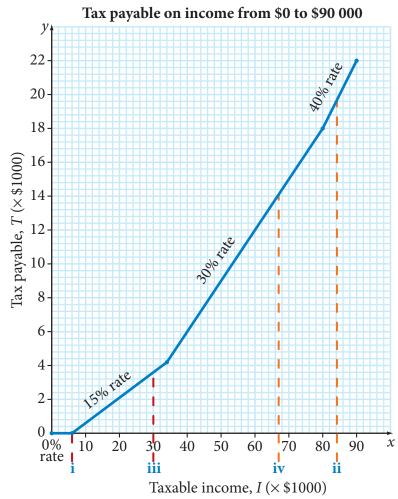
i \$6000 ii \$84 000 iii \$30 000 iv \$67 000.

Solution

a Study the graph. There are 4 line segments showing different rates of change.
Comment on your observations.

There are 4 line segments in this piecewise graph. The gradient of the line segments increases as the amount of taxable income increases. No tax is charged for the first \$6000.
Between \$6000 and \$34 000 the tax rate is 15%.
Between \$34 000 and \$80 000 the tax rate is 30%.
Between \$80 000 and \$90 000 the tax rate is 40%.

b **i** Locate $6000 on the horizontal axis. Draw a broken line to the graph and then read the y value at this point.



y value = 0

Write your answer. The tax payable on an income of $6000 is $0.

ii Locate $84 000 on the horizontal axis. Draw a broken line to the graph and then read the y value at this point.

y value = $19 600

Write your answer. The tax payable on an income of $84 000 is $19 600.

iii Locate $30 000 on the horizontal axis. Draw a broken line to the graph and then read the y value at this point.

y value = $3600

Write your answer. The tax payable on an income of $30 000 is $3600.

iv Locate $67 000 on the horizontal axis. Draw a broken line to the graph and then read the y value at this point.

y value = $14 100

Write your answer. The tax payable on an income of $67 000 is $14 100.

Sometimes a piecewise graph is not provided but instead we are given data or information from which we can construct a graph.

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Example 19

A car salesman's commission is 5% for the first \$18 000 worth of sales and 3.5% for the remaining amount.

a Copy and complete this table of values using the rates given.

Car price, $\$P$	0	18 000	25 000	50 000
Commission, $\$C$				

b Use the table of values to help you draw a piecewise linear graph that represents the car salesman's commission for values of P from \$0 to \$50 000.

Solution

a Copy the table.
If there are no sales, the commission will be \$0.

$P = 0, C = 0$

Car sales to the value of \$18 000 earn the salesman a commission of 5%.

$$5\% \text{ of } \$18\,000 = \frac{5}{100} \times 18\,000$$
$$= \$900$$

Car sales of \$25 000 give the salesman 5% of the first \$18 000 and 3.5% of the remaining amount. Calculate the remaining amount.

$\$25\,000 - \$18\,000 = \$7000$
$P = 18\,000, C = 900$

Calculate the commission on the remaining amount.

$$3.5\% \text{ of } \$7000 = \frac{3.5}{100} \times 7000$$
$$= \$245$$

Add this amount to the commission paid on the first \$18 000.

$\$900 + \$245 = \$1145$
$P = 25000, C = 1145$

Repeat the steps above for a commission of \$50 000.

$\$50\,000 - \$18\,000 = \$32\,000$

$$3.5\% \text{ of } \$32\,000 = \frac{3.5}{100} \times 32\,000$$
$$= \$1120$$

$\$900 + \$1120 = \$2020$
$P = 50\,000, C = 2020$

Enter the values into a table. This can be done as each one is calculated.

Car price, $\$P$	0	18 000	25 000	50 000
Commission, $\$C$	0	900	1145	2020

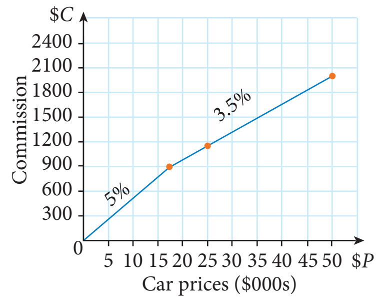
b Draw a set of axes. Label the horizontal axis 'Car price ($\$P$)' and the vertical axis 'Commission ($\$C$)'.

Plot the points from the table.

There are two segments to this graph.

Join the first two points (0, 0) and (18 000, 900) to give the first line segment.

Join the points (18 000, 900), (25 000, 1145) and (50 000, 2020) to give the second line segment.



At times we may be given a piecewise rule to show how an object is moving. We can now build on the work of the previous section to sketch the graph of the given rule.

Example 20

A moving object follows the path given by the piecewise linear function

$$d = \begin{cases} t+4, & 0 \le t < 6 \\ 10, & 6 \le t < 8 \\ 50-5t, & 8 \le t \le 10 \end{cases}$$

where d is the distance travelled in metres and t is the time taken in seconds. Sketch the path followed by the object.

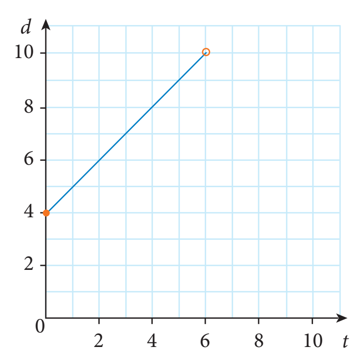
Solution

Write the equation of the first line segment. $d = t + 4$

The graph of $d = t + 4$ is only required for the x values between 0 and 6.
Find the value of d at the end points, that is, where $t = 0$ and $t = 6$.

When $t = 0$, $d = 4$
When $t = 6$, $d = 10$
The end points are (0, 4) and (6, 10).

Use the points (0, 4) and (6, 10) to sketch the first line segment.



Write the equation of the second line. $d = 10$

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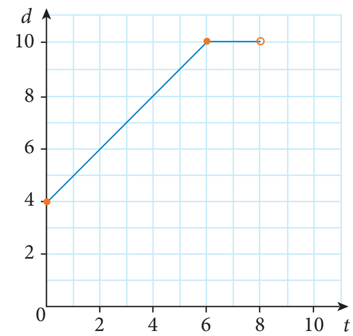
The graph of $d = 10$ is only required for t values between 6 and 8.
State the coordinates of the points at $t = 6$ and $t = 8$.

The end points are (6, 10) and (8, 10).

Use the points (6, 10) and (8, 10) to sketch the second line segment on the same set of axes.



Write the equation of the third line segment.

$d = 50 - 5t$

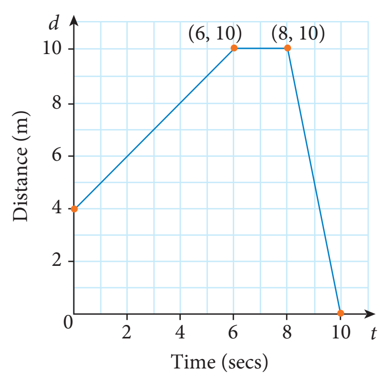
The graph of $d = 50 - 5t$ is only required for t values between 8 and 10.
Find the value of d at the end points, that is, where $t = 8$ and $t = 10$.

When $t = 8$,
$$d = 50 - 40$$
$$= 10$$
When $t = 10$,
$$d = 50 - 50$$
$$= 0$$

State the coordinates of the points at $t = 8$ and $t = 10$.

The end points are (8, 10) and (10, 0).

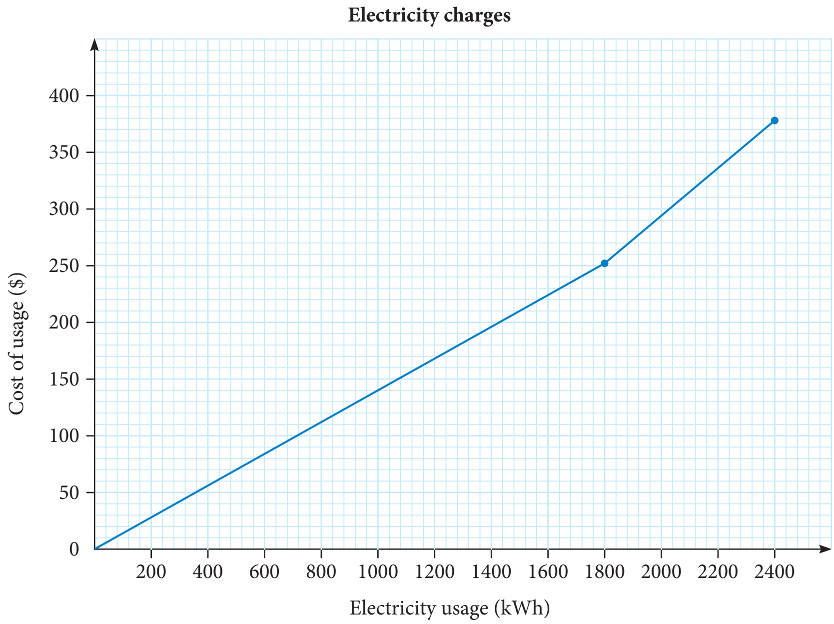
Use the points (8, 10) and (10, 0) to sketch the third line segment on the same set of axes.



EXERCISE 12.07 Applications of piecewise graphs

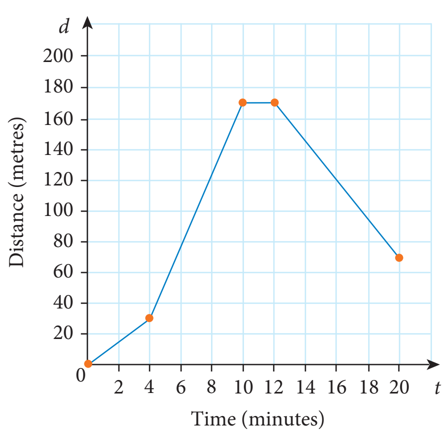
Reasoning and communication

1 Example 18 This piecewise linear graph shows the cost of electricity for different amounts of usage in kWh (kilowatt-hours).



a How many different cost rates are shown on the graph?
b What is the rate for the first 1800 kWh in c/kWh?
c Is the second rate higher or lower than the first rate?
d Use the graph to find the cost of using the following amounts of electricity.
i 880 kWh ii 2100 kWh iii 1920 kWh
e Use the graph to find the amount of electricity used if the cost was:
i $100 ii $220 iii $370

The distance-time graph below shows the different movements of an object over a period of time. Use the graph to answer questions **2**, **3** and **4**.



2 The object changes speed

A 0 times
B 1 time
C 2 times
D 3 times
E 4 times

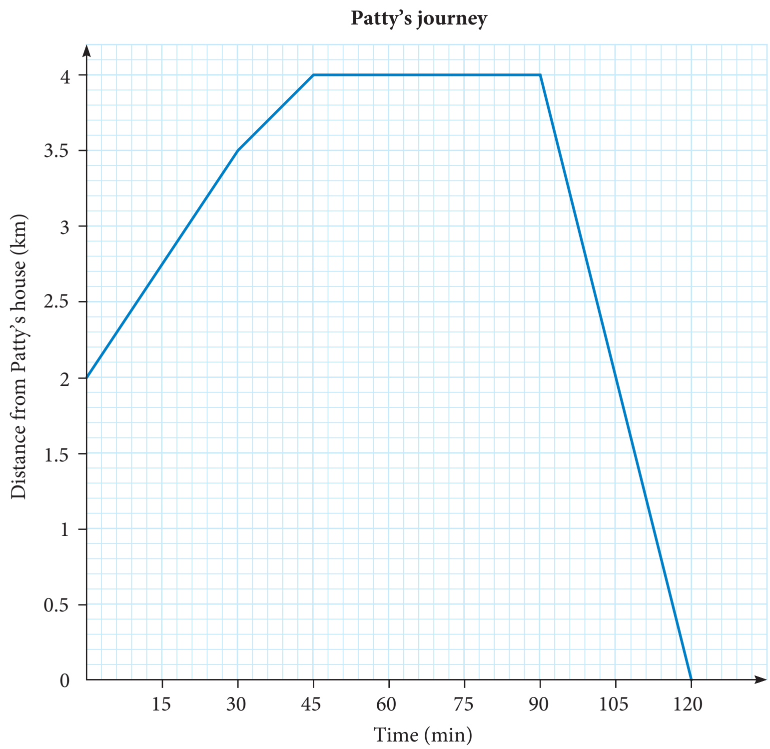
3 The total distance travelled by the object is:

A 70 metres
B 80 metres
C 150 metres
D 170 metres
E 270 metres

4 For how many minutes is this object falling?

A 2 minutes
B 4 minutes
C 8 minutes
D 10 minutes
E 20 minutes

5 Patty walked to Susan's house, stayed there for a while, then borrowed Susan's bike and cycled home. This travel graph illustrating Patty's movements has four distinct sections.



a How long did Patty stay at Susan's house?

b The vertical intercept of this graph is 2 km. What does this mean?

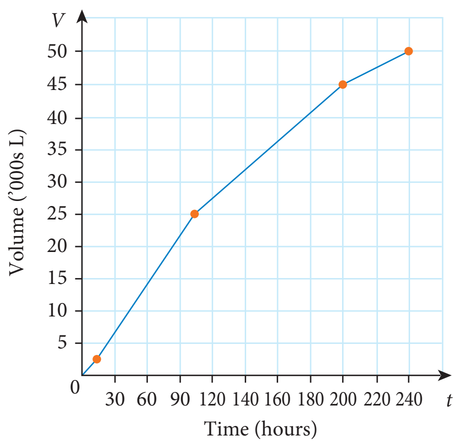
c What is the distance between Patty's house and Susan's house?

d Describe briefly what happened at each section of the graph.

e Calculate Patty's speed in kilometres/hour at each section of her journey.

Remember: speed = distance divided by time.

6 A rain water tank holds 50 000 litres. The piecewise linear graph below shows the rate at which the rain water tank fills with water.



a Does the rain water tank fill at the same rate until it reaches capacity? Give reasons for your answer.

b How many litres does the rain water tank hold after:

i 10 hours? ii 100 hours? iii 200 hours?

c After how many hours is the rain water tank filled to capacity?

d After how many hours does the rate at which the water is filling the rain water tank change? (Note: The rate changes three times.)

7 Damien is paid the following weekly commission rates for selling mobile phone plans.

- 5% of the first \$3000
- 8% of the next \$3000
- 14% of the remaining amount.

The piecewise linear graph below shows Damien's commission based on the value of the plans he sells.

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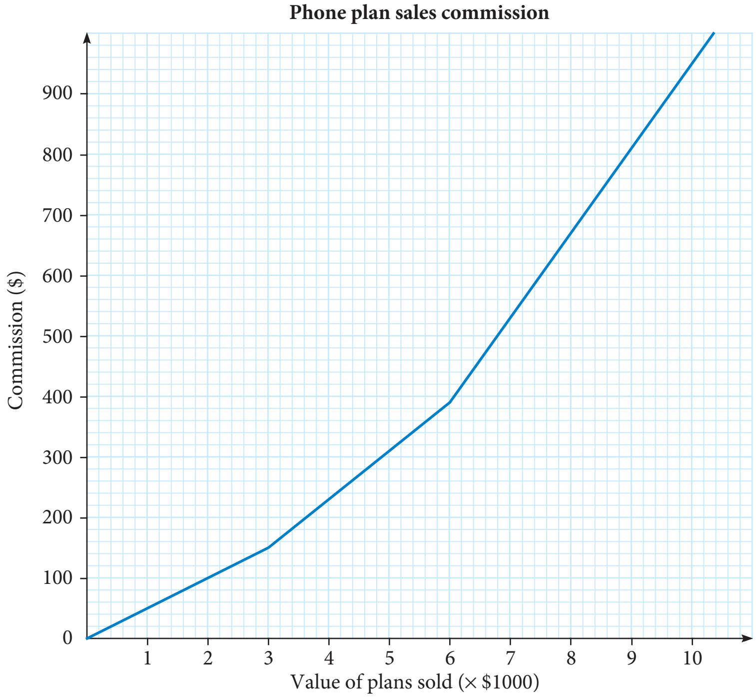

a How much commission would Damien earn for selling phone plans at each of the following amounts per week?

i $5000 **ii** $1500 **iii** $8000

b If Damien earned $500 in commission this week, what was the value of the phone plans he sold?

c As Damien sells more plans, his rate of commission increases. How is this shown on the graph?

8 Example 19 A real estate agent charges the following commission rates for selling a property.

- 5% of the first $85 000 of the property's price
- 3% for the next $60 000
- 2.5% of the remaining amount.

a Copy and complete this table of values using the rates given above.

Property price, $*P*	0	85 000	145 000	500 000
Commission, $*C*				

b Use the table from part **a** to help you draw a piecewise linear graph that illustrates the real estate agent's commission for values of P from $0 to $500 000.

c What happens to the gradient of the graph as the property prices increase? Why?

d Use the graph to find the commission earned for selling:

i a block of land for $210 000

ii an apartment for $330 000

iii a house for $450 000.

9 The new Australian taxation rates are given in the table below.

Taxable income	Tax on this income
0 – \$18 200	Nil
\$18 201 – \$37 000	19c for each \$1 over \$18 200
\$37 001 – \$80 000	\$3572 plus 32.5c for each \$1 over \$37 000
\$80 001 – \$180 000	\$17 547 plus 37c for each \$1 over \$80 000
\$180 001 and over	\$54 547 plus 45c for each \$1 over \$180 000

Source: Australian Taxation Office. Go to http://www.ato.gov.au/content/12333.htm

a Copy and complete the table below.

Taxable income (\$*I*)	0	30 000	50 000	150 000	200 000
Tax payable (\$*T*)					

b Use the table from part **a** to draw a piecewise linear graph that shows the tax payable on income up to \$200 000.

10 Example 20 A yacht race has 2 legs that need to be completed for the yachts to reach the finish line. One yacht takes 5 hours to complete the race. Each leg is defined by the equation:

$$d = \begin{cases} 8t + 4, & 0 \le t < 2 \\ 30 - 5t, & 2 \le t \le 5 \end{cases}$$

where d is the distance travelled by the yacht in kilometres and t is the time taken in hours.

a Draw a piecewise linear graph to represent this information.

b How many kilometres has the yacht sailed over the two legs?

11 Joseph rides his bike from his home to the shops 12 kilometres away. His distance from home is given by the equation

$$d = \begin{cases} 6t, & 0 \le t < 1 \\ 6, & 1 \le t < 1.5 \\ 12 - 4t, & 1.5 \le t \le 3 \end{cases}$$

where d is the distance Joseph rides on his bike and t is the time taken in hours.

a Draw a piecewise linear graph to represent his journey to the shops.

b During which part of the journey was he travelling the fastest? Explain your answer.

c Between $t = 1$ and $t = 1.5$ the graph is a horizontal line. What is the gradient of this line? Explain what you think happened between $t = 1$ and $t = 1.5$.

d How many kilometres did Joseph cycle in total?

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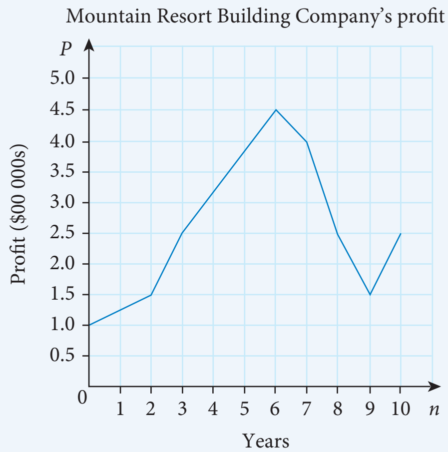
INVESTIGATION Piecewise linear rules in business

Companies often chart their profit results over a number of years to help them analyse how their company is performing and to help them predict how their company may perform in the future. At times they use line graphs that resemble piecewise linear graphs.

The Mountain Resort Building Company's performance over 10 years is shown in the graph below.



a How many line segments form this piecewise linear graph?

b What profit was the company showing at the start of this 10-year period?

c What profit was the company showing at the end of the 10 years?

d Use the graph to determine the company's profit at:

i 2.5 years **ii** 4 years **iii** 7.5 years

e After how many years did the company show a profit of $420 000?

f For how many years was the company's profit:

i increasing? **ii** decreasing?

g Between which years did the company grow the fastest? Give a reason to explain why there might have been such a growth.

h Between which years did the company show a fall in profits? How much did their profits fall by during this time?

i How can you explain this drop in profits?

j State the gradient of each of the seven line segments that form this piecewise linear graph.

k Using the trend shown in the graph, what do you think the company may do in the next five years?

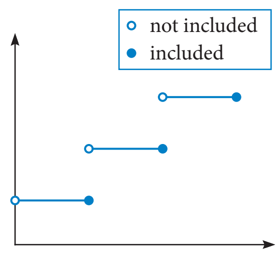
The Australian Stock Exchange lists public companies and provides information to the pubic about these companies. It shows the value of stocks and may represent the value of these stocks in a graph that resembles a piecewise linear graph. Visit the ASX website: http://www.asx.com.au/

Weblink

Research some of the companies listed there to see how their share prices are presented to the public.

12.08 STEP GRAPHS

A **step graph** is made up of horizontal line segments that look like steps. The ends of each line segment in a step graph are labelled with circles. These can be open (hollow) or closed (shaded). An open circle means that the point is not included as part of the graph while a closed circle means that the point is included as part of the graph. A step graph is said to be **discontinuous** because the line segments do not connect.



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IMPORTANT

A step graph is a type of piecewise graph.

Example 21

This step graph shows the parking charges at a car park.

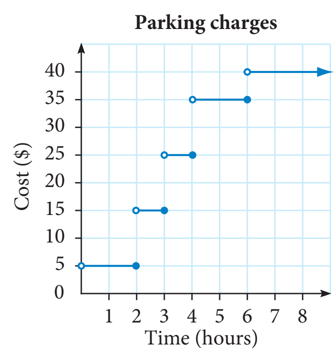
a Find the charge for parking for:

i 3 hours ii 55 minutes

iii $3\frac{1}{2}$ hours iv 4 hours 1 minute.

b For what range of times can a driver park for $25?

c What does the arrow on the $40 step mean?



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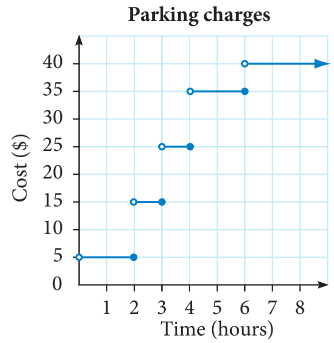
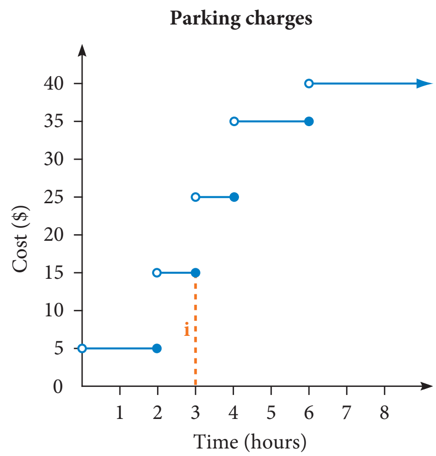
Solution

a The costs will vary according the amount of time a driver parks in the car park.
For example, if a driver parks in the car park for more than 0 hours but less than or equal to 2 hours, they will pay $5.
If a driver parks in the car park for more than 2 hours but less than or equal to 3 hours, they will pay $15.
1f a driver parks in the car park for more than 3 hours but less than or equal to 4 hours, they will pay $25, and so on.



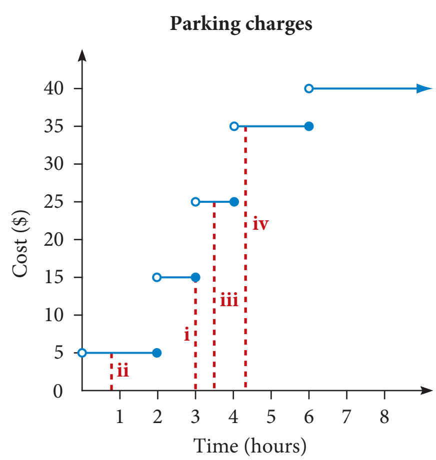
i Find 3 hours on the horizontal axis and draw a vertical line from this point until you reach one of the steps. The vertical line meets the second step at the end of the interval. The circle at the end of the step is closed, so 3 hours is included in this interval.



Read the answer from the graph as indicated and answer the question.

3 hours of parking will cost $15.

Repeat the process from part **i** for each of questions **ii**, **iii** and **iv**.



ii Read the answer from the graph as indicated and answer the question.

55 minutes of parking will cost $5.

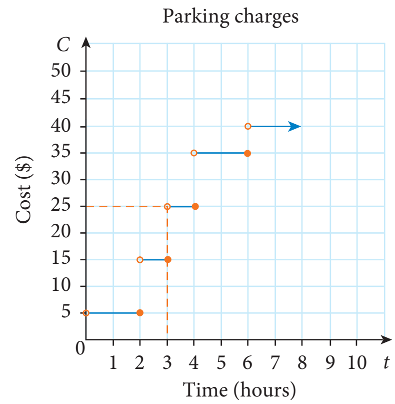
iii

$3\frac{1}{2}$ hours of parking will cost $25.

iv

4 hours 1 minute of parking will cost $35.

b Using the graph, find $25 dollars on the vertical axis.
Move across horizontally until you reach the step which is in line with 25.
The step ranges from 3 to 4. The open circle at the start means that 3 is not included. The closed circle at the end means that 4 is included.



Write the answer.

The driver can park for more than 3 hours but less than or equal to 4 hours if he pays $25.

c The $40 step has an arrow at the end of the interval. This means that the parking charge is the same for any time after 6 hours.

Write the answer.

After 6 hours, the parking charge remains constant at $40. The maximum daily charge is $40.

At times we may be given a rule and be required to draw a step graph.

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Example 22

Graph the following stepwise linear function.

$$y=\begin{cases}1, & \text{for } -2<x\le 1\\ 3, & \text{for } 1<x\le 4\\ 5, & \text{for } 4<x\le 7\\ 7, & \text{for } 7<x\le 10\end{cases}$$

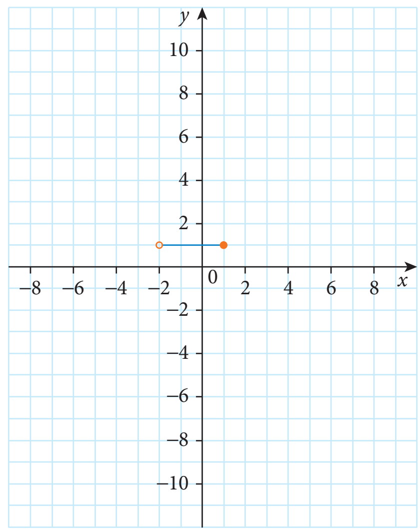
Solution

The graph of this equation will resemble a step graph. There are 4 steps in this graph. Consider each part separately.

The first step or interval has the equation $y = 1$ when x is greater than −2 but less than or equal to 1.

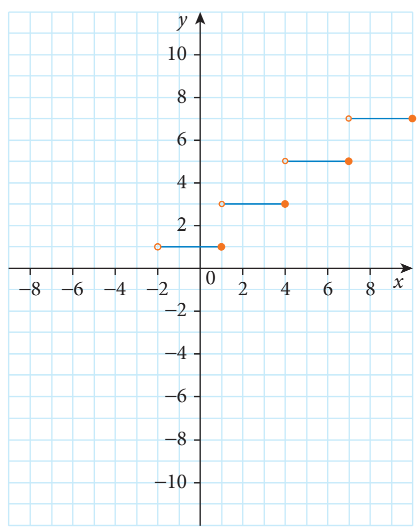
Draw a horizontal line from −2 to 1 at $y = 1$.

−2 is not included, so place an open circle at this end. 1 is included, so place a closed circle at this end.



Repeat this process for the remaining 3 steps to complete the graph.

Each step should have the open circle at the start and the closed circle at the end.



Step graphs are used in real-life situations, such as the cost of hiring items, the cost of sending parcels or car parking costs.

Example 23

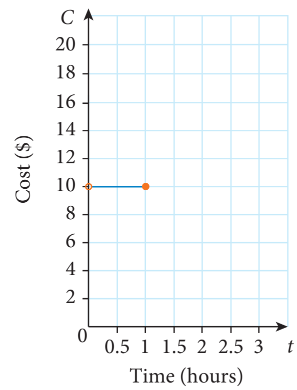
The cost of hiring a bicycle from Happy Cycling Rentals varies according to the number of hours the bicycles are rented for. The rates are as follows:

\$10 up to and including 1 hour, then \$2.50 for each additional half hour or part thereof.

Draw a step graph to represent the cost of hiring a bicycle for up to 3 hours from Happy Cycling Rentals.

Solution

The graph representing this information will be a step graph. Draw a set of axes. Label the horizontal axis 'Number of hours' (t) and the vertical axis 'Hiring cost' (\$$C$).
The bike will cost \$10 to hire for the first hour.
Draw a horizontal line from 0 to 1 at $C = 10$.
As this is the cost of hiring the bicycle for up to and including 1 hour, draw a closed circle at the right end of the line interval.
An open circle is required at the left end of this line segment as you don't pay \$10 if you don't use the bicycle at all.



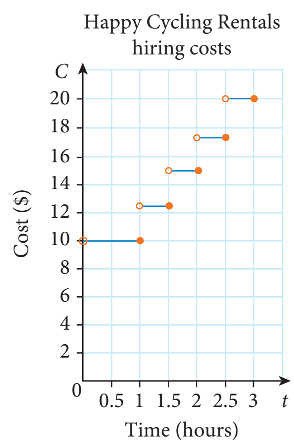
It will cost an additional \$2.50 for the next half hour. Add this cost to \$10.

$\$10 + \$2.50 = \$12.50$

Draw a horizontal line from 1 to 1.5 at $C = 12.5$.
Place an open circle at the start of this interval since the cost for 1 hour is \$10, not \$12.50.
This is the price for the next half hour, hence we need to place a closed circle at the end of this interval.

Repeat this process until you reach 3 hours on the horizontal axis.
Remember to place an open circle at the start and a closed interval at the end of each line segment.

$12.5 + 2.5 = 15$
$15 + 2.5 = 17.5$
$17.5 + 2.5 = 20$



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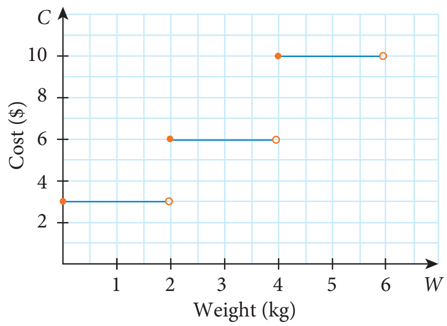
EXERCISE 12.08 Step graphs

WS
Step graphs

Concepts and techniques

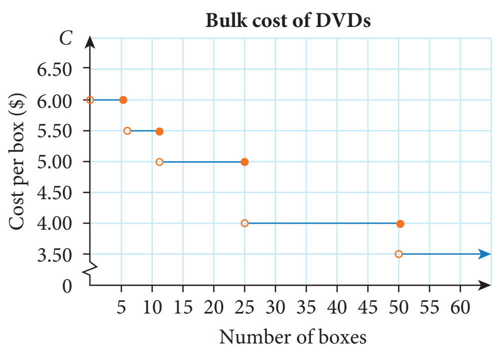
1 a In your own words, describe why the graphs above are called 'step graphs'.
b List 3 features of a step graph.

2 Example 21 A step graph shows the cost of sending parcels of different weights by air freight.



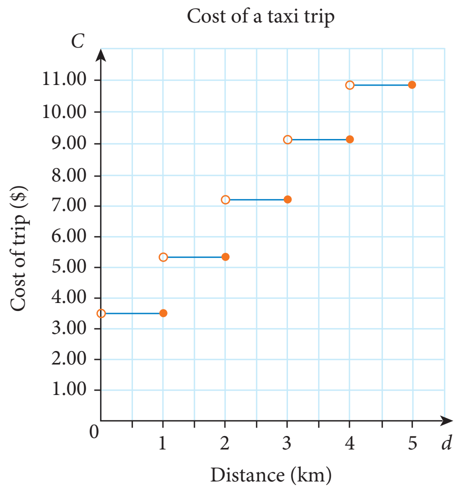
How much does it cost to send:
a one parcel weighing 3.4 kg?
b one parcel weighing 1.7 kg?
c two parcels, weighing 2 kg and 4 kg?

3 This step graph shows the cost per box of blank DVDs when they are bought in bulk.



a What happens to the cost per box as the number of boxes increases? Why?
b Find the cost per box when purchasing:
i 40 boxes ii 8 boxes iii 50 boxes iv 1 box.
c What is the maximum number of boxes that can be bought for $4 each?
d What does the arrow on the $3.50 step mean?

4 The cost of a trip in a taxi depends on the distance travelled, as shown by this step graph.



a Find the cost of travelling a distance of:
   i 4 km ii 2.7 km iii 1.1 km.

b For what range of trip lengths is the cost $9.05?

c What is the 'flag fall' (the charge for starting a trip)?

d Which of the following is the amount charged, after flag fall, for every additional kilometre?
   A $1.85 B $2.68 C $3.50 D $3.60 E $7.40

5 This step graph shows the annual premiums (costs) of life insurance policies for males of different ages.

a What is life insurance?

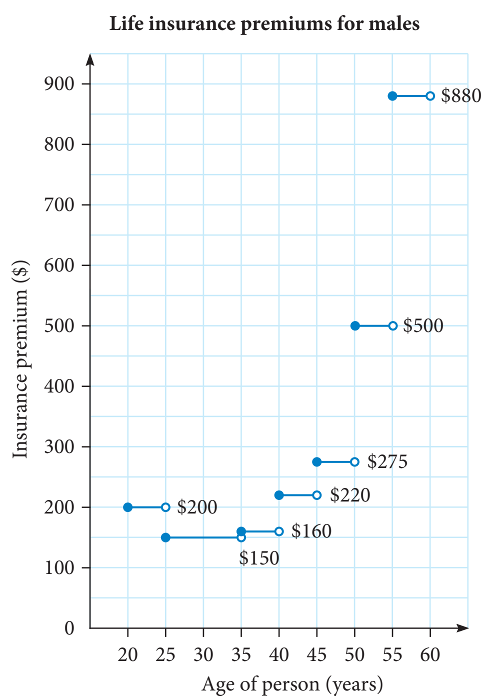
b What is the annual premium for a man aged:
   i 35 years?
   ii 48 years?
   iii 22 years?

c Why do you think the cost of life insurance is lowest for men aged 25 to 34?

d As the age of a man increases, what generally happens to the insurance premium? Why?

e For what age group is the annual premium $200?

f What is the age of the youngest man who can pay a premium of:
   i $220? ii $500?



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6 Example 22 Graph the following equations.

a $y = \begin{cases} -3, & \text{for } -5 \le x < -1 \\ 1.5, & \text{for } -1 \le x < 1 \\ 4, & \text{for } 1 \le x < 6 \end{cases}$

b $y = \begin{cases} -3.5, & \text{for } -3 < x \le -1 \\ -1, & \text{for } -1 < x \le 2.5 \\ 3.5, & \text{for } 2.5 < x \le 5.5 \\ 6.5, & \text{for } 5.5 < x \le 8 \end{cases}$

c $C = \begin{cases} \$11, & \text{for } 0 < n \le 3 \\ \$3.80, & \text{for } 3 < n \le 5 \\ \$4.20, & \text{for } 5 < n \le 8 \\ \$6.60, & \text{for } n > 8 \end{cases}$

Reasoning and communication

7 Example 23 A taxi driver charges $5 flag fall and $2 per kilometre per hour or part thereof. Draw a step graph showing the cost of hiring the taxi for the first 5 kilometres.

8 The size of the speeding fine imposed on a driver depends upon the number of km/h by which the speed limit was exceeded. The table below shows the approximate cost of a fine depending on the number of km/h over the speed limit.
Represent this data on a step graph.

Speed in excess of limit (km/h)	Fine ($)
Less than 10	176
10 or more but less than 25	282
25 or more but less than 30	387
30 or more but less than 35	458
35 or more but less than 40	528
40 or more but less than 45	599
45 or more	704

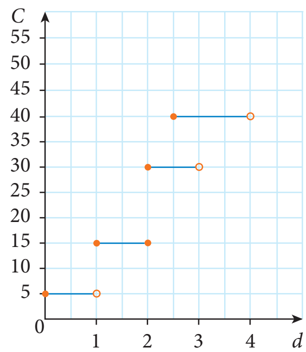
Source: Vicroads

9 The rates for hiring a fishing boat from 'Hook, Line and Sinker' are as follows.

- \$20 for the first hour
- \$5 for each additional half hour or part thereof
- \$40 for the day

Represent these costs on a step graph.

10 Find the faults in the step graph below.



11 A step graph is a type of piecewise graph, which you studied in the previous section. List some of the differences between the two types of graphs.

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CHAPTER SUMMARY

APPLICATIONS OF SIMULTANEOUS EQUATIONS, PIECEWISE AND STEP GRAPHS

12

- The graph of a **linear equation** is a straight line (or **linear graph**).
- The **general form of a linear equation** is $y = a + bx$, where a represents the y-intercept and b represents the **gradient** of the straight line.
- The gradient of a straight line is the slope of the line and is defined as $\frac{\text{rise}}{\text{run}}$. A linear graph can be sketched using the gradient and y-intercept.
- A positive gradient slopes up to the right. A negative gradient slopes down to the right. The gradient will determine the steepness of the line.
- The gradient between two points (x_1, y_1) and (x_2, y_2) can be found using the formula:

  $\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$
- The x- and y-intercepts are the points where a straight line graph cuts the axes. To find the ***x*-intercept** let $y = 0$. To find the ***y*-intercept** let $x = 0$.
- Straight line graphs can be drawn using the gradient and y-intercept method or by using the x- and y-intercepts method.
- Lines of equations in the form $y = a$, where a is a constant, are horizontal lines.
- Lines of equations in the form $x = a$, where a is a constant, are vertical lines.
- When reading graphs we can find answers within the given values or range of the graph. This is called **interpolating**.
- If we predict answers that fall outside the range of the graph we are **extrapolating**.
- **Simultaneous equations** are equations that can be solved at the same time.
- The point where two or more graphs intersect is called the **point of intersection**.
- Simultaneous linear equations can be solved graphically or algebraically. There are two algebraic methods for solving simultaneous linear equations – **substitution method** and **elimination method**. Answers should be checked by substituting the solution into the original linear equations.
- A **piecewise linear equation** is a combination of two or more linear equations.
- The graph of a piecewise linear equation is called a **piecewise graph** and is comprised of separate line segments. The critical points in a piecewise linear graph are the end points of each line segment.
- Piecewise linear graphs are useful for displaying information that has different rates of change.
- A **step graph** is made up of horizontal segments. The ends of each step in a step graph are labelled with circles. An open circle means that the point is not included, while a closed circle means that the point is included.
- A step graph is said to be **discontinuous** because the line segments do not connect.

12 CHAPTER REVIEW
APPLICATIONS OF SIMULTANEOUS EQUATIONS, PIECEWISE AND STEP GRAPHS

Multiple choice

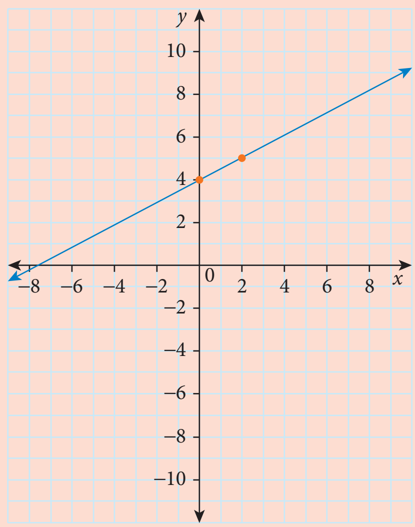
1 Example 1 The equation $3x + y = 5$ transposed into the form $y = a + bx$ is

A $x = \frac{y}{3} - \frac{5}{3}$ B $y = -5 + 3x$ C $y = -5 - 3x$
D $y = 5 + 3x$ E $y = 5 - 3x$

2 Example 1 The gradient of the line $2y - 4x = 6$ is:

A -6 B -3 C $\frac{1}{2}$ D 2 E 4

3 Example 2 The graph of the equation $y = -\frac{1}{2}x + 4$ is:

A


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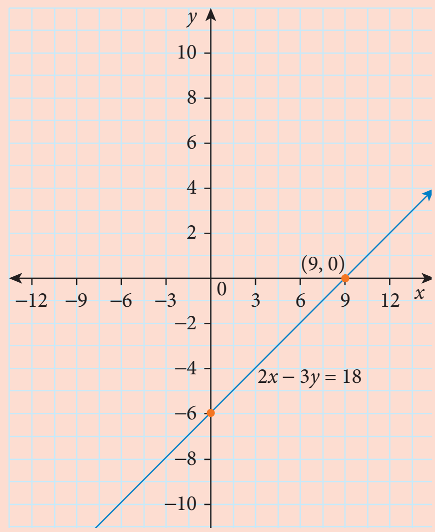

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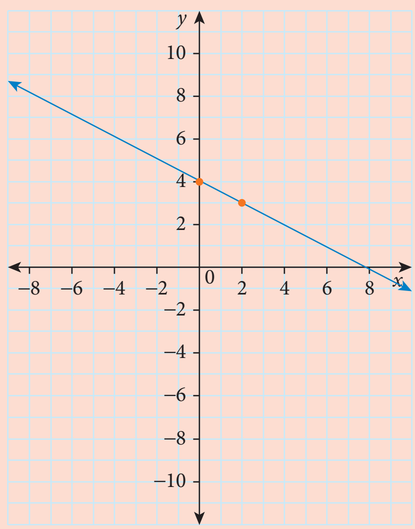

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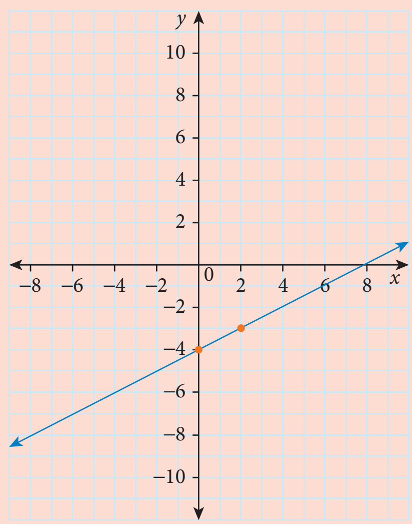

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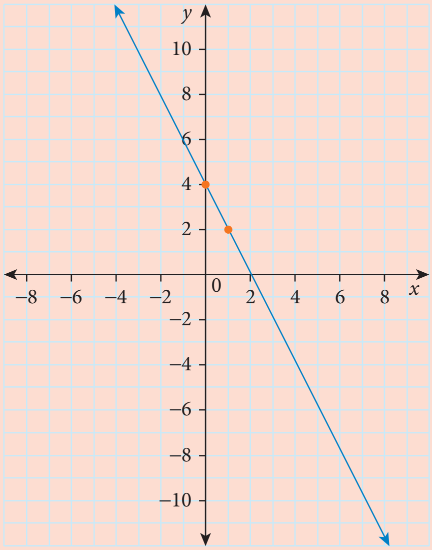
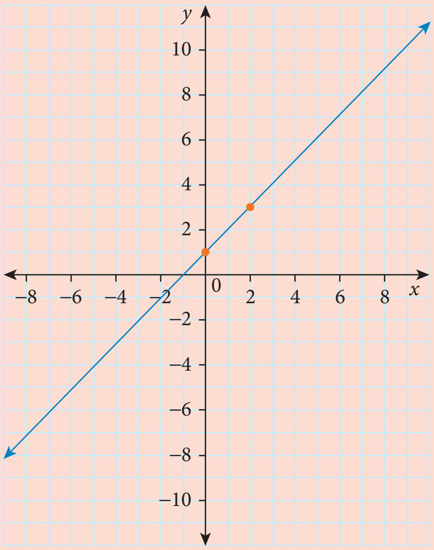
E



4 Example 2 The equation of the line on the right is:



A $y = 1 + 2x$

B $y = -1 - 2x$

C $y = 1 - x$

D $y = 1 - \frac{1}{2}x$

E $y = 1 + x$

5 Example 3 The line with the equation $5x - 3y = 30$ intersects the axes at which of the following points?

A (6, 0) and (0, –10)
B (–6, 0) and (0, 10)
C (0, 6) and (–10, 0)
D (–6, 0) and (0, –10)
E (–6, 0) and (10, 0)

6 Example 7 The solution to the simultaneous equations, $y = x - 5$ and $3x + 2y = -5$ is:

A $x = -1, y = 4$
B $x = -4, y = 1$
C $x = -1, y = -4$
D $x = 1, y = -4$
E $x = 4, y = -1$

For questions **7** and **8** use the simultaneous equations

$5x + 3y = 7$

$-2x - 3y = -37$

7 Example 9 The equation which results when one variable is eliminated is:

A $7x = -30$ B $7x = -44$ C $3x = 30$

D $3x = -30$ E $-3x = 30$

8 Example 9 The point of intersection of the two lines is:

A $\left(10, -\frac{17}{3}\right)$ B $(-10, 19)$ C $\left(-10, -\frac{41}{3}\right)$

D $(10, 19)$ E $(10, -19)$

9 Example 10 For the simultaneous equations

$7x - 2y = 25$

$-4x + 3y = -5$

the solution is:

A $x = -5, y = -5$ B $x = 5, y = 5$ C $x = \frac{55}{13}, y = \frac{15}{13}$

D $x = -\frac{55}{13}, y = -\frac{15}{13}$ E $x = \frac{35}{13}, y = \frac{155}{13}$

10 Example 15 The equation which best represents the piecewise linear graph below is:

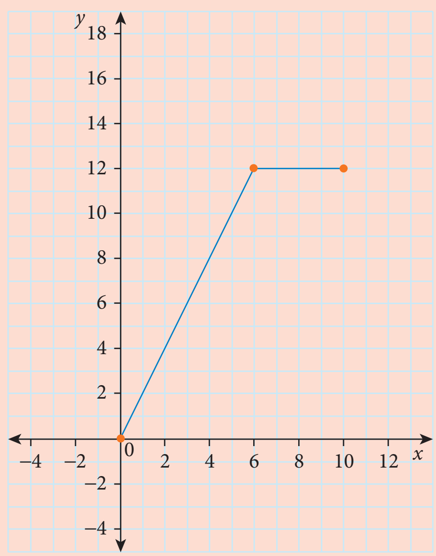
A $y = \begin{cases} -2x, & 0 \le x \le 6 \\ 12, & 6 \le x \le 10 \end{cases}$

B $y = \begin{cases} 2x, & 0 \le x < 6 \\ 12x, & 6 \le x \le 10 \end{cases}$

C $y = \begin{cases} -2x, & 0 \le x < 6 \\ 12, & 6 \le x \le 10 \end{cases}$

D $y = \begin{cases} 2x, & 0 < x < 6 \\ 12, & 6 < x < 10 \end{cases}$

E $y = \begin{cases} 2x, & 0 \le x < 6 \\ 12, & 6 \le x \le 10 \end{cases}$



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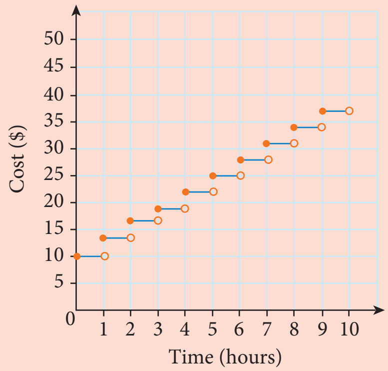
11 Example 21 The step graph below shows the charge for hiring a taxi to travel up to 10 kilometres.



The fare for travelling 6 km is:

A $3 B $10 C $18 D $25 E $28

Short answer

12 Example 1 Find the gradient and y-intercept of the line with the equation $3x + 2y + 10 = 0$

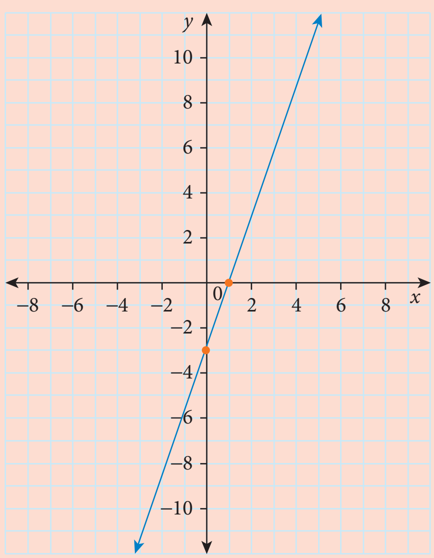
13 Example 1 A promotions company are selling theatre tickets for $75 each. There is also a flat fee of $10 for postage and handling for each booking (no matter how many tickets are bought). Write an equation for the cost (C) of n tickets.

14 Example 1 **a** Find the gradient of the line passing through the points (4, 3) and (0, –5).

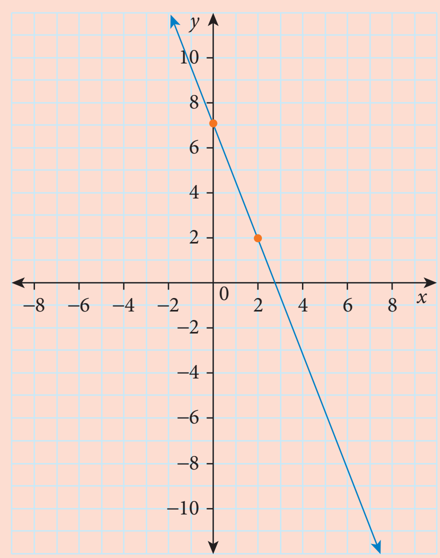
b Find the equation of the line.

15 Example 2 Find the equation of each of these lines.

a



b



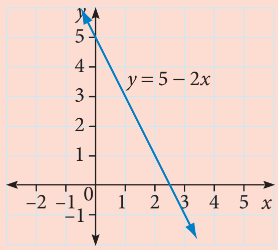
16 Example 2 a Use the gradient and y-intercept method to sketch the lines $y = 2x$ and $y = 2x + 3$ on the same set of axes.
b Comment on the slope of the lines.

17 Example 3 a Find the coordinates of the x-and y-intercepts for the line with the equation $2x - y = 4$.
b Sketch the graph of the line.

18 Example 3 Sketch the graphs of $x = -1$ and $y = 5$ on the same set of axes.

19 Example 4 a Graph the equations $3x - y = 11$ and $2x + y = 4$ on the same set of axes.
b Use the graph to find the point of intersection of $3x - y = 11$ and $2x + y = 4$.

20 Example 5 a Copy the graph below and graph the equation $y = x - 1$ on the same axes.



b Use your graph from part **a** to solve the simultaneous equations $y = 5 - 2x$ and $y = x - 1$.

21 Example 6 Using your CAS calculator, sketch the graphs $y = x + 2$ and $2x - y = 1$. State the point of intersection.

22 Example 7 Solve the simultaneous linear equations below using the substitution method.
$y = 2x + 9$
$3x + 2y = 4$

23 Example 8 a Express $x + y + 6 = 0$ in the form $y = a + bx$.
b Use your answer to part **a** to solve the following pair of simultaneous linear equations by substitution.
$x + y + 6 = 0$
$2x - 5y = 9$

24 Example 9 The equations below are to be solved simultaneously using the method of elimination.
$x - 2y = 11$
$x + 2y = 9$
a Which variable would be the most appropriate to eliminate first?
b Solve the simultaneous equations by elimination.

25 Example 10 Using the elimination method, solve the following simultaneous linear equations.
$3x + 4y = 5$
$2x - 3y = -8$

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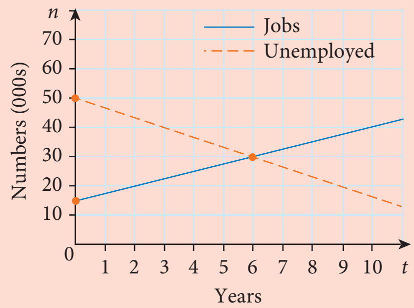
26 Example 12 The number of unemployed people is declining at a steady rate, while the number of jobs available is rising, according to the graph.



a How many people were initially unemployed?

b When will the number of unemployed people equal the number of jobs? How many people were unemployed at this point?

27 Example 13 The perimeter of a rectangle is 72 cm. The length of the rectangle is 8 cm longer than its width.

a Write a set of two of linear equations using the information above.

b Solve the equations simultaneously to find the length and the width of the rectangle.

28 Example 14 A screen printing company has income given by $A = 15p$ and expenses of $A = 54 + 6p$, where p is the number of T-shirts printed. Solve the equations simultaneously to find the break-even point.

29 Example 16 a Sketch the graph of $y = 4x + 1$ for $-3 \le x < 1$.

b On the same set of axes, sketch the graph of $y = 7 - 2x$ for $1 \le x \le 5$.

30 Example 17 Sketch the graph of

$$y = \begin{cases} -x-3, & -10 \le x < -5 \\ 2, & -5 \le x < 1 \\ 2x, & 1 \le x \le 8. \end{cases}$$

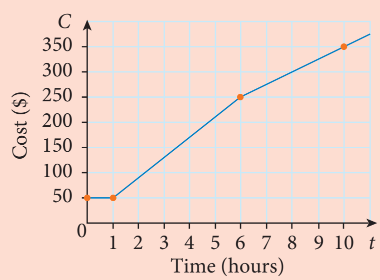
31 Example 18 The graph below shows the cost of servicing a washing machine.

How much will it cost to service a machine for:

a 1 hour?

b 2 hours?

c 8 hours?



32 Example 22 Sketch the step graph given by the equation

$$y = \begin{cases} -5, & -6 \le x < -2 \\ -1, & -2 \le x < 2 \\ 3, & 2 \le x < 6. \end{cases}$$

Application

33 A shipping container holds 5 pallets of washing powder and 3 pallets of sugar. The weight of the contents is 1880 kg. The contents of another shipping container, holding 3 pallets of washing powder and 5 pallets of sugar, weighs 1640 kg.

a Write a pair of equations to represent the contents of each container. Let w = the weight of each washing powder pallet and s = weight of each sugar pallet.

b Find the weight of a pallet of washing powder and a pallet of sugar.

34 Salma is organising an end-of-year outdoor lunch for her staff. The cost of hiring a tent is $133 and the food is $7.50 per guest. To cover these costs, Salma is charging each guest $11. The cost and revenue functions may be represented by the following linear rules.

Cost: $C = 7.5n + 133$

Revenue: $R = 11n$

where C and R are in dollars and n represents the number of guests.

a Graph both the C and R functions on the same axes, for values of n from 0 to 50.

b Use the graph to find the cost and revenue for running the lunch for 32 guests.

c For what value of n does the revenue from the lunch equal the cost?

d Why is the value you found in part **c** called the 'break-even point'?

e What is the profit made when the lunch caters for 50 guests?

35 The step graph below shows the cost of mobile phone calls, for call durations from 0 minutes to 4 minutes.

a What is the cost of a mobile phone call that lasts:

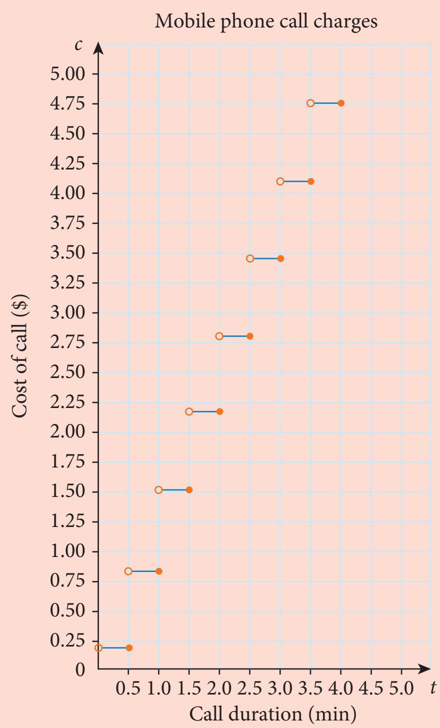
i 3 min 10 s? **ii** 15 s? **iii** 1 min 59 s?

b What is the longest call that can be made for $2.15?

c One particular call cost $4.10. What was the duration of the call?

d By what rate does the call cost increase?

Practice quiz



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