

Name: Marking Key

Score: \_\_\_\_\_ / 22

Section 1 is worth approximately 42% of your final test mark.

No calculators or notes are to be used.

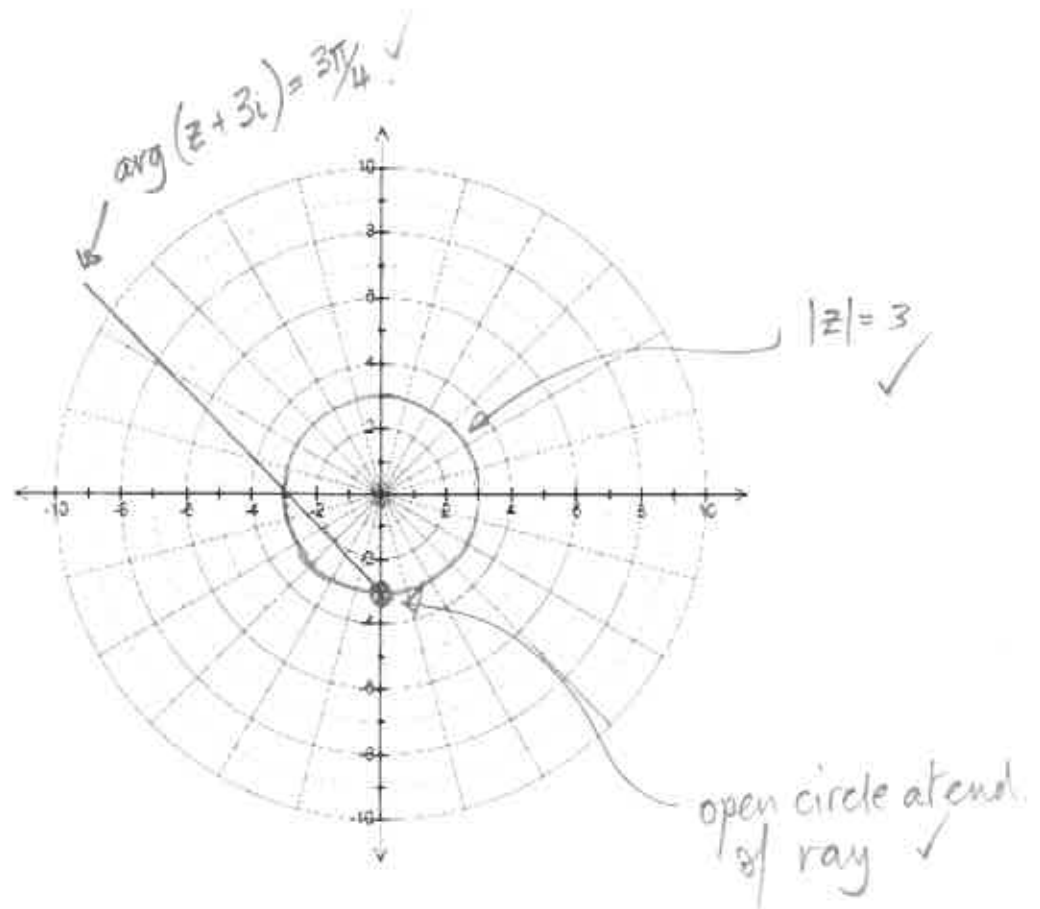
Access to approved Mathematics Specialist formulae sheet is permitted.

Simplify answers where possible.  
minutes.

Time limit = 25

1. [4 marks: 3, 1]

- a) On the axes below, draw the graphs of
- $\arg(z + 3i) = \frac{3\pi}{4}$
- and
- $|z| = 3$
- , for
- $0 \leq \theta \leq 2\pi$
- .

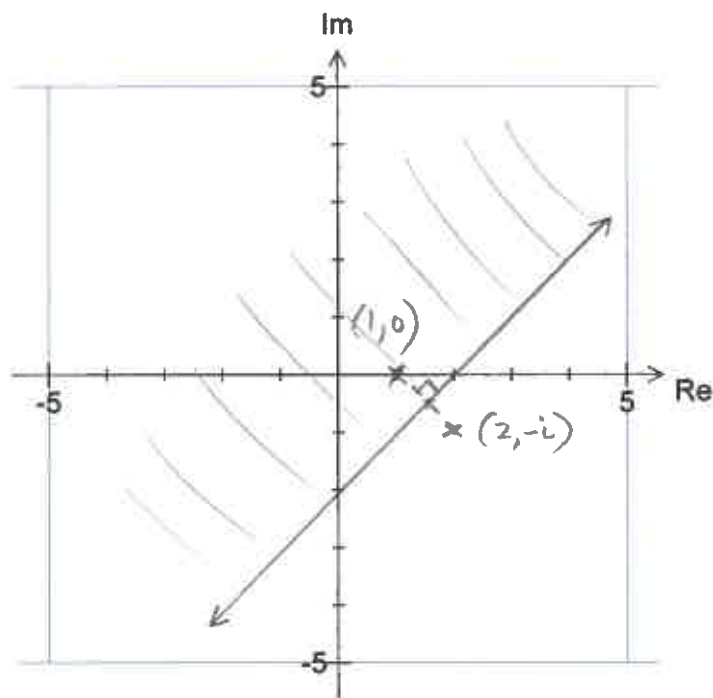


- b) Write down the coordinates of the point where the graphs intersect.

$(-3, 0)$  or  $3\text{cis}\pi$  or  $3\text{cis}(-\pi)$  ✓

2. [7 marks: 3, 4]

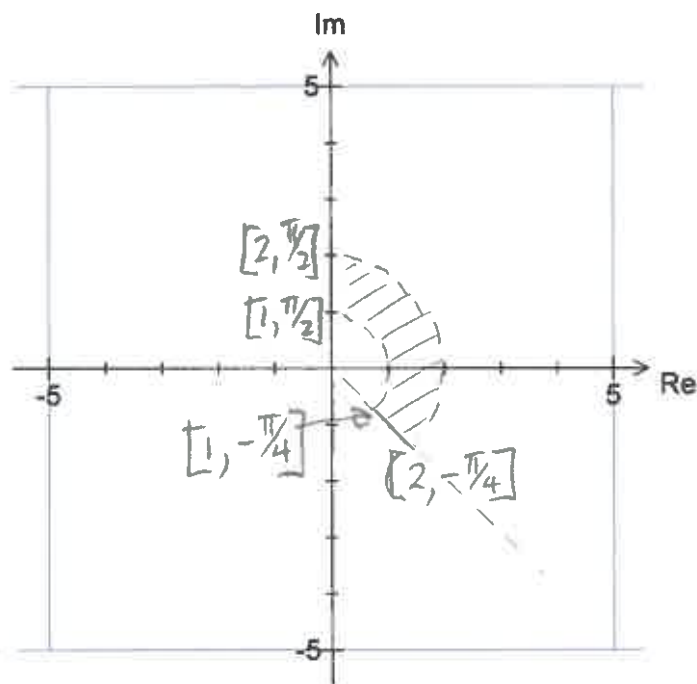
- (a) Sketch  $\{z : |1 - z| \leq |z + i - 2|\}$  on the Argand diagram. Identify important features.



shading ✓  
points (1,0) and (2,-i) ✓  
perpendicular bisecting line ✓

- (b) Sketch the region of the complex plane defined by (4 marks)

$$\{z : -\frac{\pi}{4} \leq \arg z \leq \frac{\pi}{2} \cap 1 < |z| < 2\} \text{ Identify important features.}$$



shading ✓  
dotted lines  $r=1$  and  $r=2$  ✓  
critical points ✓  
shape of region ✓

3. [6 marks: 1, 1, 1, 3]

The complex numbers of  $z$  and  $w$  are such that:

$$z = 2 + 5i$$

and

$$w = 4 - 3i.$$

Express each of the following in the form  $a + bi$ .

a)  $\bar{z} + w$

$$2 - 5i + 4 - 3i \\ = 6 - 8i \checkmark$$

c)  $wz$   $(2 + 5i)(4 - 3i)$   
 $= 8 - 6i + 20i - 15i^2$   
 $= 23 + 14i \checkmark$

b)  $w^2$

$$(4 - 3i)(4 - 3i) \\ = 16 - 12i - 12i + 9i^2 \\ = 7 - 24i \checkmark$$

d)  $\frac{z}{w}$

$$= \frac{2 + 5i}{4 - 3i} \times \frac{4 + 3i}{4 + 3i} \checkmark \\ = \frac{-7 + 26i}{25} \checkmark$$

4. [5 marks]

Factorise the polynomial  $x^4 - 2x^3 - 7x^2 + 8x + 12$ .

$$f(1) \neq 0, f(-1) = 0 \text{ \& } f(2) = 0$$

so  $(x+1)$  and  $(x-2)$  are factors  $\checkmark$

$$\therefore (x+1)(x-2)(ax^2+bx+c) = f(x)$$

By inspection  $a=1$   $c=-6$   $\checkmark$

$$\text{so } (x^2-x-2)(x^2+bx-6) = f(x)$$

Equating  $x^3$  terms:  $b-1 = -2$

$$\text{so } b = -1 \checkmark$$

$$\therefore (x+1)(x-2)(x^2-x-6) = f(x)$$

$$\therefore (x+1)(x-2)(x-3)(x+2) = f(x) \checkmark$$

END OF PAPER

Name: Marking Key

Score: \_\_\_\_ / 31

Section 2 is worth 58% of your final test mark.

Calculators allowed and 1 page of A4 notes, writing on both sides.

Access to approved Mathematics Specialist formulae sheet is permitted.

Time limit = 30 minutes

5. [4 marks: 2, 2]

Use De Moivre's Theorem to find the exact value in polar form of

a)  $(1 - \sqrt{3}i)^{11}$

$$= (2 \operatorname{cis}(-\pi/3))^{11} \checkmark$$

$$= 2^{11} \operatorname{cis}(\pi/3) \checkmark$$

$$\text{or } (1024 + 1024\sqrt{3}i)$$

b)  $(-1 + i)^{-15}$

$$= (\sqrt{2} \operatorname{cis}(3\pi/4))^{-15}$$

$$= \frac{\sqrt{2}}{256} \operatorname{cis}(-15 \times 3\pi/4) \checkmark$$

$$= \frac{\sqrt{2}}{256} \operatorname{cis}(3\pi/4) \checkmark$$

$$\text{or } \left(-\frac{1}{256} + \frac{1}{256}i\right)$$

6. [5 marks]

By finding  $z^3$  if  $z = \operatorname{cis} \theta$ , show that  $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$

$$(\cos \theta + i \sin \theta)^3 = \cos^3 \theta + 3 \cos^2 \theta i \sin \theta + 3 \cos \theta i^2 \sin^2 \theta + i^3 \sin^3 \theta$$

$$= \cos^3 \theta - 3 \cos \theta \sin^2 \theta + 3 \cos^2 \theta i \sin \theta - i \sin^3 \theta \checkmark$$

$$\text{By De Moivre's Theorem } = \cos 3\theta + i \sin 3\theta$$

Equating Imaginary parts  $\checkmark$

$$\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta \checkmark$$

$$= 3 \sin \theta (1 - \sin^2 \theta) - \sin^3 \theta \checkmark$$

$$= 3 \sin \theta - 3 \sin^3 \theta - \sin^3 \theta \checkmark$$

$$= 3 \sin \theta - 4 \sin^3 \theta$$

7. [3 marks]

If  $z = 4\text{cis}\theta$  and  $z = A\text{cis}\left(\frac{2\pi}{3}\right)$  are two of the four, fourth roots of  $w$ .

Find, exactly, all values of  $A$  and  $\theta$ , where  $-\pi \leq \theta \leq \pi$ , for the four roots.

$$A = 4 \checkmark$$

$$\text{Let } z_1 = 4 \text{cis}\left(\frac{2\pi}{3}\right) \sqrt[4]{2}$$

$$\text{then } z_2 = 4 \text{cis}\left(\frac{2\pi}{3} + \frac{\pi}{2}\right) = 4 \text{cis}\left(-\frac{5\pi}{6}\right) \sqrt[4]{2}$$

$$z_3 = 4 \text{cis}\left(-\frac{5\pi}{6} + \frac{\pi}{2}\right) = 4 \text{cis}\left(-\frac{\pi}{3}\right) \sqrt[4]{2}$$

$$z_4 = 4 \text{cis}\left(-\frac{\pi}{3} + \frac{\pi}{2}\right) = 4 \text{cis}\left(\frac{\pi}{6}\right) \sqrt[4]{2}$$

8. [7 marks: 2, 1, 4]

For  $z$ , a complex number where  $z = \cos\theta + i\sin\theta$ ;

a) Show that  $\frac{1}{z^n} = \cos n\theta - i\sin n\theta$

$$\begin{aligned} \text{LHS} &= \frac{1}{\cos n\theta + i\sin n\theta} \times \frac{\cos n\theta - i\sin n\theta}{\cos n\theta - i\sin n\theta} \checkmark \\ &= \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta} = \frac{\cos n\theta - i\sin n\theta}{1} \checkmark \\ &= \text{RHS} \end{aligned}$$

b) Hence, or otherwise, show that  $2\cos n\theta = z^n + \frac{1}{z^n}$

$$\begin{aligned} \text{RHS} &= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta \quad (\text{from above}) \checkmark \\ &= 2\cos n\theta \end{aligned}$$

c) Hence show that  $\cos^5 \theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos \theta)$

$$\left(z + \frac{1}{z}\right)^5 = (2\cos \theta)^5 \quad \checkmark$$

$$\begin{aligned} \text{LHS} &= z^5 + \frac{5z^4}{z} + 10\frac{z^3}{z^2} + \frac{10z^2}{z^3} + \frac{5z}{z^4} + \frac{1}{z^5} \\ &= z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5} \quad \checkmark \end{aligned}$$

$$= \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right) \quad \checkmark$$

$$= 2(\cos 5\theta + 5\cos 3\theta + 10\cos \theta)$$

$$\text{RHS} = 2^5 \cos^5 \theta \quad \checkmark \therefore \cos^5 \theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos \theta)$$

9. [4 marks]

$2x^n + ax^2 - 6$  leaves a remainder of -7 when divided by  $x-1$ , and 129 when divided by  $x+3$ . Find  $a$  and  $n$ , given that  $a, n \in \mathbb{Z}$

$$\begin{aligned} f(1) &= 2 \times 1^n + a - 6 = -7 \\ \Rightarrow a &= -3 \quad \checkmark \end{aligned}$$

$$f(-3) = 2(-3)^n + -3(-3)^2 - 6 = 129 \quad \checkmark$$

$$\Rightarrow 2(-3)^n - 27 - 6 = 129$$

$$\Rightarrow (-3)^n = 81 \quad \checkmark$$

$$\Rightarrow n = 4 \quad \checkmark$$

10.

11. [8 marks]

$3+i$  is a root of  $z^4 - 2z^3 + az^2 + bz + 10$ , where  $a$  and  $b$  are real. Determine  $a$  and  $b$  and the other roots of the equation.

$3-i$  is also a root since all coefficients are real ✓

So  $z - 3+i$  and  $z - 3-i$  are both factors

$$\text{So } (z^2 - 6z + 10)(cz^2 + dz + e) = z^4 - 2z^3 + az^2 + bz + 10$$

By inspection  $c=1$   $e=1$ . ✓

$$\text{So } (z^2 - 6z + 10)(z^2 + dz + e) = f(z)$$

Comparing  $z^3$  terms

$$d - 6 = -2 \Rightarrow d = 4 \quad \checkmark$$

$$\text{So } (z^2 - 6z + 10)(z^2 + 4z + 1) = 0$$

$$\text{Therefore } z^4 - 2z^3 - 13z^2 + 34z + 10 = 0$$

$$a = -13 \quad b = 34 \quad \checkmark$$

Roots are  $-3 \pm i$  and  $-2 \pm \sqrt{3}$  ✓

$$\left( \frac{-4 \pm \sqrt{16-4}}{2} \right) = -2 \pm \frac{2\sqrt{3}}{2}$$