MARKING KEY – TEST 2 2017 MATHEMATICS SPECIALIST CALCULATOR FREE

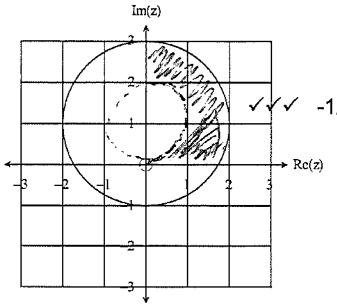
1. (13 marks)

(a) $\left(3\sqrt{3}+\sqrt{2}i\right)\left(2\sqrt{3}-2\sqrt{2}i\right)$.

$=18+2i\sqrt{6}-6i\sqrt{6}-4i^2$ ✓✓

$=22-4i\sqrt{6}$ ✓ (3)

(b)



✓✓✓ -1/error

(3)

(c)

$$(1-i)^{10}=\left(\sqrt{2}\,cis\left(-\frac{\pi}{4}\right)\right)^{10}$$

$$=\left(\sqrt{2}\right)^{10} cis\left(-\frac{10\pi}{4}\right)$$

$$=2^5 cis\left(-\frac{5\pi}{2}\right)$$ ✓

$$=32 cis\left(-\frac{\pi}{2}\right)$$

$$=32(0-i)$$

$$(1-i)^{10}=-32i$$ ✓

(2)

(d) $Re\left(\frac{3-4i}{1+2i}\right)=Re(-1-2i)=-1$

✓ ✓ (2)

(e) (i) $z=cis\left(\frac{2\pi}{3}\right)=-\frac{1}{2}+\frac{\sqrt{3}}{2}i \Rightarrow \bar{z}=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ ✓ (1)

(ii) $z^2=cis\left(\frac{2\pi}{3}\right)^2=cis\left(\frac{4\pi}{3}\right)=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ ✓ (1)

(ii) $\frac{1}{z^2}=\frac{1}{\left(cis\left(\frac{2\pi}{3}\right)\right)^2}=\frac{1}{cis\left(\frac{4\pi}{3}\right)}=cis\left(-\frac{4\pi}{3}\right)=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ ✓ (1)

Question 2 **(9 marks)**

(a) If $(a-3i)^2=-5-bi$ find the values of a and b, where a and b are real constants.

(3 marks)

Solution
$(a-3i)^2=a^2-9-6ai$ Equating the real parts: $a^2-9=-5 \Rightarrow a=\pm 2$ Equating the imaginary parts: $-6a=-b$: So, for $a=2,\ b=12$ and for $a=-2,\ b=-12$
Specific behaviours
✓ correctly expands $(a-3i)^2$ ✓ equates real and imaginary parts ✓ correctly states corresponding values of a and b

(b) The complex number $z=1-\sqrt{3}i$ is transformed to its reciprocal $\frac{1}{1-\sqrt{3}i}$.

(i) What is the reciprocal of z in the form $a+bi$? (2 marks)

Solution
$\frac{1}{1-\sqrt{3}i}=\frac{1}{1-\sqrt{3}i}\cdot\frac{1+\sqrt{3}i}{1+\sqrt{3}i}=\frac{1+\sqrt{3}i}{4}=\frac{1}{4}(1+\sqrt{3}i)$
Specific behaviours

✓ multiplies by $\frac{\bar{z}}{\bar{z}}$
✓ simplifies to arrive at the correct result

(ii) State the reciprocal of $z = 1-\sqrt{3}i$ in polar form. (2 marks)

Solution
$mod\left(\frac{1}{1-\sqrt{3}i}\right)=\frac{1}{2}$ and $arg\left(\frac{1}{1-\sqrt{3}i}\right)=arg\left(\frac{1+\sqrt{3}i}{4}\right) = arctan\left(\sqrt{3}\right)=\frac{\pi}{3}$ $\therefore \frac{1}{1-\sqrt{3}i}=\frac{1}{2}cis\left(\frac{\pi}{3}\right)$ in polar form.
Specific behaviours
✓ determines $arg\left(\frac{1}{1-\sqrt{3}i}\right)=\frac{\pi}{3}$ ✓ correctly states $\frac{1}{z}$ in polar form

(c) Given z is a complex number, express the modulus and argument of $\frac{1}{z}$ in terms of $\mathrm{mod}\,z$ and $\arg z$. (2 marks)

Solution
$mod\left(\frac{1}{z}\right)=\frac{1}{mod\,z}$ or $\left\lvert\frac{1}{z}\right\rvert=\frac{1}{\lvert z\rvert}$ $arg\left(\frac{1}{z}\right)=-arg(z)$
Specific behaviours
✓✓states the correct relationship for the modulus and the argument

Question 3

Since $P(x)$ is monic it can be factorised as $P(x) = x\left(x^2-1\right)\left(x^2+bx+c\right)$ ✓

Then, by expanding and comparing coefficients:

$\therefore P(x) = x^5 + bx^4 + (c-1)x^3 - bx^2 - cx = x^5 + x^3 - 2x$

$\therefore b = 0$ and $c = 2$ ✓✓

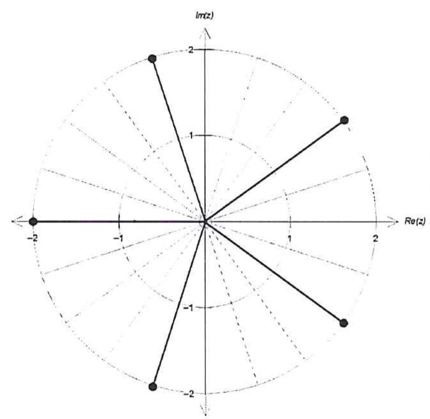
$\therefore P(x) = x\left(x^2-1\right)\left(x^2+2\right)$ ✓

(Long division of polynomials is also an alternative)

Also, $x^5 - x = x - x^3$ becomes $x^5 + x^3 - 2x = 0 = P(x)$ and therefore the

solutions are: $x = 0, \pm 1$ and $\pm\sqrt{2}\,i$ ✓ [5]

4 (a)



✓ magnitude = 2

✓ $\frac{2\pi}{5}$ radians apart

(b) $z_0 = 2\,\text{cis}\left(\frac{\pi}{5}\right)$, hence $z_0^5 = \left(2\,\text{cis}\left(\frac{\pi}{5}\right)\right)^5 = 32\,\text{cis}(\pi) = -32$ ✓

$\therefore a = -32$ and $b = 0$ ✓

(c) Other solutions can be obtained using $Z_n = 2\,\text{cis}\left[2n\pi + \frac{\pi}{5}\right]$ with $n = 0, 1, 2, 3, 4$

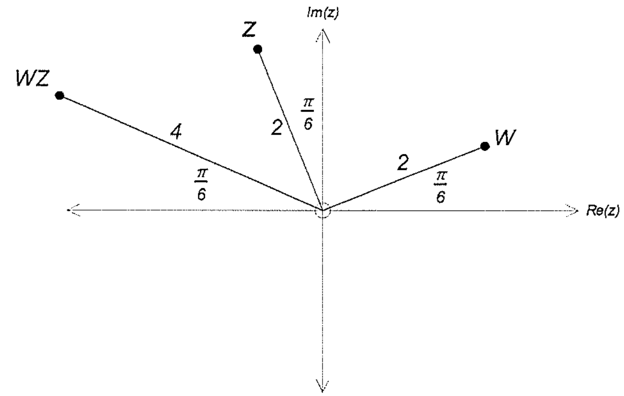
or graphically. Therefore, the solutions are:

$2\,\text{cis}\left(\pm\frac{\pi}{5}\right)$, $2\,\text{cis}\left(\pm\frac{3\pi}{5}\right)$ and $2\,\text{cis}(\pi)$ ✓✓✓ [7]

5. (a) $z = 2\,\text{cis}\left(\frac{2\pi}{3}\right)$ ✓

$wz = 4\,\text{cis}\left(\frac{2\pi}{3} + \frac{\pi}{6}\right) = 4\,\text{cis}\left(\frac{5\pi}{6}\right)$ ✓✓

(b)



✓ correct magnitudes

✓ correct arguments

Question6 **(7 marks)**

A complex number z, is defined by $|z| = \sqrt{2}$ and $\arg z = \frac{\pi}{6}$.

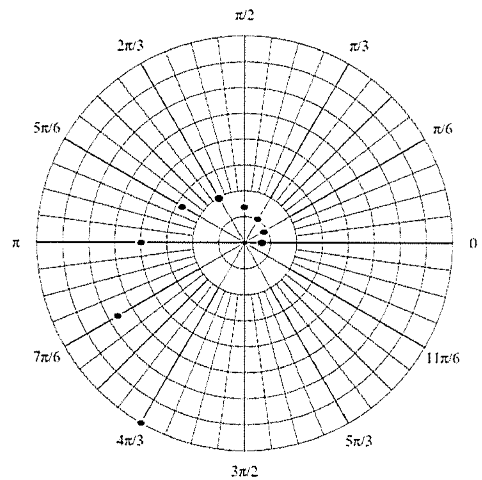
(a) On the polar grid below, graph the sequence z^n for integers, $0 \le n \le 8$. (4 marks)

Solution

Evaluate $r = |z^n|$ and $\theta = \arg(z^n)$ for each value of n within the given domain.

n	0	1	2	3	4	5	6	7	8
$\lvert z^n \rvert$	1	$\sqrt{2}$	2	$2\sqrt{2}$	4	$4\sqrt{2}$	8	$8\sqrt{2}$	16
$\arg(z^n)$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$

Plotting the points gives ...



Specific behaviours

✓ Calculates correct values r of z^n for $0 \le n \le 8$

✓ Calculates correct values θ of z^n for $0 \le n \le 8$

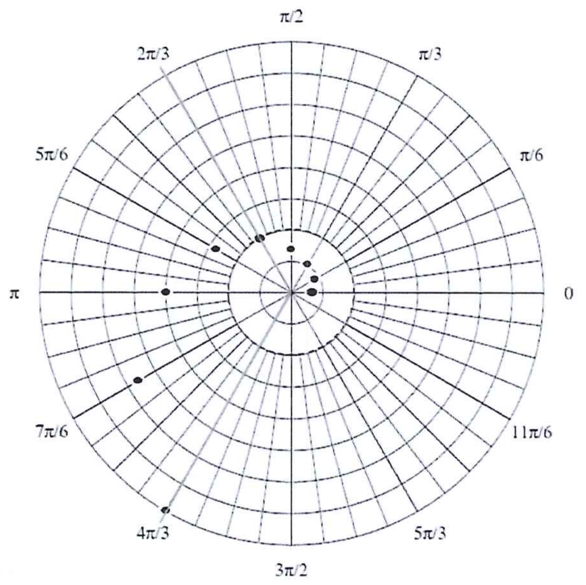
✓ Plots points correctly : mod

✓ Plots points correctly : arg

(b) Hence or otherwise find the value(s) of k, where k is an integer $-6 \le k \le 6$,

such that $-\frac{2\pi}{3} < Arg(z^k) < \frac{2\pi}{3}$ and $|z^k| < 4$. (3 marks)

Solution



For $-3 \le k \le -1$, $|z^k| < 4$ since $|z^{-k}| = \left|\frac{1}{z^k}\right|$ for $k < 0$ and $-\frac{2\pi}{3} < Arg(z) < 0$

For $0 \le k \le 3$, $|z^k| < 4$ and $0 < Arg(z) < \frac{2\pi}{3}$

Solution: $-3 \le k \le 3$

Specific behaviours

✓ Recognises that $|z^k| < 4$ for $-3 \le k \le -1$

✓ Recognises that $|z^k| < 4$ for $0 \le k \le 3$

✓ Recognises that $-\frac{2\pi}{3} < Arg(z^k) < \frac{2\pi}{3}$ and $|z| < 4$ for $-3 \le k \le 3$

Question 7

$(cos(x) + i\,sin(x))^3 = cos^3x + 3cos^2x\,isinx - 3cosxsin^2x - isin^3x$ ✓

$(cos(x) + i\,sin(x))^3 = cos3x + isin3x$ ✓

Equating real parts

$cos3x = cos^3x - 3cosxsin^2x$ ✓

$= cos^3x - 3cosx(1 - cos^2x)$ ✓

$= cos^3x - 3cosx + 3cos^3x$

$= 4cos^3x - 3cosx$ ✓