Year 11 Specialist Mathematics

MELVILLE
SENIOR HIGH SCHOOL

Semester 2, October 2021

Test 6: Complex Numbers and Number Proofs

Calculator Free Section

Weighting: 6%

[Australian Curriculum Reference Numbers: 2.3.1 - 2.3.16]

Total Time: 25min **Total Marks =** $\frac{\quad}{26}$

Student Name: SOLUTIONS

Teacher: ______________________

INSTRUCTIONS TO STUDENTS:

- You **are not allowed** a calculator or any notes.
- A formula booklet will be provided.

You are required to attempt ALL questions.

Write answers in the spaces provided beneath each question.

Marks are shown with the questions.

Show all working clearly, in sufficient detail to allow your answers to be checked readily and for marks to be answered for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks.

1. Given that $z = 2 - i$ and $w = 2i + 5$, determine the following:
   (express your answers in the form $a + bi$, where a and b are real)

   a) $\overline{w}$

   $\overline{w} = 5 - 2i$

   b) $w\overline{z}$

   $= (2i+5)(2+i) = 4i + 2i^2 + 10 + 5i$
   $= 8 + 9i$

   c) $Re(w) + Im(\overline{z})$

   $= 5 + 1 = 6$ or $6 + 0i$

   d) $\frac{w}{zi}$

   $= \frac{5+2i}{(2-i)i} = \frac{5+2i}{1+2i} \times \frac{1-2i}{1-2i}$
   $= \frac{5 - 10i + 2i - \cancel{4}i^2}{5} = \frac{9}{5} - \frac{8}{5}i$

   e) $|w\overline{z} - w|$

   $w\overline{z} - w = 3 + 7i$

   $|3+7i| = \sqrt{3^2+7^2} = \sqrt{58}$

$2 = 9$
[1,2,1,3,~~3~~ = ~~10~~ marks]

2.

a) Solve $z^2 + 3z + 3 = 0$ over the complex plane.

$a = 1$

$z = \frac{-3 \pm \sqrt{9 - 4(1)(3)}}{2}$ ✓ $\quad b = 3$

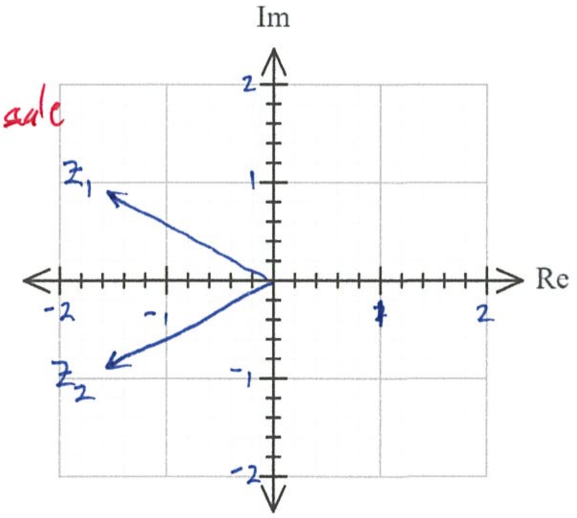
$c = 3$

$z = \frac{-3 \pm \sqrt{-3}}{2}$ ✓ $\Rightarrow z = \frac{-3 \pm \sqrt{3}i}{2}$ ✓ $\rightarrow z = \frac{-3 \pm i\sqrt{3}}{2}$

b) Sketch the solutions to $z^2 + 3z + 3 = 0$ on an argand diagram, labelling them z_1 *and* z_2

✓ accurate for scale

✓ Labels



c) What geometric property do the complex roots of quadratic equations with real coefficients display?

Roots are complex conjugates & are reflections of each other in the real (x) axis. ✓

[3,2,1 = 6 marks]

3. Sketch the complex numbers as vectors on the Argand Diagram below.

$$z_1 = 2 + i, \quad z_2 = \frac{3}{-i}, \quad z_3 = i^7 z_1, \quad z_4 = \overline{z_2}, \quad z_5 = \frac{1}{9} z_1 z_2 z_3 z_4$$

$z_2 = \frac{3}{-i} \times \frac{i}{i}$

$= 3i$

$z_3 = i^3 z_1$

$= -i(2+i)$

$= -2i - i^2$

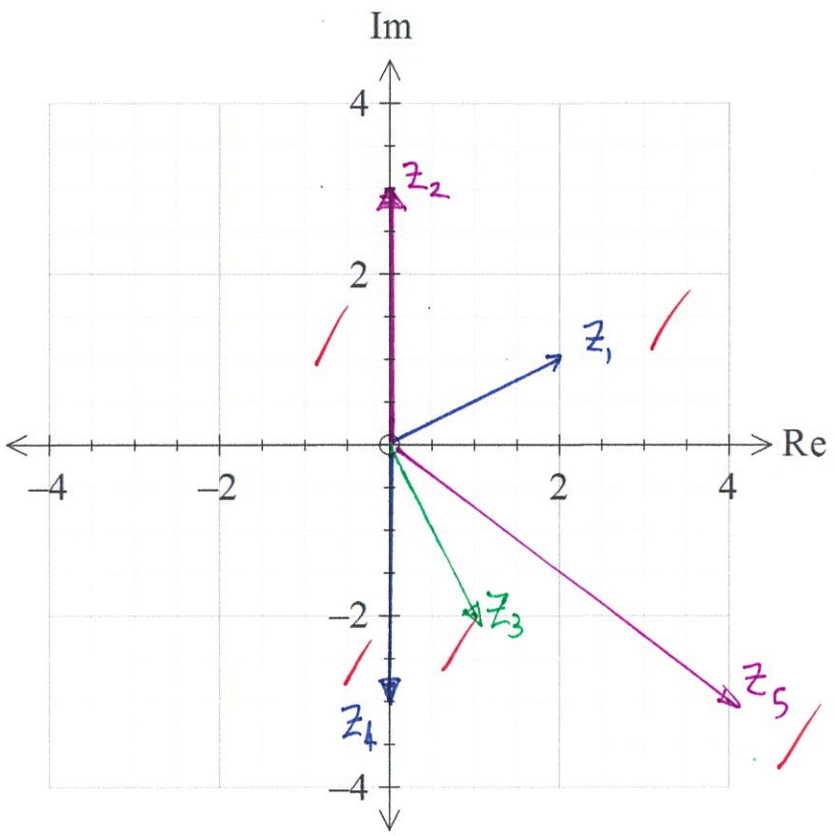
$= 1 - 2i$

$z_4 = -3i$

$z_5 = \frac{1}{9}(2+i)\boxed{(3i)}(1-2i)\boxed{(-3i)}$

$= \frac{1}{\cancel{9}}(\cancel{-9}i^2)(2+i)(1-2i) = 2 - 4i + i - 2i^2$

$= 4 - 3i$



[8 marks]

4. Convert the decimal $2.3\overline{7}$ into the form $\frac{a}{b}, a, b \in \mathbb{Z}^+$ where a *and* b have no common factors.

Let $x = 2.3\overline{7}$

$100x = 237.\overline{7}$

$10x = 23.\overline{7}$

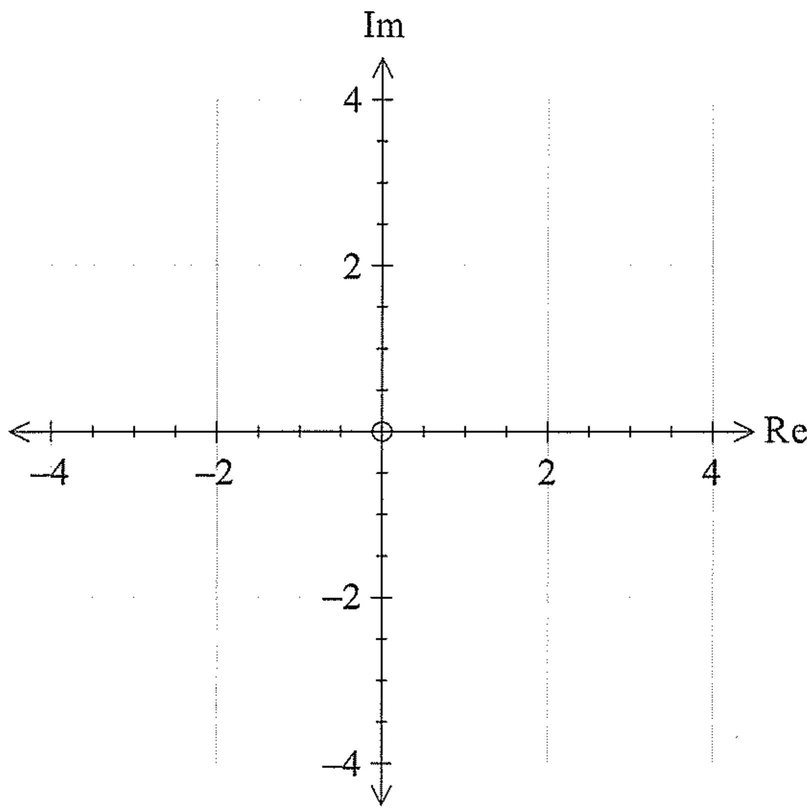
$\therefore 90x = 214$

$x = \frac{214}{90} = \frac{107}{45}$

[3 marks]

*** End of Test ***

Extra space for working out



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Total Time: 25min **Total Marks =** ~~25~~ 22

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1. Use proof by exhaustion to prove that:

$$\textit{The cube of any integer is always either}$$
$$\textit{a multiple of } 9, \textit{or } 1 \textit{ away from being a multiple of } 9$$

Hint: Consider integers as multiples of 3.

The integer cubed could be either:

① · a multiple of 3 ie $n = 3k$, $k \in \mathbb{Z}$

② · one more than multiple of 3 ie. $3k+1$ cases

③ · one less than multiple of 3 ie $3k+2$.

Consider each case:

① $n = 3k \therefore (3k)^3 = 27k^3 = 9M$, $M \in \mathbb{Z}$
(multiple of 9)

② $n = 3k+1 \therefore (3k+1)^3 = 27k^3 + 27k^2 + 9k + 1$
$= 9(3k^3 + 3k^2 + k) + 1$
$= 9M + 1$, $M \in \mathbb{Z}$
(1 more than multiple of 9)

③ $n = (3k+2) \therefore (3k+2)^2 = 27k^3 + 54k^2 + 36k + 8$
$= 27k^3 + 54k^2 + 36k + 9 - 1$
$= 9(3k^3 + 6k^2 + 4k + 1) - 1$
$= 9M - 1$, $M \in \mathbb{Z}$
(1 less than a multiple of 9)

All cases considered

[8 Marks]

$\therefore$ cube of any integer is always either a multiple of 9, or 1 away from being a multiple of 9

QED

5. Prove that any complex number multiplied by its conjugate will always give a real number.

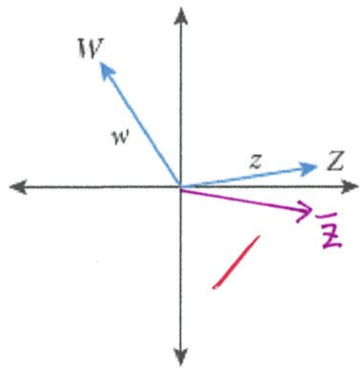
Let $z = a + bi$ where $a, b \in \mathbb{R}$

then $\bar{z} = a - bi$

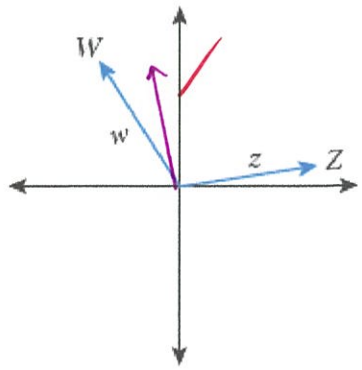
$$(z\bar{z}) = (a+bi)(a-bi)$$
$$= a^2 - iab + iab - b^2 i^2$$
$$= a^2 + b^2, \text{ which is real}$$

[3 marks]

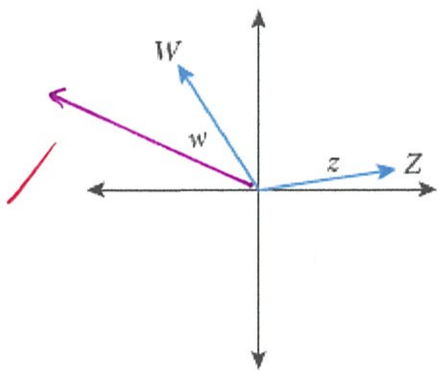
6. For the Argand Diagrams below:



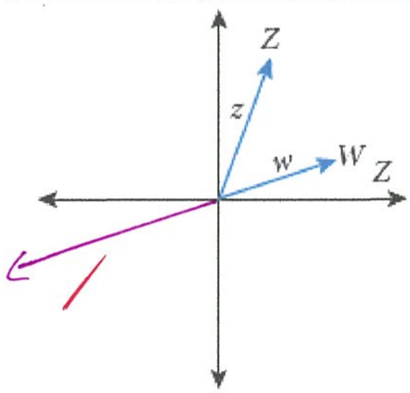
a) Sketch $\bar{z}$



b) Sketch iz



c) Sketch $w - z$



d) Sketch $2i^{26}w$

[4 marks]

7. Prove that:

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}, \textit{for all positive integers } n$$

RTP: $\frac{1}{1\times2}+\frac{1}{2\times3}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}, \ \forall n\in\mathbb{Z}^+$

When $n=1$, $LHS=\frac{1}{1\times2}=\frac{1}{2}$, $RHS=\frac{1}{1+1}=\frac{1}{2}$ } True for $n=1$

Assume true for $n=k$ ie. $\frac{1}{1\times2}+\frac{1}{2\times3}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$

Consider case $n=k+1$

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

* Aiming for $\frac{k+1}{(k+1)+1}$

$$=\frac{k(k+2)}{(k+1)(k+2)}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{(k+1)^2}{(k+1)(k+2)}=\frac{k+1}{k+2}=\frac{k+1}{(k+1)+1}$$

ie. if its true for $n=k$ it is also true for $n=k+1$
& it was true in the initial case

$\therefore$ By mathematical induction: $\frac{1}{1\times2}+\frac{1}{2\times3}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}, \ \forall n\in\mathbb{Z}^+$

QED

7
[~~9~~ Marks]

** End of Test **