MINDARIE
SENIOR COLLEGE
WHERE YOUR FUTURE BEGINS NOW

MATHEMATICS: SPECIALIST 3 & 4

SEMESTER 1 2018

TEST 3

Resource Free

Reading Time: 2 minutes
Time Allowed: 23 minutes

Total Marks: ~~27~~ 19

1. [~~4~~, 1 marks]

The lines with equations $\boldsymbol{r} = \begin{pmatrix} 4-\lambda \\ -2+3\lambda \\ 1+\lambda \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 3-5\lambda \\ 5-\lambda \\ 5-7\lambda \end{pmatrix}$ both lie in the same plane.

(a) Determine the equation of the plane in the form $\boldsymbol{r} \cdot \boldsymbol{n} = c$.

$$\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} -5 \\ -1 \\ -7 \end{pmatrix} = \begin{pmatrix} -20 \\ -12 \\ 16 \end{pmatrix}$$

✓ Determines two vectors

$$\underset{\sim}{a} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$

✓ Determines cross product

$$\underset{\sim}{r} \cdot \begin{pmatrix} -20 \\ -12 \\ 16 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -20 \\ -12 \\ 16 \end{pmatrix}$$

✓ Selects pt in plane

✓ Determines -40

$$= -80 + 24 + 16$$
$$= -40$$

(b) Determine the Cartesian equation of the plane.

$$-20x - 12y + 16z = -40$$

✓ solution

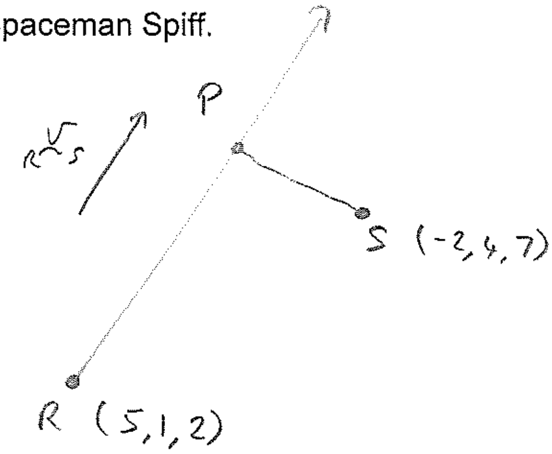
2. [7 marks]

On a beautiful Sunday morning on a bleak, desolate planet, Spaceman Spiff is flying in a straight line with vector equation $\boldsymbol{r}(t) = (-2\boldsymbol{i} + 4\boldsymbol{j} + 7\boldsymbol{k}) + t(5\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k})$ kilometres per minute. Unbeknownst to Spaceman Spiff, a Mangzarr Beast fires a rocket that travels according to the vector equation $\boldsymbol{r}(t) = (5\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) + t(2\boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k})$ kilometres per minute.

Determine the closest distance that the rocket gets to Spaceman Spiff.

Make Spiff stand still.

$${}_R\underset{\sim}{V}_S = -3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$$



$$\overrightarrow{SP} \cdot {}_R\underset{\sim}{V}_S = 0$$

$$\overrightarrow{SP} = (7\underset{\sim}{i} - 3\underset{\sim}{j} - 5\underset{\sim}{k}) + t(-3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k})$$

$$= (7-3t)\underset{\sim}{i} + (-3-t)\underset{\sim}{j} + (2t-5)\underset{\sim}{k}$$

$$\overrightarrow{SP} \cdot {}_R\underset{\sim}{V}_S = -3(7-3t) - 1(-3-t) + 2(2t-5)$$

$$= -21 + 9t + 3 + t + 4t - 10$$

$$= 14t - 28$$

$$t = 2$$

$$\overrightarrow{SP} = \underset{\sim}{i} - 5\underset{\sim}{j} - \underset{\sim}{k}$$

$$|\overrightarrow{SP}| = \sqrt{1+25+1}$$

$$= \sqrt{27} \text{ km}$$

✓ makes one stand still

✓ Determines $\overrightarrow{SP}(t)$

✓ States dot product required

✓ Determines dot product.

✓ Calculates t

✓ Determines $\overrightarrow{SP}$

✓ Determines $|\overrightarrow{SP}|$

3. [~~4~~ 3 marks]

(a) Solve the system of equations

$$\begin{aligned} 4x + 3y + z &= 1 \\ 2x - y + 3z &= 3 \\ x + 3y - z &= 1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 4 & 3 & 1 & 1 \\ 2 & -1 & 3 & 3 \\ 1 & 3 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 4 & 3 & 1 & 1 \\ 0 & 5 & -5 & -5 \\ 0 & -7 & 5 & 1 \end{array}\right] \begin{array}{l} R2 \to R1 - 2R2 \\ R3 \to R2 - 2R3 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 3 & 1 & 1 \\ 0 & 5 & -5 & -5 \\ 0 & -2 & 0 & -4 \end{array}\right] R3 \to R2 + R3$$

✓ eliminates one variable
✓ eliminates some variable
✓ eliminates another variable

✓ Solves for x, y, z

$x = -2$

$y = 2$

$z = 3$

(b) Consider the three planes defined by the equations:

$4x + 3y + z = 1$	Plane 1
$2x - y + 3z = 3$	Plane 2
$kx + 3y - z = 1$	Plane 3

Determine the value for k that would represent the situation where Plane 2 is parallel to Plane 3, with Plane 1 intersecting both planes.

$$\left[\begin{array}{ccc|c} 4 & 3 & 1 & 1 \\ 2 & -1 & 3 & 3 \\ k & 3 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 4 & 3 & 1 & 1 \\ 10 & 0 & 10 & 10 \\ 6+k & 0 & 8 & 10 \end{array}\right] \begin{array}{l} R2 \to R1 + 3R2 \\ R3 \to 3R2 + R3 \end{array}$$

$k = 2$

✓ eliminates one variable
✓ eliminates some variable
✓ determines $k = 2$

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Resource Assumed

Reading Time: 2 minutes
Time Allowed: 33 minutes

Total Marks: ~~32~~ 31

4. [1, ~~1~~ 2, 3, 3 marks]

Given the vectors $\boldsymbol{p} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}$, determine

-1 mk if not nearest degree

(a) the angle between the two vectors, to the nearest degree,

$85°$

✓ Determines angle

(b) the angle, to the nearest degree, between vector $\boldsymbol{p}$ and the y-axis,

$\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $43°$

✓ Determines angle.

(c) ~~the~~ a ~~unit~~ length 12 units vector in the direction $\boldsymbol{p} + \boldsymbol{q}$,

$\underset{\sim}{p} + \underset{\sim}{q} = \begin{pmatrix} 0 \\ 7 \\ 1 \end{pmatrix} = \underset{\sim}{r}$ $\quad \hat{\underset{\sim}{r}} = \frac{1}{\sqrt{50}} \begin{pmatrix} 0 \\ 7 \\ 1 \end{pmatrix}$

$\therefore \frac{12}{\sqrt{50}} \begin{pmatrix} 0 \\ 7 \\ 1 \end{pmatrix}$

✓ determines $\underset{\sim}{p}+\underset{\sim}{q}$
✓ determines unit vector.
✓ determines correct vector.

(d) the exact area of the triangle formed by the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$.

$\underset{\sim}{p} \times \underset{\sim}{q} = \begin{pmatrix} 17 \\ -2 \\ 14 \end{pmatrix}$

$|\underset{\sim}{p} \times \underset{\sim}{q}| = \sqrt{489}$

$\therefore \text{Area } \triangle = \frac{\sqrt{489}}{2}$

✓ determines Cross Prod.
✓ determines magnitude
✓ determines area of triangle exactly

5. [3, 3 marks]

A particle travels according to the equation $\boldsymbol{r} = 2\cos t\,\boldsymbol{i} + (2 - 3\sin t)\boldsymbol{j}$.

(a) State the Cartesian equation of the path of the particle.

$x = 2\cos t$

$\frac{x}{2} = \cos t$

$\frac{x^2}{4} = \cos^2 t$

$y = 2 - 3\sin t$

$3\sin t = 2 - y$

$\sin t = \frac{2-y}{3}$

$\sin^2 t = \frac{(2-y)^2}{9}$

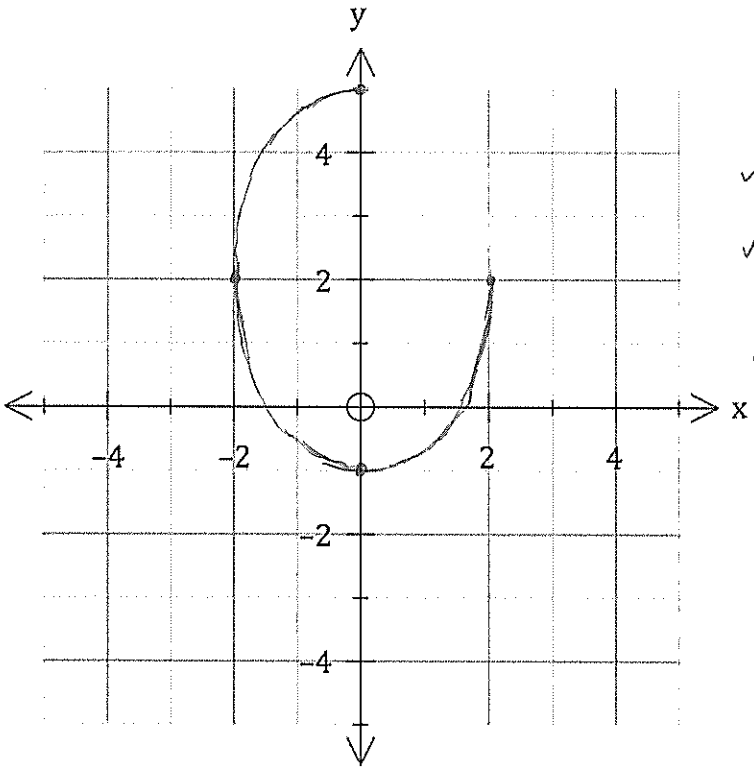
$$\frac{x^2}{4} + \frac{(2-y)^2}{9} = 1$$

✓ Determines cost

✓ Determines sint

✓ Determines equation

(b) Sketch the path of the particle for $0 \le t \le \frac{3\pi}{2}$ on the axes below.



✓ Good graph

✓ Correct position of max/min

✓ Correct domain

6. [1, 5, 2 marks]

A plane has Cartesian equation $5x + 3y - z = 7$.

(a) Determine the normal equation for the plane.

$$\underset{\sim}{r} \cdot \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} = 7$$

✓ Normal equation.

A sphere has equation $|\boldsymbol{r} - 3\boldsymbol{i} + 6\boldsymbol{j} + 2\boldsymbol{k}| = 7$.

(b) Determine the exact distance from the plane to the centre of the sphere.

Distance plane to $C(3, -6, -2)$

Point on plane $A(1, 1, 1)$

$$\text{Distance} = \left| \frac{\overrightarrow{AC} \cdot \underset{\sim}{n}}{|\underset{\sim}{n}|} \right|$$

$$= \left| \frac{\begin{pmatrix} 2 \\ -7 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}}{\sqrt{25+9+1}} \right|$$

$$= \left| \frac{10 - 21 + 3}{\sqrt{35}} \right|$$

$$= \frac{8}{\sqrt{35}} \text{ units.}$$

✓ Determines centre of sphere.

✓ Determines point on plane.

✓ Determines vector from point to centre.

✓ Calculates dot product

✓ Determines distance

(c) Determine the Cartesian equation of a second plane that is parallel to the plane above, and contains the centre of the sphere.

$$\begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} = 15 - 18 + 2$$

$$= -1$$

$$\therefore 5x + 3x - z = -1$$

✓ calculates $\underset{\sim}{c} \cdot \underset{\sim}{n}$

✓ Gives equation

7. [5, 4 marks]

A sphere has the vector equation $|\boldsymbol{r} - (4\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k})| = \sqrt{83}$.

(a) Determine the coordinates of the points, P_1 and P_2, where the line given by the equation $\boldsymbol{r} = (\lambda - 4)\boldsymbol{i} + (2\lambda - 12)\boldsymbol{j} + (2\lambda - 16)\boldsymbol{k}$ intersects the sphere.

$$|(\lambda-8)\underset{\sim}{i} + (2\lambda-15)\underset{\sim}{j} + (2\lambda-17)\underset{\sim}{k}| = \sqrt{83}$$

$$(\lambda-8)^2 + (2\lambda-15)^2 + (2\lambda-17)^2 = 83$$

$$\lambda = 5 \text{ or } 11$$

$$\underset{\sim}{r}(5) = \underset{\sim}{i} - 2\underset{\sim}{j} - 6\underset{\sim}{k}$$

$$\underset{\sim}{r}(11) = 7\underset{\sim}{i} + 10\underset{\sim}{j} + 6\underset{\sim}{k}$$

$\therefore (1, -2, -6)$ & $(7, 10, 6)$

✓ substitutes line into sphere.

✓ Sets up equation

✓ Finds λ

✓ Substitutes for λ

✓ Finds Coordinates

(b) Points $P_3(-5, 4, 2)$ and $P_4(-1. -4, 4)$ also lie on the sphere. Determine the angle that the points P_3 and P_4 subtend at the centre of the sphere.
Give your answer in radians to 2 decimal places.

$$\overrightarrow{CP_3} = -9\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$$

$$\overrightarrow{CP_4} = -5\underset{\sim}{i} - 7\underset{\sim}{j} + 3\underset{\sim}{k}$$

angle = 1.05 radians

✓ calculates $\overrightarrow{CP_3}$

✓ calculates $\overrightarrow{CP_4}$

✓ calculates angle in radians

✓ 2 d.p.