MINDARIE SENIOR COLLEGE

WHERE YOUR FUTURE BEGINS NOW

MATHEMATICS: SPECIALIST 3 & 4

SEMESTER 2 2019

TEST 5

Resource Free

Reading Time: 2 minutes
Time Allowed: 15 minutes

Total Marks: 15

Question 1 **(7 marks)**

The relationship $2y^2 + 3xy + 5x = 3$ is graphed below.

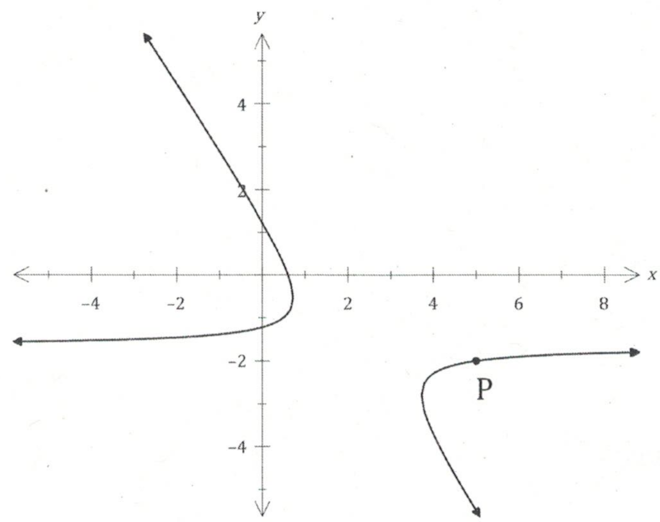
(a) Find an expression for $\frac{dy}{dx}$ for the given relationship. (3 marks)

$4y\frac{dy}{dx} + 3y + 3x\frac{dy}{dx} + 5 = 0$

$\frac{dy}{dx}(4y + 3x) = -5 - 3y$

$\frac{dy}{dx} = \frac{-5-3y}{4y+3x}$

✓ Differentiates $3xy$ correctly
✓ Differentiates rest correctly
✓ Gives $\frac{dy}{dx}$



(b) Determine the gradient of the graph at the point $P(5, -2)$. (2 marks)

$\frac{dy}{dx} = \frac{-5+6}{-8+15}$ ✓ substitutes correctly

$= \frac{1}{7}$ ✓ solution

(c) Hence, determine the approximate value of y when $x = 5.1$. (2 marks)

$\delta y \approx \frac{1}{7} \times \frac{1}{10}$ ✓ Finds δy

$\approx \frac{1}{70}$

$\therefore y \approx -1\frac{69}{70}$ ✓ Finds new y.

Question 2 **(3 marks)**

A function is defined parametrically by $x = 3 - \sin t$, $y = \cos 2t$.

Determine an equation for the gradient function, $\frac{dy}{dx}$, simplifying your answer.

$$\frac{dx}{dt} = -\cos t$$

✓ Differentiates x & y correctly

$$\frac{dy}{dt} = -2\sin 2t$$

✓ Determines equation for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{+2\sin 2t}{+\cos t}$$

✓ Determines simplified equation

$$= \frac{4\sin t \cos t}{\cos t}$$

$$= 4\sin t$$

Question 3 **(5 marks)**

Determine a general solution (with y in terms of x) to the following differential equations.

(a) $\frac{dy}{dx} = 3x(y + 1)$ (3 marks)

$$\int \frac{dy}{y+1} = \int 3x\,dx$$

✓ Separates variables

$$\ln|y+1| = \tfrac{3}{2}x^2 + C$$

✓ Integrates correctly

$$y + 1 = e^{\frac{3}{2}x^2 + C}$$

✓ Gives general solution

$$y = e^{\frac{3}{2}x^2 + C} - 1$$

(b) $\frac{dy}{dx} = \frac{x+2}{x^2 + 4x + 2}$ (2 marks)

$$\int dy = \int \frac{x+2}{x^2+4x+2}\,dx$$

✓ Integrates to get $\ln|x^2+4x+2|$

$$= \frac{1}{2}\int \frac{2x+4}{x^2+4x+2}\,dx$$

✓ correct multiplier ($\frac{1}{2}$)

$$y = \frac{1}{2}\ln|x^2+4x+2| + C$$

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Resource Assumed

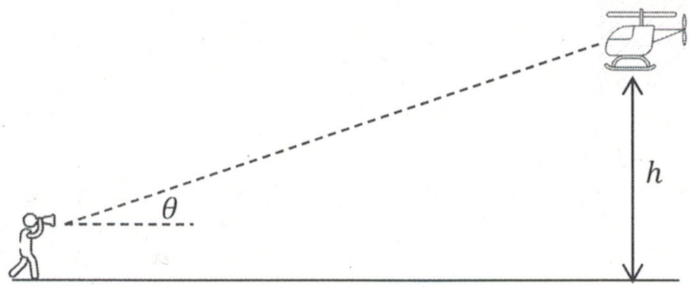
Reading Time: 2 minutes
Time Allowed: 35 minutes

Total Marks: 28

Question 4 **(6 marks)**

An observer on the ground is watching a helicopter land on a landing pad. The landing pad is level with the observer and 100 metres away. The helicopter is dropping vertically at a constant 3 metres per second.

Determine the rate at which the angle of elevation, θ, of the helicopter is changing when the helicopter is 50 metres above the landing pad.
Give your answer to three ~~significant figures~~ decimal places.



100

$\frac{dh}{dt} = -3 \text{ m/s}$

$\frac{d\theta}{dt} = \frac{dh}{dt} \times \frac{d\theta}{dh}$

$= -3 \times \frac{1}{100\sec^2\theta}$

$= \frac{-3}{100} \times \cos^2\theta$

$= \frac{-3}{100} \times 0.8$

$= 0.024$ rad/sec

$\tan\theta = \frac{h}{100}$

$h = 100\tan\theta$

$\frac{dh}{d\theta} = 100\sec^2\theta$

when $h = 50$

$\theta = \tan^{-1}\left(\frac{50}{100}\right)$

$= 0.4636476$

✓ Using chain rule to get $\frac{d\theta}{dt}$.

✓ Relationship between h and θ.

✓ Finding $\frac{dh}{d\theta}$.

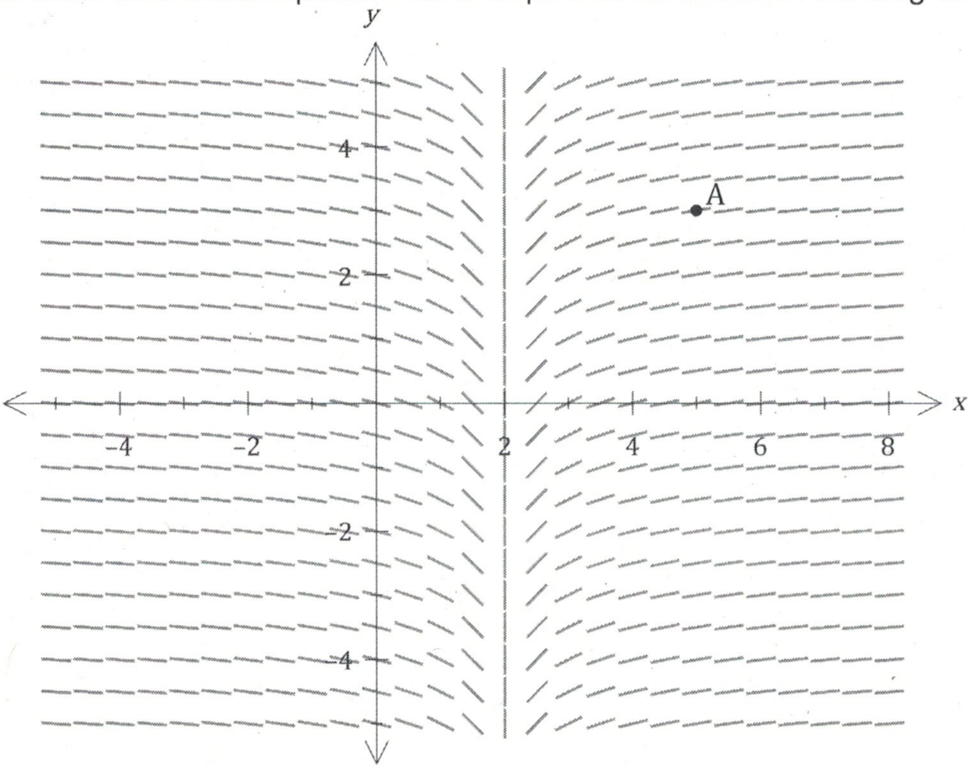
✓ Finding $\frac{d\theta}{dt}$ in terms of θ

✓ Finding θ when $h = 50$m

✓ Finding $\frac{d\theta}{dt}$ to 3 d.p.

Question 5 **(9 marks)**

A first-order differential equation has a slope field as shown in the diagram below.



(a) Determine the general differential equation that would yield this slope field. (3 marks)

$\frac{dy}{dx} = \frac{a}{x-b}$, $a, b > 0$

✓ $\frac{dy}{dx}$ as reciprocal
✓ states $a>0$
✓ states $b>0$

The slope field at point $A(5, 3)$ has a value of $\frac{1}{6}$.

(b) Determine another point on the slope field that has a value of $\frac{1}{6}$. (1 mark)

$(5, *)$

✓ another point with $x=5$

(c) Determine the equation of the curve $y = f(x)$ containing point A. (5 marks)

$\frac{dy}{dx} = \frac{a}{x-2}$

$\frac{1}{6} = \frac{a}{5-2}$

$= \frac{a}{3}$

$\frac{3}{6} = a$

$a = \frac{1}{2}$

$\frac{dy}{dx} = \frac{\frac{1}{2}}{x-2}$

$y = \frac{1}{2}\ln|x-2| + c$

✓ $b=2$
✓ $x=5$ $\frac{dy}{dx} = \frac{1}{6}$
✓ $a = \frac{1}{2}$
✓ integrates correctly
✓ Finds c
✓ Finding F(x)

$3 = \frac{1}{2}\ln 3 + c$

$c = 3 - \frac{1}{2}\ln 3$

$\therefore y = \frac{1}{2}\ln|x-2| + 3 - \frac{1}{2}\ln 3$

$= \frac{1}{2}\ln\left|\frac{x-2}{3}\right| + 3$

$= \ln\sqrt{\frac{x-2}{3}} + 3$

Question 6 **(13 marks)**

The MEG mobile phone company introduces a new phone into the market, the Spec H-P². The marketing department in a capital city estimate that approximately 1.5 million people use mobile phones. In order to get the new mobile phone out into the community the marketing department begins an advertising campaign that involves immediately giving away 2000 phones through prizes and give-aways on mainstream and social media.

Let $N(t)$ = the number of people who have the new Spec H-P² phone t weeks after it is released into the market. Research shows that the rate at which new customers are purchasing the new phone is given by the equation:

$$\frac{dN}{dt} = k\,N(150\,000 - N)$$

It is estimated that at the start of the advertising campaign, 740 people per week are buying the new phone.

(a) Show that $k = 0.0000025$. (2 marks)

$N = 2000 \quad \frac{dN}{dt} = 740$

$740 = k(2000)(150000 - 2000)$

$k = \frac{740}{2000 \times 148000}$

$= 0.0000025$

✓ Uses $N = 2000$ and $\frac{dN}{dt} = 740$

✓ Solves to get k.

(b) Six weeks after the new phone had been released it was estimated that approximately 17 000 people were using the phone. Using the incremental formula, determine the approximate number of people who bought a Spec H-P² in the first day of the seventh week. (3 marks)

$\delta N \approx \frac{dN}{dt} \times \delta t$

$\approx 0.0000025 \times 17000 \times (150000 - 17000) \times \frac{1}{7}$

≈ 807.5

So ~ 810

✓ uses $N = 17\,000$

✓ uses $\delta t = \frac{1}{7}$

✓ gets 810

(c) Given that $\frac{1}{x(k-x)} = \frac{1}{k}\left(\frac{1}{x} + \frac{1}{k-x}\right)$, use the separation of variables technique to show that

$$N(t) = \frac{150\,000}{1 + 74e^{-0.375t}}$$

(7 marks)

$$\frac{dN}{N(150000-N)} = 0.0000025\,dt \quad ✓$$

$$\frac{1}{150\,000}\int\left(\frac{1}{N} + \frac{1}{150000-N}\right)dN = \int 0.0000025\,dt \quad ✓$$

$$\frac{1}{150\,000}\left(\ln N - \ln\left|150\,000-N\right|\right) = 0.0000025t + C \quad ✓$$

$$\ln\left(\frac{N}{150\,000-N}\right) = 0.375t + C$$

$t = 0$, $N = 2000$ ✓

$$\ln\frac{2000}{148\,000} = C$$

$$C = \ln\frac{1}{74} \quad ✓$$

$$\ln\left(\frac{N}{150\,000-N}\right) = 0.375t + \ln\frac{1}{74}$$

$$\frac{N}{150\,000-N} = e^{0.375t}\cdot\frac{1}{74} \quad ✓$$

$$74N = (150\,000-N)e^{0.375t}$$

$$N(74 + e^{0.375t}) = 150000e^{0.375t}$$

$$N = \frac{150\,000e^{0.375t}}{74 + e^{0.375t}} \quad \begin{matrix} \div e^{0.375t} \\ \div e^{0.375t} \end{matrix}$$

$$= \frac{150\,000}{74e^{-0.375t} + 1} \quad ✓$$

✓ separates variables
✓ Correct partial Fractions
✓ Integrates correctly
✓ uses (0, 2000)
✓ Determines C
✓ Removes ln
✓ Gives N(t) as required

(d) What proportion of the mobile phone users in the city do the marketing team estimate will eventually be using the Spec H-P^2? (1 mark)

Saturation @ 150 000 i.e. 10%

✓ Determines 10% (0.1)