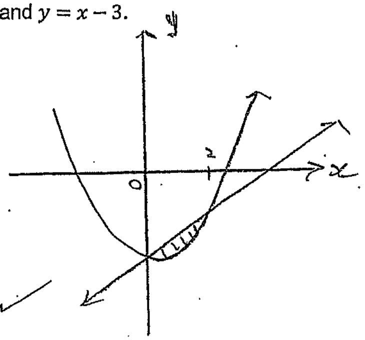
QUESTION ONE (4 MARKS)

Determine the area of the region bounded by $y = x^2 - x - 3$ and $y = x - 3$.



$x^2 - x - 3 = x - 3$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0$ or 2 ✓

Area $= \int_0^2 (x-3) - (x^2 - x - 3)\,dx$ ✓

$= \int_0^2 -x^2 + 2x \, dx$

$= \left[\frac{-x^3}{3} + x^2\right]_0^2$ ✓

$= \left(\frac{-8}{3} + 4\right) - (0 - 0)$

$= \frac{4}{3}$ units^2 ✓

QUESTION TWO (5 MARKS)

Determine the $x -$ coordinate(s) of the stationary point(s) for $y = \frac{e^x}{\sin x}$ over the domain $0 \le x \le \pi$.

$\frac{dy}{dx} = \frac{\sin x\,(e^x) - e^x \cos x}{\sin^2 x}$ ✓ applying quotient rule

$= \frac{e^x(\sin x - \cos x)}{\sin^2 x} = 0$ ✓ ✓

$e^x(\sin x - \cos x) = 0$

$e^x = 0$ or $\sin x - \cos x = 0$

$x = $ N/A ✓

$\sin x = \cos x$

$\frac{\sin x}{\cos x} = 1$

$\tan x = 1$

$x = \frac{\pi}{4}$ ✓

QUESTION THREE (3, 1, 3 = 7 MARKS)

Consider $f'(x) = \cos x \sin^2 x$.

a) Determine $f''\left(\frac{\pi}{6}\right)$.

use product rule

$f''(x) = \cos x \; 2\sin x \cos x + \sin^2 x \; (-\sin x)$ ✓

$= 2\sin x \cos^2 x - \sin^3 x$ ✓

$f''\left(\frac{\pi}{6}\right) = 2\sin\left(\frac{\pi}{6}\right)\cos^2\left(\frac{\pi}{6}\right) - \sin^3\left(\frac{\pi}{6}\right)$

$= 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^3$

$= \frac{3}{4} - \frac{1}{8}$

$= \frac{5}{8}$ ✓

b) Explain the significance of your answer in part(a).

f(x) concave up when $x = \frac{\pi}{6}$ ✓

c) Determine $f(x)$ given that it passes through the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$.

$f(x) = \int \cos x \sin^2 x \, dx$

$= \frac{\sin^3 x}{3} + c$ ✓

$f\left(\frac{\pi}{6}\right) = \frac{1}{2} = \frac{\sin^3\left(\frac{\pi}{6}\right)}{3} + c$ ✓

$\frac{1}{2} = \frac{1}{3}\left(\frac{1}{2}\right)^3 + c$

$\frac{1}{2} = \frac{1}{24} + c$

$\frac{11}{24} = c$

$\therefore f(x) = \frac{\sin^3 x}{3} + \frac{11}{24}$ ✓

QUESTION FOUR (1, 2, 2 = 5 MARKS)

The table below shows the values of $f(x)$ and $g(x)$, and their derivatives $f'(x)$ and $g'(x)$ respectively, when $x = 1, x = 2, x = 3$ and $x = 4$.

	$x = 1$	$x = 2$	$x = 3$	$x = 4$
$f(x)$	-1	0	3	8
$g(x)$	1	3	4.5	5.5
$f'(x)$	0	2	4	6
$g'(x)$	2	2	1	1

Determine:

a) $\int_1^4 g'(x)\,dx$

$= g(4) - g(1)$

$= 5.5 - 1$

$= 4.5$ ✓

b) $\int_2^3 f'(x) - 1\,dx$

$= \int_2^3 f'(x)\,dx - \int_2^3 1\,dx$

$= f(3) - f(2) - [x]_2^3$ ✓

$= 3 - 0 - (3-2)$

$= 2$ ✓

c) $\frac{d}{dx}\int_1^3 2f'(x)\,dx$

$= \frac{d}{dx}[2f(3) - 2f(1)]$ ✓

$= \frac{d}{dx}[2\times 3 - 2(-1)]$

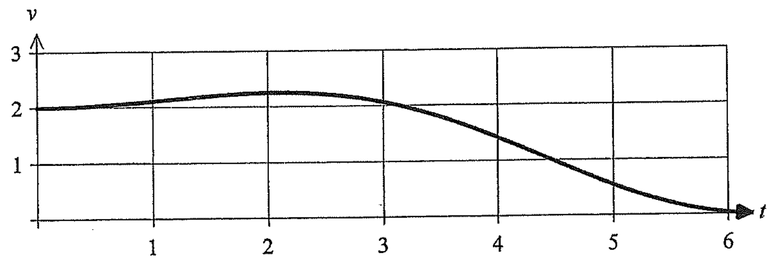
$= \frac{d}{dx}(8)$

$= 0$ ✓

$\boxed{\text{OR}}$ because differentiation of a definite integral is the same as differentiating a constant $\Rightarrow 0$ ✓✓

QUESTION FIVE (2, 3, 1, 2 = 8 MARKS)

The speed of an object in metres per second can be modelled by the equation $v = 1 + cos(0.5t) + sin^2(0.5t)$ where t represents the time in seconds and its graph is as shown below.



The area under the curve for any time interval represents the distance travelled by the object.

a) Complete the table below, rounding to two decimal places.

t	0	2	4	6
v	2	2.25	1.41	0.03

b) Estimate the distance travelled by the object during the first six seconds by calculating the circumscribed areas using three rectangular strips of width 2 seconds.

i.e. by overestimation.

correctly using area formula with height correct

$2 \times f(2) + 2 \times f(2) + 2 \times f(4)$

all heights correct

$= 2 \times 2.25 + 2 \times 2.25 + 2 \times 1.41$

$= 11.82$ m

c) Suggest one way to improve the accuracy of the estimation.

Any one of
- use a smaller width for all rectangles or use more strips (rects)
- calculate an underestimate and averaging the over & under estimate distance
- use trapezium strips instead of rectangular strips

d) Determine the actual distance travelled in the first six seconds.

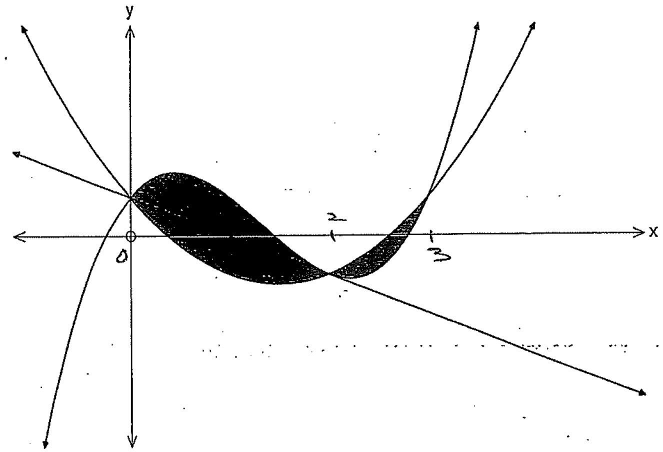
$\int_0^6 |v|\, dt$

$= \int_0^6 |1 + \cos(0.5t) + \sin^2(0.5t)|\, dt$

$= 9.42$ m/s

QUESTION SIX (4 MARKS)

Two curves with equations $y = x^3 - 4x^2 + 3x + 1$ and $y = x^2 - 3x + 1$ intersect as shown below. The line passing through the points of intersection of the curves has the equation $y = 1 - x$. Determine the fraction of the shaded area which lies above the line $y = 1 - x$.



$x^3 - 4x^2 + 3x + 1 = x^2 - 3x + 1$

$x = 0$ or 2 or 3

$A(\text{shaded}) = \int_0^3 |x^3 - 4x^2 + 3x + 1 - (x^2 - 3x + 1)|\, dx$

$= \int_0^3 |x^3 - 5x^2 + 6x|\, dx$

$= \frac{37}{12}\ u^2$

$A(\text{shaded under line}) = \int_0^2 |1 - x - (x^2 - 3x + 1)|\, dx$

$= \int_0^2 |-x^2 + 2x|\, dx$

$= \frac{4}{3}\ u^2$

Fraction $= \dfrac{\frac{37}{12} - \frac{4}{3}}{\frac{37}{12}}$

$= \frac{7}{[illegible]} = \frac{21}{[illegible]}$

QUESTION SEVEN (2, 1, 1, 1, ~~1, 2~~ = 9 MARKS)

A group of medical researchers is investigating the spread of a virus in a province by keeping track of P, the population of people infected, t weeks after the virus is first discovered.

Initially, 60 people are infected with the virus. It is suspected that P and t are related by the differential equation:

$\frac{dP}{dt} = -6e^{-\frac{t}{5}} + K$, where K is a constant.

a) Determine an expression for the population of people infected, P, in terms of t and K.

$P = \int -6e^{-t/5} + K \; dt$

$= 30e^{-t/5} + Kt + d$ ✓

$t = 0,\ P = 60 \Rightarrow 60 = 30e^{0} + 0 + d$

$30 = d$

$\therefore P = 30e^{-t/5} + Kt + 30$ ✓

b) In the long-term, ~~w~~what happens to the population of infected people when:

i) $K = 1$? $P = 30e^{-t/5} + t + 30$

As $t \to \infty$, $P \to \infty$

i.e. the population of infected people increases indefinitely. ✓

ii) $K = 0$?

$P = 30e^{-t/5} + 30$

As $t \to \infty$, $P \to 30$

i.e. the population of infected people stabilises to 30 ✓

c) If $K = 0$,

i) write an integral which can be used to determine the exact change in the population of people infected in the fifth week.

$\int_4^5 -6e^{-t/5} \, dt$ ✓

ii) hence, determine the exact change in the population of people infected in the fifth week.

$= -2.44$ ✓

A decrease of $\simeq 2$ people ✓ must have decrease & 2.

d) Given that one week later the population of infected people reduces to half of its initial number, ~~predict what will eventually happen~~ determine when ~~to~~ the population of infected people~~?~~ will reduce to zero by first determining the value of K.

$P = 30e^{-t/5} + Kt + 30$

$t = 1,\ P = 30 \Rightarrow 30 = 30e^{-1/5} + K + 30$

$K = -30e^{-1/5}$ or -24.56 ✓

$\therefore P = 30e^{-t/5} - 30e^{-1/5}t + 30 = 0$

From CAS, this is a decreasing function

$P = 0,\ t = 2.03$

$\therefore$ Population of infected people will reduce to zero after 2.03 weeks ✓

QUESTION EIGHT (2, 1, 5 = 8 MARKS)

The volume of a liquid (V in cubic metres) in a mixing tank varies depending on the height of the liquid (h in metres) in the mixing tank and is given by:

$$V = \int_0^h e^{-0.01x^2}\,dx$$

The height of the liquid in the mixing tank depends on time (t in minutes) as follows:

$$h = 2t^2 - t + 4$$

a) Determine $\frac{dV}{dh}$ when the height is 1.5 m.

$$\frac{dV}{dh} = e^{-0.01h^2} \quad ✓$$

$h = 1.5$, $\frac{dV}{dh} = 0.98$ m³/min ✓

b) Explain the significance of your answer in part a).

The rate of change of the volume with respect to height when h = 1.5 m is 0.98 m³/min ✓

c) Determine $\frac{dV}{dt}$ when the height is 5 m.

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$= e^{-0.01h^2} \times (4t-1) \quad ✓ ✓$$

$h = 5 \Rightarrow 2t^2 - t + 4 = 5$

$t = -0.5$ (N.A) or 1 ✓

$$\left.\frac{dV}{dt}\right|_{t=1} = e^{-0.01(5)^2}(4(1)-1) \quad ✓$$

$$= 3e^{-0.25}$$

$$= 2.34 \text{ m}^3/\text{min} \quad ✓$$

Alternate solution to 8c.

$$V = \int_0^h e^{-0.01x^2}\,dx$$

$$= \int_0^{2t^2-t+4} e^{-0.01x^2}\,dx \quad ✓$$

$$\frac{dV}{dt} = \frac{d}{dt}\int_0^{2t^2-t+4} e^{-0.01x^2}\,dx$$

$$= e^{-0.01(2t^2-t+4)^2}(4t-1) \quad ✓$$

$h = 5 = 2t^2 - t + 4$

$t = -\frac{1}{2}$ (N.A) or 1 ✓

$$\left.\frac{dV}{dt}\right|_{t=1} = e^{-0.01[2(1)^2-(1)+4]^2}[4(1)-1] \quad ✓$$

$$= 3e^{-0.25}$$

$$= 2.34 \quad ✓$$