Name: ....

Calculator Free	/23	%
Calculator Allowed	/35	%
Total	/58	%

Mathematics Methods, Year 12, 2018

Test 5 – CRVs and the Normal Distribution (& logs)

25 minutes working time.
Calculator Free Section (no notes, no calculators)
SCSA Formula sheet allowed

Question 1 **[8 marks: 2, 2, 2, 2]**

(a) Simplify $\log_2 32$.

$\log_2 2^5 = 5\log_2 2 = 5$

(b) Solve $2^{x-1} = 5$.

$\log 2^{x-1} = \log 5$

$(x-1)\log 2 = \log 5$

$x - 1 = \dfrac{\log 5}{\log 2}$

$x = \dfrac{\log 5}{\log 2} + 1$

(c) If $x = \log_n 3$ and $y = \log_n 4$

(i) express $\log_n \left(\frac{4}{9}\right)$ in terms of x and/or y.

$\log_n 4 - \log_n 9$

$y - 2x$

(ii) evaluate n^{2x}.

$x = \log_n 3 \Rightarrow n^x = 3$

$n^{2x} = (n^x)^2 = 9$

Question 2 **[9 marks: 2, 1, 1, 3, 2]**

The continuous random variable X has probability density function

$$f(x) = \begin{cases} \frac{2x}{9} & 0 \le x \le 3 \\ 0 & elsewhere \end{cases}.$$

(a) Determine $E(X)$.

$$= \int_0^3 \left(x \times \frac{2x}{9}\right) dx$$
$$= \left[\frac{2x^3}{27}\right]_0^3 \quad ✓$$
$$= 2 \quad ✓$$

(b) The variance of X, $Var(X)$, is $\frac{1}{2}$.

(i) Determine $E(4X + 3)$.

$$4 \times 2 + 3 = 11 \quad ✓$$

(ii) Determine $Var(4X + 3)$.

$$4^2 \times \frac{1}{2} = 8 \quad ✓$$

(c) Determine the cumulative distribution function $F(X)$.

$$F(t) = \int_0^t \frac{2x}{9}\, dx$$
$$= \left[\frac{2x^2}{2\times 9}\right]_0^t = \frac{t^2}{9}$$

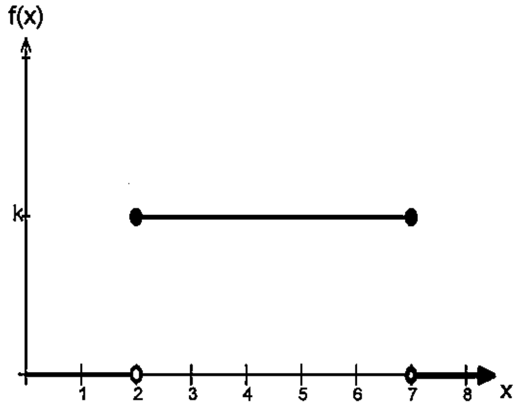
$$F(x) = \begin{cases} 0 & x < 0 \quad ✓ \\ \frac{x^2}{9} & 0 \le x \le 3 \quad ✓ \\ 1 & x > 3 \quad ✓ \end{cases}$$

(d) Calculate $P(1 < X < 2)$.

$$P(1 < X < 2) = F(2) - F(1)$$
$$= \frac{4}{9} - \frac{1}{9} = \frac{3}{9} = \frac{1}{3} \quad ✓✓$$

Question 3 **[6 marks: 1, 2, 1, 1, 1]**

This graph is of a uniform random variable, X.



(a) Determine the value of k.

$\frac{1}{7-2} = \frac{1}{5} = 0.2$ ✓

(b) Complete the rule:

$$f(x) = \begin{cases} 0.2 & 2 \leq x \leq 7 \checkmark \\ 0 & \text{elsewhere} \checkmark \end{cases}$$

(c) Calculate $P(3 \leq X \leq 6)$

$= 0.6$ ✓

(d) Calculate $P(X < 5 \mid X > 3)$

$= \frac{0.4}{0.8} = 0.5$ ✓

(e) Calculate $\boldsymbol{m}$ if $P(X > \boldsymbol{m} \mid X > 5) = 0.25$

$m = 6.5$ ✓

Name: ....

Total	/35	%

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Calculator Assumed Section (notes allowed)

SCSA Formula sheet and calculators allowed

Question 4 **[8 marks: 2, 2, 2, 2]**

X is a continuous random variable with probability density function given by:

$$f(x) = \frac{x}{50} \qquad 0 \le x \le 10$$

(a) Show that $f(x)$ is a probability density function.

$$\int_0^{10} \left(\frac{x}{50}\right) dx$$

$$\left[\frac{x^2}{100}\right]_0^{10} = 1 \quad ✓✓$$

(b) Find:

(i) $P(1 \le X \le 5)$ $\quad \int_1^5 \left(\frac{x}{50}\right) dx = \left[\frac{x^2}{100}\right]_1^5 = \frac{24}{100} = 0.24$ ✓✓

(ii) $P(X \le 5)$ $\quad \int_0^5 \left(\frac{x}{50}\right) dx = \left[\frac{x^2}{100}\right]_0^5 = \frac{25}{100} = 0.25$ ✓✓

(iii) $P(X \ge 1 | X \le 5)$ $\quad = \frac{0.24}{0.25} = 0.96$ ✓✓

$$\frac{P(1 \le x \le 5)}{P(0 \le x \le 5)}$$

Question 5 **[8 marks: 1, 1, 2, 2, 2]**

A bakery packages a loaf of bread as a Standard if it weighs between 450 g and 500 g. The weights of all loaves produced by the bakery are normally distributed with a mean of 470 g and a standard deviation of 16 g.

(a) What is the probability that a randomly selected loaf produced by the bakery

(i) weighs 450 g? $P(450 < x < 450) = 0$ ✓

(ii) is a Standard loaf? $P(450 < x < 500) = 0.8640$ ✓

(b) In a batch of 250 loaves, how many would be expected to weigh less than a Standard loaf?

$P(x < 450) = 0.1056$

$250 \times 0.1056 = 26.4$ ✓

≈ 26 loaves. ✓

(c) Determine the probability that a randomly selected Standard loaf weighs less than 470 g.

$$\frac{P(450 < x < 470)}{P(450 < x < 500)} = \frac{0.3944}{0.8640}$$ ✓

$$= 0.4564 \quad (0.4565)$$ ✓

(d) A bakery worker randomly selects loaves of bread until the loaf chosen is not a Standard. Determine the probability that the sixth loaf chosen is the first that is not a Standard.

$$(0.8640)^5 \times (1 - 0.8640) = 0.0655$$ ✓✓

Question 6 **[7 marks: 4, 3]**

(a) The continuous random variable X has probability density function

$$f(x) = \begin{cases} ax + b & 1 \le x \le 3 \\ 0 & elsewhere \end{cases}.$$

It is also know that $P(X < 2) = \frac{5}{8}$.

Determine the values of the constants a and b.

$$\int_1^3 (ax+b)\,dx = 1$$
$$\left[\frac{ax^2}{2} + bx\right]_1^3 = 1$$
$$4a + 2b = 1 \quad ✓$$

$$\int_1^2 (ax+b)\,dx = \frac{5}{8}$$
$$\left[\frac{ax^2}{2} + bx\right]_1^2 = \frac{5}{8}$$
$$3a + 2b = \frac{5}{4} \quad ✓$$

$$4a + 2b = 1$$
$$-\left(3a + 2b = \frac{5}{4}\right)$$
$$a = -\frac{1}{4} \quad ✓$$

$$4\left(-\frac{1}{4}\right) + 2b = 1$$
$$2b = 2$$
$$b = 1 \quad ✓$$

(b) Another continuous random variable Y has probability density function

$$g(y) = \begin{cases} \frac{6-y}{18} & 0 \le y \le 6 \\ 0 & elsewhere \end{cases}.$$

Determine the mean and standard deviation of Y.

$$E(Y) = \int_0^6 y\left(\frac{6-y}{18}\right) dy = 2 \quad ✓$$
$$Var(Y) = \int_0^6 \frac{6-y}{18}(y-2)^2\,dy = 2 \quad ✓$$
$$SD(Y) = \sqrt{2} \quad ✓$$

Question 7 **[7 marks: 1, 1, 2, 3]**

The train service from Perth to Mandurah opened on December 23, 2007. It averages 48 minutes for the 72 km journey.

(a) What is the average speed for the trip, in km/hr?

$$\frac{72}{48} \times 60 = 90 \text{ km/hr}$$

The completion times for the journey have since been analysed, and seem to be normally distributed. The range of completion times (Max – Min) is approximately 30 minutes. The standard deviation of completion times is estimated to be 5 minutes.

(b) Explain how this estimate can be obtained.

Nearly 100% of times fall within 3 standard deviations on either side of the mean, so $30 \div 6$ is equal to 5 $\therefore$ 5 minutes.

(c) What percentage of trips would take more than 58 minutes?

$X \sim N(48, 5^2)$

$P(X > 58) = 0.0228$

$\therefore$ 2.3% of trips.

(d) Given that the interquartile range (IQR) is found by subtracting the Lower Quartile value from the Upper Quartile value, determine the IQR of the times. ***(Use a labelled diagram of a normal curve to help explain your answer if necessary.)***

LQ = 44.6 min ✓ (25%)

UQ = 51.4 min ✓ (75%)

IQR = 6.8 min = 6 minutes and 48 seconds ✓

Question 8 **[5 marks: 3, 2]**

A Gallup Poll establishes that 75% of people interviewed are in favour of a certain proposal. If 32 people are interviewed, find, using the normal approximation to the binomial distribution, the probability that:

(a) there will be exactly 25 in favour of the proposal.

$n = 32, \ p = 0.75 \qquad \mu = 32 \times 0.75 = 24$

$\sigma = \sqrt{32 \times 0.75 \times 0.25} = \sqrt{6}$ ✓

$P(24.5 < X < 25.5) = 0.1490$ ✓

✓

(b) there will be more than 20 but fewer than 25 in favour of the proposal.

$P(20.5 < X < 24.5) = 0.5044$

✓ ✓