2024

TEST 3

SPECIALIST MATHEMATICS Year 12

Section Two:

Calculator-assumed

Your name ________________________________

Teacher's name ________________________________

Time and marks available for this section

Working time for this section: 30 minutes
Marks available: 39 marks

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. The rules of conduct of the CCGS assessments are detailed in the Reporting and Assessment Policy. Sitting this assessment implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet using a blue/black pen. Do not use erasable or gel pens.

3. Answer all questions.

4. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

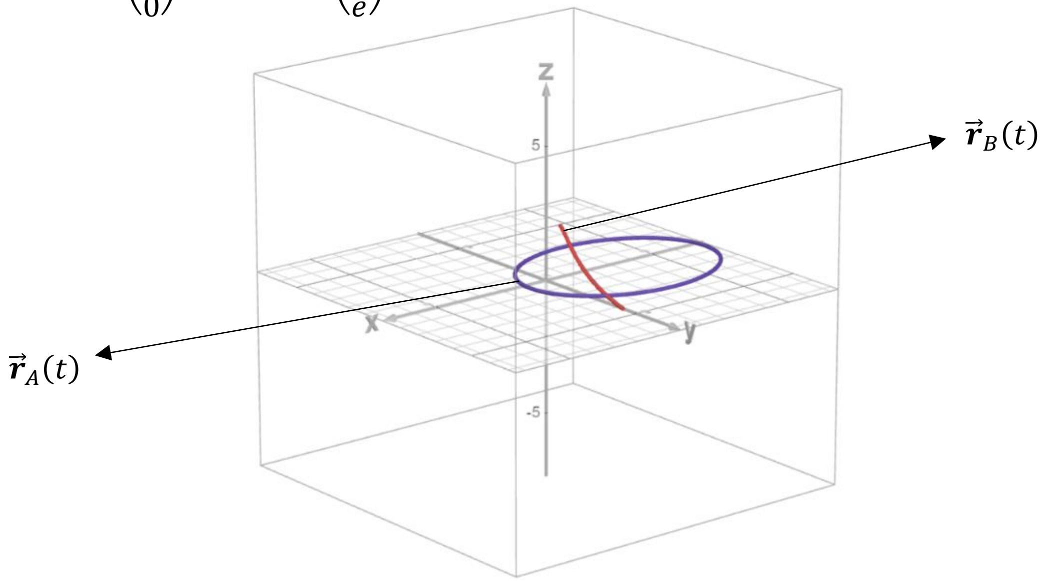
5. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

6. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

7. It is recommended that **you do not use pencil**, except in diagrams.

Question 1 **(5 + 5 + 4 = 14 marks)**

Two particles A and B, are travelling within the same Cartesian plane, as shown below. The acceleration vectors for both particles are $\vec{\boldsymbol{a}}_A(t) = \begin{pmatrix} -4\cos(t) \\ -3\sin(t) \\ 0 \end{pmatrix}$ and $\vec{\boldsymbol{a}}_B(t) = \begin{pmatrix} e^{\sin(t)}(1-\sin^2(t)-\sin(t) \\ 2e^t \\ -\cos(t)e^{\cos(t)} + (-\sin^2(t)e^{\cos(t)}) \end{pmatrix}$, where A initially had a velocity and displacement of $\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} ms^{-1}$ and $\begin{pmatrix} 4 \\ 0 \\ -\frac{3}{2} \end{pmatrix} m$ and B initially had a velocity and displacement of $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} ms^{-1}$ and $\begin{pmatrix} 1 \\ 2 \\ e \end{pmatrix} m$.



a) Hence, show that the displacement function of particle A, is represented as

$$\vec{\boldsymbol{r}}_B(t) = \begin{pmatrix} e^{\sin t} \\ 2e^t \\ e^{\cos t} \end{pmatrix}$$

Solution

$$\int \vec{\boldsymbol{a}}_B \, dt = \int \begin{pmatrix} e^{\sin(t)}(1 - \sin^2(t) - \sin(t) \\ 2e^t \\ -\cos(t)e^{\cos(t)} + (-\sin^2(t)e^{\cos(t)}) \end{pmatrix} dt$$

$$\to \int \begin{pmatrix} e^{\sin(t)}(\cos^2 t - \sin t) \\ 2e^t \\ -\cos(t)e^{\cos(t)} + (-\sin^2(t)e^{\cos(t)}) \end{pmatrix} dt = \int \begin{pmatrix} e^{\sin(t)}\cos^2 t - e^{\sin(t)}\sin t) \\ 2e^t \\ -\cos(t)e^{\cos(t)} + (-\sin^2(t)e^{\cos(t)}) \end{pmatrix} dt$$

$$\to \int \begin{pmatrix} (\cos(t))(\cos(t)\, e^{\sin(t)}) + (-\sin(t))(e^{\sin(t)}) \\ 2e^t \\ -\cos(t)e^{\cos(t)} + (-\sin^2(t)e^{\cos(t)}) \end{pmatrix} dt$$

The $\vec{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{k}}$ component of the velocity function, resembles the form of the product rule.
Hence, let $h(x) = f(x)g(x)$
(1 mark)

$$h'(x) = f(x)g'(x) + f'(x)g(x)$$
$$\to h'(x) = (\cos(t))(\cos(t)\, e^{\sin(t)}) + (-\sin(t))(e^{\sin(t)})$$
$$\to h(x) = \cos(t)e^{\sin(t)}$$ (1 mark)

$$h(x) = (-\sin(t))(-\sin(t)e^{\cos(t)}) + (-\cos(t))(e^{\cos(t)})$$
$$\to h(x) = -\sin(t)\, e^{\cos(t)}$$ (1 mark)

$$\to \boldsymbol{v}_B = \begin{pmatrix} \cos(t)e^{\sin(t)} \\ 2e^t \\ -\sin(t)\, e^{\cos(t)} \end{pmatrix} + C$$

Sub in $\boldsymbol{v}_B(0) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \to C = 0 \to \boldsymbol{v}_B = \begin{pmatrix} \cos(t)e^{\sin(t)} \\ 2e^t \\ -\sin(t)\, e^{\cos(t)} \end{pmatrix}$ (1 mark)

$$\int \boldsymbol{v}_B \, dt = \int \begin{pmatrix} \cos(t)e^{\sin(t)} \\ 2e^t \\ -\sin(t)\, e^{\cos(t)} \end{pmatrix} dt = \boldsymbol{r}_B = \begin{pmatrix} e^{\sin(t)} \\ 2e^t \\ e^{\cos(t)} \end{pmatrix} + C$$

Sub in $\boldsymbol{r}_B(0) = \begin{pmatrix} 1 \\ 2 \\ e \end{pmatrix} \to C = 0 \to \boldsymbol{r}_B = \begin{pmatrix} e^{\sin(t)} \\ 2e^t \\ e^{\cos(t)} \end{pmatrix}$ (QED) (1 mark)

$$\therefore \boldsymbol{r}_B = \begin{pmatrix} e^{\sin(t)} \\ 2e^t \\ e^{\cos(t)} \end{pmatrix}$$

b) Determine if the particles A and B will collide in 2π seconds, and if so, where, and when?

Solution

$$\int \vec{\boldsymbol{a}}_A dt = \int \begin{pmatrix} -4\cos(t) \\ -3\sin(t) \\ 0 \end{pmatrix} dt = \begin{pmatrix} -4\sin(t) \\ 3\cos(t) \\ 0 \end{pmatrix} + C$$

$\rightarrow$ Sub in $\vec{\boldsymbol{v}}_A(0) = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \rightarrow C = 0 \rightarrow \vec{\boldsymbol{v}}_A = \begin{pmatrix} -4\sin(t) \\ 3\cos(t) \\ 0 \end{pmatrix}$ (1 mark)

$\int \vec{\boldsymbol{v}}_A dt = \int \begin{pmatrix} -4\sin(t) \\ 3\cos(t) \\ 0 \end{pmatrix} dt = \begin{pmatrix} 4\cos(t) \\ 3\sin(t) \\ 0 \end{pmatrix} + K \rightarrow$ Sub in $\vec{\boldsymbol{r}}_A(0) = \begin{pmatrix} 4 \\ 0 \\ -\frac{3}{2} \end{pmatrix} \rightarrow K = \begin{pmatrix} 0 \\ 0 \\ \frac{3}{2} \end{pmatrix}$

$\therefore \vec{\boldsymbol{r}}_A(t) = \begin{pmatrix} 4\cos(t) \\ 3\sin(t) \\ \frac{3}{2} \end{pmatrix}$ (1 mark)

For A and B to collide

$$\vec{\boldsymbol{r}}_A(t) = \boldsymbol{r}_B(t)$$

$\begin{pmatrix} 4\cos(t) \\ 3\sin(t) \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} e^{\sin(t)} \\ 2e^t \\ e^{\cos(t)} \end{pmatrix}$ (1 mark)

(equating i component)

$4\cos t = e^{\sin t} \mid 0 \le t \le 2\pi$

$\therefore t = 0.9655, 4.8048$

(equating j component)

$3\sin(t) = 2e^t \mid 0 \le t \le 2\pi$

$\therefore$ No solution

$\therefore$ Since, there is no common value for t, then A and B don't collide.

(1 mark)

c) Prove that particle A is not in circular motion.

Solution

<u>Method One:</u>

$$\vec{\boldsymbol{r}}_A(t) = \begin{pmatrix} 4\cos(t) \\ 3\sin(t) \\ \frac{3}{2} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4\cos(t) \\ 3\sin(t) \\ \frac{3}{2} \end{pmatrix}$$ (1 mark)

$\rightarrow x = 4\cos(t)$

$\therefore \cos(t) = \frac{x}{4}$

$\rightarrow y = 3\sin(t)$

$\therefore \sin(t) = \frac{y}{3}$

$\cos^2 t + \sin^2 t = 1$ (1 mark)

$\rightarrow \left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$\therefore \frac{x^2}{16} + \frac{y^2}{9} = 1$ translated $\frac{3}{2}$ units to the z-axis.

(1 mark)

Hence, the Cartesian equation describes the shape of an ellipse, which shows it's going in elliptical motion, not circular motion.

(1 mark)

<u>Method Two:</u>

$\vec{\boldsymbol{a}}.\vec{\boldsymbol{v}} = 0$ or $\vec{\boldsymbol{r}}.\vec{\boldsymbol{v}} = 0$ (1 mark)

$$\vec{\boldsymbol{r}}.\vec{\boldsymbol{v}} = \begin{pmatrix} 4\cos(t) \\ 3\sin(t) \\ \frac{3}{2} \end{pmatrix} . \begin{pmatrix} -4\sin(t) \\ 3\cos(t) \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 4\cos(t) \\ 3\sin(t) \\ \frac{3}{2} \end{pmatrix} . \begin{pmatrix} -4\sin(t) \\ 3\cos(t) \\ 0 \end{pmatrix} = 0$$

$\rightarrow -16\cos(t)\sin(t) + 9\sin(t)\cos(t) = 0$

$\rightarrow -7\sin(t)\cos(t) = 0$ (1 mark)

(By using CAS)

$\therefore t = \frac{n\pi}{2}$ where $n \in \mathbb{Z}^+$ (1 mark)

$\therefore$ This shows that $a.v$ and $r.v$ are not always perpendicular to each other, whereas for an object in circular motion, they're always perpendicular to each other.

(1 mark)

OR

But for circle

$\vec{\boldsymbol{r}}.\vec{\boldsymbol{v}} = \vec{\boldsymbol{a}}.\vec{\boldsymbol{v}} = 0 \;\forall\; t$ where $t \geq 0$

$\therefore$ Thus, particle A isn't in circular motion.

(1 mark)

Question 2 **(5 + 1 = 6 marks)**

Three objects have the given equations represented and graphed below.

$$2x^2 + y^2 + z^2 = 5$$
$$x^2 + y^2 + z^2 = 3$$
$$x^2 + 2y^2 - z^2 = 4$$

a) Hence, by showing all necessary steps, find the coordinates of all intersections of the three objects.

Solution

Can use either Gaussian elimination or algebra
(In this marking key, we're using Gaussian elimination)

Define the equations into different variables.

$$x^2 = a$$
$$y^2 = b$$
$$z^2 = c$$
$$2a + b + c = 5$$
$$a + b + c = 3$$
$$a + 2b - c = 4$$

(1 mark)

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$$R_2 - R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & -1 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$$2R_2 - R_1 \rightarrow R_2$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 2 & -1 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$$R_2 + R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 0 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

(1 mark)

$R_3 \rightarrow 3c = 0$

$\therefore c = 0$

$R_2 \rightarrow b + c = 1$

$\therefore b = 1$

$R_1 \rightarrow 2a + b + c = 5$

$\rightarrow 2a + 1 = 5$

$\therefore a = 2$ (1 mark)

Solution (continued..)

$$x^2 = 2$$
$$\rightarrow \therefore x = \pm\sqrt{2}$$
$$y^2 = 1$$
$$\rightarrow \therefore y = \pm 1$$
$$\therefore z = 0$$

(1 mark)

$\therefore$ Hence, the coordinates are listed as

$$(\sqrt{2}, 1, 0)$$
$$(\sqrt{2}, -1, 0)$$
$$(-\sqrt{2}, 1, 0)$$
$$(-\sqrt{2}, -1, 0)$$

(1 mark)

b) Hence, find the Cartesian equation of the plane, containing all those points.

Solution

All the coordinates are in the x, y-plane.

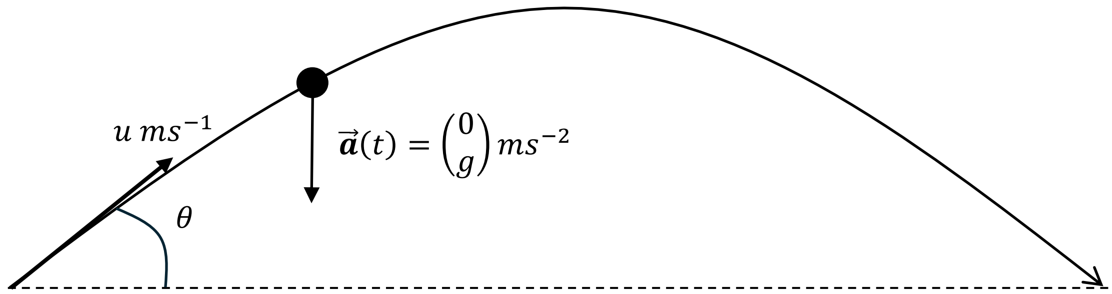
Hence, the normal vector is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\therefore$ Thus, the equation is $z = 0$

(1 mark)

Question 3 **(5 + 3 + 4 = 12 marks)**

A man threw a tennis ball, with a speed of $u\ ms^{-1}$ at an angle of θ^o above the horizontal. The force acted on the ball is its gravitational force, which will have a constant acceleration of $g\ ms^{-2}$.



a) Show that the maximum height h_{max} is expressed as follows.

$$h_{max} = \frac{-u^2 \sin^2 \theta}{2g}$$

Solution

$$\int \begin{pmatrix} 0 \\ g \end{pmatrix} dt = \begin{pmatrix} 0 \\ gt \end{pmatrix} + C \rightarrow v(t) = \begin{pmatrix} u\cos\theta \\ gt + u\sin\theta \end{pmatrix} \quad \text{(1 mark)}$$

The maximum height, is when the j component of $v(t) = 0$

$$gt + u\sin\theta = 0$$

$$\rightarrow t = \frac{u\sin\theta}{g} \quad \text{(1 mark)}$$

$$\int \begin{pmatrix} u\cos\theta \\ gt + u\sin\theta \end{pmatrix} dt = \begin{pmatrix} u\cos\theta t \\ \frac{gt^2}{2} + u\sin\theta t \end{pmatrix} + K \rightarrow r(t) = \begin{pmatrix} u\cos\theta t \\ \frac{gt^2}{2} + u\sin\theta t \end{pmatrix}$$

(1 mark)

Sub in $t = \frac{u\sin\theta}{g}$ in the j component of $r(t)$, which will give the maximum height h_{max}

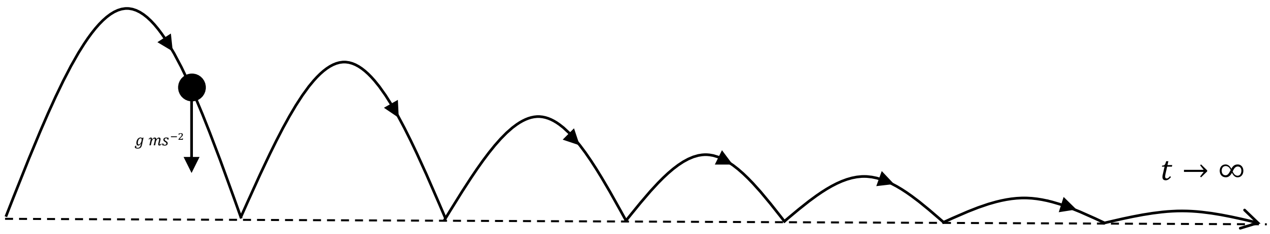
$r\left(\frac{u\sin\theta}{g}\right)$ in j component

(1 mark)

$$h_{max} = \frac{g\left(\frac{u\sin\theta}{g}\right)^2}{2} + u\sin\theta\left(\frac{u\sin\theta}{g}\right) \rightarrow h_{max} = \frac{-u^2\sin^2\theta}{2g} \quad \text{(1 mark)}$$

(By using CAS)

As the tennis ball hits the ground once, it continuously bounces until the ball is fully at rest. This is depicted at the diagram below.



As $t \to \infty$, the maximum height of the tennis ball is going to decrease gradually. It is estimated that for each bounce, it vertically rebounds to 75% of the height from which it fell.

b) Hence, show that the total height travelled of the tennis ball before coming to rest, d_{total} is the expression shown below.

$$d_{total} = -\frac{4u^2 \sin^2 \theta}{g}$$

Solution

Refer back to Year 11 concepts, in terms of geometric series and sequences

Common ratio $r = \frac{3}{4}$

First term $a = \frac{-u^2 \sin^2 \theta}{2g}$, as it's the max height for the first bounce.

The total height as the ball bounces is S_∞

$$S_\infty = \frac{a}{1-r}$$

(1 mark)

$$\rightarrow\ S_\infty = \frac{\frac{-u^2 \sin^2 \theta}{2g}}{1 - 0.75}$$

$$\rightarrow\ S_\infty = \frac{-2u^2 \sin^2 \theta}{g}$$

(1 mark)

As the ball bounces at height twice

Hence, $2 \times S_\infty$

$$\rightarrow \therefore d_{total} = -\frac{4u^2 \sin^2 \theta}{g}$$

(1 mark)

c) Show that the velocity increases for each bounce, assuming the angle θ is constant for each bounce.

Solution

Assume the velocity initially at the second bounce is $a\ ms^{-1}$ with angle of elevation θ

Hence, by the formula derived at a)

$$h_{max\ 2nd\ bounce} = \frac{-a^2 \sin^2 \theta}{2g}$$

(1 mark)

By identifying the geometric sequence in b), the height of the 2nd bounce can be expressed in terms of the general rule $T_n = ar^{n-1}$

Let n be number of bounces.

$$T_2 = ar \rightarrow \frac{-u^2 \sin^2 \theta}{2g} \times 0.75$$

$$T_2 = \frac{-3u^2 \sin^2 \theta}{8g}$$

(1 mark)

$$h_{max\ 2nd\ bounce} = \frac{-3u^2 \sin^2 \theta}{8g}$$

$$\frac{-a^2 \sin^2 \theta}{2g} = \frac{-3u^2 \sin^2 \theta}{8g}$$

$$\rightarrow a^2 = 3u^2$$

$$\therefore a = \sqrt{3}u\ ms^{-1}$$

(1 mark)

Hence, as the speed for the second bounce is increased by a factor $\sqrt{3}$, then this trend will continue on, and that the velocity of ball increases for each bounce.

(1 mark)

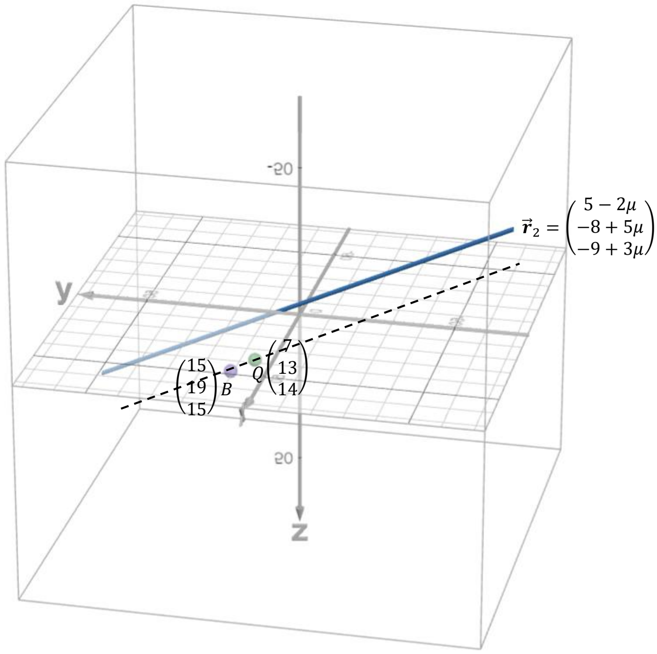
Question 4 **(7 marks)**

The position vectors are follows as $\overrightarrow{OQ} = \begin{pmatrix} 7 \\ 13 \\ 14 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 15 \\ 19 \\ 15 \end{pmatrix}$. The line $\overrightarrow{\boldsymbol{r_1}}$ passes through these coordinates, and the line $\overrightarrow{\boldsymbol{r_2}}$ has the vector equation.

$$\overrightarrow{\boldsymbol{r_2}} = \begin{pmatrix} 5 - 2\mu \\ -8 + 5\mu \\ -9 + 3\mu \end{pmatrix}$$

All the information provided is presented in a Cartesian plane.

By proving that $\overrightarrow{\boldsymbol{r_2}}$ and $\overrightarrow{\boldsymbol{r_1}}$ are non-intersecting, find the shortest distance between both lines without using cross-product and only using the scalar projection method *(Anyone who does otherwise will get one mark).*



Solution

Find the parallel vector of $\overrightarrow{\boldsymbol{r_1}}$

$$QB = OB - OQ$$

$$\rightarrow \begin{pmatrix} 15 \\ 19 \\ 15 \end{pmatrix} - \begin{pmatrix} 7 \\ 13 \\ 14 \end{pmatrix}$$

$$QB = \begin{pmatrix} 8 \\ 6 \\ 1 \end{pmatrix}$$

$$\overrightarrow{\boldsymbol{r_1}} = \begin{pmatrix} 7 \\ 13 \\ 14 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 6 \\ 1 \end{pmatrix}$$ (1 mark)

To prove their non-intersecting

$$\overrightarrow{\boldsymbol{r_1}} = \overrightarrow{\boldsymbol{r_2}}$$

$$\begin{pmatrix} 8\lambda + 7 \\ 6\lambda + 13 \\ \lambda + 14 \end{pmatrix} = \begin{pmatrix} 5 - 2\mu \\ -8 + 5\mu \\ -9 + 3\mu \end{pmatrix}$$

By equating and forming equations from i and j components, by using CAS

$\lambda = -1$ and $\mu = 3$

(1 mark)

Solution

Check from the k component.

$$-1 + 14 = -9 + 3(3)$$

$$\rightarrow 13 \neq 0$$

Hence, $\vec{\boldsymbol{r}}_1$ and $\overrightarrow{\boldsymbol{r_2}}$ are non-intersecting.

(1 mark)

Finding closest distance, can be able to find the vector perpendicular of the parallel vectors of the lines.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 8 \\ 6 \\ 1 \end{pmatrix} = 0 \text{ and } \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix} = 0$$

$$8x + 6y + z = 0$$

$$-2x + 5y + 3z = 0$$

(1 mark)

Solution

By using CAS

$x = k, y = -2k, z = 4k$ where $k \in \mathbb{R}$

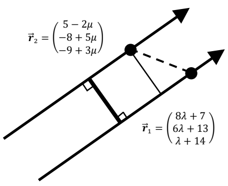
Let $k = 1$

$x = 1, y = -2, z = 4$

(1 mark)

Finding closest distance

(Drawing a diagram helps)



Find vector between two points in both lines

Solution

$$\begin{pmatrix} 7 \\ 13 \\ 14 \end{pmatrix} - \begin{pmatrix} 5 \\ -8 \\ -9 \end{pmatrix} = \begin{pmatrix} 2 \\ 21 \\ 23 \end{pmatrix}$$ (1 mark)

Finding closest distance.

$$\frac{1}{\sqrt{(1)^2 + (-2)^2 + (4)^2}} \left| \begin{pmatrix} 2 \\ 21 \\ 23 \end{pmatrix} . \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \right|$$

$\rightarrow \frac{1}{\sqrt{21}} |52|$ (1 mark)

$$d = \frac{52}{\sqrt{21}} \rightarrow \frac{52\sqrt{21}}{21}$$

(1 mark)

$\therefore$ Hence, the closest distance between $\overrightarrow{\boldsymbol{r_1}}$ and $\overrightarrow{\boldsymbol{r_2}}$ is $\frac{52\sqrt{21}}{21}$ $units$.

END OF CALCULATOR-ASSUMED