SCOTCH COLLEGE

12 Mathematics Methods 2020

Test 1 – Differentiation and Logarithms

Section 1: Calculator-free

Time allowed: 25 minutes **Maximum marks: 26**

Name: Solutions **Teacher:** Foster | Giese

Instructions:

- Show all working clearly.
- Sufficient detail must be shown for marks to be awarded for reasoning.
- A formula sheet will be provided.
- No calculators or personal notes are permitted.

Question 1 **[1, 1, 3, 3 = 8 marks]**

Solve the following equations.

(a) $\log_{10} x = -2$

$x = 10^{-2}$ ✓

$x = \frac{1}{100}$

(b) $\log_x x^2 = x$

$x = 2\log_x x$

$x = 2$ ✓

(c) $2^{x+1} = 3^{x-1}$

$\log 2^{x+1} = \log 3^{x-1}$ ✓

$(x+1)\log 2 - (x-1)\log 3 = 0$ ✓

$x\log 2 - x\log 3 + \log 2 + \log 3 = 0$

$x(\log 2 - \log 3) = -\log 3 - \log 2$

$x = \frac{-\log 3 - \log 2}{\log 2 - \log 3}$ ✓

$x = \frac{\log 2 + \log 3}{\log 3 - \log 2}$

(d) $e^{2x} = e^x + 6e^0$

$e^{2x} - e^x - 6 = 0$ ✓

$(e^x)^2 - e^x - 6 = 0$

$(e^x - 3)(e^x + 2) = 0$ ✓

$e^x = 3, \cancel{-2}$ $e^x > 0$

$\therefore x = \ln 3$ ✓

Question 2 **[3 marks]**

$\log_2 7 \approx 2.8$ and $\log_2 3 \approx 1.6$. Calculate the approximate value of $\log_2 24 - \log_2 14$.

$\log_2 24 - \log 14 = \log_2(2^3 \times 3) - \log_2(2 \times 7)$ ✓

$= 3\log_2 2 + \log_2 3 - (\log_2 2 + \log_2 7)$ ✓

$= 3 + 1.6 - 1 - 2.8$

$= 0.8$ ✓

Question 3 **[3, 3 = 6 marks]**

Differentiate the following (do not simplify your answers).

(a) $f(x) = \frac{3(x^4-10)^5}{x^2}$

(b) $y = (2+x^2)\sqrt{x} + \frac{3}{x^3}$

$$f'(x) = \frac{x^2(5)(4x^3)(3(x^4-10)^4) - 2x(3(x^4-10)^5)}{x^4}$$

$$y' = 2x\sqrt{x} + (2+x^2)(\tfrac{1}{2}x^{-\frac{1}{2}}) - \frac{3(3)}{x^4}$$

Question 4 **[4 marks]**

Consider the quadratic function $y = ax^2 + bx + 5$. This function has a tangent that is $y = 4x + 6$ at the point $(1,10)$. Find the values of a and b.

$y' = 2ax + b$

At $(1,10)$ $y' = 4$

$4 = 2a(1) + b$

$4 = 2a + b \ldots$ ①

and $10 = a(1)^2 + b(1) + 5$

$5 = a + b \ldots$ ②

① - ② $\Rightarrow$ $\left.\begin{array}{l} -1 = a \\ b = 6 \end{array}\right\}$

Question 5 **[5 marks]**

The cost, $\$C$, to construct a water tank in the shape of cylinder with a height of h m and a radius of r m is given by the formula $C = 120(2\pi rh + 2\pi r^2)$. The cost of constructing a water tank with a height of 10m and radius of 5m is approximately \$56 550.

Use the incremental formula to calculate the approximate cost of a water tank with a height of 10m and a radius of $(5 + \frac{1}{\pi})$m.

$$C = 120(2\pi r(10) + 2\pi r^2)$$
$$= 2400\pi r + 240\pi r^2$$
$$\frac{dC}{dr} = 2400\pi + 480\pi r \checkmark$$
$$\delta r = \frac{1}{\pi} \checkmark$$

$$\delta C = \frac{dC}{dr} \times \delta r$$
$$= (2400\pi + 480\pi(5))\left(\frac{1}{\pi}\right) \checkmark \checkmark$$
$$= 4800$$

New Cost:

56550 + 4800 = \$61 350 ✓

END OF SECTION 1

12 Mathematics Methods 2020

Test 1 – Differentiation and Logarithms

Section 2: Calculator-assumed

Time allowed: 20 minutes **Maximum marks: 19**

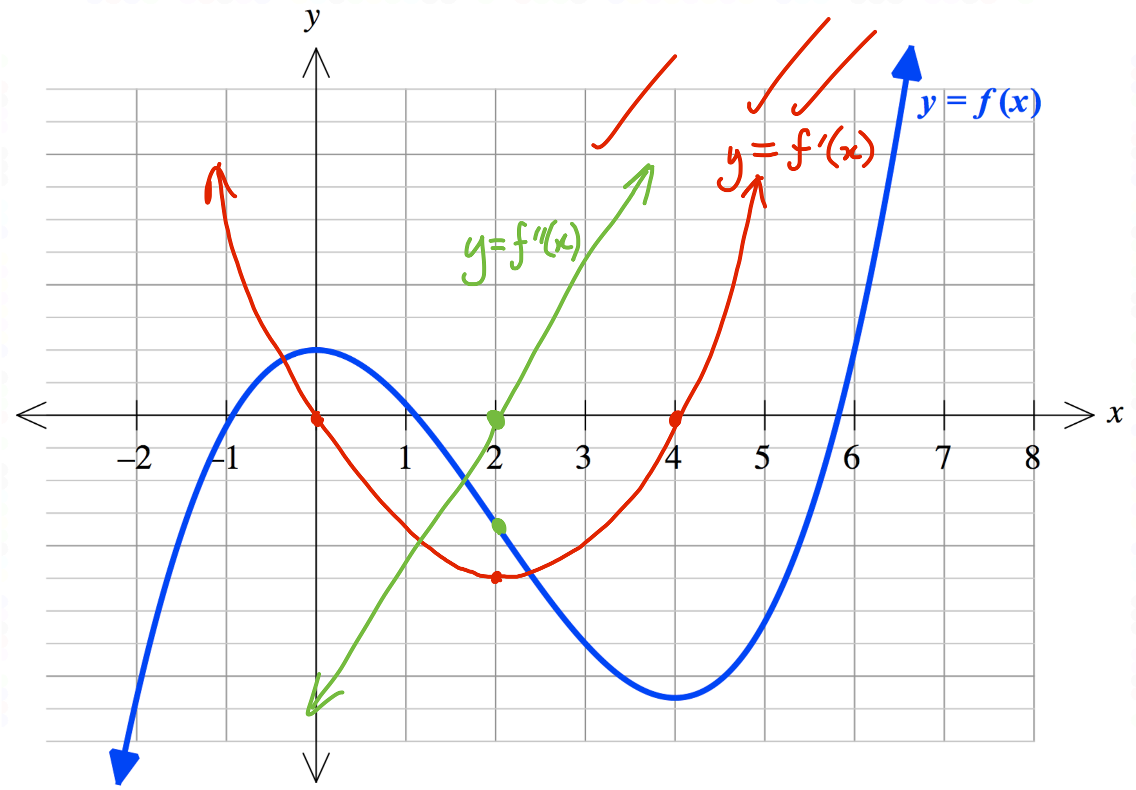
Name:______________________________ **Teacher:** Foster | Giese

Instructions:

- Show all working clearly.
- Sufficient detail must be shown for marks to be awarded for reasoning.
- A formula sheet will be provided.
- Calculators and 1xA4 double-sided page of personal notes are permitted.

Question 6 **[3 marks]**

A graph of the function $y = f(x)$ is given below.
Sketch the graphs of the functions $y = f'(x)$ and $y = f''(x)$ on the number plane below.



Question 7 **[1, 3, 3, 2 = 9 marks]**

A particle is initially at rest before it moves in a straight line. Its displacement, x mm, from the origin after t seconds can be described by the following equation.

$$x = \frac{t^3}{3} - 4.5t^2 + 8t + 22, \quad 0 \le t \le 12$$

(a) What is the initial displacement of the particle?

22 mm ✓

(b) Use calculus to show that the particle is at rest twice in the first 12 seconds.

$v(t) = t^2 - 9t + 8$ ✓

$0 = t^2 - 9t + 8$ ✓

$(t-8)(t-1) = 0$

$t = 1, 8$ ✓

(c) Is the particle travelling faster the first time it returns to the origin or the second time?

$x(t) = 0$

$0 = t^3 - 4.5t^2 + 8t + 22$ ✓

$t = \cancel{-1.45}, 4.28, 10.67$

$t > 0$

$v(4.28) = -12.20$
$v(10.67) = 25.84$ ✓

∴ The particle is travelling faster the second time it passes through the origin. ✓

(d) What is the maximum distance that this particle is from the origin?

Stat pts @ $v(t) = 0$

$t^2 - 9t - 8 = 0$

$(t-8)(t-1) = 0$ ✓

$t = 1, 8$

t	0	1	8	12
x	22	25.8	-31.3	46

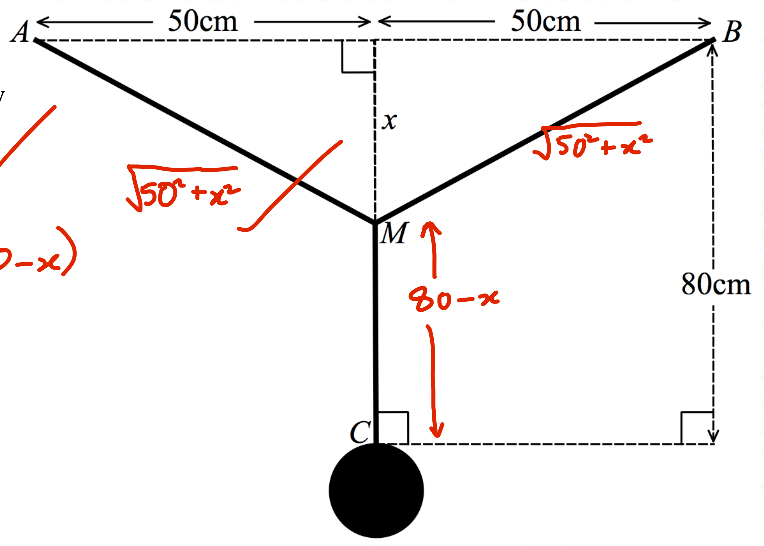
✓

∴ Max. distance from the origin is 46m.

Question 8 **[2, 5 = 7 marks]**

Two sparrows are flying level with each other 1m apart and are each carrying one end of a piece of string. One end of a second piece of string is tied at M to the string carried by the birds while the other end is attached to a hook on the surface of a small coconut. The coconut is 80cm lower than the sparrows.

(a) Show that the total length of all the string, L cm, is given by $L = 2\sqrt{50^2 + x^2} - x + 80$.



$$L = \sqrt{50^2 + x^2} + \sqrt{50^2+x^2} + (80 - x)$$

$$L = 2\sqrt{50^2 + x^2} - x + 80$$

(b) Using calculus techniques, show that there is a minimum length of string that can be achieved and justify it is a minimum. Determine the length of both pieces of string when this occurs.

$$\frac{dL}{dx} = \frac{2x - \sqrt{x^2+50^2}}{\sqrt{x^2+50^2}}$$

Stat. pts @ $\frac{dL}{dx} = 0$,

$$0 = \frac{2x - \sqrt{x^2+50^2}}{\sqrt{x^2+50^2}}$$

$x = 28.87$, $\frac{d^2L}{dx^2} > 0$, $\therefore$ Min. length

1st string: $2\sqrt{50^2 + 28.87^2}$
$= 115.47$cm

2nd string: $80 - 28.87$
$= 51.13$cm

END OF TEST