GREENWOOD COLLEGE

Mathematics Methods Units 3 & 4

Test 1 Applications of Differentiation 2019

Name: Marking Key Mark: / ~~25~~ 24

All electronic devices must be switched off and in bags.
Access to Formulae Sheet allowed. No notes.
No calculators allowed in this section. Time limit 25 minutes.

1. [4 marks]

Giving your answer in factored form, use the product rule to differentiate

$$f(x) = (1 \cancel{+} - x^2)(1 + 2x)^3$$

$f(x) = (1 + x^2)(1 + 2x)^3$

$u = 1 - x^2$ $u' = -2x$

$v = (1 + 2x)^3$ $v' = 3(1 + 2x)^2 \times 2 = 6(1 + 2x)^2$

$f'(x) = -2x.(1+2x)^3 + 6(1+2x)^2(1-x^2)$ ✓ ✓

$= (1+2x)^2\left[-2x(1+2x) + 6(1-x^2)\right]$ ✓

$= (1+2x)^2\left[-2x - 4x^2 + 6 - 6x^2\right]$

$= (1+2x)^2(-10x^2 - 2x + 6)$ ✓

or $2(1+2x)^2(-5x^2 - x + 3)$

2. [4, ~~2~~ 1, 1 = ~~7~~ 6 marks]

a) If $y = \frac{2\sqrt{x}}{1-x}$ give a simplified expression (in fraction form) for $\frac{dy}{dx}$.

$u = 2x^{1/2}$ $v = 1 - x$

$u' = x^{-1/2}$ $v' = -1$

$y' = \frac{(1-x).x^{-1/2} + 2\sqrt{x}}{(1-x)^2}$ ✓ ✓

$= \frac{(1 - x + 2x)}{\sqrt{x}(1-x)^2} = \frac{1+x}{\sqrt{x}(1-x)^2}$ ✓

b) For what values of x is y

i) undefined ?

$x = 1$ ✓

ii) zero ?

$x = 0$ ✓

3. **[5,4 = 9 marks]**

A curve has the equation $y = 3x^4 - 8x^3 + 6x^2$.

a) Determine the coordinates of the stationary point(s) of this curve. Use the second derivative to determine their nature.

$y' = 12x^3 - 24x^2 + 12x$ ✓

$y' = 0 \Rightarrow 12x(x^2 - 2x + 1) = 0$

$\Rightarrow x = 0$ or $x = 1$ ✓

Coordinates are $(0,0)$ and $(1,1)$ ✓

$y'' = 36x^2 - 48x + 12$

$y''(0) = 12$ so $(0,0)$ is minimum point ✓

$y''(1) = 0$ so $(1,1)$ is a horizontal point of inflection ✓

b) The curve has two inflection points, one of which is located at $x = 1$.

Determine the coordinates of the second point of inflection.

$36x^2 - 48x + 12 = 0$ ✓ $= 12(3x^2 - 4x + 1)$

$= 12(3x-1)(x-1)$ ✓

$y''(x) = 0 \Rightarrow x = 1/3$ or 1 ✓

So 2nd point of inflection is $(1/3, \frac{11}{27})$ ✓

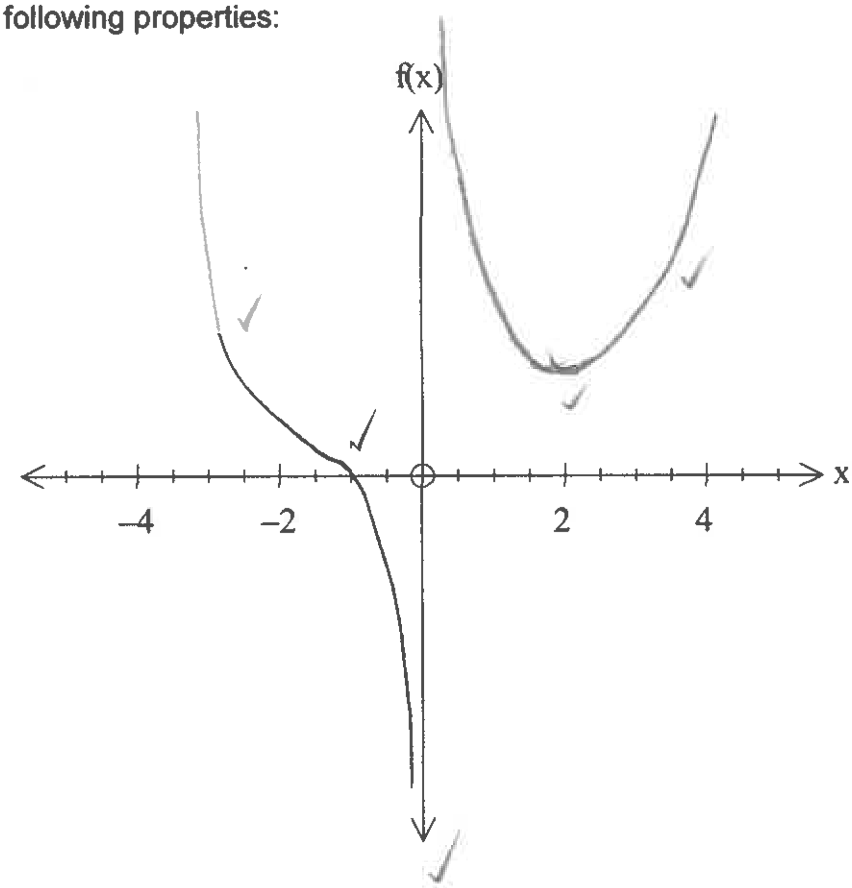
$$\left(\frac{\cancel{3}^{1}}{\cancel{81}_{27}} - \frac{8}{27} + \frac{\cancel{6}^{18}}{\cancel{9}_{27}} = \frac{11}{27} \right)$$

$\begin{array}{ccc} 3 & -1 & -1 \\ +1 & -1 & -3 \\ & & -4 \end{array}$

4. **[5 marks]**

Sketch the graph of $y = f(x)$ given the following properties:

- $\frac{dy}{dx} < 0$ for $x < 0$ and $0 < x < 2$
- $\frac{dy}{dx} = 0$ for $x = 2$
- $\frac{dy}{dx} > 0$ for $x > 2$
- $y = \frac{d^2y}{dx^2}$=0 for $x = -1$
- y is undefined for $x = 0$



GREENWOOD COLLEGE
Mathematics Methods Units 3 & 4
Test 1 Applications of Differentiation 2019

Name Marking Key Mark ____/30

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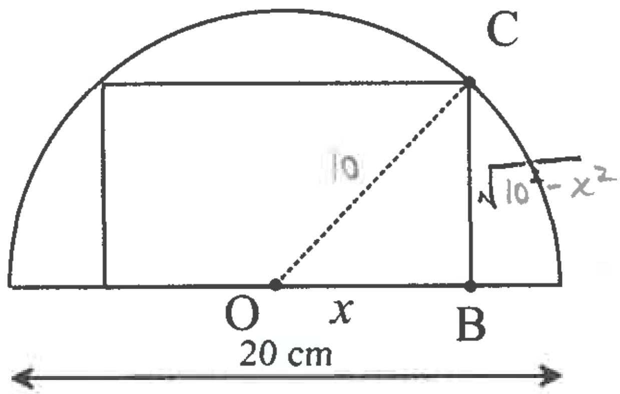
Access to Formulae Sheet and one sheet of A4 notes allowed. Use of approved calculators is assumed in this section. Time limit 30 minutes.

5. **[7,1 = 8 marks]**

Infinitely many rectangles can be inscribed in a semi-circle of diameter 20 cm.

a) Use calculus techniques to determine the dimensions of the rectangle with the largest area.
(Hint: Consider triangle OBC with $x = OB$)



$A(x) = 2x\sqrt{10^2 - x^2}$ ✓ ✓

$A'(x) = \dfrac{-4x^2 - 200}{\sqrt{100 - x^2}}$ ✓

$A'(x) = 0 \Rightarrow x = \pm 5\sqrt{2}$ ✓ $(-5\sqrt{2}$ not applicable$)$

$A''(5\sqrt{2}) = -8$ so $x = 5\sqrt{2}$ is a maximum point ✓

Dimensions are $10\sqrt{2}$ cm by $5\sqrt{2}$ cm ✓ ✓

b) State the area of your selected triangle — should have been rectangle!

Area $= 5\sqrt{2} \times 10\sqrt{2}$

$= 100\text{ cm}^2$ ✓ or $\dfrac{5\sqrt{2} \times 5\sqrt{2}}{2}$

$= 25\text{ cm}^2$ ✓

6. **[4,2 = 6 marks]**

Helium gas is being pumped into a balloon. The balloon maintains a spherical shape as it inflates, and its volume increases at a constant rate of 600 cubic centimetres per minute.

a) At what rate is the radius of the balloon increasing when the volume of the balloon is 20 litres ? (1 litre = 1000 cm^3 and $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$)

Find $\frac{dr}{dt}$ $\qquad \frac{dV}{dt} = 600 \text{ cm}^3/\text{min}$

$$600 = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$= 4\pi r^2 \times \frac{dr}{dt} \checkmark$$

$V = 20\,000 \text{ cm}^3$ so $20\,000 = \frac{4}{3}\pi r^3$

$$\Rightarrow r = 16.83890301 \text{ cm} \checkmark$$

$\therefore$ solve $600 = 4\pi \times 16.8389^2 \times \frac{dr}{dt} \checkmark$

$$\Rightarrow \frac{dr}{dt} = 0.1683890301 \text{ cm/min} \checkmark$$

b) Use the small increment formula to estimate the amount by which the radius will increase in the next second.

$\partial r = \frac{dr}{dt} \times \delta t$ $\qquad \partial t = \frac{1}{60} \text{ min} \checkmark$

$$\text{so } \delta r = 0.1683890301 \times \frac{1}{60}$$

$$= 0.002806483 83 \text{ cm} \checkmark$$

7. **[2,4,2,2 = 10 marks]**

A particle moves in a straight line with a displacement function $x(t) = 12t - 2t^3 - 1$ cm, where t is in seconds, $t \geq 0$.

a) Give velocity and acceleration functions for the particle's motion.

$v(t) = 12 - 6t^2$ ✓

$a(t) = -12t$. ✓

b) State the initial conditions of this particle's motion, and interpret their meaning

$x(0) = -1$ cm

$v(0) = 12$ cm/s ✓

$a(0) = 0$ cm/s^2

The particle starts 1 cm to the left of O, the origin, ✓ moving at a constant ✓ velocity to the right of the origin. ✓

c) When and where does the particle change direction ?

$x(\sqrt{2}) = 8\sqrt{2} - 1$

or 10.3137085

$v'(t) = 0 \Rightarrow 12 - 6t^2 = 0$

$\Rightarrow t = \sqrt{2}$ s.

So changes direction after $\sqrt{2}$ s ✓, at $8\sqrt{2} - 1$ cm from the origin ✓

At what time is the particle's

i) speed increasing

V is negative for $t > \sqrt{2}$ + most when $t = 0$

So speed increasing from $t > \sqrt{2}$ ✓

ii) velocity increasing

never. ✓ since max $v(t)$ happens when $t = 0$

8. **[4,2 = 6 marks]**

The sales manager of a car yard estimates that the number of cars sold by his staff will be 80 next month. Bonuses and other incentives mean that the profit function for car sales is given by

$$P(n) = 2000n + 10n^2$$

In dollars, where n is the number of cars sold in a month.

a) Use derivatives to find the error in profit if the manager's estimate is out by 5%.

$\frac{\delta n}{n} = 0.05$ Find $\frac{\partial P}{P}$ $n = 80$

$\frac{\partial P}{n} = \frac{dP}{dn} \times \frac{\partial n}{n}$ ✓

$= 2000 + 20n \times 0.05 \; \big|_{n=80}$

$= 3600 \times \underline{0.05}$ ✓

$\frac{\partial P}{80} = 180$ ✓

$\Rightarrow \partial P = 14400$ So Profit error is \$ 14400 ✓

b) What percentage error in profit does this represent ?

$\frac{\partial P}{P} = \frac{14400}{224\,000} \times 100$ ✓

$= 6.428571429\,\%$ ✓