Mathematics Specialist Unit 4

Test 4: Calculus

Student Name: Solutions

Section One – calculator-free section

Time allowed for this task: 30 minutes, in class, under test conditions

Materials Provided: SCSA Formula Sheet

Materials required: (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters.

Marks available: 31 marks

1. **[2, 3, 4, 4 = 13 marks]**

Determine the following definite integrals; using a judicious substitution, if necessary.

(a) $\int \frac{4x+10}{x^2+5x+4}\,dx$

$= 2\int \frac{2x+5}{x^2+5x+4}\,dx$ ✓

$= 2\ln|x^2+5x+4| + c$ ✓

or $\int \frac{2}{x+4} + \frac{2}{x+1}\,dx.$

$= 2\ln|x+4| + 2\ln|x+1| + c$

(b) $\int \frac{\sin(\ln x)}{x}\,dx$ using the substitution, $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$x\,du = dx$

$\int \frac{\sin u}{\cancel{x}}\,\cancel{x}\,du$ ✓

$\int \sin u\,du$ ✓

$= -\cos u + c$

$= -\cos(\ln|x|) + c$ ✓

(c) $\int \frac{2}{16+x^2}\,dx$ using the substitution, $x = 4\tan u$

$\frac{dx}{du} = 4\sec^2 u$

$dx = 4\sec^2 u\,du$

$\frac{x}{4} = \tan u$

$\tan^{-1}\left(\frac{x}{4}\right) = u$

$\int \frac{2}{16+16\tan^2 u}\,4\sec^2 u\,du$ ✓

$\int \frac{2}{16(1+\tan^2 u)}\,4\sec^2 u\,du$

$\frac{1}{2}\int \frac{\sec^2 u}{\sec^2 u}\,du$ ✓

$\frac{1}{2}\int 1\,du$ ✓

$= \frac{1}{2}u + c$ ✓

$= \frac{1}{2}\tan^{-1}\left(\frac{x}{4}\right) + c$

or

(d) $\int \frac{3x-1}{(3x-4)^2}\,dx$

$\frac{3x-1}{(3x-4)^2} = \frac{A}{3x-4} + \frac{B}{(3x-4)^2}$ ✓

$3x-1 = A(3x-4) + B$

let $x = \frac{4}{3}$ let $x = 0$

$3 = B$ ✓ $-1 = -4A + 3$

$1 = A$

$\int \left(\frac{1}{3x-4} + \frac{3}{(3x-4)^2}\right) dx$ ✓

$= \frac{1}{3}\ln|3x-4| - \frac{1}{3x-4} + C$ ✓

let $u = 3x-4$

$\therefore \frac{du}{dx} = 3$ $x = \frac{u+4}{3}$

$\therefore dx = \frac{du}{3}$

$\int \frac{3\left(\frac{u+4}{3}\right) - 1}{u^2}\,\frac{du}{3}$ ✓

$= \frac{1}{3}\int \frac{u+3}{\cancel{3}u^2}\,du$ ✓

$= \frac{1}{3}\left(\ln|u| - 3u^{-1} + C\right)$ ✓

$= \frac{1}{3}\ln|3x-4| - \frac{1}{3x-4} + C$ ✓

2. [4, 5, 5 = 14 marks]

Evaluate

(a) $\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$ using the trig substitution x = 3 sin θ

$\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{9-9\sin^2\theta}}\,3\cos\theta\,d\theta$ ✓✓ $\frac{dx}{d\theta} = 3\cos\theta$

$dx = 3\cos\theta\,d\theta$

Correct substitution and bounds

$\int_0^{\frac{\pi}{2}} \frac{3\cos\theta}{3\cos\theta}\,d\theta$

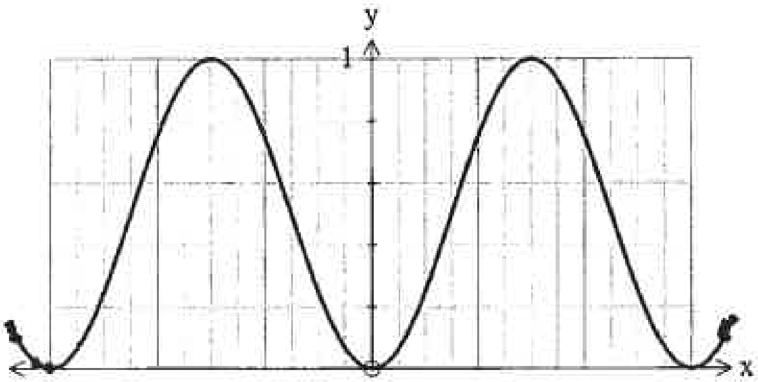
$\int_0^{\frac{\pi}{2}} 1\,d\theta$ ✓

$= \theta \Big]_0^{\pi/2}$

$= \frac{\pi}{2}$ ✓

3. **[4 marks]**

The graph below shows the function $y = sin^2\left(\frac{x}{6}\right)$. Determine the area of the region trapped between the function $y = sin^2\left(\frac{x}{6}\right)$, the lines $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$ and $y = 0$



$A = 2\int_0^{\frac{\pi}{2}} \sin^2\left(\frac{x}{6}\right) dx$ ✓

$A = 2\int_0^{\frac{\pi}{2}} \frac{1-\cos\left(\frac{x}{3}\right)}{2} dx$ ✓

$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(1 - \cos\left(\frac{x}{3}\right)\right) dx$

$A = \frac{1}{2}\left[x - 3\sin\left(\frac{x}{3}\right)\right]_0^{\pi/2}$ ✓

$A = \frac{1}{2}\left(\frac{\pi}{2} - 3\sin\frac{\pi}{6}\right)$

$A = \frac{\pi - 3}{2}$ units^2 ✓

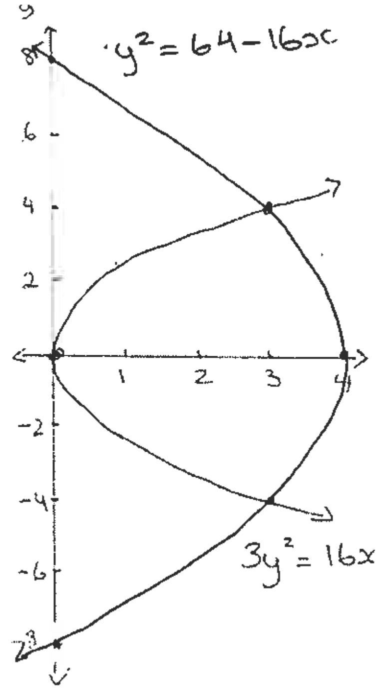
4. **[4, 4 = 8 marks]**

For the curves with equations

$$3y^2 = 16x \qquad \text{and} \qquad y^2 = 64 - 16x,$$

determine:

(a) the area of the region enclosed between the two curves,



$$\int_{-4}^{4} \left(\frac{64-y^2}{16} - \frac{3y^2}{16} \right) dy \quad \checkmark\checkmark\checkmark$$

— 1 correct bounds
— 1 rearrange eq^ns
— 1 subtract curves

$$= 21\tfrac{1}{3} \text{ units}^2 \quad \checkmark$$

or

$$2\left[\int_0^3 \sqrt{\frac{16x}{3}}\, dx + \int_3^4 \sqrt{64-16x}\, dx \right] \quad \checkmark\checkmark\checkmark$$

$$= 21\tfrac{1}{3} \text{ units}^2 \quad \checkmark$$

(b) the volume of the solid of revolution about the x-axis formed by this region

$$V = \pi \left[\int_0^3 \frac{16x}{3}\, dx + \int_3^4 (64 - 16x)\, dx \right]$$

✓ correct $\int \pi y^2 \, dx$
✓ correct 1st integral
✓ correct 2nd integral

$$= \pi \left[24 + 8 \right]$$

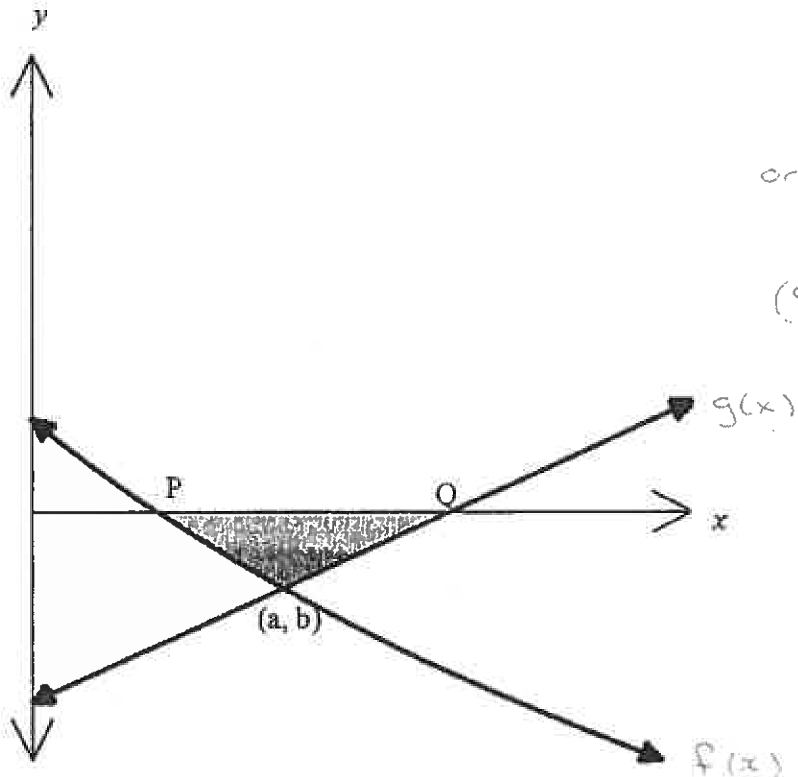
$$= 32\pi \text{ units}^3$$

or 100.53 units^3

5. [6 marks]

Below is shown the cross-section of a river (see shaded), the bed of which is described by curves; $f(x) = -10\ln(0.1x + 0.8)$ and $g(x) = 2.5e^{(0.1x+0.8)} - 10$, where x and y are in metres. If Karan is located at P, gazing into the river, contemplating the meaning of life, and the river current is 2m/s then how many kilolitres (to the nearest kL) of water will pass him by in five minutes of his thinking?

Note: 1 m^3 = 1kL



$P = (2, 0)$

$Q = (5.8629436, 0)$

or $Q = (10\ln 4 - 8, 0)$

$(a, b) = (3.92573, -1.76$

Area of cross section

$$= \int_2^{3.92573} -10\ln(0.1x + 0.8)\,dx - \int_{3.92573}^{5.86294} (2.5e^{0.1x+0.8} - 10)\,dx$$

$= 1.745 + 1.761$

$= 3.506\ m^2$

Quantity of water $= 3.506 \times 2 \times 5 \times 60$

≈ 2104 kL

(b) $\int_0^4 \frac{1}{9-\sqrt{x}}\, dx$ using the substitution u = 9 − $\sqrt{x}$

$\frac{du}{dx} = \frac{-1}{2\sqrt{x}}$

$-2\sqrt{x}\, du = dx$

$\int_9^7 \frac{1}{u}\left(-2(9-u)\right) du$ ✓✓

correct bounds and substitution.

$2\int_7^9 \frac{9-u}{u}\, du$

$2\int_7^9 \left(\frac{9}{u} - 1\right) du$ ✓

$= 2\left(9\ln|u| - u\right)_7^9$ ✓

$= 18\ln 9 - 18 - 18\ln 7 + 14$

$= 18\ln\left(\frac{9}{7}\right) - 4$ ✓

(c) $\int_1^4 \frac{x}{\sqrt{5-x}} \quad dx$

Let $u = 5 - x$

$\frac{du}{dx} = -1$

$-du = dx$

$\int_4^1 \frac{5-u}{\sqrt{u}}\,(-du)$ ✓✓

correct bounds & substitution

$= \int_1^4 \left(5u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right) du$ ✓

$= \left[2\left(5u^{\frac{1}{2}}\right) - \frac{2}{3}u^{\frac{3}{2}}\right]_1^4$ ✓

$= \left(10(2) - \frac{16}{3}\right) - \left(10 - \frac{2}{3}\right)$

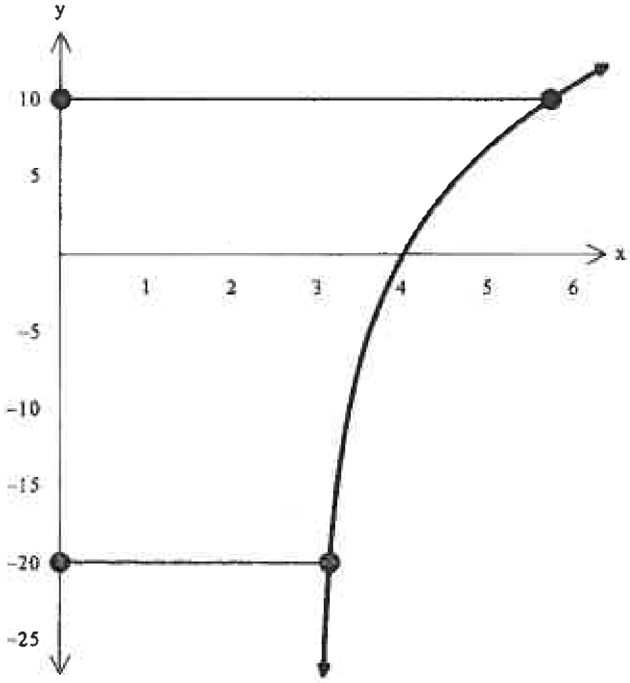
$= \frac{44}{3} - \frac{28}{3}$

$= \frac{16}{3}$ ✓

6. **[3 marks]**

Below is shown the graph of the curve of y = 10ln(x–3) where both x and y are in centimetres.

A vase is to be created by rotating the curve about the y-axis from y = -20 to y = 10



Determine the *volume* of the vase, correct to the nearest cm³.

$y = \ln(x-3)$

$\frac{y}{10} = \ln(x-3)$ ✓

$e^{0.1y} + 3 = x$

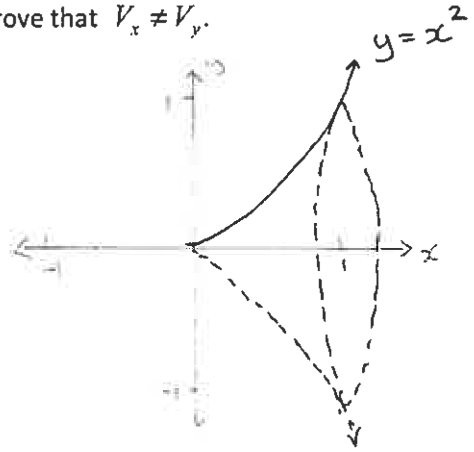
$$V = \pi \int_{-20}^{10} \left(e^{0.1y} + 3\right)^2 \, dy$$ ✓

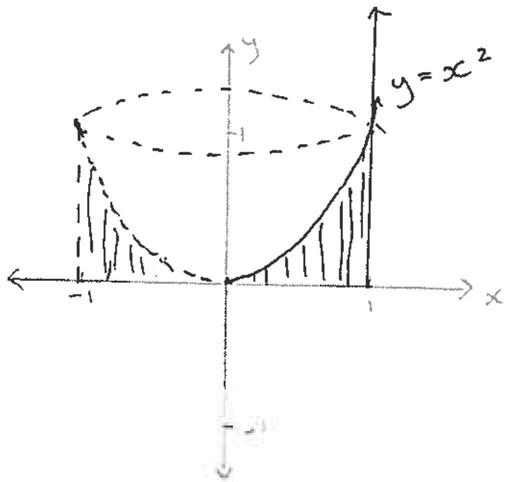
$V \approx 1451 \text{ cm}^3$ ✓

7. [6 marks]

The region define by $\{(x,y): x \geq 0 \cap y \geq 0 \cap y \leq x^2\}$ for $0 \leq x \leq 1$ is rotated about the x axis to generate V_x and then rotated about the y axis to generate V_y.

Prove that $V_x \neq V_y$.





$V_x = \pi \int_0^1 (x^2)^2 \, dx$

$V_x = \frac{\pi}{5}$ units^2

$V_y = \pi \int_0^1 (1^2 - y)\, dy$

$V_y = \frac{\pi}{2}$ units^2

$\therefore V_x \neq V_y$.