Name: ______________________________ **Score:** _____ / 27

No calculators or notes are to be used.
Access to approved Mathematics Specialist formulae sheet is permitted.
Simplify answers where possible. **Time limit = 25 minutes.**

1. [5 marks]

Determine the value of a given the three planes below meet along a common line:

$$x + 3y + 3z = a - 1$$
$$\boldsymbol{r}.\langle 2, -1, 1\rangle = 7$$
$$3x - 5y + az = 16$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 2 & -1 & 1 & 7 \\ 3 & -5 & a & 16 \end{array}\right] \checkmark$$

$$\begin{array}{l} \\ R2 - 2R1 \\ R3 - 3R1 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 0 & -7 & -5 & 9-2a \\ 0 & -14 & a-9 & 19-3a \end{array}\right] \begin{array}{l} \\ \checkmark \\ \checkmark \end{array}$$

$$\begin{array}{l} \\ \\ R3 - 2R2 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 0 & -7 & -5 & 9-2a \\ 0 & 0 & a+1 & 1+a \end{array}\right] \checkmark$$

$$1 + a = 0$$
$$a = -1 \checkmark$$

2. [3, 2, 1, 2, 1, 2 = 11 marks]

The motion of a particle at time t hours is given by:

$$\boldsymbol{r}(t) = \left(10\cos\left(\frac{\pi}{6}t\right)\right)\boldsymbol{i} + \left(20\sin\left(\frac{\pi}{6}t\right)\right)\boldsymbol{j} \quad \text{km,} \qquad \text{for } t \geq 0$$

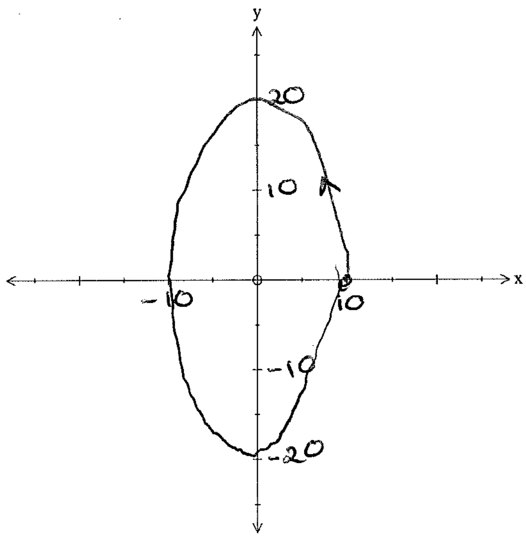
a) Find the Cartesian equation of the path

$x = 10\cos\left(\frac{\pi}{6}t\right)$ ✓ $\qquad y = 20\sin\left(\frac{\pi}{6}t\right)$ ✓

$\frac{x}{10} = \cos\left(\frac{\pi}{6}t\right) \qquad \frac{y}{20} = \sin\left(\frac{\pi}{6}t\right)$

$\frac{x^2}{100} + \frac{y^2}{400} = 1$ ✓

b) On the axes below, sketch the path of the particle. Indicate clearly the direction of motion of the particle.



✓ ellipse
✓ direction of motion

c) How long does it take the particle to return to its initial position?

$\frac{2\pi}{\frac{\pi}{6}} = 12$ hours ✓

d) Determine at what times the particle is moving perpendicular to its displacement vector.

Perpendicular at $(10,0)$, $(-10,0)$ $\qquad$ $(0,20)$ & $(0,-20)$

$t=0, 6, 12$, etc $\qquad$ $t=3, 9$, etc.

$\therefore t = 6n, \; n \in Z^{+}$ $\qquad$ $t = 3n, \; n \in Z^{+}$

e) Find $v(t)$, the velocity vector of the particle at any time t hours.

$$\underset{\sim}{v}(t) = -\frac{5\pi}{3}\sin\left(\frac{\pi}{6}t\right)\underset{\sim}{i} + \frac{10\pi}{3}\cos\left(\frac{\pi}{6}t\right)\underset{\sim}{j}$$

f) Prove the acceleration of the particle is parallel to the path of the particle.

$$\underset{\sim}{a}(t) = -\frac{5\pi^2}{18}\cos\left(\frac{\pi}{6}t\right)\underset{\sim}{i} - \frac{5\pi^2}{9}\sin\left(\frac{\pi}{6}t\right)\underset{\sim}{j}$$

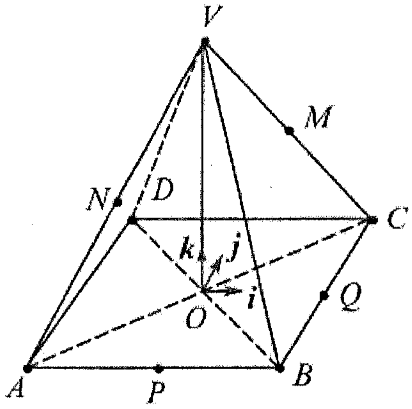
$$\underset{\sim}{a}(t) = -\frac{\pi^2}{36} \cdot \underset{\sim}{r}(t)$$

$\therefore$ $\underset{\sim}{a}(t)$ is parallel to the path of the particle

3. [2, 4, 5 = 11 marks]

VABCD is a square based pyramid.

- The origin O is the centre of the base.
- The unit vectors **i**, **j** and **k** are in the directions of $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{OV}$ respectively.
- $AB = BC = CD = DA$ = 4cm
- OV = 2h cm, where h is a positive real number
- P, Q, M and N are the midpoints of AB, BC, VC and VA respectively.
- $\overrightarrow{PM} = \langle 1, 3, h\rangle$ and $\overrightarrow{QN} = \langle -3, -1, h\rangle$



a) Determine the position vectors of P and Q

$P = \langle 0, -2, 0\rangle$ ✓ $\quad Q = \langle 2, 0, 0\rangle$ ✓

b) Determine the position vector $\overrightarrow{OX}$, where X is the point of intersection of QN and PM.
Hint: Determine the equations of the lines containing vectors $\overrightarrow{PM}$ and $\overrightarrow{QN}$

Line $\overrightarrow{PM}$: $\underline{r} = \langle 0, -2, 0\rangle + \lambda\langle 1, 3, h\rangle$ ✓

Line $\overrightarrow{QN}$: $\underline{r} = \langle 2, 0, 0\rangle + \lambda\langle -3, -1, h\rangle$ ✓

$\langle \lambda, -2+3\lambda, \lambda h\rangle = \langle 2-3\lambda, -\lambda, h\lambda\rangle$ ✓

$\lambda = 2 - 3\lambda \qquad -2+3\lambda = -\lambda$

$4\lambda = 2 \qquad -2 = -4\lambda$

$\lambda = \frac{1}{2}$ ✓ $\qquad \frac{1}{2} = \lambda$

$\overrightarrow{OX} = \langle \frac{1}{2}, \frac{-1}{2}, \frac{h}{2}\rangle$ ✓

c) Given OX is perpendicular to VB, determine the value of h.

$V = \langle 0, 0, 2h\rangle$ ✓ $\qquad B = \langle 2, -2, 0\rangle$ ✓

$\overrightarrow{VB} = \langle 2, -2, -2h\rangle$ ✓

$\langle 2, -2, -2h\rangle \cdot \langle \frac{1}{2}, -\frac{1}{2}, \frac{h}{2}\rangle = 0$

$\frac{1}{2}(2) - \frac{1}{2}(-2) + \frac{h}{2}(-2h) = 0$ ✓

$1 + 1 - h^2$

$2 = h^2$

$\therefore h = \sqrt{2}$ ✓ only +ive value.

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Up to two approved calculators and one A4 page of notes (front and back) are permitted.
Access to approved Mathematics Specialist formulae sheet is permitted.
Simplify answers where possible. **Time limit = 30 minutes.**

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4. [5, 2 = 7 marks]

The line that passes through point A, position vector $\langle 2, -1, 3\rangle$, is perpendicular to the plane $x - y + 2z = 27$.

a) Determine the point of intersection of the line and the plane.

Plane: $\underset{\sim}{r} \cdot \langle 1, -1, 2\rangle = 27$ ✓

Line: $\underset{\sim}{r} = \langle 2, -1, 3\rangle + \lambda \langle 1, -1, 2\rangle$ ✓

$\langle 2+\lambda, -1-\lambda, 3+2\lambda\rangle \cdot \langle 1, -1, 2\rangle = 27$ ✓

$2+\lambda + 1 + \lambda + 6 + 4\lambda = 27$

$6\lambda = 18$

$\lambda = 3$ ✓

$\underset{\sim}{I} = \langle 5, -4, 9\rangle$ ✓

b) Calculate the shortest distance from the point A to the plane.

$|\langle 5, -4, 9\rangle - \langle 2, -1, 3\rangle|$

$= |\langle 3, -3, 6\rangle|$ ✓

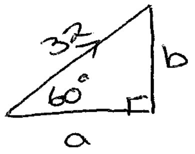
$\sqrt{3^2 + (-3)^2 + 6^2}$

$= 3\sqrt{6}$ ✓

5. [1, 3, 4, 4 = 12 marks]

A baseball player strikes a ball with initial velocity $\begin{pmatrix} a \\ b \end{pmatrix}$ ms^{-1}, the bat connects with the ball at $\begin{pmatrix} 0.3 \\ 1.3 \end{pmatrix}$ m and acceleration due to gravity is -9.8 ms^{-2}.

a) The ball was hit at a speed of 32 ms^{-1} at an angle of 60° above the horizontal. Calculate the values of a and b.



$b = 32\sin 60°$ $\quad$ $a = 32\cos 60°$

$b = 16\sqrt{3}$ $\quad$ $a = 16$

b) Find the speed of the ball after 3 seconds.

$\underline{a}(t) = -9.8$

$\underline{v}(t) = -9.8t\,\underline{j} + c \text{ ms}^{-1}$, $\quad \underline{v}(0) = 16\underline{i} + 16\sqrt{3}\underline{j}$

$16\underline{i} + 16\sqrt{3}\,\underline{j} = 0\underline{j} + c$.

$16\underline{i} + 16\sqrt{3}\underline{j} = c$

$\underline{v}(t) = 16\underline{i} + (16\sqrt{3} - 9.8t)\underline{j}$

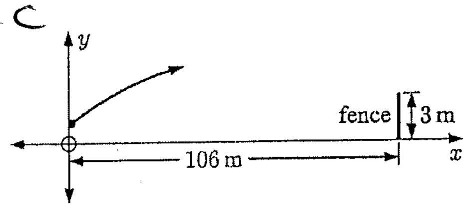
$\underline{v}(3) = 16\underline{i} - 1.6872\underline{j}$

$|\underline{v}(3)| = \sqrt{16^2 + (-1.6872)^2}$

$= 16.09 \text{ ms}^{-1}$.

c) A fence 3 m tall is located 106 m horizontally from the baseball player. Will the ball clear the fence? Justify your answer.

$\underline{r}(t) = 16t\underline{i} + (16\sqrt{3}t - 4.9t^2)\underline{j} + c$



$\underline{r}(0) = \langle 0.3, 1.3 \rangle$

$\langle 0.3, 1.3 \rangle = 0\underline{i} + 0\underline{j} + c$.

$\langle 0.3, 1.3 \rangle = c$

$\underline{r}(t) = (16t + 0.3)\underline{i} + (16\sqrt{3}t - 4.9t^2 + 1.3)\underline{j}$

$16\sqrt{3}t - 4.9t^2 + 1.3 > 3$

$0.06 < t < 5.59$

at $t = 5.59$

i component: $16(5.59) + 0.3$

$= 89.74\text{m}$

the ball is > 3 between 1.26 m & 89.74 m from the player, ∴ won't make it.

d) How far did the ball travel before hitting the ground?

$$16\sqrt{3}t - 4.9t^2 + 1.3 = 0 \checkmark$$

$$t = 5.7 \text{ seconds.} \checkmark$$

$$\int_0^{5.7} \sqrt{(16)^2 + (16\sqrt{3} - 9.8t)^2}\, dt \checkmark$$

$$= 126.32\text{m} \checkmark$$

6. [4 marks]

Determine the vector equation of the sphere with diameter AB. Point A is located at position vector $\langle -1, 4, -3\rangle$ and the position vector of point B is $\langle 4, -2, 1\rangle$.

$$\text{Centre} = \left\langle \frac{-1+4}{2}, \frac{4+(-2)}{2}, \frac{-3+1}{2} \right\rangle$$

$$C = \left\langle \frac{3}{2}, 1, -1 \right\rangle \checkmark$$

$$\overrightarrow{CB} = \langle 4, -2, 1\rangle - \left\langle \frac{3}{2}, 1, -1 \right\rangle$$

$$= \left\langle \frac{5}{2}, -3, 2 \right\rangle \checkmark$$

$$|\overrightarrow{CB}| = \sqrt{\left(\frac{5}{2}\right)^2 + (-3)^2 + 2^2}$$

$$= \frac{\sqrt{77}}{2}. \checkmark$$

$$\therefore \left| \underline{r} - \left\langle \frac{3}{2}, 1, -1 \right\rangle \right| = \frac{\sqrt{77}}{2}. \checkmark$$

7. [7 marks]

The vector equation a plane is given by $\boldsymbol{r}.(\boldsymbol{i} - 7\boldsymbol{j} + 5\boldsymbol{k}) = 5$. The vector $2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ lies on the plane. Determine a second vector on the plane which is at an angle of 60 degrees to the vector $2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$.

let the second vector $= \langle a, b, c \rangle$.

$\langle 2,1,1 \rangle \cdot \langle a,b,c \rangle = \sqrt{6}\sqrt{a^2+b^2+c^2}\cos 60°$ ✓

$2a+b+c = \dfrac{\sqrt{6}\sqrt{a^2+b^2+c^2}}{2}$ --- ① ✓

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{matrix} 1 \\ -7 \\ 5 \end{matrix}$$ ✓

$c - b = 1$ --- ② ✓

$2c - a = 7$ ---- ③ ✓

$2b - a = 5$ ---- ④ ✓

solve equations simultaneously

$a = -\frac{1}{3}$ $b = \frac{7}{3}$ $c = \frac{10}{3}$ ✓

<u>or</u>

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -7 \\ 5 \end{pmatrix}$$

$b - c = 1$

$a - 2c = 7$

$a - 2b = 5$

$\therefore$ $a = \frac{11}{3}$, $b = -\frac{2}{3}$, $c = -\frac{5}{3}$