KELMSCOTT SENIOR HIGH SCHOOL

YR 12 MATHEMATICS: METHODS

Test 1, Sem.1, 2022 DATE:

Section 1: ***CALCULATOR FREE***

Name: SOLUTIONS | MARKING KEY

MATHEMATICS DEPARTMENT

Time allowed for this paper:

SECTION	WORKING
CALCULATOR FREE	20 MINUTES
CALCULATOR ASSUMED	40 MINUTES

Permissible Items:

Section One (Calculator free): 29 marks

Standard items: Pens, pencils, pencil sharpener, highlighter, eraser, ruler, correction tape/fluid, formula sheet

TOTAL for Section 1: ____________________

Section Two (Calculator assumed): 45 marks

Standard items: Pens, pencils, pencil sharpener, highlighter, eraser, ruler, correction tape/fluid, formula sheet

Special items: Drawing instruments, templates, up to three calculators – CAS, graphic or scientific, satisfying the conditions set by the School Curriculum and Standards Authority for this course.

TOTAL for Section 2: ____________________

FINAL MARK: ____________________

Write your answers in the spaces provided. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily, and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.
It is recommended that you **do not use pencil**, except in diagrams.

(1) (1, 1, 2, 2 = 6 marks)

Determine the derivative of the following functions

(i) $y = 6e^{3x} - \sqrt{e} + \cos 2x$

$y' = 18e^{3x} - 2\sin 2x$ ✓

(ii) $f(x) = 4x^2 . \sin 5x$

$f'(x) = 8x . \sin 5x + 4x^2(5\cos 5x)$ ✓

(iii) $f(x) = (3\sqrt{x} - 5x^2)^4$

$f(x) = (3x^{1/2} - 5x^2)^4$

$f'(x) = 4(3x^{1/2} - 5x^2)^3 . (\frac{3}{2}x^{-1/2} - 10x)$

$= 4(3\sqrt{x} - 5x^2)^3 \left(\frac{3}{2\sqrt{x}} - 10x\right)$ ✓ ✓

(iv) $y = (\frac{1}{e^x} + 1)(e^x - 1)$

e^{-x}

$y = 1 - \frac{1}{e^x} + e^x - 1$

$y = -e^{-x} + e^x$ ✓

$y' = e^{-x} + e^x \Rightarrow \frac{1}{e^x} + e^x$ ✓

OR

$y' = u'v + uv'$

$= -e^{-x}(e^x - 1) + (\frac{1}{e^x} + 1) . e^x$ ✓

$\Rightarrow -1 + e^{-x} + 1 + e^x$

$\Rightarrow \frac{1}{e^x} + e^x$ ✓

(2) (4 marks)

Find the equation of the tangent to the function $f(x) = x^2 e^{2x}$ when $x = 1$

$f'(x) = 2x . e^{2x} + x^2 . 2e^{2x}$ ✓

$f'(1) = 2e^2 + 1 . 2e^2$

$= 4e^2$ ✓

pt $(1, e^2)$ ✓

$y = mx + b$

$e^2 = 4e^2(1) + b$

$b = -3e^2$

$y = 4e^2 - 3e^2$ ✓

10

(3) (5 marks)

Show that the curve with equation $y = \frac{\cos x}{1+\sin x}$ does not have any horizontal tangents.

$$y' = \frac{u}{v} = \frac{u'v - uv'}{v^2} \Rightarrow \frac{-\sin x(1+\sin x) - [\cos x . \cos x]}{(1+\sin x)^2}$$

$$\Rightarrow \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$\Rightarrow \frac{-\sin x - 1(\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

$$\Rightarrow \frac{-\sin x - 1}{(1+\sin x)^2}$$

$$\Rightarrow \frac{-1(1+\sin x)}{(1+\sin x)^2} \Rightarrow \frac{-1}{(1+\sin x)}$$

horizontal tangent $y' = 0$, not possible.

5

(4) (1,1,1,3,3 = 9 marks)

Given $f(x) = - x^2 + 2x + 3$

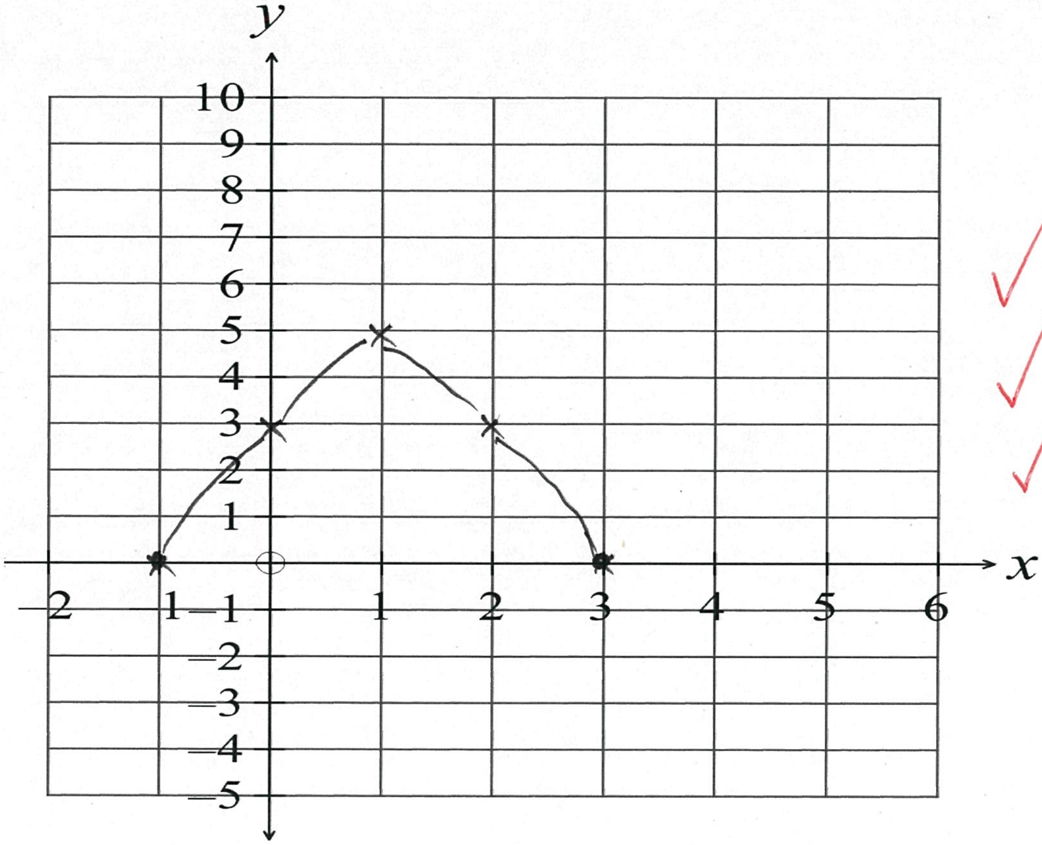
a) Evaluate

(i) f (1) $-(1)^2+2(1)+3 \Rightarrow 5$ ✓

(ii) f ' (1) $-2(1)+2 \Rightarrow 0$ ✓

(iii) f '' (1) -2 ✓

b) (i) Sketch the graph of f (x) over the domain $-1 \le x \le 4$ on the axes below



✓ shape
✓ pts
✓ ave correct domain.

(ii) With reference to your sketch, explain the significance of each answer from part a

(i) $f(1)$ – y-value Turning pt ✓

(ii) $f'(1)$ – stationary pt ✓

(iii) $f''(1)$ – concave down, max Turning pt ✓

9

(5) (3, 2 = 5 marks)

A particle is moving in a straight line such that its position x cm, relative to point O at time t is given by

$$x = \frac{2}{3}t^3 - 2t^2 - 6t + 5$$

Find (i) The position of the particle when the velocity is 0.

$x' = v = 2t^2 - 4t - 6$

$= 2(t^2 - 2t - 3)$

when $v=0$

$0 = 2(t-3)(t+1)$

$t = 3$ or $\cancel{-1}$

$x(3) = \frac{2}{3}(3)^3 - 2(3)^2 - 6(3) + 5$

$= -13m$

(ii) At what time will the acceleration of the particle be 8 m/s^2?

$a = x'' \Rightarrow 4t - 4$

$\therefore 4t - 4 = 8$

$t = 3 sec$

5

KELMSCOTT SENIOR HIGH SCHOOL

MATHEMATICS METHODS (Y12)

Test 1, Sem.1, 2022 **DATE:**

Section 2: ***CALCULATOR ASSUMED***

MATHEMATICS DEPARTMENT

Name: SOLUTIONS | MARKING KEY Mark: /45

(6) (4 marks)

A sphere has surface area of 500 cm^2. Use the small change formula and the determined radius (to 3 d.p) to find the approximate change necessary in the radius of the sphere to cause the surface area to change from 500 cm^2 to 520 cm^2.

$SA = 4\pi r^2 = 500$

$r = 6.308\text{cm}$ ✓

$\frac{dA}{dr} = 8\pi r$

we want δr thus $\delta r = \frac{dr}{dA} \cdot \delta a$ ✓

$\Rightarrow \frac{1}{8\pi(6.308)} \times 20$ ✓

$\Rightarrow 0.126\text{cm}$ ✓

4

(7) (1, 2, 3, 2, 2, 2 = 12 marks)

The population of frogs in a colony after t days is given by $N(t) = 80\, e^{0.02t}$

(i) How many frogs are in the colony after 10 days

$N(10) = 80e^{0.02(10)} = 97.7 \approx 98$

(ii) In how many days will the population double?

$160 = 80e^{0.02t}$

$2 = e^{0.02t}$ $t = 34.6$

$\sim t = 35$ days.

(iii) What is the rate of change of population after 20 days?

$N' = 1.6e^{0.02t}$ @ $t = 20$

$N' = 2.39$

≈ 2.4

In the same pond is a population of fish (that are a food source for the frogs). The growth of the population is modelled by the equation

$$\frac{dF}{dt} = -\,0.07t, \text{ where t is in days}$$

(iv) If the initial population of fish was 250, determine the function F(t) that satisfies the differential equation given.

$F(t) = 250e^{-0.07t}$

(v) When would you expect the population of fish to double? Explain.

negative implies decay ie will never happen.

(vi) Using the above two models of population, when would you expect the number of fish and frogs to be the same?

$80e^{0.02t} = 250e^{-0.07t}$

$t = 12.6$ days

12

(8) (2, 2, 2, 4 = 10 marks)

A wheel is mounted on a wall and rotates such that the distance , *d* cm, of a particular point *P* on the wheel from the ground is given by the rule

$d = 80 - 50\cos\left(\frac{4\pi}{3}t\right)$, where t is the time in seconds.

(i) How far is the point *P* above the ground when t = 0

$d = 80 - 50\cos 0$ ✓

$= 30m$ ✓

(ii) How long does it take for the wheel to rotate once?

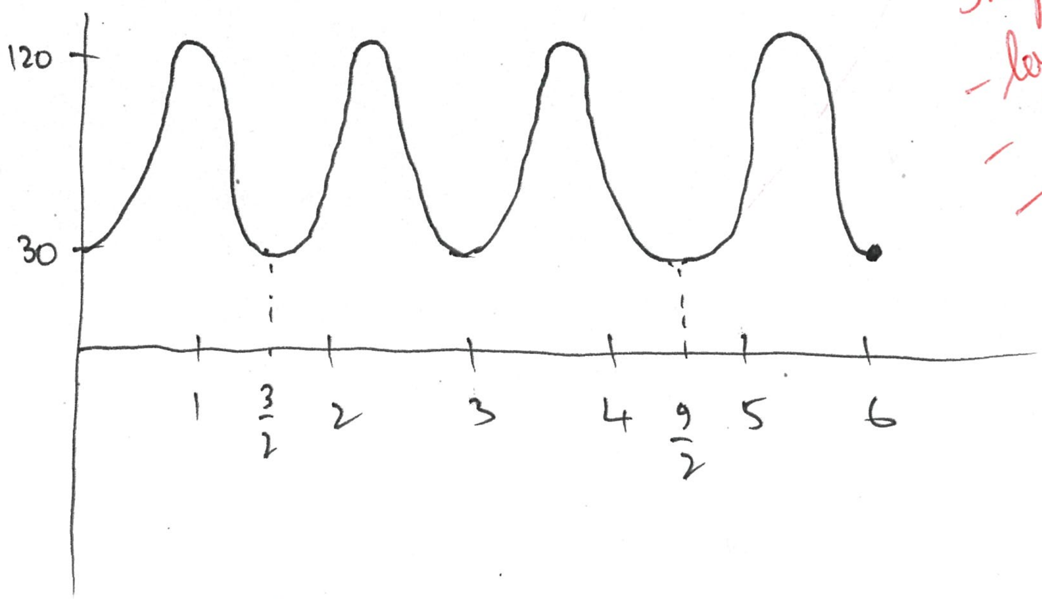
Period $\Rightarrow \frac{2\pi}{\frac{4\pi}{3}}$ ✓ $\Rightarrow \frac{2\pi}{1} \times \frac{3}{4\pi} = \frac{3}{2}$ sec ✓

(iii) Find the minimum and maximum distances of the point *P* above the ground.

min $\Rightarrow \cos\frac{4\pi t}{3} = 1 \Rightarrow 30m$ ✓

max $\Rightarrow \cos\frac{4\pi}{3}t = -1 \Rightarrow 120m$ ✓

(iv) Sketch the graph of *d* against t. (0 ≤ t ≤ 6 seconds)



shape ✓
- lowest ✓
- highest ✓
- scale/placement ✓

10

(9) (2, 8, 5 = 15 marks)

Given the function $y = 9x^2 - x^3 - 15x + 11$

(i) State the intercepts for this function $(7.1, 0)$ $(0, 11)$

(ii) Use calculus to find all stationary point(s) and verify what type of point(s) they are.

$y' = 18x - 3x^2 - 15$

$= -3(x^2 - 6x + 5)$

stationary pt $y' = 0$ $(x^2 - 6x + 5) = 0$

$(x-5)(x-1) = 0$

$x = 5$ or 1

$y'' = 18 - 6x$

@ $x = 5 \Rightarrow -$ (max) @ $x = 1 \Rightarrow +$ (min)

$y'' = 0$ $18 - 6x = 0$

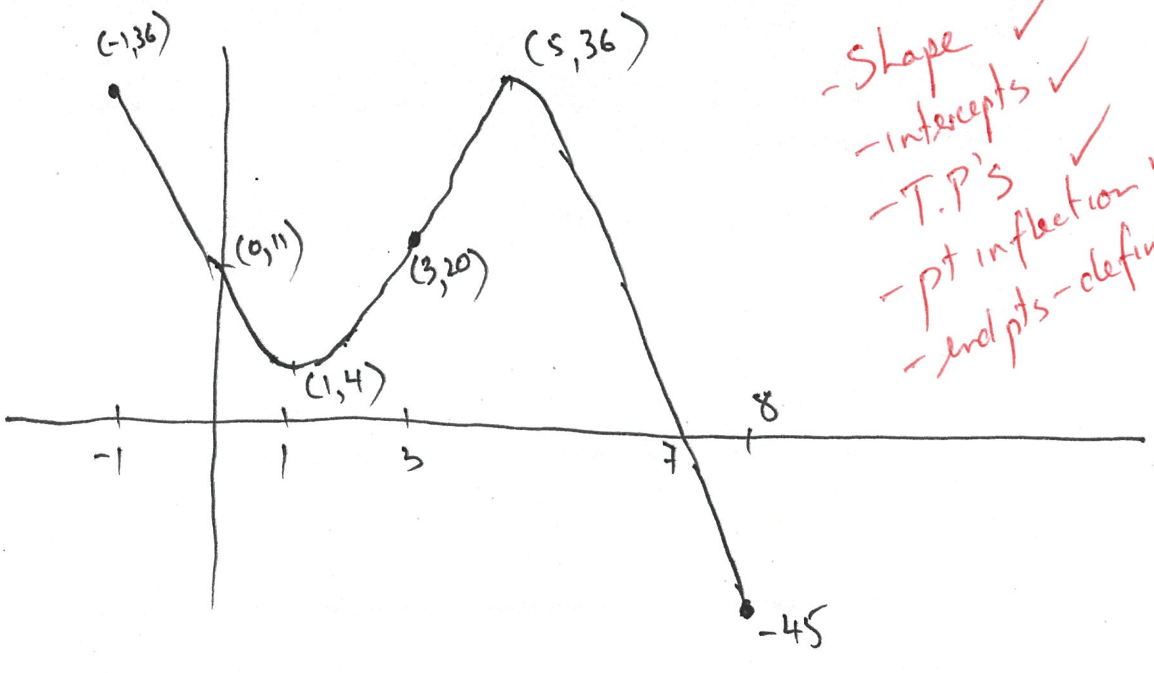
$x = 3 \rightarrow$ sign Test

$\left.\begin{array}{l} y' @ 2 \Rightarrow + \\ y' @ 4 \Rightarrow + \end{array}\right)$ pt inflection.

max pt $(5, 36)$ inflection $(3, 20)$

min pt $(1, 4)$

(iii) Sketch the graph of the function ($-1 \le x \le 8$), indicating and labelling all important points



- Shape
- intercepts
- T.P's
- pt inflection
- end pts - defined

15

(10) (4 marks)

For each week a banana grower delays her delivery of bananas to the market, her profit from the selling of the bananas is given by

$$P = (160 + 80x)(3 - 0.5x)$$

Where P is the profit in $ after the delay of x weeks

Use calculus techniques to find how many weeks the banana grower should delay in order to maximize profits and find this maximum profit.

$P' = 80(3-0.5x) + (160+80x)(-0.5)$

$\Rightarrow 240 - 40x + 80 - 40x$

$\Rightarrow 160 - 80x$ ✓

$P' = 0 \quad 160 - 80x = 0$

$x = 2$ ✓

$P'' \Rightarrow -80$

$\Rightarrow$ max ✓

Profit $\Rightarrow (160+160)2$

$\Rightarrow \$640$ ✓

$\frac{}{4}$