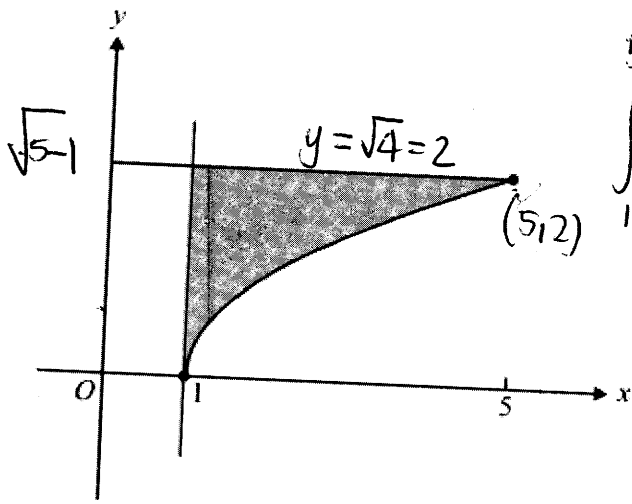
ANSWERS

Question 16

The graph of $f: [1, 5] \to R, f(x) = \sqrt{x-1}$ is shown below.



$$\int_1^5 \left(2 - \sqrt{x-1}\right) dx$$

Which one of the following definite integrals could be used to find the area of the shaded region?

A. $\int_1^5 \left(\sqrt{x-1}\right) dx$

B. $\int_0^2 \left(\sqrt{x-1}\right) dx$

C. $\int_0^5 \left(2-\sqrt{x-1}\right) dx$

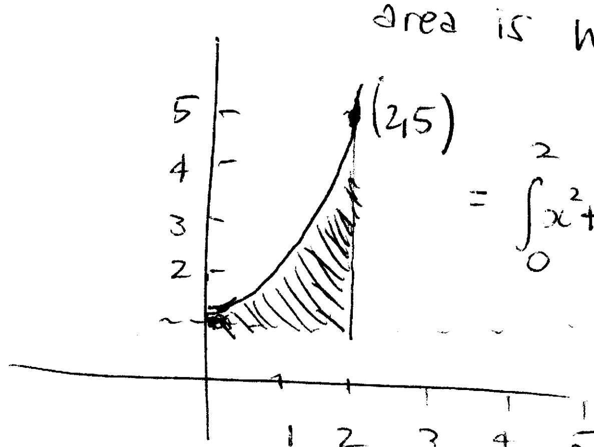
D. $\int_0^2 \left(x^2+1\right) dx$

E. $\int_0^2 \left(x^2\right) dx$

$y = \sqrt{x-1}$

$x = \sqrt{y-1}$

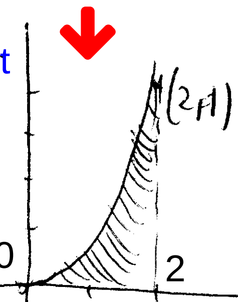
$y = x^2+1$ where the shaded area is what we require



$= \int_0^2 x^2+1\, dx - \int_0^2 1\, dx$

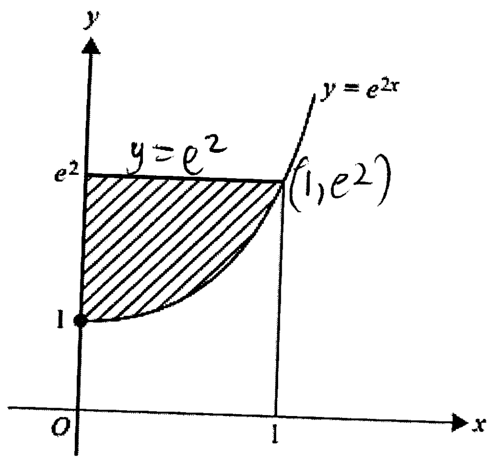
$= \int_0^2 x^2 dx$

Shift this area 1 unit down



$= \int_0^2 x^2 dx$

Question 14



$$\int_0^1 (e^2 - e^{2x})\,dx$$

$$= \left[e^2 x\right]_0^1 - \int_0^1 e^{2x}\,dx$$

$$= e^2 - \int_0^1 e^{2x}\,dx$$

(ii) is correct

To find the area of the shaded region in the diagram shown, four different students proposed the following calculations.

~~i.~~ $\int_0^1 e^{2x}\,dx$

ii. $e^2 - \int_0^1 e^{2x}\,dx$

~~iii.~~ $\int_1^{e^2} e^{2y}\,dy$

iv. $\int_1^{e^2} \frac{\log_e(x)}{2}\,dx$

$y = e^{2x}$

$x = e^{2y}$

$y = \frac{1}{2}\log_e x$

Which of the following is correct?

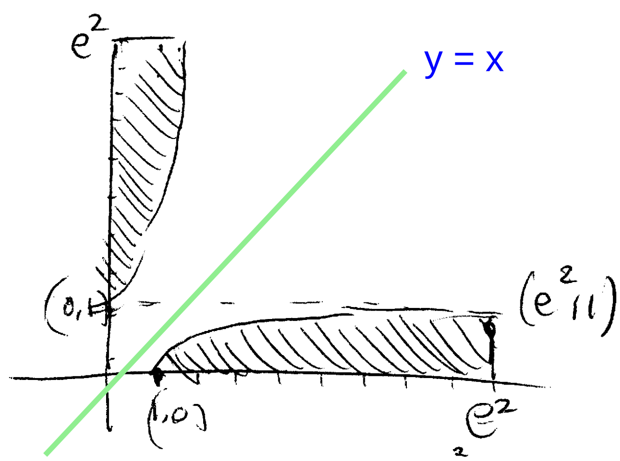
A. ii. only

B. ii. and iii. only

C. i., ii., iii. and iv.

D. ii. and iv. only

E. i. and iv. only



Required area $= \int_1^{e^2} \frac{1}{2}\log_e x\,dx$

So (iv) is correct

b. Let $f: R\backslash\{-2\} \to R,\ f(x) = \dfrac{2x+1}{x+2}$.

$$x+2 \overline{)\,2x+1\,}\quad 2$$
$$\underline{-\,2x+4}$$
$$-3$$

i. Find the rule and domain of f^{-1}, the inverse function of f. **2 marks**

$$f(x) = 2 - \frac{3}{x+2}$$

$$x = 2 - \frac{3}{y+2}$$

$$\Rightarrow x - 2 = \frac{-3}{y+2}$$

$$\therefore y + 2 = \frac{-3}{x-2}$$

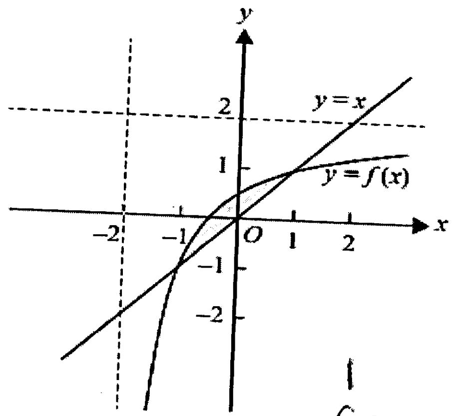
$$f^{-1}(x) = \frac{-3}{x-2} - 2$$

where $x \in R\backslash\{2\}$

Domain/Range Table

$dom(f)$	$ran(f)$
$R\backslash\{-2\}$	$R\backslash\{2\}$
$dom(f^{-1})$	$ran(f^{-1})$
$R\backslash\{2\}$	$R\backslash\{-2\}$

ii. Part of the graphs of f and $y = x$ are shown in the diagram below.

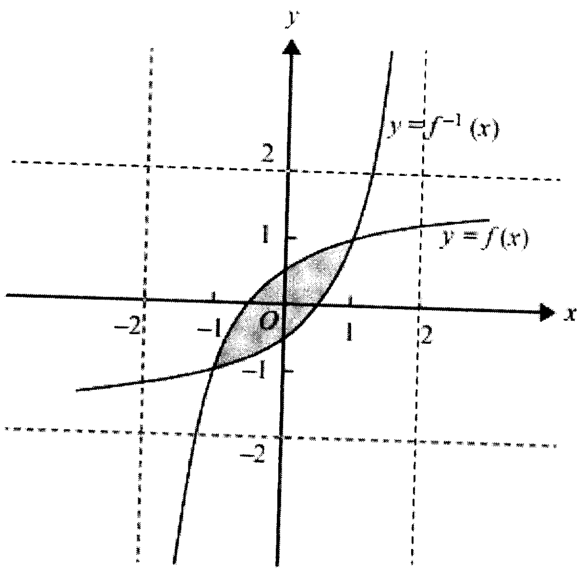


Find the area of the shaded region. **1 mark**

$$\int_{-1}^{1} \frac{2x+1}{x+2} - x \, dx$$

$$= 4 - 3\log_e 3 \quad \text{(sq units)}$$

iii. Part of the graphs of f and f^{-1} are shown in the diagram below.



Find the area of the shaded region.

By symmetry $= 2 \times (4 - 3\log_e 3)$

$= 8 - 6\log_e 3$

$= 8 - 6\log_e 3$

Question 15

If $\int_0^5 g(x)dx = 20$ and $\int_0^5 (2g(x)+ax)dx = 90$, then the value of a is

A. 0
B. 4
C. 2
D. −3
E. 1

$$\int_0^5 (2g(x)+ax)dx$$
$$= 2\int_0^5 g(x)dx + \int_0^5 ax\,dx$$
$$= 2\times 20 + \left[\frac{ax^2}{2}\right]_0^5$$
$$= 40 + \left(\frac{25a}{2} - 0\right)$$
$$\therefore 40 + \frac{25a}{2} = 90$$
$$\therefore \frac{25a}{2} = 50$$
$$\therefore a = 4$$

Question 4

If $\int_1^3 f(x)dx = 5$, then $\int_1^3 (2f(x)-3)dx$ is equal to

A. 4
B. 5
C. 7
D. 10
E. 16

$$\int_1^3 (2f(x)-3)dx$$
$$= 2\int_1^3 f(x)dx - \int_1^3 3\,dx$$
$$= 2\times 5 - [3x]_1^3$$
$$= 10 - (9-3) = 10-6 = 4$$

Question 10

If $\int_1^3 g(x)dx = 2$ then $2\int_3^1 (g(x)+1)dx$ equals

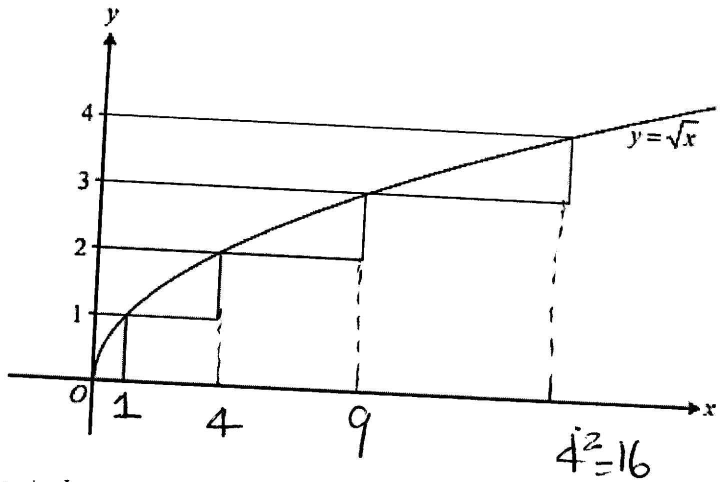
A. 6
B. 5
C. −8
D. −6
E. −4

$$2\int_3^1 (g(x)+1)dx$$
$$= -2\int_1^3 (g(x)\cancel{dx}+1)dx$$
$$= -2\int_1^3 g(x)dx - 2\int_1^3 1\,dx$$

$= -2 \times 2 - 2 \times [x]_1^3$
$= -4 - 2 \times (3 - 1)$
$= -8$

Question 14

The graph of $f: R^{+} \cup \{0\} \to R$, $f(x) = \sqrt{x}$ is shown below.
In order to find an approximation to the area of the region bounded by the graph of f, the y-axis and the line $y = 4$, Zoe draws four rectangles, as shown, and calculates their total area.



Zoe's approximation to the area of the region is

A. 14

B. 21

C. 29

D. 30

E. $\frac{64}{3}$

Area of rectangles =

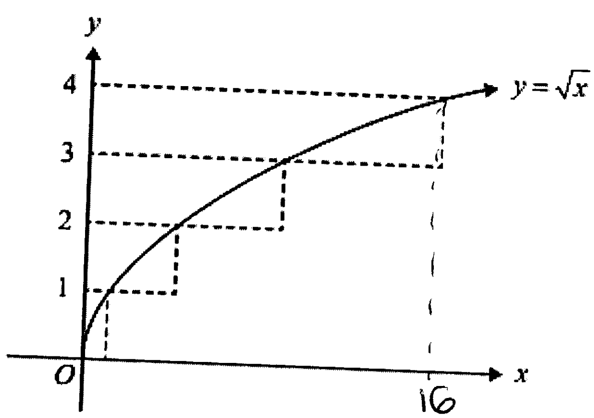
$16 \times 1 + 9 \times 1 + 4 \times 1 + 1 \times 1$

$= 16 + 9 + 4 + 1$

$= 30$

Question 19

Jake and Anita are calculating the area between the graph of $y = \sqrt{x}$ and the y-axis between $y = 0$ and $y = 4$. Jake uses a partitioning, shown in the diagram below, while Anita uses a definite integral to find the exact area.



Jake gets 30 as above

Anita:

$\int_0^{16} 4 - \sqrt{x}\, dx$

$= \left[4x - \frac{3}{2} x^{3/2} \right]_0^{16}$

$= 4 \times 16 - \frac{2}{3} \times 16^{3/2}$

$= 64 - \frac{2}{3} \times 64 = \frac{64}{3}$

The difference between the results obtained by Jake and Anita is

A. 0

B. $\frac{22}{3}$

C. $\frac{26}{3}$

D. 14

E. 35

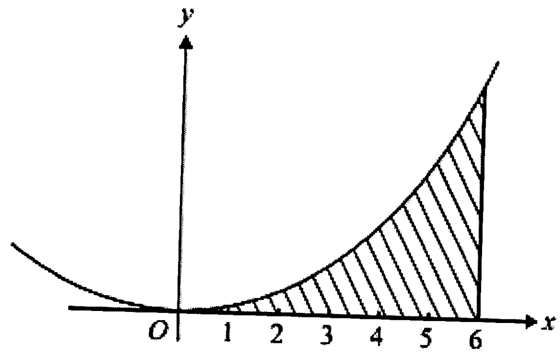
Difference

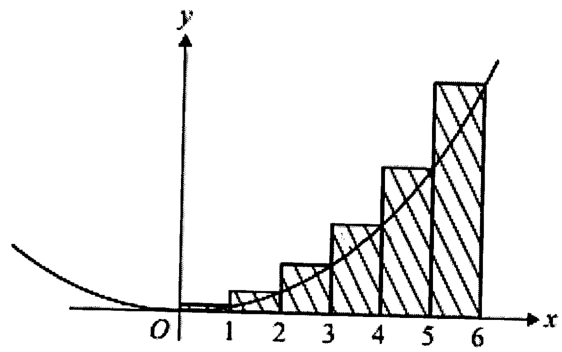
$= 30 - \frac{64}{3}$

$= \frac{90}{3} - \frac{64}{3} = \frac{26}{3}$

Question 19

A part of the graph of $f: R \rightarrow R, f(x) = x^2$ is shown below. Zoe finds the approximate area of the shaded region by drawing rectangles as shown in the second diagram.





Zoe's approximation is p% more than the exact value of the area.
The value of p is closest to

A. 10
B. 15
C. 20
D. 25
E. 30

Sum of rectangles

$= 1 \times f(1) + 1 \times f(2) + 1 \times f(3) + 1 \times f(4) + 1 \times f(5) + 1 \times f(6)$

$= 1 + 4 + 9 + 16 + 25 + 36$

$= 91$

Exact area $= \int_0^6 x^2 dx = \left[\frac{x^3}{3}\right]_0^6$

$= \frac{216}{3} = 72$

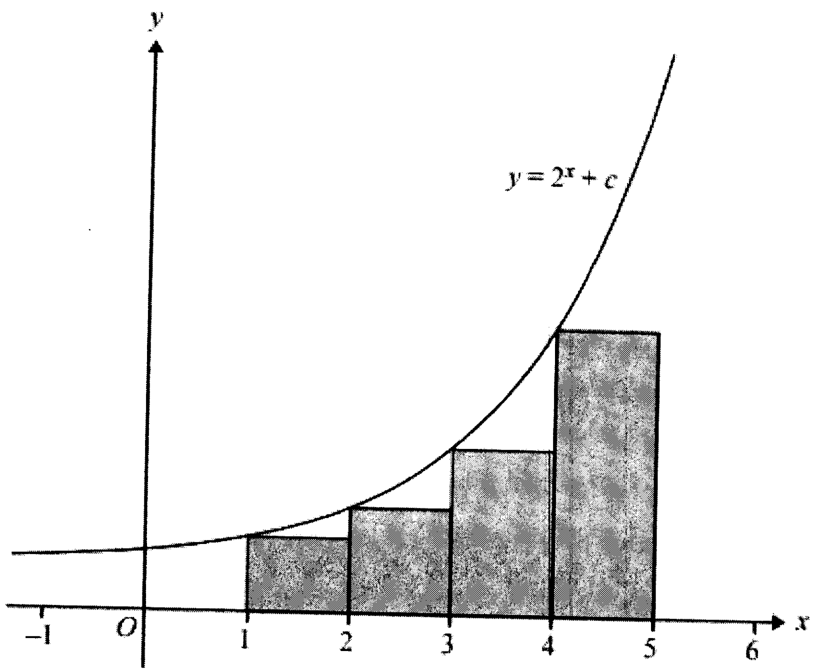
$\therefore$ Required percentage

$= \frac{91-72}{72} \times 100\%$

$\simeq 26\%$

Question 14

Consider the graph of $y = 2^x + c$, where c is a real number. The area of the shaded rectangles is used to find an approximation to the area of the region that is bounded by the graph, the x-axis and the lines $x = 1$ and $x = 5$.



If the total area of the shaded rectangles is 44, then the value of c is

A. 14

B. –4

C. $\frac{14}{5}$

D. $\frac{7}{2}$

E. $-\frac{16}{5}$

Shaded area

$= (2^1 + c) \times 1 + (2^2 + c) \times 1$

$+ (2^3 + c) \times 1 + (2^4 + c) \times 1$

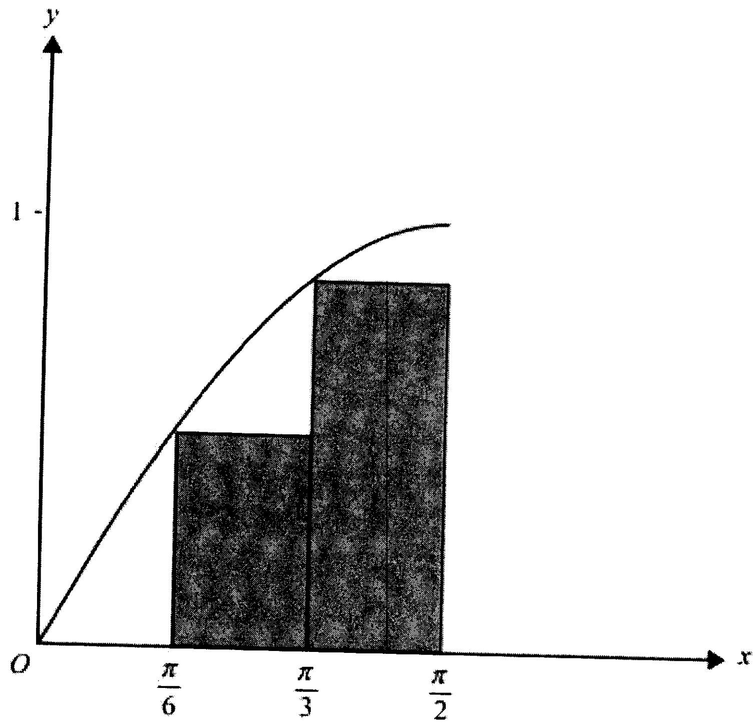
$= (2 + c) + (4 + c) + (8 + c) + (16 + c)$

$= 30 + 4c$

$\therefore 30 + 4c = 44$

$\therefore 4c = 14$ so c = 7/2

Question 1



The area under the curve $y = \sin(x)$ between $x = 0$ and $x = \frac{\pi}{2}$ is approximated by two rectangles as shown. This approximation to the area is

A. 1

B. $\frac{\pi}{2}$

C. $\frac{(\sqrt{3}+1)\pi}{12}$

D. 0.5

E. $\frac{(\sqrt{3}+1)\pi}{6}$

$$f\left(\frac{\pi}{6}\right) \times \left(\frac{\pi}{3} - \frac{\pi}{6}\right) + f\left(\frac{\pi}{3}\right) \times \left(\frac{\pi}{2} - \frac{\pi}{3}\right)$$

$$= \sin\left(\frac{\pi}{6}\right) \times \frac{\pi}{6} + \frac{\pi}{6} \times \sin\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{2} \times \frac{\pi}{6} + \frac{\pi}{6} \times \frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{12} + \frac{\pi\sqrt{3}}{12}$$

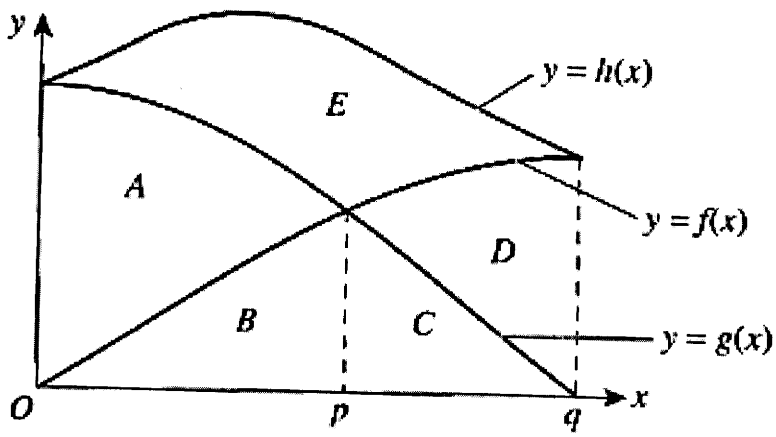
$$= \frac{\pi}{12}\left(1 + \sqrt{3}\right)$$

Question 3

The diagram below shows the graphs of three functions f, g and h on the domain $[0, q]$ where

$$f(x) = 2\sin(2x)$$
$$g(x) = 3\cos(2x)$$
$$h(x) = f(x) + g(x)$$

where q is the smallest positive integer for which $g(x) = 0$.



p is the x-coordinate of the point of intersection of $y = f(x)$ and $y = g(x)$.

A, B, C, D and E represent the areas of the respective regions as shown on the graph.

a. Show that $q = \frac{\pi}{4}$.

1 mark

b. Write an appropriate equation to show that, correct to four decimal places, $p = 0.4914$.

2 marks

c. Find the approximate area of region $(A + B + C + D + E)$ using the left endpoint rule with four equal intervals, correct to three decimal places.

2 marks

d. **i.** Write definite integrals whose value is the area of region A.

ii. Calculate the area of region A correct to three decimal places.

2 + 1 = 3 marks

e. Calculate the area of region E correct to three decimal places.

3 marks

Total 11 marks

Q3

(a) At $x = q$, $g(x) = 0$

$\therefore 3\cos(2x) = 0$

$\therefore 2x = \frac{\pi}{2}$

$\therefore x = \frac{\pi}{4}$ $\therefore q = \frac{\pi}{4}$

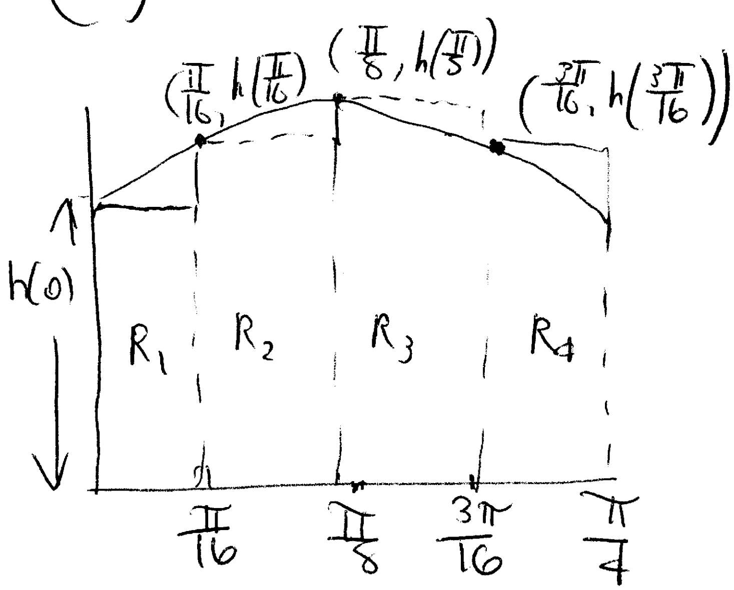
(b) At $x = p$, $f(x) = g(x)$

$\therefore 2\sin(2x) = 3\cos(2x), \quad 0 < x < \frac{\pi}{4}$

$\therefore \tan(2x) = \frac{3}{2}, \quad 0 < x < \frac{\pi}{4}$

Solving: $x \simeq 0.4914$ (0.49139686)

(c)



Left endpoint estimate means that all rectangles have the left top corner on the curve.

$R_1 + R_2 + R_3 + R_4$

$= \frac{\pi}{16} \times h(0) + \frac{\pi}{16} \times h\left(\frac{\pi}{16}\right) + \frac{\pi}{16} \times h\left(\frac{\pi}{8}\right) + \frac{\pi}{16} \times h\left(\frac{3\pi}{16}\right)$

$$= \frac{\pi}{16}\left(h(0) + h\left(\frac{\pi}{16}\right) + h\left(\frac{\pi}{8}\right) + h\left(\frac{3\pi}{16}\right)\right)$$

≈ 1.447 sq. units

(d) A: $\int_0^p g(x) - f(x)\, dx$

A : $\int_0^{0.49139686} (3\cos(2x) - 2\sin(2x))\, dx$

≈ 0.803 sq. units

(e)

E : $\int_0^p (h(x) - g(x))\, dx + \int_p^q h(x) - f(x)\, dx$

$= \int_0^p (f(x) + g(x) - g(x))\, dx + \int_p^q f(x) + g(x) - f(x)\, dx$

$= \int_0^p f(x)\, dx + \int_p^q g(x)\, dx$

$= \int_0^{0.49139686} 2\sin(2x)\, dx + \int_{0.49139686}^{\pi/4} 3\cos(2x)\, dx$

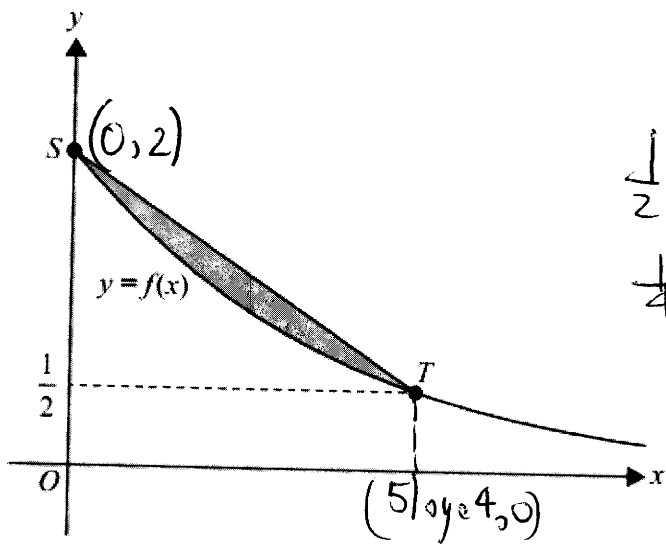
≈ 0.697 sq. units

NO CAS

Let $f: [0, \infty) \to R$, $f(x) = 2e^{-\frac{x}{5}}$.

Let S be the point on the graph of f on the y-axis and let T be the point on the graph of f with the y-coordinate $\frac{1}{2}$.

Find the area of the region bounded by the graph of f and the line segment ST. 3 marks



$$\frac{1}{2} = 2e^{-\frac{x}{5}}$$

$$\frac{1}{4} = e^{-\frac{x}{5}}$$

$$\therefore 4 = e^{\frac{x}{5}}$$

$$\frac{x}{5} = \log_e 4$$

$$x = 5\log_e 4$$

$$\text{Area} = A_{\text{trapezium}} - \int_0^{5\log_e 4} 2e^{-x/5}\,dx$$

$$= \frac{25}{4}\log_e 4 - \left[-10e^{-x/5}\right]_0^{5\log_e 4}$$

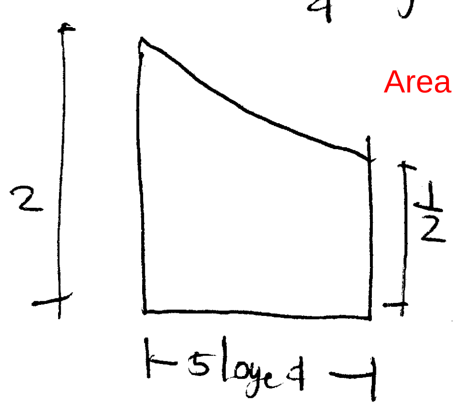
$$= \frac{25}{4}\log_e 4 - \left(-10e^{-\log_e 4} + 10\right)$$

$$= \frac{25}{4}\log_e 4 + 10e^{-\log_e 4} - 10 = \frac{25}{4}\log_e 4 + \frac{10}{4} - 10$$

$$= \boxed{\frac{25}{4}\log_e 4 - \frac{15}{2}}$$

Final answer

Note: $e^{\log_e P} = P$



Area of trapezium:

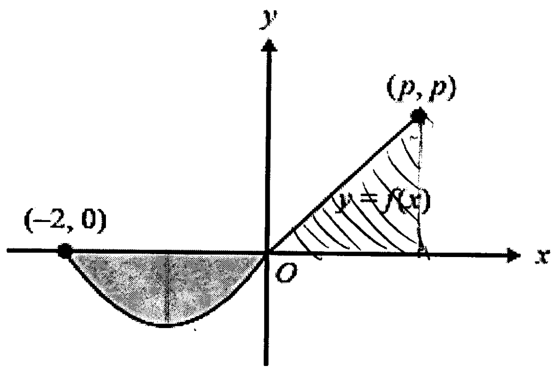
$$A = \frac{1}{2}\left(2 + \frac{1}{2}\right) \times 5\log_e 4$$

$$= \frac{1}{2} \times \frac{5}{2} \times 5\log_e 4$$

$$= \frac{25}{4}\log_e 4$$

Question 8

The graph of a function $f: [-2, p] \to R$ is shown below.



The average value of f over the interval $[-2, p]$ is zero.

The area of the shaded region is $\frac{25}{8}$.

If the graph is a straight line, for $0 \le x \le p$, then the value of p is

A. 2

B. 5

C. $\frac{5}{4}$

D. $\frac{5}{2}$

E. $\frac{25}{4}$

If average value is zero, then the area shaded = area triangle

$$\therefore \frac{25}{8} = \frac{p \times p}{2}$$

$$\frac{25}{8} = \frac{p^2}{2}$$

$$p^2 = \frac{25}{4}$$

$$\therefore p = \frac{5}{2}$$

Question 18

The area of the region bounded by the x-axis, the y-axis and the curve $y = e^{2x} - 2e^x - 3$ is

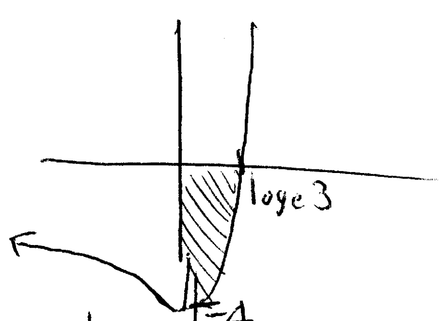
A. $4\log_e(3)$

B. 3

C. $3\log_e(3)$

D. $4 - \log_e(3)$

E. $4 - 3\log_e(3)$



Let $e^x = p$

$p^2 - 2p - 3 = 0$

$(p-3)(p+1) = 0$

$p = 3, -1$

$\therefore e^x = 3 \quad (e^x > 0)$

$\therefore x = \log_e 3$

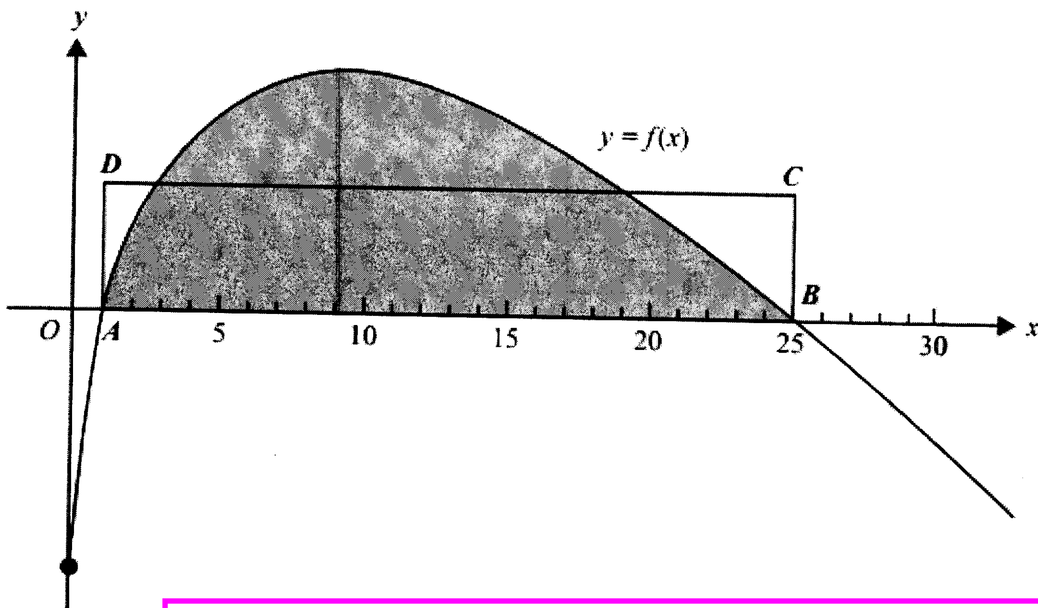
$$\text{Area} = -\int_0^{\log_e 3} (e^{2x} - 2e^x - 3)\,dx = 3\log_e 3$$

Question 1

Let $f: R^{+} \cup \{0\} \rightarrow R$, $f(x) = 6\sqrt{x} - x - 5$.

c. Points A and B are the points of intersection of $y = f(x)$ with the x-axis. Point A has coordinates $(1, 0)$ and point B has coordinates $(25, 0)$.

Find the length of AD such that the area of rectangle $ABCD$ is equal to the area of the shaded region.



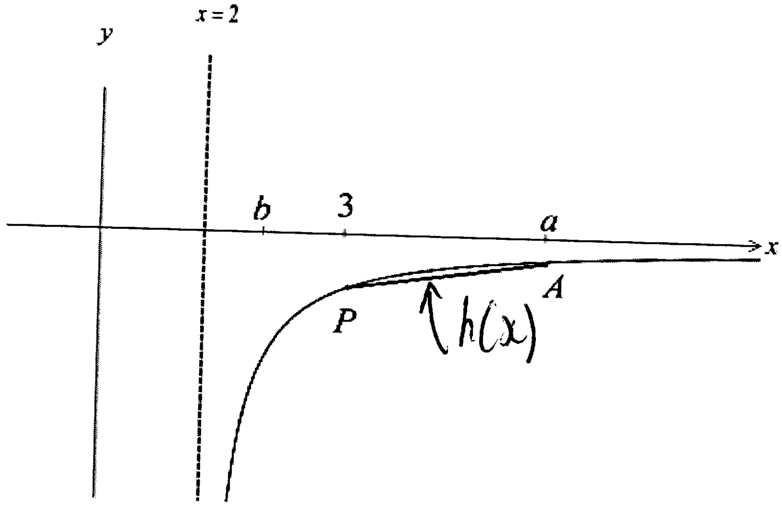
AD = Average value of function over interval [1, 25]

$$= \frac{1}{25-1} \int_{1}^{25} \left(6\sqrt{x} - x - 5\right) dx$$

$$= \frac{64}{24} = \frac{8}{3}$$

Question 3 (12 marks)

The diagram below shows part of the graph of the function $f:(2,\ \infty)\to R,\ f(x)=\frac{6}{2-x}$.



b. **i.** Find $-\int_{3}^{e+2}\frac{6}{2-x}\,dx$.

6

1 mark

ii. Find the value of b for $b\in(2,3)$ such that $-\int_{b}^{3}\frac{6}{2-x}\,dx=6$.

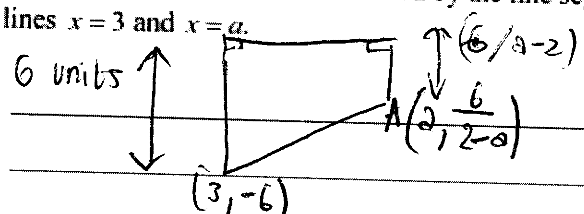
2 marks

$-\int_{b}^{3}\frac{6}{2-x}\,dx = 6$

$\therefore\ b = 2+\frac{1}{e}$ (since $b\in(2,3)$)

c. **i.** Find, in terms of a, the area bounded by the line segment PA, the x-axis and the lines $x=3$ and $x=a$.

2 marks



$a>3$

$\text{Area} = \frac{1}{2}\left(6+\frac{6}{a-2}\right)\times(a-3) = (a-3)\left(3+\frac{3}{a-2}\right)$

ii. For what value of a does this area equal 6?

2 marks

$(a-3)\left(3+\frac{3}{a-2}\right)=6$

$\therefore\ a=\sqrt{2}+3$ (since $a>3$)

Explanation for iii. is on the bottom of the next page.

Question 22

The function g is a smooth, continuous function for $x \in R$ and $g(x) \geq 0$ for $x \in [0, k]$.

Given that $\int_0^k g(x)dx = 2k$, then $4\int_0^{3k}\left(g\left(\frac{x}{3}\right)-1\right)dx$ is equal to

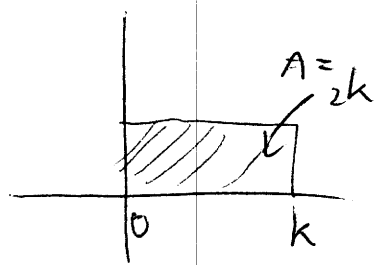
A. 0
B. $12k$
C. $20k$
D. $12k - 4$
E. $24k - 4$

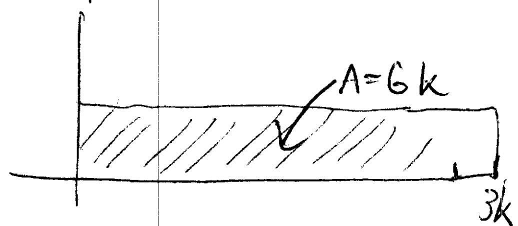
$$4\int_0^{3k}\left(g\left(\frac{x}{3}\right)-1\right)dx$$

$$=4\int_0^{3k} g\left(\frac{x}{3}\right)dx - \int_0^{3k} 4\,dx$$

$$= 4 \times 6k - [4x]_0^{3k}$$

$$= 24k - (12k - 0) = 12k$$





Question 20

Let f be a differentiable function defined for all real x, where $f(x) \geq 0$ for all $x \in [0, a]$.

If $\int_0^a f(x)dx = a$, then $2\int_0^{5a}\left(f\left(\frac{x}{5}\right)+3\right)dx$ is equal to

A. $2a + 6$
B. $10a + 6$
C. $20a$
D. $40a$
E. $50a$

$$2\int_0^{5a}\left(f\left(\frac{x}{5}\right)+3\right)dx$$

$$= 2\int_0^{5a} f\left(\frac{x}{5}\right)dx + \int_0^{5a} 6\,dx$$

$$= 2 \times 5a + [6x]_0^{5a}$$

$$= 10a + 30a - 0 = 40a$$

c.iii (from previous page)

We know:

$\int_3^{e+2} \frac{6}{x-2}\,dx = 6$ and $\int_3^{\sqrt{2}+3} -h(x)\,dx = 6$

But the graph of $y = -h(x)$ always lies above the curve $y = \frac{6}{x-2}$ so in order for the two integrals above to both equal 6, it means that the interval of integration of the first one must be longer than for the second. Therefore: $e + 2 > \sqrt{2} + 3$, so $e > \sqrt{2} + 1$

Question 1

Two ships, the Elsa and the Violet, have collided. Fuel immediately starts leaking from the Elsa into the sea. The captain of the Elsa estimates that at the time of the collision his ship has 6075 litres of fuel on board and he also forecasts that it will leak into the sea at a rate of $\frac{t^2}{5}$ litres per minute, where t is the number of minutes that have elapsed since the collision.

a. At this rate how long, in minutes, will it take for all the fuel from the Elsa to leak into the sea?

Let V = amount of fuel left in ship

$\frac{dV}{dt} = \frac{-t^2}{5}$ (Since the volume of fuel in the boat is decreasing dv/dt is negative)

$\therefore V = \int \frac{-t^2}{5}\,dt$

$V = \frac{-t^3}{15} + c$

When $t=0$, $V=6075$

$\therefore 6075 = c$

$\therefore V = \frac{-t^3}{15} + 6075$

When $V=0$

$6075 - \frac{t^3}{15}$

$t^3 = 15 \times 6075$

3 marks

$t^3 = 91{,}125$

$t = \sqrt[3]{91{,}125} = 45$ minutes

NO CAS

Question 9 (3 marks)

In a certain town the **rate of deaths** due to a particular disease has been found to closely fit the mathematical model $\frac{dN}{dt} = \frac{10\,000}{(t+3)^3}$, where t is the time in months after the disease was first detected and N is the number of deaths.

Calculate the total number of deaths predicted by the model, giving your answer to the nearest whole number of deaths.

$\frac{dN}{dt} = 10000(t+3)^{-3}$

$N = \int 10000(t+3)^{-3}\,dt$

$= \frac{10000 (t+3)^{-2}}{-2} + c$

When $t=0$, $N=0$, $c = \frac{10000}{2} \times \frac{1}{9}$

$N = \frac{10000}{2 \times 9} - \frac{10000}{2(t+3)^2}$

As $t \to \infty$, $N \to \frac{10000}{18} \simeq 556$ deaths