BALDIVIS SECONDARY COLLEGE
YEAR 11 SPECIALIST MATHEMATICS 2020
INVESTIGATION 1
VALIDATION

INSTRUCTIONS:

Calculator allowed

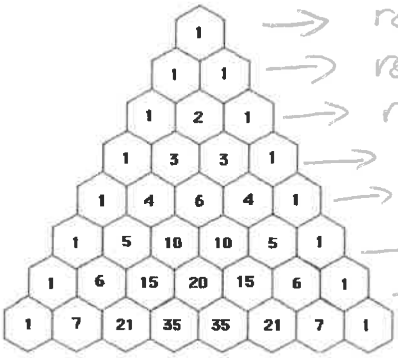
Notes not allowed

Take-home section NOT allowed

Full working must be shown for all questions (or parts) worth more than 2 marks.

Marks will be deducted for rounding and unit errors.

Name: Solutions **Time: 40 minutes Total______ / 25**



→ row 0
→ row 1
→ row 2
→ row 3
→ row 4
→ row 5
→ row 6
→ row 7

Question 1 **(5 marks)**

(a) Expand $(x+1)^5$ ← row 5 gives coefficients (1)

$(x+1)(x+1)(x+1)(x+1)(x+1)$

$= 1x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$ ✓

(b) Comment on how the coefficients of x^k in $(x+1)^n$, $0 \le k \le n$ are related to the numbers in Pascal's triangle (2)

$x^k, x^{k-1}, x^{k-2}, x^{k-3} \ldots x$

In Pascal's triangle the coefficient of x^k in $(x+1)^n$ are the values in the n^{th} row

(c) Find, leaving your answer in the form ${}^nC_r$, the coefficient of x^{16} when $(x+1)^{100}$ is expanded r is not stipulated so (2)

${}^nC_k$ $\binom{100}{16}$ or $\binom{100}{84}$ is ok

$(x+1)^{100}$ ← 99th row

$k=16$ $(x+1)^{100}$

Question 2 **(12 marks)**

Pick any number inside Pascal's Triangle (i.e. excluding the 1s). Let this number be the centre of a flower. You will notice there are six numbers surrounding the centre and these form the petals of a flower. For example, if the 2 in line three is taken as the centre of the flower, then the petals are formed by 1, 1, 1, 1, 3 and 3, the numbers surrounding the centre.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

If the number 15 in the 7th row is considered as the centre of the flower then the petals are 5, 10, 20, 35, 21 and 6.

It has been formulated that the product of one set of alternating petals is equal to the product of the other set of alternating petals. Starting with any petal, the product of the first, third and fifth "petals" is equal to the product of the second, fourth and sixth "petals". For example $5\times20\times21 = 10\times35\times6$

$2100 = 2100$

(a) Choose a centre and let it be equal to ${}^nC_r$. (You cannot choose 1 as the centre.)

Determine the expression for all six petals in terms of n and r in the form ${}^aC_b$. (4)

${}^{n-1}C_{r-1}$ 5 10 ${}^{n-1}C_r$

${}^nC_{r-1}$ 6 ${}^nC_r$ 20 ${}^nC_{r+1}$

${}^{n+1}C_r$ 21 35 ${}^{n+1}C_{r+1}$

✓/ row numbers n

✓✓ term numbers r

(b) Prove that the product of three alternate petals is equal to the product of the other three alternate petals. That is, ${}^{n-1}C_{r-1} \times {}^nC_{r+1} \times {}^{n+1}C_r = {}^{n-1}C_r \times {}^{n+1}C_{r+1} \times {}^nC_{r-1}$ In any petal (8)

In above Petal
$5\times21\times20 = 10\times35\times6$
$2100 = 2100$

$$\frac{(n-1)!}{(n-1-(r-1))!\,(r-1)!} \times \frac{n!}{(n-r+1)!\,(r+1)!} \times \frac{(n+1)!}{(n+1-r)!\,r!}$$ ✓LHS

✓✓

$$= \frac{(n-1)!}{(n-r)!\,(r-1)!} \times \frac{n!}{(n-r-1)!\,(r+1)!} \times \frac{n!}{(n-(r-1))!\,(r-1)!}$$

✓✓

$$= \frac{(n-1)!}{(n-r-1)!\,r!} \times \frac{(n+1)!}{(n+1-(r+1))!\,(r+1)!} \times \frac{n!}{(n-(r-1))!\,(r-1)!}$$

$= \text{RHS} = {}^{n-1}C_r \times {}^{n+1}C_{r+1} \times {}^nC_{r-1}$ ✓

Question 3 **(8 marks)**

The triangular numbers are 1, 3, 6, 10, 15…..because the units can be arranged to form triangles as in the diagram below

```
                      o
            o        o o
      o    o o     o o o
o    o o  o o o   o o o o

1     3     6       10
```

Locate the triangular numbers on Pascal's Triangle.

$^{0}C_0$ $\quad$ $^{1}C_0$ $\quad$ $^{1}C_1$

```
              1
            1   1
          1   2   1
        1   3   3   1
      1   4   6   4   1
    1   5  10  10   5   1
  1   6  15  20  15   6   1
1   7  21  35  35  21   7   1
```

$T_1 = {}^{2}C_0$ $\quad$ $n+1$ $\quad$ $n-1$

$T_2 = {}^{3}C_1$

$T_3 = {}^{4}C_2$

$T_4 = {}^{5}C_3$

$n+1-(n-1)$

$n+1-n+1$

$= 2$

$^{2}C_1$

(a) (i) Using $n = 1$ for the second row; the one containing 1 1, determine an expression for T_n, i.e. the nth triangular number, in terms of $^{n}C_k$. (2)

$\frac{n!}{r!(n-r)!}$

$1!$

$T_n = {}^{n+1}C_{n-1}$ ✓ $= \frac{(n+1)!}{2!\,(n-1)!} = \frac{n(n+1)}{2}$ ✓

if $n=1$ $\quad {}^{1+1}C_{1-1} = {}^{2}C_0$ $\qquad {}^{n+1}C_2$ $\quad n=3$

(ii) Test your rule by obtaining the value for T_6, the 6[th] triangular number, 21 (1)

$^{n+1}C_K$ $\quad n=6$ $\quad K=2$

$T_6 = {}^{7}C_5 = \frac{(6+1)!}{2!\,(6-1)!} = \frac{7!}{2!\,5!} = \frac{(7)(6)}{2!} = \frac{42}{2} = 21$

$n+1 \quad n-1$

$= \frac{n(n+1)}{2}$

(b) Prove that the sum of any two consecutive triangular numbers is equal to a square number (5)

$T_n = {}^{n+1}C_{n-1} \qquad T_K,\ T_{K+1}$

$T_K + T_{K+1} = {}^{K+1}C_{K-1} + {}^{K+2}C_K$ ✓

$= \frac{(K+1)! \times K}{2!\,(K-1)! \times K} + \frac{(K+2)!}{2!\,K!}$ ✓

$= \frac{K(K+1)! + (K+2)!}{2 \times K!}$ simplify ✓ $\frac{(K+1)!\,(K+2)}{}$ $\quad K! \times (K+1)$ ✓ $= \frac{(K+1)!\,(K+K+2)}{2 \times K!}$ ✓ factorise

$= \frac{(K+1)(2K+2)}{2} = \frac{2(K+1)(K+1)}{2} = (K+1)^2$

$(K+1)^2$ is a square number ✓

The following diagrams may be useful:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

${}^{0}C_{0}$

${}^{1}C_{0}$ ${}^{1}C_{1}$

${}^{2}C_{0}$ ${}^{2}C_{1}$ ${}^{2}C_{2}$

${}^{3}C_{0}$ ${}^{3}C_{1}$ ${}^{3}C_{2}$ ${}^{3}C_{3}$

${}^{4}C_{0}$ ${}^{4}C_{1}$ ${}^{4}C_{2}$ ${}^{4}C_{3}$ ${}^{4}C_{4}$

${}^{5}C_{0}$ ${}^{5}C_{1}$ ${}^{5}C_{2}$ ${}^{5}C_{3}$ ${}^{5}C_{4}$ ${}^{5}C_{5}$

${}^{6}C_{0}$ ${}^{6}C_{1}$ ${}^{6}C_{2}$ ${}^{6}C_{3}$ ${}^{6}C_{4}$ ${}^{6}C_{5}$ ${}^{6}C_{6}$

${}^{7}C_{0}$ ${}^{7}C_{1}$ ${}^{7}C_{2}$ ${}^{7}C_{3}$ ${}^{7}C_{4}$ ${}^{7}C_{5}$ ${}^{7}C_{6}$ ${}^{7}C_{7}$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1