Name: SOLUTIONS Date: ___________

Year 11 Methods

Test 6, 2015

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=____ %

Topics – Properties of derivatives, Applications of Differentiation

Total Time:	*60 minutes*
Total Reading:	*5 minutes*
Total Working:	*55 minutes*
Weighting:	*4.2% of the year.*

BONUS MARKS (look at the whole assessment):

☐ *Notation*: appropriate (+1). ☐ 1st (+1/2) ☐ 2nd (0)

☐ *Units*: appropriate (+1). ☐ 1st (+1/2) ☐ 2nd (0)

*This test comprises of **TWO sections**. The **first section** is **calculator free** where no calculators of any kind are to be used. The **second section** is **calculator assumed** where a CAS calculator may be used. All questions must be answered in both sections. **Answers should be rounded appropriately**. All working should be shown in the space provided. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks.*

No pens, pencils, highlights etc. may be used during reading time. This time is to be used to read through the assessment and check that you understand what is being asked of you. You may speak with the teacher/supervisor during this time (by putting up your hand and waiting patiently for them to approach you) but you may only ask clarification questions and not how to solve the problems. After reading time has ended, you may not ask any more questions.

SECTION 1: CALCULATOR FREE

Time:	*25 minutes*	**Marks for Section 1:**	*23*
Reading:	*2 minutes*	**Equipment Allowed:**	*Nil*
Working:	*23 minutes*		

1. [4 marks: 2, 2]
Differentiate the following with respect to x.

a) $f(x) = -4x^3(2x+3)$

$f(x) = -8x^4 - 12x^3$

$f'(x) = -32x^3 - 36x^2$

b) $f(x) = \sqrt{x} + \dfrac{3}{x}$

$f(x) = x^{\frac{1}{2}} + 3x^{-1}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} + 3\times(-1)x^{-2}$

$f'(x) = \dfrac{1}{2\sqrt{x}} - \dfrac{3}{x^2}$

2. [6 marks: 4, 1, 1]

If $f(x) = x^3 + 3x^2 - 4$:

a) Show, using the first principle, that a derivative of a given function is $f'(x) = 3x^2 + 6x$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 + 3(x+h)^2 - 4 - (x^3 + 3x^2 - 4)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 + \cancel{3x^2} + 6xh + 3h^2 \cancel{-4} \cancel{-x^3} \cancel{-3x^2} \cancel{+4}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 + 6x + 3h)}{\cancel{h}} = 3x^2 + 6x$$

b) Calculate $-5f'(x)$.

$$-15x^2 - 30x$$

c) Evaluate $f'(3)$.

$$f'(3) = 3 \times 3^2 + 6 \times 3 = 27 + 18 = 45$$

3.

[5 marks: 2, 3]

a) Find the antiderivative of:

$$\int (3x^3 - 4x)\, dx = \frac{3x^4}{4} - \frac{4x^2}{2} + C = \frac{3x^4}{4} - 2x^2 + C$$

b) Find the integral of the function $f(x) = 3x^2 + 2x - 5$ where $f(2) = 5$.

$$F(x) = \int (3x^2 + 2x - 5)\, dx = x^3 + x^2 - 5x + C$$

For $F(2) = 5$

$$5 = 2^3 + 2^2 - 5 \times 2 + C$$

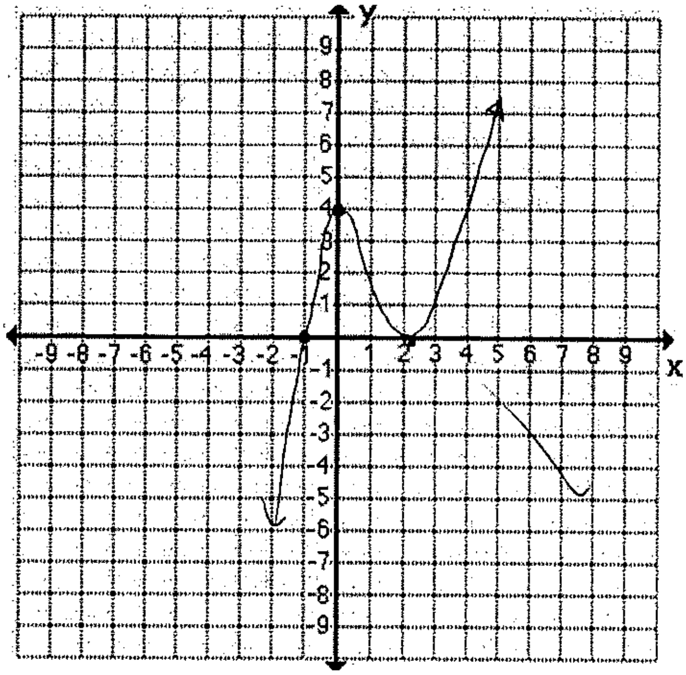
$$5 = 8 + 4 - 10 + C$$

$$5 = 2 + C$$

$$C = 3 \implies F(x) = x^3 + x^2 - 5x + 3$$

4. [8 marks]

a) Sketch the graph of function $y = (x-3)^2(x+1)$.
Note: remember to include zeros, sing of function, stationary points, increasing and decreasing intervals, y-intercept and behavior as $x \to \pm\infty$.



ZEROS: $x = 2$
$x = -1$

STATIONARY POINTS:
$y' = 3x^2 - 6x = 0$
$3x(x-2) = 0$
$x = 0 \quad x = 2$

x	-1	0	1	2	3
y'	9	0	-3	0	9
SIGN	+	0	-	0	+

$x = 0$ is MAX $\quad$ $x = 2$ is MIN
$y = 4$ $\quad$ $y = 0$

INCREASING & DECREASING:
y is INCREASING BEFORE $x = 0$, DECREASING BETWEEN $x = 0$ AND $x = 2$ AND INCREASING AGAIN AFTER $x = 2$

y- INTERCEPT: FOR $x = 0$ $y = 4$

BEHAVIOUR:
AS $x \to \infty$ $y \to \infty$
AS $x \to -\infty$ $y \to -\infty$

~ **END OF TEST SECTION 1** ~

Name: ______________________________ Date: __________

SECTION 2: CALCULATOR ASSUMED

Time:	*35 minutes*	**Marks for Section 2:**	*30*
Reading:	*3 minutes*	**Equipment Allowed:**	*½ page notes (A4 one side), CAS calculator*
Working:	*32 minutes*		

5. **[8 marks: 4, 4]**

a) Find the equation of the tangent to the function $y = 2x^3 + x^2 + 5x - 1$ at $x = 3$.

$m = y' = 6x^2 + 2x + 5$

FOR $x = 3$ $\quad y' = 6 \times 3^2 + 2 \times 3 + 5$

$= 54 + 6 + 5$

$= 65$

FOR $x = 3$

$y = 2 \times 3^3 + 3^2 + 5 \times 3 - 1$

$y = 54 + 9 + 15 - 1$

$= 77$

$\Rightarrow y = 65x + c$

FOR $x = 3$, $y = 77$ $\quad 77 = 65 \times 3 + c$

$77 = 195 + c$

$c = 118$

$\Rightarrow y = 65x + 118$

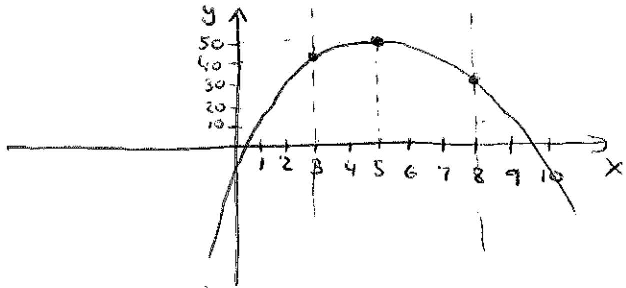
b) Find the greatest and least values of the function $y = 20x - 2x^2$ over the domain $3 < x < 8$.

$y' = 20 - 4x$

$20 - 4x = 0$

$x = 5$

$y = 20 \times 5 - 2 \times 5^2 = 50$



WHEN $x < 5$ $\quad f'(x) > 0$

WHEN $x > 5$ $\quad f'(x) < 0$

$\Rightarrow$ FUNCTION HAS A LOCAL MAXIMUM AT $(5, 50)$

FOR $3 < x < 8$ $\quad f(3) = 20 \times 3 - 2 \times 3^2 = 60 - 18 = 42$

$f(8) = 20 \times 8 - 2 \times 8^2 = 160 - 128 = 32$

BOUNDRY VALUES ARE $(3, 42)$ AND $(8, 32)$

$\Rightarrow$ GREATEST VALUE IS 50 AND SMALLEST VALUE IS 32.

6. **[3 marks]**

Find coordinates of a point for which the gradient of the tangent to the curve $y = 3x^2 - 5x + 2$ is equal to 19.

$y' = 6x - 5$

$6x - 5 = 19$

$6x = 24$

$x = 4$ $\frac{1}{2}$

$y = 3 \times 4^2 - 5 \times 4 + 2$ $(4, 30)$

$= 48 - 20 + 2$

$= 30$ $\frac{1}{2}$

7. 5 **[~~6~~ marks: 1, 2, 1, 1]**

On a calm day a small stone is dropped into a lake causing a circular wave to radiate outwards. The radius of this circle r (in cms) is given by the equation r = 35t, where t is the time in seconds after the stone has broken the smooth surface of the water.

a) Write an equation for the area of the circle in terms of t.

$A = r^2\pi$ $r = 35t$

$A = 1225t^2\pi$

b) Calculate the exact area of this circle after 2 seconds giving your answer in square meters.

$A = 1225 \times 2^2\pi$

$= 4900\pi \text{ cm}^2$

$= 0.49\pi \text{ m}^2$

c) In terms of t, what is the rate at which the area of this circle is increasing?

$A = 2250t\pi$

d) What is the instantaneous rate of increase of the area of this circle when t = 2?

$A = 4500\pi$

8. **[9 marks: 3, 3, 3]**

If $y = 4x^2 - 6x + 5$

a) Find the acute angle that the tangent at x = 1 makes with the x – axis.

$m = \tan\theta$

$m = y' = 8x - 6$

For $x = 1$ $\quad m = 8 \times 1 - 6 = 2$

$\tan\theta = 2$

$\theta = \tan^{-1} 2$

$\theta = 63.4^\circ$

b) Find the equation of the tangent at x = 1.

$y = 2x + c$ $\qquad$ For $x = 1$ $\quad y = 4 \times 1^2 - 6 \times 1 + 5$

$y = 3$

For $x = 1$ $\quad y = 3$

$3 = 2 \times 1 + c$

$c = 1$ $\qquad \Rightarrow y = 2x + 1$

b) Find the length of the tangent between x = 1 and x = 5.

$x = 1 \quad y = 3$ $\qquad (1, 3)$

$x = 5 \quad y = 4 \times 5^2 - 6 \times 5 + 5$ $\qquad (5, 75)$

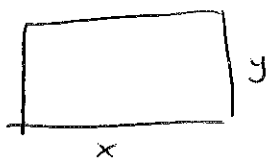
$= 75$

$d = \sqrt{(5-1)^2 + (75-3)^2}$

$d = \sqrt{5200} = 72.11$

8. **[~~6~~ 5 marks]**

The perimeter of a rectangle is 200m. Find what length and width will give the maximum area.



$2x + 2y = 200$

$x + y = 100$

$y = 100 - x$

$A = xy$

$= x(100 - x)$

$= 100x - x^2$

$A' = 100 - 2x = 0$

$x = 50$

$\Rightarrow y = 50$

~ END OF TEST SECTION 2 ~