Full Name: SOLUTIONS

Ellenbrook
CHRISTIAN COLLEGE

# MATHEMATICS METHODS
# YEAR 11

## Investigation – Conjectures, Reasoning and Proof[1]

**Semester 1 2016**

**Time allowed:** 60 minutes

**Materials required:** Writing implements, correction fluid/tape or eraser, ruler, CAS calculator

**Structure:**

| Task | Marks available | Marks awarded |
|---|---|---|
| 1 | 6 | |
| 2 | 10 | |
| 3 | 6 | |
| 4 | 8 | |
| 5 | 10 | |
| Total | 40 | |

**Instructions:**

1. Write your answers in the spaces provided in this Question/Answer Booklet.
2. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

[1] Some content taken from *Reasoning with Algebraic Contexts*, Mathematics Teaching in the Middle School, April 2009.

## Introduction

In this investigation you will,

- identify and correct mistakes;
- decide whether statements are always true, sometimes true or always false;
- find counterexamples;
- investigate conjectures; and
- make and prove a conjecture.

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You may wish to refer to the information on this page as you work through the tasks.

- An **integer** is a positive or negative whole number. Zero is also an integer.
- The set of **whole numbers** is {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...}
- The set of **multiples of 2** is $\{n : n = 2k, \text{ where } k \text{ is a whole number}\}$
- The set of **multiples of 3** is $\{n : n = 3k, \text{ where } k \text{ is a whole number}\}$
- The set of **multiples of 4** is $\{n : n = 4k, \text{ where } k \text{ is a whole number}\}$
- For any whole number $k$,
    - $2k$ is an **even number**, whereas $2k+1$ and $2k-1$ are **odd numbers**
    - $k,\ k+1,\ k+2, \ldots$ are **consecutive whole numbers**
    - $2k,\ 2k+2,\ 2k+4, \ldots$ are **consecutive even numbers**
    - $2k+1,\ 2k+3,\ 2k+5, \ldots$ are **consecutive odd numbers**
- Any two digit whole number can be written in expanded form as $10a+b$.
    - e.g. $38 = 10\times3+8$
- Any three digit whole number can be written in expanded form as $100a+10b+c$.
    - e.g. $529 = 100\times5+10\times2+9$

**Task 1: Identify and correct mistakes** **[6 marks]**

Each of the following problems contains a mistake. Identify the error **and** correct the solution.

1. Jennifer was asked to simplify $\frac{5^2 - 15\times 4}{5\times 7}$.

   She calculated her answer without using a calculator as $\frac{25-60}{35} = \frac{-35}{35} = -1$.

   When she checked her work on a calculator, she typed 5^2 – 15*4/5*7 and got -59.

   Describe the error Jennifer made when using the calculator.

   She divided 4 by 5.

   She should have typed

   (5^2-15*4)/(5*7)

2. Julian was asked to simplify $\frac{16y^{12}}{8y^4}$.

   Describe Julian's error in his answer of $\frac{16y^{12}}{8y^4} = 2y^3$ and correct the answer.

   Julian divided the indices $(12 \div 4 = 3)$

   He should have subtracted them.

   $$\frac{16y^{12}}{8y^4} = 2y^8$$

3. Carl was asked to simplify $5(2x+6)-4(3x-1)$. Find the error that Carl made in his work and correct the solution.

$$5(2x+6)-4(3x-1) = 10x+30-12x-4$$ error
$$= -2x+26$$

He should have written

$5(2x+6)-4(3x-1)$

$= 10x+30-12x+4$

$= -2x+34$

**Task 2: Always true, sometimes true, or always false** **[10 marks]**

Decide whether the following statements are ***always true, sometimes true*** or ***always false***. For each statement, circle the relevant information in italics and **JUSTIFY** your decision.

1. Tomorrow is Monday.

*always true* ***sometimes true*** *always false*

Tomorrow will be Monday if today is Sunday.

2. The product of two primes is never a prime.

***always true*** *sometimes true* *always false*

By definition a prime number only has factors of one and itself. If multiplying two primes they are both therefore factors of the new number.

3. If you add one to an odd number, you obtain an odd number.

*always true* *sometimes true* ***always false***

Adding one to an odd number produces an even number (eg: $(2k+1)+1 = 2k+2$ (even)).

4. The sum of the interior angles of a quadrilateral is 270°.

*always true* *sometimes true* ***always false***

Sum of interior angles is always 360°.

5. For any real number $x$, $x^2 > 0$. ***always true*** *sometimes true* *always false*

The square of any number produces a positive number (negative times negative is positive)

**Task 3: Find counterexamples** **[6 marks]**

Sometimes you want to prove that a statement is *not* true. A statement that proves a statement is not true is called a *counterexample*.

For example, if Anna says that every number whose last digit is 3 is divisible by 3, then to prove her wrong you need to find just one example that makes her statement false. Since 13 is a number whose last digit is 3 but 13 is not divisible by 3, then Anna's statement is not true.

For each statement, find a counterexample that shows that the statement is false.

1. Statement: Drinks are brown in colour.

   *Counterexample*:

   Orange juice is orange.

2. Statement: Every month has 30 days.

   *Counterexample*:

   ~~Decem~~ February has either 28 or 29 days.

Sometimes mathematics statements are false. For example, 6 + 14 = 24 is a false statement.

If a mathematics statement contains a variable, it can be true for some values of the variable and false for others. For example, $13 - y > 10$ is true if $y$ < 3, but is false if $y \geq 3$.

For each statement, choose a value for $x$ such that the statement is false. Justify your choice.

3. Statement: $9(x-3) = 9x - 3$

   False when $x = 0$, because: $9(0-3) \neq 9(0) - 3$

   $-27 \neq -3$

* any example would suffice

4. Statement: $12 - 2x = 2x - 12$

False when $x = 1$, because: $12 - 2(1) \neq 2(1) - 12$

$10 \neq -10$

* any value that works.

5. Statement: $(x+3)^2 = x^2 + 9$

False when $x = 2$, because: $(2+3)^2 \neq 2^2 + 9$

$25 \neq 13$

* any value that works.

6. Statement: $(3x)^2 = 3x^2$

False when $x = 1$, because: $(3 \times 1)^2 \neq 3(1)^2$

$9 \neq 3$

* any value that works

**Task 4: Investigate conjectures** **[8 marks]**

A *conjecture* is a mathematical statement which appears likely to be true, but has not been formally proven to be true under the rules of mathematical logic.

To test the truth of a conjecture, start by checking several examples; if the examples are true, then you need to find a **general argument** to justify the conjecture for all possible examples. If you find one example that gives a false statement (a counterexample), then the conjecture is not true.

1. Conjecture: *The sum of any three consecutive even numbers is always a multiple of 6.*

   Test this conjecture using sets of three consecutive even numbers,

   e.g. $2 + 4 + 6 =$ 12 $2 \times 6$

   $4 + 6 + 8 =$ 18 $3 \times 6$

   $6 + 8 + 10 =$ 24 $4 \times 6$

   $8 + 10 + 12 =$ 30 $5 \times 6$

   **If** all of these tests support the conjecture, then use the three consecutive even numbers $2k$, $2k+2$, and $2k+4$, where $k$ is a whole number, to prove that the conjecture is ***always true***.

$2k + 2k+2 + 2k+4 = 6k+6$

6k is a multiple of 6 adding 6 remains so

OR

$= 6(k+1)$

6 times any number results in a multiple of 6

2. Conjecture: *If you multiply any integer by 6, the result will end in 2, 4, 6, or 8.*

Test this conjecture by multiplying several integers by 6. Try different integers, including positive and negative integers. Does the result always end in 2, 4, 6, or 8?

$6 \times 1 = 6$ 6

$6 \times 2 = 12$ 2

$6 \times 3 = 18$ 8

$6 \times 4 = 24$ 4

$6 \times 5 = 30$ 0 $\Leftarrow$ false.

**If** all of your tests support the conjecture, then prove that the conjecture is ***always true***.

Conjecture is false.

3. Conjecture: *Two less than the sum of the squares of three consecutive whole numbers is always a multiple of 3.*

Test this conjecture using sets of three consecutive whole numbers,

e.g. $1^2 + 2^2 + 3^2 - 2 =$ $1+4+9-2 = 12$ $4 \times 3$

$2^2+3^2+4^2-2 = 4+9+16-2 = 27$ $9 \times 3$

$3^2+4^2+5^2-2 = 9+16+25-2 = 48$ $16 \times 3$

$4^2+5^2+6^2-2 = 16+25+36-2 = 75$ $25 \times 3$

**If** all of your tests support the conjecture, then prove that the conjecture is ***always true***.

$$k^2 + (k+1)^2 + (k+2)^2 - 2$$

$$= k^2 + k^2 + 2k + 1 + k^2 + 4k + 4 - 2$$

$$= 3k^2 + 6k + 3$$

$$= 3(k^2 + 2k + 1)$$

3 times any number is a multiple of 3

∴ always true

**Task 5: Make and prove a conjecture** **[10 marks]**

In our base 10 number system, we use the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 to form numbers.

Follow the example below using the number 305.

(a) Choose any three digit number which contains no repeated digit.

**305**

(b) Write down all the two digit numbers that can be formed using the digits from the chosen three digit number. Do not include numbers with repeated digits.

**30, 35, 03, 05, 53, 50**

(c) Find the sum of the digits of the three digit number.

$$3 + 0 + 5 = 8$$

(d) Find the sum of all of the two digit numbers listed in (b).

$$30 + 35 + 03 + 05 + 53 + 50 = 176$$

(e) Divide your answer in (d) by your answer in (c).

$$176 \div 8 = 22$$

Repeat steps (a) to (e) **twice** using two different three digit numbers of your choice.

(a) My three digit number is: 102

(b) 10 12 20 21 01 02

(c) 3 (1+0+2)

(d) 66

(e) $\frac{66}{3} = 22$

(a) My three digit number is: 403

(b) 03 04 30 34 40 43

(c) 7

(d) 154

(e) 22

Use your results from step (e) to make a conjecture.

I conjecture that this process will always produce a result of 22.

Now prove your conjecture.

a) 3 digits $k$, $k+1$, $k+2$

b) $k, k+1$ $\quad$ $k+1, k$ $\quad$ $k+2, k$

$k, k+2$ $\quad$ $k+1, k+2$ $\quad$ $k+2, k+1$

c) $3k+3$

d) $k+k+k+1+k+1+k+2+k+2 = 6k+6$ ⇐ tens digit.

$6k+6$ ⇐ ones digit. (exactly the same)

$\frac{6k+6}{3k+3} = 2$ (tens) $\quad \frac{6k+6}{3k+2} = 2$ (ones)

$\therefore 22$

true

**End of Investigation**