2022 Calc free (25 mins)

Question 1 **(9 marks)**

Fully simplify each of the following, leaving your answers with a positive index.

a) $1000^{-\frac{4}{3}}$ (2 marks)

$= 10^{-4} = \frac{1}{10^4}$

b) $\sqrt[3]{x^6\,y^{\frac{3}{2}}} \div x\,y^{\frac{1}{2}}$ (2 marks)

$= x^2 y^{1/2} \div xy^{1/2}$

$= x$

c) $\frac{12^x - 6^x}{2^{2x} - 2^x}$ (3 marks)

$= \frac{3^x(2^2)^x - 3^x(2^x)}{(2^2)^x - (2^x)}$ ✓

$= \frac{3^x(2^{2x} - 2^x)}{(2^{2x} - 2^x)}$ ✓

$= 3^x$ ✓

d) $\left(\frac{1}{\sqrt{x-1}} + \frac{\sqrt{x-1}}{1}\right)^2$ (2 marks)

$= \left(\frac{1}{\sqrt{x-1}} + \frac{\sqrt{x-1}\,.\,\sqrt{x-1}}{1\,.\,\sqrt{x-1}}\right)^2$

$= \left(\frac{1}{\sqrt{x-1}} + \frac{x-1}{\sqrt{x-1}}\right)^2$ ✓

$= \left(\frac{x}{\sqrt{x-1}}\right)^2 = \frac{x^2}{x-1}$ ✓

Question 2 **(4 marks)**

Given that $\sqrt{25} = 5$ and $\sqrt{2.5} = 1.581$, determine each of the following to 2 significant figures.

a) $\sqrt{2.5 \times 10^9}$ (2 mark)

$= \sqrt{25 \times 10^8}$ ✓ $= 5 \times 10^4$

$= 5.0 \times 10^4$ ✓

b) $\sqrt{0.000\,002\,5}$ (2 mark)

$= \sqrt{2.5 \times 10^{-6}}$ ✓ $= 1.581 \times 10^{-3}$

$= 1.6 \times 10^{-3}$ ✓

Question 3 **(5 marks)**

Solve each of the following equations.

a) $4^{x-1} = 32^{-2}$ (2 marks)

$2^{2(x-1)} = 2^{5(-2)}$ ✓

$\therefore\ 2x - 2 = -10$

$2x = -8$

$x = -4$ ✓

b) $(18x)^{\frac{2}{3}} = 36$ (3 marks)

$(18x)^{1/3} = \pm 6$ ✓ $\quad 18x = \pm 6^3$ ✓ $\quad x = \pm\frac{6^3}{3 \times 6}$ $\quad x = \pm 12$ ✓

QUESTION 1 *[2, 2 = 4 marks]*

State the next three terms for each of the sequences below:

a. $T_n = -2T_{n-1}\ ,\quad T_1 = -2$

$T_2 = -2(-2) = 4$

$T_3 = -2(4) = -8$

$T_4 = -2(-8) = 16$

$4, -8, 16$ ✓✓

b. $T_{n+1} = T_n + 2n\ ,\quad T_1 = -5$

$T_2 = -5 + 2(1) = -3$

$T_3 = -3 + 2(2) = 1$

$T_4 = 1 + 2(3) = 7$

$-3, 1, 7$ ✓✓

QUESTION 2 *[2 marks]*

Write the recursive formula for the following sequence: $\frac{1}{2}, \frac{5}{4}, \frac{4}{2}, \frac{11}{4}, \ldots$

$T_{n+1} = T_n + \frac{3}{4}$ ✓ $\quad T_1 = \frac{1}{2}$ ✓

$\frac{5}{4} - \frac{1}{2} = \frac{3}{4}$

$\frac{4}{2} - \frac{5}{4} = \frac{3}{4}$

QUESTION 3 *[1, 3 = 4 marks]*

The general term of a sequence is given by $T_n = 4n + 8$. Calculate:

a. T_5

$T_5 = 4(5) + 8$

$= 28$ ✓

b. Which term of the sequence is the first to exceed 217?

$217 = 4n + 8$

$209 = 4n$

$52\frac{1}{4} = n \quad \therefore T_{53}$

QUESTION 4 *[4, 2 = 6 marks]*

For a geometric sequence; $\quad T_1 = x - 2, \quad T_2 = x + 1, \quad T_3 = x + 5$

Determine:

a. The first three terms.

$\frac{x+1}{x-2} = \frac{x+5}{x+1}$ ✓

$(x+1)^2 = (x+5)(x-2)$ ✓

$x = 11$ ✓

$\therefore T_1 = 11 - 2 = 9$

$T_2 = 11 + 1 = 12$

$T_3 = 11 + 5 = 16$ ✓

b. The general rule of the sequence.

$T_n = 9\left(\frac{4}{3}\right)^{n-1}$ ✓ ✓

$r = \frac{12}{9} = \frac{4}{3}$

Calc Assumed (25 mins)

Question 7 **(10 marks)**

The amount of water in a rainwater tank, W litres, was observed at the end of each day, over a ten-day period. Some of these observations are recorded in the following table.

Days since the start of the observations (d)	4	6	10
Amount of water in the rainwater tank (W)	244	381	931

a) Show that the exponential rate of increase is consistently about 1.25 litres/day. (2 marks)

$244 \times r^2 = 381$ ✓ $\qquad r = \sqrt{\frac{381}{244}} = 1.25$

$381 \times r^4 = 931 \qquad r = \sqrt[4]{\frac{931}{381}} = 1.25$ ✓

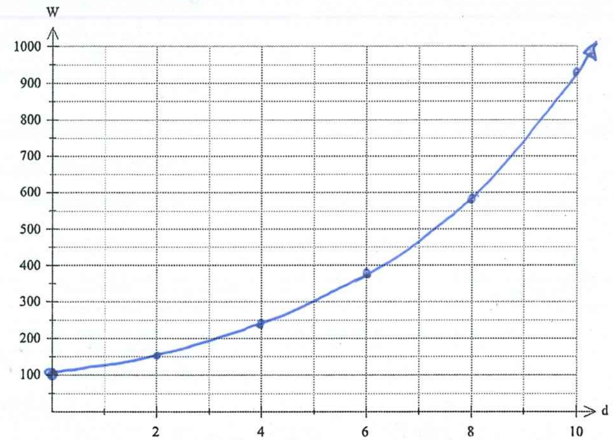
b) State the initial amount of water in the tank to the nearest litre. (2 marks)

$a \times r^4 = 244$ ✓ $\qquad a = \frac{244}{1.25^4} = 100\text{ L}$ ✓

c) State the amount of water in the tank at the end of the eighth day. (1 mark)

$W = 100 \times 1.25^8 = 596.046$ or 596 L.

d) On the axes below, draw a graph of the volume of water in the rainwater tank, W, given the number of days, d. (3 marks)



✓ ✓ ✓

It was later discovered that the observations were incorrectly recorded. The observations recorded on the fourth day was actually the first day's observations, the sixth day was actually the third day's observations, and the tenth day was actually the seventh day's observations.

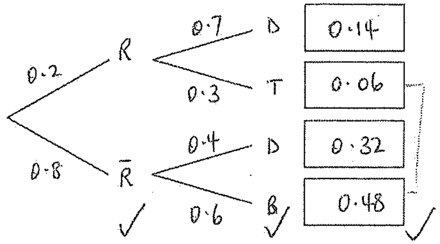
e) Determine the revised equation for W. (2 marks)

$W = 100 \times 1.25^{(d+3)}$ ✓ ✓

Question 8 (6 marks)

The probability that it will rain is 0.2. If it rains, Ryan will either drive (D) or catch the train (T) to work. If it rains, the probability that he catches the train is 0.3. If it does not rain, Ryan will either drive or ride his bicycle (B). The probability that Ryan rides his bicycle when it is fine is 0.6.

a) Complete the probability tree below. (3 marks)



b) What is the probability that Ryan drives to work? (1 mark)

$0.14 + 0.32 = 0.46$ ✓

c) What is the probability that it is a fine day when he did not drive to work? (2 marks)

$P(\bar{R} \mid \bar{D}) = \dfrac{0.48}{0.48+0.06} = \dfrac{48}{54}$ or $\dfrac{8}{9}$ ✓ ✓

0.89 ✓

Question 9 (7 marks)

For two events, X and Y, it is known that P(X) = 0.48 and P(Y) = 0.35. Determine the following probabilities, using any additional information only within that part of the question.

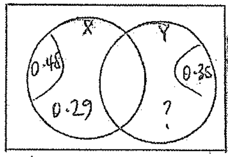
(a) $P(\bar{Y})$. 0.65 ✓ (1 mark)

(b) $P(Y \cap \bar{X})$ when $P(X \cap \bar{Y}) = 0.29$. (2 marks)

$P(Y \cap \bar{X}) = 0.35 - (0.48 - 0.29)$ ✓

$= 0.35 - 0.19$

$= 0.16$ ✓



OR

	X	$\bar{X}$	T
Y	0.19	?	0.35
$\bar{Y}$	0.29	0.36	0.65
T	0.48	0.52	1.0

OR $0.52 - (0.65 - 0.29)$

(c) $P(X|Y)$ when $P(Y|X) = 0.7$. (2 marks)

$P(X \cap Y) = P(X) \times P(Y|X)$

$= 0.48 \times 0.7$

$= 0.336$ ✓

$P(X|Y) = \dfrac{0.336}{0.35}$

$= 0.96$ ✓

(d) $P(X \cup Y)$ when X and Y are independent. (2 marks)

$P(X \cup Y) = P(X) + P(Y) - P(X) \times P(Y)$ ✓

$= 0.48 + 0.35 - \underbrace{0.48 \times 0.35}_{0.168}$

$= 0.662$ ✓

/13

Question 10 (5 marks)

The game of poker is played with a standard pack of 52 cards with four different suits as listed below.

Hearts	A♥, 2♥, 3♥, 4♥, 5♥, 6♥, 7♥, 8♥, 9♥, 10♥, J♥, Q♥, K♥
Diamonds	A♦, 2♦, 3♦, 4♦, 5♦, 6♦, 7♦, 8♦, 9♦, 10♦, J♦, Q♦, K♦
Spades	A♠, 2♠, 3♠, 4♠, 5♠, 6♠, 7♠, 8♠, 9♠, 10♠, J♠, Q♠, K♠
Clubs	A♣, 2♣, 3♣, 4♣, 5♣, 6♣, 7♣, 8♣, 9♣, 10♣, J♣, Q♣, K♣

Each player is dealt five cards at random.

a) In factorial form, state the number of sets of five cards that are possible. (1 mark)

$\binom{52}{5} = \dfrac{52!}{5!\,47!}$ ✓

b) State the probability that a set of five cards contain each of the following, leaving your answers in combination notation. No simplifying necessary. (4 marks)

(i) Four kings

$\dfrac{\binom{4}{4}\binom{48}{1}}{\binom{52}{5}}$ ✓

(ii) An ace, a king, a queen, a jack and a ten

$\dfrac{\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{32}{0}}{\binom{52}{5}}$ ✓

(iii) Five cards that are all the same suit

$\dfrac{\binom{13}{5}\binom{39}{0} + \binom{13}{5}\binom{39}{0} + \binom{13}{5}\binom{39}{0} + \binom{13}{5}\binom{39}{0}}{\binom{52}{5}}$ ✓

(iv) At least one heart.

$\dfrac{\binom{52}{5} - \binom{39}{5}}{\binom{52}{5}}$ $\quad 1 - \dfrac{\binom{13}{0}\binom{39}{5}}{\binom{52}{5}}$ ✓ $\quad \dfrac{\binom{13}{1}\binom{39}{4} + \binom{13}{2}\binom{39}{3} + \binom{13}{3}\binom{39}{2} + \binom{13}{4}\binom{39}{1} + \binom{13}{5}\binom{39}{0}}{\binom{52}{5}}$

$1 - \dfrac{\binom{39}{5}}{\binom{52}{5}}$

QUESTION 1 *[4 marks]*

The sum to infinity of a geometric sequence is equal to 25, while the first two terms of this sequence add up to 9. Find the value(s) of T_1 and r which satisfy these conditions.

$$\left.\begin{aligned} 25 &= \frac{a}{1-r} \\ a + ar &= 9 \end{aligned}\right\}$$

Solve simultaneous equations

$a = 5 \quad r = 0{\cdot}8$ ✓

$a = 45 \quad r = -0{\cdot}8$ ✓

End of paper