MATHEMATICS METHODS 4

SEMESTER 2 2016

INVESTIGATION 4

The exponential PDF

Eastern Goldfields College

MARKING KEY

Marks: 33 **Time: 40 minutes**

One continuous probability density function we have encountered is the exponential function. This function is defined as

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & for\ x \geq 0 \\ 0 & for\ x < 0 \end{cases}$$

The exponential function has an mean or expected value E[X] = $\frac{1}{\lambda}$

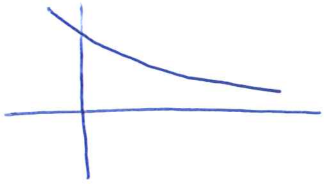
Suppose that the random variable X is amount of time a person spends waiting in a bank to be served and is exponentially distributed with a mean of 5 minutes. If E[X] = 5, then λ = 0.2. Thus if we are asked to determine the probability that a person will need to wait more than 10 minutes to be served in the bank the answer will be

$$P(X > 10) = \int_{10}^{\infty} 0.2e^{-0.2x} \quad dx = 0.1353$$

1. [5, 1, 2 marks]

Another way to determine P(X > 10) is to use a cumulative density function.

a) Show the cumulative density function when the mean time is 5 minutes will be P(X < t) = 1 – $e^{--0.2t}$ and verify that P(X > 10) = 0.1353.



$\int_0^t 0.2e^{-0.2x} dx$ ✓

$= \left[-e^{-0.2x}\right]_0^t$ ✓

$= -e^{-0.2t} - (-1)$

$= 1 - e^{-0.2t}$ ✓

$P(X>10) = 1 - P(X<10)$ ✓

$= 1 - (1 - e^{-0.2\times 10})$

$= 0.1353.$ ✓

b) Use the c.d.f. (or otherwise) to determine the probability that

i) A person waits in the queue for less than 8 minutes

$P(X<8) = 1 - e^{-0.2\times 8} = 0.7981.$ ✓

ii) Given a person already waits for at least 10 minutes, will wait for at least 15 minutes.

$P(X>15 | X>10) = \frac{P(X>15)}{P(X>10)} = \frac{0.0498}{0.1353}$ ✓

$= 0.3679.$ ✓

2. [2 marks]

The median time m will be such that $1 - e^{-0.2m} = 0.5$. Determine the median time, correct to 2 d.p.

$m = 3.47$ ✓✓

3. [4 marks]

Determine the mean and median times for the p.d.f.

$$f(x) = \begin{cases} 0.4e^{-0.4x} & for\ x \geq 0 \\ 0 & for\ x < 0 \end{cases}$$

$E(X) = \frac{1}{0.4} = \frac{10}{4} = 2.5$ ✓✓

$1 - e^{-0.4m} = 0.5$ ✓

$m = 1.73$ ✓

4. [5 marks]

It would appear that the mean time exceeds the median time in both examples. We can determine the median time by evaluating m in terms of λ.

If we start by saying that $1 - e^{-\lambda m} = 0.5$, find m in terms of λ. Explain why the mean will always exceed the median.

$0.5 = e^{-\lambda m}$

$\ln \frac{1}{2} = -\lambda m$ ✓

✓ $-\frac{1}{\lambda}(\ln 1 - \ln 2) = m$

$\frac{1}{\lambda} \ln 2 = m$

mean ✓✓ less than 1

$\therefore m < \frac{1}{\lambda}$ (mean) ✓

or similar

5. [1, 1, 1, 2, 2, 2, 3, 2 marks]

Another property of the exponential density function is that it is what we call the **memoryless** property. We can demonstrate this with the following example:

The mean waiting time in a bank queue is 10 minutes.

a) Write the c.d.f. for this distribution.

$$f(x) = \begin{cases} \frac{1}{10} e^{-\frac{1}{10}x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

cdf: $P(X < t) = 1 - e^{-0.1t}$ ✓

b) Determine the probability that a person waits at least 5 minutes.

$$P(X > 5) = 1 - (1 - e^{-\frac{1}{2}}) = e^{-\frac{1}{2}} = 0.6065. \checkmark$$

c) Determine the probability that a person waits at least 10 minutes.

$$P(X > 10) = e^{-1} = 0.3679 \checkmark$$

d) Given the person waits at least 10 minutes, determine the probability the person waits at least 5 more minutes. [i.e. P(X > 15|X > 10)]

$$\frac{P(X>15)}{P(X>10)} = \frac{e^{-1.5}}{e^{-1}} = e^{-0.5} = 0.6065.$$

✓ ✓

e) Given the person waits at least 15 minutes, determine the probability the person waits at least 5 more minutes.

$$\frac{P(X>20)}{P(X>15)} = \frac{e^{-2}}{e^{-1.5}} = e^{-0.5} = 0.6065.$$

✓ ✓

f) Given the person waits at least 20 minutes, determine the probability the person waits at least 5 more minutes.

$$\frac{P(X>25)}{P(X>15)} = \frac{e^{-2.5}}{e^{-2}} = e^{-0.5} = 0.6065.$$

✓ ✓

g) Comment on your answers, and explain why this is called the memoryless property.

All the answers for (d)→(f) were the same ✓
and same for (b) ✓
Independent of how much time already elapsed
– past has no bearing on future. ✓

h) We summarise this property as P(X > t + s | X > t) = ? Complete this statement.

$$P(X>t+s \mid X>t) = P(X>s)$$

✓✓