# Mathematics Specialist Unit 4 2017

## Test 4: Calculus Solutions

**Student Name:** ____________________

**Time allowed for this task:** 55 minutes, in class, under test conditions

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|---|---|
| Section One – calculator-free section | 35 minutes (35 marks) |
| Section Two – calculator-assumed section | 20 minutes (20 marks) |

**Materials required:** (to be provided by the student)

Calculator with CAS capability.

Formula Sheet

Notes on one unfolded sheet of A4 paper.

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlights.

Special items: Drawing instruments, templates, and up to three calculators approved for use in WACE examinations.

**Marks available:** 55 marks

**Task weighting:** 7%

## Section One – calculator-free section (35 marks)

**Question 1. (10 marks)**

(a) Find $\int tan^2 4xdx$ using a suitable trigonometric formula. (2 marks)

$$\int tan^2 4xdx = \int (sec^2 4x - 1)dx \quad ✓$$

$$= \frac{1}{4} tan4x - x + c \quad ✓$$

(b) Find $\int (4x+3)(2x+1)^5\, dx$ using a suitable substitution. (4 marks)

$$\int (4x+6)(2x+1)^5\, dx = \int (2u+4)u^5 \frac{du}{2} \qquad u = 2x+1 \Rightarrow u - 1 = 2x \quad ✓$$

$$= \int (u^6 + 2u^5)du \qquad \Rightarrow 2u - 2 = 4x \quad ✓$$

$$= \frac{u^7}{7} + \frac{u^6}{3} + c \qquad du = 2dx \Rightarrow \frac{du}{2} = dx \quad ✓$$

$$= \frac{(2x+1)^7}{7} + \frac{(2x+1)^6}{3} + c \quad ✓$$

$$= \frac{(6x+10)(2x+1)^6}{21} + c$$

(c) Find $\int \frac{x+4}{(x+2)(x+1)}\, dx$ using partial fractions. (4 marks)

Put $\frac{x+4}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1} = \frac{A(x+1)+B(x+2)}{(x+2)(x+1)} \Rightarrow A + B = 1$ and $A + 2B = 4$

$\therefore A = -2, B = 3$ ✓✓

$$\therefore \int \frac{x+4}{(x+2)(x+1)}\, dx = \int \left(\frac{-2}{x+2} + \frac{3}{x+1}\right) dx \quad ✓$$

$$= -2ln|x+2| + 3ln|x+1| + c \quad ✓$$

**Question 2. (9 marks)**

(a) Determine the volume of the solid formed when the area in the first quadrant and enclosed by $y = x^2$ , the line $y = 3$ and the $y$ axis is rotated through one revolution about the $y$ axis. (4 marks)

$V_y = \pi \int_m^n x^2 dy$ ✓

$V_y = \pi \int_0^3 y \, dy$ ✓

$V_y = \pi \left[\frac{y^2}{2}\right]_{y=0}^{y=3}$ ✓

$V_y = \frac{9\pi}{2}$ $units^3$ ✓

(b) The area in the first quadrant enclosed by the curve $y = \frac{1}{x^2}$, the lines $x = 1, x = k, k > 1$ and the $x$-axis is rotated 360° about the $x$-axis. If the volume of the solid generated is $\frac{21\pi}{64}$ units$^3$ determine the value of the constant k. (5 marks)

$V_x = \pi \int_a^b y^2 \, dx$

$\frac{21\pi}{64} = \pi \int_1^k \frac{1}{x^4} dx$ ✓

$\frac{21}{64} = \left[\frac{-1}{3x^3}\right]_{x=1}^{x=k}$ ✓

$\frac{21}{64} = \frac{-1}{3k^3} + \frac{1}{3}$ ✓

$\frac{63}{64} = \frac{-1}{k^3} + 1 \Rightarrow k^3 = 64$ ✓

$\therefore k = 4$ ✓

**Question 3. (8 marks)**

(a) Determine $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} cos^3xdx$ using the substitution $u = sinx$. (4 marks)

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} cos^3xdx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 - sin^2x)cosxdx \qquad u = sinx \Rightarrow du = cosx \quad ✓$$

$$= \int_{\frac{1}{2}}^{1} (1 - u^2)du \qquad x = \frac{\pi}{2} \Rightarrow u = 1, x = \frac{\pi}{6} \Rightarrow u = \frac{1}{2} \quad ✓$$

$$= \left[u - \frac{1}{3}u^3\right]_{u=\frac{1}{2}}^{u=1} \quad ✓$$

$$= 1 - \frac{1}{3} - \left(\frac{1}{2} - \frac{1}{24}\right)$$

$$= \frac{2}{3} - \frac{11}{24}$$

$$= \frac{5}{24} \quad ✓$$

(b) Show that $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{sinx - cosx}{sinx + cosx} dx = \frac{1}{2}ln2$ (4 marks)

$$\frac{d}{dx}(sinx + cosx) = cosx - sinx = -(sinx - cosx)$$

$$\therefore \int \frac{sinx - cosx}{sinx + cosx} dx = -\ln(sinx + cosx) + c$$

$$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{sinx - cosx}{sinx + cosx} dx = [-\ln(sinx + cosx)]_{x=\frac{\pi}{4}}^{x=\frac{\pi}{2}} \quad ✓$$

$$= -\ln(1 + 0) - \left(-\ln\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)\right) \quad ✓$$

$$= -ln1 + ln\sqrt{2} \quad ✓$$

$$= 0 + ln2^{\frac{1}{2}} \quad ✓$$

$$= \frac{1}{2}ln2$$

**Question 4. (8 marks)**

(a) Use the substitution $tan\theta = x + 2$ to determine $\int \frac{4}{x^2+4x+5} dx$ (4 marks)

$\int \frac{4}{x^2+4x+5} dx = \int \frac{4}{(x+2)^2+1} dx$ $\qquad tan\theta = x + 2$ ✓

$= \int \frac{4sec^2\theta}{tan^2\theta+1} d\theta$ $\qquad sec^2\theta d\theta = dx$ ✓

$= \int \frac{4sec^2\theta}{sec^2\theta} d\theta$ ✓

$= \int 4d\theta$

$= 4\theta + c$

$= 4tan^{-1}(x + 2) + c$ ✓

(b) Find $\int (sin2x + cos2x)cos2xdx$ (4 marks)

$\int (sin2x + cos2x)cos2xdx = \int (sin2xcos2x + cos^2 2x)dx$ ✓

$= \int (\frac{1}{2}sin4x + \frac{1}{2}cos4x + \frac{1}{2})dx$ ✓✓

$= -\frac{1}{8}cos4x + \frac{1}{8}sin4x + \frac{1}{2}x + c$ ✓

**End of Section One**

# Section Two – calculator-assumed section (20 marks)

**Question 5. (4 marks)**

The area enclosed by the $x$ axis, the lines $x = 1$ and $x = 4$ and the curve $y = 1 + \sqrt{x}$ is rotated 360° about the $x$ axis. Calculate the volume of the solid generated to an accuracy of two decimal places. (4 marks)

$$V_x = \pi \int_a^b y^2 \, dx \quad ✓$$

$$V_x = \pi \int_1^4 (1 + \sqrt{x}\,)^2 \, dx \quad ✓$$

$$V_y = \frac{119\pi}{6} \approx 62.31 unit^3 \quad ✓✓$$

**Question 6. (6 marks)**

(a) Find $\int \frac{5x^2-10x-3}{(x+1)(x-1)^2} \, dx$ using partial fractions. (3 marks)

$$\int \frac{5x^2-10x-3}{(x+1)(x-1)^2} \, dx = \int \left(\frac{3}{x+1} + \frac{2}{x-1} - \frac{4}{(x-1)^2}\right) dx \quad ✓$$

$$= 3ln|x+1| + 2\ln|x-1| + \frac{4}{x-1} + c \quad ✓✓$$

(b) Hence express $\int_2^5 \frac{5x^2-10x-3}{(x+1)(x-1)^2} \, dx$ as a single logarithm. (3 marks)

$$\int_2^5 \frac{5x^2-10x-3}{(x+1)(x-1)^2} \, dx = 3ln6 + 2ln4 + 1 - (3ln3 + 2ln1 + 4) \quad ✓$$

$$= 7ln2 - 3 \quad ✓$$

$$= ln2^7 - lne^3$$

$$= ln\left(\frac{128}{e^3}\right) \quad ✓$$