SCOTCH
COLLEGE

12 Mathematics Methods 2019

Test 1 – Differentiation and Logarithms

Section 1: Calculator-free

Time allowed: 25 minutes **Maximum marks: 24**

Name: SOLUTIONS **Teacher:** Foster | Giese | Reyhani

Instructions:

- Show all working clearly.
- Sufficient detail must be shown for marks to be awarded for reasoning.
- A formula sheet will be provided.
- No calculators or personal notes are permitted.

Question 1 **[2, 2, 3 = 7 marks]**

Differentiate the following with respect to x (do not simplify):

(a) $\frac{3}{x} - \frac{5}{x^3}$ $= 3x^{-1} - 5x^{-3}$

$\therefore \frac{d}{dx}\left(\frac{3}{x} - \frac{5}{x^3}\right) = -3x^{-2} + 15x^{-4}$

(b) $(2 - x^5)^4$

$\frac{d}{dx}(2-x^5)^4 = 4(2-x^5)^3(-5x^4)$

(c) $x^3\sqrt{1 - x^2}$ $= x^3(1-x^2)^{\frac{1}{2}}$

$\frac{d}{dx}\left(x^3\sqrt{1-x^2}\right) = x^3 \times \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) + 3x^2(1-x^2)^{\frac{1}{2}}$

Question 2 **[4 marks]**

Determine the equation of the tangent to $y = \frac{x^3}{x+1}$ at $x = 1$.

$\frac{dy}{dx} = \frac{(x+1)(3x^2) - x^3}{(x+1)^2}$

$y = \frac{1}{2}$

$x = 1$ $\frac{dy}{dx} = \frac{5}{4}$

$\therefore y = \frac{5}{4}x + c$

$\frac{1}{2} = \frac{5}{4} + c$

$\therefore c = -\frac{3}{4}$

$\therefore y = \frac{5}{4}x - \frac{3}{4}$

Question 3 **[4 marks]**

Solve for x:

$$2\ln(x) - 3\ln\left(\frac{1}{x}\right) = 10$$

$\ln x^2 + \ln x^3 = 10$

$\ln x^5 = 10$

$5 \ln x = 10$

$\ln x = 2$

$e^2 = x$

use of log laws.

Question 4 **[2, 3 = 5 marks]**

Consider the function $f(x) = \log_2(x-1) + 1$.

(a) State the domain and range of f.

$x > 1$

$y \in \mathbb{R}$

(b) Find any asymptotes and co-ordinates of axes intercepts.

$x = 1$ is an asymptote.

no y-intercept.

$0 = \log_2(x-1) + 1$

$2^{-1} = x - 1$

$\therefore x = 1\frac{1}{2}$

x-intercept at $(1\frac{1}{2}, 0)$

Question 5 **[4 marks]**

If $y = \sqrt[3]{x}$, use the incremental formula to determine the approximate value of $\sqrt[3]{1001}$.

$x = 1000$

$\delta x = 1$

$\frac{dy}{dx} = \frac{1}{3} x^{-\frac{2}{3}}$

$\therefore \quad \delta y \approx \frac{dy}{dx} \times \delta x$

$\approx \frac{1}{3}(1000)^{-\frac{2}{3}} = \frac{1}{300} \approx 0.00\overline{33}$

$\therefore \quad y = 10$

$\sqrt[3]{1001} = 10 + 0.00\overline{33} \approx 10.003$.

or $10\frac{1}{300}$.

END OF SECTION 1

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12 Mathematics Methods 2019

Test 1 – Differentiation and Logarithms

Section 2: Calculator-assumed

Time allowed: 25 minutes **Maximum marks: 21**

Name: SOLUTIONS **Teacher:** Foster | Giese | Reyhani

Instructions:

- Show all working clearly.
- Sufficient detail must be shown for marks to be awarded for reasoning.
- A formula sheet will be provided.
- Calculators and 1A4 double-sided page of personal notes are permitted.

Question 6 [2, 1, 2 = 5 marks]

A particle moves in a straight line with its position from the origin given by

$$x(t) = 15t - \frac{60}{(t+1)^2} \quad \text{cm}$$

where t is the time in seconds, $t \geq 0$.

(a) What is the initial velocity of the particle?

$x'(0) = v(0) = 135$ cm/s.

(b) Determine the acceleration function for the particle's motion.

$$a(t) = -\frac{360}{(t+1)^4}$$

(c) For what values of t is the particle's velocity increasing? Justify your answer.

$x''(t) > 0$

but $a(t) < 0$ for $t > 0$

$\therefore$ the particle's velocity is never increasing

Question 7 [2, 3 = 5 marks]

If $A = \log_5 2$ and $B = \log_5 3$, write the following in terms of A and B:

(a) $\log_5 1.5$

$\log_5 \frac{3}{2}$

$B - A$

(b) $\log_5 60$

$\log_5 2^2 \times 3 \times 5$

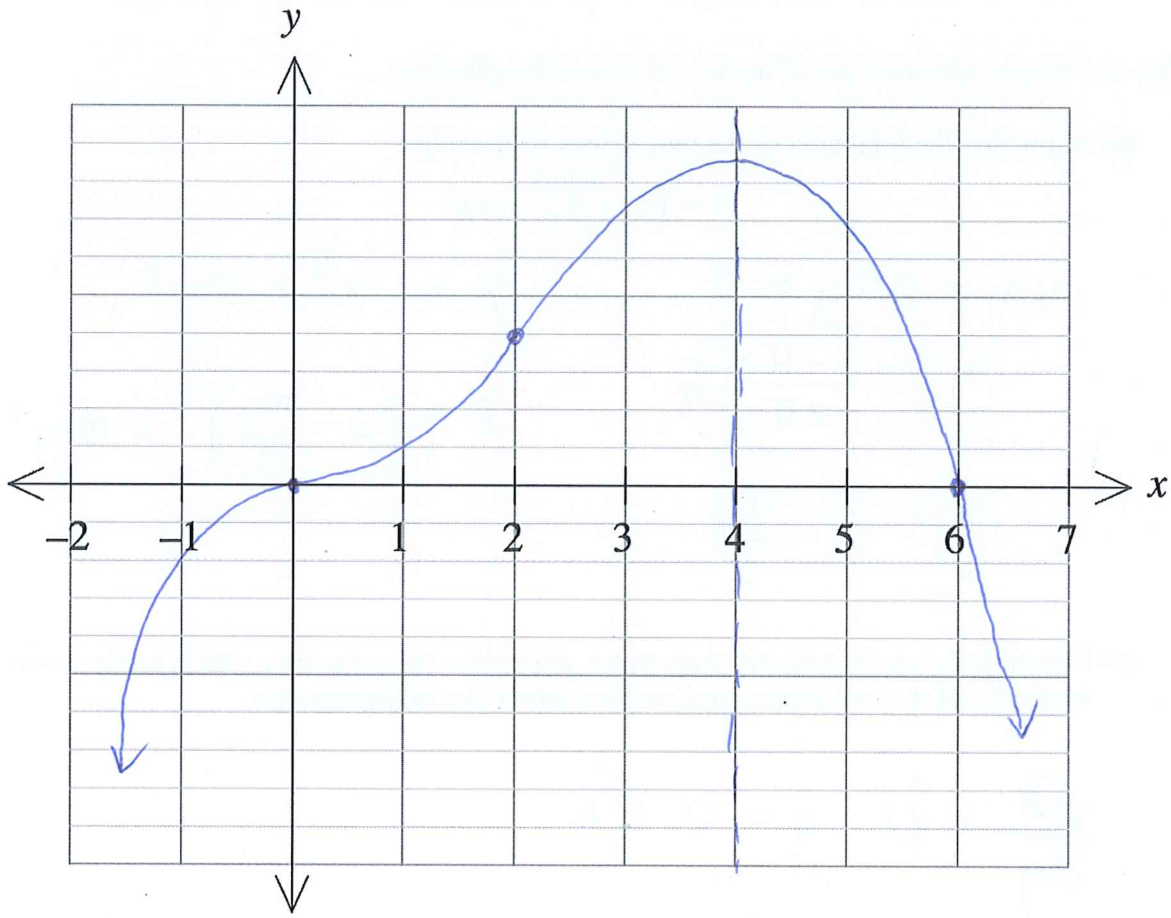
$2\log_5 2 + \log_5 3 + \log_5 5$

$2A + B + 1$

Question 8 **[5 marks]**

Sketch a function $y = f(x)$ with all of the following features, clearly indicating any stationary points and points of inflection:

- $f(0) = f(6) = 0$
- $f'(0) = f'(4) = 0$
- $f'(x) \geq 0$ strictly for $x < 4$
- $f''(x) > 0$ strictly for $0 < x < 2$



✓ x-intercepts

✓ maximum

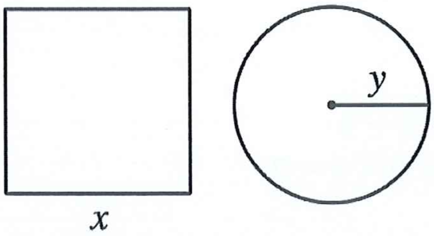
✓ horizontal point of inflection

✓ vertical point of inflection

✓ general shape

Question 9 **[2, 4 = 6 marks]**

The diagram below shows a square with side x cm and a circle with radius y cm.



The two shapes are made out of a piece of wire of length 8 cm.

(a) Show that the total area of the two shapes is given by:

$$A = \left(2 - \frac{\pi y}{2}\right)^2 + \pi y^2$$

$4x + 2\pi y = 8$

$\therefore x = \dfrac{8 - 2\pi y}{4}$ ✓

$x = 2 - \dfrac{\pi y}{2}$

$A = x^2 + \pi y^2$ ✓

$\therefore A = \left(2 - \dfrac{\pi y}{2}\right)^2 + \pi y^2$

(b) Showing the use of calculus techniques, determine the value of y which minimises the total area of the two shapes and confirm that it is a minimum area.

$\dfrac{dA}{dy} = 0$ ✓ $\quad y = 0.56$ ✓

$\dfrac{d^2A}{dy^2} = 11.22 > 0$ ✓ $\quad \therefore$ minimum ✓

END OF TEST