Greenwood College
Year 12 Maths Methods
Test 4 2017
Calculator-Free Section

Name……………………………………………………………………………………

No notes nor calculator allowed.
Formula sheet allowed.
Working time 30 minutes.

/~~30~~ MARKS
29

1. [3 marks: 1, 1, 1]

Let $\log_{10} 9 = x$ and $\log_{10} 11 = y$. Express each of the following in terms of x and y.

(a) $\log_{10} 99$ $= \log(9\times 11)$
$= \log 9 + \log 11$
$= x + y$ ✓

(b) $\log_{10} 0.09$ $= \log \frac{9}{100}$
$= \log 9 - 2\log 10$
$= x - 2$

(c) $\log_{10} (1100)^{\frac{1}{3}}$ $= \frac{1}{3}\log(11\times 100)$
$= \frac{1}{3}\log 11 + \frac{1}{3}\log 100$
$= \frac{y}{3} + \frac{2}{3}$ ✓

2. [3 marks]

For what values of "x" is $x^2 \ln x$ concave downwards? Express your answer in exact form.

$y = x^2 \ln x$
$y' = (2x)\ln x + x$ ✓
$y'' = 2\ln x + 3$ ✓

$y'' < 0 \rightarrow 2\ln x + 3 < 0$
$\ln x < -\frac{3}{2}$
$x < e^{-3/2}$ ✓

6

3. **[2 marks]**

If $\log_b y = 3\log_b x - 5$, express y in terms of x.

$$\log y = \log x^3 - 5\log b$$
$$= \log x^3 + \log \frac{1}{b^5}$$
$$\log y = \log\left(\frac{x^3}{b^5}\right)$$
$$y = \frac{x^3}{b^5} \checkmark\checkmark$$

4. **[4 marks]**

Solve $2\log_3 x + 1 = \log_3(4x - 1)$

$$\log_3 x^2 + \log_3 3 = \log_3(4x-1)$$
$$\log_3(3x^2) = \log_3(4x-1) \checkmark$$
$$3x^2 = 4x - 1 \checkmark$$
$$3x^2 - 4x + 1 = 0 \checkmark$$
$$x = 1,\ x = 1/3 \checkmark$$

5. **[2 marks]**

Using $\log_a b = x$, prove that $\log_a b = \frac{\log b}{\log a}$.

$\downarrow$

$$a^x = b \checkmark$$
$$x\log a = \log b$$
$$x = \frac{\log b}{\log a}$$
$$\log_a b = \frac{\log b}{\log a} \checkmark$$

8

6. [3 marks]

List 3 special features of the graph of $y = \ln x$

① Vertical asymptote $x=0$
② Passes through $(1,0)$
③ No horizontal asymptote.
④ $x > 0$ domain

7. [3 marks]

Find $\int \frac{2-x}{3x^2 - 12x + 1}\, dx$ $= -\frac{1}{6} \ln(3x^2 - 12x + 1) + c$ ✓ ✓ ✓

8. [4 marks]

The random variable X has its PDF as $p(x) = \frac{x}{2}$ for $0 \le x \le 2$. Determine the mean and variance for X.

$$\text{Mean} = \int_0^2 x \cdot \frac{x}{2}\, dx$$
$$= \left[\frac{x^3}{6} + c\right]_0^2$$
$$= \frac{8}{6}$$
$$= \frac{4}{3}$$
✓

$$\text{Var} = \int_0^2 \left(x - \frac{4}{3}\right)^2 \frac{x}{2}\, dx$$
$$= \int_0^2 \left(x^2 - \frac{8}{3}x + \frac{16}{9}\right) \cdot \frac{x}{2}\, dx$$
$$= \int_0^2 \frac{x^3}{2} - \frac{8x^2}{6} + \frac{16x}{18}\, dx$$
$$= \int_0^2 \left(\frac{x^3}{2} - \frac{4}{3}x^2 + \frac{8}{9}x\right) dx \checkmark$$
$$= \left[\frac{x^4}{8} - \frac{4x^3}{9} + \frac{8x^2}{18} + c\right]_0^2 \checkmark$$
$$= 16 - \frac{32}{9} + \frac{16}{9} = \frac{2}{9} \checkmark$$

10

9. [~~6~~ 5 marks: 1, 2, ~~3~~ 2]

X is a random variable from a PDF p(x) over the domain $c \le x \le d$. The expected value of X is $E(X) = \int_c^d x \, . \, p(x) \, dx$ by definition.

(a) What is $\int_c^d p(x) \, dx$ equal to? 1 ✓

(b) Show that E(b) = b, where "b" is a constant using integration.

$$E(b) = \int_c^d b \, . \, p \, dx$$
$$= b \int_c^d p \, dx \checkmark$$
$$= b \times 1$$
$$= b \checkmark$$

(c) $Y = aX + b$ is a linear transformation of the random variable X. Using integration with respects to the PDF p(x), show that $E(Y) = a\,E(X) + b$.

$$E(aX+b) = \int_c^d (ax+b) \, p(x) \, dx$$
$$= \int_c^d (ax) \, p(x) \, dx + \int_c^d b \, p(x) \, dx$$
$$= a \int_c^d x \, p(x) dx + b \int_c^d p(x) dx$$
$$= a\,E(X) + b$$

6

Greenwood College
Year 12 Maths Methods
Test 4 2017
Calculator-Assumed Section

Name…………………………………………………………………………………………

1 page /1 side A4 notes + Formula sheet + Calculator allowed. **/25 MARKS**
Working time 25 minutes.

10. [4 marks]

Solve $3^{x-y} = \frac{1}{9}$

$2^{y-3x} = \sqrt{4^x}$

$3^{x-y} = \frac{1}{9}$

$3^{x-y} = 3^{-2}$

$x - y = -2$ ✓

$2^{y-3x} = \sqrt{4^x}$

$2^{y-3x} = (2^{2x})^{1/2}$

$y - 3x = x$

$y = 4x$ ✓

$x - 4x = -2$

$-3x = -2$ ✓

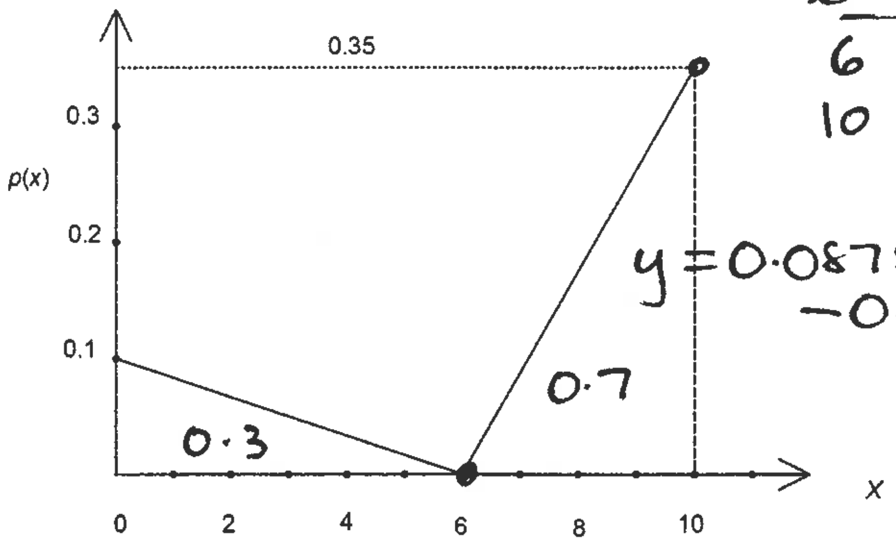
$x = 2/3 \rightarrow y = 8/3$ ✓

4

11. [6 marks: 4, 2]

(a) Determine the CDF for values of random variable X between 6 and 10 using the PDF below. Use FOUR decimal places for each stage of calculation.

x	y
6	0
10	0.35



$y = 0.0875x - 0.525$

$$P(x) = 0.3 + \int_6^x (0.0875t - 0.525)\,dt \checkmark \checkmark$$

$$= 0.3 + \left[\frac{0.0875t^2}{2} - 0.525t\right]_6^x$$

$$= 0.3 + 0.04375x^2 - 0.525x + 1.575$$

$$= 1.875 + 0.04375x^2 - 0.525x \checkmark\checkmark$$

(b) The median of the above PDF occurs when the CDF function value is exactly equal to 0.5 (50% of recorded data is below it and 50% of recorded data is above it). For what value of "x" does this occur?

$$0.3 + \frac{(x-6) \times y}{2} = 0.5$$

$$(x-6)y = 0.4$$

Solve $(x-6)(0.0875x - 0.525) = 0.4$ ✓

$$x = 8.138 \checkmark$$

OR solve $(1.875 + 0.04375x^2 - 0.525x) = 0.5$

6

12. **[4 marks]**

Find the boundaries "a" and "b" if the PDF is $p(x) = 0.125$ for $a \le x \le b$ and
$P(X \ge 15$ given $X \le 17) = 0.2857$

$$\frac{P(15 \le X \le 17)}{P(X \le 17)} = 0.2857$$

$$\frac{2 \times 0.125}{(17-a) \times 0.125} = 0.2857$$

$a = 10$

$b = 18$

13. **[6 marks: 2, 2, 2]**

The **apparent magnitude** (***m***) of a celestial object is a number that is a measure of its brightness as seen by an observer on Earth. The brighter an object appears, the lower its magnitude value (i.e. inverse relation). The Sun, at apparent magnitude of −26.74, is the brightest object in the sky. It is adjusted to the value it would have in the absence of the atmosphere.

The ratio of brightness (or intensity) $\frac{I_A}{I_B}$ of two objects A and B, of apparent magntiudes m_A and m_B respectively, satisfies the equation

$$\ln\left(\frac{I_A}{I_B}\right) = m_B - m_A$$

-26.74 -12.74

(a) Determine the ratio of brightness of the Sun to the Moon if the Moon has an apparent ~~brightness~~ Magnitude of -12.74.

$$\ln\left(\frac{I_S}{I_M}\right) = -12.74 - (-26.74) = 14$$

$$\frac{I_S}{I_M} = e^{14} = 1.20 \times 10^6$$

6

(b) If the ratio I_{MOON}/I_{SIRIUS} is 2.1393, determine the apparent magnitude of Sirius.

$$\ln\left(\frac{I_M}{I_S}\right) = m_S - m_M$$

$$\ln(2.1393) = m_S + 12.74$$

$$m_S = -11.98 \checkmark\checkmark$$

(c) Determine the intensity ratio for a difference of one in apparent magnitude.

$$\ln\left(\frac{I_A}{I_B}\right) = 1 \checkmark$$

$$\frac{I_A}{I_B} = e \checkmark$$

14. [5 marks: 1, 1, 1, ~~1, 1~~ 2]

The length of time Mary is late to school in the morning may be modelled by a uniform distribution with a minimum late time of zero minutes and a maximum late time of 20 minutes.

(a) Write the probability distribution function (PDF).

$$f(x) = \frac{1}{20}, \quad 0 \le x \le 20$$
$$0, \text{ otherwise}$$

(b) Find the probability that Mary is exactly 10 minutes late.

$0 \checkmark$

(c) Find the probability that Mary is no more than 15 minutes late given she is at least 10 minutes late. $\checkmark$

$$P(X \le 15 / X \ge 10) = \frac{5 \times 0.05}{10 \times 0.05} = 0.5$$

(d) On a school week of five days, find the probability that Mary is late by at least 5 minutes at most 3 times. ← $p = 0.75$

$$1 - P(X=4) - P(X=5)$$
$$= 1 - 0.3955 - 0.2373$$
$$= 0.3672 \checkmark\checkmark$$

9

Greenwood College
Year 12 Maths Methods
Test 4 2018
Calculator-Free Section

Name: Marking guide $\;\; /47$

No notes nor calculator allowed!
Formula sheet allowed.
Working time 25 minutes.

/24 MARKS

Question 1 **[8 marks: 2, 2, 2, 2]**

(a) Simplify $\log_2 64$

$$= \log_2 2^6$$
$$= 6\log_2 2$$
$$= 6 \times 1$$
$$= 6 \quad ✓✓$$

(b) Solve $2^{x-1} = 5$

$$\log 2^{x-1} = \log 5$$
$$(x-1)\log 2 = \log 5$$
$$x \log 2 = \log 5 + \log 2$$
$$x = \frac{\log 5 + \log 2}{\log 2} \quad ✓✓ \quad = \frac{1}{\log 2}$$

(c) If $x = \log_n 3$ and $y = \log_n 4$

(i) express $\log_n \left(\frac{4}{9}\right)$ in terms of x and/or y.

$$\log_n\left(\frac{4}{9}\right) = \log_n 4 - \log_n 9$$
$$= \log_n 4 - \log_n 3^2$$
$$= \log_n 4 - 2\log_n 3 \quad ✓✓$$
$$= y - 2x$$

(ii) evaluate n^{2x}

$$x = \log_n 3 \rightarrow n^x = 3$$

$$n^{2x} = (n^x)^2$$
$$= 3^2$$
$$= 9 \quad ✓✓$$

8

Question 2 **[7 marks: 2, 3, 2]**

The continuous random variable X has a probability density function

$$f(x)\begin{cases} \frac{2x}{9} & 0 \le x \le 3 \\ 0 & \textit{Elsewhere} \end{cases}$$

(a) Determine E(X).

$$E(X) = \int_0^3 \left(x \cdot \frac{2x}{9}\right) dx$$
$$= \left[\frac{2x^3}{27} + c\right]_0^3$$
$$= 2 \quad ✓✓$$

(b) Determine the cumulative distribution function F(x).

$$F(x) = \int_0^x \frac{2t}{9}\, dt$$
$$= \left[\frac{2t^2}{18} + c\right]_0^x$$
$$= \frac{x^2}{9}, \quad 0 \le x \le 3 \; ✓$$
✓✓

(c) Calculate P(1 < X < 2)

$$P(1 < X < 2) = F(2) - F(1)$$
$$= \frac{4}{9} - \frac{1}{9}$$
$$= \frac{1}{3} \quad ✓✓$$

7

Question 3 **[5 marks: 3, 2]**

A small storage tank, initially holding 20 litres of water, is being filled so that the rate of change of volume of water in the tank t minutes after filling began is given by $\frac{dV}{dt} = \frac{10t}{t^2 + 1}$ for $t \geq 0$, where V is the volume of water in the tank, in litres.

(a) Determine an expression for V in terms of t.

$$V = \int \frac{10t}{t^2+1}\, dt$$

$$= 5\ln(t^2+1) + c \checkmark$$

$$t = 0,\ V = 20 \rightarrow c = 20 \checkmark$$

$$V = 5\ln(t^2+1) + 20 \checkmark$$

(b) The tank has a maximum capacity of 80 litres. Determine an exact expression for the time it will take to reach this capacity.

$$5\ln(t^2+1) + 20 = 80 \checkmark$$

$$\ln(t^2+1) = 12$$

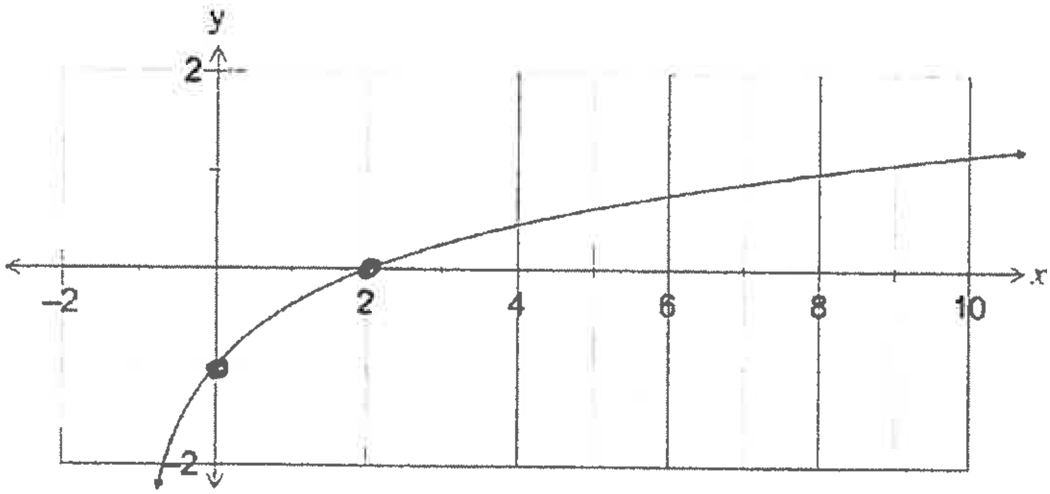
$$t = \sqrt{e^{12} - 1} \text{ min} \checkmark$$

5

Question 4 **[2 marks]**

The graph of $y = f(x)$, where $f(x) = \log_a(x + 1) + b$, is shown below.

Determine the values of a and b.



$f(0) = \log_a(1) + b = -1 \rightarrow b = -1$ ✓

$f(2) = \log_a(3) - 1 = 0 \rightarrow a = 3$ ✓

Question 5 **[2 marks]**

The discrete random variable X is defined by $P(X = x) = \frac{1}{2}\log x$ for x = 2, 5 and 10.

Show that this is a probability distribution.

$\frac{1}{2}\log 2 + \frac{1}{2}\log 5 + \frac{1}{2}\log 10 = 1$ ✓

$\frac{1}{2}\log(2 \times 5 \times 10) = 1$

$\frac{1}{2}\log 100 = 1$ ✓

$1 = 1$

4

Greenwood College
Year 12 Maths Methods
Test 4 2018
Calculator-Assumed Section

Name..

1 page /1 side A4 notes + Formula sheet + Calculator allowed! **/23 MARKS**
Working time 25 minutes.

Question 6 **[7 marks: 1, 2, 4]**

The moment magnitude scale M_w is used by seismologists to measure the size of earthquakes in terms of the energy released. It was developed to succeed the 1930's-era Richter magnitude scale.

The moment magnitude has no units and is defined as $M_w = \frac{2}{3}\log_{10}(M_0) - 10.7$,

where M_0 is the total amount of energy that is transformed during an earthquake, measured in dyn·cm.

(a) On 28 June 2016, an estimated 2.82×10^{21} dyn·cm of energy was transformed during an earthquake near Norseman, WA. Calculate the moment magnitude for this earthquake.

$M_w = 3.6$ ✓

Question 6 cont.

(b) A few days later, on 8 July 2016, there was another earthquake with moment magnitude 5.2 just north of Norseman. Calculate how much energy was transformed during this earthquake.

$$5.2 = \frac{2}{3}\log(x) - 10.7$$

$$x = 7.08\times10^{23}\ \text{dyn.cm} \checkmark\checkmark$$

(c) Show that an increase of 2 on the moment magnitude scale corresponds to the transformation of 1000 times more energy during an earthquake.

1 $M_w = \frac{2}{3}\log(x) - 10.7$ ✓ 2 $M_w + 2 = \frac{2}{3}\log(y) - 10.7$

1 − 2 ✓

$$2 = \frac{2}{3}(\log y - \log x)$$

$$\log\left(\frac{y}{x}\right) = 3 \checkmark$$

$$\frac{y}{x} = 10^3 \checkmark$$

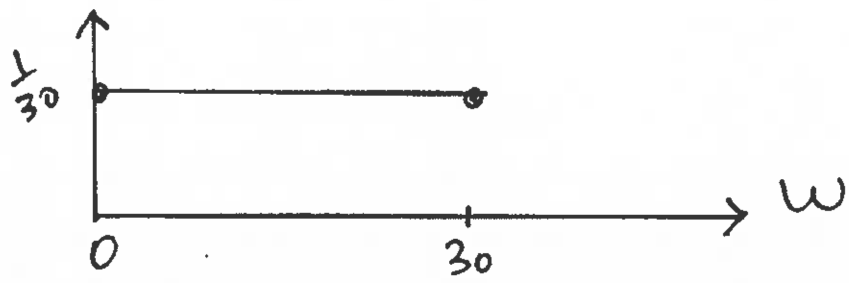
1000 times

6

Question 7 **[8 marks: 2, 1, 2, 1, 2]**

A bus service departs from a terminus every 30 minutes throughout the day. If a passenger arrives at the terminus at a random time to catch the bus, their waiting time in minutes, W, until the next bus departs is a uniformly distributed random variable.

(a) Sketch the graph of the probability density function of W.



✓✓

(b) What is the probability that a passenger who arrives at the terminus at a random time has to wait no more than 25 minutes for the bus to depart?

$$\frac{25-0}{30} = \frac{5}{6}$$ ✓

3

Question 7 cont.

(c) What is the probability that fewer than four passengers, out of a random selection of ten, have to wait at least 25 minutes for the bus to depart?

$$X \sim Bin\left(10, \frac{1}{6}\right) \quad ✓✓$$

$$P(X \leq 3) = 0.9303$$

(d) Determine the probability that a passenger who waits for at least 12 minutes has to wait for no more than 20 minutes.

$$\frac{20-12}{30-12} = \frac{8}{18} \quad ✓$$

$$= \frac{4}{9}$$

(e) Determine the value of w for which . $P(W < w) = P(W > 3w)$.

$$w = 30 - 3w \quad ✓$$

$$4w = 30$$

$$w = 7.5 \quad ✓$$

5

Question 8 **[8 marks: 2, 2, 4]**

The random variable X denotes the number of hours that a business telephone line is in use per nine hour working day.

The probability density function of X is given by

$$f(x) = \begin{cases} \dfrac{(x-a)^2 + b}{k} & 0 \le x \le 9 \\ 0 & \textit{Elsewhere} \end{cases}$$

where a, b and k are constants.

(a) If $a = 15$ and $b = 3$, determine the value of k.

$$\checkmark \int_0^9 \frac{(x-15)^2+3}{k}\,dx = 1 \rightarrow k = 1080 \checkmark$$

(b) Let $a = 16$, $b = 1$ and $k = 1260$.

(i) The business is open for work for 308 days per year. On how many of these days can the business expect the phone line to be in use for more than eight hours?

$$\int_8^9 \frac{(x-16)^2+1}{1260}\,dx = 0.0455 \checkmark$$

$308 \times 0.0455 = 14$ days ✓

4

Question 8 cont.

(ii) Determine, correct to two decimal places, the mean and variance of X.

$$E(X) = \int_0^9 \frac{x[(x-16)^2+1]}{1260}\,dx = 3.39$$

$$Var(X) = \int_0^9 (x-3.39)^2 \frac{[(x-16)^2+1]}{1260}\,dx$$

$$= 5.78$$

4

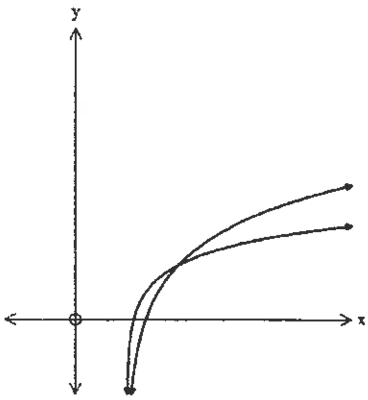
Greenwood College
Year 12 Maths Methods
Test 5 2017
Chapters 1 to 6 of Sadler Unit 4

Name: Solutions

1 page / 2 sides A4 notes + Formula sheet + Calculator allowed.
Working time 60 minutes.
59 marks total.

Question 1 **[3 marks: 2, 1]**

Given $f(x) = \ln x$ and $g(x) = \log_a x$



(a) Determine the possible values of "a".

$\ln x = \frac{\ln x}{\ln a}$

$1 < a < e$ ✓ ✓

$\ln a = 0$

$\frac{\ln x}{\ln e} = \ln x$

(b) State the co-ordinate of intersection.

$(1, 0)$ ✓

3

Question 2 **[2 marks]**

The brightness of a star from Earth can be calculated using the formula:

$m - M = 5\,(\log_{10} d \; - \; 1)$

Where

m = apparent magnitude
M = absolute magnitude
d = distance to the star in parsecs

Calculate the distance to Earth from Betelgeuse if it has an absolute magnitude of 0.9 and an apparent magnitude of 7.4.

M m

Solve $7.4 - 0.9 = 5(\log d - 1)$ ✓

$d = 200$ parsecs ✓

Question 3 **[11 marks: 2, 2, 2, 2, 1, 2]**

$f(x) = \dfrac{k}{x}$ for $1 \le x \le 5$ is a continuous probability density function.

(a) Determine the value of k, to four decimal places, to ensure it is a PDF.

$\int_1^5 \frac{k}{x}\,dx = 1$

$k\left[\ln x + c\right]_1^5 = 1$

$k\ln 5 = 1 \rightarrow k = \frac{1}{\ln 5}$ ✓✓

(b) Determine the mean of f(x). Round it to four decimal places.

$\int_1^5 x \cdot \frac{1}{x\ln 5}\,dx = 2.4853$ ✓✓

(c) Detemine the variance of f(x). Round it to four decimal places.

$\int_1^5 (x - 2.4853)^2 \frac{1}{x\ln 5}\,dx$ ✓

$= 1.279$ ✓

8

Question 5 **[7 marks: 2, 2, 3]**

A particle moves in a straight line such that its displacement from point O, at time "t" seconds is given by $x = 15\ln(2t-5) - 5t$, where "x" is in metres.

(a) Find the time/s when the numerical value of the velocity is equal to its acceleration.

Done graphically.

Solve $\underbrace{\frac{d}{dx}(15\ln(2x-5)-5x)}_{\text{velocity}} = \underbrace{\frac{d^2}{dx^2}(15\ln(2x-5)-5x)}_{\text{acceleration}}$

$t = 1.71\text{ s}$ ✓✓

(b) When is the particle at rest.

Velocity $= 0$

$t = 5.5\text{ s}$ ✓✓

(c) What is the total distance travelled between 4 and 8 seconds?

Max at $t = 5.5\text{ s}$

t	x
4	-3.52
5.5	-0.62
8	-4.03

2.9 ✓, 3.41 ✓] 6.31 ✓

OR $\int_4^8 \frac{d}{dx}\left(\left|15\ln(2x-5)-2x\right|\right)dx$

7