MINDARIE
SENIOR COLLEGE
WHERE YOUR FUTURE BEGINS NOW

MATHEMATICS: SPECIALIST 3 & 4

SEMESTER 1 2016

TEST 3

Resource Free

Reading Time: 2 minutes
Time Allowed: 16 minutes

Total Marks: 16

1. [1, 1, 1, 3 marks]

A system of linear equations in x, y and z has been partially reduced to the following form:

$$\begin{aligned} 2x + 7y - z &= 2 \qquad R_1 \\ 3y + z &= 3 \qquad R_2 \\ 9y + b^2 z &= 10 \qquad R_3 \end{aligned}$$

(a) Perform one further operation so that the coefficient of y in R_3 is zero.

$9y + b^2 z = 10$ $R3$
$-\ 9y + 3z = 9$ $R_2 \times 3$
$(b^2 + 3)z = 1$ ✓

State, if possible, all the values of *b* where by the system of equations will have

(b) no solutions $b = \pm\sqrt{3}$ ✓

(c) many solutions   Not possible ✓

If *b* was given the value 2

(d) give the solution to this system of linear equations.

$z = 1$ ✓
$3y + 1 = 3$
$3y = 2$
$y = 2/3$ ✓

$2x + \frac{14}{3} - 1 = 2$
$2x = 3 - \frac{14}{3}$
$= \frac{-5}{3}$
$x = \frac{-5}{6}$ ✓

2. [1, 2, 2 marks]

Given that $\boldsymbol{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$

Determine

(a) the scalar product, $\boldsymbol{a} \cdot \boldsymbol{b}$ $= 8 + 3 - 2$

$= 9$ ✓

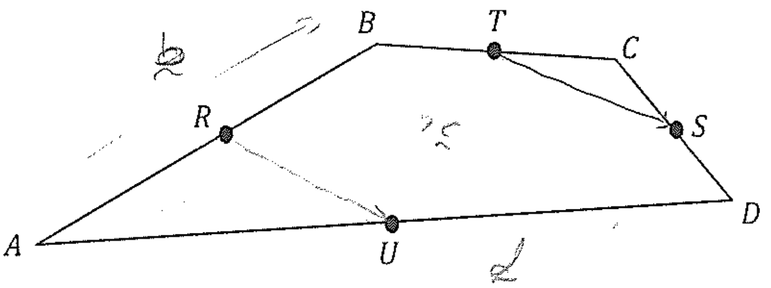
(b) the cross product, $\boldsymbol{a} \times \boldsymbol{b}$ $= \begin{pmatrix} 7 \\ -8 \\ -10 \end{pmatrix}$ ✓✓ -1/error

(c) the vector equation of the line that is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ and passes through the point $(5, 1, 3)$.

$$\underset{\sim}{r} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -8 \\ -10 \end{pmatrix}$$ ✓✓

3. [6 marks]

In the quadrilateral $ABCD$ shown below, $\overrightarrow{AB} = \boldsymbol{b}$, $\overrightarrow{AC} = \boldsymbol{c}$ and $\overrightarrow{AD} = \boldsymbol{d}$. The points R, T, S and U are the midpoints of the sides shown in the diagram.



Show, using vectors $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ that $\overrightarrow{TS}$ is parallel to $\overrightarrow{RU}$.

$\overrightarrow{TS} = \frac{1}{2}\overrightarrow{BC} + \frac{1}{2}\overrightarrow{CD}$

$= \frac{1}{2}(-\underset{\sim}{b} + \underset{\sim}{c}) + \frac{1}{2}(-\underset{\sim}{c} + \underset{\sim}{d})$ ✓

$= \frac{1}{2}(-\underset{\sim}{b} + \underset{\sim}{d})$

$= \frac{1}{2}(\underset{\sim}{d} - \underset{\sim}{b})$ ✓

$\overrightarrow{RU} = -\frac{1}{2}\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{d}$ ✓

$= \frac{1}{2}(\underset{\sim}{d} - \underset{\sim}{b})$

$= \overrightarrow{TS}$ ✓

$\therefore \overrightarrow{RU} \parallel \overrightarrow{TS}$ ✓

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Resource Assumed

Reading Time: 4 minutes
Time Allowed: 42 minutes

Total Marks: 42

4. [3, 1 marks]

Sally has the ability to do 3 jobs: window cleaning, aerobics coaching, and school teaching. She finds that she has only 36 hours available to work. She knows that there is only half as much aerobics as teaching and window cleaning combined. She must do 2 hours less aerobics than window cleaning.

(a) Write these restrictions as simultaneous equations.

$$w + a + t = 36$$

$$a = \frac{1}{2}(t + w)$$

$$a = w - 2$$

(b) Solve this system of equations.

aerobics = 12 hrs

windows = 14 hours

teaching = 10 hours

5. [5 marks]

A particle is moving in the plane such that its position at any time t is given by

$$\boldsymbol{r}(t) = 4\cos t\,\boldsymbol{i} + 2\sin t\,\boldsymbol{j}$$

Determine the vector equation of the line that is perpendicular to the motion of the particle at the point when $t = \frac{\pi}{3}$.

$\underset{\sim}{r}\left(\frac{\pi}{3}\right) = 2\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j}$ ✓

$\frac{d\underset{\sim}{r}}{dt} = -4\sin t\,\underset{\sim}{i} + 2\cos t\,\underset{\sim}{j}$ ✓

$\left.\frac{d\underset{\sim}{r}}{dt}\right|_{t=\frac{\pi}{3}} = -2\sqrt{3}\,\underset{\sim}{i} + \underset{\sim}{j}$ ✓

$\therefore$ normal vector $= \underset{\sim}{i} + 2\sqrt{3}\,\underset{\sim}{j}$ ✓

$\therefore \underset{\sim}{r} = 2\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j} + \lambda\left(\underset{\sim}{i} + 2\sqrt{3}\,\underset{\sim}{j}\right)$ ✓

6~~5~~. [2, 2, 3]

(a) An ellipse has vector equation $\boldsymbol{r} = 5\cos\theta\,\boldsymbol{i} + 4\sin\theta\,\boldsymbol{j}$.
Determine the Cartesian equation of the ellipse.

$x = 5\cos\theta$ $\qquad$ $y = 4\sin\theta$

$\frac{x}{5} = \cos\theta$ $\qquad$ $\frac{y}{4} = \sin\theta$

$\frac{x^2}{25} + \frac{y^2}{16} = 1$ ✓✓

(b) Determine the acute angle between the lines with equations

$\boldsymbol{r} = (3 + 2\lambda)\boldsymbol{i} + (1 - 3\lambda)\boldsymbol{j} + (4 + \lambda)\boldsymbol{k}$ and
$\boldsymbol{r} = (2 - \lambda)\boldsymbol{i} + (5 + 2\lambda)\boldsymbol{j} + (7 - 2\lambda)\boldsymbol{k}$

$\underset{\sim}{r}_1$ dir'n $2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$

$\underset{\sim}{r}_2$ dir'n $-\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$ ✓

angle $= 153^\circ$ ✓ $\therefore 27^\circ$ ✓

(c) Points O, A and B are such that $\overrightarrow{OA} = 3\boldsymbol{i} + a\boldsymbol{j} + 4\boldsymbol{k}$ and $\overrightarrow{OB} = \boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}$.
Determine the area, in terms of a, of triangle OAB.

$$\text{Area }(\triangle) = \frac{|\overrightarrow{OA} \times \overrightarrow{OB}|}{2}$$ ✓

$$= \frac{|(a-20)\underset{\sim}{i} + \underset{\sim}{j} + (15-a)\underset{\sim}{k}|}{2}$$ ✓

$$= \frac{\sqrt{2(a^2-35a+313)}}{2}$$ ✓

7. [4, 2, 3 marks]

A particle is moving in a vertical plane such that its acceleration, $\boldsymbol{a}$, is given by $\boldsymbol{a} = -9.8\boldsymbol{j}$ m/s^2 where $\boldsymbol{i}$ is a horizontal unit vector in the plane, and $\boldsymbol{j}$ is a vertical unit vector in the plane. Initially the particle is at the origin, travelling with velocity, $\boldsymbol{v} = 20\boldsymbol{i} + 100\boldsymbol{j}$ m/s.

Point P has coordinates $(120, y)$.

(a) Given that the particle passes through point P, determine the value for y.

$\underset{\sim}{a} = -9.8\underset{\sim}{j}$

$\underset{\sim}{v} = C_1\underset{\sim}{i} + (C_2 - 9.8t)\underset{\sim}{j}$

$t=0 \quad 20\underset{\sim}{i} + 100\underset{\sim}{j} = C_1\underset{\sim}{i} + C_2\underset{\sim}{j}$

$\therefore \underset{\sim}{v} = 20\underset{\sim}{i} + (100 - 9.8t)\underset{\sim}{j}$ ✓

$\underset{\sim}{s} = (20t + C_3)\underset{\sim}{i} + (100t - 4.9t^2 + C_4)\underset{\sim}{j}$

$t=0 \Rightarrow C_3 \,\&\, C_4 = 0$

$\therefore \underset{\sim}{s} = 20t\underset{\sim}{i} + (100t - 4.9t^2)\underset{\sim}{j}$ ✓

$120\underset{\sim}{i} + y\underset{\sim}{j} = 20t\underset{\sim}{i} + (100t - 4.9t^2)\underset{\sim}{j}$ ✓

$t = 6$ ✓

$\therefore y = 423.6$

(b) Determine the speed of the particle when it passes through point P.
Give your answer to one decimal place.

$\underset{\sim}{v}(6) = 20\underset{\sim}{i} + 41.2\underset{\sim}{j}$ ✓

Speed $= 45.8$ m/s. ✓

(c) Determine the maximum height of the particle, and its horizontal distance from the origin at that point.

$100 - 9.8t = 0$

$t = \frac{100}{9.8}$ ✓

$\underset{\sim}{s}\left(\frac{100}{9.8}\right) = 204.1\underset{\sim}{i} + 510.2\underset{\sim}{j}$

max hgt $= 204.1$m ✓

@ 510.2 m ✓

8. [4, 3, 3 marks]

Consider two particles, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, that move along the lines defined by

$$\boldsymbol{r}_1 = -\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k} + \overset{a}{t}(\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}) \text{ and}$$
$$\boldsymbol{r}_2 = -3\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k} + \underset{b}{t}(2\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k})$$

(a) Show that the particles cross paths, *but do not meet*, at the point $(2, 7, -3)$.

$\underset{\sim}{i}: -1 + a = -3 + 2b$ ✓

$\underset{\sim}{j}: -2 + 3a = -3 + 4b$

$a = 3 \quad b = 2.5$ ✓

$\underset{\sim}{r}_1(3) = -\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k} + 3\underset{\sim}{i} + 9\underset{\sim}{j} - 6\underset{\sim}{k}$ ✓
$= 2\underset{\sim}{i} + 7\underset{\sim}{j} - 3\underset{\sim}{k}$

$\underset{\sim}{r}_2(2.5) = -3\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k} + 5\underset{\sim}{i} + 10\underset{\sim}{j} - 5\underset{\sim}{k}$ ✓
$= 2\underset{\sim}{i} + 7\underset{\sim}{j} - 3\underset{\sim}{k}$

$\therefore$ cross @ $(2, 7, -3)$
but at $t = 3$ & $t = 2.5$

A sphere has Cartesian equation $x^2 + y^2 + z^2 - 10x - 4y + 2z - 6 = 0$

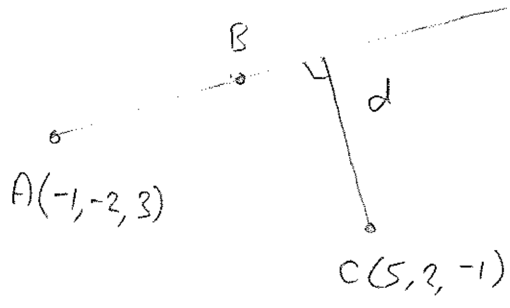
(b) Determine the exact distance between the point at which the particles' paths cross and the centre of the sphere,

$C(5, 2, -1)$

$d = \sqrt{3^2 + 5^2 + 2^2}$
$= \sqrt{38}$ units

(c) Determine the minimum distance that the particle r_1 gets to the centre of the sphere.

$$d = \frac{|\vec{AB} \times \vec{AC}|}{|\vec{AB}|}$$



$$\vec{AB} = \underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$$

$$\vec{AC} = 6\underset{\sim}{i} + 4\underset{\sim}{j} - 4\underset{\sim}{k}$$

$$d = \frac{|-4\underset{\sim}{i} - 8\underset{\sim}{j} - 14\underset{\sim}{k}|}{|\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}|} \checkmark \checkmark$$

$$= \frac{2\sqrt{69}}{\sqrt{14}}$$

$$= \frac{\sqrt{966}}{7}$$

$$= 4.44 \text{ units} \checkmark$$

8. [7 marks]

The acceleration of a particle moving in the $x - y$ plane is $-g\boldsymbol{j}$, where g is a constant. At time $t = 0$ the particle leaves the point with position vector $h\boldsymbol{j}$, with velocity

$$(V\cos\alpha)\boldsymbol{i} + (V\sin\alpha)\boldsymbol{j}$$

By integration, find expressions for the velocity $\boldsymbol{v}$ and the position vector $\boldsymbol{s}$, of the particle, at time t.

Deduce that the particle follows a path whose Cartesian equation is

$$y = h + x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha$$

$\underset{\sim}{v} = C_1\underset{\sim}{i} + (C_2 - gt)\underset{\sim}{j}$ ✓

$t=0$ $(V\cos\alpha)\underset{\sim}{i} + (V\sin\alpha)\underset{\sim}{j} = C_1\underset{\sim}{i} + (C_2)\underset{\sim}{j}$

$C_1 = V\cos\alpha$

$C_2 = V\sin\alpha.$ ✓

$\therefore \underset{\sim}{v} = V\cos\alpha\underset{\sim}{i} + (V\sin\alpha - gt)\underset{\sim}{j}$

$\underset{\sim}{s} = (Vt\cos\alpha + C_3)\underset{\sim}{i} + (Vt\sin\alpha - \frac{1}{2}gt^2 + C_4)\underset{\sim}{j}$ ✓

$t=0$ $h\underset{\sim}{j} = C_3\underset{\sim}{i} + C_4\underset{\sim}{j}$

$\therefore C_4 = h \quad C_3 = 0$

$\underset{\sim}{s} = Vt\cos\alpha\underset{\sim}{i} + (Vt\sin\alpha - \frac{1}{2}gt^2 + h)\underset{\sim}{j}$ ✓

$x = Vt\cos\alpha$ ✓ $y = Vt\sin\alpha - \frac{1}{2}gt^2 + h$

$t = \frac{x}{V\cos\alpha}$

$y = V \cdot \frac{x}{V\cos\alpha} \cdot \sin\alpha - \frac{1}{2} \cdot g \cdot \frac{x^2}{V^2\cos^2\alpha} + h$ ✓

$= x\tan\alpha - \frac{gx^2}{2V^2} \cdot \sec^2\alpha + h$ ✓