Specialist Mathematics: Unit 3

Investigation 1 – Complex Plane Dynamics

Name: Solutions

Part 2: Validation Test **Marks: /55**

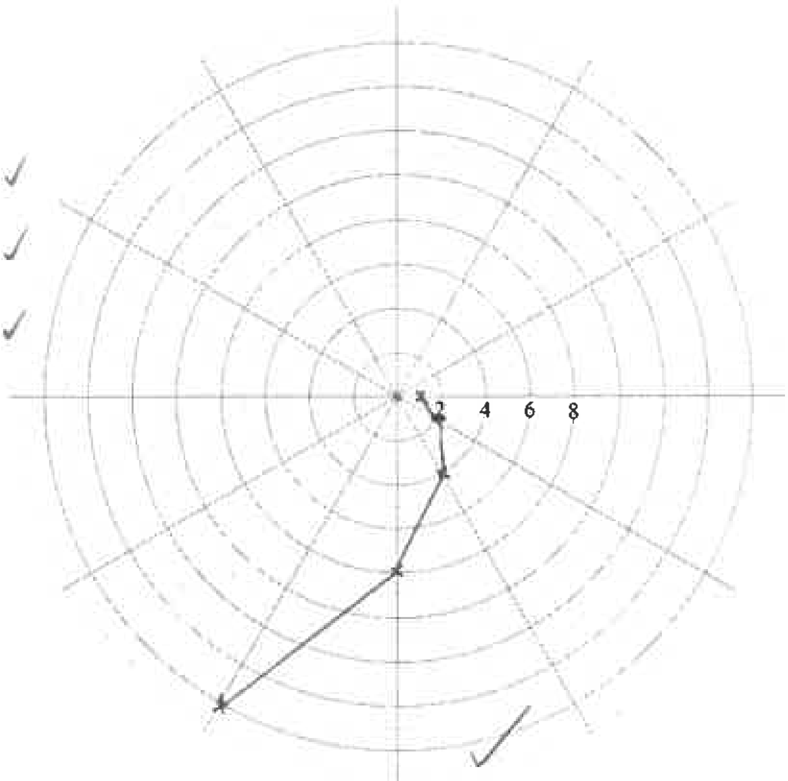
Write your responses in the space provided.

SECTION A: 25 Marks

1. **(4, 2, 1, 1, 1, 2 = 11 marks)**

(a) Complete the table and plot the first 5 points generated by the sequence 1, z, z^2, z^3, z^4, ... where $Z = \sqrt{3} - i$ and connect them with straight lines. (4 marks)

Cartesian	Polar
$Z^0 = 1$	$Z^0 = cis(0)$
$Z = \sqrt{3} - i$	$Z = 2cis\left(\frac{-\pi}{6}\right)$
$Z^2 = 2 - 2\sqrt{3}i$	$Z^2 = 4\,cis\left(-\frac{\pi}{3}\right)$ ✓
$Z^3 = -8i$	$Z^3 = 8\,cis\left(-\frac{\pi}{2}\right)$ ✓
$Z^4 = -8\sqrt{3} - 8i$	$Z^4 = 16\,cis\left(-\frac{2\pi}{3}\right)$ ✓



(b) Explain, using polar form, how the location (modulus and argument) of each point (excluding the first) may be calculated from the one before it. (2 marks)

Modulus is found by multiplying by 2 ✓

Argument is found by adding $\left(-\frac{\pi}{6}\right)$. ✓

(c) How can you determine $|z^5|$ if you know $|z^1|$? (1 mark)

$|z^5| = |z_1|^5$ ✓

(d) How can you determine $\arg(z^5)$ if you know $\arg(z^1)$? (1 mark)

$\arg(z^5) = 5\arg(z^1)$ ✓

(e) Why is 1 the logical T_0 term in such a sequence? (1 mark)

Anything to the power of zero is 1 ✓

(f) One of the terms in the sequence will be **w cis (-π)**. State **which term it is** and **give the value of w.** (2 marks)

It will be z^6 ✓ as $n(-\pi/6) = -\pi$

$w = 2^6$

$= 64.$ ✓

2. **(3, 1, 1, 1, 2 = 8 marks)**

Consider the sequence 1, z, z^2, z^3 , z^4, ... where z = 0.48 + 0.2i

(a) Give the polar coordinates, $\left[\,|z|,\ \theta\,\right]$ of the first four terms of the sequence. (Give the argument in radians accurate to 2 d.p. and the modulus accurate to 4 decimal places)

(3 marks)

$z_1 = [0.52, 0.39]$

$z_2 = [0.2704, 0.79]$

$z^3 = [0.1406, 1.18]$

(b) State the two major differences between the spiral in Q.4 and the spiral that would be created here. (No need to draw)

(1 mark)

This is spiralling inward ✓
and moving anti-clockwise ✓

(c) How can you determine $|z^3|$ if you know $|z|$?

(1 mark)

$|z^3| = |z|^3$ ✓

(d) How can you determine $\arg(z^3)$ if you know $\arg(z)$?

(1 mark)

$\arg(z^3) = 3\arg(z)$ ✓

(e) Find the lowest value of n such that $\text{Arg}(z^n) > 4\pi$.

(2 marks)

$n > 32$ $\quad 4\pi/0.39 = 32.22146311$ ✓

so 33. ✓

3. **(1, 1 = 2 marks)**

(a) Determine a formula to calculate $|z^n|$ given you know $|z|$. (1 mark)

$|z^n| = |z|^n$ ✓

(b) Determine a formula to calculate $\arg(z^n)$ given you know $\arg(z)$. (1 mark)

$\arg(z^n) = n\arg(z)$ ✓

4. **(4 marks)**

A spiral of the sequence 1, z, z^2, z^3 ... z^{10} finishes with [243,0]. That is $z^{10} = 243cis(0)$.

Using exact values, find $|z|$ and give the smallest possible positive value for Arg(z).

$\sqrt[10]{243} = 1.732050808 = \sqrt{3}$ ✓

$360/10 = \frac{2\pi}{10} = \frac{\pi}{5}$ ✓

So.

6. **(2, 2, 1, 1 = 6 marks)**

An equilateral triangle, drawn on an Argand diagram and centred at the origin has one of its vertices at $z = 1$. cis 0

$0 \leq \arg z \leq 2\pi$

(a) Give the polar form of the complex numbers that define the other two vertices. (2 marks)

$[1, \frac{2\pi}{3}]$ and $[1, -\frac{2\pi}{3}]$

(b) Convert your answers above to Cartesian form (2 marks)

$z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ $\quad z_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

(c) Raise each of the complex numbers that form the vertices of the triangle to the power of 3. (1 mark)

$1^3, \underbrace{\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3}_{=1}, \underbrace{\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3}_{=1}$

(d) Hence state the complex solutions to $z^3 = 1$ (1 mark)

$z = 1 \quad z = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3 = 1 \quad z = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3 = 1$

7. **(1, 5, 1 = 7 marks)**

$z = 1 + i$ is one of the solutions to the equation $z^5 = w$, where w is a complex number.

(a) Find w. $(1+i)^5 = w = -4 - 4i$ (1 mark)

(b) $z = 1 + i$ is one vertex of a regular pentagon. Give the polar form of the complex numbers that define the five vertices of the pentagon. Give $0 \leq \arg(z) \leq 2\pi$. (5 marks)

$z_1 = \sqrt{2} \operatorname{cis} \frac{\pi}{4} = \frac{5\pi}{20}$

$z_2 = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{4} + \frac{2\pi}{5}\right) = \sqrt{2} \operatorname{cis} \frac{13\pi}{20}$

$z_3 = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{4} + \frac{4\pi}{5}\right) = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{20}\right)$

$z_4 = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{4} + \frac{6\pi}{5}\right) = \sqrt{2} \operatorname{cis}\left(\frac{9\pi}{20}\right)$

$z_5 = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{4} + \frac{8\pi}{5}\right) = \sqrt{2} \operatorname{cis}\left(\frac{17\pi}{20}\right)$

(c) Hence solve $z^5 = w$. (1 mark)

Solutions as above

SECTION B: 30 Marks

5. **(2, 4, 4, 4, 1, 2 = 17 marks)**

Consider the sequence generated by multiplying a complex number, z, by itself:

$$z, z^2, z^3, z^4 \ldots$$

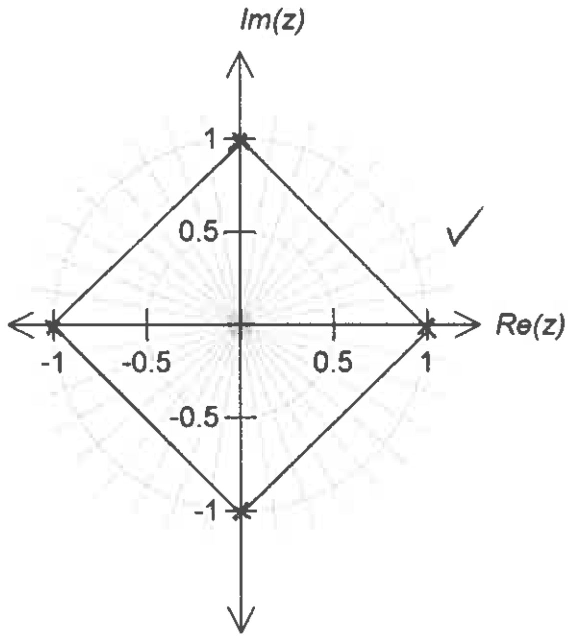
In the Take Home Component you should have discovered that if the modulus of z is 1, the points generated neither spiral in or out. In this question you will investigate the role of the argument of z in these sequences.

(a) <u>List</u> and <u>plot</u> the points generated by the sequence $z, z^2, z^3, z^4 \ldots$ for the following complex numbers and <u>connect</u> them with straight lines. You need to continue the sequence for as many terms as necessary to view the resulting pattern.

For the next two parts $0 \le \arg(z) \le 360°$

i. $z = \cancel{i}$ cis 90 (2 marks)

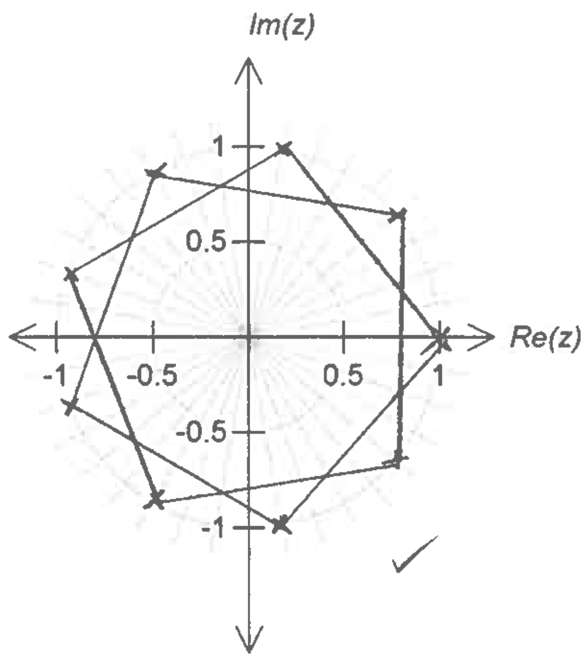
z	i cis 90°
z^2	-1 cis 180°
z^3	$-i$ cis 270° ✓
z^4	1 cis 360° or 0°
z^5	



ii. $z = cis80°$ (4 marks)

z	$cis80°$
z^2	$cis\ 160°$
z^3	$cis\ 240°$
z^4	$cis\ 320°$
z^5	$cis\ 40°$
z^6	$cis\ 120°$
z^7	$cis\ 200°$
z^8	$cis\ 280°$
z^9	$cis\ 0°$
z^{10}	$cis\ 80°$

✓✓✓



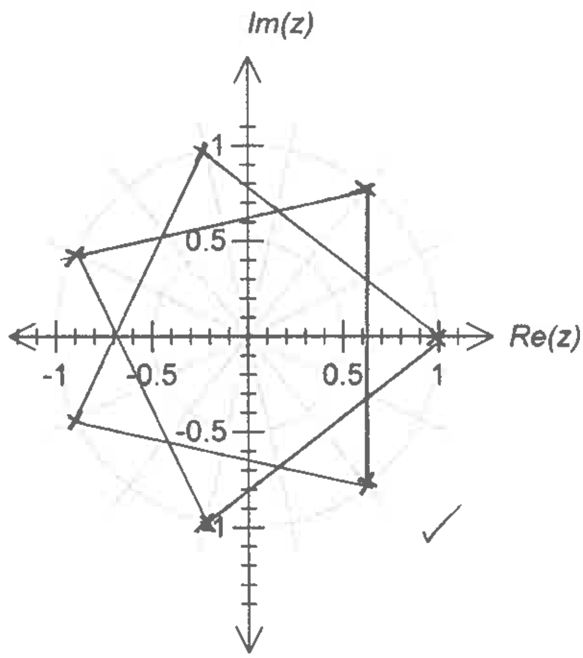
✓

For the next two parts $0 \le \arg(z) \le 2\pi$

iii. $z = cis\frac{4\pi}{7}$ (4 marks)

z	$cis\ \frac{4\pi}{7}$
z^2	$cis\ \frac{8\pi}{7}$
z^3	$cis\ \frac{12\pi}{7}$
z^4	$cis\ \frac{2\pi}{7}$
z^5	$cis\ \frac{6\pi}{7}$
z^6	$cis\ \frac{10\pi}{7}$
z^7	$cis\ 2\pi$
z^8	$cis\ \frac{4\pi}{7}$

✓✓✓

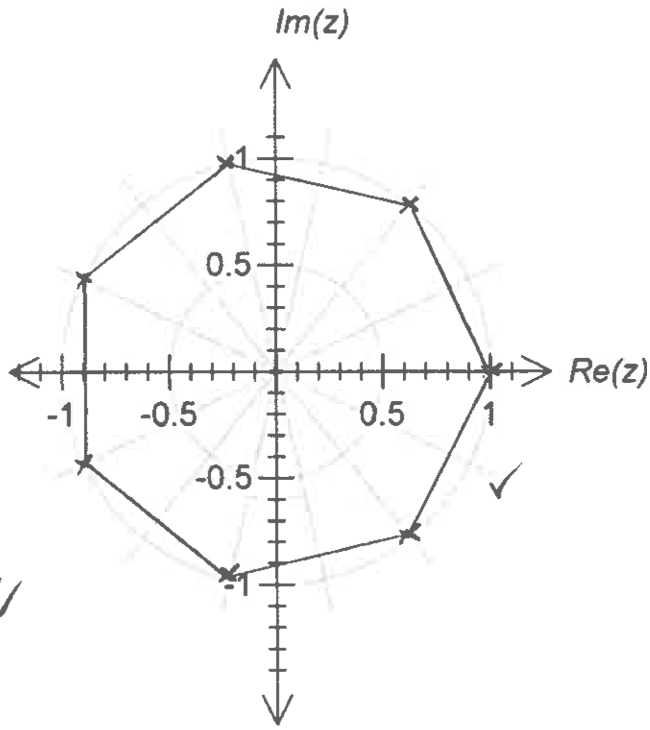


✓

iv. $z = cis\frac{2\pi}{7}$ (4 marks)

z	$cis\ 2\pi/7$
z^2	$cis\ 4\pi/7$
z^3	$cis\ 6\pi/7$
z^4	$cis\ 8\pi/7$
z^5	$cis\ 10\pi/7$
z^6	$cis\ 12\pi/7$
z^7	$cis\ 2\pi$
z^8	$cis\ 2\pi/7$

✓✓✓



(b) Generalise your results. What values of $\arg(z)$ will produce closed polygons in one revolution? (1 mark)

$cis\ \pi/2$ and $cis\ 2\pi/7$ produce closed polygons.

Any value that is factor of 2π will produce a closed polygon ✓

(c) All of the above sequences have rotated <u>anticlockwise</u>. Determine a complex number that would produce a closed polygon by rotating <u>clockwise</u>. State the type of polygon that would be formed. It should be a different polygon to those drawn above. (2 marks)

$cis\left(-\frac{2\pi}{3}\right)$ ✓ $\rightarrow$ equilateral triangle ✓

or $cis\left(-\frac{2\pi}{5}\right)$ etc.