Year 11 Specialist Mathematics

MELVILLE
SENIOR HIGH SCHOOL

Semester 1, March 2021

Test 1: Combinatorics and Introduction to Proofs

Weighting: 6%

[Australian Curriculum Reference Numbers: 1.3.1-1.3.5, 1.1.1-1.1.9, 2.1.5]

Total Time: 50min **Total Marks = $\frac{\quad}{50}$**

Student Name: SOLUTIONS

Teacher: ______________________________

TO BE PROVIDED BY THE STUDENT

Standard Items: Pens, pencils, eraser, sharpener, correction tape/fluid, highlighters, ruler.

Special Items:

- Drawing instruments, templates
- A maximum of three CAS calculators satisfying the conditions set by the Curriculum Council ***for use in the Calculator Allowed section only***

TO BE PROVIDED TO THE STUDENT

- A formula sheet will be provided

INSTRUCTIONS TO STUDENTS:

You are required to attempt ALL questions.

Write answers in the spaces provided beneath each question.

Marks are shown with the questions.

Show all working clearly, in sufficient detail to allow your answers to be checked readily and for marks to be answered for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks.

It is recommended that students **do not use a pencil,** except in diagrams

Part A – Calculator - Free (30 minutes)	/30
Part B – Calculator Assumed (20 minutes)	/20
Final mark	**/50**

1. Consider the statement:

"All squares have two equal pairs of parallel sides"

a) State the inverse of the statement.

$Any\ shape\ that\ is\ not\ a\ square\ does\ not\ have\ two\ equal\ pairs\ of\ parallel\ sides$

b) State the converse of the statement.

$Any\ shape\ that\ has\ two\ equal\ pairs\ of\ parallel\ sides\ is\ a\ square$

c) State the contrapositive of the original statement.

$Contrapositive$:
$A\ shape\ that\ does\ not\ have\ two\ pairs\ of\ parallel\ is\ not\ a\ square$

d) Is the original statement necessary and sufficient? Justify your answer.

$NO, as\ a\ rhombus\ has\ two\ equal\ pairs\ of\ parallel\ sides\ but\ is\ not\ a\ square.$

[1,1,1,2 = 5 marks]

2. A pencil case contains a variety of red, blue, black, and green pens.

a) How many pens do you need to have to be certain of having six pens of the same colour?

$$5 + 5 + 5 + 5 + 1 more = 21\ pens$$

What is the smallest number of pens you can have in the pencil case to ensure you have at least 4 red pens, or 1 green pen, or 2 blue pens, or 5 black pens?

$$3 + 0 + 1 + 4 + 1 more = 9\ pens$$

[2,2 = 4 marks]

3.

a) Prove the following by proving its contrapositive.
Note: (same parity means both odd or both even)

$If\ x\ and\ y\ are\ two\ integers\ for\ which\ x + y\ is\ even, then\ x\ and\ y\ have\ the\ same\ parity.$

$RTP: If\ x\ and\ y\ have\ different\ parity, then\ x + y\ is\ odd$

Proof: $Let\ x\ be\ even\ and\ y\ be\ odd\ (different\ parity)$
$Then\ x = 2k, y = 2m + 1, k, m \in \mathbb{Z}\ (definition\ of\ odd\ and\ even)$

$x + y = 2k + 2m + 1$
$x + y = 2(k + m) + 1 = 2(Some\ integer) + 1, \quad which\ is\ odd$

$Same\ is\ true\ if\ y\ was\ even\ and\ x\ was\ odd.$
$The\ contraposotive\ has\ been\ proved,$
$therefore\ the\ original\ statement\ is\ also\ true\ ie.$
$for\ x, y\ \in \mathbb{Z}, \quad if\ x + y\ is\ even, then\ x\ and\ y\ have\ same\ parity$

b) Prove the following statement by contrapositive:

$$\forall\ p \in \mathbb{Z}, p^2 odd\ \Rightarrow p\ odd$$

$RTP: p\ even\ \Rightarrow\ p^2\ even$

$By\ definition\ of\ even\ we\ have, p = 2k, k \in\ \mathbb{Z}$

$\therefore p^2 = (2k)^2 = 4k^2 = 2(Some\ integer), which\ is\ even\ by\ definition$

$We\ have\ proved\ the\ contrapositive\ so\ the\ original\ statement\ is\ true$
$ie. For\ all\ integers\ p, if\ p^2 is\ odd, then\ p\ is\ odd.$

[5,5 = 10 marks]

4. Evaluate the following:

a) ${}^{9}P_4$

$$= \frac{9!}{5!} = 9 \times 8 \times 7 * 6 = 3024$$

b) ${}^{12}C_9$

$$\frac{12!}{3!\,9!} = \frac{12 \times 11 \times 10}{3!} = 220$$

[1,2 = 3 marks]

5. Determine the value of ${}^{n-1}P_2$ if it is known that $7 \times {}^{n}P_2 = 6 \times {}^{n+1}P_2$ for $n > 0$.

$$7 \times \frac{n!}{(n-2)!} = 6 \times \frac{(n+1)!}{(n+1-2)!}$$

$$7 \times \frac{n(n-1) \times (n-2)!}{(n-2)!} = 6 \times \frac{(n+1)(n) \times (n-1)!}{(n-1)!}$$

$$7 \times n(n-1) = 6 \times n(n+1)$$

$$7(n-1) = 6(n+1)$$

$$7n - 7 = 6n + 6$$

$$n = 13$$

$$\therefore {}^{12}P_2 = 12 \times 11 = 132$$

[5 marks]

6. To prove that $\sqrt{3}$ is irrational, we first assume $\sqrt{3}$ is rational and then show there is a contradiction. Use mathematical notation to write the assumption that $\sqrt{3}$ is rational.

$$\textit{Assume } \exists\, p, q \in \mathbb{Z}: \sqrt{3} = \frac{p}{q}, q \neq 0, p \textit{ and } q \textit{ are coprime.}$$

[3 marks]

***** End of Calculator - Free Section *****

Year 11 Specialist Mathematics

MELVILLE
SENIOR HIGH SCHOOL

Semester 1, March 2021

Part B: Calculator Assumed Section

Time Allowed: 20 minutes

Marks = $\overline{}$
$\mathbf{20}$

Student Name: ______________________________

Teacher: ______________________________

INSTRUCTIONS TO STUDENTS:

- You **are allowed** a CAS calculator
- You **are not allowed** any notes
- A formula sheet will be provided

You are required to attempt ALL questions.

Write answers in the spaces provided beneath each question.

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Show all working clearly, in sufficient detail to allow your answers to be checked readily and for marks to be answered for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks.

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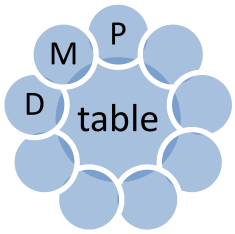
1. A large circular table at a restaurant has 9 people from an office seated at random around it.

a) What is the probability of Mark being seated between David and Peter?

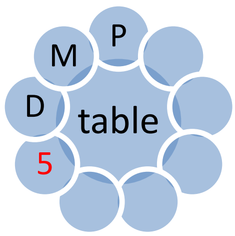


$$\textit{Grouping D M P together}: 2\ \textit{ways with M in middle}$$

$$\textit{Arrangements} = 2 \times \frac{7!}{7} = 2! \times 6! = 1440\ \textit{ways}$$

$$\Pr(\text{DMP together}) = \frac{1440}{8!} = \frac{1}{28}$$

b) Fiona and David do not like each other very much and do not wish to sit next to each other. How many seating arrangements are possible provided Mark is seated between David and Peter, ***and*** Fiona and David do not sit together?



$$\textit{Having DMP together and not Fiona next to D} = 10\ \textit{possibilities and 6 objects}$$

$$\textit{Arrangements} = 5 * 2 \times \frac{6!}{6} = 5 * 2! \times 5! = 1200\ \textit{ways}$$

[3,3 = 6 marks]

2.

a) How many "words" can be formed using the letters from the word ENTERPRISES if you must use every letter?

$$Total\ words = \frac{11!}{3!\,2!\,2!} = 1663200$$

b) How many 4-letter "words" can be formed using the letters from the word ENTERPRISES?

Choose and arrange approach

$$All\ Different: \binom{7}{4} \times 4! = 840$$

$$Two\ E's: \binom{6}{2} \times \frac{4!}{2!} = 180$$

$$Two\ R's: \binom{6}{2} \times \frac{4!}{2!} = 180$$

$$Two\ S's: \binom{6}{2} \times \frac{4!}{2!} = 180$$

$$Two\ R'sand\ 2\ E's: \frac{4!}{2!\,2!} = 6$$

$$Two\ R'sand\ 2\ S's: \frac{4!}{2!\,2!} = 6$$

$$Two\ E'sand\ 2\ S's: \frac{4!}{2!\,2!} = 6$$

$$Three\ E's: \binom{6}{1}\frac{4!}{3!} = 24$$

$$\boldsymbol{Total\,ways\ =\ 1422}$$

[2,7 = 9 marks]

3. There are 255 Year 11 students at Melville SHS. Of these students, 68 study Physics and 44 study Computer Science. 29 students study Chemistry and Physics, 14 study Physics and Computer Science and 19 study Computer Science and Chemistry. 9 students study all 3 subjects and 126 students study none of these subjects.

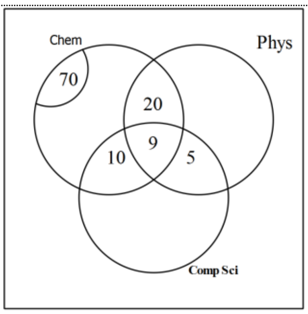
Use the inclusion-exclusion principle to determine the probability that a Year 11 student chosen at random studies chemistry but not Physics or Computer Science.

$$255 \; students - 126 = 129 \; studying \; atleast \; one \; of \; the \; subjects$$

$$|Chem \cup P \cup CS| = |Chem| + |P| + |CS| - |Chem \cup P| - |P \cup CS| - |Chem \cup CS| + |Chem \cap P \cap CS|$$

$$129 = |Chem| + 68 + 44 - 29 - 14 - 19 + 9$$

$$|Chem| = 70$$



∴ 31 study Chem only

$$\Pr(\text{Chem only}) = \frac{31}{255}$$

**Deduct 2 Marks of Inclusion Exclusion Principle not used **

[5 marks]

***** End of Calculator Assumed Section *****