West Coast Collaborative
Test 4 2016
Calculator Free Section

Name: Marking Key   Score: _____ / 40

Section 1 is worth 75% of your final test mark.
<u>No</u> calculators or notes are to be used.
Access to approved Sample Mathematics Specialist formulae sheet is permitted.   Time limit = 40 minutes.

Question 1 (8 marks)

Find the following antiderivatives. Simplify answers, where possible.

(a) $\int \frac{x-1}{x+1} dx$ (2 marks)

$= \int \frac{x+1-2}{x+1} dx$

$= \int \left(1 - \frac{2}{x+1}\right) dx$

$= x - 2\ln|x+1| + c$ ✓ ✓

(b) $\int \frac{\tan^3 2x}{\cos^2 2x} dx$ (2 marks)

$= \int \sec^2 2x \tan^3 2x \, dx$

$= \frac{\tan^4 2x}{8} + c$ ✓ ✓

(c) $\int 5\tan x dx$ (2 marks)

$= 5\int \frac{\sin x}{\cos x} dx$ ✓

$= -5\ln|\cos x| + c$ ✓

(d) $\int \frac{2x^2+1}{6x^3+9x} dx$ (2 marks)

$= \frac{1}{9}\ln|6x^3+9x| + c$ ✓ ✓

Question 2 8 **(9 marks)**

Antidifferentiate 4

(a) $\cos^3 5x$ **(5 marks)**

$$\int \cos^3 5x = \int \cos 5x \cos^2 5x \, dx \checkmark$$

$$= \int (\cos 5x (1 - \sin^2 5x)) dx$$

$$= \int (\cos 5x - \cos 5x \sin^2 5x) dx \checkmark$$

$$= \frac{\sin 5x}{5} \checkmark - \frac{\sin^3 5x}{15} \checkmark + c$$

(b) $(2x + 4) \sqrt[3]{(x - 2)}$ **using the substitution u =** $x - 2$. **(4 marks)**

$$\int (2x+4) \sqrt[3]{x-2} \, dx$$

Let $u = x - 2$

$u' = 1 \Rightarrow du = dx$.

$x = u + 2$

$$= \int (2(u+2) + 4) u^{1/3} du \checkmark$$

$$= \int (2u + 8) u^{1/3} du \checkmark$$

$$= \int (2u^{4/3} + 8u^{1/3}) du$$

$$= \frac{6u^{7/3}}{7} + 6u^{4/3} + c \checkmark$$

$$= \frac{6(x-2)^{7/3}}{7} + 6(x-2)^{4/3} + c$$

$$= 6(x-2)^{4/3} \left(\frac{x-2}{7} + 1 \right) + c$$

$$= 6(x-2)^{4/3} \left(\frac{x-2}{7} + \frac{7}{7} \right) + c$$

$$= 6(x-2)^{4/3} \left(\frac{x+5}{7} \right) + c \checkmark$$

4.

Question 3 **(5 marks)**

Given that $a = \ln 2$, $b = \ln 3$ and $c = \ln 5$, evaluate $\int_1^2 \frac{5x}{5x^2+7}\,dx$ in terms of *a*, *b* and/or *c*.

$$\int_1^2 \frac{5x}{5x^2+7}\,dx = \frac{1}{2}\Big[\ln|5x^2+7|\Big]_1^2 ✓$$

$$= \frac{1}{2}(\ln 27 - \ln 12) ✓$$

$$= \frac{1}{2}\ln\left(\frac{27}{12}\right)$$

$$= \frac{1}{2}\ln\left(\frac{9}{4}\right) ✓$$

$$= \frac{1}{2}(2\ln 3 - 2\ln 2)$$

$$= b - a ✓$$

Question 4 **(3 marks)**

Evaluate $\int_1^4 \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx$ exactly.

$$= \Big[e^{\sqrt{x}}\Big]_1^4 ✓$$

$$= e^2 - e ✓$$

$$= e(e-1) ✓$$

Question 5 **(1 mark)**

Explain why $\int_{-2}^3 \frac{1}{x}\,dx$ is undefined.

because there is no value for $\int \frac{1}{x}\,dx$ when $x = 0$. ✓

or you can't integrate across an infinite discontinuity or for negative values of $\ln x$.

Question 6 **(4 marks)**

Show that $\int_0^{\frac{\pi}{4}} \cos 2x \cos x \, dx = \frac{\sqrt{2}}{3}$

$$LHS = \int_0^{\pi/4} \left(\frac{\cos 3x}{2} + \frac{\cos x}{2} \right) dx \checkmark$$

$$= \left[\frac{\sin 3x}{6} + \frac{\sin x}{2} \right]_0^{\pi/4} \checkmark$$

$$= \frac{1}{6\sqrt{2}} + \frac{1}{2\sqrt{2}} - 0 \checkmark$$

$$= \frac{4}{6\sqrt{2}} \checkmark = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3}$$

or

$$\int_0^{\pi/4} (1 - 2\sin^2 x) \cos x \, dx$$

$$= \int_0^{\pi/4} (\cos x - 2\sin^2 x \cos x) \, dx \checkmark$$

$$= \left[\sin x - \frac{2}{3} \sin^3 x \right]_0^{\pi/4} \checkmark$$

$$= \frac{1}{\sqrt{2}} - \frac{\cancel{2}}{3} \times \frac{1}{\cancel{2}\sqrt{2}} \checkmark$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}} = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3} \checkmark$$

Question 7 **(5 marks)**

Showing full algebraic reasoning determine the following definite integral giving your answer as an exact value.

$$\int_1^2 \frac{3x^2 + 5x - 1}{(x+2)(x+1)^2} = \int \left(\frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \right) dx$$

$$= \int \frac{A(x+1)^2 + B(x+1)(x+2) + C(x+2)}{(x+2)(x+1)^2} \, dx$$

$$= \int \frac{Ax^2 + 2Ax + A + Bx^2 + 3Bx + 2B + Cx + 2C}{(x+2)(x+1)^2} \checkmark$$

$A + B = 3$ — ①

$2A + 3B + C = 5$ — ②

$A + 2B + 2C = -1$ — ③

② × 2: $4A + 6B + 2C = 10$

− ③: $A + 2B + 2C = -1$

$3A + 4B = 11$

− 3 × ①: $3A + 3B = 9$

$B = 2$

$\therefore A = 1$ ✓

$1 + 4 + 2C = -1$

$\Rightarrow C = -3$

$$= \int \left(\frac{1}{x+2} + \frac{2}{x+1} - \frac{3}{(x+1)^2} \right) dx \checkmark$$

$$= \left[\ln|x+2| + 2\ln|x+1| + \frac{3}{x+1} \right]_1^2 \checkmark$$

$$= (\cancel{\ln 4} + 2\ln 3 + 1) - (\ln 3 + \cancel{2\ln 2} + 3/2)$$

$$= \ln 3 - \frac{1}{2} \checkmark$$

Question 8 **(4 marks)**

Use the substitution $u = \ln x$ to determine $\int \frac{\sqrt{\ln x} + \ln\sqrt{x}}{x} dx$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$\therefore du = \frac{dx}{x}$

$= \int \frac{\sqrt{\ln x} + \ln x^{\frac{1}{2}}}{x} dx$

$= \int \frac{\sqrt{\ln x} + \frac{1}{2}\ln x}{x} dx$ ✓

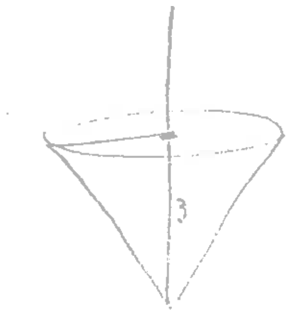
$= \int \left(u^{\frac{1}{2}} + \frac{u}{2}\right) du$ ✓

$= \frac{2u^{3/2}}{3} + \frac{u^2}{4} + C \quad = \frac{2}{3}\sqrt{(\ln x)^3} + \frac{(\ln x)^2}{4} + C$ ✓ ✓

Question 9 **(6 marks)**

The curve $y = \sqrt{11}x$ is rotated one revolution about the y – axis in order to make a cone. Determine the volume of the cone from y = 0 to y = 3 and show that we can get the same result by using the volume of a cone formula

$$V = \frac{\pi r^2 h}{3}$$



$y = \sqrt{11}\,x$

$\therefore x = \frac{y}{\sqrt{11}}$

$\therefore x^2 = \frac{y^2}{11}$ ✓

$V = \int_0^3 \pi x^2 dy$

$= \int_0^3 \pi \left(\frac{y^2}{11}\right) dy$ ✓

$= \pi \left[\frac{y^3}{33}\right]_0^3$

$= \pi\left(\frac{27}{33} - 0\right)$ ✓

$= \frac{9\pi}{11}$ units^3 ✓

$V = \frac{\pi r^2 h}{3}$

when $y = 3$, $x = \frac{3}{\sqrt{11}}$ ✓

$= \frac{\pi \times \left(\frac{3}{\sqrt{11}}\right)^2 \times \cancel{3}}{\cancel{3}}$

$= \frac{9\pi}{11}$ units^3 ✓

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Name: Marking Key   Score: _____ / 13

Section 2 is worth 25% of your final test mark.
Calculators allowed and 1 page of A4 notes, writing on both sides.
Access to approved Sample Mathematics Specialist formulae sheet is permitted. **Time limit = 15 minutes.**

Question 10 **(3,2 marks)**

a) Given $f(x) = \dfrac{x-1}{\sqrt{x^2-2x+2}}$ and $g(x) = 2-x$, determine the area enclosed above by $f(x)$ and $g(x)$ and by the x-axis below.

Roots $x=1$ and $x=2$ ✓ Intersection at $x = 1.5310101$

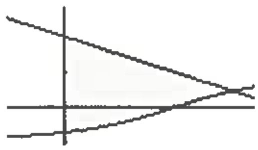
(not to scale)

$$\text{Area} = \int_1^{1.53} \frac{x-1}{\sqrt{x^2-2x+2}} + \int_{1.53}^{2} (2-x)\,dx \checkmark$$

$$= 0.1322419027 + 0.1099757632$$

$$= 0.2422176659 \text{ units}^2 \checkmark$$

b) For the same two functions as part (a), determine the area enclosed on the left by the Y-axis and on the right by $f(x)$ and $g(x)$.

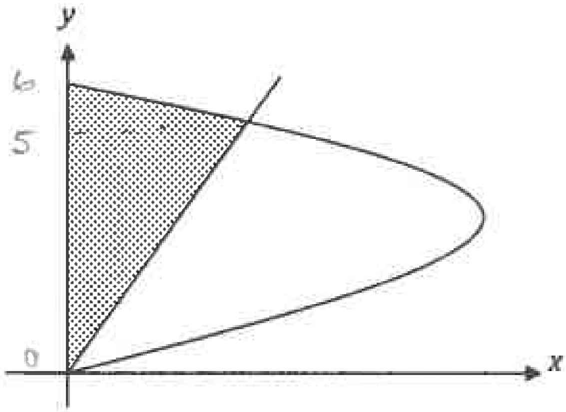

(not to scale)

$$\text{Area} = \int_0^{1.5310101} |g(x) - f(x)|\,dx \checkmark$$

$$= 2.171995896 \text{ units}^2 \checkmark$$

Question 11 **(4,1 marks)**

The diagram below shows the graph of $x = 6y - y^2$ and the line $y = x$.



a) Write an integral that can be used to find the area shaded. **(4 marks)**

$= \int_0^5 y\,dy + \int_5^6 (6y - y^2)\,dy$

b) Determine this area. **(1 mark)**

$\frac{91}{6}$ or $15\frac{1}{6}$ units^2

Question 12 **(3 marks)**

Calculate the area between the functions $y_1 = x - 0.1$ and $y_2 = x^2 \ln x$, giving your answers to 4 decimal places.

Intersection at $x = 1.7632228$ ✓ and 0.0828901 ✓

$$\text{Area} = \int_0^{1.7632228} |x - 0.1 - x^2 \ln x| \, dx$$

$$= 0.9570845656$$

so 0.9571 units^2 ✓