



MINDARIE
SENIOR COLLEGE

WHERE YOUR FUTURE BEGINS NOW

MATHEMATICS: SPECIALIST 1 & 2

EXTENDED PIECE OF WORK 2

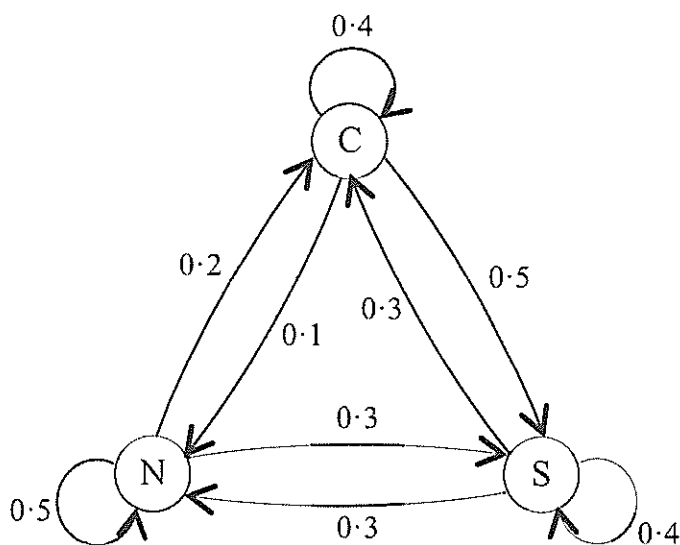
Time Allowed: 60 minutes

Total Marks: 25

Emu Taxi Company have divided Perth into three regions: North, Central, and South. By keeping track of pick-ups and deliveries, the company has found that of the fares picked up in the North, 50% stay in that region, 20% are taken to Central, and 30% go to the South region.

Of the fares picked up in Central, only 10% go to the North, 40% stay in Central, and the remainder go to the South. Of the fares picked up in the South, 30% go to each of North and Central, and the rest stay in the South.

The information can be illustrated with a network diagram that shows the probabilities along each arc:



From

Or by 'transition matrix'

$$T = \begin{matrix} & \begin{matrix} N & C & S \end{matrix} \\ \begin{matrix} N \\ C \\ S \end{matrix} & \begin{bmatrix} 0.5 & 0.1 & 0.3 \\ 0.2 & 0.4 & 0.3 \\ 0.3 & 0.5 & 0.4 \end{bmatrix} \end{matrix}$$

1. [1 mark]

Explain the entry t_{21} ($= 0.2$)

20% will start in the north & finish at central ✓

For future planning purposes, the company would like to know what the distribution of taxis will be over time as they pick up and drop off successive fares. For example, if a taxi starts in Central, what is the probability it will be in Central after letting off its third fare? Its fourth fare?

A tree diagram could be used but as the number of fares increases, the number of branches and subsequent calculations become very tedious.

Matrix multiplication is far quicker.

2. [2, 1, 3 marks]

(a) Determine T^2 and explain its meaning.

$T = \begin{bmatrix} 0.36 & 0.24 & 0.3 \\ 0.27 & 0.33 & 0.3 \\ 0.37 & 0.43 & 0.4 \end{bmatrix}$ ✓ The probability of ending up in any region based on initial pick up region after Two fares. ✓

(b) What is the probability that a taxi that starts in South region will be in North region after dropping off its second fare?

0.3 or 30% ✓

(c) Determine the sums of the entries in each row and column of T^2 and comment on your results.

$\sum \text{rows} = \begin{matrix} 0.9 \\ 0.9 \\ 1.0 \end{matrix}$ A population drain ✓
From N & C into S. ✓

$\sum \text{col} = \begin{matrix} 1 & 1 & 1 \end{matrix}$

A taxi will end up in N, C or S after 2 trips no matter where it started ✓

anything reasonable.

3. [1, 1 marks]

- (a) Determine T^3 and hence state the probability that a taxi that starts in Central will still be in Central after dropping off its third fare.

$$\begin{bmatrix} 0.318 & 0.282 & 0.3 \\ 0.291 & 0.309 & 0.3 \\ 0.391 & 0.409 & 0.4 \end{bmatrix} \quad 0.309 \checkmark$$

- (b) Where should a taxi start from to have the best chance of being in North region after 3 fares?

Start from North (31.8%)
✓

4. [1, 1, 1 marks]

- (a) Explain the meaning of entries in T^n .

The probability of ending up in any region based on pick up region after n fares. ✓

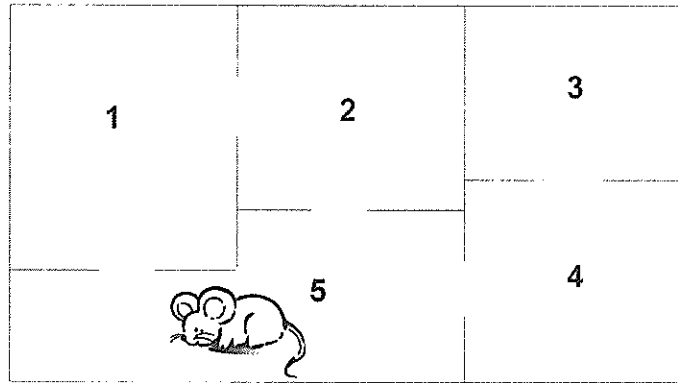
- (b) Investigate larger values of n . You should notice that the entries in the matrix begin to stabilise. Use appropriate rounding to state the 'Stable State Matrix'.

$$\begin{bmatrix} 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 \\ 0.4 & 0.4 & 0.4 \end{bmatrix} \quad \checkmark$$

- (c) Hence determine the probability that a taxi that starts in North region will be in South region at the end of the day after n fares.

0.4 - 40%
✓

Systems like the Taxi problem that involve transitions (picking up and dropping off fares) that occur at regular intervals are called Markov Chains.



A mouse is placed in the maze shown above. During a fixed time interval the mouse randomly chooses one of the doors available to it (depending on which room it is in) and moves to the next room – it does not remain in the room it occupies.

Each movement is considered a transition in a *Markov Chain*.

If the mouse started in Room 1, the probabilities that the mouse will be in each room after 1 transition are:

- Room 1: 0 (since it must move to another room)
- Room 2: 0.5 (since it has two equally likely choices, room 2 or 5)
- Room 3: 0 (Since it is impossible to get to room 3 from room 1)
- Room 4: 0 (Since it is impossible to get to room 4 from room 1)
- Room 5: 0.5 (since it has two equally likely choices, room 2 or 5)

Thus the first row of the transition matrix T would be:

		From room
		1
To room	1	0
	2	0.5
	3	0
	4	0
	5	0.5

5. [2, 1, 2, 1, 1 marks]

(a) Construct the entire transition matrix T for this process.

$$\begin{bmatrix}
 0 & 0.5 & 0 & 0 & 1/3 \\
 0.5 & 0 & 0 & 0 & 1/3 \\
 0 & 0 & 0 & 0.5 & 0 \\
 0 & 0 & 1 & 0 & 1/3 \\
 0.5 & 0.5 & 0 & 0.5 & 0
 \end{bmatrix}$$

✓
✓ -1/error

(b) If the mouse starts in room 1, what is the probability that it is in room 3 after 4 transitions?

$$\frac{1}{24} = 0.041\bar{6} \quad \checkmark$$

- (c) Using appropriate rounding, determine the 'Stable State Matrix'.

$$\begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \end{bmatrix} \begin{matrix} 1 \\ 1 \\ 0.5 \\ 1 \\ 1.5 \end{matrix}$$

✓✓

- (d) After a large number of transitions, what is the probability that the mouse will be in room 3?

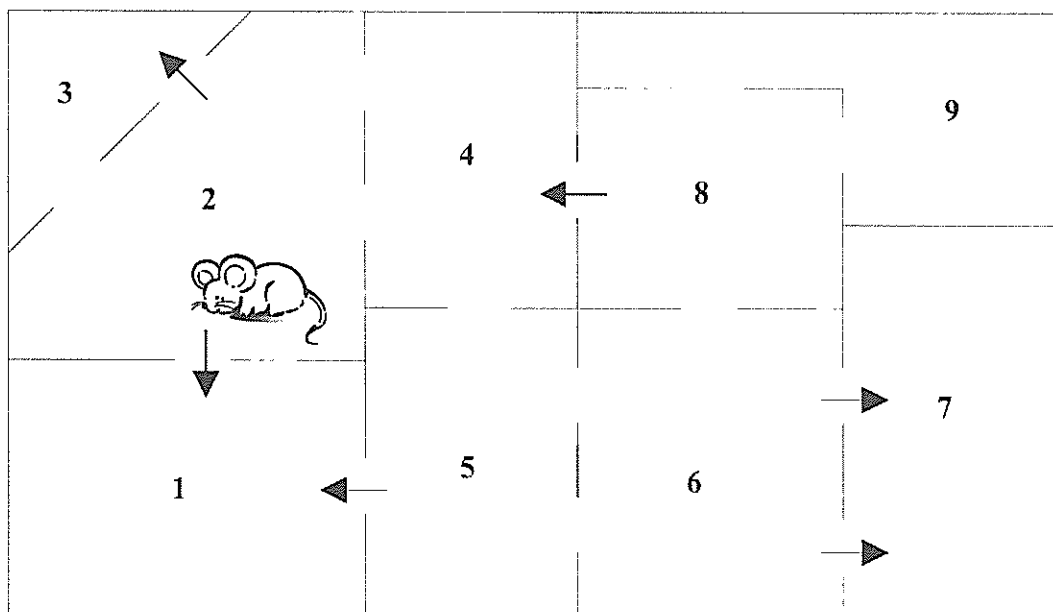
$$10\% = 0.1$$

✓

- (e) In the long run, what percentage of the time will the mouse spend in rooms 4 or 5?

$$50\%$$

✓



A mouse is in the maze shown above. Doors are shown by openings between rooms. Arrows indicate one-way doors and the direction of passage through the one-way doors. The mouse does not have to change rooms at each transition, but can stay in a room. Notice that some of the rooms are impossible to leave once they are entered. During each transition, the mouse has an equal chance of leaving a room by a particular door or staying in the room. For example, during a single transition, a mouse in room 2 has a one in six chance of moving into room 1, a one in six chance of staying in room 2, a one in three (two in six) chance of moving to room 3, and a one in three (two in six) chance of moving to room 4.

6. [3, 2, 1 marks]

(a) Find the transition matrix which describes the movement of the mouse.

$$\begin{bmatrix} 1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{5} & \frac{1}{6} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{5} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{5} & \frac{1}{3} & 0 \end{bmatrix}$$

(b) If the mouse starts in room 4, what is the probability that it will eventually be trapped in room 1?

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✓✓

(c) In which room, besides room 7, should the mouse be started to have the best chance of being trapped in room 7?

Room 6 (642)