Specialist Mathematics Units 3 & 4

Investigation 2 2017

Vector Applications

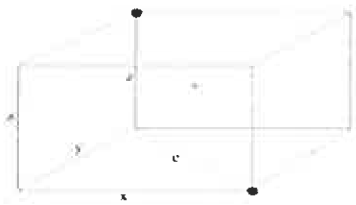
Name: Marking Key **Marks:** ___ / ~~55~~ 6

Validation Section1

Write your responses in the space provided. CAS calculators will be allowed in Section 2 only but no other notes will be permitted. Other electronic devices must be switched off and in bags.

Many vector procedures in two dimensions can be meaningfully applied in three in exactly the same way, just with a third dimension.

The distance between two points in three dimensions is complicated by the third dimension, and needs to be calculated in the same way you would find the length of the longest diagonal in a box.



$c^2 = x^2 + y^2 \qquad s^2 = c^2 + z^2$

$\therefore \quad s^2 = x^2 + y^2 + z^2$

1. **[1, 2, 1, 2 marks = ~~7~~ 6 marks]**

For the points A = 3**i**+**j**, denoted<3,1,0> and P = -**i**+**j**+2**k**, denoted <-1,1,2> , where **k** is the unit vector parallel to the *z* axis,

a) Determine the vector $\overrightarrow{AP}$

$\overrightarrow{AP} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix}$ ✓

b) the exact distance between A and P

$\sqrt{(-4)^2 + 0^2 + 2^2}$ ✓

$= \sqrt{20}$

$= 2\sqrt{5}$ units ✓

c) the unit vector $\overrightarrow{AP}$

$$= \frac{1}{2\sqrt{5}}\begin{pmatrix}-4\\0\\2\end{pmatrix} \checkmark$$

$$\text{or} \quad \begin{pmatrix}\frac{-2}{\sqrt{5}}\\0\\\frac{1}{\sqrt{5}}\end{pmatrix}$$

The format of the vector equation of a line $\mathrm{r} = \underset{\sim}{a}+\lambda \underset{\sim}{b}$ remains the same in 3D.

d) State a vector equation for the line containing both points.

$$\underset{\sim}{r} = \begin{pmatrix}3\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}-4\\0\\2\end{pmatrix} \quad \checkmark\checkmark$$

$$\text{or} \quad \begin{pmatrix}-1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}-4\\0\\2\end{pmatrix}$$

END OF PAPER

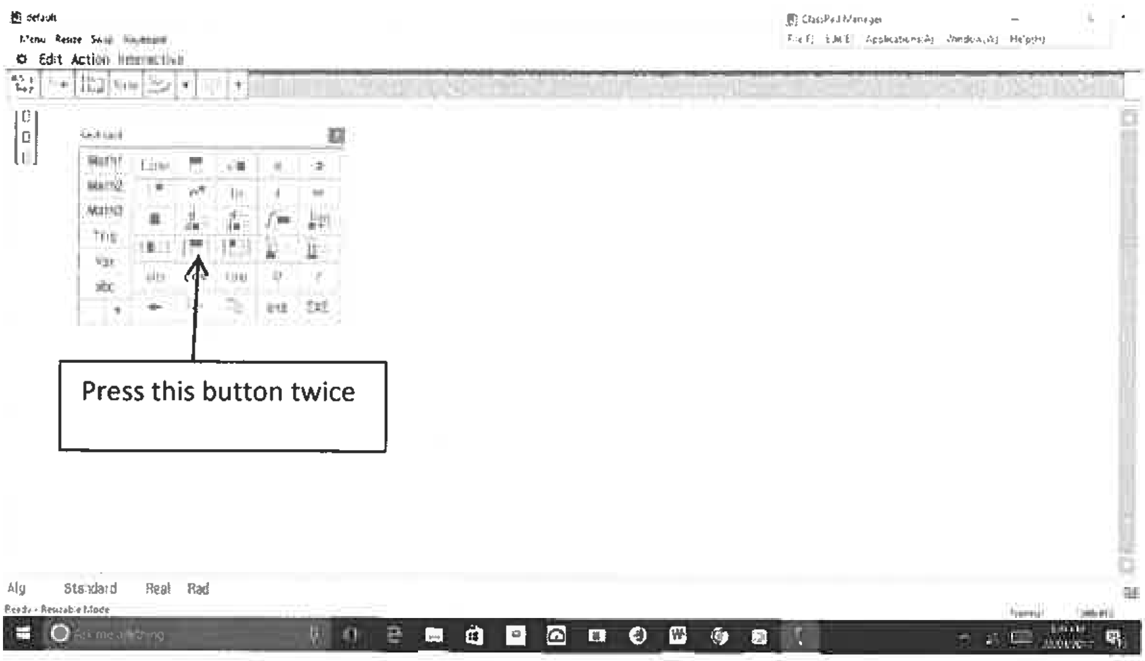
Vector Applications

Name: Marking Key **Marks:** ___ /~~55~~ 43

Validation Section 2

Write your responses in the space provided. CAS calculators allowed in this section only but no other notes will be permitted. Other electronic devices must be switched off and in bags.

Graphic calculator vector operations can be duplicated by converting 2D vector shapes as indicated below.



Otherwise the work done in the Take Home Section can be reproduced using 3D vectors.

2. [~~5~~ 6 marks]

Use a scalar product method to determine the distance from point A <2,-3,4> to the point P on the line $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$, where P is the point closest to A.

$$\overrightarrow{AP} = \begin{pmatrix} 1+4\lambda \\ -3+2\lambda \\ 2-\lambda \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4\lambda-1 \\ 2\lambda \\ -\lambda-2 \end{pmatrix} \checkmark \checkmark$$

$$\overrightarrow{AP} \cdot \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = 0 \checkmark$$

Solve gives $\lambda = 2/21$ ✓

$\therefore P = -\frac{13}{21}\underset{\sim}{i} + \frac{4}{21}\underset{\sim}{j} - \frac{44}{21}\underset{\sim}{k}$ ✓

$|\overrightarrow{AP}| = 2.193$ units (3dp) ✓

3. **[marks]**

During army war games a targeted drone T is being shot at using a non-guided missile A. The position vectors are in multiples of 100m at the time t = 0 and the velocities in 100m/s.

$$\mathbf{R_T} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \qquad \mathbf{v_T} = \begin{pmatrix} 7 \\ 10 \\ -3 \end{pmatrix} \qquad \mathbf{r_A} = \begin{pmatrix} 5 \\ 28 \\ -6 \end{pmatrix} \qquad \mathbf{v_A} = \begin{pmatrix} 6 \\ 1 \\ -2 \end{pmatrix}$$

a) Assuming both maintain the same velocities show that the target will be hit by the missile. State the position coordinates and the time of that collision.

$$r_T(t) = \begin{pmatrix} 2+7t \\ 1+10t \\ -3-3t \end{pmatrix} \checkmark \qquad r_A(t) = \begin{pmatrix} 5+6t \\ 28+t \\ -6-2t \end{pmatrix} \checkmark$$

Equating $\underset{\sim}{i}, j, \underset{\sim}{k}$ components

$$2+7t = 5+6t \quad \text{—①}$$
$$1+10t = 28+t \quad \text{—②} \quad \checkmark$$
$$-3-3t = -6-2t \quad \text{—③}$$

Using simultaneous equations - solve gives $t = 3$ ✓

Position of collision $= \begin{pmatrix} 2+7t \\ 1+10t \\ -3-3t \end{pmatrix} \Big| \, t=3$ ✓

$= 23\underset{\sim}{i} + 31\underset{\sim}{j} - 12\underset{\sim}{k}$ ✓

b) Missile B is also fired at the drone. Its position and velocity vectors are given below.

$$\mathbf{R_B} = \cancel{\begin{pmatrix} 15 \\ 36 \\ 16 \end{pmatrix}} \begin{pmatrix} 9.5 \\ 27 \\ 13.5 \end{pmatrix} \qquad \mathbf{v_B} = \begin{pmatrix} 3 \\ 1 \\ -7 \end{pmatrix}$$

Show that the missiles' paths will cross but that they will not collide. Give the point of their crossing and difference in time between them.

$$\text{Let } \underset{\sim}{r}_A(t_1) = \begin{pmatrix} 2+7t_1 \\ 1+10t_1 \\ -6-2t_1 \end{pmatrix} \qquad \underset{\sim}{r}_B(t_2) = \begin{pmatrix} 9.5+3t_2 \\ 27+t_2 \\ 13.5-7t_2 \end{pmatrix} \checkmark$$

Equating $\underset{\sim}{i}, j + \underset{\sim}{k}$ components

$$5+6t_1 = 9.5+3t_2 \quad \text{—①}$$
$$28+t_1 = 27+t_2 \quad \text{—②} \quad \checkmark \checkmark$$
$$-6-2t_1 = 13.5-7t_2 \quad \text{—③}$$

Solving by simultaneous equations gives $t_1 = 2.5 \quad t_2 = 3.5$ ✓

Point of crossing is $\begin{pmatrix} 20 \\ 30.5 \\ -11 \end{pmatrix}$ ✓ their crossing is 1 second apart. ✓

5 marks

c) Determine the angle between the missiles' direction vectors, to the nearest degree

$$\underset{\sim}{v}_A = \begin{pmatrix} 6 \\ 1 \\ -2 \end{pmatrix} \quad \underset{\sim}{v}_B = \begin{pmatrix} 3 \\ 1 \\ -7 \end{pmatrix}$$

$$\cos\theta = \frac{\begin{pmatrix} 6 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -7 \end{pmatrix}}{\left|\begin{pmatrix} 6 \\ 1 \\ -2 \end{pmatrix}\right| \times \left|\begin{pmatrix} 3 \\ 1 \\ -7 \end{pmatrix}\right|} \checkmark \checkmark \checkmark$$

$$= \frac{33}{\sqrt{41} \times \sqrt{59}}$$

$$= 0.6709590355 \checkmark$$

$$\theta = \cos^{-1} ans = 47.85887317, \quad \text{So } 48^\circ \checkmark$$

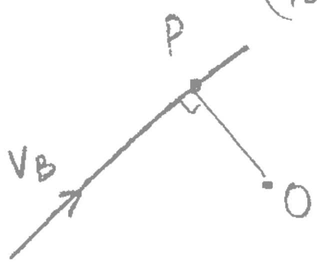
Missile B's path takes it past an observation platform, situated at <42,~~70~~ 79,2>.

8 marks

d) How close does the missile get to the platform to the nearest 100m ?

Point P, point of closest approach

$$= \begin{pmatrix} 9.5 + 3t \\ 27 + t \\ 13.5 - 7t \end{pmatrix} - \begin{pmatrix} 42 \\ 79 \\ 2 \end{pmatrix} = \begin{pmatrix} 3t - 32.5 \\ t - 43 \\ -7t + 11.5 \end{pmatrix} \checkmark \checkmark$$



$$\begin{pmatrix} 3 \\ 1 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} 3t - 32.5 \\ t - 43 \\ -7t + 11.5 \end{pmatrix} = 0 \checkmark$$

Solving gives $t = 3.745762712$ ✓

$$\text{So distance} = \left|\begin{pmatrix} 3t - 32.5 \\ t - 43 \\ -7t + 11.5 \end{pmatrix}\right| t = ans = \left|\begin{pmatrix} -21.26 \\ -39.25 \\ -14.72 \end{pmatrix}\right| \checkmark \text{ (2dp)}$$

$= \text{norm } ans$ ✓

$= 47.00730199$ ✓

so 4.7 km ✓

10 marks

e) We have already established that Missiles A and B smoke trails crossed.

Assuming both were fired at exactly the same time, How far apart were they at their closest point to the nearest 100m and when did they come closest to each other to the nearest 0.1 seconds ?

Stop everything with respect to B.

$$ {}_A\underset{\sim}{r}_B = \begin{pmatrix} 5 \\ 28 \\ -6 \end{pmatrix} - \begin{pmatrix} 9.5 \\ 27 \\ 13.5 \end{pmatrix} = \begin{pmatrix} -4.5 \\ 1 \\ -19.5 \end{pmatrix} \checkmark $$

$$ {}_A\underset{\sim}{V}_B = \begin{pmatrix} 6 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ -7 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix} \checkmark $$

Closest point P relative to B $= \begin{pmatrix} -4.5 + 3t \\ 1 \\ -19.5 + 5t \end{pmatrix} \checkmark$

$$ \overrightarrow{PB} \cdot {}_A\underset{\sim}{V}_B = 0 \quad \text{ie} \quad \begin{pmatrix} 3t - 4.5 \\ 1 \\ 5t - 19.5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix} = 0 \checkmark $$

Solving gives $t = \frac{111}{34}$ s. or 3.264705882 ✓

So 3.3 seconds ✓

$$ \text{Distance} = \left| \begin{pmatrix} 3t - 4.5 \\ 1 \\ 5t - 19.5 \end{pmatrix} \right| t = \text{ans} \checkmark \checkmark $$

$= 6.254$ km (3dp) ✓

So 6.3 km (nearest 100m) ✓

END OF PAPER