Specialist Mathematics Units 3 & 4
Test 1 2017
Calculator Free Section

Name: Marking Key **Score:** _____ **/ 22**

Section 1 is worth approximately 42% of your final test mark.
<u>No</u> calculators or notes are to be used.
Access to approved Mathematics Specialist formulae sheet is permitted.
Simplify answers where possible. **Time limit = 25 minutes.**

1. **[4 marks: 3, 1]**

a) On the axes below, draw the graphs of $\arg(z+3i)=\frac{3\pi}{4}$ and $|z|=3$, for $0 \leq \theta \leq 2\pi$

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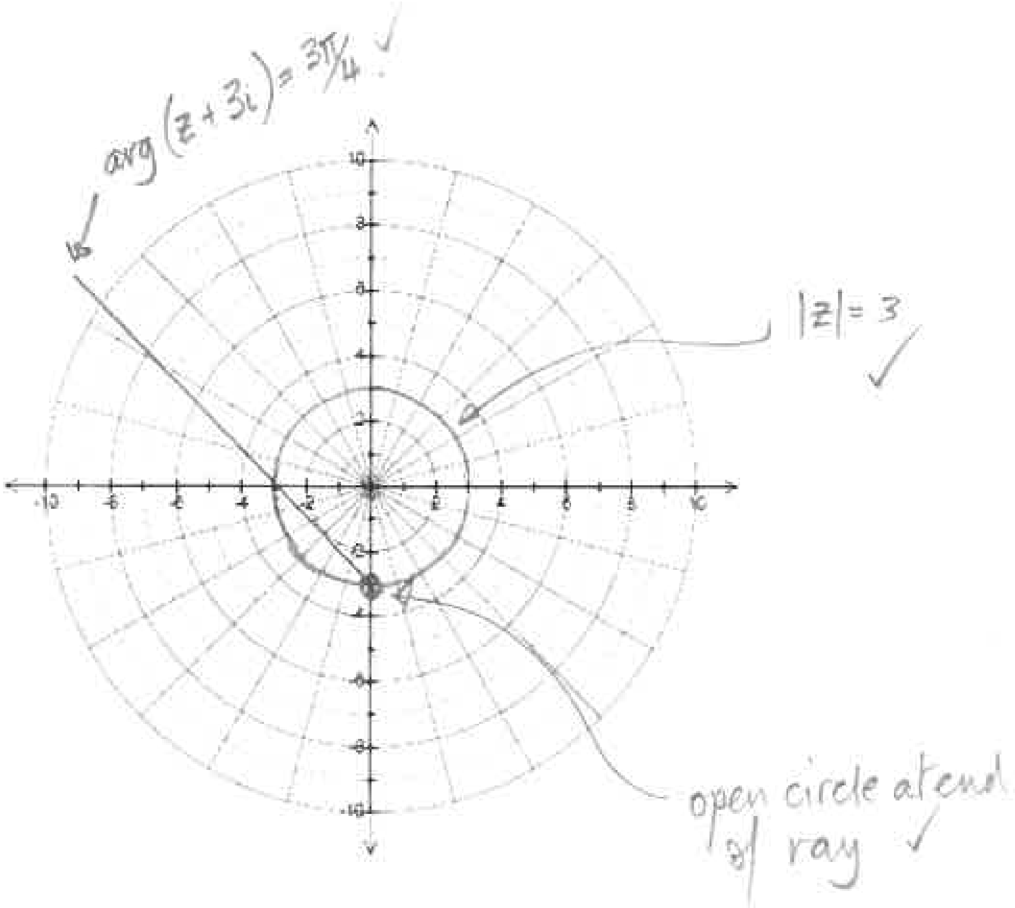
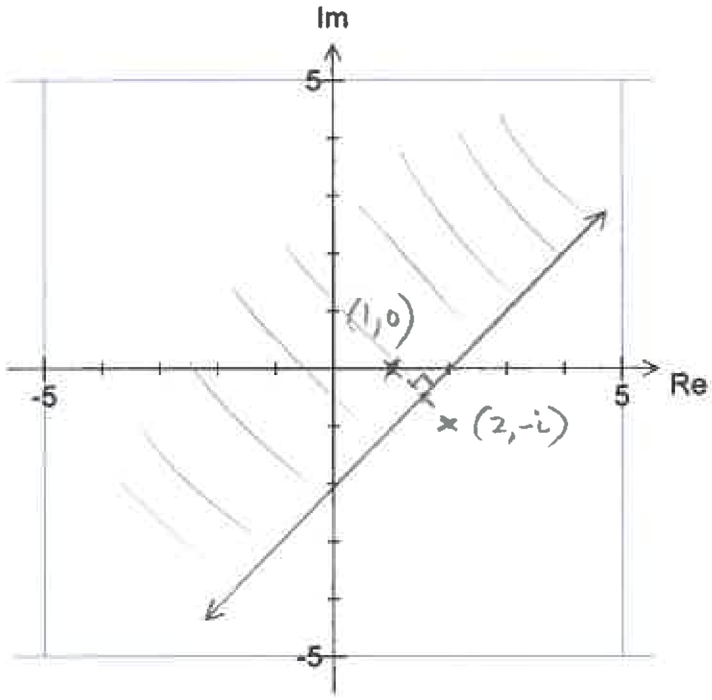


b) Write down the coordinates of the point where the graphs intersect.

$(-3,0)$ or $3\,\mathrm{cis}\,\pi$ or $3\,\mathrm{cis}(-\pi)$ ✓

2. **[7 marks: 3, 4]**

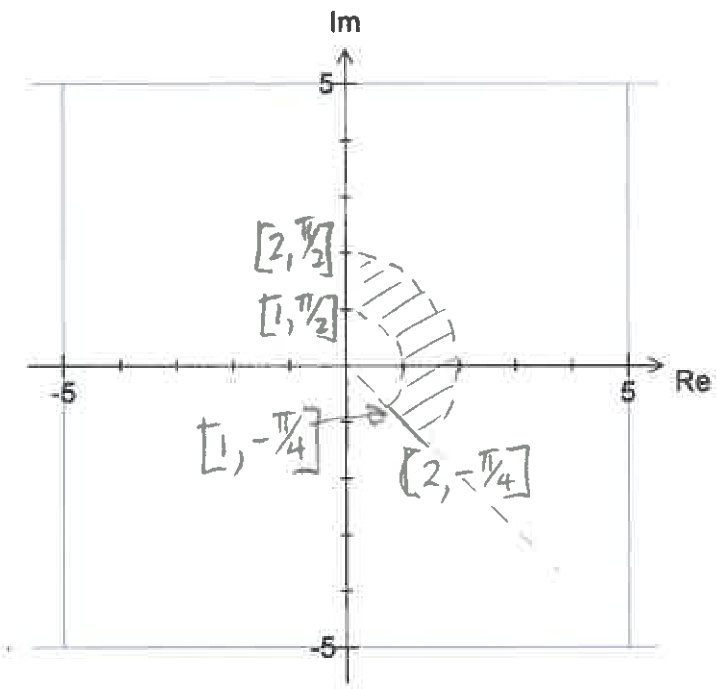
(a) Sketch $\{z : |1-z| \le |z+i-2|\}$ on the Argand diagram.
Identify important features.



shading ✓
points (1,0) and (2,−i) ✓
perpendicular bisecting line ✓

(b) Sketch the region of the complex plane defined by (4 marks)

$\{z: -\frac{\pi}{4} \le \arg z \le \frac{\pi}{2} \quad \cap \quad 1< |z| <2\}$ Identify important features.



shading ✓
dotted lines r = 1 and r = 2 ✓
critical points ✓
shape of region ✓

3. **[6 marks: 1, 1, 1, 3]**

The complex numbers of z and w are such that:

$z = 2 + 5i$ and $w = 4 - 3i$.

Express each of the following in the form a + bi.

a) $\bar{z} + w$

$2 - 5i + 4 - 3i$

$= 6 - 8i$ ✓

b) w^2

$(4-3i)(4-3i)$

$= 16 - 12i - 12i + 9i^2$

$= 7 - 24i$ ✓

c) wz $(2+5i)(4-3i)$

$= 8 - 6i + 20i - 15i^2$

$= 23 + 14i$ ✓

d) $\frac{z}{w} = \frac{2+5i}{4-3i} \times \frac{4+3i}{4+3i}$ ✓

$= \frac{-7+26i}{25}$ ✓ ✓

4. **[5 marks]**

Factorise the polynomial $x^4 - 2x^3 - 7x^2 + 8x + 12$.

$f(1) \neq 0$, $f(-1) = 0$ & $f(2) = 0$

so $(x+1)$ and $(x-2)$ are factors ✓ ✓

$\therefore (x+1)(x-2)(ax^2+bx+c) = f(x)$

By inspection. $a = 1$ $c = -6$ ✓

so $(x^2-x-2)(x^2+bx-6) = f(x)$

Equating x^3 terms: $b - 1 = -2$

so $b = -1$ ✓

$\therefore (x+1)(x-2)(x^2-x-6) = f(x)$

$\therefore (x+1)(x-2)(x-3)(x+2) = f(x)$ ✓

END OF PAPER

West Coast Collaborative
Specialist Mathematics Units 3 & 4
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Calculator Section

Name: Marking Key. **Score: _____ / 31**

Section 2 is worth 58% of your final test mark.
Calculators allowed and 1 page of A4 notes, writing on both sides.
Access to approved Mathematics Specialist formulae sheet is permitted.
Time limit = 30 minutes

5. **[4 marks: 2, 2]**

Use De Moivre's Theorem to find the exact value in polar form of

a) $(1-\sqrt{3}i)^{11}$

$= \left(2\,cis\left(-\frac{\pi}{3}\right)\right)^{11}$ ✓

$= 2^{11}\,cis\left(\frac{\pi}{3}\right)$ ✓

or $\left(1024 + 1024\sqrt{3}\,i\right)$

b) $(-1+i)^{-15}$

$= \left(\sqrt{2}\,cis\left(\frac{3\pi}{4}\right)\right)^{-15}$

$= \frac{\sqrt{2}}{256}\,cis\left(-15\times\frac{3\pi}{4}\right)$ ✓

$= \frac{\sqrt{2}}{256}\,cis\left(\frac{3\pi}{4}\right)$ ✓

or $\left(-\frac{1}{256} + \frac{1}{256}i\right)$

6. **[5 marks]**

By finding z^3 if $z = cis\theta$, show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta\, i^2\sin^2\theta + i^3\sin^3\theta$

$= \cos^3\theta - 3\cos\theta\sin^2\theta + 3\cos^2\theta\, i\sin\theta - i\sin^3\theta$ ✓

By De Moivre's Theorem $= \cos 3\theta + i\sin 3\theta$

Equating Imaginary parts ✓

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ ✓

$= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta$ ✓

$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$ ✓

$= 3\sin\theta - 4\sin^3\theta$

7. **[3 marks]**

If $z = 4cis\theta$ and $z = Acis\left(\frac{2\pi}{3}\right)$ are two of the four, fourth roots of w.

Find, exactly, all values of A and θ, where $-\pi \le \theta \le \pi$, for the four roots.

$A = 4$ ✓

Let $z_1 = 4\ cis\left(\frac{2\pi}{3}\right)$ ✓½

then $z_2 = 4\ cis\left(\frac{2\pi}{3} + \frac{\pi}{2}\right) = 4\ cis\left(-\frac{5\pi}{6}\right)$ ✓½

$z_3 = 4\ cis\left(-\frac{5\pi}{6} + \frac{\pi}{2}\right) = 4\ cis\left(-\frac{\pi}{3}\right)$ ✓½

$z_4 = 4\ cis\left(-\frac{\pi}{3} + \frac{\pi}{2}\right) = 4\ cis\left(\frac{\pi}{6}\right)$ ✓½

8. **[7 marks: 2, 1, 4]**

For z, a complex number where $z = \cos\theta + i\sin\theta$;

a) Show that $\frac{1}{z^n} = \cos n\theta - i\sin n\theta$

$$LHS = \frac{1}{\cos n\theta + i\sin n\theta} \times \frac{\cos n\theta - i\sin n\theta}{\cos n\theta - i\sin n\theta} \checkmark$$

$$= \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta} = \frac{\cos n\theta - i\sin n\theta}{1} \checkmark$$

$$= RHS$$

b) Hence, or otherwise, show that $2\cos n\theta = z^n + \frac{1}{z^n}$

$RHS = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$ (from above) ✓

$= 2\cos n\theta$

c) Hence show that $\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$

$$\left(z+\frac{1}{z}\right)^5 = (2\cos\theta)^5 \checkmark$$

$$LHS = z^5 + \frac{5z^4}{z} + 10\frac{z^3}{z^2} + \frac{10z^2}{z^3} + \frac{5z}{z^4} + \frac{1}{z^5} \checkmark$$

$$= z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$$

$$= \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right) \checkmark$$

$$= 2(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$$

$$RHS = 2^5\cos^5\theta \checkmark \quad \therefore \quad \cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$$

9. **[4 marks]**

$2x^n + ax^2 - 6$ leaves a remainder of -7 when divided by $x-1$, and 129 when divided by $x+3$. Find a and n, given that $a, n \in Z$

$$f(1) = 2\times 1^n + a - 6 = -7$$

$$\Rightarrow a = -3 \checkmark$$

$$f(-3) = 2(-3)^n + -3(-3)^2 - 6 = 129 \checkmark$$

$$\Rightarrow 2(-3)^n - 27 - 6 = 129$$

$$\Rightarrow (-3)^n = 81 \checkmark$$

$$\Rightarrow n = 4 \checkmark$$

10.
~~11.~~ **[8 marks]**

$3+i$ is a root of $z^4-2z^3+az^2+bz+10$, where a and b are real. Determine a and b and the other roots of the equation.

3 − i is also a root since all coefficients are real ✓

So $z-3+i$ and $z-3-i$ are both factors

So $(z^2-6z+10)(cz^2+dz+e) = z^4-2z^3+az^2+bz+10$ ✓

By inspection $c=1$ $e=1$. ✓

So $(z^2-6z+10)(z^2+dz+e) = f(z)$ ✓

Comparing z^3 terms

$d-6=-2 \Rightarrow d=4$ ✓

So $(z^2-6z+10)(z^2+4z+1)=0$ ✓

$$\left(\frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \frac{2\sqrt{3}}{2}\right)$$ ✓

Therefore $z^4-2z^2-13z^2+34z+10=0$ ✓

$a=-13$ $b=34$ ✓

Roots are $-3 \pm i$ and $-2 \pm \sqrt{3}$ ✓