Year 12 Methods Solutions to Test 6 Past Questions

Question 3 **[4 marks]**

Determine the co-ordinates and nature of the stationary points for $f(x) = x\ln(x^2)$

$$f'(x) = x \cdot \frac{2x}{x^2} + 1 \cdot \ln(x^2)$$

$$= 2 + \ln(x^2) \checkmark$$

$$f''(x) = \frac{2x}{x^2} = \frac{2}{x}$$

$$f'(x) = 0 \rightarrow x = +e^{-1}, -e^{-1} \checkmark$$

$$f''(e^{-1}) = 2e^{1} > 0 \text{ min}$$

$$f''(-e^{-1}) = -2e^{1} < 0 \text{ max} \checkmark$$

$$\underset{\text{max}}{\left(e^{-1}, -\frac{2}{e}\right)} \quad \underset{\text{min}}{\left(-e^{-1}, +\frac{2}{e}\right)} \checkmark$$

Question 4 **[4 marks: 2, 2]**

The continuous random variable X is defined by the probability density function

$$f(x) = \begin{cases} \frac{q}{x} & 1 \le x \le 3 \\ 0 & \textit{Elsewhere} \end{cases}$$

(a) Determine the exact value of q.

$$\int_1^3 \frac{q}{x}\,dx = 1 \checkmark$$

$$\left[q\ln(x) + c\right]_1^3 = 1$$

$$q\ln(3) - q\ln(1) = 1$$

$$q\ln(3) = 1$$

$$q = \frac{1}{\ln 3} \checkmark$$

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Question 4 cont.

(b) Determine the cumulative distribution function (CDF) for $1 \le t \le 3$.

$$\checkmark \int_1^t \frac{1}{x \ln 3}\,dx = \left[\frac{\ln(x)}{\ln 3} + c \right]_1^t$$

$$= \frac{\ln(t)}{\ln(3)} \checkmark$$

Question 5 **[3 marks: 2, 1]**

A political party is keen to know the proportion of voters in Australia who support a policy. A census is not possible. A far more efficient approach would be to sample voters and then form a reliable estimate for the proportion p.

(a) List two things that you can do to make the policy support proportion estimate more reliable.

> 30 per sample ✓
Many samples ✓
of voters!

(b) What statistical effect are you trying to reduce?

Bias. ✓

Question 3 **[8 marks: 1, 1, 2, 2, 2]**

A bakery packages a loaf of bread as a Standard if it weighs between 450g and 500g. The weights of all loaves produced by the bakery are normally distributed with a mean of 470g and a standard deviation of 16g.

(a) What is the probability that a randomly selected loaf produced by the bakery.

(i) weighs 450g? $P(X=450)=0$ ✓

(ii) is a Standard loaf? $P(450<X<500)=0.8640$ ✓

(b) In a batch of 250 loaves, how many would be expected to weigh less than a Standard loaf?

$P(X<450) = 0.1056$ ✓

$250 \times 0.1056 = 26.4$

≈ 26 ✓

(c) Determine the probability that a randomly selected Standard loaf weighs less than 470g.

$$P(X<470 / 450<X<500)$$

$$= \frac{P(450<X<470)}{P(450<X<500)}$$

$$= \frac{0.3944}{0.8640} = 0.4565$$ ✓✓

(d) A bakery worker randomly selects loaves of bread until the loaf chosen is not a Standard. Determine the probability that the sixth loaf chosen is the first that is not a Standard.

$$P = 0.8460^5 \times (1-0.8640)$$

$$= 0.0655$$ ✓✓

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Question 2 **[7 marks: 2, 3, 2]**

The continuous random variable X has a probability density function

$$f(x)\begin{cases} \dfrac{2x}{9} & 0 \le x \le 3 \\ 0 & \textit{Elsewhere} \end{cases}$$

(a) Determine E(X).

$$E(X) = \int_0^3 \left(x \cdot \frac{2x}{9}\right) dx$$

$$= \left[\frac{2x^3}{27} + c\right]_0^3$$

$$= 2 \quad ✓✓$$

(b) Determine the cumulative distribution function F(x).

$$F(x) = \int_0^x \frac{2t}{9}\, dt$$

$$= \left[\frac{2t^2}{18} + c\right]_0^x$$

$$= \frac{x^2}{9}, \quad 0 \le x \le 3 \quad ✓✓✓$$

(c) Calculate P(1 < X < 2)

$$P(1 < X < 2) = F(2) - F(1)$$

$$= \frac{4}{9} - \frac{1}{9}$$

$$= \frac{1}{3} \quad ✓✓$$

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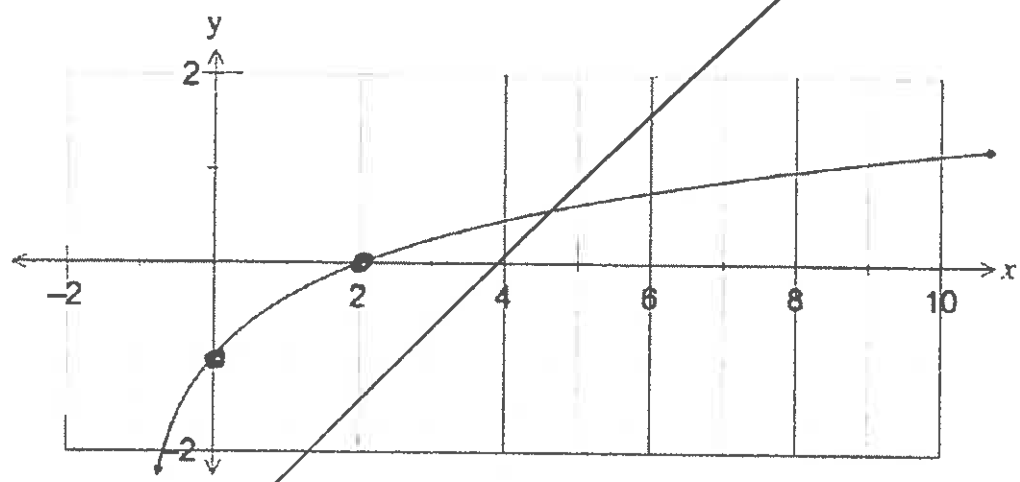
Question 4 **[2 marks]**

The graph of $y = f(x)$, where $f(x) = \log_a (x + 1) + b$, is shown below.

Determine the values of a and b.



$f(0) = \log_a(1) + b = -1 \rightarrow b = -1$ ✓

$f(2) = \log_a(3) - 1 = 0 \rightarrow a = 3$ ✓

Question 5 **[2 marks]**

The discrete random variable X is defined by $P(X = x) = \frac{1}{2}\log x$ for x = 2, 5 and 10.

Show that this is a probability distribution.

$\frac{1}{2}\log 2 + \frac{1}{2}\log 5 + \frac{1}{2}\log 10 = 1$ ✓

$\frac{1}{2}\log(2 \times 5 \times 10) = 1$

$\frac{1}{2}\log 100 = 1$ ✓

$1 = 1$

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Question 8 **[4 marks: 2, 2]**

$f(x) = \dfrac{1}{x \ln 5}$ for $1 \le x \le 5$ is a continuous probability density function.

(a) Determine the mean of f(x). Round it to four decimal places.

$$\int_1^5 x \cdot \frac{1}{x \ln 5} dx = 2.4853 \checkmark$$

$\checkmark$

(b) Detemine the variance of f(x). Round it to four decimal places.

$$\int_1^5 (x - 2.4853)^2 \frac{1}{x \ln 5} dx \checkmark$$

$$= 1.279 \checkmark$$

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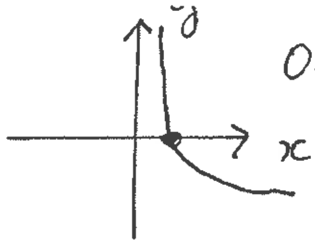

$0<k<1$

Question 10 **[3 marks]**

Determine the range of possible values of *k* so that $f(x) = \log_{10} x$ and $g(x) = \log_k x$ can intersect at one point. Hint: graph both log functions.

✓ $k>0 \rightarrow \log_k x = \frac{\ln x}{\ln k} = \frac{\ln x}{\ln(-ve)}$ ⟵ undefined

✓ $k \neq 1 \rightarrow \frac{\ln x}{\ln 1}$ ⟵

✓ $k \neq 10 \rightarrow f(x) = g(x)$. You will see only one curve, Not a single pt of intersection

$\{0<k<1\} \cup \{1<k<10\} \cup \{k>10\}$

Question 11 **[3 marks: 1, 2]**

Karen achieves a result of 67% for a test where the class mean for the test was 40% and the standard deviation was 18%.

(a) What is Karen's standard normal distribution Z-score?

$$Z = \frac{67-40}{18}$$
$$= 1.5$$ ✓

(b) The class teacher wants to scale the the test results so that the class test mean becomes 58% with a standard deviation of 12%.

What scaled % did the Karen achieve for her test?

$58 + 1.5 \times 12$

$= 76\%$ ✓✓

6

2018

Question 4 **[8 marks: 3, 1, 3, 1]**

It is known that 15% of households in a large city own a cat.

(a) Let X be the random variable that represents the number of cat owning households chosen in a random sample of 60 households.

(i) Describe the distribution of X and state its mean and standard deviation.

$X \sim Bin(60, 0.15)$ ✓

$\bar{x} = 60 \times 0.15 = 9$ ✓ $sd = \sqrt{60 \times 0.15 \times 0.85} = 2.766$ ✓

(ii) Determine the probability that 12 or more cat owning households will be chosen in a random sample of 60.

$P(X \geqslant 12) = 0.1806$ ✓

(b) A large number of random samples of 60 households are taken and each sample is used to calculate a point estimate for the proportion of cat owning households in the city.

(i) Describe the distribution of these sample proportions and state the mean and standard deviation of the distribution of the distribution.

$X \sim Norm(p, s^2)$ ✓

$p = 0.15$ ✓

$s = \sqrt{\frac{0.15 \times 0.85}{60}} = 0.0461$ ✓

(ii) Determine the probability that a point estimate for the proportion of cat owning households in the city exceeds 0.2.

$P(X > 0.2) = 0.1390$ ✓

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Question 5 **[6 marks: 4, 2]**

Vlad runs a live Christmas tree farm. He has 243 suitable trees for sale. Vlad measures the height of each tree. He then collated the information into a frequency histogram. Vlad discovers that the histogram shows a normal distribution with a mean tree height of 195cm and a standard deviation of 32cm.

The trees are sold at the following prices. The tree heights range from 5 foot (153cm) to 8 foot (244 cm). Any thing less than 5 foot or greater then 8 foot is not sold.

Probs.

Tree height in cm.	Price per tree
0.2592 153 to 183	$30
0.3698 183 to 214	$40
0.2135 214 to 244	$50

Estimate the total revenue if the trees are sold to the public. How many trees will not be sold due to their height?

$$243 \times (1 - 0.2592 - 0.3698 - 0.2135) \cong 38$$

$$\$ = 0.2592 \times 243 \times 30 + 0.3698 \times 243 \times 40 + 0.2135 \times 243 \times 50$$

$$= 8078.05$$

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2018

Question 6 **[4 marks]**

Let the proportion of parents/guardians in a college that support a four-day school week be p^.

A random sample of 200 parents/guardians was selected and 78 indicated that they support the proposal.

Find the % confidence level if the margin of error is 0.1.

$$\hat{p} = \frac{78}{200} = 0.39 \checkmark$$

$$\text{Solve } k\sqrt{\frac{0.39 \times 0.61}{200}} = 0.1$$

$$k = 2.8995 \checkmark$$

$$X \sim N(0, 1^2)$$

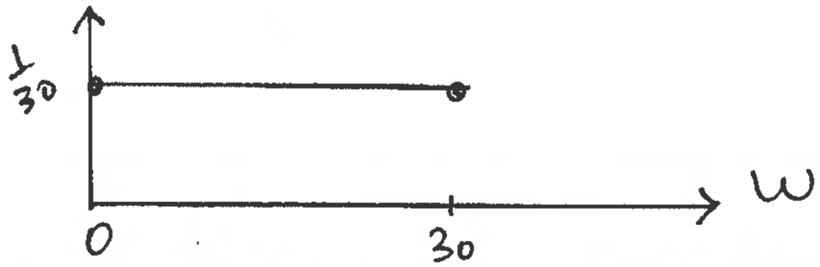
$$P(-2.8995 \leq X \leq 2.8995) = 0.9963 \checkmark$$

$$99.63\% \checkmark$$

Question 7 **[8 marks: 2, 1, 2, 1, 2]**

A bus service departs from a terminus every 30 minutes throughout the day. If a passenger arrives at the terminus at a random time to catch the bus, their waiting time in minutes, W, until the next bus departs is a uniformly distributed random variable.

(a) Sketch the graph of the probability density function of W.



✓✓

(b) What is the probability that a passenger who arrives at the terminus at a random time has to wait no more than 25 minutes for the bus to depart?

$$\frac{25-0}{30} = \frac{5}{6}$$ ✓

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Question 7 cont.

(c) What is the probability that fewer than four passengers, out of a random selection of ten, have to wait at least 25 minutes for the bus to depart?

$X \sim Bin(10, \frac{1}{6})$ ✓✓

$P(X \leq 3) = 0.9303$

(d) Determine the probability that a passenger who waits for at least 12 minutes has to wait for no more than 20 minutes.

$\frac{20-12}{30-12} = \frac{8}{18}$ ✓

$= \frac{4}{9}$

(e) Determine the value of w for which . $P(W < w) = P(W > 3w)$.

$w = 30 - 3w$ ✓

$4w = 30$

$w = 7.5$ ✓

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Question 8 **[8 marks: 2, 2, 4]**

The random variable X denotes the number of hours that a business telephone line is in use per nine hour working day.

The probability density function of X is given by

$$f(x) = \begin{cases} \dfrac{(x-a)^2 + b}{k} & 0 \le x \le 9 \\ 0 & \textit{Elsewhere} \end{cases}$$

where a, b and k are constants.

(a) If $a = 15$ and $b = 3$, determine the value of k.

✓ $\int_0^9 \dfrac{(x-15)^2+3}{k}\,dx = 1 \rightarrow k = 1080$ ✓

(b) Let $a = 16$, $b = 1$ and $k = 1260$.

(i) The business is open for work for 308 days per year. On how many of these days can the business expect the phone line to be in use for more than eight hours?

$\int_8^9 \dfrac{(x-16)^2+1}{1260}\,dx = 0.0455$ ✓

$308 \times 0.0455 = 14$ days ✓

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Question 8 cont.

(ii) Determine, correct to two decimal places, the mean and variance of X.

$$E(X) = \int_0^9 \frac{x[(x-16)^2+1]}{1260}\,dx = 3.39 \checkmark$$

$$Var(X) = \int_0^9 (x-3.39)^2 \frac{[(x-16)^2+1]}{1260}\,dx$$

$$= 5.78 \checkmark$$

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