Specialist Mathematics Units 3 & 4

Investigation 2 2017

Vector Applications

Solutions

Take Home Section

Complete this Take Home Component on file paper showing all working out and reasoning. Use of CAS calculator to aid calculation is assumed, particularly for Questions 2 to 5. On completion of Part 1 there will be a Validation Task (Part 2). For Part 2, CAS calculators will be allowed in the calculator section of the validation, but no other notes will be permitted.

Part 1: Take Home Component

1. Given two points **A**<-9,3> and **P** <3,-3> determine

a) the vector $\overrightarrow{AP}$

$\overrightarrow{AP} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} - \begin{pmatrix} -9 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ -6 \end{pmatrix}$ ✓ ✓ or $12\underset{\sim}{i} - 6\underset{\sim}{j}$

b) the exact distance between **A** and **P**

$\sqrt{12^2 + (-6)^2}$ ✓ $= \sqrt{180}$ or $6\sqrt{5}$ ✓

c) the unit vector $\overrightarrow{AP}$

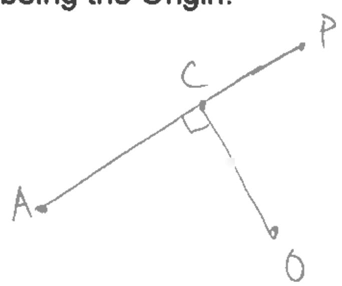
$= \dfrac{1}{6\sqrt{5}} \begin{pmatrix} 12 \\ -6 \end{pmatrix}$ or $\begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}$ ✓

d) the vector equation of the line containing both points

$\underset{\sim}{r} = \begin{pmatrix} -9 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 12 \\ -6 \end{pmatrix}$ or $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 12 \\ -6 \end{pmatrix}$ ✓

Point C lies on the line AP such that $\overrightarrow{CO}$ is perpendicular to $\overrightarrow{AP}$, *O* being the Origin.

e) Determine the coordinates of point C.



$C = \begin{pmatrix} -9 + 12\lambda \\ 3 - 6\lambda \end{pmatrix}$ ✓

$\overrightarrow{AP} \cdot \overrightarrow{CO} = 0$

$\begin{pmatrix} 12 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} -9 + 12\lambda \\ 3 - 6\lambda \end{pmatrix} = 0$ ✓

Solve gives $\lambda = 7/10$ ✓

So $C = \begin{pmatrix} -9 + 12 \times 7/10 \\ 3 - 6 \times 7/10 \end{pmatrix} = \begin{pmatrix} -3/5 \\ -6/5 \end{pmatrix}$ ✓

f) Determine $|\overrightarrow{CQ}|$. $\left|\begin{matrix}-3/5\\ 6/5\end{matrix}\right| = \sqrt{\dfrac{3^2+6^2}{25}}$ ✓

$= \dfrac{3\sqrt{5}}{5}$ or 1.341640786 ✓

2. The position vectors (**r**) and velocity vectors (**v**) of two ships A and B at 9.00 a.m. on a particular day were as follows:

$r_A = 30\underset{\sim}{i} + 30\underset{\sim}{j}$ km $\quad v_A = 12\underset{\sim}{i} - 4\underset{\sim}{j}$ km/h.

$r_B = -10\underset{\sim}{i} + 35\underset{\sim}{j}$ km $\quad v_B = 20\underset{\sim}{i} - 5\underset{\sim}{j}$ km/h.

Show that if the two ships continue with these velocity vectors they will collide.

$r_A(t) = \begin{pmatrix}30+12t\\30-4t\end{pmatrix}$ ✓ $\quad r_B(t) = \begin{pmatrix}-10+20t\\35-5t\end{pmatrix}$ ✓

Equating $\underset{\sim}{i}$ components $\quad 30+12t = -10+20t$ ✓

$\Rightarrow t = 5$ ✓

Equating j components $\quad 30-4t = 35-5t$ ✓

$\Rightarrow t = 5$ ✓

Since both times are the same, the ships will collide ✓

3. The position vectors (**r**) and velocity vectors (**v**) of two ships A and B at 9.00 a.m. on a particular day were as follows:

$r_A = 10\underset{\sim}{i} + 15\underset{\sim}{j}$ km $\quad v_A = 12\underset{\sim}{i} + 4\underset{\sim}{j}$ km/h.

$r_B = -10\underset{\sim}{i} + 35\underset{\sim}{j}$ km $\quad v_B = 20\underset{\sim}{i} + 5\underset{\sim}{j}$ km/h.

a) Show that if the two ships continue with these velocity vectors their *paths will cross* but they will *not collide.*

$r_A(t_1) = \begin{pmatrix}10+12t_1\\15+4t_1\end{pmatrix}$ ✓ $\quad r_B(t_2) = \begin{pmatrix}-10+20t_2\\35+5t_2\end{pmatrix}$ ✓

Equating $\underset{\sim}{i}$ components $\quad 10+12t_1 = -10+20t_2$

$\Rightarrow 12t_1 - 20t_2 = -20$ —— ① ✓

Equating j components $\quad 15+4t_1 = 35+5t_2$

$\Rightarrow 4t_1 - 5t_2 = 20$ —— ②. ✓

$\Rightarrow t_1 = 25,\ t_2 = 16$ ✓

Both ships are at the same point, but at different times, so their paths will cross but not collide. ✓

3b) ~~4.~~ Determine the angle between the ships' respective direction vectors, giving your answer to the nearest degree. Full working must be shown for full. marks

$V_A = \begin{pmatrix} 12 \\ 4 \end{pmatrix} \quad V_B = \begin{pmatrix} 20 \\ 5 \end{pmatrix}$

$\cos\theta = \frac{\underset{\sim}{V_A} \cdot \underset{\sim}{V_B}}{|\underset{\sim}{V_A}||\underset{\sim}{V_B}|} = \frac{\begin{pmatrix} 12 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 20 \\ 5 \end{pmatrix}}{\left|\begin{pmatrix} 12 \\ 4 \end{pmatrix}\right| \times \left|\begin{pmatrix} 20 \\ 5 \end{pmatrix}\right|}$

$= \frac{260 \checkmark}{4\sqrt{10} \times 5\sqrt{17}} \checkmark$ ✓

$\therefore \theta = 4.398705355^\circ$ ✓

So 4° (0dp) ✓

4. ~~5.~~ Particle P starts moving from a point with position vector < 10, 14 > metres with constant velocity < 5, 2 > metres per second. P continues with this velocity passing a stationary object at A< 34,12 > metres. Determine the closest distance between P and A. *State the position of the closest point to A and when this occurs.*

$P(t) = \begin{pmatrix} 10 + 5t \\ 14 + 2t \end{pmatrix} \checkmark \quad \underset{\sim}{r_A} = \begin{pmatrix} 34 \\ 12 \end{pmatrix}$

$\overrightarrow{AP} = \begin{pmatrix} 10 + 5t - 34 \\ 14 + 2t - 12 \end{pmatrix} \checkmark = \begin{pmatrix} 5t - 24 \\ 2t + 2 \end{pmatrix} \checkmark$

$\underset{\sim}{V_P} \cdot \overrightarrow{AP} = 0$

so $\begin{pmatrix} 5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 5t - 24 \\ 2t + 2 \end{pmatrix} = 0$

Solving gives $t = 4$ ✓

So $\overrightarrow{AP} = \begin{pmatrix} 5\times4 - 24 \\ 2\times4 + 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 10 \end{pmatrix} \checkmark$

$|\overrightarrow{AP}| = 2\sqrt{29}$ or 10.77032961 m ✓

Point P is $\begin{pmatrix} 10 + 5\times4 \\ 14 + 2\times4 \end{pmatrix} = \begin{pmatrix} 30 \\ 22 \end{pmatrix} \checkmark$

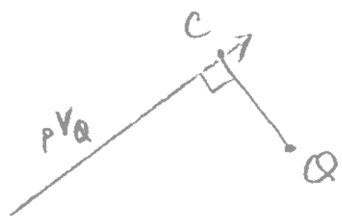
Closest point to A is $30\underset{\sim}{i} + 22\underset{\sim}{j}$ m. after 4 seconds ✓

5. ~~6.~~ Objects P and Q start moving from points with position vectors < —5, —15 > m and < 10, 20 > m with constant velocities < 3, −4 > m/s and < 1, —5 > m/s respectively. By using relative positions and relative velocities and a scalar product method, determine the closest distance between P and Q. State the positions of P and Q at the time and when this occurs.

$$\underset{\sim}{r}_P = \begin{pmatrix} -5 \\ -15 \end{pmatrix} \quad \underset{\sim}{r}_Q = \begin{pmatrix} 10 \\ 20 \end{pmatrix} \quad \underset{\sim}{V}_P = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad \underset{\sim}{V}_Q = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

$${}_P r_Q = \begin{pmatrix} -5 \\ -15 \end{pmatrix} - \begin{pmatrix} 10 \\ 20 \end{pmatrix} = \begin{pmatrix} -15 \\ -35 \end{pmatrix} ✓$$ Stopping everything relative to Q.

$${}_P \underset{\sim}{V}_Q = \begin{pmatrix} 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} ✓$$



C the closest point is at $\begin{pmatrix} -15 + 2t \\ -35 + t \end{pmatrix}$. ✓

$$\text{So} \quad \begin{pmatrix} -15 + 2t \\ -35 + t \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0 ✓$$

Solve gives $t = 13$ ✓

$$\text{So} \quad \underset{\sim}{r}_P = \begin{pmatrix} -5 + 3 \times 13 \\ -15 - 4 \times 13 \end{pmatrix} = \begin{pmatrix} 34 \\ -67 \end{pmatrix} ✓ \quad \underset{\sim}{r}_Q = \begin{pmatrix} 10 + 1 \times 13 \\ 20 - 5 \times 13 \end{pmatrix} = \begin{pmatrix} 23 \\ -45 \end{pmatrix} ✓$$

$$\text{Closest distance} = \left| \begin{pmatrix} 34 \\ -67 \end{pmatrix} - \begin{pmatrix} 23 \\ -45 \end{pmatrix} \right| ✓$$

$$\left| \begin{pmatrix} 11 \\ -22 \end{pmatrix} \right| ✓$$

$$= 11\sqrt{5} \quad \text{or } 24.59674775 \text{ m} ✓$$

P is at $34\underset{\sim}{i} - 67\underset{\sim}{j}$ and Q at $23\underset{\sim}{i} - 45\underset{\sim}{j}$ when the distance is least, at 24.597m (3dp) after 13 seconds ✓