# 12 Mathematics Methods 2022

## Test 3 – Calculus of trig, exponential and log functions

## Section 1: Calculator-free

**Time allowed: 25 minutes** **Maximum marks: 24**

**Name:** Solutions **Teacher:** Foster | Kelly

**Instructions:**

- Show all working clearly.
- Sufficient detail must be shown for marks to be awarded for reasoning.
- A formula sheet will be provided.
- No calculators or personal notes are permitted.

**Question 1** **[2, 2 = 4 marks]**

Differentiate the following functions with respect to $x$.

a) $y = \cos[\ln x]$

$$\frac{dy}{dx} = -\sin(\ln x) \cdot \frac{1}{x}$$

$$= \frac{-\sin(\ln x)}{x}$$

b) $y = \int_0^{4x} 0.5 \tan 4t \, dt$

$$\frac{dy}{dx} = 0.5 \tan 4(4x) \cdot 4$$

$$= 2 \tan 16x$$

**Question 2** **[3, 3 = 6 marks]**

a) Calculate the following.

i) $\int \frac{2-\cancel{7}5x^2}{5x^3-6x-9} dx$

$$= -\frac{1}{3} \int \frac{15x^2-6}{5x^3-6x-9} dx$$

$$= -\frac{1}{3} \ln |5x^3-6x-9| + c$$

ii) $\int_0^{\pi/6} 4 \sin(3x) \, dx$

$$= \left[ -\frac{4}{3} \cos 3x \right]_0^{\pi/6}$$

$$= -\frac{4}{3} \cos \frac{3\pi}{6} - \left( -\frac{4}{3} \cos 0 \right)$$

FT

$$= 0 + \frac{4}{3}$$

$$= \frac{4}{3}$$

FT

**Question 3** **[1, 3, 4 = 8 marks]**

Cedric the cyclist travels up and down hills with his velocity ($kmh^{-1}$) after $t$ hours given by

$v(t) = 20 + 5\sin(3\pi t)$

a) Determine a function for Cedric's acceleration.

$a(t) = v'(t) = 15\pi\cos(3\pi t)$ ✓

b) Determine his maximum acceleration and the first time this occurs for $t > 0$.

max acceleration $= 15\pi \ kmh^{-2}$ ✓✓

period $= \frac{2\cancel{\pi}}{3\cancel{\pi}} = \frac{2}{3}$ ✓

$\therefore$ if $t > 0$; $t = \frac{2}{3}$ hrs

c) Calculate Cedric's average speed over first 2 hours.

$\int_0^2 v(t)\, dt = \left[ \frac{-5}{3\pi}\cos(3\pi t) + 20t \right]_0^2$ ✓

$= \left(\frac{-5}{3\pi}\cdot 1 + 40\right) - \left(\frac{-5}{3\pi}\cdot 1 + 0\right) = 40$ ✓✓

$\therefore$ average speed $= \frac{40}{2} = 20 km/h$ ✓

**Question 4** **[6 marks]**

Find the area enclosed, in the first quadrant, between $y = 2e^x$, $x = \frac{1}{2}\log_e y$ and the $y$-axis.

$x = \frac{1}{2}\log_e y$ ; $2x = \ln y$

$y = e^{2x}$

$2e^x = e^{2x}$ ✓

$\ln(2e^x) = \ln e^{2x}$

$\ln 2 + x = 2x$

$x = \ln 2$ ✓

$\therefore \int_0^{\ln 2} (2e^x - e^{2x})\,dx = \left[2e^x - \frac{1}{2}e^{2x}\right]_0^{\ln 2}$ ✓

$= \left(2e^{\ln 2} - \frac{1}{2}e^{2\ln 2}\right) - \left(2e^0 - \frac{1}{2}e^0\right)$

$= \left(4 - \frac{1}{2}\cdot 4\right) - \left(2\times 1 - \frac{1}{2}\right)$ ✓

$= 2 - 1.5$

$= \frac{1}{2}$ units$^2$ ✓

**END OF SECTION 1**

**12 Mathematics Methods 2022**

**Test 3 – Calculus of trig, exponential and log functions**

**Section 2: Calculator-assumed**

**Time allowed: 25 minutes** **Maximum marks: 25**

**Name:**______________________________ **Teacher:** Foster | Kelly

**Instructions:**

- Show all working clearly.
- Sufficient detail must be shown for marks to be awarded for reasoning.
- A formula sheet will be provided.
- Calculators and 1xA4 double-sided page of personal notes are permitted.

**Question 5** **[3 marks]**

If $y = 3\sin 2x + 2\ \cos 2x$, determine the value of $k$ for which $ky = \frac{d^2y}{dx^2}$

$y' = 6\cos 2x - 4\sin 2x$ ✓

$y'' = -12\sin 2x - 8\sin 2x$ ✓

$\therefore\ k = -4$ ✓

**Question 6** **[4 marks]**

Determine the values of $a$ and $b$ where $f(x) = ax\ln(bx)$, $f(1) = 12$ and $f'(1) = 16$.

$f(1) = a\ln b = 12$ ✓

$f'(x) = a\ln bx + \frac{1}{\cancel{b}\cancel{x}} \cdot \cancel{b}\, a\, \cancel{x}$

$f'(1) = a\ln b + a = 16$ ✓

$\therefore\ a = 16 - 12 = 4$ ✓

$\ln b = 3$

$b = e^3$ ✓ $(\doteq 20.086)$

**Question 7** **[2, 3 = 5 marks]**

Let $g(x)$ be a function such that $g(-14) = g(16) = -72$, $g(-2) = g(4) = 0, g(1) = 3$ and $g'(1) = 0$.

If $g'(x) > 0\ for - 14 \leq x < 1$ and $g'(x) < 0\ for\ 1 < x \leq 16$, evaluate:

a) $\int_1^{16} g'(x)dx$

$= g(16) - g(1) = -72 - 3$
$= -75$

b) What is the area bounded by the graph of $g'(x)$ and the $x$-axis between $x = -14$ and $x = 16$? Justify your answer.

$\text{Area} = |g(1) - g(-14)| + |g(16) - g(1)|$
$= 75 + 75 = 150 \text{ units}^2$

(as $g'(x)$ is below axis at $x < 1$ then above at $x > 1$)

**Question 8** **[2, 2, 2 = 6 marks]**

$V$ is the volume of water left in a 350Litre container, $t$ minutes after a leak occurs. The rate of change of $V$ can be modelled by $\frac{dV}{dt} = kV$. After 3 minutes, 43 litres of water were lost.

a) Write an equation to model the amount of water left in the tank after $t$ mins given that the container was full when the leak started.

$307 = 350e^{3k}$; $k = -0.043695$

$\therefore V = 350e^{-0.043695t}$

b) To the nearest litre, how much water was lost in the first 20mins?

$t = 20$; $350 - 350e^{kt} = 204$ litres

c) Determine the rate at which water is leaving the tank, when the tank has 10Litres of water remaining.

$\frac{dV}{dt} = -0.043695 \times 10 = -0.43695 L/min$

**Question 9** **[2, 3, 3 = 8 marks]**

An AFL match starts at 2:40pm.

The rate at which spectators enter a stadium is given by $S'(t) = \frac{9000e^{-0.05t}}{(1+4e^{-0.05t})^2}$ spectators per minute, where $t$ is the time from 1pm.

a) At what rate were spectators entering the stadium when the match started?

$S'(100) \simeq 57.5$ ectators / mi.

b) When, to the nearest minute, were spectators entering at the fastest rate and what was this rate?

$S''(t) = 0$ ; $t = 28$ 1:28

$S'(28) \simeq 562$ ecl / mi.

1

c) If 5000 spectators were already in the stadium at 1pm, how many spectators were in the stadium when the match starts?

$5000 + \int_0^{100} S'(t)\,dt = 3\ 819$ spectators.

FT FT

**END OF TEST**