

NAME: Marking Key

Investigation 2 – Vector Product (Cross Product)

IN-CLASS VALIDATION TEST

Time allowed: 50 MINUTES

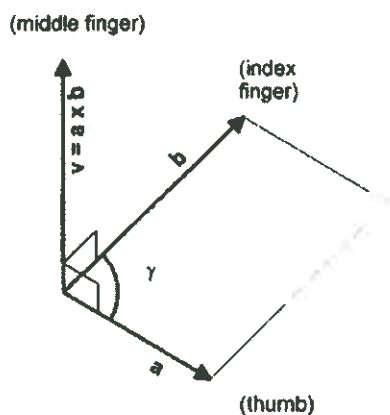
Total Marks

47 MARKS

NOTES are NOT permitted in this in-class validation test. An approved graphic calculator is allowed to be used. The definitions reproduced below are for your reference:

The vector product or cross product of two vectors **a** and **b** is a vector **v** which is written as:

$$\mathbf{v} = \mathbf{a} \times \mathbf{b} \quad \text{and is defined as follows:}$$



If **a** and **b** have the same or opposite direction or either **a** or **b** is zero then $\mathbf{v} = \mathbf{a} \times \mathbf{b} = \mathbf{0}$, otherwise;

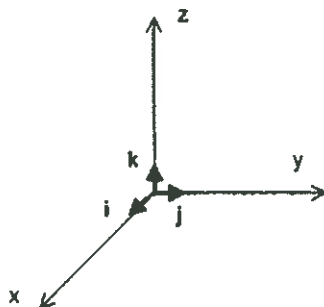
$\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is the vector whose length is equal to the area of the parallelogram with **a** and **b** as adjacent sides and whose direction is perpendicular to both **a** and **b** and is such that **a**, **b**, **v**, in this order, form a right-handed triple or right handed triad, as shown in the diagram.

The term right-handed is derived from the fact that the vectors **a**, **b**, **v**, in this order, assume the same orientation as the thumb, index finger and middle finger of the right hand when held as indicated in the

diagram.

The parallelogram with **a** and **b** as adjacent sides has the area $|\mathbf{v}| = |\mathbf{a}||\mathbf{b}|\sin(\gamma)$

Where γ is the angle between **a** and **b**.



Suppose that the vectors **a**, **b**, and **v** are represented in component form with respect to a right-handed, three dimensional Cartesian coordinate system, with unit vectors **i**, **j** and **k** in the directions **x**, **y** and **z** respectively.

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

Then

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Finally, if $\mathbf{v} = \mathbf{a} \times \mathbf{b}$, and the component vectors of **v** are v_1 , v_2 and v_3 , then the magnitude of **v** is $|\mathbf{v}| = |\mathbf{v}_1\mathbf{i} + \mathbf{v}_2\mathbf{j} + \mathbf{v}_3\mathbf{k}|$

$$\text{where } |\mathbf{v}|^2 = v_1^2 + v_2^2 + v_3^2$$

QUESTION 1

[1 Mark]

Is $\mathbf{i} \times \mathbf{j}$ a scalar or a vector quantity? **ANSWER:**

Vector ✓

QUESTION 2

[5, 1, 5, 2 Marks]

If, $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$ then determine the following, showing full working of each element of the calculation, where appropriate:

2a) $\mathbf{a} \times \mathbf{b}$

$$\begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ 2 & 3 & -5 \\ 1 & 4 & -1 \end{vmatrix} \checkmark$$

$$= (-3 + 20)\underline{\mathbf{i}} - (-2 + 5)\underline{\mathbf{j}} + (8 - 3)\underline{\mathbf{k}}$$

$$= \begin{pmatrix} 17 \\ -3 \\ 5 \end{pmatrix} \checkmark$$

2b) $\mathbf{b} \times \mathbf{a}$

$$\underline{\mathbf{b}} \times \underline{\mathbf{a}} = - \begin{pmatrix} 17 \\ -3 \\ 5 \end{pmatrix}$$

$$\text{so } \begin{pmatrix} -17 \\ 3 \\ -5 \end{pmatrix} \checkmark$$

2c) $\mathbf{a} \times \mathbf{c}$

$$\begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ 2 & 3 & -5 \\ 2 & 2 & 5 \end{vmatrix} \checkmark$$

$$= (15 + 10)\underline{\mathbf{i}} - (10 + 10)\underline{\mathbf{j}} + (4 - 6)\underline{\mathbf{k}}$$

$$= \begin{pmatrix} 25 \\ -20 \\ -2 \end{pmatrix} \checkmark$$

2d) $|\mathbf{a} \times \mathbf{c}|$, exactly

$$|\mathbf{a} \times \mathbf{c}| = \sqrt{25^2 + (-20)^2 + (-2)^2} \checkmark$$

$$= 7\sqrt{21} \checkmark$$

QUESTION 3

[7,7 Marks]

Determine the **exact** area to three decimal places of the parallelogram, of which the given vectors are adjacent sides. Show full working for each set of calculations.

$\underline{r} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, $\underline{s} = 4\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, by two different methods.

Method 1, using the cross product .

$$\begin{aligned}
 |\underline{r} \times \underline{s}| &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -1 & -2 \\ 4 & 5 & -2 \end{vmatrix} \checkmark \\
 &= (2+10)\underline{i} - (-6+8)\underline{j} + (15+4)\underline{k} \checkmark \\
 &= \begin{pmatrix} 12 \\ -2 \\ 19 \end{pmatrix} \checkmark \\
 \text{norm } \underline{r} \times \underline{s} &= \sqrt{12^2 + (-2)^2 + 19^2} \checkmark \\
 &= 22.561 \text{ units}^2 \text{ (3dp)} \checkmark
 \end{aligned}$$

Method 2, using the dot product

$$\underline{r} \cdot \underline{s} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix} = 12 - 5 + 4 = 11 \checkmark$$

$$|\underline{r}| = \sqrt{14} \checkmark$$

$$|\underline{s}| = 3\sqrt{5} \checkmark$$

$$\cos \theta = \frac{\underline{r} \cdot \underline{s}}{|\underline{r}| |\underline{s}|} = \frac{11}{\sqrt{14} \times 3\sqrt{5}} \checkmark$$

$$\therefore \theta = 64.0076913^\circ \checkmark$$

$$\begin{aligned}
 \text{area of parallelogram} &= \sqrt{14} \times 3\sqrt{5} \sin \theta \checkmark \\
 &= 22.561 \text{ units}^2 \text{ (3dp)} \checkmark
 \end{aligned}$$

QUESTION 4

[7,3 Marks]

- a) For the vectors \underline{a} and \underline{b} , such that $\underline{a} = \langle 11, 2, -9 \rangle$ and $\underline{b} = \langle -1, 5, -6 \rangle$,
Show that the cross product $\underline{a} \times \underline{b}$ is perpendicular to both vectors \underline{a} and \underline{b} .
Show full working for your calculations.

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 11 & 2 & -9 \\ -1 & 5 & -6 \end{vmatrix}$$

$$= (-12 + 45)\underline{i} - (-66 - 9)\underline{j} + (55 + 2)\underline{k}$$

$$= \begin{pmatrix} 33 \\ 75 \\ 57 \end{pmatrix}$$

Alternatively - since $\sin 90^\circ = 1$

$$|\underline{a} \times (\underline{a} \times \underline{b})| = |\underline{a}| \times |\underline{a} \times \underline{b}|$$

$$9\sqrt{25338} = \sqrt{206} \times 9\sqrt{123}$$

and $|\underline{b} \times (\underline{a} \times \underline{b})| = |\underline{b}| \times |\underline{a} \times \underline{b}|$

$$45\sqrt{74} = \sqrt{62} \times 9\sqrt{123}$$

$$\begin{pmatrix} 33 \\ 75 \\ 57 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ 2 \\ -9 \end{pmatrix} = 363 + 150 - 513 = 0 \quad \therefore \text{perpendicular}$$

$$\begin{pmatrix} 33 \\ 75 \\ 57 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ -6 \end{pmatrix} = -33 + 375 - 342 = 0$$

\therefore perpendicular

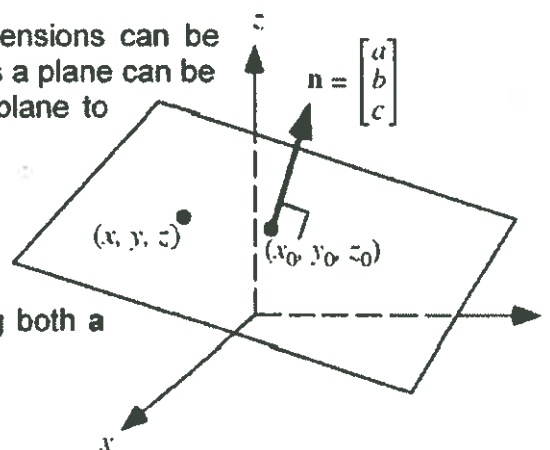
- b) In the same way a vector equation in two dimensions can be defined by the normal to that vector, in three dimensions a plane can be defined by applying the normal to the vectors on the plane to any point on the plane.

It is defined by $\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$
where \underline{a} is the position vector of a point on the plane

If the point $\underline{p} = 3\mathbf{i} - 4\mathbf{j} - \mathbf{k}$ lies on the plane containing both \underline{a} and \underline{b} , state the equation of the plane.

$$\underline{r} \cdot \begin{pmatrix} 33 \\ 75 \\ 57 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 33 \\ 75 \\ 57 \end{pmatrix}$$

$$= -258$$



QUESTION 5

[3 Marks]

If $\mathbf{a} \times \mathbf{b} = 9\mathbf{k}$ and $\mathbf{a} = -5\mathbf{i} - 9\mathbf{j}$ and $\mathbf{b} = x\mathbf{i} + 9\mathbf{j}$, then determine the value of x .

$$\mathbf{a} = \begin{pmatrix} -5 \\ -9 \\ 0 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} x \\ 9 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & -9 & 0 \\ x & 9 & 0 \end{vmatrix}$$

$$0\mathbf{i} + 0\mathbf{j} + (-45 + 9x)\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}$$

$$\therefore 9x - 45 = 9 \quad \checkmark$$

$$\Rightarrow x = 6. \quad \checkmark$$

QUESTION 6**[5, 1 Marks]**

5 a) Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. Prove that $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$ by expanding both sides of the equation.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{pmatrix} a_2b_3 - a_3b_2 \\ -(a_1b_3 - a_3b_1) \\ a_1b_2 - a_2b_1 \end{pmatrix} \checkmark$$

$$\mathbf{b} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \begin{pmatrix} a_3b_2 - a_2b_3 \\ -(a_3b_1 - a_1b_3) \\ a_2b_1 - a_1b_2 \end{pmatrix} \checkmark$$

$$= - \mathbf{b} \times \mathbf{a} \checkmark$$

5b) Given that $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$, sketch $\mathbf{b} \times \mathbf{a}$ in the following drawing:

