TRIAL TEST 2: *Complex Numbers II*

1. $z\bar{z} + z + \bar{z} = 24$, let $z = x + yi$, $\bar{z} = x - yi$

$$(x + yi)(x - yi) + x + yi + x - yi = 24 \checkmark$$
$$x^2 + y^2 + 2x = 24 \checkmark$$
$$x^2 + 2x + 1 + y^2 = 24 + 1 \checkmark$$
$$(x + 1)^2 + y^2 = 25 \checkmark$$

The locus is a circle of radius 5 units with centre $z = -1 + 0i$. ✓ [5]

2. $|a + z|^2 = a^2 + a(z + \bar{z}) + |z|^2$

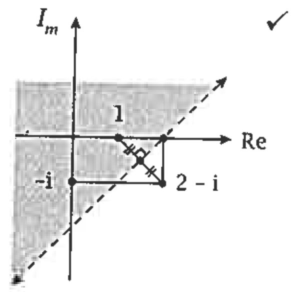
let $z = x + yi$

L.H.S. $= |a + z|^2$
$= |a + x + yi|^2$
$= (a + x)^2 + y^2$ ✓

R.H.S. $= a^2 + a(z + \bar{z}) + |z|^2$
$= a^2 + a(x + yi + x - yi) + x^2 + y^2$ ✓
$= a^2 + 2ax + x^2 + y^2$ ✓
$= (a + x)^2 + y^2$
= L.H.S. proved ✓ [4]

3. (a) $|1 - z| < |z + i - 2|$

$|z - 1| < |z - (2 - i)|$ ✓



✓

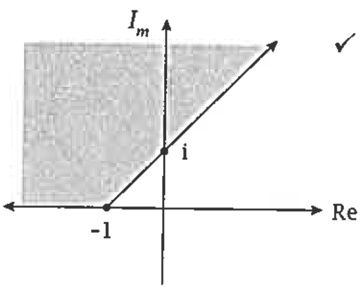
Test $z = 0 + 0i$ True ✓

Required region is shaded

(b) The required locus is the locus of

$\text{Arg } z \geq \frac{\pi}{4}$ ✓

shifted one unit to the left



✓

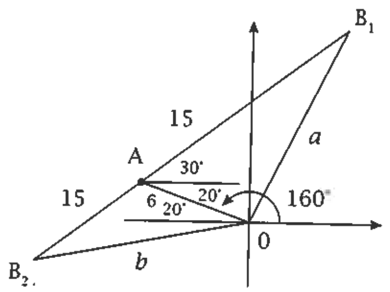
Required region is shaded [5]

4. $\frac{z - i}{z + i} = 1 + i$

$$z - i = (z + i)(1 + i)$$
$$z - i = z + zi + i - 1 \checkmark$$
$$1 - 2i = zi$$
$$z = \frac{1 - 2i}{i} \checkmark$$
$$z = \frac{(1 - 2i) \times i}{i \times i}$$
$$= \frac{i + 2}{-1}$$
$$= -2 - i \checkmark$$

[3]

5.



✓✓

$a^2 = 6^2 + 15^2 - 2 \times 6 \times 15 \times \cos 50°$ ✓
$a = 12.054$ cm 3dp ✓

$\cos \angle AOB_1 = \frac{6^2 + a^2 - 15^2}{2 \times 6 \times a}$ ✓

$\therefore \angle AOB_1 = 107.585°$ ✓

$160° - 107.585°$
$= 52.415°$

$\therefore B_1$ is $(12.054, 52.415°)$ 3dp ✓

$\angle OAB_2 = 180° - 50°$
$= 130°$

$b^2 = 6^2 + 15^2 - 2 \times 6 \times 15 \times \cos 130°$ ✓
$b = 19.409$ cm 3dp ✓

$\cos \angle AOB_2 = \frac{6^2 + b^2 - 15^2}{2 \times 6 \times b}$ ✓

$\therefore \angle AOB_2 = 36.301°$

and $360° - 160° - 36.301°$
$= 163.699°$ ✓

$\therefore B_2$ is $(19.409, -163.699°)$ 3dp [11]

6. L.H.S. $= \frac{z - z^{-1}}{i(z + z^{-1})}$

$$= \frac{\text{cis } \theta - \text{cis}(-\theta)}{i(\text{cis } \theta + \text{cis}(-\theta))} \checkmark$$
$$= \frac{\cos \theta + i \sin \theta - \cos(-\theta) - i \sin(-\theta)}{i(\cos \theta + i \sin \theta + \cos(-\theta) + i \sin(-\theta))} \checkmark$$
$$= \frac{\cos \theta + i \sin \theta - \cos \theta + i \sin \theta}{i(\cos \theta + i \sin \theta + \cos \theta - i \sin \theta)} \checkmark$$
$$= \frac{2i \sin \theta}{2i \cos \theta} \checkmark$$
$$= \tan \theta = \text{R.H.S.}$$

[4]

7. (a) $|z| = \sqrt{2}$ ✓

(b) $w = 2 \text{ cis } \frac{\pi}{6}$

$= 2 \cos \frac{\pi}{6} + 2i \sin \frac{\pi}{6}$ ✓

$= 2 \cdot \frac{\sqrt{3}}{2} + 2i \cdot \frac{1}{2}$

$= \sqrt{3} + i$ ✓

(c) $\bar{z} = 1 + i = \sqrt{2} \text{ cis } \frac{\pi}{4}$ ✓

$\bar{z}w = \sqrt{2} \text{ cis } \frac{\pi}{4} \cdot 2 \text{ cis } \frac{\pi}{6}$

$= \overset{\checkmark}{2\sqrt{2}} \text{ cis } \left(\frac{5\pi}{12}\right)$ ✓

(d) $z^2 = \left(\sqrt{2} \text{ cis } \left(\frac{-\pi}{4}\right)\right)^2$

$= 2 \text{ cis } \left(\frac{-\pi}{2}\right)$

$= 2e^{-i\frac{\pi}{2}}$ ✓

$\frac{w}{z^2} = \frac{2e^{i\frac{\pi}{6}}}{2e^{-i\frac{\pi}{2}}}$

$= e^{i\frac{2\pi}{3}}$ ✓

(e) $z^5 = \left(\sqrt{2} \text{ cis } \left(\frac{-\pi}{4}\right)\right)^5$ ✓

$= (\sqrt{2})^5 \text{ cis } \left(\frac{-5\pi}{4}\right)$

$= 4\sqrt{2} \text{ cis } \left(\frac{3\pi}{4}\right)$ ✓ [10]

8. (a) $A = 4$ ✓

$\frac{2\pi}{4} = \frac{\pi}{2}$

$\frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$

$\frac{\pi}{6} - \frac{\pi}{2} = -\frac{\pi}{3}$ ✓

$-\frac{\pi}{3} - \frac{\pi}{2} = -\frac{5\pi}{6}$

$\therefore \theta = \frac{\pi}{6}, \frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{5\pi}{6}$ ✓

(b) $z = \left(4 \text{ cis } \frac{\pi}{6}\right)^4$

$= 4^4 \text{ cis } \left(\frac{4\pi}{6}\right)$ ✓

$= 256 \text{ cis } \left(\frac{2\pi}{3}\right)$ ✓ [5]

9. $(1 + i)^5 - (1 - i)^5$

$(\sqrt{2} \text{ cis } 45°)^5 - (\sqrt{2} \text{ cis } (-45°))^5$ ✓

$= 2^{\frac{5}{2}} \text{ cis } 225° - 2^{\frac{5}{2}} \text{ cis } (-225°)$

$= 2^{\frac{5}{2}} (\text{cis } 225° - \text{cis } (-225°))$ ✓

$= 2^{\frac{5}{2}}(\cos 225° + i \sin 225°$
$- (\cos (-225°) + i \sin (-225°))$ ✓

$= 2^{\frac{5}{2}}(\cos 225° + i \sin 225°$
$- \cos 225° + i \sin 225°)$ ✓

$= 2^{\frac{5}{2}}(2 i \sin 225°)$

$= 2^{\frac{7}{2}} \cdot i\left(-\frac{1}{\sqrt{2}}\right)$ ✓

$= -8i$ [5]

10. (a) $w^3 = -2 + 2i$

$= \sqrt{8} \text{ cis } 135°$

$= 2^{\frac{3}{2}} \text{ cis } 135°$ ✓

$w = \left(2^{\frac{3}{2}} \text{ cis } 135°\right)^{\frac{1}{3}}$

$= 2^{\frac{1}{2}} \text{ cis } 45°$ ✓

$= 1 + i$ ✓

(b) $(z - 1)^3 = -2(z + 1)^3 + 2i (z + 1)^3$

$= (z + 1)^3 (-2 + 2i)$ ✓

$\frac{(z - 1)^3}{(z + 1)^3} = -2 + 2i$

$\left(\frac{z - 1}{z + 1}\right)^3 = -2 + 2i$ ✓

$\therefore \frac{z - 1}{z + 1} = \overset{\checkmark}{1 + i}$ as $(1 + i)^3 = -2 + 2i$

$z - 1 = (1 + i)(z + 1)$

$z - 1 = z + 1 + iz + i$ ✓

$-2 - i = iz$

$z = \frac{-2 - i}{i} \times \frac{i}{i}$

$= \frac{-2i + 1}{-1}$

$z = -1 + 2i$ ✓ [8]