Name: ____________________________ **Score: _____ / 17**

<u>No</u> calculators or notes are to be used.
Access to approved Mathematics Specialist formulae sheet is permitted.

Simplify answers where possible. **Time limit = 15 minutes.**

1. [3 marks]

The equation of a curve is $y^4 + x^2y^2 = 2a^3(x + 2a)$, where *a* is a constant. Find the gradient of the curve at the point (3a, a).

$4y^3\frac{dy}{dx} + 2xy^2 + x^2(2y)\frac{dy}{dx} = 2a^3$ ✓

$4a^3\frac{dy}{dx} + 2(3a)a^2 + (3a)^2(2a)\frac{dy}{dx} = 2a^3$

$\frac{dy}{dx} = \frac{2a^3 - 6a^3}{4a^3 + 18a^3}$ ✓

$\frac{dy}{dx} = \frac{-4a^3}{22a^3} = -\frac{2}{11}$ ✓

or $\frac{dy}{dx} = \frac{2a^3 - 2xy^2}{4y^3 + 2x^2y}$

$\frac{dy}{dx} = \frac{2a^3 - 2(3a)(a)^2}{4a^3 + 2(3a)^2(a)}$ ✓

$\frac{dy}{dx} = \frac{-4a^3}{22a^3} = -\frac{2}{11}$ ✓

2. [3 marks]

Find the equation relating x and y, given the differential equation $cosy\frac{dy}{dx} = 2x + 1$ and $y = \frac{\pi}{2}$ when $x = 3$.

✓ $\int \cos y \, dy = \int (2x+1) dx$

✓ $\sin y = x^2 + x + c$; $(3, \frac{\pi}{2})$

$\sin\frac{\pi}{2} = 9 + 3 + c$

$1 = 12 + c$

$-11 = c$

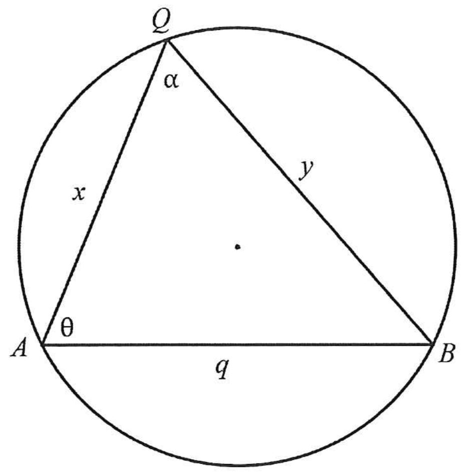
$\therefore \sin y = x^2 + x - 11$ ✓

or

$y = \sin^{-1}(x^2 + x - 11)$

3. [1, 2, 2 = 5 marks]

Consider the diagram provided below. $\overline{AB}$ is a chord of fixed length and Q can be located anywhere on the major arc AB, meaning $_s\angle AQB$ (α) will be constant no matter where exactly Q is.



a) Write an expression for q in terms of x, y and α.

$$q^2 = x^2 + y^2 - 2xy\cos\alpha \checkmark$$

b) Write an expression for $\frac{dy}{dx}$ in terms of x, y and α, remembering that q and α are constants.

$$0 = 2x + 2y\frac{dy}{dx} - \left(2y\cos\alpha + 2x\cos\alpha\frac{dy}{dx}\right) \checkmark$$

$$\frac{dy}{dx} = \frac{2x - 2y\cos\alpha}{2x\cos\alpha - 2y} \checkmark$$

c) Justify that y is a stationary point where $\cos\alpha = \frac{x}{y}$

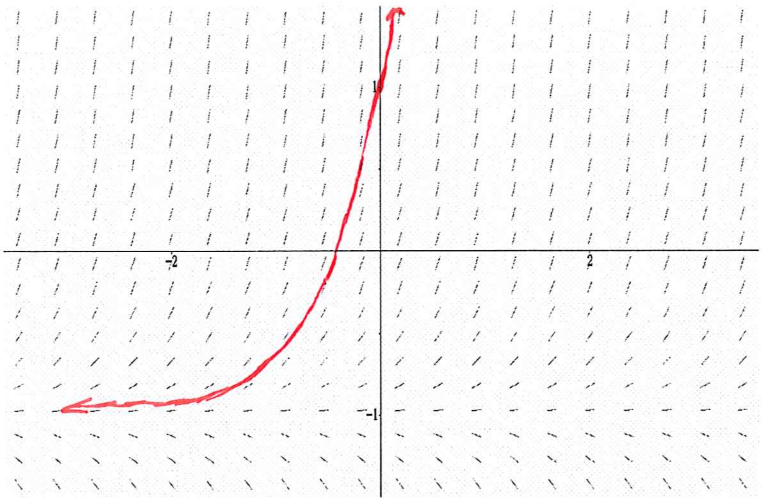
$$0 = 2x - 2y\cos\alpha \checkmark$$

$$\frac{-2x}{-2y} = \cos\alpha$$

$$\frac{x}{y} = \cos\alpha \checkmark$$

4. [2, 4 = 6 marks]

The slope field for the differential equation $\frac{dy}{dx} = 2(y+1)$ is shown below.



Correct Shape ✓

Passes through (0,1) and asymptote at $y=-1$ ✓

a) Sketch the solution curve which passes through (0, 1).

b) Determine the equation of the solution curve drawn in part a.

$$\int \frac{1}{y+1}\,dy = \int 2\,dx \ ✓$$

$$\ln|y+1| = 2x + c \ ✓$$

$$y+1 = e^{2x+c}$$

$$y+1 = e^{2x} \cdot e^{c} \quad \text{at } (0,1)$$

$$2 = e^{c} \ ✓$$

$$\therefore \ y = 2e^{2x} - 1 \ ✓$$

Test 5 2017 Calculator Section

Name: ______________________ Score: _____ / 38

Calculators allowed and 1 page of A4 notes, writing on both sides.
Access to approved Mathematics Specialist formulae sheet is permitted.

Time limit = 40 minutes.

5. [1, 2, 3, 2 = 8 marks]

A particle P is projected from the top of a cliff of height 100 m. Define *x* and *y* metres as the horizontal and vertical distance travelled by the particle measured from the base of the cliff after *t* seconds. Its equation of motion at any time t seconds is given by $\frac{dx}{dt} = 20\sqrt{2}$ and $\frac{dy}{dt} = 20\sqrt{2} - 10t$.

a) Determine the equation of the horizontal distance travelled by the particle after *t* seconds.

$x = 20\sqrt{2}t + c$ at $t=0, x=0$

$\therefore x = 20\sqrt{2}t$ ✓

b) Determine the equation of the vertical distance travelled by the particle after t seconds.

$y = 20\sqrt{2}t - 5t^2 + c$ ✓ at $t=0, y=100$

$\therefore y = 20\sqrt{2}t - 5t^2 + 100$ ✓

c) The equation of the path of P is given by $y = ax^2 + bx + c$. Determine *a*, *b* and *c*.

$x = 20\sqrt{2}t$

$\frac{x}{20\sqrt{2}} = t$ ✓

$y = 20\sqrt{2}t - 5t^2 + 100$

$y = x - 5\left(\frac{x}{20\sqrt{2}}\right)^2 + 100$ ✓

$y = x - \frac{5x^2}{800} + 100$

$y = -\frac{1}{160}x^2 + x + 100$

$\therefore a = -\frac{1}{160}$

$b = 1$ ✓

$c = 100$

d) Determine, using calculus methods, the approximate change in the vertical distance when the horizontal distance changes from 10m to 10.1m.

$\frac{dy}{dx} = -\frac{1}{80}x + 1$

$\delta y \approx \left(-\frac{1}{80}x + 1\right)\delta x$ ✓

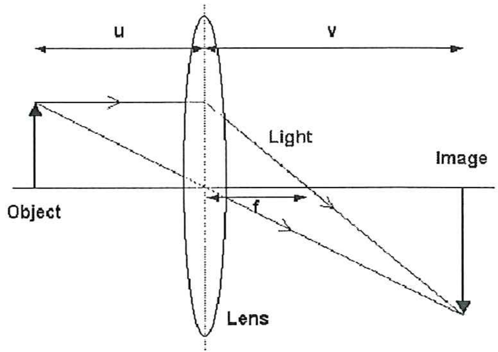
$\delta y \approx \left(-\frac{1}{80}(10) + 1\right)0.1$

$\delta y \approx 0.0875m$ ✓

6. [4 marks]

When an object is at a distance of u cm from a lens of focal length, $f = 20$ cm, an image is created at a distance of v cm from the lens.

The variables are related by the formula $\frac{1}{20} = \frac{1}{u} + \frac{1}{v}$



An object is moving with a constant speed of 2 cm/s towards the lens. At the instant when the image is 30 cm from the lens, in what direction and with what speed is the image moving?

$\frac{du}{dt} = 2cms^{-1}$

$v = 30cm$

$\frac{1}{20} = \frac{1}{u} + \frac{1}{30}$

$u = 60cm$ ✓

$\frac{d}{dt}\left(\frac{1}{20} = \frac{1}{u} + \frac{1}{v}\right)$

$0 = -\frac{1}{u^2}\frac{du}{dt} - \frac{1}{v^2}\frac{dv}{dt}$ ✓

$0 = -\frac{1}{60^2}(2) - \frac{1}{30^2}\frac{dv}{dt}$

$\frac{dv}{dt} = \frac{1}{2} cms^{-1}$ ✓ away from the lens ✓

7. [1, 2 = 3 marks]

A leaking water tank initially contains water to a depth of 1.6m. The depth of water, h cm, is decreasing at a rate (in cm/s) equal to one tenth of the depth of the water in the tank.

a) Write a differential equation that describes the rate of change of the depth of the water in relation to the depth.

$\frac{dh}{dt} = -\frac{1}{10}h$ ✓

b) Determine the solution to the differential equation and calculate, to the nearest tenth of a second, the time taken for the depth to decrease to just 5 cm.

$\int \frac{1}{h}\,dh = \int -0.1\,dt$

$\ln|h| = -0.1t + c$

$h = e^{-0.1t} \cdot e^{c}$

$h = 160e^{-0.1t}$ ✓

$5 = 160e^{-0.1t}$

$t = 34.7$ seconds ✓

8. [3, 5 = 8 marks]

The motion of a body moving in a straight line is modelled by the differential equation $\frac{dx}{dt} = 3 + 4x,$ where x metres is the displacement from a fixed point at time t seconds for $t \geq 0$. After 1 second the body is 2m to the right of the origin.

a) What is the acceleration of the body when t = 1s?

$\frac{d}{dt}\left(v = 3 + 4x\right)$

$a = 4\frac{dx}{dt}$ ✓

$a = 4(3 + 4x)$ ✓ at $x = 2m$

$a = 44 \; ms^{-2}$ ✓

b) Show that the displacement is given by $x = \frac{11e^{4t-4} - 3}{4}$.

$\frac{dx}{dt} = 3 + 4x$

$\int \frac{1}{3+4x}\, dx = \int 1 \, dt$ ✓

$\frac{1}{4}\ln|3 + 4x| = t + c$ ✓

$\ln|3 + 4x| = 4t + c$

$3 + 4x = e^{4t} \cdot e^{c}$ ✓ at $t = 1$ $x = 2$

$11 = e^{4} \cdot e^{c}$

$\frac{11}{e^4} = e^{c}$ ✓

$3 + 4x = e^{4t}\frac{11}{e^4}$

$x = \frac{\frac{11}{e^4}e^{4t} - 3}{4}$ ✓

$x = \frac{11e^{4t-4} - 3}{4}$

9. **[7, 2 = 9 marks]**

The population P of a colony of marsupials at a remote island is currently 20. The colony's expected growth rate is given by $\frac{dP}{dt} = 0.1P(1 - \frac{P}{200})$, where t is in years.

a) By separation of variables and use of partial fractions express P as a function of t, in the form $P = \frac{A}{1+Be^{-kt}}$

$$\frac{dP}{dt} = \frac{0.1P}{200}(200-P) \checkmark$$

$$\int \frac{1}{P(200-P)}\,dP = \int \frac{1}{2000}\,dt \checkmark$$

$$\frac{1}{200}\int \left[\frac{1}{P} + \frac{1}{200-P}\right] dP = \frac{1}{2000}t + c \checkmark$$

$$\ln\left|\frac{P}{200-P}\right| = 0.1t + c \checkmark$$

$$\ln\left|\frac{200-P}{P}\right| = -0.1t + c$$

$$\frac{200-P}{P} = e^{-0.1t} \cdot e^{c} \checkmark \quad \text{at } t=0,\ P=20$$

$$9 = e^{c} \checkmark$$

$$\frac{200}{P} - 1 = 9e^{-0.1t}$$

$$\frac{200}{P} = 9e^{-0.1t} + 1$$

$$\frac{200}{9e^{-0.1t}+1} = P \checkmark$$

b) Determine the time it takes for the population to reach half its limiting value.

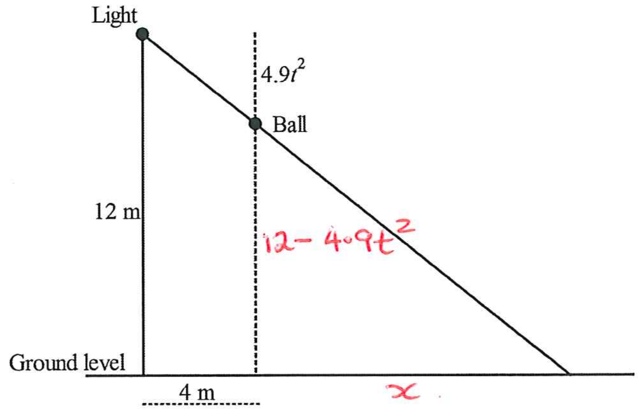
$$100 = \frac{200}{1+9e^{-0.1t}} \checkmark$$

$$t = 21.97$$

$$\approx 22 \text{ yrs} \checkmark$$

10. [6 marks]

A light is positioned at the top of a vertical post 12m high. A small ball is dropped vertically from the same height as the light but at a point 4m away. The vertical distance travelled by the ball t seconds after release is given by $4.9t^2$.



How fast is the shadow of the ball moving along the horizontal ground half a second after the ball is dropped?

$$\frac{12-4.9t^2}{12} = \frac{x}{x+4} \checkmark\checkmark$$

$$(12-4.9t^2)(x+4) = 12x$$

$$\cancel{12x} + 48 - 4.9t^2x - 19.6t^2 = \cancel{12x}$$

$$\frac{d}{dt}\left(48 - 4.9t^2x - 19.6t^2 = 0\right) \checkmark$$

$$-9.8tx - 4.9t^2\frac{dx}{dt} - 39.2t = 0 \checkmark$$

at $t = 0.5$

$$\frac{12-4.9(0.5)^2}{12} = \frac{x}{x+4}$$

$x = 35.18\text{m} \checkmark$

$$0 = -9.8(0.5)(35.18) - 4.9(0.5)^2\frac{dx}{dt} - 39.2(0.5)$$

$$\frac{dx}{dt} = -156.7\ \text{ms}^{-1} \checkmark$$