SCOTCH
COLLEGE

12 Mathematics Methods 2019

Test 2 – Integration and Area

Section 1: Calculator-free

Time allowed: 20 minutes **Maximum marks: 17**

Name: Marking guide **Teacher:** Foster | Giese | Reyhani

Instructions:

- Show all working clearly.
- Sufficient detail must be shown for marks to be awarded for reasoning.
- A formula sheet will be provided.
- No calculators or personal notes are permitted.

Question 1 (9 marks)

a) Perform the following indefinite integral, leaving your answer in simplest form. [3]

$$\int \frac{1}{x^2} + 4x^5\, dx = \int x^{-2} + 4x^5 \, dx$$

$$= -x^{-1} + \frac{4x^6}{6} + c$$

$$= -\frac{1}{x} + \frac{2}{3}x^6 + c$$

b) Determine the following definite integrals.

i) $\int_0^2 (2x-3)^3 dx$ [3]

$$= \left[\frac{(2x-3)^4}{2\times 4}\right]_0^2$$

$$= \frac{1}{8} - \left(\frac{(-3)^4}{8}\right)$$

$$= \frac{1}{8} - \frac{81}{8}$$

$$= -\frac{80}{8}$$

$$= -10$$

ii) $\int_0^1 \frac{6x}{\sqrt{3x^2+1}} dx$ [3]

$$= \int_0^1 6x(3x^2+1)^{-\frac{1}{2}} dx$$

$$= \left[2(3x^2+1)^{\frac{1}{2}}\right]_0^1$$

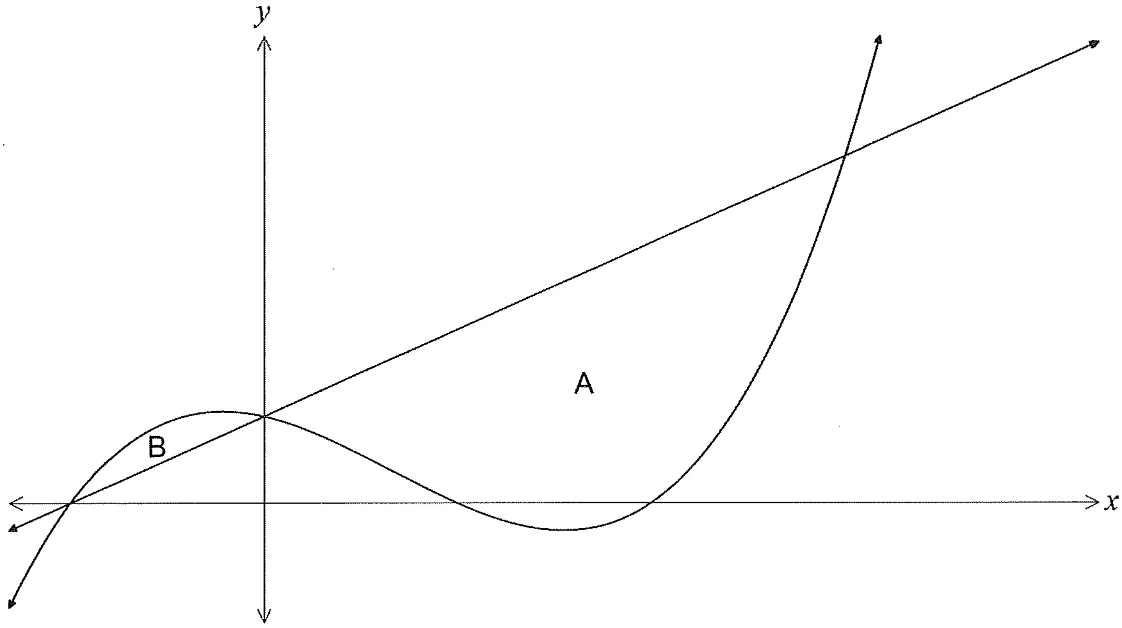
$$= 2\sqrt{4} - 2\sqrt{1}$$

$$= 4 - 2$$

$$= 2$$

Question 2 **(8 marks)**

The graphs of $y = 2x + 2$ and $y = f(x)$ are shown below, where $f(x) = x^3 - 2x^2 - x + 2$.



a) Use calculus techniques to show and justify that a point of inflection exists on the graph of $y = f(x)$ when $x = \frac{2}{3}$ [3]

$f'(x) = 3x^2 - 4x - 1$ ✓

$f''(x) = 6x - 4$ ✓

$f''(x) = 0$ at inflection

$\therefore 0 = 6x - 4$ ✓

$x = 2/3$

b) Two regions are trapped between the linear and cubic functions, marked A and B on the diagram. Show that the difference in the areas of these two regions is $10\frac{2}{3}$ square units. [5]

$x^3 - 2x^2 - x + 2 = 2x + 2$

$x^3 - 2x^2 - 3x = 0$

$x(x^2 - 2x - 3) = 0$

$x(x-3)(x+1) = 0$

$x = 0, 3, -1$ ✓

P.T.O.

Extra working space

Area B: $\int_{-1}^{0} x^3-2x^2-x+2-(2x+2)\,dx$.

$= \int_{-1}^{0} x^3-2x^2-3x\,dx$

$= \left[\frac{x^4}{4}-\frac{2}{3}x^3-\frac{3}{2}x^2\right]_{-1}^{0} = 0-\left(\frac{1}{4}+\frac{2}{3}-\frac{3}{2}\right)$

Area A: $\int_{0}^{3} 2x+2-(x^3-2x^2-x+2)\,dx$

$= \int_{0}^{3} -x^3+2x^2+3x\,dx$

$= \left[-\frac{x^4}{4}+\frac{2}{3}x^3+\frac{3}{2}x^2\right] = -\frac{81}{4}+\frac{54}{3}+\frac{27}{2}$

Area A − Area B

$-\frac{81}{4}+\frac{54}{3}+\frac{27}{2}+\frac{1}{4}+\frac{2}{3}-\frac{3}{2} = 10\,^2\!/_3$.

END OF SECTION

SCOTCH
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12 Mathematics Methods 2019

Test 2 – Integration and Area

Section 2: Calculator-assumed

Time allowed: 25 minutes **Maximum marks: 23**

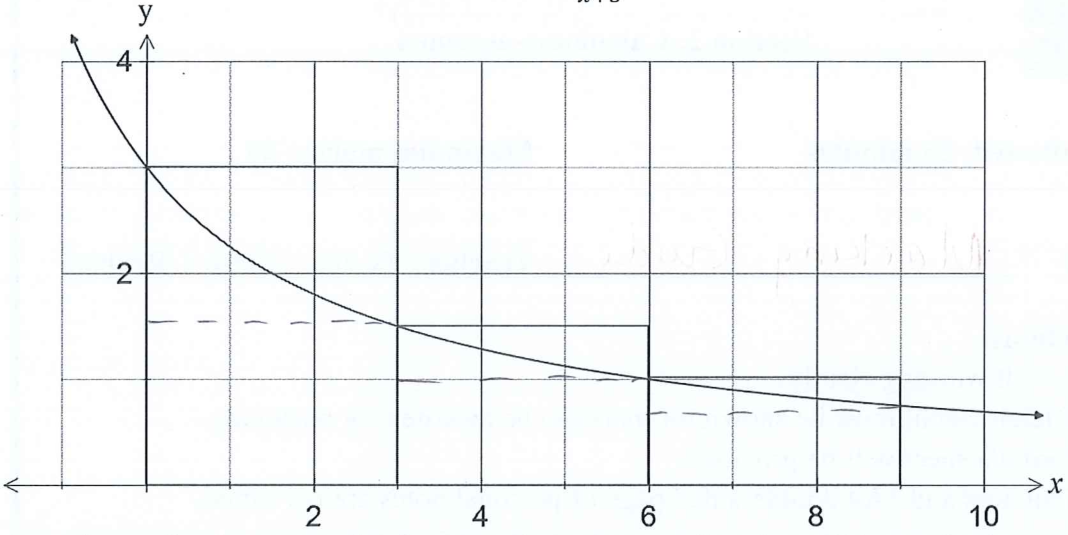
Name: Marking Guide **Teacher:** Foster | Giese | Reyhani

Instructions:

- Show all working clearly.
- Sufficient detail must be shown for marks to be awarded for reasoning.
- A formula sheet will be provided.
- Calculators and 1A4 double-sided page of personal notes are permitted.

Question 3 **(6 marks)**

The graph below shows the function $f(x) = \frac{9}{x+3}$



An estimate for the area under the curve between $x = 0$ and $x = 9$ is required.

a) Three circumscribed rectangles are shown on the diagram. Use these rectangles to calculate an over-estimate for the area. [2]

$3 \times 3 + 3 \times 1.5 + 3 \times 1$

$= 16.5 \text{ units}^2$

b) Use three inscribed rectangles to calculate an under-estimate for the area. [2]

$3 \times 1.5 + 3 \times 1 + 3 \times \frac{9}{12}$

$= 9.75 \text{ units}^2$

c) Use your over- and under- estimates to calculate a better estimate for the area under the curve between $x = 0$ and $x = 9$. [1]

$\frac{16.5 + 9.75}{2} = 13.125 \text{ units}^2$

d) The exact area is $18\log_e 2$. Calculate the error in the best estimate above as a percentage of the exact area. [1]

$\frac{18\ln 2 - 13.125}{18\ln 2} = -5.2\%$

Question 4 (5 marks)

A motor vehicle slows down from an initial velocity of 25 ms^{-1} until it is stationary. During this interval, its acceleration t seconds after the brakes were applied is given by;

$$a(t) = \frac{t}{2} - 5 \text{ ms}^{-2}$$

a) Determine the velocity of the vehicle after four seconds. [3]

$$v(t) = \frac{t^2}{4} - 5t + c$$

$$c = 25 \text{ ms}^{-1}$$

$$\therefore v(t) = \frac{t^2}{4} - 5t + 25$$

$$v(4) = 9 \text{ ms}^{-1}$$

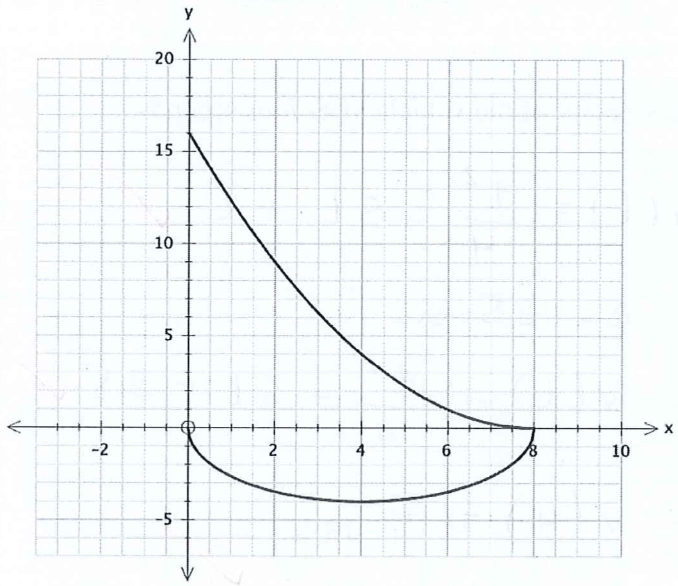
b) Calculate the distance travelled by the vehicle in the first two seconds after the brakes were applied. [2]

$$\int_0^2 \left| \frac{t^2}{4} - 5t + 25 \right| dt$$

$$= 40\tfrac{2}{3} \text{ m}.$$

Question 5 **(5 marks)**

The graph below shows parts of the relationships $y = 0.25(x-8)^2$ and $(x-4)^2 + y^2 = 16$ That are being used to model a new fin for a surfboard. The picture below shows a cross-section of the fin. Both the x and y axis have the scale 1 unit = 2cm.



The fin is to be 1.5cm thick.

Determine the exact volume of a prototype of this fin.

$$\text{Area} = \int_0^8 0.25(x-8)^2\,dx + \frac{1}{2}\pi(4^2)$$

$$= \frac{128}{3} + 8\pi$$

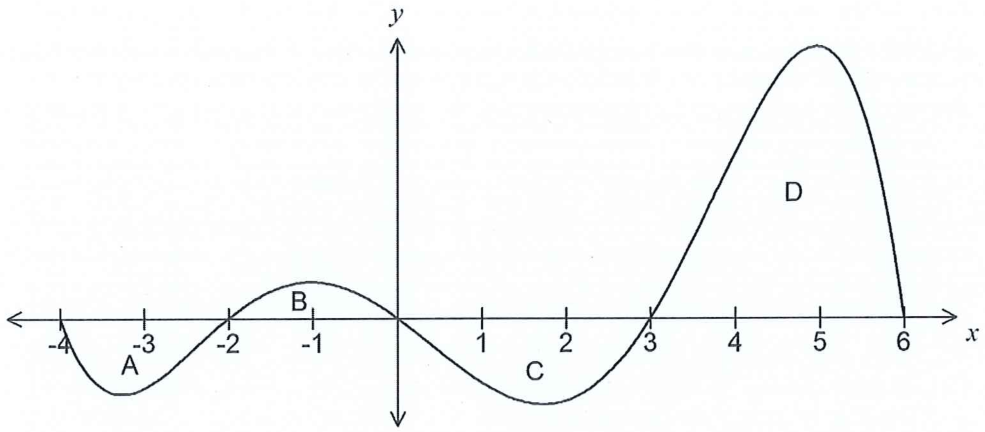
exact value.

$$\text{Volume} = \left(\frac{128}{3} + 8\pi\right) \times 4 \times 1.5 = (256 + 48\pi)\ cm^3$$

Question 6 (7 marks)

The graph of the function $y = f(x)$ is shown below for $-4 \leq x \leq 6$.

The area of each region enclosed by the curve and the x -axis is shown in the table below the graph.



Region	A	B	C	D
Area of region	5	3	11	25

a) Determine the area enclosed between the graph of $y = f(x)$ and the x -axis, from $x = -4$ to $x = 6$. [1]

$5 + 3 + 11 + 25 = 44$ units^2

b) Determine the value of

i) $\int_{-2}^{6} f(x)\,dx$ [1]

$= 3 - 11 + 25$

$= 17$

ii) $\int_{-2}^{3} 3f(x)\,dx$ [2]

$= 3(3 - 11)$

$= -24$

iii) $\int_{0}^{6} 6 - f(x)\,dx$ [3]

$= \int_0^6 6\,dx - \int_0^6 f(x)\,dx$

$= 36 - (-11 + 25)$

$= 22$

Extra working space

END OF TEST