MINDARIE SENIOR COLLEGE

WHERE YOUR FUTURE BEGINS NOW

MATHEMATICS: SPECIALIST 3 & 4

SEMESTER 1 2017

TEST 3

Resource Free

Reading Time: 2 minutes
Time Allowed: 18 minutes

Total Marks: 17

1. [3, 4 marks]

(a) Solve the system of equations

$$x + y - z = 4 \quad - ①$$
$$x - y + 3z = 10 \quad - ②$$
$$2x - y + z = 11 \quad - ③$$

①+③ $3x = 15$

$x = 5$

$y - z = -1$
$-y + 3z = 5$

$2z = 4$

$z = 2$

$y = 1$

$x = 5, \ y = 1, \ z = 2$

✓ ✓ ✓

(b) State the number of solutions for each of the following systems of equations. In each case, interpret the system of equations geometrically.

(i)
$$2x + 3y - z = 5 \quad - ①$$
$$3x - 2y + 4z = 7$$
$$4x + 6y - 2z = 8 \quad - ③$$

① & ③ are parallel

No solution. ✓

Two parallel planes with 3rd plane cutting both of them. ✓

(ii)
$$2x + 4y - 6z = 10 \quad - ①$$
$$x + 2y - 3z = 5 \quad - ②$$
$$5x + 3y + z = 7$$

Infinite n° solutions ✓

① & ② are same plane, being intersected by other plane. ✓

Sol'ns form a line

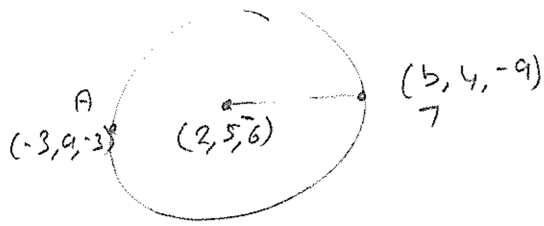
2. [2, 2 marks]

A sphere has centre $C(2, 5, -6)$. The points $A(-3, a, -3)$ and $B(b, 4, -9)$ form the diameter of the sphere.

(a) Determine the values of a and b.

$a = 6$ ✓

$b = 7$ ✓



(b) Determine the vector equation of the sphere.

$r = \sqrt{25 + 1 + 9}$

$= \sqrt{35}$

$\left| \underset{\sim}{r} - \begin{pmatrix} 2 \\ 5 \\ -6 \end{pmatrix} \right| = \sqrt{35}$ ✓ ✓

3. [2, 4 marks]

The points $P(3, 5, 2)$, $Q(6, 2, 5)$ and $R(2, 6, 3)$ all lie in the plane Π.

(a) Determine the vectors PQ and PR.

$\overrightarrow{PQ} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$ ✓ $\quad \overrightarrow{PR} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ ✓

(b) Determine the normal equation of the plane, Π, in the form $\boldsymbol{r} \cdot \boldsymbol{n} = c$.

$\underset{\sim}{n} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ ✓ Knows to use cross product

$= \begin{pmatrix} -6 \\ -6 \\ 0 \end{pmatrix}$ ✓ calculates cross product.

$\underset{\sim}{r} \cdot \begin{pmatrix} -6 \\ -6 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ -6 \\ 0 \end{pmatrix}$

✓ $= -48$ ✓ calculates dot product

$\underset{\sim}{r} \cdot \underset{\sim}{n} = c$

MATHEMATICS: SPECIALIST 3 & 4

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TEST 3

Resource Assumed

Reading Time: 2 minutes
Time Allowed: 33 minutes

Total Marks: 32

4. [3, 3 marks]

A particle travels according to the equation $\boldsymbol{r} = (1 - 2\cos t)\boldsymbol{i} + 3\sin t\,\boldsymbol{j}$.

(a) State the Cartesian equation of the path of the particle.

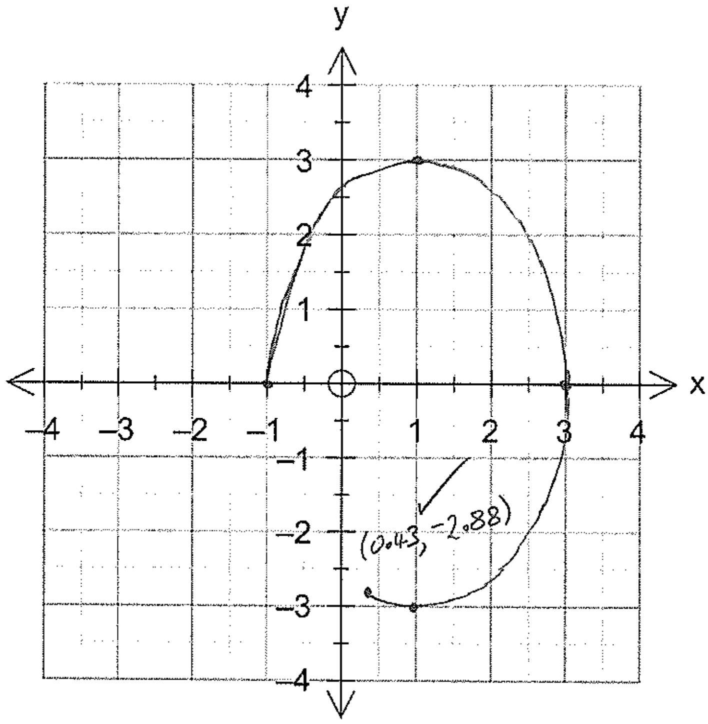
$x = 1 - 2\cos t$ $\qquad$ $y = 3\sin t$

$\cos t = \frac{1-x}{2}$ ✓ $\qquad$ $\sin t = \frac{y}{3}$ ✓

$\left(\frac{1-x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$\frac{(1-x)^2}{4} + \frac{y^2}{9} = 1$ ✓

(b) Sketch the path of the particle for $0 \le t \le 5$ on the axes below.



Smooth ✓

$-1 \le x \le 3$ ✓

$-3 \le y \le 3$

5. [2, 3, 2, 2 marks]

The points $A(4,1,2)$, $B(1,1,-3)$ and $C(-1,5,2)$ form a triangle.

(a) Determine vectors AB and AC.

$$\overrightarrow{AB} = \begin{pmatrix} -3 \\ 0 \\ -5 \end{pmatrix} \checkmark \quad \overrightarrow{AC} = \begin{pmatrix} -5 \\ 4 \\ 0 \end{pmatrix} \checkmark$$

(b) Using the scalar product, determine the angle between vectors AB and AC. Give your answer in radians to two decimal places.

$$\begin{pmatrix} -3 \\ 0 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 4 \\ 0 \end{pmatrix} = \sqrt{34}\sqrt{41}\cos\theta \qquad \checkmark \text{ uses Formula}$$

$$\checkmark \rightarrow 15 = \sqrt{34}\sqrt{41}\cos\theta$$

$$\theta = 1.16^{r} \checkmark$$

(c) Using your answer from (b), determine the area of the triangle ABC.

$$\text{Area} = \tfrac{1}{2}\sqrt{34}\sqrt{41}\sin 1.16^{r} \checkmark$$

$$= 17.1 \text{ sq. units} \checkmark$$

(d) Determine the exact value of $\frac{|\overrightarrow{AB}\times\overrightarrow{AC}|}{2}$, and state the geometric significance of your answer.

$$\frac{\overrightarrow{AB}\times\overrightarrow{AC}}{2} = \frac{\sqrt{1169}}{2} \checkmark$$

It is the area of the triangle ABC $\checkmark$

6. [6, 4 marks]

A rocket is fired with velocity $\begin{pmatrix} 6.5 \\ 11 \\ 2.5 \end{pmatrix}$ km/min from a point with position vector $\begin{pmatrix} -8 \\ 2 \\ 5 \end{pmatrix}$ km from some reference point O. At exactly the same time a second rocket is fired from the point $\begin{pmatrix} 24 \\ -10 \\ 7 \end{pmatrix}$ km with velocity $\begin{pmatrix} -1.5 \\ 14 \\ 2 \end{pmatrix}$ km/~~h~~. min

(a) Show that the rockets collide, and determine the time and position at which the collision takes place.

$$\underset{\sim}{r}_A = \begin{pmatrix} 6.5t - 8 \\ 11t + 2 \\ 2.5t + 5 \end{pmatrix}$$

$$\underset{\sim}{r}_B = \begin{pmatrix} 24 - 1.5t \\ 14t - 10 \\ 7 + 2t \end{pmatrix}$$

$\underset{\sim}{i}$: $6.5t - 8 = 24 - 1.5t$

$t = 4$

$\underset{\sim}{j}$: $11t + 2 = 14t - 10$

$t = 4$

$\underset{\sim}{k}$: $2.5t + 5 = 7 + 2t$

$t = 4$

(b) If the second rocket ~~had have~~ had a velocity of $\begin{pmatrix} -2 \\ 12 \\ 2 \end{pmatrix}$ km/min, determine the minimum distance that would occur between the two rockets. Give your answer to the nearest 10 metres.

$$\underset{\sim}{r}_A = \begin{pmatrix} 6.5t - 8 \\ 11t + 2 \\ 2.5t + 5 \end{pmatrix}$$

$$\underset{\sim}{r}_B = \begin{pmatrix} 24 - 2t \\ 12t - 10 \\ 7 + 2t \end{pmatrix}$$

$${}_A\underset{\sim}{r}_B(t) = \begin{pmatrix} 8.5t - 32 \\ 12 - t \\ 0.5t - 2 \end{pmatrix}$$

$$\text{dist} = \sqrt{(8.5t - 32)^2 + (12 - t)^2 + (\tfrac{1}{2}t - 2)^2}$$

min dist = 8.18 km

7. [1, 4, 2 marks]

A plane has Cartesian equation $3x - 2y + 4z = 6$.

(a) Determine the normal equation for the plane.

$$\underset{\sim}{r} \cdot \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = 6 \quad \checkmark$$

(b) Determine the exact distance that the point $(8, -6, 7)$ is from the plane.

$$\underset{\sim}{r} = \begin{pmatrix} 8+3\lambda \\ -6-2\lambda \\ 7+4\lambda \end{pmatrix} \quad \checkmark$$

$$\left|\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}\right| = \sqrt{29}$$

$$\begin{pmatrix} 8+3\lambda \\ -6-2\lambda \\ 7+4\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = 6 \quad \checkmark$$

$$\therefore \text{dist} = 2\sqrt{29} \quad \checkmark$$

$$24 + 9\lambda + 12 + 4\lambda + 28 + 16\lambda = 6$$

$$29\lambda = {}^-58$$

$$\lambda = {}^-2 \quad \checkmark$$

(c) Determine the Cartesian equation of a second plane that is parallel to the plane above, and contains the point $(8, -6, 7)$.

$$\begin{pmatrix} 8 \\ -6 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = 24 + 12 + 28$$

$$= 64 \quad \checkmark$$

$$\underset{\sim}{r} \cdot \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = 64 \quad \checkmark$$