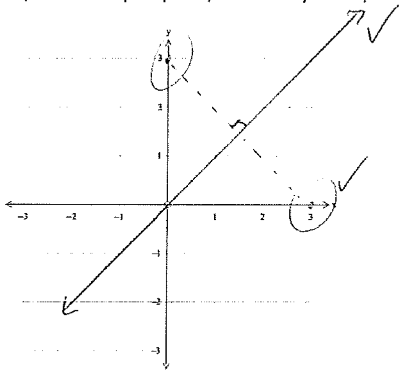
Mathematics Specialist Test 2 2019 Marking Key

Question 1. (2 marks)

Sketch the set of points z, in the complex plane, that satisfy the equation $|z - 3i| = |z - 3|$



Question 2. (4 marks)

Evaluate $(1 + i)^7 + (1 - i)^7$ using de Moivre's rule.

$\left(\sqrt{2}\,\text{cis}\frac{\pi}{4}\right)^7 + \left(\sqrt{2}\,\text{cis}\left(\frac{-\pi}{4}\right)\right)^7$ ✓

$= 8\sqrt{2}\,\text{cis}\left(\frac{7\pi}{4}\right) + 8\sqrt{2}\,\text{cis}\left(\frac{-7\pi}{4}\right)$ ✓

$= 8\sqrt{2}\,\text{cis}\left(\frac{-\pi}{4}\right) + 8\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$

$= 8\sqrt{2}\left(\cos\left(\frac{-\pi}{4}\right) + i\cancel{\sin\left(\frac{-\pi}{4}\right)} + \cos\frac{\pi}{4} + i\cancel{\sin\left(\frac{\pi}{4}\right)}\right)$

$= 8\sqrt{2}\left(2\cos\frac{\pi}{4}\right)$

$= 8\cancel{\sqrt{2}}\cdot 2\cdot\frac{1}{\cancel{\sqrt{2}}}$

$= \underline{16}$ ✓

Question 3(a)

Solution
$\left(\frac{z+i}{\bar{z}}\right) = \frac{a+(b+1)i}{a-bi} \times \frac{a+bi}{a+bi}$ **....(1)** $= \frac{a^2-b(b+1)+ai}{a^2+b^2}$ **...(2)**   **Hence if the real part is zero then it must be true that:** $a^2 - b(b+1)$**=0**   **i.e.** $a^2 - b^2 - b$ **= 0**
Specific Behaviours
✓ **Forms correct expression equivalent to (1)**   ✓ **Forms the correct expression equivalent to (2)**   ✓ **Forms the equation relating** a, b **stating the real part is zero**

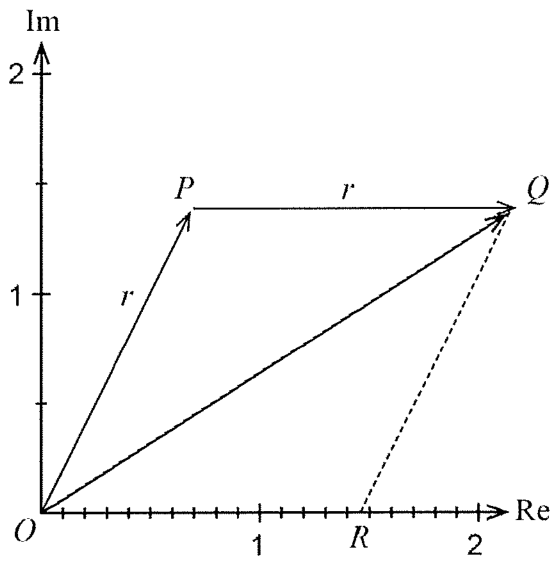
Question 3(b)(i)

Solution
$\frac{-2i}{\bar{z}} = \frac{2cis(-\frac{\pi}{2})}{rcis(-\theta)} = \frac{2}{r}cis\left(\theta - \frac{\pi}{2}\right)$
Specific Behaviours
✓ **converts -3i into polar from correctly** ✓ **writes the correct expression for $\bar{z}$ in polar form** ✓ **divides polar forms correctly in terms of r, θ**

Question 3(b)(ii)

(ii) $arg(z+r)$ in terms of θ. (3 marks)

Solution



Let $z = \overline{OP} = rcis\theta$ and $r = \overline{PQ}$. $OPQR$ is a rhombus with side length r.

Then the complex number $z + r = \overline{OQ}$ is a diagonal in the rhombus.

It is a property that a diagonal bisects the angles in a rhombus.

$\therefore \; s\angle QOR = \frac{1}{2}s\angle POR$ i.e. $\arg(z+r) = \frac{\theta}{2}$

Specific behaviours

✓ indicates the vector position for $z+r$ correctly
✓ identifies $z+r$ as the diagonal of a rhombus
✓ writes the correct expression for $arg(z+r)$

Question 4(a)

Solution
$z^3 = 8cis(0)$ $z_0 = 2cis0$ $z_1 = 2cis\frac{2\pi}{3}$ $z_2 = 2cis(-\frac{2\pi}{3})$
Specific Behaviours
✓ expresses z^3 as $8cis0$ ✓ states first root in correct polar form   ✓ states the other 2 roots correctly using the argument separation $\frac{2\pi}{3}$

Question 4(b)

Solution
$P(z) = (z^3 - 8)(z^5 + z + 3)$
Specific Behaviours
✓ determines $Q(z)$ correctly

Question4(c)

Solution
$z = -\frac{1}{2} \pm \frac{\sqrt{11}}{2}i$ and $z = -\frac{1}{2} \pm \frac{\sqrt{11}}{2}i$
Specific Behaviours
✓ states $z = 2, -1 \pm \sqrt{3}i$ as solutions   ✓ states $z = -\frac{1}{2} \pm \frac{\sqrt{11}}{2}i$ as solutions

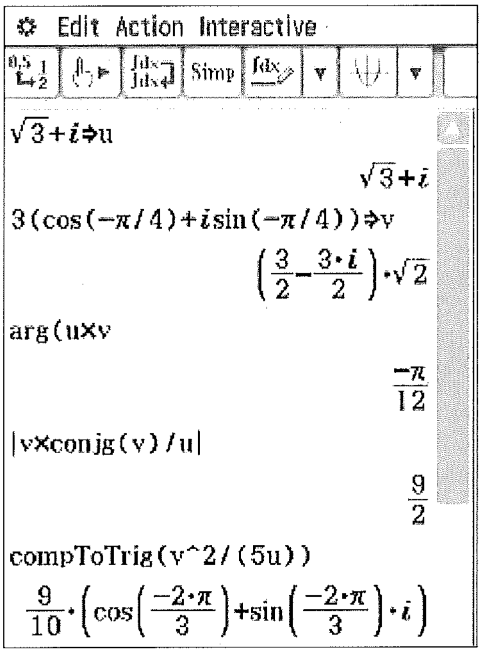
Question 5

(6 marks)

Two complex numbers are $u = \sqrt{3} + i$ and $v = 3\,\text{cis}\left(-\frac{\pi}{4}\right)$.

(a) Determine the argument of uv. (2 marks)

Solution
$\arg(u) = \frac{\pi}{6}$ $\arg(uv) = \frac{\pi}{6} - \frac{\pi}{4} = -\frac{\pi}{12}$
Specific behaviours
✓ argument of u ✓ argument



(b) Simplify $|v \times \bar{v} \times u^{-1}|$. (2 marks)

Solution
$\lvert v \times \bar{v}\rvert \times \left\lvert\frac{1}{u}\right\rvert = \lvert v\rvert^2 \times \left\lvert\frac{1}{u}\right\rvert = 9 \times \frac{1}{2}$ $= \frac{9}{2}$
Specific behaviours
✓ evaluates $\lvert v \times \bar{v}\rvert$ ✓ correct magnitude

(c) Determine z in polar form if $5zu = v^2$. (2 marks)

Solution
$z = \frac{v^2}{5u} = \frac{1}{5} \times \frac{9\,\text{cis}\left(-\frac{\pi}{2}\right)}{2\,\text{cis}\left(\frac{\pi}{6}\right)}$ $z = \frac{9}{10}\text{cis}\left(-\frac{2\pi}{3}\right)$
Specific behaviours
✓ indicates v^2 in polar form   ✓ simplifies z

Question 6 **(8 marks)**

(a) Let $r \operatorname{cis} \theta$ be a point in the complex plane. Determine, in terms of r and θ, the polar form of this point after it is rotated by $\frac{\pi}{4}$ about the origin and then reflected in the real axis.

(2 marks)

Solution
$z \to z_1 = r \operatorname{cis}\left(\theta + \frac{\pi}{4}\right)$ $z_1 \to \overline{z_1} = r \operatorname{cis}\left(-\theta - \frac{\pi}{4}\right)$
Specific behaviours
✓ rotation   ✓ conjugate

(b) Let $f(w) = -i\overline{w} + 1 + i$.

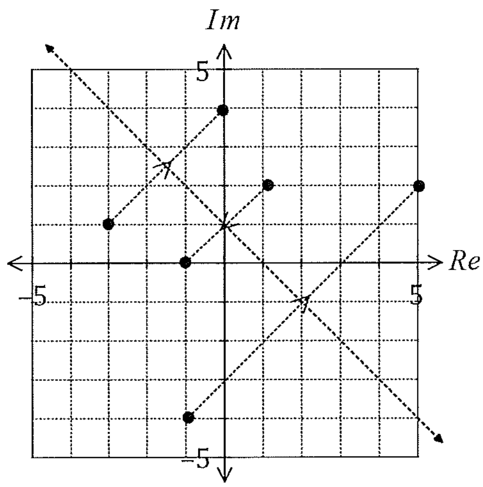
(i) Complete the following table. (3 marks)

w	$1+2i$	$-3+i$	$-1-4i$
$f(w)$	-1	$4i$	$5+2i$

Solution
See table
Specific behaviours
✓✓✓ each point

(ii) Sketch each point, w, and join it with a dotted line to its image, $f(w)$, on the diagram below.

(1 mark)



Solution
See diagram
Specific behaviours
✓ plots points

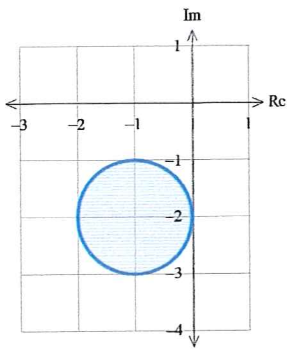
(iii) Describe the geometric transformation that $f(w)$ represents. (2 marks)

Solution
Reflection in the line $\mathrm{Re}(z) + \mathrm{Im}(z) = 1$
Specific behaviours
✓ reflection   ✓ line of reflection

Question 7 **(8 marks)**

A sketch of the locus of a complex number z is shown below. Write equations or inequalities in terms of z (without using $x = \text{Re}(z)$ or $y = \text{Im}(z)$) for each of the following:

(a) (3 marks)

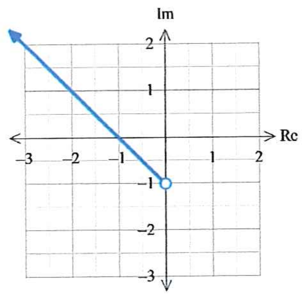


$|z-(-1-2i)| \leq 1$

OR $|z+1+2i| \leq 1$

✓ forms inequality using the modulus.
✓ uses a difference between z and $-1-2i$.
✓ uses radius as 1 unit (constant on RHS)

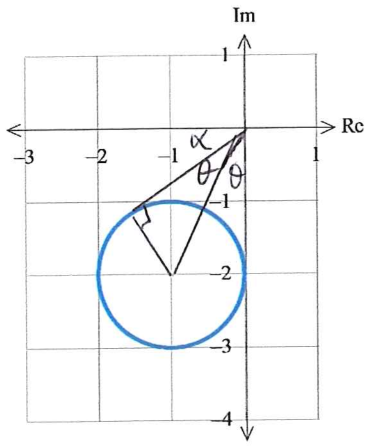
(b) (2 marks)



$\text{Arg}(z+i) = \frac{3\pi}{4}$

✓ forms an equation stating argument $= \frac{3\pi}{4}$
✓ states the argument of $z+i$

The sketch in (a) is repeated below, with only the circle indicated.



$\sin\theta = \frac{1}{\sqrt{5}}$

$\theta = 0.4636.$

$\alpha = \frac{\pi}{2} - 2\theta = 0.6435$

$\therefore \text{Arg}(z) = \pi + 0.6435 = 3.79$ (nearest 0.01)

(c) From the locus from part(a), determine the minimum value for $\arg(z)$ correct to 0.01, where

$0 \leq \arg(z) \leq 2\pi.$ (3 marks)

✓ indicates how max value occurs using tangent.
✓ uses approp. trig to determine θ correctly
✓ evaluates minimum value for $\arg(z)$ correctly

Question 8(a) **(4 marks)**

Solution	
By De Moivre's theorem we have $$\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4.$$ Expanding and taking real parts gives $$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\ &= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2 \\ &= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta \\ &= 8\cos^4\theta - 8\cos^2\theta + 1\end{aligned}$$	
Mathematical behaviours	**Marks**
• uses De Moivre's theorem appropriately	1
• expands the fourth power of the expression correctly	1
• takes the real parts of each side	1
• replaces $\sin^2\theta = 1 - \cos^2\theta$ and simplifies to obtain the result	1

Question 8(b) **(5 marks)**

Solution	
Let $x = \cos\theta$ where $0 \le \theta \le \pi$. Then by part (a) we have $p(x) = p(\cos\theta) = \cos 4\theta$ Hence the maximum and minimum values of $p(x)$ are ± 1. At maximum values $\cos 4\theta = 1 \Rightarrow 4\theta = 2n\pi$ so that $\theta = 0, \frac{\pi}{2}$ or π within the range. Since $x = \cos\theta$ we have $x = 0, \pm 1$. At minimum values $\cos 4\theta = -1 \Rightarrow 4\theta = (2n+1)\pi$ so that $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ within range whence $x = \pm\frac{1}{\sqrt{2}}$. In summary, the maximum value of $p(x)$ is 1, and occurs at $x = 1, 0$ or -1 and the minimum value of $p(x)$ is -1, and occurs at $x = \pm 1/\sqrt{2}$	
Mathematical behaviours	**Marks**
• derives 1 and -1 as the extreme values of $p(x)$	1
• determines the correct values of θ at the maximum	1
• infers the corresponding correct values of x	1
• determines the correct values of θ at the minimum	1
• infers the corresponding correct values of x	1