Year 11 Mathematics Methods Investigation 3 Out of Class Section

Name:

Score: out of

This investigation is about **number patterns and sequences.** Make sure that you understand this out of class section and how to use the CAS calculator in determining answers. An in class validation will follow.

Part One:

The following diagrams are made of matchsticks. The first diagram shows a shape that uses 8 matches.

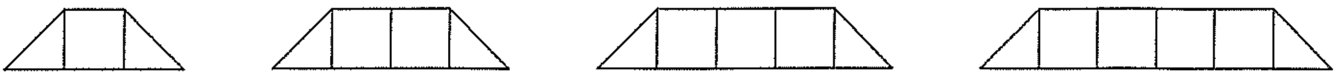


diagram 1 diagram 2 diagram 3 diagram 4

(a) Complete the table showing the number of matchsticks needed for each shape.

Diagram	1	2	3	4	5
Matchsticks	8	11	14	17	20

(b) What type of pattern do these numbers form? Linear

Statistics Regression Model
Recursive Rule Explicit Rule
Sequence Define Solve
Σ display Spreadsheet

(c) Determine a ***recursive*** rule for the sequence of matchsticks of the form

$T_{n+1} =$ $T_n + 3$, $T_1 =$ 8

(d) Use the Recursive tab in ***Sequence*** and your rule from part c to determine

(i) T_{15} $= 50$ (ii) T_{53} $= 164$

(e) Which diagram number could you make if you had

(i) 300 matches $T_{98} = 299$ $\Rightarrow 98$ (ii) 3000 matches $T_{998} = 2999$ $\Rightarrow 998$

(f) If you had to make

(i) diagram 1 and diagram 2; how many matches would you need in total? 19

(ii) diagram 1, diagram 2 and diagram 3; how many matches would you need in total? 33

(iii) diagram 1, diagram 2, diagram 3 and diagram 4; how many matches would you need in total? 50

Complete this table

Diagram	1	1 and 2	1, 2 and 3	1, 2, 3 and 4	1, 2, 3, 4 and 5
Total Matches (S_n)	8	19	33	50	70

11 14 17 20

3 3 3

Determine an explicit rule for S_n Quadratic CAS statistics

$S_n = \frac{3n^2}{2} + \frac{13n}{2}$

(h) If you were to make the first 15 diagrams, how many matches in total would you need?

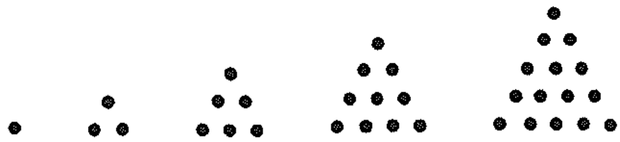
$S_{15} = 435$

(i) How many diagrams in total could you make if you had 5 000 000 matches?

Solve $\frac{3x^2}{2} + \frac{13x}{2} = 5\,000\,000$, $x = 1823.576\ldots$ $\Rightarrow 1823$

Part Two:

The first five triangular numbers are shown in the diagram below.



1 3 6 10 15

x	1	2	3	4	5
y	1	3	6	10	15

2 3 4 5

1 1 1

Quadratic

(a) What type of pattern do these numbers form?

Quadratic

(b) Determine the rule for this set of numbers by using an appropriate regression model in ***Statistics***.
Write down your rule.

$T_n = \frac{n^2}{2} + \frac{n}{2}$

(c) Use the Explicit tab in ***Sequence*** and your rule from part b to generate the first 20 triangular numbers.
Write down your values in the table below.

1	3	6	10	15	21	28	36	45	55
66	78	91	105	120	136	153	171	190	210

(d) Consider the table shown below.

n	1	2	3	4	5	6	7	8
T_n	1	3	6	10	15	21	28	36
S_n	1	4	10	20	35	56	84	120

(i) Describe how the numbers in the S_n row are found.

adding together successive terms (adding on the next term each time)

(ii) Complete the remaining entries in the table.

(iii) Determine rules, in terms on n, for S_n

$S_n = \frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3}$

x	1	2	3	4	5
y	1	4	10	20	35

3 6 10 15

3 4 5

1 1

Cubic

(e) Use your rules from above to determine

(i) T_{50} = 1275 (ii) S_{50} = 22 100

(f) Use your rules from part d to determine n where

(i) $T_n = 2016$ (ii) $S_n = 246905$

n = 63 n = 113

(g) The 14th triangular number is the smallest to exceed 100 (i.e. $T_{14} = 105$).

Which is the smallest triangular number to exceed

(i) 1 thousand — T_{45}

(ii) 1 million — T_{1414}

(iii) 1 billion ? — $T_{44\,720}$

(h) The sum of the first 18 triangular numbers is 1140 i.e. $S_{18} = 1140$.

What is the smallest value of n for the sum to first exceed

(i) 1 million — S_{181}

(ii) 1 billion — S_{1817}

(iii) 1 trillion ? — S_{18171}

Part Three:

The Fibonacci sequence begins as follows 1, 1, 2, 3, 5, 8, 13, 21, . . .

(a) The Fibonacci sequence can be defined recursively by the rule $F_n = F_{n-1} + F_{n-2}$, $F_1 = F_2 = 1$

Use the Recursive tab in ***Sequence*** to enter this rule and complete the table below.

NOTE: You may need to consider the rule $F_{n+2} = F_{n+1} + F_n$, $F_1 = F_2 = 1$ on the calculator.

n	27	28	29	30
F_n	196 418	317 811	514 229	832040

(b) What is the sum of the first 30 Fibonacci numbers?

2 178 308

(c) The table below lists the first ten Fibonacci numbers and some values in a new sequence. This new sequence T_n is defined as

$T_n = F_n + 2F_{n-1}$ for $n \geq 2$

n	1	2	3	4	5	6	7	8	9	10
F_n	1	1	2	3	5	8	13	21	34	55
T_n	dne	3	4	7	11	18	29	47	76	123

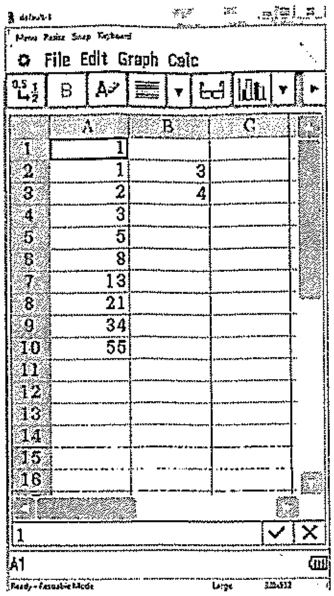
Complete the table.

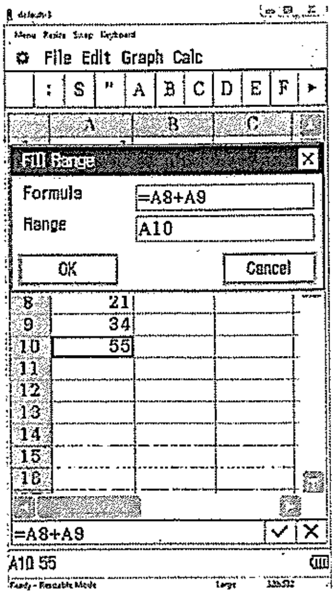
eg $T_4 = F_4 + 2 \times F_3 = 3 + 2 \times 2 = 7$

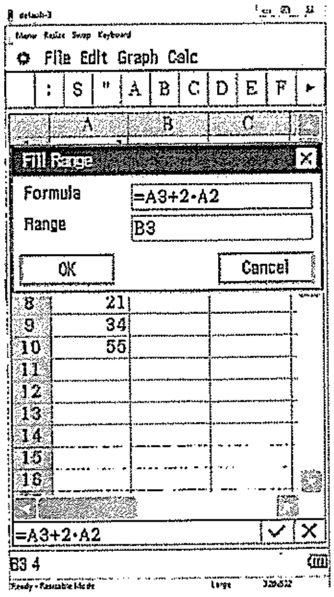
$T_5 = F_5 + 2 \times F_4 = 5 + 2 \times 3 = 11$

$T_6 = F_6 + 2 \times F_5 = 8 + 2 \times 5 = 18$

(d) You can write appropriate formula in ***Spreadsheet*** on the calculator to determine numbers in the sequences F_n and T_n. Part of a sample output and some formula instructions are shown below.







Use ***Spreadsheet*** on the calculator to help you complete the table below.

n	27	28	29	30
F_n	196 418	317 811	514 229	832040
T_n	439 204	710 647	1 149 851	1 860 498

(e) A student makes a conjecture that $T_n = F_{n+1} + F_{n-1}$ for $n \geq 2$

Test this conjecture for n = 28, n = 29 and n = 30.

$n = 28$

$T_{28} = F_{29} + F_{27}$

$= 514\,229 + 196\,418$

$= 710\,647$ True

$n = 29$

$T_{29} = F_{30} + F_{28}$

$= 832\,040 + 317\,811$

$= 1\,149\,851$ True

$n = 30$

$T_{30} = F_{31} + F_{29}$

$= 1\,346\,269 + 514\,229$

$= 1\,860\,498$ True.

(f) The squares of the Fibonacci numbers sum as follows:

$1^2 + 1^2 = 2 = 1 \times 2$

$1^2 + 1^2 + 2^2 = 6 = 2 \times 3$

$1^2 + 1^2 + 2^2 + 3^2 = 15 = 3 \times 5$

$1^2 + 1^2 + 2^2 + 3^2 + 5^2 = 40 = 5 \times 8$

Write down the next two lines in the pattern

$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 = 104 = 8 \times 13$

$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 + 13^2 = 273 = 13 \times 21$

Use your pattern to complete the conjecture $(F_1)^2 + (F_2)^2 + (F_3)^2 + \ldots + (F_n)^2 = F_n \times F_{n+1}$

Use your conjecture to determine $(F_1)^2 + (F_2)^2 + (F_3)^2 + \ldots + (F_{30})^2$

$= F_{30} \times F_{31}$

$= 832\,040 \times 1\,346\,269$

$= 1\,120\,149\,658\,760$

Part Four:

Shown below are rows 0 to 8 of the famous Pascal's triangle. The Frenchman Pascal (1623 – 1662) did not discover it first. The existence of this triangle can be traced back to China and Persia about 1100 AD.

	Row
1	0
1 1	1
1 2 1	2
1 3 3 1	3
1 4 6 4 1	4
1 5 10 10 5 1	5
1 6 15 20 15 6 1	6
1 7 21 35 35 21 7 1	7
1 8 28 56 70 56 28 8 1	8

(a) Calculate the total of the rows for each row 0 to row 8. Place your answers in a table.

Row	0	1	2	3	4	5	6	7	8
Total	1	2	4	8	16	32	64	128	256

(b) Find a rule to predict the sum of the numbers in the nth row of Pascal's triangle. $S_n = 2^n$

(c) Calculate the cumulative sums of the numbers for rows 0 to row 8 and place in the table below.

1   1+2   1+2+4   1+2+4+8   etc.

Row	0	1	2	3	4	5	6	7	8
Sum	1	3	7	15	31	63	127	255	511

Find a rule for the cumulative sum of the first n rows of Pascal's Triangle $C_n = 2^{n+1} - 1$

or

$C_n = 2(2^n) - 1$

(d) Use your rules to determine

(i) the sum of the numbers in the 20th row

$S_{20} = 2^{20}$

$= 1048576$

(ii) the sum of all the numbers in the 20th row

$C_{20} = 2^{21} - 1$

$= 2\ 097\ 151$

Part Five:

Wayne borrows money to purchase a share of a race horse called "Broken Promises". He starts paying off the loan in January 2015. The first four months of Wayne's loan details are shown below.

Month	Amount owing	Interest	Monthly payment	Balance owing
1	$20 000.00	$200.00	$350.00	$19 850.00
2	$19 850.00	$198.50	$350.00	$19 698.50
3	$19698.50	$196.99	$350.00	$19 545.49
4	$19545.49	195.45	350	19390.94

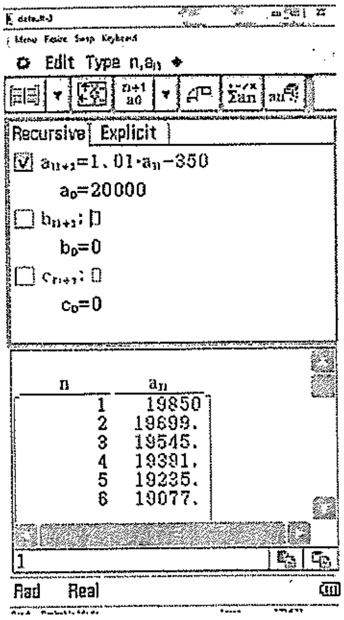
(a) How much did Wayne borrow form the bank? $20 000

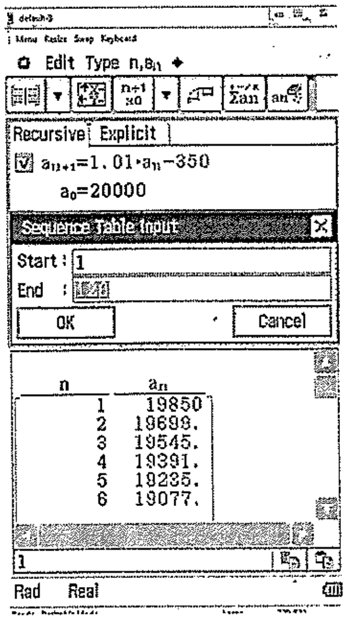
(b) How much interest did Wayne pay in the first month? What is this rate as a percentage per month? What is this an as annual percentage rate? $200 $\Rightarrow \frac{200}{20\,000} \times 100 = 1\%$ per month

$= 12\%$ p.a.

(c) Complete the table above.

(d) Wayne's loan can be generated on the ***Sequence*** application on the calculator as follows:





Find the amount Wayne owes after 1 year $18 097·62

How much does Wayne owe after 85 month? Explain why Wayne will not pay back his usual $350 in the 86th month.
$53·15 only needs to pay back $53·15 + interest

Calculate the final (86th) payment Wayne makes.
$53.15 \times 1.01 = \$53.68$

Calculate the extra money (interest) that Wayne has paid overall.
$85 \times \$350 + \$53.68 - \$20\,000 = \$9803.68$
$(29750 + 53.68 - 20\,000)$

(e) Tony is a friend of Wayne. He borrows the same amount for the racehorse share and makes the same monthly payment as Wayne. However Tony is a credit risk and must borrow at a higher rate of 24% per annum.
24% p.a. ⇒ 2% per month

Explain what terrible thing is happening to Tony in this loan by looking at the table in ***Sequence***.
$T_{n+1} = 1.02 \times T_n - 350, \quad T_0 = 20\,000$
Loan balance is alway increasing,

Use a repayment of $550 per month to answer these questions:

(i) How many months did Tony take to pay off the loan? 66 months

(ii) What was Tony's payment in the last month? $331.08 \times 1.02 = \$337.70$

(iii) How much did Tony pay back overall in payments? $36087·70
$65 \times \$550 + 337.70$

(iv) How much interest did Tony pay back over the course of the loan?
$36\,087.70 - 20\,000$
$= \$16\,087.70$

8% p.a. $\Rightarrow$ $\frac{2}{3}$% per month

(f) Julia is Wayne's boss. She also takes a share in the horse. She has a friend in the bank and gets a cheaper rate of 8% per annum. She makes the same \$350 payment that Wayne did. Change the details on your Sequence application to match Julia's loan. $T_{n+1} = (1 + 0.08 \div 12) \times T_n$, $T_0 = 20\,000$

(i) How quickly did Julia pay off her loan? 73 months

(ii) How much interest did she have to pay? $72 \times \$350 + 61.18 \times (1 + 0.08 \div 12) - 20\,000$
$= \$5261.59$

(iii) How much less interest did she save compared to Wayne?

9803.68
− 5261.59

\$4542.09

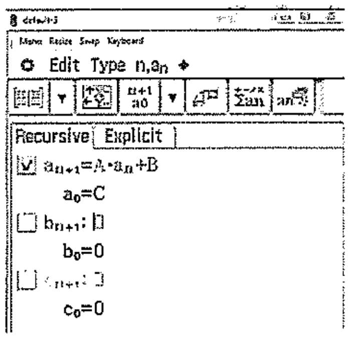
Part Six:

The parents of a newborn, Jacinta, are financially very astute. They placed \$500 in an account when she was born and then agree to deposit another \$500 on her birthday until she turns 21. The account earns 5.6% p.a. compounding yearly. Part of this situation is shown in the table below.

Year	Balance at start	Interest	Deposit	Balance at end
1	\$500.00	\$28.00	\$500	\$1028.00
2	\$1028.00	\$57.57	\$500	\$1585.57
3	\$1585.57	\$88.79	\$500	\$2174.36
4	2174.36	121.76	500	2796.12

(a) Complete the table for year 4.

This situation can be replicated in ***Sequence***.



(b) What are the values for A, B and C that you will need to use?

A = 1.056

B = 500

C = 500

(c) How much will be in Jacinta's account on her 21[st] birthday?

\$ 20 678.00 (includes the last \$500 deposit)

Explain/Justify your answers for the questions that follow.

(d) Jacinta's parents have put in 22 payments (0[th] – 21[st] birthdays) of \$500 each. How much interest has this account earned?

$20\,678 - 22 \times \$500 = \$9678$

(e) If instead of 5.6% p.a. Jacinta's parents could have found an account paying 6%, how much more money would she have by her 21[st] birthday?

$T_{n+1} = 1.06 \times T_n + 500$, $T_0 = 500$

\$21 696.15 Extra \$1018.15

20678

(f) How much more would be in the account on her 21st birthday if they deposited $600 instead of $500? (Use the original interest rate).

$T_{n+1} = 1.056 \times T_n + 600$ T_{21} $24813.60 Extra $4135.60

$T_0 = 600$

(g) How much more would be in the account on her 21st birthday if they deposited $50 per month and the interest was compounded monthly? (Use the original interest rate).

$T_{n+1} = (1 + 0.056 \div 12) \times T_n + 50$ T_{252} $25 859.46 Extra $5181.46

$T_0 = 600$

(h) To the nearest dollar, determine how much Jacinta's parents would need to deposit annually at 5.6% p.a. for the total amount to accrue to $50 0000 by her 21st birthday.

$T_{n+1} = 1.056 \times T_n + 1209$ $\Rightarrow$ $49 999

$T_{n+1} = 1.056 \times T_n + 1210$ $\Rightarrow$ 50 040

$1209

Part Seven:

Sylvia borrows $85 000 to start a business organising end of year leavers parties for lazy Year Twelve students. The interest rate that she pays on her loan is 8.2% p.a. and she wishes to repay the money in exactly 10 years.

Find the repayment that she must make each month in order to pay the loan off in 10 years (answer correct to the nearest 10 cents). Explain clearly how you did this. Show any working.

Include a table showing the money owing for the first four months under the correct repayment for Sylvia.

Month	Start	Interest	Balance
1	85 000.00	1040.29	84 540.54
2	84 540.54	1040.29	84 077.95
3	84 077.95	1040.29	83 612.19
4	83 612.19	1040.29	83 143.25

$1040.29 $\Rightarrow$ $1040.30

To help banks and customers calculate repayments for different loans there is a formula. It is shown below.

$$A_0 = \frac{Q[1-(1+i)^{-n}]}{i}$$

Number of regular repayment needed

Amount borrowed

Regular repayment

Interest rate per payment period

Given Sylvia's details from above state the values of A_0, i and n

A_0 = 85 000 i = 0.00683 ($8.2 \div 100 \div 12$) and n = 120

solve / eActivity

Use the formula above to calculate the regular repayment needed. Indicate any working and the method used. Does this rule yield the same answer as before?

YES

Find the regular monthly repayment needed to pay off a car loan of $15 250 in six years at an interest rate of 8.5% p.a.

$A_0 = 15250$

$i = 8.5 \div 100 \div 12 = 0.00708\dot{3}$

$n = 6 \times 12 = 72$

Solve

$$15250 = \frac{Q\left(1 - (1 + 8.5 \div 100 \div 12)^{-72}\right)}{8.5 \div 100 \div 12}$$

$Q = 271.12$

$271.12 payment

Use the same formula to calculate the amount Mary borrowed if she pays back a 9.7% p.a. loan in 48 months at $193 per month.

$Q = 193$

$i = 9.7 \div 12 \div 100 = 0.00808\dot{3}$

$n = 48$

Solve

$$A_0 = \frac{193\left(1 - (1 + 9.7 \div 12 \div 100)^{-48}\right)}{9.7 \div 12 \div 100}$$

$A_0 = 7653.03$

$\Rightarrow$ $7653 loan