Manjimup S.H.S Year 12 2016
Mathematical Methods Investigation 3
Out of Class Section

NORMAL DARTBOARDS

Name: ANSWERS .

DATE: Handout: Tuesday June 7th 2016
Due: Tuesday June 14th 2016
Test day: Friday June 17th 2016

- *The Out of Class Investigation is designed for you to learn the essentials needed for the In-Class validation.*
- *This is the "Take Home" part of the Investigation. It does not count towards your mark for this investigation.*

This investigation covers three main ideas:

- Continuous Probability Distributions
- The Normal distribution Curve

Part One: Dartboards

A game developer is experimenting with different dart board shapes. He is looking at certain dartboard designs and is considering the chance of hitting certain parts of the board. In each problem the area of certain sections can be calculated using calculus or by using geometrical shape rules.

Assume the dart always hits the board in these problems.

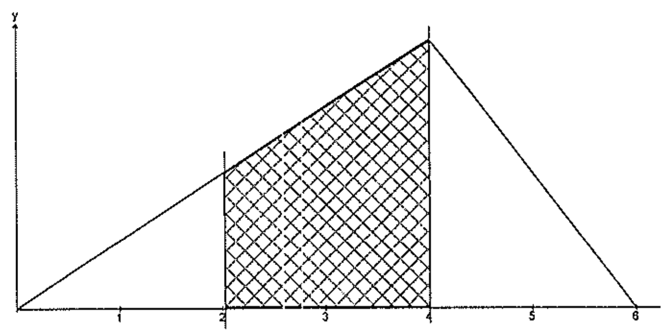
1. For each dartboard following you bare required to calculate the following probabilities.
   - P(x<1)
   - P(2<x<4)
   - P(x≥3/x<4)
   - P(1<x<2 or x≥4)

   To get you started an example has been given to you below.

Example Dartboard

$$y = 0.25x \quad \text{for } 0 \le x < 4$$
$$y = -0.5x + 3 \quad \text{for } 4 \le x \le 6$$

Sketch



$$P(x<1) = \frac{\text{Area under line from } x = 0 \text{ to } x = 1}{\text{Total area of dartboard}}$$

$$= \frac{\int_0^1 0.25x\,dx}{\int_0^4 0.25x\,dx + \int_4^6 (3-0.5x)\,dx}$$

$$\frac{0.125}{2+1} = \frac{1}{24}$$

OR use the formula for area of a triangle

$$\text{Area} = \frac{1}{2}\text{base} \times \text{height}$$

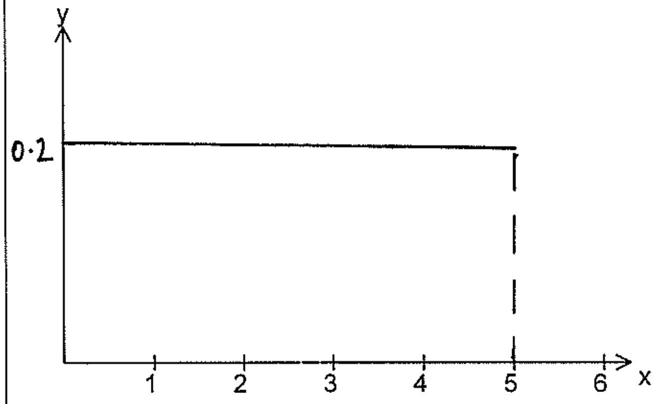
$$P(2<x<4) = \frac{\text{Area under line from } x = 2 \text{ to } x = 4}{\text{Total area of dartboard}}$$

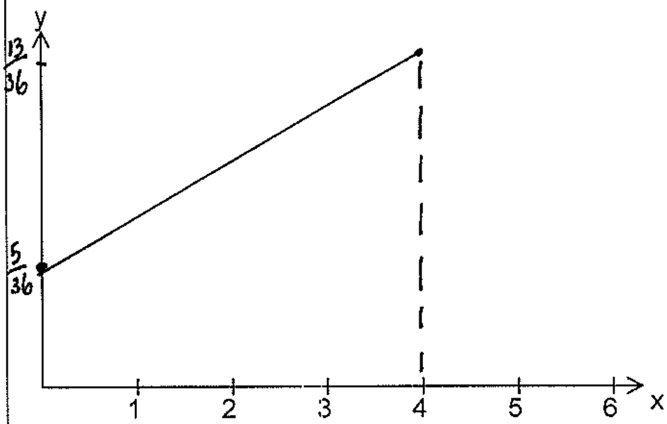
$$= \frac{\int_2^4 0.25x\,dx}{\int_0^4 0.25x\,dx + \int_4^6 (3-0.5x)\,dx}$$

$$= \frac{1.5}{2+1} = \frac{1}{2}$$

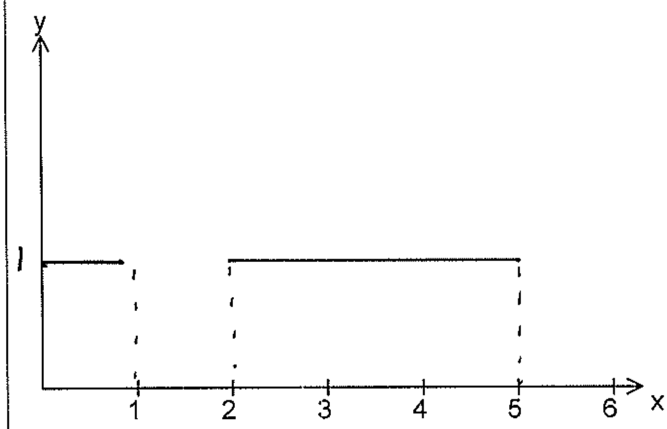
OR use the formula for area of a trapezium

$$\text{Area} = \frac{1}{2}\text{ sum of the parallel sides} \times \text{perpendicular distance between them.}$$

Dartboard 1	**Sketch**
$y = 0.2 \quad \text{for } 0 \le x \le 5$	
$P(x<1)$ $= \frac{1}{5}$	$P(2<x<4)$ $= \frac{2}{5}$
$P(x \ge 3 \mid x<4)$ $\frac{\frac{1}{5}}{\frac{4}{5}} = \frac{1}{4}$	$P(1<x<2 \text{ or } x \ge 4)$ $\frac{1}{5} + \frac{1}{5} = \frac{2}{5}$.

Dartboard 2	**Sketch**
$y = \frac{2x+5}{36} \quad \text{for } 0 \le x \le 4$	
$P(x<1)$ $= \frac{1}{6}$	$P(2<x<4)$ $\frac{11}{18}$.
$P(x \ge 3 \mid x<4)$ $\frac{\frac{1}{3}}{1} = \frac{1}{3}$	$P(1<x<2 \text{ or } x \ge 4)$ $\frac{2}{9} + 0 = \frac{2}{9}$

<table>
<tr><td>Dartboard 3

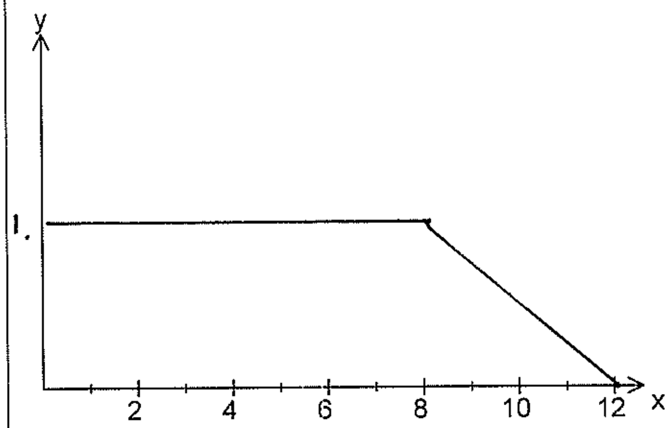
$y=1$ for $0 \le x \le 1$ or $2 \le x \le 5$</td><td>Sketch

</td></tr>
<tr><td>$P(x<1)$

$\frac{1}{4}$</td><td>$P(2<x<4)$

$\frac{2}{4} = \frac{1}{2}$.</td></tr>
<tr><td>$P(x \ge 3 \mid x<4)$

$\frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$</td><td>$P(1<x<2 \text{ or } x \ge 4)$

$0 + \frac{1}{4} = \frac{1}{4}$</td></tr>
<tr><td>Dartboard 4

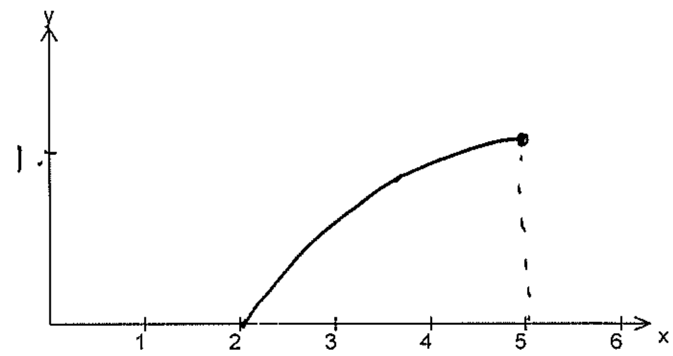
$y=1$ for $0 \le x \le 8$
and
$y=-\frac{1}{4}x+3$ for $8 \le x \le 12$</td><td>Sketch

</td></tr>
<tr><td>$P(x<1)$

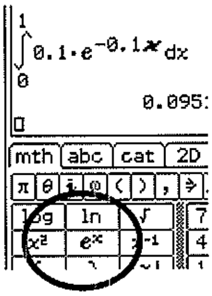
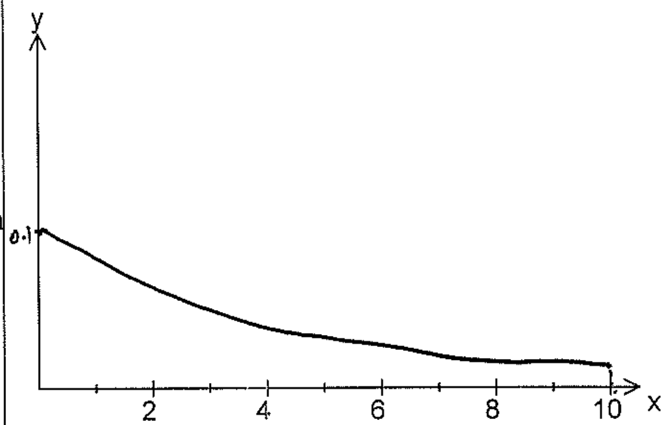
$\frac{1}{10}$</td><td>$P(2<x<4)$

$\frac{2}{10} = \frac{1}{5}$</td></tr>
<tr><td>$P(x \ge 3 \mid x<4)$

$\frac{1}{4}$</td><td>$P(1<x<2 \text{ or } x \ge 4)$

$\frac{7}{10}$.</td></tr>
</table>

Dartboard 6	**Sketch**
$y=\dfrac{10x-x^2-16}{9}$ for $2<x<5$	
$P(x<1)$ $\dfrac{0}{2}=0$	$P(2<x<4)$ $\dfrac{\frac{28}{27}}{2}=\dfrac{14}{27}=0.5185.$
$P(x\geq 3 \mid x<4)$ $\dfrac{\frac{20}{27}}{\frac{28}{27}}=\dfrac{20}{28}=\dfrac{10}{14}=\dfrac{5}{7}=0.7143.$	$P(1<x<2 \text{ or } x\geq 4)$ $\dfrac{\frac{26}{27}}{2}=\dfrac{26}{54}=\dfrac{13}{27}=0.4815.$

Dartboard 5	**Sketch**
$y=0.1e^{(-0.1x)}$ for $0<x<10$ where $e=2.71828$ (a constant) *Remember:* The e symbol is on your virtual keyboard and you can do integrals on the calculator. 	
$P(x<1)$ $\dfrac{0.0952}{0.6321}=0.1505.$	$P(2<x<4)$ $\dfrac{0.2348}{0.6321}=0.3714$
$P(x\geq 3 \mid x<4)$ $\dfrac{0.0704}{0.3297}=0.2135$	$P(1<x<2 \text{ or } x\geq 4)$ $\dfrac{0.0861+0.3024}{0.6321}=0.6146.$

2 The game developer wishes to establish a gambling club so that the patrons can bet on a machine which throws a dart at random at a board. He decides that it would be easier for calculation purposes if the total area of each board was ***1 square unit***, thus ensuring that the integrals give probabilities directly.

(a) Which of the dartboards would satisfy this requirement? Justify your answers using calculus.

Dartboards 1, and 2. Both have Areas = 1.

$\int_1^5 0.2\,dx = 1$. $\int_0^4 \frac{2x+5}{36}\,dx = 1$.

(b) He plans to design boards according to the following specifications. The boundaries of the board are the given function over the stated domain and the positive x axis. Calculate the value of k so that the total area of each board is 1 square unit.

(i) $y = k(5 - x)$ for $-2 \le x \le 4$

$K = \frac{1}{24}$

(ii) $y = kx^2(1 - x)$ for $0 < x < 1$

$K = 12$

(iii) $y = kx \quad 0 < x < 2$

$K = \frac{1}{2}$

(iv) $y = kx^3$ for $0 < x < 1$

$k = \frac{1}{20}$

(v) $y = \dfrac{2x+5}{k}$ for $0 < x < 4$

$k = 36$

Part Two: The Normal Distribution

Analysing Heights

The data given here is the heights (in centimetres) of a random group of high school students taken from Census on Line on the Australian Bureau of Statistics website on 6th June 2016.

Boys:

183	175	186	185	188	174	175	181	183	180	185
160	184	179	185	174	180	194	172	176	171	175
178	172	165	175	187	189	182	180	182	178	
179	173	181	190	180	175	186	175	162	192	
171	187	174	186	180	185	178	195	165	183	

$\Sigma B = 9330$

Girls:

156	181	173	150	180	165	169	173	169	168
170	160	165	160	173	197	154	172	165	168
165	163	170	153	171	152	163	163	164	155
180	160	163	160	157	170	164	167	171	
162	173	168	146	163	157	171	163	170	

$\Sigma G = 7952.$

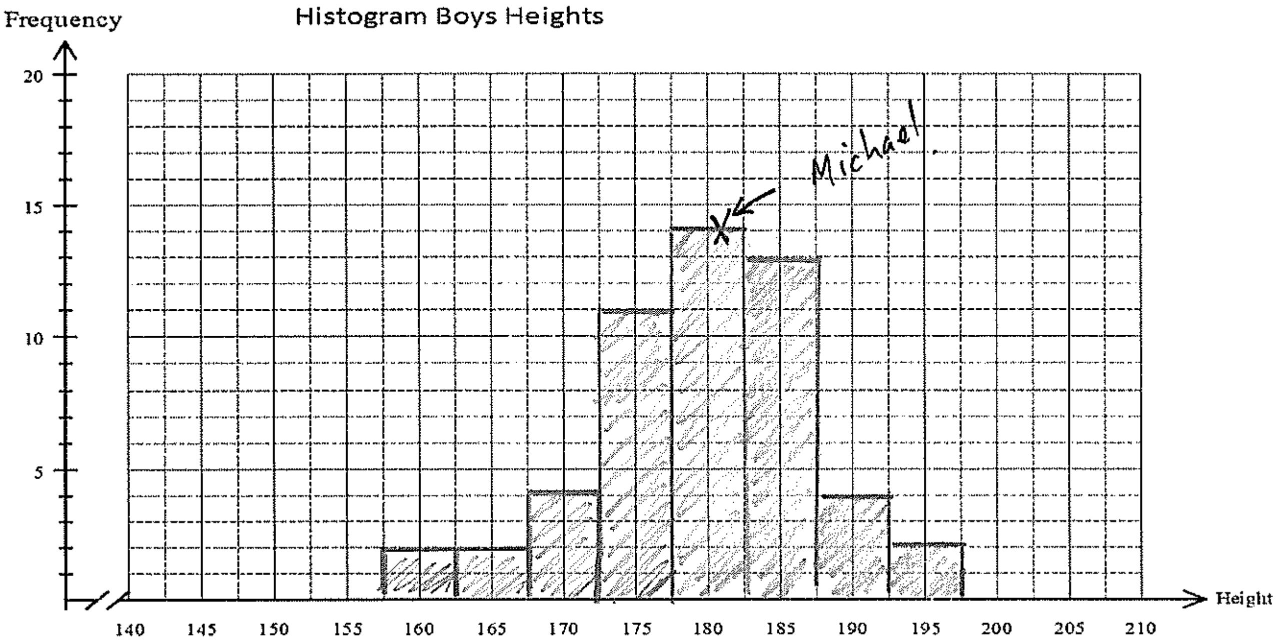
3. For each distribution use your calculator to determine the number, the mean and standard deviation.

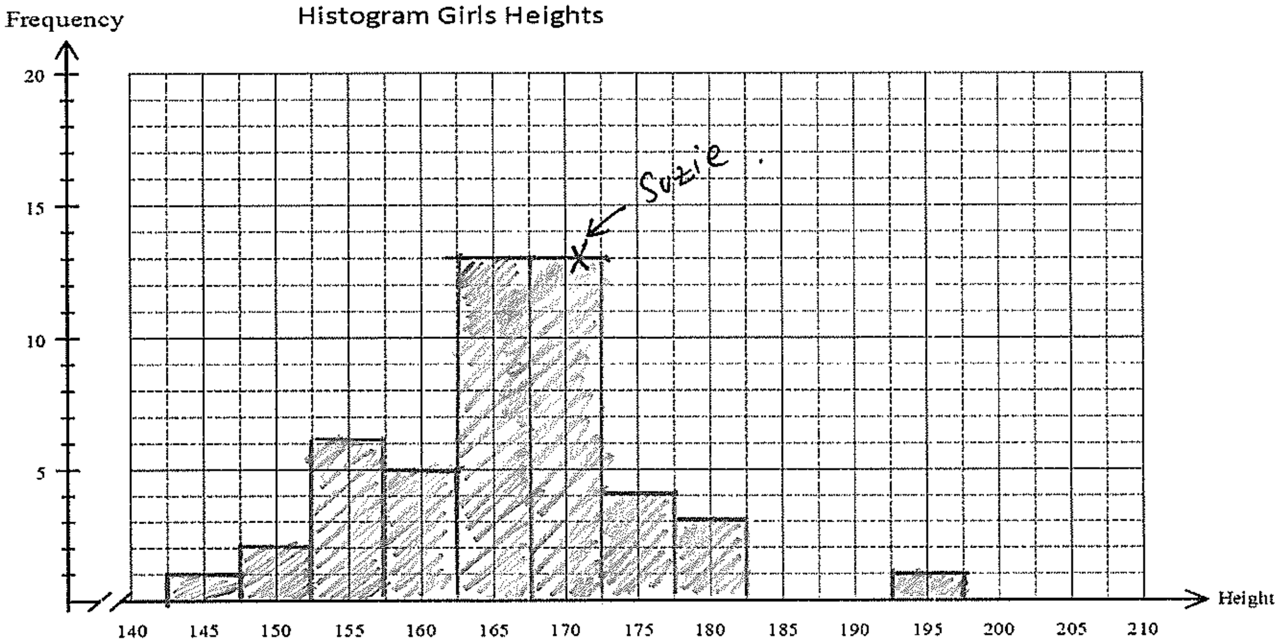
	Boys	Girls
Number	52	48
Mean height	179.42	165.67
Standard deviation	7.54	8.84

4. Grouping the data in 5cm intervals, starting from 127.5cm to 132.5 cm, draw histograms for each set of data.

Class interval	Frequency Boys	Frequency Girls
142.5 – 147.5	–	1
147.5 – 152.5	–	2
152.5 – 157.5	–	6
157.5 – 162.5	2	5
162.5 – 167.5	2	13
167.5 – 172.5	4	13
172.5 – 177.5	11	4
177.5 – 182.5	14	3
182.5 – 187.5	13	–
187.5 – 192.5	4	–
192.5 – 197.5	2	1
197.5 – 202.5	–	–

Histograms:

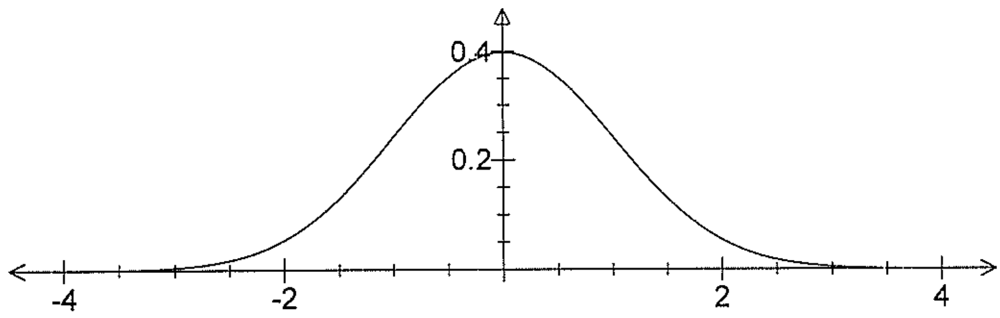




Comment on the shape of each histogram.

Both distributions are approximately symmetrical. The majority of scores are in the middle, with not many scores on the extremes

Both sets of data collected would be from normal distributions. A normal distribution has a symmetrical spread of data about the mean. The sketch here shows a standard normal distribution:

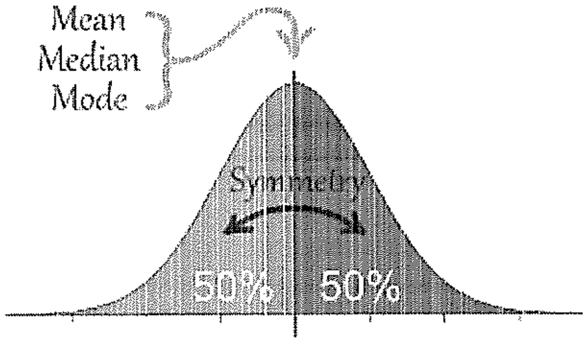


The values on the horizontal axis are standard deviations from the mean which is at zero.
As you can see from the sketch, a normal distribution spreads for about three standard deviations either side of the mean.

5. Comment on the shape of your two histograms in comparison with the shape of a normal distribution. Do they seem to match a normal distribution? Explain your answer.

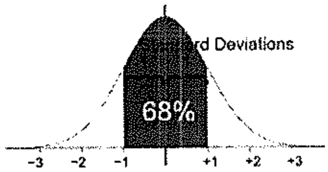
Both distributions have a similar shape to the Normal Curve. Both exhibit Normal characteristics – majority of scores in the middle, tapering off to the sides.

The "bell shaped" Normal curve is a very common distribution.

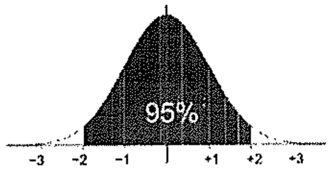


The Normal Distribution has:

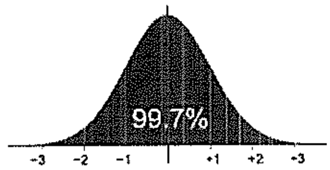
- <u>mean</u> = <u>median</u> = <u>mode</u>
- symmetry about the centre
- 50% of values less than the mean and 50% greater than the mean



68% of values are within 1 standard deviation of the mean



95% are within 2 standard deviations



99.7% are within 3 standard deviations

In a normal distribution 68% of the data should be within one standard deviation of the mean i.e. $\mu \mp 1 \times \sigma$

6. Find the interval of heights that is within one standard deviation of the mean for the two sets of Data, boys and girls heights.

Boys 172 - 186cm Girls 157 - 174 cm.

7. Calculate the percentage of heights within these intervals for the boys and girls data.

Boys $\frac{38}{52} = 73.0\%$ Girls $\frac{32}{48} = 66.7\%$.

8. Comment on the values for the two data sets compared to the expected percentage.

Both are quite close to the 68% Normal model.

In a normal distribution 95% of the data should be within two standard deviations of the mean i.e. $\mu \mp 2 \times \sigma$

9. Find the interval of heights that is within two standard deviations of the mean for the two sets of data, boys and girls heights.

Boys 164 - 194 cm Girls 148 - 183 cm.

10. Calculate the percentage of heights within these intervals for the boys and girls data.

Boys $\frac{49}{52} = 94.2\%$ Girls $\frac{46}{48} = 95.8\%$.

11. Comment on the values for the two data sets compared to the expected percentage.

Both are very close to expected 95%.

In a normal distribution 99.7% of the data should be within three standard deviations of the mean i.e. $\mu \mp 3 \times \sigma$

12. Find the interval of heights that is within three standard deviations of the mean for the two sets of data, boys and girls heights.

Boys 157-202 cm Girls 139-192 cm.

13. Calculate the percentage of heights within these intervals for the boys and girls data.

Boys $\frac{52}{52} = 100\%$ Girls $\frac{47}{48} = 97.9\%$.

14. Comment on the values for the two data sets compared to the expected percentage.

Again these percentages are close to 99.7% expected.

The heights of one girl and one boy are measured to be; Suzie 173 cm and Michael 182 cm. Relative to their gender group it appears that these heights are similar.

15. Indicate on the histograms where these two heights are.

16. Show that Suzie's height is 0.83 standard deviations above the mean relative to the girls' data.

Suzie $\frac{173 - 165.66}{8.84} = 0.83$ sds.

17. Calculate how many standard deviations more than the mean Michael's height is. For this calculation use the boys' mean and standard deviation.

Michael $\frac{182 - 179.4}{7.54} = 0.34$ sd

18. Relative to their gender group, who is the tallest? Explain your answer.

Suzie, she is more standard deviation above the mean.

19. Do a similar comparison between Angie at 155 cm and Brad at 171 cm. Explain your result.

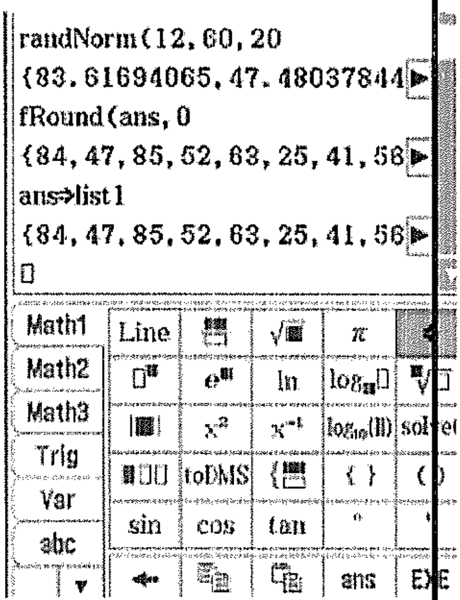
Angie : $\frac{155-165.67}{8.84} = -1.21$ Brad $\frac{171-179.42}{7.54} = -1.12$ Brad is very slightly relatively taller. (Closer to the mean)

20. Is there any height which is more than 3 standard deviations away from either the boy's or girl's mean? Justify your answer

Yes, 197 cm female. 3.54 sd above the mean.

21. Your Casio can draw a sample from a Normal Distribution.
You must feed in the mean, standard deviation and number of sample points you need.

Shown below is an example of a sample of 20 being drawn from a Normal distribution with a mean of 60 and a standard deviation of 12.



The sample points have been rounded and stored into List 1 on Statistics.

Your task: assuming Australian male heights are 176 cm and the standard deviation 7 cm use your calculator to produce on your calculator

- 200 sample points and store them into List 1.
- A histogram of these points starting from 150 cm going up in steps of 5 cm. How closely does this match the Normal curve?
- Find what percentage of your sample points are greater than 2 standard deviations above the mean.

Mr Chapman's results $\bar{x} = 176.9$, $sd = 7.07$ $\Rightarrow$ Above 191 cm. $\frac{3}{200} = 1.5\%$

- Use Normal CDF to see how closely this result for your sample matches the theoretical Normal result.

$X \sim N(7^2, 176) \Rightarrow p(x > 190) = 2.3\%$