NAME:________________________________

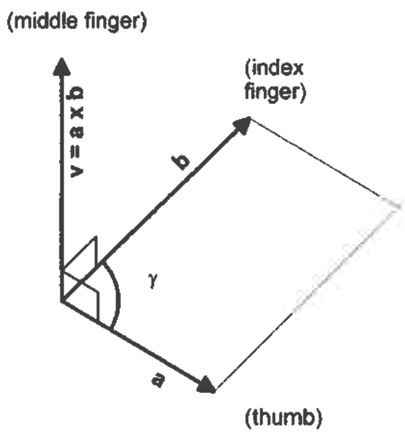
Investigation 2 – Vector Product (Cross Product)

IN-CLASS VALIDATION TEST

Time allowed: 50 MINUTES Total Marks 47 MARKS

NOTES are NOT permitted in this in-class validation test. An approved graphic calculator is allowed to be used. The definitions reproduced below are for your reference:

The **vector product** or **cross product** of two vectors **a** and **b** is a vector **v** which is written as: **v = a x b** and is defined as follows:

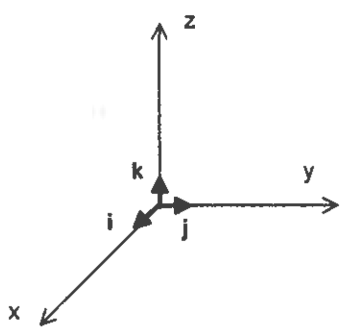


If **a** and **b** have the same or opposite direction or either **a** or **b** is zero then **v = a x b = 0**, otherwise;

v = a x b is the vector whose length is equal to the area of the parallelogram with **a** and **b** as adjacent sides and whose direction is perpendicular to both **a** and **b** and is such that **a, b, v**, in this order, form a right-handed triple or right handed triad, as shown in the diagram.

The term right-handed is derived from the fact that the vectors **a, b, v**, in this order, assume the same orientation as the thumb, index finger and middle finger of the right hand when held as indicated in the diagram.

The parallelogram with **a** and **b** as adjacent sides has the area $|\mathbf{v}| = |\mathbf{a}||\mathbf{b}|\sin(\gamma)$
Where γ is the angle between **a** and **b**.



Suppose that the vectors **a, b**, and **v** are represented in component form with respect to a right-handed, three dimensional Cartesian coordinate system, with unit vectors **i, j** and **k** in the directions x, y and z respectively.

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

Then

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Finally, if **v = a x b** , and the component vectors of **v** are v_1, v_2 and v_3, then the magnitude of **v** is $|\mathbf{v}| = |v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}|$

where $|\mathbf{v}|^2 = v_1^{\,2} + v_2^{\,2} + v_3^{\,2}$

QUESTION 1 [1 Mark]

Is **i** x **j** a scalar or a vector quantity? **ANSWER:** Vector ✓

QUESTION 2 [5, 1, 5, 2 Marks]

If, a = 2i + 3j - 5k, b = i + 4j – k and c = 2i + 2j + 5k then determine the following, showing full working of each element of the calculation, where appropriate:

2a) **a x b**

$$\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 3 & -5 \\ 1 & 4 & -1 \end{vmatrix} \checkmark$$

$$= (-3+20)\underset{\sim}{i} - (-2+5)\underset{\sim}{j} + (8-3)\underset{\sim}{k}$$

$$= \begin{pmatrix} 17 \\ -3 \\ 5 \end{pmatrix} \checkmark$$

2b) **b x a**

$$\underset{\sim}{b} \times \underset{\sim}{a} = -\begin{pmatrix} 17 \\ -3 \\ 5 \end{pmatrix}$$

$$\text{so } \begin{pmatrix} -17 \\ 3 \\ -5 \end{pmatrix} \checkmark$$

2c) **a x c**

$$\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 3 & -5 \\ 2 & 2 & 5 \end{vmatrix} \checkmark$$

$$= (15+10)\underset{\sim}{i} - (10+10)\underset{\sim}{j} + (4-6)\underset{\sim}{k}$$

$$= \begin{pmatrix} 25 \\ -20 \\ -2 \end{pmatrix} \checkmark$$

2d) **|a x c|, exactly**

$$|a \times c| = \sqrt{25^2 + (-20)^2 + (-2)^2} \checkmark$$

$$= 7\sqrt{21} \checkmark$$

QUESTION 3 [7,7 Marks]

Determine the **exact** area to three decimal places of the parallelogram, of which the given vectors are adjacent sides. Show full working for each set of calculations.

r = 3**i** – **j** – 2**k** , **s** = 4**i** + 5**j** – 2**k** , by two <u>different</u> methods.

<u>Method 1</u>, using the cross product .

$$|\underset{\sim}{r} \times \underset{\sim}{s}| = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & -1 & -2 \\ 4 & 5 & -2 \end{vmatrix} \checkmark$$

$$= (2+10)\underset{\sim}{i} - (-6+8)\underset{\sim}{j} + (15+4)\underset{\sim}{k} \checkmark \checkmark \checkmark$$

$$= \begin{pmatrix} 12 \\ -2 \\ 19 \end{pmatrix} \checkmark$$

$$\text{norm } \underset{\sim}{r} \times \underset{\sim}{s} = \sqrt{12^2 + (-2)^2 + 19^2} \checkmark$$

$$= 22.561 \text{ units}^2 \ (3\text{dp}). \checkmark$$

<u>Method 2</u>, using the dot product

$$\underset{\sim}{r} \cdot \underset{\sim}{s} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix} = 12 - 5 + 4 = 11 \checkmark$$

$$|\underset{\sim}{r}| = \sqrt{14}. \checkmark$$

$$|\underset{\sim}{s}| = 3\sqrt{5} \checkmark$$

$$\cos\theta = \frac{\underset{\sim}{r} \cdot \underset{\sim}{s}}{|\underset{\sim}{r}| \times |\underset{\sim}{s}|} = \frac{11}{\sqrt{14} \times 3\sqrt{5}} \checkmark$$

$$\therefore \theta = 64.0076913^\circ \checkmark$$

$$\text{area of parallelogram} = \sqrt{14} \times 3\sqrt{5} \sin\theta \checkmark$$

$$= 22.561 \text{ units}^2 \ (3\text{dp}) \checkmark$$

QUESTION 4 [7,3 Marks]

a) For the vectors **a** and **b**, such that **a** =< 11, 2, -9 > and **b** = < -1, 5, -6 >, Show that the cross product **a x b** is perpendicular to both vectors **a** and **b**. Show full working for your calculations.

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 11 & 2 & -9 \\ -1 & 5 & -6 \end{vmatrix}$$

$$= (-12+45)\underset{\sim}{i} - (-66-9)\underset{\sim}{j} + (55+2)\underset{\sim}{k}$$

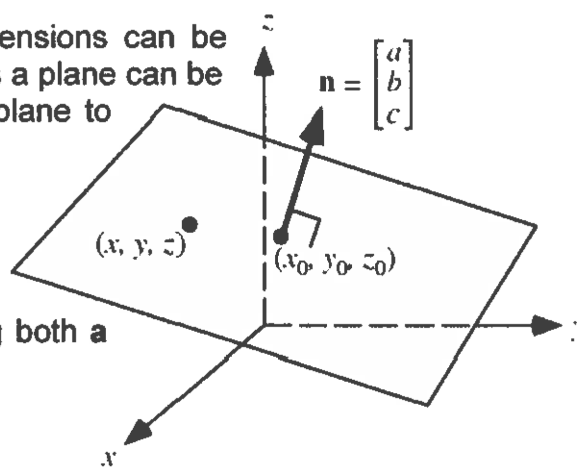
$$= \begin{pmatrix} 33 \\ 75 \\ 57 \end{pmatrix}$$

$$\begin{pmatrix} 33 \\ 75 \\ 57 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ 2 \\ -9 \end{pmatrix} = 363 + 150 - 513 = 0 \quad \therefore \text{perpendicular}$$

$$\begin{pmatrix} 33 \\ 75 \\ 57 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ -6 \end{pmatrix} = -33 + 375 - 342 = 0$$

$\therefore$ perpendicular

b) In the same way a vector equation in two dimensions can be defined by the normal to that vector, in three dimensions a plane can be defined by applying the normal to the vectors on the plane to any point on the plane.



It is defined by **r.n = a.n**
where **a** is the position vector of a point on the plane

If the point **p** = 3**i** – 4**j** – **k** lies on the plane containing both **a** and **b**, state the equation of the plane.

$$\underset{\sim}{r} \cdot \begin{pmatrix} 33 \\ 75 \\ 57 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 33 \\ 75 \\ 57 \end{pmatrix}$$

$$= -258$$

QUESTION 5

[3 Marks]

If $\mathbf{a} \times \mathbf{b} = 9\mathbf{k}$ and $\mathbf{a} = -5\mathbf{i} - 9\mathbf{j}$ and $\mathbf{b} = x\mathbf{i} + 9\mathbf{j}$, then determine the value of x.

$$\underset{\sim}{a} = \begin{pmatrix} -5 \\ -9 \\ 0 \end{pmatrix} \qquad \underset{\sim}{b} = \begin{pmatrix} x \\ 9 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -5 & -9 & 0 \\ x & 9 & 0 \end{vmatrix}$$

$$0\underset{\sim}{i} + 0\underset{\sim}{j} + (-45 + 9x)\underset{\sim}{k} = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix} \checkmark$$

$$\therefore \quad 9x - 45 = 9 \checkmark$$

$$\Rightarrow \quad x = 6. \checkmark$$

QUESTION 6 [5, 1 Marks]

5 a) Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. Prove that $\mathbf{a}\times\mathbf{b} = -(\mathbf{b}\times\mathbf{a})$ by expanding both sides of the equation.

$$\underset{\sim}{a} \times \underset{\sim}{b} \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{pmatrix} a_2b_3 - a_3b_2 \\ -(a_1b_3 - a_3b_1) \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

$$\underset{\sim}{b} \times \underset{\sim}{a} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \begin{pmatrix} a_3b_2 - a_2b_3 \\ -(a_3b_1 - a_1b_3) \\ a_2b_1 - a_1b_2 \end{pmatrix}$$

$$= - \underset{\sim}{b} \times \underset{\sim}{a}$$

5b) Given that $\mathbf{a}\times\mathbf{b} = -(\mathbf{b}\times\mathbf{a})$, sketch **b** x **a** in the following drawing: