Year 11 Mathematics-Specialist
Test 6 - 2017

MELVILLE
SENIOR HIGH SCHOOL

This test is Resource Free
It contains 13 questions worth 65 marks in total

Time Allowed : 60 minutes

Student Name : _______________________________________

Question 1 — 5 marks

Use Proof by Exhaustion to prove p^2 is only a multiple of three if p is a multiple of 3.

Identifies 2 cases : multiple of 3 or not multiple of 3 – 1 mark

If $p = 3n$,
$p^2 = (3n)^2 = 9n^2 = 3(3n^2)$ which is a multiple of 3 ✓

If $p = 3n + 1$,
$p^2 = (3n+1)^2 = 9n^2 + 6n + 1$ which is not a multiple of 3 ✓

If $p = 3n + 2$,
$p^2 = (3n+2)^2 = 9n^2 + 12n + 4$ which is not a multiple of 3 ✓

$\therefore p^2$ is only a multiple of 3 if p is a multiple of 3 ✓

Question 2 — 1 mark

If m is divisible by six and n is divisible by 15, then which of the following statements might be false?

a) $m \times n$ is divisible by 90

b) $m \times n$ is divisible by 30

c) $m \times n$ is divisible by 15

d) $m + n$ is divisible by 3

e) $m + n$ is divisible by 15

Question 3 **5 marks**

Prove by induction that $7^n - 4$ is divisible by 3 for all $n \in \mathbb{N}$.

Step 1 : When $n = 1$, $7 - 4 = 3 \therefore$ the statement is true for $n = 1$. ✓

Step 2 : Assume it is true when $n = k$. ✓

$ie.$ $7^k - 4$ is divisible by 3 for $k \in \mathbb{N}$, $7^k - 4 = 3m, m \in \mathbb{N}$

Step 3: Prove true when $n = k + 1$.

$$
\begin{aligned}
7^{k+1} - 4 &= 7^k.7 - 4 \\
&= 7(3m + 4) - 4 \quad ✓✓ \\
&= 21m + 28 - 4 \\
&= 21m + 24 \\
&= 3(7m + 8) \text{ which is a multiple of 3.}
\end{aligned}
$$

Step 4 : The statement is true for $n = k + 1$ if it is true for $n = k$,

and as it is true for $n = 1$, by the principles of ✓

mathematical induction $7^n - 4$ is divisible by 3, $n \in \mathbb{N}$

Question 4 — 1, 3 - 4 marks

Consider the statement " If n is odd, then $n^2 - 1$ is divisible by eight".

a) Show using 3 examples it might be true.

Substitutes 3 odd numbers into statement ✓

b) Prove the statement to be true.

If n is odd, $n = 2m + 1$, $m \in \mathbb{N}$ ✓

$n^2 - 1 = (2m+1)^2$

$= 4m^2 + 4m + 1 - 1$

$= 4m^2 + 4m$

$= 4m(m+1)$ ✓

States that of m & $m+1$ must be even and can be written as $2p$, so $4m(m+1) = 8pq$ which is a multiple of 8 ✓

Question 5 **1, 4, 5 - 10 marks**

Let $n \in \mathbb{Z}$. Consider the statement "If n^3 is even, then n is even."

a) Write the contrapositive of this statement.

If n is odd, the n^3 is odd. ✓

b) Prove the contrapositive.

If n is odd, $n = 2m+1, m \in \mathbb{Z}$ ✓

$$\begin{aligned} n^3 &= (2m+1)^3 \\ &= 8m^3 + 12m^2 + 6m + 1 \\ &= 2(4m^3 + 6m^2 + 3m) + 1 \quad ✓✓ \end{aligned}$$

n^3 is odd. ✓

c) Hence, prove by contradiction that $\sqrt[3]{6}$ is irrational.

Assume $\sqrt[3]{6}$ is rational. ✓

$\sqrt[3]{6}$ can be written as $\frac{a}{b}$, $a, b \in \mathbb{Z}, b \neq 0$ & a & b are coprime ✓

$$\sqrt[3]{6} = \frac{a}{b}$$

$$6 = \frac{a^3}{b^3} \Rightarrow 6b^3 = a^3$$

a^3 is a multiple of 6 and therefore even, ie $a = 2k, c \in \mathbb{Z}$ ✓

$$(2k)^3 = 8k^3 = 6b^3$$

$$4k^3 = 3b^3$$

$3b^3$ is even as $4k^3$ is even.

If $3b^3$ is even, b^3 is even and therefore b is even. ✓

We see that a and b are both even, which contradicts our original assumption that a and b are coprime. ✓

Thus $\sqrt[3]{6}$ is irrational.

Question 6 — 2, 4 - 6 marks

a) Prove that $0.15\overline{537}$ is rational.

$$\text{Let } x = 0.15\overline{537}$$
$$100x = 15.\overline{537}$$
$$100000x = 15537.\overline{537}$$ ✓ - Recognises appropriate multiples of x
$$99900x = 15522$$
$$x = \frac{15522}{99900}$$ ✓ - Provides fraction equivalent to x

b) Prove that $3.\dot{6}$ is rational by first expressing it as an infinite sum.

$$3.\dot{6} = 3 + 0.6 + 0.06 + 0.006 + \ldots$$
$$= 3 + \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} \ldots$$

Identifies infinite sum ✓

$\frac{6}{10} + \frac{6}{100} + \frac{6}{1000} \ldots$ is a geometric series with $a = \frac{6}{10}$ and $r = \frac{1}{10}$

Identifies a & r ✓

$$S_\infty = \frac{a}{1-r}$$
$$= \frac{6}{10} \div \frac{9}{10}$$
$$= \frac{2}{3}$$

Determines S_∞ ✓

$$3.\dot{6} = 3 + \frac{2}{3}$$
$$= \frac{11}{3}$$

✓ Calculates correct fraction

Question 7 — 1,1, 2 - 4 marks

Given $z = -11 + 7i$ state

a) $\text{Re}(z)$

-11

b) $\text{Im}(z)$

7

c) Evaluate $1 + i + i^6 - i^{11} + i^{16} + i^5$.

$1 + i + i^2 - i^3 + 1 + i$
$= 1 + i - 1 + i + 1 + i$
$= 1 + 3i$

Question 8 **1, 1, 2, 1 - 5 marks**

Given that $z = 4 + 3i$, express in the form $a + bi$.

a) $\bar{z}$

$4 - 3i$

b) $z + \bar{z}$

$4 + 3i + 4 - 3i = 8$

c) $z\bar{z}$

$(4 + 3i)(4 - 3i)$
$= 16 - 9i^2$ ✓
$= 25$ ✓

d) $\dfrac{1}{\bar{z}}$

$\dfrac{1}{4 - 3i} \times \dfrac{(4 + 3i)}{(4 + 3i)}$
$= \dfrac{4 + 3i}{25}$
$= \dfrac{4}{25} + \dfrac{3}{25}i$
or $(0.16 + 0.12i)$

Question 9 **2 marks**

Find the complex number, w, formed when $z = 3 - 5i$ is rotated on the Argand diagram by

a) 180° clockwise

$-3 + 5i$

b) 270° anticlockwise

$-5 - 3i$

Question 10 **1, 1, 2, 2 - 6 marks**

Consider the Argand diagram shown.

a) State the complex number represented by z.

$-4 + 3i$

b) Show and label on the diagram the location of $\overline{z}$

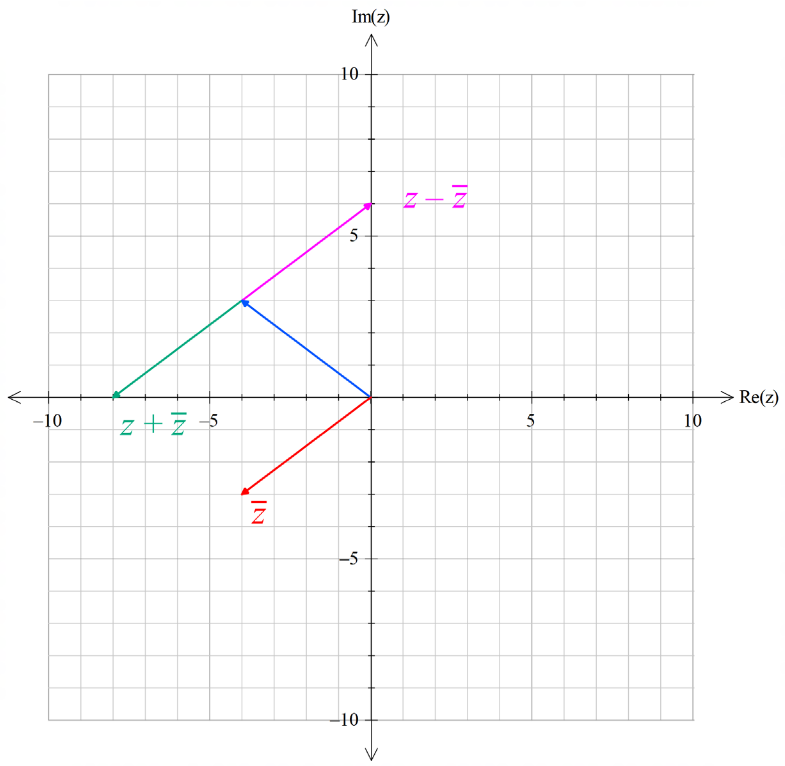
c) Show how to determine the value of the following expressions on the diagram given.
State the value of each.

ii) $z + \overline{z}$

-8

iv) $z - \overline{z}$

$6i$



Question 11 **1,3, 3 - 7 marks**

a) Solve for x if $x^2 = -81$

$x = \pm\sqrt{-1 \times 81}$

$= \pm 9i$

b) Find the values of x and y in $w = x + yi$ if:

i) $w\overline{w} = 13$ and $w + \overline{w} = -6$.

$(x+yi)+(x-yi) = -6$

$2x = -6$

$x = -3$ ✓

$(x+yi)(x-yi) = 13$

$x^2 - y^2 i^2 = 13$

$9 + y^2 = 13$

$y = \pm 2$ ✓✓

ii) $5w - 7\overline{w} = 14 + 24i$

$5(x+yi) - 7(x-yi) = 14 + 24i$ ✓

$-2x = 14$

$x = -7$ ✓

$5y + 7y = 24$

$y = 2$ ✓

Question 12 **4 marks**

If one of the solutions to the equation $z^2 + bz + c = 0$ is $-6 + i$, find the values of b and c.

Identifies other root ✓

Other root: $-6 - i$

$(-6 - i)(-6 + i) = 36 - i^2 = 37$

$-6 + i + (-6 - i) = -12$

Correctly determines equation ✓✓

$\therefore$ equation is $z^2 + 12z + 37$

States b & c ✓

$\Rightarrow b = +12, c = 37$

Question 13 4, 2 - 6 marks

a) Find the four roots of $f(x) = (4x^2 + 9)(x^2 - 6x + 34)$, giving your answers in the form $x = a + bi$, where a and b are real.

$4x^2 = -9$ or $x^2 - 6x + 34 = 0$

$x^2 = -\frac{9}{4}$ $(x-3)^2 + 25 = 0$

$x = \pm\frac{3}{2}i$ $x - 3 = \pm 5i$

$x = 3 \pm 5i$

b) Show these four roots on a single Argand diagram.