SCOTCH COLLEGE

12 Mathematics Methods 2021

Test 3 – Calculus of Trig, Exponential and Log functions

Section 1: Calculator-free

Time allowed: 25 minutes **Maximum marks: 25**

Name: Solutions **Teacher:** Foster | Kelly

Instructions:

- Show all working clearly.
- Sufficient detail must be shown for marks to be awarded for reasoning.
- A formula sheet will be provided.
- No calculators or personal notes are permitted.

Question 1 **[5 marks]**

Calculate;

a) $\frac{d}{dx}\sqrt{\cos^3 x}$ $= \frac{d}{dx}(\cos x)^{3/2}$ [2]

$= \frac{3}{2}\cos x^{1/2} \cdot -\sin x$

(accept alternate correct solutions)

$= -\frac{3}{2}\sin x\sqrt{\cos x}$

b) $\int\left(\frac{4x}{x^2-2}\right)dx$ [3]

$= 2\int\frac{2x}{x^2-2}\,dx = 2\ln(x^2-2) + c$

Question 2 **[5 marks]**

a) Evaluate: $\int_{-1}^{1}\frac{d}{dx}e^{x^3}x^2\,dx$ [2]

$= e^{1^3}1^2 - e^{(-1)^3}(-1)^2 = e - \frac{1}{e}$

b) Find $\frac{dy}{dx}$ given that $y = \int_2^{x^3}\left(3-\frac{4}{t^3}\right)^6 \mathrm{dt} \; - \frac{1}{e^x}$ [3]

$= \left(3 - \frac{4}{(x^3)^3}\right)^6(3x^2) - e^{-x}(-1)$

$= \left(3 - \frac{4}{x^9}\right)^6(3x^2) + \frac{1}{e^x}$

Question 3 **[5 marks]**

Find the equation of the tangent, in the form $y = mx + c$, to $y = e^{5x}$ when $x = -2$.

$\frac{dy}{dx} = 5e^{5x}$ ✓ ; $x = -2$, $m = 5e^{-10}$ ✓FT

$y - e^{-10} = 5e^{-10}(x+2)$ ✓ ✓FT

$y = 5e^{-10}x + 11e^{-10}$ ✓FT

Question 4 **[5 marks]**

Determine the area enclosed between the functions:

$f(x) = sinx$ and $g(x) = -1$ for $-\pi < x < 2\pi$.

$\sin x = -1$; $x = -\frac{\pi}{2}, \frac{3\pi}{2}$ ✓

$\int_{-\pi/2}^{3\pi/2} \left[(\sin x) - (-1)\right] dx = \left[-\cos x + x\right]_{-\pi/2}^{3\pi/2}$ ✓FT ✓FT

$= \left(-\cos\frac{3\pi}{2} + \frac{3\pi}{2}\right) - \left(-\cos\left(-\frac{\pi}{2}\right) - \frac{\pi}{2}\right)$ ✓FT

$= \left(-0 + \frac{3\pi}{2}\right) - \left(-0 - \frac{\pi}{2}\right) = 2\pi$ ✓FT

Question 5 **[5 marks]**

Let $F(x) = \int_0^x f(t)dt$, where $F(3) = -e^3$ and $F''(x) = -e^x$.

Find $f(x)$.

$$F'(x) = \frac{d}{dx}\int_0^x f(t)\,dt = f(x)$$

$$F''(x) = f'(x)$$

$$f'(x) = -e^x$$

$$f(x) = \int -e^x\,dx = -e^x + c$$

$$F(x) = \int_0^x (-e^t + c)\,dt$$

$$F(3) = \int_0^3 (-e^t + c)\,dt = -e^3$$

$$\left[-e^t + ct\right]_0^3 = -e^3$$

$$(-e^3 + 3c) - (-e^0 + 0) = -e^3 \;;\; 3c = -1 \;;\; c = -\frac{1}{3}$$

END OF SECTION 1

$$\therefore f(x) = -e^x - \frac{1}{3}$$

SCOTCH COLLEGE

12 Mathematics Methods 2021

Test 1 – Differentiation and Logarithms

Test 3 – Calculus of Trig, Exponential and Log functions

Time allowed: 20 minutes **Maximum marks: 20**

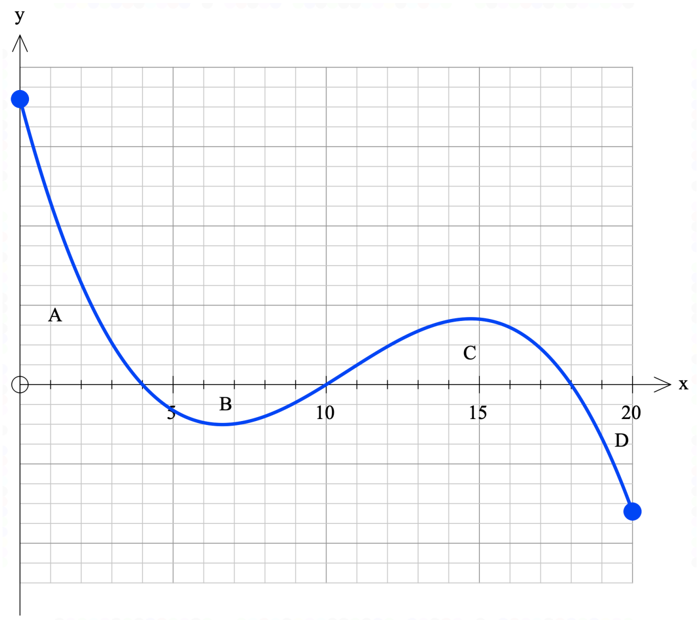
Name:_______________________________ **Teacher:** Foster | Kelly

Instructions:
- Show all working clearly.
- Sufficient detail must be shown for marks to be awarded for reasoning.
- A formula sheet will be provided.
- Calculators and 1xA4 double-sided page of personal notes are permitted.

Question 6 **[6 marks]**

The graph of the function $y = f(x)$ is shown below for $0 \leq x \leq 20$.

The areas of regions A, B, C and D enclosed by the function and the $x - axis$ are 11.6, 4, 8.5 and 2.9 cm^2 respectively.



a) State the area enclosed by the function and the $x - axis$ for $0 \leq x \leq 20$. [1]

$11.6 + 4 + 8.5 + 2.9 = 27 cm^2$

b) Determine the value of; [5]

i) $\int_0^{20} f(x)\,dx$

$= 11.6 - 4 + 8.5 - 2.9$

$= 13.2$

ii) $\int_{18}^{4} f(x)\,dx$

$= 4 - 8.5$

$= -4.5$

iii) $\int_0^{10} 3f(x)\,dx$

$= 3(11.6 - 4)$

$= 22.8$

iv) $\int_{18}^{20} (3 - f(x))\,dx$

$= \int_{18}^{20} 3\,dx - \int_{18}^{20} f(x)\,dx$

$= \left[3x\right]_{18}^{20} + 2.9$

$= 8.9$ FT

Question 7 **[6 marks]**

A study found that the rate of change in the population of a colony of ants was found to be proportional to the current population of the colony.

At the time of the study, the population was estimated to be 5000 and growing at 4% per annum.

a)

i) Estimate the population of the ant colony three months after the study. [2]

$P = 5000e^{0.04\left(\frac{3}{12}\right)} \approx 5050$ ants

✓ ✓ FT

ii) If this growth rate continued, when is the population expected to reach 1 million? [1]

Solve$(1000000 = 5000e^{0.04t}$

$t \approx 132$ yrs ✓ FT

b) It is expected that when the rate of change in the population of the colony reaches 500 ants/year, the colony will then decrease continuously at a rate of 6% of the population per year.

How long (after the original study began) will it take for the colony to return to its initial population? [3]

$\frac{dP}{dt} = 500$; $t = 22.907$

✓ FT

$t = 22.907$; $P = 12500$

$\therefore 12500e^{-0.06t} = 5000$; $t = 15.27$

✓ FT

$\therefore 22.907 + 15.27 \approx 38.18$ yrs ✓ FT

Question 8 **[4 marks]**

An object has an initial velocity of $-70\ ms^{-1}$ as it moves in a straight line with acceleration $4e^{-\frac{t}{20}}\ ms^{-2}$

Find the total distance travelled by the object in the **first minute**.

$$v(t) = \int 4e^{-t/20}\,dt = -80e^{-t/20} + c$$

$$v(0) = -70 \quad ; \quad c = 10$$

$$\int_0^{60} \left| -80e^{-t/20} + 10 \right| dt = 1047.88\text{m}$$

FT FT

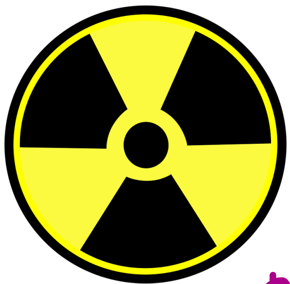
Question 9 **[4 marks]**

When talking about continuous decay, half-life is the time required for a quantity to reduce to half of its initial value.

A radioactive substance has a half-life of 50 days.

After 40 days, 260g of the substance remains.

Calculate the initial amount of the substance to the nearest gram.



$$P = P_0 e^{kt}$$

$$\frac{P}{P_0} = \frac{1}{2}$$

$$\frac{1}{2} = e^{50k}$$

$$\ln \frac{1}{2} = 50k$$

$$k = -0.01386\ldots$$

$$\therefore 260 = P_0 e^{-0.01386(40)}$$

$$P_0 = 452.68$$

$$\approx 453\text{g}$$

Alternative methods:

Solve $\begin{cases} P_0 e^{50k} = \frac{P_0}{2} \\ P_0 e^{40k} = 260 \end{cases}$

or

Solve $\left(260 = P_0 \left(\frac{1}{2}\right)^{\frac{40}{50}}\right)$

END OF TEST