MATHEMATICS SPECIALIST 1 & 2

SEMESTER 2 2019
TEST 4

Name________________________

Calculator Free

Reading time: 2 mins
Time allowed: 25 mins

Total marks: 25

1. [1, 2, 3 = 6 marks]

$A = \begin{bmatrix} 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 0 & -4 \\ 1 & 3 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

a) Which pairs of matrices can be added?

none ✓

b) Which pairs of matrices can be multiplied? (Since order counts, make sure you list them as pairs in the correct order)

AB BC AC CD DA. ✓✓ (−1 per error missed or extra)

c) Complete a multiplication of any two of the above matrices in which the product will be a 2 by 1 matrix.

✓
$CD = \begin{bmatrix} 2 & 0 & -4 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$ ✓✓

2. [3 marks]

Given the matrix $A = \begin{bmatrix} x & 1 \\ 2 & 3x \end{bmatrix}$ find the value(s) of x for which $A^2 = \begin{bmatrix} 6 & 8 \\ 16 & 38 \end{bmatrix}$

$$\begin{bmatrix} x & 1 \\ 2 & 3x \end{bmatrix}\begin{bmatrix} x & 1 \\ 2 & 3x \end{bmatrix}$$

$$\begin{bmatrix} 6 & 8 \\ 16 & 38 \end{bmatrix} = \begin{bmatrix} x^2+2 & 4x \\ 8x & 2+9x^2 \end{bmatrix} \checkmark\checkmark$$

$$x = 2 \checkmark$$

3. [3, 1 = 4 marks]

Given that $ABC = \begin{bmatrix} 4 & 5 \\ -2 & 1 \end{bmatrix}$ and $(BC)^{-1} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$

a) Find A.

$$A = ABC(BC)^{-1} \quad \checkmark$$

$$= \begin{bmatrix} 4 & 5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \quad \checkmark$$

$$= \begin{bmatrix} 3 & 5 \\ -5 & 1 \end{bmatrix} \checkmark$$

b) Find $C^{-1}B^{-1}$ $= (BC)^{-1} = \begin{bmatrix} -2 & 0 \\ -1 & 1 \end{bmatrix}. \checkmark$

Question 4(a) (2 marks)

Solution

Answer 1

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{cccc} B & W & M & N \end{array} \\ \begin{bmatrix} 20 & 10 & 4 & 5 \\ 10 & 8 & 4 & 3 \\ 5 & 3 & 4 & 6 \end{bmatrix} \end{array}$$

Answer 2

$$\begin{array}{c} \\ B \\ W \\ M \\ N \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 20 & 10 & 5 \\ 10 & 8 & 3 \\ 4 & 4 & 4 \\ 5 & 3 & 6 \end{bmatrix} \end{array}$$

Marking key/mathematical behaviours	Marks
• constructs the matrix accurately	1
• labels the rows and columns	1

Question 4(b) (1 mark)

Solution

Answer 1

$$\begin{array}{c} B \\ W \\ M \\ N \end{array}\begin{bmatrix} 1.20 \\ 1.00 \\ 1.50 \\ 2.00 \end{bmatrix}$$

Answer 2

$$\begin{array}{c} \begin{array}{cccc} B & W & M & N \end{array} \\ \begin{bmatrix} 1.2 & 1.0 & 1.5 & 2.0 \end{bmatrix} \end{array}$$

Marking key/mathematical behaviours	Marks
• defines the correct dimensions to suit answer in part (a)	1

Question 4(c) (2 marks)

Solution

Answer 1

$$\begin{bmatrix} 20 & 10 & 4 & 5 \\ 10 & 8 & 4 & 3 \\ 5 & 3 & 4 & 6 \end{bmatrix} \times \begin{bmatrix} 1.20 \\ 1.00 \\ 1.50 \\ 2.00 \end{bmatrix} = \begin{array}{c} A \\ B \\ C \end{array}\begin{bmatrix} \$50 \\ \$38 \\ \$27 \end{bmatrix}$$

Answer 2

$$\begin{bmatrix} 1.2 & 1.0 & 1.5 & 2.0 \end{bmatrix} \times \begin{bmatrix} 20 & 10 & 5 \\ 10 & 8 & 3 \\ 4 & 4 & 4 \\ 5 & 3 & 6 \end{bmatrix} = \begin{bmatrix} A & B & C \\ \$50 & \$38 & \$27 \end{bmatrix}$$

Marking key/mathematical behaviours	Marks
• shows the appropriate matrix multiplication	1
• gives the correct solution to the matrix	1

Question 5 **(7 marks)**

(a) The work done, in joules, by a force of $\mathbf{F}$ Newtons in changing the displacement of an object by $\mathbf{s}$ metres, is given by the scalar product of $\mathbf{F}$ and $\mathbf{s}$. Determine the work done by

(i) force $\mathbf{F} = (5\mathbf{i} + 10\mathbf{j})$ N that moves a small body from $(16\mathbf{i} - 2\mathbf{j})$ m to $(22\mathbf{i} + 8\mathbf{j})$ m.

(2 marks)

Solution
$\begin{pmatrix}22\\8\end{pmatrix} - \begin{pmatrix}16\\-2\end{pmatrix} = \begin{pmatrix}6\\10\end{pmatrix}$ $w = \begin{pmatrix}5\\10\end{pmatrix} \cdot \begin{pmatrix}6\\10\end{pmatrix} = 30 + 100 = 130\text{ J}$
Specific behaviours
✓ displacement vector   ✓ correct work done

(ii) a horizontal force of 45 N that pushes a small body 0.4 m up a slope inclined at 45° to the horizontal. (2 marks)

Solution
$w = 45 \times 0.4 \times \cos 45$ $= 45 \times 0.4 \times \dfrac{\sqrt{2}}{2}$ $= 9\sqrt{2}\text{ J}$
Specific behaviours
✓ uses correct expression   ✓ correct work done

(b) Determine the vector projection of $(-\mathbf{i} - 4.5\mathbf{j})$ on $(3\mathbf{i} - 4\mathbf{j})$. (3 marks)

Solution
$\begin{pmatrix}-1\\-4.5\end{pmatrix} \cdot \begin{pmatrix}3\\-4\end{pmatrix} = 15$ $\begin{pmatrix}3\\-4\end{pmatrix} \cdot \begin{pmatrix}3\\-4\end{pmatrix} = 25$ $\dfrac{15}{25}\begin{pmatrix}3\\-4\end{pmatrix} = \begin{pmatrix}9/5\\-12/5\end{pmatrix}$
Specific behaviours
✓ scalar products   ✓ substitutes into expression   ✓ correct vector projection

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TEST 4

Name__________________________

Calculator Assumed

Reading time: 2 mins
Time allowed: 29 mins

Total marks: 29

6. **[3, 2 = 5 marks]**

(a) A, B, C and X are all 2 x 2 matrices. Write an expression for X in terms of A, B, C and A^{-1} given:

$$A(X + B) = C$$

$AX + AB = C$ ✓

$AX = C - AB$ ✓

$X = A^{-1}(C - AB)$ ✓ or $A^{-1}C - B$.

(b) Solve for X, given $A(X + B) = C$ and $A = \begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$

$$X = \frac{1}{5}\begin{bmatrix} -1 & -3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -12 \\ 7 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{22}{5} & -\frac{18}{5} \\ \frac{9}{5} & -\frac{14}{5} \end{bmatrix}$$ ✓✓

7. [3, 3, 1 = 7 marks]

(a) For what value of x is $\begin{bmatrix} 3 & x \\ -5 & x+2 \end{bmatrix}$ singular?

$3(x+2) + 5x = 0$ ✓

$3x + 6 + 5x = 0$ ✓

$8x = -6$

$x = -\frac{3}{4}$ ✓

(b) Find the inverse matrix that would enable you to solve the system of equations

$$\begin{aligned} 5x + ky &= 3 \\ 7x - 8y &= 37 \end{aligned}$$

$\begin{bmatrix} 5 & k \\ 7 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 37 \end{bmatrix}$

✓ $\frac{1}{-40-7k}\begin{bmatrix} -8 & -k \\ -7 & 5 \end{bmatrix}$ ✓✓

(c) If k = 5, find the solution to the system of equations.

$\begin{pmatrix} 2.7866... \\ -2.1866... \end{pmatrix}$

$x = 2.79$ ✓

$y = -2.19$ ✓

−1 if not rounded correctly

Question 8 **(8 marks)**

(a) Show that the vectors $(12, -4)$ and $(3, 9)$ are perpendicular. (2 marks)

Solution
$\begin{pmatrix} 12 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 9 \end{pmatrix} = 36 - 36 = 0$ Hence perpendicular as scalar (dot) product is 0.
Specific behaviours
✓ uses dot product   ✓ explains result

(b) Determine, to the nearest degree, the angle between the vectors $(-1, 3)$ and $(-2, 2)$.

(2 marks)

Solution
Using CAS: $\theta = 26.56 \approx 27°$ Or: $\theta = \cos^{-1}\left(\frac{8}{\sqrt{10}\times\sqrt{8}}\right)$
Specific behaviours
✓ indicates method   ✓ correct angle

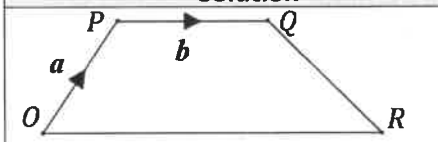
(c) The vectors $(a, a-2)$ and $(a-6, 4)$ are perpendicular, where a is a constant. Determine the value(s) of a and the corresponding pair(s) of vectors. (4 marks)

Solution
$\begin{pmatrix} a \\ a-2 \end{pmatrix} \cdot \begin{pmatrix} a-6 \\ 4 \end{pmatrix} = a^2 - 6a + 4a - 8 = 0$ $(a+2)(a-4) = 0 \Rightarrow a = -2, a = 4$ $a = -2 \Rightarrow \begin{pmatrix} -2 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} -8 \\ 4 \end{pmatrix}$ $a = 4 \Rightarrow \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$
Specific behaviours
✓ uses dot product to form equation   ✓ solves equation   ✓ states one pair of vectors   ✓ states both pairs of vectors

Question 9 **(9 marks)**

Trapezium $OPQR$ has parallel sides PQ and OR such that $|\overrightarrow{OR}| = k|\overrightarrow{PQ}|$. Let $\overrightarrow{OP} = \mathbf{a}$ and $\overrightarrow{PQ} = \mathbf{b}$.

(a) Sketch the trapezium. (1 mark)

Solution

Specific behaviours
✓ draws labelled trapezium

(b) Determine vectors for $\overrightarrow{OQ}$ and $\overrightarrow{PR}$ in terms of k, $\mathbf{a}$ and $\mathbf{b}$. (2 marks)

Solution
$\overrightarrow{OQ} = \mathbf{a} + \mathbf{b}$ and $\overrightarrow{PR} = k\mathbf{b} - \mathbf{a}$.
Specific behaviours
✓ states first vector

(c) Show that the scalar product of $\overrightarrow{OQ}$ and $\overrightarrow{PR}$ is $k|\mathbf{b}|^2 - |\mathbf{a}|^2 + (k-1)\mathbf{a}\cdot\mathbf{b}$. (2 marks)

Solution
$(\mathbf{a}+\mathbf{b})\cdot(k\mathbf{b}-\mathbf{a}) = k\mathbf{a}\cdot\mathbf{b} - \mathbf{a}.\mathbf{a} + k\mathbf{b}.\mathbf{b} - \mathbf{a}.\mathbf{b}$ $= k|\mathbf{b}|^2 - |\mathbf{a}|^2 + (k-1)\mathbf{a}\cdot\mathbf{b}$
Specific behaviours
✓ expands scalar product
✓ simplifies scalar product

(d) Simplify your result from (c) if $k = 1$, $\mathbf{a} = \mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - 2\sqrt{2}\mathbf{j}$. (2 marks)

Solution
$$\begin{aligned} k\lvert\mathbf{b}\rvert^2 - \lvert\mathbf{a}\rvert^2 + (k-1)\mathbf{a}\cdot\mathbf{b} &= \lvert\mathbf{b}\rvert^2 - \lvert\mathbf{a}\rvert^2 + (1-1)\mathbf{a}\cdot\mathbf{b} \\ &= (9+8) - (1+16) + 0 \\ &= 0 \end{aligned}$$
Specific behaviours
✓ substitutes $k = 1$ to eliminate $\mathbf{a}\cdot\mathbf{b}$ ✓ determines magnitudes and simplifies expression to zero

(e) Explain the geometric significance of your result from (d). (2 marks)

Solution
The values of k, **a** and **b** have turned the trapezium into a rhombus and as the scalar product is zero, the diagonals must intersect at right angles.
Specific behaviours
✓ identifies significance of values ✓ uses scalar product to conclude that diagonals intersect at right angles