West Coast Collaborative
Specialist Mathematics Investigation 4
Simple Harmonic Motion
In Class Validation

No Calculators or notes allowed in the validation. All electronic devices must be switched off and in bags.

Time Allowed: 55 minutes **Total Marks:** 49

Name: Marking Key

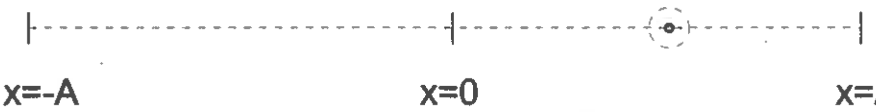
You should be familiar with the relationship between displacement, velocity and acceleration:

Displacement ⟷ Velocity ⟷ Acceleration

$x \longleftrightarrow \frac{dx}{dt} = v \longleftrightarrow \frac{d^2x}{dt^2} = \frac{dv}{dt} = a$

A special type of motion, called simple harmonic motion, occurs when an object oscillates on a straight line and at any time its acceleration is related to its position x by the differential equation:

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -n^2x$, for some constant n and $-A \le x \le A$, where x is the distance of the body from the origin (x=0) and at points $x = \pm A$ the body is stationary (that is v=0).



Part 1: Acceleration

1. [1, 2, 2 = 5 marks]

Using the fact that $a = -n^2x$, determine the magnitude and direction of the acceleration for the object at:

i) $x = 0$ $a = 0$ ✓ ✓

ii) $x = A$ $a = -n^2A$ ✓, acceleration acting to left ✓

iii) $x = -A$ $a = n^2A$ ✓, acceleration acting to the right ✓

2. [1, 1, 1 = 3 marks]

Use the information from question 1 to complete the following:

i) The acceleration always points towards the centre of the motion ✓

ii) The magnitude of the acceleration is at a maximum at the extremes of the motion ✓

iii) The magnitude of the acceleration is at a minimum at the centre of the motion ✓

Part 2: Velocity

Using the differential equation $a = -n^2 x$, we can solve this to obtain an expression for the velocity of an object in simple harmonic motion.

$$a = -n^2 x$$
$$\frac{dv}{dt} = -n^2 x$$
$$\frac{dv}{dx} \times \frac{dx}{dt} = -n^2 x$$
$$\frac{dv}{dx} \times v = -n^2 x$$
$$v\ dv = -n^2 x\ dx$$
$$\int v\ dv = \int -n^2 x\ dx$$

3. **[6 marks]**

Complete the solution to the differential equation above to show that $v = \pm n\sqrt{A^2 - x^2}$. (Remember that at points $x = \pm A$ the body is stationary, that is v=0)

$$\Rightarrow \frac{v^2}{2} = -\frac{1}{2}n^2x^2 + c \quad ✓$$

When $x = a$, $v = 0$

$$\Rightarrow 0 = -\frac{1}{2}n^2a^2 + c$$
$$\Rightarrow c = \tfrac{1}{2}n^2a^2 \quad ✓$$
$$\Rightarrow \frac{v^2}{2} = -\frac{1}{2}n^2x^2 + \frac{1}{2}n^2a^2 \quad ✓$$
$$\Rightarrow v^2 = -n^2x^2 + n^2a^2 \quad ✓$$
$$= n^2(a^2 - x^2) \quad ✓$$
$$\Rightarrow v = \pm n\sqrt{a^2 - x^2} \quad ✓$$

Part 3: Displacement

Using the differential equation $v = \pm n\sqrt{A^2 - x^2}$, we can solve this to obtain an expression for the displacement of an object in simple harmonic motion.

$$v = n\sqrt{A^2 - x^2}$$
$$\frac{dx}{dt} = n\sqrt{A^2 - x^2}$$
$$\frac{1}{\sqrt{A^2 - x^2}}dx = n\ dt$$
$$\int \frac{1}{\sqrt{A^2 - x^2}}dx = \int n\ dt$$

5

4. **[~~9~~ marks]**
Use the substitution $x = A\sin\theta$ to solve the differential equation above to show the solution to the integral is $x = A\sin(nt + \varepsilon)$, where ε = constant of integration.

$$x = A\sin\theta$$
$$\frac{dx}{d\theta} = A\cos\theta \checkmark$$

$$\int \frac{A\cos\theta\, d\theta}{\sqrt{A^2 - A^2\sin^2\theta}} = \int n\, dt \checkmark$$

$$\Rightarrow \int \frac{A\cos\theta\, d\theta}{A\sqrt{\cos^2\theta}} = \int n\, dt \checkmark$$

$$\Rightarrow \int d\theta = \int n\, dt \checkmark$$

$$\Rightarrow \theta = nt + \varepsilon \quad \text{where } \varepsilon = \text{constant}$$
$\checkmark$

1

5. **[~~2~~ marks]**

The displacement $x = A\sin(nt + \varepsilon)$, can be differentiated to give another expression for the velocity.

Determine $v = \frac{dx}{dt}$

$$\frac{dx}{dt} = An\cos(nt + \varepsilon) \checkmark$$

In conclusion for a body in simple harmonic motion we know:

i) $\frac{d^2x}{dt^2} = a = -n^2 x$,

for some constant n and $-A \le x \le A$, where x is the distance of the body from the origin (x=0) and at points $x = \pm A$ the body is stationary (that is v=0).

ii) $\frac{dx}{dt} = v = \pm n\sqrt{A^2 - x^2}$

$\frac{dx}{dt} = v = An\cos(nt + \varepsilon)$

iii) $x = A\sin(nt + \varepsilon)$

Use these to answer the following questions:

6. **[3, 2, 2 = 7 marks]**

A body is moving with Simple harmonic motion on a straight line and the acceleration is given by the equation: $\frac{d^2x}{dt^2} = -9x$, where x is the distance of the body from the origin x=0 and also the body is stationary at points $x = \pm 5$.

(a) Hence give an equation for the velocity of the body in terms of *x*, its position on the number line.

$v^2 = n^2(A^2 - x^2)$

$\Rightarrow v^2 = 3^2 \checkmark (5^2 \checkmark - x^2)$

$\Rightarrow v^2 = \pm 3\sqrt{5^2 - x^2} \checkmark$

(b) Given that: when t=0 the body is at x= +2.5 and the velocity is <u>positive</u> give the equation in the form $x = A\sin(nt + \varepsilon)$ which shows the position x as a function time.

At $t = 0$ $\quad 2.5 = 5\sin\varepsilon$.

$\Rightarrow \frac{1}{2} = \sin\varepsilon$.

$\Rightarrow \varepsilon = \pi/6$ or $5\pi/6$ $\checkmark$

$\frac{dx}{dt} = 5n\cos(3t + \varepsilon)$ $\quad$ $\cos \pi/6$ is +ve, $\cos 5/6$ is −ve, but v is positive

So. $x = 5\sin(3t + \pi/6) \checkmark$

(c) Alternatively: when t=0 the body is at x= +2.5 and the velocity is <u>negative</u> give the equation in the form $x = A\sin(nt + \varepsilon)$ which shows the position x as a function time.

$\frac{dx}{dt} = 5n\cos(3t + \varepsilon)$

when $t = 0$, $x = 2.5$ and velocity is negative $\checkmark$

$\Rightarrow 2.5 = 5\sin\varepsilon$ $\quad \therefore \varepsilon = 5\pi/6$

So $x = 5\sin(3t + 5\pi/6) \checkmark$

7 3

7. [3, ~~8~~, ~~2~~ = 13 marks]

A body is moving with Simple harmonic motion on a straight line and the acceleration is given by the equation: $\dfrac{d^2x}{dt^2} = -16x$ and $-3 \le x \le 3$

(a) Hence give an equation for the velocity of the body in terms of x, its position on the number line.

$v^2 = n^2(A^2 - x^2)$ $\quad n = 4,\ A = 3$ ✓

$\Rightarrow v^2 = 4^2(3^2 - x^2)$ ✓

$\Rightarrow v = \pm 4\sqrt{3^2 - x^2}$ ✓

(b) Use the substitution $x = A\cos(\theta)$ to show the solution to the integral $\dfrac{dx}{dt} = n\sqrt{A^2 - x^2}$ is: $x = A\cos(nt + \varepsilon)$.

$\dfrac{dx}{dt} = 4\sqrt{3^2 - x^2}$ ✓

$\Rightarrow \displaystyle\int \frac{dx}{\sqrt{3^2 - x^2}} = \int 4\,dt$ ✓

Substitute $x = 3\cos\theta \quad \therefore \dfrac{dx}{d\theta} = -3\sin\theta$ ✓

$\Rightarrow \displaystyle\int \frac{-3\sin\theta\,d\theta}{\sqrt{3^2 - 3^2\cos^2\theta}} = \int 4\,dt$ ✓

$\Rightarrow -\displaystyle\int d\theta = \int 4\,dt$ ✓

$\Rightarrow \theta = -(4t + \varepsilon)$ ✓

$\Rightarrow x = 3\cos(4t + \varepsilon)$ ✓

OR $\displaystyle\int \frac{dx}{\sqrt{A^2 - x^2}} = \int n\,dt$ ✓

Substitute $x = A\cos\theta$

$\therefore \dfrac{dx}{d\theta} = -A\sin\theta$ ✓

$\Rightarrow \displaystyle\int \frac{-A\sin\theta\,d\theta}{\sqrt{A^2 - A^2\cos^2\theta}} = \int n\,dt$ ✓

$\Rightarrow \displaystyle\int \frac{-A\sin\theta}{-A\sin\theta}\,d\theta = \int n\,dt$ ✓

$\Rightarrow -\displaystyle\int d\theta = \int n\,dt$ ✓

$\Rightarrow \theta = -nt - \varepsilon$ ✓

So $x = A\cos(nt + \varepsilon)$ ✓

Since $\cos(-\theta) = \cos\theta$

(c) Given that: when t=0 the body is at x= +1.5 and the velocity is <u>positive</u> find the solution to the equation $x = A\cos(nt + \varepsilon)$ which shows the position x as a function time.

$\dfrac{dx}{dt} = -12\sin(4t + \varepsilon)$ ✓

At $t = 0$ $x = 1.5$ $v > 0$.

So $1.5 = 3\cos(\varepsilon)$ and $-12\sin\varepsilon > 0$

$\Rightarrow 0.5 = \cos\varepsilon$ $\quad$ only negative when $\varepsilon = -\pi/3$

$\Rightarrow \varepsilon$ is $\pi/3$ or $-\pi/3$ ✓

$\therefore x = 3\cos(4t - \pi/3)$ ✓

8. **[2, 2, 2, 3 = 9 marks]**

A body is moving with Simple harmonic motion on a straight line and the acceleration is given by the equation: $\frac{d^2x}{dt^2} = -4(x-4)$............(1) where x is the distance of the body from the origin x=0 and also the body is stationary at points $x = -1$ and $x = 9$.

(a) Using a change of origin where $u = (x-4)$ give the equation for the velocity of the body in the form $v = \pm n\sqrt{A^2 - u^2}$.................(2a)

$$V = \pm 2\sqrt{5^2 - u^2}$$

✓ ✓

(b) Also express the velocity in the form $v = \pm n\sqrt{ax^2 + bx + c}$.................(2b)

$$V = \pm 2\sqrt{25 - (x-4)^2}$$ ✓

$$= \pm 2\sqrt{-x^2 + 8x + 9}$$ ✓

(c) From part (a) express the displacement equation in the form $u = A\sin(nt + \alpha)$

$$u = 5\sin(2t + \alpha)$$

✓ ✓

(d) Also given that when t=0 the body is at x= +6.5 and the velocity is positive give the equation which shows the position x as a function time in the form: $x = B + A\sin(nt + \alpha)$

$$x - 4 = 5\sin(2t + \alpha)$$ ✓

$$\Rightarrow \quad x = 4 + 5\sin(2t + \alpha)$$

When $t = 0$ $x = 6.5$ and $v > 0$

So $6.5 = 4 + 5\sin\alpha$ also $10\cos\alpha > 0$

$$\Rightarrow \quad 2.5 = 5\sin\alpha$$

$$\Rightarrow \quad 0.5 = \sin\alpha$$

$\therefore \quad \alpha = \pi/6$ or $5\pi/6$ only $\pi/6$ ✓ since $\cos 5\pi/6 < 0$.

$$\Rightarrow \quad x = 5 + 4\sin(2t + \pi/6)$$ ✓