Mathematics Specialist Units 3 & 4
Test 3 2016
Calculator Free Section

Name: Solutions **Score:** _____ / 21

Section 1 is worth 50% of your final test mark.
No calculators or notes are to be used.
Access to approved Sample Mathematics Specialist formulae sheet is permitted.
Time limit = 20 minutes.

Question 1 [6 marks]

A plane passes through the point P, Q, R with position vectors <1, 2, 1> , <-2, -1, 4> and <2, 1, -2> respectively. Find the vector equation of the plane in the form r.n = k.

$PQ = \langle -3, -3, 3 \rangle$ ✓

$PR = \langle 1, -1, -3 \rangle$ ✓

$PQ \times PR = 12\underline{i} + -6\underline{j} + 6\underline{k}$ ✓✓

$$\begin{array}{cccc} & i & j & k \\ 3 & -3 & -3 & 3 \\ -3 & 1 & -1 & -3 \end{array}$$

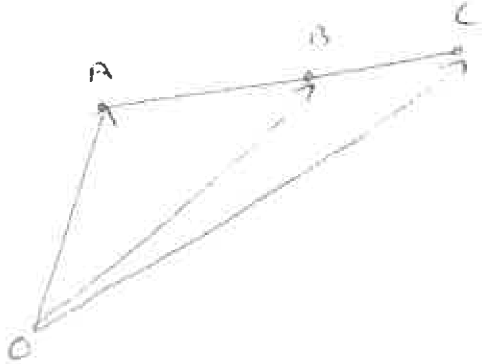
$r . \langle 12, -6, 6 \rangle = \langle 1, 2, 1 \rangle . \langle 12, -6, 6 \rangle$ ✓

$\underline{r} . \langle 12, -6, 6 \rangle = 6.$ ✓

Question 2 [5 marks]

A and B have position vectors of ⟨2, 3, -1⟩ and ⟨-1, 15, 5⟩ respectively. Find point C such that

AB : AC = 3 : 5



$$\vec{OC} = \vec{OA} + \vec{AC}$$ ✓

$$= \vec{OA} + \frac{5}{3}\vec{AB}$$ ✓

$$= \vec{OA} + \frac{5}{3}(\vec{OB} - \vec{OA})$$ ✓

$$= -\frac{2}{3}\vec{OA} + \frac{5}{3}\vec{OB}$$

$$= -\frac{2}{3}\langle 2, 3, -1\rangle + \frac{5}{3}\langle -1, 15, 5\rangle$$ ✓

$$= \langle -\frac{4}{3}, -2, \frac{2}{3}\rangle + \langle -\frac{5}{3}, 25, \frac{25}{3}\rangle$$

$$= \langle -\frac{9}{3}, 23, \frac{27}{3}\rangle$$

$$= \langle -3, 23, 9\rangle$$ ✓

Question 3 [2, 2, 3 marks]

$r(t) = (3\,t\,i - 2t^2\,j)$ meters represents the position of a particle at time t seconds.

(a) Find expressions for the velocity and acceleration.

$\underset{\sim}{v}(t) = 3\underset{\sim}{i} - 4t\,\underset{\sim}{j}$ ✓

$\underset{\sim}{a}(t) = -4\underset{\sim}{j}\ ms^{-2}$ ✓

(b) Find the speed when t = 1 seconds.

$\underset{\sim}{v}(1) = 3\underset{\sim}{i} - 4\underset{\sim}{j}$ ✓

$|v(1)| = 5\ ms^{-1}$ ✓

(c) Write the Cartesian equation of the displacement.

$x = 3t$ ✓ $\frac{x}{3} = t$

$y = -2t^2$

$y = -2\left(\frac{x}{3}\right)^2$ ✓

$y = -\frac{2}{9}x^2$ ✓

Question 4 [3 marks]

Find the shortest distance from a point (2, 3, -1) to a plane 5x + 2y + 3z = 4

$(2, 3, -1)$ to $\underset{\sim}{r} \cdot \langle 5, 2, 3 \rangle = 4$

Point on plane $\langle -1, 3, 1 \rangle$ ✓

$|\underset{\sim}{n}| = \sqrt{5^2 + 2^2 + 3^2}$

$= \sqrt{38}$

$\therefore \hat{\underset{\sim}{n}} = \frac{1}{\sqrt{38}} \langle 5, 2, 3 \rangle$

$\left[\langle 2, 3, -1 \rangle - \langle -1, 3, 1 \rangle\right] \cdot \frac{1}{\sqrt{38}} \langle 5, 2, 3 \rangle$ ✓

$\frac{1}{\sqrt{38}} \langle 3, 0, -2 \rangle \cdot \langle 5, 2, 3 \rangle$

$\frac{1}{\sqrt{38}} (9)$

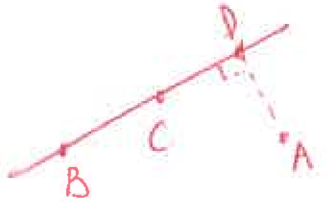
$= \frac{9}{\sqrt{38}}$ /

$= \frac{9\sqrt{38}}{38}$

Mathematics Specialist Units 3 & 4
Test 3 2016
Calculator Section

Name: ____________________ Score: _____ / 29

Section 2 is worth 50% of your final test mark.
Calculators allowed and 1 page of A4 notes, writing on both sides.
Access to approved Sample Mathematics Specialist formulae sheet is permitted.
Time limit = 40 minutes.



Question 5 [6 marks]

Find the closest distance from (1, 4, -2) to line containing (3, 1, -2) and (6, -2, 1).

A B C

$\vec{AB} = \langle 2, -3, 0 \rangle$ ✓

$\vec{BC} = \langle 3, -3, 3 \rangle$ ✓

$\vec{AB} \times \vec{BC} = -9\underline{i} - 6\underline{j} + 3\underline{k}$ ✓

i	j	k
2	-3	0
3	-3	3

Cross Product

$|(a-b) \times \hat{d}|$

$\text{distance} = \dfrac{|-9\underline{i} - 6\underline{j} + 3\underline{k}|}{|3\underline{i} - 3\underline{j} + 3\underline{k}|}$ ✓

$= \dfrac{\sqrt{(-9)^2 + (-6)^2 + 3^2}}{\sqrt{3^2 + (-3)^2 + 3^2}}$

$= \dfrac{\sqrt{126}}{\sqrt{27}} = \dfrac{3\sqrt{14}}{3\sqrt{3}} = \dfrac{\sqrt{14}}{\sqrt{3}} = 2.16\text{m}$ ✓ ✓

Dot Product $\vec{BD} \cdot \vec{AD} = 0$

$\therefore (\lambda \vec{BC} - \vec{BA}) \cdot \vec{BC} = 0$ ✓

$\therefore \lambda \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} = 0$ ✓

$\Rightarrow \begin{pmatrix} 2 + 3\lambda \\ -3 - 3\lambda \\ 3\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} = 0$ ✓

$\lambda = -5/9$ ✓

So $\vec{AD} = \begin{pmatrix} 1/3 \\ -4/3 \\ -5/3 \end{pmatrix}$ ✓

$|\vec{AD}| = \dfrac{\sqrt{42}}{3}$ or $\dfrac{\sqrt{14}}{\sqrt{3}}$ ✓

(2.16024689)

Question 6 [9 marks]

Consider x + y + z =3, x – 2y + z = 6, x – y + kz = m
Find the values(s) of k and m so that the given system has:
(a) a unique set of solutions
(b) more than one solution
(c) no solutions.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -2 & 1 & 6 \\ 1 & -1 & k & m \end{array}\right]$$

$$\begin{array}{c} \\ R_2-R_1 \\ R_3-R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & 0 & 3 \\ 0 & -2 & k-1 & m-3 \end{array}\right] \checkmark$$

$$\begin{array}{c} \\ \\ 2R_2-3R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & 0 & 3 \\ 0 & 0 & 3-3k & 15-3m \end{array}\right] \checkmark$$

a) $3-3k \neq 0$
$3 \neq 3k$
$1 \neq k$ ✓

no condition on m

$y = -1$
$z = \dfrac{m-5}{k-1}$
$x = 4 - \dfrac{m-5}{k-1}$ ✓

b) $3-3k=0$
$k=1$ ✓
$15-3m=0$
$m=5$ ✓

c) $3-3k=0$
$k=1$ ✓
$15-3m \neq 0$
$m \neq 5$ ✓

Question 7 [3, 4 marks]

The equation of a sphere is given by $2x^2 + 2y^2 + 2z^2 - 4y + 8x + 12z = 0$.

(a) Determine the vector equation of the sphere.

$x^2 + y^2 + z^2 - 2y + 4x + 6z = 0$ ✓

$(x+2)^2 - 4 + (y-1)^2 - 1 + (z+3)^2 - 9 = 0$

$(x+2)^2 + (y-1)^2 + (z+3)^2 = 14$ ✓

$|\underset{\sim}{r} - \langle -2, 1, -3 \rangle| = \sqrt{14}$ ✓

(b) Determine the position vector(s) of the points of intersection between the sphere and the line

$r = -3\mathbf{i} + 5\mathbf{j} + \mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$.

$|\langle -3-2\lambda, 5+\lambda, 1-2\lambda \rangle - \langle -2, 1, -3 \rangle| = \sqrt{14}$

$|-1-2\lambda, 4+\lambda, 4-2\lambda| = \sqrt{14}$ ✓

$(-1-2\lambda)^2 + (4+\lambda)^2 + (4-2\lambda)^2 = 14$ ✓

$1 + 4\lambda + 4\lambda^2 + 16 + 8\lambda + \lambda^2 + 16 - 16\lambda + 4\lambda^2 = 14$

$9\lambda^2 - 4\lambda + 19 = 0$

No Solution ✓

$\therefore$ Don't intersect ✓

Question 8[17 marks]

A jet travelling at -200, 150, 0.5 km/hr passes through point 50, -20, 3.8 km at 2pm one day. At the same time a light plane is at -238, 460, 4.52 km travelling at -80, -50, 0.2 km/hr.

(a) If the aircraft continue as above, prove that the planes will collide and find the time and place of the collision. (7 marks)

Let J=Jet P=Plane at 2pm let t=0

and initial displacement JP = OP − OJ

$JP = OP - OJ$

$= \langle -238, 460, 4.52\rangle - \langle 50, -20, 3.8\rangle = \langle -288, 480, 0.72\rangle$ ✓

also $V = V_j - V_p$

$= \langle -200, 150, 0.5\rangle - \langle -80, -50, 0.2\rangle = \langle -120, 200, 0.3\rangle$ ✓

For collision $JP = t_j V_p$

$\langle -288, 480, 0.72\rangle = t\langle -120, 200, 0.3\rangle$ ✓

$-288 = t(-120)$ $480 = 200t$ $0.72 = t(0.3)$

$t = \frac{-288}{-120}$ $t = \frac{480}{200}$ $t = \frac{0.72}{0.3}$

$t = 2.4$ $t = 2.4$ $t = 2.4$ ✓✓

So $JP = 2.4\,v$ Which means that the aircraft do collide at 4:24 pm ✓

$r_j = \langle 50, -20, 3.8\rangle + 2.4\langle -200, 150, 0.5\rangle$

$= \langle -430, 340, 5\rangle$

So the aircraft would collide at $\langle -430, 340, 5\rangle$ km ✓