Name: Marking Key   Score: _____ / 27

No calculators or notes are to be used.
Access to approved Mathematics Specialist formulae sheet is permitted.
Simplify answers where possible.   Time limit = 30 minutes.

1. [5 marks]

Points A (-1,2,3), B (x,y,-1) and C (14,-4,-9) are collinear. Determine the values of x and y, and state the vector equation of this line.

If A, B + C are collinear $\overrightarrow{AB} = k\overrightarrow{AC}$

$$\overrightarrow{AB} = \begin{pmatrix} x \\ y \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} x+1 \\ y-2 \\ -4 \end{pmatrix} \checkmark$$

$$\overrightarrow{AC} = \begin{pmatrix} 14 \\ -4 \\ -9 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 15 \\ -6 \\ -12 \end{pmatrix} \checkmark$$

For K components $\overrightarrow{AC} = 3\overrightarrow{AB}$

so $15 = 3(x+1)$ and $-6 = 3(y-2)$

$\Rightarrow 5 = x+1$ $\Rightarrow -2 = y-2$

$\Rightarrow x = 4$ ✓ $\Rightarrow y = 0$ ✓

Vector equation $= \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix}$, or equivalent ✓

2. [5 marks]

Find the vector equation of the plane in the form **r.n** = k passing through the lines with equation:

$\mathbf{r} = \langle 1,1,1\rangle + \lambda\langle 2,7,1\rangle$ and $\mathbf{r} = \langle 4-\lambda,\ 2+7\lambda,\ 7-3\lambda\rangle$

$$\underset{\sim}{n} = \begin{pmatrix}2\\7\\1\end{pmatrix} \times \begin{pmatrix}-1\\7\\-3\end{pmatrix} = \begin{pmatrix} \det\begin{bmatrix}7&7\\1&-3\end{bmatrix} \\ -\det\begin{bmatrix}2&-1\\1&-3\end{bmatrix} \\ \det\begin{bmatrix}2&-1\\7&7\end{bmatrix}\end{pmatrix} = \begin{pmatrix}-28\\5\\21\end{pmatrix} \checkmark$$

✓ ✓

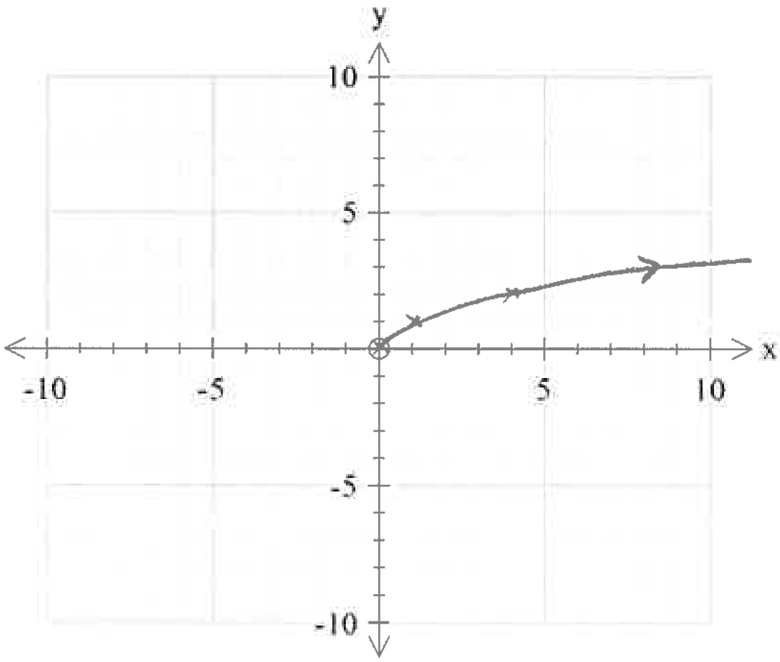
$$\begin{pmatrix}1\\1\\1\end{pmatrix} \text{ lies on plane so } \underset{\sim}{r}\cdot\begin{pmatrix}-28\\5\\21\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}-28\\5\\21\end{pmatrix} \checkmark$$

$$\underset{\sim}{r}\cdot\begin{pmatrix}-28\\5\\21\end{pmatrix} = -2 \checkmark$$

3. [3 marks]

State the Cartesian equation traced by the point P with position vector $\mathbf{r} = t^2\,\boldsymbol{i} + t\,\boldsymbol{j}$, where t represents time. Sketch the path on the axes below, indicating the direction of motion.

$x = t^2$
$y = t$ so $y = \sqrt{x}$ or $y^2 = x$ ✓



$t=0 \Rightarrow x=0,\ y=0$

$t=1 \Rightarrow x=1,\ y=1$

$t=2 \Rightarrow x=4,\ y=2$

graph including key points ✓

direction ✓

4. [9 marks: 5, 1, 1, 2]

Determine the possible values of *m* so that the following system of equations has
a) a unique solution,
b) no solution,
c) or more than one solution.

$2x + y - z = 3$
$mx - 2y + z = 1$
$x + 2y + mz = -1$

$$\begin{array}{cl} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ m & -2 & 1 & 1 \\ 1 & 2 & m & -1 \end{array}\right] & \begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array} \end{array}$$

$$= \begin{array}{cl} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & m+4 & -m-2 & 3m-2 \\ 0 & -3 & -1-2m & 5 \end{array}\right] & \begin{array}{l} \\ mr_1 - 2r_2 \ \checkmark \\ r_1 - 2r_3 \ \checkmark \end{array} \end{array}$$

$$\checkmark = \begin{array}{cl} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & m+4 & -m-2 & 3m-2 \\ 0 & 0 & -2(m+1)(m+5) & 14(m+1) \end{array}\right] & \begin{array}{l} \\ \\ \checkmark\ 3r_2 + (m+4)r_3 \end{array} \end{array}$$

$$\begin{aligned} &(-3m-6+(m+4)(-1-2m) \\ &= -3m-6+m-2m^2-4-8m \\ &= -2m^2-12m-10 \\ &= -2(m^2+6m+5) \\ &= -2(m+1)(m+5)\ \checkmark\ ) \end{aligned}$$

a) Unique solution when $m \neq -1$ or -5 ✓

b) No solution when $m = -5$ ✓

c) More than one solution when $m = -1$ ✓

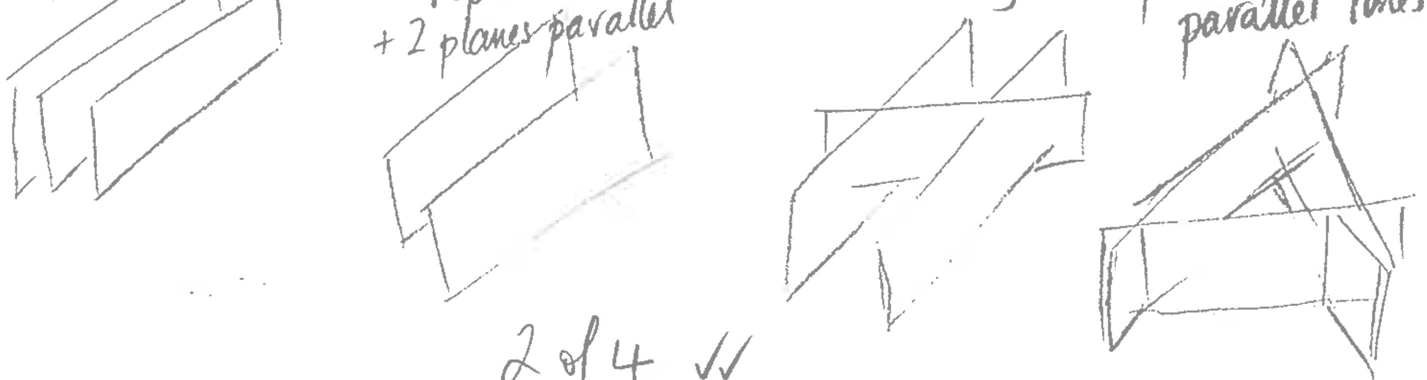
d) Using geometric planes, sketch and/or describe two different situations where a system of linear equations will have **no** solutions.

3 parallel planes

~~one plane repeated~~ + 2 planes parallel

2 parallel and one crossing

Intersecting planes form parallel lines



2 of 4 ✓✓

5. **[3 marks]**

State the vector equation of the sphere with the Cartesian equation
$x^2 + y^2 + z^2 - 2x + 4y = 44$

$$x^2 - 2x + y^2 + 4y + z^2 = 44$$

$$\Rightarrow (x-1)^2 + (y+2)^2 + z^2 = 44 + 1 + 4 \checkmark$$

$$\Rightarrow (x-1)^2 + (y+2)^2 + z^2 = 49 \quad \checkmark$$

Vector equation is $\left| \underset{\sim}{r} - \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \right| = 7 \checkmark$

6. **[3 marks]**

Prove the property:

"If a and b are non-zero vectors then a x b = 0 means a is parallel to b."

$$|\underset{\sim}{a} \times \underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta \checkmark$$

if $\underset{\sim}{a} \times \underset{\sim}{b} = 0 \Rightarrow \sin\theta = 0$, since $|\underset{\sim}{a}|, |\underset{\sim}{b}| \neq 0 \checkmark$

$$\Rightarrow \theta = 0^\circ, 180^\circ, 360^\circ, \ldots \checkmark$$

$\therefore$ parallel lines

END OF PAPER

Name: Marking Key **Score:** _____ / 30

Section 2 is worth 58% of your final test mark.
Calculators allowed and 1 page of A4 notes, writing on both sides.
Access to approved Mathematics Specialist formulae sheet is permitted.
Time limit = 30 minutes

7. [7 marks: 4, 3]

a) Determine the position vector of the point where the line $\mathbf{r} = <3,0,6> + \lambda <-6,3,9>$ meets the plane $\mathbf{r}.<6,2,4> = -6$.

$\underset{\sim}{r} = \begin{pmatrix} 3-6\lambda \\ 3\lambda \\ 6+9\lambda \end{pmatrix}$ ✓ so $\begin{pmatrix} 3-6\lambda \\ 3\lambda \\ 6+9\lambda \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix} = -6$ ✓

Solve gives $\lambda = -8$ ✓

so point of intersection $= \begin{pmatrix} 51 \\ -24 \\ -66 \end{pmatrix}$ ✓

b) Determine the angle between the line and the plane in degrees to 1 decimal place.

$$\cos\theta = \frac{\begin{pmatrix} -6 \\ 3 \\ 9 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}}{\left|\begin{pmatrix} -6 \\ 3 \\ 9 \end{pmatrix}\right| \times \left|\begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}\right|}$$ or angle $\begin{bmatrix} -6 \\ 3 \\ 9 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}$ ✓

$\Rightarrow \theta = 85.9°$ ✓

So α, angle between line + plane, is $4.1°$ ✓

8. **[6 marks]**

Two rockets are moving at a constant velocity in straight lines. The first starts at (2,2, 1) and has velocity (3,-1,-7) and the second has velocity (2,1,-5), starting from (2, -3,2). Show whether their paths cross and determine if they collide ?
Give the coordinates of the point of crossing or collision.

$$\underset{\sim}{r}_A = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + t_1 \begin{pmatrix} 3 \\ -1 \\ -7 \end{pmatrix} \qquad \underset{\sim}{r}_B = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$$

Equating $\underline{i}, \underline{j} + \underline{k}$ components.

$$2 + 3t_1 = 2 + 2t_2 \quad \text{— } 1$$

$$2 - t_1 = t_2 - 3 \quad \text{— } 2 \quad \checkmark\checkmark$$

$$1 - 7t_1 = 2 - 5t_2 \quad \text{— } 3$$

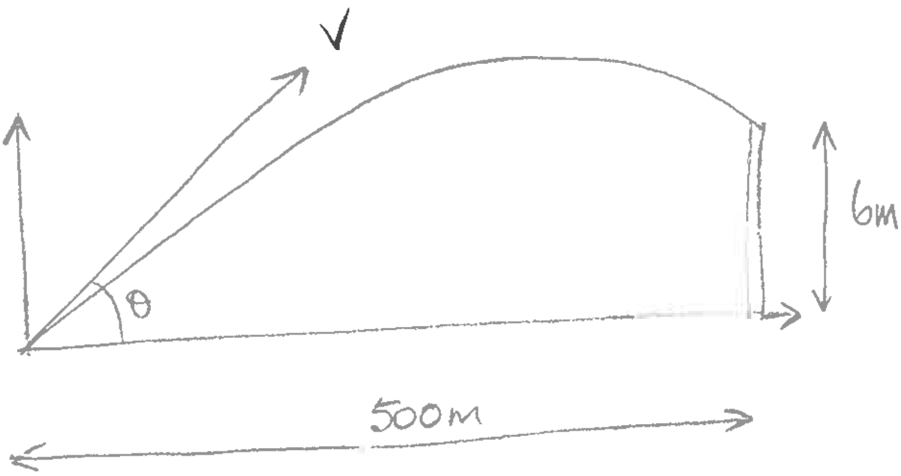
giving $t_1 = 2$ ✓ , $t_2 = 3$ ✓

therefore their paths cross, but they do not collide, since t_1 and t_2 are not the same ✓

Point of crossing $= \begin{pmatrix} 8 \\ 0 \\ -13 \end{pmatrix}$ ✓

9. [7 marks]

In order to bring about a prison break a grappling hook is fired from a crossbow from a position level with the base of a wall 500m away. Two-and-a-half seconds later the grappling hook just clears the wall. If the wall is 6 m high, calculate the initial velocity to the nearest m/s and the projection angle of the ball-bearing to the nearest degree.
Assume gravity to be 9.8 m/s^2. It may help to draw a diagram first.



$x(t) = 2.5\, V \cos\theta$

So $500 = 2.5\, V \cos\theta$ —— (1) ✓

$y(t) = 2.5\, V \sin\theta - \frac{1}{2} \times 9.8 \times 2.5^2$ ✓

$\Rightarrow \quad 6 = 2.5\, V \sin\theta - 30.625$

$\Rightarrow \quad 36.625 = 2.5\, V \sin\theta$ —— (2) ✓

(2) ÷ (1) gives $\tan\theta = \dfrac{36.625}{500}$ ✓

$\Rightarrow \theta = 4.189^\circ$, so 4° ✓

in (1) $\quad V = \dfrac{500}{2.5 \times \cos 4.189^\circ}$ ✓

$= 200.5358384$ m/s

So 201 m/s ✓

10. [9 marks: 2, 2, 2, 3]

The velocity vector of a particle P, at time t seconds after projection from < 0, 0, 0 > is given by $\mathbf{v}(t) = \langle t - 1, 0, t - t^2 \rangle$ where the components are measured in cms^{-1}.

a) Determine when P is instantaneously at rest.

$$v(t) = 0 \quad \Rightarrow \quad \left.\begin{array}{l} t - 1 = 0 \\ \text{and } t - t^2 = 0 \end{array}\right\} \Rightarrow t = 1$$

b) Calculate the distance travelled by P between the time it starts moving and when it is instantaneously at rest.

$$r(t) = \int_0^1 \left| \langle t-1, 0, t-t^2 \rangle \right| dt$$

$$= \int_0^1 \sqrt{(t+1)^2 + (t-t^2)}\, dt$$

$$= 0.538 \text{ m } (3dp)$$

c) What is its position when it comes to rest ?

$$r(t) = \begin{pmatrix} \frac{t^2}{2} - t \\ 0 \\ t^2/2 - t^3/3 \end{pmatrix} + c \quad r(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{so} \quad r(t) = \begin{pmatrix} t^2/2 - t \\ 0 \\ t^2/2 - t^3/3 \end{pmatrix}$$

$$\therefore r(1) = \begin{pmatrix} -1/2 \\ 0 \\ 1/6 \end{pmatrix}$$

d) Determine when, if ever, the velocity is perpendicular to the acceleration.

$$a(t) = \begin{pmatrix} 1 \\ 0 \\ 1 - 2t \end{pmatrix}$$

$$\text{Solve} \quad \begin{pmatrix} 1 \\ 0 \\ 1 - 2t \end{pmatrix} \cdot \begin{pmatrix} t - 1 \\ 0 \\ t - t^2 \end{pmatrix} = 0$$

gives $t = 1$

END OF TEST