Section One: Calculator-free **35% (50 Marks)**

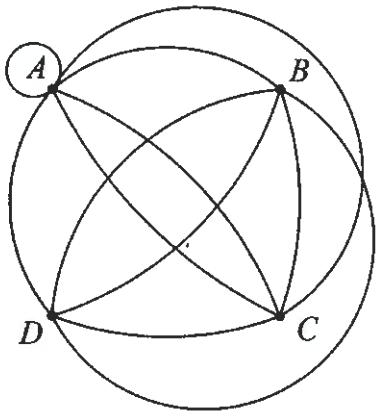
This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

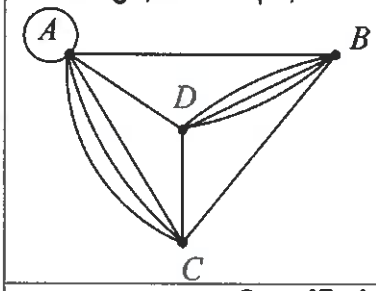
Test 3 2019

Working time for this section is 50 minutes.

Q1 **Question 1** **(7 marks)**

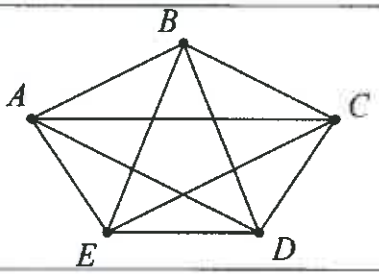
(a) Re-draw the following graph to clearly demonstrate that it is planar. (3 marks)



Solution
Tip: Remove loops and multiple edges, redraw without crossings, add loops, etc back

Specific behaviours
✓ has no edges crossing ✓ AC and BD both have three connecting edges ✓ correct equivalent graph

-1 no loop at A

(b) Draw a complete graph with five vertices. (2 marks)

Solution

Specific behaviours
✓ draws graph with five vertices ✓ connects each vertex to all others

(c) Do **all** complete graphs obey Euler's formula? Justify your answer. (2 marks)

Solution
No - complete graphs with more than 4 vertices are not planar and Euler's rule only applies to connected planar graphs.
Specific behaviours
✓ states no ✓ states valid reason

I was tough on marking this one

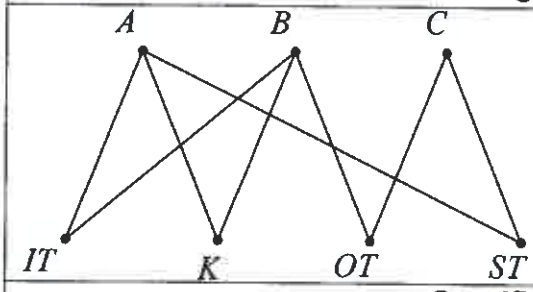
0 or 2.

Q2 **Question 3** **(6 marks)**

(a) Three hikers were sharing tips about walking on four famous routes. Ahn had walked the Inca Trail, Kilimanjaro and the Snowman Trek. Bi had walked the Inca Trail, Kilimanjaro and the Overland Track. Chris had walked the Overland Track and the Snowman Trek.

Display this information as a bipartite graph. (2 marks)

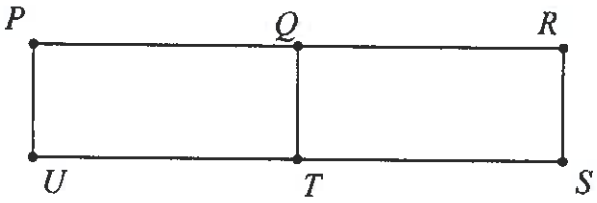
Solution



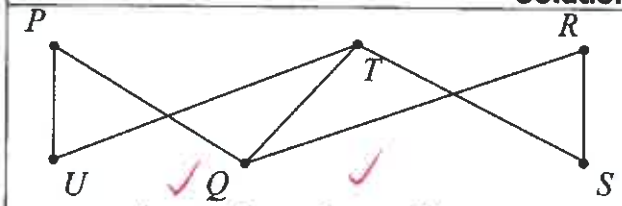
Specific behaviours

✓ displays two sets of vertices
✓ joins vertices correctly

(b) A bipartite graph is shown below, joining vertices in two disjoint sets. Clearly list the vertices belonging to each of the two sets. (2 marks)



Solution



Sets are {P, T, R} and {U, Q, S}

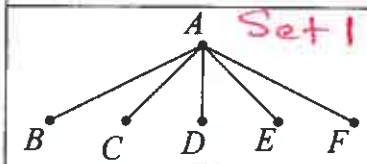
Specific behaviours

✓ redraws graph
✓ lists both sets

(c) A complete bipartite graph has six vertices. Determine the smallest possible number of edges. (2 marks)

See SCSA glossary.

Solution



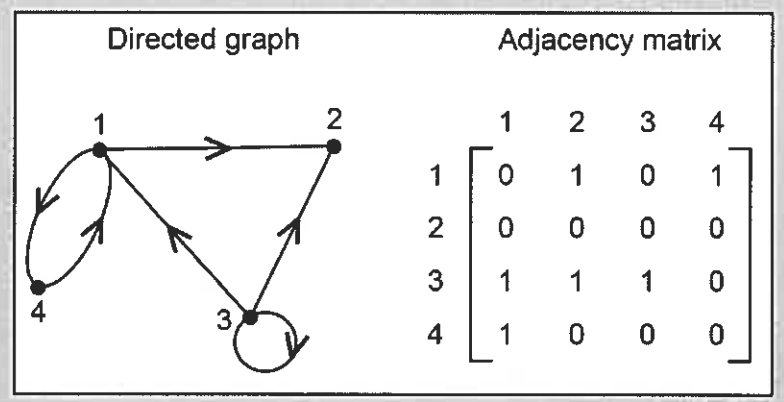
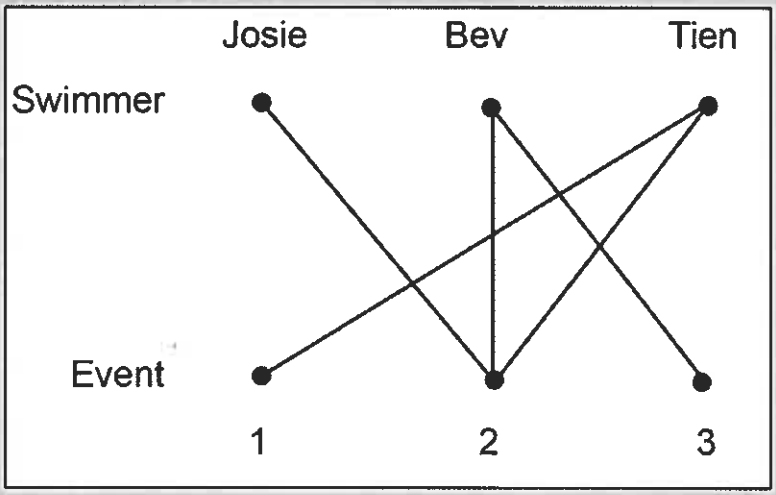
Do not have to been the same size.

Answer: 5 edges

Specific behaviours

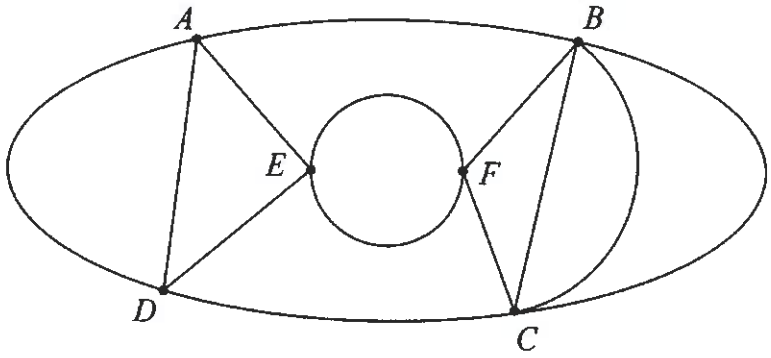
✓ sketches possible graphs
✓ states correct minimum number

See next page

Adjacency matrix continued	For a directed graph the entry in row i and column j is the number of directed edges (arcs) joining the vertex i and j in the direction i to j. Example: 
Arc	See Directed graph.
Bipartite graph	A bipartite graph is a graph whose set of vertices can be split into two distinct groups in such a way that each edge of the graph joins a vertex in the first group to a vertex in the second group. Example: 
Bridge	See Connected graph.
Closed path	See Path
Closed trail	See Trail.
Closed walk	See Walk.
Complete graph	A complete graph is a simple graph in which every vertex is joined to every other vertex by an edge. The complete graph with n vertices is denoted K_n. A complete bipartite graph is a bipartite graph where every vertex of the first set is connected to every vertex of the second set.
Connected graph	A graph is connected if there is a path between each pair of vertices. A bridge is an edge in a connected graph that, if removed, leaves a graph disconnected.

Q3 **Question 5** **(7 marks)**

The edges in the graph below represent the roads in a park that meet at the given vertices.



(a) Show that Euler's formula applies to this graph. (2 marks)

Solution
$v = 6, f = 9, e = 13$ $v + f - e = 6 + 9 - 13$ $= 2 \Rightarrow$ formula applies
Specific behaviours
✓ states correct values for edges, vertices and faces ✓ substitutes into Euler's formula correctly

(b) Record the degree of each vertex shown in the graph in the table below. (2 marks)

Vertex	A	B	C	D	E	F
Degree	**4**	**5**	**5**	**4**	**4**	**4**

Solution
See table
Specific behaviours
✓ at least five correct ✓ all six correct

(c) Without referring to the information in (b), clearly explain why the graph is semi-Eulerian. (2 marks)

Solution
The graph is semi-Eulerian because it contains an open (rather than closed) trail that includes each edge just once.
Specific behaviours
✓ states trail using all edges just once ✓ states trail is open (start finish different vertex) *NB Do not accept explanations based on degree of vertices*

(d) A park ranger has to inspect every road in the park. List all possible starting points so that the ranger can complete this task without driving on the same road more than once. (1 mark)

Solution
B or C
Specific behaviours
✓ states both possible solutions

See next page

Q4

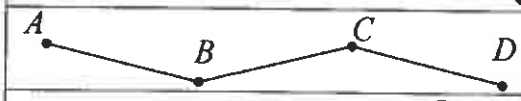
Question 7 **(9 marks)**

(a) A connected planar graph has one face and three edges.

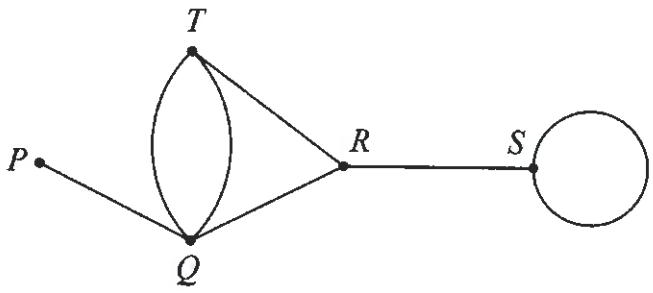
(i) Determine the number of vertices the graph has. (2 marks)

Solution
$v+1-3=2 \Rightarrow v=4$
Specific behaviours
✓ uses Euler's formula ✓ states correct number

(ii) Sketch a graph with these properties. (2 marks)

Solution

Specific behaviours
✓ sketches a tree ✓ sketch has all other required properties

(b) Consider the graph below.



State the length of the longest possible walk in this graph and list, in order from first to last, the vertices visited, so that the walk

(i) is an open trail. (2 marks)

Solution
PQTQRSS - length 6
Specific behaviours
✓ lists vertices with no edges repeated, starting and finishing at different vertices ✓ states correct length

gave no ½ mark

(ii) is a closed path. (3 marks)

Solution
QRTQ (or similar cycle) - length 3
Specific behaviours
✓ lists vertices with no repeats, except for first and last, and no edges repeated ✓ start and finish at same vertex ✓ states correct length

End of questions

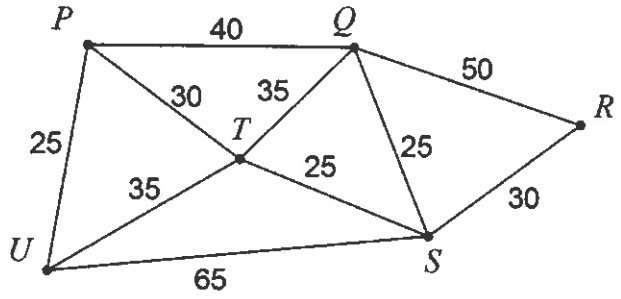
Q5

~~Question 13~~ **(9 marks)**

A business has branches in six cities. The table below shows the time, in minutes, it takes for a package received at one branch to be transported to a branch in another city, where a direct route exists.

Q	40				
R	-	50			
S	-	25	30		
T	30	35	-	25	
U	25	-	-	65	35
	P	Q	R	S	T

(a) Construct a weighted graph to show this information, using the cities placed below. (3 marks)



Solution
See diagram
Specific behaviours
✓ correctly adds at least 7 edges
✓ adds all edges correctly
✓ labels all edges correctly

(b) Determine the shortest transport time for a package to travel from

(i) P to S. (1 mark)

Solution
55 minutes
Specific behaviours
✓ states correct time

(ii) Q to U. (1 mark)

Solution
65 minutes
Specific behaviours
✓ states correct time

(c) A document needs to be sent from branch U via branch R, where a customer will sign the document, to branch P. Determine the minimum transport time for the document to make this journey, listing all branches on the way. (2 marks)

Solution
U - T - S - R - S - T - P = 175 minutes
Specific behaviours
✓ states minimum time ✓ lists branches

(d) Another business document requires signing by the manager of each branch. In planning a route for this document, would finding a Eulerian trail be more appropriate than finding a Hamiltonian trail? Explain your answer. (2 marks)

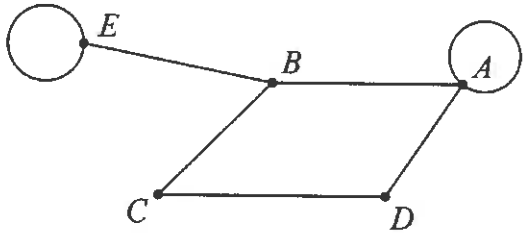
Solution
No. Hamiltonian is needed, as every vertex must be visited just once.
Specific behaviours
✓ Answers no ✓ Explains Hamiltonian trail

See next page

Q6

~~Question 16~~ **(9 marks)**

An airline has flights between six cities as shown in the graph below. Two of the flights are sightseeing flights that return to the city from which they departed.



(a) Determine M, the adjacency matrix for this graph. (3 marks)

Solution
$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$
Specific behaviours
✓ draws a 5x5 matrix ✓ completes at least three correct rows ✓ completes all rows correctly

(b) Calculate M^2 and explain the significance of the elements in this matrix that are zero. (3 marks)

Solution
$M^2 = \begin{bmatrix} 3 & 1 & 2 & 1 & 1 \\ 1 & 3 & 0 & 2 & 1 \\ 2 & 0 & 2 & 0 & 1 \\ 1 & 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 & 2 \end{bmatrix}$ A zero indicates that not possible to travel between these cities by taking exactly two flights.
Specific behaviours
✓ shows M^2 is a 5x5 matrix ✓ calculates matrix correctly ✓ explains zeros

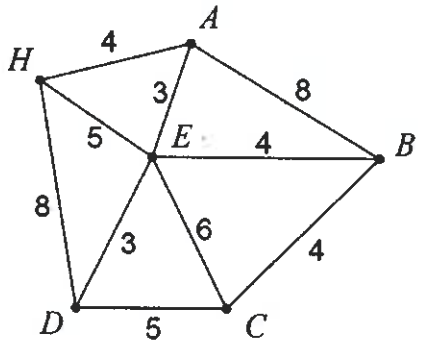
(c) Determine the number of zero elements in the matrix $M + M^2$ and explain their significance in terms of specific flight(s). (3 marks)

Solution
There are two zeros, $M_{4,5}$ and $M_{5,4}$. There is no way to travel between D and E taking either one or two flights.
Specific behaviours
✓ states number of zeros ✓ ~~states cities involved~~ ✓ states significance

See next page

Q7 **Question 19** **(9 marks)**

The vertex H on the graph below represents a hotel and vertices A to E represent tourist attractions. The numbers on the edges of the graph below represent the walking times, in minutes, between the various attractions.



A group of tourists plan to leave the hotel at 10 am and visit all the attractions, spending 15 minutes at each one.

(a) Given that the hotel bus will pick them up from the last attraction they visit,

(i) determine the route they should take that involves the least possible walking time. (2 marks)

Solution
H - A - E - D - C - B 19
Specific behaviours
✓ lists vertices that form a Hamilton path ✓ lists shortest Hamilton path

(ii) determine the time the bus should meet them at their last attraction. (2 marks)

Solution
$4+3+3+5+4+5\times15=19+75=94$ minutes Bus should meet them at 11:34 am.
Specific behaviours
✓ calculates total walking and viewing time ✓ states correct pick up time

(b) One member of the group knows a little about graph theory and suggests that the route that the group plan should be a Hamiltonian cycle.

(i) Explain what is meant by a Hamiltonian cycle. (2 marks)

Solution
A closed walk that starts and ends at the same vertex and visits all vertices once.
Specific behaviours
✓ states walk is closed ✓ states visits all vertices just once

(ii) Determine the Hamiltonian cycle the group of tourists should walk and state the time they will arrive back at their hotel. (3 marks)

Solution
H - A - E - B - C - D - H HEDCBAH - 29 more than one way. $4+3+4+4+5+8+5\times15=28+75=103$ minutes Arrive back at 11:43 am.
Specific behaviours
✓ states correct cycle ✓ calculates total walking and viewing time ✓ states correct return time

End of questions