Wanneroo Secondary College Year 12 Mathematics Specialist Unit 4 Test 5 2019 Calculator-Free Section

You are only allowed the SCSA formula sheet for this section. Any question worth more than 2 marks requires sufficient working to justify your solutions.

Time : 20 Minutes **Marks : 17**

1. [2, 3 = 5 marks]

A particle moves with velocity $v = 2\sqrt{x}\ \ ms^{-1}$, where x is it's displacement. The particle is initially 2 m to the right of the origin.

a) Show that the particle has constant acceleration

$a = v\dfrac{dv}{dx}$

$a = 2\sqrt{x}\dfrac{dv}{dx}$ ✓

$a = 2\sqrt{x}\left(\dfrac{1}{\sqrt{x}}\right)$

$a = 2\ ms^{-2}$ ✓

b) Determine the displacement, x, as a function of time, t.

$\dfrac{dx}{dt} = 2\sqrt{x}$

$\int \dfrac{1}{\sqrt{x}}\,dx = \int 2\,dt$ ✓

$\int x^{-1/2}\,dx = \int 2\,dt$

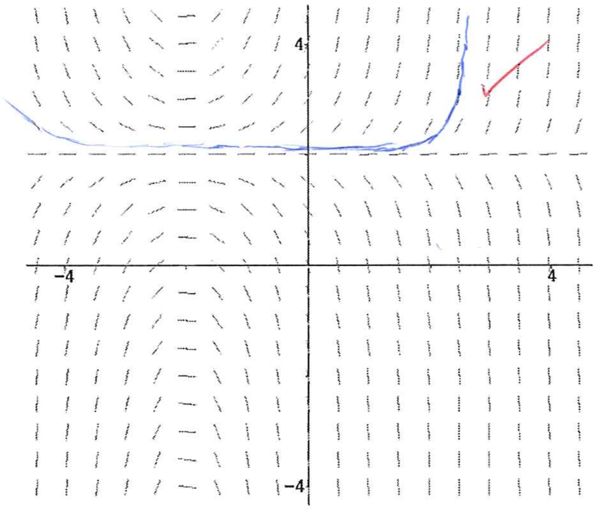
$2x^{1/2} = 2t + c$ ✓ at $t=0$, $x=2$.

$2\sqrt{2} = c$.

$2\sqrt{x} = 2t + 2\sqrt{2}$

$x = (t+\sqrt{2})^2$ ✓

2. **[1, 2, 4 = 7 marks]**



a) Circle the differential equation that best matches the slope field shown:

i) $\frac{dy}{dx} = \frac{2-x}{y-2}$

ii) $\frac{dy}{dx} = (x+2)(y-2)$

iii) $\frac{dy}{dx} = \frac{2+x}{y+2}$

iv) $\frac{dy}{dx} = (x-2)(y+2)$

b) Determine the value of the slope field at the point (2, 3) and draw the solution on the slope field above.

$$\frac{dy}{dx} = (2+2)(3-2)$$
$$= 4.$$

c) Determine the equation of the particular solution curve that passes through the point (2, 3)

$$\int \frac{1}{y-2}\, dy = \int (x+2)\, dx$$

$$\ln|y-2| = \frac{x^2}{2} + 2x + C.$$ at (2, 3).

$$\ln 1 = 6 + C$$
$$0 = 6 + C$$
$$-6 = C$$

$$\ln|y-2| = \frac{x^2}{2} + 2x - 6$$

$$y = e^{\frac{x^2}{2}+2x-6} + 2$$

3. **[5 marks]**

Determine the gradient of the tangent to the curve $\frac{2x^3 - y^2}{xy} = 1$ at the point (1, 1)

$$\frac{xy\left(6x^2 - 2y\frac{dy}{dx}\right) - (2x^3 - y^2)\left(y + x\frac{dy}{dx}\right)}{x^2y^2} = 0$$

$$1\left(6 - 2\frac{dy}{dx}\right) - 1\left(1 + \frac{dy}{dx}\right) = 0$$

$$6 - 2\frac{dy}{dx} - 1 - \frac{dy}{dx} = 0$$

$$-3\frac{dy}{dx} = -5$$

$$\frac{dy}{dx} = \frac{5}{3}$$

Wanneroo Secondary College	
Year 12 Mathematics Specialist	Unit 4
Test 5 2019	
Calculator-Assumed Section	

You are allowed the SCSA formula sheet, approved calculator(s) and one page of A4 notes (back and front) for this section. Any question worth more than 2 marks requires sufficient working to justify your solutions.

Time : 35 Minutes **Marks : 33**

4. **[3, 2 = 5 marks]**

After t seconds, the displacement x centimetres of a small mass attached to a spring, oscillates about a fixed point O according to the differential equation $\frac{d^2x}{dt^2} = -25x \ cms^{-2}$.

The initial velocity is 15 centimetres per second and the initial displacement is zero.

a) Determine the function $x(t)$ that gives the displacement of the mass at time t

$$\frac{d^2x}{dt^2} = -k^2 x$$

$$\therefore k = 5. \checkmark$$

$$x = A\sin(5t)$$

$$\frac{dx}{dt} = 5A\cos(5t)$$

$$15 = 5A\cos(0) \checkmark$$

$$3 = A$$

$$\therefore x = 3\sin(5t). \checkmark$$

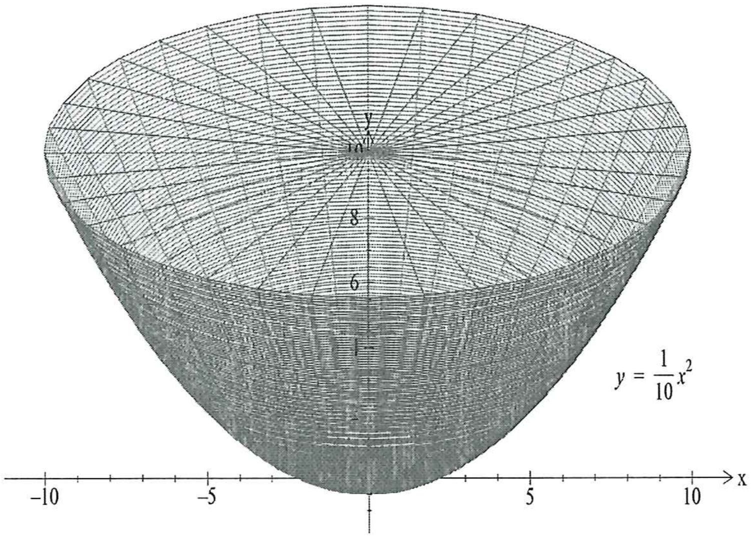
b) Calculate the distance the mass travels during the first 3 seconds.

$$\int_0^3 |15\cos(5t)| \, dt \checkmark$$

$$= 28.05 \text{ cm} \checkmark$$

5. **[2, 3 = 5 marks]**

A dog's drinking bowl is formed by revolving the curve $y = \frac{1}{10}x^2$, $0 \le x \le 10$, through 360° about the y-axis. The units of x and y are in centimetres.



a) Show the expression connecting the height h of water in the bowl and the volume V of water in the bowl is given by $V = 5\pi h^2$

$10y = x^2$

$V = \pi \int_0^h 10y \, dy$

$V = \pi \left[5y^2\right]_0^h$

$V = 5\pi h^2$

b) The bowl is filled with water at a rate of 50mL per second. At what rate is the height of the water increasing at the time when the bowl contains 600 mL of water?

$600 = 5\pi h^2$

$6.18 \text{ cm} = h$

$\frac{dV}{dt} = 50 \text{cm}^3/\text{sec}$

$\frac{d}{dt}(V = 5\pi h^2)$

$\frac{dV}{dt} = 10\pi h \frac{dh}{dt}$

$50 = 10\pi(6.18)\frac{dh}{dt}$

$0.26 \text{ cm/sec} = \frac{dh}{dt}$

6. **[7 marks]**

Determine the values of the constants a and b if $y = e^{4x}(2x+1)$ is a solution of the differential equation $\frac{d^2y}{dx^2} - a\frac{dy}{dx} + by = 0$

$$\frac{dy}{dx} = 4e^{4x}(2x+1) + e^{4x}(2)$$

$$\frac{dy}{dx} = 2e^{4x}(4x+3)$$

$$\frac{d^2y}{dx^2} = 8e^{4x}(4x+3) + 2e^{4x}(4)$$

$$\frac{d^2y}{dx^2} = 8e^{4x}(4x+4) = 32e^{4x}(x+1)$$

$$32e^{4x}(x+1) - a2e^{4x}(4x+3) + be^{4x}(2x+1) = 0$$

Divide by e^{4x}

$$32(x+1) - 2a(4x+3) + b(2x+1) = 0.$$

$$32x + 32 - 8ax - 6a + 2bx + b = 0$$

$$(32 - 8a + 2b)x + (32 - 6a + b) = 0.$$

$$32 - 8a + 2b = 0$$

$$32 - 6a + b = 0$$

$$a = 8$$

$$b = 16.$$

7. [6, 4 = 10 marks]

An ecology student is studying the repopulation of wild emus in Western Australia after a drought. She notices that the population growth rate is approximately logistic so that $\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right)$ where $P(t)$ is the population t years after her study begins.

The carrying capacity A is known to be 2550 emus, since this was the population before the drought.

Initially, the student finds that there are 310 emus. After 2 years she finds there are 390 emus.

a) Given that $\frac{1}{P(2550-P)} = \frac{1}{2550}\left(\frac{1}{P} + \frac{1}{2550-P}\right)$, use the separation of variables technique to determine the equation to find the population $P(t)$ after t years

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{2550}\right)$$

$$\frac{dP}{dt} = \frac{k}{2550} P(2550 - P)$$

$$\int \frac{1}{P(2550-P)}\, dP = \int \frac{k}{2550}\, dt$$

$$\frac{1}{2550}\int \left(\frac{1}{P} + \frac{1}{2550-P}\right) dP = \int \frac{K}{2550}\, dt$$

$$\frac{1}{2550}\Big[\ln|P| - \ln|2550-P|\Big] = \frac{K}{2550}t + c$$

$$\ln\left|\frac{2550-P}{P}\right| = -Kt + d$$

$$\frac{2550}{P} - 1 = e^{d} \cdot e^{-kt}$$

$$\frac{2550}{1 + e^{d} \cdot e^{-Kt}} = P$$

at $t = 0$ $\quad P = 310$

$$\frac{2550}{1 + e^{d}} = 310$$

$$e^{d} = 7.226$$

at $t = 2$ $\quad P = 390$

$$\frac{2550}{1 + 7.226e^{-2K}} = 390$$

$K = 0.133$

$$\therefore P = \frac{2550}{1 + 7.226e^{-0.133t}}$$

b) Find the time at which the population growth rate is a maximum.

$$\frac{dP}{dt} = 0.133P\left(\frac{2550 - P}{2550}\right)$$

Quadratic with roots at $P=0$, $P=2550$

$\therefore$ los $x = 1275$

$$1275 = \frac{2550}{1 + 7.226e^{-0.133t}}$$

$t = 14.9$ years.

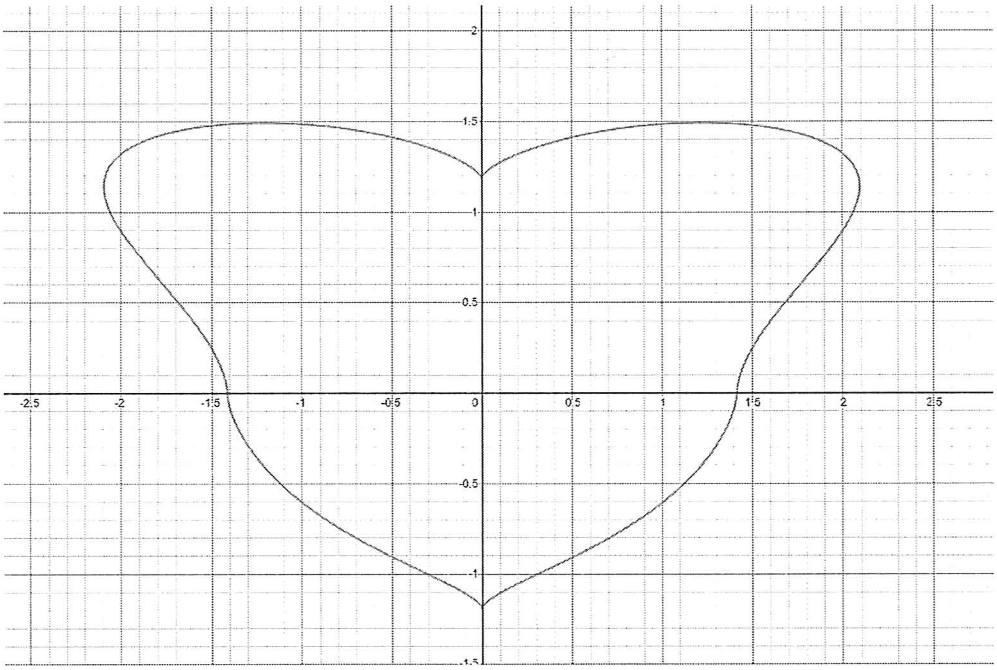
or $$\frac{d^2}{dt^2}\left(\frac{2550}{1 + 7.226e^{-0.133t}}\right)$$

$$0 = -\frac{0.00227\left(1.79 \times 10^{13}\left(2.718^{0.266t}\right) - 1.3 \times 10^{14}\left(2.718^{0.133t}\right)\right)}{\left(500\left(2.718^{0.133t}\right) + 3613\right)^3}$$

$t = 14.9$ years

8. **[3, 3 = 6 marks]**

The graph of $(x^2 + y^4 - 2)^3 = 8x^2y^5$ is shown below:



a) By implicitly differentiating the given equation, obtain an equation relating x, y and $\frac{dy}{dx}$.

Note: Do not attempt to obtain $\frac{dy}{dx}$ as the subject of this equation.

$$3(x^2+y^4-2)^2\left(2x+4y^3\frac{dy}{dx}\right) = 16xy^5 + 40x^2y^4\frac{dy}{dx}$$

b) Determine the coordinates of the point on the graph in the first quadrant $(x > 0 \; and \; y > 0\}$ where the gradient is horizontal.

$\frac{dy}{dx} = 0.$

$3(x^2+y^4-2)^2(2x) = 16xy^5 \quad \text{----} \; 1$

$(x^2+y^4-2)^3 = 8x^2y^5 \quad \text{----} \; 2$

Solve 1 & 2 simultaneously

$(1.22, 1.49)$