Name: Marking Key Score: ____ / 21

Section 1 is worth 35% of your final test mark.
No calculators or notes are to be used.
Access to approved Mathematics Specialist formulae sheet is permitted.
Simplify answers where possible. **Time limit = 20 minutes.**

Question 1 (6 marks)

Determine the gradient of $2xy^2 - 3x^2y = 16$ at x = 2.

$$2y^2 + 2x(2y)\frac{dy}{dx} - 6xy - 3x^2\frac{dy}{dx} = 0 \quad ✓$$

$$\frac{dy}{dx} = \frac{6xy - 2y^2}{4xy - 3x^2} \quad ✓$$

at $x = 2$

$$4y^2 - 12y = 16$$
$$4y^2 - 12y - 16 = 0 \quad ✓$$
$$y^2 - 3y - 4 = 0$$
$$(y-4)(y+1) = 0 \quad ✓$$
$$y = 4, \quad y = -1$$

at $(2, 4)$

$$\frac{dy}{dx} = \frac{48 - 32}{32 - 12}$$
$$= \frac{16}{20} = \frac{4}{5} \quad ✓$$

at $(2, -1)$

$$\frac{dy}{dx} = \frac{-12 - 2}{-8 - 12}$$
$$= \frac{-14}{-20} = \frac{7}{10} \quad ✓$$

Question 2 **(5 marks)**

The production of a chemical in a laboratory can be modelled by the differential equation $\frac{dm}{dt} = e^{2t-m}$, where m kg is the total mass of the chemical produced after t hours.

Given that $m(0) = 0$, determine an exact value for the total mass of substance produced after three hours.

$$\frac{dm}{dt} = \frac{e^{2t}}{e^{m}}$$

$$\int e^{m}\,dm = \int e^{2t}\,dt \checkmark$$

$$e^{m} = \frac{1}{2}e^{2t} + c \checkmark$$

at $t = 0$, $m = 0$

$$1 = \frac{1}{2} + c$$

$$\frac{1}{2} = c \checkmark$$

$$e^{m} = \frac{1}{2}e^{2t} + \frac{1}{2}$$

$$m = \ln\left|\frac{1}{2}e^{2t} + \frac{1}{2}\right| \checkmark \quad \text{at } t = 3$$

$$m = \ln\left|\frac{e^{6}+1}{2}\right| \checkmark$$

Question 3 **(2, 1, 3 marks)**

The tip of a needle in a sewing machine moves vertically with simple harmonic motion, and the distance between the highest and lowest positions of the tip is 8mm.

The height of the tip of the needle above the mid-position t seconds after it starts to move is x(t) which satisfies the differential equation:

$$\frac{d^2x}{dt^2} = -16\pi^2 x$$

a) Determine the equation for x(t), given the tip of the needle starts at its highest point.

$x(t) = 4\cos(4\pi t)$ ✓ ✓

$A = 4\text{mm}$

$k = \sqrt{16\pi^2}$

$k = 4\pi$

b) How long does it take for the tip of the needle to first return to its highest point?

$T = \frac{2\pi}{4\pi} = \frac{1}{2}$ second ✓

c) How far does the tip of the needle travel in the first $\frac{1}{3}$ of a second?

$x(\frac{1}{3}) = 4\cos(\frac{4\pi}{3})$

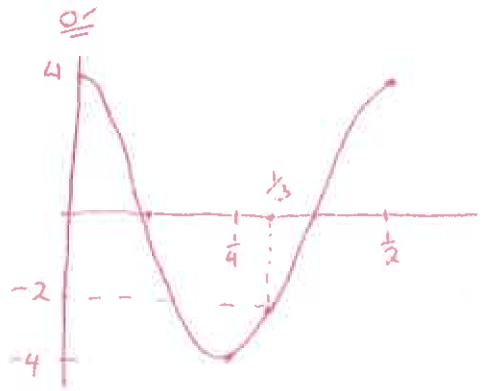
$= 4(-\frac{1}{2})$

$= -2\text{mm}$ ✓

In $\frac{1}{4}$ second needle has travelled from top to bottom $= 8\text{mm}$ ✓

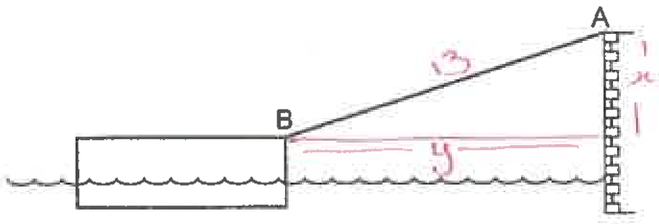
$\therefore$ Distance $= 8 + 2$

$= 10\text{mm}$ ✓



Question 4 **(4 marks)**

A floating pontoon at a tidal marina is connected to the top of the harbour wall by a hinged walkway AB of length 13 metres.



When the top of the pontoon, B, is 5 metres below the top of the wall, A, the sea is rising at a rate of 2 cm per minute.

At this instant, calculate the rate at which the barge is moving away from the wall.

when $x = 5m$

$y = \sqrt{13^2 - 5^2}$

$= 12m$ ✓

$\frac{dx}{dt} = \frac{-1}{50}$ m/ minute

$x^2 + y^2 = 169$ ✓

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ ✓

$2(5)\left(\frac{-1}{50}\right) + 2(12)\frac{dy}{dt} = 0$

$24\frac{dy}{dt} = \frac{1}{5}$

$\frac{dy}{dt} = \frac{1}{120}$ m/minute ✓

or $\frac{5}{6}$ cm/minute

Test 5 2016
Calculator Section

Name: ________________________ Score: _____ / 39

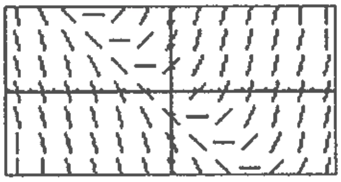
Section 2 is worth 65% of your final test mark.
Calculators allowed and 1 page of A4 notes, writing on both sides.
Access to approved Mathematics Specialist formulae sheet is permitted.
Time limit = 40 minutes

Question 5 **(2, 4 marks)**

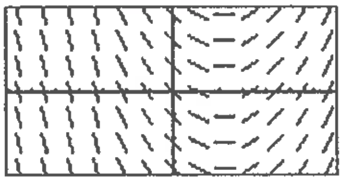
a) Match the differential equations with their corresponding slope field:

i) $\frac{dy}{dx} = 0.5x - 1$ B ii) $\frac{dy}{dx} = 0.5y$ C iii) $\frac{dy}{dx} = \frac{-x}{y}$ D iv) $\frac{dy}{dx} = x + y$ A

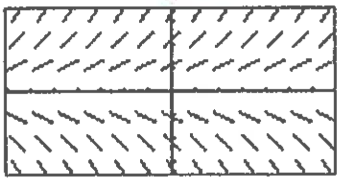
(A) (iv)



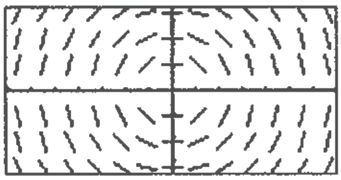
(B) (i)



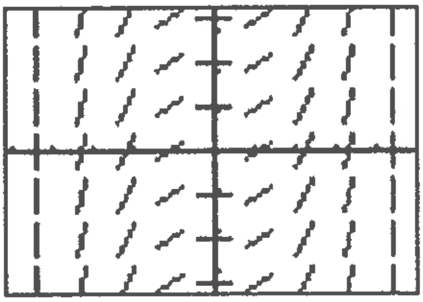
(C) (ii)



(D) (iii)



b) The slope field for a certain differential equation is shown below, state a possible solution for this differential equation. Give 3 reasons to justify your answer.



$y = ax^3 + b$ ✓

Gradient is always positive ✓

Horizontal point of inflection at $x = 0$ ✓

Gradient $\to +\infty$ as $x \to \pm\infty$ ✓

Question 6 (2 marks)

A closed cylinder 10 cm long has a cross-sectional radius of 2 cm. Use the incremental formula to find the approximate error in the volume of the cylinder corresponding to an error of 0.05 cm in the measurement of its radius.

$V = \pi r^2 h$

$\frac{dV}{dr} = 2\pi r h$ ✓

$\delta V \approx 2\pi r h \, \delta r$

$\approx 2\pi(2)(10)(0.05)$

$\approx 2\pi \text{ cm}^3$ ✓

Question 7 (5, 1 marks)

The velocity v metres per second of particle, P, is related to its displacement, x metres, from a fixed point O by the equation $v = 2x + 4$. It is known that P starts from O and $x \geq 0$ when $t \geq 0$.

a) Determine x in terms of time t.

$v = \frac{dx}{dt} = 2x + 4$ ✓

$\int \frac{1}{2x+4}\, dx = \int dt$ ✓

$\frac{1}{2}\ln|x+2| = t + c$ ✓

$\ln|x+2| = 2t + c$

$e^{2t} \cdot e^{c} = x + 2$ ✓

at $t = 0$, $x = 0$

$e^{c} = 2$

$2e^{2t} - 2 = x$ ✓

or $\frac{1}{2}\ln|2x+4| = t + c$ ✓

$\Rightarrow \ln|2x+4| = 2t + c$

$\Rightarrow 2x + 4 = e^{2t} \cdot e^{c}$ ✓

at $t = 0$, $x = 0$

$\Rightarrow e^{c} = 4$ ✓

So $\cancel{2e^{2t}}$

$2x + 4 = 4e^{2t}$

$\Rightarrow x + 2 = 2e^{2t}$ ✓

$\Rightarrow x = 2e^{2t} - 2$ ✓

b) Determine its acceleration, a, in terms of time t.

$\frac{dx}{dt} = 4e^{2t}$

$a = \frac{d^2x}{dt^2} = 8e^{2t}$ ✓

Question 9 **(3, 1, 2, 3 marks)**

A small particle, P, describes simple harmonic motion along a straight line with centre O. Two points, A and B, lie on this straight line with A between O and B such that OA = 4 cm and AB = 3.5 cm. At A the speed of the particle is 22.5 cm per second and at B its speed is 12 cm per second.

Determine:

a) the amplitude of the motion

at A $\quad 22.5^2 = k^2(a^2 - 4^2)$ ✓

at B $\quad 12^2 = k^2(a^2 - 7.5^2)$ ✓

$\therefore a = 8.5, \quad k = 3$

Amplitude is 8.5 cm ✓

b) the period of the motion

$P = \frac{2\pi}{3}$ seconds ✓

c) the maximum speed of P

$v^2 = 3^2(8.5^2 - 0^2)$ ✓

$v = 25.5 \text{ cms}^{-1}$ ✓

d) the time to travel from A to B

$x = 8.5 \sin(3t)$

$4 = 8.5 \sin(3t)$

$t = 0.1633$ seconds to A ✓

$7.5 = 8.5 \sin(3t)$

$t = 0.3603$ seconds to B ✓

$t \approx 0.3603 - 0.1633$

$= 0.197$ seconds ✓

Question 10 (3, 4 marks)

Let A be the area of the region between two concentric circles of radii x and y, $x \geq y$, at any time t seconds. x is increasing at a constant rate of 4 centimetres per second and when t = 0, x = 60 cm and y = 20 cm.

a) If y is increasing at a constant rate of 5 centimetres per second, determine the rate of increase of A when t =0.

$A = \pi (x^2 - y^2)$ ✓

$\frac{dA}{dt} = \pi \left(2x \frac{dx}{dt} - 2y \frac{dy}{dt}\right)$ ✓

$= \pi (2 \times 60 \times 4 - 2 \times 20 \times 5)$

$= 280\pi \ cms^{-1}$ ✓

b) If the area A is fixed, determine the rate of increase of y when x = 90 cm.

$\frac{dA}{dt} = 0$

$0 = \pi \left(2x \frac{dx}{dt} - 2y \frac{dy}{dt}\right)$

$0 = \pi \left((2 \times 90 \times 4) - 2y \frac{dy}{dt}\right)$ ✓ .... ①

at $t = 0$

$A_0 = \pi (60^2 - 20^2) = \pi (90^2 - y^2)$ ✓

$60^2 - 20^2 = 90^2 - y^2$

$70 = y$ ✓

Subst. $y = 70$ into ①

$0 = 720 - 2(70) \frac{dy}{dt}$

$\frac{dy}{dt} = 5.14286$ cm/sec ✓

Question 8 **(7, 2 marks)**

The number of people infected by a strain of influenza is modelled by $\frac{dP}{dt} = 10P - 0.005P^2$, where t is time in months after the detection of the virus strain among 10 influenza patients.

a) Use the variables separable method to write P as a function of t, in the form $P = \frac{A}{1+Be^{-kt}}$

$$\frac{dP}{dt} = \frac{1}{200}P(2000-P) \checkmark$$

$$\int \frac{1}{P(2000-P)}\,dP = \int \frac{1}{200}\,dt \checkmark$$

$$\frac{1}{2000}\int \left(\frac{1}{P} + \frac{1}{2000-P}\right)dP = \frac{1}{200}\int dt \checkmark$$

$$\ln|P| - \ln|2000-P| = 10t + c \checkmark$$

$$\ln\left|\frac{P}{2000-P}\right| = 10t + c$$

$$\ln\left|\frac{2000-P}{P}\right| = -10t - c$$

$$\frac{2000-P}{P} = e^{-10t} \cdot e^{-c} \checkmark$$

at $t = 0$, $P = 10$

$$\frac{1990}{10} = e^{-c}$$

$$199 = e^{-c}$$

$$\frac{2000-P}{P} = 199e^{-10t} \checkmark$$

$$2000 = P(199e^{-10t} + 1)$$

$$\frac{2000}{1+199e^{-10t}} = P \checkmark$$

$$\frac{1}{P(2000-P)} = \frac{a}{P} + \frac{b}{2000-P}$$

$$1 = a(2000-P) + bP$$

let $P = 0$ $\quad$ $P = 2000$

$1 = 2000a$ $\quad$ $1 = 2000b$

$\frac{1}{2000} = a$ $\quad$ $\frac{1}{2000} = b$

b) Determine the time it takes for the infection to reach 80% of its limiting value.

$$1600 = \frac{2000}{1+199e^{-10t}} \checkmark$$

$t = 0.66796$ month $\checkmark$