MINDARIE
SENIOR COLLEGE
WHERE YOUR FUTURE BEGINS NOW

MATHEMATICS: SPECIALIST 1 & 2

SEMESTER 2 2015

TEST 6

Resource Free

Time Allowed: 20 minutes

Total Marks: 18

1. [1, 1, 2, 3 marks]

Given that $z_1 = 3 + i4$ and $z_2 = 5 - i2$ determine

(a) $\bar{z}_2 = 5 + i2$ ✓

(b) $z_1 + z_2 = 8 + i2$ ✓

(c) $z_1 \times \bar{z}_2 = (3+i4)(5+i2)$

$= 15 + i6 + i20 - 8$

$= 7 + i26$

✓ ✓

(d) $\frac{z_1}{z_2} = \frac{3+i4}{5-i2} \times \frac{5+i2}{5+i2}$ ✓

$= \frac{7+i26}{25+4}$ ✓

$= \frac{7+i26}{29}$ ✓

2. [3 marks]

Show that a reflection about the line $y = x$ followed by a reflection about the line $y = -x$ is the same as a rotation of 180° about the origin.

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = R_{180°}$$

✓ ✓ ✓

3. [1, 3, 4 marks]

(a) One factor of $x^2 + 6x + 13$ is $(x + 3 + i2)$. State the second factor.

$x + 3 - i2$ ✓

(b) Solve for w, leaving your answer in exact form: $w^2 + 2w + 5 = 0$

$$w = \frac{-2 \pm \sqrt{4-20}}{2}$$ ✓

$$= \frac{-2 \pm \sqrt{-16}}{2}$$ ✓

$$= \frac{-2 \pm i4}{2} \qquad = -1 \pm i2$$ ✓

(c) Determine the complex number $w = a + bi$, given that $2w - 3i\bar{w} = -18 + 22i$.

$$2(a+bi) - 3i(a-bi) = -18 + 22i$$

$$2a + 2bi - 3ai - 3b =$$ ✓

$$2a - 3b = -18 \qquad \times 2$$
$$-3a + 2b = 22 \qquad \times 3$$ ✓

$$4a - 6b = -36$$
$$-9a + 6b = 66$$
$$-5a = -30$$

$$a = 6$$ ✓

$$12 - 3b = -18$$
$$3b = 30$$
$$b = 10$$ ✓

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Resource Assumed

Time Allowed: 40 minutes

Total Marks: 36

4. [4 marks]

Consider the complex numbers $z_1 = a + i2$ and $z_2 = b - i5$, where $a, b > 0$.

Determine a and b given that $z_1 z_2 = 22 - i7$.

$z_1 z_2 = ab - i5a + i2b + 10$ ✓

$= (ab+10) + i(2b - 5a)$

$\therefore ab + 10 = 22$ ✓

$2b - 5a = {}^{-}7$

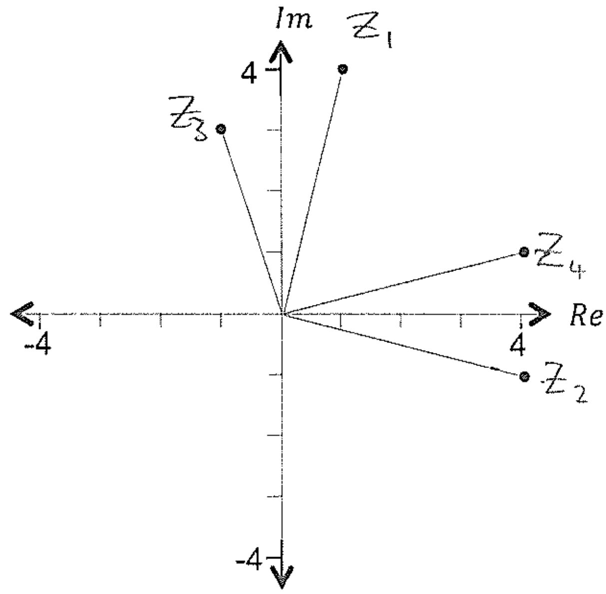
$a = 3$ & $b = 4$ ✓✓ (only 1 if also $a = {}^{-}1.6$ & $b = {}^{-}7.5$)

5. [3 marks]

Label the diagram to right with z_1, z_2, z_3 and z_4 given that

$z_2 = \overline{z_4}$
$z_1 = iz_2$
$Im(z_1) > Im(z_3)$

4 correct ✓✓✓
2 correct ✓✓
1 correct ✓



6. [3, 1 marks]

A set of points X are transformed to the points X' by the following transformations:

- a dilation of scale factor 3 parallel to the y-axis, followed by
- a reflection in the line $y = \frac{1}{2}x$

(a) Determine the single matrix that would transform the points X directly to X'.

$$\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0.6 & 2.4 \\ 0.8 & -1.8 \end{bmatrix}$$

✓ ✓order ✓

Ref mx ✓
correct order ✓
soln mx ✓

(b) Determine the single matrix that would return the points X' back to X.

$$\frac{-1}{3}\begin{bmatrix} -1.8 & -2.4 \\ -0.8 & 0.6 \end{bmatrix}$$ ✓

7. [3, 3 marks]

(a) Show that the transformation matrix $\begin{bmatrix} 3 & 1 \\ 12 & 4 \end{bmatrix}$ maps all the points in the plane to a line, and determine the equation of this line.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 12 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 3x + y \\ 12x + 4y \end{bmatrix}$$ ✓

$$y' = 4(3x+y) \quad \checkmark \qquad y = 4x \quad \checkmark$$
$$= 4x'$$ ✓

(b) Show that the transformation matrix $\begin{bmatrix} 9 & -3 \\ 3 & -1 \end{bmatrix}$ maps all the points in the line $y = 3x - 2$ to a single point in the plane, and determine this point.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ 3x-2 \end{bmatrix}$$ ✓

$$= \begin{bmatrix} 9x - 9x + 6 \\ 3x - 3x + 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 2 \end{bmatrix} \quad \checkmark \quad \therefore (6, 2)$$ ✓

8. [1, 1, 3, 2 marks]

The quadrilateral with vertices $A(0,0)$, $B(1,2)$, $C(-3,1)$ and $D(-2,-1)$ is transformed by the matrix $M = \begin{bmatrix} 0 & -1 \\ -3 & 0 \end{bmatrix}$ on to the quadrilateral $A'B'C'D'$.

(a) Determine the coordinates of C'.

$$\begin{bmatrix} 0 & -1 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \end{bmatrix}$$

$(-1, 9)$ ✓

(b) The area of quadrilateral $A'B'C'D'$ is 18 square units. What is the area of quadrilateral $ABCD$?

$|M| = 0 - 3$
$= -3$

$\therefore$ Area $(ABCD) = \frac{18}{3}$
$= 6$ sq. units ✓

(c) Matrix M represents the combination of transformation P followed by transformation Q. If the matrix for transformation $P = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$, determine the matrix for transformation Q and describe the geometric transformation Q represents.

$QP = M$
$QPP^{-1} = MP^{-1}$ ✓
$Q = \begin{bmatrix} 0 & -1 \\ -3 & 0 \end{bmatrix} \cdot \frac{1}{-3}\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ ✓
$= \frac{-1}{3}\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$
$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ✓

(d) The quadrilateral $A'B'C'D'$ is then transformed by the matrix $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ such that the image of C' lies on the y-axis. Determine the value of k.

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 + 9k \\ 9 \end{bmatrix}$$ ✓

$9k - 1 = 0$
$9k = 1$
$k = 1/9$ ✓

9. [5 marks]

By considering a rotation of angle A followed by another rotation of angle A, prove that

$$\sin 2A = 2\sin A\, cosA$$

and

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$R_A R_A = \begin{bmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{bmatrix}\begin{bmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{bmatrix} \checkmark$$

$$= \begin{bmatrix} \cos^2 A - \sin^2 A & -\sin A\cos A - \sin A \cos A \\ \sin A\cos A + \sin A\cos A & -\sin^2 A + \cos^2 A \end{bmatrix} \checkmark$$

$$= \begin{bmatrix} \cos^2 A - \sin^2 A & -2\sin A\cos A \\ 2\sin A\cos A & \cos^2 A - \sin^2 A \end{bmatrix} \checkmark$$

$$R_{2A} = \begin{bmatrix} \cos 2A & -\sin 2A \\ \sin 2A & \cos 2A \end{bmatrix} \checkmark$$

$$\left.\begin{aligned} \therefore\ \cos 2A &= \cos^2 A - \sin^2 A \\ \&\ \sin 2A &= 2\sin A\cos A \end{aligned}\right\} \checkmark$$

10. [1, 2, 2, 2 marks]

Consider the point $(5 + i2)$ in the Argand plane.

(a) Determine the value of $(5 + i2)(\frac{3\sqrt{3}}{2} + \frac{3i}{2})$. $= \left(\frac{15\sqrt{3}}{2} - 3\right) + i\left(3\sqrt{3} + \frac{15}{2}\right)$ ✓

The matrix $C = \begin{bmatrix} \frac{3\sqrt{3}}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{3\sqrt{3}}{2} \end{bmatrix}$ is a transformation matrix in the complex plane such that $Z' = CZ$.

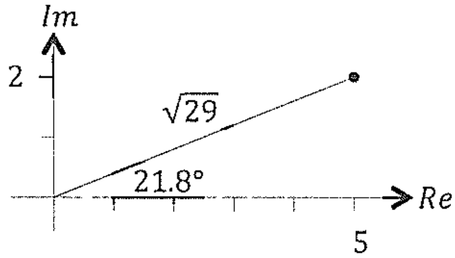
NOTE: C is a rotation of 30° combined with a dilation of scale factor 3 about the origin.

(b) Determine the coordinates in the Argand plane of the point $(5 + i2)$ after it has been transformed by C.

$$\begin{bmatrix} \frac{3\sqrt{3}}{2} & -3/2 \\ 3/2 & \frac{3\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{15\sqrt{3}}{2} - 3 \\ 3\sqrt{3} + \frac{15}{2} \end{bmatrix}$$ ✓✓

The point $(5 + i2)$ has a distance of $\sqrt{29}$ units from the origin and makes an angle of 21.8° with the positive real axis. Hence it can be written as

$$\begin{aligned}(5 + i2) &= \sqrt{29}\cos 21.8° + i\sqrt{29}\sin 21.8° \\ &= \sqrt{29}(\cos 21.8° + i\sin 21.8°) \\ &= \sqrt{29}\, cis(21.8°)\end{aligned}$$



Similarly, $(\frac{3\sqrt{3}}{2} + \frac{3i}{2})$ can be written as $3cis(30°)$

(c) Write your answer to (b) in cis form.

$3\sqrt{29}$ cis $51.8°$ ✓ ✓

(d) Using what you have learned from (a), (b) and (c), or otherwise, determine in cis form the position of the point z_3 where

$$z_3 = z_1 \times z_2$$
$$z_1 = \frac{5\sqrt{2}}{2} + i\frac{5\sqrt{2}}{2} = 5\, cis(45°)$$
and
$$z_2 = \left(\frac{3\sqrt{3}}{2} + \frac{3i}{2}\right) = 3cis(30°)$$

$z_3 = 15$ cis $(75°)$ ✓ ✓