Mathematics Specialist Unit 4

Test 4: Calculus

Student Name: _____

Section One – calculator-free section

Time allowed for this task: 30 minutes, in class, under test conditions

Materials Provided: SCSA Formula Sheet

Materials required: (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters.

Marks available: 31 marks

1. **[2, 3, 4, 4 = 13 marks]**

Determine the following definite integrals; using a judicious substitution, if necessary.

(a) $\int \frac{4x+10}{x^2+5x+4}\,dx$

$= 2\int \frac{2x+5}{x^2+5x+4}\,dx$ ✓

$= 2\ln|x^2+5x+4| + C$ ✓

(b) $\int \frac{\sin(lnx)}{x}\,dx$ using the substitution, $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$x\,du = dx$

$\int \frac{\sin u}{x}\,x\,du$ ✓

$\int \sin u\,du$ ✓

$= -\cos u + C$

$= -\cos(\ln|x|) + C$ ✓

(c) $\int \frac{2}{16+x^2}\,dx$ using the substitution, $x = 4\tan u$

$\frac{dx}{du} = 4\sec^2 u$

$dx = 4\sec^2 u\,du$

$\int \frac{2}{16+16\tan^2 u}\,4\sec^2 u\,du$ ✓

$\int \frac{2}{16(1+\tan^2 u)}\,4\sec^2 u\,du$

$\frac{x}{4} = \tan u$

$\tan^{-1}\left(\frac{x}{4}\right) = u$

$\frac{1}{2}\int \frac{\sec^2 u}{\sec^2 u}\,du$ ✓

$\frac{1}{2}\int 1\,du$

$= \frac{1}{2}u + C$ ✓

$= \frac{1}{2}\tan^{-1}\left(\frac{x}{4}\right) + C$

(d) $\int \frac{3x-1}{(3x-4)^2}\,dx$

$$\frac{3x-1}{(3x-4)^2} = \frac{A}{3x-4} + \frac{B}{(3x-4)^2}$$ ✓

$$3x-1 = A(3x-4) + B$$

let $x = \frac{4}{3}$ ✓

$3 = B$

let $x = 0$

$-1 = -4A + 3$

$1 = A$

$$\int \left(\frac{1}{3x-4} + \frac{3}{(3x-4)^2}\right) dx$$ ✓

$$= \frac{1}{3}\ln|3x-4| - \frac{1}{3x-4} + C.$$ ✓

2. [4, 5, 5 = 14 marks]

Evaluate

(a) $\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$ using the trig substitution x = 3 sin θ

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{9-9\sin^2\theta}}\, 3\cos\theta\, d\theta$$

$\frac{dx}{d\theta} = 3\cos\theta$

$dx = 3\cos\theta\, d\theta$.

✓✓ correct substitution and bounds

$$\int_0^{\frac{\pi}{2}} \frac{3\cos\theta}{3\cos\theta}\, d\theta$$

$$\int_0^{\frac{\pi}{2}} 1\, d\theta$$ ✓

$$= \theta\Big]_0^{\pi/2}$$

$$= \frac{\pi}{2}$$ ✓

(b) $\int_0^4 \frac{1}{9-\sqrt{x}}\,dx$ using the substitution $u = 9 - \sqrt{x}$

$\frac{du}{dx} = \frac{-1}{2\sqrt{x}}$

$-2\sqrt{x}\,du = dx$.

$\int_9^7 \frac{1}{u}(-2(9-u))\,du$ ✓✓

correct bounds and substitution.

$2\int_7^9 \frac{9-u}{u}\,du$

$2\int_7^9 \left(\frac{9}{u} - 1\right)du$ ✓

$= 2\left(9\ln|u| - u\right)_7^9$ ✓

$= 18\ln 9 - 18 - 18\ln 7 + 14$

$= 18\ln\left(\frac{9}{7}\right) - 4$ ✓

(c) $\int_1^4 \frac{x}{\sqrt{5-x}}\,dx$

Let $u = 5 - x$

$\frac{du}{dx} = -1$

$-du = dx$

$\int_4^1 \frac{5-u}{\sqrt{u}}(-du)$ ✓✓

correct bounds & substitution

$= \int_1^4 \left(5u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right)du$ ✓

$= 2\left(5u^{\frac{1}{2}}\right) - \frac{2}{3}u^{\frac{3}{2}}\Big]_1^4$ ✓

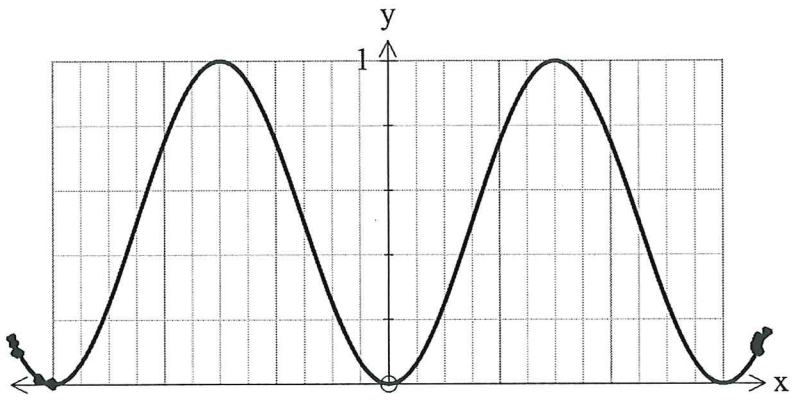
$= \left(10(2) - \frac{16}{3}\right) - \left(10 - \frac{2}{3}\right)$

$= \frac{44}{3} - \frac{28}{3}$

$= \frac{16}{3}$ ✓

3. [4 marks]

The graph below shows the function $y = sin^2\left(\frac{x}{6}\right)$. Determine the area of the region trapped between the function $y = sin^2\left(\frac{x}{6}\right)$, the lines $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$ and $y = 0$



$$A = 2\int_0^{\frac{\pi}{2}} \sin^2\left(\frac{x}{6}\right) dx \quad \checkmark$$

$$A = 2\int_0^{\frac{\pi}{2}} \frac{1-\cos\left(\frac{x}{3}\right)}{2}\, dx \quad \checkmark$$

$$A = \int_0^{\frac{\pi}{2}} \left(1-\cos\left(\frac{x}{3}\right)\right) dx$$

$$A = x - 3\sin\left(\frac{x}{3}\right)\Big]_0^{\pi/2} \quad \checkmark$$

$$A = \frac{\pi}{2} - 3\sin\frac{\pi}{6}$$

$$A = \frac{\pi - 3}{2} \text{ units}^2 \quad \checkmark$$

End of Section One

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You may use this space to extend or re-attempt an answer to a question or questions and should you do so then number the question(s) attempted and cross out any previous unwanted working.

Mathematics Specialist Unit 4

Test 4: Calculus

Student Name: _____

Section Two – calculator-assumed section

Time allowed for this task: 25 minutes, in class, under test conditions

Materials Provided: SCSA Formula Sheet

Materials required: (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters.

Notes on one unfolded sheet of A4 paper.

Special items: Drawing instruments, templates, and up to three calculators approved for use in WACE examinations.

Marks available: 23 marks

4. [4, 4 = 8 marks]

For the curves with equations

$$3y^2 = 16x \quad \text{and} \quad y^2 = 64 - 16x,$$

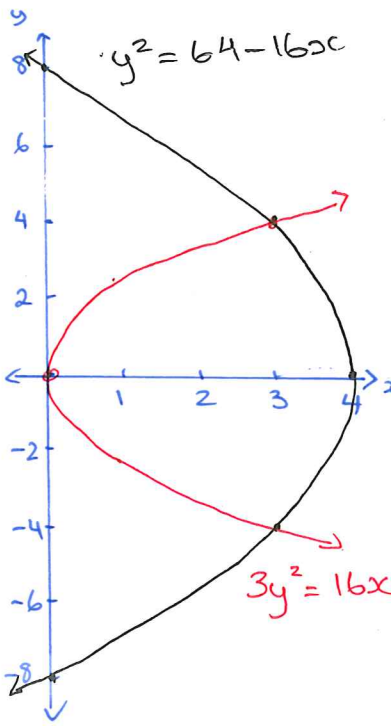
determine:

(a) the area of the region enclosed between the two curves,

$$\int_{-4}^{4} \left(\frac{64 - y^2}{16} - \frac{3y^2}{16} \right) dy \quad \checkmark\checkmark$$

−1 correct bounds
−1 rearrange eq'ns
−1 subtract curves

$$= 21\tfrac{1}{3} \text{ units}^2 \quad \checkmark$$



or

$$2\left[\int_0^3 \sqrt{\frac{16x}{3}}\, dx + \int_3^4 \sqrt{64 - 16x}\, dx \right] \checkmark\checkmark\checkmark$$

$$= 21\tfrac{1}{3} \text{ units}^2 \quad \checkmark$$

(b) the volume of the solid of revolution about the x-axis formed by this region

$$V = \pi \left[\int_0^3 \frac{16x}{3}\, dx + \int_3^4 (64 - 16x)\, dx \right]$$

✓ correct $\int \pi y^2\, dy$
✓ correct 1st integral
✓ correct 2nd integral

$$= \pi \left[24 + 8 \right]$$

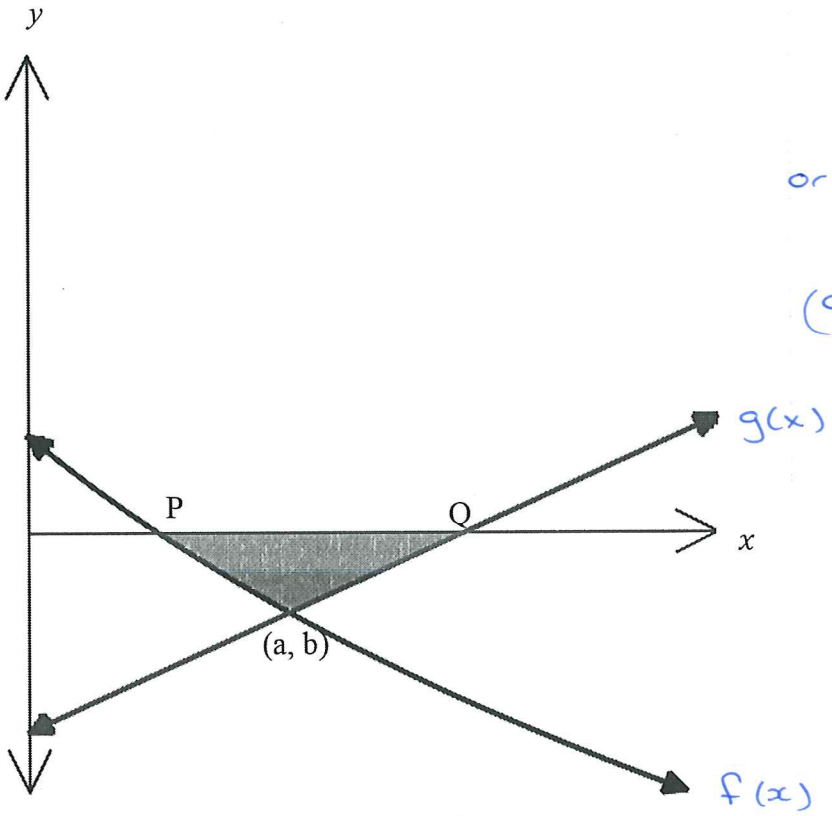
$$= 32\pi \text{ units}^3 \quad \checkmark$$

$$\text{or} \quad 100.53 \text{ units}^3$$

5. [6 marks]

Below is shown the cross-section of a river (see shaded), the bed of which is described by curves; $f(x) = -10\ln(0.1x + 0.8)$ and $g(x) = 2.5e^{(0.1x+0.8)} - 10$, where x and y are in metres. If Karan is located at P, gazing into the river, contemplating the meaning of life, and the river current is 2m/s then how many kilolitres (to the nearest kL) of water will pass him by in five minutes of his thinking?

Note: 1 m^3 = 1kL



P = (2, 0)

Q = (5.8629436, 0)

or Q = (10ln4 − 8, 0)

(a, b) = (3.92573, −1.76

✓

Area of Cross Section

$$-\int_{2}^{3.92573} -10\ln(0.1x+0.8)\,dx - \int_{3.92573}^{5.86294} (2.5e^{0.1x+0.8} - 10)\,dx$$

✓ ✓

$= 1.745 + 1.761$

$= 3.506\ m^2$ ✓

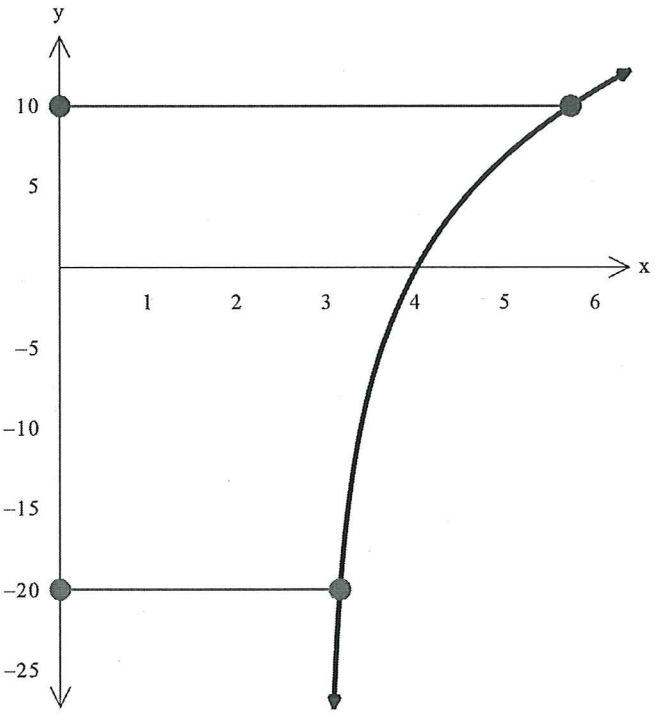
Quantity of water = $3.506 \times 2 \times 5 \times 60$ ✓

≈ 2104 kL ✓

6. [3 marks]

Below is shown the graph of the curve of y = 10ln(x–3) where both x and y are in centimetres.

A vase is to be created by rotating the curve about the y-axis from y = -20 to y = 10



Determine the *volume* of the vase, correct to the nearest cm^3.

$y = \ln(x-3)$

$\frac{y}{10} = \ln(x-3)$

$e^{0.1y} + 3 = x$ ✓

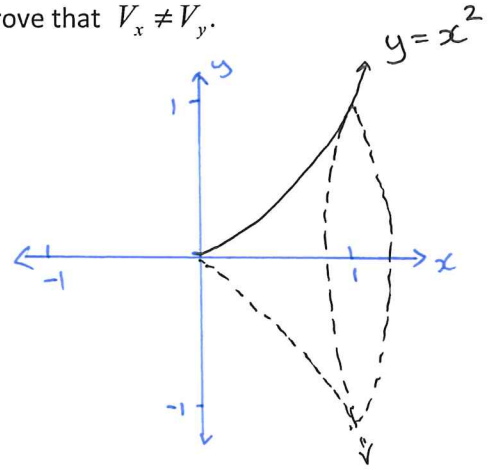
$V = \pi \int_{-20}^{10} (e^{0.1y} + 3)^2 \, dy$ ✓

$V \approx 1451 \, cm^3$ ✓

7. [6 marks]

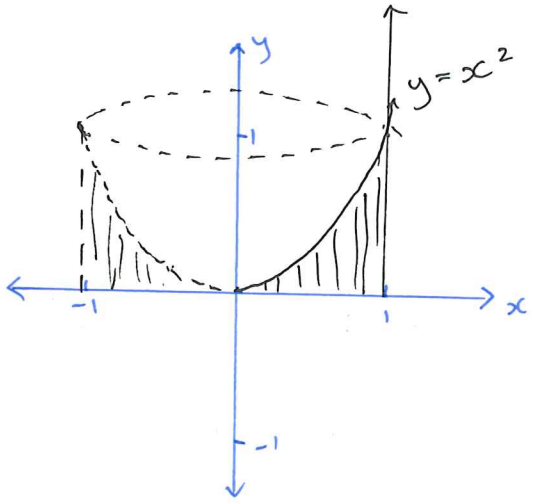
The region define by $\{(x,y): x \geq 0 \cap y \geq 0 \cap y \leq x^2\}$ for $0 \leq x \leq 1$ is rotated about the x axis to generate V_x and then rotated about the y axis to generate V_y.

Prove that $V_x \neq V_y$.



$V_x = \pi \int_0^1 (x^2)^2 \, dx$ ✓

$V_x = \frac{\pi}{5}$ units^2 ✓



$V_y = \pi \int_0^1 (1^2 - y) \, dy$ ✓ ✓

$V_y = \frac{\pi}{2}$ units^2 ✓

$\therefore V_x \neq V_y$.

End of Section Two

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You may use this space to extend or re-attempt an answer to a question or questions and should you do so then number the question(s) attempted and cross out any previous unwanted working.