Name: ______________________

Mathematics Specialist Units 3 & 4
Test 1 2018
Calculator Free

Time Allowed: 20 minutes **Total Marks**: 20
Materials Allowed: SCSA formula booklet
Instructions: Where a question or part of a question is worth more than 2 marks sufficient working to justify your solution is required.

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1. **[7, 1, 2, 1 marks]**
One of the three cube roots of a complex number ***Z*** is z_1, where $z_1 = 2(i - \sqrt{3})$.
a) On the argand plane, accurately sketch the 3 cube roots of ***Z***. $z_1 = -2\sqrt{3} + 2i$

$|z_1| = \sqrt{2^2 + (2\sqrt{3})^2}$
$= 4$

2, $2\sqrt{3}$, α

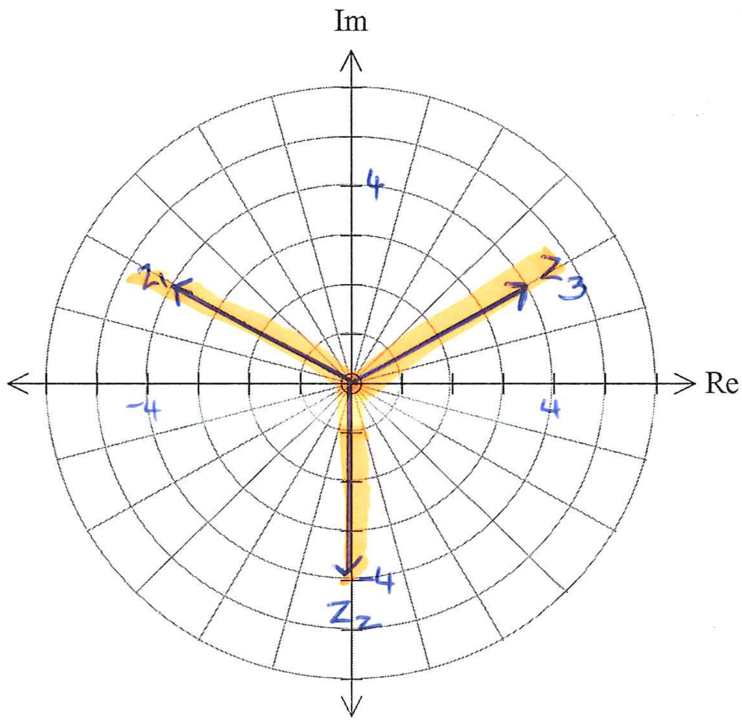
$\alpha = \tan^{-1}\left(\frac{2}{2\sqrt{3}}\right)$

$\alpha = \frac{\pi}{6}$

$z_1 = 4\text{cis}\left(\frac{5\pi}{6}\right)$ ✓ ✓

$z_2 = 4\text{cis}\left(-\frac{\pi}{2}\right)$ ✓

$z_3 = 4\text{cis}\left(\frac{\pi}{6}\right)$ ✓



✓ scale
✓ correct location z_1
✓ correct location other roots

b) Determine as exact values:
I. $z_1 + z_2 + z_3$

0 ✓

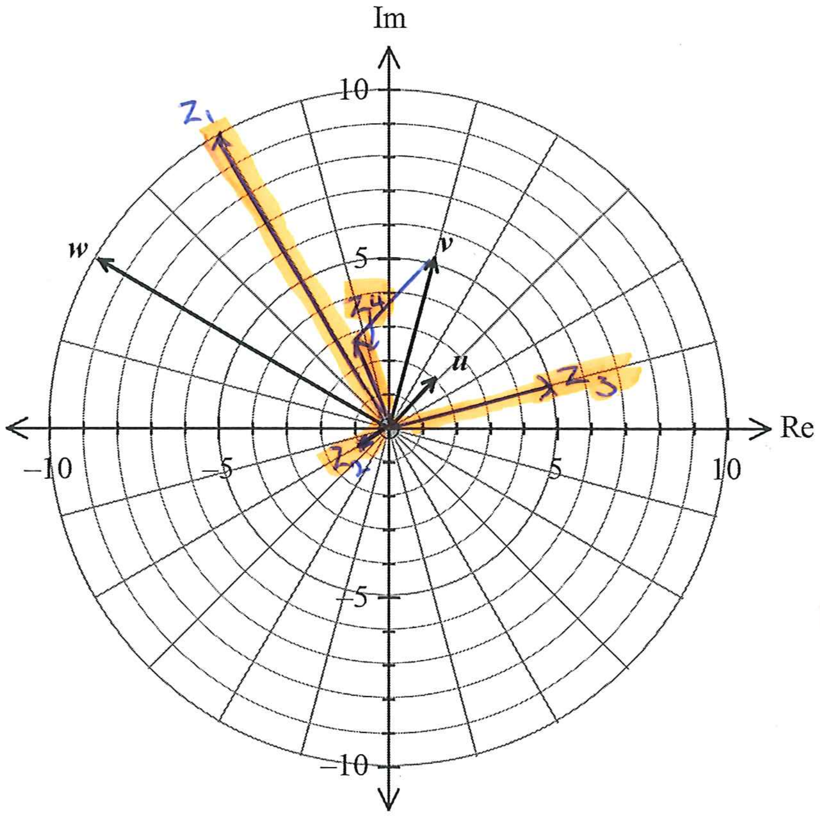
II. $z_1 \times z_2 \times z_3$ in polar form

$64\text{cis}\frac{\pi}{2}$ ✓ ✓

III. Z in polar form

$64\text{cis}\frac{\pi}{2}$ ✓

2. [4 marks]
The diagram below shows the unit circle in the complex plane and the position of two complex numbers w and u.



1each.

On the diagram above plot the following:

a. $z_1 = uv$

b. $z_2 = \frac{1}{w}$

c. $z_3 = i\bar{v}$

d. $z_4 = v - 2u$

3. [5 marks]
The polynomial $P(z) = 3z^4 - 3z^3 + az^2 + bz - 3$ has a factor of $z + 1$ and leaves a remainder of 51 when $P(z)$ is divided by $z - 2$. Determine the values of a and b.

$P(-1) = 3 + 3 + a - b - 3 = 0$

$a - b = -3 \quad \text{....} \; (1)$

$P(2) = 48 - 24 + 4a + 2b - 3 = 51$

$4a + 2b = 30 \quad \text{....} \; (2)$

$(2) + 2(1)$

$4a + 2b = 30$
$2a - 2b = -6$

$6a = 24$

$a = 4$

$a - b = -3$
$4 - b = -3$
$b = 7$

Name: ___________________________

Mathematics Specialist Units 3 & 4
Test 1 2018
Calculator Assumed

Time Allowed: 35 minutes **Total Marks**: 37
Materials Allowed: SCSA formula booklet, SCSA approved calculators and one A4 page of notes (both sides).
Instructions: Where a question or part of a question is worth more than 2 marks sufficient working to justify your solution is required.

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4. [6 marks]
Solve the equation $z^3 + z^2 - z + 15 = 0$.

$P(z) = z^3 + z^2 - z + 15$

$P(-3) = 0$

$\Rightarrow$ $z+3$ is factor ✓

$z^3 + z^2 - z + 15 = (z+3)(az^2 + bz + c)$ ✓

$a = 1, \ c = 5$ ✓

$1z^2 = bz^2 + 3az^2$

$1 = b + 3$

$-2 = b$ ✓

$(z+3)(z^2 - 2z + 5) = 0$

$z = -3$ ✓ $(z-1)^2 + 4 = 0$

$z - 1 = \pm\sqrt{-4}$

$z = 1 \pm 2i$ ✓

5. [4, 3 marks]

a. Given $z = rcis\theta$, prove that $\bar{z} + \frac{1}{z} = \frac{(r^2+1)}{r^2}\bar{z}$

$\bar{z} = r\,cis(-\theta)$

$\frac{\bar{z}}{r} = cis(-\theta)$ ✓

$r\,cis(-\theta) + (r\,cis\theta)^{-1}$

$= r\,cis(-\theta) + \frac{1}{r}cis(-\theta)$ ✓

$= \frac{r^2+1}{r}\,cis(-\theta)$ ✓

$= \frac{r^2+1}{r}\,\frac{\bar{z}}{r}$ ✓

$= \frac{r^2+1}{r^2}\,\bar{z}$

b. Hence, show that $\left(\sqrt{3} + i + \frac{1}{\sqrt{3}-i}\right)^n = \left(\frac{5}{2}\right)^n cis\frac{n\pi}{6}$

$z = 2cis\left(-\frac{\pi}{6}\right)$ ✓

$\left(\bar{z} + \frac{1}{z}\right)^n$

$= \left(\frac{2^2+1}{2^2}\,2cis\left(\frac{\pi}{6}\right)\right)^n$ ✓

$= \left(\frac{5}{2}\right)^n cis\frac{n\pi}{6}$. ✓

6. **[4, 4 marks]**

Two complex numbers are given by $u = 3i$ and $v = \frac{-3\sqrt{2}-3\sqrt{2}i}{2}$

a. Express u^3v in the form $rcis\theta$, where $-\pi < \theta \leq \pi$ and $r \geq 0$.

$u = 3cis\frac{\pi}{2}$ ✓ $\quad v = 3cis\left(-\frac{3\pi}{4}\right)$ ✓

$u^3 = 27cis\left(-\frac{\pi}{2}\right)$ ✓

$$u^3v = 27cis\left(-\frac{\pi}{2}\right)3cis\left(-\frac{3\pi}{4}\right)$$

$$= 81cis\left(-\frac{5\pi}{4}\right)$$

$$= 81cis\left(\frac{3\pi}{4}\right)$$ ✓

b. Use de Moivre's theorem to determine all the solutions for z in polar form, given $z^4 = u^3v$.

$z^4 = 81cis\left(\frac{3\pi}{4} + 2n\pi\right), \quad n = 0, 1, 2, 3$ ✓

$Z = \left[81cis\left(\frac{3\pi}{4} + 2n\pi\right)\right]^{1/4}$

$Z = 81^{1/4}cis\left(\frac{3\pi}{16} + \frac{n\pi}{2}\right), \quad n = 0, 1, 2, 3$ ✓

$Z_1 = 3cis\left(\frac{3\pi}{16}\right)$ ✓

$Z_2 = 3cis\left(\frac{11\pi}{16}\right)$

$Z_3 = 3cis\left(-\frac{13\pi}{16}\right)$

$Z_4 = 3cis\left(-\frac{5\pi}{16}\right)$ ✓

7. [5, 2 marks]

a. Sketch in the complex plane the region satisfying the two inequalities given by

$$|z - \sqrt{3} - i| \leq 2 \quad \cap \quad |z| \geq |z - \sqrt{3} - 3i|$$

$|z - (\sqrt{3} + i)| \leq 2$

circle ✓

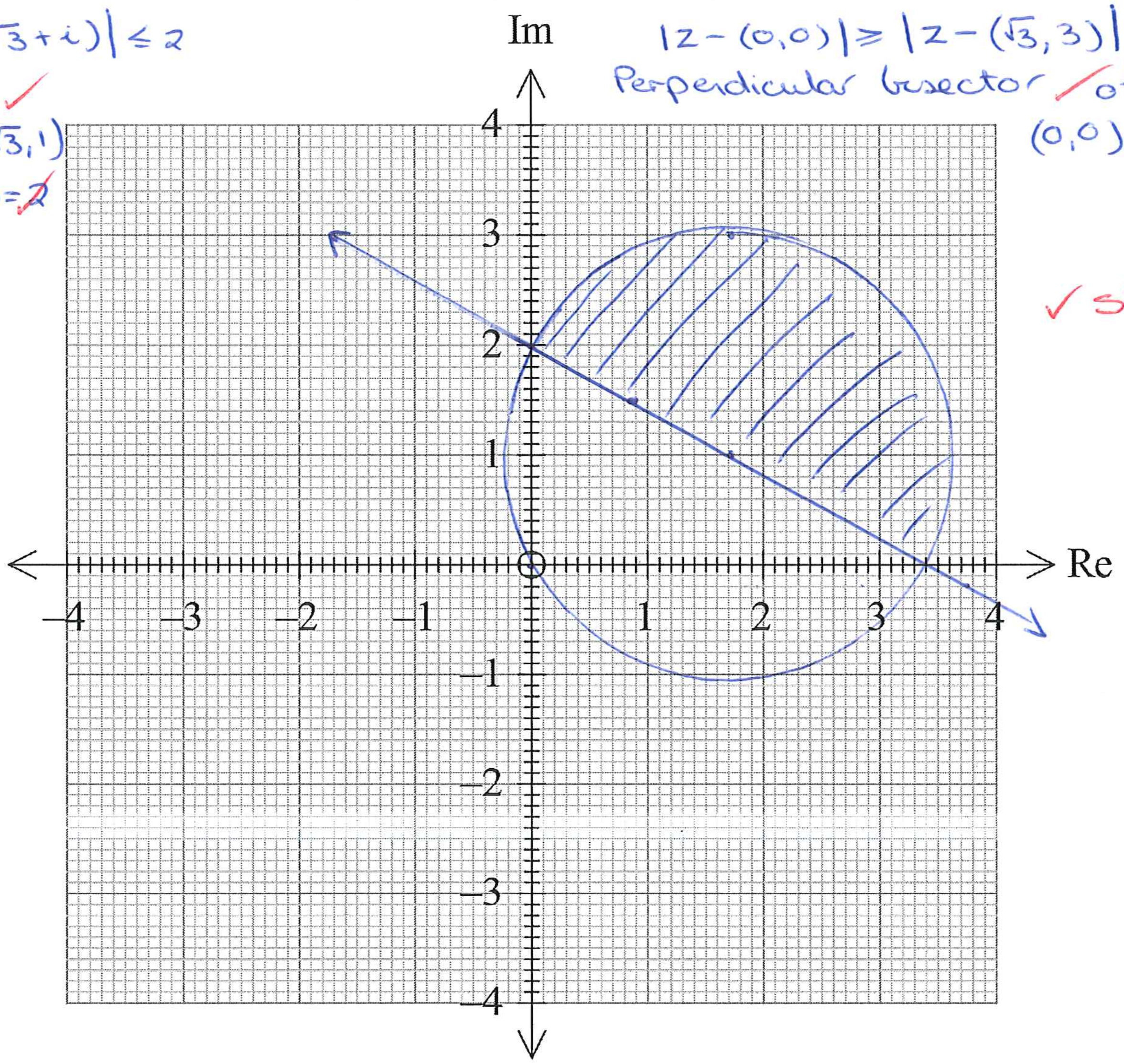
centre $(\sqrt{3}, 1)$

radius $= 2$

$|z - (0,0)| \geq |z - (\sqrt{3}, 3)|$

Perpendicular bisector ✓ of

$(0,0)$ & $(\sqrt{3}, 3)$



✓ shaded

b. If z is the complex number that satisfies both inequalities in part **a**, determine the maximum value of $|z|$.

Distance to centre of circle $= \sqrt{(\sqrt{3})^2 + 1^2}$

$= 2$ ✓

Max $|z| = 2 + 2$

$= 4$ ✓

8. [3 marks]

Use de Moivre's theorem to prove $\cos(3\theta) = cos^3\theta - 3cos\theta sin^2\theta$

$(\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta)$

LHS

$\cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + i^3\sin^3\theta$

Equate real components

$\cos^3\theta - 3\cos\theta\sin^2\theta = \cos 3\theta$

9. [6 marks]

Determine the centre and radius of the circle described by z, where $arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$.

Hint: Investigate the Cartesian equation by letting $z = x + yi$.

$$\frac{x + yi - 1}{x + yi + 1}$$

$$= \left[\frac{(x-1) + yi}{(x+1) + yi}\right]\left[\frac{(x+1) - yi}{(x+1) - yi}\right]$$

$$= \frac{(x-1)(x+1) - (x-1)yi + (x+1)yi - y^2 i^2}{(x+1)^2 + y^2}$$

$$= \frac{x^2 + y^2 - 1 + [(x+1)y - (x-1)y]i}{(x+1)^2 + y^2}$$

$$= \frac{x^2 + y^2 - 1 + 2yi}{(x+1)^2 + y^2}$$

$$\arg\left(\frac{x^2 + y^2 - 1}{(x+1)^2 + y^2} + \frac{2y}{(x+1)^2 + y^2} i\right) = \frac{\pi}{4}$$

For argument $= \frac{\pi}{4}$ Real component = Imaginary component

$x^2 + y^2 - 1 = 2y$

$x^2 + y^2 - 2y = 1$

$x^2 + (y-1)^2 - 1 = 1$

$x^2 + (y-1)^2 = 2$

$\therefore$ Centre $(0, 1)$

radius $= \sqrt{2}$.