Name : ____________________

Esperance Senior High School
Year 12 *MATHEMATICS METHODS*

TEST 1 2019

CALCULATOR AND RESOURCE FREE

Total Marks: 28

Reading: 2 minutes Time Allowed: 30 minutes

Question 1. **(5 marks)**

A function defined by $f(x) = 39 + 24x - 3x^2 - x^3$ has stationary points at $(-4, -41)$ and $(2, 67)$.

(a) Use the second derivative to show that one of the stationary points is a local maximum and the other a local minimum. (3 marks)

$f'(x) = 24 - 6x - 3x^2$ ✓

$f''(x) = -6 - 6x$

$f''(-4) = -6 - 6(-4)$

$= 18$ $\quad 18 > 0 \therefore (-4, -41)$ is a min ✓

$f''(2) = -6 - 6(2)$

$= -18$ $\quad -18 < 0 \therefore (2, 67)$ is a max ✓

(b) Determine the coordinates of the point of inflection of the graph of $y = f(x)$. (2 marks)

$f''(x) = 0$ when $x = -1$ ✓

$f(-1) = 39 - 24 - 3 + 1$

$= 13$

At $(-1, 13)$ ✓

Question 2 **(4 marks)**

If the radius of a sphere is measured with an error of at most 4%, estimate the percentage error in the volume of the sphere.

to find $\frac{\delta V}{V}$ $\qquad \frac{\delta r}{r} = 4\%$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$ ✓

$\delta V = \frac{dV}{dr} \times \delta r$

$\frac{\delta V}{V} = \frac{4\pi r^2 \times \delta r}{\frac{4}{3}\pi r^3}$ ✓

$= 3 \times \frac{\delta r}{r}$ ✓

$= 3 \times 4\%$

$= 12\%$ ✓

Question 3 **(12 marks)**

Determine the gradient function $\frac{dy}{dx}$ for each of the following.

Leave your answers with positive indices, where necessary. Do not simplify.

(a) $y = ax^a$ (Where *a* is a positive constant) (1 mark)

$\frac{dy}{dx} = a^2 x^{a-1}$

(b) $y = \frac{5x^3 + 4x^4 + \pi^2}{2x}$ let $y = \frac{u}{v}$ where $u = 5x^3 + 4x^4 + \pi^2$ (2 marks)

$u' = 15x^2 + 16x^3$

and $v = 2x$

$v' = 2$

$\therefore \frac{dy}{dx} = \frac{2x(15x^2 + 16x^3) - 2(5x^3 + 4x^4 + \pi^2)}{4x^2}$

(c) $y = (x^3 - 4x)^3(3x^2)$ let $y = uv$ where $u = (x^3 - 4x)^3$ (3 marks)

$u' = 3(x^3 - 4x)^2(3x^2 - 4)$

and $v = 3x^2$

$v' = 6x$

$\frac{dy}{dx} = 3x^2 \times 3(x^3 - 4x)^2(3x^2 - 4) + 6x(x^3 - 4x)^3.$

(d) y = [3 + cos(*x*/2)]^4 (3 marks)

$\frac{dy}{dx} = -4 \times \frac{1}{2}\sin\left(\frac{x}{2}\right) \left(3 + \cos\left(\frac{x}{2}\right)\right)^3.$

(e) $y = \frac{3x^2}{e^{5x} - 2}$ (3 marks)

let $y = \frac{u}{v}$ $u = 3x^2$ $v = e^{5x} - 2$

$u' = 6x$ $v' = 5e^{5x}$

$\frac{dy}{dx} = \frac{(e^{5x} - 2)6x - 3x^2(5e^{5x})}{(e^{5x} - 2)^2}$

Question 4 **(7 marks)**

$$f(x) = -x^2 + 4x + 5$$

(a) Evaluate:

(i) $f(2)$ $= -(2)^2 + 4(2) + 5$

$= 9$ (1 mark)

(ii) $f'(2)$ $f'(x) = -2x + 4$

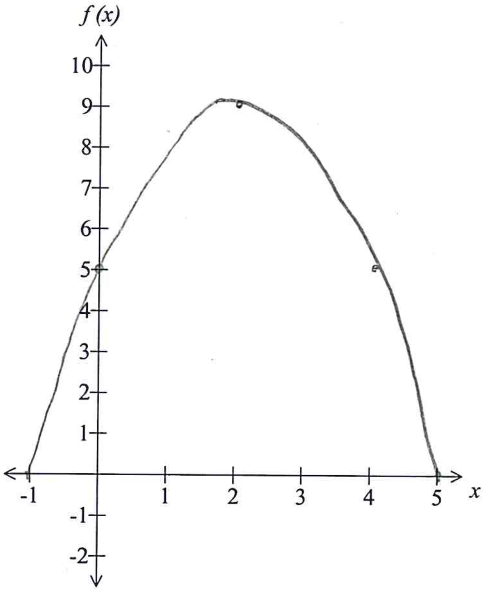
$f'(2) = -2(2) + 4$

$= 0$ (1 mark)

(iii) $f''(2)$ $f''(x) = -2$

$f''(2) = -2$ (1 mark)

(b) (i) Sketch the graph of $f(x)$ over the domain $-1 \le x \le 5$ on the axes provided. (1 mark)



(ii) With reference to your sketch, explain the significance of each answer from part (a). (3 marks)

- Maximum value (turning point) at (2, 9)
- Concave down

Stationary point ✓

max ✓

Value ✓

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Question 5 **(4 marks)**

A spherical balloon is being inflated by pumping gas into it at a rate of 5 m³/minute. Determine the rate at which the diameter is increasing when the radius is 1 m.

$V = \frac{4}{3}\pi r^3$ $\quad \frac{dV}{dr} = 4\pi r^2$ $\quad \frac{dV}{dt} = 5$ ✓

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ $\quad \therefore \frac{dr}{dt} = \frac{5}{4\pi r^2}$ ✓ $\quad$ When $r = 1$ $\quad \frac{dr}{dt} = \frac{5}{4\pi}$ ✓

$\therefore$ diameter $= \frac{10}{4\pi}$ ✓

Question 6 **(4 marks)**

A population, y, increases according to the differential equation:

$$\frac{dy}{dt} = 0.04\,y \quad \text{where } t \text{ is the time, in years, after the start of 2000}$$

The population at the start of 2000 has size 1 000.

(a) State the equation for population, y, in terms of t. (1 mark)

$y = 1000e^{0.04t}$

(b) State the population size when $t = 5$. (1 mark)

1221

(c) Determine the doubling time for the population. (2 marks)

$2 = e^{0.04t}$

$t = 17.33$ years.

Question 7 **(11 marks)**

The function $h(x) = \frac{2\pi}{3\sqrt{x}}(3x+2)$ has a global minimum value over the domain $0 < x \le 2$.

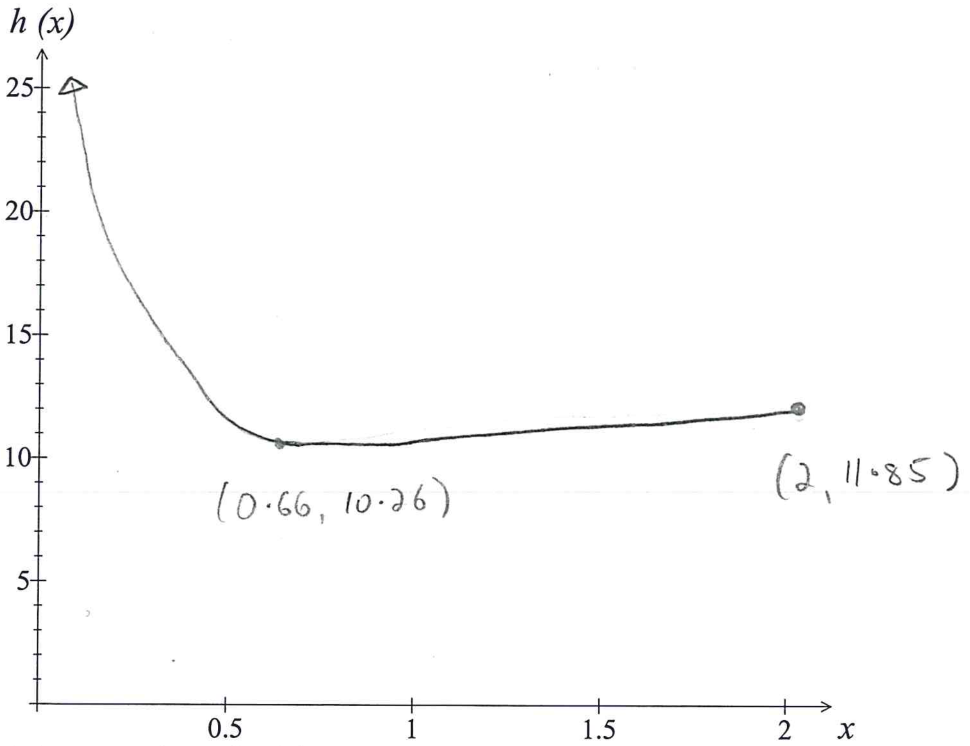
(a) Complete the following table. (2 marks)

x	$h(x) = \frac{2\pi}{3\sqrt{x}}(3x+2)$
0.5	10.367
0.6	10.275
0.7	10.263
0.8	10.303

(b) Explain why the table shows that a local minimum exists in the domain. (1 mark)

Decreases 0.5 → 0.7 but increases 0.7 → 0.8

(c) Sketch the graph of $y = h(x)$ over the domain given on the axes provided, labelling clearly any important points. (2 marks)



(d) State the minimum value of $h(x)$, correct to three decimal places, over the stated domain. (1 mark)

10.260.

Question 7 continued

(e) Use your calculator to determine the derivative $h'(x)$, giving your answer in fractional form, and with positive indices. (1 mark)

$$h'(x) = \frac{3\pi x^{\frac{3}{2}} - 2\pi\sqrt{x}}{3x^2}$$

(f) Explain how to use the derivative $h'(x)$ and your calculator to determine any stationary point of $h(x)$. State the x value of any stationary point. (2 marks)

when $h'(x) = 0$

$x = \frac{2}{3}.$

(g) Use the "Sign Test", by completing the following table, to prove that the stationary point found in (f) is a local minimum. (1 mark)

x	$x < \frac{2}{3}$	$x = \frac{2}{3}$	$x > \frac{2}{3}$
$h'(x)$	$-ve$	0	$+ve$.

(h) State the nature of concavity of $h(x)$. (1 mark)

Concave up.

Question 8 **(8 marks)**

A storage container of volume 36π cm^3 is to be made in the form of a right circular cylinder with one end open. The material for the circular end costs 12c per square centimetre and for the curved side costs 9c per square centimetre.

(a) Show that the cost of materials for the container is $12\pi r^2 + \dfrac{648\pi}{r}$ cents, where r is the radius of the cylinder. (4 marks)

$V = \pi r^2 h \qquad h = \dfrac{V}{\pi r^2} \checkmark \qquad h = \dfrac{36\pi}{\pi r^2} = \dfrac{36}{r^2} \checkmark$

$$\text{S.A cylinder} = \pi r^2 + 2\pi r h$$

$$\text{Cost} = 12(\pi r^2) + 9(2\pi r h) \checkmark$$

$$= 12(\pi r^2) + 18\pi r\left(\frac{36}{r^2}\right)$$

$$= 12\pi r^2 + \frac{648\pi}{r} \checkmark$$

(b) Use calculus techniques to determine the dimensions of the container that minimise its material costs and state this minimum cost. (4 marks)

$$C'(r) = \frac{24\pi r^3 - 648\pi}{r^2} \checkmark$$

$C'(r) = 0$ when $r = 3$ cm. $\checkmark$

$C(3) = 324\pi$ cents $= \$10.18$ $\checkmark$

$h = \dfrac{36}{3^2} = 4$ cm. $\checkmark$

Minimum cost 324π cents when $r = 3$ cm and height $= 4$ cm.