Mathematics Methods Unit 3 2019
Investigation 2: Applications of Integration

This investigation comprises three parts.

Part A is an exercise from a previous Sadler publication which deals with an aspect of integration hitherto unexplored: centres of gravity. This exercise will be started in class on Friday 12 April.

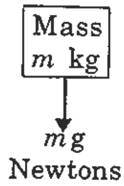
Part B is the Take Home Section of this assignment. Students will complete the exercise in time for a review of the questions on Tuesday 30 April. No marks will be awarded for the Take Home section of this investigation.

Part C is the validation section of the report and will take place in class on Friday 3 May and will present a real life context to an application of integration for students to comment on and provide solutions under test conditions. This mark will constitute 100% of the total investigation mark. Notes will not be allowed in this section, however calculators will be allowed.

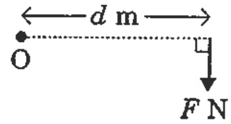
Extension: Centres of gravity

Necessary background information:

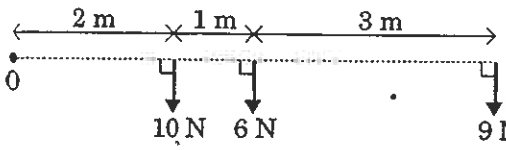
- According to Newton's second law of motion
  Force = mass × acceleration.
  Thus the weight of a body of mass m kg is mg N (Newtons) where $g = 9{\cdot}8$ m/s^2, the acceleration due to gravity.

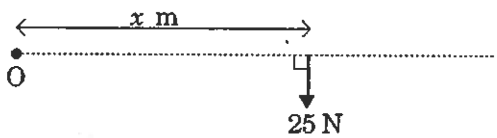


- A force F N produces a turning effect about some point O equal to Fd Nm (Newton metres) where d metres is the perpendicular distance from the line of action of the force to the point O.
  Thus for the diagram on the right, F has a clockwise turning effect, or *moment*, of Fd Nm about O.



- To find the centre of gravity of a system of forces, i.e. to find the point where a single force would act to exactly replace the system of forces, we use the fact that
  the sum of the moments = the moment of the sum
  For example:
  Suppose the single force below right is equivalent to the system below left.





For the turning effect to be the same, taking moments about O:

$$10 \times 2 + 6 \times 3 + 9 \times 6 = 25 \times x$$
$$20 + 18 + 54 = 25x$$
$$\therefore \quad x = 3{\cdot}68$$

We can use these ideas, together with our ability to determine definite integrals, to locate the centre of gravity of various shapes.

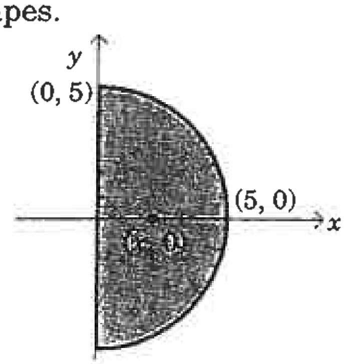
Example

Consider a flat semicircular lamina (see note below) of uniform mass per unit area and radius 5 units.

Such a shape is shown on the right with its axis of symmetry along the x-axis.

By symmetry the centre of gravity of such a shape must lie somewhere on the x-axis.

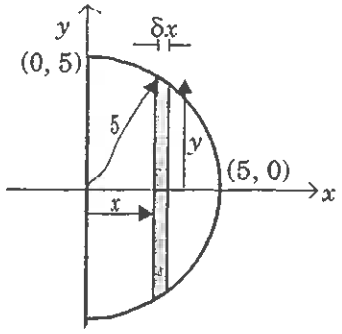
If this centre of gravity is at $(\bar{x}, 0)$ determine $\bar{x}$.

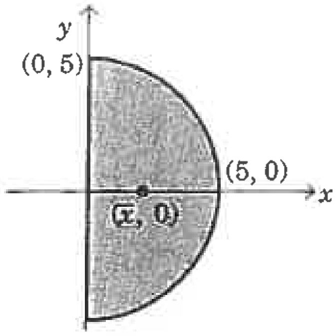


Note: A *lamina* is a flat body of negligible thickness.

Extension activity continued ⇨

Consider this shape both as a complete semicircle with centre of gravity $(\bar{x}, 0)$, as shown below right, and divided into strips of width δx, as shown below left.





If the mass per unit area is m then

mass of strip $\approx 2ym\delta x$	mass of semicircle $= 0{\cdot}5\pi 5^2 m$
weight of strip $\approx 2ymg\delta x$	weight of semicircle $= 12{\cdot}5\pi mg$
moment about y-axis $\approx 2xymg\delta x$	moment about y-axis $= 12{\cdot}5\pi mg\bar{x}$

Now: the sum of the moments = the moment of the sum

Hence $$\lim_{\delta x \to 0} \sum_{x=0}^{x=5} 2xymg\delta x = 12{\cdot}5\pi mg\bar{x}$$

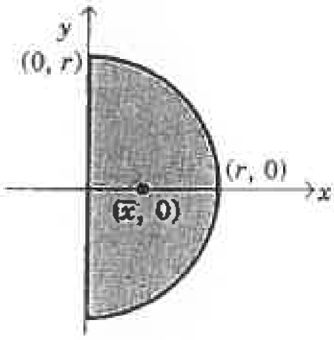
$$\therefore \quad \int_0^5 2xymg\,dx = 12{\cdot}5\pi mg\bar{x}$$

$$\int_0^5 2x\sqrt{5^2 - x^2}\,dx = 12{\cdot}5\pi\bar{x}$$

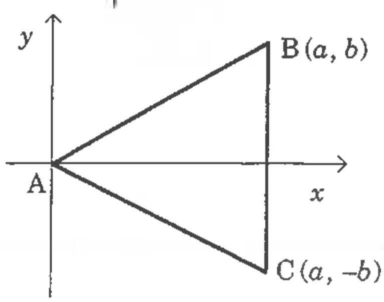
Which gives $$\bar{x} = \frac{20}{3\pi}$$

Exercise.

1. A semicircular lamina of radius r has constant mass per unit area. With the lamina positioned as shown in the diagram, use calculus to prove that the centre of gravity is at the point $\left(\frac{4r}{3\pi}, 0\right)$.

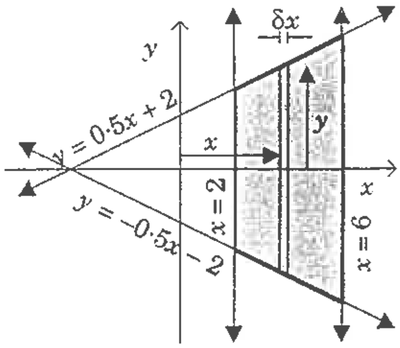


2. The diagram on the right shows a uniform isosceles triangular lamina ABC with A, B and C having coordinates (0, 0), (a, b) and $(a, -b)$ respectively. Use calculus to prove that the centre of gravity of this triangle will lie at the point $\left(\frac{2a}{3}, 0\right)$.



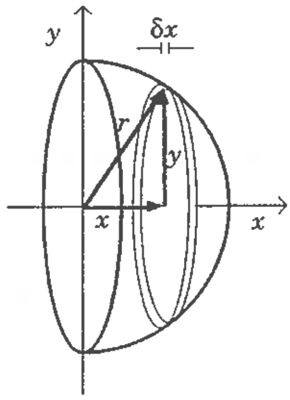
Extension activity continued ⇨

3. A lamina with uniform mass per unit area has the shape shown shaded in the diagram on the right. By considering the lamina positioned as shown in the diagram, and divided up into vertical strips of thickness δx, use calculus to prove that the centre of gravity of the lamina is at the point $\left(\frac{25}{6}, 0\right)$.

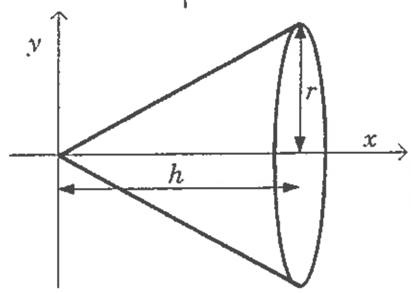


4. Use calculus to prove that the coordinates of the centre of gravity of the uniform lamina bounded by $y = kx^2$ and $y = p$, for positive constants k and p, has its centre of gravity at the point $\left(0, \frac{3p}{5}\right)$.

5. By considering a solid hemisphere of constant mass per unit volume to consist of a large number of circular discs of thickness δx use calculus to prove that with the hemisphere of radius r and positioned as shown on the right the centre of gravity of the hemisphere has coordinates $\left(\frac{3r}{8}, 0\right)$.



6. A solid right circular cone of constant mass per unit volume has base radius r and height h. With the cone positioned as shown on the right use calculus to prove that the centre of gravity of the cone has coordinates $\left(\frac{3h}{4}, 0\right)$.



7. (Challenge)
   (a) Use calculus to prove that the centre of gravity of the uniform lamina enclosed by the x-axis, the y-axis, the curve $y = x^2 + 2$ and the straight line $x = 4$ has coordinates $\left(\frac{30}{11}, \frac{287}{55}\right)$.

   Hint: The centre of gravity of a vertical strip of height y and width δx will be $0{\cdot}5y$ from x-axis.

   (b) Use calculus to prove that the centre of gravity of the solid formed by rotating the area of part (a) through one revolution about the x-axis has coordinates $\left(\frac{130}{41}, 0\right)$.