MATHEMATICS:SPECIALIST UNIT 3

TEST 2 2016

Calculator Free

Name: MARKING KEY

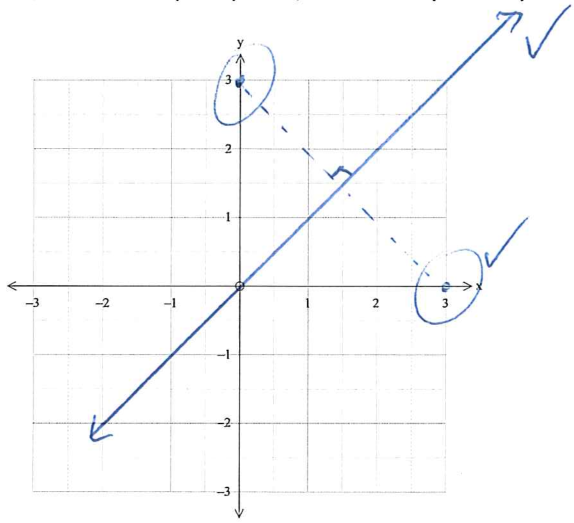
Reading Time: 2 minutes

Total Marks: 40 marks

Time Allowed: 40 minutes

Question 1. (2 marks)

Sketch the set of points z, in the complex plane, that satisfy the equation $|z-3i| = |z-3|$

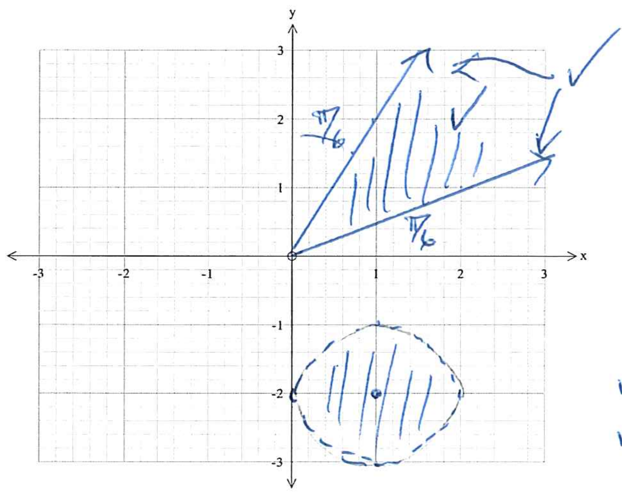


Question 2. (4 marks; 2,2)

Sketch, on the same set of axes below, the region in the complex plane defined by

a) $|z-1+2i| < 1$

b) $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$



✓ circle correct location

✓ shading (must be dotted line and shaded inside)

Question 3. **(7 marks: 1, 1, 1, 1, 1, 2)**

Given that $z = 3\ cis\ \frac{\pi}{3}$ and $w = 2\ cis\left(-\frac{\pi}{4}\right)$ determine using cis notation:

a) $z.w$ $= 6\ cis\ \frac{\pi}{2}$ ✓

b) $\frac{z}{w}$ $= \frac{3}{2}\ cis\ \frac{7\pi}{12}$ ✓

c) $i.w$ $cis\frac{\pi}{2} . 2cis\left(\frac{-\pi}{4}\right)$

$= 2 cis \frac{\pi}{4}$ ✓

d) $\overline{w}$ $= 2\ cis\ \frac{\pi}{4}$ ✓

e) $\frac{i}{z}$ $= \dfrac{cis\frac{\pi}{2}}{3cis\frac{\pi}{3}} = \frac{1}{3} cis \frac{\pi}{6}$ ✓

f) Determine $Re\ (z + w)$ exactly.

$Re\ (z+w) = 3\cos\frac{\pi}{3} + 2\cos\left(\frac{-\pi}{4}\right)$ ✓

$= \frac{3}{2} + \sqrt{2}$ ✓

Question 4. **(11 marks: 3, 2, 4, 2)**

a) Express $\dfrac{2-6\sqrt{3}\,i}{1-\sqrt{3}\,i}$ in the form a + bi where a and b are real numbers.

$\dfrac{(2-6\sqrt{3}i)}{(1-\sqrt{3}i)} \dfrac{(1+\sqrt{3}i)}{1+\sqrt{3}i}$ ✓ $= \dfrac{2+2\sqrt{3}i-6\sqrt{3}i+18}{1+3}$

-1 per error.

$= \dfrac{20-4\sqrt{3}i}{4}$

$= 5 - \sqrt{3}i$ ✓✓

b) If (a + 5i) (2 - i) = b where a and b are real numbers, determine a and b

$2a - ai + 10i + 5 = b$

$2a + 5 = b$ $10 - a = 0$

$b = 25$ ✓ $a = 10$ ✓

Question 4.

c) Given z = $2+2\sqrt{3}\,i$ and w = $-\sqrt{8}-\sqrt{8}\,i$

i) express z and w in polar form.

$z = 4\,\text{cis}\,\frac{\pi}{3}$ ✓ ✓ $\quad W = 4\,\text{cis}\,\frac{-3\pi}{4}$ ✓ ✓

ii) Find $\sqrt{z}$ in cartesian form.

$2\,\text{cis}\,\frac{\pi}{6}$ ✓

$= \sqrt{3} + i$ ✓

Question 5. **(5 marks)**

The function f(x) = $x^3 + ax^2 + bx - 2$ has (x - 2) as a factor but a remainder of -6 is left when f(x) is divided by (x + 1). Find a and b.

$f(2) = 8 + 4a + 2b - 2 = 0$ ✓

$f(-1) = -1 + a - b - 2 = -6$ ✓

$6 + 4a + 2b = 0$

$a = b - 3$ ✓

$6 + 4b - 12 + 2b = 0$

$6b = 6$

$b = 1$ ✓

$a = -2$ ✓

Question 6. **(6 marks)**

Find all the real and complex roots of $x^3 + 3x^2 + 3x + 2 = 0$.

$P(-2) = -8 + 12 - 6 + 2 = 0$ ✓

$x^3 + 3x^2 + 3x + 2 = (x+2)(x^2 + x + 1)$ ✓

$x = \frac{-1 \pm \sqrt{1-4.1}}{2}$

$= \frac{-1 \pm \sqrt{3}}{2}$ ✓

$\therefore$ roots $-2,\ \frac{-1+\sqrt{3}i}{2},\ \frac{-1-\sqrt{3}i}{2}$ ✓✓

Question 7. **(5 marks)**

Evaluate $(1 + i)^7 + (1 - i)^7$ using de Moivre's rule.

$$\left(\sqrt{2}\operatorname{cis}\frac{\pi}{4}\right)^7 + \left(\sqrt{2}\operatorname{cis}\left(\frac{-\pi}{4}\right)\right)^7$$

$$= 8\sqrt{2}\operatorname{cis}\left(\frac{7\pi}{4}\right) + 8\sqrt{2}\operatorname{cis}\left(\frac{-7\pi}{4}\right)$$

$$= 8\sqrt{2}\operatorname{cis}\left(\frac{-\pi}{4}\right) + 8\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$$

$$= 8\sqrt{2}\left(\cos\left(\frac{-\pi}{4}\right) + i\cancel{\sin\left(\frac{-\pi}{4}\right)} + \cos\frac{\pi}{4} + i\cancel{\sin\left(\frac{\pi}{4}\right)}\right)$$

$$= 8\sqrt{2}\left(2\cos\frac{\pi}{4}\right)$$

$$= 8\cancel{\sqrt{2}}.2.\frac{1}{\cancel{\sqrt{2}}}$$

$$= \underline{16}$$

MATHEMATICS:SPECIALIST UNIT 3

TEST 2 2016

Calculator Assumed

Eastern Goldfields College

Name: ______________________________

Reading Time: 2 minutes

Total Marks: ~~2~~6 26 marks

Time Allowed: 28 minutes

Question 8 **(5 marks; 1,4)**

One cube root of z is $\frac{5\sqrt{2}}{2}+\frac{5\sqrt{2}}{2}i$

a) Write z in exact Cartesian form.

$z = -\frac{125\sqrt{2}}{2} + \frac{125\sqrt{2}}{2}i$ ✓

(Doesn't matter if they don't rationalise denominator)

b) State the other two cube roots of z in exact polar form.

$z_1 = \frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i$

$= 5\,cis\,\frac{\pi}{4}$ ✓

roots $\frac{2\pi}{3}$ apart. ✓

$z_2 = 5\,cis\,\frac{11\pi}{12}$ ✓

$z_3 = 5\,cis\left(\frac{-5\pi}{12}\right)$ ✓

Question 9 (10 marks; 5, 3, 2)

a) Determine all the roots of the equation $z^6 = \sqrt{3} + i$, expressing them in exact polar form $r\text{cis}\theta$ where $r \geq 0$ and $-\pi < \theta \leq \pi$.

$z^6 = 2\,\text{cis}\frac{\pi}{6}$ ✓

$z_1 = \left(2\,\text{cis}\frac{\pi}{6}\right)^{\frac{1}{6}}$

$= 2^{\frac{1}{6}}\text{cis}\frac{\pi}{36}$ ✓

$z_2 = 2^{\frac{1}{6}}\text{cis}\frac{13\pi}{36}$

$z_3 = 2^{\frac{1}{6}}\text{cis}\frac{25\pi}{36}$

$z_4 = 2^{\frac{1}{6}}\text{cis}\frac{-11\pi}{36}$

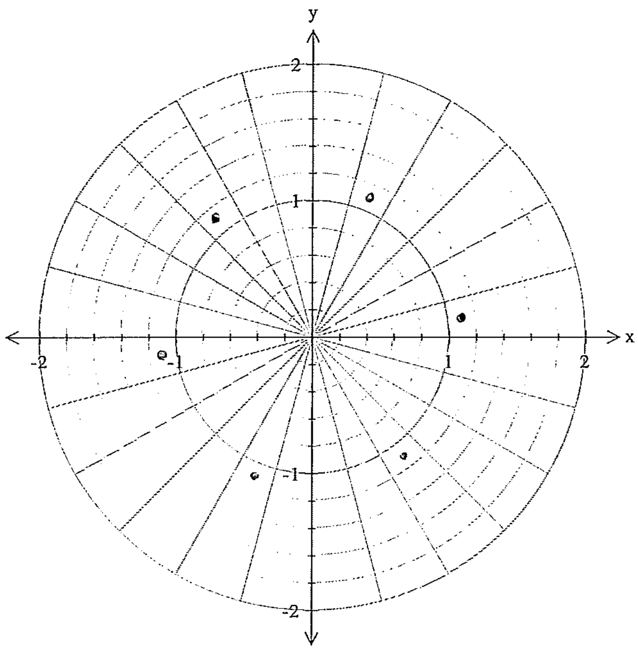
$z_5 = 2^{1/6}\text{cis}\left(\frac{-23\pi}{36}\right)$

$z_6 = 2^{1/6}\text{cis}\left(\frac{-35\pi}{36}\right)$

roots $\frac{\pi}{3}$ apart ✓

✓✓

b) Sketch all the roots from (a) on the diagram below.



≈ 1.12 ✓
first root ✓
spaces ✓
roots
$\frac{\pi}{3}$ apart

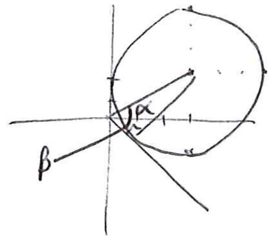
c) The roots form the vertices of a hexagon. Determine the exact value for the perimeter of the hexagon.

Triangles equil. ⇒ sides $2^{1/6}$ ✓

perimeter $2^{1/6} \times 6$. units. ✓

Question 9: (6 marks; 2, 4]

For $\{z : |z-3-2i| = 3\}$ determine:



a) The maximum and minimum value of $|z|$

$\max |z| = \sqrt{13} + 3$ ✓

$\min |z| = \sqrt{13} - 3$ ✓

b) the maximum and minimum values of arg z.

$\max \arg = \frac{\pi}{2}$ ✓

$\tan \alpha = \frac{2}{3}$

$\alpha = 0.5880$ ✓

$\sin \beta = \frac{3}{\sqrt{13}}$

$\beta = 0.9828$ ✓

$\alpha - \beta = -0.3948$

$\therefore \min \arg = -0.3948$ ✓

Question 11. **(5 marks)**

Use the identity $2i \sin n\theta = z^n - \frac{1}{z^n}$ to prove that $\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$.

$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ ✓

$LHS = \sin^3\theta = \left(\frac{1}{2i}\right)^3\left(z - \frac{1}{z}\right)^3 = -\frac{1}{8i}\left(z^3 + 3z^2\left(-\frac{1}{z}\right) + 3z\left(-\frac{1}{z}\right)^2 + \left(-\frac{1}{z}\right)^3\right)$ ✓

$= -\frac{1}{8i}\left(z_3 - \frac{1}{z_3} - 3z - \frac{1}{z^3}\right)$ ✓

$= -\frac{1}{8i}\left(2i\sin 3\theta - 3(2i\sin\theta)\right)$ ✓

$= -\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta$ ✓

$= \frac{3\sin\theta - \sin 3\theta}{4}$

$= RHS.$

ok similar