CAMBRIDGE

GENERAL MATHEMATICS

VCE UNITS 3 & 4

CAMBRIDGE SENIOR MATHEMATICS **VCE**

SECOND EDITION

PETER JONES | KAY LIPSON | MICHAEL EVANS
ROSE HUMBERSTONE | PETER KARAKOUSSIS | KYLE STAGGARD

INCLUDES INTERACTIVE TEXTBOOK POWERED BY CAMBRIDGE HOTMATHS

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Contents

Online appendices accessed through the Interactive Textbook or PDF Textbook

Introduction and overview

Cambridge General Mathematics VCE Units 3&4 Second Edition provides a complete teaching and learning resource for the VCE Study Design **to be first implemented in 2023**. It has been written with understanding as its chief aim, and with ample practice offered through the worked examples and exercises. The work has been trialled in the classroom, and the approaches offered are based on classroom experience and the responses of teachers to earlier editions of this book and the requirements of the new Study Design.

The textbook and its accompanying digital resources provide comprehensive coverage of the assumed knowledge and skills required.

General Mathematics Units 3 and 4 provide the following four Areas of Study:

- Data analysis
- Recursion and financial modelling
- Matrices and their applications
- Networks and decision mathematics

Separate **revision chapters** containing Multiple-choice (Exam 1-style questions) and Written-response (Exam 2-style questions) are included for each study area. Additional Multiple-choice and Written-response questions, **covering the year's course**, are included in the final revision chapter.

Other key features of *Cambridge General Mathematics VCE Units 3&2 Second Edition*
For each topic within a chapter:

- The topic starts with a clear outline of its **Learning intentions**
- Exam 1-style questions are included at the end of every exercise.

The **Review** section at the end of each chapter includes:

- **Key ideas and a chapter summary**
- A **Skills checklist**, which provides a question to check mastery of each skill listed in the **Learning intentions**
- **Multiple-choice** and **written-response** questions review the skills and concepts of the chapter.

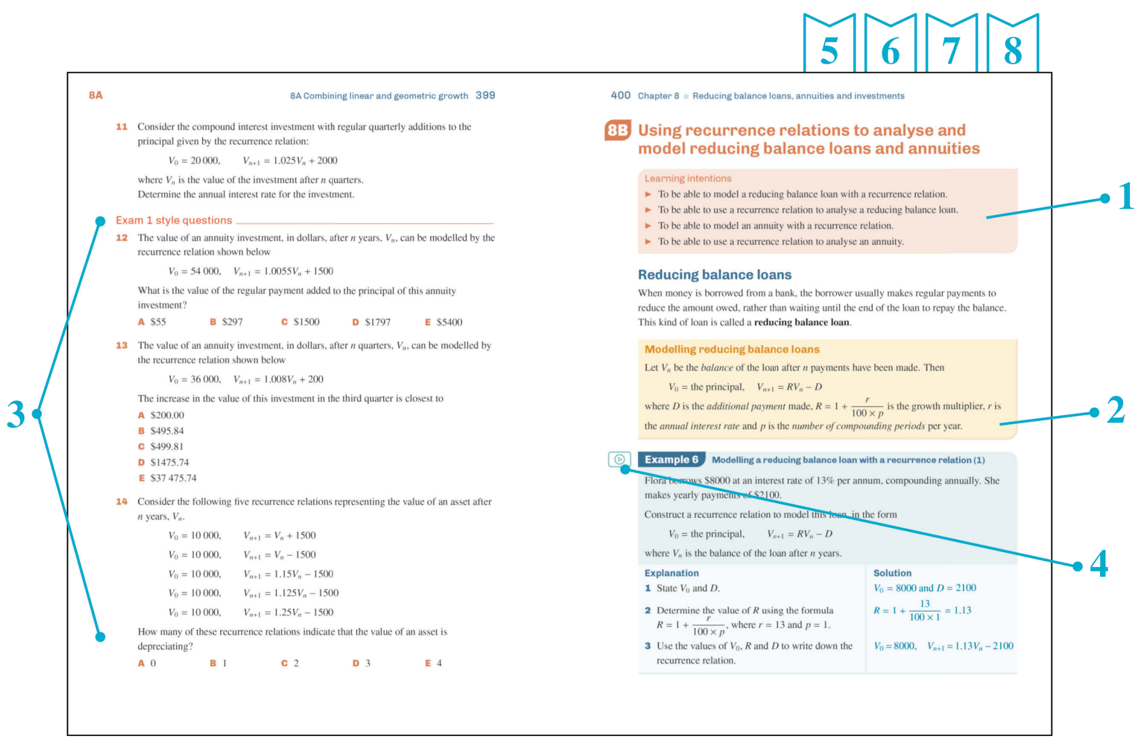
The textbook is supported by an extensive range of online teacher and student resources, as outlined in the 'How to use this resource' section on the following pages.

The TI-Nspire calculator examples and instructions have been completed by Peter Flynn, and those for the Casio ClassPad by Mark Jelinek, and we thank them for their helpful contributions.

Overview of the print book

1. Learning Intentions added at the beginning of each section to clearly outline the goals of the lesson.
2. Graded step-by-step worked examples with precise explanations (and video versions) encourage independent learning, and are linked to exercise questions.
3. Exam 1-style questions added to every exercise.
4. Additional linked resources in the Interactive Textbook are indicated by icons, such as skillsheets and video versions of examples.
5. End-of-chapter content includes a Skills Checklist and a summary of the key terminology.
6. Chapter reviews contain multiple-choice and written-response questions.
7. Revision chapters include Exam 1-style and Exam 2-style questions, providing comprehensive revision and preparation for assessment.
8. The glossary includes page numbers of the main explanation of each term.

Numbers refer to descriptions above.



8A   8A Combining linear and geometric growth 399

11 Consider the compound interest investment with regular quarterly additions to the principal given by the recurrence relation:

$$V_0 = 20\,000, \qquad V_{n+1} = 1.025V_n + 2000$$

where V_n is the value of the investment after n quarters.
Determine the annual interest rate for the investment.

Exam 1 style questions

12 The value of an annuity investment, in dollars, after n years, V_n, can be modelled by the recurrence relation shown below

$$V_0 = 54\,000, \quad V_{n+1} = 1.0055V_n + 1500$$

What is the value of the regular payment added to the principal of this annuity investment?

A \$55 **B** \$297 **C** \$1500 **D** \$1797 **E** \$5400

13 The value of an annuity investment, in dollars, after n quarters, V_n, can be modelled by the recurrence relation shown below

$$V_0 = 36\,000, \quad V_{n+1} = 1.008V_n + 200$$

The increase in the value of this investment in the third quarter is closest to

A \$200.00
B \$495.84
C \$499.81
D \$1475.74
E \$37 475.74

14 Consider the following five recurrence relations representing the value of an asset after n years, V_n.

$$V_0 = 10\,000, \qquad V_{n+1} = V_n + 1500$$
$$V_0 = 10\,000, \qquad V_{n+1} = V_n - 1500$$
$$V_0 = 10\,000, \qquad V_{n+1} = 1.15V_n - 1500$$
$$V_0 = 10\,000, \qquad V_{n+1} = 1.125V_n - 1500$$
$$V_0 = 10\,000, \qquad V_{n+1} = 1.25V_n - 1500$$

How many of these recurrence relations indicate that the value of an asset is depreciating?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

400 Chapter 8 ■ Reducing balance loans, annuities and investments

8B Using recurrence relations to analyse and model reducing balance loans and annuities

Learning intentions
- To be able to model a reducing balance loan with a recurrence relation.
- To be able to use a recurrence relation to analyse a reducing balance loan.
- To be able to model an annuity with a recurrence relation.
- To be able to use a recurrence relation to analyse an annuity.

Reducing balance loans

When money is borrowed from a bank, the borrower usually makes regular payments to reduce the amount owed, rather than waiting until the end of the loan to repay the balance. This kind of loan is called a **reducing balance loan**.

Modelling reducing balance loans

Let V_n be the *balance* of the loan after n payments have been made. Then

$$V_0 = \text{the principal}, \quad V_{n+1} = RV_n - D$$

where D is the *additional payment* made, $R = 1 + \frac{r}{100 \times p}$ is the growth multiplier, r is the *annual interest rate* and p is the *number of compounding periods* per year.

Example 6 Modelling a reducing balance loan with a recurrence relation (1)

Flora borrows \$8000 at an interest rate of 13% per annum, compounding annually. She makes yearly payments of \$2100.

Construct a recurrence relation to model this loan, in the form

$$V_0 = \text{the principal}, \qquad V_{n+1} = RV_n - D$$

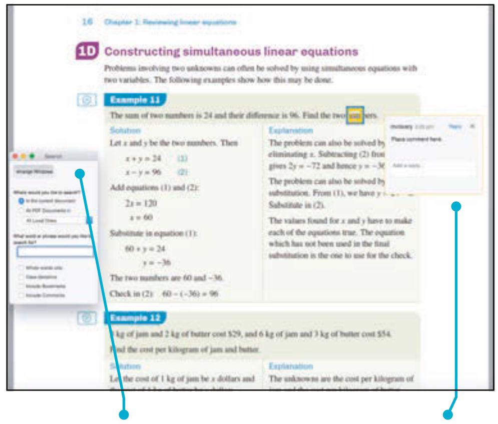
where V_n is the balance of the loan after n years.

Explanation	Solution
1 State V_0 and D.	$V_0 = 8000$ and $D = 2100$
2 Determine the value of R using the formula $R = 1 + \frac{r}{100 \times p}$, where $r = 13$ and $p = 1$.	$R = 1 + \frac{13}{100 \times 1} = 1.13$
3 Use the values of V_0, R and D to write down the recurrence relation.	$V_0 = 8000, \quad V_{n+1} = 1.13V_n - 2100$

Overview of the downloadable PDF textbook

9 The convenience of a downloadable PDF textbook has been retained for times when users cannot go online.

10 PDF annotation and search features are enabled.



10 10

Overview of the Interactive Textbook

The **Interactive Textbook (ITB)** is an online HTML version of the print textbook powered by the HOTmaths platform, included with the print book or available as a separate purchase.

11 The material is formatted for on screen use with a convenient and easy-to-use navigation system and links to all resources.

12 **Workspaces** for all questions, which can be enabled or disabled by the teacher, allow students to enter working and answers online and to save them. Input is by typing, with the help of a symbol palette, handwriting and drawing on tablets, or by uploading images of writing or drawing done on paper.

13 **Self-assessment tools** enable students to check answers, mark their own work, and rate their confidence level in their work. This helps develop responsibility for learning and communicates progress and performance to the teacher. Student accounts can be linked to the learning management system used by the teacher in the Online Teaching Suite, so that teachers can review student self-assessment and provide feedback or adjust marks.

14 All worked examples have **video versions** to encourage independent learning.

15 **Worked solutions** are included and can be enabled or disabled in the student ITB accounts by the teacher.

16 An expanded and revised set of **Desmos interactives** and activities based on embedded graphics calculator and geometry tool windows demonstrate key concepts and enable students to visualise the mathematics.

17 The **Desmos graphics calculator, scientific calculator**, and **geometry tool** are also embedded for students to use for their own calculations and exploration.

18 **Revision of prior knowledge** is provided with links to diagnostic tests and Year 10 HOTmaths lessons.

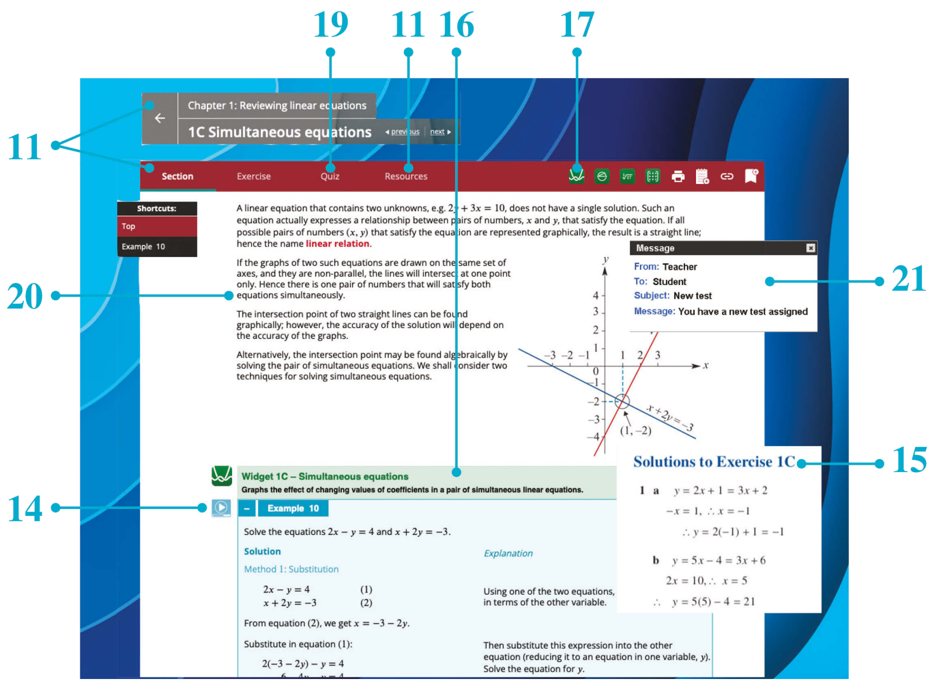
19 **Quick quizzes** containing automarked multiple-choice questions have been thoroughly expanded and revised, enabling students to check their understanding.

20 **Definitions** pop up for key terms in the text, and are also provided in a dictionary.

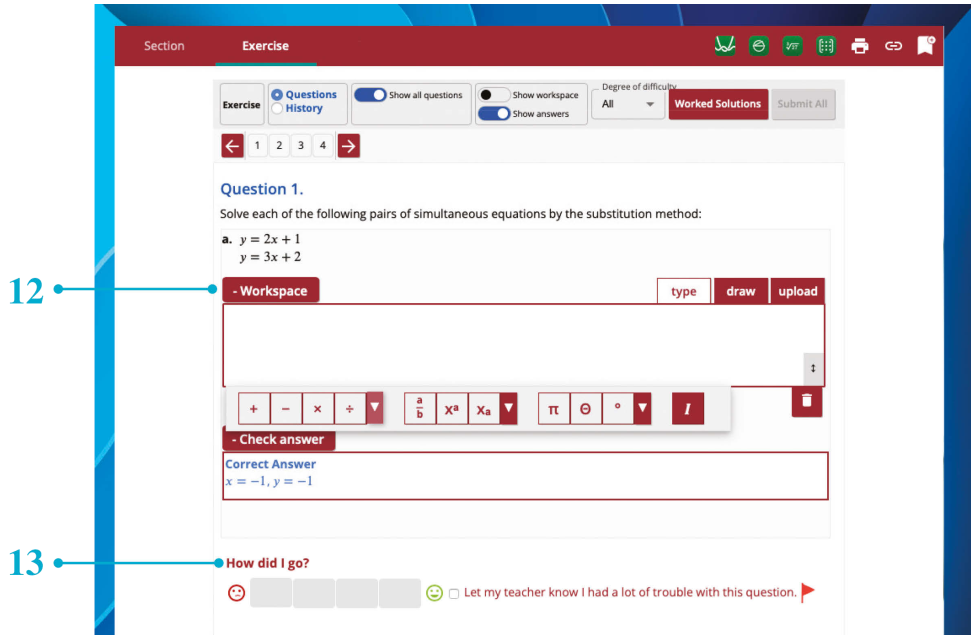
21 Messages from the teacher assign tasks and tests.

INTERACTIVE TEXTBOOK POWERED BY THE HOTmaths PLATFORM

A selection of features is shown. Numbers refer to the descriptions on pages xi–xii. HOTmaths platform features are updated regularly



WORKSPACES AND SELF-ASSESSMENT



Overview of the Online Teaching Suite powered by the HOTmaths platform

The Online Teaching Suite is automatically enabled with a teacher account and is integrated with the teacher's copy of the Interactive Textbook. All the teacher resources are in one place for easy access. The features include:

22 The HOTmaths learning management system with class and student analytics and reports, and communication tools.
23 Teacher's view of a student's working and self-assessment which enables them to modify the student's self-assessed marks, and respond where students flag that they had diffculty.
24 A HOTmaths-style test generator.
25 An expanded and revised suite of chapter tests, assignments and sample investigations.
26 Editable curriculum grids and teaching programs.
27 A brand-new **Exam Generator**, allowing the creation of customised printable and online trial exams (see below for more).

More about the Exam Generator

The Online Teaching Suite includes a comprehensive bank of VCAA exam questions, augmented by exam-style questions written by experts, to allow teachers to create custom trial exams.

Custom exams can model end-of-year exams, or target specific topics or types of questions that students may be having difficulty with.

Features include:

- Filtering by question-type, topic and degree of difficulty
- Searchable by key words
- Answers provided to teachers
- Worked solutions for all questions
- VCAA marking scheme
- Multiple-choice exams can be auto-marked if completed online, with filterable reports
- All custom exams can be printed and completed under exam-like conditions or used as revision.

Acknowledgements

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Chapter 1

Investigating data distributions

Chapter questions

- What is categorical data?
- What is numerical data?
- How do we summarise and display categorical data?
- How do we use the distribution of a categorical variable to answer statistical questions?
- What is a dot plot?
- What is a stem plot?
- What is a histogram?
- What is a boxplot?
- What are summary statistics, and how do we choose which ones to use?
- How do we use the distribution of a numerical variable to answer statistical questions?
- What is the normal distribution?
- What is the 68-95-99.7% rule and why is it useful?
- What are standardised values and why are they useful?

We can think of data as factual information about a person, object or situation which has been collected and recorded. In General Mathematics Units 1&2 we learned a range of statistical procedures to help us analyse such sets of data. In this chapter, we will review and extend our knowledge of those procedures for data which has been collected from a **single variable**, which is called **univariate** data.

1A Types of data

Learning intentions

- ▶ To be able to classify data as categorical or numerical.
- ▶ To be able to further classify categorical data as nominal or ordinal.
- ▶ To be able to further classify numerical data as discrete or continuous.

A group university of students were asked to complete a survey, and the information collected from eight of these students is shown in the following table:

Height (cm)	*Weight (kg)*	*Age (years)*	*Study mode (C on-campus, O online)*	*Fitness level (1 high, 2 medium, 3 low)*	*Pulse rate (beats/min)*
173	57	18	C	2	86
179	58	19	C	2	82
167	62	18	C	1	96
195	84	18	O	1	71
173	64	18	C	3	90
184	74	22	O	3	78
175	60	19	O	3	88
140	50	34	C	3	70

Since the answers to each of the questions in the survey will vary from student to student, each question defines a different **variable** namely:

- *height* (in centimetres)
- *weight* (in kilograms)
- *age* (in years)
- *study mode* (C = on-campus, O = online)
- *fitness level* (1 = high, 2 = medium, 3 = low)
- *pulse rate* (beats/minute).

The values we collect about each of these variables are called **data**.

Types of variables

Variables come in two general types, **categorical** and **numerical**:

Categorical variables

Categorical variables classify or name a quality or attribute– for example, a person's eye colour, study mode, or fitness level.

Data generated by a categorical variable can be used to organise individuals into one of several groups or categories that characterise this quality or attribute.

For example, a 'C' in the *Study mode* column indicates that the student studies on-campus, while a '3' in the *Fitness level* column indicates that their fitness level is low.

Categorical variables can be further classified as one of two types: **nominal** and **ordinal**.

- **Nominal variables**

  Nominal variables have data values that are simply names.

  The variable *Study mode* is an example of a nominal variable because the values of the variable, on-campus or online, simply name the group to which the students belong.

- **Ordinal variables**

  Ordinal variables have data values that can be used to both name and order.

  The variable *Fitness level* is an example of an ordinal variable. The data generated by this variable contains two pieces of information. First, each data value can be used to group the students by fitness level. Second, it allows us to logically order these groups according to their fitness level – in this case, as 'low', 'medium' or 'high'.

Numerical variables

Numerical variables have data values which are quantities, generally arising from counting or measuring.

For example, a '179' in the *Height* column indicates that the person is 179 cm tall, while an '82' in the *Pulse rate* column indicates that they have a pulse rate of 82 beats/minute.

Numerical variables can be further classified as one of two types: **discrete** and **continuous**.

- **Discrete variables**

  Discrete variables are those which may take on only a countable number of distinct values such as 0, 1, 2, 3, 4, . . .

  Discrete random variables often arise when the situation involves counting. The number of mobile phones in a house is an example of a discrete variable.

  As a guide, discrete variables arise when we ask the question 'How many?'

- **Continuous variables**

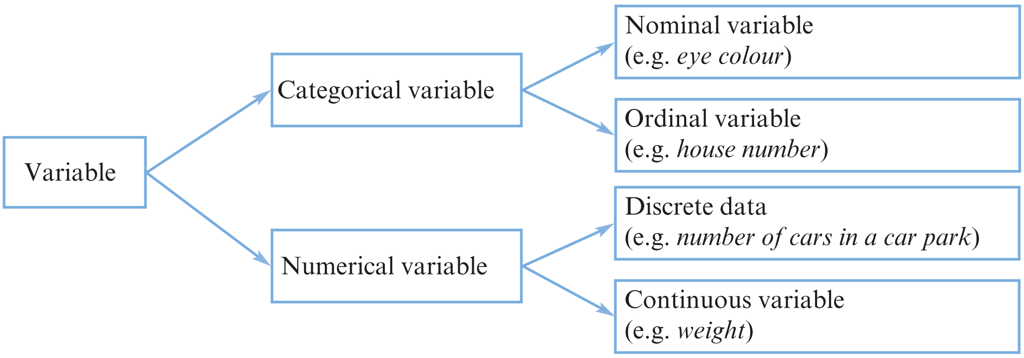
  Continuous variables are ones which take an infinite number of possible values, and are often associated with measuring rather than counting.

  Thus, even though we might record a person's height as 179 cm, their height could be any value between 178.5 and 179.4 cm. We have just rounded to 179 cm for convenience, or to match the accuracy of the measuring device.

As a guide, continuous variables arise when we ask the question ‘How much?’

Comparing numerical and categorical variables

The interrelationship between categorical (nominal and ordinal) and numerical variables (discrete and continuous) is displayed in the diagram below.



Numerical or categorical?

Deciding whether data are numerical or categorical is not an entirely trivial exercise. Two things that can help your decision-making are:

1 Numerical data can always be used to perform arithmetic computations. This is not the case with categorical data. For example, it makes sense to calculate the average weight of a group of individuals, but not the average house number in a street. This is a good test to apply when in doubt.

2 It is not the variable name alone that determines whether data are numerical or categorical; it is also the way the data are recorded. For example, if the data for variable *weight* are recorded in kilograms, they are numerical. However, if the data are recorded as ‘underweight’, ‘normal weight’, ‘overweight’, they are categorical.

Example 1 **Types of data**

Classify the following data as nominal, ordinal, discrete or continuous.

a The number of chocolate chips in each of 10 cookies is counted.

b The time taken for 20 students to complete a puzzle is recorded in seconds.

c Member of a football club were asked to rate how they felt about the current coach, 1= Very satisfied, 2 = Satisfied, 3 = Indifferent, 4 = Dissatisfied, 5 = Very dissatisfied.

d Students are asked to each choose their preferred colour from the list 1= Blue, 2 = Green, 3 = Red, 4 = Yellow.

e Students weights were classified as ‘less than 60kg’, ‘60kg - 80kg’ or ‘more than 80kg’.

Solution

a **Discrete**, as the number of chocolate chips will only take whole number values.

b **Continuous**, as the data can take any value, limited only by the accuracy to which the time can be measured.

c **Ordinal**, as the numbers in this data do not represent quantities, they represent each person's level of approval of the coach.

d **Nominal**, as the numbers in this data again do not represent quantities, they represent colours.

e **Ordinal**, as each student's weight has been recorded into three categories which can be ordered.

Skill-sheet

Exercise 1A

Types of variables: categorical or numerical

Example 1

1 Classify each of the following variables (in *italics*) as categorical or numerical:

a *time* (in minutes) spent exercising each day
b *number* of frogs in a pond
c *bank account numbers*
d *height* (short, average, tall)
e *time* spent playing computer games (hours)
f *number of people* in a bus
g *eye colour* (brown, blue, green)
h *post code*

Categorical variables: nominal or ordinal

2 Classify the categorical variables identified below (in italics) as nominal or ordinal.

a The *colour* of a pencil
b The different *types of animals* in a zoo
c The *floor levels* in a building (0, 1, 2, 3 . . .)
d The *speed* of a car (less than 50 km/hr, 50 to 80 km/hr, above 80 km/hr)
e *Shoe size* (6, 8, 10, . . .)
f Family names

Numerical variables: discrete or continuous

3 Classify the numerical variables identified below (in italics) as discrete or continuous.

a The *number of pages* in a book
b The *cost* (to the nearest dollar) to fill the tank of a car with petrol
c The *volume* of petrol (in litres) used to fill the tank of a car
d The *speed* of a car in km/h
e The *number* of people at a football match
f The air *temperature* in degrees Celsius

Exam 1 style questions

4 Respondents to a survey question "Are you concerned about climate change?" were asked to select from the following responses
1 = not at all, 2 = a little, 3 = somewhat, 4 = extremely
The data which was collected in response to this question is:

A nominal **B** ordinal **C** discrete
D continuous **E** numerical

5 The variables *weight* (light, medium, heavy) and *age* (under 25 years, 25-40 years, over 40 years) are:

A a nominal and an ordinal variable respectively
B an ordinal and a nominal variable respectively
C both nominal variables
D both ordinal variables
E both continuous variables

6 Data relating to the following five variables were collected from a group of university students:

- *course*
- *study mode* (1= on campus, 2 = online)
- *study load* (1= full-time, 2-part-time)
- *number of contact hours per week*
- *number of subject needed to complete degree*

The number of these variables that are discrete numerical variables is:

A 1 **B** 2 **C** 3 **D** 4 **E** 5

1B Displaying and describing the distributions of categorical variables

Learning intentions

- ► To be able to construct frequency and percentage frequency tables for categorical data.
- ► To be able to construct bar charts and percentage bar charts from frequency tables.
- ► To be able to interpret and describe frequency tables and bar charts.
- ► To be able to answer statistical questions about a categorical variable.

The frequency table

With a large number of data values, it is difficult to identify any patterns or trends in the raw data.

For example, the following set of categorical data, listing the study mode (C = on-campus, O = online) for 60 individuals, is hard to make sense of.

O O C O O O C O C C C O C O O O C C C O
C O C O C C C O C O C O C O O O C O C O
C O O O O O O C C O C O O O C O C C C O

To help make sense of the data, we first need to organise them into a **frequency table**.

The frequency table

A frequency table is a listing of the values a variable takes in a data set, along with how often (frequently) each value occurs.

Frequency can be recorded as a:

- number: the number of times a value occurs, or
- percentage: the percentage of times a value occurs (**percentage frequency**):

$$\text{percentage frequency} = \frac{\text{number of times a value occurs}}{\text{total number of values}} \times 100\%$$

Example 2 Frequency table for a categorical variable

A group of 11 preschool children were asked to choose between chocolate and vanilla ice-cream (C = chocolate, V = vanilla):

C V V C C V C C C V V

Construct a frequency table (including percentage frequencies) to display the data.

Explanation

1. Set up a table as shown.
2. Count up the number of chocolate (6) and vanilla (5), and record in the Number column.
3. Add to find the total number, 11 (6 + 5).
4. Convert the frequencies into percentage frequencies. e.g. percentage chocolate= $\frac{6}{11} \times 100\% = 54.5\%$
5. Finally, total the percentages and record.

Solution

	Frequency	
flavour	Number	Percentage
chocolate	6	54.5
vanilla	5	45.5
Total	11	100.0

Note that the total should always equal the total number of observations – in this case, 11, and that the percentages should add to 100%. However, if percentages are rounded to one decimal place a total of 99.9 or 100.1 is sometimes obtained. This is due to rounding error. Totalling the count and percentages helps check on your tallying and percentaging.

The bar chart

Once categorical data have been organised into a frequency table, it is common practice to display the information graphically to help identify any features that stand out in the data.

The statistical graph we use for this purpose is the **bar chart**.

The bar chart represents the key information in a frequency table as a picture. The bar chart is specifically designed to display categorical data.

In a bar chart:

- frequency (or percentage frequency) is shown on the vertical axis
- the variable being displayed is plotted on the horizontal axis
- the height of the bar (column) gives the frequency (number or percentage)
- the bars are drawn with gaps to show that each value is a separate category
- there is one bar for each category.

Example 3 Constructing a bar chart from a frequency table

The *climate type* of 23 countries is classified as 'cold', 'mild' or 'hot'. The results are summarised in the table opposite.

a What is the level of measurement of the variable *climate type*?

b Construct a frequency bar chart to display this information.

	Frequency	
Climate type	Number	Percentage
Cold	3	13.0
Mild	14	60.9
Hot	6	26.1
Total	23	100.0

Explanation

a The data enable us to both group the countries by *climate type* and put these groups in some sort of natural order according to the 'warmth' of the different climate types. The variable is ordinal.

b **1** Label the horizontal axis with the variable name, *Climate type*. Mark the scale off into three equal intervals and label them 'Cold', 'Mild' and 'Hot'.

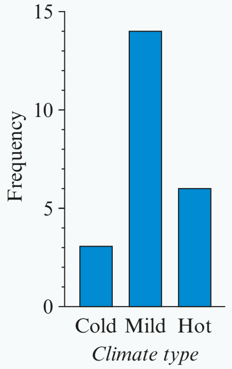
2 Label the vertical axis 'Frequency'. Scale allowing for the maximum frequency, 14.

3 For each climate type, draw a bar. There are gaps between the bars to show that the categories are separate. The height of the bar is made equal to the frequency (given in the 'Number' column).

Solution

a Ordinal

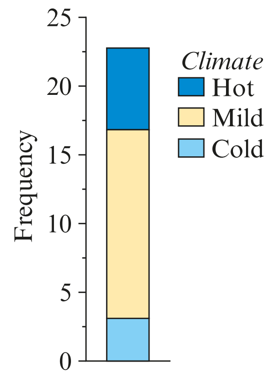
b



The segmented bar chart

A variation on the standard bar chart is the segmented bar chart. It is a compact display that is particularly useful when comparing two or more categorical variables.

In a **segmented bar chart**, the bars are stacked one on top of another to give a single bar with several parts or segments. The lengths of the segments are determined by the frequencies. The height of the bar gives the total frequency. A legend is required to identify which segment represents which category (see opposite). The segmented bar chart opposite was formed from the climate data used in Example 3. In a **percentage segmented bar chart**, the lengths of each segment in the bar are determined by the percentages. When this is done, the height of the bar is 100%.



Example 4 Constructing a percentage segmented bar chart from a frequency table

The climate type of 23 countries is classified as ‘cold’, ‘mild’ or ‘hot’.

Construct a percentage frequency segmented bar chart to display this information.

	Frequency	
Climate type	Number	Percentage
Cold	3	13.0
Mild	14	60.9
Hot	6	26.1
Total	23	100.0

Explanation

1 In a segmented bar chart, the horizontal axis has no label.

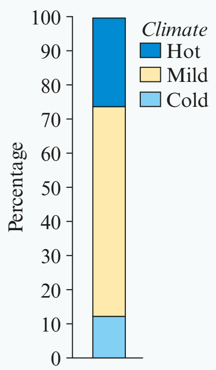
2 Label the vertical axis ‘Percentage’. Scale allowing for the maximum of 100 (%), Mark the scale in tens.

3 Draw a single bar of height 100. Divide the bar into three by inserting dividing lines at 13% and 73.9% (13 + 60.9%).

4 Shade (or colour) the segments differently.

5 Insert a legend to identify each shaded segment by climate type.

Solution



The mode

One of the features of a data set that is quickly revealed with a frequency table or a bar chart is the **mode** or **modal category**.

The **mode** is the most frequently occurring value or category.

In a bar chart, the mode is given by the category with the tallest bar or longest segment. For the previous bar charts, the modal category is clearly 'mild'. That is, for the countries considered, the most frequently occurring climate type is 'mild'.

Modes are particularly important in 'popularity' polls. For example, in answering questions such as 'Which is the most watched TV station between 6:00 p.m and 8:00 p.m.?' or 'When is the time a supermarket is in peak demand: morning, afternoon or night?'

Note, however, that the mode is only of real interest when a single category stands out from the others.

Answering statistical questions for categorical variables

A **statistical question** is a question that depends on data for its answer.

Statistical questions that are of most interest when working with a single categorical variable are of these forms:

- Is there a dominant category into which a large percentage of individuals fall or are the individuals relatively evenly spread across all of the categories? For example, are the shoppers in a department store predominantly male or female, or are there roughly equal numbers of males and females?
- How many and/or what percentage of individuals fall into each category? For example, what percentage of visitors to a national park are 'day-trippers' and what percentage of visitors are staying overnight?

A short written report is the standard way to answer these questions.
The following guidelines are designed to help you to produce such a report.

Guidelines for writing a report describing a categorical variable

- Briefly summarise the context in which the data were collected including the number of individuals involved in the study.
- If there is a clear modal category, ensure that it is mentioned.
- Include frequencies or percentages in the report. Percentages are preferred.
- If there are a lot of categories, it is not necessary to mention every category, but the modal category should always be mentioned.

Example 5 Describing the distribution of a categorical variable

In an investigation of the variation of climate type across countries, the climate types of 23 countries were classified as 'cold', 'mild' or 'hot'. The data are displayed in a frequency table to show the percentages.

	Frequency	
Climate type	Number	%
Cold	3	13.0
Mild	14	60.9
Hot	6	26.1
Total	23	100.0

Use the information in the frequency table to write a concise report on the distribution of climate types across these 23 countries.

Solution

Report

The climate types of 23 countries were classified as being, 'cold', 'mild' or 'hot'. The majority of the countries, 60.9%, were found to have a mild climate. Of the remaining countries, 26.1% were found to have a hot climate, while 13.0% were found to have a cold climate.

Exercise 1B

Constructing a frequency table for categorical data

Example 2

1 Construct appropriately labelled frequency tables showing both frequencies and percentage frequencies for each of the following data sets:

a *Grades*: A A C B A B B B B C C

b *Shoe size*: 8 9 9 10 8 8 8 9 8 10 12 8

Example 3

2 The following data identify the *state of residence* of a group of people where 1 = Victoria, 2 = South Australia and 3 = Western Australia.

2 1 1 1 3 1 3 1 1 3 3

a Is the variable *state of residence*, categorical or numerical?

b Construct a frequency table (with both numbers and percentages) to show the distribution of *state of residence* for this group of people.

c Construct a bar chart of the percentaged frequency table.

3 The *size* (S = small, M = medium, L = large) of 20 cars was recorded as follows.

S S L M M M L S S M

M S L S M M M S S M

a Is the variable *size* in this context numerical or categorical?

b Construct a frequency table (with both numbers and percentages) to show the distribution of size for these cars.

c Construct a percentage bar chart.

Constructing a percentage segmented bar chart from a frequency table

Example 4

4 The table shows the frequency distribution of the place of birth for 500 Australians.

a Is *place of birth* an ordinal or a nominal variable?

b Display the data in the form of a percentage segmented bar chart.

Place of birth	Percentage
Australia	78.3
Overseas	21.8
Total	100.1

5 The table records the number of new cars sold in Australia during the first quarter of one year, categorised by *type of vehicle* (private, commercial).

Type of vehicle	Frequency	
	Number	Percentage
Private	132 736	
Commercial	49 109	
Total		

a Is *type of vehicle* an ordinal or a nominal variable?

b Copy and complete the table giving the percentages correct to the nearest whole number.

c Display the data in the form of a percentage segmented bar chart.

Using the distribution of a categorical variable to answer statistical questions

Example 5

6 The table shows the frequency distribution of *school type* for a number of schools. The table is incomplete.

School type	Frequency	
	Number	Percentage
Catholic	4	20
Government	11	
Independent	5	25
Total		100

a Write down the information missing from the table.

b How many schools are categorised as 'independent'?

c How many schools are there in total?

d What percentage of schools are categorised as 'government'?

e Use the information in the frequency table to complete the following report describing the distribution of school type for these schools.

> **Report**
>
> ____ schools were classified according to school type. The majority of these schools, ____%, were found to be ____. Of the remaining schools, ____ were ____ while 20% were ____.

7 Twenty-two students were asked the question, 'How often do you play sport?', with the possible responses: 'regularly', 'sometimes' or 'rarely'. The distribution of responses is summarised in the frequency table.

Plays sport	Frequency	
	Number	Percentage
Regularly	5	22.7
Sometimes	10	
Rarely		31.8
Total	22	

a Write down the information missing from the table.

b Use the information in the frequency table to complete the following report describing the distribution of student responses to the question, 'How often do you play sport?'

Report

When ☐ students were asked the question, 'How often do you play sport', the dominant response was 'Sometimes', given by ☐ % of the students. Of the remaining students, ☐ % of the students responded that they played sport ☐ while ☐ % said that they played sport ☐.

8 The table shows the frequency distribution of the eye colour of 11 preschool children.

Use the information in the table to write a brief report describing the frequency distribution of eye colour.

	Frequency	
Eye colour	Number	Percentage
Brown	6	54.5
Hazel	2	18.2
Blue	3	27.3
Total	11	100.0

Exam 1 style questions

9 In a survey people were asked to select how much of the time they felt happy from the alternatives 'almost none of the time', 'some of the time', 'most of the time', or 'almost all of the time'. Their responses are summarised in the following barchart of the variable *happiness*.

The percentage of people who chose the modal response to this question is closest to:

A 30% **B** 43% **C** 50% **D** 57% **E** 70%

1C Displaying and describing numerical data

Learning intentions

- ▶ To be able to construct frequency tables for discrete numerical data.
- ▶ To be able to construct frequency tables for grouped numerical data (discrete and continuous).
- ▶ To be able to construct a histogram from a frequency table for numerical data.
- ▶ To be able to construct a histogram from numerical data using a CAS calculator.
- ▶ To be able to describe the distribution of numerical data according to its key characteristics of shape, centre, spread and outliers.

Frequency tables can also be used to organise numerical data. For a discrete variable which only takes a small number of values the process is the same as that for categorical data, as shown in the following example.

Example 6 **Constructing a frequency table for discrete numerical data taking a small number of values**

The number of bedrooms in each of the 24 properties sold in a certain area over a one month period are as follows:

2 3 4 3 3 4 3 4 4 1 3 2 1 2 2 2 4 5 3 4 4 5 3 4

Construct a table for these data showing both frequency and percentage frequency, giving the values of the percentage frequency rounded to one decimal place.

Explanation

1 Find the maximum and the minimum values in the data set. Here the minimum is 1 and the maximum is 5.

2 Construct a table as shown, including all the values between the minimum and the maximum.

3 Count the number of 1s, 2s, etc. in the data set. Record these values in the number column and add the frequencies to find the total.

4 Convert the frequencies to percentages, and record in the per cent (%) column. For example, percentage of 1s equals $\frac{2}{24} \times 100 = 8.3\%$.

5 Total the percentages and record.

Solution

Number of bedrooms	Frequency	
	Number	%
1	2	8.3
2	5	20.8
3	7	29.2
4	8	33.3
5	2	8.3
Total	24	99.9

The grouped frequency distribution

When the variable can take on a large range of values (e.g., age from 0 to 100 years) or when the variable is continuous (e.g. response times measured in seconds to 2 decimal places), we group the data into a small number of convenient intervals.

These grouping intervals should be chosen according to the following principles:

- Every data value should be in an interval.
- The intervals should not overlap.
- There should be no gaps between the intervals.

The choice of intervals can vary but there are some guidelines.

- A division which results in about 5 to 15 groups, is preferred.

- Choose an interval width that is easy for the reader to interpret such as 10 units, 100 units or 1000 units (depending on the data).
- By convention, the beginning of the interval is given the appropriate exact value, rather than the end. As a result, intervals of 0–49, 50–99, 100–149 would be preferred over the intervals 1–50, 51–100, 101–150 etc.

When we then organise the data into a frequency table using these data intervals we call this table a **grouped frequency table**.

Example 7 **Constructing a grouped frequency table**

The data below give the average hours worked per week in 23 countries.

35.0	48.0	45.0	43.0	38.2	50.0	39.8	40.7	40.0	50.0	35.4	38.8
40.2	45.0	45.0	40.0	43.0	48.8	43.3	53.1	35.6	44.1	34.8	

Construct a grouped frequency table with five intervals.

Explanation

1. Set up a table as shown. Use five intervals: 30.0–34.9, 35.0–39.9, . . . , 50.0–54.9.
2. List these intervals, in ascending order, under *Average hours worked*.
3. Count the number of countries whose average working hours fall into each of the intervals. Record these values in the 'Number' column.
4. Convert the counts into percentages and record in the 'Percentage' column.
5. Total the number and percentage columns.

Solution

Average hours worked	Frequency	
	Number	Percentage
30.0–34.9	1	4.3
35.0–39.9	6	26.1
40.0–44.9	8	34.8
45.0–49.9	5	21.7
50.0–54.9	3	13.0
Total	23	99.9

The histogram and its construction

As with categorical data, we would like to construct a visual display of a frequency table for numerical data. The graphical display of a frequency table for a numerical variable is called a **histogram**. A histogram looks similar to a bar chart but, because the data is numerical, there is a natural order to the plot and the bar widths depend on the data values.

Constructing a histogram from a frequency table

In a frequency histogram:

- frequency (count or per cent) is shown on the vertical axis
- the values of the variable being displayed are plotted on the horizontal axis

- each bar in a histogram corresponds to a data interval
- the height of the bar gives the frequency (or the percentage frequency).

Example 8 Constructing a histogram from a frequency table

Construct a histogram for the frequency table opposite.

Average hours worked	Frequency
30.0–34.9	1
35.0–39.9	6
40.0–44.9	8
45.0–49.9	5
50.0–54.9	3
Total	23

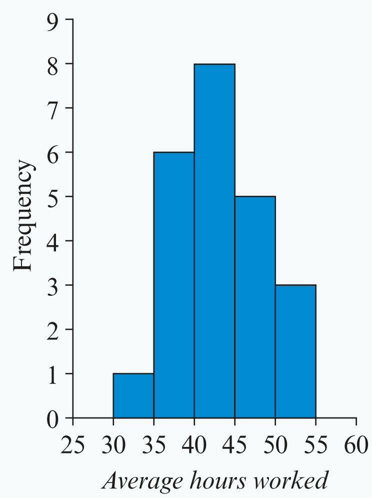
Explanation

1 Label the horizontal axis with the variable name, *Average hours worked*. Mark the scale using the start of each interval: 30, 35, . . .

2 Label the vertical axis 'Frequency'. Scale allowing for the maximum frequency, 8.

3 Finally, for each interval draw a bar, making the height equal to the frequency.

Solution



Constructing a histogram from raw data

It is relatively quick to construct a histogram from a frequency table. However, if you have only raw data (as you mostly do), it is a very slow process because you have to construct the frequency table first. Fortunately, a CAS calculator will do this for you.

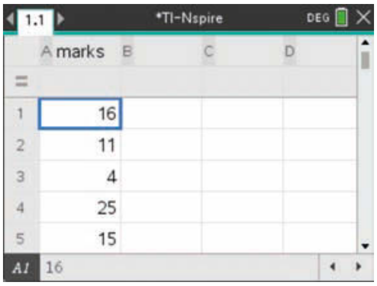
CAS 1: How to construct a histogram using the TI-Nspire CAS

Display the following set of 27 marks in the form of a histogram.

16	11	4	25	15	7	14	13	14	12	15	13	16	14
15	12	18	22	17	18	23	15	13	17	18	22	23	

Steps

1 Start a new document by pressing `ctrl` + `N` (or `⌂ on`>**New**. If prompted to save an existing document, move the cursor to **No** and press `enter`.

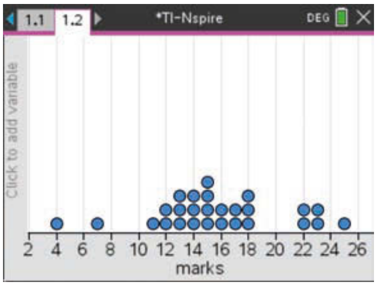


2 Select **Add Lists & Spreadsheet**.
Enter the data into a list named *marks*.

a Move the cursor to the name cell of column A and type in *marks* as the list variable. Press `enter`.

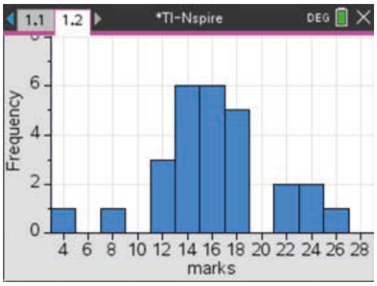
b Move the cursor down to row 1, type in the first data value and press `enter`. Continue until all the data have been entered. Press `enter` after each entry.

3 Statistical graphing is done through the **Data & Statistics** application. Press `ctrl` + `I` (or alternatively press `ctrl` `doc▾`) and select **Add Data & Statistics**.



a Press `tab` `enter` (or click on the **Click to add variable box** on the x-axis) to show the list of variables. Select *marks*.
Press `enter` to paste *marks* to that axis.

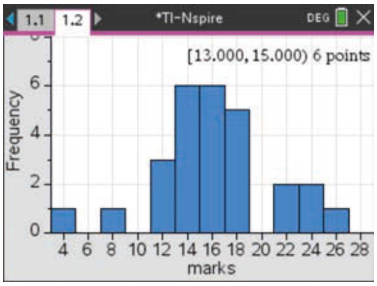
b A dot plot is displayed as the default. To change the plot to a histogram, press `menu`>**Plot Type>Histogram**. The histogram shown opposite has a column (or bin) width of 2, and a starting point (alignment) of 3. See Step 5 below for instructions on how to change the appearance of a histogram.



4 Data analysis

a Move the cursor over any column; a ☝ will appear and the column data will be displayed as shown opposite.

b To view other column data values, move the cursor to another column.



Note: If you click on a column, it will be selected.

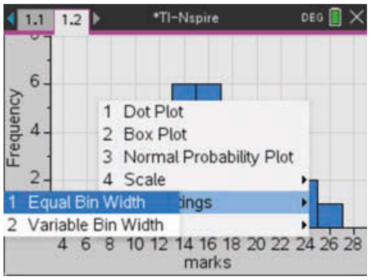
Hint: If you accidentally move a column or data point, `ctrl` + `esc` `enter` will undo the move.

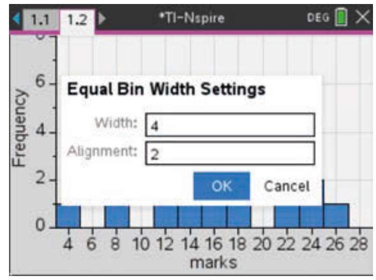
5 Change the histogram column (bin) width to **4** and the starting point to **2**.

a Press `ctrl` + `menu` to access the context menu as shown (below left).
Hint: Pressing `ctrl` + `menu` `enter` with the cursor on the histogram gives you a context menu that relates only to histograms. You can access the commands through `menu`>**Plot Properties**.

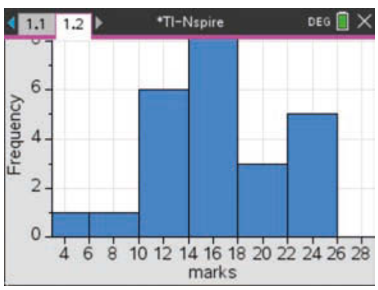
b Select **Bin Settings**>**Equal Bin Width**.

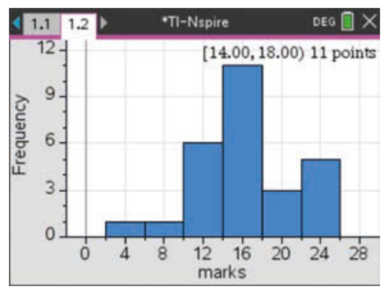
c In the settings menu (below right) change the Width to 4 and the Starting Point (Alignment) to 2 as shown. Press enter.



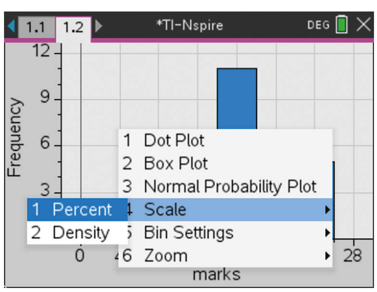


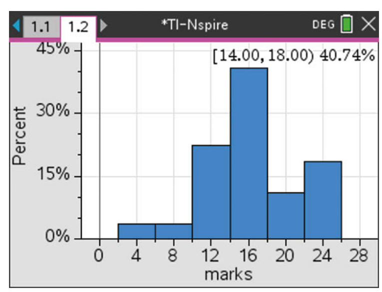
d A new histogram is displayed with column width of 4 and a starting point of 2 but it no longer fits the window (below left). To solve this problem, press ctrl + menu>**Zoom>Zoom-Data** and enter to obtain the histogram as shown below right.





6 To change the frequency axis to a percentage axis, press ctrl + menu»**Scale>Percent** and then press enter.





CAS 1: How to construct a histogram using the ClassPad

Display the following set of 27 marks in the form of a histogram.

16 11 4 25 15 7 14 13 14 12 15 13 16 14

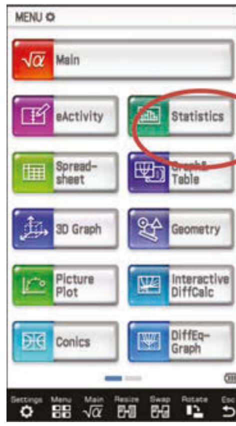
15 12 18 22 17 18 23 15 13 17 18 22 23

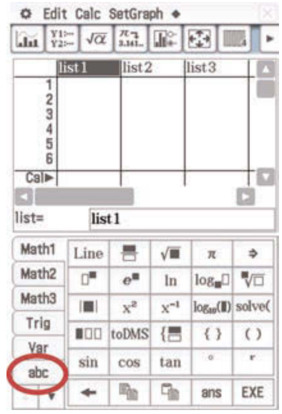
Steps

1 From the application menu screen, locate the built-in **Statistics** application.

Tap Statistics to open.

Tapping Menu from the icon panel (just below the touch screen) will display the application menu if it is not already visible.





2 Enter the data into a list named *marks*.

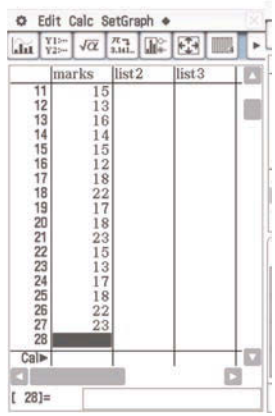
To name the list:

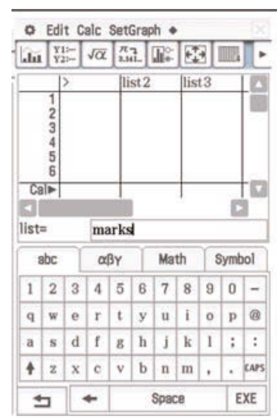
a Highlight the heading of the first list by tapping it.

b Press Keyboard on the front of the calculator and tap the abc tab.

c To enter the data, type the word **marks** and press EXE. Tap ⮌ and Keyboard to return to the list screen.

d Type in each data value and press EXE or ⊙ (which is found on the cursor button on the front of the calculator) to move down to the next cell.





The screen should look like the one shown above right.

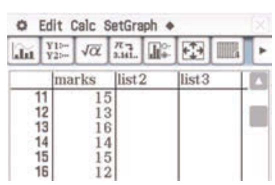
3 Set up the calculator to plot a statistical graph.

a Tap the SetGraph icon from the toolbar. This opens the **Set StatGraphs** dialog box.

b Complete the dialog box as given below.

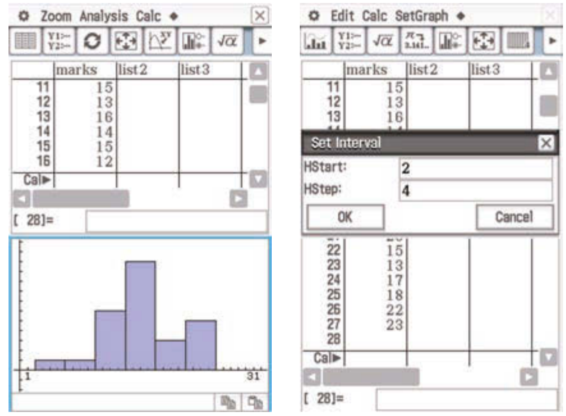
- **Draw:** select **On**.
- **Type:** select **Histogram** (▼).
- **XList**: select **main\marks** (▼).
- **Freq**: leave as **1**.

c Tap **Set** to confirm your selections.



Note: To make sure only this graph is drawn, select **SetGraph** from the menu bar at the top and confirm that there is a tick only beside **StatGraph1** and no others.

4 To plot the graph:



a Tap in the toolbar.

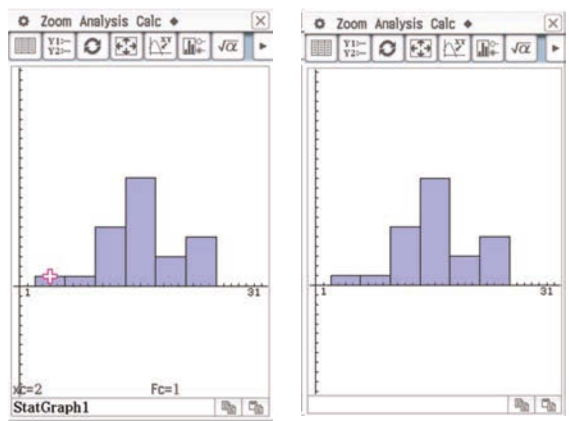
b Complete the **Set Interval** dialog box as follows.

- **HStart:** type **2** (i.e. the starting point of the first interval)
- **HStep:** type **4** (i.e. the interval width).

Tap OK to display histogram.

Note: The screen is split into two halves, with the graph displayed in the bottom half, as shown above. Tapping Resize from the icon panel allows the graph to fill the entire screen. Tap Resize again to return to half-screen size.

5 Tapping from the toolbar places a marker (+) at the top of the first column of the histogram (see opposite) and tells us that:



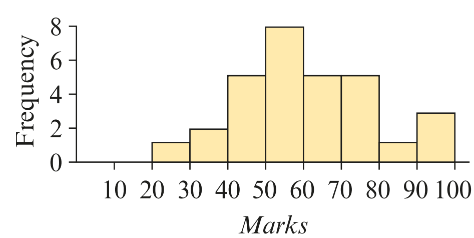
a the first interval begins at 2 **($x_c = 2$)**

b for this interval, the frequency is 1 **($F_c = 1$)**.

To find the frequencies and starting points of the other intervals, use the cursor key arrow (▶) to move from interval to interval.

What to look for in a histogram

A histogram provides a graphical display of a data distribution. For example, the histogram opposite displays the distribution of test marks for a group of 32 students.



The purpose of constructing a histogram is to help understand the key features of the data distribution. These features are:

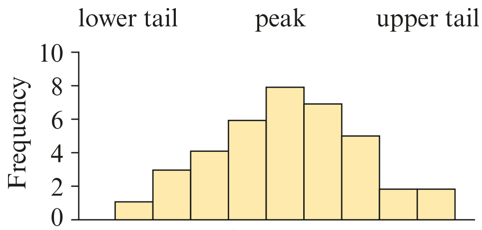
- **shape**
- **centre**
- **spread**
- **outliers**

Shape

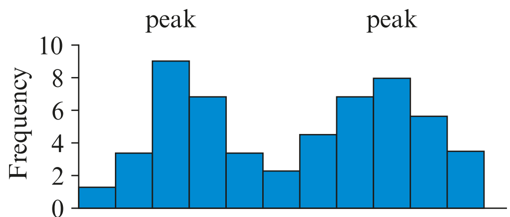
How are the data distributed? Is the histogram peaked? That is, do some data values tend to occur much more frequently than others, or is the histogram relatively flat, showing that all values in the distribution occur with approximately the same frequency?

Symmetric distributions

If a histogram is single-peaked, does the histogram region tail off evenly on either side of the peak? If so, the distribution is said to be **symmetric** (see Histogram 1).



Histogram 1



Histogram 2

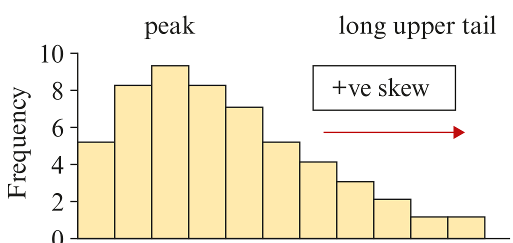
A single-peaked **symmetric distribution** is characteristic of the data that derive from measuring variables such as intelligence test scores, weights of oranges, or any other data for which the values vary evenly around some central value.

The double-peaked distribution (histogram 2) is symmetric about the dip between the two peaks. A histogram that has two distinct peaks indicates a **bimodal** (two modes) distribution.

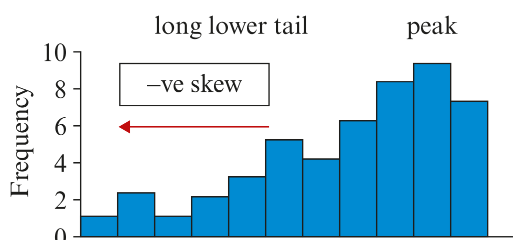
A bimodal distribution often indicates that the data have come from two different populations. For example, if we were studying the distance the discus is thrown by Olympic-level discus throwers, we would expect a bimodal distribution if both male and female throwers were included in the study.

Skewed distributions

Sometimes a histogram tails off primarily in one direction. If a histogram tails off to the right, we say that it is **positively** skewed (Histogram 3). The distribution of salaries of workers in a large organisation tends to be positively skewed. Most workers earn a similar salary with some variation above or below this amount, but a few earn more and even fewer, such as the senior manager, earn even more. The distribution of house prices also tends to be positively skewed.



Histogram 3

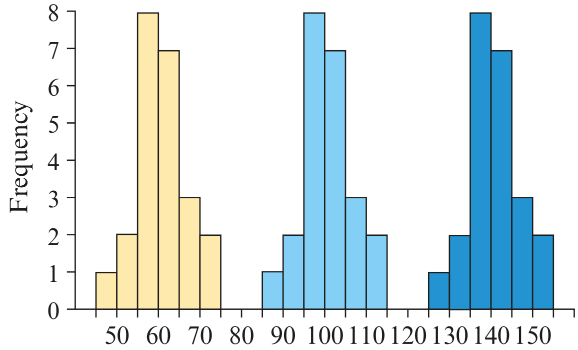


Histogram 4

If a histogram tails off to the left, we say that it is **negatively** skewed (Histogram 4). The distribution of age at death tends to be negatively skewed. Most people die in old age, a few in middle age and fewer still in childhood.

Centre

Histograms 6 to 8 display the distribution of test scores for three different classes taking the same subject. They are identical in shape, but differ in where they are located along the axis. In statistical terms we say that the distributions are 'centred' at different points along the axis. But what do we mean by the **centre of a distribution**?



Histograms 5 to 7

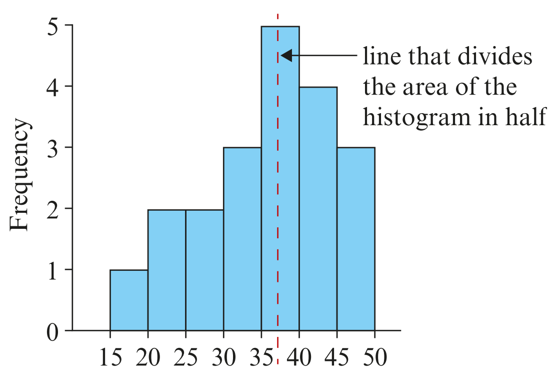
This is an issue we will return to in more detail later in the chapter. For the present we will take the centre to be the middle of the distribution.

The middle of a symmetric distribution is reasonably easy to locate by eye. Looking at histograms 5 to 7, it would be reasonable to say that the centre or middle of each distribution lies roughly halfway between the extremes; half the observations would lie above this point and half below. Thus we might estimate that histogram 5 (yellow) is centred at about 60, histogram 6 (light blue) at about 100, and histogram 7 (dark blue) at about 140.

For skewed distributions, it is more difficult to estimate the middle of a distribution by eye. The middle is not halfway between the extremes because, in a skewed distribution, the scores tend to bunch up at one end.

However, if we imagine a cardboard cut-out of the histogram, then the middle lies on the line that divides the histogram into two equal areas (Histogram 8).

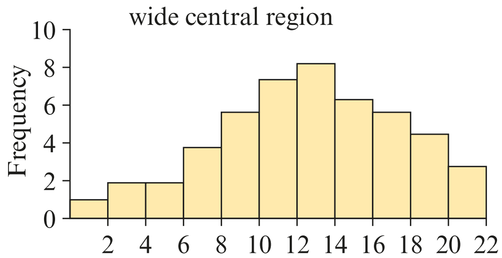
Using this method, we would estimate the centre of the distribution to lie somewhere between 35 and 40, but closer to 35, so we might opt for 37. However, remember that this is only an estimate.



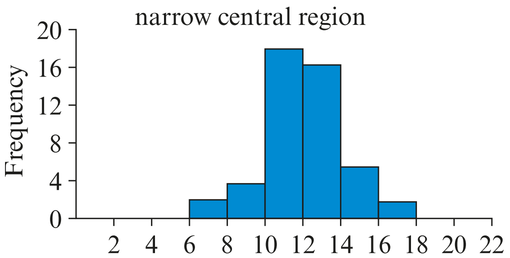
Histogram 8

Spread

If the histogram is single-peaked, is it narrow? This would indicate that most of the data values in the distribution are tightly clustered in a small region. Or is the peak broad? This would indicate that the data values are more widely spread out. Histograms 9 and 10 are both single-peaked. Histogram 9 has a broad peak, indicating that the data values are not very tightly clustered about the centre of the distribution. In contrast, Histogram 10 has a narrow peak, indicating that the data values are tightly clustered around the centre of the distribution.



Histogram 9

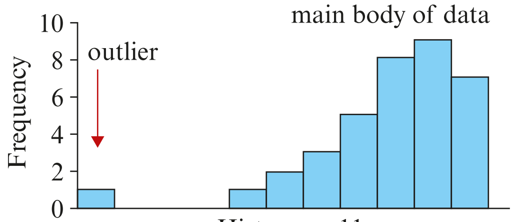


Histogram 10

Outliers

Outliers are any data values that stand out from the main body of data. These are data values that are atypically high or low. See, for example, Histogram 11, which shows an outlier. In this case it is a data value that is atypically low compared to the rest of the data values.

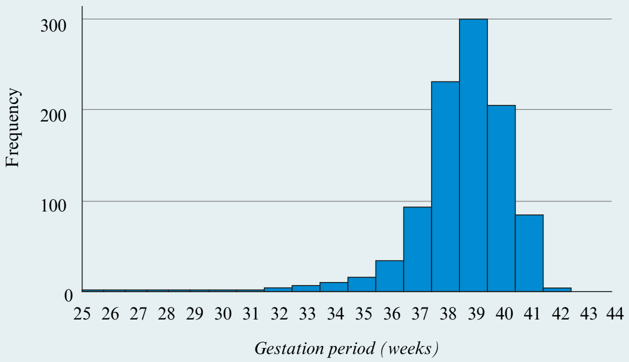
In the histogram shown there appears to be an outlier, a data value which is lower than than rest of the data values. Such values should be checked, they may indicate an unusually low value, but they may also indicate an error in the data.



Histogram 11

Example 9 Describing the features of a distribution from a histogram

The histogram shows the gestation period (completed weeks) for a sample for 1000 babies born in Australia one year. Describe this histogram in terms of shape, centre, spread and outliers.



Explanation	Solution
1 Determine the shape of the distribution.	**1** The distribution is clearly negatively skewed, with a long lower tail.
2 Locate the (approximate) centre of the distribution, the value seems to divide the area of the histogram in half.	**2** The centre of the distribution is around 38-39 weeks.

3 Consider the spread of the distribution, are the majority of the values close to the centre, or quite spread out?	**3** The data values range from 25-42 weeks, but most of the data values are close to the centre, in the range of 36-41 weeks.
4 Can we identify any outliers?	**4** Those values which are less than 34 weeks seem to be small in comparison to the rest of the data, but which values are outliers cannot be determined from the histogram.

We can see from this example that it is very difficult to give an exact values for centre and spread, and to clearly identify outliers, from the histogram. We will return to this example later in the chapter once we have discussed which measures of centre and spread are appropriate for this distribution, and when we have an exact definition on an outlier.

Exercise 1C

Constructing a frequency table for discrete numerical data taking a small number of values

Example 6

1 The number times a sample of 20 people bought take-away food over a one week period is as follows:

0 5 3 0 1 0 2 4 3 1 0 2 1 2 1 5 3 0 0 4

- **a** Construct a frequency table for the data, including both the frequency and percentage frequency.
- **b** What percentage of people bought take away food more than 3 times?
- **c** What is the mode of this distribution?

2 The number of chocolate chips per biscuit in a sample 40 biscuits was found to be as as follows:

2 5 4 4 5 4 6 4 4 4 5 4 4 5 6 6 5 5 4 6
4 5 5 4 5 4 6 4 6 4 5 4 5 4 6 5 5 6 4 6

- **a** Construct a frequency table for the data, including both the frequency and percentage frequency.
- **b** What percentage of biscuits contained three or less chocolate chips?
- **c** What is the mode of this distribution?

Constructing a grouped frequency table

Example 7

3 The following are the heights of the 25 players in a women's football team, in centimetres.

188	175	176	161	183
169	171	177	165	166
162	170	174	168	178
169	181	173	164	179
163	170	164	175	182

Height (cm)	Frequency
160–164	
165–169	
170–174	
175–179	
180–184	
185–189	
Total	25

a Use the data to complete the grouped frequency table.

b What is the model height for this group of players?

c What percentage of the players are 180 cm or more in height?

Constructing a histogram from a frequency table

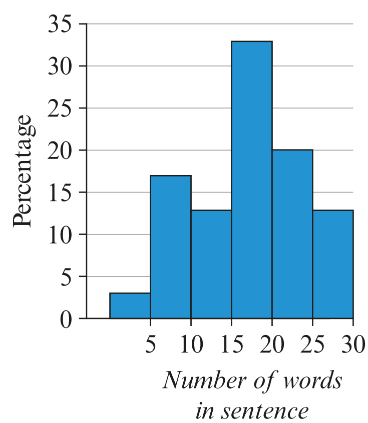
Example 8

4 Construct a histogram to display the information in the frequency table opposite. Label axes and mark scales.

Population density	Frequency
0–199	11
200–399	4
400–599	4
600–799	2
800–999	1
Total	22

Reading information from a histogram

5 The histogram opposite displays the distribution of the number of words in 30 randomly selected sentences.



a What percentage of these sentences contained:

i 5–9 words?

ii 25–29 words?

iii 10–19 words?

iv fewer than 15 words?

Write answers correct to the nearest per cent.

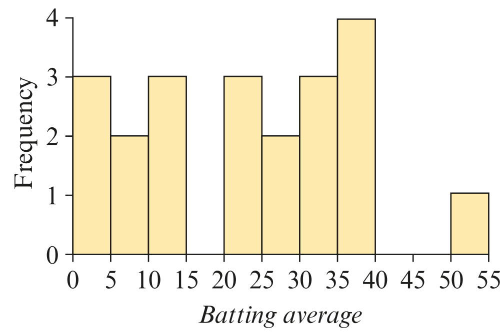
b How many of these sentences contained:

i 20–24 words?

ii more than 25 words?

c What is the modal interval?

6 The histogram opposite displays the distribution of the average batting averages of cricketers playing for a district team.



a How many players have their averages recorded in this histogram?

b How many of these cricketers had a batting average:

i 20 or more?

ii less than 15?

iii at least 20 but less than 30?

iv of 45?

c What percentage of these cricketers had a batting average:

i 50 or more? **ii** at least 20 but less than 40?

Constructing a histogram from raw data using a CAS calculator

7 The pulse rates of 23 students are given below.

86	82	96	71	90	78	68	71	68	88	76	74
70	78	69	77	64	80	83	78	88	70	86	

a Use a CAS calculator to construct a histogram so that the first column starts at 63 and the column width is two.

b **i** What is the starting point of the third column?

ii What is the 'count' for the third column? What are the *actual* data values?

c Redraw the histogram so that the column width is five and the first column starts at 60.

d For this histogram, what is the count in the interval '65 to <70'?

8 The numbers of children in the families of 25 VCE students are listed below.

1	6	2	5	5	3	4	1	2	7	3	4	5
3	1	3	2	1	4	4	3	9	4	3	3	

a Use a CAS calculator to construct a histogram so that the column width is one and the first column starts at 0.5.

b What is the starting point for the fourth column and what is the count?

c Redraw the histogram so that the column width is two and the first column starts at 0.

d **i** What is the count in the interval from 6 to less than 8?

ii What actual data value(s) does this interval include?

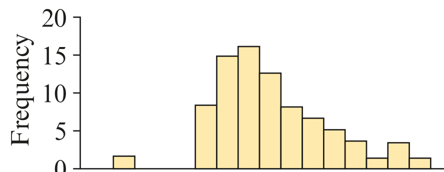
Determining shape, centre and spread from a histogram

Example 9

9 Identify each of the following histograms as approximately symmetric, positively skewed or negatively skewed, and mark the following.

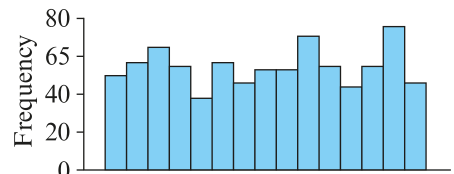
- **i** The mode (if there is a clear mode)
- **ii** Any potential outliers
- **iii** The approximate location of the centre

a



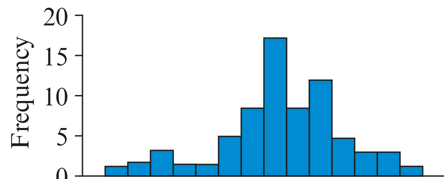
Histogram A

b



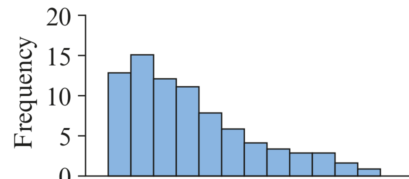
Histogram B

c



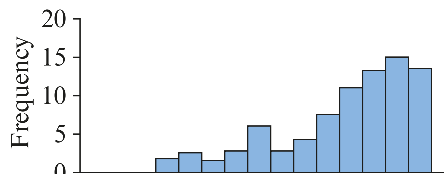
Histogram C

d



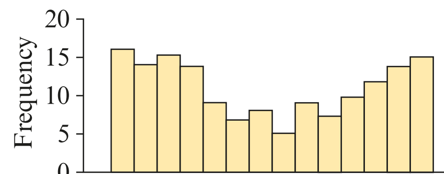
Histogram D

e



Histogram E

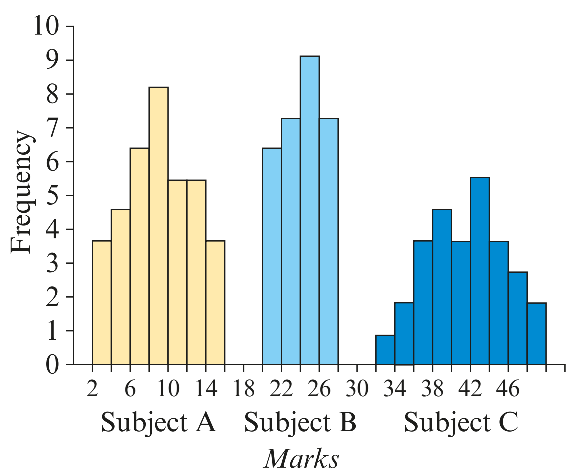
f



Histogram F

10 These three histograms show the marks obtained by a group of students in three subjects.

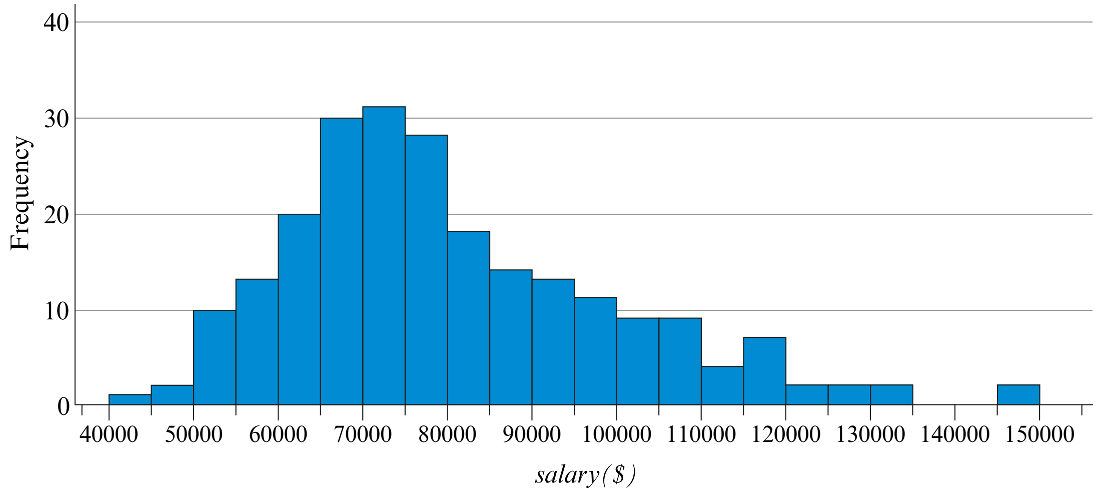
- **a** Are each of the distributions approximately symmetric or skewed?
- **b** Are there any clear outliers?
- **c** Determine the interval containing the central mark for each of the three subjects.
- **d** In which subject was the spread of marks the least?



Exam 1 style questions

Use the following information to answer questions 11 and 12

The annual salaries for all the sales staff in a large company are summarised in the following histogram.



11 The number of people in the company who earn from \$65,000 to less than \$70,000 per year is equal to:

A 20 **B** 30 **C** 50 **D** 32 **E** 62

12 The shape of this histogram is best described as:

A positively skewed with possible outlliers
B positively skewed with no outliers
C approximately symmetric
D negatively skewed with no outliers
E negatively skewed with outliers

1D Dot plots and stem plots

Learning intentions

- ▶ To be able to construct a dot plot for numerical data.
- ▶ To be able to construct a stem plot for numerical data.

Dot plots and stem plots are two simple plots used to display numerical data. They are generally constructed by hand (that is, without using a calculator), from a data set that is reasonably small.

The dot plot

The simplest way to display numerical data is to construct a **dot plot**. A dot plot is particularly suitable for displaying discrete numerical data and provides a very quick way to order and display a small data set.

A dot plot consists of a number line with each data point marked by a dot. When several data points have the same value, the points are stacked on top of each other.

Example 10 Constructing a dot plot

The ages (in years) of the 13 members of a cricket team are:

22 19 18 19 23 25 22 29 18 22 23 24 22

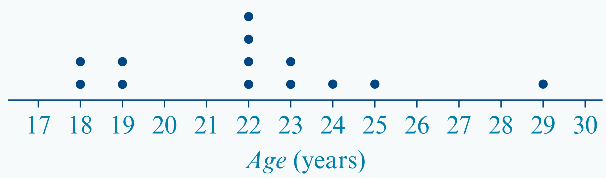
Construct a dot plot.

Explanation

1 Draw a number line, scaled to include all data values. Label the line with the variable being displayed.

2 Plot each data value by marking a dot above the corresponding value on the number line.

Solution



While some CAS calculators will construct a stem plot, they were designed to be a quick and easy way of ordering and displaying a small data set by hand.

The stem plot

The **stem-and-leaf plot**, or **stem plot** for short, is another quick and easy way to display numerical data. Stem plots work well for both discrete and continuous data. They are particularly useful for displaying small- to medium-sized sets of data (up to about 50 data values). Like the dot plot, they are designed to be a pen and paper technique.

In a stem plot, each data value is separated into two parts: its leading digits, which make up the 'stem' of the number, and its last digit, which is called the 'leaf'.

For example, in the stem-and-leaf plot opposite, the data values 21 and 34 are displayed as follows:

	Stem	Leaf
21 is displayed as	2	1
34 is displayed as	3	4

A key should always be included to show how the numbers in the plot should be interpreted.

Key: 1|2 = 12

0	8
1	2 4 9 9
2	1 1 1 1 2 2 2 3 6 6 9 9
3	2 4
4	4
5	9

Example 11 Constructing a stem plot

University participation rates (%) in 23 countries are given below.

26	3	12	20	36	1	25	26	13	9	26	27
15	21	7	8	22	3	37	17	55	30	1	

Display the data in the form of a stem plot.

Explanation

1 The data set has values in the units, tens, twenties, thirties, forties and fifties. Write the stems 0, 1, 2, 3, 4, and 5 in ascending order, followed by a vertical line. Now attach the leaves. The first data value is '26'. The stem is '2' and the leaf is '6'. Opposite the 2 in the stem, write down the number 6, as shown.

2 Continue systematically working through the data, following the same procedure until all points have been plotted. You will then have the stem plot, as shown.

3 To complete the task, write the leaves on each stem in ascending order, then add the variable name and a key.

Solution

```
0 |
1 |
2 | 6
3 |
4 |
5 |
```

```
0 | 3  1  9  7  8  3  1
1 | 2  3  5  7
2 | 6  0  5  6  6  7  1  2
3 | 6  7  0
4 |
5 | 5
```

rate (%) Key: 1|2 = 12

```
0 | 1  1  3  3  7  8  9
1 | 2  3  5  7
2 | 0  1  2  5  6  6  6  7
3 | 0  6  7
4 |
5 | 5
```

Stem plots with split stems

In some instances, using the simple process outlined above produces a stem plot that is too cramped to give an good overall picture of the variation in the data. This happens when the data values all have the same one or two first digits.

For example, consider the marks obtained by 17 VCE students on a statistics test.

2 12 13 9 18 17 7 16 12 10 16 14 11 15 16 15 17

If we use the process described in Example 11 to form a stem plot, we end up with a 'bunched-up' plot like the below.

0	2 7 9
1	0 1 2 2 3 4 5 5 6 6 6 7 7 8

We can solve this problem by 'splitting' the stems.

Generally the stem is split into halves or fifths as shown below.

Key: 1|6 = 16

0	2 7 9
1	0 1 2 2 3 4 5 5 6 6 6 7 7 8

Single stem

Key: 1|6 = 16

0	2
0	7 9
1	0 1 2 2 3 4
1	5 5 6 6 6 7 7 8

Stem split into halves

Key: 1|6 = 16

0	
0	2
0	
0	7
0	9
1	0 1
1	2 2 3
1	4 5 5
1	6 6 6 7 7
1	8

Stem split into fifths

Splitting the stems is useful when there are only a few different values for the stem.

Exercise 1D

Constructing a dot plot

Example 10

1 The following data gives the number of rooms in 11 houses.

4 6 7 7 8 4 4 8 8 7 8

a Is the variable *number of rooms* discrete or continuous?

b Construct a dot plot.

2 The following data give the number of children in the families of 14 VCE students:

1 6 2 5 5 3 4 4 2 7 3 4 3 4

a Is the variable *number of children* discrete or continuous?

b Construct a dot plot.

c Write down the value of the mode. What does the mode represent in the context of the data?

3 The following data give the average life expectancies in years of 13 countries.

76 75 74 74 73 73 75 71 72 75 75 78 72

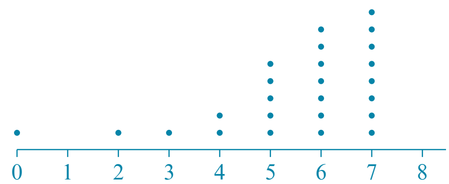
a Is the variable *life expectancy* discrete or continuous?

b Construct a dot plot.

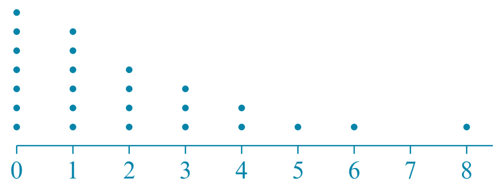
c Write down the value of the mode. What does the mode represent in the context of the data?

4 Describe the shape of each of the following distributions (negatively skewed, positively skewed, or approximately symmetric).

a



b



5 The ages of each member of a running club are as follows:

22 20 20 23 21 22 21 25 21 24 18

20 19 22 23 25 19 21 20 21 21 22

a Construct a dot plot of the ages of the players.

b What is the mode of this distribution?

c What is the shape of the distribution of runners ages?

d What percentage of runners are younger than 20? Give your answer to the nearest whole percentage.

Constructing a stem plot

Example 11

6 The data below give the urbanisation rates (%) in 23 countries.

54 99 22 20 31 3 22 9 25 3 56 12

16 9 29 6 28 100 17 99 35 27 12

a Is the variable *urbanisation rate* discrete or continuous?

b Construct a stem plot with an appropriate key.

Constructing a stem plot with split stems

7 The data below give the wrist circumference (in cm) of 15 men.

16.9 17.3 19.3 18.5 18.2 18.4 19.9 16.7

17.7 16.5 17.0 17.2 17.6 17.1 17.6

a Is the variable *wrist circumference* discrete or continuous?

b Construct a stem plot for wrist circumference using:

i stems: 16, 17, 18, 19

ii these stems split into halves, that is: 16, 16, 17, 17, . . .

Interpreting a stem plot

8 Describe the shape of each of following distributions (negatively skewed, positively skewed, or approximately symmetric).

a key: 4|1 represents 4.1

0	2 3
0	5 6 6 6 7 7 8 8 9 9
1	0 0 1 2 2 2 3 3 3 3 3
1	5 5 6 7 8
2	0 2 3
2	2 5
3	0 0
3	7

b key: 3|1 represents 31

3	2
4	2 7
5	0 1 5 9
6	1 3 3 5 7 7 7 9
7	0 2 3 3 4 4 6 7 8 9 9
8	2 5

9 The stem plot on the right shows the ages, in years, of all the people attending a meeting.

a How many people attended the meeting?

b What is the shape of the distribution of ages?

c How many of these people were less than 33 years old?

Age (years) key: 2|0 represents 20

2	2 3
2	5 6 6 7 7 8 9
3	0 1 3 3 4 4 4 4
3	5 5 5 6 7 7 7 8 8 8 9 9
4	0 2 3 3 4 4
4	5 5 6 8
5	0

Exam 1 style questions

Use the following information to answer questions 10 and 11

The following stem plot shows the distribution of the time it took (in minutes) for each of a group of 25 people to solve a complex task.

Time (minutes) key: 4|0 represents 4.0

4	2 6
5	1 3 6 8
6	0 1 5 6 7
7	1 3 4 5 7 8 9
8	0 2 5 9
9	5 5
10	6

10 The shape of this distribution is best described as:

A positively skewed with a possible outlier

B positively skewed with no outliers

C approximately symmetric

D negatively skewed with no outliers

E negatively skewed with outliers

11 The time taken by the slowest 20% of people was:

A more than 8.5 minutes **B** 8.5 minutes or more **C** less than 5.6 minutes
D 5.6 minutes or less **E** more than 5.6 minutes

1E Using a logarithmic (base 10) scale to display data

Learning intentions

- ▶ To be able to revise the concept of $\log_{10}x$.
- ▶ To be able to investigate the effect of the logarithmic scale on the features of a distribution.

Many numerical variables that we deal with in statistics have values that range over several orders of magnitude. For example, the populations of countries range from a few thousand to hundreds of thousands, to millions, to hundreds of millions to just over 1 billion. Constructing a histogram that effectively locates every country on the plot is impossible.

One way to solve this problem is to use a scale that spreads out the countries with small populations and 'pulls in' the countries with huge populations.

A scale that will do this is called a logarithmic scale (or, more commonly, a **log scale**).

Consider the numbers:

0.01, 0.1, 1, 10, 100, 1000, 10 000, 100 000, 1 000 000

Such numbers can be written more compactly as:

10^{-2}, 10^{-1}, 10^{0}, 10^{1}, 10^{2}, 10^{3}, 10^{4}, 10^{5}, 10^{6}

In fact, if we make it clear we are only talking about powers of 10, we can merely write down the powers:

−2, −1, 0, 1, 2, 3, 4, 5, 6

These powers are called the **logarithms** of the numbers or 'logs' for short.

When we use logarithms to write numbers as powers of 10, we say we are working with logarithms to the base 10. We can indicate this by writing $\log_{10}$.

$\log_{10}x$

If $\log_{10}x = b$, then $10^b = x$

Thus we can say for example that:

- $\log_{10}(100) = 2$, since $10^2 = 100$
- $\log_{10}(1000) = 3$, since $10^3 = 1000$
- $\log_{10}(1000000) = 6$, since $10^6 = 1000000$

Properties of logarithms to the base 10

1. If a number is greater than one, its log to the base 10 is greater than zero.
2. If a number is greater than zero but less than one, its log to the base 10 is negative.
3. If the number is zero, then its log is undefined.

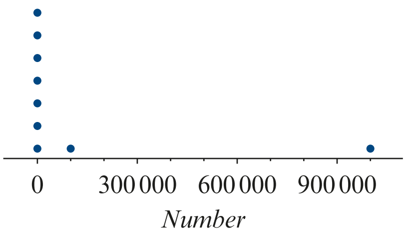
The effect of the the logarithmic scale

The set of numbers

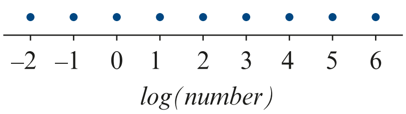
0.01, 0.1, 1, 10, 100, 1000, 10 000, 100 000, 1 000 000

ranges from 0.01 to 1 million.

Thus, if we wanted to plot these numbers on a scale, the first seven numbers would cluster together at one end of the scale, while the eighth (1 million) would be located at the far end of the scale.



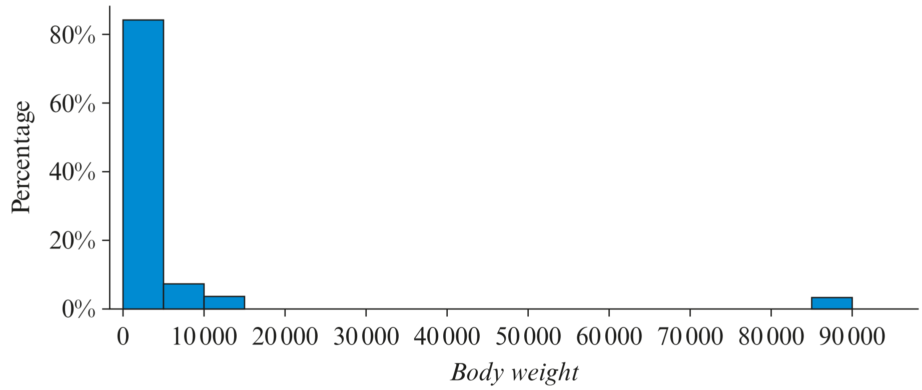
By contrast, if we plot the logs of these numbers, they are evenly spread along the scale. We use this idea to display a set of data whose values range over several orders of magnitude. Rather than plot the data values themselves, we plot the logarithms of their data values.



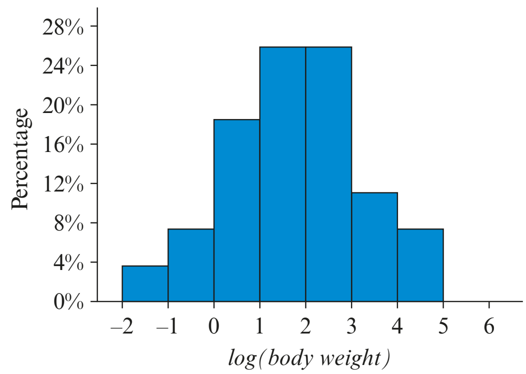
Logarithmic transformation

A **logarithmic transformation** involves changing the scale on the horizontal axis from x to $\log_{10}(x)$, and replacing each of the data values with its logarithm.

For example, the histogram below displays the body weights (in kg) of a number of animal species. Because the animals represented in this data set have weights ranging from around 1 kg to 90 tonnes (a dinosaur), most of the data are bunched up at one end of the scale and much detail is missing. The distribution of weights is highly positively skewed, with an outlier.



However, when a logarithmic transformation is used, their weights are much more evenly spread along the scale. The distribution is now approximately symmetric, with no outliers, and the histogram is considerably more informative.



We can now see that the percentage of animals with weights between 10 and 100 kg is similar to the percentage of animals with weights between 100 and 1000 kg.

Working with logarithms

To construct and interpret a log data plot, like the one above, you need to be able to:

1 Work out the log for any number. So far we have only done this for numbers such as 10, 100, 1000 that are exact powers of 10; for example, $100 = 10^2$, so $\log 100 = 2$.

2 Work backwards from a log to the number it represents. This is easy to do in your head for logs that are exact powers of 10 – for example, if the log of a number is 3 then the number is $10^3 = 1000$. But it is not a sensible approach for numbers that are not exact powers of 10.

Your CAS calculator is the key to completing both of these tasks in practice.

Example 12 Using a CAS calculator to find logs

a Find the log of 45, correct to two significant figures.

b Find the number with log equal to 2.7125, correct to the nearest whole number.

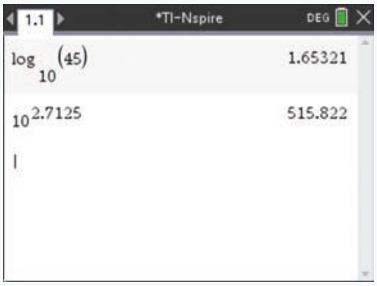
Explanation

a Open a calculator screen, type log (45) and press enter. Write down the answer correct to two significant figures.

b If the log of a number is 2.7125, then the number is $10^{2.7125}$.

Enter the expression $10^{2.7125}$ and press enter. Write down the answer correct to the nearest whole number.

Solution



a $\log 45 = 1.65\ldots$

$= 1.7$ (to 2 sig. figs)

b $10^{2.7125} = 515.82\ldots$

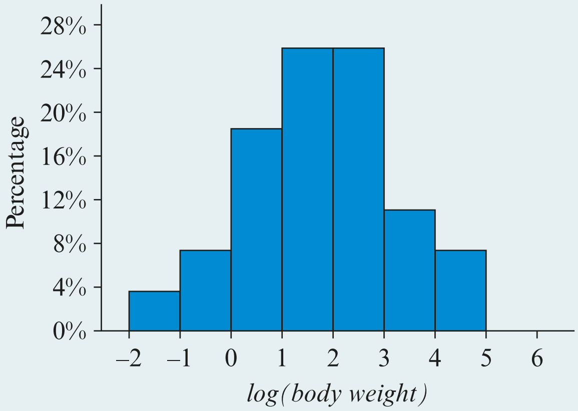
$= 516$ (to the nearest whole number)

Analysing data displays with a logarithmic scale

Now that you know how to work out the log of any number and convert logs back to numbers, you can analyse a data plot using a log scale.

Example 13 Interpreting a histogram with a log scale

The histogram shows the distribution of the weights of 27 animal species plotted on a log scale.



a What body weight (in kg) is represented by the number 4 on the log scale?

b How many of these animals have body weights more than 10 000 kg?

c The weight of a cat is 3.3 kg. Use your calculator to determine the log of its weight correct to two significant figures.

d Determine the weight (in kg) of the animal with a log(*body weight*) of 3.4 (the elephant). Write your answer correct to the nearest whole number.

Explanation

a If the log of a number is 4 then the number is $10^4 = 10\,000$.

Solution

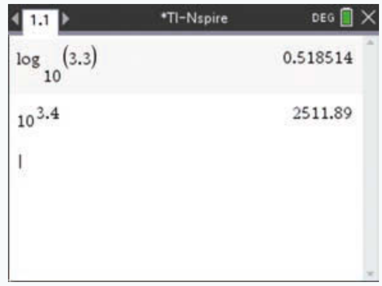
a $10^4 = 10\,000$ kg

b On the log scale, 10 000 is shown as 4.

Thus, the number of animals with a weight greater than 10 000 kg corresponds to the number of animals with a log weight of greater than 4.

This can be determined from the histogram which shows there are two animals with log weights greater than 4.

b Two animals



c The weight of a cat is 3.3 kg. Use your calculator to find log 3.3. Write the answer correct to two significant figures.

c Cat: $\log 3.3 = 0.518...$

$= 0.52$ kg (to 2 sig. figs)

d The log weight of an elephant is 3.4. Determine its weight in kg by using your calculator to evaluate $10^{3.4}$. Write the answer correct to the nearest whole number.

d Elephant: $10^{3.4} = 2511.88...$

$= 2512$ kg

Constructing a histogram with a log scale

The task of constructing a histogram is also a CAS calculator task.

CAS 2: Using a TI-Nspire CAS to construct a histogram with a log scale

The weights of 27 animal species (in kg) are recorded below.

1.4	470	36	28	1.0	12 000	2600	190	520
10	3.3	530	210	62	6700	9400	6.8	35
0.12	0.023	2.5	56	100	52	87 000	0.12	190

Construct a histogram to display the distribution:

a of the body weights of these 27 animals and describe its shape

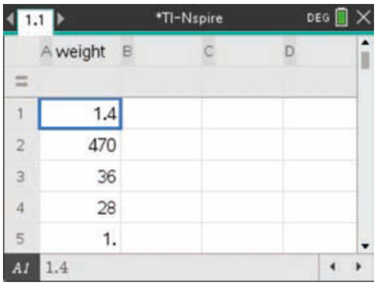
b of the log of the body weights of these animals and describe its shape.

Steps

1 a Start a new document by pressing ctrl + N.

b Select **Add Lists & Spreadsheet**.

Enter the data into a column named *weight*.

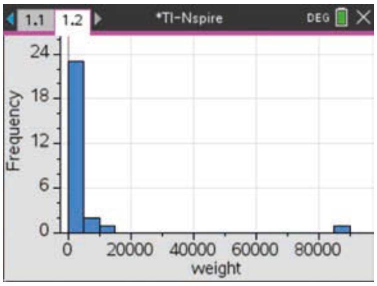


2 a Press ctrl + I and select **Add Data & Statistics**.

Click on the **Click to add variable** on the x-axis and select the variable *weight*. A dot plot is displayed.

b Plot a histogram using menu **>Plot Type>Histogram**.

c Describe the shape of the distribution.

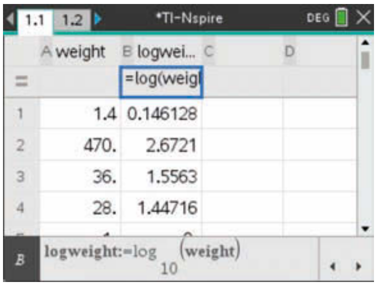


Shape: positively skewed with outliers

3 a Return to the **Lists & Spreadsheet** screen.

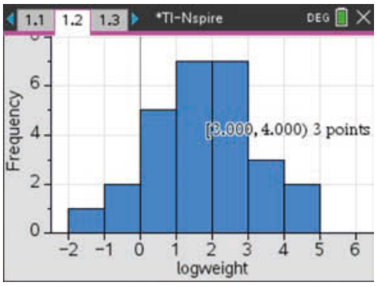
b Name another list *logweight*.

c Move the cursor to the formula cell below the *logweight* heading. Type in = **log(*weight*)**. Press enter to calculate the values of *logweight*.



4 a Plot a histogram using a log scale. That is, plot the variable *logweight*.

Note: Use menu>**Plot Properties**>**Histogram Properties**>**Bin Settings**>**Equal Bin Width** and set the column width (bin) to 1 and alignment (start point) to −2 and use menu>**Window/Zoom**>**Zoom-Data** to rescale.



b Describe the shape of the distribution.

Shape: approximately symmetric

CAS 2: Using a ClassPad to construct a histogram with a log scale

The weights of 27 animal species (in kg) are recorded below.

1.4	470	36	28	1.0	12 000	2600	190	520
10	3.3	530	210	62	6700	9400	6.8	35
0.12	0.023	2.5	56	100	52	87 000	0.12	190

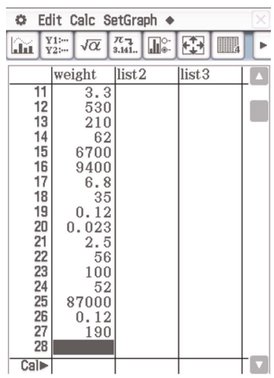
Construct a histogram to display the distribution:

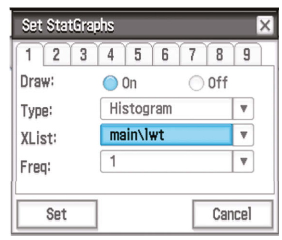
a of the body weights of these 27 animals and describe its shape

b of the log of the body weights of these animals and describe its shape.

Steps

1 In the statistics application Statistics enter the data into a column named *weight* as shown.





2 Plot a histogram of the data.

a Tap from the toolbar.

b Complete the dialog box.

- **Draw:** select **On**.
- **Type:** select **Histogram** (▼)
- **XList:** select **main****weight**(▼).
- **Freq:** leave as **1**.

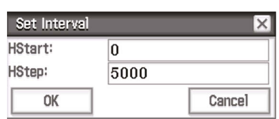
Tap **Set** to confirm your selections.

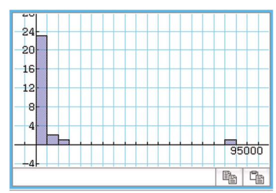
c Tap in the toolbar.

d Complete the **Set Interval** dialog box as follows:

HStart: 0

HStep: 5000





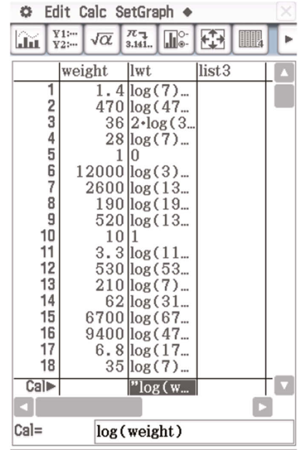
Describe the shape of the distribution. Shape: positively skewed with outliers

3 **a** Return to the data entry screen.

b Name another column ‘lwt’, short for log(*weight*).

c Tap in the calculation cell at the bottom of this column.

Type log(*weight*) and tap EXE.



4 Plot a histogram to display the distribution of weights on a log scale. That is, plot the variable lwt.

a Tap the graph icon from the toolbar.

b Complete the dialog box.

- **Draw**: select **On**.
- **Type**: select **Histogram** (▼).
- **XList**: select **main\lwt** (▼).
- **Freq**: leave as **1**.

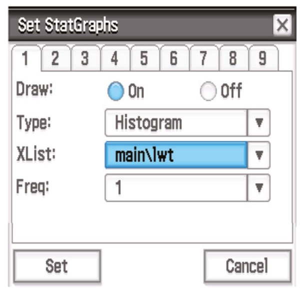
Tap **Set** to confirm your selections.

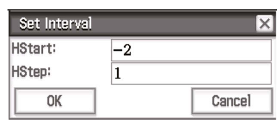
c Tap the graph icon in the toolbar.

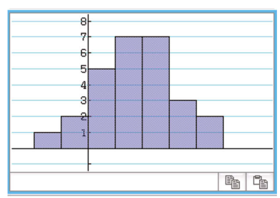
d Complete the **Set Interval** dialog box as follows:

- **HStart**: **type -2**
- **HStep**: type **1**

Tap **OK** to display histogram.







Describe the shape of the distribution. Shape: approximately symmetric

Skill-sheet

Exercise 1E

Determining the log of a number

Example 12

1 Using a CAS calculator, find the logs of the following numbers correct to one decimal place.

a 2.5	**b** 25	**c** 250	**d** 2500
e 0.5	**f** 0.05	**g** 0.005	**h** 0.0005

Determining a number from its log

2 Find the numbers with the following logs:

a −2.5 **b** −1.5 **c** −0.5 **d** 0

Write your decimal answers correct to two significant figures.

Constructing a histogram with a log scale

3 The brain weights of the same 27 animal species (in g) are recorded below.

465	423	120	115	5.5	50	4600	419	655
115	26	680	406	1320	5712	70	179	56
1.0	0.4	12	175	157	440	155	3.0	180

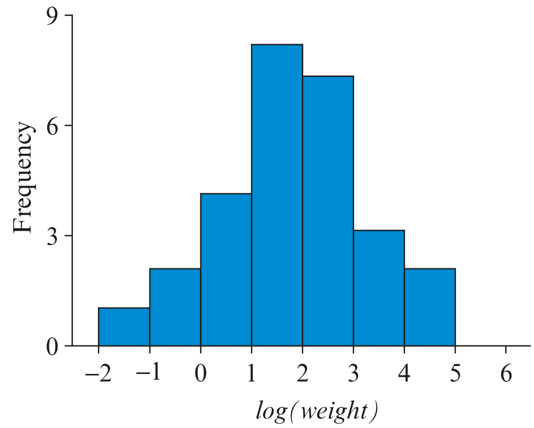
a Construct a histogram to display the distribution of brain weights and comment on its shape.

b Construct a histogram to display the log of the brain weights and note the shape of the distribution.

Interpreting a histogram with a log scale

Example 13

4 The histogram opposite shows the distribution of brain weights (in g) of 27 animal species plotted on a log scale.



a The brain weight (in g) of a mouse is 0.4 g. What value would be plotted on the log scale?

b The brain weight (in g) of an African elephant is 5712 g. What is the log of this brain weight (to two significant figures)?

c What brain weight (in g) is represented by the number 2 on the log scale?

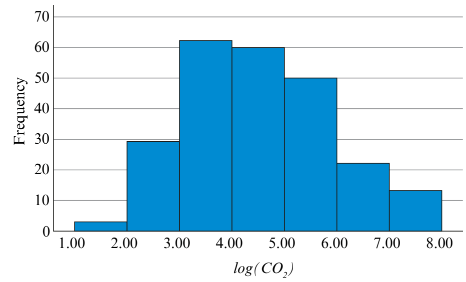
d What brain weight (in g) is represented by the number –1 on the log scale?

e Use the histogram to determine the number of these animals with brain weights:

i 1000 g or more **ii** from 1 to less than 100 g **iii** 1 g or more

Exam 1 style questions

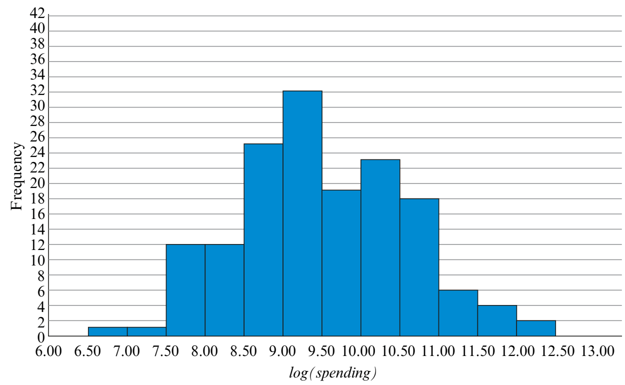
5 The histogram shows the carbon dioxide emissions (in thousands of metric tons) for 239 different countries, plotted on a $\log_{10}$ scale.



Based on this histogram, the percentage of countries with carbon dioxide emissions (in thousands of metric tons) from 10 000 to less than 100 000 is equal to:

A 21 **B** 25 **C** 26 **D** 50 **E** 60

6 The following histogram shows the amount spent by tourists from several countries in one year (*spending*), plotted on a $\log_{10}$ scale.



The number of countries where tourists spent from \$100 000 000 to less than \$1 000 000 000 per year is equal to:

A 12 **B** 25 **C** 33 **D** 37 **E** 51

1F Measures of centre and spread

Learning intentions

- To be able to understand the mean and the median as measures of centre.
- To be able to understand the range, interquartile range and standard deviation as measures of spread.
- To be able to know whether to use the median and interquartile range, or the mean and standard deviation, for a particular distribution.
- To be able to use a CAS calculator to calculate these summary statistics.

The median, range and interquartile range

The most versatile statistical tools for numerically describing the centre and spread of a distribution are:

- the **median** (the middle value) as a measure of **centre**;
- the **range** (the maximum spread of the data values), and the **interquartile range** (the spread of the middle half of data values) as measures of **spread**.

While these statistical values could be estimated only approximately from a histogram, they can be determined exactly when we use either a dot or stem plot.

Determining the median

We begin by revisiting the rule for locating the median of a data set.

The median

The median is the middle value in an ordered data set.

For n data values the median is located at the $\left(\frac{n+1}{2}\right)$th position.

When:

- n is odd, the median will be the middle data value
- n is even, the median will be the average of the two middle data values.

Example 14 Finding the median value in a data set

Order each of the following data sets, locate the median, and then write down its value.

a 2 9 1 8 3 5 3 8 1 **b** 10 1 3 4 8 6 10 1 2 9

Explanation	Solution
a For an odd number of data values, the median will be the middle data value.	
1 Write down the data set in order.	1 1 2 3 3 5 8 8 9
	1 1 2 3 3 5 8 8 9
2 Locate the middle data value by eye or use the rule.	Median is the $\left(\frac{9+1}{2}\right)$th or fifth value.
3 Write down the median.	Median = 3
b For an even number of data values, the median will be the average of the two middle data values.	

1 Write down the data set in order.	1 1 2 3 4 6 8 9 10 10
	1 1 2 3 4 6 8 9 10 10
2 Locate the two middle data values and find their average or use the rule.	Median is the average of the 5th and 6th values.
Write down the median.	Median = $\left(\frac{4+6}{2}\right) = 5$

Note: You can check that you are correct by counting the number of data values each side of the median. They should be equal.

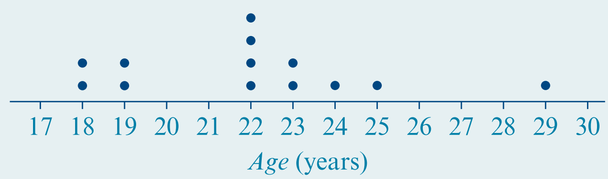
Using a dot plot to help locate medians

The process of calculating a median, as outlined above, is very simple in theory but can be time-consuming in practice. This is because you have to spend most of your time ordering the data set. For a very large data set this is a calculator task.

However, even for a reasonably large data set, locating a median in a dot or stem plot requires no more than counting because the data are already ordered for you.

Example 15 Finding the median value from a dot plot

The dot plot opposite displays the age distribution (in years) of the 13 members of a local cricket team.



Determine the median age of these cricketers and mark its location on the dot plot.

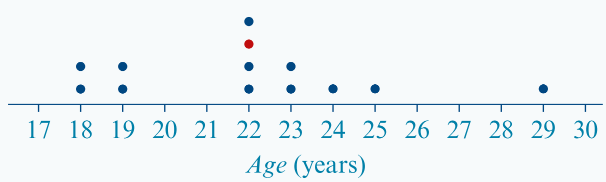
Explanation

The median value is the middle data value in the dot plot.

1 Locate the middle data value (or use the rule) and identify it on the dot plot.

2 Write down its value.

Solution



Median = 22 years

Example 16 Finding the median value from a stem plot

The stem plot opposite displays the maximum temperature (in °C) for 12 days in January.

Determine the median maximum temperature for these 12 days.

Key: 0|8 = 8°C

1	8 9 9
2	0 2 5 7 8 9 9
3	1 3

Explanation	Solution
1 For an even number of data values, as in this example, the median will be the average of the two middle data values.	Key: 0|8 = 8°C
2 Locate the two middle data values in the dot plot by eye (or use the rule) and identify them on the plot.	1 | 8 9 9 2 | 0 2 5 7 8 9 9 3 | 1 3
3 Determine the median by finding the average of these two data values.	$M = \dfrac{25 + 27}{2} = 26°\text{C}$

Having found the median value in a dot plot or stem plot, we now look at ways of doing the same with the first measure of spread, the range.

The range

The range

The range, R, is the simplest measure of spread of a distribution. It is the difference between the largest and smallest values in the data set.

$$R = \text{largest data value} - \text{smallest data value}$$

Example 17 Finding the range from a stem plot

The stem plot opposite displays the maximum temperature (in °C) for 12 days in January.

Determine the temperature range over these 12 days.

Key: 0|8 = 8°C

1	8 9 9
2	0 2 5 7 8 9 9
3	1 3

Explanation	Solution
1 Identify the lowest and highest values in the stem plot and write them down.	Key: 0|8 = 8°C 1 | 8 9 9 2 | 0 2 5 7 8 9 9 3 | 1 3
2 Substitute into the rule for the range and evaluate.	Lowest = 18, highest = 33, range = 33 − 18 = 15°C

Because the range depends only on the two extreme values in the data, it is not always an informative measure of spread. For example, one or other of these two values might be an outlier. Furthermore, any data with the same highest and lowest values will have the same range, irrespective of the way in which the data are spread out in between.

A more refined measure of spread that overcomes these limitations of the range is the interquartile range (*IQR*).

The interquartile range (*IQR*)

Quartiles

To determine the value of the *IQR,* we first need to determine the quartiles.

Just as the median is the point that divides a distribution in half, **quartiles** are the points that divide a distribution into quarters. We use the symbols Q_1, Q_2 and Q_3 to represent the quartiles. Note that the second quartile, Q_2, is the median.

Determining the interquartile range

To find the interquartile range of a distribution:

- arrange all observations in order according to size
- divide the observations into two equal-sized groups, and if n is odd, omit the median from both groups
- locate Q_1, the **first quartile**, which is the median of the lower half of the observations
- locate Q_3, the **third quartile**, which is the median of the upper half of the observations.

The interquartile range *IQR* is then: $IQR = Q_3 - Q_1$

We can interpret the interquartile range as follows:

- Since Q_1, the first quartile, is the median of the lower half of the observations, then it follows that 25% of the data values are less than Q_1, and 75% are greater than Q_1.
- Since Q_3, the third quartile, is the median of the upper half of the observations, then it follows that 75% of the data values are less than Q_3, and 25% are greater than Q_3.
- Thus, the interquartile range (*IQR*) gives the spread of the middle 50% of data values.

Example 18 Finding the *IQR* from an ordered stem plot when *n* is even

Find the interquartile range of the weights of the 18 cats whose weights are displayed in the ordered stem plot below.

Weight (kg) 1 | 2 represents 1.2 kg

1	9
2	1 3 5 8
3	0 0 4 9 9
4	0 4 5 8
5	0 3
6	3 4

Explanation	Solution
1 There are 18 values in total. This means that there are nine values in the lower 'half', and nine in the upper 'half'.	

2 The median of the lower half (Q_1) is the middle of lower nine values, which is the 5th value from the bottom.

Lower half:
1.9 2.1 2.3 2.5 2.8 3.0 3.0 3.4 3.9
$Q_1 = 2.8$

3 The median of the upper half (Q_3) is the middle of the upper nine values, which is the 5th value from the top.

Upper half:
3.9 4.0 4.4 4.5 4.8 5.0 5.3 6.3 6.4
$Q_3 = 4.8$

4 Determine the IQR using $IQR = Q_3 - Q_1$.

$IQR = Q_3 - Q_1 = 4.8 - 2.8 = 2.0$

Example 19 Finding the IQR from an ordered stem plot when n is odd

The stem plot shows the life expectancy (in years) for 23 countries. Find the IQR for life expectancies.

Stem: 5|2 = 52 years

5	2
5	5 6
6	4
6	6 6 7 9
7	1 2 2 3 3 4 4 4 4
7	5 5 6 6 7 7

Explanation

1 Since there are 23 values, the median is the 12th value from either end which is 73. Mark the value 73 on the stem plot.

2 Since n is odd, to find the quartiles the median value is excluded. This leaves 11 values below the median and 11 values above the median. Then:

- Q_1 = midpoint of the bottom 11 data values
- Q_3 = midpoint of the top 11 data values.

Write these values down.

Solution

Stem: 5|2 = 52 years

5	2
5	5 6
6	4
6	6 6 7 9
7	1 2 2 ~~3~~ 3 4 4 4 4
7	5 5 6 6 7 7

Q1
median
Q3

$Q_1 = 66$, $Q_3 = 75$

3 Determine the IQR using $IQR = Q_3 - Q_1$.

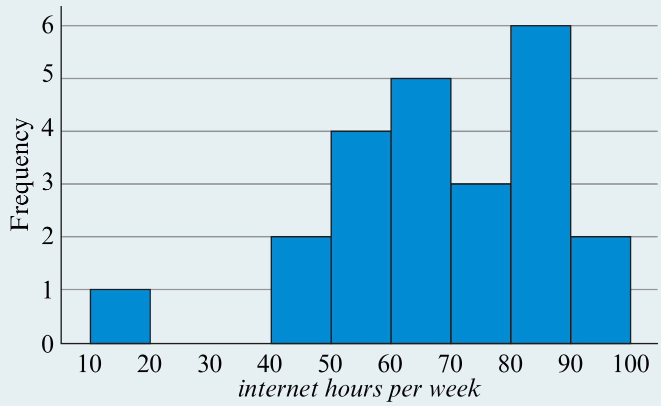
$IQR = Q_3 - Q_1 = 75 - 66 = 9$

To check that these quartiles are correct, write the data values in order, and mark the median and the quartiles. If correct, the median divides the data set up into four equal groups.

$$\underbrace{52\ \ 55\ \ 56\ \ 64\ \ 66}_{5\text{ values}}\ \ \overset{Q_1}{\mathbf{66}}\ \ \underbrace{67\ \ 69\ \ 71\ \ 72\ \ 72}_{5\text{ values}}\ \ \overset{Q_2\ (=M)}{\mathbf{73}}\ \ \underbrace{73\ \ 74\ \ 74\ \ 74\ \ 74}_{5\text{ values}}\ \ \overset{Q_3}{\mathbf{75}}\ \ \underbrace{75\ \ 76\ \ 76\ \ 77\ \ 77}_{5\text{ values}}$$

Example 20 Finding the median and quartiles from a histogram

The histogram shows the average number of hours per week a group of 23 people spent on the internet. Find possible values for the median and quartiles of this distribution.



Explanation

1 Since there are 23 values, we can locate which interval contains the median by adding the number of values in each interval moving from left to right.

- M = the 12th value from the bottom (adding from left to right)

Solution

There is 1 value in the interval 10-20 (total 1), 2 values in the interval 40-50 (total 3), 4 values in the interval 50-60 (total 7), 5 values in the interval 60-70 (total 12). Thus the median is in the interval 60-70.

Explanation

2 Similarly

- Q_1 = the 6th value from the bottom (adding from left to right)
- Q_3 = the 6th value from the top (adding from right to left)

Solution

Q_1 is in the interval 50-60

Q_3 is in the interval 80-90

Why is the *IQR* a more useful measure of spread than the range?

The *IQR* is a measure of spread of a distribution that includes the middle 50% of observations. Since the upper 25% and lower 25% of observations are discarded, the interquartile range is generally not affected by the presence of outliers.

The mean and standard deviation

So far, we have looked at methods for describing the centre and spread for distributions of any shape. We used the median, *IQR* and range for this purpose. In this section, we will look at alternative measures of centre (the mean) and spread (the standard deviation) that are only useful when working with symmetric distributions without outliers. While this may seem unnecessarily restrictive, these two measures have the advantage of being able to fully describe the centre and spread of a symmetric distribution with only two numbers.

The mean

The **mean** of a set of data is what most people call the 'average'. The mean of a set of data is given by:

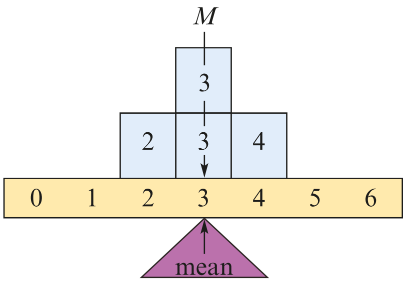
$$\text{mean} = \frac{\text{sum of data values}}{\text{total number of data values}}$$

For example, consider the set of data:

2 3 3 4

The mean of this set of data is given by:

$$\text{mean} = \frac{2+3+3+4}{4} = \frac{12}{4} = 3$$



From a pictorial point of view, the mean is the balance point of a distribution (see above).

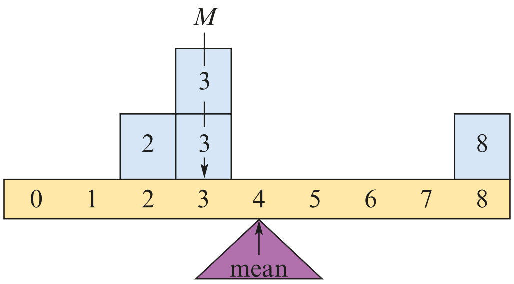
Note that in this case, the mean and the median coincide; the balance point of the distribution is also the point that splits the distribution in half. That is, there are two data points to the left of the mean and two to the right. This is a general characteristic of symmetric distributions.

However, consider the data set

2 3 3 8

The median remains at $M = 3$, but:

$$\text{mean} = \frac{2+3+3+8}{4} = \frac{16}{4} = 4$$



Note that the mean is affected by changing the largest data value but that the median is not.

Some notation

Because the rule for the mean is relatively simple, it is easy to write in words. However, later you will meet other rules for calculating statistical quantities that are extremely complicated and hard to write out in words.

To overcome this problem, we introduce a shorthand notation that enables complex statistical formulas to be written out in a compact form. In this notation, we use:

- the Greek capital letter sigma, $\sum$, as a shorthand way of writing 'sum of'
- a lower case x to represent a data value
- a lower case x with a bar, $\bar{x}$ (pronounced '*x bar*'), to represent the mean of the data values
- an n to represent the total number of data values.

The rule for calculating the mean then becomes: $\bar{x} = \dfrac{\sum x}{n}$

Example 21 **Calculating the mean from the formula**

The following is a set of reaction times (in milliseconds): 38 36 35 43 46 64 48 25

Write down the values of the following, correct to one decimal place.

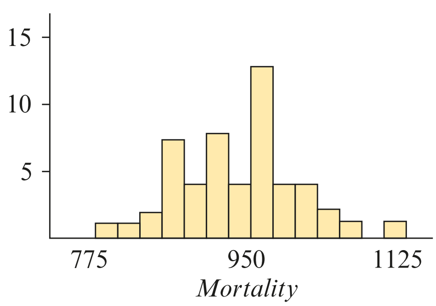
a n **b** $\sum x$ **c** $\bar{x}$

Explanation	Solution
a n is the number of data values.	$n = 8$
b $\sum x$ is the sum of the data values.	$\sum x = 38 + 36 + 35 + 43 + 46 + 64 + 48 + 25$ $= 335$
c $\bar{x}$ is the mean. It is defined by $\bar{x} = \dfrac{\sum x}{n}$.	$\bar{x} = \dfrac{\sum x}{n} = \dfrac{335}{8} = 41.9$

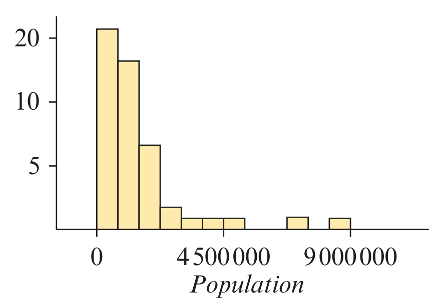
The relationship between the mean and the median

Whereas the median lies at the midpoint of a distribution, the mean is the balance point of the distribution. For approximately symmetric distributions, both the median and mean will be approximately equal in value.

An example of a symmetric distribution is the distribution of mortality rates for 60 US cities shown opposite. Calculations reveal that the mean mortality rate for the cities is 940 per 100 000 while the median mortality rate is 944 per 100 000 people. As expected, the mean and median are approximately equal in value.



An example of a highly skewed distribution is the population distribution of different cities, shown opposite. This distribution is clearly positively skewed with two outliers. The mean population is 1.4 million, while the median population is 0.9 million. They are quite different in value. The mean has been affected by the extreme values in the tail and no longer represents the typical city.



When to use the median rather than the mean

Because the value of the median is relatively unaffected by the presence of extreme values in a distribution, it is said to be a **resistant** statistic. For this reason, the median is frequently used as a measure of centre when the distribution is known to be clearly skewed and/or likely to contain outliers.

For example, median house prices are used to compare housing prices between capital cities in Australia because the distribution of house prices tends to be positively skewed. There are always a small number of very expensive houses sold for much higher prices than the rest of houses sold.

However, if a distribution is symmetric, there will be little difference in the value of the mean and median and we can use either. In such circumstances, the mean is often preferred because:

- it is more familiar to most people
- more can be done with it theoretically, particularly in the area of statistical inference (which you will learn about if you are doing Mathematics Methods).

Choosing between the mean and the median

The mean and the median are both measures of the centre of a distribution. If the distribution is:

- symmetric and there are no outliers, either the mean or the median can be used to indicate the centre of the distribution
- clearly skewed and/or there are outliers, it is more appropriate to use the median to indicate the centre of the distribution.

The standard deviation

To measure the spread of a data distribution around the median (M) we use the interquartile range (IQR). To measure the spread of a data distribution about the mean ($\bar{x}$) we use the **standard deviation** (s).

The standard deviation

The formula for the standard deviation, s, is: $s = \sqrt{\dfrac{\sum(x-\bar{x})^2}{n-1}}$

Although not easy to see from the formula, the standard deviation is an average of the squared deviations of each data value from the mean. We work with the squared deviations because the sum of the deviations around the mean (the balance point) will always be zero.

Calculating the standard deviation

Normally, you will use your calculator to determine the value of a standard deviation. Instructions for the TI-Nspire or ClassPad follow.

CAS 3: How to calculate the mean and standard deviation using the TI-Nspire CAS

The following are the heights (in cm) of a group of women.

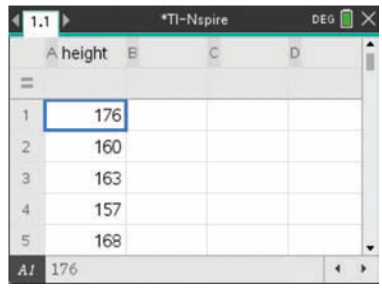
176 160 163 157 168 172 173 169

Determine the mean and standard deviation of the women's heights. Give your answers correct to two decimal places.

Steps

1 Start a new document by pressing ctrl + N.

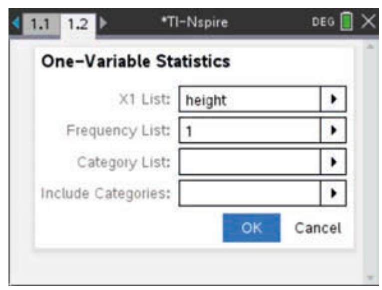
2 Select **Add Lists & Spreadsheet**.
Enter the data into a list named *height*, as shown.



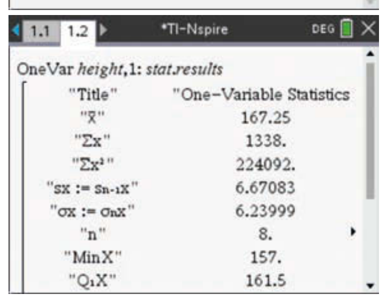
3 Statistical calculations can be done in either the **Lists & Spreadsheet** application or the **Calculator** application (used here).
Press ctrl + I and select **Add Calculator**.

a Press menu>**Statistics>Stat Calculations>One-Variable Statistics**. Press enter to accept the **Num of Lists** as 1.

b **i** To complete this screen, use the ▸ arrow and enter to paste in the list name *height*.



ii Pressing enter exits this screen and generates the results screen shown opposite.



4 Write down the answers to the required degree of accuracy (i.e. two decimal places).

The mean height of the women is $\bar{x} = 167.25$ cm and the standard deviation is $s = 6.67$ cm.

Notes: **a** The sample standard deviation is **sx**.

b Use the ▲ ▼ arrows to scroll through the results screen to obtain values for additional statistical values.

CAS 3: How to calculate the mean and standard deviation using the ClassPad

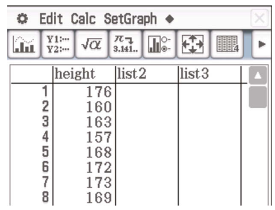
The following are all heights (in cm) of a group of women.

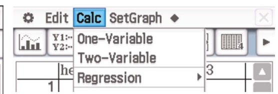
176 160 163 157 168 172 173 169

Determine the mean and standard deviation of the women's heights correct to two decimal places.

Steps

1 Open the **Statistics** application  and enter the data into the column labelled *height*.



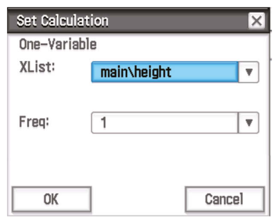


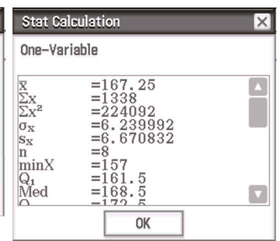
2 To calculate the mean and standard deviation, select **Calc** from the menu One-Variable from the drop-down menu to open the **Set Calculation** dialog box shown below.

3 Complete the dialog box as shown.

- **XList**: select **main\height** (▼).
- **Freq**: leave as **1**.

4 Tap **OK** to confirm your selections and calculate the required statistics, as shown.





5 Write down the answers to two decimal places.

The mean height of the women is $\bar{x} = 167.25$ cm.

The standard deviation is $s_x = 6.67$ cm.

Notes: **a** The value of the standard deviation is given by s_x.

b Use the side-bar arrows to scroll through the results screen to obtain values for additional statistical values (i.e. median, Q_3 and the maximum value) if required.

Exercise 1F

Determining the median from data

Example 14

1 Locate the medians of the following data sets. For each set of data, check that the median divides the ordered data set into two equal groups.

a 4 9 3 1 8 6 **b** 10 9 12 20 14

2 The prices of nine second-hand mountain bikes advertised for sale were as follows.

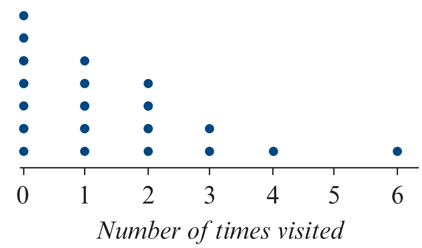
$650 $3500 $750 $500 $1790 $1200 $2950 $430 $850

What is the median price of these bikes?

Determining the median from a dot plot

Example 15

3 The dot plot opposite displays the number of times 20 shoppers visited their supermarket in a week. Find the median number of visits.



Determining the median and range from a stem plot

4 The following stem plot shows the distribution of the time it took (in minutes) for each of a group of 25 people to solve a complex task.

Time (minutes) key: 4|0 represents 4.0

4	2 6
5	1 3 6 8
6	0 1 5 6 7
7	1 3 4 5 7 8 9
8	0 2 5 9
9	5 5
10	6

Example 16
Example 17

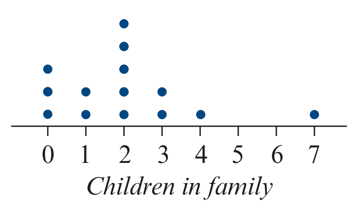
a Find the median time taken.

b Find the range of the time taken.

Determining the median and quartiles from a dot plot

Example 18

5 The dot plot shows the distribution of the number of children in each of 14 families.



a Determine the median, M.

b Determine the quartiles Q_1 and Q_3.

c Calculate the IQR.

d Calculate the range, R.

e By writing the data values in a line, check that the quartiles and the median have divided the data set up into four equal groups.

Determining the median and quartiles from a stem plot

6 The stem plot displays the infant mortality rates (deaths per 1000 live births) in 14 countries.

a Determine the median, M.

b Determine the quartiles Q_1 and Q_3.

c Calculate the IQR and the range, R.

Key: 0|7 = 7

0	7 7 9
1	0 0 0 0 2 2 4
1	5
2	0 1
2	5

7 The stem plot displays the test scores for 20 students.

a Describe the shape of the distribution.

b Determine the median, M.

c Determine the quartiles, Q_1 and Q_3.

d Calculate the IQR and the range, R.

Key: 1|0 = 10

1	0 2
1	5 5 6 9
2	3 3 4
2	5 7 9 9 9
3	0 1 2 4
3	5 9

Example 19

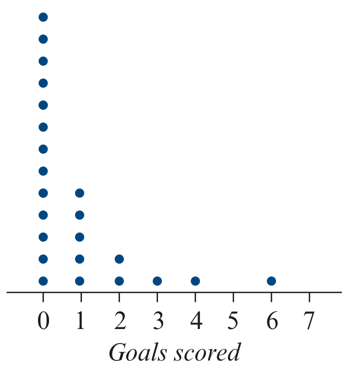
8 The dot plot displays the number of goals scored in 23 games.

a Describe the shape of the distribution and note outliers (if any).

b Without using your calculator determine:

i the median, M.

ii the IQR.



9 The stem plot displays the university participation rates (%) in 17 countries.

a Determine the median, M.

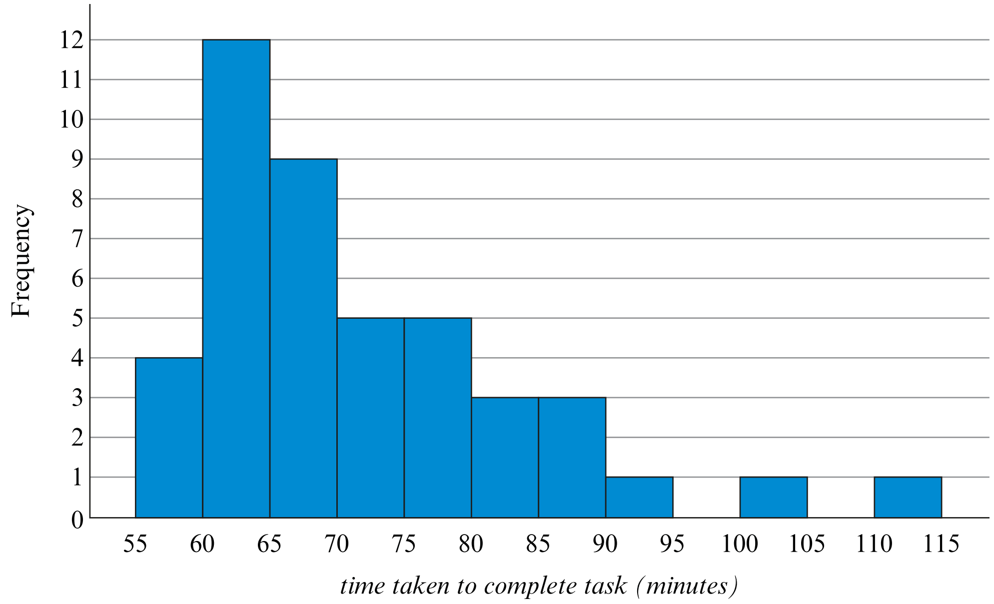
b Determine the quartiles Q_1 and Q_3.

c Calculate the IQR and the range, R.

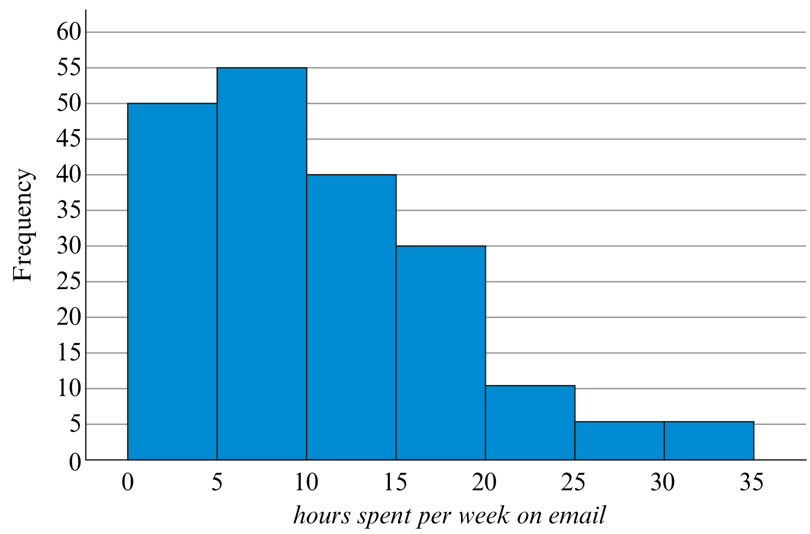
Key: 0|1 = 1

0	1 3 8 9
1	2 3 7
2	0 1 2 5 6 6
3	0 6 7
4	
5	5

Example 20 **10** The histogram shows the time taken to complete a complex task by a group of students. Find possible values for the median and quartiles of this distribution.



11 A group of 195 people were asked to record (to one decimal place) the average number of hours they spent on email each week over a 10 week period. The data are shown in the following histogram:



a Find possible values for the median.

b Find the maximum value for the IQR.

Determining the mean, median and mode from data

Example 21

12 For each of the following data sets, write down the value of n, the value of Σx and hence evaluate $\bar{x}$.

a 2 5 2 3 **b** 12 15 20 32 25 **c** 2 1 3 2 5 3 5

13 Calculate the mean and locate the median and modal value(s) of the following scores.

a 1 3 2 1 2 6 4 5 4 3 2

b 3 12 5 4 3 2 6 5 4 5 5 6

14 The temperature of a hospital patient (in degrees Celsius) taken at 6-hourly intervals over 2 days was as follows.

35.6 36.5 37.2 35.5 36.0 36.5 35.5 36.0

a Calculate the patient's mean and median temperature over the 2-day period.

b What do these values tell you about the distribution of the patient's temperature?

15 The amounts (in dollars) spent by seven customers at a corner store were:

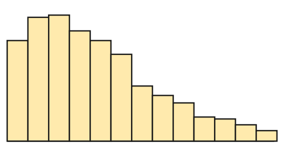
0.90 0.80 2.15 16.55 1.70 0.80 2.65

a Calculate the mean and median amount spent by the customers.

b Does the mean or the median give the best indication of the typical amount spent by customers? Explain your answer.

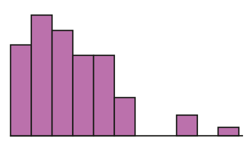
16 For which of the following distributions might you question using the mean as a measure of the centre of the distribution? Justify your selection.

a



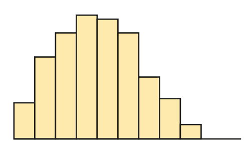
Age distribution in a country

b



Urban car accident rates

c



Blood cholesterol levels

17 The stem plot shows the distribution of weights (in kg) of 22 footballers.

a Name the shape of the distribution. Which measure of centre, the mean or the median, do you think would best indicate the typical weight of these footballers?

b Determine both the mean and median to check your prediction.

Weight (kg)

6	9
7	0 2
7	6 6 7 8
8	0 0 1 2 3 3 4
8	5 5 5 6
9	1 2
9	8
10	3

The concept of standard deviation

18 Which measure of spread:

a always incorporates 50% of the scores?

b uses only the smallest and largest scores in the distribution?

c gives the average variation around the mean?

19 Without using the statistical capabilities of your calculator, write down the mean and standard deviation of the following six data values: 7.1 7.1 7.1 7.1 7.1 7.1

20 For which of the following variables does it *not* make sense to calculate a mean or standard deviation?

a Speed (in km/h) **b** Sex **c** Age (in years)

d Post code **e** Neck circumference (in cm)

f Weight (underweight, normal, overweight)

Calculating the mean and standard deviation using a CAS calculator

21 A sample of 10 students were given a general knowledge test with the following results.

20 20 19 21 21 18 20 22 23 17

a Calculate the mean and standard deviation of the test scores, correct to one decimal place.

b The median test score is 20, which is similar in value to the mean. What does this tell you about the distribution of test scores?

22 Calculate the mean and standard deviation for the variables in the table.
Give answers to the nearest whole number for cars and TVs, and one decimal place for alcohol consumption.

Number of TVs/ 1000	*Number of cars/ 1000*	*Alcohol consumption (litres)*
378	417	17.6
404	286	12.5
471	435	16.0
354	370	24.1
539	217	9.9
381	357	9.5
624	550	14.6

Exam 1 style questions

Use the following information to answer questions 23 and 24

The stem plot displays the number of times each person in a sample of 16 people bought a take-away coffee in the last month.

Key: 0|1 = 1

0	0	0	8	9		
1	2	3	7			
2	0	1	2	5	6	6
3	0	6	7			

23 The median, M, of the number of take away coffees bought is equal to:

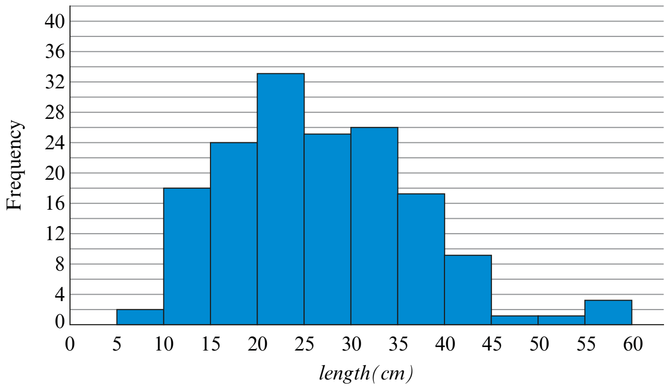
A 10.5 **B** 18.5 **C** 20.5 **D** 21 **E** 26

24 The interquartile range, IQR, of the number of take away coffees bought is equal to:

A 10.5 **B** 15.0 **C** 15.5 **D** 18.5 **E** 20.5

Use the following information to answer questions 25 and 26

The following histogram shows the length (in cm) for each of 159 fish.



25 The median fish length (in cm) could be

A 15.8 **B** 24.3 **C** 25.2 **D** 31.4 **E** 80.0

26 The first quartile (Q_1) for this distribution could be

A 12.2 **B** 16.7 **C** 29.0 **D** 20.2 **E** 25.0

1G The five-number summary and the boxplot

Learning intentions

- To be able to construct the boxplot for displaying the distribution of a numerical data.
- To be able to define and identify **outliers**.
- To be able to construct both **simple boxplots** and **boxplots with outliers**.
- To be able to determine the features of a distribution from a boxplot.
- To be able to use a CAS calculator to construct a boxplot.

The five-number summary

Knowing the median and quartiles tells us quite a lot about the centre and spread of the distribution. If we also knew something about the tails (ends) we would have a good picture of the whole distribution. This can be achieved by recording the smallest and largest values of the data set. Putting all this information together gives the **five-number summary**.

Five-number summary

A listing of the median, M, the quartiles Q_1 and Q_3, and the smallest and largest data values of a distribution, written in the order

$$\text{minimum},\ Q_1,\ M,\ Q_3,\ \text{maximum}$$

is known as a five-number summary.

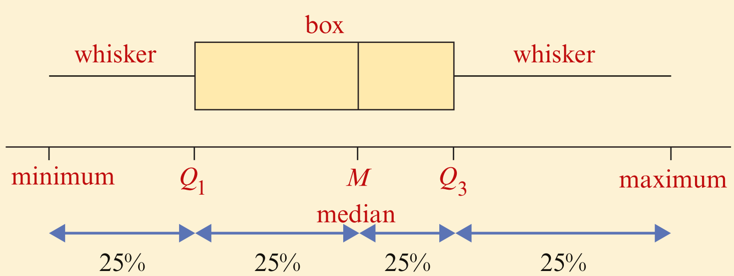
The five-number summary is the starting point for constructing one of the most useful graphical tools in data analysis, the boxplot.

The boxplot

The **boxplot** (or box-and-whisker plot) is a graphical display of a five-number summary. The essential features of a boxplot are summarised below.

The boxplot

A boxplot is a graphical display of a five-number summary.



In a boxplot:

- a box extends from Q_1 to Q_3, locating the middle 50% of the data values
- the median is shown by a vertical line drawn within the box
- lines (called whiskers) are extended out from the lower and upper ends of the box to the smallest and largest data values of the data set respectively
- 25% of the data values are from the minimum to Q_1
- 25% of the data values are from Q_1 to the median M
- 25% of the data values are from the median M to Q_3
- 25% of the data values are from Q_3 to the maximum

Example 22 Constructing a boxplot from a five-number summary

The stem plot shows the distribution of life expectancies (in years) in 23 countries.

The five-number summary for these data is:

minimum	52
first quartile (Q_1)	66
median (M)	73
third quartile (Q_3)	75
maximum	77

Key: 5|2 = 52 years

5 | 2 ← minimum
5 | 5 6
6 | 4
6 | 6 6 7 9
7 | 1 2 2 3 3 4 4 4 4
7 | 5 5 6 6 7 7 ← maximum

Q_1, median, Q_3

Use the five-number summary to construct a boxplot.

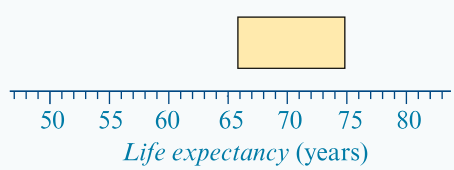
Explanation

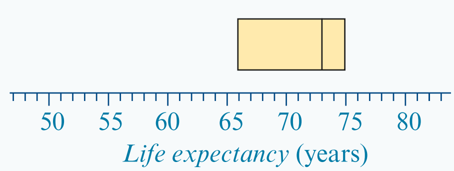
1 Draw a labelled and scaled number line that covers the full range of values.

2 Draw a box starting at $Q_1 = 66$ and ending at $Q_3 = 75$.

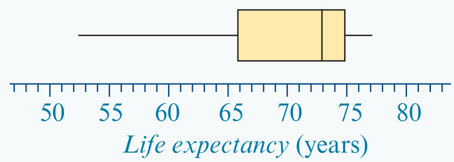
3 Mark the median value with a vertical line segment at $M = 73$.

Solution





4 Draw the whiskers: lines joining the midpoint of the ends of the box to the minimum and maximum values, 52 and 77.



Boxplots with outliers

An extension of the boxplot can also be used to identify possible outliers in a data set.

Sometimes it is difficult to decide whether or not an observation is an outlier. For example, a boxplot might have one extremely long whisker. How might we explain this?

- One explanation is that the data distribution is extremely skewed with lots of data values in its tail.
- Another explanation is that the long whisker hides one or more outliers.

By modifying the boxplots, we can decide which explanation is most likely, but firstly we need a more exact definition of an outlier.

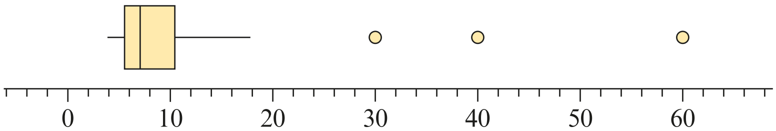
Defining outliers

Outlier

An **outlier** in a distribution is any data point that lies more than 1.5 interquartile ranges below the first quartile or more than 1.5 interquartile ranges above the third quartile.

To be more informative the boxplot can be modified so that the outliers are plotted individually in the boxplot with a dot or cross, and the whisker now ends only to the largest or smallest data value that is not outside these limits.

An example of a boxplot with outliers is shown opposite.



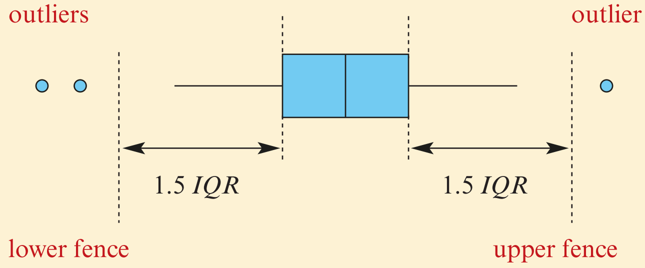
Three of the data values 30, 40, and 60 are possible outliers.

To display outliers on a boxplot, we must first determine the location of what we call the **upper** and **lower fences**. These are imaginary lines drawn one and a half interquartile ranges (or box widths) above and below the box ends, as shown in the diagram following. Data values outside these fences are then classified as possible outliers and plotted separately.

Using a boxplot to display outliers

In a boxplot, possible outliers are defined as being those values that are:

- greater than $Q_3 + 1.5 \times IQR$ (upper fence)
- less than $Q_1 - 1.5 \times IQR$ (lower fence).



When drawing a boxplot, any observation identified as an outlier is shown by a dot. The whiskers end at the smallest and largest values that are not classified as outliers.

While we have used a five-number summary as the starting point for our introduction to boxplots, in practice the starting point for constructing a boxplot is raw data. Constructing a boxplot from raw data is a task for your CAS calculator.

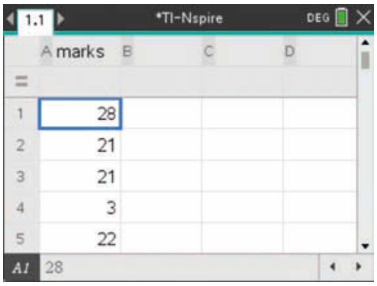
CAS 4: How to construct a boxplot with outliers using the TI-Nspire CAS

Display the following set of 19 marks in the form of a boxplot with outliers.

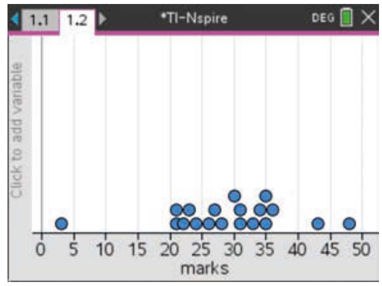
28	21	21	3	22	31	35	26	27	33
43	31	30	34	48	36	35	23	24	

Steps

1. Start a new document by pressing ctrl + N.
2. Select **Add Lists & Spreadsheet**. Enter the data into a list called ***marks*** as shown.
3. Statistical graphing is done through the **Data & Statistics** application. Press ctrl + I and select **Add Data & Statistics**.

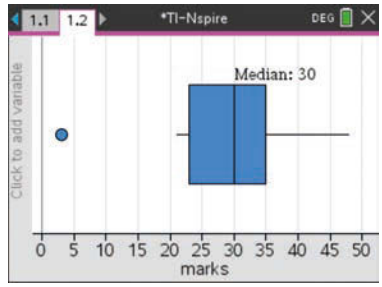


Note: A random display of dots will appear – this indicates that list data are available for plotting. Such a dot is not a statistical plot.



a Click on the **Click to add variable** on the x-axis and select the variable ***marks***. A dot plot is displayed by default as shown opposite.

b To change the plot to a boxplot press menu>**Plot Type**>**boxplot**. Your screen should now look like that shown opposite.



4 Data analysis

Key values can be read from the boxplot by moving the cursor over the plot or using menu>**Analyze>Graph Trace**.

Starting at the far left of the plot, we see that the:

- minimum value is 3 (an outlier)
- first quartile is 23 (**Q_1 = 23**)
- median is 30 (**Median = 30**)
- third quartile is 35 (**Q_3 = 35**)
- maximum value is 48.

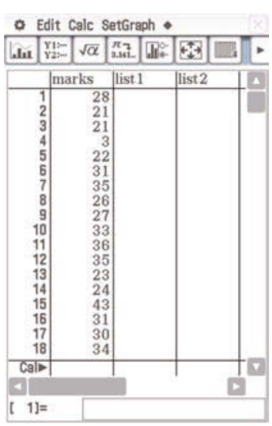
CAS 4: How to construct a boxplot with outliers using the ClassPad

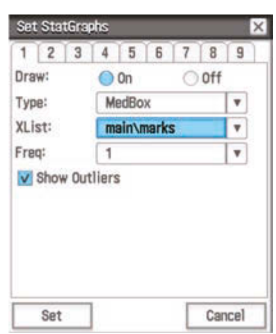
Display the following set of 19 marks in the form of a boxplot with outliers.

28	21	21	3	22	31	35	26	27	33
43	31	30	34	48	36	35	23	24	

Steps

1 Open the **Statistics** application Statistics and enter the data into the column labelled **marks**.



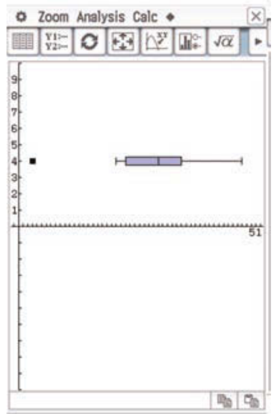


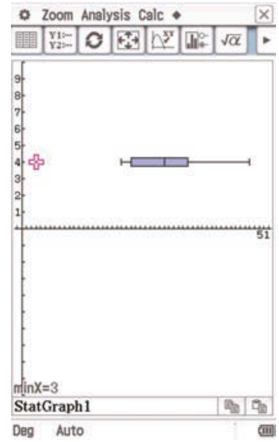
2 Open the **Set StatGraphs** dialog box by tapping in the toolbar. Complete the dialog box as shown below.

- **Draw**: select **On**.
- **Type**: select **MedBox** (▼).
- **XList**: select **main\marks** (▼).
- **Freq**: leave as **1**.

Tap the **Show Outliers** box to tick (☑).

3 Tap **Set** to confirm your selections and plot the boxplot b tapping . The graph is drawn in an automatically scaled window, as shown.



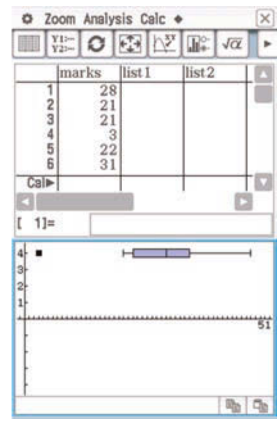


4 Tap the Resize icon at the bottom of the screen for a full-screen graph.

Note: If you have more than one graph on your screen, tap the data screen, select **StatGraph** and turn off any unwanted graphs.

5 Tap to read key values. This places a marker on the boxplot (+), as shown. Use the horizontal cursor arrows (◀) and (▶) to move from point to point on the boxplot. We see that the:

- minimum value is 3 (**minX** = **3**; an outlier)
- first quartile is 23 (**Q_1** = **23**)
- median is 30 (**Med** = **30**)
- third quartile is 35 (**Q_3** = **35**)
- maximum value is 48 (**maxX** = **48**).

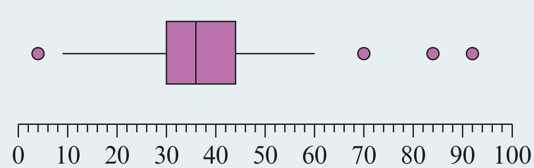


Interpreting boxplots

Constructing a boxplot is not an end in itself. The prime reason to construct boxplots is to help us answer statistical questions. To do this, you need to know how to read values from a boxplot and use them to determine statistics such as the median, the interquartile range and the range. We also use boxplots to identify possible outliers.

Example 23 Reading values from a boxplot

For the boxplot shown, write down the values of:



a the median

b the quartiles Q_1 and Q_3

c the interquartile range (IQR)

d the minimum and maximum values

e the values of any possible outliers

f the smallest value in the upper end of the data set that will be classified as an outlier

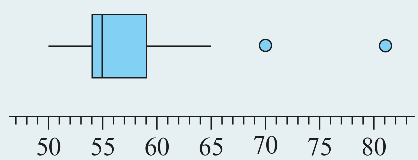
g the largest value in the lower end of the data set that will be classified as an outlier.

Explanation	Solution
a The median (the vertical line in the box)	$M = 36$
b Quartiles Q_1 and Q_3 (end points of box)	$Q_1 = 30$, $Q_3 = 44$
c Interquartile range ($IQR = Q_3 - Q_1$)	$IQR = Q_3 - Q_1 = 44 - 30 = 14$
d Minimum and maximum values (extremes)	$Min = 4$, $Max = 92$
e The values of the possible outliers (dots)	4, 70, 84 and 92 are possible outliers
f Upper fence (given by $Q_3 + 1.5 \times IQR$)	Upper fence $= Q_3 + 1.5 \times IQR$ $= 44 + 1.5 \times 14 = 65$ Any value above 65 is an outlier.
g Lower fence (given by $Q_1 - 1.5 \times IQR$)	Lower fence $= Q_1 - 1.5 \times IQR$ $= 30 - 1.5 \times 14 = 9$ Any value below 9 is an outlier.

Once we know the location of the quartiles, we can use the boxplot to estimate percentages.

Example 24 Estimating percentages from a boxplot

For the boxplot shown, estimate the percentage of values:



a less than 54 **b** less than 55

c less than 59 **d** greater than 59

e between 54 and 59 **f** between 54 and 86.

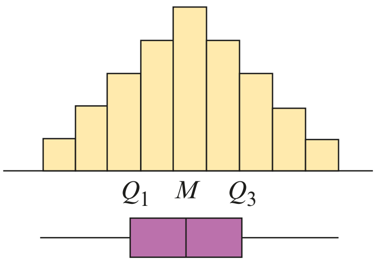
Explanation	Solution
a 54 is the first quartile (Q_1); 25% of values are less than Q_1.	**a** 25%
b 55 is the median or second quartile (Q_2); 50% of values are less than Q_2.	**b** 50%
c 59 is the third quartile (Q_3); 75% of values are less than Q_3.	**c** 75%
d 75% of values are less than 59 and 25% are greater than 59.	**d** 25%
e As 75% of values are less than 59 and 25% are less than 54, 50% of values are between 54 and 59.	**e** 50%
f As 100% of values are less than 86 and 25% of values are less than 54, 75% of values are between 54 and 86.	**f** 75%

Relating a boxplot to shape

When there are a reasonable number of data values, the shape of a distribution can be identified from a boxplot.

A symmetric distribution

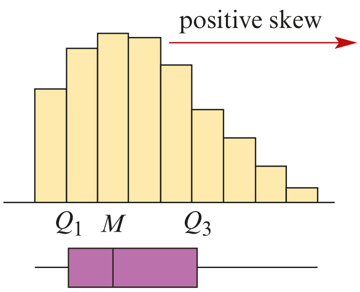
A symmetric distribution tends to be centred on its median and have values evenly spread around the median. As a result, its boxplot will also be symmetric, its median is close to the middle of the box and its whiskers are approximately equal in length.



Positively skewed distributions

Positively skewed distributions are characterised by a cluster of data values around the median at the left-hand end of the distribution with a gradual tailing off to the right.

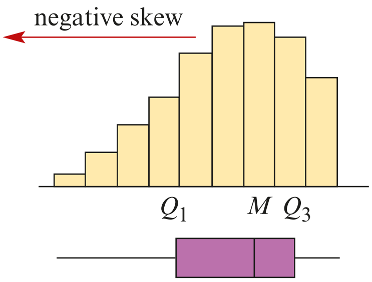
As a result, the boxplot of a positively skewed distribution will have its median off-centre and to the left-hand side of its box. The left-hand whisker will be short, while the right-hand whisker will be long, reflecting the gradual tailing off of data values to the right.



Negatively skewed distributions

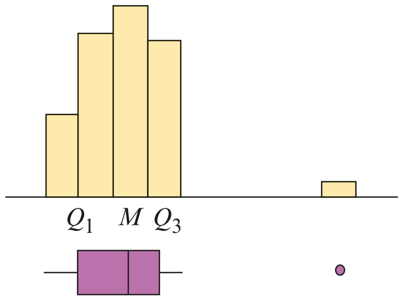
Negatively skewed distributions are characterised by a clustering of data values around the median at the right-hand end of the distribution, with a gradual tailing off of data values to the left.

As a result, the boxplot of a negatively skewed distribution has the median off-centre and in the right-hand side of its box. The right-hand whisker will be short, while the left-hand whisker will be long, reflecting the gradual tailing off of data values to the left.



Distributions with outliers

Distributions with outliers are characterised by large gaps between the main body and data values in the tails. The histogram opposite displays a distribution with an outlier. In the corresponding boxplot, the box and whiskers represent the main body of data and the dot, separated by a gap from the box and whiskers, an outlier.

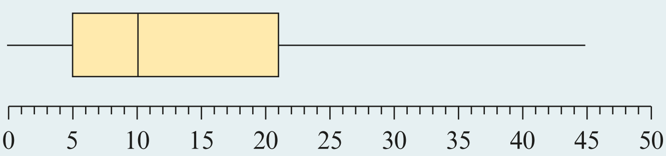


Using boxplots to describe a distribution

Because of the wealth of information contained in a boxplot, it is an extremely powerful tool for describing the features of distribution in terms of shape, centre, spread and outliers.

Example 25 **Using a boxplot to describe the features of distribution without outliers**

Describe the distribution represented by the boxplot in terms of shape, centre and spread. Give appropriate values.

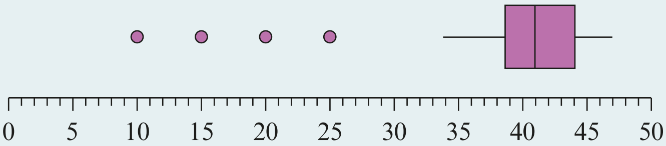


Solution

The distribution is positively skewed with no outliers. The distribution is centred at 10, the median value. The spread of the distribution, as measured by the *IQR*, is 16 and, as measured by the range, 45.

Example 26 **Using a boxplot to describe the features of a distribution with outliers**

Describe the distributions represented by the boxplot in terms of shape and outliers, centre and spread. Give appropriate values.



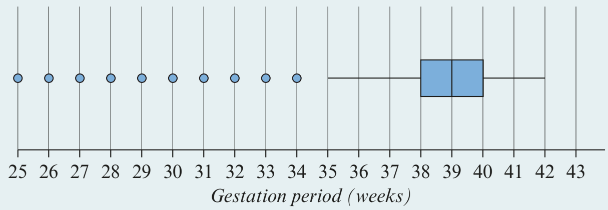
Solution

The distribution is symmetric but with outliers. The distribution is centred at 41, the median value. The spread of the distribution, as measured by the *IQR*, is 5.5 and, as measured by the range, 37. There are four outliers: 10, 15, 20 and 25.

Earlier in this chapter we attempted to describe the feature of a distribution from a histogram. We found that from the histogram it was difficult to give exact values for centre and spread, and to clearly identify outliers. This is much easier to do from a boxplot.

Example 27 Using a boxplot to answer statistical questions

The boxplot shows the gestation period (completed weeks) for a sample for 1000 babies born in Australia one year. Describe the distribution of gestation period in terms of shape, centre, spread and outliers.



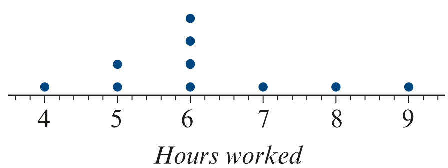
Solution

The distribution of gestational period is negatively skewed with several outliers. The distribution is centred at 39 weeks, the median value. The range of the distribution is 17 weeks, but the interquartile range is only 2 weeks. Any gestational period less than 35 weeks or less is considered unusual, with outliers at 25, 26, 27, 28, 29, 30, 31, 32, 33 and 34 weeks.

Exercise 1G

Constructing a five-number summary from a dot or stem plot

1 Construct a five-number summary for the dot plot opposite.



2 Construct a five-number summary for the stem plot opposite.

Key: 13|6 = 136

13	6 7
14	3 6 8 8 9
15	2 5 8 8 8
16	4 5 5 6 7 9
17	8 8 9
18	2 9

Constructing a boxplot from a five-number summary

Example 22

3 Use the following five-number summaries to construct boxplots.

a Min = 1, Q_1 = 4, M = 8, Q_3 = 13.5, Max = 24

b Min = 136, Q_1 = 148, M = 158, Q_3 = 169, Max = 189

4 **a** Construct a boxplot from the following five-number summary:

Min = 10, Q_1 = 22, M = 40, Q_3 = 70, Max = 70

b Explain why the box has no upper whisker.

5 University participation rates (%) in 21 countries are listed below.

3	3	7	8	9	12	13	15	17	20	21
22	25	26	26	26	27	30	36	37	55	

a Show that the five number summary for this data is:
Min =3, Q_1 = 10.5, M = 21, Q_3 = 26.5, Max = 55

b Show that the upper fence is equal to 50.5.

c Explain why this boxplot will show at least one outlier.

d Construct a boxplot showing the outlier.

6 The five-number summary for a data set is:

Min = 14 Q_1 = 40 M = 55 Q_3 = 62 Max = 99

a Determine the values of the upper and lower fences.

b The smallest three values in the data set are 6, 18, 34 and the largest are 90, 94, 99. Which of these are outliers?

Constructing a boxplot using a CAS calculator

7 The reaction times (in milliseconds) of 18 people are listed below.

38 36 35 35 43 46 42 64 40 48 35 34 40 44 30 25 39 31

a Use a CAS calculator to construct a boxplot with outliers for the data. Name the variable *rtime*.

b Use the boxplot to construct a five-number summary. Identify the outlier.

Reading values from a boxplot

Example 23

8 For each of the boxplots below, estimate the values of:

i the median, M

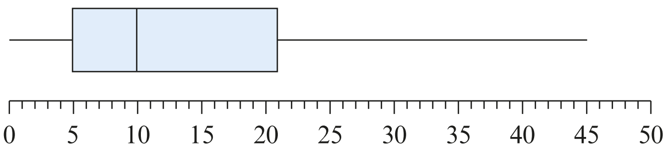
ii the quartiles Q_1 and Q_3

iii the interquartile range, IQR

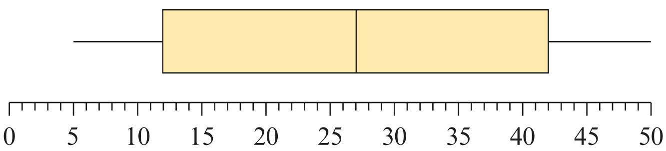
iv the minimum and maximum values

v the values of possible outliers.

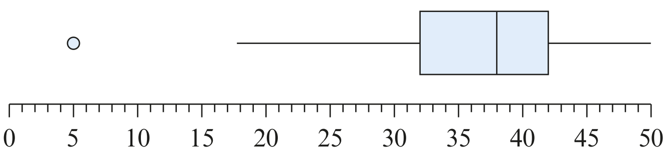
a



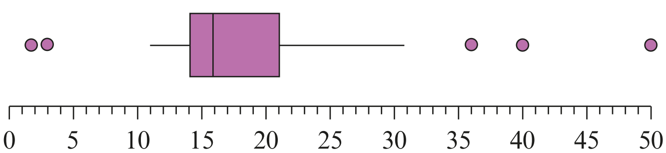
b



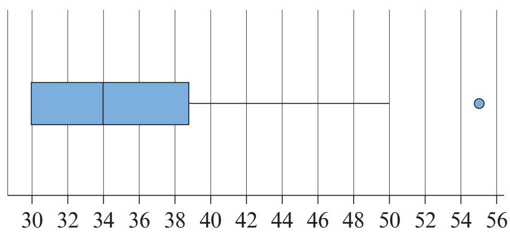
c



d



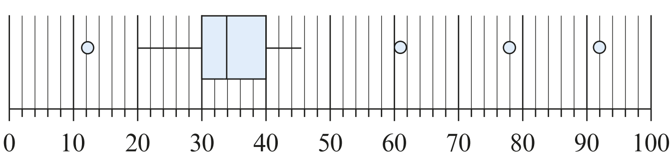
e



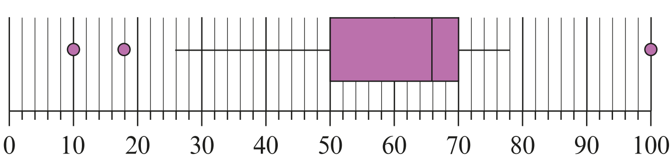
9 For the boxplots below, determine the location of:

i the upper fence **ii** the lower fence.

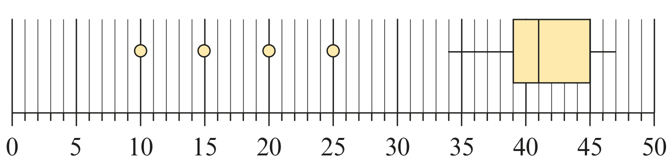
a



b



10 **a** Determine the lower fence for the boxplot opposite.

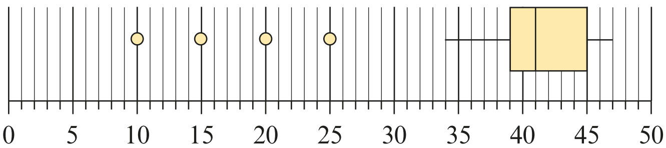


b When the data were originally entered, a value of 31 was incorrectly entered as 35. Would the 31 be shown as an outlier when the error is corrected? Explain your answer.

Reading percentages from a boxplot

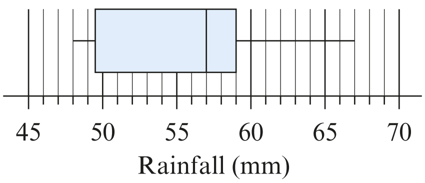
Example 24

11 Use the boxplot opposite to estimate the percentage of values that are:



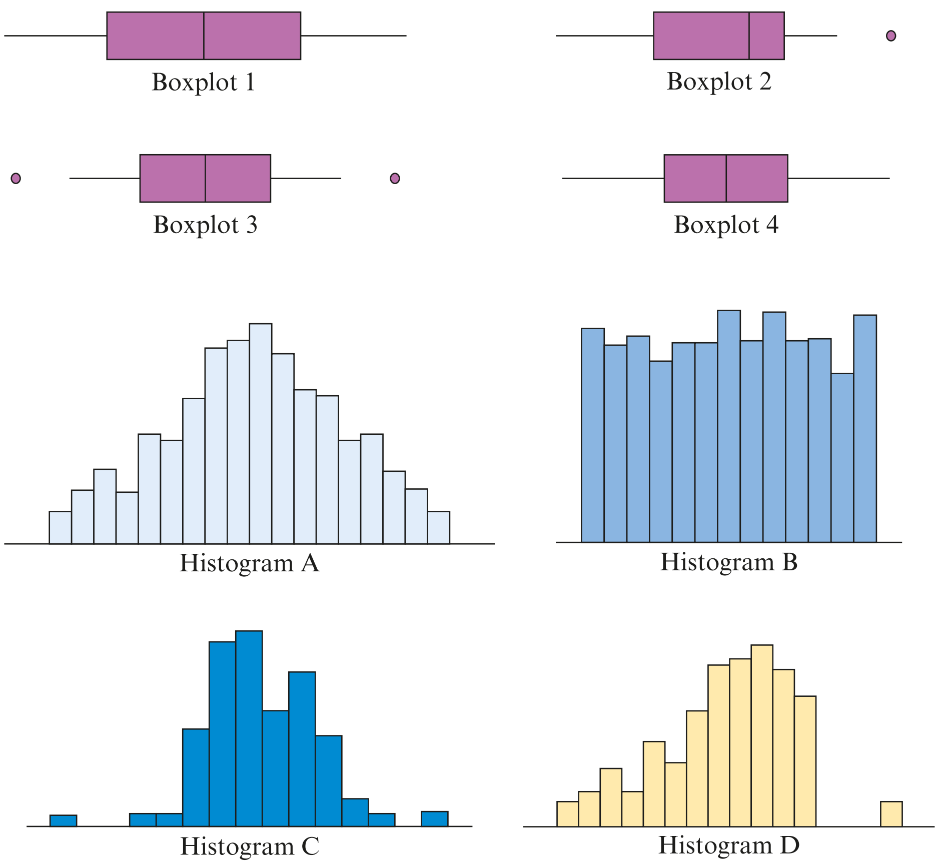
a less than 39 **b** less than 45 **c** greater than 45
d between 39 and 45 **e** between 5 and 45.

12 The boxplot displays the monthly rainfall (in mm) for 12 months.
Use the boxplot to estimate the percentage of months in which the monthly rainfall was:



a greater than 59 mm **b** less than 49.5 mm **c** between 49.5 and 59 mm
d between 57 and 59 mm **e** less than 59 mm **f** between 57 and 70 mm.

13 Match these boxplots with their histograms.

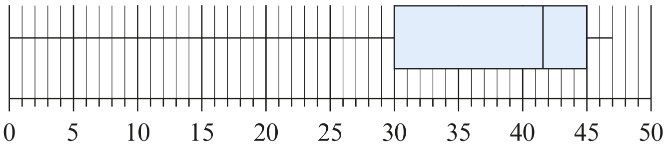


Describing the features of a distribution from a boxplot

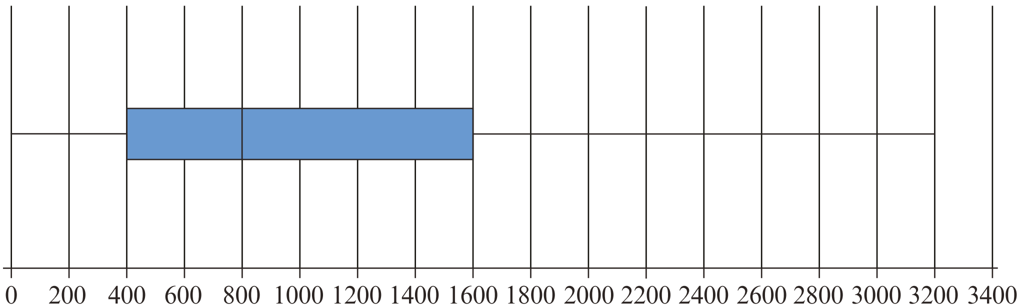
Example 25

14 Describe the distributions represented by the following boxplots in terms of shape, centre, spread. Give appropriate values.

a



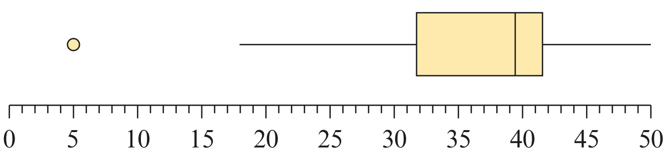
b



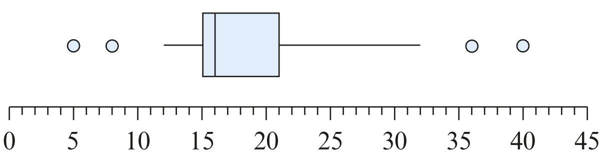
Example 26

15 Describe the distributions represented by the following boxplots in terms of shape, centre, spread and outliers (if any). Give appropriate values.

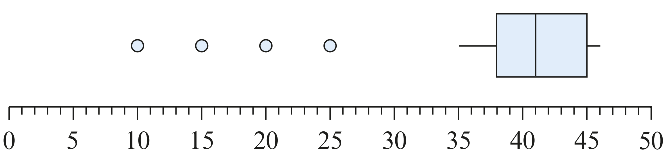
a



b



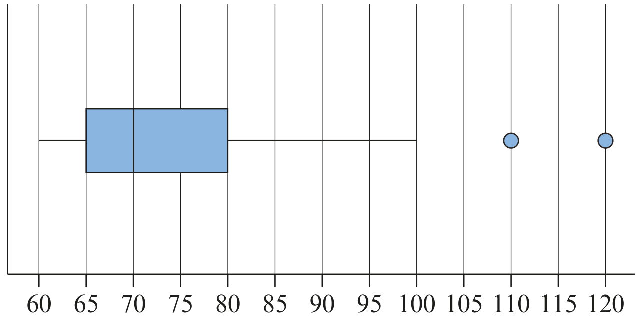
c



Using a boxplot to answer statistical questions

Example 27

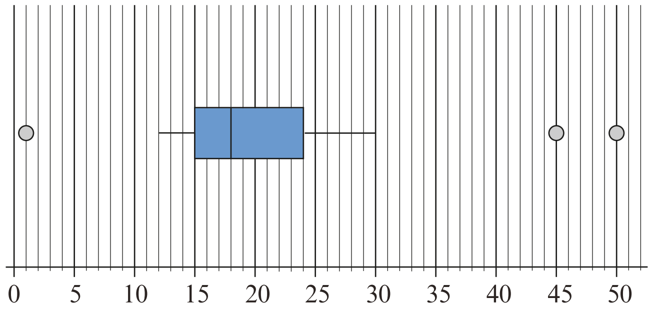
16 Taj recorded his travel time to university (in minutes) each day for 60 days, and summarised the data in the following boxplot. Write a brief report describing the distribution of his travel time.



Exam 1 style questions

Use the following information to answer questions 17 and 18

The boxplot below shows the distribution of marks scored by 200 students in a test.



17 The percentage of students who scored more than 24 marks is closest to:

A 15% **B** 25% **C** 50% **D** 75% **E** 100%

18 The five-number summary for the test scores is closest to:

A 1, 15, 18, 24, 50

B 12, 15, 18, 24, 30

C 1, 12, 15, 18, 24

D 12, 15, 18, 24, 50

E 12, 15, 24, 30, 50

Use the following information to answer questions 19 and 20

The weights (in gm) of 159 fish were measured, and the table gives the mean and the five-number summary for this data.

mean	398
minimum	20
first quartile (Q_1)	120
median (M)	273
third quartile (Q_3)	650
maximum	1650

19 The shape of the distribution is best described as

A approximately symmetric

B positively skewed

C symmetrically skewed

D uniform

E negatively skewed

20 The largest five values in the data set are 1100gm, 1250gm, 1550gm, 1600gm and 1650gm. Which of these are outliers?

A {1100, 1250, 1550, 1600,1650} **B** {1650}

C {1250, 1550, 1600,1650} **D** {1550, 1600,1650}

E {1600,1650}

1H The normal distribution and the 68–95–99.7% rule

Learning intentions

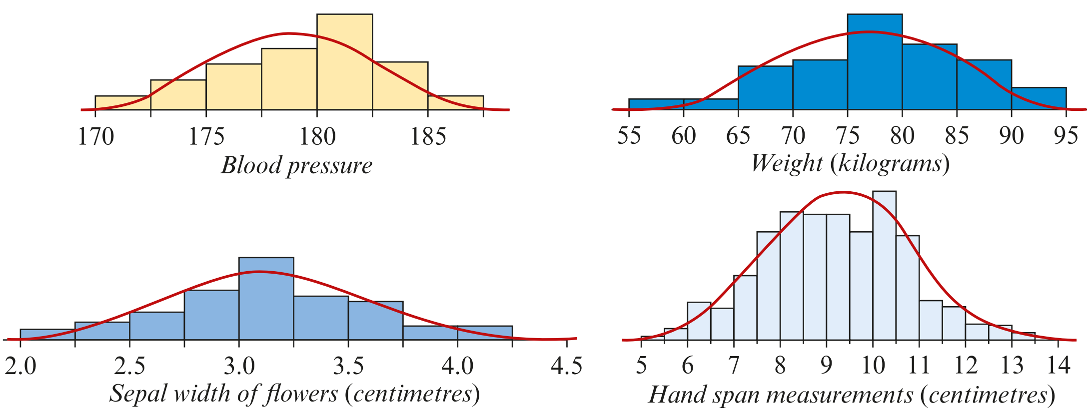
- To be able to introduce the normal model for bell-shaped distributions.
- To be able to use the 68 -95 - 99.7% rule to estimate percentages and give meaning to the standard deviation.
- To be able to calculate standardised scores and use them to compare values across distributions.

We know that the interquartile range is the spread of the middle 50% of the data set. Can we find some similar way in which to interpret the standard deviation?

It turns out we can, but we need to restrict ourselves to symmetric distributions that have an approximate bell shape. Again, while this may sound very restrictive, many of the data distributions we work with in statistics (but not all) can be well approximated by this type of distribution. In fact, it is so common that it is called the **normal distribution**.

The normal distribution

Many data sets that arise in practice are roughly symmetrical and have approximate bell shapes, as shown in the four examples below.



Data distributions that are bell-shaped can be modelled by a normal distribution.

The 68–95–99.7% rule

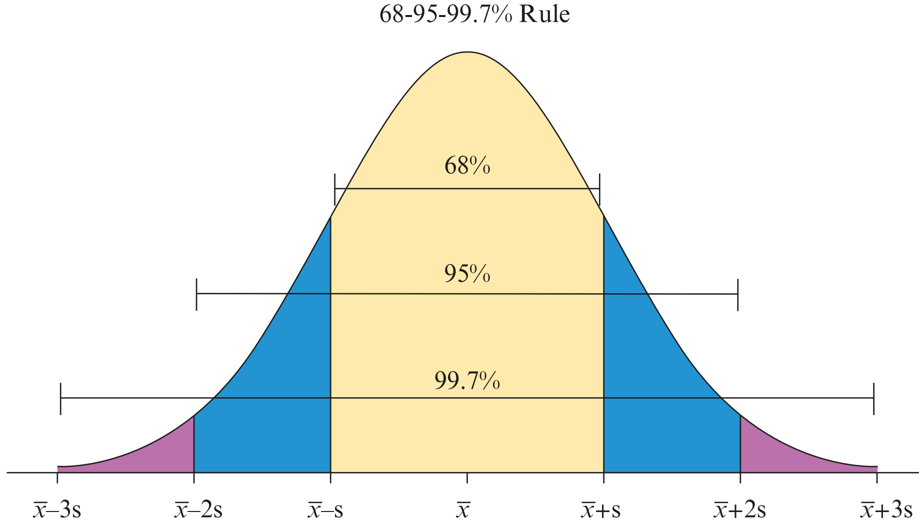
In normal distributions, the percentage of observations that lie within a certain number of standard deviations of the mean can always be determined. In particular, we are interested in the percentage of observations that lie within one, two or three standard deviations of the mean. This gives rise to what is known as the **68–95–99.7% rule**.

The 68–95–99.7% rule

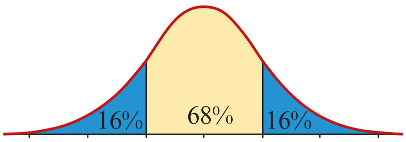
For a normal distribution, approximately:

- *68%* of the observations lie within **one** standard deviation of the mean, that is in the interval $(\bar{x} - s, \bar{x} + s)$.
- *95%* of the observations lie within **two** standard deviations of the mean, that is in the interval $(\bar{x} - 2s, \bar{x} + 2s)$.
- *99.7%* of the observations lie within **three** standard deviations of the mean, that is in the interval $(\bar{x} - 3s, \bar{x} + 3s)$.

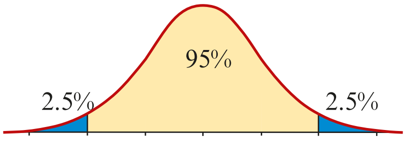
To give you an understanding of what this rule means in practice, it is helpful to view this rule graphically.



Since the normal distribution is symmetrical, and 100% of the observations are within the normal curve, we can use the 68–95–99.7% rule to allocate percentages to the tails of the distribution in each instance.

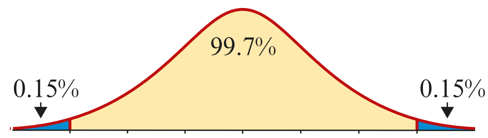


- Since around *68%* of the data values will lie within one standard deviation (SD) of the mean, then we can also say that around 16% of values lie in each of the tails.

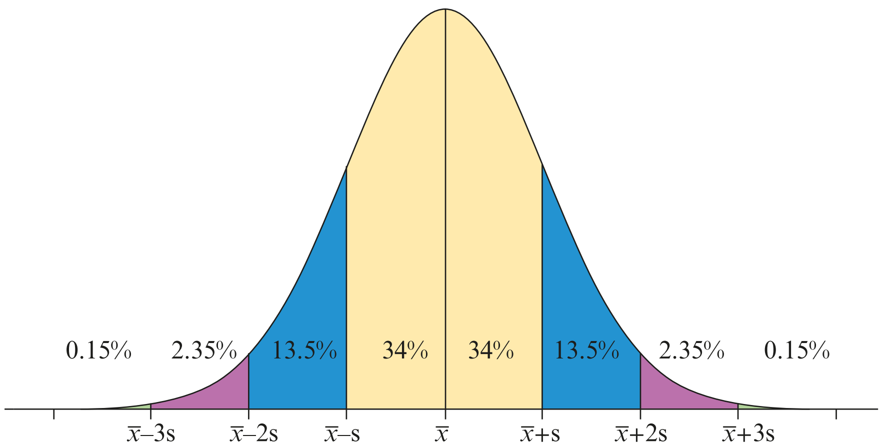


- Since around 95% of the data values will lie within two standard deviations of the mean, then we can also say that around 2.5% of values lie in each of the tails.

- Since around 99.7% of the data values will lie within three standard deviations of the mean we can also say that around 0.15% of values lie in each of the tails.



Putting together all of this information, we can then allocate percentages of the data which fall into each section of the normal curve, as shown in the following diagram:



Example 28 **Applying the 68–95–99.7% rule**

The distribution of delivery times for pizzas made by House of Pizza is approximately normal, with a mean of 25 minutes and a standard deviation of 5 minutes.

a What percentage of pizzas have delivery times of between 15 and 35 minutes?

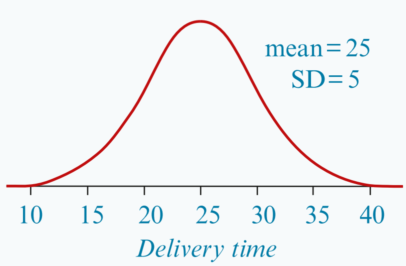
b What percentage of pizzas have delivery times of greater than 30 minutes?

c In 1 month, House of Pizza delivers 2000 pizzas. Approximately many of these pizzas are delivered in less than 10 minutes?

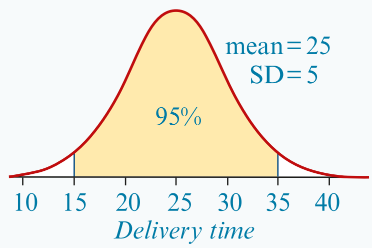
Explanation

a **1** Sketch, scale and label a normal distribution curve with a mean of 25 and a standard deviation of 5.

Solution



2 Shade the region under the normal curve representing delivery times of between 15 and 35 minutes.

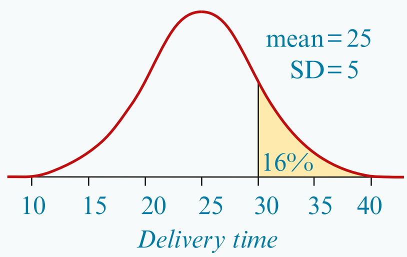


3 Note that delivery times of between 15 and 35 minutes lie within *two* standard deviations of the mean. ($15 = 25 - \mathbf{2} \times 5$ and $35 = 25 + \mathbf{2} \times 5$)

4 95% of values are within two standard deviations of the mean. Use this information to write your answer.

95% of pizzas will have delivery times of between 15 and 35 minutes.

b **1** As before, draw, scale and label a normal distribution curve with a mean of 25 and a standard deviation of 5. Shade the region under the normal curve representing delivery times of greater than 30 minutes.



2 Delivery times of greater than 30 minutes are more than *one* standard deviation above the mean. ($30 = 25 + \mathbf{1} \times 5$)

3 16% of values are more than one standard deviation above the mean. Write your answer.

16% of pizzas will have delivery times of greater than 30 minutes.

c **1** Write down the number of pizzas delivered.

Number = 2000

2 Delivery times of less than 10 minutes are more than *three* standard deviations below the mean. ($10 = 25 - \mathbf{3} \times 5$).

Percentage delivered in less than 10 minutes = 0.15%

3 0.15% of values are more than *three* standard deviations below the mean. Record this.

4 Therefore, the number of pizzas delivered in less than 10 minutes is 0.15% of 2000.

Number of pizzas delivered in less than 10 minutes ≈ 0.15% of 2000

$$= \frac{0.15}{100} \times 2000 = 3$$

Standard scores

The 68–95–99.7% rule makes the standard deviation a natural measuring stick for normally distributed data.

For example, a person who obtained a score of 112 on an IQ test with a mean of 100 and a standard deviation of 15 has an IQ score less than one standard deviation from the mean. Her score is typical of the group as a whole, as it lies well within the middle 68% of scores. In contrast, a person who scores 133 stands out; her score is more than two standard deviations from the mean and this puts her in the top 2.5%.

Because of the additional insight provided by relating the standard deviations to percentages, it is common to transform data into a new set of units that show the number of standard deviations a data value lies from the mean of the distribution. This is called **standardising** and these transformed data values are called **standardised** or ***z*-scores**.

Calculating standardised (z) scores

To obtain a standard score for an actual score, subtract the mean from the score and then divide the result by the standard deviation. That is:

$$\text{standard score} = \frac{\text{actual score} - \text{mean}}{\text{standard deviation}} \quad \text{or} \quad z = \frac{x - \bar{x}}{s}$$

Let us check to see that the formula works.

We already know that an IQ score of 115 is one standard deviation above the mean, so it should have a standard or z-score of 1. Substituting into the formula above we find, as we had predicted, that:

$$z = \frac{115 - 100}{15} = \frac{15}{15} = 1$$

Standard scores can be both positive and negative:

- a positive z-score indicates that the actual score it represents lies above the mean
- a z-score of zero indicates that the actual score is equal to the mean
- a negative z-score indicates that the actual score lies below the mean.

Example 29 Calculating standard scores

The heights of a group of young women have a mean of $\bar{x} = 160$ cm and a standard deviation of $s = 8$ cm. Determine the standard or z-scores of a woman who is:

a 172 cm tall **b** 150 cm tall **c** 160 cm tall.

Explanation	Solution
1 Write down the data value (x), the mean ($\bar{x}$) and the standard deviation (s).	**a** $x = 172,\ \bar{x} = 160,\ s = 8$ $z = \frac{x - \bar{x}}{s} = \frac{172 - 160}{8} = \frac{12}{8} = 1.5$
2 Substitute the values into the formula $z = \frac{x - \bar{x}}{s}$ and evaluate.	**b** $x = 150,\ \bar{x} = 160,\ s = 8$ $z = \frac{x - \bar{x}}{s} = \frac{150 - 160}{8} = -\frac{10}{8} = -1.25$
	c $x = 160,\ \bar{x} = 160,\ s = 8$ $z = \frac{x - \bar{x}}{s} = \frac{160 - 160}{8} = \frac{0}{8} = 0$

Using standard scores to compare performance

Standard scores are also useful for comparing groups that have different means and/or standard deviations. For example, consider a student who obtained a mark of 75 in Psychology and a mark of 70 in Statistics. In which subject did she do better?

Calculating standard scores

Subject	*Mark*	*Mean*	*Standard Deviation*
Psychology	75	65	10
Statistics	70	60	5

We could take the marks at face value and say that she did better in Psychology because she got a higher mark in that subject. The assumption that underlies such a comparison is that the marks for both subjects have the same distribution with the same mean and standard deviation. However, in this case the two subjects have very different means and standard deviations, as shown in the table above.

If we assume that the *marks* are normally distributed, then standardisation and the 68–95–99.7% rule give us a way of resolving this issue.

Let us standardise the marks.

Psychology: standardised mark $z = \frac{75 - 65}{10} = 1$

Statistics: standardised mark $z = \frac{70 - 60}{5} = 2$

What do we see? The student obtained a higher score for Psychology than for Statistics. However, relative to her classmates she did better in Statistics.

- Her mark of 70 in Statistics is equivalent to a z-score of 2. This means that her mark was two standard deviations above the mean, placing her in the top 2.5% of students.
- Her mark of 75 for Psychology is equivalent to a z-score of 1. This means that her mark was only one standard deviation above the mean, placing her in the top 16% of students. This is a good performance, but not as good as for statistics.

Example 30 **Using standardised scores to make comparisons**

Another student studying the same two subjects obtained a mark of 55 for both Psychology and Statistics. Does this mean that she performed equally well in both subjects? Use standardised marks to help you arrive at your conclusion.

Explanation

1 Write down her mark (x), the mean ($\bar{x}$) and the standard deviation (s) for each subject and compute a standardised score for both subjects.

Solution

Pyschology: $x = 55, \bar{x} = 65, s = 10$

$$z = \frac{x - \bar{x}}{s} = \frac{55 - 65}{10} = \frac{-10}{10} = -1$$

Statistics: $x = 55, \bar{x} = 60, s = 5$

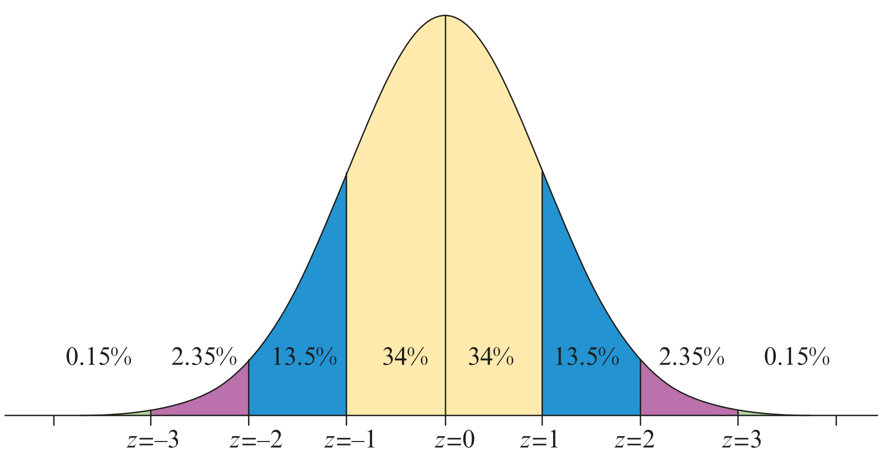
$$z = \frac{x - \bar{x}}{s} = \frac{55 - 60}{5} = \frac{-5}{5} = -1$$

2 Write down your conclusion.

Yes, her standardised score, $z = -1$, was the same for both subjects. In both subjects she finished in the bottom 16%.

Using standard scores to determine percentages

Now we know how to determine standard scores we can revisit the diagram showing the percentages associated with each section of the normal curve, replacing the horizontal scale with the values of Z (the standard normal scores).



Example 31 **Using standard scores to determine percentages**

Suppose the weight of a certain species of bird is normally distributed with a mean of 42 grams with a standard deviation of 3 grams.

a If a bird selected at random from this population has a standardised weight of $z = -1$, what percentage of birds in this population weigh more than this bird?

b Approximately what percentage of birds would weigh between 39 and 48 grams?

Explanation	Solution
a Locate $z = -1$ on the graph above.	We can see that the percentage of the distribution below $z = -1$ is 16%, so the percentage above $z = -1$ is 84%.
b 1 Substitute the values of x into the formula $z = \frac{x - \bar{x}}{s}$ and evaluate.	$\bar{x} = 42,\ s = 3$ $x = 39 \quad z = \frac{x - \bar{x}}{s} = \frac{39 - 42}{3} = -1$ $x = 48 \quad z = \frac{x - \bar{x}}{s} = \frac{48 - 42}{3} = 2$
2 Locate $z = -1$ and $z = 2$ on the graph above, and determine the percentage of the distribution between these values.	The percentage of the distribution between $z = -1$ and $z = 2$ is 81.5%.

Converting standard scores into actual scores

Having learnt how to calculate standard scores, you also need to be able to convert a standardised score back into an actual score. The rule for converting a standardised score into an actual score is given below.

Converting standardised scores into actual scores

By making the actual score the subject of the rule for calculating standard scores, we arrive at:

actual score = mean + standard score × standard deviation or $x = \bar{x} + z \times s$

Example 32 Converting standard scores into actual scores

A class test (out of 50) has a mean mark of $\bar{x} = 34$ and a standard deviation of $s = 4$. Joe's standardised test mark was $z = -1.5$. What was Joe's actual mark?

Explanation	Solution
1 Write down mean ($\bar{x}$), the standard deviation (s) and Joe's standardised score (z).	$\bar{x} = 34,\ s = 4,\ z = -1.5$
2 Write down the rule for calculating the actual score and substitute these values into the rule.	$x = \bar{x} + z \times s$ $= 34 + (-1.5) \times 4 = 28$ Joe's actual mark was 28.

If we know something about the percentages associated with a normal distribution, we can use this information to find the values of the mean, or standard deviation, or both.

In the following example we are given the value of the mean, and one percentage associated with the distribution. From this we can determine the value of the standard deviation.

Example 33 Finding the value of the standard deviation given the mean and one percentage

Suppose the heights of red flowering gum trees have a mean of 10.2 metres, and 2.5% of these trees grow to more than 11.4 metres tall. Assuming that the heights of these trees are approximately normally distributed, what is the standard deviation of the height of the red flowering gum trees?

Explanation	Solution
1 Since 2.5% of the trees are taller than 11.4 metres, this height corresponds to a z-score of 2.	$\bar{x} = 10.2,\ z = 2$
2 Write down the rule for calculating the actual score and substitute these values into the rule.	$x = \bar{x} + z \times s$ $11.4 = 10.2 + 2 \times s$
3 Solve this equation for s.	$2 \times s = 1.2$ $s = 0.6$ metres

Example 34 Finding the value of the standard deviation given the mean and two percentages

The marks scored in an examination are known to be approximately normally distributed. If 16% of students score more than 80 marks, and 2.5% of students score less than 20 marks, estimate the mean and standard deviation of this distribution.

Explanation	Solution
1 Since 2.5% of the students score less than 20, this value corresponds to a z-score of -2.	$\bar{x} - 2 \times s = 20 \qquad (1)$
2 Since 16% of the students score more than 20, this value corresponds to a z-score of 1.	$\bar{x} + 1 \times s = 80 \qquad (2)$
3 To solve these equations for $\bar{x}$ and s, subtract equation (2) from equation (1).	(1)-(2) $\quad 3 \times s = 60$ Hence $\quad s = 20$
4 To find $\bar{x}$ substitute the value of s in equation 1.	$\bar{x} - 2 \times 20 = 20$ Hence $\bar{x} = 60$

Skill-sheet

Exercise 1H

The 68–95–99.7% rule

Example 28

1 The blood pressure readings for executives are approximately normally distributed with a mean systolic blood pressure of 134 and a standard deviation of 20.
Given this information it can be concluded that:

a about 68% of the executives have blood pressures between ☐ and ☐
b about 95% of the executives have blood pressures between ☐ and ☐
c about 99.7% of the executives have blood pressures between ☐ and ☐
d about 16% of the executives have blood pressures above ☐
e about 2.5% of the executives have blood pressures below ☐
f about 0.15% of the executives have blood pressures below ☐
g about 50% of the executives have blood pressures above ☐.

2 The weight of a bag of 10 blood plums picked at U-Pick Orchard is normally distributed with a mean of 1.88 kg and a standard deviation of 0.2 kg.
Given this information the percentage of the bags of 10 plums that weigh:

a between 1.68 and 2.08 kg is approximately ☐%
b between 1.28 and 2.48 kg is approximately ☐%
c more than 2.08 kg is approximately ☐%
d more than 2.28 kg is approximately ☐%
e less than 1.28 kg is approximately ☐%
f more than 1.88 kg is approximately ☐%.

3 The distribution of times taken for walkers to complete a circuit in a park is normal, with a mean time of 14 minutes and a standard deviation of 3 minutes.

a What percentage of walkers complete the circuit in:
i more than 11 minutes?
ii less than 14 minutes?
iii between 14 and 20 minutes?

b In a week, 1000 walkers complete the circuit. Approximately how many will take less than 8 minutes?

4 The distribution of heights of 19-year-old women is approximately normal, with a mean of 170 cm and a standard deviation of 5 cm.

a What percentage of these women have heights:
i between 155 and 185 cm?
ii greater than 180 cm?
iii between 160 and 175 cm?

b In a sample of 5000 of these women, approximately how many have heights greater than 175 cm?

5 The distribution of resting pulse rates of a sample of 2000 20-year-old men is approximately normal, with a mean of 66 beats/minute and a standard deviation of 4 beats/minute.

a What percentage of these men have pulse rates of:

i higher than 66? **ii** between 66 and 70? **iii** between 62 and 74?

b Approximately how many of this sample of 2000 men have pulse rates between 54 and 78 beats/minute?

Calculating standard scores

Example 29

6 A set of scores has a mean of 100 and a standard deviation of 20. Calculate standardised scores for each of the following test scores:

a 120 **b** 140 **c** 80 **d** 100 **e** 40 **f** 110

7 A set of scores has a mean of 30 and a standard deviation of 7. Calculate standardised scores for each of the following test scores. Give your answers to one decimal place.

a 37 **b** 23 **c** 40 **d** 20

Applying standardised scores

Example 30

8 The table below contains the scores a student obtained in a practice test for each of their VCE subjects. Also shown are the mean and standard deviation for each subject.

Subject	*Mark*	*Mean*	*Standard deviation*
English	69	60	4
Biology	75	60	5
Chemistry	55	55	6
Further Maths	55	44	10
Psychology	73	82	4

a Calculate the standardised score for each subject.

b Use the standardised score to rate the student's performance in each subject, assuming a normal distribution of marks and using the 68–95–99.7% rule.

Using standardised scores to determine percentages

Example 31

9 The volume of soft drink in a small can is normally distributed with a mean of 300 mL and a standard deviation of 2 mL.

a If a can selected at random from this population has a standardised volume of $z = 2$, what percentage of cans in this population contain more soft drink than this can?

b Approximately what percentage of cans contain between between 302 mL and 306 mL?

10 To be considered for a special training program applicants are required to sit for an aptitude test. Suppose that 2000 people sit for the test, and their scores on the aptitude test are approximately normally distributed with a mean of 45 and a standard deviation of 2. People who score more than 49 are selected for the special training program. People who are not chosen for the training program, but score more than 47, are invited to resit the aptitude test at a later date.

- **a** What percentage of people who sat for the test are eligible for the training program?
- **b** Approximately how many people would be invited to resit the aptitude test at a later date?

Calculating actual scores from standardised scores

Example 32 **11** A set of scores has a mean of 100 and a standard deviation of 20. Calculate the actual score if the standardised score was:

a 1 **b** 0.8 **c** 2.1 **d** 0 **e** −1.4 **f** −2.5

Find the values of the mean and standard deviation

Example 33 **12** The mean salary for retail assistants is \$27/hr. If 2.5% of retail assistants earn more than \$30/hr, what is the standard deviation of the salary for retail assistants? Assume that the salaries are approximately normally distributed.

13 The weights of bananas from a certain grower are approximately normally distributed. If the standard deviation of the weight of these bananas is 5 g, and 16% of the bananas weigh less than 96 g, what is the mean weight of the bananas?

Example 34 **14** The birth weights of babies are known to be approximately normally distributed. If 16% of babies weigh more than 4.0 kg, and 0.15% of babies weigh more than 5.0 kg, estimate the mean and standard deviation of this distribution. Give your answers to one decimal place.

15 The marks scored in an examination are known to be approximately normally distributed. If 99.7% of students score between 43 and 89 marks, estimate the mean and standard deviation of this distribution. Give your answers to one decimal place.

Applications

16 The body weights of a large group of 14-year-old girls have a mean of 54 kg and a standard deviation of 10.0 kg.

- **a** Kate weighs 56 kg. Determine her standardised weight.
- **b** Lani has a standardised weight of –0.75. Determine her actual weight.
- **c** Find:
  - **i** percentage of these girls who weigh more than 74 kg
  - **ii** percentage of these girls who weigh between 54 and 64 kg

iii percentage of these girls who have standardised weights less than –1

iv percentage of these girls who have standardised weights greater than –2.

17 Suppose that IQ scores are normally distributed with mean of 100 and standard deviation of 15.

a What percentage of people have an an IQ:

i greater than 115?

ii less than 70?

b To be allowed to join an elite club, a potential member must have an IQ in the top 2.5% of the population. What IQ score would be necessary to join this club?

c One student has a standardised score of 2.2. What is their actual score?

18 The heights of women are normally distributed with a mean of 160 and a standard deviation of 8.

a What percentage of women would be:

i taller than 152 cm?

ii shorter than 176 cm?

b What height would put a woman among the tallest 0.15% of the population?

c What height would put a woman among the shortest 2.5% of the population?

d One woman has a standardised height of -1.2. What is her actual height? Give your answer to one decimal place.

Exam 1 style questions

Use the following information to answer questions 19 - 22

A total of 16,000 students sat for a statewide mathematics exam. Their results are are normally distributed with mean 50 and standard deviation 7.

19 The percentage of students in the state who scored more than 71 marks is closest to:

A 0.15% B 2.5% C 5% D 15% E 0.3%

20 The top 2.5% of the state are to be awarded a distinction. What would be the lowest mark required to gain a distinction in this exam?

A 36 B 43 C 57 D 64 E 71

21 Approximately how many students gained a mark of 57 or more?

A 400 B 800 C 2560 D 5120 E 15200

22 Approximately how many students gained a mark between 43 and 64?

A 10888 B 11120 C 13040 D 13440 E 15200

23 The table below shows Miller's swimming times (in seconds) for 50 metres in each of butterfly, backstroke, breaststroke and freestyle. Also shown are the mean and standard deviation of the times recorded for all of the swimmers in his swimming club. In how many of these swimming styles is he in the fastest 2.5% of swimmers at his swimming club?

Style	*Miller's time*	*Mean*	*Standard deviation*
Butterfly	38.8	46.2	3.2
Breaststroke	51.4	55.1	4.1
Backstroke	53.5	48.3	2.5
Freestyle	33.3	38.2	2.3

A 0 **B** 1 **C** 2 **D** 3 **E** 4

1I Populations and samples

This material is available in the Interactive Textbook.

Key ideas and chapter summary

Assignment

Univariate data

Univariate data are generated when each observation involves recording information about a single variable.

Types of data

Data can be classified as numerical or categorical.

Categorical variables

Categorical variables are used to represent characteristics of individuals. Categorical variables come in two types: nominal and ordinal. **Nominal variables** generate data values that can only be used by name, e.g. eye colour. **Ordinal variables** generate data values that can be used to both name and order, e.g. house number.

Numerical variables

Numerical variables have data values which are quantities. Numerical variables come in two types: discrete and continuous. **Discrete variables** are those which may take on only a countable number of distinct values such as 0 1 2 3 4 . . . and are often associated with counting. **Continuous variables** are ones which take an infinite number of possible values, and are often associated with measuring rather than counting.

Frequency table

A **frequency table** lists the values a variable takes, along with how often (frequently) each value occurs. Frequency can be recorded as:

- the number of times a value occurs – e.g. the number of Year 12 students in the data set is 32.
- the percentage of times a value occurs – e.g. the percentage of Year 12 in the data set is 45.5%.

Bar chart

Bar charts are used to display frequency distribution of categorical data. Each value of the variable is represented by a bar showing the frequency, or the percentage frequency.

Segmented bar chart

A **segmented bar chart** is like a bar chart, but the bars are stacked one on top of another to give a single bar with several segments.

Mode, modal category

The **mode** (or modal interval) is the value of a variable (or the interval) that occurs most frequently.

Histogram

A **histogram** uses columns to display the frequency distribution of a numerical variable: suitable for medium to large-sized data sets.

Describing the distribution of a numerical variable

The distribution of a numerical variable can be described in terms of:

- shape: symmetric or skewed (positive or negative)
- outliers: values that seem unusually small or large.
- centre: the median or mean.
- spread: the IQR , range or the standard deviation.

Dot plot

A **dot plot** consists of a number line with each data point marked by a dot. A dot plot is particularly suitable for displaying a small data set of discrete numerical data.

Stem plot

The **stem plot** is particularly suitable for displaying a small to medium sized data sets of numerical data.It shows each data value separated into two parts: the leading digits, which make up the stem of the number, and its last digit, which is called the leaf.

Log scales

Log scales can be used to transform a skewed histogram to symmetry.

Summary statistics

Summary statistics are numerical values for special features of a data distribution such as centre and spread.

Mean

The **mean** ($\bar{x}$) is a summary statistic that can be used to locate the centre of a symmetric distribution. The value of the mean is determined from the formula: $\bar{x} = \dfrac{\sum x}{n}$

Range

The **range** ***(R)*** is the difference between the smallest and the largest data values. It is the simplest measure of spread.

Standard deviation

The **standard deviation** (s) is a summary statistic that measures the spread of the data values around the mean. The value of the standard deviation is determined from the formula:

$$s = \sqrt{\frac{\sum(x - \bar{x})^2}{n - 1}}$$

Median

The **median** ***(M)*** is a summary statistic that can be used to locate the centre of a distribution. It is the midpoint of a distribution, so that 50% of the data values are less than this value and 50% are more. It is sometimes denoted as Q_2

Quartiles

Quartiles are summary statistics that divide an ordered data set into four equal groups.

Interquartile range

The **interquartile range (IQR)** gives the spread of the middle 50% of data values in an ordered data set. It is found by evaluating $IQR = Q_3 - Q_1$

Five-number summary

The median, the first quartile, the third quartile, along with the minimum and the maximum values in a data set, are known as a **five-number summary**.

Outliers

Outliers are data values that appear to stand out from the rest of the data set. They are values that are less than the **lower fence** or more than the **upper fence**.

Lower and upper fences

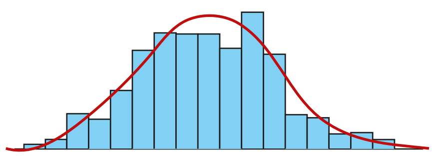
The **lower fence** is equal to $Q_1 - 1.5 \times IQR$.
The **upper fence** is equal to $Q_3 + 1.5 \times IQR$.

Boxplot

A **boxplot** is a visual display of a five-number summary with adjustments made to display outliers separately when they are present.

The normal distribution

Data distributions that have a bell shape can be modelled by a **normal distribution.**



The 68-95-99.7%rule

For a data distribution which is approximately normally distributed approximately:

- 68% of the data will lie within one standard deviation of the mean.
- 95% of the data will lie within two standard deviations of the mean.
- 99.7% of the data will lie within three standard deviations of the mean.

Standardised scores

The value of the standard score gives the distance and direction of a data value from the mean in terms of standard deviations.
The rule for calculating a **standardised score** is:

$$\text{standardised score} = \frac{\text{actual score} - \text{mean}}{\text{standard deviation}}$$

Review

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

1A **1** I can identify types of data.

See Example 1, and Exercise 1A Question 1

1B **2** I can construct a frequency table for categorical data.

See Example 2, and Exercise 1B Question 1

1B **3** I can construct a bar chart from a frequency table.

See Example 3, and Exercise 1B Question 2

1B **4** I can construct a percentage segmented bar chart from a frequency table.

See Example 4, and Exercise 1B Question 4

1B **5** I can describe the distribution of a categorical variable.

See Example 5, and Exercise 1B Question 6

1C **6** I can construct a frequency table for discrete numerical data.

See Example 6, and Exercise 1C Question 1

1C **7** I can construct a grouped frequency table.

See Example 7, and Exercise 1C Question 3

1C **8** I can construct a histogram from a grouped frequency table.

See Example 8, and Exercise 1C Question 4

1C **9** I can describe the features of a distribution from a histogram.

See Example 9, and Exercise 1C Question 9

1D **10** I can construct a dot plot.

See Example 10, and Exercise 1D Question 1

1D **11** I can construct a stem plot.

See Example 11, and Exercise 1D Question 6

1E **12** I can use a CAS calculator to find logs.

See Example 12, and Exercise 1E Question 1

1E **13** I can interpret a histogram with a log scale. ☐

See Example 13, and Exercise 1E Question 4

1F **14** I can find the median value in a data set. ☐

See Example 14, and Exercise 1F Question 1

1F **15** I can find the median value from a dot plot. ☐

See Example 15, and Exercise 1F Question 3

1F **16** I can find the median value from a stem plot. ☐

See Example 16, and Exercise 1F Question 4

1F **17** I can find the range from a stem plot. ☐

See Example 17, and Exercise 1F Question 4

1F **18** I can find the *IQR* from a stem plot when n is even. ☐

See Example 18, and Exercise 1F Question 5

1F **19** I can find the *IQR* from a stem plot when n is odd. ☐

See Example 19, and Exercise 1F Question 8

1F **20** I can calculate the mean using the formula. ☐

See Example 21, and Exercise 1F Question 10

1F **21** I can calculate the mean and standard deviation using the CAS calculator. ☐

See CAS 3, and Exercise 1F Question 12

1G **22** I can construct a boxplot from a five number summary. ☐

See Example 22, and Exercise 1G Question 3

1G **23** I can construct a boxplot with outliers from data using the CAS calculator. ☐

See CAS 4, and Exercise 1G Question 7

1G **24** I can read values from a boxplot. ☐

See Example 23, and Exercise 1G Question 8

1G **25** I estimate percentages from a boxplot. ☐

See Example 24, and Exercise 1G Question 11

1G **26** I can use boxplots for describe a distribution without outliers. ☐

See Example 25, and Exercise 1G Question 14

1G **27** I can use boxplots to describe a distribution with outliers. ☐

See Example 26, and Exercise 1G Question 15

1G **28** I can use boxplots to answer statistical questions. ☐

See Example 27, and Exercise 1G Question 16

1H **29** I can apply the 68-95-99.7% rule. ☐

See Example 28, and Exercise 1H Question 1

1H **30** I can calculate standardised scores. ☐

See Example 29, and Exercise 1H Question 6

1H **31** I can use standardised scores to make comparisons. ☐

See Example 30, and Exercise 1H Question 8

1H **32** I can use standard scores to determine percentages. ☐

See Example 31, and Exercise 1H Question 9

1H **33** I can convert standardised scores into actual scores. ☐

See Example 32, and Exercise 1H Question 11

1H **34** I can solve for the values of the mean and standard deviation. ☐

See Example 33 and 34, Exercise 1H Question 12 and Question 14.

Multiple-choice questions

The following information relates to Questions 1 and 2.

Data pertaining to the following five variables was collected about secondhand cars:

- *colour*
- *model*
- *number of seats*
- *age* (1 = less than 2 years, 2 = from 30,000-60,000km, 3 = more than 80,000km)
- *mileage* (in kilometres)

1 The number of these variables that are discrete numerical is:

A 1 **B** 2 **C** 3 **D** 4 **E** 5

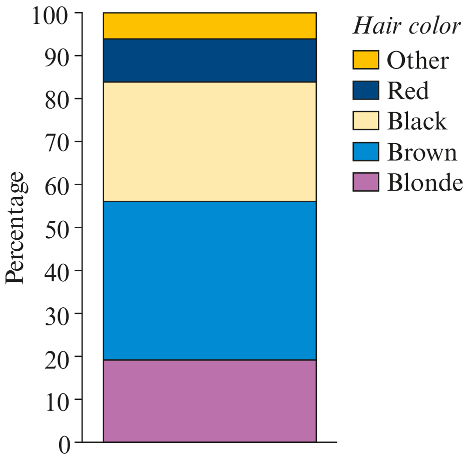
2 The number of ordinal variables is:

A 0 **B** 1 **C** 2 **D** 3 **E** 4

Review

The following information relates to Questions 3 and 4.

The percentage segmented bar chart shows the distribution of hair colour for 200 students.



3 The number of students with brown hair is closest to:

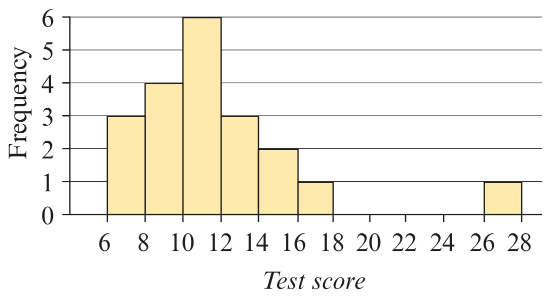
A 4 **B** 34 **C** 57
D 72 **E** 114

4 The most common hair colour is:

A black **B** blonde
C brown **D** red **E** other

Questions 5 to 7 relate to the histogram shown below.

The histogram below displays the test scores of a class of students.



5 The number of students in the class who obtained a test score less than 14 is:

A 4 **B** 10 **C** 14 **D** 16 **E** 28

6 The shape of the histogram is best described as:

A negatively skewed **B** positively skewed with an outlier
C approximately symmetric **D** approximately symmetric with an outlier
E negatively skewed with an outlier

7 The value of the first quartile could be:

A 5.5 **B** 6.8 **C** 8.9 **D** 10.5 **E** 11.4

8 $\log_{10} 100$ equals:

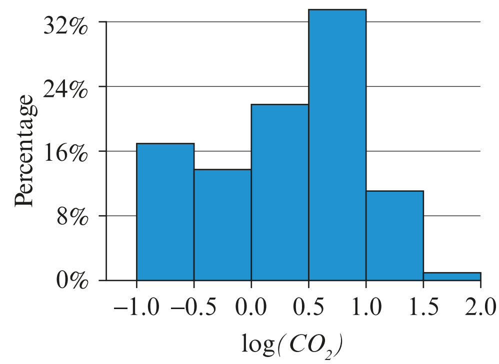
A 0 **B** 1 **C** 2 **D** 3 **E** 100

9 Find the number with log equal to 2.314; give the answer to the nearest whole number.

A 2 **B** 21 **C** 206 **D** 231 **E** 20606

The following information relates to Questions 10 and 11.

The percentage histogram opposite displays the distribution of the log of the annual per capita CO_2 emissions (in tonnes) for 192 countries in 2011.



10 Australia's per capita CO_2 emissions in 2011 were 16.8 tonnes. In which column of the histogram would Australia be located?

A −0.5 to <0.0 **B** 0.0 to <0.5 **C** 0.5 to <1.0 **D** 1.0 to <1.5 **E** 1.5 to < 2.0

11 The percentage of countries with per capita CO_2 emissions of under 10 tonnes is closest to:

A 14% **B** 17% **C** 31% **D** 69% **E** 88%

12 The following is an ordered set of 10 daily maximum temperatures (in degrees Celsius):

22 22 23 24 24 25 26 27 28 29

The five-number summary for these temperatures is:

A 22, 23, 24, 27, 29 **B** 22, 23, 24.5, 27, 29 **C** 22, 24, 24.5, 27, 29
D 22, 23, 24.5, 27.5, 29 **E** 22, 24, 24.5, 27, 29

The following information relates to Questions 13 to 15.

The stem plot opposite displays the distribution of the marks obtained by 25 students.

Key: 1|5 means 15 marks

0	2
1	5 9 9 9
2	0 4 4 5 5 8 8 8
3	0 3 5 5 6 8 9
4	1 2 3 5
5	
6	0

13 The median mark is:

A 20 **B** 27 **C** 28 **D** 29 **E** 30

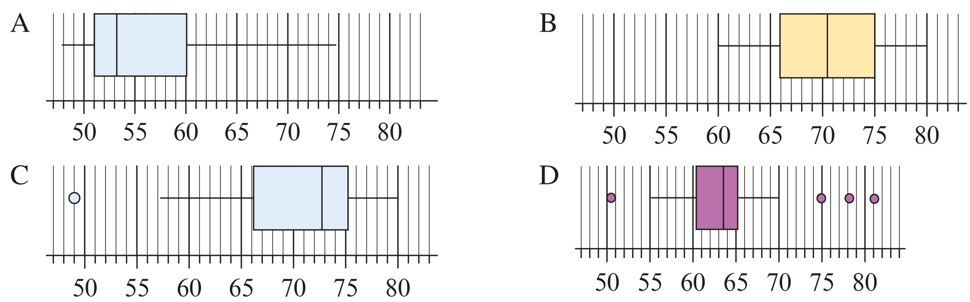
14 The interquartile quartile range (IQR) of the marks is:

A 12 **B** 16.5 **C** 20 **D** 30.5 **E** 31.5

15 The shape of the data distribution displayed by this stem plot is best described as:

A approximately symmetric
B approximately symmetric with an outlier
C negatively skewed with an outlier
D negatively skewed
E positively skewed with an outlier

The following information relates to Questions 16 to 23.



16 The median of boxplot A is closest to:

A 5 **B** 53 **C** 54.5 **D** 55 **E** 60

17 The *IQR* of boxplot B is closest to:

A 9 **B** 20 **C** 25 **D** 65 **E** 75

18 The range of boxplot C is closest to:

A 4 **B** 13 **C** 20 **D** 31 **E** 80

19 The description that best matches boxplot A is:

A symmetric
B symmetric with outliers
C negatively skewed
D positively skewed
E positively skewed with outliers

20 The description that best matches boxplot B is:

A symmetric
B negatively skewed with an outlier
C negatively skewed
D positively skewed
E positively skewed with outliers

21 The description that best matches boxplot D is:

A symmetric
B symmetric with outliers
C negatively skewed
D positively skewed
E positively skewed with outliers

22 For the data represented by boxplot D, the percentage of data values greater than 65 is:

A 2.5% **B** 25% **C** 50% **D** 75% **E** 100%

23 To be an outlier in boxplot D, a score must be:

A either less than 52.5 or greater than 72.5 **B** greater than 72.5
C either less than 55 or greater than 70 **D** greater than 70
E less than 55

24 The mean ($\bar{x}$) and standard deviation (s) for the following set of test marks

1 1 10 15 16 25 8 10 12

are closest to:

A 7.1, 10.9 **B** 7.5, 10.9 **C** 10.9, 7.1 **D** 10.9, 7.5 **E** 10.8, 7.5

25 It would *not* be appropriate to determine the mean and standard deviation of a group of peoples':

A ages **B** phone numbers **C** heights
D weights **E** family sizes

26 The median is a more appropriate measure of the centre of a distribution than the mean when the distribution is:

A symmetric **B** symmetric with no outliers **C** bell-shaped
D clearly skewed **E** normal

27 The mean length of 10 garden stakes is 180.5 cm, and the standard deviation is 2.9 cm. If the length of each garden stake is reduced by exactly 5 cm, the mean and standard deviation of the 10 stakes will be:

A 175.5 cm and 2.4 cm **B** 180.5 cm and 2.4 cm **C** 175.5 cm and 2.9 cm
D 175.5 cm and 3.4 cm **E** 185.5 and 2.9 cm

28 A student's mark on a test is 50. The mean mark for their class is 55 and the standard deviation is 2.5. Their standard score is:

A −2.5 **B** −2.0 **C** 0 **D** 2 **E** 2.5

The following information relates to Questions 29 to 33.

Each week, a bus company makes 200 trips between two large country towns. The time taken to make a trip between the two towns is approximately normally distributed with a mean of 78 minutes and a standard deviation of 4 minutes.

29 The percentage of trips each week that take 78 minutes or more is:

A 16% **B** 34% **C** 50% **D** 68% **E** 84%

30 The number of trips each week that take between 70 and 82 minutes is approximately:

A 4 **B** 32 **C** 68 **D** 127 **E** 163

Review

31 A trip that takes 71 minutes has a standardised time (z-score) of:

A −1.75 **B** −1.5 **C** −1.25 **D** 1.5 **E** 1.75

32 A standardised time for a trip is $z = -0.25$. The actual time (in minutes) is:

A 77 **B** 77.25 **C** 77.75 **D** 78.25 **E** 79

33 The time of a bus trip has a standardised time of $z = 2.1$. This time is:

A very much below average **B** just below average **C** around average
D just above average **E** very much above average

34 The table shows the time taken to run one kilometre (in minutes) by three runners. To be invited to join the athletics team the standardised score for their time needs to be no more than than 0.5.

Runner	*Time(mins)*
Albie	7.5
Lincoln	4.9
Wendy	8.0

If the mean running time for one kilometre is 7.0 minutes and the standard deviation is 1.2 minutes, who will be invited to join the athletics team?

A Only Wendy **B** Lincoln and Albie **C** Only Lincoln
D Albie and Wendy **E** All three runners

35 The diameter of bolts produced by a machine is normally distributed. If 2.5% of bolts the bolts have a diameter of more than 4.94 mm, and 0.15% have a diameter less than 4.84 mm, the mean and standard deviation of this distribution in millimetres are closest to:

A mean = 4.88 standard deviation =0.02
B mean = 4.92 standard deviation = 0.01
C mean = 4.90 standard deviation = 0.001
D mean = 4.90 standard deviation = 0.1
E mean = 4.90 standard deviation = 0.02

Written response questions

1 A group of 52 teenagers were asked, ‘Do you agree that the use of marijuana should be legalised?’ Their responses are summarised in the table.

	Frequency	
Legalise	Number	Percentage
Agree	18	
Disagree	26	
Don’t know	8	
Total	52	

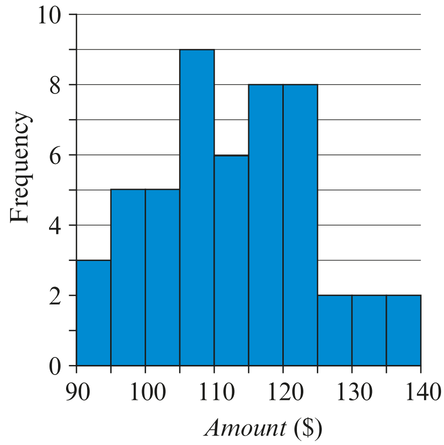
a Construct a labelled and scaled frequency bar chart for the data.

b Complete the table by calculating the percentages to one decimal place.

c Use the percentages to construct a percentage segmented bar chart for the data.

d Write a short report describing the distribution of responses.

2 Students were asked how much they spent on entertainment each month. The results are displayed in the histogram. Use the histogram to answer the following questions.



a How many students:

   i were surveyed?

   ii spent from \$100 to less than \$105 per month?

b What is the mode?

c How many students spent \$110 or more per month?

d What percentage spent less than \$100 per month?

e **i** What is the shape of the distribution displayed by the histogram?

   ii In which interval is the median of the distribution?

   iii In which interval is the upper quartile of the distribution (Q_3)?

3 The amount of weight lost in one week by 32 people who participated in a weight loss program was recorded and displayed in the ordered stem plot below.

Weight loss (kg)   key: 2|0 represents 2.0

1	5	5	7	8	9	9	
2	2	2	2	3	3	4	4
2	5	6	6	7	7	8	
3	0	1	3	3	4		
3	5	5	5	7			
4	1	2	2				
4							
5	0						

a Describe the shape of the distribution.

b Determine the median weight loss. Give your answer to 2 decimal places.

c Find the value of the interquartile range. Give your answer to 2 decimal places.

d What percentage of this group had a weight loss of more than 3.5 kg? Give your answer to 2 decimal places.

e Is the weight loss of 5.0 kg an outlier for this data set? Justify your answer.

4 The systolic blood pressure (measured in mmHg) for a group of 2000 people was measured. The results are summarised in the five-number summary below:
Min = 75, Q_1 = 110, M = 125, Q_3 = 140, Max = 180

a Use the five-number summary to construct a simple boxplot.

b Indicate on your plot where the lower and upper fences would be, and hence if there would be any outliers.

c Assume that the distribution of systolic blood pressure for this sample of 2000 people is approximately normally distributed, with a mean of 128 mmHg and a standard deviation of 20 mmHg.

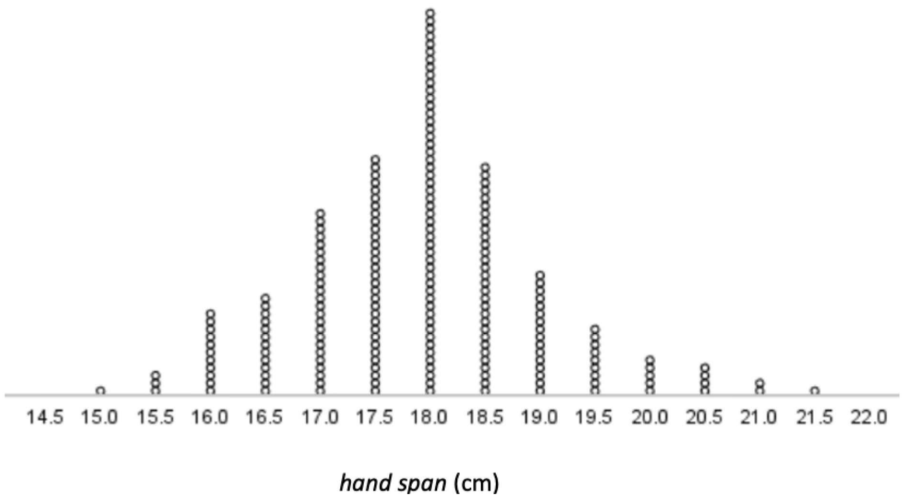
i Approximately what percentage of people of people have a systolic blood pressure between 108 mmHg and 148 mmHg?

ii Suppose a person has a blood pressure three standard deviations below the mean, what would be their actual blood pressure?

iii Of the 2000 people measured, how many could we expect to have a blood pressure three standard deviations below the mean?

iv Of the 2000 people measured, how many actually did have a blood pressure three standard deviations below the mean?

5 The *hand span* in centimetres of 200 women was recorded and displayed in the dot plot below.



a Write down the modal *hand span*, in centimetres, for this group of 200 women.

b The mean *hand span* for this group of 200 women is 17.9 cm, and the standard deviation is 1.1 cm. Use the information in the dot plot to determine the percentage of women in this group who had an actual hand span more than two standard deviations above or below the mean. Round your answer to one decimal place.

c The five-number summary for this sample of hand spans, in centimetres, is given below:

Min =15.0, Q_1 = 17.0, M = 18.0, Q_3 =18.5, Max = 21.5

Use the five-number summary to construct a boxplot showing outliers.

Chapter 2

Investigating associations between two variables

Chapter questions

- What are bivariate data?
- What are explanatory and response variables?
- What are two-way frequency tables and how do we interpret them?
- How do we construct and interpret segmented bar charts from two-way frequency tables?
- How do we construct and interpret parallel dot plots?
- How do we construct and interpret back-to-back stem plots?
- How do we construct and interpret parallel boxplots?
- What is a scatterplot, how is it constructed and what does it tell us?
- What do we mean when we describe the association between two numerical variables in terms of direction, form and strength?
- What is the difference between observation and experimentation?
- What is the difference between association and causation?

In this chapter we begin our study of **bivariate data**, data which is recorded on two variables from the same subject.

2A Bivariate data – Classifying the variables

Learning intentions

- ► To introduce bivariate data.
- ► To be able to classify data as categorical or numerical.
- ► To be able to identify explanatory and response variables.

So far you have learned how to display, describe and compare the distributions of single variables. In the process you learned how to use data to answer questions like 'What is the favourite colour of prep-grade students?' or 'How do the weights of tuna fish vary?' In each case we concentrated on investigating the statistical variables individually.

However, questions like 'Does the new treatment for headache work more quickly than the old treatment?', 'Are city voters more likely to vote for the Greens party than country voters?' or 'Can we predict a student's test score from the time (in hours) they spent studying for the test?' cannot be answered by considering variables separately. All of these questions relate to situations where the two variables are linked in some way (associated) so that they vary together. The data generated in these circumstances is called **bivariate data**.

Analysing bivariate data requires a new set of statistical tools. Developing and applying these tools is the subject of the next four chapters.

Categorical and numerical variables

You will recall from Chapter 1 we defined two classifications of variables, categorical and numerical variables:

- **Categorical variables** generate data values that are names or labels, such as *favourite pet* (dog, cat, rabbit, bird) or *coffee size* (small, medium, large).
- **Numerical variables** generate data values that are numbers, usually resulting from counting or measuring, such as *number of brothers* $(0, 1, 2, \ldots)$ or *hand span* (cm).

The first step in investigating the association between two variables is to classify each variable as either categorical or numerical. Consider again the previous questions:

- Does the new treatment for headache work more quickly than the old treatment?

The two variables in this question are *type of treatment*, a categorical variable taking the values 'new' and 'old', and *time taken for the headache to be relieved*, a numerical variable, measured in minutes. Thus, investigation of a question like this can be classified as **investigating the association between a categorical variable and a numerical variable**.

- Are city voters more likely to vote for the Greens party than country voters?

  This question involves two variables, *place of residence*, which is a categorical variable taking the values 'city' and 'country', and *vote for the Greens*, which also is a categorical variable taking the values 'yes' and 'no'. Investigation of a question like this can be classified as **investigating the association between two categorical variables**.

- Can we predict a student's test score (%) from time (in hours) spent studying for the test?

  The variable *test score* is a numerical variable, as is *time spent studying for the test*. Investigation of a question like this can be classified as **investigating the association between two numerical variables**.

As discussed in Chapter 1, categorical variables can be further classified as nominal or ordinal, and numerical variables can be further classified as discrete or continuous.

Example 1 **Identifying associations as categorical or numerical**

For each of the following questions, determine if they involve investigating associations between

- one numerical variable and one categorical variable or
- two categorical variables or
- two numerical variables.

a Are younger people (age measured in years) more likely to believe in astrology (measured as 'yes' or 'no') than older people?

b Do people who weigh more (weight measured in kg) tend to have higher blood pressure (blood pressure measured in mmHg)?

c Are people who have a driver's licence (measured as 'yes' or 'no') more likely to be in favour of lowering the driving age (measured as 'yes' or 'no')?

Solution

a One numerical variable (*age*) and one categorical variable (*belief in astrology*)

b Two numerical variables (*weight* and *blood pressure*)

c Two categorical variables (*have a driver's licence* and *support for lowering the driving age*)

Identifying response and explanatory variables

When investigating associations between variables, it is helpful to think of one of the variables as the **explanatory variable**. The other variable is then called the **response variable**. We use the explanatory variable to explain changes that might be observed in the response variable.

For example, the question, 'Are city voters more likely to vote for the Greens party than country voters?', suggests that knowing a person's place of residence might be useful in explaining voting preference. In this situation *place of residence* is the explanatory variable and *vote for Greens* is the response variable.

It is important to be able to identify the explanatory and response variables before you explore the association between them. Consider the following examples.

Example 2 Identifying the response and explanatory variables

We wish to investigate the question, 'Does the time it takes a student to get to school depend on their mode of transport?' The variables here are *time* and *mode of transport*. Which is the response variable (RV) and which is the explanatory variable (EV)?

Explanation

In asking the question in this way we are suggesting that a student's *mode of transport* might explain the differences we observe in the time it takes students to get to school.

Solution

EV: *mode of transport*

RV: *time*

Example 3 Identifying the response and explanatory variables

Can we predict people's height (in cm) from their wrist measurement? The variables in this investigation are *height* and *wrist measurement*. Which is the response variable (RV) and which is the explanatory variable (EV)?

Explanation

Since we wish to predict height from wrist circumference, we are using *wrist measurement* as the predictor or explanatory variable. *Height* is then the response variable.

Solution

EV: *wrist measurement*

RV: *height*

It is important to note that, in Example 3, we could have asked the question the other way around; that is, 'Can we predict people's wrist measurement from their height?' In that case *height* would be the explanatory variable, and *wrist measurement* would be the response variable. The way we ask our statistical question is an important factor when there is no obvious explanatory variable.

Response and explanatory variables

When investigating the association between two variables the *explanatory variable (EV)* is the variable we expect to explain or predict the value of the *response variable (RV)*.

Note: The explanatory variable is sometimes called the independent variable (IV) and the response variable the dependent variable (DV).

Exercise 2A

Identifying variables as categorical or numerical

Example 1

1 For each of the following questions, determine if they involve investigating associations between:

- one numerical and one categorical variable or
- two categorical variables or
- two numerical variables.

a Are full-time and part-time students equally likely travel to university by car?

b Do Year 11 students watch more hours of television each week than Year 12 students?

c Do countries with higher household incomes ($) tend to have lower infant mortality rates (deaths/1000 births)?

d Is there a relationship between attitude to gun control and country of birth?

Identifying explanatory and response variables

Example 2
Example 3

2 For each of the following situations identify the explanatory variable (EV) and response variable (RV). In each situation the variable names are *italicised.*

a We wish to investigate whether a fish's *toxicity* can be predicted from its *colour*. We want to be able to predict *toxicity* from *colour*.

b The relationship between *weight loss* and *type of diet* is to be investigated.

c We wish to investigate the relationship between a used car's *age* and its *price*.

d It is suggested that the *cost* of heating in a house depends on the type of *fuel* used.

e The relationship between the *price* of a house and its *location* is to be investigated.

3 The following pairs of variables are related. In each case identify which is likely to be the explanatory variable and which is the response variable, and the level of measurement of each variable (categorical or numerical). The variable names are italicised.

a *exercise level* (1 = light, 2 = moderate, 3 = a lot) and *age* (years).

b *years of education* (years) and *salary level* ($ per annum).

c *comfort level* (0 = uncomfortable, 1 = comfortable) and *temperature* (°C).

d *time of year* (summer, autumn, winter, spring) and *incidence of hay fever* (1 = never, 2 = sometimes, 3 = regularly).

e *age group* (less than 25, 25 - 40, more than 40) and *musical taste* (classical, rock, rap, country, indie, dance, jazz).

f *AFL team supported* and *state of residence*.

Exam 1 style questions

4 Respondents to a survey question "How concerned are you about climate change?" were asked to select from the following responses:

1 = not at all, 2 = a little, 3 = moderately, 4 = extremely

The data which was collected in response to this question is:

A nominal **B** ordinal **C** discrete
D continuous **E** numerical

5 The variables *weight* (light, medium, heavy) and *height* (less than 160cm, 160-175cm, over 175cm) are:

A both nominal variables
B both ordinal variables
C a nominal and an ordinal variable respectively
D an ordinal and a nominal variable respectively
E both continuous variables

6 Researchers believe that reaction time might be lower in cold temperatures. They devise an experiment where *reaction time* in seconds is measured at three different *temperature* levels (1 = less than 8°C, 2 = from 8°C to 18 °C, 3 = more than 18°C). The response variable, and its classification are:

A *reaction time*, categorical **B** *temperature*, categorical
C *reaction time*, numerical **D** *temperature*, numerical
E *temperature*, ordinal

2B Investigating associations between categorical variables

Learning intentions

- ▶ To be able to summarise data from two categorical variables using two-way frequency tables.
- ▶ To be able to appropriately percentage two-way frequency tables.
- ▶ To be able to use a percentaged two-way frequency table to identify and describe an association between two categorical variables.

If two variables are related or linked in some way, we say they are associated. To begin the investigation of an association between two categorical variables we create a **contingency table** or a **two-way frequency table**. It is called a two-way frequency table because it is summarising data from two variables.

Constructing a two-way frequency table

It has been suggested that city and country people have differing attitudes to gun control; that is, that support for gun control depends on where a person lives. How might we investigate this relationship? Suppose we ask a sample of three people about their *attitude to gun control*, and we also record their *residence*. The resulting data for the three people might look like this:

Subject no.	*Residence*	*Attitude to gun control*
1	City	For
2	Country	For
3	City	Against

The first thing to note is that these two variables, *attitude to gun control* (for or against) and *residence* (city or country), are both categorical variables. Categorical data are usually presented in the form of a frequency table.

Suppose we continue until we have interviewed a sample of 100 people, and we find that there are 58 who live in the country and 42 who live in the city. We can present this result in a frequency table as shown to the right.

Residence	Frequency
Country	58
City	42
Total	100

From this table, we can see that there were more country than city people in our sample.

Suppose also when we record the attitude to gun control, we might have 62 'for' and 38 'against' gun control. Again, we could present these results in a frequency table as shown to the right.

Attitude to gun control	Frequency
For	62
Against	38
Total	100

From this table, we can see that more people in the sample were for gun control than against gun control. However, we cannot tell from the information contained in the tables whether *attitude to gun control* depends on the *residence* of the person. To do this we need to construct a **two-way frequency table**, which gives both the *attitude to gun control* and the *residence* for each person in the sample.

We begin by counting the number of people in the sample who are:

- from the country and for gun control
- from the city and for gun control
- from the country and against gun control
- from the city and against gun control.

Suppose again from our sample of 100 people we find the following frequencies:

- 32 country people are for gun control
- 30 city people are for gun control
- 26 country people are against gun control
- 12 city people are against gun control.

Explanatory and response variables in two-way frequency tables

Before we set up the two-way frequency table, we need to decide which is the explanatory variable and which is the response variable of the two variables. Since we think that a person's attitude to gun control might depend on their place of residence, but not the other way around, then:

- *residence* is the explanatory variable (EV)
- *attitude to gun control* is the response variable (RV).

In two-way frequency tables, it is conventional to let the categories of the *response variable* label the *rows* of the table and the categories of the *explanatory variable* label the *columns* of the table. Following this convention, we can create the following two-way frequency table.

	Residence	
Attitude to gun control	Country	City
For	32	30
Against	26	12

To complete the table, it is usual to calculate the row and column sums, as shown below.

	Residence			
Attitude to gun control	Country	City	Total	
For	32	30	62	Row sum
Against	26	12	38	Row sum
Total	58	42	100	
	Column sum	Column sum		

The shaded regions in the table are called the **cells** of the table. It is the numbers in these cells that we look at when investigating the relationship between the two variables.

Example 4 **Constructing a two-way frequency table**

The following data were obtained when a sample of ten Year 9 students were asked if they intended to go to university (*university*). The gender of the student was also recorded.

Student	*Gender*	*University*	Student	*Gender*	*University*
1	Female	Yes	6	Male	Yes
2	Male	Yes	7	Female	Yes
3	Female	No	8	Male	No
4	Female	Yes	9	Female	No
5	Male	No	10	Female	Yes

Create a two-way frequency table from these data.

Explanation

1 We first need to identify the explanatory variable and the response variable.

Solution

It is possible that a student's intention to go to university may depend on their gender, but not the other way around. Thus, *gender* is the explanatory variable and *university* is the response variable.

2 Create the table showing the values of *Gender* labelling the columns, and *University* labelling the rows.

	Gender	
University	Male	Female
Yes		
No		

3 Consider Student 1, who is female and indicated yes to go to university. Place a mark in the corresponding cell of the table.

	Gender	
University	Male	Female
Yes		|
No		

4 Go through the data set one person at a time, placing a mark in the appropriate cell for each person.

	Gender	
University	Male	Female
Yes	||	||||
No	||	||

5 Finally, tally the marks in each cell, and then calculate the row and columns sums. Make sure the total adds to the number of students in the sample.

	Gender		
University	Male	Female	Total
Yes	2	4	6
No	2	2	4
Total	4	6	10

Consider again the two-way frequency table created to investigate the association between place of residence and attitude to gun control. This table tells us that more country people are in favour of gun control than city people. But is this just due to the fact that there were more country people in the sample? To help us answer this question we need to express the frequencies in each cell as **percentage frequencies**.

Percentaged two-way frequency table

When the two-way frequency table has been constructed so that the values of the explanatory variable label the rows, then we calculate **column percentages** to help us investigate the association. This will give us the percentage of country and the percentage of city people for and against gun control, which can then be compared.

Column percentages are determined by dividing each of the cell frequencies by the relevant column sums. Thus, the percentage of:

- country people who are for gun control is: $\frac{32}{58} \times 100 = 55.2\%$
- country people who are against gun control is: $\frac{26}{58} \times 100 = 44.8\%$
- city people who are for gun control is: $\frac{30}{42} \times 100 = 71.4\%$
- city people who are against gun control is: $\frac{12}{42} \times 100 = 28.6\%$

Note: Unless small percentages are involved, it is usual to round percentages to one decimal place in tables.

	Residence	
Attitude to gun control	Country	City
For	55.2%	71.4%
Against	44.8%	28.6%
Total	100.0%	100.0%

Using percentages to identify relationships between variables

Calculating the values in the table as percentages enables us to compare the attitudes of city and country people on an equal footing. From the table, we see that 55.2% of country people in the sample were for gun control compared to 71.4% of the city people. This means that the city people in the sample were more supportive of gun control than the country people. This reverses what the frequencies showed.

The fact that the percentage of 'country people for gun control' differs from the percentage of 'city people for gun control' indicates that a person's attitude to gun control depends on their residence. Thus, we can say that the variables *attitude to gun control* and *residence* are **associated**.

If the variables *attitude to gun control* and *residence* were not associated, we would expect approximately equal percentages of country people and city people to be 'for' gun control. Finding a single row in the two-way frequency distribution in which percentages are clearly different is sufficient to identify a relationship between the variables.

We could have also arrived at this conclusion by focusing our attention on the percentages 'against' gun control. We might report our findings as follows.

Report

In this sample of 100 people, a higher percentage of city people were for gun control than country people: 71.4% to 55.2%. This indicates that a person's attitude to gun control is associated with their place of residence.

We will now consider a two-way percentage frequency table that shows no evidence of a relationship. Consider the following table that summarises responses to the question 'Should mobile phones be banned in cinemas?' These responses were obtained from 100 students in Year 10 and Year 12 – we are interested in investigating whether there is an association between these variables.

	Year level	
Should mobile phones be banned in cinemas?	Year 10	Year 12
Yes	87.9%	86.8%
No	12.1%	13.2%
Total	100.0%	100.0%

When we look across the first row of the table, we see that the percentages in favour are very similar. In this case, we might report our findings as follows.

Report

In this sample of 100 Year 10 and Year 12 students, we see that the percentage of Year 10 and Year 12 students in support of banning mobile phones in cinemas is similar: 87.9% to 86.8%. This indicates that a person's support for banning mobile phones in cinemas is not associated with their year level.

Example 5 **Identifying and describing an association from a percentaged two-way table (2 × 2)**

Are males and females in Year 9 equally likely to indicate an intention to go to university? Data from interviews with 200 Year 9 students are summarised in the following table. Write a brief report addressing this question and quoting appropriate percentages.

	Gender		
University	Male	Female	Total
Yes	50	54	104
No	55	41	96
Total	105	95	200

Explanation

1 Determine the column percentages and complete the table as shown.

Solution

	Gender	
University	Male	Female
Yes	47.6%	56.8%
No	52.4%	43.2%
Total	100.0%	100.0%

2 Select an appropriate row to compare the male and female percentages.

We can see from the top row that a greater proportion of females than males (56.8% compared with 47.6%) were intending to go to university.

3 Construct a report.

Report: In this sample of 200 Year 9 students, a greater proportion of females than males (56.8% compared with 47.6%) were intending to go to university. There is an association between gender and intention to go to university.

Two-way frequency tables for categorical variables taking more than two values

The table below displays the *smoking status* for a group of adults (smoker, past smoker, never smoked) by *educational level* (Year 9 or less, Year 10 or 11, Year 12, university). This is still a two-way frequency table (because involves two variables), each of these variables can take three values, and so we call this a 3×3 table.

Smoking status	*Education level (%)*			
	Year 9 or less	Year 10 or 11	Year 12	University
Smoker	34.0	31.7	26.5	18.4
Past smoker	36.0	33.8	30.9	28.0
Never smoked	30.0	34.5	42.6	53.6
Total	100.0	100.0	100.0	100.0

Again, we look for an association between variables by comparing the percentages across one of the rows. The following report has been prepared using the percentages in the 'Smoker' row.

Report

From Table 3.3 we see that the percentage of smokers steadily decreases with education level, from 33.9% for Year 9 or below to 18.4% for university. This indicates that smoking is associated with level of education.

Example 6 Identifying and describing associations from a percentaged two-way table (3 × 3)

A survey was conducted with 1000 males under 50 years old. As part of this survey, they were asked to rate their interest in sport as 'high', 'medium', or 'low'. Their age group was also recorded as 'under 18','19–25', '26–35' and '36–50'. The results are displayed in the table.

Interest in sport	*Age group (%)*			
	Under 18 years	19–25 years	26–35 years	36–50 years
High	56.5	50.2	40.7	35.0
Medium	30.1	34.4	36.8	44.7
Low	13.4	15.4	22.5	20.3
Total	100.0	100.0	100.0	100.0

a Which is the explanatory variable, *interest in sport* or *age group*?

b Is there an association between *interest in sport* and *age group*? Write a brief response quoting appropriate percentages.

Explanation

a Age is a possible explanation for the level of interest in sport, but interest in sport cannot explain age.

Solution

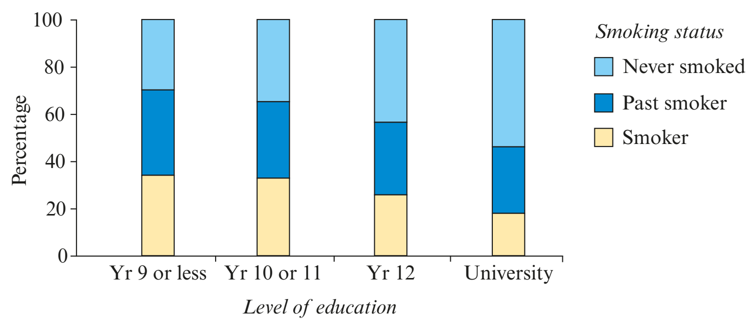
Age group is the EV.

b If we look across all rows, we can see that the percentages are different for each age group. Select one row to compare and discuss – here we have chosen 'high'.	There is an association between the level of interest in sport and age. A high level of interest in sport is seen to decrease steadily across the age categories from 56.5% for under 18 years, 50.2% for 19–25 years, 40.7% for 26–35 years to, at its lowest, 35% for 36–50 years.

The segmented bar chart

A visual display which which can be used to display the information in a two-way frequency table is a **segmented bar chart**. A segmented bar chart consists of separate bars for each value of the explanatory variables, with each bar separated into parts (segments) that show the percentage for each value of the response variable.

The following segmented bar chart below that has been constructed from the table displaying the smoking status of adults (smoker, past smoker, never smoked) by level of education (Year 9 or less, Year 10 or 11, Year 12, university).

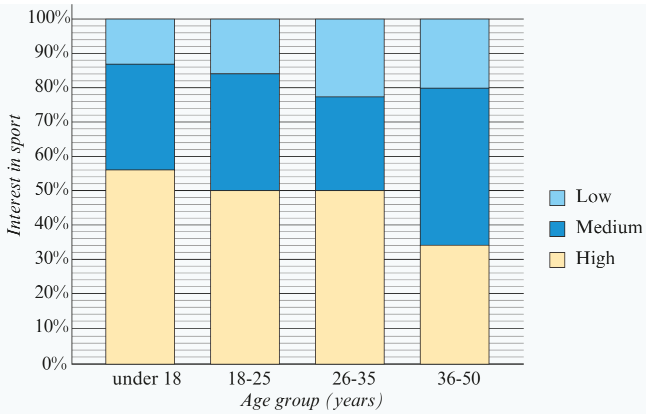


Example 7 Constructing a segmented bar chart

Construct a segmented bar chart to display the association *interest in sport* and *age group* displayed in the table in Example 6.

Solution

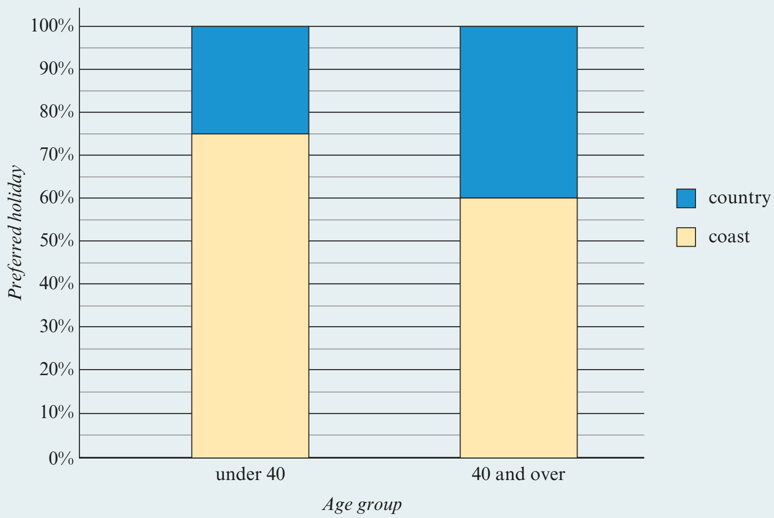
1 Since *age group* is the EV, this variable will label the horizontal axis.
2 The vertical axis should be scaled from 0% to 100%, in intervals of 10%.
3 There will be a bar for each value of *age group*, that is, a bar for each column of the table.
4 Mark off, and colour, with each value of *interest in sport* assigned in the same colour.
5 Add a Key showing which colour has been assigned to each value of *interest in sport*.



The percentaged segmented bar chart allows an easier visual comparison of the percentages than does the percentaged two-way table, and can be used to investigate the association between two categorical variables, as shown in the following example.

Example 8 Identifying and describing associations from a segmented bar chart

The percentaged segmented bar chart below shows the association between *preferred holiday* (country or coast) and *age group* (under 40, 40 or over) for a sample of 800 visitors to a travel website.



Does the percentaged segmented bar chart support the contention there is an association between *preferred holiday* and *age group*?

Explanation

If we look across all rows, we can see that the heights of the segments, and thus the percentages, are different for each age group. Select one row to compare and discuss – here we have chosen 'coast'.

Solution

There is an association between the holiday preference and age. Those aged under forty are more likely to choose a coastal holiday (75%) than those aged forty or over (60%).

Skill-sheet

Exercise 2B

Constructing a two-way frequency table

Example 4

1 The following data were obtained when a sample of 20 Year 12 students were asked if they intended to go to university (*university*). The *gender* of the student was also recorded.

Student No.	*Gender*	*Intends to go to university*	Student No.	*Gender*	*Intends to go to university*
1	F	Yes	11	F	Yes
2	M	Yes	13	M	Yes
3	F	No	13	F	No
4	F	Yes	14	F	Yes
5	M	No	15	M	No
6	M	Yes	16	M	Yes
7	F	Yes	17	F	Yes
8	M	No	18	M	No
9	F	No	19	F	No
10	F	Yes	20	F	Yes

a Identify which is the explanatory and which is the response variable.

b Create a two-way frequency table from the data, with the values of the explanatory variable labelling the columns.

Example 5

2 The following data were obtained when a sample of 30 adults were asked if they supported *reducing university fees*. They were also classified by their *age group*: 17–18 years, 19–25 years, or 26 years or more. The results are given in the table below.

Age group	*Reduce fees*	*Age group*	*Reduce fees*	*Age group*	*Reduce fees*
17–18	Yes	26 or more	Yes	26 or more	No
19–25	Yes	17–18	Yes	19–25	Yes
26 or more	No	19–25	Yes	17–18	No
17–18	Yes	17–18	Yes	26 or more	Yes
19–25	Yes	17–18	Yes	17–18	No
26 or more	Yes	26 or more	No	26 or more	Yes
17–18	Yes	19–25	Yes	19–25	Yes
19–25	No	26 or more	Yes	17–18	Yes
26 or more	No	17–18	No	19–25	No
19–25	No	17–18	Yes	26 or more	Yes

a Identify which variable is the explanatory variable and which is the response variable.

b Create a two-way frequency table from these data, with the values of the explanatory variable labelling the columns.

c Calculate the column percentages for the table.

Using two-way tables to identify associations between two categorical variables

Example 6

3 A survey was conducted with 242 university students. For this survey, data were collected on the students' *enrolment status* (full-time, part-time) and whether or not each *drinks alcohol* ('Yes' or 'No'). Their responses are summarised in the table opposite.

Drinks alcohol	*Enrolment status (%)*	
	Full-time	Part-time
Yes	80.5	81.8
No	19.5	18.2
Total	100.0	100.0

a Which variable is the explanatory variable?

b Is there an association between drinking alcohol and enrolment status? Write a brief report quoting appropriate percentages.

4 The table opposite was constructed from data collected to see if *handedness* (left, right) was associated with *gender* (male, female).

	Gender	
Handedness	Male	Female
Left	22	16
Right	222	147

a Which variable is the response variable?

b Convert the table to percentages by calculating column percentages.

c Is *handnesss* is associated with *gender*? Write a brief explanation using appropriate percentages.

5 A survey was conducted with 59 students studying Business and 51 students studying Arts at university to determine whether they exercise, 'regularly', 'sometimes' or 'rarely'. Their responses are summarised in the percentaged two-way frequency table.

	Course (%)	
Exercise	Business	Arts
Rarely	28.8	39.2
Sometimes	52.5	54.9
Regularly	18.6	5.9
Total	99.9	100.0

a Which is the explanatory variable?

b Is the variable *exercise* nominal or ordinal?

c What percentage of Arts students exercised sometimes?

d Is there an association between how regularly these students exercise and their course? Write a brief response quoting appropriate percentages.

Example 7

6 It was suggested that students in Dr Evans' mathematics class would achieve higher grades than students in Dr Smith's mathematics class. The following table shows the results for each class that year.

	Class		
Exam grade	Dr Evans	Dr Smith	Total
Fail	2	3	5
Pass	11	20	31
Credit or above	5	9	14
Total	18	32	50

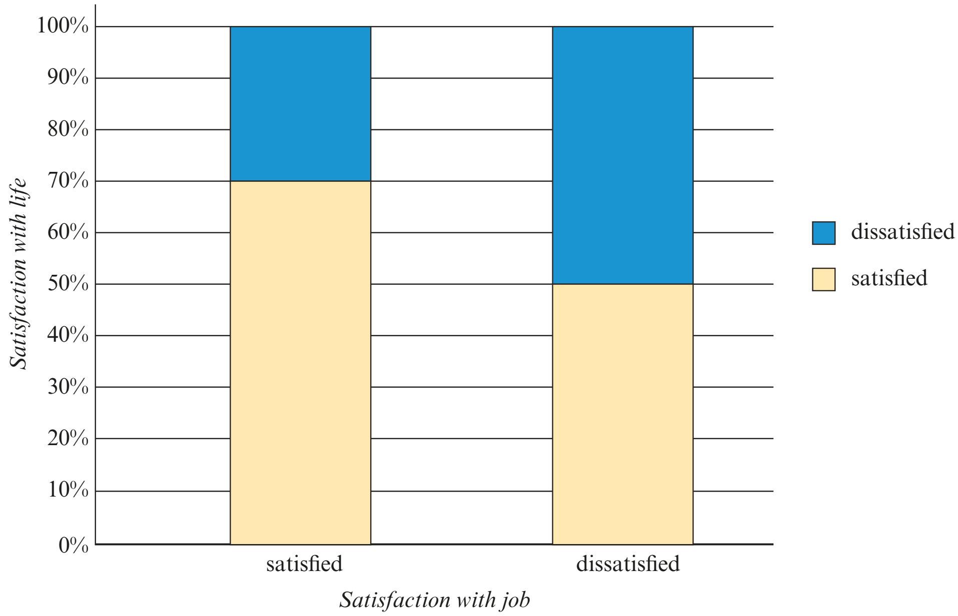
a Construct a percentaged two-way frequency table.

b Construct a percentaged segmented bar chart.

c Write a brief report on the association between teacher and grade.

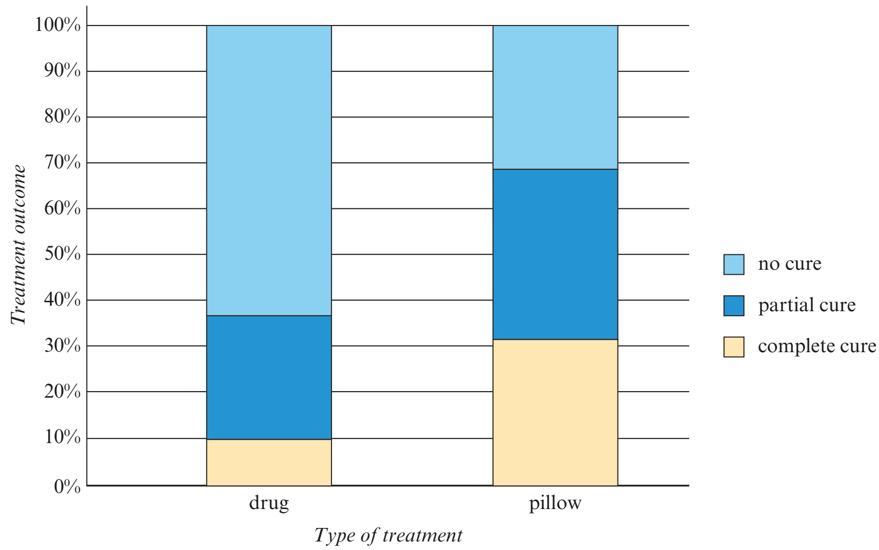
Example 8

7 Are those people who are satisfied with their job more likely to be satisfied with their life? Data collected from a survey of 200 adults are summarised in the following percentaged segmented bar chart.



Does the data support the contention that people who are satisfied with their job are more likely to be satisfied with their life? Write a brief report quoting appropriate percentages.

8 Researchers predicted that using a special pillow would be more effective in curing snoring than treatment with drugs. The association between the outcome of treatment and type of treatment is shown in the following percentaged segmented bar chart.



a Identify which variable is the explanatory variable and which is the response variable.

b Does the data support the contention the special pillow is more effective at treating snoring than the drug treatment? Write a brief report quoting appropriate percentages.

9 As part of the General Social Survey conducted in the US, respondents were asked to say whether they found life *exciting*, *pretty routine* or *dull*. Their marital status was also recorded as married, widowed, divorced, separated or never married. The results are organised into a table as shown.

	Marital status (%)				
Attitude to life	*Married*	*Widowed*	*Divorced*	*Separated*	*Never*
Exciting	47.6	33.8	46.7	45.9	52.3
Pretty routine	48.7	54.3	47.6	44.6	44.4
Dull	3.7	11.9	6.7	9.5	3.2
Total	100.1	100.0	100.0	100.0	99.9

a What percentage of widowed people found life ‘dull’?

b What percentage of people who were never married found life ‘exciting’?

c What is the likely explanatory variable in this investigation?

d Is the variable *attitude to life* nominal or ordinal?

e Does the information you have been given support the contention that a person’s attitude to life is related to their marital status? Justify your argument by quoting appropriate percentages.

Exam 1 style questions

Use the following information to answer Questions 10-12

The data in the following table was collected to investigate the association between tertiary qualifications and happiness.

	Tertiary qualification		
Happy with life	Yes	No	Total
Yes	116	138	254
No	12	34	46
Total	128	172	300

10 The percentage of participants in the study who do not have a tertiary education is closest to:

A 57.3% **B** 80.2% **C** 54.3% **D** 11.3% **E** 19.8%

11 Of those people in the study who did not have a tertiary education, the percentage who are happy with their lives is closest to:

A 57.3% **B** 80.2% **C** 54.3% **D** 11.3% **E** 19.8%

12 The data in the table supports the contention that there is an association between *tertiary qualifications* and *happiness* because:

A 84.7% of people are happy.

B more people without a tertiary qualification are happy than people with a tertiary qualification.

C 90.6% of people with a tertiary qualification are happy, compared to 80.2% of those without a tertiary qualification.

D 54.3% of happy people do not have a tertiary qualification.

E 57.3% of people do not have a tertiary qualification, compared to 42.7% who do.

2C Investigating the association between a numerical and a categorical variable

Learning intentions

- ▶ To be able to use parallel dot plots to identify and describe the association between a numerical variable and a categorical variable for small data sets.
- ▶ To be able to use back-to-back stem plots to display and describe the association between a numerical variable and a categorical variable for small data sets.
- ▶ To be able to use parallel boxplots to display the association between a numerical variable and a categorical variable which can take two or more values.

In the previous section, we learned how to identify and describe associations between two categorical variables. In this section, we will learn to identify and describe associations between a numerical variable and a categorical variable. Suppose, for example, we wish to investigate the association between attendance at a revision class, and test score. Here we can actually identify two variables. One is the variable *test score*, a numerical variable, and the other is the variable *attended revision class*, which is a categorical variable taking the values 'yes' or 'no'.

The outcome of such an investigation will be a brief written report that compares the distribution of the numerical variable across two or more groups, the number of groups equal to the number of values which the categorical variable can take. The starting point for these investigations will be, as always, a graphical display of the data. Here our options are **parallel dot plots**, **back-to-back stem plots** or the **parallel boxplots**.

Using a graphical display of the data, as well as the values of the relevant summary statistics, we can compare the distributions of the numerical variable for each value of the categorical variable according to:

- shape
- centre
- spread

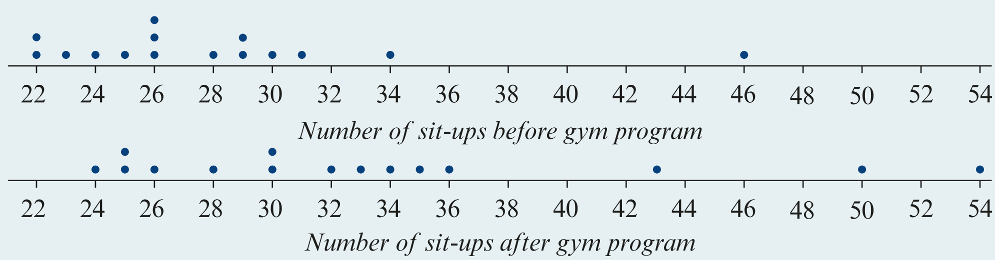
If any of these are noticeably different for differing values of the categorical variable we will conclude that the two variables are associated. Because it is often difficult to clearly identify the shape of a distribution with a small amount of data, we usually confine ourselves to comparing centre and spread, using the medians and *IQR*s, when using dot plots and back-to-back stem plots.

Using parallel dot plots and back-to-back stem plots to identify and describe associations

For small data sets, parallel dot plots and back-to-back stem plots are ideal displays for identifying and describing associations between a numerical and a categorical variable.

Example 9 **Using a parallel dot plot to identify and describe associations**

The parallel dot plots below display the distribution of the number of sit-ups performed by 15 people before and after they had completed a gym program.



Do the parallel dot plots support the contention that the number of sit-ups performed is associated with completing the gym program? Write a brief explanation that compares the distributions.

Solution

Here the numerical variable *sit-ups* is the response variable and the categorical variable *gym program*, taking the values 'before' and 'after', is the explanatory variable.

1 Locate the median number of sit-ups performed before and after the gym program, $M = 26$ and $M = 32$ sit-ups respectively.

2 Determine the IQR of sit-ups performed before and after the gym program, $IQR = 6$ and $IQR = 10$ sit-ups respectively.

3 There are reasonable difference in both the median and IQR of the number of sit-ups performed before and after the gym program, evidence that the number of sit-ups performed is associated with completing the gym program. Report your conclusion, backed up by a brief explanation.

The median number of sit-ups performed after attending the gym program ($M = 32$) is higher than the median number of sit-ups performed before attending the gym program ($M = 26$). The variability in the number of sit-ups has also increased from $IQR = 6$ to $IQR = 10$. Thus we can conclude that the number of sit-ups performed is associated with completing the gym program.

Example 10 Using a back-to back stem plot to identify and describe associations

The back-to-back stem plot below displays the distribution of life expectancy (in years) for the same 13 countries in 1970 and 2010.

1970		2010
2|5 = 52 years		6|7 = 67 years
8	3	
	4	
9 4 2	5	8
9 9 9 8 7 7 1 0	6	9
4	7	1 2 4 4 6 8 9 9
	8	0 0 0

Do the back-to-back stem plots support the contention that life expectancy has changed between these two time periods?

Solution

Here the numerical variable *life expectancy* is the response variable and the variable *year* is the explanatory variable. While *year* can be considered a numerical variable, because it is only taking two values (1970 and 2010) we are treating it as a categorical variable in this example.

1 Determine the median life expectancies for 1970 and 2010. You should find them to be 67 and 76 years, respectively.

2 Determine the quartiles, and hence the values of the IQR for 1970 and 2010. You should find them to be 12.5 years and 8 years respectively.

3 These differences in median and IQR between 1970 and 2010 are sufficient to conclude that the distribution of life expectancy had changed over this time period. Report your conclusion, supported by a brief explanation.

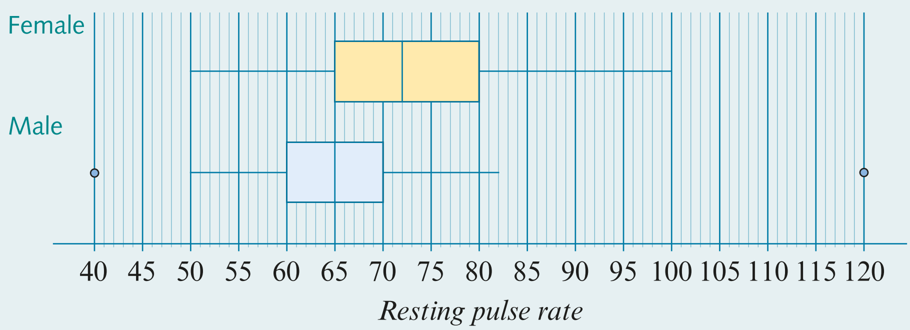
Report: There is an association between year and life expectancy. The median life expectancy has increased between 1970 and 2010, from $M = 67$ years to $M = 76$ years. Over the same period life expectancy has also become less variable (IQR in 1970 = 12.5 years; IQR in 2010 = 8 years).

Using parallel boxplots to identify and describe associations

The statistical tool most commonly used for investigating associations between a numerical and a categorical variable is the **parallel boxplot**. In a parallel boxplot, there is one boxplot for each category of the categorical variable. Associations can then be identified by comparing the way in which the distribution of the numerical variable changes between categories in terms of shape, centre and spread. We should also mention outliers when describing the distributions.

Example 11 Comparing distributions across two groups using parallel boxplots

Use the following parallel boxplots to compare the pulse rates (in beats/minute) for a group of 70 male students and 90 female students.



Solution 0.5

Here the numerical variable *resting pulse rate* is the response variable and the categorical variable *gender* is the explanatory variable.

1 Compare the medians: The median for females is about 72, which is higher than that for males, which is about 65.

2 Compare the spread: The IQR for females is 15, which is more than the IQR for males, which is 10.

3 Compare the shape: Both distributions are approximately symmetric.

4 Locate any outliers. There are two outliers for the males, one at 40 and one at 120.

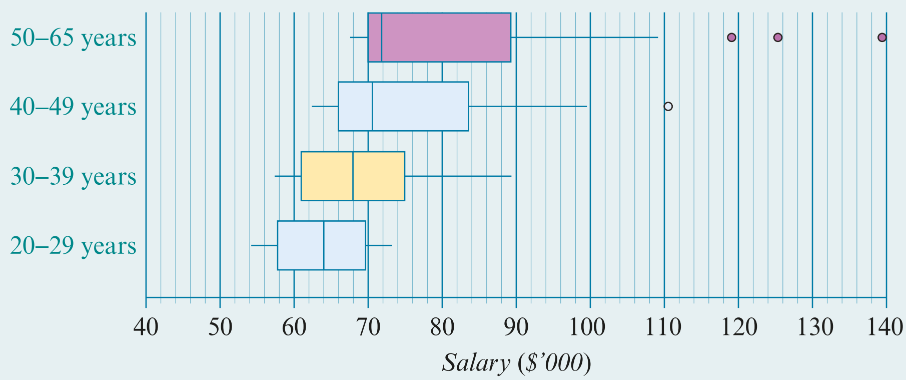
5 Write the report comparing the distributions.

Report

There is an association between resting pulse rate and gender. On average, the resting pulse rate for males is lower (median: male = 65, female = 72) and less variable than that for females (*IQR*: male = 10, female = 15). The distributions of resting pulse rates for both male and female students were approximately symmetric. One male was found to have an extremely low pulse rate of 40, while another had an extremely high pulse rate of 120.

Example 12 **Comparing distributions across more than two groups using parallel boxplots**

Use the parallel boxplots below to compare the salary distribution for workers in a certain industry across four different age groups: 20–29 years, 30–39 years, 40–49 years and 50–65 years.



Solution

Here the numerical variable *salary* is the *response* variable and the categorical variable *age group* is the *explanatory* variable.

1 Compare the medians: The median salary increased from \$64 000 for 20–29 year-olds to \$72 000 for 50–65 year-olds.

2 Compare the *IQR*s: The *IQR* increased from around \$12 000 for 20–29-year-olds to around \$20 000 for 50–65-year-olds.

3 Comparing the shapes: The shape of the distribution of salaries changes with the age group, from symmetric to positively skewed.

4 Locate the outliers: There are no outliers in the 20-29 and 30-39 age group. Outliers also begin to appear at \$110 000 for the 40-49 age group, and at \$119 000, \$126 000 and \$140 000 for the 50-65 age group.

5 Write the report comparing the distributions.

Report

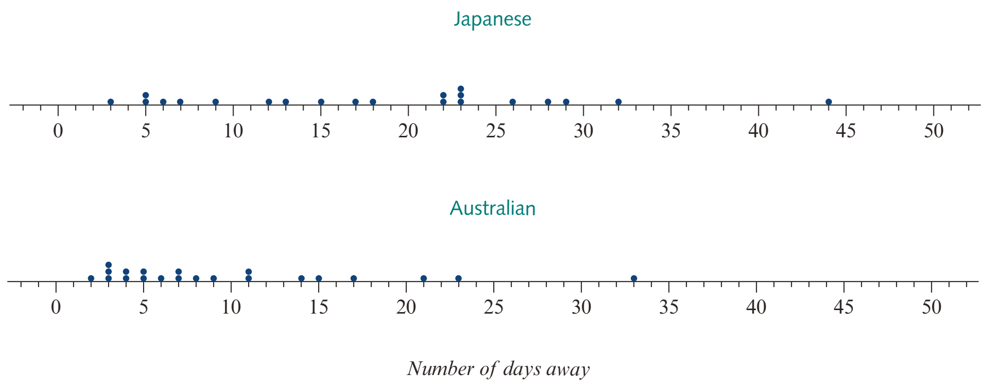
In this industry there is an association between salary and age group. The median salaries increase across the age groups, from \$64 000 for 20–29 year-olds to \$72 000 for 50–65 year-olds. The salaries also became more variable, with the *IQR* increasing from around \$12 000 for 20–29-year-olds to around \$20 000 for 50–65-year-olds.The shape of the distribution of salaries changes with age group, from symmetric for 20–29-year-olds, to progressively more positively skewed as age increases. There are no outliers in the 20-29 and 30-39 age group. Outliers also begin to appear at \$110 000 for the 40-49 age group, and at \$119 000, \$126 000 and \$140 000 for the 50-65 age group.

Skill-sheet

Exercise 2C

Example 9

1 Data was collected to compare the the number of days spent away from home (*number of days away*) by 21 tourists from each of Japan and Australia (*country of origin*). The data collected is displayed in the parallel dot plots below.



a Identify each of the variables and classify each as categorical or numerical.

b Use the parallel dot plots to compare the distributions of *number of days away* for the two different nationalities.

Example 10

2 The back-to-back stem plot shown compares the distribution of the *age* patients (in years) admitted to a small hospital during one week, and their *gender*.

Females		Males
9	0	
5 0	1	3 6
7	2	1 4 5 6 7
7 1	3	4
3 0	4	0 7
0	5	
	6	
9	7	
0|4 = 40 years		4|0 = 40 years

a Classify each variable as categorical or numerical.

b Do the back-to-back plots support the contention that the age of the patients is associated with their gender? Write a brief explanation that compares these distributions in terms of centre and spread.

3 The following back-to-back stem plot displays the distributions of *the number of hours per week spent online* by a group of students, and their *year level*.

Number of hours per week online

1|2 = 21 hours 1|6 = 16 hours

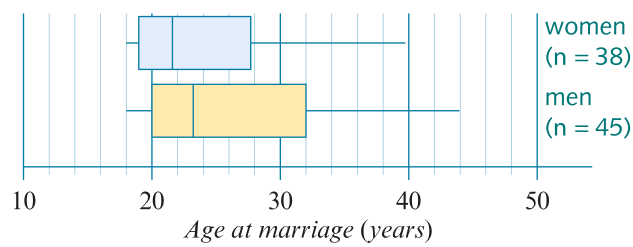
Year 10		Year 11
8 0	0	5 6 7
3 1	1	2 4
8 7 6 5	1	5 5 6 7
4 4 3 1 1 0 0	2	0 1 2 3 3 4
8 5	2	7

a Classify each of the variables as categorical or numerical.

b Use the stem plots to compare these distributions in terms of centre and spread. Draw an appropriate conclusion about the association between year level and the number of hours students spend online each week.

Example 11

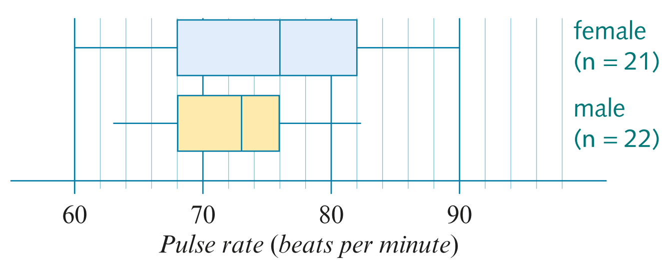
4 The parallel boxplots show the distribution of ages of 45 men and 38 women when first married.



a Identify each of the variables and classify as categorical or numerical.

b Use the boxplots to compare these distributions in terms of shape, centre and spread and draw an appropriate conclusion about the association between gender and the age when first married.

5 The parallel boxplots show the distribution of pulse rates of 21 females and 22 males.

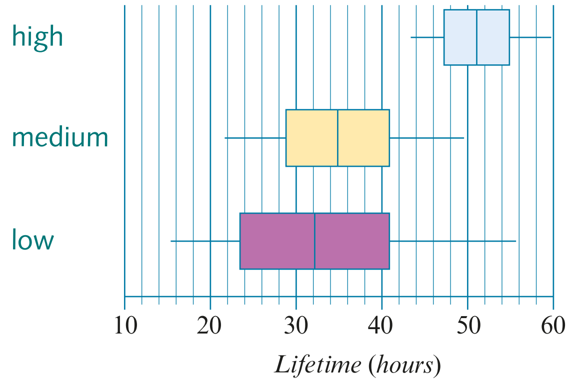


a Identify each of the variables and classify as categorical or numerical.

b Use the boxplots to compare these distributions, and draw an appropriate conclusion about the association between gender and pulse rate.

Example 12

6 The parallel boxplots show the distribution of the lifetime (in hours) of three differently priced batteries (low, medium, high).



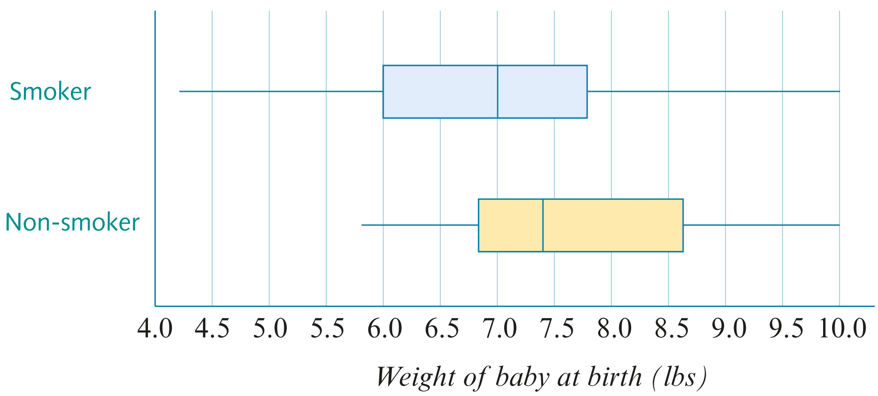
a The two variables displayed here are battery *lifetime* and battery *price* (low, medium, high). Which is the numerical and which is the categorical variable?

b Do the parallel boxplots support the contention that battery lifetime depends on price? Write a brief explanation.

Exam 1 style questions

Use the following information to answer Questions 7 and 8

The data in the following boxplots was collected to investigate the association between smoking and the birth weight of babies.



Use the information in the boxplots to answer the following questions.

7 Which of the following statements is true:

A 75% of babies born to non-smokers weigh more than the lightest 50% of babies born to smokers.

B 50% of babies born to non-smokers weigh more than the heaviest 25% of babies born to smokers.

C 25% of the babies born to smokers weigh less than all the babies born to non-smokers

D All of the babies born to non-smokers weigh more than the heaviest 75% of the babies born to smokers

E The range of baby weights for smokers is less than the range of baby weights for non smokers.

8 The information in the boxplots supports the contention that there is an association between *smoking* and *weight of baby at birth* because:

A The IQRs of birthweight for both groups are approximately the same.

B The median birthweight for smokers is more than the median birthweight for non-smokers.

C The IQRs of birthweight for both groups are very different.

D The median birthweight for smokers is less than the median birthweight for non-smokers.

E Both distributions are approximately symmetric.

2D Investigating associations between two numerical variables

Learning intentions

- To be able to introduce the scatterplot for displaying data from two numerical variables.
- To be able to construct a scatterplot using a CAS calculator.

In this section, we will learn to identify and describe associations between two numerical variables. Suppose, for example, we wish to investigate the association between university *participation rate* (the EV) and average *hours worked* (the RV) in nine countries. The starting point for this investigation is again a graphical display of the data. Here our options are to construct a scatterplot. The data for 9 countries are shown below.

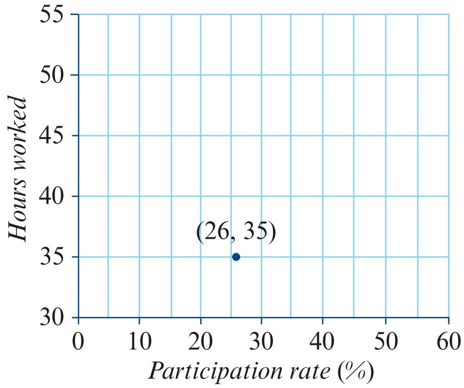
Participation rate (%)	26	20	36	1	25	9	30	3	55
Hours worked	35	43	38	50	40	50	40	53	35

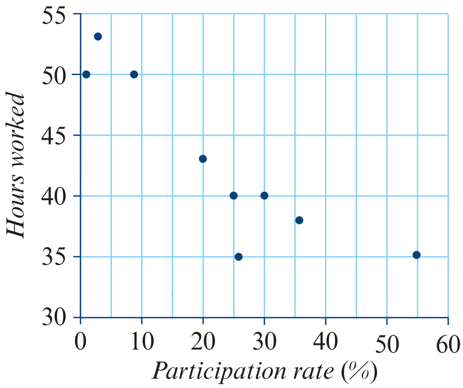
The first step in investigating an association between two numerical variables is to construct a visual display of the data, which we call a **scatterplot**.

The scatterplot

- A scatterplot is a plot which enables us to display bivariate data when **both of the variables are numerical**.
- In a scatterplot, each point represents a single case.
- When constructing a scatterplot, it is conventional to use the **vertical** or **y-axis** for the response variable (RV) and the **horizontal** or **x-axis** for the explanatory variable (EV).

The scatterplot below left shows the point for a country for which the university participation rate is 26% and average hours worked is 35, and the scatterplot below right is the completed scatterplot when each of the remaining countries are plotted.





CAS 1: How to construct a scatterplot using the TI-Nspire CAS

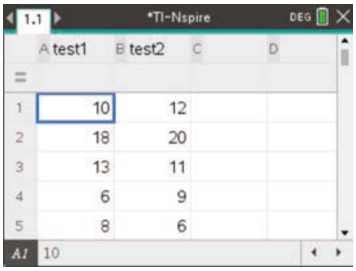
Construct a scatterplot for the set of test scores given below.
Treat *test 1* as the explanatory (i.e. x) variable.

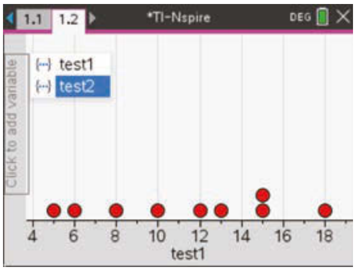
Test 1	10	18	13	6	8	5	12	15	15
Test 2	12	20	11	9	6	6	12	13	17

Steps

1 Start a new document by pressing ctrl + N.

2 Select **Add Lists & Spreadsheet**. Enter the data into lists named *test1* and *test2*.

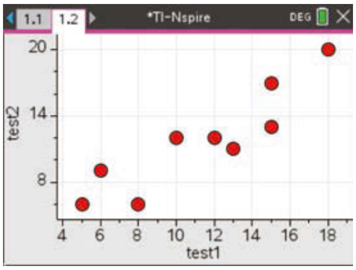




3 Press ctrl + I and select **Add Data & Statistics**.

4 **a** Click on **Click to add variable** on the x-axis and select the explanatory variable *test1*.

b Click on **Click to add variable** on the y-axis and select the response variable *test2*. A scatterplot is displayed. The plot is scaled automatically.



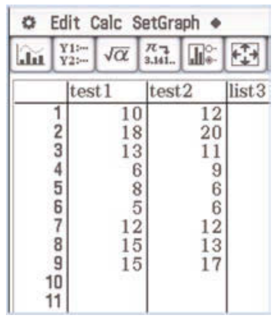
CAS 1: How to construct a scatterplot using the ClassPad

Construct a scatterplot for the set of test scores given below.
Treat *test 1* as the explanatory (i.e. x) variable.

Test 1	10	18	13	6	8	5	12	15	15
Test 2	12	20	11	9	6	6	12	13	17

Steps

1 Open the **Statistics** application and enter the data into the columns named *test1* and *test2*.



2 Tap to open the **Set StatGraphs** dialog box and complete as given below.

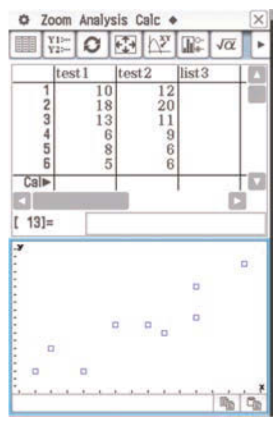
- **Draw:** select **On**.
- **Type:** select **Scatter** (▼).
- **XList:** select **main\test1**(▼).
- **YList:** select **main\test2**(▼).
- **Freq:** leave as **1**.
- **Mark:** leave as **square**.

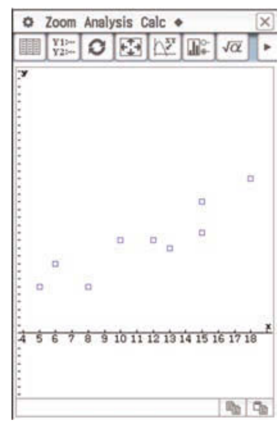
Tap **Set** to confirm your selections.

3 Tap in the toolbar at the top of the screen to plot the scatterplot in the bottom half of the screen.

4 To obtain a full-screen plot, tap Resize from the icon panel.

Note: If you have more than one graph on your screen, tap the data screen, select StatGraph and turn off any unwanted graphs.





Exercise 2D

1 The scatterplot opposite has been constructed to investigate the association between the airspeed (in km/h) of commercial aircraft and the number of passenger seats.

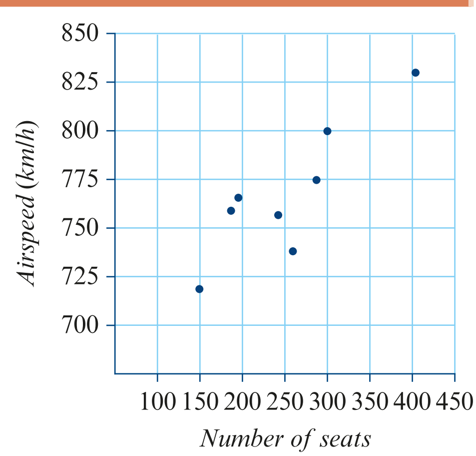
Use the scatterplot to answer the following questions.

a Which is the explanatory variable?

b What type of variable is airspeed?

c How many aircraft were investigated?

d What was the airspeed of the aircraft that has 300 seats?



2

Minimum temperature (x)	17.7	19.8	23.3	22.4	22.0	22.0
Maximum temperature (y)	29.4	34.0	34.5	35.0	36.9	36.4

The table above shows the maximum and minimum temperatures (in °C) during a hot week in Melbourne.

a Enter the data into your calculator, naming the variables *mintemp* and *maxtemp*.

b Construct a scatterplot with minimum temperature as the EV.

3

Balls faced	29	16	19	62	13	40	16	9	28	26	6
Runs scored	27	8	21	47	3	15	13	2	15	10	2

The table above shows the number of runs scored and the number of balls faced by batsmen in a 1-day international cricket match. Use a calculator to construct an appropriate scatterplot.

4

Temperature (°C)	0	10	50	75	100	150
Diameter (cm)	2.00	2.02	2.11	2.14	2.21	2.28

The table above shows the changing diameter of a metal ball as it is heated. Use a calculator to construct an appropriate scatterplot, with temperature as the EV.

5

Number in theatre	87	102	118	123	135	137
Time (minutes)	0	5	10	15	20	25

The table above shows the number of people in a movie theatre at 5-minute intervals after the advertisements started. Use a calculator to construct an appropriate scatterplot.

Exam 1 style questions

6 For which one of the following pairs of variables would it be appropriate to construct a scatterplot?

A *eye colour* (blue, green, brown, other) and *hair colour* (black, brown, blonde, other)

B *test score* and *sex* (male, female)

C *political party preference* (Labor, Liberal, Other) and *age* in years

D *age* in years and *blood pressure* in mmHg

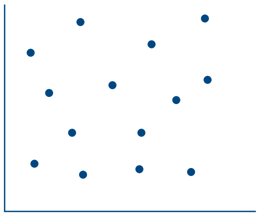
E *height* in cm and *sex* (male, female)

2E How to interpret a scatterplot

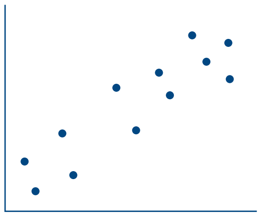
Learning intentions

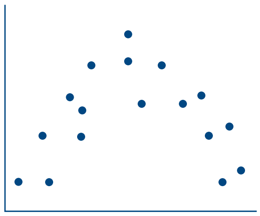
- To be able to use a scatterplot to identify an association between two variables.
- From the the scatterplot, be able to classify an association according to:
  - Direction, which may be positive or negative.
  - Form, which may be linear or non-linear.
  - Strength, which may be weak, moderate or strong.

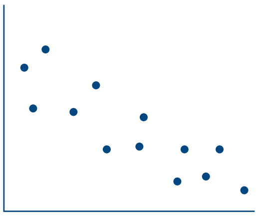
What features do we look for in a scatterplot to help us identify and describe any associations present? First we look to see if there is a **clear pattern** in the scatterplot. In the scatterplot opposite, there is **no clear pattern** in the points. The points are **randomly scattered** across the plot, so we conclude that there is **no association**.



For the three examples below, there is a clear (but different) pattern in each set of points, so we conclude that there is an association in each case.





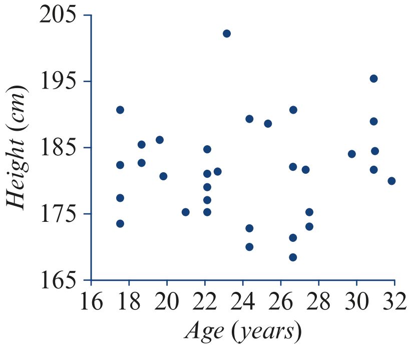


Having found a clear pattern, we need to be able to describe these associations clearly, as they are obviously quite different. The three features we look for in the pattern of points are **direction**, **form** and **strength.** Having found a clear pattern, there are several things we look for in the pattern of points. These are:

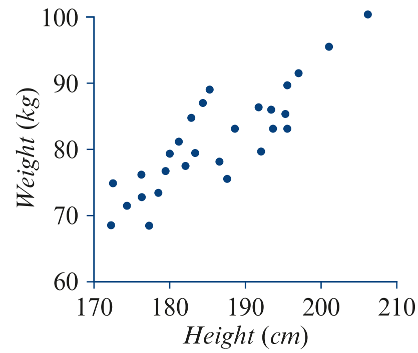
- direction and outliers (if any)
- form
- strength.

Direction and outliers

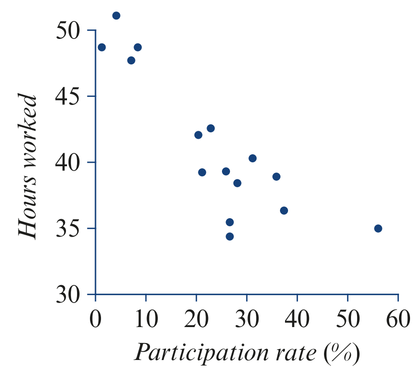
The scatterplot of height against age for a group of footballers (shown opposite) is just a random scatter of points. This suggests that there is **no association** between the variables *height* and *age* for this group of footballers. However, there is a possible outlier for *height*; a footballer who is 201 cm tall.



In contrast, there is a clear pattern in the scatterplot of *weight* against *height* for these footballers (shown opposite). The two variables are associated. If the points in the scatterplot trend upwards as we go from left to right we say there is a **positive association** between the variables. In this example the positive association means that taller players tend to be heavier. In this scatterplot, there are no outliers.



Likewise, the scatterplot of working hours against university participation rates for 15 countries shows a clear pattern. The two variables are associated. If the points in the scatterplot trend downwards as we go from left to right we say there is a **negative association** between the variables. In this example the negative association means that countries with university participation rate tend to work fewer hours. In this scatterplot, there are no outliers.



In general terms, we can classify the direction of an association as follows.

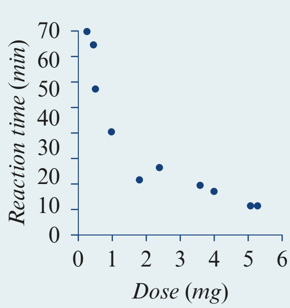
Direction of an association

- Two variables have a **positive association** when the value of the response variable tends to increase as the value of the explanatory variable increases.
- Two variables have a **negative association** when the value of the response variable tends to decrease as the value of the explanatory variable increases.
- Two variables have **no association** when there is no consistent change in the value of the response variable when the values of the explanatory variable increase.

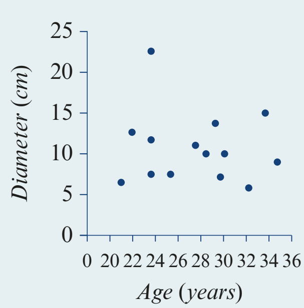
Example 13 Direction of association

Classify each of the following scatterplots as exhibiting positive, negative or no association. Where there is an association, describe the direction of the association in terms of the variables in the scatterplot and what it means in terms of the variables involved.

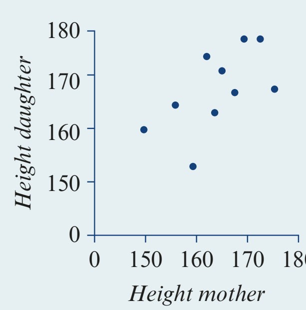
a



b



c

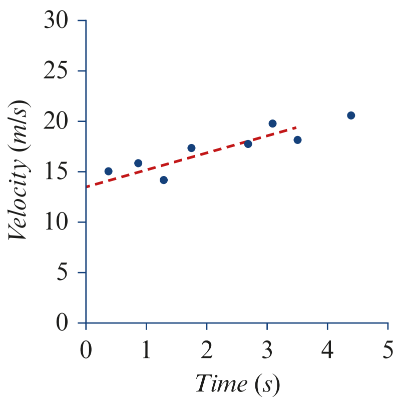


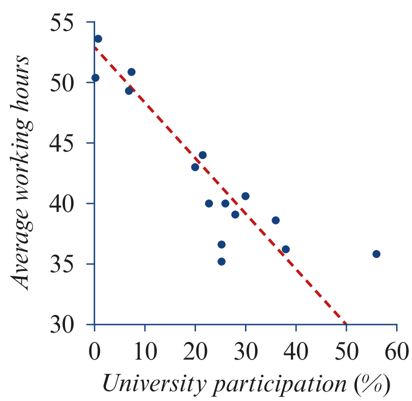
Explanation	Solution
a There is a clear pattern in the scatterplot. The points in the scatterplot trend *downwards* from left to right.	The direction of the association is negative. Reaction times tend to decrease as the drug dose increases.
b There is no pattern in the scatterplot of *diameter* against *age*.	There is no association between diameter and age.
c There is a clear pattern in the scatterplot. The points in the scatterplot trend *upwards* from left to right.	The direction of the association is positive. Taller mothers tend to have taller daughters.

Form

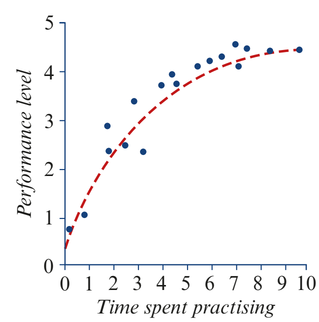
The next feature that interests us in an association is its general form. Do the points in a scatterplot tend to follow a linear pattern or a curved pattern? If the scatterplot has a linear form then we say that the association between the variables is **linear**.

For example, both of the scatterplots below can be described as having a linear form; that is, the scatter in the points can be thought of as scattered around a straight line. (The dotted lines have been added to the graphs to make it easier to see the linear form.)





By contrast, consider the scatterplot opposite, plotting performance level against time spent practising a task. There is an association between performance level and time spent practising, but it is clearly non-linear.



This scatterplot shows that while level of performance on a task increases with practice, there comes a time when the performance level will no longer improve substantially with extra practice.

While non-linear relationships exist (and we must always check for their presence by examining the scatterplot), many of the relationships we meet in practice are linear or can be made linear by transforming the data (a technique you will meet in Chapter 4). For this reason we will restrict ourselves to the analysis of scatterplots with linear forms for now.

In general terms, we can describe the **form of an association** as follows.

Form

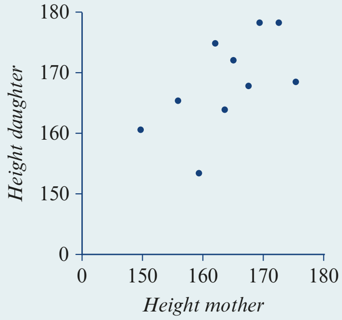
A scatterplot is said to have a **linear form** when the points tend to follow a straight line.

A scatterplot is said to have a **non-linear form** when the points tend to follow a curved line.

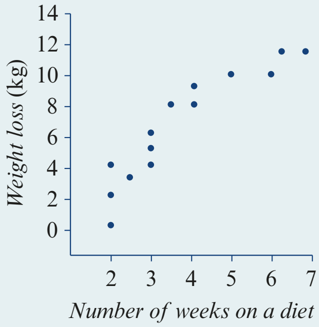
Example 14 Form of an association

Classify the form of the association in each of scatterplot as linear or non-linear.

a



b

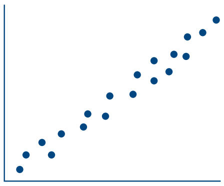


Explanation	Solution
a There is a clear pattern. The points in the scatterplot can be imagined to be scattered around a straight line.	The association is linear.
b There is a clear pattern. The points in the scatterplot can be imagined to be scattered around a curved line rather than a straight line.	The association is non-linear.

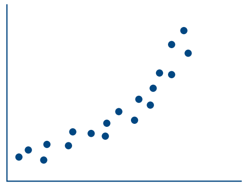
Strength

The **strength of an association** is a measure of how much scatter there is in the scatterplot.

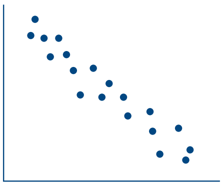
When there is a **strong association** between the variables, there is only a small amount of scatter in the plot, and a pattern is clearly seen.



Strong positive association

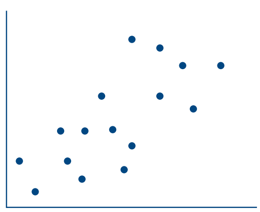


Strong positive association

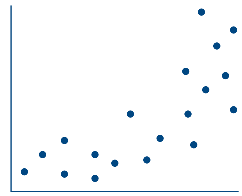


Strong negative association

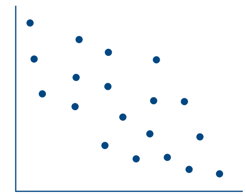
As the amount of scatter in the plot increases, the pattern becomes less clear. This indicates that the association is less strong. In the examples below, we might say that there is a **moderate association** between the variables.



Moderate positive association

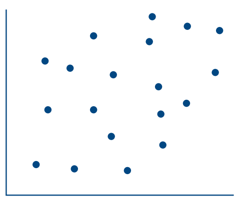


Moderate positive association

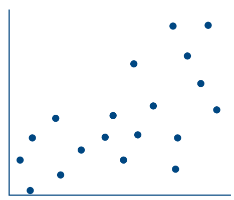


Moderate negative association

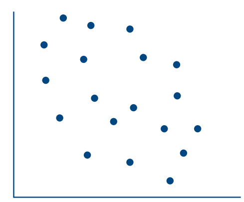
As the amount of scatter increases further, the pattern becomes even less clear. The scatterplots below are examples of **weak association** between the variables.



Weak positive association



Weak positive association



Weak negative association

In general terms, we can describe the **strength of an association** as follows.

Strength

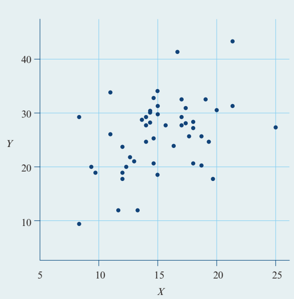
An association is classified as:

- **Strong** if the points on the scatterplot tend to be tightly clustered about a trend line.
- **Moderate** if the points on the scatterplot tend to be broadly clustered about a trend line.
- **Weak** if the points on the scatterplot tend to be loosely clustered about a trend line.
- When no pattern can be seen we say that there is **no association**.

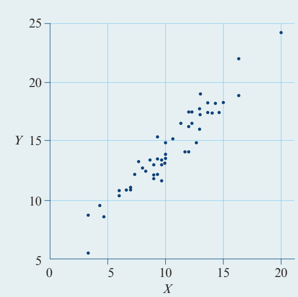
Example 15 Strength of an association

Classify the **strength** of the association in each of the following scatterplots.

a



b



Explanation	Solution
a The points are loosely clustered.	The association is weak.
b The points are tightly clustered.	The association is strong.

Exercise 2E

Assessing the direction of an association from the variables

1 For each of the following pairs of variables:

a Indicate whether you expect an association to exist between the variables.

b If associated, say which variable you would expect to be the EV and which would be the RV, and whether you would expect the variables to be positively or negatively associated.

i *intelligence* and *height*
ii *level of education* and *salary level*
iii *salary* and *tax paid*
iv *frustration* and *aggression*
v *population density* and *distance from the city centre*
vi *time using social media* and *time spent studying*

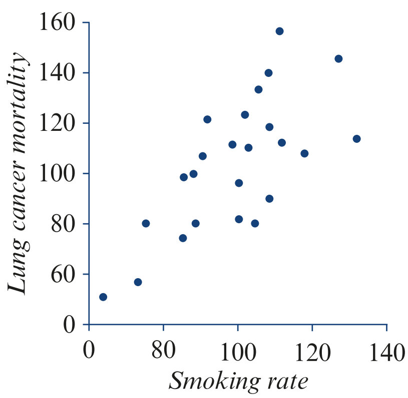
Using a scatterplot to assess the direction, form and strength of an association

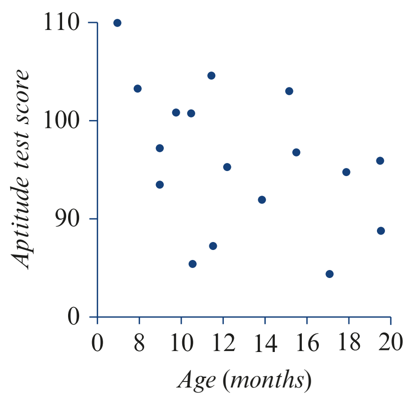
Example 13
Example 14
Example 15

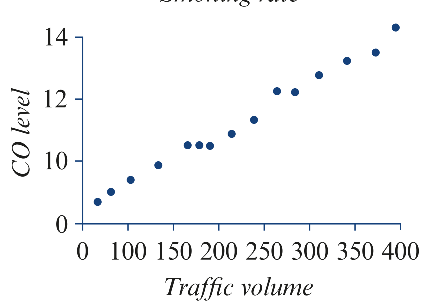
2 For each of the following scatterplots, state whether the variables appear to be associated. If the variables appear to be associated:

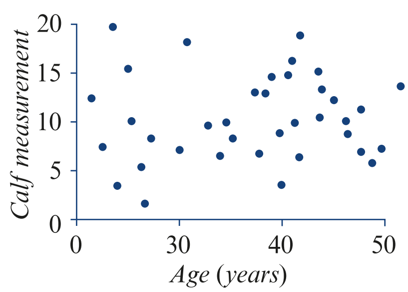
a Describe the association in terms of its direction (positive/negative), form (linear/non-linear) and strength (strong/moderate/weak).

b Write a sentence describing the direction of the association in terms of the variables in the scatterplot.

i


ii


iii


iv


2F Strength of a linear relationship: the correlation coefficient

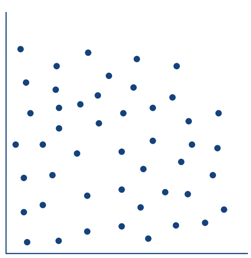
Learning intentions

- ▶ To introduce Pearson's correlation coefficient r as a measure of the strength of a linear association between two variables.
- ▶ To be able to use technology to determine the value of Pearson's correlation coefficient r.
- ▶ To be able to classify the strength of a linear association as weak, moderate or strong based on the value of Pearson's correlation coefficient r.

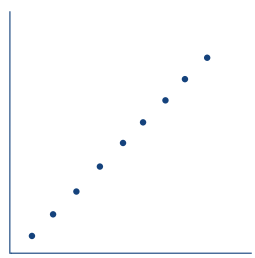
The strength of a linear association is an indication of how closely the points in the scatterplot fit a straight line. If the points in the scatterplot lie exactly on a straight line, we say that there is a perfect linear association. If there is no fit at all we say there is no association. In general, we have an imperfect fit, as seen in all of the scatterplots to date.

To measure the **strength of a linear relationship**, a statistician called Carl Pearson developed a **correlation coefficient**, ***r***, which has the following properties.

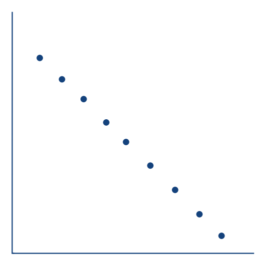
- If there is *no linear* association, $r = 0$.
- If there is a *perfect positive linear* association, $r = +1$.
- If there is a *perfect negative linear* association, $r = -1$.



$r = 0$

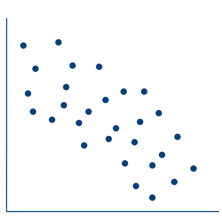


$r = +1$

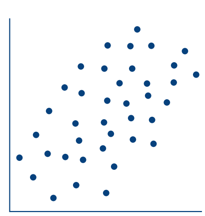


$r = -1$

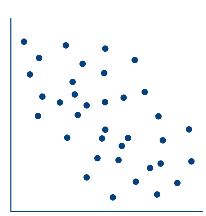
If there is a less than perfect linear association, then the correlation coefficient, r, has a value between –1 and +1, or $-1 < r < +1$. The scatterplots below show approximate values of r for linear associations of varying strengths.



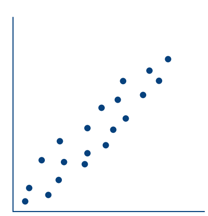
$r = -0.7$



$r = +0.5$



$r = -0.3$



$r = +0.9$

Pearson's correlation coefficient

The Pearson's correlation coefficient, r:

- measures the **strength** of **a linear association**, with larger values indicating stronger relationships
- has a value between –1 and +1
- is positive if the direction of the linear association is positive.
- is negative if the direction of the linear association is negative.
- is close to zero if there is no association.

Calculating the correlation coefficient

Pearson's correlation coefficient, r, gives a numerical measure of the degree to which the points in the scatterplot tend to cluster around a straight line.

Formally, if we call the two variables we are working with x and y, and we have n observations, then r is given by:

$$r = \frac{\sum(x-\bar{x})(y-\bar{y})}{(n-1)s_x s_y}$$

In this formula, $\bar{x}$ and s_x are the mean and standard deviation of the x-values, and $\bar{y}$ and s_y are the mean and standard deviation of the y-values.

Calculating *r* using the formula (optional)

In practice, you can always use your calculator to determine the value of the correlation coefficient. However, to understand what is involved when you use your calculator, it is best that you know how to calculate the correlation coefficient from the formula first.

How to calculate the correlation coefficient using the formula

Use the formula to calculate the correlation coefficient, r, for the following data.

x	1	3	5	4	7
y	2	5	7	2	9

$\bar{x} = 4,\ s_x = 2.236$

$\bar{y} = 5,\ s_y = 3.082$

Give the values rounded to two decimal places.

Steps

1 Write down the values of the means, standard deviations and n.

$\bar{x} = 4 \quad s_x = 2.236$

$\bar{y} = 5 \quad s_y = 3.082 \quad n = 5$

2 Set up a table like that shown opposite to calculate $\sum(x-\bar{x})(y-\bar{y})$.

x	$(x-\bar{x})$	y	$(y-\bar{y})$	$(x-\bar{x})\times(y-\bar{y})$
1	−3	2	−3	9
3	−1	5	0	0
5	1	7	2	2
4	0	2	−3	0
7	3	9	4	12
Sum	0		0	23

$\therefore \sum(x-\bar{x})(y-\bar{y}) = 23$

3 Write down the formula for r.
Substitute the appropriate values and evaluate, rounding the answer to two decimal places.

$$r = \frac{\sum(x-\bar{x})(y-\bar{y})}{(n-1)s_x s_y}$$

$$\therefore r = \frac{23}{(5-1)\times 2.236\times 3.082}$$

$$= 0.834... = 0.83 \text{ (2 d.p.)}$$

CAS 2: How to calculate the correlation coefficient using the TI-Nspire CAS

The following data show the per capita income (in $'000) and the per capita carbon dioxide emissions (in tonnes) of 11 countries.

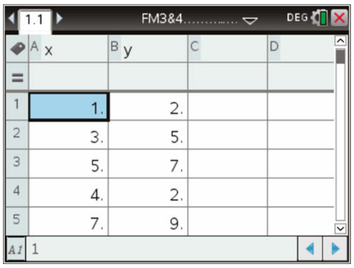
Determine the value of Pearson's correlation coefficient rounded to two decimal places.

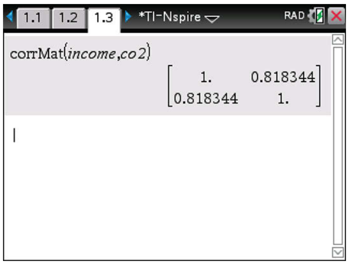
Income (000)	8.9	23.0	7.5	8.0	18.0	16.7	5.2	12.8	19.1	16.4	21.7
CO_2 (tonnes)	7.5	12.0	6.0	1.8	7.7	5.7	3.8	5.7	11.0	9.7	9.9

Steps

1 Start a new document by pressing ctrl + N.

2 Select **Add Lists & Spreadsheet.**
Enter the data into lists named *income* and *co*2.





3 Press ctrl + I and select **Add Calculator**.
Using the **correlation matrix** command: type in **corrmat(*income*, *co2*)** and press enter.
Alternatively: Press 📖 1 C to access the **Catalog**, scroll down to **corrMat(** and press enter. Complete the command by typing in ***income*, *co2*** and press enter.
The value of the correlation coefficient is $r = 0.8342\ldots$ or 0.83 (2 d.p.)

CAS 2: How to calculate the correlation coefficient using the ClassPad

The following data show the per capita income (in $'000) and the per capita carbon dioxide emissions (in tonnes) of 11 countries.

Determine the value of Pearson's correlation coefficient rounded to two decimal places.

Income ($'000)	8.9	23.0	7.5	8.0	18.0	16.7	5.2	12.8	19.1	16.4	21.7
CO_2 *(tonnes)*	7.5	12.0	6.0	1.8	7.7	5.7	3.8	5.7	11.0	9.7	9.9

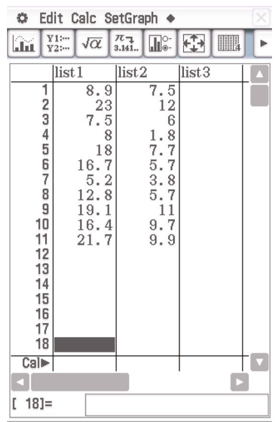
Steps

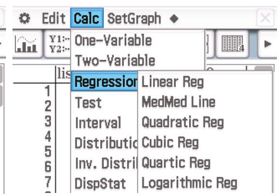
1 Open the **Statistics** application .

2 Enter the data into the columns:

- *Income* in List1
- CO_2 in List2.

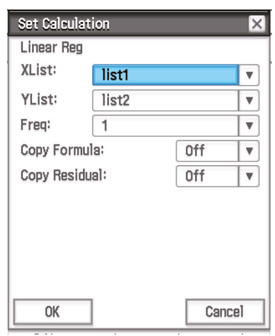
3 Select **Calc>Regression>Linear Reg** from the menu bar.

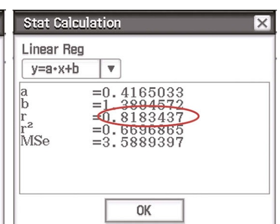




4 Press EXE.

5 Tap **OK** to confirm your selections. The value of the correlation coefficient is $r = 0.818\ldots$ or 0.82 (to 2 d.p.).





Classifying the strength of a linear association

Pearson's correlation coefficient, r, can be used to classify the strength of a linear associations as follows:

$0.75 \le r \le 1$	**strong** positive association
$0.5 \le r < 0.75$	**moderate** positive association
$0.25 \le r < 0.5$	**weak** positive association
$-0.25 < r < 0.25$	**no** association
$-0.5 < r \le -0.25$	**weak** negative association
$-0.75 < r \le -0.5$	**moderate** negative association
$-1 \le r \le -0.75$	**strong** negative association

Example 16 Classifying the strength of a linear association

Classify the strength of each of the following linear associations using the previous table:

a $r = 0.35$ **b** $r = -0.507$

c $r = 0.992$ **d** $r = -0.159$

Explanation	Solution
a The value 0.35 is more than 0.25 and less than 0.5. That is, $0.25 \le r < 0.5$	weak, positive
b The value –0.507 is more than –0.75 and less than –0.5. That is, $-0.75 < r \le -0.5$	moderate, negative
c The value 0.992 is more than 0.75 and less than 1. That is, $0.75 \le r \le 1$	strong, positive
d The value –0.159 is more than –0.25 and less than 0.25. That is, $-0.25 < r < 0.25$	no association

Warning!

If you use the value of the correlation coefficient as a measure of the strength of an association, you should ensure that:

1 the variables are **numeric**

2 the association is **linear**

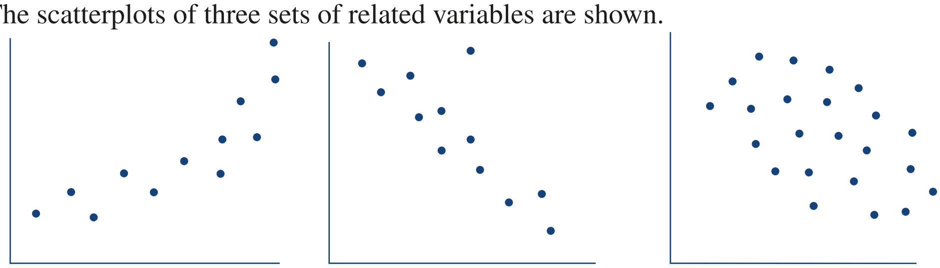
3 there are **no outliers** in the data (the correlation coefficient can give a misleading indication of the strength of the linear association if there are outliers present)

Skill-sheet

Exercise 2F

Basic ideas

1 The scatterplots of three sets of related variables are shown.



Scatterplot A Scatterplot B Scatterplot C

a For each scatterplot, describe the association in terms of strength, direction, form and outliers (if any).

b For which of these scatterplots would it be inappropriate to use the correlation coefficient, r, to give a measure of the strength of the association between the variables? Give reasons.

Calculating r using the formula (optional)

2 Use the formula to calculate the correlation coefficient, r, correct to two decimal places.

x	2	3	6	3	6
y	1	6	5	4	9

$\bar{x} = 4, \; s_x = 1.871$
$\bar{y} = 5, \; s_y = 2.915$

Calculating r using a CAS calculator

3 **a** The table below shows the maximum and minimum temperatures during a heat-wave. The *maximum* and *minimum* temperature each day are linearly associated. Use your calculator to show that $r = 0.818$, correct to three decimal places.

Day	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday
Maximum (°*C*)	29.4	34.0	34.5	35.0	36.9	36.4
Minimum (°*C*)	17.7	19.8	23.3	22.4	22.0	22.0

b This table shows the number of runs scored and balls faced by batsmen in a cricket match. *Runs scored* and *balls faced* are linearly associated. Use your calculator to show that $r = 0.8782$, correct to four decimal places.

Batsman	1	2	3	4	5	6	7	8	9	10	11
Runs scored	27	8	21	47	3	15	13	2	15	10	2
Balls faced	29	16	19	62	13	40	16	9	28	26	6

c This table shows the hours worked and university participation rate (%) in six countries. *Hours worked* and university *participation rate* are linearly associated. Use your calculator to show that $r = -0.6727$, correct to four decimal places.

Country	Australia	Britain	Canada	France	Sweden	US
Hours worked	35.0	43.0	38.2	39.8	35.6	34.8
Participation rate (%)	26	20	36	25	37	55

Classifying the strength of the association based on the value of r

Example 16

4 Use the guidelines on page 143 to classify the strength of the linear associations for each of the linear associations in Question 3.

2G The coefficient of determination

Learning intentions

- To be able to calculate the value of the coefficient of determination.
- To be able to use the coefficient of determination to assess the strength of the association in terms of the explained variation.

If two variables are associated, it is possible to estimate the value of one variable from that of the other. For example, people's weights and heights are associated. Thus, given a person's height, we can roughly predict their weight. The degree to which we can make such predictions depends on the value of r. If there is a perfect linear association ($r = 1$) between two variables, we can make an exact prediction.

For example, when you buy cheese by the gram there is an exact association between the weight of the cheese and the amount you pay ($r = 1$). At the other end of the scale, there is no association between an adult's height and their IQ ($r \approx 0$). So knowing an adult's height will not enable you to predict their IQ any better than guessing.

The coefficient of determination

The degree to which one variable can be predicted from another linearly related variable is given by a statistic called the **coefficient of determination**.

The coefficient of determination is calculated by squaring the correlation coefficient:

$$\text{coefficient of determination} = r^2$$

Example 17 Calculating the coefficient of determination

If the correlation between weight and height is $r = 0.8$, find the value of the coefficient of determination. Express your answer as a percentage.

Solution

The coefficient of determination $= r^2 = 0.8^2 = 0.64 = 64\%$

Note: We have converted the coefficient of determination into a percentage (64%) as this is the most useful form when we come to interpreting the coefficient of determination.

We now know how to calculate the coefficient of determination, but what does it tell us?

Interpreting the coefficient of determination

The coefficient of determination (as a percentage) tells us the **variation in the response variable** that is **explained** by the **variation in the explanatory variable**.

Example 18 Interpreting the coefficient of determination

In the previous example we found the coefficient of determination between height and weight to be 0.64 (or 64%). Interpret this value in terms of the variables *weight* and *height*.

Solution

The coefficient of determination tells us that 64% of the variation in people's *weight* is explained by the variation in their *height*.

What do we mean by 'explained'?

If we take a group of people, their weights and heights will vary. One explanation is that taller people tend to be heavier and shorter people tend to be lighter. The coefficient of determination tells us that 64% of the variation in people's weights can be explained by the variation in their heights. The rest of the variation (36%) in their weights will be explained by other factors, such as diet, lifestyle, build. We could say that 36% of the variation in weight is NOT explained by the variation in height.

Example 19 Calculating and interpreting the coefficient of determination

The level of carbon monoxide (CO) in the air measured at the roadside, and the traffic volume at the same location are linearly related, with $r = +0.985$. Determine the value of the coefficient of determination, write it in percentage terms and interpret. In this relationship, *traffic volume* is the explanatory variable.

Solution

The coefficient of determination is:

$$r^2 = (0.985)^2 = 0.9702$$

Written as a percentage: $0.9702 \times 100 = 97.0\%$ rounded to one decimal place.

Therefore, 97.0% of the variation in carbon monoxide levels in the air can be explained by the variation in traffic volume.

Clearly, traffic volume is a very good predictor of carbon monoxide levels in the air. Thus, knowing the traffic volume enables us to predict carbon monoxide levels with a high degree of accuracy. This is not the case with the next example.

Example 20 Calculating and interpreting the coefficient of determination

Scores on tests of verbal and mathematical ability are linearly related with correlation coefficient $r = +0.275$. Determine the value of the coefficient of determination, write it in percentage terms, and interpret. In this relationship, *verbal ability* is the explanatory variable.

Solution

The coefficient of determination is:

$$r^2 = (0.275)^2 = 0.0756$$

Written as a percentage: $0.0756 \times 100 = 7.6\%$ rounded to one decimal place.

Therefore, only 7.6% of the variation observed in scores on the mathematical ability test can be explained by the variation in scores obtained on the verbal ability test.

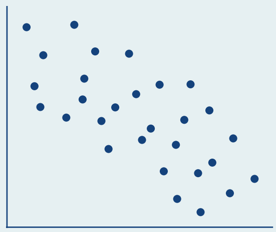
Clearly, scores on the verbal ability test are not good predictors of the scores on the mathematical ability test; 92.4% of the variation in mathematical ability is explained by other factors.

Given the value of the coefficient of determination we can reverse the calculation and find the value of the correlation coefficient. However, since the square root of a number can be positive or negative, we need more information to be able to do this correctly, such as a scatterplot.

Example 21 Calculating the correlation coefficient from the coefficient of determination

For the relationship described by this scatterplot, the coefficient of determination = 0.5210.

Determine the value of the correlation coefficient, r, rounded to four decimal places.



Explanation	Solution
1 Since we know the value of the coefficient of determination (= r^2), we need to find the square root of this value to find r.	$r^2 = 0.5210$ $\therefore r = \pm\sqrt{0.5210} = \pm 0.7218$
2 There are two solutions, one positive and the other negative. Use the scatterplot to decide which applies.	Scatterplot indicates a negative association.
3 Write down your answer.	$\therefore r = -0.7218$

Exercise 2G

Calculating the coefficient of determination from r

Example 17

1 For each of the following values of r, calculate the value of the coefficient of determination and convert to a percentage (correct to one decimal place).

a $r = 0.675$ **b** $r = 0.345$ **c** $r = -0.567$ **d** $r = -0.673$ **e** $r = 0.124$

Calculating and interpreting the coefficient of determination

Example 18
Example 19

2 For each of the following, determine the value of the coefficient of determination, write it in percentage terms, and interpret.

a *Scores on a hearing test* and *age* (EV) are linearly related, with $r = -0.611$.

b *Mortality rate* and *smoking rate* (EV) are linearly related, with $r = 0.716$.

c *Life expectancy* and *birth rate* (EV) are linearly related, with $r = -0.807$.

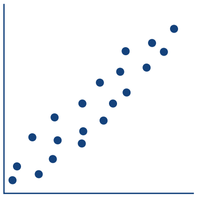
d Daily *maximum* (RV) and *minimum* temperatures are linearly related, with $r = 0.818$.

e *Runs scored* (RV) and *balls faced* by a batsman are linearly related, with $r = 0.8782$.

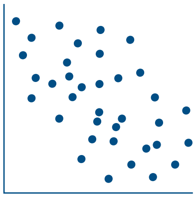
Calculating *r* from the coefficient of determination given a scatterplot

Example 21

3 **a** For the relationship described by the scatterplot shown, the coefficient of determination, $r^2 = 0.8215$.
Determine the value of the correlation coefficient, r (rounded to three decimal places).



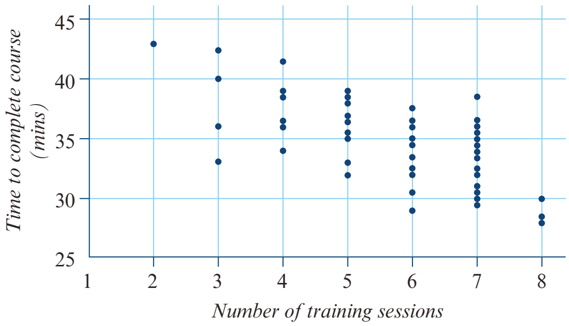
b For the relationship described by the scatterplot shown, the coefficient of determination $r^2 = 0.1243$.
Determine the value of the correlation coefficient, r (rounded to three decimal places).



Exam 1 style questions

Use the following information to answer Questions 4 to 6

The association between the *number of training sessions* attended by participants before undertaking an an obstacle course, and the *time* in minutes it took them to complete the course, is described by the scatterplot shown. The coefficient of determination is 0.3969.



4 The value of the correlation coefficient, r (rounded to two decimal places) is closest to.

A 0.16 **B** 0.40 **C** 0.63 **D** –0.40 **E** –0.63

5 The percentage of variation in *time* explained by the variation in the *number of training sessions* is closest to:

A 39.7% **B** 63.0% **C** 15.8% **D** 37.0% **E** 60.3%

6 The percentage of variation in *time* NOT explained by the variation in the *number of training sessions* is closest to:

A 39.7% **B** 63.0% **C** 15.8% **D** 37.0% **E** 60.3%

7 Suppose that in a certain industry the correlation between *years spent studying* and *income* for employees is 0.73, and the correlation between *age* and *income* is 0.45. Given this information, which one of the following statements is true?

A Older employees tend to have spent more years studying.

B The correlation between *age* and *years spent studying* is 0.32.

C *Age* explains a higher percentage of the variation in *income* than *years spent studying*.

D *Years spent studying* explains a higher percentage of the variation in *income* than *age*.

E Together *age* and *years spent studying* explain 100% of the variation in *income*.

8 Which of the following statements could be true?

A The correlation coefficient between *height* (in centimetres) and *weight* (1 = light, 2 = medium, 3 = heavy) was found to be 0.68.

B The correlation coefficient between *height* (in centimetres) and *head circumference* (in centimetres) was found to be 1.45.

C The correlation coefficient between *blood pressure* (in mmHg) and *weight* (in kg) was found to be -0.3, and the coefficient of determination was found to be $r^2 = -0.09$.

D The correlation coefficient between *age* (in years) and *salary* (in $000's) was found to be 0.68.

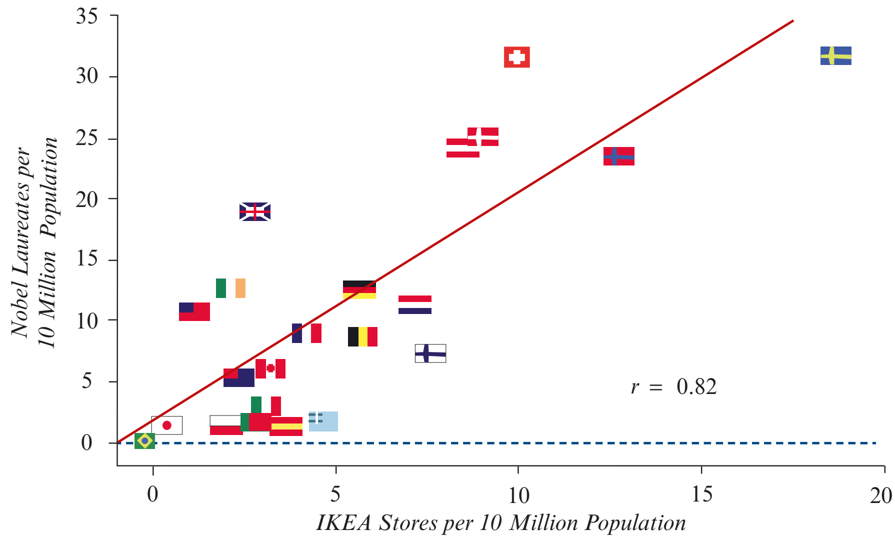
E The correlation coefficient between *height* (in centimetres) and *head circumference* (in centimetres) was found to be 0.49, and the coefficient of determination was found to be 70%.

2H Correlation and causality

Learning intentions

- To be able to define and differentiate the concepts of association and causation.

Recently there has been interest in the strong association between the number of Nobel prizes a country has won and the number of IKEA stores in that country ($r = 0.82$). This strong association is evident in the scatterplot below. Here country flags are used to represent the data points.



Does this mean that one way to increase the number of Australian Nobel prize winners is to build more IKEA stores?

Almost certainly not, but this association highlights the problem of assuming that a strong correlation between two variables indicates the association between them is **causal**.

Correlation does not imply causality

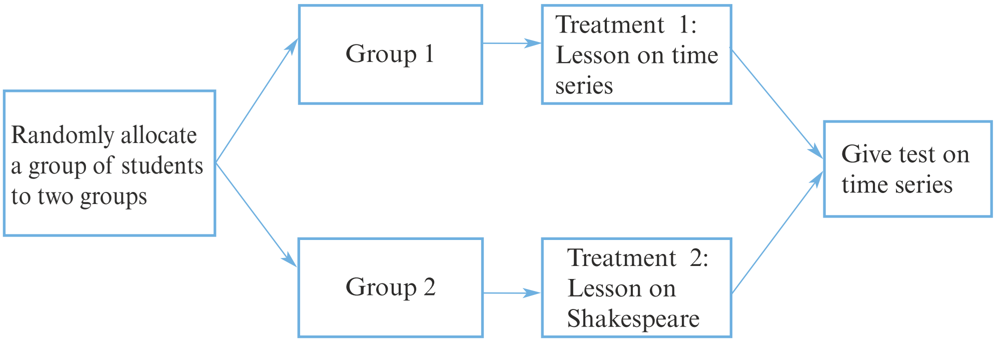
A correlation tells you about the strength of the association between the variables, but no more. It tells you nothing about the source or cause of the association.

Video

To help you with this concept, you should watch the video 'The Question of Causation', which can be accessed through the link below. It is well worth 15 minutes of your time. http://cambridge.edu.au/redirect/?id=6103

Establishing causality

To establish causality, you need to conduct an **experiment**. In an experiment, the value of the explanatory variable is deliberately manipulated, while all other possible explanatory variables are kept constant or controlled. A simplified version of an experiment is displayed below.



In this experiment, a class of students is randomly allocated into two groups. Random allocation ensures that both groups are as similar as possible.

Next, group 1 is given a lesson on time series (treatment 1), while group 2 is given a lesson on Shakespeare (treatment 2). Both lessons are given under the same classroom conditions. When both groups are given a test on time series the next day, group 1 does better than group 2.

We then conclude that this was because the students in group 1 were given a lesson on time series.

Is this conclusion justified?

In this experiment, the students' test score is the response variable and the type of lesson they were given is the explanatory variable. We randomly allocated the students to each group while ensuring that all other possible explanatory variables were controlled by giving the lessons under the same classroom conditions. In these circumstances, the observed difference in the response variable (*test score*) can reasonably be attributed to the explanatory variable (*lesson type*).

Unfortunately, it is extremely difficult to conduct properly controlled experiments, particularly when the people involved are going about their everyday lives.

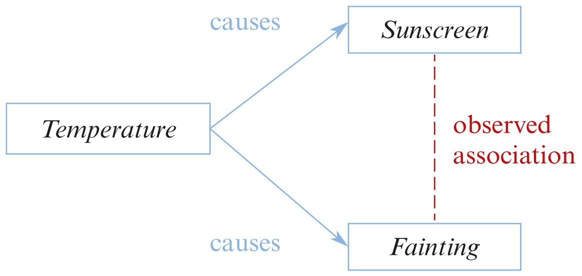
When data are collected through observation rather than experimentation, we must accept that strong association between two variables is insufficient evidence by itself to conclude that an observed change in the response variable has been caused by an observed change in the explanatory variable. It may be, but unless all of the relevant variables are under our control, there will always be alternative non-causal explanations to which we can appeal. We will now consider the various ways this might occur.

Possible non-causal explanations for an association

Common response

Consider the following. There is a strong positive association between the number of people using sunscreen and the number of people fainting. Does this mean that applying sunscreen causes people to faint?

Almost certainly not. On hot and sunny days, more people apply sunscreen and more people faint due to heat exhaustion. The two variables are associated because they are both strongly associated with a common third variable, *temperature*. This phenomenon is called a **common response**. See the diagram below.

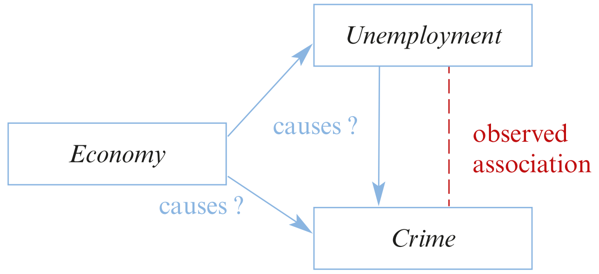


Unfortunately, being able to attribute an association to a single third variable is the exception rather than the rule. More often than not, the situation is more complex.

Confounding variables

Statistics show that *crime* rates and *unemployment* rates in a city are strongly correlated. Can you then conclude that a decrease in unemployment will lead to a decrease in crime rates?

It might, but other possible causal explanations could be found. For example, these data were collected during an economic downturn. Perhaps the state of the economy caused the problem. See the diagram below.



In this situation, we have at least two possible causal explanations for the observed association, but we have no way of disentangling their separate effects. When this happens, the effects of the two possible explanatory variables are said to be **confounded**, because we have no way of knowing which is the actual cause of the association.

Coincidence

It turns out that there is a strong correlation ($r = 0.99$) between the consumption of margarine and the divorce rate in the American state of Maine. Can we conclude that eating margarine causes people in Maine to divorce?

A better explanation is that this association is purely coincidental.

Occasionally, it is almost impossible to identify any feasible confounding variables to explain a particular association. In these cases we often conclude that the association is 'spurious' and it has happened just happened by chance. We call this **coincidence**.

Conclusion

However suggestive a strong association may be, this alone does not provide sufficient evidence for you to conclude that two variables are causally related. Unless the association is totally spurious and devoid of meaning, it will always be possible to find at least one variable 'lurking' in the background that could explain the association.

Association (correlation) and causation

By itself, an observed association between two variables is never enough to justify the conclusion that two variables are causally related, no matter how obvious the causal explanation may appear to be.

Exercise 2H

1 A study of primary school children aged 5 to 11 years finds a strong positive correlation between height and score on a test of mathematics ability. Does this mean that taller people are better at mathematics? What common cause might counter this conclusion?

2 There is a clear positive correlation between the number of churches in a town and the amount of alcohol consumed by its inhabitants. Does this mean that religion is encouraging people to drink? What common cause might counter this conclusion?

3 There is a strong positive correlation between the amount of ice-cream consumed and the number of drownings each day. Does this mean that eating ice-cream at the beach is dangerous? What common cause might explain this association?

4 The number of days a patient stays in hospital is positively correlated with the number of beds in the hospital. Can it be said that bigger hospitals encourage patients to stay longer than necessary just to keep their beds occupied? What common cause might counter this conclusion?

5 Suppose we found a high correlation between smoking rates and heart disease across a group of countries. Can we conclude that smoking causes heart disease? What confounding variable(s) could equally explain this correlation?

6 There is a strong correlation between cheese consumption and the number of people who died after becoming tangled in their bed sheets. What do you think is the most likely explanation for this correlation?

7 There is a strong positive correlation between the number of fire trucks attending a house fire and the amount of damage caused by the fire. Is the amount of damage in a house fire caused by the fire trucks? What common cause might explain this association?

Exam 1 style questions

8 There is a positive correlation between the Gross Domestic Product (GDP), a measure of a country's wealth, and the country's carbon dioxide emissions. From this information it can be concluded that:

A increasing a country's GDP will increase the carbon dioxide emissions of that country.

B decreasing a country's GDP will increase the carbon dioxide emissions of that country.

C increasing a country's carbon dioxide emissions will increase the GDP of that country.

D countries with higher GDP also tend to have lower carbon dioxide emissions.

E countries with higher GDP also tend to have higher carbon dioxide emissions.

2I Which graph?

When investigating associations your first decision is choosing an appropriate graph to display and understand the data you have been given. This decision depends on the type of variables involved – that is, whether they are both categorical, one categorical and one numerical, or both numerical.

The following guidelines might help you make your decision. They are guidelines only, because in some instances there may be more than one suitable graph.

Type of variables		
Response variable	*Explanatory variable*	*Graph*
Categorical	Categorical	Segmented bar chart.
Numerical	Categorical	Parallel boxplots, parallel dot plots
Numerical	Categorical (two categories only)	Back-to-back stem plot, parallel dot plots or parallel boxplots
Numerical	Numerical	Scatterplot

Exercise 2I

1 Which graphical display (parallel boxplots, parallel dot plots, back-to-back stem plot, a segmented bar chart or a scatterplot) would be appropriate to display the relationships between the following? There may be more than one appropriate graph.

a vegetarian (yes, no) and sex (male, female)

b mark obtained on a statistics test and time spent studying (in hours)

c number of hours spent at the beach each year and state of residence

d number of CDs purchased per year and income (in dollars)

e runs scored in a cricket game and number of 'overs' faced

f attitude to compulsory sport in school (agree, disagree, no opinion) and school type (government, independent)

g income level (high, medium, low) and place of residence (urban, rural)

h number of cigarettes smoked per day and sex (male, female)

2 A back-to-back stem plot would be an appropriate graphical tool to investigate the association between a car's *speed*, in kilometres per hour, and the

A driver's *age*, in years

B car's *colour* (white, red, grey, other)

C car's *fuel consumption*, in kilometres per litre

D average *distance* travelled, in kilometres

E driver's *type of licence* (probationary licence, full licence)

Exam 1 style questions

3 The relationship between *height* (in centimetres) and *weight* (1 = light, 2 = medium, 3 = heavy) is best displayed using:

A a histogram

B segmented bar charts

C a scatterplot

D parallel boxplots

E a percentaged two-way frequency table

Key ideas and chapter summary

Assign-ment

Bivariate data	**Bivariate data** are generated when information about two variables is recorded for each subject.
Explanatory and response variables	When investigating associations (relationships) between two variables, the **explanatory** variable (EV) is the variable we expect to explain or predict the value of the **response** variable (RV).
Two-way frequency tables	**Two-way frequency tables** are used as the starting point for investigating the association between two categorical variables.
Segmented bar charts	A **segmented bar chart** can be used to graphically display the information contained in a two-way frequency table. It is a useful tool for identifying relationships between two categorical variables.
Identifying associations between two categorical variables	Associations between two categorical variables are described by comparing appropriate percentages in a **percentaged two-way frequency table** or **percentaged segmented bar chart**.
Identifying associations between a numerical and a categorical variable	Associations between a numerical and a categorical variable are identified using **parallel dot plots**, **parallel boxplots** or a **back-to-back stem plot**. Associations between a numerical and a categorical variable are described by comparing the shape, centre and spread for the distributions.
Scatterplots	A **scatterplot** is used to help identify and describe an association between two numerical variables. In a scatterplot, the **response variable** (**RV**) is plotted on the vertical axis and the **explanatory variable** (**EV**) is plotted on the horizontal axis.
Identifying associations between two numerical variables	Associations between two numerical variables are identified using a **scatterplot**. Associations are classified according to: ■ **Direction**, which may be positive or negative. ■ **Form**, which may be linear or non-linear. ■ **Strength**, which may be weak, moderate or strong.
Correlation coefficient, *r*	The **correlation coefficient**, r, gives a measure of the strength of a linear association.

The coefficient of determination

Coefficient of determination = r^2

The coefficient of determination gives the percentage of variation in the response variable that can be explained by the variation in the explanatory variable.

Correlation and causation

A correlation between two variables does not automatically imply that the association is causal. Alternative non-causal explanations for the association include a **common response** to a common third variable, a **confounded** variable or simply **coincidence**.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ✓

2A **1** I can identify categorical and numerical variables in bivariate data. ☐

See Example 1, and Exercise 2A Question 1

2A **2** I can identify explanatory and response variables. ☐

See Example 2, and Exercise 2A Question 2

2B **3** I can construct a two-way frequency table. ☐

See Example 4, and Exercise 2B Question 1

2B **4** I can percentage a two-way frequency table. ☐

See Example 5, and Exercise 2B Question 2

2B **5** I can describe an association from a percentaged two-way frequency table. ☐

See Example 6, and Exercise 2B Question 3

2B **6** I can construct a segmented bar chart from a percentaged two-way frequency table. ☐

See Example 7, and Exercise 2B Question 6

2B **7** I can describe an association from a percentaged segmented bar chart. ☐

See Example 8, and Exercise 2B Question 7

2C **8** I can use parallel dot plots to identify and describe the association between a numerical variable and a categorical variable. ☐

See Example 9, and Exercise 2C Question 1

2C **9** I can use back-to-back stem plots to display and describe the association between a numerical variable and a categorical variable. ☐

See Example 10, and Exercise 2C Question 2

2C **10** I can use parallel boxplots to display and describe the association between a numerical variable and a categorical variable. ☐

See Example 11, and Exercise 2C Question 4

2D **11** I can construct a scatterplot using a CAS calculator. ☐

See CAS 1, and Exercise 2D Question 2

2E **12** I can classify the direction, form and strength of an association from a scatterplot. ☐

See Example 13, Example 14, Example 15, and Exercise 2E Question 2

2F **13** I can use technology to determine the value of the correlation coefficient r. ☐

See CAS 2, and Exercise 2F Question 3

2F **14** I can classify the strength of a linear association as weak, moderate or strong based on the value of the correlation coefficient r. ☐

See Example 16, and Exercise 2F Question 4

2G **15** I can calculate the value of the coefficient of determination. ☐

See Example 17, and Exercise 2G Question 1

2G **16** I can use the coefficient of determination to assess the strength of the association in terms of the the explained variation. ☐

See Example 18, and Exercise 2G Question 2

2H **17** I understand that correlation does not imply causation. ☐

See Exercise 2H Question 1

Review

Multiple-choice questions

The information in the following frequency table relates to Questions 1 to 4.

	Gender	
Plays sport	Male	Female
Yes	68	79
No	34	
Total	102	175

1 The variables *plays sport* and *gender* are:

- **A** both categorical variables
- **B** a categorical and a numerical variable, respectively
- **C** a numerical and a categorical variable, respectively
- **D** both numerical variables
- **E** neither numerical nor categorical variables

2 The number of females who do not play sport is:

A 21 **B** 45 **C** 79 **D** 96 **E** 175

3 The percentage of males who do not play sport is:

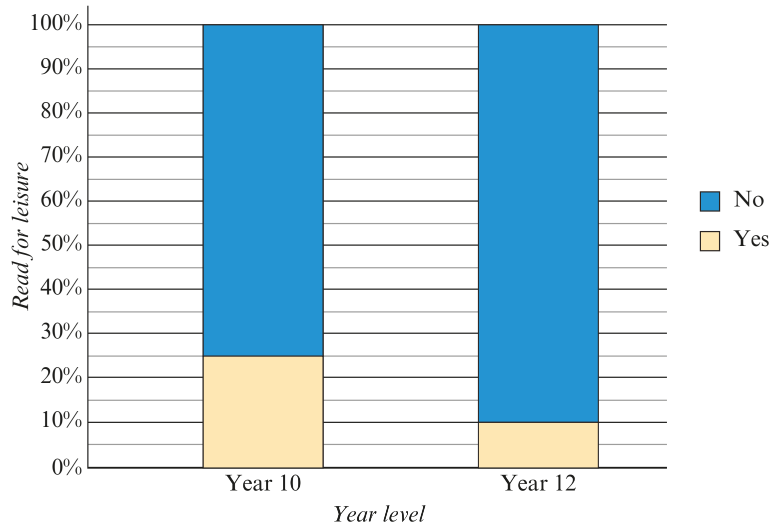
A 19.4% **B** 33.3% **C** 34.0% **D** 66.7% **E** 68.0%

4 The variables *plays sport* and *gender* appear to be associated because:

- **A** more females play sport than males
- **B** fewer males play sport than females
- **C** a higher percentage of females play sport compared to males
- **D** a higher percentage of males play sport compared to females
- **E** both males and females play a lot of sport

Questions 5 to 7 relate to the following information

Students in Year 10 and Year 12 in a certain school were asked whether they read for leisure (*read*). Their responses are summarised in the percentaged segmented bar chart shown.



5 The percentage of Year 12 students who do not read for leisure is closest to:

A 10% **B** 25% **C** 30% **D** 75% **E** 90%

6 The results could be summarised in a two-way frequency table. Which of the following frequency tales could match the percentaged segmented bar chart?

A

	Year Level	
Read	Year 10	Year 12
Yes	31	45
No	47	66

B

	Year Level	
Read	Year 10	Year 12
Yes	45	11
No	135	99

C

	Year Level	
Read	Year 10	Year 12
Yes	75	90
No	25	10

D

	Year Level	
Read	Year 10	Year 12
Yes	75	25
No	90	10

E

	Year Level	
Read	Year 10	Year 12
Yes	40	8
No	38	5

7 The variables *read* and *year level* appear to be associated because:

A very few students in either year level read for leisure

B 75% of Year 10 students do not read for leisure

C only 10% of Year 12 students read for leisure

D 25% of Year 10 students read for leisure, while only 10% of Year 12 students read for leisure

E a higher percentage of Year 12 students read for leisure than Year 10 students

8 The stem plots displays the *time taken* (in minutes) for two groups of 12 people to solve a complex puzzle. Before commencing the puzzle the people were divided into two groups and assigned a different *acitvity*. Group A were asked to exercise vigorously for 10 minutes, while Group B were asked to meditate for 10 minutes.

time

Group A: 1|2 = 21 minutes Group B: 1|6 = 16 minutes

Group A	Stem	Group B
9	0	5 6 7
3 1	1	2 4
8 6 5	1	5 5 6
4 4 3 1	2	0 1 2 3
8 5	2	

The information in the stem plots supports the contention that there is an association between *time* and *activity* because:

A The median time for Group A is more than the median time for Group B.

B The range of times for both groups are approximately equal.

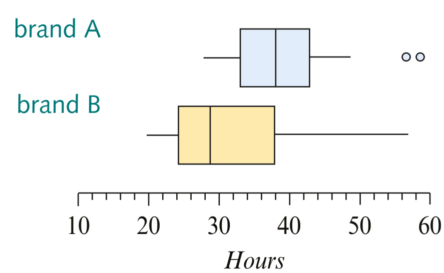
C The median time for Group B is more than the median time for Group A.

D Both distributions are approximately symmetric.

E Both distributions are negatively skewed.

The information in the following parallel boxplots relates to Questions 9 and 10.

The parallel boxplots shown display the distribution of battery life (in hours) for two brands of batteries (brand A and brand B).



9 The variables *battery life* and *brand* are:

A both categorical variables

B a categorical and a numerical variable respectively

C a numerical and a categorical variable respectively

D both numerical variables

E neither a numerical nor a categorical variable

10 Which of the following statements (there may be more than one) support the contention that *battery life* and *brand* are related?

I the median battery life for brand A is clearly higher than for brand B

II battery lives for brand B are more variable than brand A

III the distribution of battery lives for brand A is symmetrical with outliers but positively skewed for brand B

A I only **B** II only **C** III only **D** I and II only **E** I, II and III

11 The association between weight at age 21 (in kg) and weight at birth (in kg) is to be investigated. The variables *weight at age 21* and *weight at birth* are:

A both categorical variables

B a categorical and a numerical variable respectively

C a numerical and a categorical variable respectively

D both numerical variables

E neither numerical nor categorical variables

12 The scatterplot shows the *weights at age 21* and *weight at birth* of 12 women. The association is best described as a:

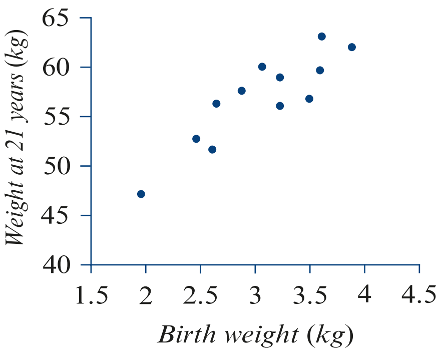
A weak positive linear

B weak negative linear

C moderate positive non-linear

D strong positive non-linear

E moderate positive linear



13 The association between *weight at age 21* and *weight at birth* for a group of males is found to be positive and linear, with a correlation coefficient of $r = 0.58$. For males, the percentage of variation in *weight at age 21* explained by the variation in *weight at birth* is closest to:

A 0.34% **B** 24% **C** 34% **D** 58% **E** 76%

14 The variables *response time* to a drug and *drug dosage* are linearly associated, with $r = -0.9$. From this information, we can conclude that:

A response times are –0.9 times the drug dosage

B response times decrease with decreased drug dosage

C response times decrease with increased drug dosage

D response times increase with increased drug dosage

E response times are 81% of the drug dosage

Review

15 The birth weight and weight at age 21 of eight women are given in the table below.

Birth weight (kg)	1.9	2.4	2.6	2.7	2.9	3.2	3.4	3.6
Weight at 21 (kg)	47.6	53.1	52.2	56.2	57.6	59.9	55.3	56.7

The value of the correlation coefficient is closest to:

A 0.536 **B** 0.6182 **C** 0.7863 **D** 0.8232 **E** 0.8954

16 The value of a correlation coefficient is $r = -0.7685$. The value of the corresponding coefficient of determination is closest to:

A −0.77 **B** −0.59 **C** 0.23 **D** 0.59 **E** 0.77

Use the following information to answer Questions 17 and 18.

The correlation coefficient between heart weight and body weight in a group of mice is $r = 0.765$.

17 Using body weight as the EV, we can conclude that:

A 58.5% of the variation in heart weight is explained by the variation in body weights
B 76.5% of the variation in heart weight is explained by the variation in body weights
C heart weight is 58.5% of body weight
D heart weight is 76.5% of body weight
E 58.5% of the mice had heavy hearts

18 Given that heart weight and body weight of mice are strongly correlated ($r = 0.765$), we can conclude that:

A increasing the body weights of mice will decrease their heart weights
B increasing the body weights of mice will increase their heart weights
C increasing the body weights of mice will not change their heart weights
D heavier mice tend to have lighter hearts
E heavier mice tend to have heavier hearts

19 We wish to investigate the association between the variables *weight* (in kg) of young children and *level of nutrition* (poor, adequate, good). The most appropriate graphical display would be:

A a histogram **B** parallel boxplots **C** a segmented bar chart
D a scatterplot **E** a back-to-back stem plot

20 We wish to investigate the association between the variables *weight* (underweight, normal, overweight) of young children and *level of nutrition* (poor, adequate, good). The most appropriate graphical display would be:

A a histogram **B** parallel boxplots **C** a segmented bar chart
D a scatterplot **E** a back-to-back stem plot

21 There is a strong linear positive correlation ($r = 0.85$) between the amount of *garbage recycled* and *salary level*.
From this information, we can conclude that:

- **A** the amount of garbage recycled can be increased by increasing people's salaries
- **B** the amount of garbage recycled can be increased by decreasing people's salaries
- **C** increasing the amount of garbage you recycle will increase your salary
- **D** people on high salaries tend to recycle less garbage
- **E** people on high salaries tend to recycle more garbage

22 There is a strong linear positive correlation ($r = 0.95$) between the marriage rate in Kentucky and the number of people who drown falling out of a fishing boat.
From this information, the most likely conclusion we can draw from this correlation is:

- **A** reducing the number of marriages in Kentucky will decrease the number of people who drown falling out of a fishing boat
- **B** increasing the number of marriages in Kentucky will increase the number of people who drown falling out of a fishing boat
- **C** this correlation is just coincidence, and changing the marriage rate will not affect the number of people drowning in Kentucky in any way
- **D** only married people in Kentucky drown falling out of a fishing boat
- **E** stopping people from going fishing will reduce the marriage rate in Kentucky

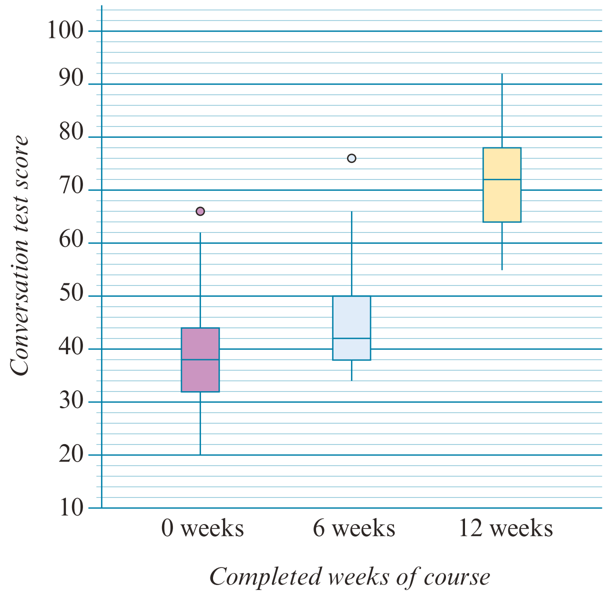
Written response questions

1 One thousand drivers who had an accident during the past year were classified according to age and the number of accidents.

Number of accidents	*Age < 30*	*Age ≥ 30*
At most one accident	130	170
More than one accident	470	230
Total	600	400

- **a** What are the variables shown in the table? Are they categorical or numerical?
- **b** Determine the response and explanatory variables.
- **c** How many drivers under the age of 30 had more than one accident?
- **d** Convert the table values to percentages by calculating the column percentages.
- **e** Use these percentages to comment on the statement: 'Of drivers who had an accident in the past year, younger drivers (age < 30) are more likely than older drivers (age ≥ 30) to have had more than one accident.'

2 In order to improve their ability in French conversation a group of 50 students who were studying French participated in a 12 weeks intensive conversation course. The students were given a test to assess their conversation ability at the start of the course, midway through the course, and at the end of the course. Their results in each of the three tests are shown in the following boxplots.



a The two variables are *Completed weeks of course* and *Conversation test score*. Which is numerical and which is categorical?

b Use the boxplots to compare these distributions, and draw an appropriate conclusion about the association between the number of weeks of the course completed and the score in the conversation test. Quote appropriate statistics in your response.

3 The data below give the hourly pay rates (in dollars per hour) of 10 production-line workers along with their years of experience on initial appointment.

Rate ($/h)	22.57	25.78	28.84	27.37	27.23	24.64	28.95	33.35	29.68	33.99
Experience (years)	1	1	2	2	3	4	5	6	8	12

a Determine which variable is the explanatory variable and which is the response variable.

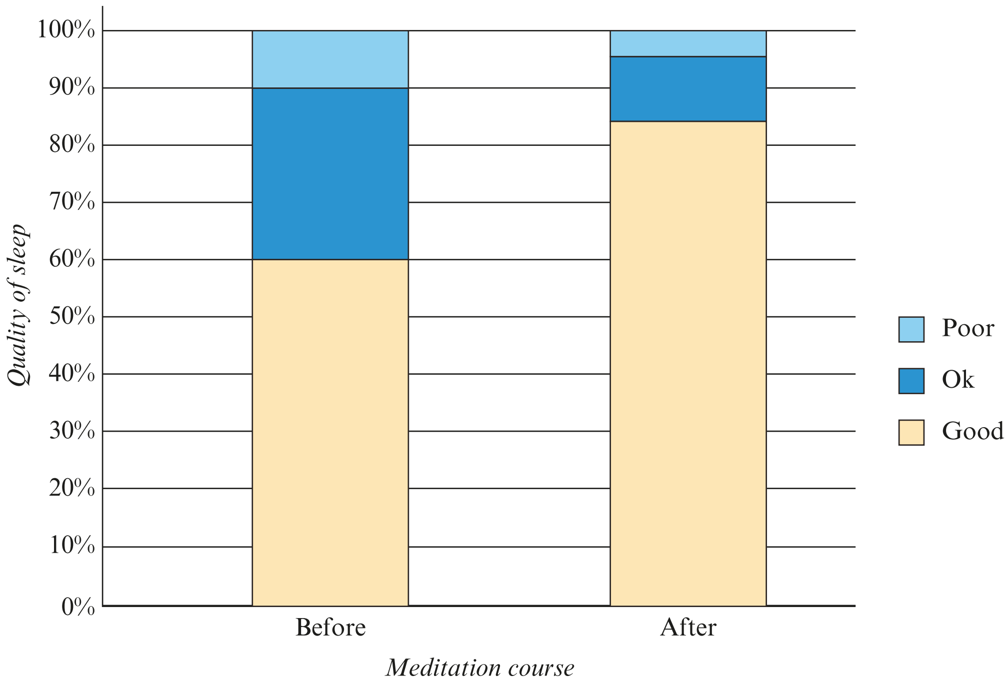
b Use a CAS calculator to construct a scatterplot of the data,

c Comment on direction, outliers, form and strength of any association revealed.

d Determine the value of the correlation coefficient (r) rounded to three decimal places.

e Determine the value of the coefficient of determination (r^2), giving your answer as a percentage rounded to one decimal place, and interpret.

4 In a study of the effects of meditation on the quality of sleep a sample of 500 people were asked to rate the quality of their sleep as 'good', 'OK', or 'poor' before and after participating in the course. Their responses are shown in the segmented bar chart below.



a What percentage of people rated the quality of their sleep as 'good' before they participated in the course?

b Does the segmented bar chart support the contention that for these people their quality of sleep is associated with participation in the course? Justify your answer by quoting appropriate percentages.

Chapter 3

Investigating and modelling linear associations

Chapter questions

- ▶ What is linear regression?
- ▶ What is a residual?
- ▶ What is a least squares line of best fit?
- ▶ How do you find the equation of the least squares line using summary statistics?
- ▶ How do you find the equation of the least squares line using technology?
- ▶ How do you interpret the intercept and slope of the least squares line?
- ▶ How do you use the equation of the least squares line to make predictions?
- ▶ How do you use the coefficient of determination in a regression analysis?
- ▶ What is a residual plot and how is it used?
- ▶ How do you report a regression analysis?

Once we identify a linear association between two numerical variables, we can fit a linear model to the data and find its equation. This equation gives us a better understanding of the nature of the relationship between the two variables, and we can also use the linear model to make predictions based on this understanding of the relationship.

3A Fitting a least squares regression line to numerical data

Learning intentions

- ▶ To be able to define linear regression.
- ▶ To be able to define a residual.
- ▶ To introduce the least squares line of best fit.
- ▶ To be able to find the equation of the least squares line using summary statistics.
- ▶ To be able to find the equation of the least squares line using technology.

The process of modelling an association with a straight line is known as **linear regression** and the resulting line is often called the **regression line**.

The equation of a line relating two variables x and y is of the form

$$y = a + bx$$

where a and b are constants. When the equation is written in this form:

- a represents the coordinate of the point where the line crosses the y-axis (the y-intercept)
- b represents the slope of the line.

In order to summarise any particular (x, y) data set, numerical values for a and b are needed that will ensure the line passes close to the data. There are several ways in which the values of a and b can be found.

The easiest way to fit a line to bivariate data is to construct a scatterplot and draw the line 'by eye'. We do this by placing a ruler on the scatterplot so that it seems to follow the general trend of the data. You can then use the ruler to draw a straight line. Unfortunately, unless the points are very tightly clustered around a straight line, the results you get by using this method will differ a lot from person to person.

The more mathematical approach to fitting a straight line to data is to use the **least squares method**. This method assumes that the variables are linearly related, and works best when there are no clear outliers in the data.

Some terminology

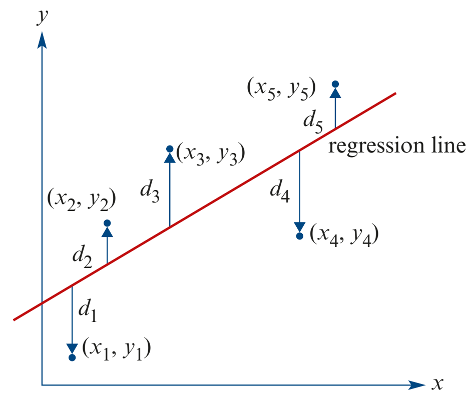
To explain the least squares method, we need to define several terms.

The scatterplot shows five data points, (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) and (x_5, y_5).

A regression line (not necessarily the least squares line) has also been drawn on the scatterplot.

The vertical distances d_1, d_2, d_3, d_4 and d_5 of each of the data points from the regression line are also shown.

These vertical distances, d, are known as **residuals**.



Residuals can be positive, negative or zero:

- Data points above the fitted regression line have a positive residual
- Data points below the fitted regression line have a negative residual
- Data points on the fitted regression line have zero residual.

The least squares line

The least squares line is the line where the sum of the squares of the residuals is as small as possible; that is, it minimises:

$$\text{the sum of the squares of the residuals} = d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$

Why do we minimise the sum of the *squares* of the residuals and not the sum of the residuals? This is because the sum of the residuals for the least squares line is always zero. The least squares line is like the mean. It balances out the data values on either side of itself. Some residuals are positive and some negative, and in the end they add to zero. Squaring the residuals solves this problem.

The least squares line

The **least squares line** is the line that minimises the sum of the squares of the residuals.

The assumptions for fitting a least squares line to data are the same as for using the correlation coefficient, r. These are that:

- the data is numerical
- the association is linear
- there are no clear outliers.

Determining the equation of the least squares regression line

To determine exactly the equation of the least squares regression line we need to determine the values of the intercept (a) and the slope (b) that define the line. The mathematics required is beyond the scope of this course, but calculus can be used to give us rules for these values:

The equation of the least squares regression line

The equation of the least squares regression line is given by $y = a + bx$, where:

the **slope** (b) is given by $b = \dfrac{rs_y}{s_x}$

and

the **intercept** (a) is then given by $a = \bar{y} - b\bar{x}$

Here:

- r is the **correlation coefficient**
- s_x and s_y are the **standard deviations** of x and y
- $\bar{x}$ and $\bar{y}$ are the **mean** values of x and y.
- In these formulas y is the response variable, and x is the explanatory variable.

Note: The formula for the slope of the least squares regression line can be used to find the value of the correlation coefficient (r), when the slope is known.

The **correlation coefficient** (r) is given by $r = \dfrac{bs_x}{s_y}$

Warning!

If you do not correctly decide which is the explanatory variable (the x-variable) and which is the response variable (the y-variable) before you start calculating the equation of the least squares regression line, you will get the wrong answer.

Example 1 Determining the equation of the least squares regression line using summary statistics and the correlation coefficient

The height and weight of 11 people have been recorded, and the values of the following statistics determined:

	height	*weight*
mean	173.3 cm	65.45 kg
standard deviation	7.444 cm	7.594 kg
correlation coefficient	$r = 0.8502$	

Use the formula to determine the equation of the least squares regression line that enables *weight* to be predicted from *height*. Calculate the values of the slope and intercept rounded to two decimal places.

Explanation	Solution
1 Identify and write down the explanatory variable (EV) and the response variable (RV). Label as x and y, respectively.	EV: *height* (x) RV: *weight* (y)
2 Write down the given information.	$\bar{x} = 173.3 \quad s_x = 7.444$ $\bar{y} = 65.45 \quad s_y = 7.594$ $r = 0.8502$
3 Calculate the slope.	Slope: $b = \frac{rs_y}{s_x} = \frac{0.8502 \times 7.594}{7.444}$ $= 0.87$ (rounded to two significant figures)
4 Calculate the intercept.	Intercept: $a = \bar{y} - b\bar{x}$ $= 65.45 - 0.87 \times 173.3$ $= -85$ (rounded to two significant figures)
5 Use the values of the intercept and the slope to write down the least squares regression line using the variable names.	$y = -85 + 0.87x$ or $weight = -85 + 0.87 \times height$

Example 2 Determining the correlation coefficient using the slope of the least squares regression line

Use the following information to find the value of the correlation coefficient r, rounded to three significant figures.

	hours studied	*exam score*
mean	5.87	68.3
standard deviation	1.34	5.42
least squares equation	*exam score* = 52.7 + 2.45 × *hours studied*	

Explanation	Solution
1 Identify and write down the explanatory variable (EV) and the response variable (RV). Label as x and y, respectively.	EV: *hours studied* (x) RV: *exam score* (y)

2 Write down the required information.	$b = 2.45 \quad s_x = 1.34 \quad s_y = 5.42$
3 Calculate the correlation coefficient.	Correlation coefficient: $r = \dfrac{bs_x}{s_y} = \dfrac{2.45 \times 1.34}{5.42}$ $= 0.61$ (rounded to two significant figures)

CAS 1: How to determine and graph the equation of a least squares regression line using the TI-Nspire CAS

The following data give the height (in cm) and weight (in kg) of 11 people.

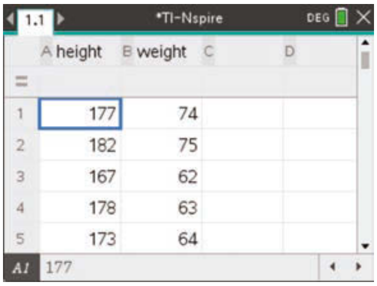
Height (x)	177	182	167	178	173	184	162	169	164	170	180
Weight (y)	74	75	62	63	64	74	57	55	56	68	72

Determine and graph the equation of the least squares regression line that will enable weight to be predicted from height. Write the intercept and slope rounded to three significant figures.

Steps

1 Start a new document by pressing ctrl + N.

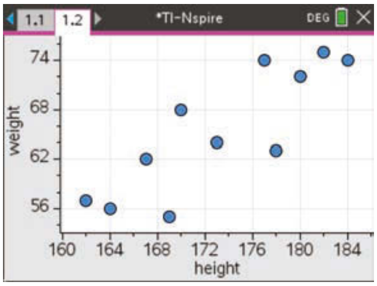
2 Select **Add Lists & Spreadsheet**. Enter the data into lists named *height* and *weight*, as shown.



3 Identify the explanatory variable (EV) and the response variable (RV).
EV: *height*
RV: *weight*

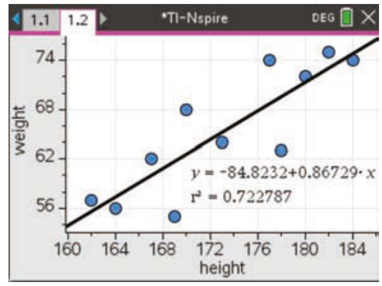
Note: In saying that we want to predict *weight* from *height*, we are implying that *height* is the EV.

4 Press ctrl + I and select **Add Data & Statistics**. Construct a scatterplot with *height* (EV) on the horizontal (or x-) axis and *weight* (RV) on the vertical (or y-) axis.
Press menu **>Settings** and click the **Diagnostics** box. Select **OK** to activate this feature for *all* future documents. This will show the coefficient of determination (r^2) whenever a regression is performed.



5 Press menu>**Analyze**>**Regression**>**Show Linear (a + bx)** to plot the regression line on the scatterplot. Note that, simultaneously, the equation of the regression line is shown on the screen.
The equation of the regression line is:

$$weight = -84.8 + 0.867 \times height$$



The coefficient of determination is $r^2 = 0.723$, rounded to three significant figures.

CAS 1: How to determine and graph the equation of a least squares regression line using the ClassPad

The following data give the height (in cm) and weight (in kg) of 11 people.

Height (x)	177	182	167	178	173	184	162	169	164	170	180
Weight (y)	74	75	62	63	64	74	57	55	56	68	72

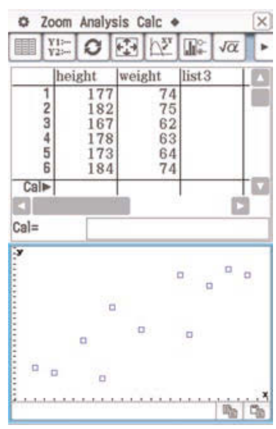
Determine and graph the equation of the least squares regression line that will enable weight to be predicted from height. Write the intercept and slope rounded to three significant figures.

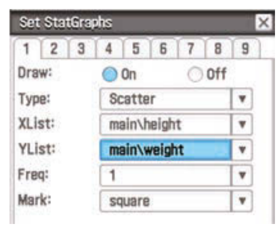
Steps

1 Open the **Statistics** application and enter the data into columns labelled **height** and **weight**.

2 Tap to open the **Set StatGraphs** dialog box and complete as shown.
Tap **Set** to confirm your selections.

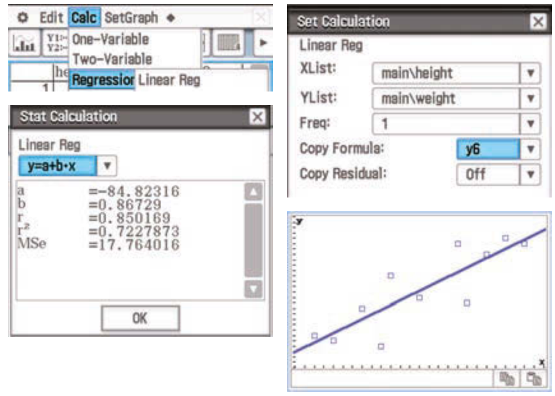
3 Tap in the toolbar at the top of the screen to plot the scatterplot in the bottom half of the screen.





4 To calculate the equation of the least squares regression line:

- Tap **Calc** from the menu bar.
- Tap **Regression** and select **Linear Reg**.
- Complete the **Set Calculations** dialog box as shown.
- Tap **OK** to confirm your selections in the **Set Calculations** dialog box. This also generates the regression results shown opposite.



- Tapping **OK** a second time automatically plots and displays the regression line.

Note: **y6** as the formula destination is an arbitrary choice.

5 Use the values of the intercept a and slope b to write the equation of the least squares line in terms of the variables *weight* and *height*.

weight = −84.8 + 0.867 × *height* (to three significant figures)

The coefficient of determination is r^2 = 0.723, rounded to three significant places.

Skill-sheet

Exercise 3A

Basic ideas

1 What is a residual?

2 The least-squares regression line is obtained by:

A minimising the residuals
B minimising the sum of the residuals
C minimising the sum of the squares of the residuals
D minimising the square of the sum of the residuals
E maximising the sum of the squares of the residuals.

3 Write down the three assumptions we make about the association we are modelling when we fit a least squares line to bivariate data.

Using a formula to calculate the equation of a least square line

Example 1

4 A least squares line $y = a + bx$ is calculated for a set of bivariate data.

a Write down the explanatory variable for this least squares line.
b Given the following information, determine the equation of the least squares line, giving the values of the intercept and slope rounded to three significant figures.

	x	y
mean	10.65	19.91
standard deviation	5.162	6.619
correlation coefficient	$r = 0.7818$	

5 We wish to find the equation of the least squares regression line that enables *pollution level* beside a freeway to be predicted from *traffic volume*.

a Which is the response variable (RV) and which is the explanatory variable (EV)?

b Use the formula to determine the equation of the least squares regression line that enables the pollution level to be predicted from the traffic volume where:

	traffic volume	*pollution level*
mean	11.4	231
standard deviation	1.87	97.9
correlation coefficient	$r = 0.940$	

Write the equation in terms of *pollution level* and *traffic volume* with the intercept and slope rounded to two significant figures.

6 We wish to find the equation of the least squares regression line that enables *life expectancy* in a country to be predicted from *birth rate*.

a Which is the response variable (RV) and which is the explanatory variable (EV)?

b Use the formula to determine the equation of the least squares regression line that enables life expectancy to be predicted from birth rate, where:

	life expectancy	*birth rate*
mean	55.1	34.8
standard deviation	9.99	5.41
correlation coefficient	$r = -0.810$	

Write the equation in terms of *life expectancy* and *birth rate* with the y-intercept and slope rounded to two significant figures.

Using a formula to calculate the correlation coefficient from the slope

Example 2

7 The equation of a least squares line $y = a + bx$ is calculated for a set of bivariate data.

a Write down the response variable for this least-squares line.

b Use the following information to find the value of the correlation coefficient r, rounded to three significant figures.

	x	y
mean	12.51	10.65
standard deviation	4.796	5.162
least squares equation	$y = 16.72 - 0.4847x$	

8 The equation of the least squares regression line that enables *distance* travelled by a car (in 1000s of km) to be predicted from its *age* (in years) was found to be:

$$distance = 15.62 + 11.08 \times age$$

a Which is the response variable (RV) and which is the explanatory variable (EV)?

b Use the following information to find the value of the correlation coefficient r, rounded to three significant figures.

	distance	*age*
mean	78.0	5.63
standard deviation	42.6	3.64

9 The following questions relate to the formulas used to calculate the slope and intercept of the least squares regression line.

a A least squares line is calculated and the slope is found to be negative. What does this tell us about the sign of the correlation coefficient?

b The correlation coefficient is zero. What does this tell us about the slope of the least squares regression line?

c The correlation coefficient is zero. What does this tell us about the intercept of the least squares regression line?

Using a CAS calculator to determine the equation of the least squares line from data

10 The table shows the number of sit-ups and push-ups performed by six students.

Sit-ups (x)	52	15	22	42	34	37
Push-ups (y)	37	26	23	51	31	45

Let the number of *sit-ups* be the explanatory (x) variable. Use your calculator to show that the equation of the least squares regression line is:

push-ups = 16.5 + 0.566 × *sit-ups* (rounded to three significant figures)

11 The table shows average hours worked and university participation rates (%) in six countries.

Hours	35.0	43.0	38.2	39.8	35.6	34.8
Rate	26	20	36	25	37	55

Use your calculator to show that the equation of the least squares regression line that enables participation *rates* to be predicted from *hours* worked is:

$$rate = 130 - 2.6 \times hours \text{ (rounded to two significant figures)}$$

12 The table shows the number of *runs* scored and *balls faced* by batsmen in a cricket match.

Runs (y)	27	8	21	47	3	15	13	2	15	10	2
Balls faced (x)	29	16	19	62	13	40	16	9	28	26	6

a Use your calculator to show that the equation of the least squares regression line enabling *runs* scored to be predicted from *balls faced* is:

$$y = -2.6 + 0.73x$$

b Rewrite the regression equation in terms of the variables involved.

13 The table below shows the number of TVs and cars owned (per 1000 people) in six countries.

Number of TVs (y)	378	404	471	354	381	624
Number of cars (x)	417	286	435	370	357	550

We wish to predict the *number of TVs* from the *number of cars*.

a Which is the response variable?

b Show that, in terms of x and y, the equation of the regression line is:

$$y = 61.2 + 0.930x \text{ (rounded to three significant figures).}$$

c Rewrite the regression equation in terms of the variables involved.

Exam 1 style questions

14 A least squares line of the form $y = a + bx$ is fitted to a scatterplot. Which of the following statements is always true:

A The line will divide the data points so that there are as many points above the line as below the line.

B The sum of the vertical distances from the line to each data point will be a minimum.

C x is the explanatory variable and y is the response variable.

D y is the explanatory variable and x is the response variable.

E Most of the data points will lie on the line.

15 The statistical analysis of the set of bivariate data involving variables x and y resulted in the information displayed in the table below:

	x	y
mean	32.5	88.1
standard deviation	3.42	6.84
least squares equation	$y = -2.56 + 1.45x$	

Using this information the value of the correlation coefficient r for this set of bivariate data is closest to

A 0.73 **B** 0.34 **C** 0.50 **D** 0.53 **E** 0.78

16 A retailer recorded the number of ice creams sold and the day's maximum temperature over 8 consecutive Saturdays one summer.

Temperature (°C)	22	25	36	34	21	28	41	31
Number of ice creams sold	145	155	200	198	150	179	230	180

The equation of the least squares regression line fitted to the data is closest to:

A *number of ice-creams* = 4.08 + 58.2× *temperature*

B *number of ice-creams* = −12.9 + 0.237× *temperature*

C *number of ice-creams* = 58.2 + 4.08× *temperature*

D *temperature* = 3.57 + 72.3× *number of ice-creams*

E *temperature* = −12.8 + 0.237× *number of ice-creams*

3B Using the least squares regression line to model a relationship between two numerical variables

Learning intentions

- To be able to interpret the intercept and slope of the least squares line.
- To be able to use the equation of the least squares line to make predictions.
- To be able to use the coefficient of determination in a regression analysis.
- To be able to use a residual plot to investigate the linearity assumption.
- To be able to report a regression analysis.

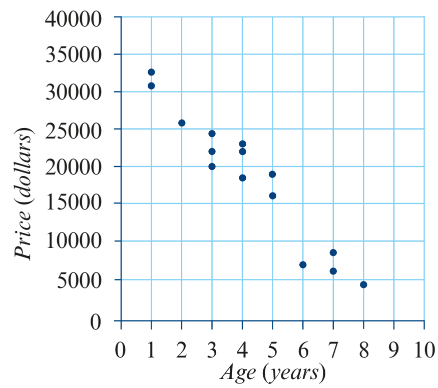
Suppose, for example, that we wish to investigate the nature of the association between the price of a secondhand car and its age. The ultimate aim is to find a mathematical model that will enable the price of a secondhand car to be predicted from its age.

The age (in years) and price (in dollars) of a selection of secondhand cars of the same brand and model have been collected and are recorded in a table (shown).

Age (years)	*Price (dollars)*	*Age (years)*	*Price (dollars)*	*Age (years)*	*Price (dollars)*
1	32 500	3	22 000	5	18 400
1	30 500	4	22 000	6	6 500
2	25 600	4	23 000	7	6 400
3	20 000	4	19 200	7	8 500
3	24 300	5	16 000	8	4 200

We start our investigation of the association between price and age by constructing a scatterplot and using it to describe the association in terms of strength, direction and form. In this analysis, *age* is the explanatory variable.

From the scatterplot, we see that there is a strong, negative, linear association between the price of the car and its age. There are no clear outliers. The correlation coefficient is $r = -0.9643$.



The equation of the least squares regression line from these data is:

$$price = 35\,100 - 3940 \times age$$

Interpreting the slope and intercept of a regression line

Interpreting the slope and intercept of a regression line

For the regression line $y = a + bx$:

- the slope (b) estimates the average change (increase/decrease) in the *response variable* (y) for each one-unit increase in the *explanatory variable* (x)
- the intercept (a) estimates the average value of the *response variable* (y) when the *explanatory variable* (x) equals 0.

Note: The interpretation of the y-intercept in a data context can be meaningless when $x = 0$ is not within the range of observed x-values.

Consider again the least squares regression line relating the *age* of a car to its *price*:

$$price = 35100 - 3940 \times age$$

The two key values in this mathematical model are the intercept (35100) and the slope (−3940). The interpretation of these values is discussed in the following example.

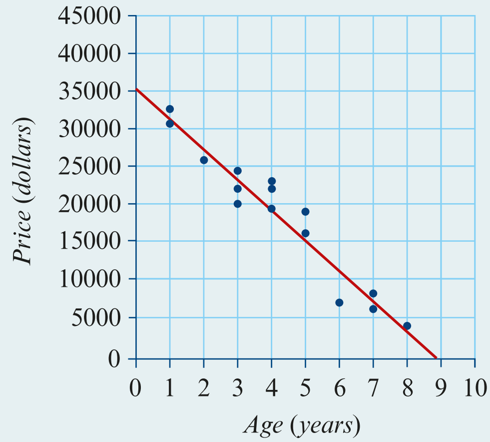
Example 3 Interpreting the slope and intercept of a regression line

The equation of a regression line that enables the *price* of a second-hand car to be predicted from its *age* is:

$$price = 35\,100 - 3940 \times age$$

a Interpret the slope in terms of the variables *price* and *age*.

b Interpret the intercept in terms of the variables *price* and *age*.



Explanation

a The slope predicts the average change (increase/decrease) in the *price* for each 1-year increase in the *age*. Because the slope is negative, it will be a decrease.

b The *intercept* predicts the value of the *price* of the car when *age* equals 0; that is, when the car is new.

Solution

On average, for each additional year of age the price of these cars decreases by \$3940.

On average, the price of these cars when new was \$35 100.

Using the regression line to make predictions

Example 4 Using the regression line to make predictions

The equation of a regression line that enables the *price* of a second-hand car to be predicted from its *age* is:

$$price = 35\,100 - 3940 \times age$$

Use this equation to predict the price of a car that is 5.5 years old.

Explanation

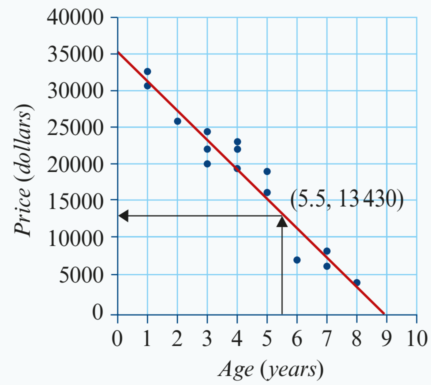
There are two ways this can be done.

One is to draw a vertical arrow at *age* = 5.5 up to the graph and then horizontally across to the *price* axis as shown, to get an answer of around $14 000.

A more accurate answer is obtained by substituting *age* = 5.5 into the equation to obtain $13 430, as shown below.

$$\text{Price} = 35\,100 - 3940 \times 5.5$$
$$= \$13\,430$$

Solution



Interpolation and extrapolation

When using a regression line to make predictions, we must be aware that, strictly speaking, the equation we have found applies only to the range of data values used to derive the equation.

For example, using the equation and rounding to the nearest dollar we would predict that:

- a car which is 2 years old would have a price of $27 220 (price = 35 100 − 3940 × 2)
- a car which is 7 years old would have a price of $7520 (price = 35 100 − 3940 × 7)
- a car which is 12 years old would have a price of $-12 180 (price = 35 100 − 3940 × 12)

This last result, −$12 180 points to one of the limitations of substituting into a regression equation without thinking carefully. Using this regression equation, we have predicted a negative price, which is clearly not correct.

The problem is that we are using the regression equation to make predictions well outside the range of values used to calculate this equation. We only have data for cars which are up to 8 years old. Without knowing that the model works equally well for cars older than this, which we don't, we are venturing into unknown territory and can have little faith in our predictions.

As a general rule, a regression equation only applies to the range of values of the explanatory variables used to determine the equation. Thus, we are reasonably safe using the line to make predictions that lie roughly within this data range, from 1 to 8 years. The process of making a prediction within the range of values of the explanatory variable used to derive the regression equation is called **interpolation** and we can have some faith in these predictions.

However, we must be extremely careful about how much faith we put into predictions made outside the range of values of the explanatory variable. Making predictions outside the data range is called **extrapolation**.

Interpolation and extrapolation

Predicting *within* the range of values of the explanatory variable is called **interpolation**. Interpolation is generally considered to give a **reliable** prediction.

Predicting *outside* range of values of the explanatory variable is called **extrapolation**. Extrapolation is generally considered to give an **unreliable** prediction.

The coefficient of determination

In the previous chapter we define the coefficient of determination as r^2, where r is the value of the correlation coefficient. The coefficient of determination can be considered a measure of the predictive power of a regression equation. While the association between the price of a second-hand car and its age does not explain all the variation in price, knowing the age of a car does give us some information about its likely price.

For a perfect relationship, the regression line explains 100% of the variation in prices. In this case, with $r = -0.964$ we have the:

$$\text{coefficient of determination} = r^2 = (-0.9643)^2 = 0.930 \text{ or } 93.0\%$$

Thus, we can conclude that:

> 93% of the variation in price of the second-hand cars can be explained by the variation in the ages of the cars.

In this case, the regression equation has good predictive power. As a guide, any relationship with a coefficient of determination greater than 30% can be regarded as having good predictive power. In practice, even much lower values of the coefficient of determination can useful.

Example 5 Using the coefficient of determination to compare associations

In a recent study across a number of countries the correlation between educational attainment and the amount spent on education was found to be 0.26, whilst the correlation between educational attainment and the student : teacher ratio was found to be −0.38.

a Find the values of the coefficient of determination between *educational attainment* and the *amount spent on education*, and *student : teacher ratio* respectively.

b Which of the variables, *amount spent on education* or *student : teacher ratio* is more important in explaining the variation in educational attainment?

Solution

a *educational attainment*: $r^2 = 0.26^2 = 6.8\%$

student : teacher ratio : $r^2 = (-0.38)^2 = 14.4\%$

b The variable *student : teacher ratio* explains 14.4% of the variation in *educational attainment*, making it a more important explanatory variable than the *amount spent on education* which explains only 6.8%.

The residual plot – assessing the appropriateness of fitting a linear model to data

So far all of our analysis has been based on the assumption that the relationship between the two variables of interest is linear. This is why it has been essential to examine the scatterplot before proceeding with any further analyses. However, sometimes the scatterplot is not sensitive enough to reveal the non-linear structure of a relationship. To gain more information we need to investigate the fit of the regression line to the data, and we do this using a **residual plot**.

Residuals are defined as the **vertical** distances between the regression line and the actual data value.

Residual plot

A residual plot is a graph of the **residuals** (plotted on the vertical axis) against the **explanatory variable** (plotted on the horizontal axis), where:

Residual value = actual data value − predicted data value

Remember residuals can be positive, negative or zero.

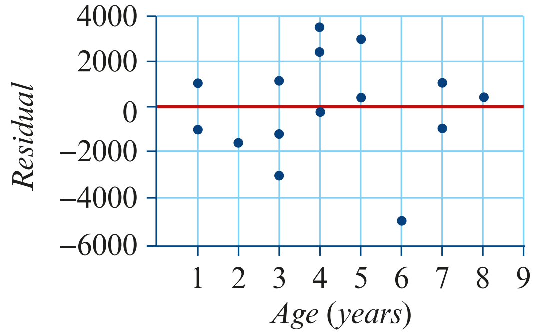
To determine the appropriateness of fitting the least squares regression line to these data we will construct a residual plot. But first, we need to calculate the residual for each value of the explanatory variable, in this case *age*.

Example 6 Calculating a residual

The actual price of the 6-year-old car is \$6500. Calculate the residual when its price is predicted using the regression equation: $price = 35100 - 3940 \times age$

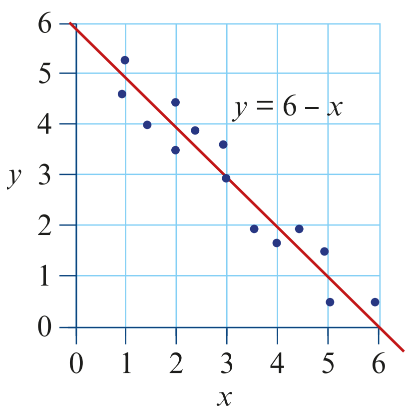
Explanation	Solution
1 Write down the actual price.	Actual price: \$6500
2 Determine the predicted price using the least squares regression equation: $price = 35100 - 3940 \times age$	Predicted price $= 35\,100 - 3940 \times 6$ $= \$\,11\,460$
3 Determine the residual.	Residual = actual - predicted $= \$6500 - \$11\,460$ $= -\$4960$

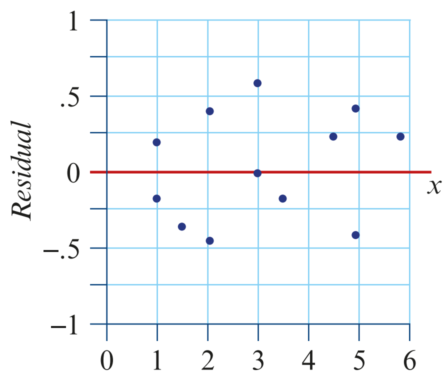
By completing this calculation for all data points, we can construct a residual plot. Because the mean of the residuals is always zero, we will construct the horizontal axis for the plot at zero (indicated by the red line) as shown.



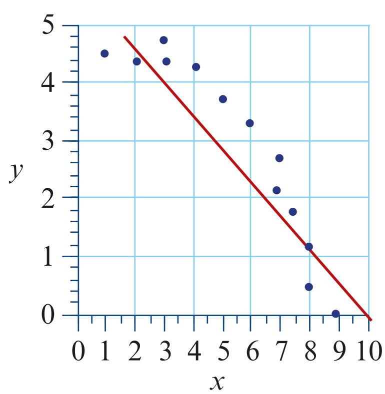
What are we looking for in a residual plot?

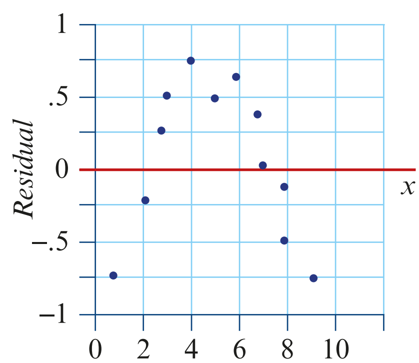
The residual plot is used to check the **linearity assumption** required for a linear regression. The scatterplot below shows a relationship that is clearly linear. When a line is fitted to the data, the resultant residual plot appears to be a random collection of points roughly spread around zero (the horizontal red line in the residual plot).





By contrast, the relationship shown in the following scatterplot is clearly non-linear. Fitting a straight line to the data results in the residual plot shown. While there is some random behaviour, there is also a clearly identifiable curve shown in the scatterplot.

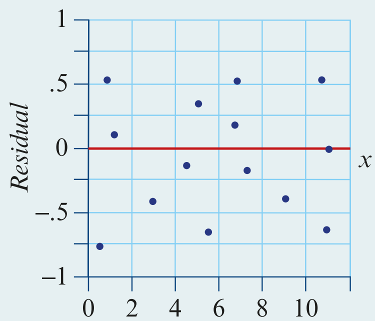




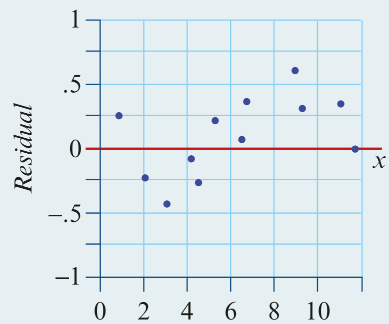
In summary, if a residual plot shows evidence of some sort of systematic behaviour (a pattern), then it is likely that the underlying relationship is non-linear. However, if the residual plot appears to be a random collection of points roughly spread around zero, then we can be happy that our original assumption of linearity was reasonable and that we have appropriately modelled the data. From a visual inspection, it is difficult to say with certainty that a residual plot is random. It is easier to see when it is not random. For present purposes, it is sufficient to say that a clear lack of a pattern in a residual plot is an indication of randomness.

Example 7 **Interpreting a residual plot**

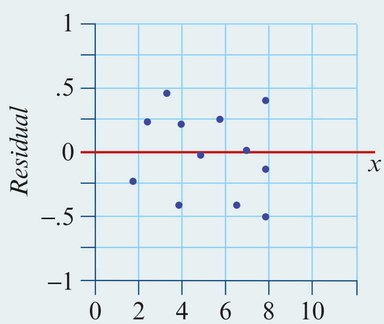
Which of the following residual plots would call into question the assumption of linearity in a regression analysis? Give reasons for your answers.



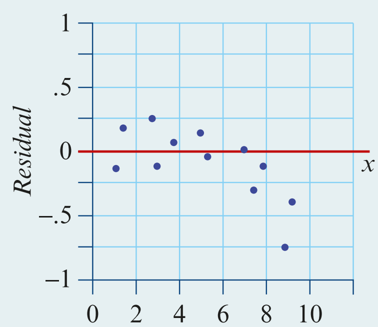
A



B



C



D

Explanation	Solution
Examine each plot, looking for a pattern or structure in the residual.	Plot A – residuals look random, so linearity assumption is met. Plot B – there is a clear curve in the residuals, the linearity assumption is not met. Plot C – residuals look random, so linearity assumption is met. Plot B – there is a clear curve in the residuals, the linearity assumption is not met.

Performing a regression analysis

A full regression analysis involves all of the following analyses, the results of which are collated in a report.

Performing a regression analysis

To carry out a **regression analysis** involves several processes, which include:

- constructing a scatterplot to investigate the nature of an association
- calculating the correlation coefficient to indicate the strength of the relationship
- determining the equation of the regression line
- interpreting the coefficients of the y-intercept (a) and the slope (b) of the least squares regression line $y = a + bx$
- calculating and interpreting the coefficient of determination
- using the regression line to make predictions
- calculating residuals and using a residual plot to test the assumption of linearity
- writing a report on your findings.

Reporting the results of a regression analysis

The final step is to construct a report which brings together all of the analyses which have been described in this section, as shown in the following example.

Example 8 **Reporting the results of a regression analysis**

Construct a report to describe the association between the price and age of secondhand cars.

Solution

From the scatterplot we see that there is a strong negative, linear association between the price of a second hand car and its age, $r = -0.964$. There are no obvious outliers.

The equation of the least squares regression line is: price = 35 100 − 3940 × age.

The slope of the regression line predicts that, on average, the price of these second-hand cars decreased by $3940 each year.

The intercept predicts that, on average, the price of these cars when new was $35 100.

The coefficient of determination indicates that 93% of the variation in the price of these second-hand cars is explained by the variation in their age.

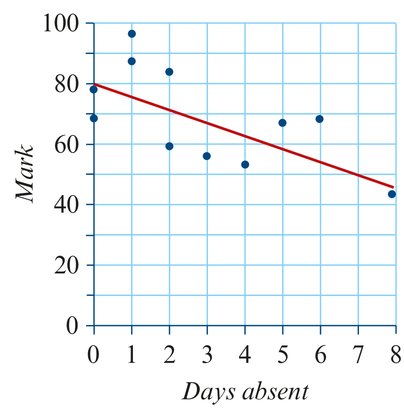
The lack of a clear pattern in the residual plot confirms the assumption of a linear association between the price and the age of these second-hand cars.

Skill-sheet

Exercise 3B

Some basics

1 Use the line on the scatterplot opposite to determine the equation of the regression line in terms of the variables, *mark* and *days absent*. Give the intercept correct to the nearest whole number and the slope correct to one decimal place.



Interpreting the intercept and slope of a regression line

Example 3

2 The equation of a regression line that enables hand span (in cm) to be predicted from height (in cm) is:

$$\textit{hand span} = 2.9 + 0.33 \times \textit{height}$$

a Write down the value of the intercept, and interpret this value in this context of the variables in the equation.

b Write down the value of the slope, and interpret this value in this context of the variables in the equation.

3 The following regression equation can be used to predict a company's weekly sales ($) from their weekly online advertising expenditure ($).

$$sales = 575 + 4.85 \times expenditure$$

a Write down the value of the intercept, and interpret this value in this context of the variables in the equation.

b Write down the value of the slope, and interpret this value in this context of the variables in the equation.

Using the regression line to make predictions

Example 4

4 For children between the ages of 36 and 60 months, the equation relating their *height* (in cm) to their *age* (in months) is:

$$height = 72 + 0.40 \times age$$

Use this equation to predict the height (to the nearest cm) of a child with the following ages. In each case indicate whether you are interpolating or extrapolating.

a 20 months old b 50 months old c 65 months old

5 When preparing between 25 and 100 meals, a hospital's cost (in dollars) is given by the equation:

$$cost = 487.50 + 6.70 \times meals$$

Use this equation to predict the cost (to the nearest dollar) of preparing the following meals. Are you interpolating or extrapolating?

a 0 meals b 80 meals c 110 meals

6 For males of heights from 150 cm to 190 cm tall cm, the equation relating a *son's height* (in cm) to his *father's height* (in cm) is:

$$son\text{'}s\ height = 83.9 + 0.525 \times father\text{'}s\ height$$

Use this equation to predict (to the nearest cm) the adult height of a male whose father is the following heights. State, with a reason, how reliable your predictions are in each case.

a 170 cm tall b 200 cm tall c 155 cm tall

Using the coefficient of determination to compare associations

Example 5

7 A teacher found the correlation between her students' scores on an IQ test (*IQ*) and their final examination score in Year 12 (*exam score*) is 0.45, whilst the correlation between the average number of hours they spend each week studying mathematics (*hours*) and their final examination score in Year 12 (*exam score*) is 0.65.

a Determine the value of the coefficient of determination between *exam score* and *IQ*, expressed as a percentage rounded to one decimal place.

b Determine the value of the coefficient of determination between *exam score* and *hours*, expressed as a percentage rounded to one decimal place.

c Which of the variables, *IQ* or *hours* is more important in explaining the variation in *exam score*?

Calculating a residual

Example 6

8 The equation of a regression line that enables hand span to be predicted from height is:

$$hand\ span = 2.9 + 0.33 \times height$$

a Using this equation, show that the predicted hand span of a person who is 160 cm is 55.7 cm.

b This person has an actual hand span of 58.5 cm. Show that the residual value for this person is 2.8 cm.

9 For a 100 km trip, the equation of a regression line that enables fuel consumption of a car (in litres) to be predicted from its weight (kg) is:

$$fuel\ consumption = -0.1 + 0.01 \times weight$$

a Use this equation to predict (to one decimal place) the fuel consumption of a car which weighs 980kg.

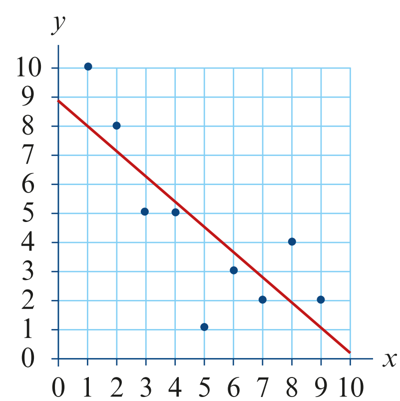
b This car has an actual fuel consumption of 8.9 litres. What is the residual value for this for this data point?

10 From the scatterplot shown determine (to the nearest whole number) the residual values when the value of x is equal to:

a 1

b 3

c 8

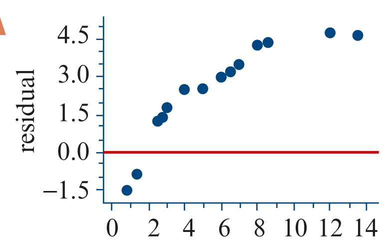


Interpreting a residual plot

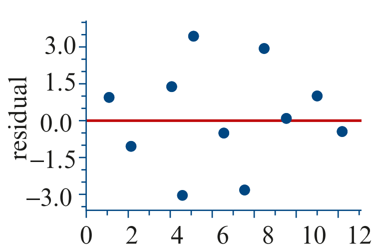
Example 7

11 Each of the following residual plots has been constructed after a least squares regression line has been fitted to a scatterplot. Which of the residual plots suggest that the use of a linear model to fit the data was inappropriate? Why?

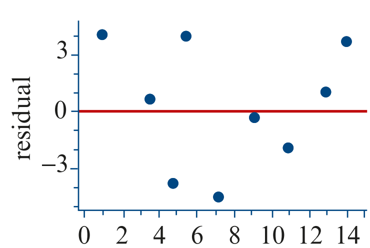
A



B



C



12 In an investigation of the association between the food energy content (in calories) and the fat content (in g) in a standard-sized packet of chips, the least squares regression line was found to be:

$$\textit{energy content} = 27.8 + 14.7 \times \textit{fat content} \quad r^2 = 0.7569$$

a Write down the value of the intercept, and interpret this value in this context of the variables in the equation.

b Write down the value of the slope, and interpret this value in this context of the variables in the equation.

c Interpret the value of the coefficient of determination in terms of the variables in *energy content* and *fat content*.

d Use this equation to predict the energy content of a packet of chips which contains 8 grams of fat.

e If the actual energy content of a packet of chips containing 8 grams of fat is 132 calories, what is the value of the residual?

13 In an investigation of the association between the success rate (%) of sinking a putt and the distance from the hole (in cm) of amateur golfers, the least squares regression line was found to be:

$$\textit{success rate} = 98.5 - 0.278 \times \textit{distance} \quad r^2 = 0.497$$

a Write down the slope of this regression equation and interpret.

b Use the equation to predict the success rate when a golfer is 90 cm from the hole.

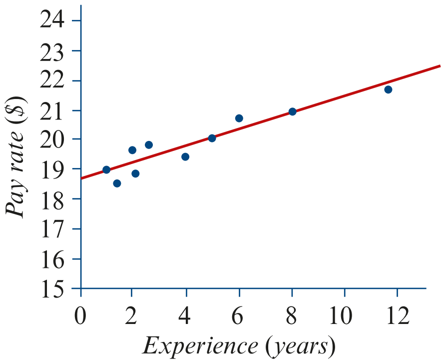
c At what distance (in metres) from the hole does the regression equation predict an amateur golfer to have a 0% success rate of sinking the putt?

d Calculate the value of r, rounded to three decimal places.

e Write down the value of the coefficient of determination in percentage terms and interpret.

14 The scatterplot opposite shows the pay rate (dollars per hour) paid by a company to workers with different years of work experience. Using a calculator, the equation of the least squares regression line is found to have the equation:

$$y = 18.56 + 0.289x \quad \text{with } r = 0.967$$



a Is it appropriate to fit a least squares regression line to the data? Why?

b Work out the coefficient of determination.

c What percentage of the variation in a person's pay rate can be explained by the variation in their work experience?

d Write down the equation of the least squares line in terms of the variables *pay rate* and years of *experience*.

e Interpret the y-intercept in terms of the variables *pay rate* and years of *experience*. What does the y-intercept tell you?

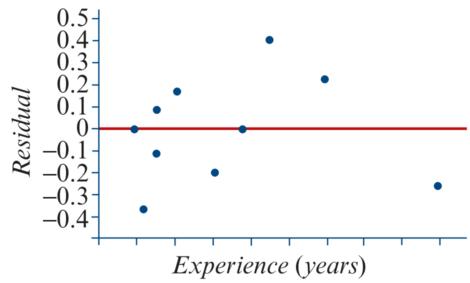
f Interpret the slope in terms of the variables *pay rate* and years of *experience*. What does the slope of the regression line tell you?

g Use the least squares regression equation to:

  i predict the hourly wage of a person with 8 years of experience

  ii determine the residual value if the person's actual hourly wage is \$21.20.

h The residual plot for this regression analysis is shown opposite. Does the residual plot support the initial assumption that the relationship between *pay rate* and years of *experience* is linear? Explain your answer.

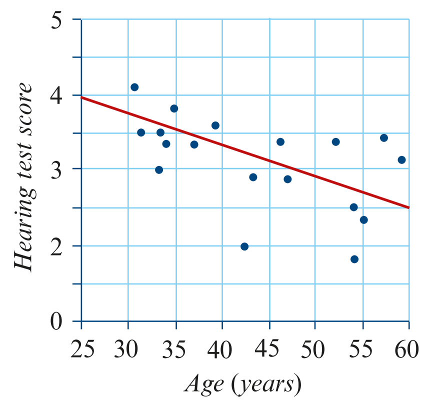


15 The scatterplot opposite shows scores on a hearing test against age. In analysing the data, a statistician produced the following statistics:

- coefficient of determination: $r^2 = 0.370$
- least squares line: $y = 4.9 - 0.043x$

a Determine the value of Pearson's correlation coefficient, r, for the data.

b Interpret the coefficient of determination in terms of the variables *hearing test score* and *age*.



c Write down the equation of the least squares line in terms of the variables *hearing test score* and *age*.

d Write down the slope and interpret.

e Use the least squares regression equation to:

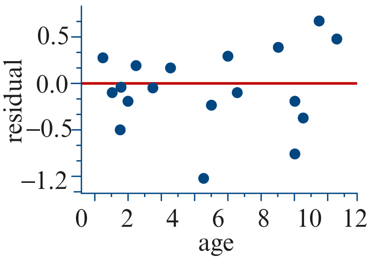
  i predict the hearing test score of a person who is 20 years old

  ii determine the residual value if the person's actual hearing test score is 2.0.

f Use the graph to estimate the value of the residual for the person aged:

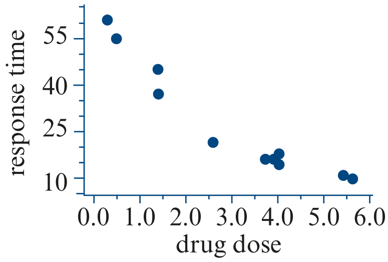
  i 35 years **ii** 55 years.

g The residual plot for this regression analysis is shown opposite.
Does the residual plot support the initial assumption that the relationship between hearing test score and age is essentially linear? Explain your answer.

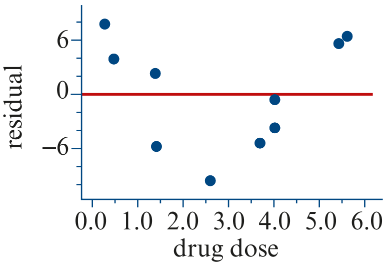


Reporting the results of a residual analysis

Example 8 **16** In a study of the effectiveness of a pain relief drug, the response time (in minutes) was measured for different drug doses (in mg). A least squares regression analysis was conducted to enable response time to be predicted from drug dose. The results of the analysis are displayed.



Regression equation: $y = a + bx$
$a = 55.8947$
$b = -9.30612$
$r^2 = 0.901028$
$r = -0.949225$

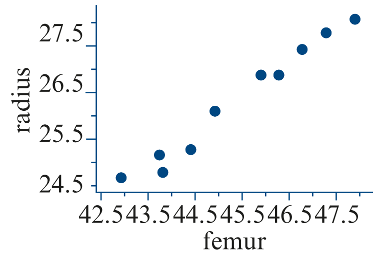


Use this information to complete the following report. Call the two variables *drug dose* and *response time*. In this analysis *drug dose* is the explanatory variable.

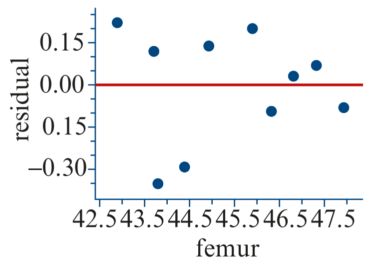
Report

From the scatterplot we see that there is a strong [ ] relationship between response time and [ ] : r = [ ]. There are no obvious outliers.
The equation of the least squares regression line is:
response time = [ ] + [ ] × drug dose
The slope of the regression line predicts that, on average, response time [increases/decreases] by [ ] minutes for a 1-milligram increase in drug dose.
The y-intercept of the regression line predicts that, on average, the response time when no drug is administered is [ ] minutes.
The coefficient of determination indicates that, on average, [ ] % of the variation in [ ] is explained by the variation in [ ].
The residual plot shows a [ ], calling into question the use of a linear equation to describe the relationship between response time and drug dose.

17 A regression analysis was conducted to investigate the nature of the association between femur (thigh bone) length and radius (the short thicker bone in the forearm) length in 18-year-old males. The bone lengths are measured in centimetres. The results of this analysis are reported below. In this investigation, femur length was treated as the independent variable.



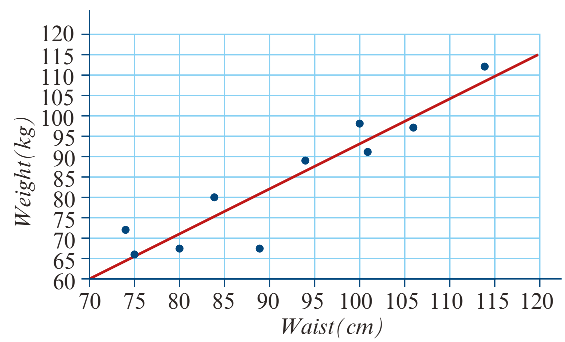
Regression equation $y = a + bx$
$a = -7.24946$
$b = 0.739556$
$r^2 = 0.975291$
$r = 0.987568$



Use the format of the report given in the previous question to summarise findings of this investigation. Call the two variables *femur length* and *radius length*.

Exam 1 style questions

18 The scatterplot shows the weight (in kg) and waist measurement (in cm) for a group of people. A least squares line had been fitted to the scatterplot with *waist* as the explanatory variable. The equation of the least squares line is closest to:



A $weight = 60.0 + 1.10 \times waist$
B $waist = 60.0 + 0.91 \times weight$
C $weight = 70.0 + 1.10 \times waist$
D $weight = -3.70 + 0.91 \times waist$
E $weight = -17.0 + 1.10 \times waist$

19 The table below shows the *life expectancy* in years and the percentage of government expenditure which is spent on health (*health*) in 10 countries.

Health	17.3	10.3	4.7	6.0	20.1	6.0	13.2	7.7	10.1	17.5
Life expectancy (years)	82	76	68	69	83	75	76	76	75	75

A least squares line which enables a country's *life expectancy* to be predicted from their expenditure on *health* is fitted to the data. The number of times that a country's predicted *life expectancy* is greater than their actual *life expectancy* is:

A 3 **B** 4 **C** 5 **D** 6 **E** 7

20 In a study of the association between the *length* in cm and *weight* in grams of a certain species of fish the following least squares line was obtained:

$$weight = -329 + 23.3 \times length$$

Which one of the following is a conclusion that can be made from this least squares line?

A On average, the *weight* of the fish increased by 23.3 grams for each centimetre increase in *length*.

B On average, the *length* of the fish increased by 23.3 cm for each one gram increase in *weight*.

C On average, the *weight* of the fish decreased by 329 grams grams for each centimetre increase in *length*.

D The equation cannot be correct as the *weight* of the fish can never be negative.

E The *weight* of the fish in grams can be determined by subtracting 305.7 from their *length*.

3C Conducting a regression analysis using data

In your statistical investigation project you will need to be able to conduct a full regression analysis from data. This section is designed to help you with this task.

CAS 2: How to conduct a regression analysis using the TI-Nspire CAS

This analysis is concerned with investigating the association between life expectancy (in years) and birth rate (in births per 1000 people) in 10 countries.

Birth rate	30	38	38	43	34	42	31	32	26	34
Life expectancy (years)	66	54	43	42	49	45	64	61	61	66

Steps

1 Write down the explanatory variable (EV) and response variable (RV). Use the variable names *birth* and *life*.

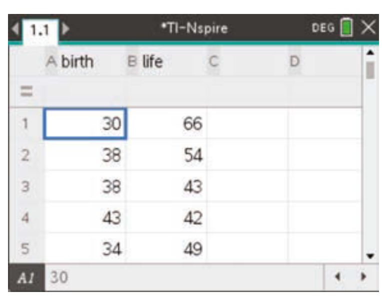
EV: *birth*

RV: *life*

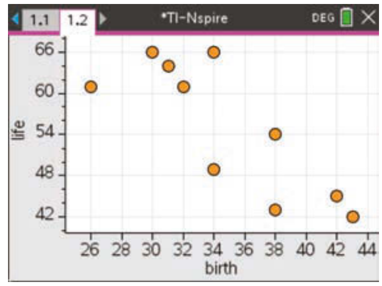
2 Start a new document by pressing ctrl + N.

Select **Add Lists & Spreadsheet**.

Enter the data into the lists named *birth* and *life*, as shown.



3 Construct a scatterplot to investigate the nature of the relationship between life expectancy and birth rate.

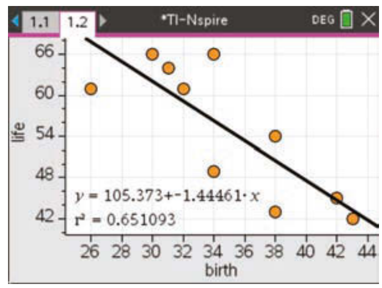


4 Describe the association shown by the scatterplot. Mention direction, form, strength and outliers.

There is a strong, negative, linear relationship between life expectancy and birth rate. There are no obvious outliers.

5 Find and plot the equation of the least squares regression line and r^2 value.

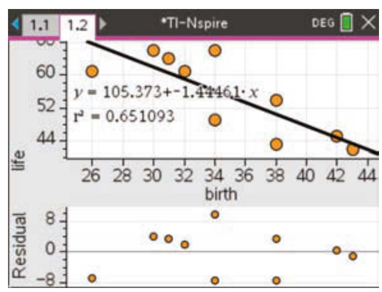
Note: Check if **Diagnostics** is activated using menu>**Settings**.



6 Generate a residual plot to test the linearity assumption.

Use ctrl + ◀ (or click on the page tab) to return to the scatterplot.

To hide the residual plot press menu>**Analyze>Residuals>Hide Residual Plot**.



7 Use the values of the intercept and slope to write the equation of the least squares regression line. Also write the values of r and the coefficient of determination.

Regression equation:

life = 105.4 – 1.445 × *birth*

Correlation coefficient: $r = -0.8069$

Coefficient of determination: $r^2 = 0.651$

CAS 2: How to conduct a regression analysis using the ClassPad

The data for this analysis are shown below.

Birth rate (per thousand)	30	38	38	43	34	42	31	32	26	34
Life expectancy (years)	66	54	43	42	49	45	64	61	61	66

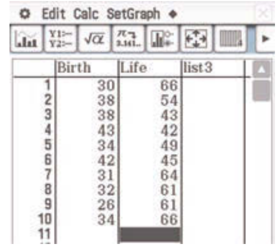
Steps

1 Write down the explanatory variable (EV) and response variable (RV). Use the variable names *birth* and *life*.

EV: *birth*

RV: *life*

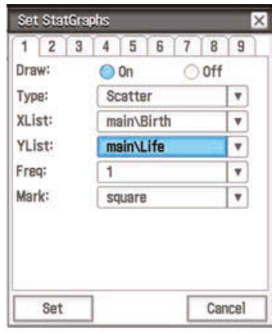
2 Enter the data into lists as shown.

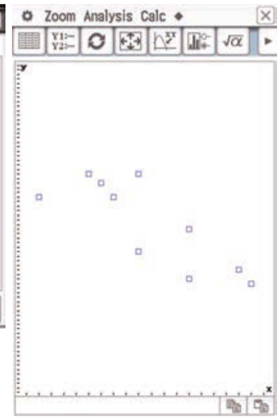


3 Construct a scatterplot to investigate the nature of the relationship between life expectancy and birth rate.

a Tap and complete the **Set Calculations** dialog box as shown.

b Tap to view the scatterplot.

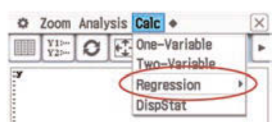


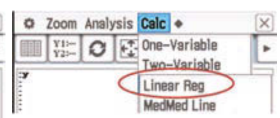


4 Describe the association shown by the scatterplot. Mention direction, form, strength and outliers.

There is a strong negative, linear association between life expectancy and birth rate. There are no obvious outliers.

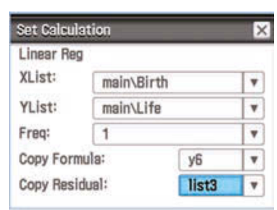
5 Find the equation of the least squares regression line and generate all regression statistics, including residuals.



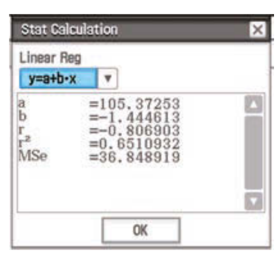


a Tap **Calc** in the toolbar.
Tap **Regression** and select **Linear Reg**.

b Complete the **Set Calculations** dialog box as shown.
Note: **Copy Residual** copies the residuals to **list3**, where they can be used later to create a residual plot.



c Tap **OK** in the Set **Calculation** box to generate the regression results.



d Write down the key results.

Regression equation:

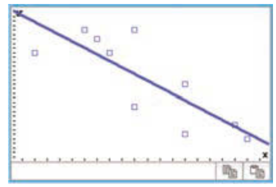
life = 105.4 – 1.445 × *birth*

Correlation coefficient:

r = −0.8069

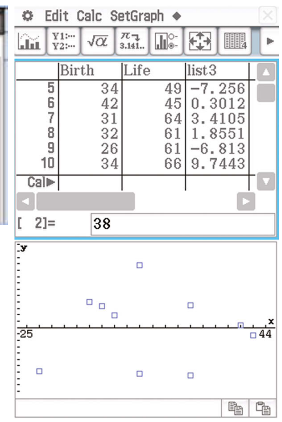
Coefficient of determination:
$r^2 = 0.651$

6 Tapping **OK** a second time automatically plots and displays the regression line on the scatterplot.



To obtain a full-screen plot, tap Resize from the icon panel.

7 Generate a residual plot to test the linearity assumption.
Tap and complete the **Set Calculations** dialog box as shown.
Tap to view the residual plot.



Inspect the plot and write your conclusion.

The random residual plot suggests linearity.

Note: When you performed a regression analysis earlier, the residuals were calculated automatically and stored in **list3**. The residual plot is a scatterplot with **list3** on the vertical axis and **birth** on the horizontal axis.

Exercise 3C

1 The table below shows the scores obtained by nine students on two tests. We want to be able to predict test B scores from test A scores.

Test A score (x)	18	15	9	12	11	19	11	14	16
Test B score (y)	15	17	11	10	13	17	11	15	19

Use your calculator to perform each of the following steps of a regression analysis.

a Construct a scatterplot. Name the variables *test a* and *test b*.

b Determine the equation of the least squares line along with the values of r and r^2.

c Display the regression line on the scatterplot.

d Obtain a residual plot.

2 The table below shows the number of careless errors made on a test by nine students. Also given are their test scores. We want to be able to predict test score from the number of careless errors made.

Test score	18	15	9	12	11	19	11	14	16
Careless errors	0	2	5	6	4	1	8	3	1

Use your calculator to perform each of the following steps of a regression analysis.

a Construct a scatterplot. Name the variables *score* and *errors*.

b Determine the equation of the least squares line along with the values of r and r^2. Write answers rounded to three significant figures.

c Display the regression line on the scatterplot.

d Obtain a residual plot.

3 How well can we predict an adult's weight from their birth weight? The weights of 12 adults were recorded, along with their birth weights. The results are shown.

Birth weight (kg)	1.9	2.4	2.6	2.7	2.9	3.2	3.4	3.4	3.6	3.7	3.8	4.1
Adult weight (kg)	47.6	53.1	52.2	56.2	57.6	59.9	55.3	58.5	56.7	59.9	63.5	61.2

a In this investigation, which would be the RV and which would be the EV?

b Construct a scatterplot.

c Use the scatterplot to:

i comment on the association between adult weight and birth weight in terms of direction, outliers, form and strength

ii estimate the value of Pearson's correlation coefficient, r.

d Determine the equation of the least squares regression line, the coefficient of determination and the value of Pearson's correlation coefficient, r. Write answers rounded to three significant figures.

e Interpret the coefficient of determination in terms of adult weight and birth weight.

f Interpret the slope in terms of adult weight and birth weight.

g Use the regression equation to predict the weight of an adult with a birth weight of:

i 3.0 kg ii 2.5 kg iii 3.9 kg.

Give answers correct to one decimal place.

h It is generally considered that birth weight is a 'good' predictor of adult weight. Do you think the data support this contention? Explain.

i Construct a residual plot and use it to comment on the appropriateness of assuming that adult weight and birth weight are linearly associated.

Review

Key ideas and chapter summary

Assign-ment

Bivariate data — **Bivariate data** are data in which each observation involves recording information about two variables for the same person or thing. An example would be the heights and weights of the children in a preschool.

Linear regression — The process of fitting a line to data is known as **linear regression**. The association can then be described by a rule of the form $y = a + bx$
In this equation:

- ***y*** is the **response variable**
- ***x*** is the **explanatory variable**
- ***a*** is the ***y*-intercept**
- ***b*** is the **slope of the line**.

Residuals — The vertical distance from a data point to the straight line is called a **residual**: residual value = data value − predicted value.

Least squares method — The **least squares method** is one way of finding the equation of a regression line. It minimises the sum of the squares of the residuals. It works best when there are no outliers.

Determining the values of *a* and *b* from the formulas — The equation of the least squares regression line is given by $y = a + bx$, where:

the **slope** (b) is given by $$b = \frac{rs_y}{s_x}$$

and

the **intercept** (a) is then given by $$a = \bar{y} - b\bar{x}$$

Here:

- r is the **correlation coefficient**
- s_x and s_y are the **standard deviations** of x and y
- $\bar{x}$ and $\bar{y}$ are the **mean** values of x and y.

Determining the value of *r* when *b* is known — The value of the correlation coefficient r is given by

$$r = \frac{bs_x}{s_y}$$

Here:

- b is the **slope** of the least squares line
- s_x and s_y are the **standard deviations** of x and y

Interpreting the intercept and slope

For the regression line $y = a + bx$:

- the slope (b) tells us on average the change in the response variable (y) for each one-unit increase or decrease in the explanatory variable (x).
- the intercept (a) tells us on average the value of the response variable (y) when the explanatory variable (x) equals 0.

Consider for example the regression line

$$cost = 1.20 + 0.06 \times number\ of\ pages$$

The slope of the regression line tells us that on average the cost of a textbook increases by 6 cents ($0.06) for each additional page.
The *intercept* of the line tells us that on average that a book with no pages costs $1.20 (this might be the cost of the cover).

Making predictions

The **regression line** $y = a + bx$ enables the value of y to be predicted for a given value of x by substitution into the equation. For example, using the previous equation

$$cost = 1.20 + 0.06 \times number\ of\ pages$$

predicts that the cost of a 100-page book is:

$$cost = 1.20 + 0.06 \times 100 = \$7.20$$

Interpolation and extrapolation

Predicting *within* the range of the values of the explanatory variable is called **interpolation**, and will give a **reliable** prediction.
Predicting *outside* the range of the values of the explanatory variable is called **extrapolation**, and will give an **unreliable** prediction.

Coefficient of determination

The **coefficient of determination** (r^2) gives a measure of the predictive power of a regression line. For example, for the regression line above, the coefficient of determination is 0.81.
From this we conclude that 81% of the variation in the cost of a textbook can be explained by the variation in the number of pages.

Residual plots

Residual plots can be used to test the linearity assumption by plotting the residuals against the EV.
A residual plot that appears to be a random collection of points clustered around zero supports the linearity assumption.
A residual plot that shows a clear pattern indicates that the association is not linear.

Review

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

3A **1** I can determine the equation of the least squares regression line using the formulas. ☐

See Example 1, and Exercise 3A Question 4

3A **2** I can determine the correlation coefficient from the slope of the least squares regression line using the formula. ☐

See Example 2, and Exercise 3A Question 7

3A **3** I can determine the equation of the least squares regression line using a CAS calculator. ☐

See CAS 1, and Exercise 3A Question 10

3B **4** I can interpret the slope and intercept of a regression line. ☐

See Example 3, and Exercise 3B Question 2

3B **5** I can use the regression line to make predictions. ☐

See Example 4, and Exercise 3B Question 4

3B **6** I can use the coefficient of determination to compare associations. ☐

See Example 5, and Exercise 3B Question 7

3B **7** I can calculate residual values. ☐

See Example 6, and Exercise 3B Question 8

3B **8** I can interpret a residual plot. ☐

See Example 7, and Exercise 3B Question 11

3B **9** I can write a report based on a regression analysis. ☐

See Example 8, and Exercise 3B Question 16

3C **10** I can use a CAS calculator to generate all of the analyses required for a regression analysis. ☐

See CAS 2, and Exercise 3C Question 1

Multiple-choice questions

1 When using a least squares line to model a relationship displayed in a scatterplot, one key assumption is that:

A there are two variables
B the variables are related
C the variables are linearly related
D $r^2 > 0.5$
E the correlation coefficient is positive

2 For the least squares regression line $y = -1.2 + 0.52x$:

A the y-intercept $= -0.52$ and slope $= -1.2$
B the y-intercept $= 0$ and slope $= -1.2$
C the y-intercept $= 0.52$ and slope $= -1.2$
D the y-intercept $= -1.2$ and slope $= 0.52$
E the y-intercept $= 1.2$ and slope $= -0.52$

3 If the equation of a least squares regression line is $y = 8 - 9x$ and $r^2 = 0.25$:

A $r = -0.5$ **B** $r = -0.25$ **C** $r = -0.0625$ **D** $r = 0.25$ **E** $r = 0.50$

4 Given that $b = 1.328$, $s_x = 1.871$ and $s_y = 3.391$, the correlation coefficient, r, is closest to:

A 0.357 **B** 0.598 **C** 0.733 **D** 0.773 **E** 1.33

5 The association between the number of *errors* made in a task, and the *time* spent practicing the task (in minutes) was found to be approximately linear, and the values of the following statistics were determined:

	time	*errors*
mean	8.00	34.5
standard deviation	2.40	12.5
correlation coefficient	$r = -0.236$	

The equation of the least squares line that enables *errors* to be predicted from *time* is given by

A *errors* = 52.2 − 1.23 × *time*
B *errors* = 10.1 − 0.99 × *time*
C *errors* = 24.7 − 1.23 × *time*
D *errors* = 32.6 + 0.24 × *time*
E *errors* = 44.3 − 1.23 × *time*

6 The *speed* at which a car is travelling (in km/hr), and the *distance* (in metres) taken by the car to come to a stop when the brakes are applied, were recorded over speeds from 60km/hr to 120km/hr.

	speed	*distance*
mean	90.5	52.7
standard deviation	1.124	1.349
correlation coefficient	$r = 0.948$	

The association was found to be approximately linear, and the values of the statistics shown were determined.

Review

On average, for each additional km/hr of *speed*, the *distance* taken to come to a stop

A decreased by 1.14 metres
B decreased by 0.79 metres
C increased by 1.14 metres
D increased by 0.95 metres
E increased by 0.79 metres

7 The least squares regression line $y = 8 - 9x$ predicts that, when $x = 5$, the value of y is:

A −45 **B** −37 **C** 37 **D** 45 **E** 53

8 A least squares regression line of the form $y = a + bx$ is fitted to the data set shown.

x	25	15	10	5
y	10	10	15	25

The equation of the line is:

A $y = -0.69 + 24.4x$ **B** $y = 24.4 - 0.69x$ **C** $y = 24.4 + 0.69x$
D $y = 28.7 - x$ **E** $y = 28.7 + x$

9 A least squares regression line of the form $y = a + bx$ is fitted to the data set shown.

y	30	25	15	10
x	40	20	30	10

The equation of the line is:

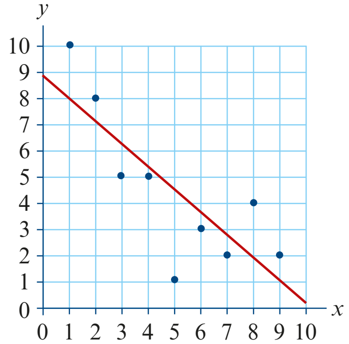
A $y = 1 + 0.5x$ **B** $y = 0.5 + x$ **C** $y = 0.5 + 7.5x$
D $y = 7.5 + 0.5x$ **E** $y = 30 - 0.5x$

10 Using a least squares regression line, the predicted value of a data value is 78.6. The residual value is –5.4. The actual data value is:

A 73.2 **B** 84.0 **C** 88.6 **D** 94.6 **E** 424.4

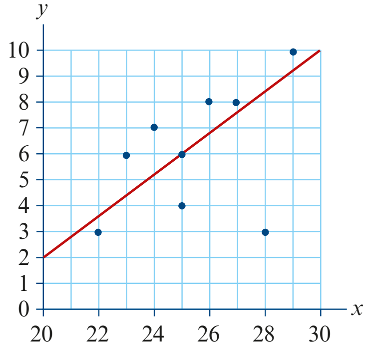
11 The equation of the least squares line plotted on the scatterplot opposite is closest to:

A $y = 8.7 - 0.9x$
B $y = 8.7 + 0.9x$
C $y = 0.9 - 8.7x$
D $y = 0.9 + 8.7x$
E $y = 8.7 - 0.1x$



12 The equation of the regression line plotted on the scatterplot opposite is closest to:

A $y = -14 + 0.8x$
B $y = 0.8 + 14x$
C $y = 2.5 + 0.8x$
D $y = 14 - 0.8x$
E $y = 17 + 1.2x$



The following information relates to Questions 13 to 16.

Weight (in kg) can be predicted from height (in cm) from the regression line:

$$weight = -96 + 0.95 \times height, \quad \text{with } r = 0.79$$

13 Which of the following statements relating to the regression line is *false*?

A The slope of the regression line is 0.95.

B The explanatory variable in the regression equation is *height*.

C The least squares line does *not* pass through the origin.

D The intercept is 96.

E The equation predicts that a person who is 180 cm tall will weigh 75 kg.

14 This regression line predicts that, on average, weight:

A decreases by 96 kg for each 1 centimetre increase in height

B increases by 96 kg for each 1 centimetre increase in height

C decreases by 0.79 kg for each 1 centimetre increase in height

D decreases by 0.95 kg for each 1 centimetre increase in height

E increases by 0.95 kg for each 1 centimetre increase in height

15 Noting that the value of the correlation coefficient is $r = 0.79$, we can say that:

A 62% of the variation in weight can be explained by the variation in height

B 79% of the variation in weight can be explained by the variation in height

C 88% of the variation in weight can be explained by the variation in height

D 79% of the variation in height can be explained by the variation in weight

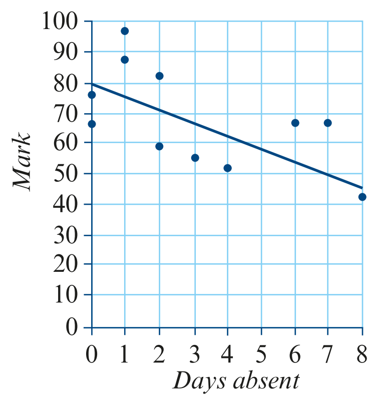
E 95% of the variation in height can be explained by the variation in weight

16 A person of height 179 cm weighs 82 kg. If the regression equation is used to predict their weight, then the residual will be closest to:

A –8 kg **B** 3 kg **C** 8 kg **D** 9 kg **E** 74 kg

The following information relates to Questions 17 to 21.

The scatterplot shows the association between a student's *mark* on a test, and the number of *days absent* during the term.



Review

17 The median *mark* for this group of students is closest to:

A 55 **B** 60 **C** 67 **D** 70 **E** 72

18 The median *days absent* for this group of students is closest to:

A 2 **B** 3 **C** 4 **D** 55 **E** 62.5

19 The coefficient of determination between *mark* and *days absent* is $r^2 = 0.5$.
The correlation coefficient is closest to:

A −0.7 **B** −0.25 **C** 0.25 **D** 0.5 **E** 0.7

20 There were two students who were absent for 2 days that term. The values of the residuals for these students are

A 0 **B** 10 **C** 60 and 80 **D** −10 and 10 **E** −10

21 Using the graph of the least squares line, we predict that a student who is absent for 4 days would receive a mark of about:

A 48 **B** 51 **C** 62 **D** 65 **E** 67

22 The table below shows the *weight* in grams and the *length* in cm for a certain species of fish.

Length(cm)	13.5	14.3	16.3	17.5	18.4	19.0	19.0	19.8	21.2	23.0
Weight(gm)	55	60	90	120	150	140	170	145	200	273

A least squares line which enables a a fish's *weight* to be predicted from their *length* is fitted to the data. The number of times that the fish's predicted *weight* is greater than their actual *weight* is:

A 3 **B** 4 **C** 5 **D** 6 **E** 7

23 The value of the correlation coefficient r for these data is equal to 0.965. The percentage of variation in fish *weight* which is not explained by the *length* of the fish is closest to:

A 96.5% **B** 93.1% **C** 9.3% **D** 6.9% **E** 3.5%

Written response questions

1 The table below shows the *age* (in years), the *number of seats*, and the *airspeed* (in km/h), of eight aircraft.

Age	3.5	3.7	4.7	4.9	5.1	7.3	8.7	8.8
number of seats	405	296	288	258	240	193	188	148
airspeed	830	797	774	736	757	765	760	718

a Determine to the nearest whole number:

  i the median *age* of these aircraft.

  ii the mean and standard deviation of the *airspeed* of these aircraft.

To investigate the association between the *number of seats* and *airspeed*, a least squares line is fitted to the data. The response variable in this investigation is *airspeed*.

b Determine and write down the equation of the least squares line in terms of *number of seats* and *airspeed*. Round the intercept and slope to 3 significant figures.

c Determine and write down the percentage of variation in the *airspeed* that is explained by the *number of seats*. Write the answer rounded to 1 decimal place.

2 In an investigation of the relationship between the hours of sunshine (per year) and days of rain (per year) for 25 cities, the least squares regression line was found to be:

$$hours\ of\ sunshine = 2850 - 6.88 \times days\ of\ rain, \text{ with } r^2 = 0.484$$

Use this information to complete the following sentences.

a In this regression equation, the explanatory variable is $\boxed{}$.

b The slope is $\boxed{}$ and the intercept is $\boxed{}$.

c The regression equation predicts that a city that has 120 days of rain per year will have $\boxed{}$ hours of sunshine per year.

d The slope of the regression line predicts that the hours of sunshine per year will $\boxed{}$ by $\boxed{}$ hours for each additional day of rain.

e $r =$ $\boxed{}$, correct to three significant figures.

f $\boxed{}$% of the variation in sunshine hours can be explained by the variation in $\boxed{}$.

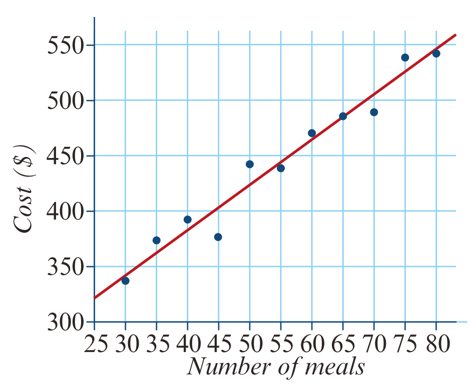
g One of the cities used to determine the regression equation had 142 days of rain and 1390 hours of sunshine.

  i The regression equation predicts that it has $\boxed{}$ hours of sunshine.

  ii The residual value for this city is $\boxed{}$ hours.

h Using a regression line to make predictions within the range of data used to determine the regression equation is called $\boxed{}$.

3 The cost of preparing meals, in dollars, and the number of meals prepared are plotted in the scatterplot shown. A least squares line has been fitted to the data which enables the cost of the meals prepared to be predicted from the number of meals prepared.



a Which is the response variable?

b Describe the association in terms of strength, direction and form.

The equation of the least squares line that relates the cost of preparing meals to the number of meals produced is:

$$cost = 222.48 + 4.039 \times number\ of\ meals$$

c **i** Use the equation to predict the cost of preparing 21 meals. Round the answer to the nearest cent.

ii In making this prediction, are you interpolating or extrapolating?

d Write down:

i the intercept of the regression line and interpret in terms of *cost* and the *number of meals* prepared.

ii the slope of the regression line and interpret in terms of *cost* and the *number of meals* prepared.

e When the number of meals prepared was 50, the cost of preparation was \$444. Show that, when the least squares line is used to predict the cost of preparing 50 meals, the residual is \$19.57, to the nearest cent.

4 We wish to find the equation of the least squares regression line that will enable height (in cm) to be predicted from femur (thigh bone) length (in cm).

a Which is the RV and which is the EV?

b Use the summary statistics shown to determine the equation of the least squares regression line that will enable *height* to be predicted from *femur length*.

	femur length	*height*
mean	24.246	166.092
standard deviation	1.873	10.086
correlation coefficient	$r = 0.9939$	

Write the equation in terms of *height* and *femur length*. Give the slope and intercept rounded to three significant figures.

c Interpret the slope of the regression equation in terms of *height* and *femur length*.

d Determine the value of the coefficient of determination and interpret in terms of *height* and *femur length*.

Arm span is also associated with height. A least squares regression line that can be used to model this association is:

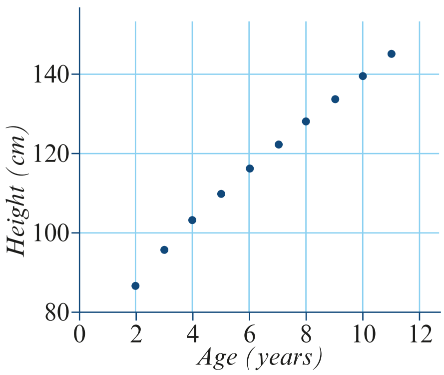
$$height = 0.498 + 0.926 \times arm\ span$$

In determining this equation, the summary statistics displayed in the table were also calculated.

	arm span	*height*
mean	169.615	166.092
standard deviation	10.761	10.086

e Determine the percentage of the variation in *height* explained by the variation in *arm span*. Write the answer as a percentage rounded to one decimal place.

5 The scatter plot shows the height (in cm) of a group of 10 children plotted against their age (in years). The data used to generate this scatterplot is shown below.



Height (cm)	86.5	95.5	103.0	109.8	116.4	122.4	128.2	133.8	139.6	145.0
Age (years)	2	3	4	5	6	7	8	9	10	11

The task is to determine the equation of a least squares regression line that can be used to predict height from age.

a In this analysis, which would be the RV and which would be the EV?

b Use the scatter plot to describe the association between *age* and *height* in terms of strength and direction.

c Use your calculator to confirm that the equation of the least squares regression line is: *height* = 76.64 + 6.366 × *age* and $r = 0.9973$.

d **i** Use the regression line to show that the predicted height of a one-year old is 83.0 cm, rounded to 3 significant figures.

ii In making this prediction are you extrapolating or interpolating?

e Interpret the slope of the least squares line in terms of *height* and *age*.

f Determine the percentage of the variation in *height* of these children explained by their *age*. Round your answer to 1 decimal place.

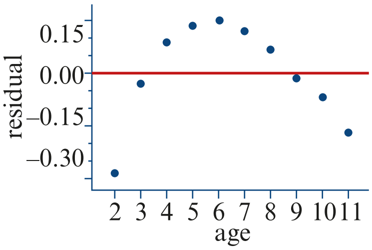
g Use the least squares regression equation to:

i predict the *height* of the 10-year-old child in this sample

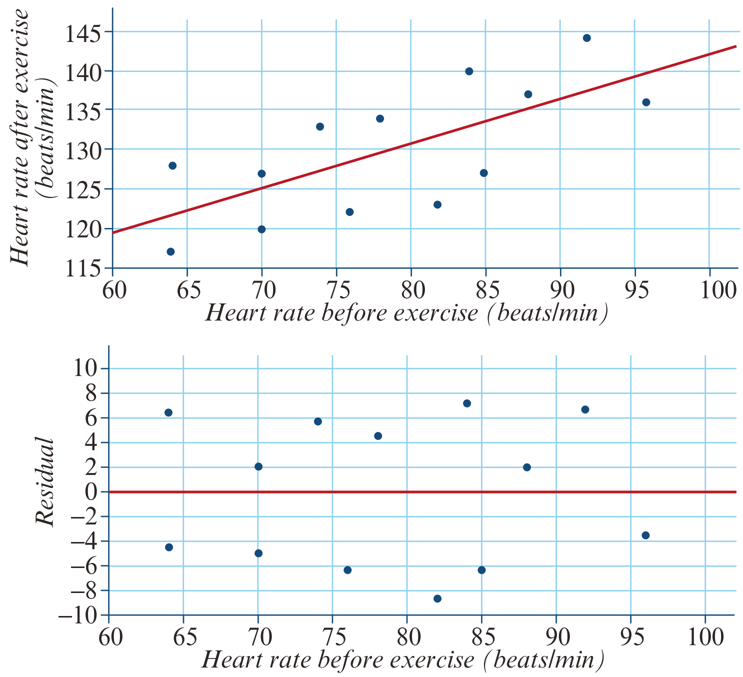
ii determine the residual value for this child.

h **i** Confirm that the residual plot for this analysis is shown opposite.

ii Explain why this residual plot suggests that a linear equation is not the most appropriate model for this association.



6 The heart rate (in beats/minute) was measured and recorded for a group of 13 students. The students then completed the same set of exercises and their heart rate measured again immediately on completion. The scatterplot below shows the students' *heart rate after exercise* plotted against their *heart rate before exercise*, with a least squares regression line fitted. Also shown is the residual plot for this line.



a Describe the association between *heart rate before exercise* and *heart rate after exercise* in terms of strength, direction and form.

The equation of the least squares line is:

heart rate after exercise = 85.671 + 0.561 × *heart rate before exercise*

b **i** Use the equation to predict heart rate after exercise when heart rate before exercise is 100 beats/minute. Round to the nearest whole number.

ii Are you extrapolating or interpolating?

c The person with a heart rate of 122 beats/minute after exercise had a heartbeat of 76 beats/minute before exercise. If the least squares line is used to predict this person's heart rate after exercise, determine the value of the residual. Give your answer rounded to one decimal place.

d **i** What assumption about the form of the association can be tested using a residual plot?

ii Referring to the residual plot, explain why this assumption is satisfied.

Chapter 4

Data transformation

Chapter questions

- What is a squared transformation and when is it used?
- What is a log transformation and when is it used?
- What is a reciprocal transformation and when is it used?
- How do I interpret a least squares line fitted to transformed data?
- How do I use a least squares line fitted to transformed data for prediction?
- How do I use a residual plot to assess the effectiveness of a data transformation?
- How do I use the coefficient of determination to assess the effectiveness of a data transformation?

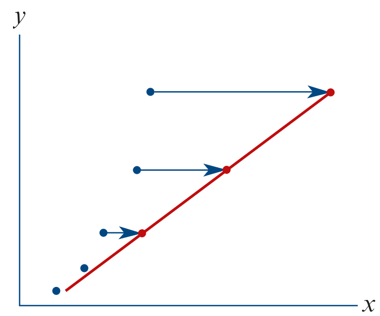
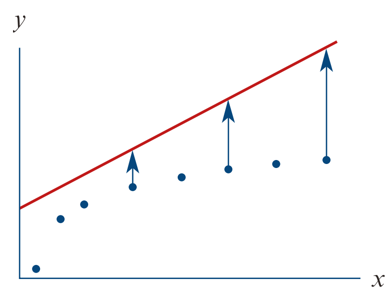
You may recall from your study of Variation in General Mathematics 12 that a non-linear association could be transformed into a linear association using **data transformation**. The transformations introduced were the squared, log and reciprocal transformations. In this chapter we consider the effect of each of these three transformations when applied to one axis only (either x or y, but not both), using them to linearise scatterplots. This is the first step towards solving problems involving non-linear associations.

4A The squared transformation

Learning intentions

- To be able to apply a squared transformation to either x or y.
- To be able to fit a least squares regression line to the transformed data.
- To be able to use the least squares regression line fitted to the transformed data for prediction.

The **squared transformation** is a *stretching* transformation. It works by *stretching* out the upper end of the scale on either the x- or y-axis. The effects of applying the x^2 and y^2 transformations (separately) to a scatterplot are illustrated graphically below.

Transformation	*Outcome*	*Graph*
x^2	Spreads out the high x-values relative to the lower x-values, leaving the y-values unchanged. This has the effect of straightening out curves like the one shown opposite.	
y^2	Spreads out the high y-values relative to the lower y-values, leaving the x-values unchanged. This has the effect of straightening out curves like the one shown opposite.	

The following example shows how the x-squared transformation works in practice.

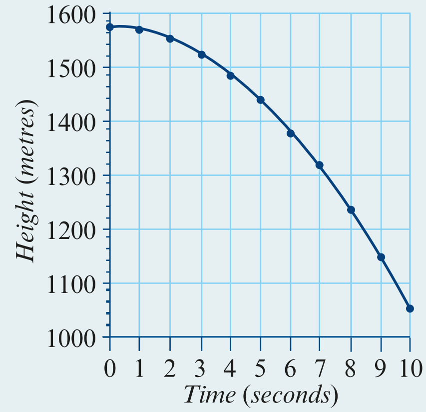
Example 1 Applying the x-squared transformation

A base jumper leaps from the top of a cliff, 1560 metres above the valley floor. The scatterplot below shows the height (in metres) of the base jumper above the valley floor every second, for the first 10 seconds of the jump.

A scatterplot shows that there is a strong negative association between the *height* of the base jumper above the ground and *time*.

a Apply a squared transformation to the variable *time*, and determine the least squares regression line for the transformed data.

b Use the least squares equation to predict to the nearest metre the height of the base jumper after 3.4 seconds.



Solution

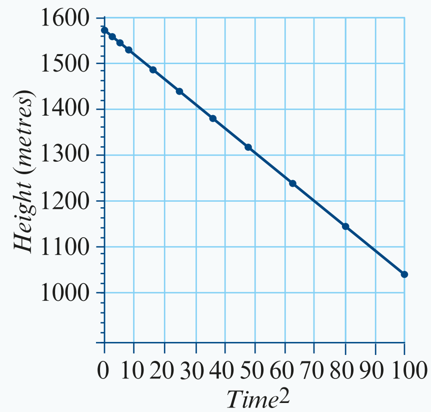
a Applying the squared transformation involves changing the scale on the *time* axis to $time^2$.

From the plot opposite we see that the association between *height* and $time^2$ is now linear.

Now that we have a linearised scatterplot, we can use a least squares line to model the association between *height* and $time^2$.

The equation of this line is:

$$height = 1560 - 4.90 \times time^2$$



b Like any regression line, we can use its equation to make predictions. After 3.4 seconds, we predict that the height of the base jumper is:

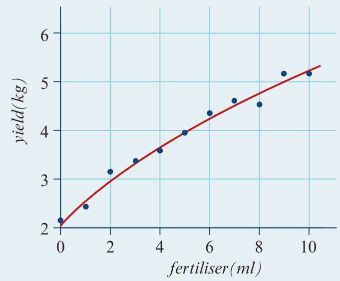
$$height = 1560 - 4.90 \times 3.4^2 = 1503 \text{ m (to nearest m)}$$

The next example shows how the *y*-squared transformation works in practice.

Example 2 **Applying the *y*-squared transformation**

In a study of the effectiveness of fertiliser on the yield of strawberry plants, differing amounts of liquid fertiliser (in mL) were given to groups of plants, and their average yield (in kg) measured.

A scatterplot shows that there is a strong positive association between the *fertiliser* and *yield*.



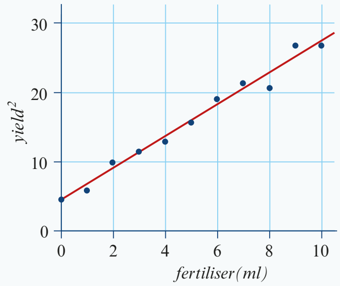
a Apply a squared transformation to the variable *yield*, and determine the least squares regression line for the transformed data.

b Use the least squares equation to predict the *yield* of a plant given 6.5 mL of fertiliser, giving your answer to 1 decimal place.

Solution

a Applying the y-squared transformation involves changing the scale on the y-axis to $yield^2$.

From the plot opposite we see that the association between $yield^2$ and *fertiliser* is now linear.



Now that we have a linearised scatterplot, we can use a least squares line to model the association between $yield^2$ and *fertiliser*.

The equation of this line is:

$$yield^2 = 4.45 + 2.29 \times fertiliser$$

b Using this equation, when we predict that:

$$yield^2 = 4.45 + 2.29 \times 6.5 = 19.34$$

and $\quad yield = \sqrt{19.34} = \pm 4.4$

Looking at the scatterplot we can see that only the positive value of the square root makes sense, so our prediction is 4.4 kg.

Performing a data transformation is quite computationally intensive, but your CAS calculator is well suited to the task.

CAS 1: Using the TI-Nspire CAS to perform a squared transformation

The table shows the height (in m) of a base jumper for the first 10 seconds of her jump.

Time	0	1	2	3	4	5	6	7	8	9	10
Height	1560	1555	1540	1516	1482	1438	1383	1320	1246	1163	1070

a Construct a scatterplot displaying *height* (the RV) against *time* (the EV).

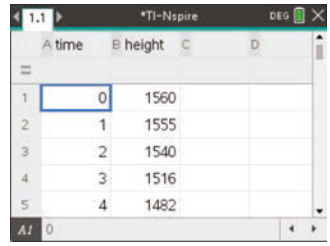
b Apply an x-squared transformation and fit a least squares line to the transformed data.

c Use the regression line to predict the height of the base jumper after 3.4 seconds.

Steps

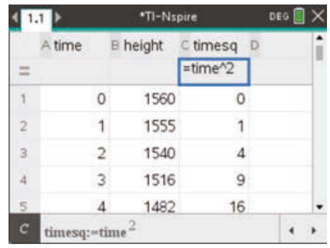
1 Start a new document by pressing ctrl + N .

2 Select **Add Lists & Spreadsheet**.
Enter the data into lists named *time* and *height*, as shown.

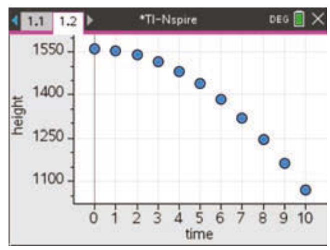


3 Name column C as *timesq* (short for 'time squared').

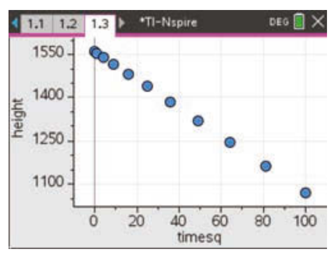
4 Move the cursor to the formula cell below *timesq*.
Enter the expression = *time*^2 by pressing (=) , then typing **time∧2**. Pressing enter calculates and displays the values of *timesq*.



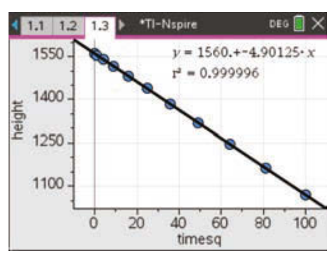
5 Press ctrl + I and select **Add Data & Statistics**.
Construct a scatterplot of *height* against *time*. Let *time* be the explanatory variable and *height* the response variable. The plot is clearly non-linear.



6 Press ctrl + I and select **Add Data & Statistics**.
Construct a scatterplot of *height* against $time^2$.
The plot is now linear.



7 Press menu**>Analyze>Regression>Show Linear (a + bx)** to plot the line on the scatterplot with its equation.
Note: The x in the equation on the screen corresponds to the transformed variable $time^2$.



8 Write down the regression equation in terms of the variables *height* and $time^2$.

$height = 1560 - 4.90 \times time^2$

9 Substitute 3.4 for *time* in the equation to find the height after 3.4 seconds.

$height = 1560 - 4.90 \times 3.4^2 = 1503$ m

CAS 1: Using the CASIO Classpad to perform a squared transformation

The table shows the height (in m) of a base jumper for the first 10 seconds of her jump.

Time	0	1	2	3	4	5	6	7	8	9	10
Height	1560	1555	1540	1516	1482	1438	1383	1320	1246	1163	1070

a Construct a scatterplot displaying *height* (the RV) against *time* (the EV).

b Apply an x-squared transformation and fit a least squares line to the transformed data.

c Use the regression line to predict the height of the base jumper after 3.4 seconds.

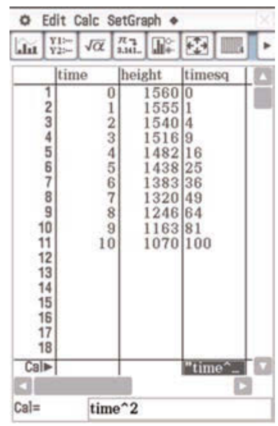
Steps

1 In the Statistics application enter the data into lists named *time* and *height*.

2 Name the third list *timesq* (short for *time* squared).

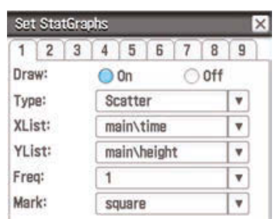
3 Place the cursor in the calculation cell at the bottom of the third column and type *time*^2. This will calculate the values of $time^2$.

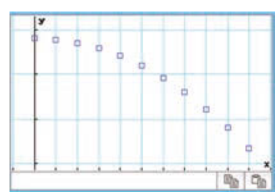
Let *time* be the explanatory variable (x) and *height* the response variable (y).



4 Construct a scatterplot of *height* against *time*.

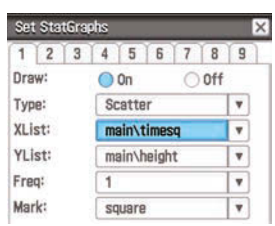
- Tap and complete the **Set StatGraphs** dialog box as shown.
- Tap to view the scatterplot. The plot is clearly non-linear.

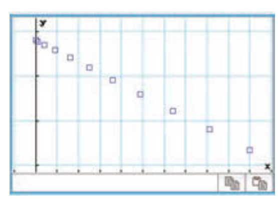




5 Construct a scatterplot of *height* against $time^2$.

- Tap and complete the **Set StatGraphs** dialog box as shown.
- Tap to view the scatterplot. The plot is now clearly linear.



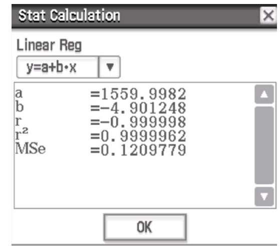


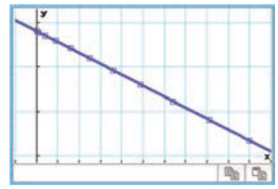
6 Fit a regression line to the transformed data.

- Go to **Calc, Regression, Linear Reg**.
- Complete the **Set Calculation** dialog box as shown and tap **OK**.

Note: The 'x' in the linear equation corresponds to the transformed variable *time*^{2}.

- Tap **OK** a second time to plot and display the regression line on the scatterplot.





7 Write down the equation in terms of *height* and *time*^{2}. — *height* = 1560 – 4.90 × *time*^{2}.

8 Substitute 3.4 for *time* in the equation. — *height* = 1560 – 4.90 × 3.4^2 = 1503 m

Skill-sheet

Exercise 4A

The x-squared transformation: some prerequisite skills

1 Evaluate y in the following expression, rounded to one decimal place.

a $y = 7 + 8x^2$ when $x = 1.25$ **b** $y = 7 + 3x^2$ when $x = 1.25$

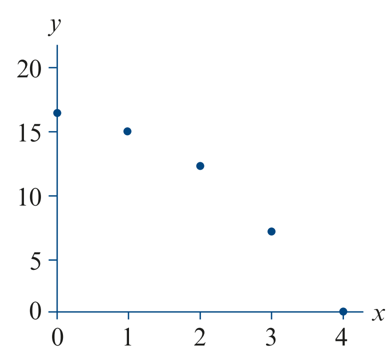
c $y = 24.56 - 0.47x^2$ when $x = 1.23$ **d** $y = -4.75 + 5.95x^2$ when $x = 4.7$

The x-squared transformation: calculator exercises

Example 1

2 The scatterplot opposite was constructed from the data in the table below.

x	0	1	2	3	4
y	16	15	12	7	0



a Linearise the scatterplot by applying an x-squared transformation and fit a least squares line to the transformed data.

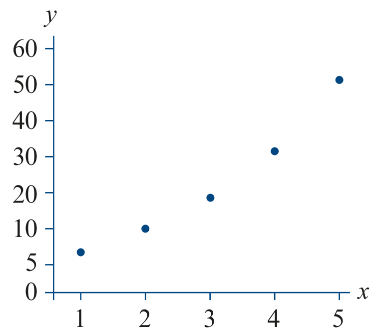
b Give its equation.

c Use the equation to predict the value of y when $x = -2$.

3 The scatterplot opposite was constructed from the data in the table below.

x	1	2	3	4	5
y	3	9	19	33	51

From the scatterplot, the association between y and x is non-linear.



a Linearise the scatterplot by applying an x-squared transformation and fit a least squares line to the transformed data.

b Give its equation.

c Use the equation to predict the value of y when $x = 6$.

The y-squared transformation: some prerequisite skills

4 Evaluate y in the following expression. Give the answers rounded to one decimal place.

a $y^2 = 16 + 4x$ when $x = 1.57$

b $y^2 = 1.7 - 3.4x$ when $x = 0.03$

c $y^2 = 16 + 2x$ when $x = 10$ $(y > 0)$

d $y^2 = 58 + 2x$ when $x = 3$ $(y < 0)$

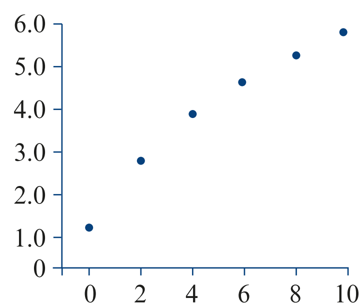
The y-squared transformation: calculator exercises

Example 2

5 The scatterplot opposite was constructed from the data in the table below.

x	0	2	4	6	8	10
y	1.2	2.8	3.7	4.5	5.1	5.7

From the scatterplot, the association between y and x is non-linear.



a Linearise the scatterplot by applying a y-squared transformation and fit a least squares line to the transformed data.

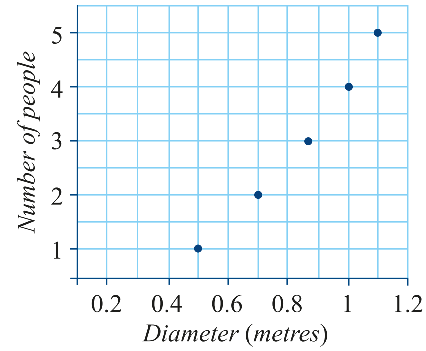
b Give its equation. Write the coefficient, rounded to two significant figures.

c Use the equation to predict the value of y when $x = 9$. Give the answer rounded to one decimal place.

Applications of the squared transformation

6 The table gives the *diameter* (in m) of five different umbrellas and the *number of people* each umbrella is designed to keep dry. A scatter plot is also shown.

Diameter	*Number*
0.50	1
0.70	2
0.85	3
1.00	4
1.10	5



a Apply the squared transformation to the variable *diameter* and determine the least squares regression line for the transformed data. *Diameter* is the EV.
Write the slope and intercept of this line, rounded to one decimal place, in the spaces provided.

$$number = \boxed{} + \boxed{} \times diameter^2$$

b Use the equation to predict the number of people who can be sheltered by an umbrella of diameter 1.3 m. Give your answer rounded to the nearest person.

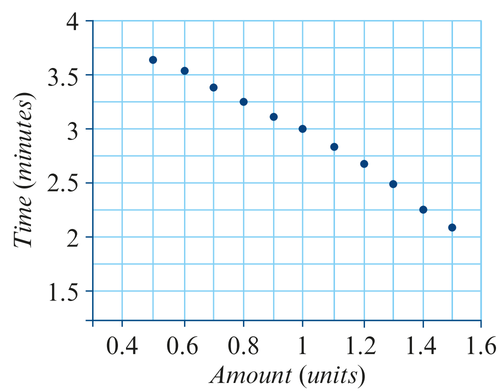
7 The time (in minutes) taken for a local anaesthetic to take effect is associated with to the amount administered (in units). To investigate this association a researcher collected the following data.

Amount	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5
Time	3.7	3.6	3.4	3.3	3.2	3.0	2.9	2.7	2.5	2.3	2.1

The association between the variables *amount* and *time* is non-linear as can be seen from the scatterplot below. A squared transformation applied to the variable *time* will linearise the scatterplot.

a Apply the squared transformation to the variable *time* and fit a least squares regression line to the transformed data. *Amount* is the EV. Write the equation of this line with the slope and intercept rounded to two significant figures.

b Use the equation to predict the time for the anaesthetic to take effect when the dose is 0.4 units. Give the answer rounded to one decimal place.

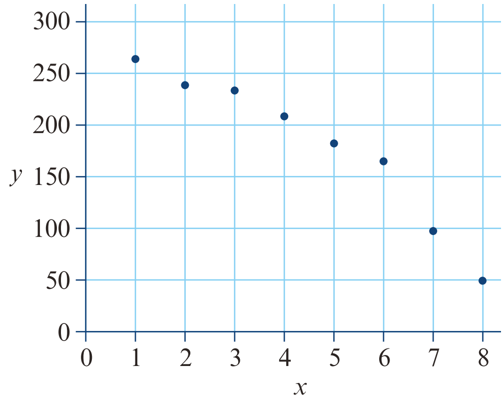


Exam 1 style questions

The following information relates to Questions 8 and 9

A student uses the data in the table below to construct the scatterplot shown:

x	y
1	264
2	239
3	234
4	208
5	182
6	164
7	98
8	49



8 A squared transformation is applied to x to linearise the association. A least squares line is fitted to the transformed data, with x^2 as the explanatory variable.
The equation of this least squares line is closest to

A $y = 310 - 29.1x^2$
B $y = 10.2 - 0.032x^2$
C $y = 1.80 - 0.106x^2$
D $y = 263 - 3.26x^2$
E $y = 79.9 - 0.303x^2$

9 A y^2 transformation could also be used to linearise this association. A least squares line is fitted to the transformed data, with y^2 as the response variable, and the equation of the least squares line is

$$y^2 = 79973 - 9533.4x$$

Using this equation, the value of y when $x = 4$ is closest to:

A 205 **B** 208 **C** 247 **D** 531 **E** 41839

10 The association between the *cost* of a certain precious stone (in \$) and its *weight* (in mg) is non-linear. A squared transformation was applied to the explanatory variable *weight*, and a least squares line fitted to the transformed data. The equation of the least squares line is:

$$cost = 2370 + 0.238 \times weight^2$$

Using this equation, the cost of a precious stone weighing 75mg is closest to:

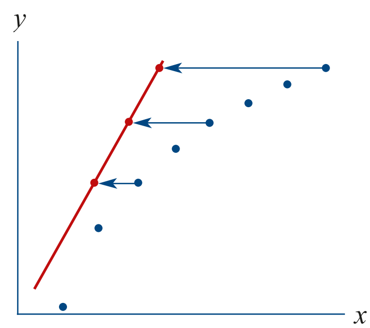
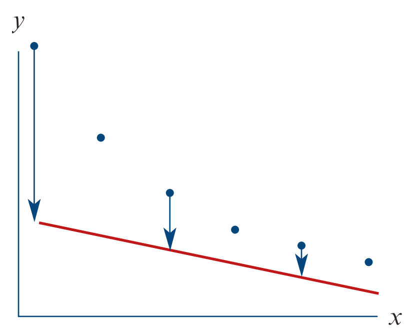
A \$2389 **B** \$3709 **C** \$2689 **D** \$7995 **E** \$177,768

4B The log transformation

Learning intentions

- ▶ To be able to apply a $\log_{10}$ transformation to either x or y.
- ▶ To be able to fit a least squares regression line to the transformed data.
- ▶ To be able to use the least squares regression line fitted to the transformed data for prediction.

You will recall from Chapter 1 that the shape of a highly skewed single variable distribution could be changed to become more symmetric by changing the scale from x to $\log_{10}x$. When applied to bivariate data, the effect of the **logarithmic transformation** is to again to *compress* the upper end of the scale on either the x- or the y-axis, potentially linearising a non-linear association. The effect of applying the $\log_{10}x$ and $\log_{10}y$ transformations (separately) to a scatterplot are illustrated graphically below.

Transformation	*Outcome*	*Graph*
$\log_{10}x$	Compresses the higher x-values relative to the lower x-values, leaving the y-values unchanged. This has the effect of straightening out curves like the one shown.	
$\log_{10}y$	Compresses larger y values relative to the smaller y values This has the effect of straightening out curves like the one shown.	

Following the normal convention, $\log x$ means $\log_{10}x$.

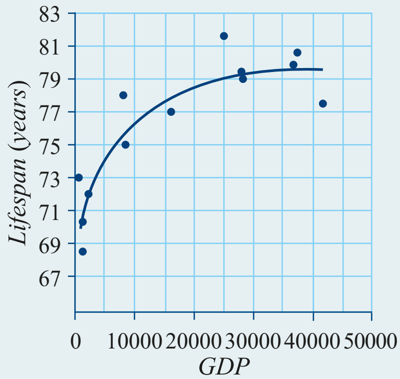
Example 3 Applying the log x transformation

The general wealth of a country, often measured by its Gross Domestic Product (*GDP*), is one of several variables associated with *lifespan* in different countries. However, the association is not linear, as can be seen in the scatterplot below which plots *lifespan* (in years) against *GDP* per person (in dollars) for 13 different countries.

The scatterplot shows that there is a strong positive association between the *lifespan* and *GDP*.

a Apply a log transformation to the variable *GDP*, and determine the least squares regression line for the transformed data.

b Use the least squares equation to predict the *lifespan* of a country with a *GDP* of \$20 000 per person, giving your answer rounded to one decimal place.



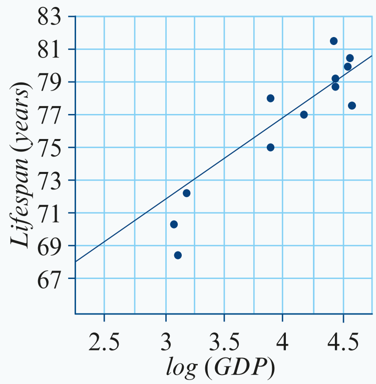
Solution

a Applying the log x transformation involves changing the scale on the x-axis to log(*GDP*).

When we make this change, we see that the association between the variables *lifespan* and log (*GDP*) is linear. See the plot opposite.

Note: On the plot, when log (*GDP*) = 4, the actual GDP is 10^4 or \$10 000.

We can now fit a least squares line to model the association between the variables *lifespan* and log(*GDP*).



The equation of this line is:

$$lifespan = 54.3 + 5.59 \times \log(GDP)$$

b Using this equation, for a country with a GDP of \$20 000, the lifespan is predicted as:

$$lifespan = 54.3 + 5.59 \times \log 20\,000 = 78.3 \text{ years (to one decimal place)}$$

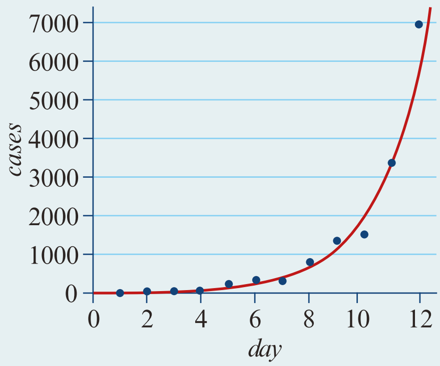
Example 4 Applying the log *y* transformation

The numbers of cases of a very infectious disease were recorded over a 12 day period. The association is not linear, as can be seen in the scatterplot below which plots *cases* against *days*.

The scatterplot shows that there is a strong positive association between the number of *case* and *day*.

a Apply a log transformation to the variable *cases*, and determine the least squares regression line for the transformed data.

b Use the least squares equation to predict the *cases* on day 13.



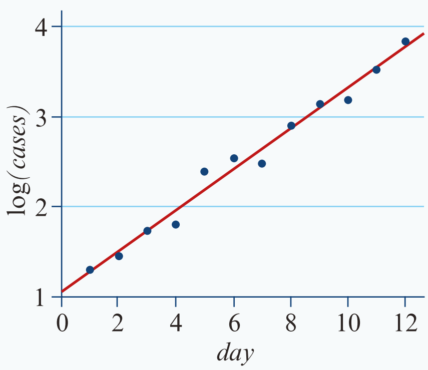
Solution

a Applying the log y transformation involves changing the scale on the y-axis to log(*cases*).

When we make this change, we see from the plot the association between the variables log(*cases*) and *day* is linear.

Note: On the plot, when log (*cases*) =3, the actual number of cases is 10^3 or 1000.

We can now fit a least squares line to model the association between the variables log(*cases*) and *day*.



The equation of this line is:

$\log(cases) = 1.046 + 0.227 \times day$

b Using this equation, on day 13 the number of cases is predicted as:

$\log(cases) = 1.046 + 0.227 \times 13 = 3.997$

To find the number of cases we use the calculator to evaluate $10^{3.997} = 9931$ cases (to the nearest whole number).

CAS 2: Using the TI-Nspire CAS to perform a log transformation

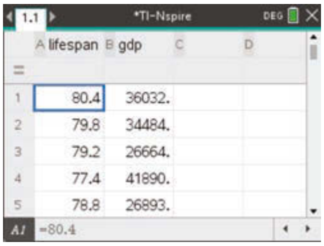
The table shows the *lifespan* (in years) and *GDP* (in dollars) of people in 12 countries. The association is non-linear. Using the log x transformation:

- linearise the data, and fit a regression line to the transformed data (*GDP* is the EV)
- write its equation in terms of the variables *lifespan* and *GDP* rounded to three significant figures.
- use the equation of the regression line to predict the lifespan in a country with a GDP of \$20 000, rounded to one decimal place.

Lifespan	*GDP*
80.4	36 032
79.8	34 484
79.2	26 664
77.4	41 890
78.8	26 893
81.5	25 592
74.9	7 454
72.0	1 713
77.9	7 073
70.3	1 192
73.0	631
68.6	1 302

Steps

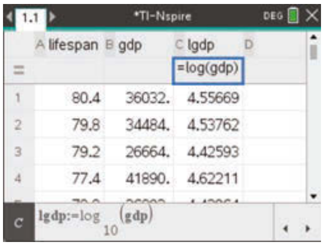
1 Start a new document by pressing ctrl + N.



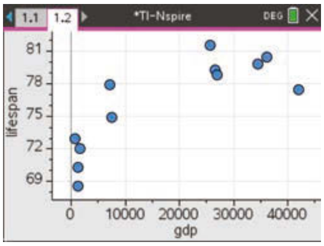
2 Select **Add Lists & Spreadsheet**.
Enter the data into lists named *lifespan* and *gdp*.

3 Name column C as *lgdp* (short for log (*GDP*)).
Now calculate the values of log (*GDP*) and store them in the list named *lgdp*.

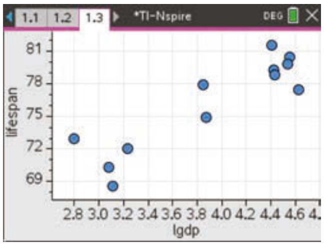
4 Move the cursor to the formula cell below the *lgdp* heading.
We need to enter the expression = **log(gdp)**.
To do this, press = then type in **log(gdp)**. Pressing enter calculates and displays the values of *lgdp*.



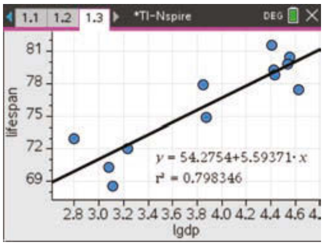
5 Press ctrl + I and select **Add Data & Statistics**.
Construct a scatterplot of *lifespan* against *GDP*. Let *GDP* be the explanatory variable and *lifespan* the response variable. The plot is clearly non-linear.



6 Press ctrl + I and select **Add Data & Statistics**. Construct a scatterplot of *lifespan* against log(*GDP*). The plot is now clearly linear.



7 Press menu **>Analyze>Regression>Show Linear (a + bx)** to plot the line on the scatterplot with its equation.
Note: The x in the equation on the screen corresponds to the transformed variable log (*GDP*).



8 Write the regression equation in terms of the variables *lifespan* and log (*GDP*).

$lifespan = 54.3 + 5.59 \times \log(GDP)$

9 Substitute 20 000 for *GDP* in the equation to find the lifespan of people in a country with GDP of \$20 000.

$lifespan = 54.3 + 5.59 \times \log 20\,000$
$= 78.3$ years

CAS 2: Using the CASIO Classpad to perform a log transformation

The table shows the *lifespan* (in years) and *GDP* (in dollars) of people in 12 countries. The association is non-linear. Using the log x transformation:

- linearise the data, and fit a regression line to the transformed data (*GDP* is the EV)
- write its equation in terms of the variables *lifespan* and *GDP* rounded to three significant figures.
- use the equation to predict the lifespan in a country with a GDP of \$20 000 rounded to one decimal place.

Lifespan	*GDP*
80.4	36 032
79.8	34 484
79.2	26 664
77.4	41 890
78.8	26 893
81.5	25 592
74.9	7 454
72.0	1 713
77.9	7 073
70.3	1 192
73.0	631
68.6	1 302

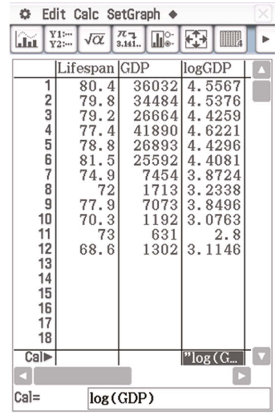
Steps

1 In the Statistics application enter the data into lists named *Lifespan* and *GDP*.

2 Name the third list *logGDP*.

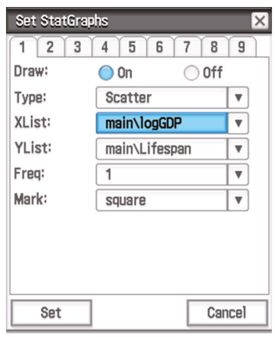
3 Place the cursor in the calculation cell at the bottom of the third column and type **log (GDP)**.

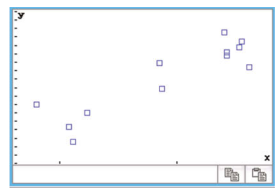
Let *GDP* be the explanatory variable (x) and *lifespan* the response variable (y).



4 Construct a scatterplot of *lifespan* against *log (GDP)*.

- Tap and complete the **Set StatGraphs** dialog box as shown.
- Tap to view the scatterplot.
- The plot is linear.



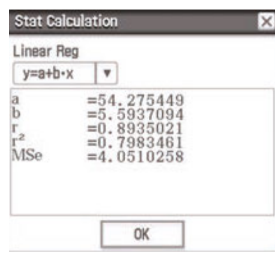


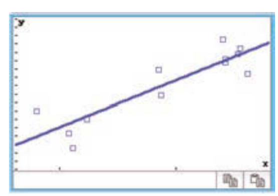
5 To find the least squares regression equation and fit a regression line to the transformed data.

- Go to **Calc, Regression, Linear Reg**.
- Complete the **Set Calculation** dialog box as shown and tap **OK**.

  This generates the regression results.

  Note: The x in the linear equation corresponds to the transformed variable log (*GDP*).





- Tap **OK** a second time to plot and display the regression line on the scatterplot.

6 Write the equation in terms of *lifespan* and log (*GDP*).

$$lifespan = 54.3 + 5.59 \times log\ (GDP)$$

7 Substitute 20 000 for *GDP* in the equation.

$$lifespan = 54.3 + 5.59 \times log\ 20\,000$$
$$= 78.3 \text{ years}$$

Skill-sheet

Exercise 4B

The log x transformation: some prerequisite skills

1 Evaluate the following expressions rounded to one decimal place.

a $y = 5.5 + 3.1 \log 2.3$

b $y = 0.34 + 5.2 \log 1.4$

c $y = -8.5 + 4.12 \log 20$

d $y = 196.1 - 23.2 \log 303$

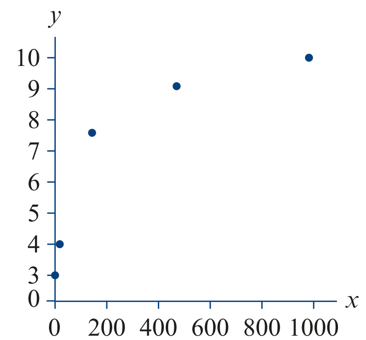
The log x transformation: calculator exercise

Example 3

2 The scatterplot opposite was constructed from the data in the table below.

x	5	10	150	500	1000
y	3.1	4.0	7.5	9.1	10.0

From the scatterplot, it is clear that the association between y and x is non-linear.



a Linearise the scatterplot by applying a log x transformation and fit a least squares line to the transformed data.

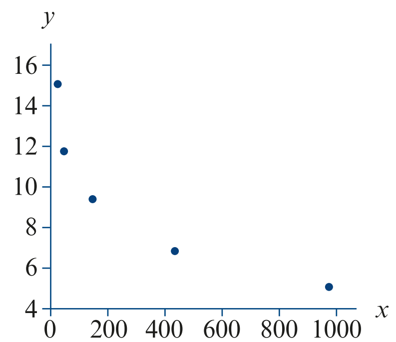
b Write down the equation, with the coefficients rounded to one significant figure.

c Use the equation to predict the value of y when $x = 100$.

3 The scatterplot opposite was constructed from the data in the table below.

x	10	44	132	436	981
y	15.0	11.8	9.4	6.8	5.0

From the scatterplot, it is clear that the association between y and x is non-linear.



a Linearise the scatterplot by applying a log x transformation and fit a least squares line to the transformed data.

b Write down the equation, with the coefficients rounded to one significant figure.

c Use the equation to predict the value of y when $x = 1000$.

The log y transformation: some prerequisite skills

4 Find the value of y in the following, rounded to one decimal place if not exact.

a $\log y = 2$

b $\log y = 2.34$

c $\log y = 3.5 + 2x$ where $x = 1.25$

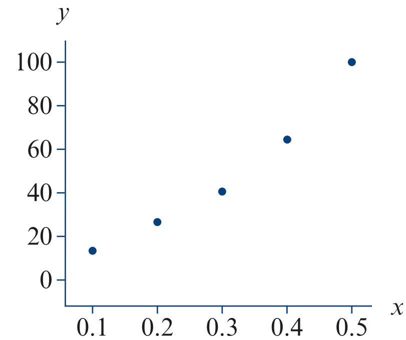
d $\log y = -0.5 + 0.024x$ where $x = 17.3$

The log *y* transformation: calculator exercise

Example 4

5 The scatterplot opposite was constructed from the data in the table below.

x	0.1	0.2	0.3	0.4	0.5
y	15.8	25.1	39.8	63.1	100.0



From the scatterplot, it is clear that the association between y and x is non-linear.

a Linearise the scatterplot by applying a log y transformation and fit a least squares line to the transformed data.

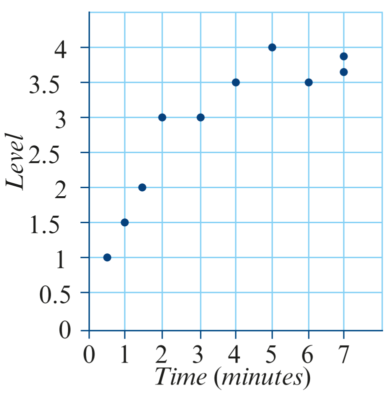
b Write down the equation, with the coefficients rounded to one significant figure.

c Use the equation to predict the value of y when $x = 0.6$, rounded to one decimal place.

Applications of the log transformation

6 The table below shows the level of performance level achieved by 10 people on completion of a task. Also shown is the time spent (in minutes) practising the task. In this situation, *time* is the EV. The association between the *level* and *time* is non-linear as seen in the scatterplot.

Time	*Level*
0.5	1
1	1.5
1.5	2
2	3
3	3
4	3.5
5	4
6	3.5
7	3.9
7	3.6



A log transformation can be applied to the variable *time* to linearise the scatterplot.

a Apply the log transformation to the variable *time* and fit a least squares line to the transformed data. log (*time*) is the EV.

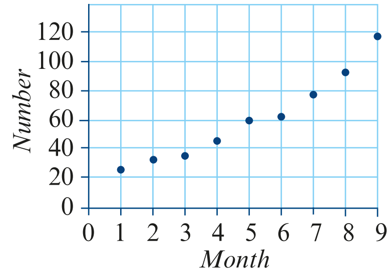
Write the slope and intercept of this line, rounded to two significant figures, in the spaces provided.

$$level = \boxed{} + \boxed{} \times \log(time)$$

b Use the equation to predict the level of performance (rounded to one decimal place) for a person who spends 2.5 minutes practising the task.

7 The table below shows the number of internet users signing up with a new internet service provider for each of the first nine months of their first year of operation. A scatterplot of the data is also shown.

Month	*Number*
1	24
2	32
3	35
4	44
5	60
6	61
7	78
8	92
9	118



The association between *number* and *month* is non-linear.

a Apply the log transformation to the variable *number* and fit a least squares line to the transformed data. *Month* is the EV.
Write the slope and intercept of this line, rounded to four significant figures, in the spaces provided.

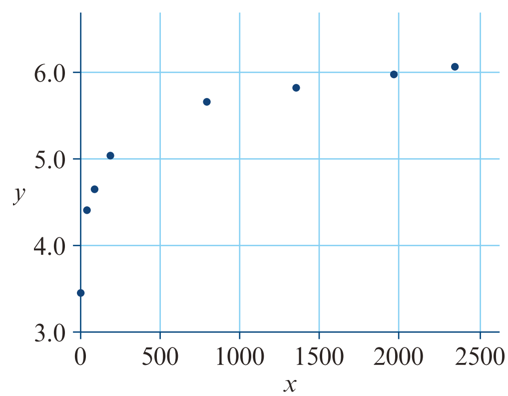
$$log\,(number) = \boxed{} + \boxed{} \times month$$

b Use the equation to predict the *number* of internet users after 10 months. Give answer to the nearest whole number.

Exam 1 style questions

8 A student uses the data in the table below to construct the scatterplot shown.

x	y
5	3.45
44	4.41
94	4.64
187	5.03
791	5.65
1350	5.81
1960	5.97
2345	6.06



A log transformation is applied to x to linearise the association. A least squares line is fitted to the transformed data, with log x as the explanatory variable.

The equation of this least squares line is closest to

A $y = 4.43 + 0.001 \log x$

B $y = -3570 + 861 \log x$

C $y = 2.78 + 0.976 \log x$

D $y = -2.85 + 1.03 \log x$

E $y = 0.642 + 0.00724 \log x$

9 The association between the power of a car (in horsepower) and the *time* it takes to accelerate from 0 to 100 km/hr (in seconds) is non-linear. A log transformation was applied to the explanatory variable *horsepower*, and a least squares line fitted to the transformed data. The equation of the least squares line is:

$$time = 42.7 - 13.9 \times \log(horsepower)$$

Using this equation, the time it would take for a car with 180 horsepower to accelerate from 0 to 100km/hr is closest to:

A 29.5 seconds **B** 65 seconds **C** 28.8 seconds **D** 11.3 seconds **E** 11.4 seconds

10 The price of shares in a newly formed technology company *price* has increased non-linearly since the company was formed 12 months ago. A log transformation was applied to the maximum share price each month (*share price*), and a least squares line fitted to the transformed data, with *month* as the explanatory variable. The equation of the least squares line is:

$$\log(shareprice) = 1.39 + 0.050 \times month$$

Using this equation, the maximum monthly share price in month 14 is closest to:

A \$123.03 **B** \$2.09 **C** \$8.08 **D** \$20.16 **E** \$25.25

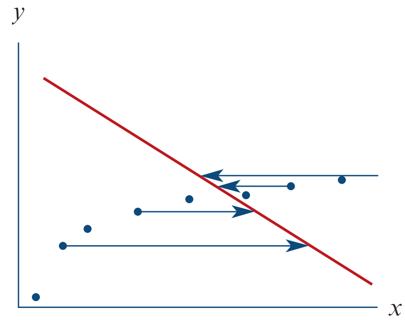
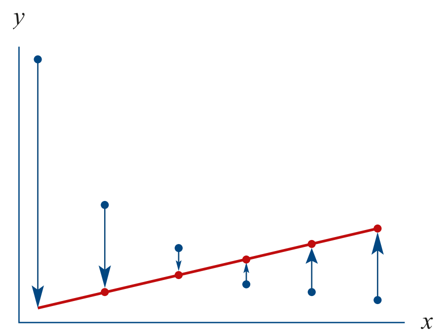
4C The reciprocal transformation

Learning intentions

- To be able to apply a reciprocal transformation to either x or y.
- To be able to fit a least squares regression line to the transformed data.
- To be able to use the least squares regression line fitted to the transformed data for prediction.

The **reciprocal transformation** is a stretching transformation that compresses the upper end of the scale on either the x- or y-axis.

The effect of applying a reciprocal y transformation to a scatterplot is as follows:

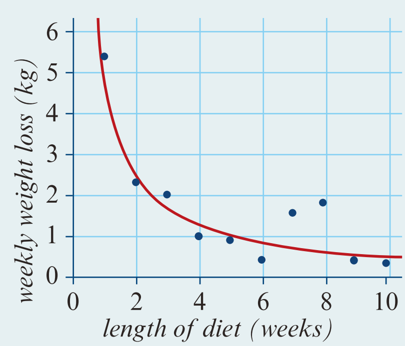
Transformation	*Outcome*	*Graph*
$\frac{1}{x}$	Compresses larger x values relative to the smaller data values, but to a greater extent than log x. This has the effect of straightening out curves like the one shown opposite. Note that values of x less than one become greater than 1, and values of x greater than 1 become less than 1, so that the order of the data values is reversed.	
$\frac{1}{y}$	Compresses larger values of y relative to lower values of y. This has the effect of straightening out curves like the one shown opposite. Again, the order of the data is reversed.	

The following example shows how the $1/x$ transformation works in practice.

Example 5 Applying the reciprocal ($1/x$) transformation

After embarking on a new healthy eating and exercise plan, Ben recorded his weekly weight loss over a 10 week. The association is not linear, as can be seen in the scatterplot below which plots *weekly weight loss* in kg against *length of diet* in weeks.

The scatterplot shows that there is a strong negative association between *weekly weight loss* and *length of diet*.



a Apply a reciprocal transformation to the variable *length of diet*, and determine the least squares regression line for the transformed data.

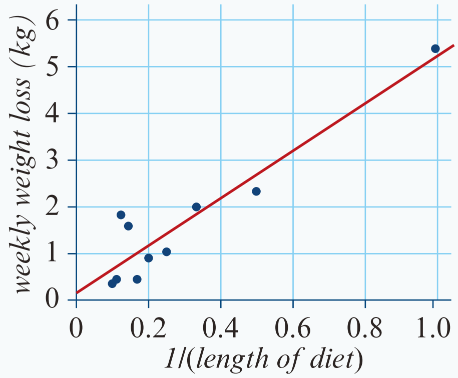
b Use the least squares equation to predict the *weekly weight loss* in week 11, giving your answer to one decimal place.

Solution

a Applying the $1/x$ transformation involves changing the scale on the x-axis to 1/(*length of diet*).

When we make this change, we see from the plot the association between the variables *weekly weight loss* and 1/(*length of diet*) is linear.

We can now fit a least squares line to model the association between the variables *weekly weight loss* and 1/(*length of diet*).



The equation of this line is:

$$\textit{weekly weight loss} = 0.13 + \frac{5.09}{\textit{length of diet}}$$

b Using this equation, in week 11 the weight loss predicted is:

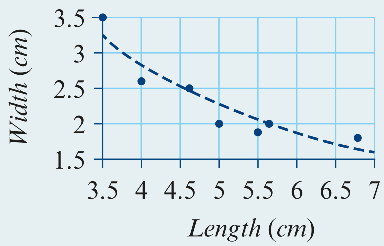
$$\textit{weekly weight loss} = 0.13 + \frac{5.09}{11} = 0.6 \text{ kg}$$

The following example shows how the $1/y$ transformation works in practice.

Example 6 Applying the reciprocal ($1/y$) transformation

A homeware company makes rectangular sticky labels with a variety of lengths and widths.

The scatterplot opposite displays the *width* (in cm) and *length* (in cm) of eight of the sticky labels.



The scatterplot shows that there is a strong negative association between the width of the sticky labels and their lengths, but it is clearly non-linear.

a Apply a reciprocal transformation to the variable *width*, and determine the least squares regression line for the transformed data.

b Use the least squares equation to predict the *width* of a sticky label which is 5 cm long, giving your answer to two decimal places.

Solution

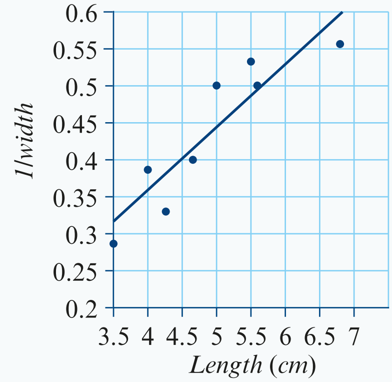
a Applying the $1/y$ transformation involves changing the scale on the y-axis from *width* to $1/(width)$.

When we make this change, we see from the scatterplot that the association between 1/*width* and *length* is linear.

We can now fit a least squares line to model the association between 1/*width* and *length*.

The equation of this line is:

$$1/width = 0.015 + 0.086 \times \ length$$



b For a sticky label of length 5 cm, we would predict that:

$$1/width = 0.015 + 0.086 \times 5 = 0.445$$

$$\text{or } width = \frac{1}{0.445} = 2.25 \text{ cm}$$

CAS 3: Using the TI-Nspire CAS to perform a reciprocal transformation

The table shows the length (in cm) and width (in cm) of eight sizes of sticky labels.

Length	6.8	5.6	4.6	4.2	3.5	4.0	5.0	5.5
Width	1.8	2.0	2.5	3.0	3.5	2.6	2.0	1.9

Using the $1/y$ transformation:

- linearise the data, and fit a regression line to the transformed data (*length* is the EV)
- write its equation in terms of the variables *length* and *width*
- use the equation to predict the width of a sticky label with a length of 5 cm.

Steps

1 Start a new document by pressing ctrl + N.

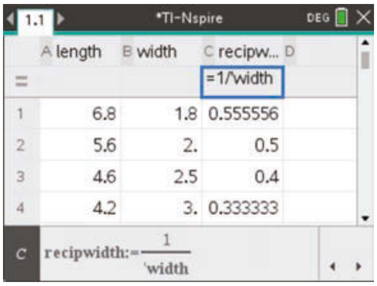
2 Select **Add Lists & Spreadsheet**.

Enter the data into lists named *length* and *width*.

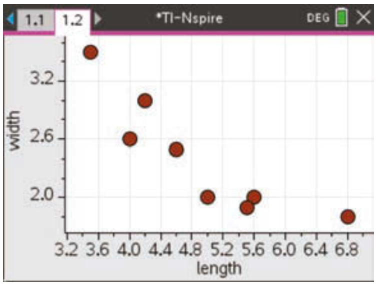
3 Name column C as *recipwidth* (short for 1/width).

Calculate the values of *recipwidth*.

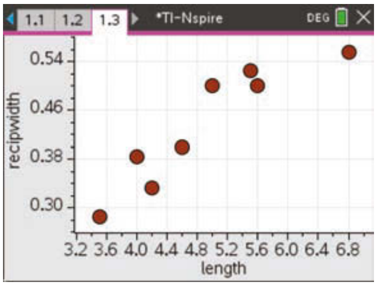
Move the cursor to the formula cell below the *recipwidth* heading. Type in **=1/width**. Press enter to calculate the values of *recipwidth*.



4 Press ctrl + I and select **Add Data & Statistics**. Construct a scatterplot of *width* against *length*. Let *length* be the explanatory variable and *width* the response variable. The plot is clearly non-linear.

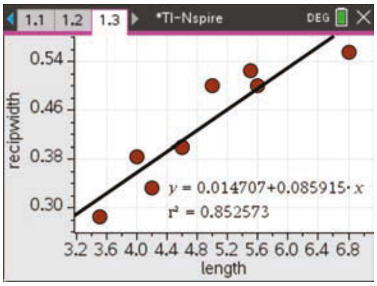


5 Press ctrl + I and select **Add Data & Statistics**. Construct a scatterplot of *recipwidth* (1/*width*) against *length*. The plot is now clearly linear.



6 Press menu **>Analyze>Regression>Show Linear (a + bx)** to plot the line on the scatterplot with its equation.

Note: The y in the equation on the screen corresponds to the transformed variable $1/width$.



7 Write down the regression equation in terms of the variables *width* and *length*.

$1/width = 0.015 + 0.086 \times length$

8 Substitute 5 cm for *length* in the equation.

$1/width = 0.015 + 0.086 \times 5 = 0.445$

Thus width = 1/0.445 = 2.25 cm (to 2 d.p.)

CAS 3: Using the CASIO Classpad to perform a reciprocal transformation

The table shows the length (in cm) and width (in cm) of eight sizes sticky labels.

Length	6.8	5.6	4.6	4.2	3.5	4.0	5.0	5.5
Width	1.8	2.0	2.5	3.0	3.5	2.6	2.0	1.9

Using the $1/y$ transformation:

- linearise the data, and fit a regression line to the transformed data. *Length* is the EV.
- write its equation in terms of the variables *length* and *width*.
- use the equation to predict the width of a sticky label with length of 5 cm.

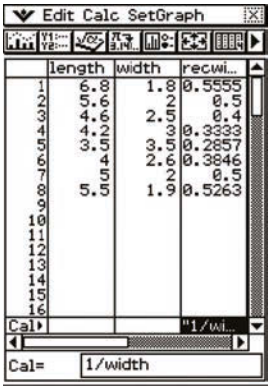
Steps

1 Open the Statistics application and enter the data into lists named *length* and *width*.

2 Name the third list *recwidth* (short for reciprocal width).

3 Place the cursor in the calculation cell at the bottom of the third column and type *1/width*. This will calculate all the reciprocal values of the width.

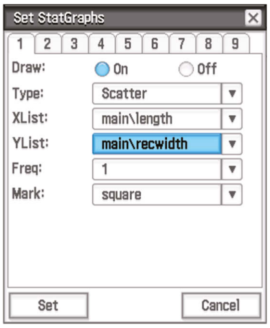
Let *length* be the explanatory variable (x) and *width* the response variable (y).

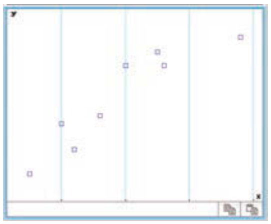


4 Construct a scatterplot of *1/width* against *length*.

- Tap and complete the **Set StatGraphs** dialog box as shown.
- Tap to view the scatterplot.

The plot is now clearly linear.



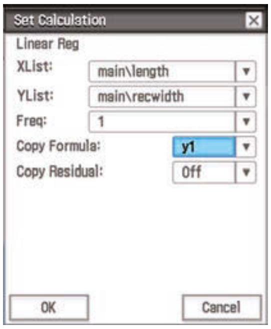


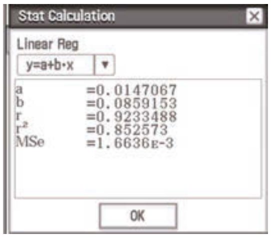
5 Fit a regression line to the transformed data.

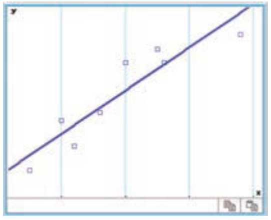
- Go to **Calc, Regression, Linear Reg**.
- Complete the **Set Calculation** dialog box as shown and tap **OK**.

This generates the regression results.

Note: The y in the linear equation corresponds to the transformed variable $1/width$; that is $1/y$.







- Tap **OK** a second time to plot and display the line on the scatterplot.

6 Write down the equation in terms of the variables *width* and *length*.

$1/width = 0.015 + 0.086 \times length$

7 Substitute 5 cm for *length* in the equation.

$1/width = 0.015 + 0.086 \times 5 = 0.445$

Thus width = 1/0.445 = 2.25 cm (to 2 d.p.)

Skill-sheet

Exercise 4C

The reciprocal (1/x) transformation: some prerequisite skills

1 Evaluate the following expressions rounded to one decimal place.

a $y = 6 + \frac{22}{x}$ when $x = 3$

b $y = 4.9 - \frac{2.3}{x}$ when $x = 1.1$

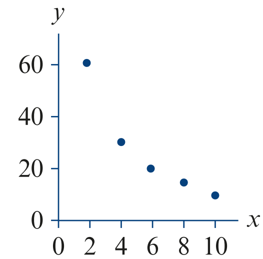
c $y = 8.97 - \frac{7.95}{x}$ when $x = 1.97$

d $y = 102.6 + \frac{223.5}{x}$ when $x = 1.08$

The reciprocal (1/x) transformation: calculator exercise

Example 5

2 The scatterplot opposite was constructed from the data in the table below.



x	2	4	6	8	10
y	60	30	20	15	12

From the scatterplot, it is clear that the association between y and x is non-linear.

a Linearise the scatterplot by applying a $1/x$ transformation and fit a least squares line to the transformed data.

b Write down its equation.

c Use the equation to predict the value of y when $x = 5$.

The reciprocal (1/y) transformation: some prerequisite skills

3 Find the value of y in the following, rounded to two decimal places.

a $\frac{1}{y} = 3x$ when $x = 2$

b $\frac{1}{y} = 6 + 2x$ when $x = 4$

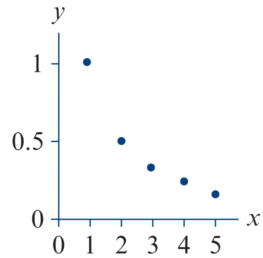
c $\frac{1}{y} = -4.5 + 2.4x$ when $x = 4.5$

d $\frac{1}{y} = 14.7 + 0.23x$ when $x = 4.5$

The reciprocal (1/y) transformation: calculator exercise

Example 6

4 The scatterplot opposite was constructed from the data in the table below.



x	1	2	3	4	5
y	1	0.5	0.33	0.25	0.20

From the scatterplot, it is clear that the association between y and x is non-linear.

a Linearise the scatterplot by applying a $1/y$ transformation and fit a least squares line to the transformed data.

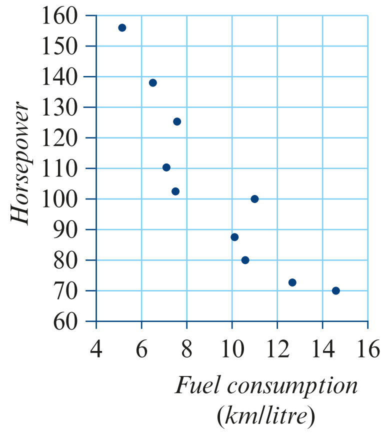
b Write down its equation.

c Use the equation to predict the value of y when $x = 0.25$.

Applications of the reciprocal transformation

5 The table shows the *horsepower* of 10 cars and their *fuel consumption*. From the scatterplot, it is clear that the association between *horsepower* and *fuel consumption* is non-linear.

Fuel consumption	*Horsepower*
5.2	155
7.3	125
12.6	75
7.1	110
6.3	138
10.1	88
10.5	80
14.6	70
10.9	100
7.7	103



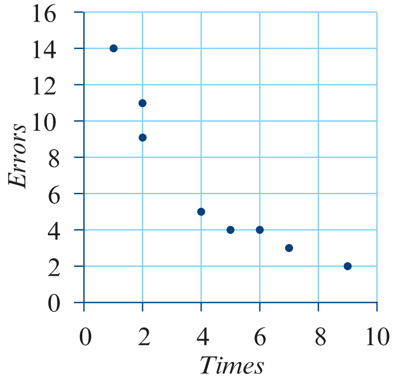
a Apply the reciprocal transformation to the variable *fuel consumption* and fit a least squares line to the transformed data. *Horsepower* is the RV.
Write the intercept and slope of this line in the boxes provided, rounded to three significant figures.

$$horsepower = \boxed{} + \boxed{} \times \frac{1}{fuel\,consumption}$$

b Use the equation to predict the horsepower of a car with a fuel consumption of 9 km/litre.

6 Ten students were given an opportunity to practise a complex matching task as often as they liked before they were assessed. The number of *times* they practised the task and the number of *errors* they made when assessed are given in the table.

Times	*Errors*
1	14
2	9
2	11
4	5
5	4
6	4
7	3
7	3
9	2



a Apply the reciprocal transformation to the variable *errors* and determine the least squares regression with the number of times the task was practiced as the EV. Write the intercept and slope of this line in the boxes provided, rounded to two significant figures.

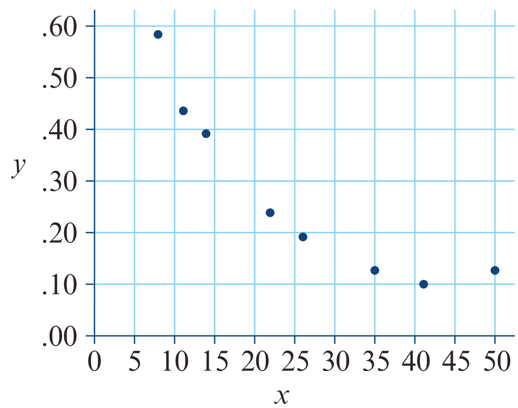
$$\frac{1}{errors} = \boxed{} + \boxed{} \times times$$

b Use the equation to predict the number of errors made when the task is practised six times.

Exam 1 style questions

7 A student used the data in the table below to construct the scatterplot shown

x	y
8	0.58
11	0.43
14	0.39
22	0.24
26	0.19
35	0.13
41	0.10
50	0.12



A reciprocal transformation is applied to y to linearise the association. A least squares line is fitted to the transformed data, with $1/y$ as the response variable. The equation of this least squares line is closest to

A $\frac{1}{y} = 0.198 + 0.196x$

B $\frac{1}{y} = -0.546 - 0.011x$

C $y = 0.013 + \frac{4.665}{x}$

D $\frac{1}{y} = 0.013 + 4.665x$

E $y = 0.546 - 0.011x$

8 The association between score on a problem solving test (*score*) and the number of attempts a person has at the test (*attempts*) is non-linear. A reciprocal transformation was applied to the explanatory variable *attempts*, and a least squares line fitted to the transformed data. The equation of the least squares line is:

$$score = 50 - 22.8 \times \frac{1}{attempts}$$

Using this equation, the score that a person achieves on their fourth attempt is closest to:

A 6.8 **B** 27.2 **C** 55.7 **D** 41.2 **E** 44.3

9 The price of shares in a newly formed technology company *price* has increased non-linearly since the company was formed 12 months ago. A reciprocal transformation was applied to the maximum share price each month (*share price*), and a least squares line fitted to the transformed data, with *month* as the explanatory variable. The equation of the least squares line is:

$$\frac{1}{share price} = 0.0349 - 0.00215 \times month$$

Using this equation, the maximum monthly share price in month 14 is closest to:

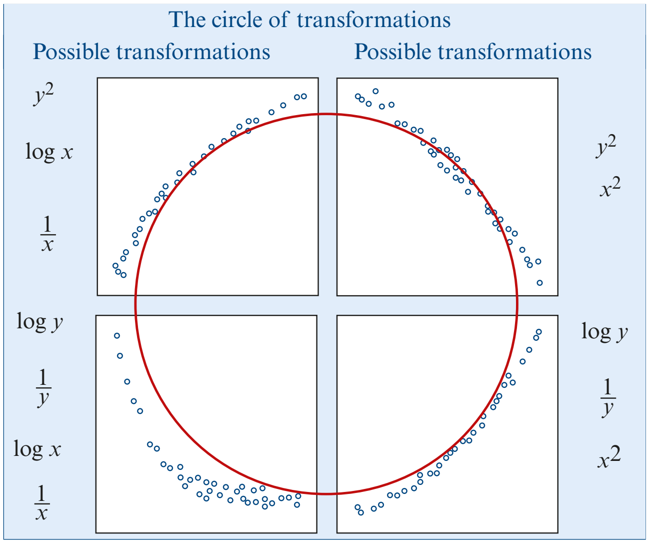
A \$2.18 **B** \$28.78 **C** \$208.33 **D** 0.48 cents **E** 48 cents

4D Choosing and applying the appropriate transformation

Learning intentions

- To be able to use the circle of transformations to determine which transformations may help linearise a non-linear association.
- To be able to use a residual plot to assess the effectiveness of a data transformation.
- To be able to use the coefficient of determination to assess the effectiveness of a data transformation.

The types of scatterplots that can be transformed by the squared, log or reciprocal transformations can be fitted together into what we call the **circle of transformations**.



The purpose of the circle of transformations is to guide us in our choice of transformation to linearise a given scatterplot.

There are two things to note when using the circle of transformations:

1. In each case, there is more than one type of transformation that might work.
2. These transformations only apply to scatterplots with a consistently increasing or decreasing trend.

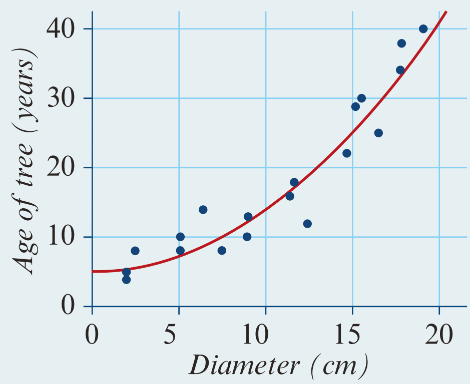
The advantage of having alternatives is that in practice, we can always try each of them to see which gives us the best result. How do we decide which transformation is the best? The best transformation is the one which results in the best linear model. To choose the best linear model we will consider for each transformation applied:

- The residual plot, in order to evaluate the linearity of the transformed association.
- The value of the coefficient of determination, r^2.

This procedure is illustrated in the following example.

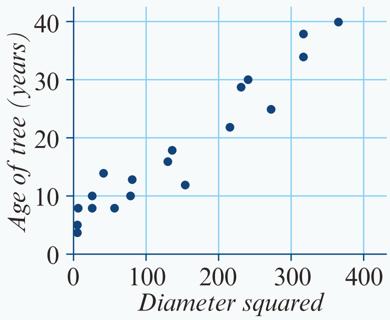
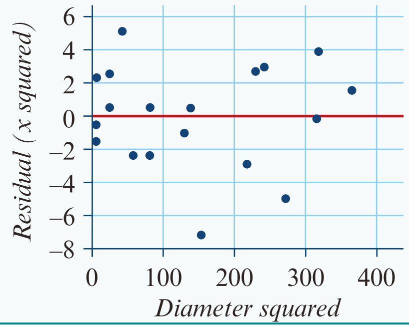
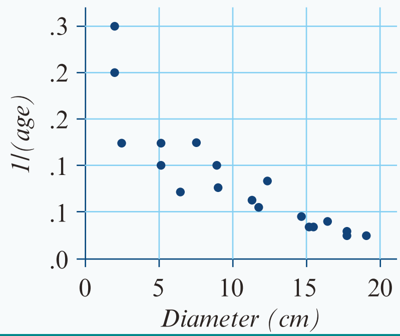
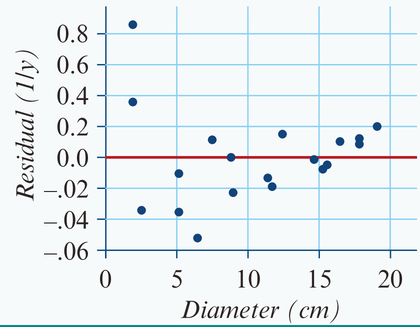
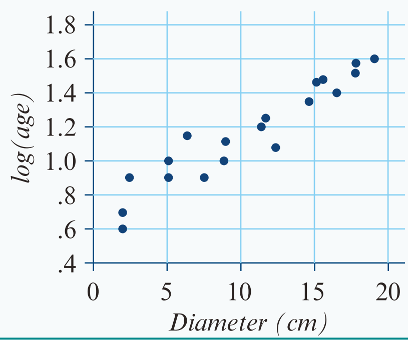
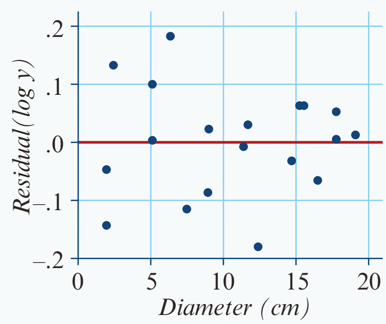
Example 7 Choosing the best transformation

The scatterplot shows the age (in years) and diameter at a height of 1.5 metre (in cm) for a sample of 19 trees of the same species. Use an appropriate transformation to find a regression model which allows the age of this species of tree to be predicted from its diameter.



Solution

The scatterplot has a consistently increasing trend so the circle of transformations applies. Comparing the scatterplot to those in the circle of transformations we see that the x^2, $1/y$ and $\log x$ transformations all have the potential to linearise this scatterplot. All of these transformations have been applied in turn, and the resulting scatterplots and residual plots are shown in the following table.

Transformation	*Scatterplot*	*Residual plot*
x^2		
$\frac{1}{y}$		
$\log y$		

Applying each of these transformations in turn we can see from the residual plots that both the x^2 and the $\log y$ transformations have been quite effective in linearising the association between the age of the tree and its diameter. There still seems to be a curve in the residual plot after the the $1/y$ transformation so that has been less effective.

To further help to choose the best transformation we can compare the values of r^2, the coefficient of determination.

- For the x^2 transformation, $r^2 = 92.7\%$
- For the $1/y$ transformation, $r^2 = 75.7\%$
- For the $\log y$ transformation, $r^2 = 90.2\%$

Both the x^2 and $\log y$ transformations have a very high explanatory power, and either would seem to be acceptable. When more than one transformation is doing a reasonable job of linearising the association, and they have similar value of r^2 then the transformation which is easier to interpret in terms of the variables is preferred. In this case $diameter^2$ makes more sense in that it tells us that the *age* of the tree relates to the cross sectional area of the tree. The log transformation does not have an equivalent meaningful interpretation.

We can now fit a least squares line to model the association between *age* and $diameter^2$

The equation of this line is:

$$age = 5.098 + 0.091 \times diameter^2$$

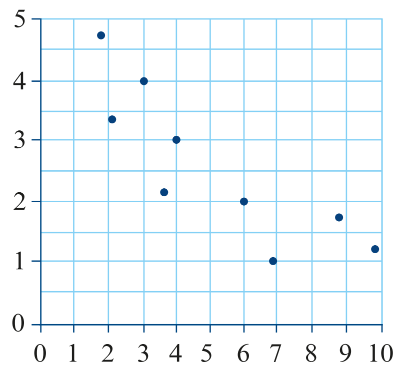
At this stage you might find it helpful to use the interactive 'Data transformation' (accessible through the Interactive Textbook) to see how these different transformations can be used to linearise scatterplots.

Exercise 4D

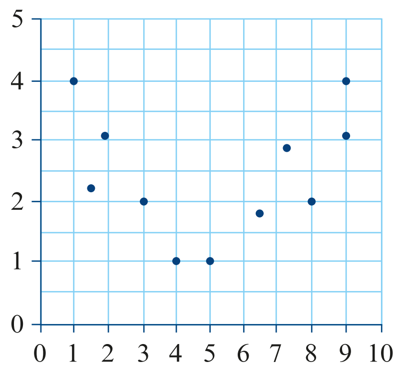
Example 7

1 The scatterplots below are non-linear. For each, identify the transformations x^2, log x, $1/x$, y^2, log y, $1/y$ or none that might be used to linearise the plot.

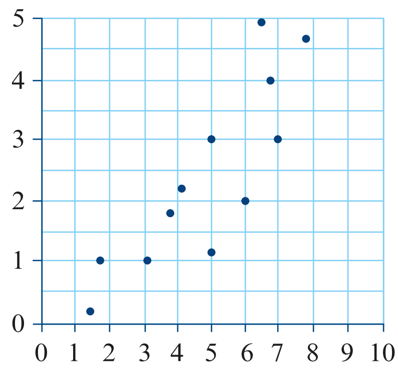
a



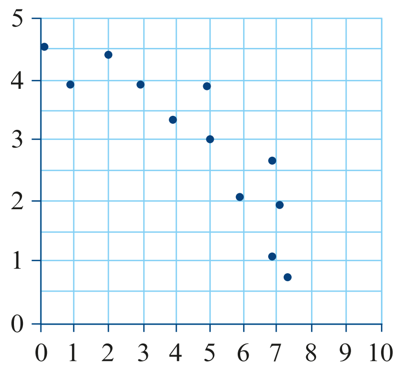
b



c



d



2 The data below gives the yield in kilograms and length in metres of 12 commercial potato plots.

yield(kg)	346	1798	152	86	436	968
length(m)	12.1	27.4	8.3	5.5	15.7	21.5
yield(kg)	686	257	2435	287	1850	1320
length(m)	19.5	9.0	34.2	14.7	31.9	25.3

a Construct a scatterplot showing the association between *yield* in kilograms (the RV) and *length* of the plot in metres (the EV).

b Fit a least squares regression line to the data. Write down the equation in terms of the variables in the question, giving the values of the intercept and slope rounded to four significant figures.

c Construct a residual plot, and comment on whether the linearity assumption has been met.

d Use the circle of transformations to select which transformations could be considered in order to linearise the association.

e Using an appropriate transformation, recommend a regression model for the association between *yield* and *length* of the plot. Write down the equation in terms of the transformed variables, giving the values of the intercept and slope rounded to four significant figures.

f What is the value of r^2 for the recommended model? Give your answer as a percentage rounded to one decimal place.

3 In order to investigate the association between the average number of cigarettes per day per smoker (*smoking*) and the cost of cigarettes in \$ per cigarette (*cost*) for a group of countries the following data was collected.

cost (\$)	0.67	0.75	0.80	0.92	1.00	1.08	1.17	1.25	1.30	1.40
smoking	16.7	15.5	14.8	13.4	12.5	12.0	11.1	10.9	10.3	9.5

a Construct a scatterplot showing the association between *smoking* (in cigarettes/day) (the RV) and *cost* (\$/cigarette) (the EV).

b Fit a least squares regression line to the data. Write down the equation in terms of the variables in the question, giving the values of the intercept and slope rounded to four significant figures.

c Construct a residual plot, and comment on whether the linearity assumption has been met.

d Use the circle of transformations to select which transformations could be considered in order to linearise the association.

e Using an appropriate transformation, recommend a regression model for the association between *smoking* and *cost*. Write down the equation in terms of the transformed variables, giving the values of the intercept and slope rounded to four significant figures.

f What is the value of r^2 for the recommended model? Give your answer as a percentage rounded to one decimal place.

4 The following data shows the population density in people per hectare (*density*) and the distance from the centre of the city in km (*distance*) for a large city.

density	307.58	294.67	283.93	270.82	234.93	175.08	101.56	49.80
distance	0	2	4	6	8	10	12	14

a Construct a scatterplot showing the association between the population *density* in people per hectare (the RV) and *distance* from the centre of the city in km (the EV).

b Fit a least squares regression line to the data. Write down the equation in terms of the variables in the question, giving the values of the intercept and slope rounded to four significant figures.

c Construct a residual plot, and comment on the whether linearity assumption has been met.

d Use the circle of transformations to select which transformations could be considered in order to linearise the association.

e Using an appropriate transformation, recommend a regression model for the association between *density* and *distance*. Write down the equation in terms of the transformed variables, giving the values of the intercept and slope rounded to four significant figures.

f What is the value of r^2 for the recommended model? Give your answer as a percentage rounded to one decimal place.

Key ideas and chapter summary

Assign-ment

Data transformation In regression analysis, **data transformation** involves changing the scale on either the *x*- or *y* axis in order to linearise an association prior to fitting a least squares line.

Squared transformation The **squared transformation** *stretches out* the upper end of the scale on an axis.

Logarithmic transformation The **logarithmic transformation** *compresses* the upper end of the scale on an axis.

Reciprocal transformation The **reciprocal transformation** *compresses* the upper end of the scale on an axis but to a greater extent than the log transformation.

Residual plots Residual plots are used to assess the effectiveness of a data transformation.

Coefficient of determination The transformation that results in a linear association (as assessed by the residual plot) and which has the highest value of the coefficient of determination is generally the preferred transformation.

The circle of transformations The **circle of transformations** provides guidance in choosing the transformations that can be used to linearise various types of scatterplots.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

4A **1** I can apply the x^2 transformation. ☐

See Example 1, and Exercise 4A Question 2

4A **2** I can apply the y^2 transformation. ☐

See Example 2, and Exercise 4A Question 5

4B **3** I can apply the $\log x$ transformation. ☐

See Example 3, and Exercise 4B Question 2

4B **4** I can apply the $\log y$ transformation. ☐

See Example 4, and Exercise 4B Question 5

4C **5** I can apply the reciprocal ($1/x$) transformation. ☐

See Example 5, and Exercise 4C Question 2

4C **6** I can apply the reciprocal ($1/y$) transformation. ☐

See Example 6, and Exercise 4C Question 4

4D **7** I can choose the best transformation to apply. ☐

See Example 7, and Exercise 4D Question 1

Multiple-choice questions

1 Select the statement that correctly completes the sentence:
'The effect of a squared transformation is to . . . '

A stretch the high values in the data
B maintain the distance between values
C stretch the low values in the data
D compress the high values in the data
E reverse the order of the data values

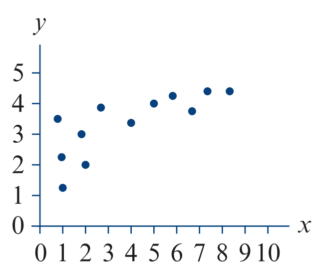
2 Select the statement that correctly completes the sentence:
'The effect of a log transformation is to . . .'

A stretch the high values in the data
B maintain the distance between values
C stretch the low values in the data
D compress the high values in the data
E maintain the order of the values in the data

3 Select the statement that correctly completes the sentence:
'The effect of a reciprocal transformation is to . . .'

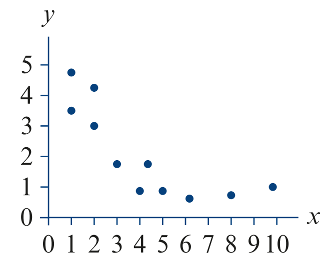
A stretch the high values in the data
B maintain the distance between values
C stretch the low values in the data
D compress the high values in the data
E reverse the order of the values in the data

4 The association between two variables y and x, as shown in the scatterplot, is non-linear. In an attempt to transform the association to linearity, a student would be advised to:



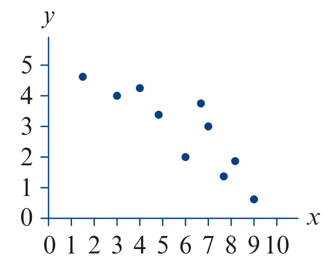
A leave out the first four points
B use a y^2 transformation
C use a log y transformation
D use a $1/y$ transformation
E use a least squares regression line

5 The association between two variables y and x, as shown in the scatterplot, is non-linear.
Which of the following sets of transformations could possibly linearise this association?



A $\log y$, $1/y$, $\log x$, $1/x$ **B** y^2, x^2
C y^2, $\log x$, $1/x$ **D** $\log y$, $1/y$, x^2
E $ax + b$

6 The association between two variables y and x, as shown in the scatterplot, is non-linear.
Which of the following transformations is most likely to linearise the association?



A a $1/x$ transformation **B** a y^2 transformation
C a $\log y$ transformation **D** a $1/y$ transformation
E a $\log x$ transformation

7 The following data were collected for two related variables x and y.

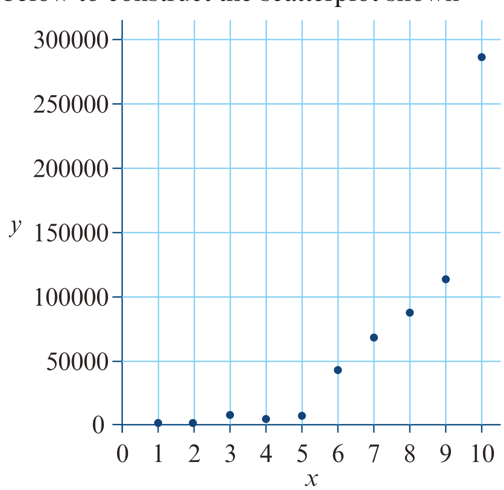
x	1	2	3	4	5	6	7	8	9	10	11
y	7	8.6	8.9	8.8	9.9	9.7	10.4	10.5	10.7	11.2	11.1

A scatterplot indicates a non-linear association. The data is linearised using a log x transformation and a least squares line is then fitted. The equation of this line is closest to:

A $y = 7.52 + 0.37 \log x$ **B** $y = 0.37 + 7.52 \log x$
C $y = -1.71 + 0.25 \log x$ **D** $y = 3.86 + 7.04 \log x$
E $y = 7.04 + 3.86 \log x$

8 A student uses the data in the table below to construct the scatterplot shown

x	y
1	2030
2	1265
3	8265
4	5654
5	6893
6	43265
7	67890
8	87803
9	113062
10	286370



A log transformation is applied to y to linearise the association. A least squares line is fitted to the transformed data, with $\log y$ as the response variable.
The equation of this least squares line is closest to

A $y = 2.88 + 0.256 \log x$
B $\log y = -10.1 + 3.64 \log x$
C $\log y = -69800 + 24000x$
D $\log y = 2.88 + 0.256x$
E $\log y = -4.84 + 4.38x$

9 The association between the total *weight* of produce picked from a vegetable garden and its *width* is non-linear. An x^2 transformation is used to linearise the data.
When a least squares line is fitted to the data, its y-intercept is 10 and its slope is 5.
Assuming that *weight* is the response variable, the equation of this line is:

A $(weight)^2 = 10 + 5 \times\ width$ **B** $width = 10 + 5 \times\ (weight)^2$
C $width = 5 + 10 \times\ (weight)^2$ **D** $weight = 10 + 5 \times\ (width)^2$
E $(weight)^2 = 5 + 10 \times\ weight$

10 A model that describes the association between the hours spent studying for an exam and the mark achieved is:

$mark = 20 + 40 \times \log\,(hours)$

From this model, we would predict that a student who studies for 20 hours would score a mark (to the nearest whole number) of:

A 80 **B** 78 **C** 180 **D** 72 **E** 140

11 A $1/y$ transformation is used to linearise a scatterplot.
The equation of a least squares line fitted to this data is:

$1/y = 0.14 + 0.045x$

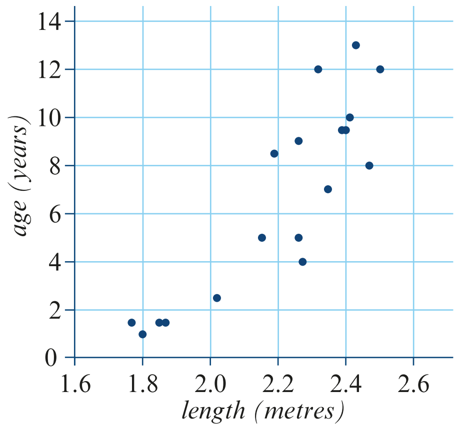
This regression line predicts that, when $x = 6$, y is closest to:

A 0.17 **B** 0.27 **C** 0.41 **D** 2.4 **E** 3.7

Written response questions

1 The table below shows the age in years (*age*) and the length in metres (*length*), for a group of 18 dugongs. A scatterplot of the data is also shown.

age	*length*	*age*	*length*
1.0	1.80	8.0	2.47
1.5	1.85	8.5	2.19
1.5	1.87	9.0	2.26
1.5	1.77	9.5	2.40
2.5	2.02	9.5	2.39
4.0	2.27	10.0	2.41
5.0	2.15	12.0	2.50
5.0	2.26	12.0	2.32
7.0	2.35	13.0	2.43



The scatterplot shows that the association is clearly non-linear. A reciprocal $\left(\frac{1}{y}\right)$ transformation can be applied to the variable *age* to linearise the association.

a Apply the reciprocal $\left(\frac{1}{y}\right)$ transformation to the data and use the transformed data to determine the equation of a least squares line that enables $\frac{1}{age}$ to be predicted from *length*. Write the values of the intercept and slope in the the appropriate boxes provided. Round to four significant figures.

$$\frac{1}{age} = \boxed{} - \boxed{} \times length$$

b The association can also be linearised by applying a log transformation to the variable *age*. When this is done, and a least squares line fitted to the transformed data, the resulting equation is:

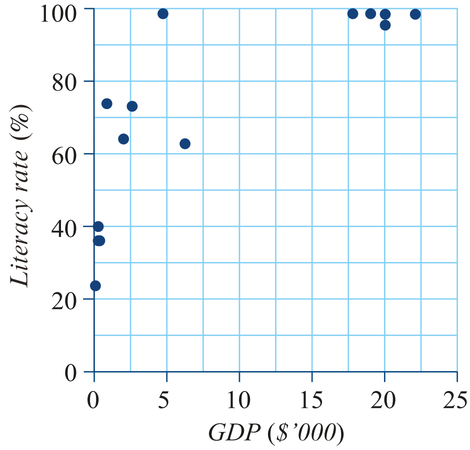
$$\log(age) = -2.443 + 1.429 \times length$$

Use this equation to predict the age of a dugong with a length of 2.00 metres. Round the answer to one decimal place.

2 The table below shows the percentage of people who can read (*literacy rate*) and the gross domestic product (*GDP*), in dollars/person, for a selection of 14 countries. A scatterplot of the data is also shown.
The scatterplot can be linearised by using a log *x* transformation.

Review

GDP	*literacy rate*
2677	72
260	35
19 904	97
122	24
18 944	99
4 500	99
17 539	99
1 030	73
19 860	99
409	40
406	35
6651	62
22 384	99
2 436	64



a Apply the log transformation to the variable *GDP*, and fit a least squares line to the transformed data. Write down its equation terms of the variables *literacy rate* and log (*GDP*). Give the slope and intercept rounded to three significant figures.

b Verify that the log transformation has linearised the association by constructing a residual plot.

c Use the regression equation to predict the literacy rate of a country with a GDP of $10 000 to the nearest percent.

d Find the value of the residual when the regression equation is used to predict the literacy rate when the GDP is equal to $19 860. Give your answer rounded to two significant figures.

3 Measurements of the *distance* travelled (metres) and *time* taken (seconds) were made on a falling body. The data are given in the table below.

time	0	1	2	3	4	5	6
distance	0	5.2	18.0	42.0	79.0	128.0	168.0

a Construct a scatterplot of the data and comment on its form.

b Determine the values of $time^2$ and complete the table.

c Construct a scatterplot of *distance* against $time^2$.

d Fit a least squares line to the transformed data, with *distance* as the RV.

e Use the regression equation to predict the distance travelled in 7 seconds.

f Obtain a residual plot and comment on whether the assumption of linearity is reasonable.

4 Is the infant mortality rate in a country associated with the number of doctors in that country? The data below gives infant mortality rate in deaths per 1000 births (*mortality*) and the number of doctors per 100 000 of population (*doctors*) for 14 countries.

mortality	12	13	12	10	10	7	111
doctors	192	222	154	182	179	204	61
mortality	15	10	20	54	75	121	71
doctors	270	271	357	79	59	27	52

a Construct a scatterplot of *mortality* against *doctors* and use it to comment on the association between infant mortality rate and doctor numbers.

b Construct a scatterplot of *mortality* against $\dfrac{1}{doctors}$.

c Determine the equation of the least squares regression line which would enable *mortality* to be predicted from $\dfrac{1}{doctors}$. Give the values of the intercept and slope rounded to four significant figures.

d Obtain a residual plot for the model fitted in part c and comment on the linearity.

e Determine the value of coefficient of determination for the model fitted in part c. Give your answer as a percentage rounded to one decimal place.

f Use the regression equation to predict the infant mortality rate in a country where there are 100 doctors per 100,000 people. Give your answer rounded to the nearest whole number.

g Comment on the reliability of the prediction made in part f.

Chapter 5

Investigating and modelling time series

Chapter questions

- What is time series data?
- How do we construct a time series plot?
- How do we recognise features such as trend, seasonality and irregular fluctuations?
- How do we smooth a time series plot using moving means?
- How do we smooth a time series plot using moving medians?
- How do we calculate and interpret seasonal indices?
- How do we calculate and interpret a trend line?
- How do we make forecasts of future values?

In this chapter we will focus on a special case of numerical bivariate data, called **time series data**. In time series data the explanatory variable is always a measure of time (for example hour, day, month or year), and we are concerned with understanding how the response variable is changing over time.

5A Time series data

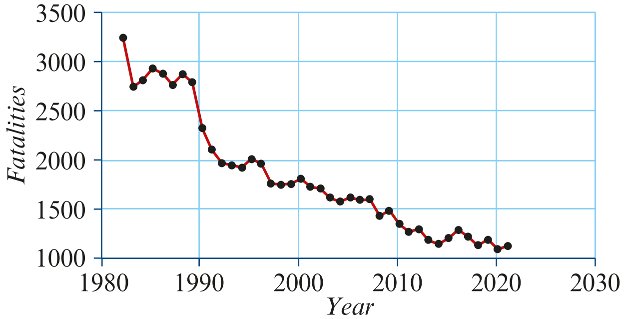
Learning intentions

- ▶ To be able to recognise time series data.
- ▶ To be able to construct a time series plot.
- ▶ To be able to recognise features in the plot such as trend, seasonality and irregular fluctuations.

When data concerned with a variable is collected, observed or recorded at successive intervals of time, it is referred to as **time series data**. An example of time series data is Annual road accident fatalities for Australia, 1982–2021, given in the following table.

Year	*Fatalities*	*Year*	*Fatalities*	*Year*	*Fatalities*	*Year*	*Fatalities*
1982	3252	1992	1974	2002	1715	2012	1300
1983	2755	1993	1953	2003	1621	2013	1187
1984	2822	1994	1928	2004	1583	2014	1150
1985	2941	1995	2017	2005	1627	2015	1209
1986	2888	1996	1970	2006	1598	2016	1293
1987	2772	1997	1767	2007	1603	2017	1225
1988	2887	1998	1755	2008	1437	2018	1135
1989	2801	1999	1764	2009	1491	2019	1195
1990	2331	2000	1817	2010	1353	2020	1095
1991	2113	2001	1737	2011	1277	2021	1127

Since time series data is just a special kind of two numerical variable example, where the explanatory variable is time, we will begin by drawing a scatterplot of the data. In this instance, the scatterplot is called a **time series plot**, with *time* always placed on the horizontal axis. A time series plot differs from a normal scatterplot in that, in general, the points will be joined by line segments in time order. An example of a time series plot, of the road accident fatality data, is given below.



Looking at the time series plot, we can readily see a clear trend of decreasing road fatalities, which is good news for drivers, as this provides some evidence that the many efforts being made to reduce the road toll across Australia have been effective.

Constructing time series plots

As previously mentioned, time series data is a special case of bivariate numerical data, where the explanatory variable is time. Consider the variable *month*, which takes values such as January, February, March and so on. For the purpose of plotting and analysing time series data, we can consider the variable *month* as numerical, taking the values $\{1, 2, 3, \ldots\}$. If we had monthly data for a two year period, then the variable *month* would take the values $\{1, 2, \ldots, 24\}$. Whether the actual value of the variable is used in the plot (January, February, March, ...) or its numerical equivalent (1, 2, 3, ...) is used, both time series plots would be considered correct. We can use a similar approach for the variables *day*, or *quarter*.

Example 1 Constructing a time series plot

Maximum temperature was recorded each day for a week in a certain town. Construct a time series plot of the data.

Day	Mon	Tues	Wed	Thur	Fri	Sat	Sun
Temperature (°C)	20	21	25	36	34	25	26

Explanation

1 In a time series plot, time (day in this case) is always the explanatory variable (EV) and is plotted on the horizontal axis.

Solution

Day is the EV – this will label the horizontal axis.

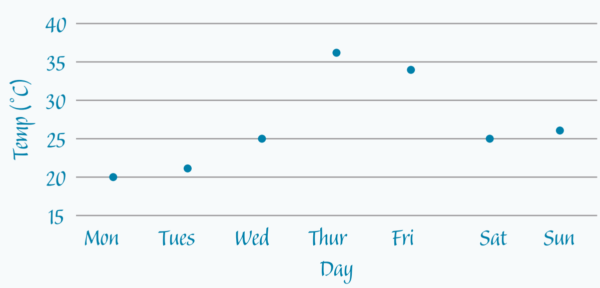
Temperature is the RV – this will label the vertical axis.

2 Determine the scales for each axis.

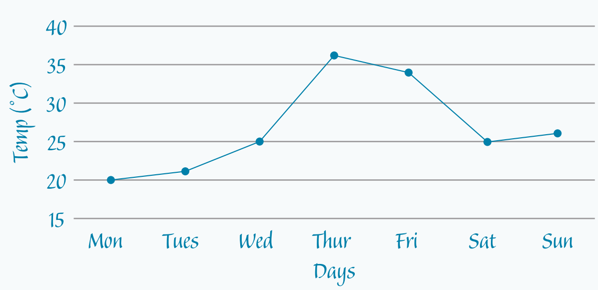
A horizontal scale from 0–7 with intervals of 1 for each day would be suitable.

Temperature ranges from 20–36. A vertical scale from 15–40 with intervals of 5 would be suitable.

3 Set up the axes, and then plot all seven data points as for a scatterplot.



4 Complete the graph, by joining consecutive data points with straight lines.



Most real-world time series data come in the form of large data sets that are best plotted with the aid of a spreadsheet or statistical package. The availability of the data in electronic form via the internet greatly helps this process. However, in this chapter, most of the time series data sets are relatively small and can be readily plotted using a CAS calculator.

CAS 1: How to construct a time series using the TI-Nspire CAS

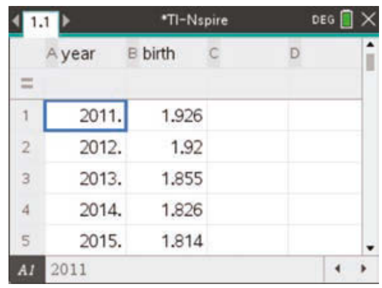
Construct a time series plot for the data presented below, which shows the birth rate in Australia (in births per woman) from 2011–2020.

year	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
birth	1.926	1.920	1.855	1.826	1.814	1.752	1.741	1.740	1.657	1.580

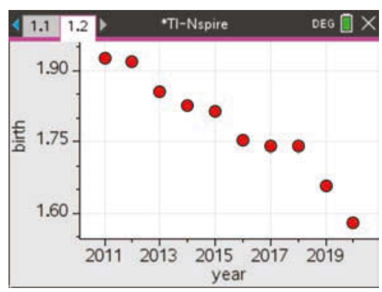
Steps

1 Start a new document by pressing ctrl + N.

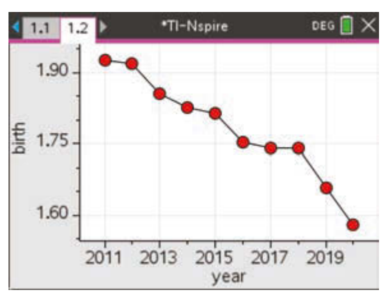
2 Select **Add Lists & Spreadsheet.** Enter the data into lists named *year* and *birth.*



3 Press ctrl + I and select **Add Data & Statistics**. Construct a scatterplot of *birth* against *year*.
As is the case for a time series plot, *year* is the explanatory variable and *birth* the response variable.



4 To display as a connected time series plot, move the cursor to the main graph area and press ctrl + menu **>Connect Data Points**. Press enter.



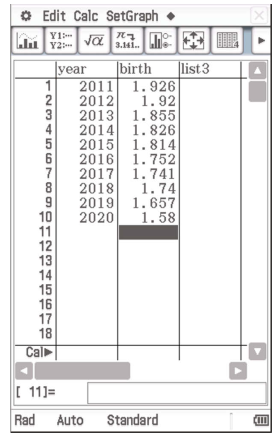
CAS 1: How to construct a time series using the ClassPad

Construct a time series plot for the data presented below, which shows the birth rate in Australia (in births per woman) from 2011–2020.

year	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
birth	1.926	1.920	1.855	1.826	1.814	1.752	1.741	1.740	1.657	1.580

Steps

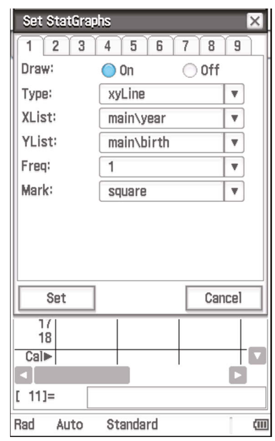
1 Open the **Statistics** application and enter the data into the columns named *year* and *birth*. Your screen should look like the one shown.



2 Tap to open the **Set StatGraphs** dialog box and complete as follows.

- **Draw:** select **On**.
- **Type:** select **xyLine** (▼).
- **XList:** select **main/year** (▼).
- **YList:** select **main/birth** (▼).
- **Freq:** leave as **1**.
- **Mark:** leave as **square**.

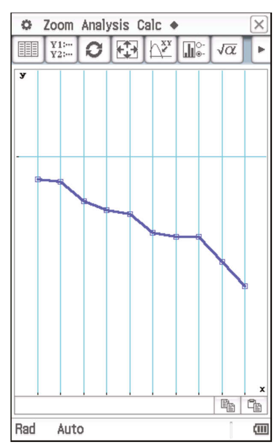
Tap **Set** to confirm your selections.



3 Tap in the toolbar at the top of the screen to display the time series plot in the bottom half of the screen.

To obtain a full-screen display, tap Resize from the icon panel.

Tap from the toolbar, and use ▶ and ◀ to move from point to point to read values from the plot.



Looking for patterns in time series plots

The features we look for in a time series are:

- trend
- cycles
- seasonality
- structural change
- possible outliers
- irregular (random) fluctuations.

We would always expect to see irregular or random fluctuations in a time series, and it is common to see one or more of the other features as well.

Trend

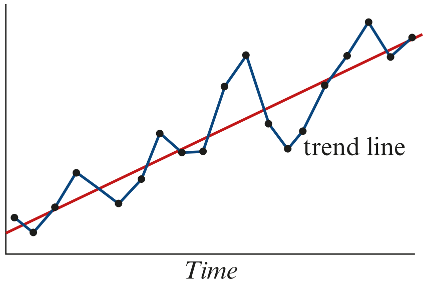
Examining a time series plot we can often see a general upward or downward movement over time. This indicates a long-term change over time that we call a trend.

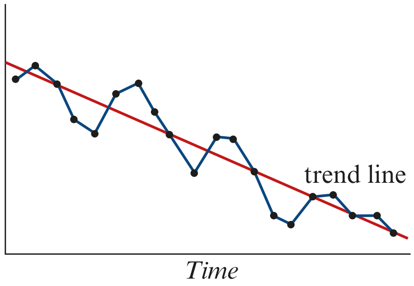
Trend

The tendency for values in a time series to generally increase or decrease over a significant period of time is called a **trend**.

One way of identifying trends on a time series graph is to draw a line that ignores the fluctuations, but which reflects the overall increasing or decreasing nature of the plot. These lines are called **trend lines**.

Trend lines have been drawn on the time series plots below to indicate an **increasing** trend (line slopes upwards) and a **decreasing** trend (line slopes downwards).

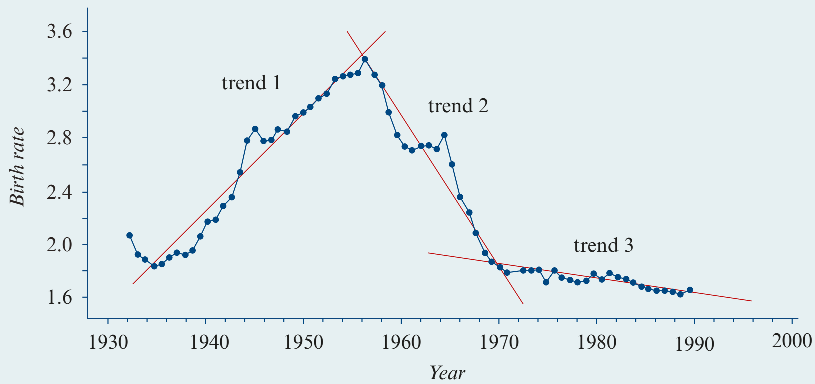




Sometimes, different trends are apparent in a time series for different time periods.

Example 2 Identifying trends

Consider the time series plot of the Australian annual birth rates over the years from 1931 to 1990, shown below. Comment on the trend shown in the plot.



Solution

There are three distinct trends, which can be seen by drawing trend lines on the plot. Each of these trends can be explained by changing socioeconomic circumstances.

Trend 1: Between 1940 and 1961 the birth rate in Australia grew quite dramatically. Those in the armed services came home from the Second World War, and the economy grew quickly. This rapid increase in the Australian birth rate during this period is known as the 'Baby Boom'.

Trend 2: From about 1962 until 1980 the birth rate declined very rapidly. Birth control methods became more effective, and women started to think more about careers. This period is sometimes referred to as the 'Baby Bust'.

Trend 3: During the 1980s, and beyond, the birth rate continued to decline slowly for a complex range of social and economic reasons.

Cycles

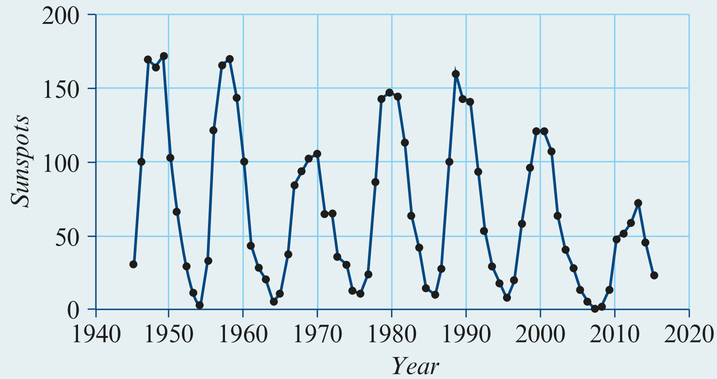
The term cycle refers to variations in time series that in general last longer than a year. These variations may not be of a regular height and they may not repeat at regular intervals.

Cycles

Cycles are recurrent movements in a time series, generally over a period greater than one year.

Example 3 Identifying cycles

Sunspots are darker, cooler area on the surface of the sun. The following plot shows the sunspot activity for the period 1945 to 2016. Comment on the cycles shown in the plot.



Solution

The recurrent pattern in the number of sunspots can be seen clearly from the time series plot. Looking at the plot the years of lowest sunspot activity look to be at approximately 1954, 1964, 1975, 1986, 1996, 2008.

Many business indicators, such as interest rates or unemployment figures, also vary in cycles, but their periods are usually less regular.

Seasonality

Cycles with calendar-related periods of one year or less are of special interest and are referred to as **seasonality**.

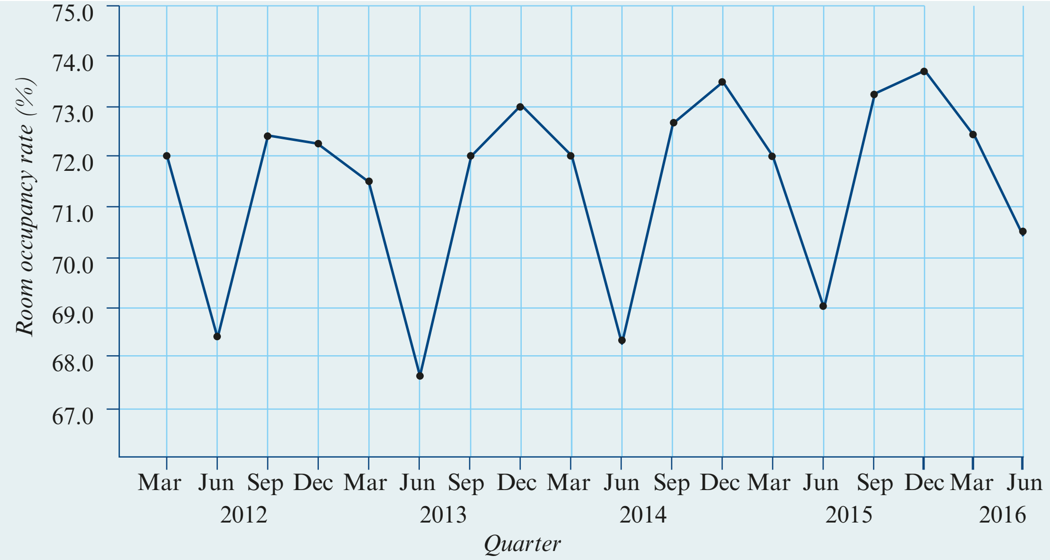
Seasonality

Seasonality is present when there is a periodic movement in a time series that is related to a calendar-related period – for example a year, a month or a week.

Seasonal movements tend to be more predictable than other time series features, and occur because of variations in the weather, such as ice-cream sales increasing in the summer, or institutional factors, like the increase in the number of unemployed people at the end of the school year.

Example 4 Identifying seasonality

The plot below shows the total percentage of hotel rooms occupied in Australia by quarter, over the years 2012–2016. Comment on the seasonality shown in the plot.



Solution

The regular peaks and troughs in the plot that occur at the same time each year signal the presence of **seasonality**. In this case, the demand for accommodation is at its lowest in the June quarter and highest in the December quarter.

This time series plot reveals both seasonality and trend in the demand for hotel rooms. The upward sloping trend line signals the presence of a general increasing **trend**. This tells us that, even though demand for accommodation has fluctuated from month to month, demand for hotel accommodation has increased over time.

Structural change

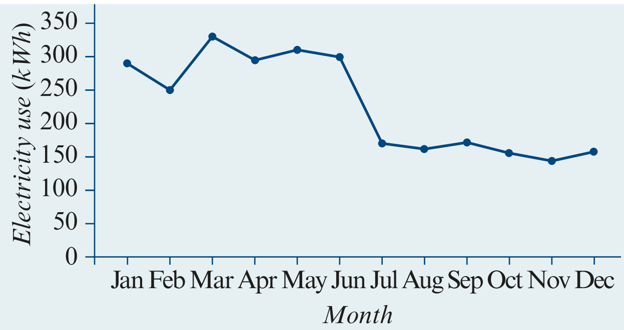
A **structural change** in a time series is a sudden change in the pattern of the time series at a point in time.

Structural change

Structural change is present when there is a sudden change in the established pattern of a time series plot.

Example 5 Identifying structural change

The time series plot below shows the power bill for a rental house (in kWh) for the 12 months of a year. Comment on any structural change in the plot.



Solution

The plot reveals an abrupt change in power usage in June to July. During this period, monthly power use suddenly decreases from around 300 kWh per month from January to June to around 175 kWh for the rest of the year. This is an example of structural change that can probably be explained by a change in circumstances, for example, from a family with children to a person living alone.

Structural change is also displayed in the birth rate time series plot we saw earlier. This revealed three quite distinct trends during the period 1931–1990. These reflect significant external events (like a war) or changes in social and economic circumstances.

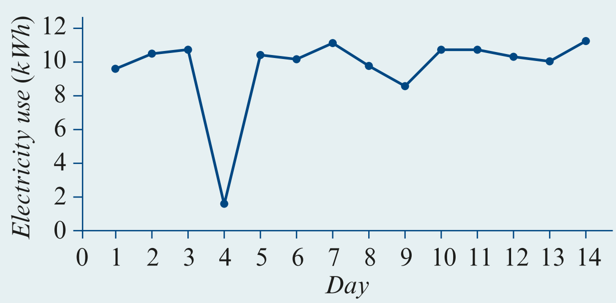
Outliers

Outliers

Outliers are individual values that stand out from the general body of data.

Example 6 Identifying outliers

The time series plot below shows the daily power bill for a house (in kWh) for a fortnight. Comment on any outliers in the plot.



Solution

For this household, daily electricity use follows a regular pattern that, although fluctuating, averages about 10 kWh per day. In terms of daily power use, day 4 is a clear outlier, with less than 2 kWh of electricity used. A follow-up investigation found that, on this day, the house was without power for 18 hours due to a storm, so much less power was used than normal.

Irregular (random) fluctuations

Irregular (random) fluctuations

Irregular (random) fluctuations include all the variations in a time series that we cannot reasonably attribute to systematic changes like trend, cycles, seasonality and structural change or an outlier. There will **always** be irregular, random variation present in any real world time series data.

There can be many sources of irregular fluctuations, mostly unknown. A general characteristic of these fluctuations is that they are unpredictable.

One of the aims of time series analysis is to develop techniques to identify regular patterns in time series plots that are often obscured by irregular fluctuations. One of these techniques is smoothing, which you will meet in the next section.

Identifying patterns in time series plots

The features we look for in a time series are:

- trend
- cycles
- seasonality
- structural change
- possible outliers
- irregular (random) fluctuations.

Trend is present when there is a *long-term* upward or downward movement in a time series.

Cycles are present when there is a periodic movement in a time series. The period is the time it takes for one complete up and down movement in the time series plot. In practice, this term is reserved for periods greater than 1 year.

Seasonality is present when there is a periodic movement in a time series that has a calendar related period – for example a year, a month or a week.

Structural change is present when there is a sudden change in the established pattern of a time series plot.

Outliers are present when there are individual values that stand out from the data.

Irregular (random) fluctuations are always present in any real-world time series plot. They include all the unexplained variations in a time series.

Exercise 5A

Constructing a time series plot

Note: A CAS calculator may be used to construct the time series plots. You may assign numerical values to the values of the time variable where convenient to do so.

Example 1

1 Construct a time series plot to display the following data.

Year	2015	2016	2017	2018	2019	2020	2021	2022
Sales	2	23	35	31	45	23	67	70

2 Researchers recorded the number of penguins present on a remote island each month for 12 months. Construct a time series plot of the data.

Month	*Number of penguins*	*Month*	*Number of penguins*
January	449	July	180
February	214	August	241
March	170	September	311
April	265	October	499
May	434	November	598
June	102	December	674

3 The following table shows the maximum temperature in Melbourne during one week in March. Construct a time series plot of the data.

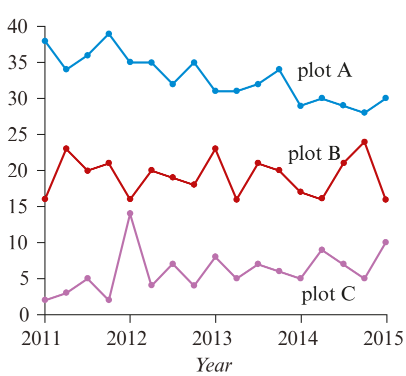
Day	Mon	Tues	Wed	Thur	Fri	Sat	Sun
Temperature (°C)	24.0	24.2	17.4	17.7	18.3	19.5	17.4

Identifying key features in a time series plot

Example 2

4 Complete the table below by indicating which of the listed features are present in each of the time series plots shown below.

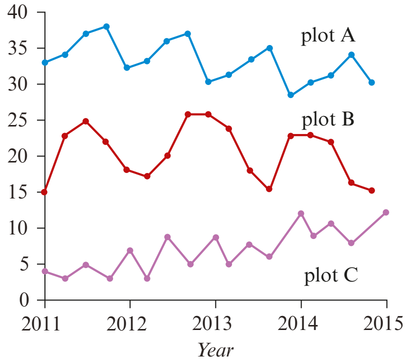
Feature	Plot		
	A	B	C
Irr fluctuations			
Increasing trend			
Decreasing trend			



Example 3
Example 4

5 Complete the table below by indicating which of the listed features are present in each of the time series plots shown below.

	Plot		
Feature	A	B	C
Irr fluctuations			
Increasing trend			
Decreasing trend			
Cycles			
Seasonality			



Example 5

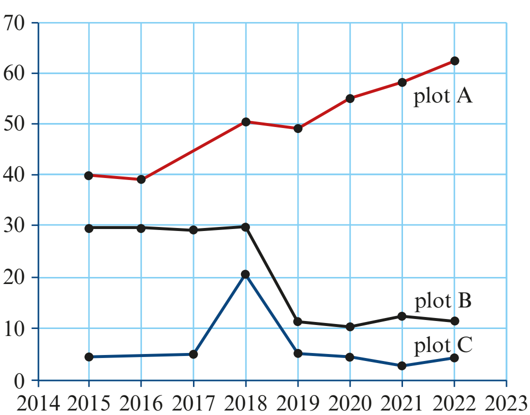
6 Complete the table below by indicating which of the listed features are present in each of the time series plots shown below.

	Plot		
Feature	A	B	C
Irr fluctuations			
Struc change			
Increasing trend			
Decreasing trend			
Seasonality			

Example 6

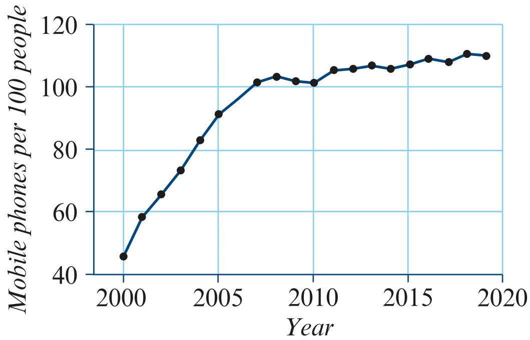
7 Complete the table below by indicating which of the listed features are present in each of the time series plots shown below.

	Plot		
Feature	A	B	C
Irr fluctuations			
Struc change			
Increasing trend			
Decreasing trend			
Outliers			



Describing time series plots

8 The time series plot for the number of mobile phones per 100 people from 2000–2019 is shown to the right. Describe the features of the time series plot.



9 The data below shows the population (in millions) in Australia over the period 2012–2021.

Year	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021
Population	22.73	23.13	23.48	23.82	24.19	24.60	24.98	25.37	25.69	25.97

a Construct a time series plot of the data.

b Describe the features of the plot.

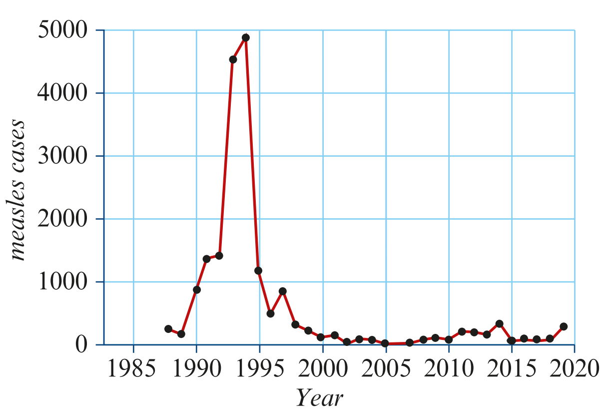
10 The table below shows the motor vehicle theft rate per 100 000 cars in Australia from 2003 to 2018.

Year	2003	2004	2005	2006	2007	2008	2009	2010
Theft rate	500.9	442.4	398.3	367.2	337.6	320.0	274.2	214.8
Year	2011	2012	2013	2014	2015	2016	2017	2018
Theft rate	220.0	228.4	204.2	191.0	194.5	231.0	213.3	214.1

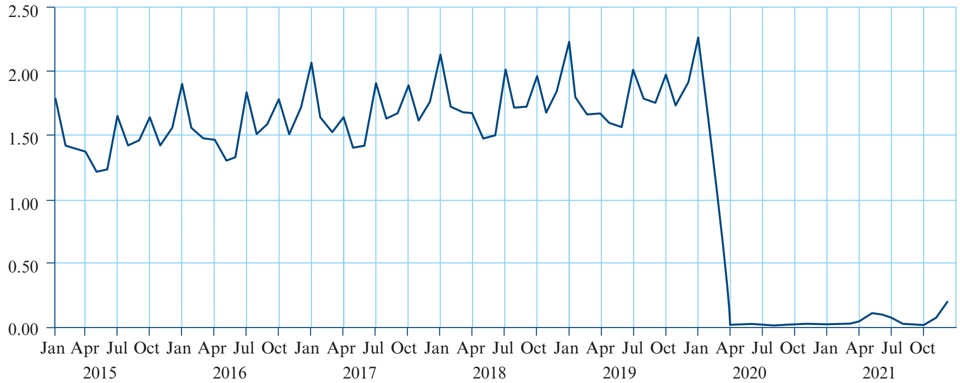
a Construct a time series plot of the data.

b Describe the features of the plot.

11 The time series plot below shows the number of measles cases reported in Australia from 1988 to 2019. Describe the features of the plot.



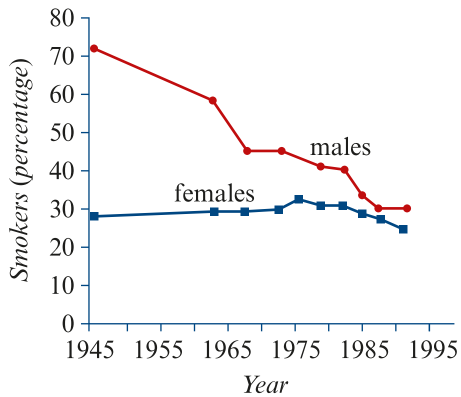
12 The time series plot below shows the number of overseas arrivals (millions of people per month) in Australia from November 2011 until December 2021. Describe the features of the plot.



13 **a** The time series plot shown shows the smoking rates (%) of Australian males and females over the period 1945–92.

i Describe any trends in the time series plot.

ii Did the *difference* in smoking rates increase or decrease over the period 1945–92?



b The table below shows the smoking rates for females and males aged 15 years at several time points from 2000–2018 (smoking rate data is not collected every year).

Year	2000	2005	2007	2010	2011	2012	2013	2018	2014	2015	2018
Female	22.4	18.9	19.9	17.9	15.4	16.6	14.4	15.6	13.5	14.5	13.6
Male	26.7	22.9	24.9	22.9	19.1	21.9	18.0	20.7	17.0	19.7	18.7

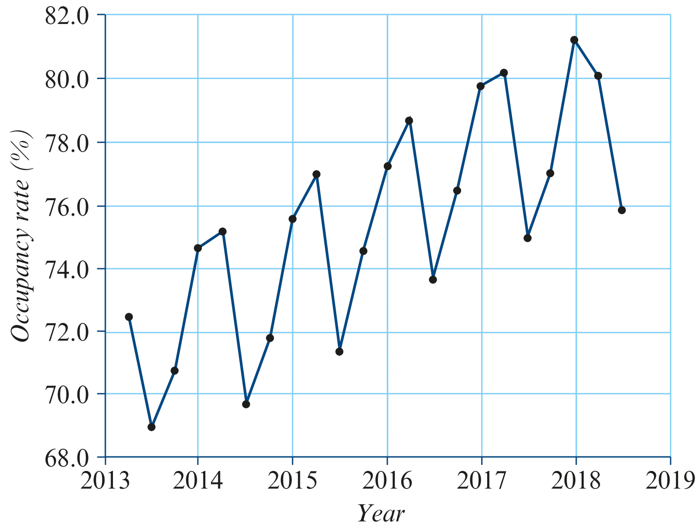
i Use a CAS calculator to construct time series plots of the male and female data.

ii Describe any trends in the time series plot.

iii Did the difference in smoking rates change over the period 2000–2018?

Exam 1 style questions

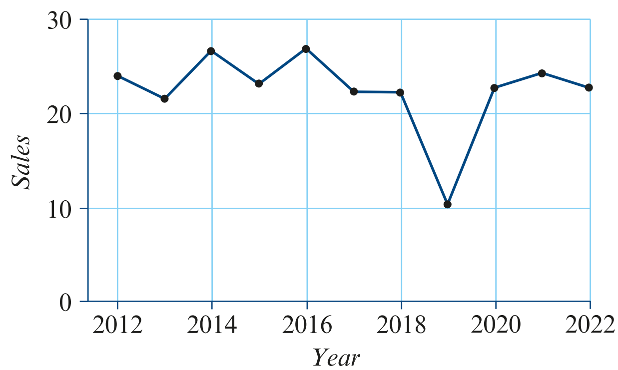
14 The time series plot below shows the quarterly room occupancy rate for a chain of hotels over the years 2013 to 2019.



The time series plot is best described as having

A seasonality only
B irregular fluctuations only
C seasonality with irregular fluctuations
D an increasing trend with irregular fluctuations
E an increasing trend with seasonality and irregular fluctuations

15 The time series plot below shows the annual sales (in $ millions) for a car sales company.



The time series plot is best described as having

A seasonality only
B irregular fluctuations only
C seasonality with irregular fluctuations
D irregular fluctuations with an outlier
E seasonality with an outlier

5B Smoothing a time series using moving means

Learning intentions

- ▶ To be able to smooth a time series plot using moving means.
- ▶ To be able to know when and how to use centring when smoothing.

A time series plot can incorporate many of the sources of variation previously mentioned: trend, cycles, seasonality, structural change, outliers and irregular fluctuations. One effect of the irregular fluctuations and seasonality can be to obscure an underlying trend. The technique of **smoothing** can sometimes be used to overcome this problem.

In this section we consider **moving mean smoothing**, which involves replacing individual data points in the time series with the mean of the data point and some adjacent points. The simplest method is to smooth over a small odd number of data points – for example, three or five, but any number of points can be used.

The three-moving mean

To use **three-moving mean smoothing**, replace each data value with the mean of that value and the one on each side. That is, if y_1, y_2 and y_3 are sequential data values, then:

$$\text{smoothed } y_2 = \frac{y_1 + y_2 + y_3}{3}$$

The first and last points in the time series do not have values on each side, so they are omitted.

The five-moving mean

To use **five-moving mean smoothing**, replace each data value with the mean of that value and the two values on each side. That is, if y_1, y_2, y_3, y_4, y_5 are sequential data values, then:

$$\text{smoothed } y_3 = \frac{y_1 + y_2 + y_3 + y_4 + y_5}{5}$$

The first two and last two points in the time series do not have two values on each side, so they are omitted.

These definitions can be readily extended for moving means involving more points.

Example 7 Three- and five-moving mean smoothing

The table below gives the temperature (°C) recorded at a weather station at 9.00 a.m. each day for a week.

Day	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Temperature	18.1	24.8	26.4	13.9	12.7	14.2	24.9

a Calculate the three-moving mean smoothed temperature for Tuesday.

b Calculate the five-moving mean smoothed temperature for Thursday.

Explanation	Solution
a **1** Write down the three temperatures centred on Tuesday.	18.1, 24.8, 26.4
2 Find their mean and write down your answer.	Mean $= \dfrac{(18.1 + 24.8 + 26.4)}{3} = 23.1$ The three-moving mean smoothed temperature for Tuesday is 23.1°C.
b **1** Write down the five temperatures centred on Thursday	24.8, 26.4, 13.9, 12.7, 14.2
2 Find their mean and write down your answer.	Mean $= \dfrac{(24.8 + 26.4 + 13.9 + 12.7 + 14.2)}{5}$ $= 18.4$ The five-moving mean smoothed temperature for Thursday is 18.4°C.

The next step is to extend these computations to smooth all terms in the time series.

Example 8 Three- and five-moving mean smoothing of a time series

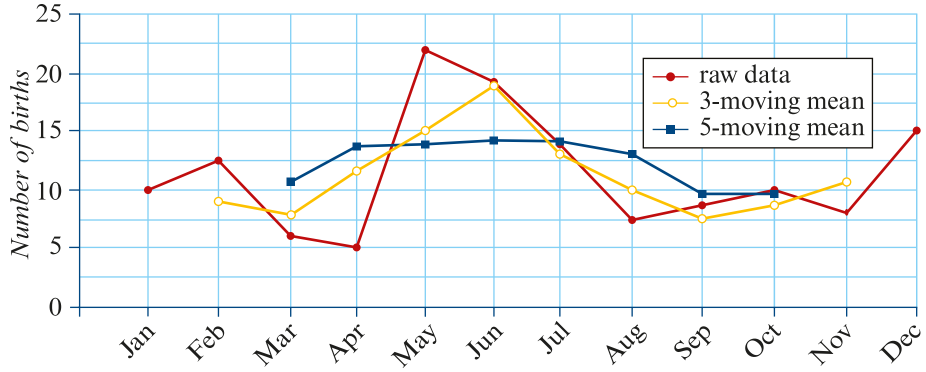
The following table gives the number of births per month over a calendar year in a country hospital. Use the three-moving mean and the five-moving mean methods, rounded to one decimal place, to complete the table.

Solution

Complete the calculations as shown below.

Month	*Number of births*	3-moving mean	5-moving mean
January	10		
February	12	$\dfrac{10+12+6}{3} = 9.3$	
March	6	$\dfrac{12+6+5}{3} = 7.7$	$\dfrac{10+12+6+5+22}{5} = 11.0$
April	5	$\dfrac{6+5+22}{3} = 11.0$	$\dfrac{12+6+5+22+18}{5} = 12.6$
May	22	$\dfrac{5+22+18}{3} = 15.0$	$\dfrac{6+5+22+18+13}{5} = 12.8$
June	18	$\dfrac{22+18+13}{3} = 17.7$	$\dfrac{5+22+18+13+7}{5} = 13.0$
July	13	$\dfrac{18+13+7}{3} = 12.7$	$\dfrac{22+18+13+7+9}{5} = 13.8$
August	7	$\dfrac{13+7+9}{3} = 9.7$	$\dfrac{18+13+7+9+10}{5} = 11.4$
September	9	$\dfrac{7+9+10}{3} = 8.7$	$\dfrac{13+7+9+10+8}{5} = 9.4$
October	10	$\dfrac{9+10+8}{3} = 9.0$	$\dfrac{7+9+10+8+15}{5} = 9.8$
November	8	$\dfrac{10+8+15}{3} = 11.0$	
December	15		

The result of this smoothing can be seen in the plot below, which shows the raw data, the data smoothed with a three-moving means and the data smoothed with a five-moving means.



Note: In the process of smoothing, **data points are lost** at the beginning and end of the time series.

Two observations can be made from this plot:

1 Five-moving mean smoothing is more effective in reducing the irregular fluctuations than three-mean smoothing.

2 The five-moving mean smoothed plot shows that there is no clear trend although the raw data suggest that there might be an increasing trend.

Moving mean smoothing with centring

If we smooth over an even number of points, we run into a problem. The centre of the set of points is not at a time point belonging to the original series. Usually, we solve this problem by using a process called **centring**.

Centring

Smoothing with centring involves taking a two-moving mean of the already smoothed values so that they line up with the original data values. Smoothing with centring is only required when smoothing using an **even** number of data values, for example 2-moving mean smoothing, or 4-moving mean smoothing.

We will illustrate the process by finding the two-moving mean, centred on Tuesday, for the daily temperature data opposite.

Day	*Temperature*
Monday	18.1
Tuesday	24.8
Wednesday	26.4

It is straightforward to calculate a series of two-moving means for this data by calculating the mean for Monday and Tuesday, followed by the mean for Tuesday and Wednesday. However, as we can see in the diagram below, these means do not align with a particular day, but lie between days. To solve this problem find the average of these two means, as shown in the following diagram.

Day	*Temperature*	*Two-moving means*	*Two-moving mean with centring*
Monday	18.1		
		$\frac{(18.1+24.8)}{2} = 21.45$	
Tuesday	24.8		$\frac{(21.45+25.6)}{2} = 23.525$
		$\frac{(24.8+26.4)}{2} = 25.60$	
Wednesday	26.4		

In practice, we do not have to draw such a diagram to perform these calculations. The purpose of doing so is to show how the centring process works. Calculating a two-moving mean with centring is illustrated in the following example.

Example 9 Two-moving mean smoothing with centring

The temperatures (°C) recorded at a weather station at 9 a.m. each day for a week are displayed in the table.

Day	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Temperature	18.1	24.8	26.4	13.9	12.7	14.2	24.9

Calculate the two-moving mean smoothed temperature with centring for Tuesday.

Explanation	**Solution**
1 For two-mean smoothing with centring, write down the **three** data values centred on Tuesday (highlighted in red).	18.1 24.8 26.4
2 Calculate the mean of the first two values (mean 1). Calculate the mean of the second two values (mean 2).	$\text{mean } 1 = \frac{(18.1 + 24.8)}{2} = 21.45$ $\text{mean } 2 = \frac{(24.8 + 26.4)}{2} = 25.60$
3 The centred mean is then the average of mean 1 and mean 2.	$\text{Centred mean} = \frac{(\text{mean } 1 + \text{mean } 2)}{2} = \frac{(21.45 + 25.60)}{2} = 23.525$
4 Write down your answer, rounded to one decimal place.	The two-moving mean smoothed temperature for Tuesday is 23.5°C.

The process of smoothing with centring across more data values is the same as two-mean smoothing except that the means are determined in larger groups. This process is illustrated in the following example with groups of four and six.

Example 10 Four- and six-moving mean smoothing with centring

The table below gives the temperature (°C) recorded at a weather station at 9.00 a.m. each day for a week.

Day	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Temperature	18.1	24.8	26.4	13.9	12.7	14.2	24.9

a Calculate the four-moving mean smoothed temperature with centring for Thursday.

b Calculate the six-moving mean smoothed temperature with centring for Thursday.

Explanation	Solution
a 1 For four-mean smoothing with centring, write down the **five** data values centred on Thursday.	24.8 26.4 13.9 12.7 14.2
2 Calculate the mean of the first four values (mean 1) and the mean of the last four values (mean 2).	$\text{mean 1} = \dfrac{(24.8 + 26.4 + 13.9 + 12.7)}{4} = 19.45$ $\text{mean 2} = \dfrac{(26.4 + 13.9 + 12.7 + 14.2)}{4} = 16.8$
3 The centred mean is then the average of mean 1 and mean 2.	$\text{centred mean} = \dfrac{(\text{mean 1} + \text{mean 2})}{2} = \dfrac{(19.45 + 16.8)}{2} = 18.125$
4 Write down your answer.	The four-mean smoothed temperature centred on Thursday is 18.1 °C (to 1 d.p.).
b 1 For six-mean smoothing with centring, write down the **seven** data values centred on Thursday.	18.1 24.8 26.4 13.9 12.7 14.2 24.9
2 Calculate the mean of the first six values (mean 1) and the mean of the last six values (mean 2).	$\text{mean 1} = \dfrac{(18.1 + 24.8 + 26.4 + 13.9 + 12.7 + 14.2)}{6} = 18.35$ $\text{mean 2} = \dfrac{(24.8 + 26.4 + 13.9 + 12.7 + 14.2 + 24.9)}{6} = 19.4833\ldots$

3 The centred mean is then the average of mean 1 and mean 2.	$\text{centred mean} = \dfrac{(\text{mean } 1 + \text{mean } 2)}{2}$ $= \dfrac{(18.35 + 19.483)}{2}$ $= 18.917$
4 Write down your answer.	The six-mean smoothed temperature centred on Thursday is 18.9 °C (to 1 d.p.).

The next step is to extend these computations to smooth all terms in the time series. This process is illustrated using four-moving mean smoothing in the following example. Setting up and using a table like the one shown in the example will help keep track of the process.

Example 11 Smoothing of a time series using four-mean smoothing with centring

The following table gives the number of births per month over a calendar year in a country hospital. Use the four moving mean with centring method to complete the table.

Solution

Complete the calculations as shown below.

Month	*Number of births*	4-moving mean	4-moving mean with centring
January	10		
February	12	$\dfrac{10+12+6+5}{4} = 8.25$	
March	6		$\dfrac{8.25+11.25}{2} = 9.75$
		$\dfrac{12+6+5+22}{4} = 11.25$	
April	5		$\dfrac{11.25+12.75}{2} = 12$
		$\dfrac{6+5+22+18}{4} = 12.75$	
May	22		$\dfrac{12.75+14.5}{2} = 13.625$
		$\dfrac{5+22+18+13}{4} = 14.5$	
June	18		$\dfrac{14.5+15}{2} = 14.75$
		$\dfrac{22+18+13+7}{4} = 15$	
July	13		$\dfrac{15+11.75}{2} = 13.375$
		$\dfrac{18+13+7+9}{4} = 11.75$	
August	7		$\dfrac{11.75+9.75}{2} = 10.75$
		$\dfrac{13+7+9+10}{4} = 9.75$	
September	9		$\dfrac{9.75+8.5}{2} = 9.125$
		$\dfrac{7+9+10+8}{4} = 8.5$	
October	10		$\dfrac{8.5+10.5}{2} = 9.5$
		$\dfrac{9+10+8+15}{4} = 10.5$	
November	8		
December	15		

Skill-sheet

Exercise 5B

Note: A CAS calculator may be used to construct the time series plots.

Calculating the smoothed values of an odd number of individual points

Example 7

1

t	1	2	3	4	5	6	7	8	9
y	5	2	5	3	1	0	2	3	0

For the time series data in the table above, find:

a the three-mean smoothed y-value for

i $t = 4$ **ii** $t = 6$ **iii** $t = 2$

b the five-mean smoothed y-value for

i $t = 3$ **ii** $t = 7$ **iii** $t = 4$

c the seven-mean smoothed y-value for

i $t = 4$ **ii** $t = 6$

d the nine-mean smoothed y-value for $t = 5$

2 The table below gives the temperature (°C) recorded at a weather station at 3.00 p.m. each day for a week.

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Temp (°C)	28.9	33.5	21.6	18.1	16.2	17.9	26.4

a Find the three-mean smoothed temperature for Wednesday.

b Find the five-mean smoothed temperature for Friday.

c Find the seven-mean smoothed temperature for Thursday.

Example 8

3 Complete the following table.

t	1	2	3	4	5	6	7	8	9
y	10	12	8	4	12	8	10	18	2
3-moving mean y	–								–
5-moving mean y	–	–						–	–

Smoothing and plotting a time series plot (odd number of points)

4 The maximum temperature of a city over a period of 10 days is given below.

Day	1	2	3	4	5	6	7	8	9	10
Temperature (°C)	24	27	28	40	22	23	22	21	25	26
3-moving mean		26.3		30.0		22.3		22.7	24.0	
5-moving mean			28.2		27			23.4		

a Construct a time series plot of the temperature data.

b Use the five-mean and seven-mean smoothing method to complete the table.

c Plot the smoothed temperature data and compare the plots.

5 The value of the Australian dollar in US dollars (exchange rate) over 10 days is given below.

Day	1	2	3	4	5	6	7	8	9	10
Exchange rate	0.743	0.754	0.737	0.751	0.724	0.724	0.712	0.735	0.716	0.711
3-moving mean		0.745	0.747		0.733			0.721	0.721	
5-moving mean				0.738	0.730		0.722			

a Construct a time series plot of the data.

b Use the three-mean and five-mean smoothing method to complete the table.

c Plot the smoothed exchange rate data and compare the plots.

Calculating the smoothed values of an even number of individual points

Example 9

6

t	1	2	3	4	5	6	7	8	9
y	5	2	5	3	1	0	2	3	0

For the time series data above, find the two-mean smoothed y-value centred at:

a $t = 3$ **b** $t = 8$

Example 10

7 Use the time series data in the table in Question 6 to find:

a the four-mean smoothed y-value centred at $t = 3$

b the four-mean smoothed y-value centred at $t = 6$

c the six-mean smoothed y-value centred at $t = 3$

d the six-mean smoothed y-value centred at $t = 6$

8 The table below gives the minimum daily temperature (°C) recorded at a weather station over a 10 day period.

Day	1	2	3	4	5	6	7	8	9	10
Temperature (°C)	8.9	3.5	11.6	14.1	12.5	13.3	6.4	8.5	9.1	4.5

a Find the two-mean smoothed temperature with centring for Day 5.

b Find the four-mean smoothed temperature with centring for Day 5.

c Find the six-mean smoothed temperature with centring for Day 5.

Smoothing and plotting a time series plot (even number of points)

Example 11

9 The table below gives the number of complaints recorded at a customer service centre over a 12 month period.

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sep	Oct	Nov	Dec
Number of complaints	10	12	6	5	22	18	13	7	9	10	8	15
2-moving mean		10.0	7.3	9.5	16.8	17.8	12.8	9.0		9.3	10.3	

a Construct a time series plot of the data.

b Show that the two-mean smoothed value with centring for September is equal to 8.8 (rounded to one decimal place).

c Plot the smoothed data and compare the plots.

10 The table below gives the amount of rain (in mm) recorded each month at a weather station.

Month	Apr	May	Jun	July	Aug	Sep	Oct	Nov	Dec
Rainfall (mm)	21.4	40.5	52.3	42.1	58.9	79.9	81.5	54.3	50.0
4-moving mean			43.8	53.4		67.1	67.5		

a Construct a time series plot of the data.

b Show that the four-mean smoothed value with centring for August is equal to 62.0 (rounded to one decimal place).

c Plot the smoothed data and compare the plots.

Exam 1 style questions

11 Hay Lam records the number of emails (*emails*) he receives over a one-week period.

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Emails	85	65	77				10

The numbers of *emails* he received on Thursday, Friday and Saturday are not shown.
The five-mean smoothed number of *emails* he received on Friday is 39.
The three-mean smoothed number of *emails* he received on Friday is:

A 36 **B** 39 **C** 40 **D** 42 **E** 45

12 The table shows the closing price (*price*) of a company's shares on the stock market over a 10 day period.

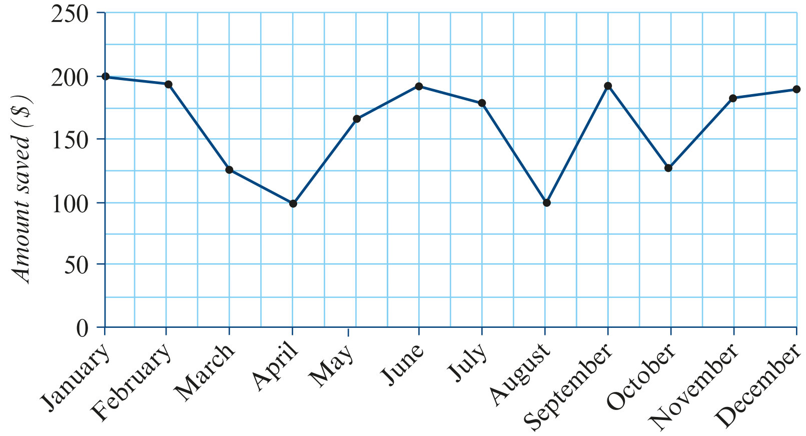
Day	1	2	3	4	5	6	7	8	9	10
Price($)	0.99	1.05	1.10	1.25	1.29	1.37	2.42	1.95	2.05	2.35

The six-mean smoothed with centring closing share *price* on Day 6 is closest to:

A \$1.56 **B** \$1.62 **C** \$1.64 **D** \$1.88 **E** \$1.72

The following information relates to Questions 13 and 14

The time series plot below shows the amount that Arnold saved each month (in dollars) over a 12 month period.



13 If he saved a total of \$831 over the period from May to September, the five-mean smoothed amount that he saved in July is closest to:

A \$277 **B** \$190 **C** \$182 **D** \$152 **E** \$166

14 If seven-mean smoothing is used to smooth this time series plot, the number of smoothed data points would be:

A 3 **B** 4 **C** 5 **D** 6 **E** 7

5C Smoothing a time series plot using moving medians

Learning intentions

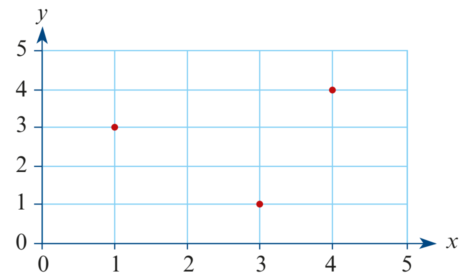
- ▶ To be able to locate the median of a data set graphically.
- ▶ To be able to smooth a time series plot using moving medians.

Another simple and convenient method of smoothing a time series is to use **moving median smoothing**. The advantage of this method is that it can be done directly on the graph without needing to know the exact values of each data point.[1] However, before smoothing a time series plot graphically using moving medians we will first need to know how to locate medians graphically.

[1] Note that, in this course, median smoothing is restricted to smoothing over an odd number of points, so centring is not required.

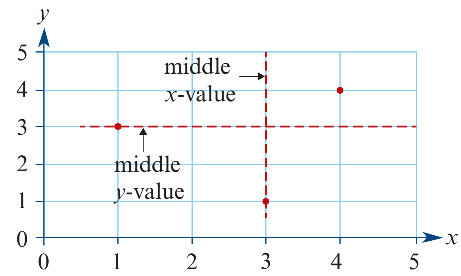
Locating medians graphically

The graph opposite shows three data points plotted on a set of coordinate axes. The task is to locate the median of these three points. The median will be a point somewhere on this set of coordinate axes. To locate this point we proceed as follows.



Step 1

Identify the middle data point moving in the x-direction. Draw a vertical line through this value as shown.



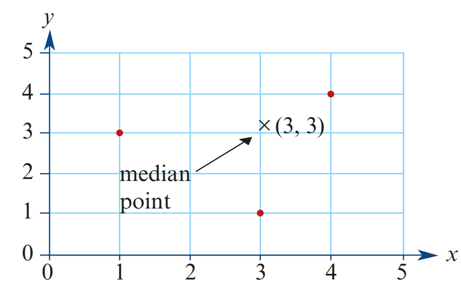
Step 2

Identify the middle data point moving in the y-direction. Draw a horizontal line through this value as shown.

Step 3

The median value is where the two lines intersect – in this case, at the point (3, 3).

Mark this point with a cross (×).

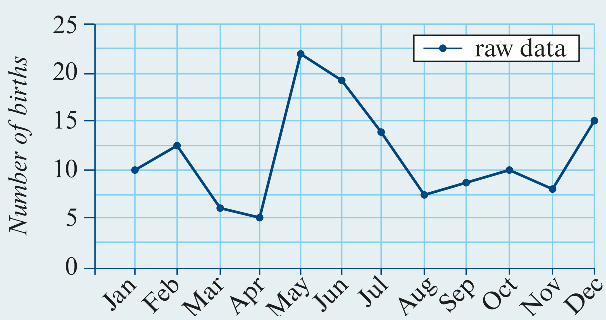


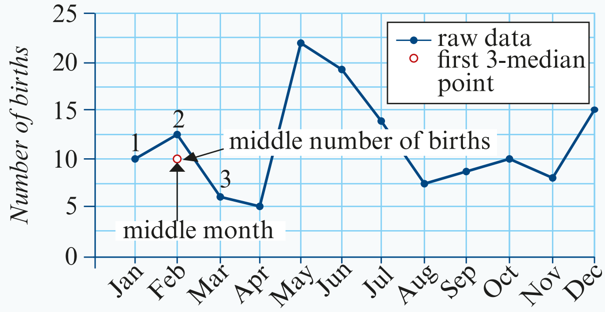
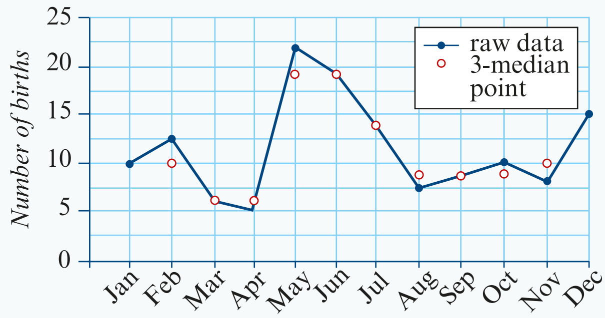
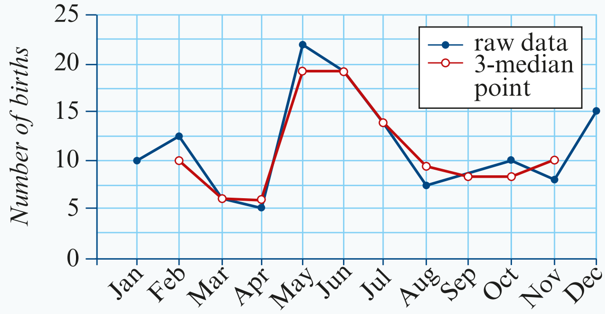
Smoothing a time series using moving median smoothing

The process of graphically smoothing a time series plot requires no more than repeating the above process for each group of three or five data points in the plot as required. The following worked examples demonstrate the process.

Example 12 **Three-moving median smoothing using a graphical approach**

Construct a three-median smoothed plot of the time series plot shown opposite.

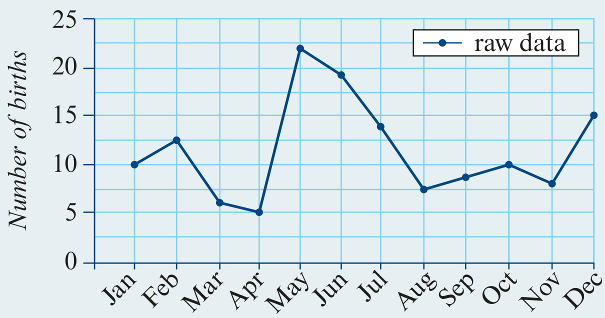


Explanation	Solution
1 Locate on the time series plot the median of the *first* three points (Jan, Feb, Mar).	
2 Continue this process by moving onto the next three points to be smoothed (Feb, Mar, Apr). Mark their medians on the graph, and continue the process until you run out of groups of three.	
3 Join the median points with a line segment – see opposite.	

Example 13 Five-moving median smoothing using a graphical approach

Construct a five-median smoothed plot of the time series plot shown opposite.

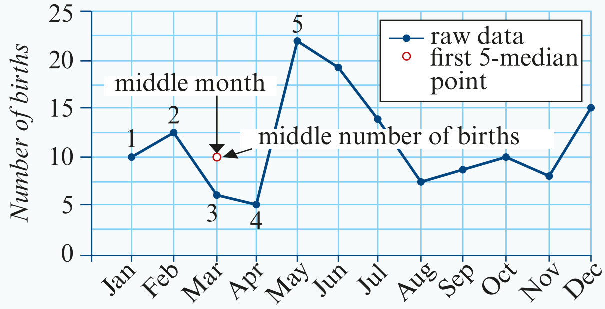
Note: The starting point for a median smoothing is a time series plot and you smooth directly onto the plot. Copies of the plots in this section can be accessed through the skillsheet icon in the Interactive Textbook.



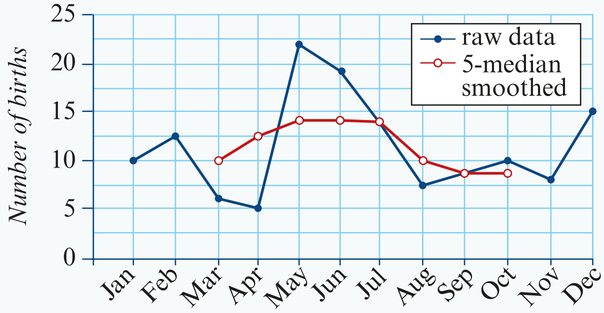
Explanation

1 Locate on the time series plot the median of the first five points (Jan, Feb, Mar, Apr, May), as shown.

Solution



2 Then move onto the next five points to be smoothed (Feb, Mar, Apr, May, Jun). Repeat the process until you run out of groups of five points. The five-median points are then joined up with line segments to give the final smoothed plot, as shown.



Note: The five-median smoothed plot is much smoother than the three-median smoothed plot.

Skill-sheet

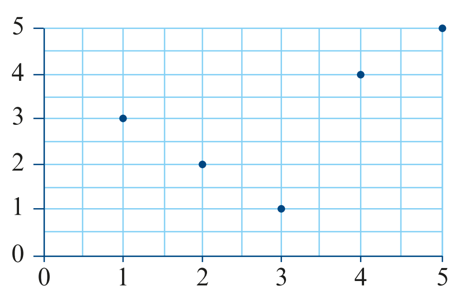
Exercise 5C

Note: Copies of all plots in this section can be accessed through the skillsheet icon in the Interactive Textbook.

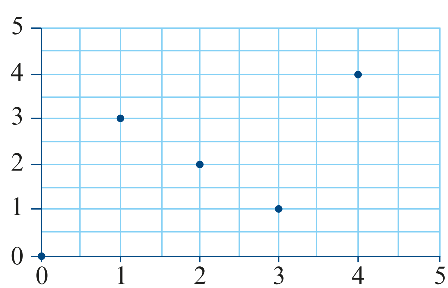
Locating the median of a set of data points graphically

1 Mark the location of the median point for each of the sets of data points below.

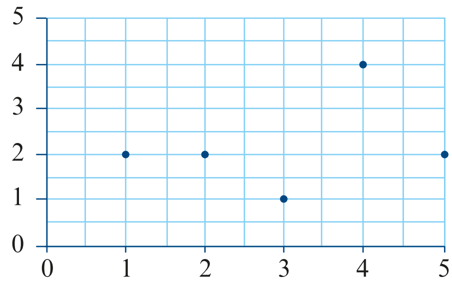
a



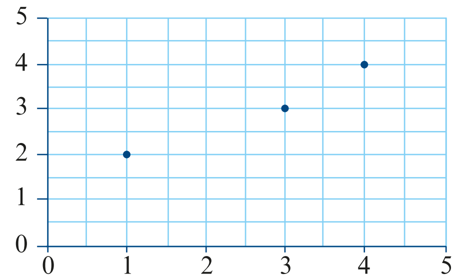
b



c



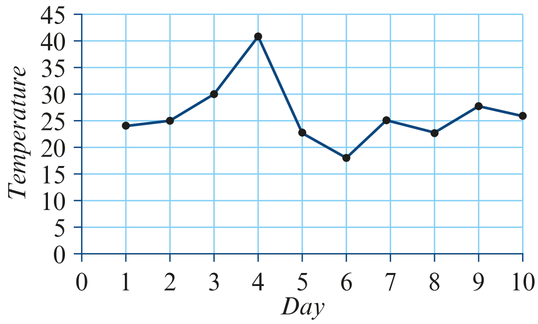
d



Smoothing a time series graphically

Example 12

2 The time series plot below shows the maximum daily temperatures (in °C) in a city over a period of 10 consecutive days.

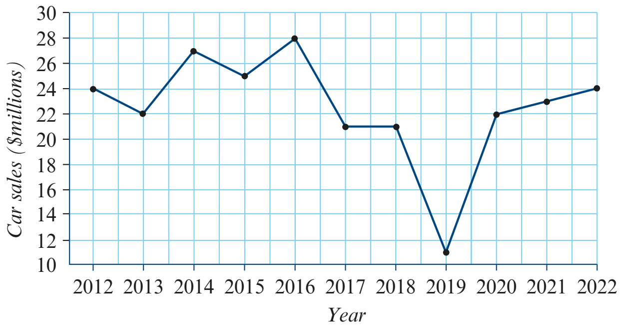


Use three-median smoothing to determine the smoothed temperature for:

a day 4

b day 8

3 The time series plot below shows the annual sales (in $ millions) for a car sales company. Use three-moving median smoothing to graphically smooth the plot and comment on the smoothed plot.



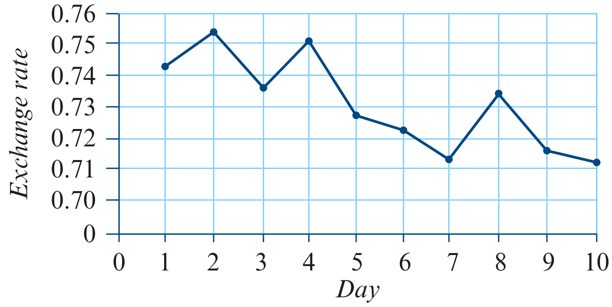
Example 13

4 Use the time series plot in Question 2 to find the five-median smoothed temperature for:

a day 4

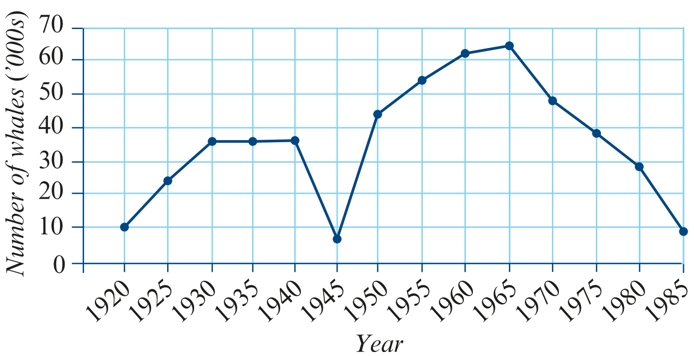
b day 8

5 The time series plot opposite shows the value of the Australian dollar in US dollars (the exchange rate) over a period of 10 consecutive days in 2009.
Use five-moving median smoothing to graphically smooth the plot and comment on the smoothed plot.

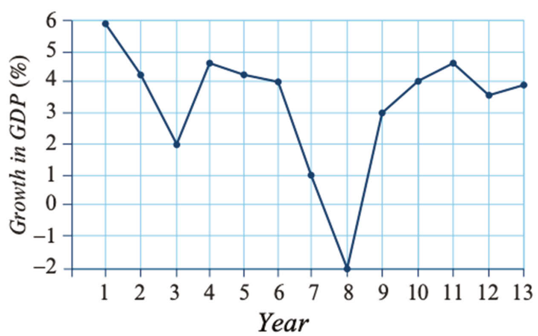


6 Use the graphical approach to smooth the time series plot below using:

a three-moving median smoothing **b** five-moving median smoothing.



7 The time series plot opposite shows the percentage growth of GDP (gross domestic product) over a 13-year period.



a Find the median value of the percentage growth in GDP over the 13 year period.

b Smooth the times series graph using:

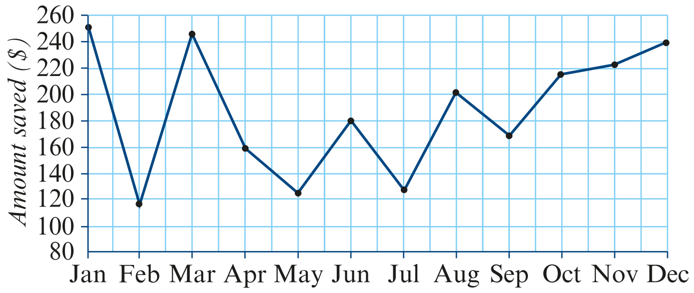
i three-moving median smoothing **ii** five-moving median smoothing.

c What conclusions can be drawn about the variation in GDP growth from these smoothed time series plots?

Exam 1 style questions

Use the following information to answer Questions 8 to 10

The time series plot below shows the amount that Lulu saved each month (to the nearest $) over a 12 month period.



8 During the year shown in the time series plot the median monthly amount Lulu saved is closest to:

A \$180 **B** \$155 **C** \$130 **D** \$190 **E** \$200

9 The five-median smoothed amount saved by Lulu in July is closest to:

A \$130 **B** \$150 **C** \$170 **D** \$190 **E** \$200

10 The nine-median smoothed amount saved by Lulu in August is closest to:

A \$132 **B** \$160 **C** \$168 **D** \$180 **E** \$218

5D Seasonal indices

Learning intentions

- ► To be able to interpret the meaning of seasonal indices.
- ► To be able to seasonally adjust data using seasonal indices.
- ► To be able to calculate seasonal indices from time series data.

When the data is considered to have a seasonal component, it is often necessary to remove this component so any underlying trend is clearer. The process of removing the seasonal component is called **deseasonalising** the data. To do this we need to calculate **seasonal indices**. Seasonal indices tells us how a particular season (generally a day, month or quarter) compares to the average season.

The concept of a seasonal index

Consider the (hypothetical) monthly seasonal indices for unemployment given in the table.

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sept	Oct	Nov	Dec	Total
1.1	1.2	1.1	1.0	0.95	0.95	0.9	0.9	0.85	0.85	1.1	1.1	12.0

Key fact 1

Seasonal indices are calculated so that their **average** is 1. This means that the sum of the seasonal indices equals the number of seasons. Thus, if the seasons are months, the seasonal indices add to 12. If the seasons are quarters, then the seasonal indices add to 4, and so on.

Key fact 2

Seasonal indices tell us how a particular season (generally a day, month or quarter) compares to the average season.

For example:

- seasonal index for unemployment for the month of February is 1.2 or 120%.
  This tells us that February unemployment figures tend to be 20% higher than the monthly average. Remember, the average seasonal index is 1 or 100%.
- seasonal index for August is 0.90 or 90%.
  This tells us that the August unemployment figures tend to be only 90% of the monthly average. Alternatively, August unemployment figures are 10% lower than the monthly average.

Example 14 Interpreting seasonal indices

Suppose that the seasonal indices (SI) for electricity usage in Esse's home are as shown in the table:

Summer	Autumn	Winter	Spring
1.16	0.94	1.26	0.64

a What does the seasonal index for Winter tell us?

b What does the seasonal index for Spring tell us?

Solution

a The seasonal index for Winter is 1.26. This tells us that Esse's electricity usage in Winter is typically 26% higher than the average season.

b The seasonal index for Spring is 0.64. This tells us that Esse's electricity usage in Spring is typically 36% lower than the average season.

Using seasonal indices to seasonally adjust a time series

We can use seasonal indices to remove the seasonal component (**deseasonalise**) from a time series, or to put it back in (**reseasonalise**).When we do this we are said to **seasonally adjust** the data.

To calculate deseasonalised figures, each entry is divided by its seasonal index as follows.

Deseasonalising data

Time series data are deseasonalised using the relationship:

$$\text{deseasonalised figure} = \frac{\text{actual figure}}{\text{seasonal index}}$$

The rule for determining deseasonalised data values can also be used to reseasonalise data – that is, convert a deseasonalised value into an actual data value.

Reseasonalising data

Time series data are reseasonalised using the rule:

$$\text{actual figure} = \text{deseasonalised figure} \times \text{seasonal index}$$

Example 15 Using seasonal indices

The seasonal indices (SI) for cold drink sales for Imogen's kiosk are as shown in the table:

Summer	Autumn	Winter	Spring
1.75	0.66	0.46	1.13

a If the actual actual cold drink sales last summer totalled \$21 653, what is the deseasonalised sales figure for that time period?

b If the deseasonalised cold drink sales last spring totalled \$10 870, what were the actual sales for that time period?

Explanation

a To deseasonalise we divide by the seasonal index for Summer (1.75)

b To find the actual sales we multiply by the seasonal index for Spring (1.13).

Solution

$$\text{Deseasonalised sales} = \frac{21\,653}{1.75} = \$12\,373.14$$

$$\text{Actual sales} = 10\,870 \times 1.13 = \$12\,283.10$$

Example 16 **Using seasonal indices to determine percentage change required to correct for seasonality**

Consider the table below which gives the seasonal indices for heater sales at a discount store:

Summer	Autumn	Winter	Spring
0.65	1.25	1.35	0.0.75

a By what percentage should the sales in summer be increased or decreased in order to deseasonalise the data? Give your answer as a percentage rounded to one decimal place.

b By what percentage should the sales in winter be increased or decreased in order to deseasonalise the data? Give your answer as a percentage rounded to one decimal place.

Explanation

a **1** Insert the seasonal index for summer into the rule

$$\text{deseasonalised sales} = \frac{\text{actual sales}}{\text{seasonal index}}$$

Solution

In general for summer:

$$\text{deseasonalised sales} = \frac{\text{actual sales}}{0.65}$$

$$= \frac{1}{0.65} \times \text{actual sales}$$

$$= 1.538 \times \text{actual sales}$$

2 Convert 1.538 into a percentage increase or decrease. Write the answer in a sentence.

Multiplying the actual sales by 1.538 is the equivalent of increasing the actual sales by 53.8%.

To correct for seasonality, the actual sales should be increased by 53.8%.

b **1** Insert the seasonal index for winter into the rule

$$\text{deseasonalised sales} = \frac{\text{actual sales}}{\text{seasonal index}}$$

In general for winter:

$$\text{deseseasonalised sales} = \frac{\text{actual sales}}{1.35}$$

$$= \frac{1}{1.35} \times \text{actual sales}$$

$$= 0.741 \times \text{actual sales}$$

2 Convert 0.741 into a percentage increase or decrease. Write the answer in a sentence.

Multiplying the actual sales by 0.741 is the equivalent of decreasing the actual sales by (100%-74.1%) = 25.9%.

To correct for seasonality, the actual sales should be increased by 25.9%.

Calculating seasonal indices

To complete this section, we will describe how to calculate a seasonal index. We will start by using only one year's data to illustrate the basic ideas and then move onto a more realistic example where several years' data are involved.

Example 17 **Calculating seasonal indices (1 year's data)**

Mikki runs a shop and she wishes to determine quarterly seasonal indices for the number of customers to her shop based on last year's figures which are shown in the table opposite.

Summer	Autumn	Winter	Spring
920	1085	1241	446

Explanation

1 The seasons are quarters. Write the formula in terms of quarters.

2 Find the quarterly average for the year.

3 The seasonal index (SI) for each quarter is the ratio of that quarter's sales to the average quarter.

4 Check that the seasonal indices sum to 4 (the number of seasons). The slight difference is due to rounding.

5 Write out your answers as a table of the seasonal indices.

Solution

$$\text{Seasonal index} = \frac{\text{value for season}}{\text{seasonal average}}$$

$$\text{Quarterly average} = \frac{920 + 1085 + 1241 + 446}{4}$$
$$= 923$$

$$SI_{Summer} = \frac{920}{923} = 0.997$$

$$SI_{Autumn} = \frac{1085}{923} = 1.176$$

$$SI_{Winter} = \frac{1241}{923} = 1.345$$

$$SI_{Spring} = \frac{446}{923} = 0.483$$

Check: $0.997 + 1.176 + 1.345 + 0.483 = 4.001$

Seasonal indices

Summer	Autumn	Winter	Spring
0.997	1.176	1.345	0.483

The next example illustrates how seasonal indices are calculated with 3 years' data. While the process looks more complicated, we just repeat what we did in the previous example three times and average the results for each year at the end.

Example 18 Calculating seasonal indices (several years' data)

Suppose that Mikki has 3 years of data, as shown. Use the data to calculate seasonal indices, rounded to two decimal places.

Year	Summer	Autumn	Winter	Spring
1	920	1085	1241	446
2	1035	1180	1356	541
3	1299	1324	1450	659

Solution

The strategy is as follows:

- Calculate the seasonal indices for years 1, 2 and 3 separately. As we already have the seasonal indices for year 1 in the previous example we will save ourselves some time by simply quoting the result.
- Average the three sets of seasonal indices to obtain a single set of seasonal indices.

Explanation

1 Write down the result for year 1.

Solution

Year 1 seasonal indices:

Summer	Autumn	Winter	Spring
0.997	1.176	1.345	0.483

2 Now calculate the seasonal indices for year 2.

a The seasons are quarters. Write the formula in terms of quarters.

$$\text{Seasonal index} = \frac{\text{value for quarter}}{\text{quarterly average}}$$

b Find the quarterly average for the year.

$$\text{Quarterly average} = \frac{1035 + 1180 + 1356 + 541}{4} = 1028$$

c Work out the seasonal index (SI) for each time period.

$$SI_{Summer} = \frac{1035}{1028} = 1.007$$

$$SI_{Autumn} = \frac{1180}{1028} = 1.148$$

$$SI_{Winter} = \frac{1356}{1028} = 1.319$$

$$SI_{Spring} = \frac{541}{1028} = 0.526$$

d Check that the seasonal indices sum to 4.

Check: $1.007 + 1.148 + 1.319 + 0.526 = 4.000$

e Write out your answers as a table of the seasonal indices.

Year 2 seasonal indices:

Summer	Autumn	Winter	Spring
1.007	1.148	1.319	0.526

3 Now calculate the seasonal indices for year 3.

a Find the quarterly average for the year.

$$\text{Quarterly average} = \frac{1299 + 1324 + 1450 + 659}{4} = 1183$$

b Work out the seasonal index (SI) for each time period.

$$SI_{Summer} = \frac{1299}{1183} = 1.098$$

$$SI_{Autumn} = \frac{1324}{1183} = 1.119$$

$$SI_{Winter} = \frac{1450}{1183} = 1.226$$

$$SI_{Spring} = \frac{659}{1183} = 0.557$$

c Check that the seasonal indices sum to 4.

Check: $1.098 + 1.119 + 1.226 + 0.557 = 4.000$

d Write out your answers as a table of the seasonal indices.

Year 3 seasonal indices:

Summer	Autumn	Winter	Spring
1.098	1.119	1.226	0.557

4 Find the 3-year averaged seasonal indices by averaging the seasonal indices for each season.

Final seasonal indices:

$$S_{Summer} = \frac{0.997 + 1.007 + 1.098}{3} = 1.03$$

$$S_{Autumn} = \frac{1.176 + 1.148 + 1.119}{3} = 1.15$$

$$S_{Winter} = \frac{1.345 + 1.319 + 1.226}{3} = 1.30$$

$$S_{Spring} = \frac{0.483 + 0.526 + 0.557}{3} = 0.52$$

5 Check that the seasonal indices sum to 4.

Check: $1.03 + 1.15 + 1.30 + 0.52 = 4.00$

6 Write out your answers as a table of the seasonal indices.

Summer	Autumn	Winter	Spring
1.03	1.15	1.30	0.52

Interpreting the seasonal indices

Having calculated these seasonal indices, what do they tell about the previous situation?

The seasonal index of:

- 1.03 tells us that in summer, customer numbers are typically 3% above average.
- 1.15 tells us that in autumn, customer numbers are typically 15% above average.
- 1.30 tells us that in winter, customer numbers are typically 30% above average.
- 0.52 tells us that in spring, customer numbers are typically 48% below average.

Using seasonal indices to deseasonalise a time series

Once we have determined the seasonal indices we can use the rule for deseasonalising the time series introduced earlier on this section

$$\text{deseasonalised figure} = \frac{\text{actual figure}}{\text{seasonal index}}$$

to deseasonalise the data.

Example 19 **Deseasonalising a time series**

The quarterly sales figures for Mikki's shop over a 3-year period are given below.

Year	Summer	Autumn	Winter	Spring
1	920	1085	1241	446
2	1035	1180	1356	541
3	1299	1324	1450	659

Use the seasonal indices shown to deseasonalise these sales figures. Write answers rounded to the nearest whole number.

Summer	Autumn	Winter	Spring
1.03	1.15	1.30	0.52

Explanation

1 To deseasonalise each sales figure in the table, divide by the appropriate seasonal index.
For example, for summer, divide the figures in the 'Summer' column by 1.03.
Round results to the nearest whole number.

2 Repeat for the other seasons.

Solution

Deseasonalised Summer sales:

Year 1: $\frac{920}{1.03} = 893$

Year 2: $\frac{1035}{1.03} = 1005$

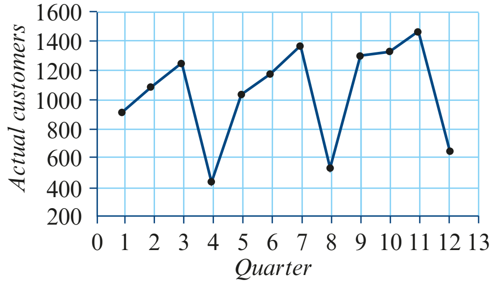
Year 3: $\frac{1299}{1.03} = 1261$

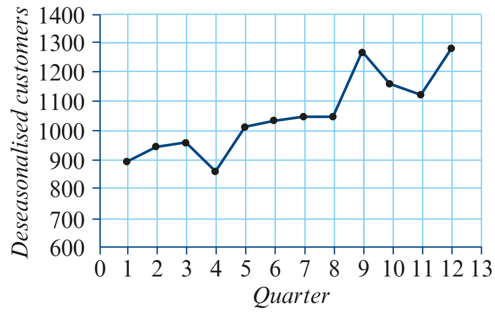
Deseasonalised sales figures

Year	Summer	Autumn	Winter	Spring
1	893	943	955	858
2	1005	1026	1043	1040
3	1261	1151	1115	1267

Why deseasonalise?

The purpose of removing the seasonality component of a time series is generally so that any trend in the time series is clearer. Consider again the actual customer data, and the deseasonalised customer data from Example 18, both of which are shown in the following time series plots.





It is hard to see from the first plot whether there has been any growth in Mikki's business, but the deseasonalised plot reveals revealed a clear underlying trend in the data.

It is common to deseasonalise time series data before you fit a trend line. We will consider this further in the next section.

Skill-sheet

Exercise 5D

Basic skills and interpretation

Use the following information to answer Questions 1 to 4.

The table below shows the monthly sales figures (in \$'000s) and seasonal indices (for January to November) for a product produced by the U-beaut company.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Sales (\$'000s)	9.6	10.5	8.6		7.1	6.0	5.4		6.4	7.2	8.3	7.4
Seasonal index	1.2	1.3	1.1	1.0	1.0	0.9	0.8	0.7	0.9	1.0	1.1	

Example 14

1 **a** Find the seasonal index for December.
b Interpret the seasonal index for February.
c Interpret the seasonal index for September.

Example 15

2 **a** Find the deseasonalised sales figure (in \$'000s) for March, giving your answer rounded to one decimal place.
b Find the deseasonalised sales figure (in \$'000s) for June, giving your answer rounded to one decimal place.

3 **a** The deseasonalised sales figure (in \$'000s) for August is 5.6. Find the actual sales (in \$'000s), giving your answer rounded to one decimal place.

b The deseasonalised sales figure (in \$'000s) for April is 6.9. Find the actual sales (in \$'000s), giving your answer rounded to one decimal place.

Example 16

4 **a** By what percentage should the sales in August be increased or decreased in order to correct for seasonality? Give your answer as a percentage rounded to one decimal place.

b By what percentage should the sales in February be increased or decreased in order to correct for seasonality? Give your answer as a percentage rounded to one decimal place.

5 The table below shows the quarterly newspaper sales (in \$'000s) of a corner store. Also shown are the seasonal indices for newspaper sales for the first, second and third quarters.

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Sales		1060	1868	1642
Seasonal index	0.8	0.7	1.3	

a Find the seasonal index for quarter 4.

b Find the deseasonalised sales (in \$'000s) for quarter 2.

c Find the deseasonalised sales (in \$'000s) for quarter 3.

d The deseasonalised sales (in \$'000s) for quarter 1 are 1256. Find the actual sales.

Deseasonalising a time series

6 The following table shows the number of students enrolled in a 3-month computer systems training course along with some seasonal indices that have been calculated from the previous year's enrolment figures. Complete the table by calculating the seasonal index for spring and the deseasonalised student numbers for each course.

	Summer	Autumn	Winter	Spring
Number of students	56	125	126	96
Deseasonalised numbers				
Seasonal index	0.5	1.0	1.3	

7 The number of waiters employed by a restaurant chain in each quarter of 1 year, along with some seasonal indices that have been calculated from the previous years' data, are given in the following table.

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Number of waiters	198	145	86	168
Seasonal index	1.30		0.58	1.10

a What is the seasonal index for the second quarter?

b The seasonal index for quarter 1 is 1.30. Explain what this means in terms of the average quarterly number of waiters.

c Deseasonalise the data.

Calculating seasonal indices

Example 17

8 The table below records quarterly sales (in \$'000s) for a shop.

Quarter 1	Quarter 2	Quarter 3	Quarter 4
60	56	75	78

Use the data to determine the seasonal indices for the four quarters. Give your results rounded to two decimal places.

9 The table below records the monthly visitors (in '000s) to a museum over one year.

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
12	13	14	17	18	15	9	10	8	11	15	20

Use the data to determine the seasonal indices for the 12 months. Give your results rounded to two decimal places.

Example 18

10 The table below records the monthly sales (in \$'000s) for a shop over a two year period.

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
22	19	25	23	20	18	20	15	14	11	23	30
21	20	23	25	22	17	19	17	16	11	25	31

Use the data to determine the seasonal indices for the 12 months. Give your results rounded to two decimal places.

Example 19

11 The daily number of cars carried on a car ferry service each day over a two-week period, together with the daily seasonal indices, are shown in the table below:

Week	Mon	Tues	Wed	Thur	Fri	Sat	Sun
1	124	110	45	67	230	134	330
2	120	108	57	74	215	150	345
Seasonal index	0.8	0.7	0.3	0.5	1.5	1.0	2.2

a Use the seasonal indices to deseasonalise the data, rounding the answers to the nearest whole number.

b Use your calculator to construct a time series plot the the deseasonalised data.

12 The numbers of retail job vacancies advertised on an online job board each quarter in each of three consecutive years are shown in the following table.

Year	Quarter 1	Quarter 2	Quarter 3	Quarter 4
1	212	194	196	227
2	220	197	196	239
3	231	205	203	245

a Construct a time series plot of the data.

b Use the data to calculate seasonal indices, rounded to two decimal places.

c Use the seasonal indices to construct a table of the deseasonalised data.

d Construct a time series plot of the deseasonalised data.

Exam 1 style questions

Use the following information to answer Questions 13 to 15.

The table below shows the number of customers each month at a restaurant and the long term seasonal indices for the number of customers at the restaurant each month of the year. The number of customers for August is missing.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Seasonal index	1.36	1.19	1.05	1.01	0.93	0.82	0.75	0.68	0.87	0.9	1.18	1.26
Number of customers	934	836	736	716	649	554	541		598	626	826	873

13 To correct the number of customers in May for seasonality, the actual number of customers should be:

A increased by 93.0% **B** decreased by 93.0% **C** decreased by 7.0%
D decreased 7.5% **E** increased by 7.5%

14 To correct the number of customers in November for seasonality, the actual number of customers should be:

A increased by 18.0% **B** decreased by 84.7% **C** increased by 15.3%
D decreased 15.3% **E** decreased by 18.0%

15 If the deseasonalised number of customers for August is 700, the actual number of customers in that month is closest to:

A 1029 **B** 768 **C** 570 **D** 607 **E** 476

16 The table below records the monthly average electricity cost (in dollars) for a home.

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
223	190	253	236	201	189	203	153	143	111	235	307

The seasonal index for August is closest to:

A 1.00 **B** 0.75 **C** 1.25 **D** 0.80 **E** 0.87

17 The table below shows the room occupancy rate for a chain of hotels over the summer, autumn, winter and spring quarters for the years 2020–2022.

Season	2020	2021	2022
summer quarter	72.0	71.5	72.0
autumn quarter	72.4	71.9	72.7
winter quarter	68.4	67.7	68.3
spring quarter	72.3	73.0	73.5
Quarterly average	71.3	71.0	71.6

The seasonal index for winter is closest to:

A 0.960 **B** 0.957 **C** 1.046 **D** 0.969 **E** 1.003

5E Fitting a trend line and forecasting

Learning intentions

- To be able to use the method of least squares to fit a trend line to a time series.
- To be able to use the trend line to make predictions.
- To be able to use seasonal indices to add seasonality to predicted values as appropriate.

Fitting a trend line

If we identify a linear trend in the time series plot, we can use the least squares method to fit a line to the data to model that trend.

The following example demonstrates fitting a trend line to times series data which shows no seasonal component.

Example 20 **Fitting a trend line**

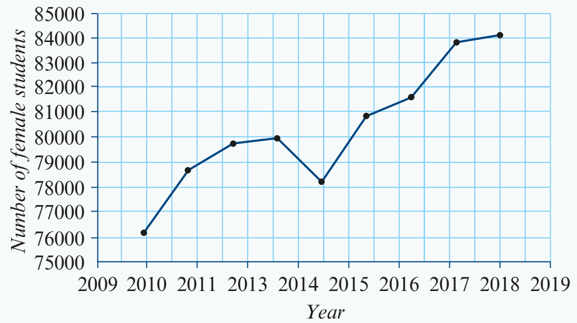
The table below shows the number of female students in Victoria enrolled in at least one subject in the Mathematics learning area at year 12 over the period 2010–18. Fit a trend line to the data, and interpret the slope.

Year	2010	2011	2012	2013	2014	2015	2016	2017	2018
Number	76260	78707	79797	79952	78237	80858	81587	83820	84069

Explanation

1 Construct a time series plot of the data to confirm that the trend is approximately linear.

Solution



2 Fit a least squares line to the data, giving the coefficients rounded to six significant figures.

Number of female students = −1 633 580 + 851.017×*year*

3 Write down the slope rounded to the nearest whole number and interpret.

Slope = 851

Over the period 2010–2018 on average the number of female students in Victoria enrolled in at least one subject in the Mathematics learning area at year 12 increased by 851 per year.

Forecasting

Using a trend line fitted to a time series plot to make predictions about future values is known as **trend line forecasting**.

Example 21 Forecasting

How many female students in Victoria do we predict being enrolled in at least one subject in the Mathematics learning area at year 12 in 2026 if the same increasing trend continues? Give your answer rounded to the nearest whole number.

Explanation

Substitute 2026 in the equation determined using least squares regression, and round to the nearest whole number.

Solution

number of female students

$= -1\,633\,580 + 851.017 \times year$

$= -1\,633\,580 + 851.017 \times 2026$

$= 90\,580$ to the nearest whole number.

Note: As with any prediction involving extrapolation, the results obtained when predicting well beyond the range of the data should be treated with caution.

Forecasting taking seasonality into account

When time series data is seasonal, it is usual to deseasonalise the data before fitting the trend line.

Example 22 Fitting a trend line (seasonality)

The deseasonalised quarterly sales data from Mikki's shop are shown below.

Quarter	1	2	3	4	5	6	7	8	9	10	11	12
Sales	893	943	955	858	1005	1026	1043	1040	1261	1151	1115	1267

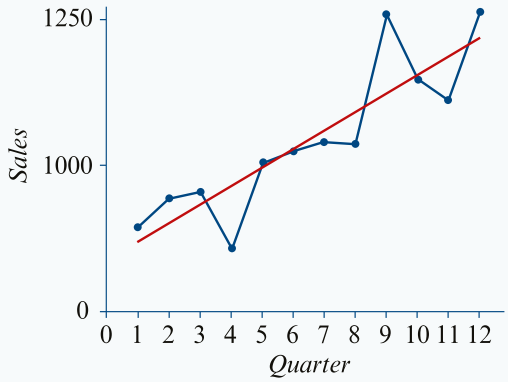
Fit a trend line and interpret the slope.

Explanation

1 Using a calculator plot the time series.

2 Fit a least squares line (with quarter the EV and sales as the RV).

Solution



3 Write down the equation of the least squares line with the intercept and slope rounded to 4 significant figures.

Sales = 838.0 + 32.07 × quarter

4 Interpret the slope in terms of the variables involved.

Over the 3-year period, on average sales at Mikki's shop increased by 32.07 sales per quarter.

Making predictions with deseasonalised data

When using deseasonalised data to fit a trend line, you must remember that the result of any prediction is a deseasonalised value. To be meaningful, this result must then be reseasonalised by multiplying by the appropriate seasonal index.

Example 23 Forecasting (seasonality)

What sales do we predict for Mikki's shop in the winter of year 4? (Because many items have to be ordered well in advance, retailers often need to make such decisions.)

Explanation	Solution
1 Substitute the appropriate value for the time period in the equation for the trend line. Since summer year 1 is quarter 1, then winter year 4 is quarter 15.	Deseasonalised sales $= 838.0 + 32.07 \times$ quarter $= 838.0 + 32.07 \times 15$ $= 1319.05$
2 To obtain the actual predicted sales figure reseasonalise the predicted value by multiplying this value by the seasonal index for winter, 1.30.	Actual sales prediction for winter of year 4 $= 1319.05 \times 1.30$ $= 1714.765$ $= 1715$ (to the nearest whole number)

Exercise 5E

Fitting a least squares line to a time series plot (no seasonality)

Example 21
Example 20

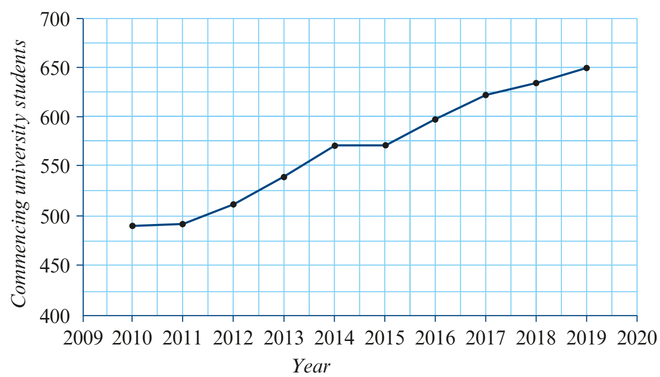
1 The data show the number of commencing university students (in thousands) in Australia for the period 2010–2019.

Year	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019
Number	488	490	510	538	569	569	595	619	632	645

The time series plot of the data is shown below.

a Comment on the plot.

b Fit a least squares regression trend line to the data, giving the values of the coefficients to 5 significant figures, and interpret the slope.



c Use this equation to predict the number of students expected to commence university in Australia in 2030 to the nearest 1000 students.

2 The table below shows the percentage of total retail sales that were made in department stores over an 11-year period:

Year	1	2	3	4	5	6	7	8	9	10	11
Sales (%)	12.3	12.0	11.7	11.5	11.0	10.5	10.6	10.7	10.4	10.0	9.4

a Using your CAS calculator, construct a time series plot.

b Comment on the time series plot in terms of trend.

c Fit a trend line to the time series plot, find its equation and interpret the slope. Give your answer rounded to 3 significant figures.

d Use the trend line to forecast the percentage of retail sales which will be made by department stores in year 15.

3 The median ages of mothers in Australia over the years 2010–2020 are shown below.

Year	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
Age	30.7	30.7	30.7	30.8	30.8	30.9	31.1	31.2	31.3	31.4	31.5

a Fit a least squares regression trend line to the data, and interpret the slope. (Give the values of the coefficients rounded to 3 significant figures.)

b Use the trend line to forecast the average ages of mothers having their first child in Australia in 2030. Explain why this prediction is not likely to be reliable.

4 The average weekly earnings (in dollars) in Australia during the period 2014–2021 are given in the following table.

Year	2014	2015	2016	2017	2018	2019	2020	2021
Earnings	1454.10	1483.10	1516.00	1543.20	1585.30	1634.80	1713.90	1737.10

a Fit a least squares regression trend line to the data, and interpret the slope. Give the values of the coefficients rounded to four significant figures.

b Use this trend relationship to forecast average weekly earnings in 2030. Explain why this prediction is not likely to be reliable.

Fitting a least-squares line to a time series with seasonality

Example 23

5 The table below shows the deseasonalised quarterly washing-machine sales of a company over 3 years.

	Year 1				Year 2				Year 3			
Quarter	1	2	3	4	5	6	7	8	9	10	11	12
Deseasonalised sales	53	51	54	55	64	64	61	63	67	69	68	66

a Use least squares regression to fit a trend line to the data.

Example 22

b Use this trend equation for washing-machine sales, with the seasonal indices below, to forecast the sales of washing machines in the fourth quarter of year 4.

Quarter	1	2	3	4
Seasonal index	0.90	0.81	1.11	1.18

6 The quarterly seasonal indices for the sales of boogie boards in a surf shop are as follows.

Seasonal index	1.13	0.47	0.62	1.77

The actual sales of the boogie boards over a 2-year period are given in the table.

Year	*Quarter 1*	*Quarter 2*	*Quarter 3*	*Quarter 4*
1	138	60	73	230
2	283	115	163	417

a Use the seasonal indices to calculate the deseasonalised sales figures for this period to the nearest whole number.

b Use a CAS calculator to plot the actual sales figures and the deseasonalised sales figures for this period and comment on the plot.

c Fit a trend line to the deseasonalised sales data. Write the slope and intercept rounded to three significant figures.

d Use the relationship calculated in **c**, together with the seasonal indices, to forecast the sales for the first quarter of year 4 (you will need to reseasonalise here).

Exam 1 style questions

7 The number of visitors to an adventure park is seasonal. A least squares regression line has been fitted to the data, and the equation is:

$$\textit{deseasonalised number of visitors} = 38345 + 286.5 \times \textit{quarter}$$

where *quarter* number one is January – March 2022.

The quarterly seasonal indices for visitors to the adventure park are shown in the table below.

Quarter	Jan-Mar	Apr-Jun	Jul-Sept	Oct-Dec
Seasonal index	1.17	0.91	0.78	1.14

The predicted number of actual visitors for the April-June quarter in 2025 is closest to:

A 42070 **B** 42356 **C** 38544 **D** 46545 **E** 37501

8 An electrical goods retailer knows that the sales of air conditioners are seasonal. A least squares regression line has been fitted to the data collected by the retailer in 2021 and 2022, and the equation is:

$$\textit{deseasonalised number of air conditioners} = 197 + 1.2 \times \textit{month}$$

where *month* number one is January 2021.

The monthy seasonal indices for air conditioner sales are shown in the table below.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Seasonal index	1.36	1.19	1.05	1.01	0.93	0.82	0.75	0.68	0.87	0.90	1.18	1.26

The predicted number of actual sales for November 2025 is closest to:

A 268 **B** 227 **C** 299 **D** 333 **E** 316

Key ideas and chapter summary

Assign-ment

Time series data	**Time series data** are a special case of bivariate data, where the explanatory variable is the time at which the values of the response variable were recorded.
Time series plot	A **time series plot** is a bivariate plot where the values of the response variable are plotted in time order. Points in a time series plot are joined by line segments.
Features to look for in a time series plot	■ Trend ■ Seasonality ■ Possible outliers ■ Cycles ■ Structural change ■ Irregular (random) fluctuations
Trend	A general increase or decrease over a significant period of time in a times series plot is called a **trend**.
Cycles	**Cycles** are present when there is a periodic movement in a time series. The period is the time it takes for one complete up and down movement in the time series plot. This term is generally reserved for periodic movements with a period greater than one year.
Seasonality	**Seasonality** is present when there is a periodic movement in a time series that has a calendar related period – for example, a year, a month, a week.
Structural change	**Structural change** is present when there is a sudden change in the established pattern of a time series plot.
Outliers	**Outliers** are present when there are individual values that stand out from the general body of data.
Irregular (random) fluctuations	**Irregular (random) fluctuations** are always present in any real-world time series plot. They include all of the variations in a time series that we cannot reasonably attribute to systematic changes like trend, cycles, seasonality, structural change or the presence of outliers.
Smoothing	**Smoothing** is a technique used to eliminate some of the irregular fluctuations in a time series plot so that features such as trend are more easily seen.

Moving mean smoothing In **moving mean smoothing**, each original data value is replaced by the mean of itself and a number of data values on either side. When smoothing over an even number of data points, centring is required to ensure the smoothed mean is centred on the chosen point of time.

Moving median smoothing **Moving median smoothing** is a graphical technique for smoothing a time series plot using moving medians rather than means.

Seasonal indices **Seasonal indices** are used to quantify the seasonal variation in a time series.

Deseasonalise The process of accounting for the effects of seasonality in a time series is called **deseasonalisation**.

Reseasonalise The process of a converting seasonal data back into its original form is called **reseasonalisation**.

Trend line forecasting **Trend line forecasting** uses the equation of a trend line to make predictions about the future.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

5A **1** I can construct a time series plot. ☐

See Example 1, and Exercise 5A Question 1

5A **2** I can identify trend in a time series plot. ☐

See Example 2, and Exercise 5A Question 4

5A **3** I can identify cycles in a time series plot. ☐

See Example 3, and Exercise 5A Question 5

5A **4** I can identify seasonality in a time series plot. ☐

See Example 4, and Exercise 5A Question 5

5A **5** I can identify structural change in a time series plot. ☐

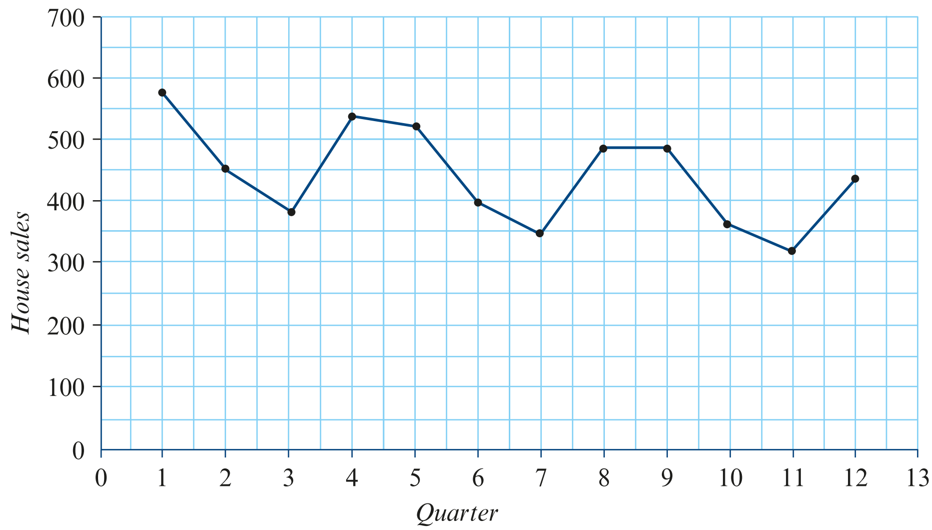
See Example 5, and Exercise 5A Question 6

5A	6	I can identify outliers in a time series plot. See Example 6, and Exercise 5A Question 7	☐
5B	7	I can smooth a time series using moving mean smoothing. See Example 7, and Exercise 5B Question 1	☐
5B	8	I can smooth a time series using moving mean smoothing with centring. See Example 9, and Exercise 5B Question 6	☐
5C	9	I can smooth a time series using moving median smoothing. See Example 12, and Exercise 5C Question 2	☐
5D	10	I can interpret seasonal indices. See Example 14, and Exercise 5D Question 1	☐
5D	11	I can use seasonal indices to deseasonalise and reseasonalise data. See Example 15, and Exercise 5D Question 2	☐
5D	12	I can use seasonal indices to determine percentage change required to correct for seasonality. See Example 16, and Exercise 5D Question 4	☐
5D	13	I can calculate seasonal indices from 1 year of data. See Example 17, and Exercise 5D Question 8	☐
5D	14	I can calculate seasonal indices from several years of data. See Example 18, and Exercise 5D Question 10	☐
5D	15	I can use seasonal indices to deseasonalise a time series. See Example 19, and Exercise 5D Question 11	☐
5E	16	I can fit a trend line to a time series plot. See Example 21, and Exercise 5E Question 1	☐
5E	17	I can use a trend line to forecast a future value (no seasonality). See Example 20, and Exercise 5E Question 1	☐
5E	18	I can fit a trend line to a time series plot with seasonality. See Example 22, and Exercise 5E Question 4	☐
5E	19	I can use a trend line to forecast a future value (with seasonality). See Example 23, and Exercise 5E Question 4	☐

Review

Multiple-choice questions

1 The time series plot below shows quarterly house sales for a real estate agency over a three year period.



The time series plot is best described as showing

A seasonality only

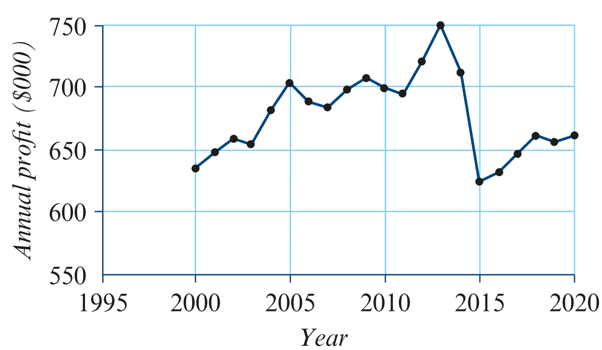
B irregular fluctuations only

C seasonality with irregular fluctuations

D an increasing trend seasonality and irregular fluctuations

E a decreasing trend with seasonality and irregular fluctuations

2 The time series plot below shows the annual profit (in $000) for a manufacturing company.



The time series plot is best described as having

A increasing trend

B decreasing trend

C seasonality with irregular fluctuations
D increasing trend with an outlier
E increasing trend with a structural change

Use the following table to answer Questions 3 to 6.

Time period	1	2	3	4	5	6
Data value	2.3	3.4	4.4	2.7	5.1	3.7

3 The three-moving mean for time period 2 is closest to:

A 3.4 **B** 3.6 **C** 3.9 **D** 4.0 **E** 4.2

4 The five-moving mean for time period 3 is closest to:

A 3.4 **B** 3.6 **C** 3.9 **D** 4.1 **E** 4.2

5 The two-moving mean for time period 5 with centring is closest to:

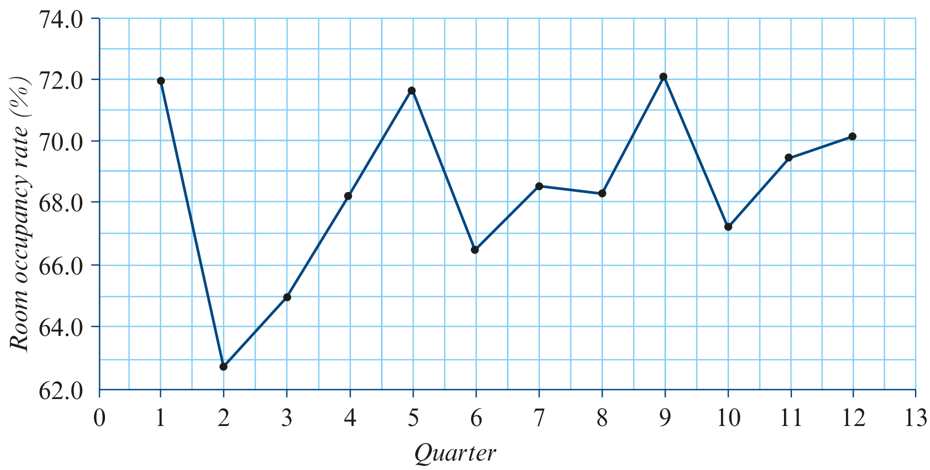
A 2.7 **B** 3.6 **C** 3.9 **D** 4.0 **E** 4.2

6 The four-moving mean for time period 4 with centring is closest to:

A 2.7 **B** 3.6 **C** 3.9 **D** 4.1 **E** 4.2

Use the following information to answer Questions 7 and 8.

The time series plot for hotel room occupancy rate (%) in a large city over a three year period is shown below.



7 The three-median smoothed value for Quarter 2 is closest to:

A 62 **B** 63 **C** 64 **D** 65 **E** 67

8 The five-median smoothed value for Quarter 3 is closest to:

A 62 **B** 64 **C** 65 **D** 68 **E** 69

9 The seasonal indices for the number of customers at a restaurant are as follows.

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
1.0	p	1.1	0.9	1.0	1.0	1.2	1.1	1.1	1.1	1.0	0.7

The value of p is:

A 0.5 **B** 0.7 **C** 0.8 **D** 1.0 **E** 1.2

The following information relates to Questions 10 and 11

10 The table shows the closing price (*price*) of a company's shares on the stock market over a 10 day period.

Day	1	2	3	4	5	6	7	8	9	10
Price($)	2.85	2.80	2.78	2.40	2.80	3.15	3.42	3.95	4.05	3.35

The six-mean smoothed with centring closing share *price* on Day 5 is closest to:

A \$2.80 **B** \$2.89 **C** \$2.91 **D** \$2.99 **E** \$3.08

11 If five-mean smoothing was used to smooth this time series, the number of smoothed values would be:

A 5 **B** 6 **C** 7 **D** 8 **E** 9

12 Suppose that Lyn spent a total of \$427 on dining out over the period from January to March, and then another \$230 over the period April-May. The five-mean smoothed amount that she spent in March is closest to:

A \$115 **B** \$129 **C** \$131 **D** \$142 **E** \$329

Use the following information to answer Questions 13 to 16.

The seasonal indices for the number of bathing suits sold at a surf shop are given in the table.

Quarter	Summer	Autumn	Winter	Spring
Seasonal index	1.8	0.4	0.3	1.5

13 The number of bathing suits sold one summer is 432. The deseasonalised number is closest to:

A 432 **B** 240 **C** 778 **D** 540 **E** 346

14 The deseasonalised number of bathing suits sold one winter was 380. The actual number was closest to:

A 114 **B** 133 **C** 152 **D** 380 **E** 1267

15 The seasonal index for spring tells us that, over time, the number of bathing suits sold in spring tends to be:

A 50% less than the seasonal average

B 15% less than the seasonal average

C the same as the seasonal average

D 15% more than the seasonal average

E 50% more than the seasonal average

16 To correct for seasonality, the actual number of bathing suits sold in Autumn should be:

A reduced by 50% **B** reduced by 40% **C** increased by 40%

D increased by 150% **E** increased by 250%

17 The number of visitors to an information centre each quarter was recorded for one year. The results are tabulated below.

Quarter	Summer	Autumn	Winter	Spring
Visitors	1048	677	593	998

Using this data, the seasonal index for autumn is estimated to be closest to:

A 0.25 **B** 1.0 **C** 1.22 **D** 0.82 **E** 0.21

Use the following information to answer Questions 18 and 19.

A trend line is fitted to a time series plot displaying the percentage change in commencing international student enrolments in Australia each year compared to the previous year (*enrolments*) for the period 2012–2019.

The equation of this line is: *% change in enrolments* $= -3480 + 1.73 \times$ *year*

18 Using this trend line, the percentage change in enrolments from the previous year forecast for 2026 is:

A 24.98 **B** -11.05 **C** 1.73 **D** 12.11 **E** 24.62

19 From the slope of the trend line it can be said that:

A on average, the number of commencing international student enrolments in Australia is increasing by by 1.73 each year.

B on average, the number of commencing international student enrolments in Australia is increasing by 1.73% each year.

C on average, the number of commencing international student enrolments in Australia is decreasing by 1.73% each month.

D on average, the number of commencing international student enrolments in Australia is decreasing by 1730 each year.

E on average, the number of commencing international student enrolments in Australia is decreasing by 1.73% each year.

20 Suppose that the seasonal indices for the wholesale price of petrol are:

Day	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Index	1.2	1.0	0.9	0.8	0.7	1.2	1.2

The equation of the least squares regression line that could enable us to predict the *deseasonalised price* per litre in cents from the *day number* is

$$\textit{deseasonalised price} = 189.9 + 0.23 \times \text{day number}$$

where day number 1 is Sunday March 20. The predicted actual price for Sunday April 3 is closest to:

A 161.1 cents **B** 193.1 cents **C** 193.4 cents **D** 231.7 cents **E** 232.0 cents

Written response questions

1 The table below shows the carbon dioxide emissions in Australia (in tonnes per capita) for the period 2013 to 2020.

Year	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018
CO_2	18.2	17.6	17.3	17.0	16.4	15.8	15.8	15.9	15.7	15.5

a Use your calculator to construct a times series plot of the data.

b Briefly describe the general trend in the data.

c Fit a least squares line to the time series plot that will enable *emissions* to be predicted from *year*. Write down the equation for the least squares line, rounding the intercept and slope to four significant figures.

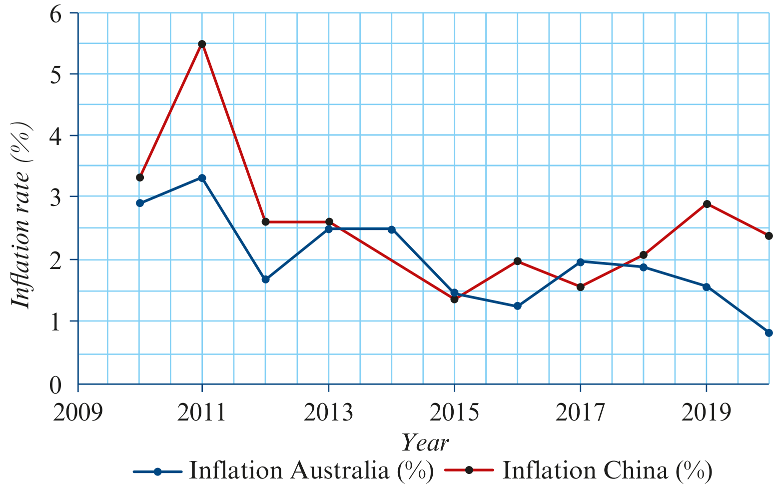
d Use the least squares equation to predict the carbon dioxide emissions in Australia in 2026. Round to three significant figures.

e Explain why the prediction you made in part d may not be reliable.

2 The table below shows the annual inflation rates in Australia and China for the period 2010–20.

Year	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
Inflation Australia (%)	2.9	3.3	1.7	2.5	2.5	1.5	1.3	2.0	1.9	1.6	0.9
Inflation China (%)	3.3	5.4	2.6	2.6	2.0	1.4	2.0	1.6	2.1	2.9	2.4

These data are plotted in the time series plot below.



a **i** Find the equation of the least squares line which allows *inflation* to be predicted from *year* for China.

ii Draw the least squares line on the time series plot.

b **i** Find the equation of the least squares line which allows *inflation* to be predicted from *year* for Australia.

ii Draw the least squares line on the time series plot.

c Explain why the equations of the least squares lines predict that the inflation rate for China will always remain higher than the inflation rate for Australia.

d Find the two-mean centred smoothed inflation rate for Australia for 2015.

3 The table below shows the number of dolphins spotted in a bay over each of the four seasons for the years 2020-2021.

Year	Summer	Autumn	Winter	Spring
2020	97	112	480	678
2021	107	145	496	730

a Use the data in the table to find seasonal indices. Give your answers rounded to two decimal places.

b The number of dolphins spotted in each of the four seasons in 2022 is shown in the table below.

Year	Summer	Autumn	Winter	Spring
2022	78	86	350	540

Use the seasonal indices from part a to deseasonalise the data. Round your answers to the nearest whole number.

Chapter 6

Revision: Data analysis

6A Exam 1 style questions: Univariate data

Use the following information to answer Questions 1–3.

The following table shows the data collected from a sample of five senior students at secondary college. The variables in the table are:

- *campus* – the campus they attend (C = city, R = regional))
- *time* – the time in minutes each students took to get to school that day
- *transport* – mode of transport (1 = walked or rode a bike, 2 = car, 3 = public transport)
- *number* – number of siblings at the school
- *postcode* – postcode of place of residence.

Campus	*Time*	*Transport*	*Number*	*Postcode*
C	12	1	0	7267
R	35	3	2	7268
C	15	2	1	7268
C	43	3	0	7250
R	27	2	3	7267

1 The variables *campus* and *transport* are:

A both nominal variables
B a nominal and an ordinal variable respectively
C an ordinal and a nominal variable respectively
D a nominal and a discrete numerical variable respectively
E a nominal and a continuous numerical variable respectively

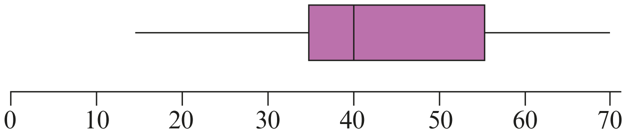
2 The number of these variables that are discrete numerical is:

A 0 **B** 1 **C** 2 **D** 3 **E** 4

3 The number of regional students who used public transport to get to school is:

A 0 **B** 1 **C** 2 **D** 3 **E** 4

4 Consider this box-and-whisker plot. Which *one* of the following statements is true?



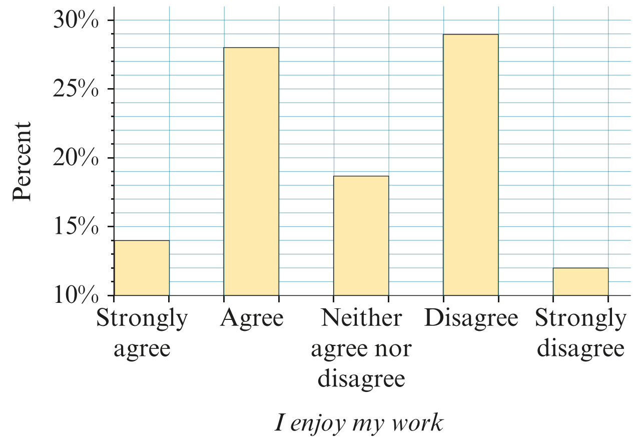
A The median is 45.
B Less than one-quarter of the observations are less than 30.
C Less than one-quarter of the observations are greater than 50.

Revision

D All of the observations are less than 60.

E More than half of the observations are less than 30.

5 In a survey people were asked to respond to the statement *I enjoy my work*, selecting from the alternatives 'strongly disagree', 'agree', 'neither agree nor disagree', 'disagree', 'strongly disagree'. Their responses are summarised in the bar chart opposite.

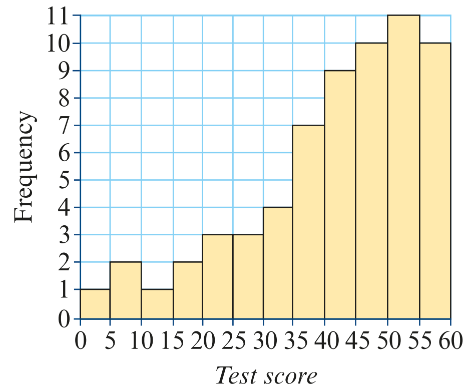


The percentage of people who chose the modal response to this question is closest to:

A 18% **B** 20% **C** 27% **D** 29% **E** 31%

Use the following information to answer Questions 6 to 11.

A group of VCE mathematics students sat for a test. There were 63 students in the group. Their test scores are summarised opposite in the form of a histogram.



6 The distribution of test scores is best described as:

A positively skewed

B negatively skewed

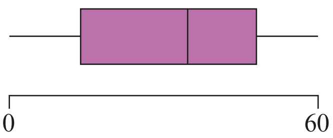
C symmetric

D symmetrically skewed

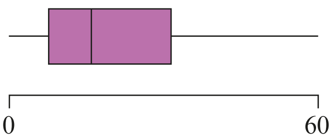
E symmetric with a clear outlier

7 Displayed in the form of a boxplot, the distribution of test scores would look like:

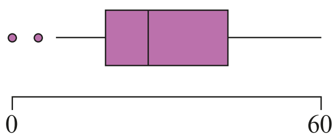
A



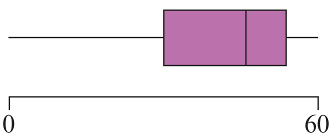
B



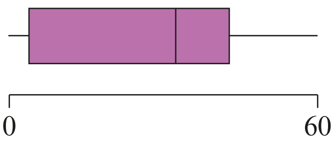
C



D



E



8 Students who scored 50 or more on the test were awarded an A on the test. The percentage of students who were awarded an A is closest to:

A 10% **B** 11% **C** 21% **D** 30% **E** 33%

9 The number of students who scored at least 30 but under 45 marks is:

A 4 **B** 7 **C** 19 **D** 11 **E** 20

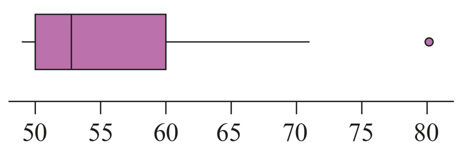
10 The modal mark lies in the interval:

A 5–10 **B** 15–20 **C** 25–30 **D** 40–45 **E** 50–55

11 The median mark lies in the interval:

A 5–10 **B** 15–20 **C** 25–30 **D** 40–45 **E** 50–55

12 For the boxplot opposite, outliers are defined as data values that are:



A less than 35 or greater than 75

B less than 48 or greater than 70

C less than 48 **D** greater than 70 **E** greater than 80

Use the following information to answer Questions 13 to 15.

The following stem plot shows the distribution of the time it took (in minutes) for each of a group of 20 people to travel to work.

Time (minutes) key: 4|0 represents 40

1	6 8
2	2 5 6 8 9
3	0 5 6
4	2 6 7 8
5	1 3 6
6	0 1
7	1

13 The median travel time is:

A 4 minutes **B** 35 minutes **C** 36 minutes **D** 39 minutes **E** 42 minutes

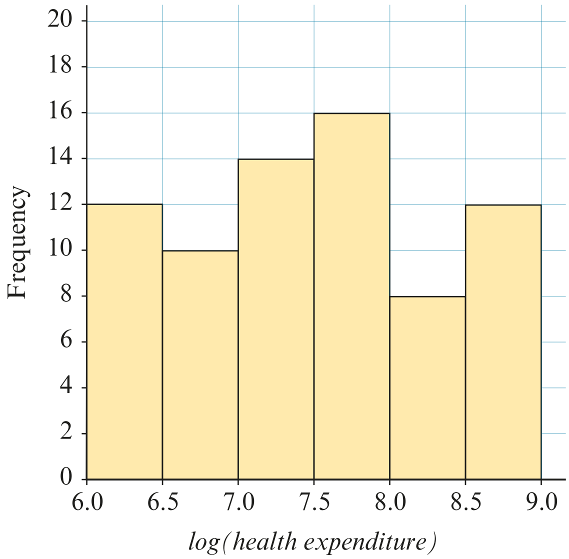
14 The interquartile quartile range (*IQR*) of the travel times is:

A 25 minutes **B** 27 minutes **C** 28 minutes **D** 30 minutes **E** 53 minutes

15 The travel time for the fastest 30% of people was:

A 28 minutes or less **B** more than 28 minutes **C** less than 25 minutes
D 51 minutes or less **E** more than 51 minutes

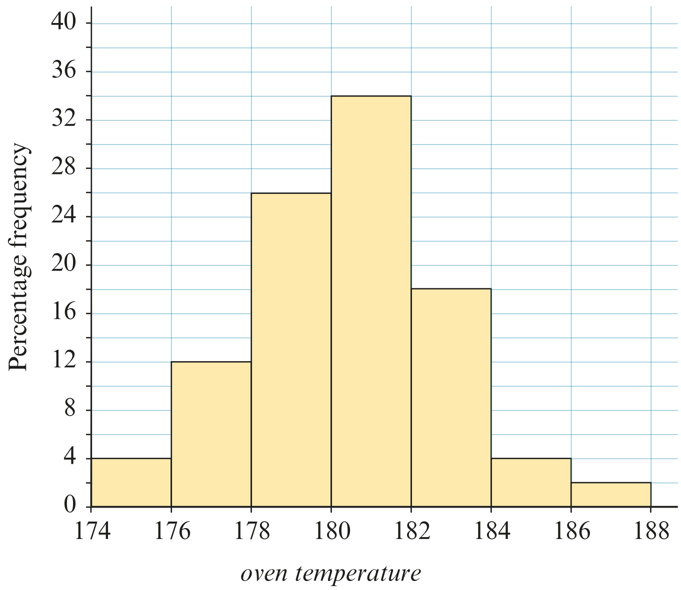
16 The following histogram shows the annual expenditure on health (in \$US) for 72 different countries, plotted on a $\log_{10}$ scale. Based on this histogram, the percentage of countries spending \$US10,000,000 or more on health is equal to:



A 11.1 **B** 16.7 **C** 20.0 **D** 30.6 **E** 69.4

Use the following information to answer Questions 17 and 18.

To test the temperature control on an oven, the control is set to 180°C and the oven is heated for 15 minutes. The temperature of the oven is then measured. The temperatures of 300 ovens tested in this way were recorded and the data displayed using the percentage frequency histogram shown.



17 The median temperature (in °C) could be

A 176.5 **B** 179.5 **C** 181.3 **D** 182.1 **E** 185.7

18 The maximum value for the interquartile range (IQR) in °C is

A 2 **B** 4 **C** 5 **D** 6 **E** 7

Use the following information to answer Questions 19–24.

The lengths of sample of 1000 ants of a particular species are approximately normally distributed with a mean of 4.8 mm and a standard deviation of 1.2 mm.

19 From this information it can be concluded that around 95% of the lengths of the ants should lie between:

A 2.4 mm and 6.0 mm **B** 2.4 mm and 7.2 mm **C** 3.6 mm and 6.0 mm
D 3.6 mm and 7.2 mm **E** 4.8 mm and 7.2 mm

20 The standardised ant length of $z = -1.2$ corresponds to an actual ant length of:

A 2.40 mm **B** 3.36 mm **C** 4.2 mm **D** 5.0 mm **E** 6.24 mm

21 The percentage of ants with lengths less than 3.6 mm is closest to:

A 2.5% **B** 5% **C** 16% **D** 32% **E** 95%

22 The percentage of ants with lengths less than 6.0 mm is closest to:

A 5% **B** 16% **C** 32% **D** 68% **E** 84%

23 The percentage of ants with lengths greater than 3.6 mm and less than 7.2 mm is closest to:

A 2.5% **B** 18.5% **C** 68% **D** 81.5% **E** 97.5%

24 In the sample of 1000 ants, the number with a length between 2.4 mm and 4.8 mm is approximately:

A 3 **B** 50 **C** 475 **D** 975 **E** 997

25 The scores on an examination are known to be approximately normally distributed. If 2.5% of students score more than 68% on the examination, and 0.15% of students score less than 16% on the exmination, estimate the mean and standard deviation of the examination scores. Give your answers to one decimal place.

A mean = 47.2, standard deviation = 10.4 **B** mean = 42.0, standard deviation = 16.0
C mean = 10.4, standard deviation = 47.2 **D** mean = 34.4, standard deviation = 16.8
E mean = 56.6, standard deviation = 10.4

26 A class of students sat for a biology test and a legal studies test. Each test had a possible maximum score of 100 marks. The table below shows the mean and standard deviation of the marks obtained in these tests.

	Subject	
	Biology	Legal Studies
Class mean	54	78
Class standard deviation	15	5

The class marks in each subject are approximately normally distributed.
Sashi obtained a mark of 81 in the biology test. The mark that Sashi would need to obtain on the legal studies test to achieve the same standard score for both legal studies and biology is:

A 81 **B** 82 **C** 83 **D** 87 **E** 95

6B Exam 1 style questions: Associations

1 Researchers believe that reaction time might be lower in cold temperatures. They devise an experiment where *reaction time* in seconds is measured at three different *temperature* levels (1 = less than 8°C, 2 = from 8°C to 18 °C, 3 = more than 18°C). The explanatory variable, and its classification are:

A *reaction time*, categorical
B *temperature*, categorical
C *reaction time*, numerical
D *temperature*, numerical
E *temperature*, ordinal

Use the following information to answer Questions 2-4

The data in the following table was collected to investigate the association between a person's age and their satisfaction with their career choice.

	Age group		
Satisfied with career choice?	Under 35	35 or more	Total
Yes	136	136	272
No	42	86	128
Total	178	222	400

2 The percentage of participants in the study aged 35 or more is closest to:

A 55.5% **B** 50.0% **C** 44.5% **D** 68.0% **E** 61.3%

3 Of those people aged under 35, the percentage who are satisfied with their career choice is closest to:

A 23.6% **B** 34.0% **C** 50.0% **D** 61.3% **E** 76.4%

4 The data in the table supports the contention that there is an association between *age group* and *satisfaction with career choice* because:

A 68.0% of people are satisfied with their career choice, compared to 32.0% who are not.

B The number of people satisfied with their career choice aged under 35 is the same as the number aged 35 or more who are satisfied with their career choice.

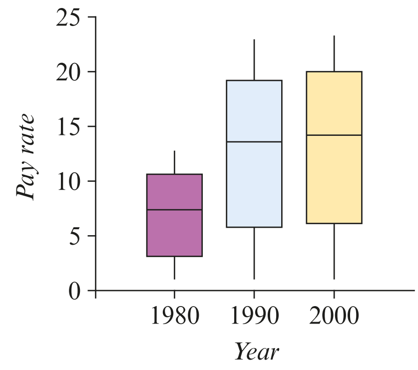
C 76.4% of people aged under 35 are satisfied with their career choice, which is more than the 61.3% of those aged 35 or more who are satisfied with their career choice.

D 50.0% of people are satisfied with their career choice are aged under 35.

E 67.2% of people who are not satisfied with their career choice are aged 35 or more.

5 The box plots opposite display the distribution of the average pay rate, in dollars per hour, of workers in 35 countries for the years 1980, 1990 and 2000.

The aim is to investigate the association between *pay rate* and *year*.



Which one of the following statements is not true?

A The IQR's of *pay rate* in 1990 and 2000 are approximately the same.

B The median *pay rate* is lower in 1980 than the median *pay rate* in 1990 and 2000.

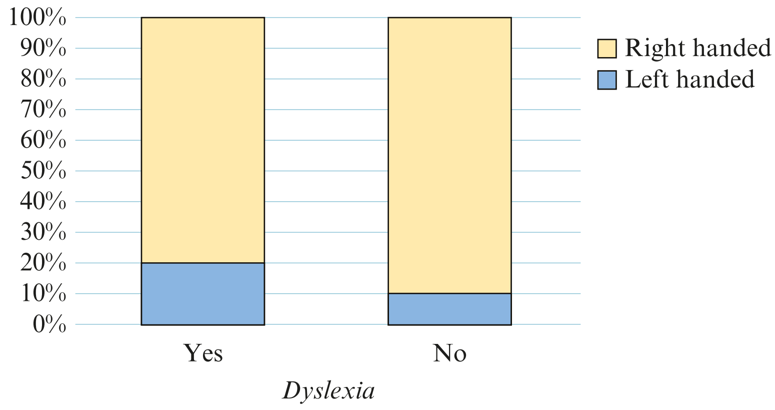
C The IQR of *pay rate* in 1980 is more than the IQR of *pay rate* in 1990 and 2000.

D The pay rate in 75% of the countries in 1980 was less than the median *pay rate* in 2000.

E All three distributions are approximately symmetric.

Questions 6 to 8 relate to the following information

In a study of the association between left-handedness and dyslexia, the dominant hand (left or right) was recorded for two groups of students, a group of students who had been diagnosed with dyslexia (yes), and a control group (no). The results are summarised in the percentaged segmented bar chart below.



6 The percentage of students diagnosed with dyslexia who are left-handed is closest to:

A 10% **B** 20% **C** 30% **D** 80% **E** 90%

Revision

7 The results could be summarised in a two-way frequency table. Which of the following frequency tales could match the percentaged segmented bar chart?

A

	Dyslexia	
Dominant hand	Yes	No
Left	80	90
Right	20	10

B

	Dyslexia	
Dominant hand	Yes	No
Left	80	20
Right	90	10

C

	Dyslexia	
Dominant hand	Yes	No
Left	20	80
Right	10	90

D

	Dyslexia	
Dominant hand	Yes	No
Left	20	10
Right	30	40

E

	Dyslexia	
Dominant hand	Yes	No
Left	10	10
Right	40	90

8 The variables *dominant hand* and *dyslexia* appear to be associated because:

A very few students are left-handed

B 20% of students diagnosed with dyslexia are left-handed, compared to only 10% of the control group

C only 30% of the students are left-handed

D 80% of students diagnosed with dyslexia are left-handed, compared to 90% of the control group

E a higher percentage of the control group are left-handed compared to the students diagnosed with dyslexia

9 For which one of the following pairs of variables would it be appropriate to construct a scatterplot?

A *eye colour* (blue, green, brown, other) and *country of birth*

B *weight* in kg and *blood pressure* in mmHg

C *number of cups of coffee drunk each day* and *stress level* (high, medium, low)

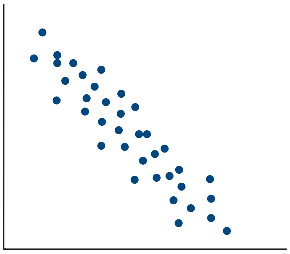
D *age* in years and *football team*

E *time spent watching TV each week* in hours and *educational level* (primary, secondary, tertiary)

10 The value of r for the scatterplot is closest to:

A 0.8 **B** 0.5 **C** 0

D −0.5 **E** −0.8



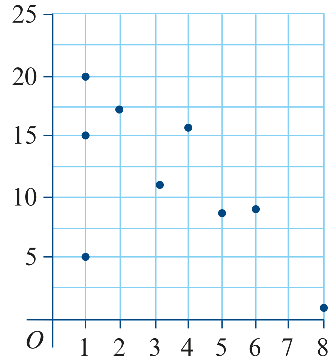
11 The association pictured in the scatterplot in the previous question is best described as:

A strong, positive, linear **B** strong, negative, linear **C** weak, negative, linear

D strong, negative, non-linear with an outlier **E** strong, negative, non-linear

12 When the correlation coefficient, r, was calculated for the data displayed in the scatterplot, it was found to be $r = -0.64$.

If the point (1, 5) was replaced with the point (6, 5) and the correlation coefficient, r, recalculated, then the value of r would be:



A unchanged **B** positive but closer to 1

C negative but closer to 0 **D** positive but closer to 0

E negative but closer to –1

The following information relates for Questions 13 to 15

For the association between between *computer ownership* (computers/1000 people) and car ownership (cars/1000 people) the coefficient of determination is equal to 0.8464.

13 If the people who own more cars also tend to own more computers, then the value of the correlation coefficient, r (rounded to two decimal places) is closest to.

A 0.64 **B** 0.72 **C** 0.85 **D** 0.92 **E** 0.96

14 The percentage of variation in *computer ownership* explained by the variation in the *car ownership* is closest to:

A 71.6% **B** 84.6% **C** 92.0% **D** 8.0% **E** 15.4%

15 The percentage of variation in *computer ownership* NOT explained by the variation in the *car ownership* is closest to:

A 71.6% **B** 84.6% **C** 92.0% **D** 8.0% **E** 15.4%

16 A back-to-back stem plot is a useful tool for displaying the association between:

A weight (kg) and handspan (cm)

B height (cm) and age (years)

C handspan (cm) and eye colour (brown, blue, green)

D height in centimetres and sex (female, male)

E meat consumption (kg/person) population and country of residence

17 To explore the association between owning an electric car (yes or no) and age group (under 25 years, 25-44 years, 44 years or more), it would be best to use the data collected to construct:

A an appropriately percentaged table **B** a back-to-back stem plot
C parallel box plots **D** a scatterplot
E a histogram

18 The association between the time taken to walk 5 km (in minutes) and fitness level (below average, average, above average) is best displayed using:

A a histogram **B** a scatterplot **C** a time series plot
D parallel box plots **E** a back-to-back stem plot

6C Exam 1 style questions: Regression and data transformation

1 A teacher collected the following statistical information about her students' scores in their mathematics examination in Year 11 and their scores in Year 12:

	Year 11	*Year 12*
mean	75.1	64.8
standard deviation	2.567	4.983
correlation coefficient	$r = 0.675$	

The slope of the least squares regression line which would allow their score in Year 12 to be predicted from their score in Year 11 is closest to:

A 0.35 **B** 0.68 **C** 1.3 **D** 1.7 **E** 3.36

2 The statistical analysis of the set of bivariate data involving variables x and y resulted in the information displayed in the table below:

	x	y
mean	123.5	38.7
standard deviation	4.65	4.78
least squares equation	$y = -140 + 0.475x$	

Using this information the value of the correlation coefficient r for this set of bivariate data is closest to

A 0.73 **B** 0.34 **C** 0.46 **D** 0.49 **E** 0.97

The following data relate to Questions 3 and 4.

Number of hot dogs sold	190	168	146	155	150	170	185
Temperature (°C)	10	15	20	15	17	12	10

3 The equation of the least squares regression line fitted to the data is closest to:

A *number of hot dogs sold* = 227 − 4.31 × *temperature*

B *number of hot dogs sold* = 48.4 − 0.206 × *temperature*

C *number of hot dogs sold* = 4.31 + 227 × *temperature*

D *number of hot dogs sold* = 0.206 − 48.4 × *temperature*

E *number of hot dogs sold* = 227 + 4.31 × *temperature*

4 The coefficient of determination will be closest to:

A −0.94 **B** −0.89 **C** 0.21 **D** 0.89 **E** 0.94

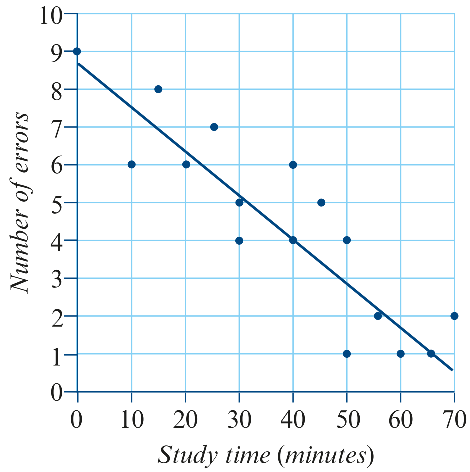
The following information relates to Questions 5 to 9.

Eighteen students sat for a 15-question multiple-choice test. In the scatterplot opposite, the number of errors made by each student on the test is plotted against the time they reported studying for the test.

A least squares regression line has been determined for the data and is also displayed on the scatterplot. The equation for the least squares regression line is:

$$number\ of\ errors = 8.8 - 0.12 \times study\ time$$

and the coefficient of determination is 0.8198.



5 The least squares regression line predicts that a student reporting a study time of 35 minutes would make:

A 4.3 errors **B** 4.6 errors **C** 4.8 errors **D** 5.0 errors **E** 13.0 errors

6 The student who reported a study time of 10 minutes made six errors. The predicted score for this student would have a residual of:

A −7.6 **B** −1.6 **C** 0 **D** 1.6 **E** 7.6

7 Which of the following statements that relate to the regression line is *not* true?

A The slope of the regression line is –0.12.

B The equation predicts that a student who spends 40 minutes studying will make around four errors.

C The least squares line does *not* pass through the origin.

D On average, a student who does not study for the test will make around 8.8 errors.

E The explanatory variable in the regression equation is *number of errors*.

8 This regression line predicts that, on average, the number of errors made:

A decreases by 0.82 for each extra minute spent studying

B decreases by 0.12 for each extra minute spent studying

C increases by 0.12 for each extra minute spent studying

D increases by 8.8 for each extra minute spent studying

E decreases by 8.8 for each extra minute spent studying

9 Given that the coefficient of determination is 0.8198, we can say that close to:

A 18% of the variation in the number of errors made can be explained by the variation in the time spent studying

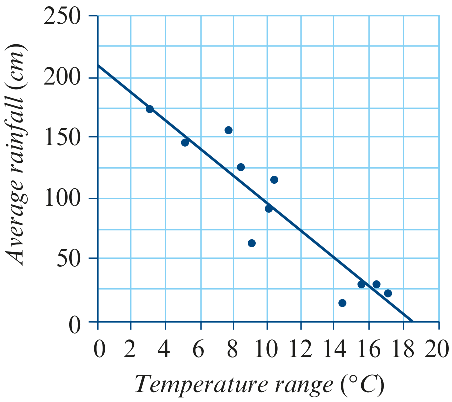
B 33% of the variation in the number of errors made can be explained by the variation in the time spent studying

C 67% of the variation in the number of errors made can be explained by the variation in the time spent studying

D 82% of the variation in the number of errors made can be explained by the variation in the time spent studying

E 95% of the variation in the number of errors made can be explained by the variation in the time spent studying

10 The average rainfall and temperature range at several locations in the South Pacific region are displayed in the scatterplot opposite.



A least squares regression line has been fitted to the data, as shown. The equation of this line is closest to:

A $average\ rainfall = 210 - 11 \times temperature\ range$

B $average\ rainfall = 210 + 11 \times temperature\ range$

C $average\ rainfall = 18 - 0.08 \times temperature\ range$

D $average\ rainfall = 18 + 0.08 \times temperature\ range$

E $average\ rainfall = 250 - 13 \times temperature\ range$

11 In a certain state, correlation coefficient between:

- university entrance score and score on a Mathematics aptitude test is $r = 0.462$
- university entrance score and score on an English aptitude test is $r = 0.662$

Given this information, which *one* of the following statements is *true*?

A Around 21.3% of the variation in score on the Mathematics aptitude test is explained by the variation in score on the English aptitude test.

B Around 43.8% of the variation in score on the English aptitude test is explained by the variation in score on the Mathematics aptitude test.

C Together the scores on the Mathematics and English aptitude tests explain 100% of the variation in university entrance score.

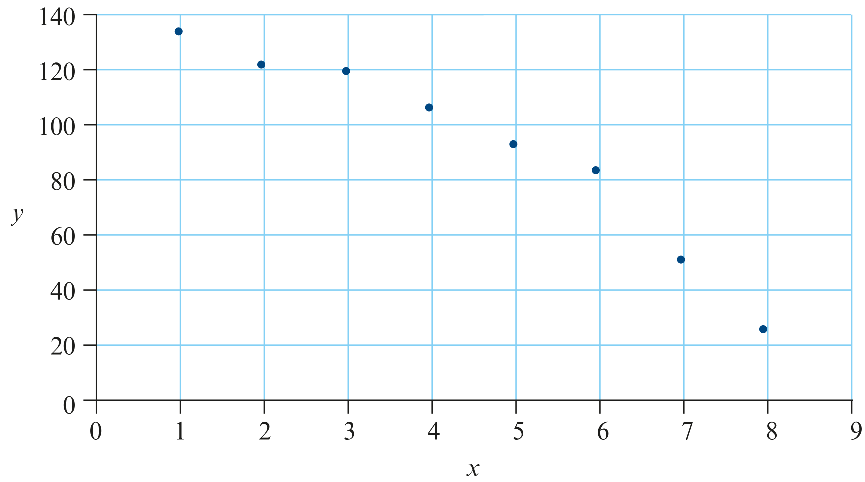
D The correlation between the scores on the Mathematics and English aptitude tests is negative.

E The score on English aptitude tests is more strongly associated with the university entrance score than is the score on the Mathematics aptitude test.

The following information relates to Questions 12 and 13

A student uses the data in the table below to construct the scatterplot shown:

x	y
1	132
2	120
3	117
4	104
5	91
6	82
7	49
8	24



12 A squared transformation is applied to x to linearise the association. A least squares line is fitted to the transformed data, with x^2 as the explanatory variable.
The equation of this least squares line is closest to

A $y = 132 - 1.64x$

B $y = 79.7 - 0.602x^2$

C $y^2 = 79.7 - 0.602x$

D $y^2 = 132 - 1.64x$

E $y = 132 - 1.64x^2$

13 A y^2 transformation could also be used to linearise this association. A least squares line is fitted to the transformed data, with y^2 as the response variable, and the equation of the least squares line is

$$y^2 = 20076 - 2397.2x$$

Using this equation, the predicted value of y when $x = 2$ is closest to:

A 102 **B** 124 **C** 120 **D** 10487 **E** 15282

Revision

14 The following data were collected for two related variables x and y.

x	0.4	0.5	1.1	1.1	1.2	1.6	1.7	2.3	2.4	3.4	3.5	4.3	4.7	5.3
y	5.8	4.7	3.3	5.5	4.2	3.4	2.3	2.8	1.8	1.3	1.9	1.2	1.6	0.9

A scatterplot indicates a non-linear relationship. The data is linearised using a $1/y$ transformation. A least squares line is then fitted to the transformed data.

The equation of this line is closest to:

A $\frac{1}{y} = 0.08 + 0.16x$ **B** $\frac{1}{y} = 0.16 + 0.08x$ **C** $\frac{1}{y} = -0.08x + 5.23x$

D $\frac{1}{y} = 5.23 - 0.08x$ **E** $\frac{1}{y} = 1.44 + 1.96x$

15 The equation of a least squares line that has been fitted to transformed data is:

$$\textit{population} = 58\,170 + 43.17 \times \textit{year}^2$$

Using this equation, the predicted value of *population* when *year* = 10 is closest to:

A 9.2 **B** 9.9 **C** 10.6 **D** 62 417 **E** 62 487

16 The equation of a least squares line that has been fitted to transformed data is:

$$\textit{weight}^2 = 52 + 0.78 \times \textit{area}$$

Using this equation, the predicted value of *weight* when *area* = 8.8 is closest to:

A −7.7 **B** ±7.7 **C** 7.7 **D** ±58 **E** 58

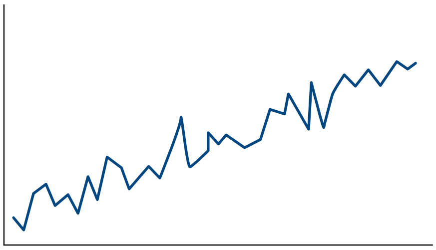
17 The equation of a least squares regression line that has been fitted to transformed data is: $\log(\textit{number}) = 1.31 + 0.083 \times \textit{month}$

Using this equation, the predicted value of *number* when *month* = 6 is closest to:

A 1.8 **B** 6.0 **C** 18 **D** 64 **E** 650

6D Exam 1 style questions: Time series

1 The pattern in the time series in the graph shown is best described as:



A trend with random variation

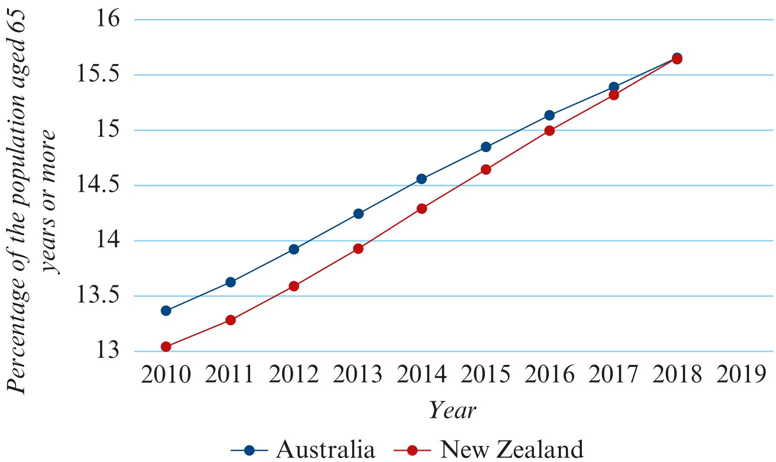
B cyclical but not seasonal

C seasonal

D random

E trend with outliers

2 The time series plot shows the percentage of the population aged 65 years or more in Australia and New Zealand, over the years from 2010 to 2018.



From the plot, it can be concluded that over the interval 2010–2018, the *difference* in the percentage of the population aged 65 years or more in the two countries has shown:

A a decreasing trend **B** an increasing trend

C seasonal variation **D** a 5-year cycle

E no trend

Use the information in the table below to answer Questions 3 to 6.

t	1	2	3	4	5	6	7	8	9	10
y	4	5	4	4	8	6	9	10	9	12

3 The three mean smoothed value for $t = 2$ is closest to:

A 4.3 **B** 6.2 **C** 6.4 **D** 6.5 **E** 7.25

4 The five mean smoothed value for $t = 5$ is closest to:

A 4.3 **B** 6.2 **C** 6.4 **D** 6.5 **E** 7.25

5 The centred two mean smoothed value for $t = 6$ is closest to:

A 4.3 **B** 4.75 **C** 6.25 **D** 6.5 **E** 7.25

6 The centred four mean smoothed value for $t = 3$ is closest to:

A 4.3 **B** 4.75 **C** 6.25 **D** 7.25 **E** 9.75

7 To help work out her staffing roster, Fleur records the number of customers who come into her cafe between 7.00 am and 8.00 am each morning for a week (*customers*).

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Customers	42	25				84	100

The numbers of *customers* on Wednesday, Thursday and Friday are not shown. The five-mean smoothed number of *customers* on Thursday is 38.
The three-mean smoothed number of *customers* on Thursday is:

A 27 **B** 29 **C** 30 **D** 38 **E** 55

8 The table shows the closing price (*price*) of a company's shares on the stock market over a 9 day period.

Day	1	2	3	4	5	6	7	8	9
Price($)	1.30	1.15	1.10	1.25	1.29	1.37	2.42	1.95	2.55

The six-mean smoothed with centring closing share *price* on Day 5 is closest to:

A \$1.43 **B** \$1.50 **C** \$1.56 **D** \$1.68 **E** \$1.81

Use the following information to answer Questions 9 and 10.

The table below records the monthly electricity cost (in dollars) for an apartment over one calendar year.

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
123	90	153	136	101	129	153	143	95	61	85	107

9 Based on this information, the seasonal index for September is closest to:

A 1.00 **B** 0.78 **C** 1.25 **D** 0.83 **E** 0.87

10 Using data collected over several years, the seasonal index for December was determined to be 0.90. To correct the cost of electricity for seasonality in December, the actual cost should be

A decreased by 11.1% **B** decreased by 9.0% **C** decreased by 10.0%
D increased by 10.0% **E** increased by 11.1%

Use the information below to answer Questions 11 and 12.

The quarterly sales figures for a soft drink company and the seasonal indices are as shown.

Quarter	1	2	3	4
Sales (\$'000s)	1200	1000	800	1200
Seasonal index	1.1	0.9	0.8	

11 The deseasonalised figure (in \$'000s) for quarter 3 is:

A 640 **B** 667 **C** 800 **D** 1000 **E** 1500

12 The seasonal index for quarter 4 is:

A 0.6 **B** 0.8 **C** 1.00 **D** 1.1 **E** 1.2

13 The deseasonalised sales (in dollars) for a company in June were \$91 564. The seasonal index for June is 1.45.

The *actual* sales for June were closest to:

A \$41 204 **B** \$61 043 **C** \$63 148 **D** \$91 564 **E** \$132 768

14 Sales for a major department store are reported quarterly. The seasonal index for the third quarter is 0.85. This means that sales for the third quarter are typically:

A 85% below the quarterly average for the year

B 15% below the quarterly average for the year

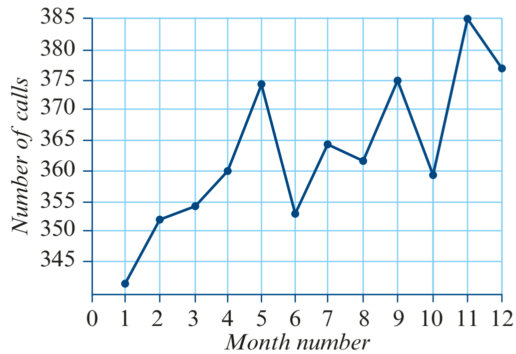
C 15% above the quarterly average for the year

D 18% above the quarterly average for the year

E 18% below the quarterly average for the year

Use the information below to answer Questions 15 and 16.

The time series plot opposite shows the number of calls each month to a call-centre over a 12-month period.



15 The three-median smoothed number of calls for month 9 is closest to:

A 358 **B** 362 **C** 371 **D** 375 **E** 377

16 The five-median smoothed number of calls for month 10 is closest to:

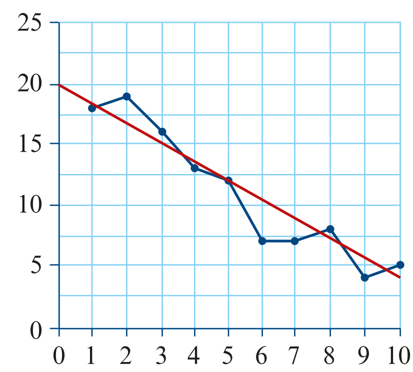
A 358 **B** 362 **C** 371 **D** 375 **E** 377

17 A time series for y is shown in the graph, where t represents time. If a linear trend line is fitted to this data, as shown, then the equation of the line is closest to:

A $y = 20 - 1.6t$ **B** $y = -1.6t$

C $y = 20 + 1.6t$ **D** $y = 20 - 0.6t$

E $y = 20 + 0.6t$



6E Exam 2 style questions

1 In a large university students in some courses were asked if they would prefer to attend on-campus lectures and tutorials, or study online and not attend the campus. Data was collected for the following variables:

- *number* student number
- *study mode* 1 = on-campus, 2 = online
- *age* age in years
- *course* 1 = Business, 2 = Health, 3 = Social Science
- *gender* F = female, M = male
- *distance* the distance the student lives from the campus, to the nearest km

The data collected for 12 students are shown in the following table:

Number	*Study mode*	*Age (years)*	*Course*	*Gender*	*Distance (kms)*
23455	1	18	1	M	2
13425	2	23	1	F	8
28445	1	18	2	M	4
19889	1	19	3	F	9
10340	2	25	2	F	13
23001	2	22	1	M	2
19968	1	19	1	F	7
20012	2	34	3	F	6
21980	1	18	3	M	12
17884	2	45	1	M	8
19456	2	22	2	F	9
21111	1	22	3	F	6

a Write down the names of the numerical variables in the table.

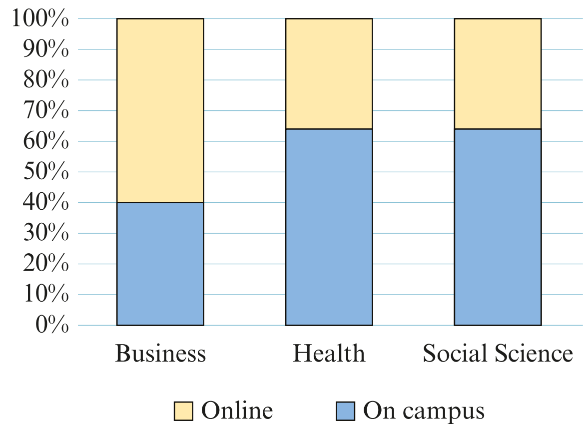
b Determine the mean and the standard deviation of the distance from their residences to the campus for these 12 students. Round your answers to two decimal places.

c One student, Charlotte, lives 13 km from the campus. Calculate the standardised score (z) for the distance Charlotte lives from campus. Round your answer to one decimal place.

d Use the data in the table to complete the following two-way frequency table.

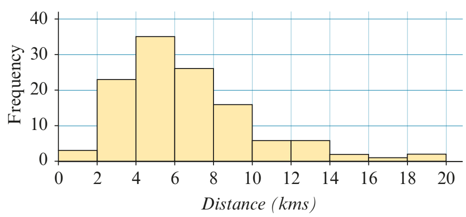
	Gender	
Study mode	Female	Male
On campus		
Online		
Total		

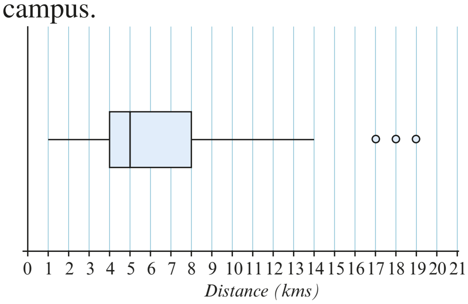
e Data was collected from a total of 120 students. The percentaged segmented bar chart shows the study mode preferences for students from each of the three courses.



- **i** What percentage of Business students chose online?
- **ii** Does the percentaged segmented bar chart support the contention that the choice of study mode (on-campus or online) is associated with course? Justify your answer by quoting appropriate percentages.

2 The histogram and the boxplot below show the distribution of the distances the 120 students surveyed in Question 1 live from the campus.





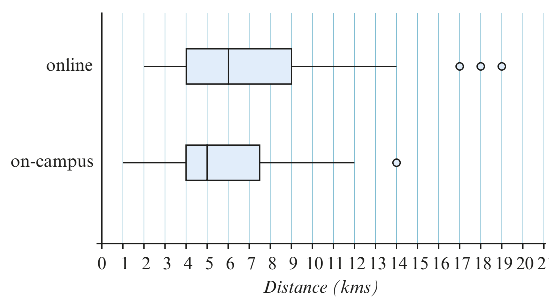
a Describe the shape of the distribution of *distance*, including the values of outliers.

b Approximately how many of these students lived from 4 to 5 km from the campus?

c **i** Determine the values of the upper and lower fences for the boxplot.

ii Use the fences to explain why a distance of 1 km would not be shown as an outlier.

d The boxplots compare the *distance* students live from the campus for students who prefer to study on-campus, and those who prefer to study online.

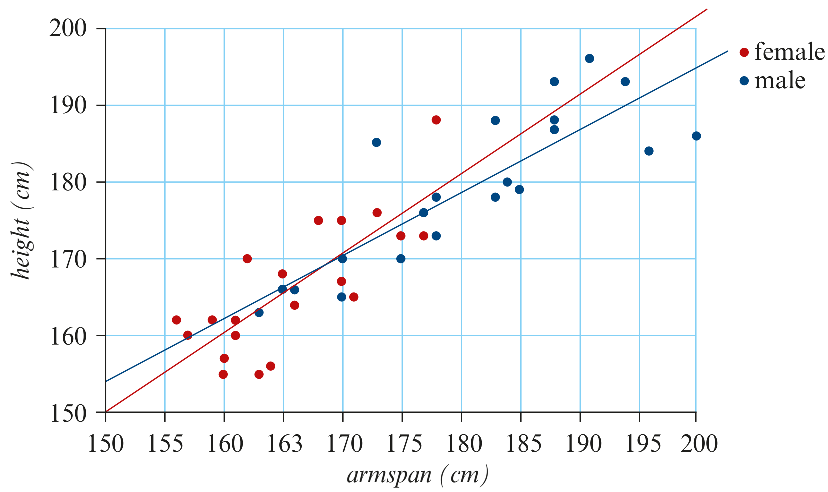


Use the information in the boxplots to answer the following questions.

i The median *distance* for the students preferring to study online was ☐ km higher than the median *distance* for students preferring to study on-campus.

ii The difference in the IQR for *distance* for the students preferring to study online and the IQR for the students preferring to study on-campus is ☐.

3 To address the question "Can we predict a person's height from their armspan?" height (in cm) and armspan (in cm) measurements were collected from 60 students in Year 11 and 12. Of the 60 students 30 were male and 30 female. The following scatterplot shows the data collected, with least squares regression lines fitted for males and females.



The equation of the least squares regression line for females is:

$$height = -4.199 + 1.028 \times armspan$$

The equation of the least squares regression line for males is:

$$height = 31.705 + 0.815 \times armspan$$

a Interpret the slope of the regression equation in terms of *height* and *armspan* for males.

In determining this equation, the summary statistics displayed in the table were also calculated.

	armspan	*height*
mean (females)	164.0	164.5
standard deviation (females)	6.319	8.083
mean (males)	178.1	177.0
standard deviation (males)	9.832	9.583

b **i** Determine the value of the coefficient of determination for females and interpret in terms of *height* and *armspan*. Give your answer as a percentage rounded to one decimal place.

ii Determine the value of the coefficient of determination for males and interpret in terms of *height* and *armspan*. Give your answer as a percentage rounded to one decimal place.

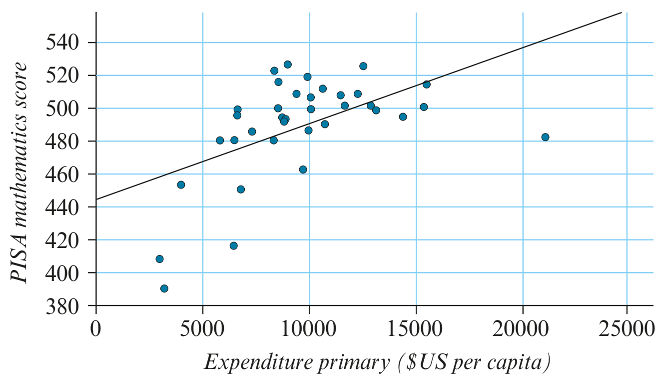
iii Explain why *armspan* is a better predictor of *height* for males than for females, quoting appropriate statistics.

c **i** Use the least squares regression line to predict the difference in height between males and females who both have armspans of 160 cm. Who is taller and by how much?

ii Use the least squares regression line to predict the difference in height between males and females who both have armspans of 190 cm. Who is taller and by how much?

iii Are the prediction made in **ci.** and **cii.** reliable? Explain.

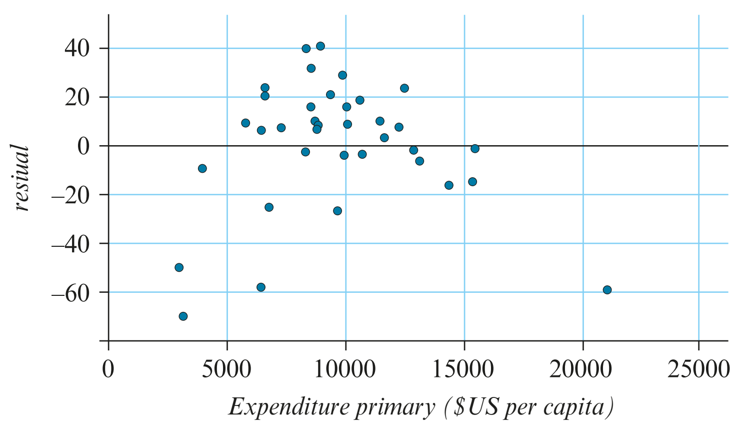
4 The average student PISA mathematics scores (*score*) for OECD countries, as well as the expenditure per primary school child in those countries in $US per capita (*expenditure*), are shown in the following scatterplot.



a Describe the association between *score* and *expenditure* in terms of form and strength.

b Which transformations could be used in order to linearise the association?

c A least squares regression line, with expenditure as the explanatory variable, was fitted to the data, and the following residual plot constructed.



i A residual plot can be used to test an assumption about the nature of the association between two numerical variables. What is this assumption?

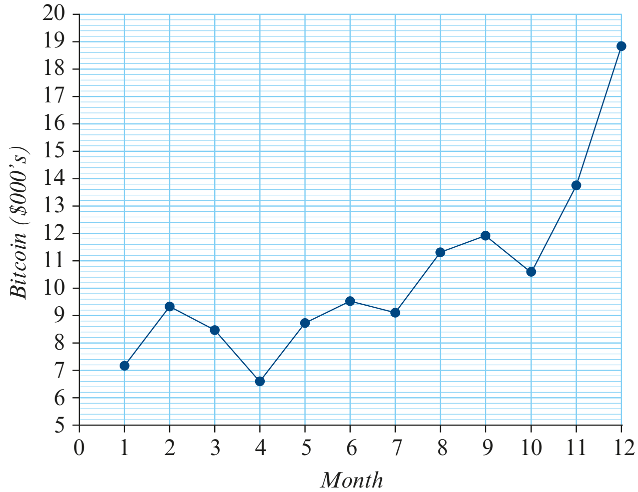
ii Does the residual plot support this assumption? Explain your answer.

d A $\log_{10}$ was applied to the variable *expenditure* to linearise the association. When a least squares line was fitted to the transformed data, it was found to have an intercept of 12.99, and a slope of 120.6.

i Write down the equation of this least squares line.

ii Use the equation from **cii** to predict the PISA mathematics score for a country which has an expenditure of \$US10 000 per capita. Round your answer to the nearest whole number.

5 The value of one bitcoin in Australian dollars at the beginning of each month in 2020 is shown in the time series plot.



a Determine:

i the 5-median smoothed value for Month 10, rounded to the nearest \$000.

ii the 7-median smoothed value for Month 9, rounded to the nearest \$000.

b The following table gives the value of bitcoin in Australian dollars at the beginning of each month for the first six months of 2021.

Month	Jan	Feb	Mar	Apr	May	Jun
Bitcoin($)	29391.78	33543.77	58726.68	57836.01	36681.74	33524.98

Find the centred two-mean smoothed value of bitcoin for the month of March, rounding your answer to the nearest cent.

c A least squares regression line fitted the monthly bitcoin data for 2021 (January 2021 is month 1), giving the following equation:

$$Bitcoin = 36382.73 + 1525.799 \times month$$

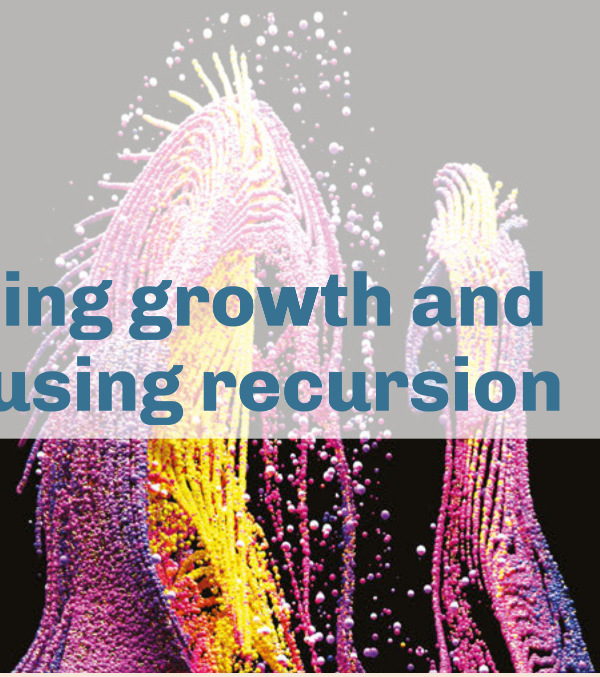
i Write down the value of the slope to the nearest cent, and interpret in terms of the variables in the question.

ii Use the equation to predict the value of bitcoin in January 2024. Give your answer rounded to the nearest dollar.

iii How reliable is the prediction made in **cii**?

Chapter 7

Modelling growth and decay using recursion



Chapter objectives

- What is a sequence?
- How do we generate a sequence of numbers from a starting value and a rule?
- How do we identify particular terms in a sequence?
- What is recursion?
- What is linear growth and decay?
- How can recurrence relations be used to model simple interest, flat rate depreciation and unit cost depreciation on assets?
- How can recurrence relations be used to model compound interest and reducing-balance depreciation on assets?
- How can the CAS calculator be used to find the length of time or the necessary interest rate required for an investment or loan to reach a particular value?
- How can investments and loans be compared using effective interest rates?

In this chapter, the notion of a sequence and recurrence relation are introduced as well as the concepts of linear growth and decay and geometric growth and decay. Taken together, these ideas are applied to financial situations including investments, loans and the depreciation of assets to investigate how much interest must be paid on a loan, how much interest an investment earns or how much an asset depreciates, under different assumptions.

7A Sequences and recurrence relations

Learning intentions

- To be able to generate a sequence of terms recursively.
- To be able to generate a sequence of numbers from a worded description using a calculator.
- To be able to generate a sequence from a recurrence relation.
- To be able to generate a sequence of numbers from a recurrence relation using a calculator.
- To be able to number and name terms in a sequence.

A list of numbers, written down in succession, is called a **sequence**. Each of the numbers in a sequence is called a **term**. We write the terms of a sequence as a list, separated by commas. If a sequence continues indefinitely, or if there are too many terms in the sequence to write them all, we use an *ellipsis*, '. . .'.

Sequences may be either generated randomly or by **recursion** using a rule. For example, this sequence

$$1, 3, 5, 7, 9, \ldots$$

has a definite pattern.

The sequence of numbers has a starting value of 1. We add 2 to this number to generate the next term, 3. Then, add 2 again to generate the next term, 5, and so on.

The rule is 'add 2 to each term'.

$$1 \xrightarrow{+2} 3 \xrightarrow{+2} 5 \xrightarrow{+2} 7 \xrightarrow{+2} 9 \quad \ldots$$

Example 1 Generating a sequence of terms recursively (1)

Write down the first five terms of the sequence with a starting value of 6 and the rule 'add 4 to the previous term'.

Explanation	Solution
1 Write down the starting value.	6
2 Apply the rule (add 4) to generate the next term.	$6 + 4 = 10$
3 Calculate three more terms.	$10 + 4 = 14$ $14 + 4 = 18$ $18 + 4 = 22$
4 Write your answer.	The first five terms are 6, 10, 14, 18, 22.

Example 2 Generating a sequence of terms recursively (2)

Write down the first five terms of the sequence with a starting value of 5 and the rule 'double the number and then subtract 3'.

Explanation	Solution
1 Write down the starting value.	5
2 Apply the rule (double **5**, then subtract 3) to generate the next term.	$5 \times 2 - 3 = 7$
3 Calculate three more terms.	$7 \times 2 - 3 = 11$ $11 \times 2 - 3 = 19$ $19 \times 2 - 3 = 35$
4 Write your answer.	The first five terms are 5, 7, 11, 19, 35.

Using a calculator to generate a sequence of numbers from a rule

All of the calculations to generate sequences from a rule are repetitive. The same calculations are performed over and over again – this is called *recursion*. A calculator can perform recursive calculations very easily because it automatically stores the answer to the last calculation it performed, as well as the method of calculation.

Example 3 Generating a sequence of numbers with a calculator

Use a calculator to generate the first five terms of the sequence with a starting value of 5 and the rule 'double and then subtract 3'.

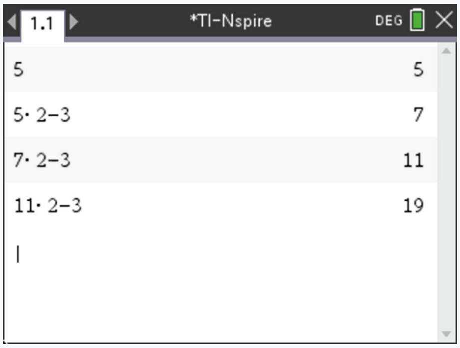
Explanation

Steps

1. Start with a blank computation screen.
2. Type 5 and press enter or EXE.
3. Next type **×2 – 3** and press enter or EXE to generate the next term in the sequence. The computation generating this value is shown as '5·2–3' on the TI-Nspire and 'ans × 2 – 3 on the ClassPad (here 'ans' represents the answer to the previous calculation).

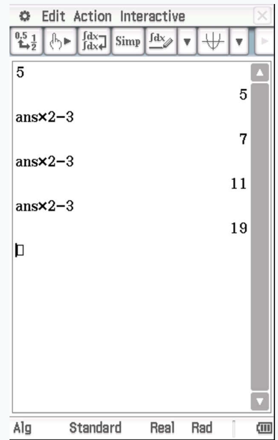
Solution

TI-Nspire



4 Pressing enter or EXE repeatedly applies the rule $\times 2 - 3$ to the last calculated value.

ClassPad



5 State your answer.

The first five terms are 5, 7, 11, 19, 35

Recurrence relations

A **recurrence relation** is a mathematical rule that we can use to generate a sequence. It has two parts:

1. a *starting value*: the value of the first term in the sequence
2. a *rule*: that can be used to generate the next term from the current term.

For example, in words, a recurrence relation that can be used to generate the sequence:

$$10, 15, 20, \ldots$$

can be written as follows:

1. Start with 10.
2. To obtain the next term, add 5 to the current term.

A more compact way of communicating this information is to translate this rule into symbolic form. We do this by defining a subscripted variable. Here we will use the variable V_n, but the V can be replaced by any letter of the alphabet.

Let V_n be the term in the sequence *after n* applications of the rule, called **iterations**.

In words	In symbols
Starting value = 10	$V_0 = 10$
Next term = current term +5	$V_{n+1} = V_n + 5$

Using this definition, we can write a formal recurrence relation where the starting value is defined, followed by the rule for generating the next term.

$$V_0 = 10, \qquad V_{n+1} = V_n + 5$$

Note: Because of the way we defined V_n, the starting value of n is 0. At the start there have been no applications of the rule.

Example 4 Generating a sequence from a recurrence relation

Write down the first five terms of the sequence defined by the recurrence relation

$$V_0 = 29, \qquad V_{n+1} = V_n - 4$$

Explanation	Solution
1 Write down the starting value.	$V_0 = 29$
2 Use the rule to find the next term, V_1.	$V_1 = V_0 - 4$ $= 29 - 4$ $= 25$
3 Use the rule to determine three more terms.	$V_2 = V_1 - 4 = 25 - 4 = 21$ $V_3 = V_2 - 4 = 21 - 4 = 17$ $V_4 = V_3 - 4 = 17 - 4 = 13$
4 Write your answer.	The first five terms are 29, 25, 21, 17, 13

Example 5 Using a calculator to generate sequences from recurrence relations

A sequence is generated by the recurrence relation

$$V_0 = 300, \qquad V_{n+1} = 0.5V_n - 9$$

Use your calculator to generate this sequence and determine how many terms at the start of the sequence are positive.

Explanation

1 Start with a blank computation screen.

2 Type **300** and press enter (or EXE).

3 Next type ×0.5 – 9 and press enter (or EXE) to generate the next term in the sequence.

4 Continue to press enter (or EXE) until the first negative term appears.

5 Write your answer.

Solution

300	300.
300 · 0.5 – 9	141.
141 · 0.5 – 9	61.5
61.5 · 0.5 – 9	21.75
21.75 · 0.5 – 9	1.875
1.875 · 0.5 – 9	–8.625

The first five terms of the sequence are positive.

Example 6 Naming terms in a sequence

Consider the recurrence relation

$$V_0 = 3, \qquad V_{n+1} = V_n + 6$$

State the values of:

a V_1 **b** V_4 **c** V_5

Explanation

1 Write the name for each term under its value in the sequence.

2 Read the value of each required term.

Solution

3,	9,	15,	21,	27,	33
V_0	V_1	V_2	V_3	V_4	V_5

$V_1 = 9, \qquad V_4 = 27 \qquad V_5 = 33$

Exercise 7A

Generating a sequence recursively

Example 1

1 Use the following starting values and rules to generate the first five terms of the following sequences recursively.

a Starting value: 2
Rule: add 6

b Starting value: 5
Rule: subtract 3

c Starting value: 1
Rule: multiply by 4

d Starting value: 64
Rule: divide by 2

Example 2

2 Use the following starting values and rules to generate the first five terms of the following sequences recursively.

a Starting value: 6
Rule: multiply by 2 then add 2

b Starting value: 24
Rule: multiply by 0.5 then add 4

c Starting value: 1
Rule: multiply by 3 then subtract 1

d Starting value: 124
Rule: multiply by 0.5 then subtract 2

Example 3

3 Use the following starting values and rules to generate the first five terms of the following sequences recursively using a calculator.

a Starting value: 4
Rule: add 2

b Starting value: 24
Rule: subtract 4

c Starting value: 2
Rule: multiply by 3

d Starting value: 50
Rule: divide by 5

e Starting value: 5
Rule: multiply by 2 then add 3

f Starting value: 18
Rule: multiply by 0.8 then add 2

Generating sequences using recurrence relations

Example 4

4 Write down the first five terms of the sequences generated by each of the recurrence relations below.

a $W_0 = 2, \quad W_{n+1} = W_n + 3$
b $D_0 = 50, \quad D_{n+1} = D_n - 5$
c $M_0 = 1, \quad M_{n+1} = 3M_n$
d $L_0 = 3, \quad L_{n+1} = -2L_n$
e $K_0 = 5, \quad K_{n+1} = 2K_n - 1$
f $F_0 = 2, \quad F_{n+1} = 2F_n + 3$
g $S_0 = -2, \quad S_{n+1} = 3S_n + 5$
h $V_0 = -10, \quad V_{n+1} = -3V_n + 5$

Example 5

5 Using your calculator, write down the first five terms of the sequence generated by each of the recurrence relations below.

a $A_0 = 12, \quad A_{n+1} = 6A_n - 15$
b $Y_0 = 20, \quad Y_{n+1} = 3Y_n + 25$
c $V_0 = 2, \quad V_{n+1} = 4V_n + 3$
d $H_0 = 64, \quad H_{n+1} = 0.25H_n - 1$
e $G_0 = 48\,000, \quad G_{n+1} = G_n - 3000$
f $C_0 = 25\,000, \quad C_{n+1} = 0.9C_n - 550$

Example 6

6 Consider the following recurrence relations. Find the required term for each.

a $A_0 = 2, \quad A_{n+1} = A_n + 2$. Find A_2.
b $B_0 = 11, \quad B_{n+1} = B_n - 3$. Find B_4
c $C_0 = 1, \quad C_{n+1} = 3C_n$. Find C_3
d $D_0 = 3, \quad D_{n+1} = 2D_n + 1$. Find D_5

7 Write a recurrence relation for each of the following worded descriptions.

a Starting value: 4
Rule: add 2

b Starting value: 24
Rule: subtract 4

c Starting value: 2
Rule: multiply by 3

8 State a recurrence relation that could be used to generate each of the following sequences.

a $5, 10, 15, 20, 25, \ldots$
b $13, 9, 5, 1, -3, \ldots$
c $1, 4, 16, 64, 256, \ldots$
d $64, 32, 16, 8, 4, \ldots$

Exploring sequences with a calculator

9 How many terms of the sequence formed from the recurrence relation below are positive?

$$F_0 = 150, \; F_{n+1} = 0.6F_n - 5$$

10 How many terms of the sequence formed from the recurrence relation below are negative?

$$Y_0 = 30, \; Y_{n+1} = 1.2Y_n + 2$$

Exam 1 style questions

11 A sequence of numbers is generated by the recurrence relation shown below

$$A_0 = 3, \qquad A_{n+1} = 4A_n + 1$$

The value of A_4 is

A 3 **B** 4 **C** 13 **D** 213 **E** 853

12 The following recurrence relation can generate a sequence of numbers

$$A_0 = 15, \qquad A_{n+1} = A_n + 4$$

The number 51 appears in this sequence as

A A_1 **B** A_7 **C** A_8 **D** A_9 **E** A_{10}

13 The first five terms of a sequence are

$$3, 7, 15, 31, 63$$

The recurrence relation that generates this sequence could be

A $B_0 = 3, \quad B_{n+1} = B_n + 4$ **B** $B_0 = 3, \quad B_{n+1} = B_n + 8$

C $B_0 = 3, \quad B_{n+1} = 3B_n - 1$ **D** $B_0 = 3, \quad B_{n+1} = 4B_n - 5$

E $B_0 = 3, \quad B_{n+1} = 2B_n + 1$

7B Modelling linear growth and decay

Learning intentions

- ▶ To be able to graph the terms of a linear growth/decay sequence.
- ▶ To be able to model simple interest loans and investments using recurrence relations.
- ▶ To be able to use a recurrence relation to analyse a simple interest investment.
- ▶ To be able to model and analyse flat rate depreciation using a recurrence relation.
- ▶ To be able to model and analyse unit cost depreciation using a recurrence relation.

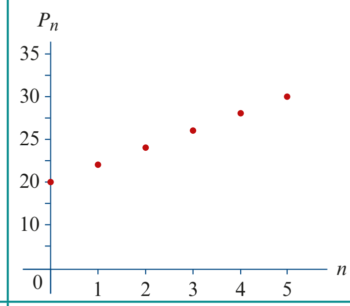
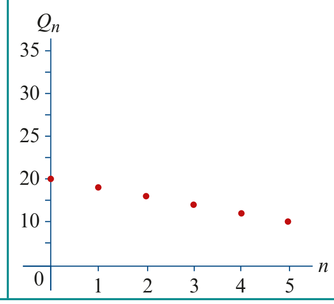
Linear growth means a value is increasing by the same amount in each unit of time. For example, if you have \$300 in your bank account and you add \$20 each week, then your savings will have linear growth. Similarly, **linear decay** is characterised as decreasing by the same amount in each unit of time. For example, the depreciation of a new car by a constant amount each year.

A recurrence model for linear growth and decay

The recurrence relations

$$P_0 = 20, \qquad P_{n+1} = P_n + 2$$
$$Q_0 = 20, \qquad Q_{n+1} = Q_n - 2$$

both have rules that generate sequences with linear patterns, as can be seen from the table below. The first generates a sequence whose successive terms have a linear pattern of growth, and the second a linear pattern of decay.

Recurrence relation	*Rule*	*Sequence*	*Graph*
$P_0 = 20, P_{n+1} = P_n + 2$	'add 2'	$20, 22, 24, \ldots$	
$Q_0 = 20, Q_{n+1} = Q_n - 2$	'subtract 2'	$20, 18, 16, \ldots$	

As a general rule, if D is a positive constant, a recurrence relation rule of the form:

- $V_{n+1} = V_n + D$ can be used to model **linear growth**.
- $V_{n+1} = V_n - D$ can be used to model **linear decay**.

We refer to D as the **common difference** and can graph the sequence to obtain a straight line graph of dots (do not join the dots). An upward slope indicates growth and a downward slope reveals decay.

Example 7 Graphing the terms of linear growth/decay sequence

For each of the following recurrence relations, list the first four terms and graph the corresponding points.

a $V_0 = 2, \quad V_{n+1} = V_n + 5$

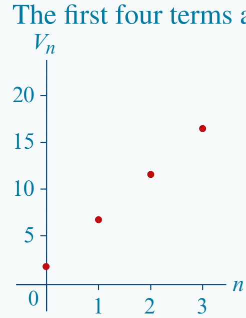
b $W_0 = 20, \quad W_{n+1} = W_n - 3$

Explanation

a From the rule, the starting value is 2.
The rule is 'add 5'.
The corresponding points can then be graphed.

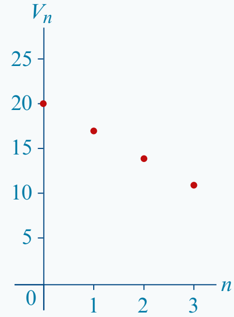
Solution

The first four terms are 2, 7, 12, 17.



b From the rule, the starting value is 20. The rule is 'subtract 3'. The corresponding points can then be graphed.

The first four terms are 20, 17, 14, 11.



Simple interest loans and investments

Simple interest is an example of linear growth in which the starting value is the amount borrowed or invested. The amount borrowed or invested is called the **principal**. The amount added at each step is the interest and is usually a percentage of this principal, found by multiplying the annual interest rate $r\%$ by the principal for each year of the loan.

Recurrence model for simple interest

Let V_n be the value of the loan or investment after n years and r be the annual percentage **interest rate**.

The recurrence relation for the value (or **balance**) of the loan or investment after n years is

$$V_0 = \text{principal}, \qquad V_{n+1} = V_n + D$$

where $D = \dfrac{r}{100} \times V_0$.

Example 8 Modelling simple interest investments with a recurrence relation

Cheryl invests \$5000 in an investment account that pays 4.8% per annum simple interest.

Model this simple investment using a recurrence relation of the form:

$$V_0 = \text{the principal}, \qquad V_{n+1} = V_n + D, \qquad \text{where } D = \frac{r}{100}V_0.$$

Let V_n be the value of the investment after n years.

Explanation	Solution
1 Write down the value of V_0.	$V_0 = 5000$
2 Write down the interest rate r and use it to determine the value of $D = \dfrac{r}{100}V_0$.	$r = 4.8$ $D = \dfrac{4.8}{100} \times 5000 = 240$
3 Use the values of V_0 and D to write down the recurrence relation.	$V_0 = 5000, \qquad V_{n+1} = V_n + 240.$

Once we have a recurrence relation, we can use it to determine the value of an investment after a given number of years.

Example 9 **Using a recurrence relation to analyse a simple interest investment**

Cheryl's simple interest investment is modelled by

$$V_0 = 5000, \qquad V_{n+1} = V_n + 240$$

where V_n is the value of the investment after n years.

a Use the recurrence relation to show that the value of Cheryl's investment after 3 years is \$5720.

b When will Cheryl's investment first exceed \$6000, and what will its value be then?

Explanation

a Calculate V_0, V_1, V_2 and V_3.

Solution

$V_0 = 5000$

$V_1 = 5000 + 240 = 5240$

$V_2 = 5240 + 240 = 5480$

$V_3 = 5480 + 240 = 5720$

Thus, after three years, the value of Cheryl's investment is \$5720.

Explanation

b **i** On a blank calculation screen, type **5000** and press enter (or EXE).

ii Type **+240** and press enter (or EXE) until the value of the investment first exceeds \$6000.

iii Count the number of times that 240 was added. Write your answer.

Solution

5000	5000.
5000. + 240	5240.
5240. + 240	5480.
5480. + 240	5720.
5720. + 240	5960.
5960. + 240	6200.

After 5 years; \$6200.

Depreciation

For some large items, their value decreases over time. This is called **depreciation**. Businesses take into account the impact of depreciation by tracking the likely value of an asset at a point in time, called the **future value**. At some point in time or at a particular value, called the **scrap value**, the item will be sold or disposed of as it is no longer useful to the business.

There are a number of techniques for estimating the future value of an asset. Two of them, **flat rate depreciation** and **unit cost depreciation**, can be modelled using linear decay recurrence relations.

Flat rate depreciation

Flat rate depreciation is an example of linear decay where a constant amount is subtracted from the value of the asset each time period. This constant amount is called the depreciation amount and is often given as a percentage of the initial purchase price of the asset. The **scrap value** is the value at which the item is no longer of use to the business.

Recurrence model for flat-rate depreciation

Let V_n be the value of the asset after n years and r be the percentage depreciation rate.

The recurrence relation for the value of the asset after n years is

$$V_0 = \text{initial value of the asset}, \qquad V_{n+1} = V_n - D$$

where $D = \dfrac{r}{100} \times V_0$.

Example 10 Modelling flat rate depreciation with a recurrence relation

A new car was purchased for \$24 000 in 2014. The car depreciates by 20% of its purchase price each year. Model the depreciating value of this car using a recurrence relation of the form:

$$V_0 = \text{initial value}, \qquad V_{n+1} = V_n - D, \qquad \text{where } D = \frac{r}{100}V_0$$

Let V_n be the value of the car after n years depreciation.

Explanation	Solution
1 Write down the value of V_0. Here, V_0 is the value of the car when new.	$V_0 = 24\,000$
2 Write down the annual rate of depreciation, r, and use it to determine the value of $D = \dfrac{r}{100}V_0$.	$r = 20$ $D = \dfrac{20}{100} \times 24\,000 = 4800$
3 Use the values of V_0 and D to write down the recurrence relation.	$V_0 = 24\,000, \qquad V_{n+1} = V_n - 4800$

Once we have a recurrence relation, we can use it to determine things such as the value of an asset after a given number of years of flat rate depreciation.

Example 11 Using a recurrence relation to analyse flat rate depreciation

The flat rate depreciation of a car is modelled by

$$V_0 = 24\,000, \qquad V_{n+1} = V_n - 4800$$

where V_n is the value of the car after n years.

a Use the model to determine the value of the car after 2 years.

b If the car was purchased in 2023, in what year will the car's value depreciate to zero?

c What was the percentage depreciation rate?

Explanation	Solution
a **i** Write down the recurrence relation.	$V_0 = 24\,000, \; V_{n+1} = V_n - 4800$
ii On a blank calculation screen, type **24 000** and press enter (or EXE). **iii** Type **–4800** and press enter (or EXE) twice to obtain the value of the car after 2 years' depreciation. Write your answer.	24000 → 24000. 24000. – 4800 → 19200. 19200. – 4800 → 14400. 14400. – 4800 → 9600. 9600. – 4800 → 4800. 4800. – 4800 → 0. **a** \$14 400
b **i** Continue pressing enter (or EXE) until the car has no value. **ii** Write your answer.	**b** In 2028
c Use the amount of depreciation and initial value.	**c** $\dfrac{4800}{24000} \times 100\% = 20\%$ The percentage depreciation rate is 20%

Unit cost depreciation

Some items lose value because of how often they are used. A photocopier that is 2 years old but has never been used could still be considered to be in 'brand new' condition and therefore worth the same as it was 2 years ago. But if that photocopier was 2 years old and had printed many thousands of pages over those 2 years, it would be worth much less than its original value.

When the future value of an item is based upon usage, we use a **unit cost** depreciation method. Unit cost depreciation can be modelled using a linear decay recurrence relation.

Recurrence model for unit-cost depreciation

Let V_n be the value of the asset after n units of use and D be the cost per unit of use.

The recurrence relation for the value of the asset after n units of use is:

$$V_0 = \text{initial value of the asset}, \qquad V_{n+1} = V_n - D$$

Example 12 Modelling unit cost depreciation with a recurrence relation

A professional gardener purchased a lawn mower for \$270. The mower depreciates in value by \$3.50 each time it is used.

a Model the depreciating value of this mower using a recurrence relation of the form:

$$V_0 = \text{initial value}, \qquad V_{n+1} = V_n - D$$

where D is the depreciation in value per use and V_n is the value of the mower after being used to mow n lawns.

b Use the model to determine the value of the mower after it has been used three times.

c How many times can the mower be used until its depreciated value is first less than \$250?

Explanation	Solution
a **1** Write down the value of V_0. Here, V_0 is the value of the mower when new.	$V_0 = 270$
2 Write down the unit cost rate of depreciation, D.	$D = 3.50$
3 Write your answer.	$V_0 = 270, \quad V_{n+1} = V_n - 3.50$
b **1** Write down the recurrence relation.	$V_0 = 270, \quad V_{n+1} = V_n - 3.50$
2 On a blank calculation screen, type **270** and press enter (or EXE). Type **–3.50** and press enter (or EXE) three times to obtain the value of the mower after three mows.	270 → 270. 270. – 3.5 → 266.5 266.5. – 3.5 → 263. 263. – 3.5 → 259.5 259.5 – 3.5 → 256. 256. – 3.5 → 252.5 252.5 – 3.5 → 249.
3 Write your answer.	\$259.50
c **1** Continue pressing enter (or EXE) until the value of the lawn mower is first less than \$250.	
2 Write your answer.	After six mows

Skill-sheet

Exercise 7B

Modelling linear growth and decay using recurrence relations

Example 7

1 For each of the following recurrence relations, write down the first four terms and graph the corresponding points.

a $V_0 = 3, \quad V_{n+1} = V_n + 2$

b $V_0 = 38, \quad V_{n+1} = V_n - 5$

Modelling and analysing simple interest with recurrence relations

Example 8

2 Ashwin invests \$8000 in an account that pays 4% per annum simple interest.

a Let V_n be the value of Ashwin's investment after n years. State the starting value, V_0, given by the principal.

b Calculate the value of D using the interest rate and the rule $D = \frac{r}{100}V_0$ to find the amount of interest paid each year.

c Model this simple investment using a recurrence relation of the form

$$V_0 = \text{starting value}, \qquad V_{n+1} = V_n + D$$

3 Huang invests \$41 000 in an account that pays 6.2% per annum simple interest.

a Let H_n be the value of Huang's investment after n years. State the value of H_0.

b Find the amount, in dollars, that Huang will receive each year from the investment.

c Complete the recurrence relation, in terms of H_0, H_{n+1} and H_n, that would model the investment over time. Write your answers in the boxes below.

$$H_0 = \boxed{}, \qquad H_{n+1} = H_n + \boxed{}$$

Example 9

4 The following recurrence relation can be used to model a simple interest investment of \$2000, paying interest at the rate of 3.8% per annum.

$$V_0 = 2000, \qquad V_{n+1} = V_n + 76$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to show that the value of the investment after 3 years is \$2228.

b Use your calculator to determine how many years it takes for the value of the investment to first be worth more than \$3000.

5 The following recurrence relation can be used to model a simple interest loan of \$7000 with interest charged at the rate of 7.4% per annum.

$$V_0 = 7000, \qquad V_{n+1} = V_n + 518$$

In the recurrence relation, V_n is the value of the loan after n years.

a Use the recurrence relation to find the value of the loan after 1, 2 and 3 years.

b Use your calculator to determine how many years it takes for the value of the loan to first have a value of more than \$10 000.

6 The following recurrence relation can be used to model a simple interest investment. In the recurrence relation, V_n is the value of the investment after n years.

$$V_0 = 15\,000, \qquad V_{n+1} = V_n + 525$$

a **i** What is the principal of this investment?

ii How much interest is earned each year?

iii Calculate 525 as a percentage of 15 000 to find the annual interest rate of this investment.

b State how many years it takes for the value of the investment to first exceed \$30 000.

Modelling flat-rate depreciation with recurrence relations

Example 10

7 Fernando purchased a cherry picker for \$82 000 in 2022. The cherry picker depreciates by 15% of its purchase price each year.

Let C_n be the value of the cherry picker n years after it was purchased.

a Calculate the value of D using the interest rate and the rule $D = \frac{r}{100}V_0$ to find the amount of depreciation each year.

b Model this simple investment using a recurrence relation of the form

$$C_0 = \text{starting value}, \qquad C_{n+1} = C_n - D$$

8 Wendy purchases a new chair for her dental surgery for \$2800. The chair depreciates by 8% of its purchase price each year.

a Show that the total amount, in dollars, that Wendy's chair will depreciate by each year is \$244.

b Let W_n be the value of Wendy's chair after n years. State the value of W_0.

c Complete the recurrence relation, in terms of W_0, W_{n+1} and W_n, that would model the investment over time by filling in the boxes below.

$$W_0 = \boxed{}, \qquad W_{n+1} = W_n + \boxed{}$$

Example 11

9 The following recurrence relation can be used to model the depreciation of a computer with purchase price \$2500 and annual depreciation of \$400.

$$V_0 = 2500, \qquad V_{n+1} = V_n - 400$$

In the recurrence relation, V_n is the value of the computer after n years.

a Use the recurrence relation to find the value of the computer after 1, 2 and 3 years.

b Use your calculator recursively to determine how many years it takes for the value of the computer to first be worth less than \$1000.

10 The following recurrence relation can be used to model the depreciation of a car purchased for \$23 000 and depreciated at 3.5% of its original value each year.

$$V_0 = 23\,000, \qquad V_{n+1} = V_n - 805$$

In the recurrence relation, V_n is the value of the car after n years.

a Use the recurrence relation to find the value of the car after 1, 2 and 3 years.

b Determine how many years it takes for the value of the car to first be worth less than \$10 000.

11 The following recurrence relation can be used to model the depreciation of a television. In the recurrence relation, V_n is the value of the television after n years.

$$V_0 = 1500, \qquad V_{n+1} = V_n - 102$$

a **i** What is the purchase price of this television?

ii What is the depreciation of the television each year?

iii What is the annual percentage depreciation of the television?

b Use your calculator to determine the value of the television after 8 years.

c If the owner of the television decides to discard the television once it is first worth less than \$100, determine how long the owner will own the television before discarding it.

Modelling unit-cost depreciation with recurrence relations

Example 12

12 A minibus was purchased for \$32 600 to take passengers to and from the airport. The minibus depreciates by \$10 on every round trip that it takes.
Let M_n be the value of the minibus after n round trips.

a State the starting value, M_0 of the minibus.

b Model the value of the minibus using a recurrence relation of the form

$$M_0 = \text{starting value}, \qquad M_{n+1} = M_n - D$$

13 The following recurrence relation can be used to model the depreciation of a printer with purchase price \$450 and depreciation of 5 cents for every page printed.

$$V_0 = 450, \qquad V_{n+1} = V_n - 0.05$$

In the recurrence relation, V_n is the value of the printer after n pages are printed.

a Write the first five terms of the sequence.

b Use your calculator to find the value of the printer after 20 pages are printed.

14 The following recurrence relation can be used to model the depreciation of a delivery van with purchase price \$48 000 and depreciation by \$200 for every 1000 kilometres travelled.

$$V_0 = 48\,000, \qquad V_{n+1} = V_n - 200$$

In the recurrence relation, V_n is the value of the delivery van after n lots of 1000 kilometres are travelled.

a Use the recurrence relation to find the value of the van after 1000, 2000 and 3000 kilometres.

b Use your calculator to determine the value of the van after 15 000 kilometres.

c Use your calculator to determine how many kilometres it takes for the value of the van to reach \$43 000.

15 Jasmine owns a cafe that sells juices. The commercial blender, purchased for \$1440, depreciates in value using the unit cost method.
The rate of depreciation is \$0.02 per juice that is produced. The recurrence relation that models the year-to-year value, in dollars, of the blender is

$$B_0 = 1440, \qquad B_{n+1} = B_n - 144$$

a Calculate the number of juices that the blender produces each year.

b Determine how many juices the blender can produce before its value becomes 0.

c Use your calculator to find the value of the blender after 36 000 juices have been produced.

d The recurrence relation above could also represent the value of the blender depreciating at a flat rate. What annual flat rate percentage of depreciation is represented?

Exam 1 style questions

16 A coffee machine was purchased for \$720.
After five years the coffee machine has a value of \$586. On average, 670 coffees were made each year during those five years.
The value of the coffee machine was depreciated using a unit cost method of depreciation. The depreciation in the value of the coffee machine, per coffee made, is, in cents, closest to

A 2 **B** 3 **C** 4 **D** 5 **E** 6

17 The value of a tandoori oven is depreciated using the flat rate method and can be modelled using the following recurrence relation where T_n is the value of the oven after n years.

$$T_0 = 4500, \qquad T_{n+1} = T_n - 405$$

The annual depreciation rate is closest to

A 8% **B** 8.5% **C** 9% **D** 9.5% **E** 10%

18 Jane purchased a motorbike for \$5500. She will depreciate the value of her motorbike by a flat rate of 10% of the purchase price per annum.
A recurrence relation that Jane can use to determine the value of the motorbike, V_n, after n years is

A $V_0 = 5500, \qquad V_{n+1} = V_n + 550$
B $V_0 = 5500, \qquad V_{n+1} = V_n - 550$
C $V_0 = 5500, \qquad V_{n+1} = 0.9V_n$
D $V_0 = 5500, \qquad V_{n+1} = 1.1V_n$
E $V_0 = 5500, \qquad V_{n+1} = 0.2(V_n - 550)$

7C Using an explicit rule for linear growth or decay

Learning intentions

- ▶ To be able to convert a recurrence relation to an explicit rule.
- ▶ To be able to model a simple interest investment using an explicit rule.
- ▶ To be able to use a rule to determine the value of a simple interest loan or investment.
- ▶ To be able to model flat rate depreciation of an asset using an explicit rule.
- ▶ To be able to use a rule for the flat rate depreciation of an asset.
- ▶ To be able to use an explicit rule for unit cost depreciation.

While we can generate as many terms of a sequence as we like through repeated addition and subtraction, the process can be tedious and so instead a rule can be used.

Consider the example of investing \$2000 in a simple interest investment paying 5% per annum. If we let V_n be the value of the investment after n years, we can use the following recurrence relation to model this investment:

$$V_0 = 2000, \qquad V_{n+1} = V_n + 100$$

Using this recurrence relation we can write out the sequence of terms generated as follows:

$$\begin{aligned}
V_0 &= 2000 & &= V_0 + 0 \times 100 & &\text{(no interest paid yet)}\\
V_1 &= V_0 + 100 & &= V_0 + 1 \times 100 & &\text{(after 1 year of interest paid)}\\
V_2 &= V_1 + 100 = (V_0 + 100) + 100 & &= V_0 + 2 \times 100 & &\text{(after 2 years of interest paid)}\\
V_3 &= V_2 + 100 = (V_0 + 2 \times 100) + 100 & &= V_0 + 3 \times 100 & &\text{(after 3 years of interest paid)}\\
V_4 &= V_3 + 100 = (V_0 + 3 \times 100) + 100 & &= V_0 + 4 \times 100 & &\text{(after 4 years of interest paid)}
\end{aligned}$$

and so on.

Following this pattern, after n years of interest has been added, we can write:

$$V_n = 2000 + n \times 100$$

This rule can be used to determine the value after n iterations in the sequence. For example, using this rule, the value of the investment after 15 years would be:

$$V_{15} = 2000 + 15 \times 100 = \$3500$$

Explicit rule for linear growth

For a recurrence rule for linear growth of the form:

$$V_0 = \text{initial value}, \qquad V_{n+1} = V_n + D \qquad (D \text{ constant})$$

the value of the term V_n in the sequence generated by this recurrence relation is:

$$V_n = V_0 + nD$$

Explicit rule for linear decay

In general, for a recurrence rule for linear decay of the form:

$$V_0 = \text{initial value}, \qquad V_{n+1} = V_n - D \qquad (D \text{ constant})$$

the value of the term V_n in the sequence generated by this recurrence relation is:

$$V_n = V_0 - nD$$

Example 13 Converting a recurrence relation to an explicit rule

Write down a rule for V_n for each of the following recurrence relations. Calculate V_{10} for each case.

a $V_0 = 8, \quad V_{n+1} = V_n + 3$

b $V_0 = 400, \quad V_{n+1} = V_n - 12$

c $V_0 = 30, \quad V_{n+1} = V_n - 7$

Explanation	Solution
a **1** Identify the starting value.	$V_0 = 8$
2 Identify the common difference, D.	$D = 3$
3 Write the rule for V_n, noting that this is an example of **linear growth**.	$V_n = 8 + 3n$
4 Calculate V_{10}.	$V_{10} = 8 + 3 \times 10 = 38$
b **1** Identify the starting value.	$V_0 = 400$
2 Identify the common difference, D.	$D = 12$
3 Write the rule for V_n, noting that this is an example of **linear decay**.	$V_n = 400 - 12n$
4 Calculate V_{10}.	$V_{10} = 400 - 12 \times 10 = 280$
c **1** Identify the starting value.	$V_0 = 30$
2 Identify the common difference, D.	$D = 7$
3 Write the rule for V_n, noting that this is an example of **linear decay**.	$V_n = 30 - 7n$
4 Calculate V_{10}.	$V_{10} = 30 - 7 \times 10 = -40$

These general rules can be applied to simple interest investments and loans, flat rate depreciation and unit cost depreciation.

Using a rule for simple interest loans or investments

Simple interest loans and investments are examples of linear growth so we use the rule

$V_n = V_0 + nD$, where $D = \dfrac{r}{100} \times V_0$.

Example 14 **Modelling simple interest investments**

Amie invests \$3000 in a simple interest investment with interest paid at the rate of 6.5% per year.

Use a rule to find the value of the investment after 10 years.

Explanation	Solution
1 Identify the starting value.	$V_0 = 3000$
2 Identify the common difference, D.	$D = \frac{6.5}{100} \times 3000 = 195$
3 Write the rule for V_n, noting that this is an example of **linear growth**.	$V_n = 3000 + 195n$
4 Calculate V_{10}.	$V_{10} = 3000 + 195 \times 10 = 4950$

Example 15 **Using a rule to determine the value of a simple interest investment**

The following recurrence relation can be used to model a simple interest investment:

$$V_0 = 3000, \quad V_{n+1} = V_n + 260$$

where V_n is the value of the investment after n years.

a What is the principal of the investment? How much interest is added each year?

b Write down the rule for the value of the investment after n years.

c Use a rule to find the value of the investment after 15 years.

d Use a rule to find when the value of the investment first exceeds \$10 000.

Explanation	Solution
a These values can be read directly from the recurrence relation.	**a** Principal: \$3000 Amount of interest = \$260
b Start with the general rule: $V_n = V_0 + nD$ and substitute $V_0 = 3000$ and $D = 260$.	**b** $V_n = 3000 + n \times 260$ $= 3000 + 260n$
c Substitute $n = 15$ into the rule to calculate V_{15}.	**c** $V_{15} = 3000 + 260 \times 15$ $= \$6900$
d Substitute $V_n = 10\,000$ into the rule, and solve for n. Write your conclusion. Note: Because the interest is only paid into the account after a whole number of years, any decimal answer will need to be *rounded up* to the next whole number.	**d** $10\,000 = 3000 + 260n$ so $7000 = 260n$ or $n = 7000/260$ $= 26.92\ldots$ years The value of the investment will first exceed \$10 000 after 27 years.

Using a rule for flat rate depreciation of assets

Flat rate depreciation is an example of linear decay so we use the rule $V_n = V_0 - nD$.

Example 16 Modelling flat rate depreciation of an asset using an explicit rule

A photocopier costs $6000 when new. Its value depreciates at the flat rate of 17.5% per year. Write a rule and use this to find its value after 4 years.

Explanation	Solution
1 Identify the starting value.	$V_0 = 6000$
2 Identify the common difference, D.	$D = \frac{17.5}{100} \times 6000 = 1050$
3 Write the rule for V_n, noting that this is an example of **linear decay**.	$V_n = 6000 - 1050n$
4 Calculate V_4.	$V_4 = 6000 - 1050 \times 4 = 1800$ The value after 4 years is $1800.

Example 17 Using a rule for the flat rate depreciation of an asset

The following recurrence relation can be used to model the flat rate of depreciation of a set of office furniture:

$$V_0 = 12\,000, \quad V_{n+1} = V_n - 1200$$

where V_n is the value of the furniture after n years.

a What is the initial value of the furniture? How much does the furniture decrease by each year?

b Write down the rule for the value of the investment after n years.

c Use a rule to find the value of the investment after 6 years.

d How long does it take for the furniture's value to decrease to zero?

Explanation	Solution
a These values can be read directly from the recurrence relation.	**a** Initial value: $12 000 Depreciation = $1200 each year
b Start with the general rule $V_n = V_0 - nD$ and substitute $V_0 = 12\,000$ and $D = 1200$.	**b** $V_n = 12\,000 - n \times 1200$ $= 12\,000 - 1200n$
c Use the rule to calculate V_6.	**c** $V_6 = 12\,000 - 1200 \times 6$ $= \$4800$
d Substitute $V_n = 0$, and solve for n. Write your conclusion.	**d** $0 = 12\,000 - n \times 1200$ so $n = 10$ The value of the furniture will depreciate to zero after 10 years.

Using a rule for unit cost depreciation of assets

Example 18 **Using an explicit rule for unit cost depreciation**

A hairdryer in a salon was purchased for \$850. The value of the hairdryer depreciates by 25 cents for every hour it is in use.

Let V_n be the value of the hairdryer after n hours of use.

a Write down a rule to find the value of the hairdryer after n hours of use.

b What is the value of the hairdryer after 50 hours of use?

c On average, the salon will use the hairdryer for 17 hours each week. How many weeks will it take for the value of the hairdryer to halve?

d The hairdryer has a scrap value of \$100 before it is disposed of. Find the number of hours of use before this occurs.

Explanation	Solution
a **1** Identify the values of V_0 and D.	$V_0 = 850$ and $D = 0.25$
2 Write down the rule for the value of the hairdryer after n hours of use.	$V_n = 850 - 0.25n$
b **1** Decide the value of n and substitute into the rule.	After 50 hours of use, $n = 50$. $V_{50} = 850 - 0.25 \times 50$ $V_{50} = 837.50$
2 Write your answer.	After 50 hours of use, the hairdryer has a value of \$837.50.
c **1** Halving the value of the hairdryer means it will have a value of \$425.	Solve $V_n = 425$
2 Write down the rule, with the value of the hairdryer, $V_n = 425$.	$425 = 850 - 0.25n$
3 Solve the equation for n.	$0.25n = 850 - 425$ $0.25n = 425$ $n = 1700$
4 Divide by 17 as the hairdryer is used for 17 hours each week.	Number of weeks = 100
5 Write your answer.	After 100 weeks, the hairdryer is expected to **halve** in value.
d Solve for $V_n = 100$.	$100 = 850 - 0.25n$ $0.25n = 750$ $n = 3000$
Write your answer.	The hairdryer can be used for 3000 hours before it reaches its scrap value.

Skill-sheet

Exercise 7C

Writing an explicit rule from a linear recurrence relation

Example 13

1 Write down a rule for A_n for each of the following recurrence relations. In each case calculate A_{20}.

a $A_0 = 4, \quad A_{n+1} = A_n + 2$ **b** $A_0 = 10, \quad A_{n+1} = A_n - 3$

c $A_0 = 5, \quad A_{n+1} = A_n + 8$ **d** $A_0 = 300, \quad A_{n+1} = A_n - 18$

Using a rule for simple interest loans and investments

Example 14

2 Webster borrows \$5000 from a bank at an annual simple interest rate of 5.4%.

a Let V_n be the value of the loan after n years. State the starting value, V_0.

b Determine how much interest is charged each year in dollars.

c Write down a rule for the value of the loan, V_n, after n years.

d Use your rule to find how much Webster will owe the bank after 9 years.

3 Anthony borrows \$12 000 from a bank at an annual simple interest rate of 7.2%.

a Let V_n be the value of the loan after n years. State the starting value, V_0.

b Determine how much interest is charged each year in dollars.

c Write down a rule for the value of the loan, V_n, after n years.

d Use your rule to find how much Anthony will owe the bank after 9 years.

Example 15

4 The value of a simple interest loan after n years, V_n, can be calculated using the rule $V_n = 8000 + 512n$.

a What is the principal of this loan?

b How much interest is charged every year in dollars?

c Use the rule to find:

i the value of the loan after 12 years

ii when the value of the loan first doubles in value.

5 The value of a simple interest investment after n years, V_n, can be calculated using the rule $V_n = 2000 + 70n$.

a What is the principal of this investment?

b How much interest is earned every year in dollars?

c Use the rule to find:

i the value of the investment after 6 years

ii when the value of the initial investment will first double in value.

Using a rule for flat rate depreciation of assets

Example 16

6 A computer is purchased for \$5600 and is depreciated at a flat rate of 22.5% per year.

- **a** State the starting value.
- **b** Determine the annual depreciation in dollars.
- **c** Write down a rule for the value of the computer, V_n, after n years.
- **d** Find the value of the computer after 3 years.

7 A machine costs \$7000 new and depreciates at a flat rate of 17.5% per annum. The machine will be written off when its value is \$875.

- **a** State the starting value.
- **b** Determine the annual depreciation in dollars.
- **c** Write down a rule for the value of the machine, V_n, after n years.
- **d** Determine the number of full years that the machine will be used (that is, has a value greater than zero).

Example 17

8 The value of a sewing machine after n years, V_n, can be calculated from the rule $V_n = 1700 - 212.5n$.

- **a** What is the purchase price of the sewing machine?
- **b** By how much is the value of the sewing machine depreciated each year in dollars?
- **c** Use the rule to find the value of the sewing machine after 4 years.
- **d** Find its value after 7 years.
- **e** Determine the number of years it takes for the sewing machine to be worth nothing.

9 The value of a harvester after n years, V_n, can be calculated from the rule $V_n = 65000 - 3250n$.

- **a** What is the purchase price of the harvester?
- **b** By how much is the value of the harvester depreciated each year in dollars?
- **c** What is the annual percentage depreciation for the harvester?
- **d** Use the rule to find the value of the harvester after 7 years.
- **e** How long does it take the harvester to reach a value of \$29 250?

Using a rule for unit cost depreciation of assets

Example 18

10 The value of a taxi after n kilometres, V_n, can be calculated from the rule $V_n = 29000 - 0.25n$.

- **a** What is the purchase price of the taxi?
- **b** By how much is the value of the taxi depreciated per kilometre of travel?
- **c** What is the value of the taxi after 20 000 kilometres of travel?
- **d** Find how many kilometres have been travelled if the taxi is valued at \$5000.

11 A car is valued at \$35 400 at the start of the year, and at \$25 700 at the end of that year. During that year, the car travelled 25 000 kilometres.

a Find the total depreciation of the car in that year in dollars.

b Find the depreciation per kilometre for this car.

c Using $V_0 = 35\,400$, write down a rule for the value of the car, V_n, after n kilometres.

d How many kilometres have been travelled if the car has a value of \$6688?

12 A printing machine costing \$110 000 has a scrap value of \$2500 after it has printed 4 million pages.

a Find:

i the unit cost of using the machine

ii the value of the machine after printing 1.5 million pages

iii the annual depreciation of the machine if it prints 750 000 pages per year.

b Find the value of the machine after 5 years if it prints, on average, 750 000 pages per year.

c How many pages has the machine printed by the time the value of the machine is \$70 053?

Exam 1 style questions

13 The value of a bicycle, purchased for \$3800, is depreciated by 10% per annum using the flat rate method.

Recursive calculations can determine the value of the bicycle after n years, B_n.

Which one of the following recursive calculations is **not** correct?

A $V_0 = 3800$

B $V_1 = 0.9 \times 3800$

C $V_2 = 0.9 \times 3420$

D $V_3 = 0.9 \times 3080$

E $V_4 = 0.9 \times 2770.20$

14 An asset is purchased for \$4280.

The value of the asset after n time periods, V_n, can be determined using the rule

$$V_n = 4280 + 25n$$

A recurrence relation that also models the value of this asset after n time periods is

A $V_0 = 4280,\ V_{n+1} = V_n + 25n$

B $V_0 = 4280,\ V_{n+1} = V_n - 25n$

C $V_0 = 4280,\ V_{n+1} = V_n + 25$

D $V_0 = 4280,\ V_{n+1} = V_n - 25$

E $V_0 = 4280,\ V_{n+1} = 25V_n + 4280$

7D Modelling geometric growth and decay

Learning intentions

- To be able to graph the terms of a geometric sequence.
- To be able to model compound interest with a recurrence relation.
- To be able to model reducing balance depreciation with recurrence relations.
- To be able to use reducing balance depreciation with recurrence relations.

A recurrence model for geometric growth and decay

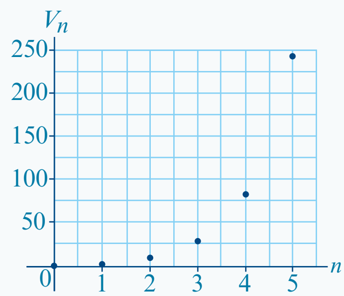
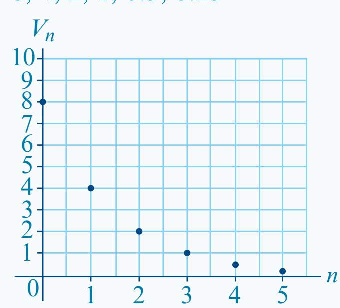
Geometric growth or **decay** occurs when quantities increase or decrease by the same percentage at regular intervals. For example, a sequence that starts with 3 and doubles is given by $3, 6, 12, 24, \ldots$ and can be written as a recurrence relation $V_0 = 3, V_{n+1} = 2V_n$.

Example 19 Graphing the terms in a geometric sequence

For each recurrence relation, state the rule, find the first 6 terms and then plot each point on a graph.

a $V_0 = 1, \quad V_{n+1} = 3V_n$

b $V_0 = 8, \quad V_{n+1} = 0.5V_n$

Explanation	Solution
a 1 Convert to words.	Starting value = 1 Next value = 3 × current value
2 Multiply each term by 3 to find the next term.	1, 3, 9, 27, 81, 243
3 Plot each of the points on the axis.	
b 1 Convert to words.	Starting value = 8 Next value = 0.5 × current value
2 Multiply each term by 0.5 to find the next term.	8, 4, 2, 1, 0.5, 0.25
3 Plot each of the points on the axis.	

As can be seen from the previous example, the first recurrence relation generates a sequence whose successive terms grow geometrically, while the second recurrence relation decays geometrically.

Modelling geometric growth and decay

As a general rule, if R is a positive constant, a recurrence relation rule of the form:

- $V_{n+1} = RV_n$ for $R > 1$, can be used to model *geometric growth*.
- $V_{n+1} = RV_n$ for $R < 1$, can be used to model *geometric decay*.

Compound interest investments and loans

More common than simple interest is **compound interest** where any interest that is earned after one time period is added to the principal and then contributes to the earning of interest in the next time period. This means that the value of the investment grows in ever increasing amounts, or grows geometrically, instead of by the same amount as in simple interest.

Consider an investment of \$5000 that pays 8% interest per annum, compounding yearly. This means that the investment's value increases by 8% each year.

We can model the investment with a recurrence relation as follows:

Let V_n be the value of the investment after n years. Initially the investment is worth \$5000 so $V_0 = \$5000$.

To find the rule between terms:

$$\begin{aligned}\text{next value} &= \text{current value } + 8\% \text{ of current value}\\ &= 108\% \text{ of current value}\\ &= 1.08 \times \text{ current value}\\ &= 1.08 \times V_n\end{aligned}$$

We now have a recurrence relation that we can use to model and investigate the growth of an investment over time. Compound interest loans and investments often accrue interest over periods of less than a year which we will consider at the end of this chapter.

A recurrence model for compound interest investments and loans that compound yearly

Let V_n be the value of the investment after n years.

Let r be the annual percentage interest rate.

The recurrence model for the value of the investment after n years is:

$$V_0 = \text{principal}, \qquad V_{n+1} = RV_n, \qquad \text{where } R = 1 + \frac{r}{100}$$

Example 20 Modelling compound interest with a recurrence relation

The following recurrence relation can be used to model a compound interest investment of \$2000 paying interest at the rate of 7.5% per annum.

$$V_0 = 2000, \qquad V_{n+1} = 1.075 \times V_n$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to show that the value of the investment after 3 years is \$2484.59.

b Determine when the value of the investment will first exceed \$2500.

Explanation	Solution
a **1** Write down the principal, V_0.	$V_0 = 2000$
2 Use the recurrence relation to calculate V_1, V_2 and V_3 and round to the nearest cent.	$V_1 = 1.075 \times 2000 = 2150$ $V_2 = 1.075 \times 2150 = 2311.25$ $V_3 = 1.075 \times 2311.25 = 2484.59$
b **1** Type '2000'. Press enter (or EXE). **2** Type **×1.075**. **3** Count how many times you press enter (or EXE) until the term value is greater than 2500.	2000 2000. 2000. · 1.075 2150. 2150. · 1.075 2311.25 2311.25 · 1.075 2484.59375 2484.59375 · 1.075 2670.93828125
4 Write your answer.	The investment will first exceed \$2500 after 4 years.

Reducing balance depreciation

Reducing balance depreciation is another method of depreciation – one where the value of an asset decays geometrically. Each year, the value will be reduced by a percentage, r%, of the previous year's value.

A recurrence model for reducing balance depreciation

Let V_n be the value of the asset after n years.

Let r be the annual percentage depreciation rate.

The recurrence model for the value of the asset after n years is:

$$V_0 = \text{initial value}, \qquad V_{n+1} = RV_n, \qquad \text{where } R = 1 - \frac{r}{100}$$

Example 21 Modelling reducing balance depreciation with recurrence relations

A sofa was purchased for \$7500 and is depreciating at a reducing balance rate of 8.4% per annum. Write down a recurrence relation where V_n is the value of the sofa after n years.

Explanation

1. Identify the value of V_0.
2. Calculate the value of R.
3. Write your answer.

Solution

$V_0 = 7500$

The depreciation rate is 8.4% per annum.

$R = 1 - \frac{8.4}{100}$ so $R = 0.916$

$V_0 = 7500, \quad V_{n+1} = 0.916 \times V_n$

Example 22 Using reducing balance depreciation with recurrence relations

The following recurrence relation can be used to model the value of office furniture with a purchase price of \$9600, depreciating at a reducing-balance rate of 7% per annum.

$$V_0 = 9600, \quad V_{n+1} = 0.93 \times V_n$$

In the recurrence relation, V_n is the value of the office furniture after n years.

a Use the recurrence relation to find the value of the office furniture, correct to the nearest cent, after 1, 2 and 3 years.

b If the office furniture was initially purchased in 2023, at the end of which year will the value of the investment first be less than \$7000?

Explanation

a 1 Write down the purchase price of the furniture, V_0.

2 Use the recurrence relation to calculate V_1, V_2 and V_3. Use your calculator if you wish.

Solution

$V_0 = 9600$

$V_1 = 0.93 \times 9600 = 8928$

$V_2 = 0.93 \times 8928 = 8303.04$

$V_3 = 0.93 \times 8303.04 = 7721.83$

b Steps

1. Type **9600** and press enter or EXE.
2. Type **× 0.93**.
3. Count how many times you press enter until the term value is less than 7000.

9600	9600.
9600. · 0.93	8928.
8928. · 0.93	8303.04
8303.04 · 0.93	7721.8272
7721.8272 · 0.93	7181.299296
7181.299296 · 0.93	6678.708345

4. Write your answer.

The value of the furniture first drops below \$7000 after 5 years. Thus, it is first worth less than \$7000 at the end of 2028.

Skill-sheet

Exercise 7D

Example 19

1 Generate and graph the first five terms of the sequences defined by the recurrence relations.

a $V_0 = 2, \quad V_{n+1} = 2V_n$

b $V_0 = 3, \quad V_{n+1} = 3V_n$

c $V_0 = 100, \quad V_{n+1} = 0.1V_n$

Modelling compound interest with recurrence relations

Example 20

2 An investment of \$6000 earns compounding interest at the rate of 4.2% per annum. A recurrence relation that can be used to model the value of the investment after n years is shown below.

$$V_0 = 6000, \qquad V_{n+1} = 1.042V_n$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to show that the value of the investment after 3 years is \$6788.20.

b Determine how many years it takes for the value of the investment to first exceed \$8000.

3 A loan of \$20 000 is charged compounding interest at the rate of 6.3% per annum. A recurrence relation that can be used to model the value of the loan after n years is shown below.

$$V_0 = 20\,000, \qquad V_{n+1} = 1.063V_n$$

In the recurrence relation, V_n is the value of the loan after n years.

a Use the recurrence relation to show that the value of the loan after 3 years is \$24 023.14.

b Determine how many years it takes for the value of the loan to first exceed \$30 000.

4 Sue invests \$5000 at a compounding rate of 6.8% per annum.
Let V_n be the value of the investment after n years. This compound interest investment can be modelled by a recurrence relation of the form

$$V_0 = \text{ principal }, \qquad V_{n+1} = R \times V_n$$

a State the value of V_0.

b Determine R using $R = 1 + \dfrac{r}{100}$.

c Write down a recurrence relation for the investment.

d Find the value of the investment after 5 years.

e Find the total interest earned over 5 years.

5 Jay takes out a loan of \$18 000 at a compounding interest rate of 9.4% per annum.

a State the principal (starting value).

b Determine R using $R = 1 + \dfrac{r}{100}$.

c Let V_n be the value of the loan after n years. Write down a recurrence relation for this loan.

d Use the recurrence relation to find the value of the loan after 4 years.

e When will the loan first be valued at more than \$25 000.

Modelling reducing balance depreciation with recurrence relations

Example 21

6 A motorcycle, purchased new for \$9800, will be depreciated using a reducing balance depreciation method with an annual depreciation rate of 3.5%. Write a recurrence relation to model the value of the motorcycle using V_n to represent the value of the motorcycle after n years.

7 Let M_n be the value of a minibus after n years. Write down a recurrence relation for a minibus that was initially valued at \$28 600 and is depreciated at a reducing-balance rate of 7.4% per annum.

Example 22

8 Office furniture was purchased new for \$18 000. It will be depreciated using a reducing balance depreciation method with an annual depreciation rate of 4.5%. Let V_n be the value of the furniture after n years.

a Write a recurrence relation to model the value of the furniture, V_n.

b Use the recurrence relation to find the value of the furniture after each of the first 5 years. Write the values of the terms of the sequence correct to the nearest cent.

c What is the value of the furniture after 3 years?

d What is the total depreciation of the furniture after 5 years?

9 A wedding gown was purchased new for \$4 000. The value of the wedding gown depreciates using a reducing balance depreciation method with an annual depreciation rate of 4.1%. Let W_n be the value of the wedding dress after n years.

a Write a recurrence relation to model the value of the wedding dress, W_n.

b Calculate the value of the wedding dress after three years.

c Determine the total amount of depreciation of the dress after five years.

10 A new computer server was purchased for \$13 420. The value of the computer server depreciates using a reducing-balance depreciation method with an annual depreciation rate of 11.2%. Let S_n be the value of the server after n years.

a Write a recurrence relation to model the value of the server, S_n.

b Use the recurrence relation to find the value of the server after each of the first 5 years. Write the values of the terms of the sequence correct to the nearest cent.

c What is the value of the server after 5 years?

d What is the depreciation of the server in the third year?

Exam 1 style questions

11 The value of a compound interest investment, in dollars, after n years, V_n can be modelled by the recurrence relation shown below.

$$V_0 = 80\,000, \qquad V_{n+1} = 1.02 \times V_n$$

The interest rate, per annum, for this investment is

A 0.02% **B** 0.2%
C 2% **D** 1.02%
E 102%

12 Raymond invests \$6000 in an account that pays interest compounding annually. The balance of his investment after n years, R_n, can be determined using the recurrence relation.

$$R_0 = 6000, \qquad R_{n+1} = 1.0384 \times R_n$$

The total interest earned by Raymond's investment after the first three years is closest to

A \$230.40 **B** \$248.43
C \$469.64 **D** \$469.65
E \$718.08

13 Mana invests \$28 000 at an interest rate of 6.2% per annum, compounding annually. Her investment will first be more than double its original value after

A 1 year **B** 2 years
C 10 years **D** 11 years
E 12 years

14 Giles purchases a jet ski for \$18 000.
The value of the jet ski is depreciated each year by 10% using the reducing balance method.
In the third year, the jet ski will depreciate in value by

A \$1312.20 **B** \$13122
C \$1458 **D** \$11809.80
E \$14580

7E Using an explicit rule for geometric growth or decay

Learning intentions

- To be able to write explicit rules for geometric growth and decay.
- To be able to use an explicit rule to find the value of an investment after n years.
- To be able to calculate the value and total depreciation of an asset after a period of reducing balance depreciation.
- To be able to use a calculator to solve geometric growth and decay problems.

As with linear growth and decay, we can derive a rule to calculate any term in a geometric sequence directly.

Assume \$2000 is invested in a compound interest investment paying 5% per annum, compounding yearly. Let V_n be the value of the investment after n years, giving the following recurrence relation to model this investment:

$$V_0 = 2000, \qquad V_{n+1} = 1.05V_n$$

Using this recurrence relation we can write out the sequence of terms generated as follows:

$$V_0 = 2000$$
$$V_1 = 1.05V_0$$
$$V_2 = 1.05V_1 = 1.05(1.05V_0) = 1.05^2V_0$$
$$V_3 = 1.05V_2 = 1.05(1.05^2V_0) = 1.05^3V_0$$
$$V_4 = 1.05V_3 = 1.05(1.05^3V_0) = 1.05^4V_0$$

and so on.

Following this pattern, after n years of interest are added, we have:

$$V_n = 1.05^nV_0$$

With this rule, we can now find the value of the investment for any specific year. For example, using this rule, the value of the investment after 18 years would be:

$$V_{18} = 1.05^{18} \times 2000 = \$4813.24 \text{ (to the nearest cent)}$$

A rule for individual terms of a geometric growth and decay sequence

For a geometric growth or decay recurrence relation

$$V_0 = \text{starting value}, \qquad V_{n+1} = RV_n$$

the value after n iterations is given by the rule:

$$V_n = R^n \times V_0$$

Exactly the same rule will work for both growth and decay, noting that $R > 1$ is used for growth and $R < 1$ for decay.

Example 23 **Writing explicit rules for geometric growth and decay**

Write down a rule for the value of V_n in terms of n for each of the following. Use the rule to find the value of V_6.

a $V_0 = 5, \quad V_{n+1} = 4V_n$

b $V_0 = 10, \quad V_{n+1} = 0.5V_n$

Explanation	Solution
a **1** $V_0 = 5, R = 4$	$V_n = 4^n \times 5$
2 Substitute $n = 6$.	$V_6 = 4^6 \times 5 = 20480$
b **1** $V_0 = 10, R = 0.5$	$V_n = 0.5^n \times 10$
2 Substitute $n = 6$.	$V_6 = 0.5^6 \times 10 = 0.15625$

Explicit rules for compound interest loans and investments and reducing balance depreciation

Since compound interest loans and investments *increase* over time, the value of R is greater than 1 and can be found using the interest rate of $r\%$ per annum using the formula $R = 1 + \dfrac{r}{100}$.

Compound interest loans and investments

Let V_0 be the amount borrowed or invested (principal).
Let r be the annual percentage interest rate, with interest compounding annually.
The value of a compound interest loan or investment after n years, V_n, is given by the rule

$$V_n = \left(1 + \frac{r}{100}\right)^n \times V_0$$

Example 24 **Using a rule to find the value of an investment after n years**

The rule for the value of the investment after n years, V_n, is shown below.

$$V_n = 1.09^n \times 10\,000$$

a State how much money was initially invested.

b Find the annual interest rate for this investment.

c Find the value of the investment after 4 years, correct to the nearest cent.

d Find the amount of interest earned over the first 4 years, correct to the nearest cent.

e Find the amount of interest earned in the fourth year, correct to the nearest cent.

f Determine if the investor has doubled their money within 10 years.

Explanation	Solution
a Recall the form of the direct rule $V_n = \left(1 + \frac{r}{100}\right)^n \times V_0$. Read off V_0.	\$10 000
b Since $R = 1.09 = 1 + \frac{r}{100}$	$r = 9$ The annual interest rate is 9%.
c **1** Substitute $n = 4$ into the rule for the value of the investment.	$V_4 = 1.09^4 \times 10\,000$ $V_4 = 14\,115.816\ldots$
2 Write your answer, rounded to the nearest cent.	After 4 years, the value of the investment is \$14 115.82, correct to the nearest cent.
d To find the total interest earned in 4 years, subtract the principal from the value of the investment after 4 years.	Amount of interest = \$14 115.82 – \$10 000 = \$4115.82 After 4 years, the amount of interest earned is \$4115.82.
e **1** Calculate V_3 to the nearest cent.	$V_3 = 1.09^3 \times 10\,000$ $V_3 = 12\,950.29$ (nearest cent)
2 Calculate $V_4 - V_3$.	$V_4 - V_3 = 14115.82 - 12950.29$ $= 1165.53$
3 Write your answer. Note: An alternate method is to calculate 9% of V_3.	Interest of \$1165.53 was earned in the fourth year.
f Calculate V_{10} and compare this to double the principal.	We require $V_n = 2 \times V_0 = 20\,000$. Note $V_{10} = 1.09^{10} \times 10\,000 = 23\,673.64$ Since 23 673.64 > 20 000, the investor has doubled their money within 10 years.

With reducing balance depreciation, the value of an asset *declines* over time. The value of R can be found using the formula $R = 1 - \frac{r}{100}$ where r% is the annual depreciation rate.

Reducing balance depreciation

Let V_0 be the purchase price of the asset.
Let r be the annual percentage rate of depreciation.
The value of an asset after n years, V_n, is given by the rule

$$V_n = \left(1 - \frac{r}{100}\right)^n \times V_0$$

Example 25 **Calculating the value and total depreciation of an asset after a period of reducing balance depreciation**

A machine costs \$9500 to buy, and decreases in value with reducing balance depreciation of 20% each year. A recurrence relation that can be used to model the value of the machine after n years, V_n, is shown below.

$$V_0 = 9500, \qquad V_{n+1} = 0.8 \times V_n$$

a Write down the rule for the value of the machine after n years.

b Use the rule to find the value of the machine after 8 years. Write your answer, correct to the nearest cent.

c Calculate the total depreciation of the machine after 8 years.

Explanation	Solution
a **1** Write down the values of V_0 and R.	$V_0 = 9500$ $R = 1 - \frac{20}{100} = 0.8$
2 Write down the rule.	$V_n = R^n \times V_0$ $V_n = 0.8^n \times 9500$
b **1** Substitute $n = 8$ into the rule.	$V_8 = 0.8^8 \times 9500$ $V_8 = 1593.835\ldots$
2 Write your answer, rounding as required.	After 8 years, the value of the machine is \$1593.84, correct to the nearest cent.
c To find the total depreciation after 8 years, subtract the value of the machine after 8 years from the original value of the machine. Write your answer.	Depreciation = \$9500 − \$1593.84 = \$7906.16 After 8 years, the machine has depreciated by \$7906.16.

Using a CAS calculator

As well as finding the value of an investment or loan with compound interest or the value of an asset with reducing balance depreciation, it is also possible to find how long it will take for an investment to reach a particular value.

While this can be done using trial and error, it is also possible to solve these types of problems using a CAS calculator. Similarly, it is also possible to find the annual rate of interest or depreciation which will lead to a particular value in a given number of years.

Example 26 **Using a calculator to solve geometric growth and decay problems to find n**

How many years will it take for an investment of \$5000, paying compound interest at 6% per annum, to grow above \$8000? Write your answer correct to the nearest year.

Explanation

1 Write down the values of V_0, V_n and R.

Solution

$V_0 = 5000, \quad R = 1 + \frac{6}{100} = 1.06$

$V_n = 8000$

2 Substitute into the rule for the particular term of a sequence.

$V_n = R^n \times V_0$

$8000 = 1.06^n \times 5000$

3 Solve this equation for n using a CAS calculator.

$$\boxed{\begin{array}{l} \text{solve}\,(8000 = (1.06)^n \cdot 5000,\ n) \\ \qquad\qquad n = 8.06611354799 \end{array}}$$

4 Write your answer, rounding up as interest is paid at the end of the year. After 8 years, the value is \$7969.24.

The value of the investment will grow above \$8000 after 9 years.

Example 27 **Using a calculator to solve geometric growth and decay problems to find r**

An industrial weaving company purchased a new loom at a cost of \$56 000. It has an estimated value of \$15 000 after 10 years of operation. If the value of the loom is depreciated using a reducing balance method, what is the annual rate of depreciation? Write your answer correct to one decimal place.

Explanation

1 Write down the values of V_0, V_n, R and n.

Solution

$V_0 = 56\,000,\ V_n = 15\,000,\ n = 10$

$R = 1 - \frac{r}{100}$

2 Substitute into the rule for the nth term of a sequence.

$V_n = R^n \times V_0$

$V_{10} = \left(1 - \frac{r}{100}\right)^{10} \times V_0$

$15\,000 = \left(1 - \frac{r}{100}\right)^{10} \times 56\,000$

3 Solve this equation for r using a CAS calculator.

$$\boxed{\begin{array}{l} \text{solve}\left(15000 = \left(1 - \frac{r}{100}\right)^{10} \cdot 56000,\ r\right) \\ r = 12.3422491484 \text{ or} \\ r = 187.657750852 \end{array}}$$

Note: there are two answers. Choose the more appropriate of the two.

4 Write your answer.

The annual rate of depreciation is 12.3%, correct to one decimal place.

Exercise 7E

Writing explicit rules for geometric recurrence relations

Example 23

1 Write down a rule for V_n in terms of n for each of the following recurrence relations. Use each rule to find the value of V_4.

a $V_0 = 6, \quad V_{n+1} = 2V_n$

b $V_0 = 10, \quad V_{n+1} = 3V_n$

c $V_0 = 1, \quad V_{n+1} = 0.5V_n$

d $V_0 = 80, \quad V_{n+1} = 0.25V_n$

Using a rule for compound interest loans and investments

Example 24

2 The value of an investment earning compound interest every year is modelled using the recurrence relation:

$$V_0 = 3000, \qquad V_{n+1} = 1.1V_n.$$

a **i** How much money was invested?

ii What is the annual interest rate for this investment?

b Write down a rule for the value of the investment after n years, V_n.

c Use the rule to find the value of the investment after 5 years. Round your answer to the nearest cent.

3 The value of a loan that is charged compound interest every year is modelled using the recurrence relation:

$$V_0 = 2000, \qquad V_{n+1} = 1.06V_n.$$

a **i** How much money was borrowed?

ii What is the annual interest rate for this loan?

b Write down a rule for the value of the loan after n years, V_n.

c Use the rule to find the value of the loan after 4 years. Round your answer to the nearest cent.

d If the loan is fully repaid after 6 years, what is the total interest that is paid? Round your answer to the nearest cent.

4 Pacey invests \$8000 in an account earning 12.5% compound interest each year. Let V_n be the value of the investment after n years.

a Write down a rule for the value of Pacey's investment after n years.

b Use the rule to find the value of the investment after 3 years. Round your answer to the nearest cent.

c How much interest has been earned after 3 years? Round your answer to the nearest cent.

d How much interest was earned in the third year of the investment? Round your answer to the nearest cent.

Using a rule for reducing balance depreciation

Example 25

5 The value of a stereo system depreciating annually using reducing balance depreciation is modelled using the recurrence relation $V_0 = 1200, \quad V_{n+1} = 0.88V_n$.

a **i** What is the purchase price of the stereo system?

ii At what percentage rate is the stereo system being depreciated?

b Write down a rule for the value of the stereo system after n years.

c Find the value of the stereo system after 7 years. Round your answer to the nearest cent.

6 A car was purchased for \$38 500 and depreciates at a rate of 9.5% per year, using a reducing balance depreciation method. Let V_n be the value of the car after n years.

a Write down a rule for the value of the car after n years.

b Find the value of the car after 5 years. Round your answer to the nearest cent.

c What is the total depreciation of the car over 5 years? Round your answer to the nearest cent.

Using a CAS calculator to solve geometric growth and decay problems

Example 26

7 Sarah invested \$3500 at 6.75% per annum, compounding annually.
If the investment now has a value of \$5179.35, for how many years was it invested?

8 After how many years would an investment of \$200 invested at 4.75% per annum, compounding annually, first exceed a value of \$20 000?

Example 27

9 An investment of \$1000 has grown to \$1601.03 after 12 years invested at r% per annum compound interest. Find the value of r to the nearest whole number.

10 What annual reducing balance depreciation rate would cause the value of a car to drop from \$8000 to \$6645 in 3 years? Give your answer to the nearest percent.

11 How much money must you deposit in a compounding interest investment at a rate of 6.8% per annum if you require \$12 000 in 4 years' time? Round your answer to the nearest cent.

12 A machine has a book value after 10 years of \$13 770.
If it depreciated at a reducing balance rate of 8.2% per annum, what was the initial value of the machine? Round your answer to the nearest cent.

Exam 1 style questions

13 A ute had an initial value of \$68 000 and was depreciated using the reducing balance method. After five years, it had a value of \$37 971.60.
The annual rate of depreciation was closest to

A 9% **B** 10% **C** 11% **D** 55% **E** 56%

14 Amber invests \$15 000 at an interest rate of 5.8% per annum, compounding annually. After how many years will her investment first be more than double its original value?

A 1 **B** 2 **C** 10 **D** 12 **E** 13

7F Interest rates over different time periods and effective interest rates

Learning intentions

- ▶ To be able to convert nominal (annual) interest rates to compounding period interest rates.
- ▶ To be able to model loans with different compounding periods using recurrence relations.
- ▶ To be able to model investments with different compounding periods using recurrence relations.
- ▶ To be able to compare loans and investments using effective interest rates.
- ▶ To be able to calculate effective interest rates using a CAS calculator.

Compound interest rates are usually quoted as annual rates, or interest rates per annum. This rate is called the **nominal interest rate** for the investment or loan. Despite this, interest can be calculated and paid according to a different time period, such as monthly. The time period which compound interest is calculated and paid upon is called the **compounding period**.

Interest rate conversions

The interest rate for the compounding period is calculated based on the following:

- 12 equal months in every year (even though some months have different numbers of days)
- 4 quarters in every year (a quarter is equal to 3 months)
- 26 fortnights in a year (even though there are slightly more than this)
- 52 weeks in a year (even though there are slightly more than this)
- 365 days in a year (ignore the existence of leap years).

A nominal interest rate is converted to a compounding period interest rate by *dividing* by these numbers, which we will refer to with the letter p.

Example 28 Converting nominal interest rates to compounding period interest rates

An investment account will pay interest at the rate of 4.68% per annum. Convert this interest rate to each of the following rates:

a monthly **b** fortnightly **c** quarterly.

Explanation	Solution
a Divide by $p = 12$.	Monthly interest rate $= \dfrac{4.68}{12} = 0.39\%$
b Divide by $p = 26$.	Fortnightly interest rate $= \dfrac{4.68}{26} = 0.18\%$
c Divide by $p = 4$.	Quarterly interest rate $= \dfrac{4.68}{4} = 1.17\%$

Recurrence relations with different compounding periods

An annual interest rate can be converted to a compounding period interest rate and then used in a recurrence relation to model a compounding investment or loan.

To do this, we update our definition of the growth multiplier, R, for compound interest loans and investments as follows:

$$R = 1 + \frac{r}{100 \times p}$$

where r is the annual nominal interest rate and p is the number of compounding periods in each year. If compounding is annual, we use $p = 1$.

Example 29 **Recurrence relations with different compounding periods**

Brian borrows \$5000 from a bank. He will pay interest at the rate of 4.5% per annum.

Let V_n be the value of the loan after n compounding periods.

Write down a recurrence relation to model the value of Brian's loan if interest is compounded:

a yearly **b** quarterly **c** monthly.

Explanation	Solution
a **1** Define the variable V_n. The compounding period is *yearly*.	Let V_n be the value of Brian's loan after n years.
2 Determine the value of R where $r = 4.5$ and $p = 1$.	The interest rate is 4.5% per annum. $R = 1 + \frac{4.5}{100 \times 1} = 1.045$
3 Write the recurrence relation.	$V_0 = 5000, \quad V_{n+1} = 1.045V_n$
b **1** Define the variable V_n. The compounding period is *quarterly*.	Let V_n be the value of Brian's loan after n quarters.
2 Determine the value of R, where $r = 4.5$ and $p = 4$.	The interest rate is 4.5% per annum. $R = 1 + \frac{4.5}{100 \times 4} = 1.01125$
3 Write the recurrence relation.	$V_0 = 5000, \quad V_{n+1} = 1.01125V_n$
c **1** Define the variable V_n. The compounding period is *monthly*.	Let V_n be the value of Brian's loan after n months.
2 Determine the value of R where $r = 4.5$ and $p = 12$.	The interest rate is 4.5% per annum. $R = 1 + \frac{4.5}{100 \times 12} = 1.00375$
3 Write the recurrence relation.	$V_0 = 5000, \quad V_{n+1} = 1.00375V_n$

Example 30 **Modelling an investment that compounds monthly using recurrence relations**

A principal value of \$10 000 is invested in an account earning compound interest monthly at the rate of 9% per annum.

Let V_n be the value of the investment after n months.

a Calculate the growth multiplier, R.

b Write down a recurrence relation for the value of the investment after n months.

c Write down a rule for the value of the investment after n months.

d Use this rule to find the value of the investment after 4 years.

Explanation	Solution
a Since interest compounds monthly, $p = 12$.	$R = 1 + \frac{9}{100 \times 12} = 1.0075$
b Substitute V_0 and R to form the recurrence relation.	$V_0 = 10\,000, \quad V_{n+1} = 1.0075V_n$
c Substitute $R = 1.0075$ and $V_0 = 10\,000$ into the rule to find the rule for V_n.	$V_n = 1.0075^n \times 10\,000$
d Substitute $n = 48$ (4 years = 48 months) into the rule.	$V_{48} = 1.0075^{48} \times 10\,000$ $= \$14314.05$

Effective interest rates

When interest compounds with different compounding periods, the total amount of interest earned in one year differs. To compare loans and investments, we can calculate the **effective interest rate** which is the percentage the value increases in one year.

$$\text{effective rate} = \frac{\text{Total interest in one year}}{\text{Principal}} \times 100\%$$

For example, if \$5000 is invested paying a nominal rate of 4.8% per annum and interest compounds quarterly, the value at the end of the year is \$5244.35 so the interest is \$244.35. Using the formula,

$$\begin{aligned}\text{effective rate} &= \frac{244.35}{5000} \times 100\% \\ &= 4.89\%\end{aligned}$$

In contrast, carrying out the same procedure when interest is compounding monthly gives an effective interest rate of 4.91% due to a value of \$5245.35 at the end of the year and hence interest of \$245.35. Thus, calculating effective interest rates with different compounding periods provides us with a good way of comparing loans and investments.

In order to calculate the effective interest rates for different loans or investments, we can use the following rule.

Effective interest rate

The effective interest rate of a loan or investment is the interest earned after one year expressed as a percentage of the amount borrowed or invested.

Let:

- r be the nominal interest rate per annum
- r_{eff} be the effective annual interest rate
- n be the number of times the interest compounds each year.

The effective annual interest rate is given by: $r_{\text{eff}} = \left(\left(1 + \dfrac{r}{100 \times n}\right)^n - 1\right) \times 100\%$

Note: n is used here in line with the VCAA formula sheet and should not be confused with other usages of n in this chapter.

Example 31 Comparing loans and investments with effective interest rates

Brooke would like to borrow \$20 000 that she will repay entirely after one year. She is deciding between two loan options:

- option A: 5.95% per annum, compounding weekly
- option B: 6% per annum, compounding quarterly.

a Calculate the effective interest rate for each investment.

b Which investment option is the best and why?

Explanation	Solution
a 1 Decide on the values of r and n for each option.	A: $r = 5.95\%$ and $n = 52$ B: $r = 6\%$ and $n = 4$
2 Apply the effective interest rate rule.	A: $r_{\text{eff}} = \left(\left(1 + \dfrac{5.95}{100 \times 52}\right)^{52} - 1\right) \times 100\% = 6.13\%$ B: $r_{\text{eff}} = \left(\left(1 + \dfrac{6}{100 \times 4}\right)^{4} - 1\right) \times 100\% = 6.14\%$
b Compare the effective interest rates.	Brooke is borrowing money, so the best option is the one with the lowest effective interest rate. She will pay less interest with option A.

While the effective interest rate can be calculated manually, the CAS calculator can also be used to quickly perform the calculation. To do this, the nominal interest rate and the number of compounding periods in a year are required.

Example 32 Calculating effective interest rates using a CAS calculator

Marissa has \$10 000 to invest. She chooses an account that will earn compounding interest at the rate of 4.5% per annum, compounding monthly.

Use a CAS calculator to find the effective interest rate for this investment, correct to three decimal places.

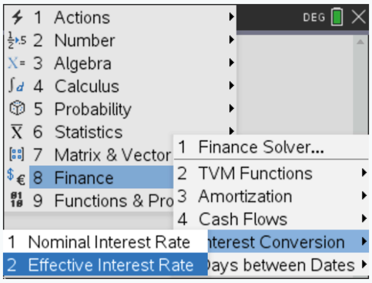
Explanation

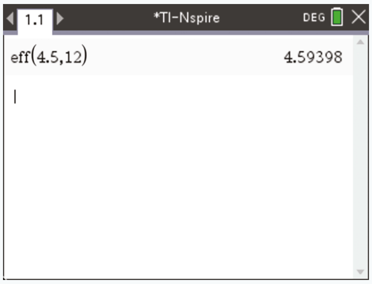
Steps

1 Press menu and then select
8: Finance ▶
5: Interest Conversion ▶
2: Effective interest rate
to paste in the **eff(...)** command.
The parameters of this function are eff(nominal rate, number of times the interest compounds each year).

2 Enter the nominal rate (4.5) and number of times the interest compounds each year (12) into the function, separated by a comma. Press enter to get the effective interest rate.

Solution

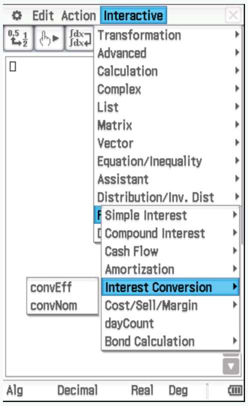


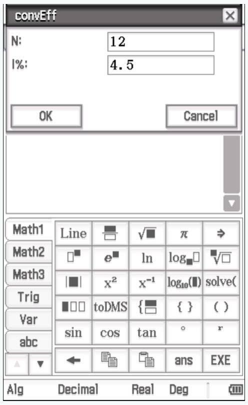


Steps

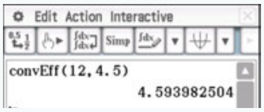
1 Select **Interactive**, **Financial**, **Interest Conversion, ConvEff**.

2 Enter the number of times the interest compounds each year (12) and the nominal rate (4.5) as shown .





3 Press **EXE** to get the effective interest rate. Write your answer.



The effective interest rate for this investment is 4.594%.

Exercise 7F

Interest rate conversions

Example 28

1 Convert each of the annual interest rates below to an interest rate for the given time period. Write your answers correct to two decimal places.

a 4.8% per annum to monthly **b** 8.3% per annum to quarterly
c 10.4% per annum to fortnightly **d** 7.4% per annum to weekly
e 12.7% per annum to daily

2 Convert each of the interest rates below to an annual interest rate.

a 0.54% monthly **b** 1.45% quarterly **c** 0.57% fortnightly
d 0.19% weekly **e** 0.022% daily

Recurrence relations with different compounding periods

Example 29

3 Verity borrows $8000 from a bank. She will pay interest at a rate of 4.8% per annum. Let V_n be the value of the loan after n compounding periods. Write down a recurrence relation to model the value of Verity's loan if interest is compounded:

a yearly **b** quarterly **c** monthly

Example 30

4 A principal value of $20 000 is invested in an account earning compound interest of 6% per annum, compounding monthly.

a Write down a recurrence relation for the value of the investment, V_n, after n months.
b Write down a rule for V_n in terms of n.
c Use this rule to find the value of the investment after 5 years (60 months). Round your answer to the nearest cent.

5 A principal value of $8 000 is invested in an account earning compound interest quarterly at the rate of 4.8% per annum.

a Write down a recurrence relation for the value of the investment, V_n, after n quarters.
b Write down a rule for V_n in terms of n.
c Use this rule to find the value of the investment after 3 years (12 quarters). Round your answer to the nearest cent.

6 Wayne invests $7600 with a bank. He will be paid interest at the rate of 6% per annum, compounding monthly. Let V_n be the value of the investment after n months.

a Write a recurrence relation to model Wayne's investment.
b Write down a rule for V_n in terms of n.
c How much is Wayne's investment worth after 5 months? Round your answer to the nearest cent.
d After how many months will Wayne's investment first double in value?

7 Jessica borrows \$3500 from a bank. She will be charged compound interest at the rate of 8% per annum, compounding quarterly. Let V_n be the value of the loan after n quarters.

a Write a recurrence relation to model the value of Jessica's loan.

b If Jessica pays back everything she owes to the bank after 1 year, how much money will she pay? Round your answer to the nearest cent.

Comparing loans and investments with effective interest rates

Example 31

8 Brenda invests \$15 000 in an account earning (nominal) compound interest of 4.6% per annum.

a Calculate the effective interest rate for the current investment when interest compounds quarterly, correct to two decimal places.

b Calculate the effective rate for this investment when interest compounds monthly, correct to two decimal places.

c Should Brenda choose quarterly or monthly compounding?

9 Stella borrows \$25 000 from a bank and pays nominal compound interest of 7.94% per annum.

a Calculate the effective rate for the current loan when interest compounds fortnightly, correct to two decimal places.

b Calculate the effective rate for this loan when interest compounds monthly, correct to two decimal places.

c Should Stella choose fortnightly or monthly compounding?

10 Luke is considering a loan of \$35 000. His bank has two compound interest rate options:

A: 8.3% per annum, compounding monthly

B: 7.8% per annum, compounding weekly.

a Calculate the effective interest rate for each of the loan options. Round your answers to two decimal places.

b Calculate the amount of interest Luke would pay in the first year for each of the loan options. Round your answers to the nearest cent.

c Which loan should Luke choose and why?

11 Sharon is considering investing \$140 000. Her bank has two compound interest investment options:

A: 5.3% per annum, compounding monthly

B: 5.5% per annum, compounding quarterly.

a Calculate the effective interest rate for each of the investment options. Round your answers to two decimal places.

b Calculate the amount of interest Sharon would earn in the first year for each of the investment options. Give your answer to the nearest dollar.

c Which investment option should Sharon choose and why?

Calculating effective interest rates using the CAS

Example 32

12 Use your calculator to determine the effective annual interest rate, correct to two decimal places, for the following nominal rates and compounding periods.

a 6.2% per annum compounding monthly

b 8.4% per annum compounding daily

c 4.8% per annum compounding weekly

d 12.5% per annum compounding quarterly

13 An account increases by 7% in one year when interest compounds monthly. Find the annual interest rate correct to 2 decimal places.

Exam 1 style questions

14 Chung invests \$3300 with a bank. He will be paid compound interest at the rate of 4.8% per annum, compounding monthly. If V_n is the value of Chung's investment after n months, a recurrence model for Chung's investment is

A $V_0 = 3300, \quad V_{n+1} = V_n + 4.8$

B $V_0 = 3300, \quad V_{n+1} = 4.8V_n$

C $V_0 = 3300, \quad V_{n+1} = 0.004V_n$

D $V_0 = 3300, \quad V_{n+1} = V_n + 158.40$

E $V_0 = 3300, \quad V_{n+1} = 1.004V_n$

15 An amount of \$4700 is invested, earning compound interest at the rate of 6.8% per annum, compounding quarterly. The effective annual interest rate is closest to

A 6.80% **B** 6.97% **C** 6.98% **D** 7.02% **E** 7.03%

16 Isabella invests \$5000 in an account that pays interest compounding monthly.
After one year, the balance of the account is \$5214.09.
The effective interest rate for this investment, rounded to two decimal places, is

A 0.35% **B** 0.42% **C** 3.50% **D** 4.20% **E** 4.28%

17 Maya invested \$25 000 in an account at her bank with interest compounding monthly.
After one year, the balance of Maya's account was \$26 253.
The difference between the rate of interest per annum used by her bank and the effective annual rate of interest for Maya's investment is closest to

A 0.112% **B** 0.2% **C** 4.89%
D 4.9% **E** 5.012%

Key ideas and chapter summary

Assign-ment

Sequence	A **sequence** is a list of numbers or symbols written in succession, for example: 5, 15, 25, . . .
Term	Each number or symbol that makes up a sequence is called a **term**.
Recurrence relation	A relation that enables the value of the next term in a sequence to be obtained by one or more current terms. Examples include 'to find the next term, add two to the current term' and 'to find the next term, multiply the current term by three and subtract five'.
Modelling	**Modelling** is the use of a mathematical rule or formula to represent or model real-life situations. Recurrence relations can be used to model situations involving the *growth* (increase) or *decay* (decrease) in values of a quantity.
Percentage growth and decay	If a quantity grows by $r\%$ each year, then $R = 1 + \frac{r}{100}$. If a quantity decays by $r\%$ each year, then $R = 1 - \frac{r}{100}$.
Principal	The **principal** is the initial amount that is invested or borrowed.
Balance	The **balance** is the value of a loan or investment at any time during the loan or investment period.
Interest	The fee that is added to a loan or the payment for investing money is called the **interest**.
Simple interest	**Simple interest** is a fixed amount of interest that is paid at regular time intervals. Simple interest is an example of linear growth.
Depreciation	**Depreciation** is the amount by which the value of an item decreases after a period of time.
Scrap value	**Scrap value** is the value of an item at which it is 'written off' or is considered no longer useful or usable.
Flat rate depreciation	**Flat rate depreciation** is a constant amount that is subtracted from the value of an item at regular time intervals. It is an example of linear decay.
Unit cost depreciation	**Unit cost depreciation** is depreciation that is calculated based on units of use rather than time. Unit-cost depreciation is an example of linear decay.

Review

Compounding period Interest rates are usually quoted as annual rates (per annum). Interest is sometimes calculated more regularly than once a year, for example each quarter, month, fortnight, week or day. The time period for the calculation of interest is called the **compounding period**.

Compound interest When interest is added to a loan or investment and then contributes to earning more interest, the interest is said to compound. **Compound interest** is an example of geometric growth.

Reducing balance depreciation When the value of an item decreases as a percentage of its value after each time period, it is said to be depreciating using a reducing balance method. **Reducing balance depreciation** is an example of geometric decay.

Nominal interest rate A **nominal interest rate** is an annual interest rate for a loan or investment.

Effective interest rate The **effective interest rate** is the interest earned or charged by an investment or loan, written as a percentage of the original amount invested or borrowed. Effective interest rates allow loans or investments with different compounding periods to be compared. Effective interest rates can be calculated using the rule $r_{\text{eff}} = \left(\left(1 + \frac{r}{100 \times n}\right)^n - 1\right) \times 100\%$ where r is the nominal annual interest rate and n is the number of compounding periods in 1 year.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

7A **1** I can generate a sequence of terms recursively. ☐

See Example 1 and 2, and Exercise 7A Question 1 and 2

7A **2** I can generate a sequence of numbers with a calculator. ☐

See Example 3, and Exercise 7A Question 3

7A **3** I can generate a sequence from a recurrence relation. ☐

See Example 4, and Exercise 7A Question 4

7A **4** I can use a calculator to generate sequences from recurrence relations. ☐

See Example 5, and Exercise 7A Question 5

7A	5	I can name terms in a sequence. See Example 6, and Exercise 7A Question 6	☐
7B	6	I can graph the terms of a linear growth/decay sequence. See Example 7, and Exercise 7B Question 1	☐
7B	7	I can model simple interest loans and investments with a recurrence relation. See Example 8, and Exercise 7B Question 2	☐
7B	8	I can use a recurrence relation to analyse a simple interest investment. See Example 9, and Exercise 7B Question 4	☐
7B	9	I can model flat rate depreciation using a recurrence relation. See Example 10, and Exercise 7B Question 7	☐
7B	10	I can use a recurrence relation to analyse flat rate depreciation. See Example 11, and Exercise 7B Question 9	☐
7B	11	I can model unit cost depreciation with a recurrence relation. See Example 12, and Exercise 7B Question 12	☐
7C	12	I can convert a recurrence relation to an explicit rule. See Example 13, and Exercise 7C Question 1	☐
7C	13	I can model a simple interest investment using an explicit rule. See Example 14, and Exercise 7C Question 2	☐
7C	14	I can use a rule to determine the value of a simple interest investment. See Example 15, and Exercise 7C Question 4	☐
7C	15	I can model flat rate depreciation of an asset with an explicit rule. See Example 16, and Exercise 7C Question 6	☐
7C	16	I can use a rule for the flat rate depreciation of an asset. See Example 17, and Exercise 7C Question 8	☐
7C	17	I can use an explicit rule for unit cost depreciation. See Example 18, and Exercise 7C Question 10	☐
7D	18	I can graph the terms in a geometric sequence. See Example 19, and Exercise 7D Question 1	☐

Review

7D **19** I can model compound interest with a recurrence relation. ☐

See Example 20, and Exercise 7D Question 2

7D **20** I can model reducing balance depreciation with recurrence relations. ☐

See Example 21, and Exercise 7D Question 6

7D **21** I can use reducing balance depreciation with recurrence relations. ☐

See Example 22, and Exercise 7D Question 8

7E **22** I can write explicit rules for geometric growth and decay. ☐

See Example 23, and Exercise 7E Question 1

7E **23** I can use a rule to find the value of an investment after n years. ☐

See Example 24, and Exercise 7E Question 2

7E **24** I can calculate the value and total depreciation of an asset after a period of reducing balance depreciation. ☐

See Example 25, and Exercise 7E Question 5

7E **25** I can use a calculator to solve geometric growth and decay problems. ☐

See Example 26 and 27, and Exercise 7E Question 7 and Question 9

7F **26** I can convert nominal (annual) interest rates to compounding period interest rates. ☐

See Example 28, and Exercise 7F Question 1

7F **27** I can use recurrence relations to model loans with different compounding periods. ☐

See Example 29, and Exercise 7F Question 3

7F **28** I can model an investment that compounds monthly using recurrence relations. ☐

See Example 30, and Exercise 7F Question 4

7F **29** I can compare loans and investments using effective interest rates. ☐

See Example 31, and Exercise 7F Question 8

7F **30** I can calculate effective interest rates using a CAS calculator. ☐

See Example 32, and Exercise 7F Question 12

Multiple-choice questions

1 Consider the following recurrence relation

$$V_0 = 5, \qquad V_{n+1} = V_n - 3$$

The sequence generated by this recurrence relation is

A 5, 15, 45, 135, 405, ...
B 5, 8, 11, 14, 17, ...
C 5, 2, −1, −4, −7, ...
D 5, 15, 45, 135, 405, ...
E 5, −15, 45, −135, 405, ...

2 Consider the following recurrence relation

$$V_0 = 2, \qquad V_{n+1} = 2V_n + 8$$

The value of the term V_4 in the sequence generated by this recurrence relation is

A 12 **B** 18 **C** 32 **D** 72 **E** 152

3 Consider the following recurrence relation

$$V_0 = 5, \qquad V_{n+1} = 3V_n - 6$$

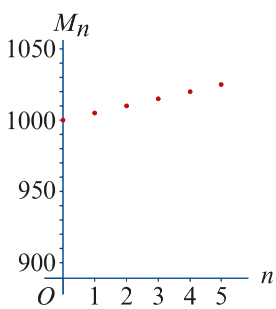
The value of the term V_3 in the sequence of numbers generated by this recurrence relation is

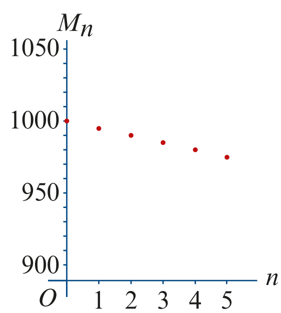
A 5 **B** 9 **C** 21 **D** 57 **E** 165

4 Brian has two trees in his backyard. Every month, he will plant three more trees. A recurrence relation for the number of trees, T_n, in Brian's backyard after n months is

A $T_0 = 2, \qquad T_{n+1} = 3T_n$
B $T_0 = 2, \qquad T_{n+1} = 3T_n + 3$
C $T_0 = 2, \qquad T_{n+1} = T_n + 3$
D $T_0 = 2, \qquad T_{n+1} = T_n - 3$
E $T_0 = 2, \qquad T_{n+1} = 3T_n - 3$

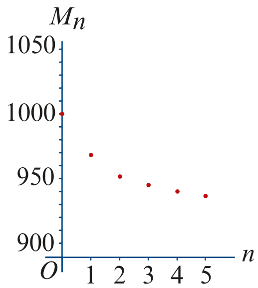
5 A graph that shows the value of a simple interest investment of \$1000, earning interest of \$5 per month is

A


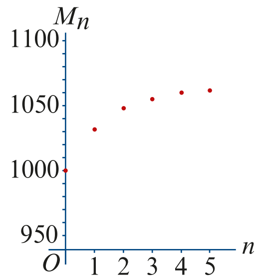
B


Review

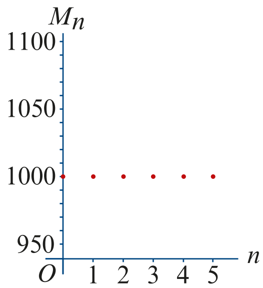
C



D



E



6 A car is depreciated using a unit cost depreciation method. It was purchased for \$18 990 and, after travelling a total of 20 000 kilometres, it has an estimated value of \$15 990. The depreciation amount, per kilometre, is

A \$0.15 B \$0.80 C \$0.95 D \$6.67 E \$3000

7 Arthur invests \$2000 with a bank. He will be paid simple interest at the rate of 5.1% per annum. If V_n is the value of Arthur's investment after n years, a recurrence relation for Arthur's investment is

A $V_0 = 2000, \quad V_{n+1} = V_n + 5.1$

B $V_0 = 2000, \quad V_{n+1} = 5.1V_n$

C $V_0 = 2000, \quad V_{n+1} = 0.051V_n + 102$

D $V_0 = 2000, \quad V_{n+1} = V_n + 102$

E $V_0 = 2000, \quad V_{n+1} = 1.051V_n + 2000$

8 An interest rate of 4.6% per annum is equivalent to an interest rate of

A 1.15% per quarter B 0.35% per month C 0.17% per week

D 0.17% per fortnight E \$0.39% per month

9 A sequence is generated from the recurrence relation

$$V_0 = 40, \qquad V_{n+1} = V_n - 16$$

The rule for the value of the term V_n is

A $V_n = 40n - 16$ B $V_n = 40 - 16n$ C $V_n = 40n$

D $V_n = 40 + 16n$ E $V_n = 40n - 16$

10 The recurrence relation that generates a sequence of numbers representing the value of a car n years after it was purchased is

$$V_0 = 18\,000, \qquad V_{n+1} = V_n - 1098$$

The car had a purchase price of \$18 000 and is being depreciated using

A flat rate depreciation at 6.1% of its value per annum
B flat rate depreciation at \$6.10 per kilometre travelled
C flat rate depreciation at \$1098 per kilometre travelled
D unit cost depreciation at \$6.10 per kilometre travelled
E unit cost depreciation at \$1098 per kilometre travelled

11 A computer is depreciated using a flat rate depreciation method. It was purchased for \$2800 and depreciates at the rate of 8% per annum. The amount of depreciation after 4 years is

A \$224 **B** \$448 **C** \$794 **D** \$896 **E** \$1904

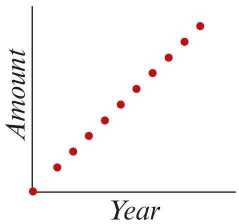
12 Sandra invests \$6000 in an account that pays interest at the rate of 4.57% per annum, compounding annually. The number of years it takes for the investment to exceed \$8000 is

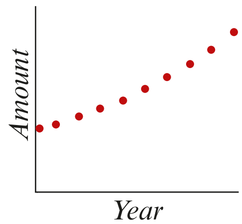
A 5 **B** 6 **C** 7 **D** 8 **E** 9

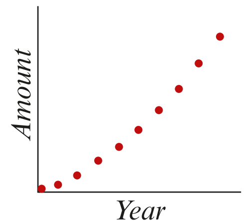
13 The value of a machine is depreciating by 8% every year. The initial value is 2700. A recurrence relation model for the value of the machine after n years, P_n, is

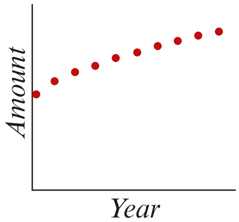
A $P_0 = 2700, \quad P_{n+1} = 1.8 \times P_n$
B $P_0 = 2700, \quad P_{n+1} = 1.08 \times P_n$
C $P_0 = 2700, \quad P_{n+1} = 0.92 \times P_n$
D $P_0 = 2700, \quad P_{n+1} = 1 + 8 \times P_n$
E $P_0 = 2700, \quad P_{n+1} = 1.08 + P_n$

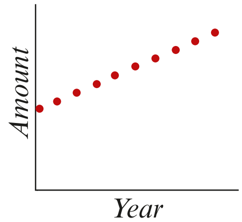
14 An investment of \$50 000 is compounding annually over a number of years. The graph that best represents the value of the investment at the end of each year is

A


B


C


D


E


15 An item is depreciated using a reducing balance depreciation method. The value of the item after n years, V_n, is modelled by the recurrence relation

$$V_0 = 4500, \quad V_{n+1} = 0.86V_n$$

The rule for the value of the item after n years is

A $V_n = 0.86^n \times 4500$

B $V_n = 1.86^n \times 4500$

C $V_n = (1 + 0.86)^n \times 4500$

D $V_n = 0.86 \times n \times 4500$

E $V_n = (1 - 0.86)^n \times 4500$

16 After 10 years, a compound interest investment of \$8000 earned a total of \$4000 in interest when compounding annually. The annual interest rate of this investment was closest to

A 2.5% **B** 4.14% **C** 5.03% **D** 7.2% **E** 50%

17 The interest rate on a compound interest loan is 12.6% per annum, compounding monthly. The value of the loan after n months, V_n, is modelled by the recurrence relation

$$V_0 = 400, \quad V_{n+1} = R \times V_n$$

The value of the growth multiplier, R, in this recurrence relation is

A 0.874 **B** 1.00 **C** 1.0105 **D** 1.126 **E** 2.05

18 An amount of \$2000 is invested, earning compound interest at the rate of 5.4% per annum, compounding quarterly. The effective annual interest rate is closest to

A 5.2% **B** 5.3% **C** 5.4% **D** 5.5% **E** 5.6%

19 A car was purchased for \$74 500. It depreciates in value at a rate of 8.5% per year, using a reducing balance depreciation method. The total depreciation of the car over 5 years is closest to

A \$4439

B \$26 718

C \$37 522

D \$47 782

E \$112 022

20 Sam invested \$6500 at 8.75% per annum with interest compounding monthly. If the investment now amounts to \$13 056, for how many years was it invested?

A 5 **B** 7 **C** 8 **D** 9 **E** 96

Written response questions

1 Jack borrows \$20 000 from a bank and is charged simple interest at the rate of 9.4% per annum. Let V_n be the value of the loan after n years.

a Write down a recurrence relation for the value of Jack's loan after n years.

b Use the recurrence relation to model how much Jack will need to pay the bank after 5 years.

The bank decides to change the loan to a compound interest loan on a yearly basis, with an annual interest rate of 9.4%. Let W_n be the value of the loan after n years.

c Write a recurrence relation to model the value of Jack's loan.

d Write a rule for W_n in terms of n.

e Use the rule to find the value of the loan after 5 years. Round your answer to the nearest cent.

2 Ilana uses a personal loan to buy a dress costing \$300. Interest is charged at 18% per annum, compounding monthly.
If she repays the loan fully after 6 months, how much will she pay? Round your answer to the nearest cent.

3 Kelly bought her current car 5 years ago for \$22 500.
Let V_n be the value of Kelly's car after n years.

a If Kelly uses a flat rate depreciation of 12% per annum:

i write down a recurrence relation for the value of Kelly's car after n years

ii use the recurrence relation to find the current value of Kelly's car.

b If Kelly uses reducing value depreciation at 16% per annum:

i write down a recurrence relation for the value of Kelly's car after n years

ii use the recurrence relation to find the current value of Kelly's car using reducing balance depreciation. Round your answer to the nearest cent.

c On the same axes, sketch a graph of the value of Kelly's car against the number of years for both flat rate and reducing balance depreciation.

4 A commercial cleaner bought a new vacuum cleaner for \$650. The value of the vacuum cleaner decreases by \$10 for every 50 offices that it cleans.

a How much does the value of the vacuum cleaner depreciate when one office is cleaned?

b Give a recurrence relation for the value of the vacuum cleaner, V_n, after n offices have been cleaned.

c The cleaner has a contract to clean 10 offices, 5 nights a week for 40 weeks in a year. What is the value of the vacuum cleaner after 1 year?

5 Meghan has \$5000 to invest.

Company A offers her an account paying 6.3% per annum simple interest.

Company B offers her an account paying 6.1% per annum compound interest.

a How much will she have in the account offered by company A at the end of 5 years?

b How much will she have in the account offered by company B at the end of 5 years? Round your answer to the nearest cent.

c Find the simple interest rate that company A should offer if the two investments are to have equal values after 5 years. Round your answer to one decimal place.

6 A sum of \$30 000 is borrowed at an interest rate of 9% per annum, compounding monthly.

Let V_n be the value of the loan after n months.

a Write a recurrence relation to model the value of this loan.

b Use the recurrence relation to find the value of the loan at the end of the first five months. Round your answer to the nearest cent.

c What is the value of the loan after 1 year? Round your answer to the nearest cent.

d If the loan is fully repaid after 18 months, how much money is paid? Round your answer to the nearest cent.

7 On the birth of his granddaughter, a man invests a sum of money at a rate of 11.65% per annum, compounding twice per year.

On her 21st birthday he gives all of the money in the account to his granddaughter.

If she receives \$2529.14, how much did her grandfather initially invest? Round your answer to the nearest cent.

8 Geoff invests \$18 000 in an investment account. After 2 years the investment account contains \$19 300.

If the account pays r% interest per annum, compounding quarterly, find the value of r, to one decimal place.

Chapter 8

Reducing balance loans, annuities and investments

Chapter objectives

- ▶ How can we combine both linear and geometric growth/decay?
- ▶ How do we model a compound interest investment where additional payments are made?
- ▶ How can recurrence relations be used to model reducing balance loans?
- ▶ How can recurrence relations be used to model annuities?
- ▶ What are amortisation tables and how can they be used?
- ▶ How can a finance solver be used to analyse reducing balance loans, annuities and investments with additional payments?
- ▶ What are interest-only loans?
- ▶ What are perpetuities?

Often loans and investments are more complex than described in the previous chapter. In particular, loans are often paid off through regular payments, investments may have additional contributions made throughout their life and interest rates may change. In this chapter, we analyse investments with additional payments, reducing balance loans, annuities, interest-only loans and perpetuities using our already developed tool of recurrence relations as well as new tools such as amortisation tables and the Finance Solver found on a CAS.

8A Combining linear and geometric growth or decay to model compound interest investments with additions to the principal

Learning intentions

- ▶ To be able to generate a sequence from a recurrence relation that combines both geometric and linear growth or decay.
- ▶ To be able to model compound interest investments with additions to the principal.
- ▶ To be able to use a recurrence relation to analyse compound interest investments with additions to the principal.
- ▶ To be able to determine the annual interest rate from a recurrence relation.

In the previous chapter, recurrence relations were used to model financial situations with linear and geometric growth/decay such as simple and compound interest and the depreciation of assets. Recurrence relations can also be used to model situations that involve elements of both linear and geometric growth/decay.

There are several examples in finance that involve both geometric and linear growth or decay. For example, an investment with compound interest grows geometrically over time but might also have linear growth if regular additions are made. Alternatively, a personal loan may be paid off with regular payments rather than at the conclusion of the loan period.

In general, a recurrence relation of the form

$$V_0 = \text{starting value}, \qquad V_{n+1} = R \times V_n \pm D$$

can be used to model situations that involve both geometric and linear growth/decay.

Example 1 **Generating a sequence from a recurrence relation of the form $V_{n+1} = R \times V_n \pm D$**

Write down the first five terms of the sequence generated by the recurrence relation

$$V_0 = 3, \qquad V_{n+1} = 4V_n - 1$$

Explanation	Solution
1 Write down the starting value.	3
2 Apply the rule (multiply by 4, then subtract 1) to generate four more terms.	$3 \times 4 - 1 = 11$ $11 \times 4 - 1 = 43$ $43 \times 4 - 1 = 171$ $171 \times 4 - 1 = 683$
3 Write your answer.	The first five terms are 3, 11, 43, 171, 683

Compound interest investments with regular additions to the principal

In the previous chapter, we considered compound interest investments. Here, we consider the option of adding to the investment by making additional payments on a regular basis. This is called an **annuity investment**.

Modelling compound interest investments with regular additions to the principal

Let V_n be the value of the compound interest investment (annuity investment) after n additional payments have been made. Then

$$V_0 = \text{the principal}, \qquad V_{n+1} = RV_n + D$$

where D is the *additional payment* made, $R = 1 + \dfrac{r}{100 \times p}$ is the growth multiplier, r is the *annual interest rate* and p is the *number of compounding periods* per year.

To start, we recall compound interest investments without additional payments as in Chapter 7. Imagine Fred has \$5000 to invest in an account paying compound interest of 4% per annum, compounding annually. The starting value is $V_0 = 5000$. To find R, we use the formula $R = 1 + \dfrac{r}{100 \times p}$ where $r = 4$ and $p = 1$ to give $R = 1.04$. Thus, the recurrence relation is

$$V_0 = 5000, \qquad V_{n+1} = 1.04V_n$$

Now we can consider the possibility of regular additions to the principal.

Example 2 Modelling compound interest investments with additions to the principal (1)

Fred has saved \$5000 and invests this in a compound interest account paying 4% per annum, compounding yearly. He also adds an extra \$1000 each year.

Model this investment using a recurrence relation of the form

$$V_0 = \text{the principal}, \qquad V_{n+1} = RV_n + D$$

where V_n is the value of the investment after n years.

Explanation	Solution
1 Write down the value of V_0 and D where D is the amount added each year.	$V_0 = 5000$ and $D = 1000$
2 Determine the value of R using $R = 1 + \dfrac{r}{100 \times p}$ where $r = 4$ and $p = 1$ because interest compounds annually.	$R = 1 + \dfrac{4}{100 \times 1} = 1.04$
3 Use the values of V_0, R and D to write down the recurrence relation.	$V_0 = 5000, \quad V_{n+1} = 1.04V_n + 1000$

When interest compounds at intervals other than a year, we need to find the interest rate for the compounding period which is calculated based on the following:

- 12 equal months in every year (even though some months have different numbers of days)
- 4 quarters in every year (a quarter is equal to 3 months)
- 26 fortnights in a year (even though there are slightly more than this)
- 52 weeks in a year (even though there are slightly more than this)
- 365 days in a year (ignore the existence of leap years).

This gives us the value of p which we use in the formula

$$R = 1 + \frac{r}{100 \times p}$$

to find the growth multiplier, R.

Example 3 Modelling compound interest investments with additions to the principal (2)

Nor invests \$1200 and plans to add an extra \$50 each month. The account pays interest at a rate of 3% per annum, compounding monthly.

Model this investment using a recurrence relation of the form

$$V_0 = \text{the principal}, \qquad V_{n+1} = RV_n + D$$

where V_n is the value of the investment after n months.

Explanation	Solution
1 Write down the value of V_0 and D where D is the amount added each month.	$V_0 = 1200$ and $D = 50$
2 Determine the value of R using the formula $R = 1 + \frac{r}{100 \times p}$ where $r = 3$ and $p = 12$ because interest compounds monthly.	$R = 1 + \frac{3}{100 \times 12} = 1.0025$
3 Use the values of V_0, R and D to write down the recurrence relation.	$V_0 = 1200, \quad V_{n+1} = 1.0025V_n + 50$

Once we have a recurrence relation, we can use it to determine the value of the investment after a given number of periods once interest has been paid and extra payments have been added to the principal. This value can be plotted on a graph so that we can see the impact of making additional payments over time.

Example 4 **Using a recurrence relation to analyse compound interest investments with additions to the principal**

Albert has an investment that can be modelled by the recurrence relation

$$V_0 = 400, \qquad V_{n+1} = 1.005V_n + 30$$

where V_n is the value of the investment after n months.

a State the value of the initial investment.

b Determine the value of the investment after Albert has made three extra payments. Round your answer to the nearest cent.

c What will be the value of his investment after 6 months? Round your answer to the nearest cent.

d Plot the points for the value of the investment after 0, 1, 2 and 3 months on a graph.

Explanation

a Note that $V_0 = 400$.

Solution

The initial investment was \$400.

Explanation

b Perform the calculations.

Solution

$V_0 = \$400$

$V_1 = 1.005 \times 400 + 30 = \$432$

$V_2 = 1.005 \times 432 + 30 = \$464.16$

$V_3 = 1.005 \times 464.16 + 30 = \$496.48$

The value of Albert's investment is \$496.48.

Explanation

c **i** Either continue performing the calculations or use your CAS by:

ii Type 400 and press enter (or EXE).

iii Type $\times$ 1.005 + 30 and press enter (or EXE) six more times.

Solution

400	400
400 · 1.005 + 30	432
432 · 1.005 + 30	464.16
464.16 · 1.005 + 30	496.48
496.4808 · 1.005 + 30	528.96
528.9632 · 1.005 + 30	561.61
561.6180 · 1.005 + 30	594.42

Explanation

ii Write your answer.

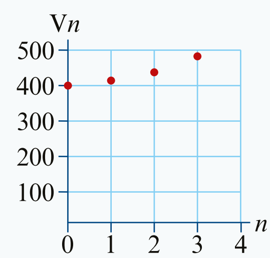
Solution

The value of Albert's investment is \$594.42.

Explanation

d Plot each of the points on the graph.

Solution



From the example above, it is clear that making additional payments on a regular basis causes the investment to increase more. In particular, making additional payments early on is beneficial as compound interest is earned on the additional payment for longer. An example of this type of investment might be saving for retirement.

Sometimes we are given a recurrence relation and asked to determine the annual interest rate. To do this, we use the formula $R = 1 + \dfrac{r}{100 \times p}$.

Example 5 Determining the annual interest rate from a recurrence relation

Determine the annual interest rates for each of the following investments.

a Consider an investment given by the recurrence relation

$$A_0 = 400, \qquad A_{n+1} = 1.005V_n + 30$$

where A_n is the value of the investment after n months.

b Consider an investment given by the recurrence relation

$$W_0 = 2000, \qquad W_{n+1} = 1.012V_n + 500$$

where W_n is the value of the investment after n quarters.

Explanation	Solution
a Solve $R = 1 + \dfrac{r}{100 \times p}$ for r where $R = 1.005$ and $p = 12$ because interest is compounded monthly.	Solve $1.005 = 1 + \dfrac{r}{100 \times 12}$. $r = 6$ Thus, the annual interest rate is 6%.
b Solve $R = 1 + \dfrac{r}{100 \times p}$ for r where $R = 1.012$ and $p = 4$ because interest is compounded quarterly.	Solve $1.012 = 1 + \dfrac{r}{100 \times 4}$. $r = 4.8$ Thus, the annual interest rate is 4.8%.

Exercise 8A

Generating a sequence using a recurrence relation

Example 1

1 Write down the first five terms of the sequences generated by the following recurrence relations.

a $A_0 = 2, \qquad A_{n+1} = 2A_n + 1$

b $B_0 = 50, \qquad B_{n+1} = 2B_n - 10$

c $C_0 = 128, \qquad C_{n+1} = 0.5C_n + 32$

Modelling compound interest investments with additions to the principal

Example 2

2 Molly has saved \$500 and plans to add an extra \$100 per year to an investment account immediately after the interest payment is calculated. The account pays 3% per annum, compounding annually.
Let V_n is the value of the investment after n years.

a State the initial value of the investment, V_0.

b State the amount added each year, D.

c Determine the value of the growth multiplier, R.

d Model this investment using a recurrence relation of the form

$$V_0 = \text{ the principal}, \quad V_{n+1} = RV_n + D$$

3 Jane has already saved \$300 000 and plans to add an extra \$50 000 per year to an investment account immediately after the interest payment is calculated. The account pays interest of 5.2% per annum, compounding annually.
Let V_n is the value of the investment after n years.

a State the initial value of the investment, V_0.

b State the amount added each year, D.

c Determine the value of the growth multiplier, R.

d Model this investment using a recurrence relation of the form

$$V_0 = \text{ the principal}, \quad V_{n+1} = RV_n + D$$

Example 3

4 Henry invests \$3500 and plans to add an extra \$150 per month after the interest is calculated. The account pays interest of 3.6% per annum, compounding monthly.
Let V_n be the value of the investment after n months.

a Determine the value of the growth multiplier, R.

b Model this investment using a recurrence relation of the form

$$V_0 = \text{ the principal}, \quad V_{n+1} = RV_n + D$$

c What is the value of the investment after two months? Round your answer to the nearest cent.

5 Lois invests \$1700 and plans to add an extra \$100 per quarter. The account pays interest of 3.2% per annum, compounding quarterly.
Let V_n be the value of the investment after n quarters.

a Model this investment using a recurrence relation of the form

$$V_0 = \text{ the principal}, \quad V_{n+1} = RV_n + D$$

b What is the value of the investment after six quarters? Round your answer to the nearest cent.

6 Sarah invests \$1500 at 7.3% per annum, compounding daily. She plans to add an extra \$4 to her investment each day, immediately after the interest is calculated.
Let V_n be the value of the investment after n days.
Write down a recurrence relation to model Sarah's investment.

7 Rachel invests \$24 000 at 6% per annum, compounding monthly. She plans to add an extra \$500 to her investment each month.
Let V_n be the value of the investment after n months.
Write down a recurrence relation to model Rachel's investment and determine the value of the investment after six months. Round your answer to the nearest cent.

Using a recurrence relation to model and analyse an investment with additions to the principal

Example 4

8 A compound interest investment with regular yearly additions to the principal can be modelled by the recurrence relation

$$V_0 = 2000, \qquad V_{n+1} = 1.08V_n + 1000$$

where V_n is the value of the investment after n years.

a What is the principal of this investment?

b How much is added to the principal each year?

c Use your calculator to determine the balance of the investment after 2 years. Round your answer to the nearest cent.

d Plot the value of the investment after 0, 1 and 2 years on a graph.

9 A compound interest investment with regular quarterly additions to the principal can be modelled by the recurrence relation

$$V_0 = 20\,000, \qquad V_{n+1} = 1.025V_n + 2000$$

where V_n is the value of the investment after n quarters.

a What is the principal of this investment?

b How much is added to the principal each quarter?

c Use your calculator to determine the balance of the investment after three quarterly payments have been made. Round your answer to the nearest cent.

d Plot the value of the investment after 0, 1, 2 and 3 quarters on a graph.

Example 5

10 Consider the compound interest investment with regular annual additions to the principal given by the recurrence relation

$$V_0 = 2000, \qquad V_{n+1} = 1.08V_n + 1000$$

where V_n is the value of the investment after n years.
Determine the annual interest rate for the investment.

11 Consider the compound interest investment with regular quarterly additions to the principal given by the recurrence relation:

$$V_0 = 20\,000, \qquad V_{n+1} = 1.025V_n + 2000$$

where V_n is the value of the investment after n quarters.
Determine the annual interest rate for the investment.

Exam 1 style questions

12 The value of an annuity investment, in dollars, after n years, V_n, can be modelled by the recurrence relation shown below

$$V_0 = 54\,000, \quad V_{n+1} = 1.0055V_n + 1500$$

What is the value of the regular payment added to the principal of this annuity investment?

A \$55 **B** \$297 **C** \$1500 **D** \$1797 **E** \$5400

13 The value of an annuity investment, in dollars, after n quarters, V_n, can be modelled by the recurrence relation shown below

$$V_0 = 36\,000, \quad V_{n+1} = 1.008V_n + 200$$

The increase in the value of this investment in the third quarter is closest to

A \$200.00
B \$495.84
C \$499.81
D \$1475.74
E \$37 475.74

14 Consider the following five recurrence relations representing the value of an asset after n years, V_n.

$$V_0 = 10\,000, \qquad V_{n+1} = V_n + 1500$$

$$V_0 = 10\,000, \qquad V_{n+1} = V_n - 1500$$

$$V_0 = 10\,000, \qquad V_{n+1} = 1.15V_n - 1500$$

$$V_0 = 10\,000, \qquad V_{n+1} = 1.125V_n - 1500$$

$$V_0 = 10\,000, \qquad V_{n+1} = 1.25V_n - 1500$$

How many of these recurrence relations indicate that the value of an asset is depreciating?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

8B Using recurrence relations to analyse and model reducing balance loans and annuities

Learning intentions

- To be able to model a reducing balance loan with a recurrence relation.
- To be able to use a recurrence relation to analyse a reducing balance loan.
- To be able to model an annuity with a recurrence relation.
- To be able to use a recurrence relation to analyse an annuity.

Reducing balance loans

When money is borrowed from a bank, the borrower usually makes regular payments to reduce the amount owed, rather than waiting until the end of the loan to repay the balance. This kind of loan is called a **reducing balance loan**.

Modelling reducing balance loans

Let V_n be the *balance* of the loan after n payments have been made. Then

$$V_0 = \text{the principal}, \quad V_{n+1} = RV_n - D$$

where D is the *additional payment* made, $R = 1 + \dfrac{r}{100 \times p}$ is the growth multiplier, r is the *annual interest rate* and p is the *number of compounding periods* per year.

Example 6 Modelling a reducing balance loan with a recurrence relation (1)

Flora borrows \$8000 at an interest rate of 13% per annum, compounding annually. She makes yearly payments of \$2100.

Construct a recurrence relation to model this loan, in the form

$$V_0 = \text{the principal}, \quad V_{n+1} = RV_n - D$$

where V_n is the balance of the loan after n years.

Explanation	Solution
1 State V_0 and D.	$V_0 = 8000$ and $D = 2100$
2 Determine the value of R using the formula $R = 1 + \dfrac{r}{100 \times p}$, where $r = 13$ and $p = 1$.	$R = 1 + \dfrac{13}{100 \times 1} = 1.13$
3 Use the values of V_0, R and D to write down the recurrence relation.	$V_0 = 8000, \quad V_{n+1} = 1.13V_n - 2100$

Example 7 Modelling a reducing balance loan with a recurrence relation (2)

Alyssa borrows \$1000 at an interest rate of 15% per annum, compounding monthly. She makes monthly payments of \$250.

Construct a recurrence relation to model this loan, in the form

$$V_0 = \text{the principal}, \qquad V_{n+1} = RV_n - D$$

where V_n is the balance of the loan after n months.

Explanation	Solution
1 State V_0 and D.	$V_0 = 1000$ and $D = 250$
2 Determine the value of R using the formula $R = 1 + \frac{r}{100 \times p}$ where $r = 15$ and $p = 12$.	$R = 1 + \frac{15}{100 \times 12} = 1.0125$
3 Use the values of V_0, R and D to write down the recurrence relation.	$V_0 = 1000, \quad V_{n+1} = 1.0125V_n - 250$

Once we have a recurrence relation, we can use it to determine things such as the balance of a loan after a given number of payments.

Example 8 Using a recurrence relation to analyse a reducing balance loan

Alyssa's loan can be modelled by the recurrence relation:

$$V_0 = 1000, \qquad V_{n+1} = 1.0125V_n - 257.85$$

a Use your calculator to find the balance of the loan after four payments.

b Find the balance of the loan after two payments have been made. Round your answer to the nearest cent.

Explanation

a **i** Write down the recurrence relation.

ii Type '1000' and press '$\boxed{\text{enter}}$' or '$\boxed{\text{EXE}}$'.

iii Type '× 1.0125-257.85' and press '$\boxed{\text{enter}}$' (or $\boxed{\text{EXE}}$) 4 times to obtain the screen opposite.

Solution

$V_0 = 1000, V_{n+1} = 1.0125V_n - 257.85$

1000	1000
1000 ·1.0125 – 257.85	754.65
754.65·1.0125 – 257.85	506.23
506.2331·1.0125 – 257.85	254.71
254.7110·1.0125 – 257.85	0.044927
|	

Balance \$0.04 (to the nearest cent).

b Read the third line of the calculator.

\$506.23 (to the nearest cent)

Annuities

An **annuity** is an investment where compound interest is earned and money is withdrawn from the investment by the individual in the form of regular payments. The calculations used to model the values of reducing balance loans and annuities are identical. The value of the annuity represents how much money is left in the investment.

An annuity can be modelled with a recurrence relation. Once we have a recurrence relation, we can use it to determine things such as the value of the annuity after a given number of payments have been received.

Modelling an annuity

Let V_n be the value of the annuity after n payments have been made. Then

$$V_0 = \text{principal}, \qquad V_{n+1} = RV_n - D$$

where D is the *payment* that has been made, $R = 1 + \dfrac{r}{100 \times p}$ is the growth multiplier, r is the *annual interest rate* and p is the *number of compounding periods* per year.

Example 9 Modelling an annuity with a recurrence relation

Reza invests \$12 000 in an annuity that earns interest at the rate of 6% per annum, compounding monthly, providing him with a monthly income of \$2035.

a Model this annuity using a recurrence relation of the form

$$V_0 = \text{the principal}, \qquad V_{n+1} = RV_n - D$$

where V_n is the value of the annuity after n months.

b Use your calculator to find the value of the annuity after the first four months. Round your answer to the nearest cent.

Explanation

a **i** State the value of V_0 and D.

ii Determine the value of R using the formula $R = 1 + \dfrac{r}{100 \times p}$.

iii Use the values of V_0, R and D to write down the recurrence relation.

b **i** Type **12000** and press enter or EXE.

ii Type **× 1.005-2035** and press enter or EXE four times to obtain the screen opposite.

Solution

$V_0 = 12\,000$ and $D = 2035$

$$R = 1 + \frac{6}{100 \times 12} = 1.005$$

$V_0 = 12\,000, \quad V_{n+1} = 1.005V_n - 2035$

	12000.
12000. · 1.005 – 2035	10025.
10025. · 1.005 – 2035	8040.125.
8040.125. · 1.005 – 2035	6045.326.
6045.326. · 1.005 – 2035	4040.55.

Skill-sheet

Exercise 8B

Modelling reducing balance loans with recurrence relations

Example 6

1 Brooke borrows \$5000 at an interest rate of 5.4% per annum, compounding annually. The loan will be repaid by making annual payments of \$1400.
Construct a recurrence relation to model this loan, in the form:

$$V_0 = \text{the principal}, \qquad V_{n+1} = RV_n - D$$

where V_n be the balance of the loan after n years.

Example 7

2 Jackson borrows \$2000 at an interest rate of 6% per annum, compounding monthly. The loan will be repaid by making monthly payments of \$339.
Let V_n be the balance of the loan after n months.

a State V_0 and D.

b Determine the value of R, using the formula $R = 1 + \dfrac{r}{100 \times p}$.

c Use the values of V_0, R and D to write down the recurrence relation in the form:

$$V_0 = \text{the principal}, \qquad V_{n+1} = RV_n - D$$

3 Benjamin borrows \$10 000 at an interest rate of 12% per annum, compounding quarterly. The loan will be repaid with quarterly payments of \$2600.
Let B_n be the balance of the loan after n quarters.

a Model this loan using a recurrence relation of the form:

$$B_0 = \text{the principal}, \qquad B_{n+1} = RB_n - D$$

b Use the recurrence relation to determine the balance of the loan after two payments have been made.

4 Write a recurrence relation to model a loan of \$3500 borrowed at 4.8% per annum, compounding monthly, with payments of \$280 per month.
Let V_n be the balance of the loan after n months.

5 Write a recurrence relation to model a loan of \$150 000 borrowed at 3.64% per annum, compounding fortnightly, with payments of \$650 per fortnight.
Let V_n be the balance of the loan after n fortnights.

6 Consider a loan of \$235 000 borrowed at 3.65% per annum, compounding daily, with payments of \$150 per day.
Let V_n be the balance of the loan after n days.

a Write a recurrence relation to model this loan.

b Find the value of the loan after 3 days. Round your answer to the nearest cent.

Using a recurrence relation to analyse a reducing balance loan

Example 8

7 A reducing balance loan is modelled by the recurrence relation

$$V_0 = 2500, \qquad V_{n+1} = 1.08V_n - 626$$

where V_n is the balance of the loan after n years.

a State the initial balance of the reducing balance loan.

b State the payment that is made each year.

c Determine the annual interest rate, r, using $1.08 = 1 + \dfrac{r}{100 \times p}$.

d Use your calculator to determine the balance of the loan after three years. Round your answer to the nearest cent.

8 A reducing balance loan can be modelled by the recurrence relation:

$$V_0 = 5000, \qquad V_{n+1} = 1.01V_n - 865$$

where V_n is the balance of the loan after n months.

a State the initial balance of the reducing balance loan.

b State the payment that is made each month.

c Determine the annual interest rate, r.

d Find the balance of the loan after two payments have been made. Round your answer to the nearest cent.

Modelling and analysing an annuity with a recurrence relation

Example 9

9 Mark invests \$20 000 in an annuity paying interest at the rate of 7.2% per annum, compounding annually. He receives a payment of \$3375 each year until the annuity is exhausted.

Let V_n be the value of the annuity after n years.

a State the value of V_0 and D.

b Determine the value of the growth multiplier, R.

c Use your values of V_0, D and R to model this annuity using a recurrence relation of the form:

$$V_0 = \text{the principal}, \qquad V_{n+1} = RV_n - D$$

10 Sandra invests \$750 000 in an annuity paying interest at the rate of 5.4% per annum, compounding monthly. She receives a payment of \$4100 per month until the annuity is exhausted.

Let V_n be the value of the annuity after n payments have been received.

a State the value of V_0 and D.

b Determine the value of the growth multiplier, R.

c Use your values of V_0, D and R to model this annuity using a recurrence relation of the form:

$$V_0 = \text{the principal}, \qquad V_{n+1} = RV_n - D$$

11 Helen invests \$40 000 in an annuity paying interest at the rate of 6% per annum, compounding quarterly. She receives a payment of \$10 380 each quarter.
Let V_n be the balance of the loan after n quarters.

a Model this loan using a recurrence relation of the form:

$$V_0 = \text{the principal}, \qquad V_{n+1} = RV_n - D$$

b Use the recurrence relation to determine the balance of the annuity after 3 quarters. Round your answer to the nearest cent.

12 An annuity is modelled by the recurrence relation

$$V_0 = 5000, \qquad V_{n+1} = 1.01V_n - 1030$$

where V_n is the balance of the annuity after n monthly payments have been received.

a State the initial balance of the annuity.

b State the payment that is received each month.

c Determine the annual interest rate, r, using $R = 1 + \dfrac{r}{100 \times p}$.

d Use your calculator to determine the balance of the annuity after three payments have been received. Round your answer to the nearest cent.

e How much will the annuity pay out in the first three months?

13 An annuity can be modelled by the recurrence relation

$$V_0 = 6000, \qquad V_{n+1} = 1.005V_n - 1500$$

where V_n is the balance of the annuity after n payments have been made.

a Use your calculator to determine the balance of the annuity after the two payments have been received.

b Assuming that payments are made quarterly, how much will the annuity pay out in the first year?

14 Jeff invests \$1 000 000 in an annuity and receives a regular monthly payment.
The balance of the annuity, in dollars, after n months, A_n, can be modelled by a recurrence relation of the form

$$A_0 = 1\,000\,000, \qquad A_{n+1} = 1.0024A_n - 4000$$

a State the initial balance of the annuity.

b State the payment that Jeff receives each month.

c Calculate the annual compound interest rate.

d Calculate the balance of this annuity after two months.

15 Esme invests \$100 000 in an annuity and receives a regular monthly payment. The balance of the annuity, in dollars, after n months, E_n, can be modelled by the recurrence relation

$$E_0 = 100\,000, \qquad E_{n+1} = 1.0055E_n - 18\,400$$

a What monthly payment does Esme receive?

b Find the annual interest rate for this annuity.

c At some point in the future, the annuity will have a balance that is lower than the monthly payment amount. What is the balance of the annuity when it **first** falls below the monthly payment amount? Round your answer to the nearest cent.

Exam 1 style questions

16 Matthew would like to purchase a new home. He will establish a loan for \$640 000 with interest charged at the rate of 4.2% per annum, compounding monthly.
Each month, Matthew will pay \$3946.05.
Let V_n be the value of Matthew's loan, in dollars, after n months.
A recurrence relation that models the value of V_n is

A $V_0 = 640\,000, \qquad V_{n+1} = 1.0035V_n$

B $V_0 = 640\,000, \qquad V_{n+1} = 1.042V_n$

C $V_0 = 640\,000, \qquad V_{n+1} = 1.042V_n - 3946.05$

D $V_0 = 640\,000, \qquad V_{n+1} = 1.0035V_n - 3946.05$

E $V_0 = 640\,000, \qquad V_{n+1} = 1.0035V_n + 3946.05$

17 Tim invests \$3800 in an annuity and receives a regular monthly payment of \$480. The balance of the annuity, in dollars, after n months, T_n, can be modelled by a recurrence relation of the form

$$T_0 = 3800, \quad T_{n+1} = 1.002T_n - 480$$

The balance of the annuity after three months is closest to

A \$3327 **B** \$3328 **C** \$2854 **D** \$2379 **E** \$2380

18 Suzanne invests \$9200 in an annuity at 4.2% per annum, compounding monthly.
Suzanne receives a regular monthly payment of \$620.
The amount of interest earned in the second month is closest to

A \$30 **B** \$588 **C** \$590 **D** \$8612 **E** \$8022

8C Amortisation tables

Learning intentions

- ▶ To be able to apply the amortisation process.
- ▶ To be able to construct an amortisation table.
- ▶ To be able to analyse an amortisation table for a reducing balance loan.
- ▶ To be able to read and interpret an amortisation table for an annuity to find the interest rate.
- ▶ To be able to interpret and construct an amortisation table for a compound interest investment with additions to the principal.

Amortisation tables provide additional information for each period, rather than just the balance after each payment.

Amortisation tables for reducing balance loans

Loans that are repaid by making regular payments until the balance of the loan is zero are called **amortising** loans. In an amortising loan, part of each of the regular payments goes towards paying the interest owed on the unpaid balance of the loan with the remainder used to reduce the principal of the loan (the amount borrowed).

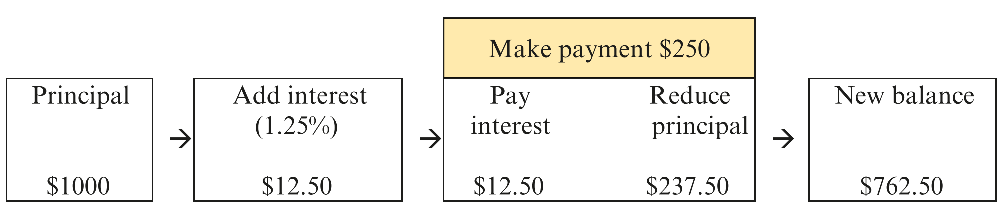
For example, consider Alyssa's loan from Example 7. Interest on the \$1000 loan was charged at the rate of 15% per year and the loan was to be repaid with monthly payments of \$250. The recurrence relation was given as

$$V_0 = 1000, \qquad V_{n+1} = 1.0125V_n - 250$$

To determine where the first payment goes:

- Calculate the interest charged ($p = 12$): $\dfrac{15\%}{12} = 1.25\%$. Thus, $1000 \times 1.25\% = \$12.50$
- Calculate the **principal reduction**: $\$250 - \$12.50 = \$237.50$
- Calculate the new balance: $\$1000 - \$237.50 = \$762.50$

This process is shown in the following diagram:



We can also represent this information in table format, showing the impact of a payment, interest and the subsequent reduction of the principal to give a new balance.

Payment number	Payment	Interest	Principal reduction	Balance
1	250.00	12.50	237.50	762.50

When this analysis is repeated, the results can be summarised in an **amortisation table**. The amortisation table shows all of the details that explain how the new balance was calculated. Note that the first line shows the initial value of the loan as the balance when no payments have been received.

The first three payments for Alyssa's loan are shown below.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	1000.00
1	250.00	12.50	237.50	762.50
2	250.00	9.53	240.47	522.03
3	250.00	6.53	243.47	278.56

Example 10 Applying the amortisation process

Flora borrows \$20 000 at an interest rate of 8% per annum, compounding annually. She makes annual payments of \$2500.

a State the principal of the loan.

b Calculate the initial interest charged on the principal.

c Determine the impact of the first annual payment to find the principal reduction.

d Calculate the new balance.

e Complete the row in the table below with your calculations.

Payment number	Payment	Interest	Principal reduction	Balance
1	2500.00			

Explanation	Solution
a Read the principal from the question or recurrence relation.	The principal is \$20 000.
b Calculate the interest paid.	Interest paid = 8% of \$20 000 = \$1600
c Principal reduction = payment – interest.	Principal reduction = 2500 – 1600 = \$900
d New balance = balance owing – principal reduction	New balance = 20 000 – 900 = \$19 100
e Place each of the numbers from the calculations into the relevant boxes.	

Interest	Principal reduction	Balance
1600.00	900.00	19 100.00

Constructing an amortisation table for a reducing balance loan

1 Interest paid = $\frac{r}{100 \times p}$ × previous balance, where r is the annual interest rate and p is the number of compounding periods each year.

2 Principal reduction = payment − interest

3 New balance = (previous) balance − reduction in balance

Example 11 Constructing an amortisation table for a reducing balance loan

Flora borrows $20 000 at an interest rate of 8% per annum, compounding annually. She makes annual payments of $2500.

Construct an amortisation table for Flora's reducing balance loan for the first three payments.

Solution

Repeat the calculations from Example 10, rounding all numbers to the nearest cent.

Once the new balance has been calculated, repeat the process for the first three payments.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	20 000.00
1	2500.00	1600.00	900.00	19 100.00
2	2500.00	1528.00	972.00	18 128.00
3	2500.00	1450.24	1049.76	17 078.24

Sometimes we are asked to fill in gaps of a given amortisation table.

Example 12 Analysing an amortisation table for a reducing balance loan

A business borrows $10 000 at a rate of 8% per annum, compounding quarterly. The loan is to be repaid by making quarterly payments of $2700.00. The amortisation table for this loan is shown below.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	10 000.00
1	2700.00		2500.00	7500.00
2	2700.00	150.00		4950.00
3	2700.00	99.00	2601.00	

a Calculate the interest paid on the initial balance.

b Calculate the principal reduction from the second payment.

c Calculate the balance of the loan after payment 3 has been made.

Explanation

a Use $\frac{r}{100 \times p}$, where $r = 8$ and $p = 4$ since interest is calculated quarterly.
Interest paid = 2% × unpaid balance
Alternatively, note that \$2700 is paid and the principal was reduced by \$250.

b Principal reduction = payment − interest

c New Balance = balance owing − principal reduction

Solution

Interest paid $= \frac{8}{100 \times 4} \times 10\,000 = \$200$

Or, Interest paid = \$2700 − \$2500 = \$200

Principal reduction = 2700.00 − 150.00
= \$2550.00

Balance of the loan after 3 payments
= 4950 − 2601 = \$2349

Amortisation tables for annuities

An amortisation table for an annuity is very similar to one for a reducing balance loan. Each row shows the payment number, the payment received, the interest earned, the principal reduction and the balance of the annuity after each payment has been received.

Example 13 **Analysing an amortisation table for an annuity to find the interest rate**

Consider the following amortisation table for an annuity after 3 monthly payments.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	12 000.00
1	2200.00	60.00	2140.00	9860.00
2	2200.00	49.30	2150.70	7709.30
3	2200.00	A	B	5547.85

a State the principal of the annuity and the amount of interest paid in the first month.
b Calculate the monthly interest rate.
c Find the value of A and B.

Explanation

a Read off from the table.

b Calculate: $\frac{\text{Interest}}{\text{Principal}} \times 100$

c A is the interest due on \$7709.30
B is the principal reduction after the third payment.

Solution

Principal: \$12 000, Interest: \$60

$\frac{60}{12000} \times 100 = 0.5\%$ per month

A: $\frac{0.5}{100} \times 7709.30 = 38.55$
B: $2200 - 38.55 = 2161.45$

Amortisation tables for compound interest investments with additions to the principal

An amortisation table for a compound interest investment with additions to the principal is similar to the previous examples but here, the payment **increases** the balance of the principal further.

Example 14 **Interpreting and constructing an amortisation table for a compound interest investment with additional payments**

Consider the following amortisation table for a compound interest investment with monthly additions to the principal. Assume that interest compounds monthly.

Payment number	Payment	Interest	Principal increase	Balance
0	0.00	0.00	0.00	1200.00
1	50.00	3.00	53.00	1253.00
2	50.00	3.13	53.13	1306.13
3	50.00	3.27	53.27	1359.40

Complete two additional lines for the table corresponding to payment 4 and payment 5.

Solution

Begin by calculating the monthly interest rate $\frac{3}{1200} \times 100 = 0.25\%$.

Now we can calculate the line associated with payment 4. The interest paid is calculated on the balance $1359.40:

$$\text{Interest} = 0.25\% \times 1359.40 = \$3.40$$

The principal increases by the interest and the additional payment:

$$\text{Principal increase} = \text{interest} + \text{payment} = 3.40 + 50 = \$53.40$$

Thus, the new balance becomes:

$$\text{New balance} = \text{previous balance} + \text{principal increase} = 1359.40 + 53.40 = \$1412.80$$

Repeating gives the following two lines of the table.

Payment number	Payment	Interest	Principal increase	Balance
4	50.00	3.40	53.40	1412.80
5	50.00	3.53	53.53	1466.33

Exercise 8C

Applying the amortisation process and constructing an amortisation table for a reducing balance loan

Example 10
Example 11

1 Walter borrows \$14 000 at an interest rate of 11% per annum, compounding annually. He makes annual repayments of \$1800 per year.

a State the principal of the loan.

b Calculate the interest charged on the principal in the first year.

c Determine the impact of the first annual payment to find the principal reduction.

d Calculate the new balance after the first year.

e Complete the row in an amortisation table corresponding to payment 1.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	14 000.00
1	1800.00			

f Complete the next two rows of the amortisation table corresponding to payment 2 and 3 for Walter.

2 Ellie borrows \$12 000 at an interest rate of 6% per annum, compounding monthly. She makes regular repayments of \$300 per month.

a State the principal of the loan.

b Calculate $\frac{r}{100 \times p}$ where r is the annual interest rate and p is the number of compounding periods each year.

c Calculate the interest charged on the principal in the first month.

d Determine the impact of the first monthly payment to find the principal reduction.

e Calculate the new balance after the first month.

f Complete the row in an amortisation table corresponding to payment 1.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	12 000.00
1	300.00			

g Complete the next two rows of the amortisation table corresponding to payment 2 and 3 for Ellie.

3 Anna borrows \$36 000 at an interest rate of 8% per annum, compounding quarterly. She makes regular repayments of \$1000 per quarter.

a State the principal of the loan.

b Calculate $\dfrac{r}{100 \times p}$ where r is the annual interest rate and p is the number of compounding periods each year.

c Construct an amortisation table corresponding to the first three payments for the loan.

Reading and interpreting an amortisation table for a reducing balance loan

Example 12

4 A student borrows \$2000 at an interest rate of 12% per annum, compounding monthly. The student makes monthly payments of \$345.
The amortisation table for this loan after 5 payments is shown below.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	2000.00
1	345.00	20.00	325.00	1675.00
2	345.00	A	328.25	1346.75
3	345.00	13.47	331.53	1015.22
4	345.00	10.15	B	680.37
5	345.00	6.80	338.20	C

a Calculate the monthly interest rate.

b Determine the values of A, B and C.

5 The amortisation table for a loan with quarterly payments is shown below.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	4000.00
1	557.65	100.00	457.65	3542.35
2	557.65	88.65	469.09	3073.26
3	557.65	76.83	480.82	2592.44
4	557.65	A	492.84	2099.60
5	557.65	52.49	B	1594.44
6	557.65	39.86	517.79	C
7	557.65	D	E	545.92

a State the principal and interest paid in the first quarter.

b Calculate the quarterly interest rate.

c Determine the values of A, B, C, D and E.

6 Ada has a reducing balance loan with an interest rate of 3.6% per annum, compounding monthly. She makes monthly payments of \$1800 as shown in the amortisation table below.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	460 000.00
1	1800.00	1380.00	420.00	459 580.00
2	1800.00	1378.74	A	459 158.74
3	1800.00			B

Calculate the value of A and B.

Analysing an amortisation table for an annuity to find the interest rate

Example 13

7 A student invested \$6000 in an annuity paying an interest rate of 3% per annum, compounding monthly.
She receives a monthly payment of \$508 as per the amortisation table shown below.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	6000.00
1	508.00	15.00	493.00	5507.00
2	508.00	13.77	494.23	5012.77
3	508.00	A	B	C

a Reading from the table, determine:
  i the interest earned when payment 1 is received,
  ii the monthly interest rate.

b Calculate the values of A, B and C.

Interpreting and constructing an amortisation table for a compound investment with additional payments

Example 14

8 The amortisation table below charts the growth of an investment which compounds monthly and has regular additions made to the balance each month.

Payment number	Payment	Interest	Principal increase	Balance
0	0.00	0.00	0.00	5000.00
1	100.00	50.00	150.00	5150.00
2	100.00	51.50	151.50	5301.50
3	100.00	A	B	C

a Calculate the monthly interest rate of the investment.

b Determine the values of A, B and C.

Exam 1 style questions

9 Edith invested \$400 000 in an annuity that provides an annual payment of \$51 801.82
Interest is calculated annually.
The first five lines of the amortisation table are shown below.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	400 000.00
1	51 801.82	20 000.00	31 801.82	368 198.18
2	51 801.82	18 409.91	33 391.91	334 806.27
3	51 801.82	16 740.31		299 744.76
4	51 801.82	14 987.24	36 814.58	262 930.18

The principal reduction associated with payment number 3 is

A \$31 801.82 **B** \$33 391.82 **C** \$35 061.50 **D** \$35 061.51 **E** \$36 814.58

10 Consider the following amortisation table for a reducing balance loan.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	200 000.00
1	3000.00	1800.00	1200.00	198 800.00
2	3000.00	1789.20	1210.80	197 589.20
3	3000.00	11778.30	1221.70	196 367.50

The annual interest rate for this loan is 3.6%.
Interest is calculated immediately before each payment.
For this loan, the repayments are made

A weekly **B** fortnightly **C** monthly **D** quarterly **E** yearly

11 Four lines of an amortisation table for an annuity investment are shown below.
The interest rate for this investment remains constant, but the amount of the additional payment may vary.

Payment number	Payment	Interest	Principal increase	Balance
1	50.00	50.00	100.00	10 100.00
2	50.00	50.50	100.50	10 200.50
3	50.00	51.00	101.00	10 301.50
4				10 553.01

The balance of the investment after payment number 4 is \$10 533.01.
The value of payment number 4 is

A \$50 **B** \$100 **C** \$150 **D** \$200 **E** \$250

8D Analysing financial situations using amortisation tables

Learning intentions

- ▶ To be able to find the final payment in a reducing balance loan and an annuity.
- ▶ To be able to find the total payment made/received and the total interest paid/earned.
- ▶ To be able to plot points using an amortisation table for a reducing balance loan, an annuity and a compound interest investment with additional payments.

Finding the final payment in a reducing balance loan and an annuity

A reducing balance loan must be paid off at some point in time and likewise, an annuity will run out after a period of time. In both cases, it is possible that the final payment might need to be different from the regular payment so that the final balance can reach zero.

Example 15 **Finding the final payment for a reducing balance loan or annuity**

Consider the following amortisation table for a reducing balance loan of $20 000 with an interest rate of 8% per annum, compounding annually. Regular payments of $5009.12 are made for the first four years as shown in the amortisation table.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	20 000.00
1	5009.12	1600.00	3409.12	16 590.88
2	5009.12	1327.27	3681.85	12 909.03
3	5009.12	1032.72	3976.40	8932.63
4	5009.12	714.61	4294.51	4638.12

Calculate the final payment required in the fifth year to pay off the loan fully.

Solution

To find the final payment, we first need to calculate the interest on $4638.12:

$$\text{Interest} = \frac{8}{100} \times \$4638.12 = \$371.05$$

Thus, the final payment can be calculated by adding the balance that is still due and the interest.

$$\begin{aligned}\text{Final payment} &= \text{Balance} + \text{interest}\\ &= 4638.12 + 371.05\\ &= 5009.17\end{aligned}$$

Thus, the final payment is $5009.17.

Finding the total payment made/received and total interest paid/earned

Once we have a completed amortisation table, we can find the total interest paid/earned and the total payment made/received for a reducing balance loan, an annuity or a compound interest investment with additional payments.

Example 16 Finding the total payment made and total interest paid

Consider the following amortisation table for a reducing balance loan of \$10 000 with an interest rate of 8% per annum, compounding quarterly. Three quarterly payments of \$2626 are made.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	10 000.00
1	2626.00	200.00	2426.00	7574.00
2	2626.00	151.48	2474.52	5099.48
3	2626.00	101.99	2524.01	2575.47
4	*A*	*B*	*C*	0.00

a Complete the amortisation table corresponding to payment four such that the final payment ensures that the balance is 0.

b Calculate the total payment made for the loan.

c Calculate the total interest paid on the loan.

Explanation

a Follow the previous example by first finding the interest applied to the loan (B), the principal reduction (C) and the final adjusted payment (A).

Solution

B: Interest $= \dfrac{8}{100 \times 4} \times \$2575.47 = \$51.51$

C: Principal reduction $= \$2575.47$

$$A\text{: Final payment} = 2575.47 + 51.51$$
$$= 2626.98$$

Payment	Interest	Principal reduction
2626.98	51.51	2575.47

Explanation

b Add up all of the payments made over the four quarters.

Solution

Payments $= 2626 \times 3 + 2626.98 = \$10\,504.98$

Explanation

c Subtract the principal from the total payments.

Alternatively, we can add up the interest column.

Solution

Interest $= 10\,504.98 - 10\,000 = \$504.98$

OR

Interest $= 200 + 151.48 + 101.99 + 51.51 = \$504.98$

Plotting points from an amortisation table

The values in an amortisation table can be plotted to give a picture of the impact of regular payments.

Example 17 Plotting from an amortisation table

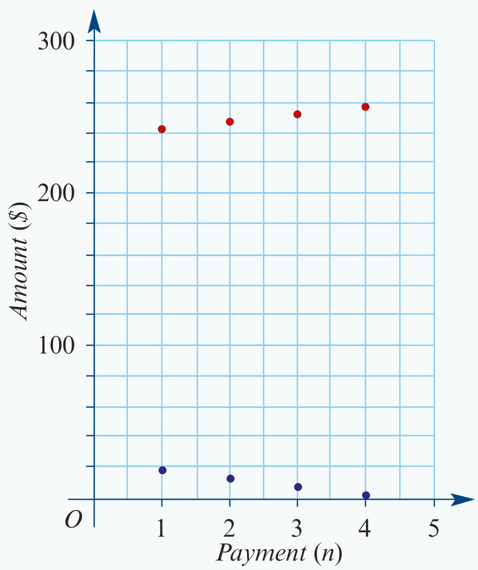
Consider the following amortisation table for a reducing balance loan.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	1000.00
1	257.85	12.50	245.35	754.65
2	257.85	9.43	248.42	506.23
3	257.85	6.33	251.52	254.71
4	257.89	3.18	254.71	0.00

Plot a graph of the interest and principal reduction on the same graph.

Solution

For this loan, we can plot a graph for each payment period to show that the amount of interest paid each payment (blue dots) declines while the amount of principal paid increases (red dots).



Note that for a loan, the graph above shows how the amount of interest paid for each payment (blue dots) decreases with the payment number, while the amount of principal paid off increases (red dots). This is because the balance is decreasing and so the interest is being calculated on a lower balance each period.

For a compound interest investment, we would expect to see the interest earned each period increase because the balance increases each time a payment is made.

Exercise 8D

Finding the final payment in a reducing balance loan and an annuity

Example 15

1 Consider the following amortisation table for a reducing balance loan of \$2000 with an interest rate of 4% per annum, compounding annually. Regular payments of \$550 are made for the first three years as shown in the amortisation table.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	2000.00
1	550.00	80.00	470.00	1530.00
2	550.00	61.20	488.80	1041.20
3	550.00	41.65	508.35	532.85
4				0.00

Calculate the value of the fourth payment to ensure the loan is repaid in full.

2 Consider the following amortisation table for a reducing balance loan of \$5000 with an interest rate of 4.8% per annum, compounding monthly. Regular payments of \$1262.50 are made for the first three months as shown in the amortisation table.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	5000.00
1	1262.50	20.00	1242.50	3757.50
2	1262.50	15.03	1247.47	2510.03
3	1262.50	10.04	1252.46	1257.57
4				0.00

Calculate the value of the fourth payment to ensure the loan is repaid in full.

3 Consider the following amortisation table for an annuity of \$3000 with an interest rate of 5% per annum, compounding annually. Regular payments of \$693 are received for the first four months as shown in the amortisation table.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	3000.00
1	693.00	150.00	543.00	2457.00
2	693.00	122.85	570.15	1886.85
3	693.00	94.34	598.66	1288.19
4	693.00	64.41	628.59	659.60

Calculate the value of the fifth payment to ensure the annuity is completely exhausted.

4 Charlie invests \$4500 into an annuity with an interest rate of 5.4% per annum, compounding monthly. He receives monthly payments of \$760 for five months. Calculate the value of the sixth payment that Charlie receives to ensure the annuity is completely exhausted.

Finding the total payment made/received and the total interest paid/earned

Example 16

5 Consider the following amortisation table for an investment of \$11 000 with an interest rate of 4.8% per annum, compounding quarterly. Regular quarterly payments of \$1200 are added each quarter as shown below for the first four quarters.

Payment number	Payment	Interest	Principal increase	Balance
0	0.00	0.00	0.00	11 000.00
1	1200.00	132.00	1332.00	12 332.00
2	1200.00	147.98	1347.98	13 679.98
3	1200.00	164.16	1364.16	15 044.14
4	1200.00	A	B	16 424.67

a Find the value of A.

b Find the value of B.

c Find the total interest earned on the investment in the first four quarters.

6 Consider the following amortisation table for a reducing balance loan of \$4000 with an interest rate of 6% per annum, compounding monthly. A monthly payment of \$344.14 is made for the first 11 months.

Payment number	Payment	Interest	Principal reduction	Balance
9	344.14	6.81	337.33	1023.71
10	344.14	5.12	339.02	684.69
11	344.14	3.42	340.72	343.97
12	A	B	C	0.00

a State the value of A, B and C in the amortisation table corresponding to payment twelve such that the final payment ensures that the balance is 0.

b Calculate the total payment made for the loan.

c Calculate the total interest paid on the loan.

7 Consider the final two lines of the amortisation table for an annuity of \$30 000 with an interest rate of 3.6% per annum, compounding quarterly. A quarterly payment of \$3903.50 is made for the first 7 quarters.

Payment number	Payment	Interest	Principal reduction	Balance
7	3903.50	69.32	3834.18	3868.37
8	A	B	C	0.00

a State the value of A, B and C in the amortisation table corresponding to payment eight such that the final payment ensures that the balance is 0.

b Calculate the total payments received from the annuity.

c Calculate the total interest paid on the loan.

8 Tania invests \$12 000 in an annuity with an interest rate of 6.6% per annum, compounding monthly. She receives regular monthly payments of \$3040 per month for three months followed by a final payment in the fourth month. Calculate the total payment and the interest for the annuity.

Plotting points from an amortisation table

Example 16

9 Consider the following amortisation table for a reducing balance loan.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	2000.00
1	345.14	20.00	325.14	1674.86
2	345.14	16.75	328.39	1346.47
3	345.14	13.46	331.68	1014.79
4	345.14	10.15	334.99	679.80
5	345.14	6.80	338.34	341.46
6	344.87	3.41	341.46	0.00

Plot a graph of the interest and principal reduction on the same graph for the first six payments.

10 The amortisation table below charts the growth of a compound interest investment with regular additions made to the principal each month.

Payment number	Payment	Interest	Principal increase	Balance
0	0.00	0.00	0.00	5000.00
1	100.00	50.00	150.00	5150.00
2	100.00	51.50	151.50	5301.50
3	100.00	53.02	153.02	5454.52

Plot a graph of the interest and principal increase on the same graph.

Exam 1 style questions

11 Francesca invests \$6000 into an annuity with an interest rate of 5.8% per annum, compounding quarterly.
She receives a quarterly payment of \$1554.13 for the first three quarters and then a final payment in the fourth quarter.
After three quarters, the value of the annuity is \$1534.48.
The payment that Francesca receives in the fourth quarter is

A \$0.04 **B** \$2.60 **C** \$1534.48 **D** \$1554.13 **E** \$1556.73

12 Ned borrows \$15 000 at an interest rate of 6% per annum, compounding monthly. He makes regular monthly payments of \$3796.99 for three months followed by a final payment in the fourth month.
The total interest that Ned pays on the loan is closest to

A \$56 **B** \$75 **C** \$188 **D** \$300 **E** \$900

8E Using a finance solver to find the balance and final payment

Learning intentions

- To be able to use the finance solver to find the value of a compounding interest investment with additional payments.
- To be able to use the finance solver to find the balance and final payment of a reducing balance loan.
- To be able to use the finance solver to find the balance of an annuity.

While the techniques used so far in this chapter are useful for performing a small number of calculations, they are tedious over a long period. For example, a typical home loan may involve monthly payments over 30 years. CAS calculators have a **Finance Solver** that allow for larger calculations to be performed with ease.

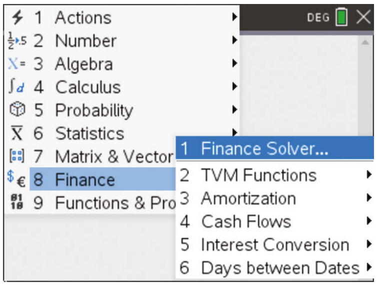
Using the Finance Solver on the TI-Nspire CAS

Steps

1 Press ctrl + N

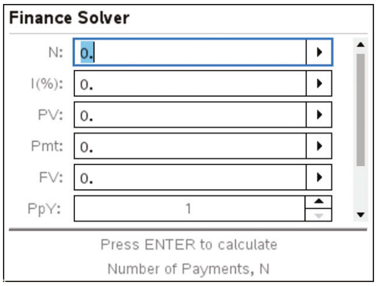
2 Select **Add Calculator**.

Press menu>**Finance**> **Finance Solver**.



3 To use Finance Solver you need to know the meaning of each of its symbols.

- **N** is the total number of payments.
- **I(%)** is the annual interest rate.
- **PV** is the present value of the loan/investment.
- **Pmt** is the amount paid at each payment.

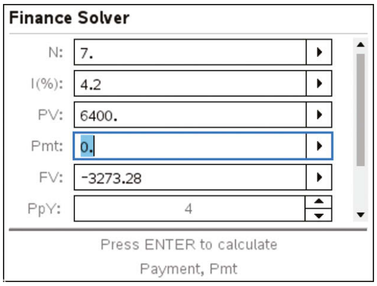


- **FV** is the future value of the loan/investment.
- **PpY** is the number of payments per year.
- **CpY** is the number of times the interest is compounded per year. (It is the same as **PpY**.)
- **PmtAt** is used to indicate whether the interest is compounded at the end or at the beginning of the time period. Ensure this is set at **END**.

Note: Use tab or ▼ to move down boxes, press ▲ to move up. For **PpY** and **CpY** press tab to move down to the next entry box.

4 When using Finance Solver to solve loan and investment problems, there will be one unknown quantity. To find its value, move the cursor to its entry field and press enter to solve.

In the example shown, pressing enter will solve for **Pmt**.

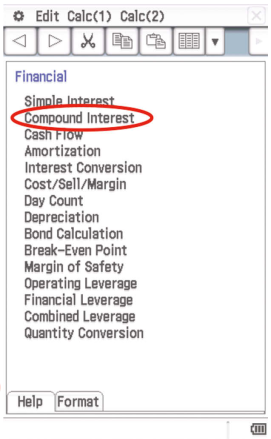


Using the Finance Solver on the Casio ClassPad

Steps

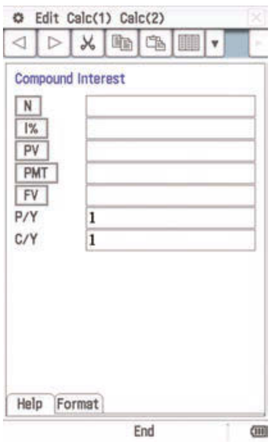
1 Tap **Financial** from the main menu screen.

2 Select the compound interest solver by tapping on **Compound Interest** from the solver screen.



3 To use Finance Solver you need to know the meaning of each of its symbols.

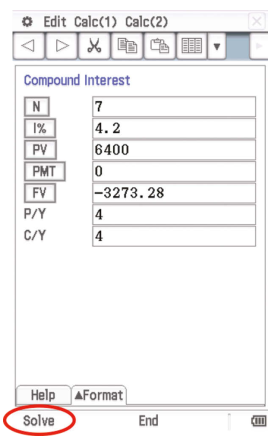
- **N** is the total number of payments.
- **I%** is the annual interest rate.
- **PV** is the present value of the loan or investment.
- **PMT** is the amount paid at each payment.
- **FV** is the future value of the loan or investment.
- **P/Y** is the number of payments per year.
- **C/Y** is the number of times interest is compounded per year. (It is the same as **P/Y**.)



4 Tap **Format** and confirm that the setting for ‘Odd Period’ is set to ‘off’ and ‘Payment Date’ is set to ‘End of period’.

5 When using Finance Solver to solve loan problems, there will be one unknown quantity. To find its value, tap its entry field and tap **Solve**.

In the example shown, tapping **Solve** will solve for **Pmt**.



Using a financial solver to analyse a compound interest investment with regular additions to the principal

A finance solver is a powerful computation tool. However, you have to be very careful in the way you enter information because it needs to know which way the money is flowing. It does this by following a **sign convention**.

In general terms:

- if you **receive** money, or someone owes you money, we treat this as a **positive** (+ve)
- if you **pay out** money or you owe someone money, we treat this as a **negative** (−ve).

Recall that a compound interest investment with regular additions to the principal is an investment where the balance increases through both the interest earned and the additional payments.

Using Finance solver for a compound interest investment with regular additions to the principal

In finance solver:

- **PV:** Negative: you make an investment by giving the bank some money.
- **PMT:** Negative: you make regular payments to the bank.
- **FV:** Positive: after the payment is made and the investment matures, the bank will give you the money.

Example 18 Determining the value of an investment with regular additions made to the principal using a financial solver

Lars invests \$500 000 at 5.5% per annum, compounding monthly. He makes a regular deposit of \$500 per month into the account. What is the value of his investment after 5 years? Round your answer to the nearest cent.

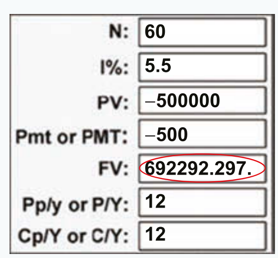
Explanation

1 Open Finance Solver and enter the information below, as shown opposite.

- **N**: 60 (5 years)
- **I%**: 5.5
- **PV**: –500 000 (you give this to the bank)
- **PMT**: –500 (you give this to the bank)
- **FV**: to be determined
- **Pp/Y**: 12 payments per year
- **Cp/Y**: 12 compounding periods per year

2 Solve for FV and write your answer, rounding to the nearest cent. Note that this is positive as the bank will give this money to you.

Solution



After 5 years, Lars' investment will be worth \$692 292.30.

Using finance solver for a reducing balance loan

Recall that for a reducing balance loan, you receive money from the bank (in the form of the loan) and then make payments until you no longer owe the bank any money. The sign convention for a reducing balance loan is summarised below.

Finance solver for a reducing balance loan

In finance solver:

- **PV:** Positive: the bank gives you money through a loan.
- **PMT:** Negative: you repay the loan by making regular repayments to the bank.
- **FV:** Negative, zero or positive: after the payment is made:
  - you still owe the bank money (FV negative),
  - the loan is fully repaid (FV zero), or
  - you have overpaid your loan and the bank now owes you money (FV positive).

Example 19 Determining the balance and final payment of a reducing balance loan after a given number of payments

Andrew borrows $20 000 at an interest rate of 7.25% per annum, compounding monthly. This loan will be repaid over 4 years with regular payments of $481.25 each month for 47 months followed by a final payment to fully repay the loan.

a How much does Andrew owe after 3 years? Round your answer to the nearest cent.

b What is the final payment amount that Andrew must make to fully repay the loan within 4 years (48 months)? Round your answer to the nearest cent.

Explanation

a 1 Open Finance Solver and enter the following:

- **N**: 36 (number of months in 3 years)
- **I%**: 7.25 (annual interest rate)
- **PV**: 20000 (positive to indicate that this is money received by Andrew from the lender)
- **Pmt** or **PMT**: −481.25 (negative as Andrew is giving this back to the lender)
- **Pp/Y**: 12 (monthly payments)
- **Cp/Y**: 12 (interest compounds monthly)

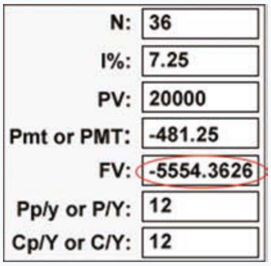
Solution

N:	36
I%:	7.25
PV:	20000
Pmt or PMT:	-481.25
FV:	
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

2 Solve for the unknown future value (FV). On the:

- *TI-NspireCAS:* Move the cursor to the **FV** entry box and press to [enter] solve.
- *ClassPad:* Tap on the **FV** entry box and tap 'Solve'. The amount –5554.3626 . . . now appears in the **FV** entry box.

Note: A negative FV indicates that Andrew will still owe the lender money after the payment has been made.

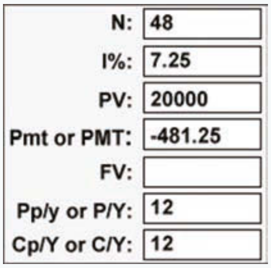


3 Write your answer, correct to the nearest cent

Andrew owes $5554.36.

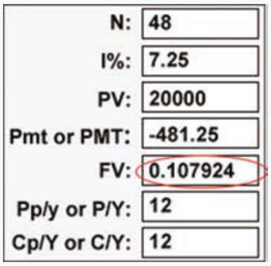
b **1** Enter the information below.

- **N**: 48 (number of months in 4 years)
- **I%**: 7.25 (annual interest rate)
- **PV**: 20000
- **Pmt** or **PMT** (the payment amount is negative): −481.25
- **Pp/Y**: 12 (monthly payments)
- **Cp/Y**: 12 (interest compounds monthly)



2 Solve for the unknown future value (FV). On the:

- *TI-NspireCAS:* Move the cursor to the **FV** entry box and press [enter] to solve.
- *ClassPad:* Tap on the **FV** entry box and tap **Solve** The amount 0.1079. . . (11 cents) now appears in the **FV** entry box.



Since FV is positive (+11 cents), the bank owes Andrew 11 cents so we **subtract** this from the regular payment.

3 Write your answer.

Final payment
= $481.25 – $0.11
= $481.14
Andrew's final payment will be $481.14.

Using a finance solver to analyse an annuity

Recall that an annuity is when an individual puts money into an account and then receives payments from the bank.

Finance solver for an annuity

In finance solver:

- **PV:** Negative: you buy an annuity by giving the bank some money.
- **PMT:** Positive: you receive regular payments from the bank.
- **FV:** Positive or zero: after the payment is made:
  - the bank still owes you money (FV positive),
  - the annuity is fully paid out (FV zero)

Note: An annuity should never have a negative FV as a bank would never overpay the individual.

Example 20 Determining the balance of an annuity using a finance solver

Charlie invests \$300 000 into an annuity, paying 5% interest per annum, compounding monthly. Over the next ten years, Charlie receives a payment of \$3182 per month from the annuity for each month except the final month.

a Find the value of the annuity after five years. Round your answer to the nearest cent.

b Find the final payment from the annuity. Round your answer to the nearest cent.

Explanation

a **1** Open Finance Solver and enter the following:

- **N**: 60 (number of monthly payments in 5 years)
- **I%**: 5.00 (annual interest rate)
- **PV**: −300000 (negative to indicate that this is money paid by Charlie to the bank)
- **Pmt** or **PMT**: 3182 (positive to indicate that the bank is paying back to Charlie)
- **Pp/Y**: 12 (monthly payments)
- **Cp/Y**: 12 (interest compounds monthly)

2 Solve for the unknown future value (FV). On the:

- *TI-NspireCAS:* Move the cursor to the **FV** entry box and press to enter solve.
- *ClassPad:* Tap on the **FV** entry box and tap 'Solve'. The amount 168612.24795 . . . now appears in the **FV** entry box.

Note: A positive FV indicates that Charlie is still owed money from the annuity.

3 Write your answer, rounding to the nearest cent.

Solution

N:	60
I%:	5.00
PV:	−300000
Pmt or PMT:	3182
FV:	
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

N:	60
I%:	5.00
PV:	−300000
Pmt or PMT:	3182
FV:	168612.247951
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

The balance of the annuity is \$168 612.25

b **1** Find the value of the annuity after 120 payments. Enter the information below, as shown opposite.

- **N**: 120 (number of monthly payments in 10 years)
- **I%**: 5.00
- **PV**: −300000
- **Pmt** or **PMT**: 3182
- **Pp/Y**: 12
- **Cp/Y**: 12

N:	120
I%:	5.00
PV:	−300000
Pmt or PMT:	3182
FV:	
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

2 Solve for the unknown future value (FV). On the:

- *TI-NspireCAS:* Move the cursor to the **FV** entry box and press enter to solve.
- *ClassPad:* Tap on the **FV** entry box and tap **Solve**

The amount −5.36 (−\$5.36) now appears in the **FV** entry box.

Note: The FV is negative (\$5.36). This means that Charlie owes the annuity \$5.36. To compensate, Charlie's final payment will be decreased by \$5.36.

N:	120
I%:	5.00
PV:	−300000
Pmt or PMT:	3182
FV:	−5.36388
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Final payment
= \$3182 − \$5.36
= \$3176.64

3 Write your answer.

Charlie's final payment will be \$3176.64.

Skillsheet

Exercise 8E

Determining the value of an investment with regular additions made to the principal using a financial solver

Example 18

1 Wanda invested \$20 000 at 7.1% per annum, compounding annually. She makes a regular deposit of \$6000 per year into the account.

a State whether the **PV** is positive or negative.

b State whether the **PMT** is positive or negative.

c Find the value of the investment after 10 years. Round your answer to the nearest cent.

d Find the value of the investment after 30 years. Round your answer to the nearest cent.

2 Ingrid invested \$20 000 at 4.9% per annum, compounding monthly. She makes a regular deposit of \$380 per month into the account.

a Find the value of the investment after 5 months. Round your answer to the nearest cent.

b Find the value of the investment after 3 years. Round your answer to the nearest cent.

Determining the balance of a reducing balance loan using a financial calculator

Example 19

3 Barry borrows \$8000 at an interest rate of 4.5% per annum, compounding monthly. This loan will be repaid with regular payments of \$350 per month, followed by a final payment.

a State whether the **PV** is positive or negative.

b State whether the **PMT** is positive or negative.

c How much does Barry owe after six months? Round your answer to the nearest cent.

4 Suzanne borrows \$25 000 at an interest rate of 7.8% per annum, compounding monthly. She repays the loan with regular payments of \$1200 per month and then a final payment to bring the balance to zero.

a How much does Suzanne owe after 3 months? Round your answer to the nearest cent.

b How much does Suzanne owe after 1 year? Round your answer to the nearest cent.

5 Rachel borrows \$240 000 at an interest rate of 8.3% per annum, compounding quarterly. She makes 119 regular payments of \$5442.90 each quarter followed by a final payment. Rachel repays the loan over thirty years.

a How much does Rachel owe after 6 years? Round your answer to the nearest cent.

b What is the final payment that Rachel must make to fully repay the loan in 30 years? Round your answer to the nearest cent.

6 David borrows \$50 000 for a new car at an interest rate of 4.6%, compounding weekly. He repays the loan over 3 years with regular payments of \$343.27 per week except for the final payment.

a How much does David owe after 1 year? Round your answer to the nearest cent.

b What is the final payment that David must make to fully repay the loan in three years? Round your answer to the nearest cent.

Determining the balance of an annuity using a financial calculator

Example 20

7 Kazou invests \$50 000 into an annuity, paying 6.1% per annum, compounding annually. Kazou receives \$6825.61 per year for nine years then a final payment so that the annuity lasts exactly 10 years.

- **a** State whether the **PV** is positive or negative.
- **b** Find the value of the annuity after five years. Round your answer to the nearest cent.
- **c** What is the final payment made to Kazou so that the value of the annuity is zero after 10 years? Round your answer to the nearest cent.

8 Eliza invests \$20 000 into an annuity, paying 7.2% per annum, compounding monthly. The annuity regularly pays \$1732.37 per month, for eleven months followed by a final payment to exhaust the annuity.

- **a** Find the value of the annuity after three months. Round your answer to the nearest cent.
- **b** What is the final payment made to Eliza so that the value of the annuity is zero after 12 months? Round your answer to the nearest cent.

9 Ezra is going backpacking around Europe and has invested \$15 000 into an annuity for this trip. The annuity pays 6.8% per annum, compounding weekly for one year. The annuity pays \$298.57 per week for each week except for the final payment.

- **a** Find the value of the annuity after twenty-six weeks. Round your answer to the nearest cent.
- **b** What is the final payment made to Ezra so that the value of the annuity is zero after 1 year? Round your answer to the nearest cent.

Exam 1 style questions

10 Josie invests \$3000 in an account that pays interest at the rate of 2.8% per annum, compounding monthly. She makes an additional payment of \$200 each month.
The value of the investment, correct to the nearest cent, after 6 years is

A \$4249.26 **B** \$4249.27 **C** \$12 112.32 **D** \$19 208.55 **E** \$19 208.56

11 Bronwyn borrows \$450 000 to buy an apartment.
The interest rate for this loan was 4.24% per annum, compounding monthly for 20 years.
Bronwyn makes regular monthly payments of \$2784 each month except for her final payment. To pay out her loan fully in 20 years, her final payment is

A \$58.57 **B** \$2725.42 **C** \$2725.43 **D** \$2784.00 **E** \$2842.57

12 Benjamin invests \$75 000 in an annuity, paying 7.3% per annum, compounding monthly.
Benjamin receives a payment of \$2326 each month from the annuity.
The value of the annuity, correct to the nearest cent after 2 years is

A \$26842.05 **B** \$26842.06 **C** \$69356.55 **D** \$69356.56 **E** \$85153.41

8F Using a finance solver to find interest rates, time taken and regular payments

Learning intentions

- ► To be able to use the finance solver to find the interest rate or time taken for a compounding interest investment with additional payments.
- ► To be able to use the finance solver to find the regular payment for a reducing balance loan.
- ► To be able to use the finance solver to find the interest rate, time taken or regular payment for an annuity.

As well as finding the future value or balance of a loan, annuity or investment, we can also use the financial solver to find the interest rate, regular payment or length of the loan, annuity or investment.

Using finance solver for investments with additional payments to find interest rates and time taken

Recall that for an investment with compound interest and additional payments:

- **PV:** Negative: you make an investment by giving the bank some money.
- **PMT:** Negative: you make regular payments to the bank.
- **FV:** Positive: when the investment matures, the bank gives you the money.

Example 21 Finding the interest rate for an investment with additional payments

Mingjia puts \$20 000 into a compound interest investment where interest compounds monthly. She adds \$50 per month. She wants her investment to reach \$40 000 in 10 years.

Find the annual interest rate required for this to occur. Round your answer to two decimal places.

Explanation

1 Open finance solver and enter the following:
   - **N**: 120 (10 years)
   - **PV**: −20000
   - **PMT**: −50
   - **FV**: 40000 (the annuity will be exhausted after 10 years)
   - **Pp/Y**: 12 (monthly payments)
   - **Cp/Y**: 12 (interest compounds monthly)

2 Solve for **I**.

3 Write your answer.

Solution

N:	120
I%:	4.807676
PV:	−20000
Pmt or PMT:	−50
FV:	40000
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Minjia would require an interest rate of 4.81% per annum.

When using a financial solver, rounding is very important so it is always a good idea to check your answer in case you need to round up or down accordingly.

Example 22 Finding the regular monthly payment and time taken for an investment with additions to the principal

Winston puts \$20 000 into an investment, paying 5.1% interest per annum, compounding monthly.

a If Winston wants his investment to be worth at least \$40 000 in 5 years, what is the minimum he will need to add each month?

b If Winston invests \$1000 each month immediately after interest is calculated, what is the minimum number of months required for his investment to at least triple in value?

Explanation

a **1** Open finance solver and enter the following:

- **N**: 60 (5 years)
- **I%**: 5.1 (annual interest rate)
- **PV**: −20000
- **FV**: 40000 (the annuity will be exhausted after 10 years)
- **Pp/Y**: 12 (monthly payments)
- **Cp/Y**: 12 (interest compounds monthly)

2 Solve for **Pmt** or **PMT**.

Note: The sign of Pmt or PMT is negative, because it is money that Winston invests.

Solution

N:	60
I%:	5.1
PV:	−20000
Pmt or PMT:	−208.341646
FV:	40000
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

3 Write your answer, noting that \$208.34 per month is insufficient as it gives a balance of \$39999.89…

Pmt or PMT:	−208.34
FV:	−39999.8877

Winston will add \$208.35 each month to the investment.

b **1** Change the payment **Pmt** or **PMT** to −1000 and the **FV** to 60 000 and solve for **N**.

N:	34.3211
I%:	5.1
PV:	−20000
Pmt or PMT:	−1000
FV:	60000
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

2 Write your answer, noting that 34 months has a **FV** of \$59598.147… so we need to round up

The value of Winston's investment will take 35 months to triple.

Analysing a reducing balance loan with financial solver

Recall that for a reducing balance loan:

- **PV:** Positive - the bank gives you money through a loan
- **PMT:** Negative - you repay the loan by making regular repayments
- **FV:** Negative, zero or positive - balance after the payment is made.

Example 23 Determining the payment amount, total repayment and total amount of interest paid for a reducing balance loan

Sipho borrows \$10 000 to be repaid in 59 equal monthly payments followed by a 60^{th} payment of less than one dollar more than the regular payment. Interest is charged at the rate of 8% per annum, compounding monthly.

a Find the regular monthly payment amount. Round your answer to the nearest cent.

b Find the final payment. Round your answer to the nearest cent.

c Find the total of the repayments on the loan. Round your answer to the nearest cent.

d Find the total amount of interest paid. Round your answer to the nearest cent.

Explanation

a 1 Open Finance Solver and enter the following:

N: 60 (number of monthly payments in 5 years, assuming 60 equal payments)

- **I%**: 8 (annual interest rate)
- **PV**: 10000
- **FV**: 0 (the balance will be zero when the loan is repaid)
- **Pp/Y**: 12 (monthly payments)
- **Cp/Y**: 12 (interest compounds monthly)

Solution

N:	60
I%:	8
PV:	10000
Pmt or PMT:	
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

2 Solve for the unknown future value (Pmt or PMT). On the:

- *TI-Nspire:* Move the cursor to the **Pmt** entry box and press enter to solve.
- *ClassPad:* Tap on the **PMT** entry box and tap **Solve**.

The amount –202.7639... now appears in the **Pmt** or **PMT** entry box.

Note: The sign of the payment is negative to indicate that this is money Sipho is giving back to the lender.

N:	60
I%:	8
PV:	10000
Pmt or PMT:	-202.7639
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

3 Write your answer.

Sipho repays \$202.76 as the regular payment.

b To find the final payment:

1 Find the final value after 60 payments of \$202.76.

2 Since **FV** is −0.289, the final payment is 0.29 more than the regular payment.

Final payment
= 202.76 + 0.29
= 203.05
Final payment is \$203.05

c Total of repayments of the loan = 59 × regular payment + final payment

Total of repayments = 59 × 202.76 + 203.05 = \$12 165.89

d Total interest = total repayments − the principal

Interest paid = 12 165.89 − 10 000 = \$2165.89

Analysing an annuity with financial solver

Recall that for an annuity:

- **PV:** Negative: you buy an annuity by giving the bank some money.
- **PMT:** Positive: you receive regular payments from the bank.
- **FV:** Positive or zero: balance after the payment is made.

Example 24 Finding the interest rate, time taken and regular payment for an annuity

Joe invests \$200 000 into an annuity, with interest compounding monthly.

a What interest rate would allow Joe to withdraw \$2500 each month for 10 years? Round your answer to one decimal place.

b Assume the interest rate is 5% per annum and that Joe receives a regular monthly payment of \$3000. For how many months will Joe receive a regular payment?

c Assume that the interest rate is 5% per annum and that Joe wishes to be paid monthly payments for 10 years. How much will he regularly receive each month?

d If Joe receives the regular monthly payment found in part **c** for 119 months, what will his final payment be? Round your answer to the nearest cent.

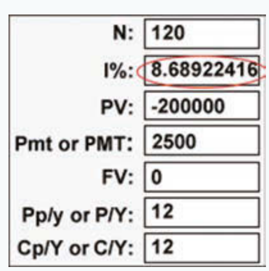
Explanation

a **1** Open finance solver and enter the following:

- **N**: 120 (10 years)
- **PV**: −200000
- **PMT**: 2500
- **FV**: 0 (exhausted after 10 years)
- **Pp/Y**: 12 (monthly payments)
- **Cp/Y**: 12 (interest compounds monthly)

Solve for I%.

Solution



2 Write your answer.

Joe would require an interest rate of 8.7% per annum to make monthly withdrawals of $2500 for 10 years.

b **1** Change the payment **Pmt** or **PMT** to 3000 and solve for **N**.

N:	78.2639745
I%:	5
PV:	-200000
Pmt or PMT:	3000
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

2 Write your answer, rounding down as we are only counting regular payments.

Joe will receive a regular payment for 78 months.

c **1** Open the finance solver on your calculator and enter the information below, as shown.

- **N**: 120 (10 years)
- **I%**: 5 (annual interest rate)
- **PV**: −200000
- **FV**: 0 (the annuity will be exhausted after 10 years)
- **Pp/Y**: 12 (monthly payments)
- **Cp/Y**: 12 (interest compounds monthly)

N:	120
I%:	5
PV:	-200000
Pmt or PMT:	
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

2 Solve for **Pmt** or **PMT**.

Note: The sign of Pmt or PMT is positive, because it is money received.

N:	120
I%:	5
PV:	-200000
Pmt or PMT:	2121.3103
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

3 Write your answer.

Joe will receive $2121.31 each month from the annuity.

d Find the final value after 120 months of $2121.31.

N:	120
I%:	5
PV:	−200000
Pmt or PMT:	2121.31
FV:	0.047
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Since **FV** is 0.047…, the final payment is 0.05 more than the regular payment.

Final payment is $2121.36

Skill-sheet

Exercise 8F

Analysing an investment with additional payments using a financial calculator

Example 21

1 Armaan puts \$15 000 into an investment that compounds annually.
Find the annual interest rate that would allow Armaan's investment to reach \$100 000 after 10 years if he invests an additional \$4500 each year for 10 years. Round your answer to two decimal places.

Example 22

2 Jemima puts \$30 000 into an investment that compounds monthly.

a What annual interest rate would allow Jemima's investment to double after 5 years if she invests an additional \$400 each month for 5 years? Round your answer to one decimal place.

Assume the investment pays 3.2% per annum, compounding monthly.

b **i** If she wants her investment to be worth at least \$40 000 in 1 year, what is the minimum she will need to add to the investment each month? Round your answer to the nearest cent.

Note: You will need to round your answer up so that it reaches \$40 000.

ii If she invests an additional monthly payment of at least \$1000, what is the minimum number of months that it will take for the investment to first reach \$100 000?

3 Kelven puts \$7500 into an investment, paying 4.7% per annum, compounding monthly. He makes regular additional contributions to the investment each month. After one year, Kelven's investment is worth \$13 991.15 to the nearest cent.

a Find the amount of Kelven's regular monthly payment. Round your answer correct to the nearest cent.

b **i** Find the amount that Kelven invested in the first year through monthly payments.

ii Find the increase in the value of the investment in the first year.

iii Hence, find how much interest was earned in the first year. Round your answer correct to the nearest cent.

c Given Kelven's monthly payment found in **a**, how many months will it take for Kelven's investment to be worth at least \$20 000?

Analysing a reducing balance loan using a financial calculator

Example 23

4 Dan arranges to make regular payments of \$450 per month followed by a single smaller final payment to repay a loan of \$20 000. Interest is charged at 9.5% per annum, compounding monthly.
Find the number of monthly payments required to pay out the loan.

5 A building society offers \$240 000 loans at an interest rate of 10.25% compounding monthly for a 30 year period.

a If payments are \$2200 per month, calculate the amount still owing on the loan after 12 years. Round your answer to the nearest cent.

b If the loan has a regular monthly payment of \$2150.64 for the first 359 payments, calculate:

i the final payment, rounding your answer to the nearest cent.

ii the total amount repaid, rounding your answer to the nearest cent.

iii the total amount of interest paid, rounding your answer to the nearest cent.

6 Rahul borrows \$17 000 at an interest rate of 6.8% per annum, compounding monthly. Rahul wishes to pay off the reducing balance loan in 30 months by making equal payments for 29 months followed by a final payment that is as close to the regular payment as possible.

a Find the regular monthly payment. Round your answer to the nearest cent.

b Find the final payment of the loan. Round your answer to the nearest cent.

c Find the total of the repayments of the loan.

d Find the total amount of interest that Rahul has paid.

7 Cale borrows \$140 000 at an interest rate of 8.6% per annum, compounding quarterly. Cale makes regular equal quarterly payments except for the final payment which is as close to the regular payment as possible.

a If Cale pays off the loan in 10 years with 39 regular quarterly payments of \$5253.39:

i find the final payment. Round your answer to the nearest cent.

ii find the total amount that Cale repays.

b Rounding each of your answers to the nearest cent, if Cale pays off the loan in 15 years, find

i the regular quarterly payment.

ii the final payment.

iii the total cost of repaying the loan.

8 Lorenzo borrows \$250 000 at 5.2% per annum, compounding fortnightly. Lorenzo makes 649 equal fortnightly payments followed by a final payment which is as close to the regular payment as possible. Find the total cost of repaying the loan. Round your answer to the nearest cent.

9 Joan takes out a loan of \$50 000 with an interest rate of 4.9% per annum, compounding monthly. She makes regular monthly payments for 23 months of \$2191.33 followed by a single final payment. How much interest does Joan pay in total over the duration of the two year loan?

Analysing an annuity using a financial calculator

Example 24

10 Olek invests \$100 000 into an annuity with interest compounding monthly.

a What is the smallest interest rate that would allow Olek to withdraw \$2500 each month for 4 years. Round your answer to two decimal places.

b Assume the interest rate is 6% and Olek wishes to be paid monthly payments for 4 years. How much will he receive each month as his regular payment? Assume that the final payment is as close as possible to the regular payment.

c Assume the interest rate is 6% and Olek receives a regular payment of \$2000. For how many months will he receive his full payment?

11 Sophia invests \$300 000 into an annuity, paying 4.3% interest per annum, compounding quarterly. She wishes to receive a payment of at least \$5000 every quarter. For how many quarters will Sophia receive at least \$5000?

12 Kai invests \$500 000 in an annuity. The annuity earns interest at the rate of 4.7% per annum, compounding monthly. The balance of Kai's annuity at the end of the first year of the investment is \$474 965.28.

a What monthly payment did Kai receive? Round your answer to the nearest cent.

b How much interest would Kai's annuity earn in the first year? Round your answer to the nearest cent.

Exam 1 style questions

13 Simone invests \$3000 in an account that pays interest at the rate of 3.1% per annum, compounding monthly. She makes an additional payment of \$250 each month. The number of months that it will take the investment to reach a balance of at least \$30 000 is

A 40 **B** 41 **C** 91 **D** 92 **E** 93

14 Lachlan borrows \$480 000 to buy an apartment using a reducing balance loan that compounds monthly. Lachlan makes regular monthly payments of \$3075.72 followed by a final payment of \$3075.53. If the loan is paid out fully in 20 years, the annual interest rate is closest to

A 0.3875% **B** 0.0465% **C** 4.65% **D** 11.55% **E** 14.67%

15 Audrey invests \$85 000 in an annuity, paying 6.3% per annum, compounding monthly. Audrey receives a regular monthly payment from the annuity. If the value of the annuity after one year is \$71 983.41, the amount of interest earned in the first year is closest to

A \$1500 **B** \$4983 **C** \$5355 **D** \$13 017 **E** \$31 016

8G Solving harder financial problems

Learning intentions

- ▶ To be able to find the value of an investment when the regular payment changes.
- ▶ To be able to analyse the impact of a change in the interest rate on a reducing balance loan, an annuity and an investment.

Sometimes the conditions of a reducing balance loan can change, requiring the regular repayment to increase or decrease for the loan to be repaid in full. Similarly, a change in the interest rate can also alter the payment received from an annuity or the balance of an investment with compound interest rates. A financial solver on the CAS calculator can help to solve for the regular payment or the new balance after a change has occurred.

Changing the regular payment to an investment

Sometimes an investor may want to change the regular additional payment to an investment. When this happens, we need to consider the investment in two parts. First, we consider the time before the change. Then, we consider the time after the change occurs.

Banks and other financial institutions do not round the value of an investment or loan until the investment is withdrawn or the loan is fully repaid. Thus, for any intermediate step, we will use unrounded values.

Example 25 Finding the value of an investment when the regular payment changes

Derek invests \$50 000 into a compound interest investment paying 6.1% per annum, compounding annually. Derek invests an additional \$8000 per year immediately after interest is calculated.

After five years, Derek increases his additional investment to \$10 000 per year.

Calculate the value of Derek's investment after twelve years (in total).

Explanation

1 Open finance solver and enter the following:
- **N**: 5 (5 years before the change)
- **I%**: 6.1 (annual interest rate)
- **PV**: -50000 (value of initial investment)
- **PMT**: -8000 (additional amount added)
- **Pp/Y**: 1 (annual payment)
- **Cp/Y**: 1 (interest compounds annually)

Solve for **FV**.

Solution

N:	5
I%:	6.1
PV:	–50000
Pmt or PMT:	–8000
FV:	112414.364
Pp/y or P/Y:	1
Cp/Y or C/Y:	1

2 Change the following in finance solver:

- **N**: 7 (7 years after the change)
- **PV**: -112414.364. . . (copied from FV above)
- **PMT**: -10000 (additional amount added)

Solve for **FV**

N:	7
I%:	6.1
PV:	−112414.3641
Pmt or PMT:	−10000
FV:	254343.79745
Pp/y or P/Y:	1
Cp/Y or C/Y:	1

3 Write your answer.

The value of Derek's investment is $254 343.80

Changing the interest rate

Example 26 **Finding the final payment of a reducing balance loan when the interest rate changes**

Adrian borrows $150 000 for 25 years at an interest rate of 6.8% per annum, compounding monthly.

For the first three years, Adrian repays $1041.11 each month.

After 3 years, the interest rate rises to 7.2% per annum. Adrian still wishes to pay off the loan in 25 years so makes 263 monthly payments of $1076.18 followed by a final payment.

Calculate the final payment to ensure the loan is fully repaid at the end of 25 years. Round your answer to the nearest cent.

Explanation

1 Open the finance solver on your calculator and enter the information below, as shown opposite.

- **N**: 36 (number of monthly payments in 3 years)
- **I%**: 6.8 (annual interest rate)
- **PV**: 150000 (initial value of loan)
- **Pmt**: −1041.11 (monthly repayments)
- **Pp/Y**: 12 (monthly payments)
- **Cp/Y**: 12 (interest compounds monthly)

Note: You can enter **N** as 3 × 12 (3 years of monthly payments). The finance solver will calculate this as 36 for you.

Solution

N:	36
I%:	6.8
PV:	150000
Pmt or PMT:	-1041.11
FV:	
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

2 Solve for **FV**.

N:	36
I%:	6.8
PV:	150000
Pmt or PMT:	-1041.11
FV:	-142391.8359
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

3 If the loan is still to be repaid in 25 years, there are still 22 years left.
Change:

- **N** to 22×12 or 264 payments
- **I(%)** to 7.2 (the new interest rate)
- **PV** to 142391.83593707 (the balance after 3 years)
- **Pmt** to -1076.18

N:	264
I%:	7.2
PV:	-142391.8359
Pmt or PMT:	-1076.81
FV:	
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

4 Solve for **FV**.

N:	264
I%:	7.2
PV:	-142391.8359
Pmt or PMT:	-1076.18
FV:	0.11735068
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

5 Since **FV** is 0.1173…, the final payment decreases by \$0.12.

Final payment =
\$1076.18 – \$0.12 = \$1076.06
Thus, the final payment will be \$1076.06

Note: It is important that you do not round prematurely or you will get the incorrect answer of \$1076.08

A similar analysis can be used for both annuities and investments with additions to the principal.

Exercise 8G

Changing the regular payment

Example 25

1 Danielle invests \$8000 into a compound interest investment paying 7.6% per annum, compounding annually.
She invests an additional \$1000 per year immediately after interest is calculated.
After five years, Danielle increases her additional investment to \$2000 per year.
Calculate the value of Danielle's investment after twelve years (in total). Round your answer to the nearest cent.

2 Peta invests \$20 000 into a compound interest investment paying 4.8% per annum, compounding monthly.
Peta invests an additional \$200 per month immediately after interest is calculated.
After ten years, Peta increases her additional investment to \$500 per month.
Caculate the value of Peta's investment after twenty years (in total). Round your answer to the nearest cent.

3 Jarrod opens an account with an initial balance of \$0 that pays interest at a rate of 6% per annum, compounding monthly.
He makes monthly deposits of \$500 to the account for 10 years.
After 10 years of making deposits, Jarrod withdraws the balance and places it in an annuity, also with an annual interest rate of 6%, compounding monthly. He withdraws \$500 each month from the account.

a How much does he invest in the annuity after the initial 10 years?

b How much will remain in the annuity after 10 years? Round your answer to the nearest cent.

Reducing balance loans with changing conditions

Example 26

4 Julien borrows \$35 000 for 20 years at an interest rate of 10.5% per annum, compounding monthly.
For four years, he pays \$349.43 each month.
After four years, the interest rate rises to 13.75% per annum. Julien still wishes to pay off the loan in a total of 20 years so he makes 191 monthly payments of \$418.66 followed by a final payment. For the loan to be fully repaid to the nearest cent, Julien's final repayment will be a smaller amount.
Calculate the final payment that Julien must make to repay the loan in 20 years. Round your answer to the nearest cent.

5 A couple negotiates a 25-year mortgage of \$500 000 at a fixed rate of 7.5% per annum compounding monthly for the first seven years.
The monthly repayment amount of \$3694.96 is paid each month for seven years.
After seven years, the interest rate rises to 8.5% per annum. The couple now pay \$3959.44 each month.
Calculate the value of the loan after a further seven years at the higher interest rate. Round your answer to the nearest cent.

6 Zian borrows \$750 000 for a new home at an interest rate of 8.5% per annum, compounding monthly.
For the first five years, he only pays the interest so the value of the loan remains at \$750 000.

a Calculate Zian's monthly repayments.

After five years the interest rate increases to 9.4%. Zian must now pay more each month in order to pay the loan in full within the original 30 years. He does this by making 299 regular monthly repayments followed by a final payment which is as close to the regular payment as possible.

b Calculate the new regular monthly payment amount Zian must make.

c Find the final payment.

d Calculate the total amount that Zian pays over 30 years.

e How much interest will Zian pay over the lifetime of the loan?

Changed conditions with annuities

7 Helen has \$80 000 to invest. She chooses an annuity that pays interest at the rate of 6.4% per annum, compounding monthly. Helen expects her investment to be fully exhausted after 15 years.

She receives a monthly payment of \$692.50 each month for two years.

After two years, the interest rate of Helen's investment was reduced to 6.2% per annum, compounding monthly.

a If Helen continues to withdraw \$692.50 each month, how many more months can she withdraw this regular amount?

b The final payment received from this annuity will be less than the regular repayments. Find the final payment that will exhaust the annuity.

8 Ethan invests \$125 000 into an annuity from which he receives a regular monthly payment of \$850. The interest rate for the annuity is 5.4% per annum, compounding monthly.

a Let V_n be the balance of the annuity after n monthly payments. Write a recurrence relation written in terms of V_0, V_{n+1} and V_n to model the value of this annuity from month to month.

b After two years, the interest rate for this annuity will fall to 4.1%.
So that Ethan will continue to receive a monthly payment of \$850 for the following 18 years, he will add an extra one-off amount to the annuity at this time.
Determine the minimum value of the one-off addition. Give your answer to the nearest dollar.

9 Sameep deposits \$150 000 into a savings account earning 6% per annum, compounding monthly, for 10 years. He makes no withdrawals or deposits during that time.

a Let S_n be the balance of Sameep's investment after n months. Write a recurrence relation to model this investment.

b What is the value of this account after 10 years? Round your answer to the nearest cent.

After 10 years Sameep withdraws the money and invests the full amount into an annuity. He will require this investment to provide monthly withdrawals of $2600.

c What is the minimum annual interest rate required if Sameep's investment is to be exhausted after 10 additional years? Round your answer to two decimal places.

Changed conditions on investments

10 Marcus' grandparents place $2000 in an investment account that pays interest at a rate of 4% per annum, compounding annually. For 18 years from Marcus' birth, they contribute an additional $1000 to the account each year.

a Find the balance of the investment after 18 years. Round your answer to the nearest cent.

When Marcus turns 18, the interest rate increases to 5% and his grandparents stop contributing.

b Give the balance of the account to the nearest dollar when Marcus turns 21.

11 When Jessica starts working, she sets up an investment account with an initial balance of $1000. Each month she deposits $200 into the account.
The account has an annual interest rate of 4.9% compounding monthly.

a Find the balance of the investment after one year. Round your answer to the nearest cent.

b How much interest has the investment earned in the first year? Round your answer to the nearest cent.

After three years, the interest rate increases to 6% per annum and Jessica will increase her monthly deposit to $350 per month.

c Find the balance of Jessica's investment account after two years at the higher interest rate. Round your answer to the nearest cent.

d How many payments of $350 will Jessica need to make until her investment first exceeds $35 000?

Exam 1 style questions

12 Cherry borrowed $500 000 to buy an apartment.
The interest rate for this loan was 4.31% per annum, compounding monthly.
Cherry paid $4200 per month for the first two years.
After these two years, the interest rate changed. Cherry was able to pay off the loan in a further 8 years by paying $5361.49 each month.
The interest rate, per annum, for the final 8 years of the loan was closest to

A 3.90% **B** 4.00% **C** 4.10% **D** 4.30% **E** 4.31%

13 Thirty years ago, Irene invested a sum of money in an account earning interest at the rate of 3.1% per annum, compounding monthly.
After 10 years, the interest rate changed.
For the next twenty years, the account earned interest at the rate of 2.7% per annum, compounding monthly.
The balance of her account today is $876 485.10.
The sum of money that Irene originally invested is closest to

A $360 300 **B** $375 000 **C** $390 000 **D** $511 100 **E** $670 000

14 Calvin plans to retire from his work in 12 years' time and hopes to have $800 000 in an annuity investment at that time.
The present value of this annuity investment is $227 727.96, where the interest rate is 3.6% per annum, compounding monthly.
To make this investment grow faster, Calvin adds $2500 at the end of each month.
Two years from now, Calvin expects the interest rate to fall to 3.3% per annum, compounding monthly, and to remain at this level until he retires.
When the interest rate changes, Calvin must change his monthly payment if he wishes to make his retirement goal.
The value of his new monthly payment will be closest to

A $1950 **B** $2500 **C** $2560 **D** $2600 **E** $2630

8H Interest-only loans

Learning intentions

- To be able to find the regular payment amount for an interest-only loan with and without a financial solver.
- To be able to find the amount borrowed for an interest-only loan.
- To be able to find the interest rate for an interest-only loan.

In an **interest-only loan**, the borrower repays only the interest that is charged. As a result, the balance of the loan remains the same for the duration of the loan. To understand how this happens, consider a loan of $1000 with an interest rate of 5% per annum, compounding yearly. The interest that is charged after 1 year will be 5% of $1000, or $50. If the borrower only repays $50, the value of the loan will still be $1000.

The recurrence relation $V_0 = 1000, \;\; V_{n+1} = 1.05V_n - D$ can be used to model this loan. The table below shows the balance of the loan over a 4-year period for three different payment amounts: $D = 40$, $D = 50$ and $D = 60$.

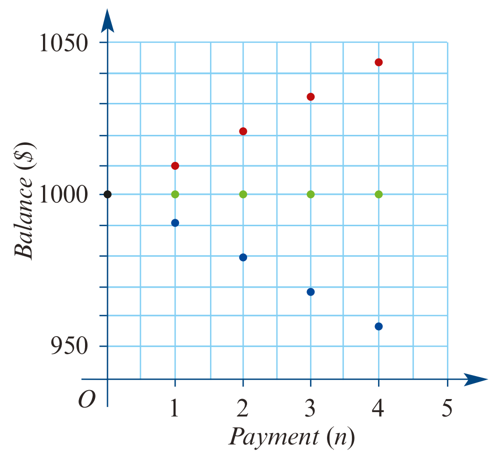
$D = 40$	$D = 50$	$D = 60$
$V_0 = 1000,$ $V_{n+1} = 1.05V_n - 40$	$V_0 = 1000,$ $V_{n+1} = 1.05V_n - 50$	$V_0 = 1000,$ $V_{n+1} = 1.05V_n - 60$
$V_0 = 1000$ $V_1 = 1010$ $V_2 = 1020.50$ $V_3 = 1031.525$ $V_4 = 1043.101\ldots$	$V_0 = 1000$ $V_1 = 1000$ $V_2 = 1000$ $V_3 = 1000$ $V_4 = 1000$	$V_0 = 1000$ $V_1 = 990$ $V_2 = 979.50$ $V_3 = 968.475$ $V_4 = 956.898\ldots$
The amount owed keeps increasing.	The amount owed stays constant.	The amount owed keeps decreasing.

We can plot these balances against the payment number.

If the periodic payments on this loan are smaller than \$50 (e.g. $D = 40$), the amount owed will increase over time. The balance of the loan is shown as red dots.

If the periodic payments on this loan are larger than \$50 (e.g. $D = 60$), the amount owed will decrease over time. The balance of the loan is shown as blue dots.

If the periodic payments on this loan are exactly \$50, then the amount owed on the loan will always be \$1000. The balance of the loan is shown as green dots.



A loan where the balance stays constant is called an **interest-only loan** and is commonly used for investment purposes.

Modelling interest-only loans

Let V_n be the value of the interest-only loan after n payments have been made. Then

$$V_0 = \text{principal}, \quad V_{n+1} = RV_n - D$$

where $R = 1 + \dfrac{r}{100 \times p}$ is the growth multiplier, r is the annual interest rate, p is the number of compounding periods per year and D is the regular payment per compounding period which is equal to the interest charged, given by

$$D = \frac{r}{100 \times p} \times V_0$$

Example 27 Finding the regular payment for an interest-only loan

Jane borrows \$50 000 to buy some shares. Jane negotiates an interest-only loan at an interest rate of 9% per annum, compounding monthly. What is the monthly amount Jane will be required to pay?

Explanation

Calculation method

Use the rule $D = \dfrac{r}{100 \times p} \times V_0$.

1 V_0 is the amount borrowed = \$50 000

2 Calculate the interest payable where $r = 9$ and $p = 12$.

3 Evaluate the rule for these values and write your answer.

Solution

$V_0 = 50\,000$

$D = \dfrac{r}{100 \times p} \times V_0$

$D = \dfrac{9}{100 \times 12} \times 50\,000$

$D = 375$

Jane will need to repay \$375 every month on this interest-only loan.

Explanation

Finance solver method

Consider one compounding period because all compounding periods will be identical.

1 Open Finance Solver and enter the following.
- **N**: 1 (one compounding period)
- **I%**: 9 (annual interest rate)
- **PV**: 50000
- **FV**: −50000 (the amount owing will be the same after one payment)
- **Pp/Y**: 12 (monthly payments)
- **Cp/Y**: 12 (interest compounds monthly)

N:	1
I%:	9
PV:	50000
Pmt or PMT:	
FV:	–50000
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

2 Solve for the unknown future value (Pmt or PMT). On the:
- *TI-Nspire:* Move the cursor to the **Pmt** entry box and press enter to solve.
- *ClassPad:* Tap on the **PMT** entry box and tap **Solve**.

The amount –375 now appears in the **Pmt** or **PMT** entry box.

N:	1
I%:	9
PV:	50000
Pmt or PMT:	–375
FV:	–50000
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Example 28 **Finding the amount borrowed for an interest-only loan**

A loan at 6% per annum, compounding monthly, requires payments of \$440 each month.

If the loan is an interest-only loan, what is the principal?

Explanation	Solution
1 Use formula $D = \dfrac{r}{100 \times p} \times V_0$, where $D = 440, r = 6, p = 12$.	Solving for the principal: $440 = \dfrac{6}{100 \times 12} \times V_0$ $V_0 = 88\ 000$
2 Write the answer.	The principal is \$88 000.

Example 29 **Finding the interest rate for an interest-only loan**

An interest-only loan of \$1 000 000 requires quarterly payments of \$4000.

What is the annual interest rate on the loan?

Explanation	Solution
1 Use formula $D = \dfrac{r}{100 \times p} \times V_0$, where $D = 4000, p = 4, V_0 = 1\ 000\ 000$.	Solving for r: $4000 = \dfrac{r}{100 \times 4} \times 1\ 000\ 000$ $r = 1.6$
2 Write the answer.	The annual interest rate is 1.6%

Exercise 8H

Finding the regular payment for an interest-only loan

Example 27

1 Georgia borrows \$100 000 to buy an investment property. If the interest on the loan is 7.2% per annum, compounding monthly, find her monthly payment on an interest-only loan.

2 In order to invest in the stockmarket, Jamie takes out an interest-only loan of \$50 000. If the interest on the loan is 8.4% per annum compounding monthly, find his monthly payment amount.

3 Robert takes out an interest-only loan for \$220 000 at an interest rate of 5.46% per annum, compounding fortnightly. Find the fortnightly payment.

4 Frannie borrows \$180 000 at an interest rate of 4.95% per annum, compounding quarterly. If Frannie only pays the interest, find the total payments made over a five year period.

5 Jackson takes out an interest-only loan of \$30 000 from the bank to buy a painting. He hopes to resell it at a profit in 12 months' time. The interest on the loan is 9.25% per annum, compounding monthly. He makes monthly payments on the loan.

a Find the total amount that Jackson pays in 12 months.

b How much will he need to sell the painting for in order not to lose money?

6 Ric takes out an interest-only loan of \$600 000 to buy an investment property. The interest on the loan is 5.11% per year, compounding monthly.

a Calculate Ric's monthly repayments if he only pays the interest.

b Ric sells the property after 10 years. Calculate the total interest paid on the loan.

c How much must Ric sell the property for if he wishes to make a profit of at least \$100 000?

7 Mindy borrows \$35 000 for 20 years at 6.24% per annum, compounding monthly. For the first five years, Mindy pays interest only.

a Calculate the monthly repayments that Mindy makes for the first two years.

b State the balance of the loan after five years.

c For the next 179 months, Mindy pays \$300 per month followed by a smaller payment to fully repay the loan. Find this final repayment. Round your answer to the nearest cent.

d Find the total amount that Mindy paid for the duration of the 20 year loan.

Find the amount borrowed for an interest-free loan

Example 28

8 An interest-only loan with an interest rate of 5.3% per annum, compounding annually, requires annual payments of \$2120. What is the principal?

9 An interest-only loan with an interest rate of 6.6% per annum, compounding monthly, requires a monthly payment of \$88. What is the principal?

10 Yianni took out an interest-only loan with an interest rate of 4.2% per annum, compounding monthly. Over a two year period, Yianni paid \$2352 in total. Find the principal of the loan.

Find the interest rate for an interest-free loan

Example 29

11 An interest-only loan of \$4000 requires annual payments of \$116. What is the annual interest rate?

12 An interest-only loan of \$12 000 compounds monthly and requires monthly payments of \$36. What is the annual interest rate?

13 Leo takes out an interest-only loan of \$35 000 which compounds monthly and requires monthly payments. Over a two year period, Leo pays a total of \$3360. What is the annual interest rate?

14 Svetlana borrows \$320 000 on an interest-only loan at an interest rate of 4.92% per annum, compounding monthly for the first 5 years. Following this, the interest rate changes.

a Calculate the monthly repayments that Svetlana makes for the first five years.

b Calculate the total amount that Svetlana paid during the first five years.

For the next five years, Svetlana pays \$86 400 on the interest-only loan.

c State the total interest that she paid during the second half of the loan.

d Calculate the monthly repayment that she made during the second half of the loan.

e Hence, find the annual interest rate during the second half of the loan.

Exam 1 style questions

15 Matthew would like to purchase a new home. He establishes a 20 year loan for \$310 000 with interest charged at the rate of 3.84% per annum, compounding monthly.
Each month, Matthew will only pay the interest charged for that month.
After three years, the amount that Matthew will owe is

A \$274 288 **B** \$277 222 **C** \$310 000 **D** \$345 712 **E** \$418 350

16 Eve borrowed \$780 000 to buy a house.
The interest rate for this loan was 4.82% per annum, compounding monthly.
A scheduled monthly repayment that allowed Eve to fully repay the loan in 20 years was determined.
Eve decided to pay interest-only for the first two years.
After these two years, the interest rate changed. Eve was still able to pay off the loan in the 20 years by paying the original monthly repayment amount each month.
The new interest rate of the loan was closest to

A 3.9% **B** 4.0% **C** 4.1% **D** 4.2% **E** 4.3%

17 Jason takes out an interest-only loan for five years.
The value of Jason's interest-only loan, V_n, after n months, can be modelled by the recurrence relation

$$V_0 = 56\,000, \qquad V_{n+1} = 1.0034V_n - D$$

The total interest paid on the loan over the five years is

A \$4.08 **B** \$190.40 **C** \$952 **D** \$11 424 **E** \$56 000

8I Perpetuities

Learning intentions

- ► To be able to calculate the regular payment from a perpetuity.
- ► To be able to calculate the investment required to establish a perpetuity.
- ► To be able to calculate the interest rate of a perpetuity.

Recall that an annuity involves money being deposited in an investment and then withdrawn over time in the form of regular payments. In our earlier analysis, we considered the case where the withdrawals were made to exhaust the annuity over a given time frame. That is, the value of the annuity eventually reached zero.

If the regular payments are *smaller* than the interest received, the annuity will continue to grow. If the payments received are exactly the same as the interest earned in each compounding period, the annuity will maintain its value indefinitely. This type of annuity is called a **perpetuity** and the payments that are equal to the interest earned can be made forever (or in Perpetuity). Perpetuities have the same relationship to annuities as interest-only loans have to reducing balance loans.

Modelling perpetuities

Let V_n be the value of the perpetuity after n payments have been made. Then

$$V_0 = \text{principal}, \qquad V_{n+1} = RV_n - D$$

where $R = 1 + \dfrac{r}{100 \times p}$ is the growth multiplier, r is the annual interest rate, p is the number of compounding periods per year and D is the regular payment per compounding period which is equal to the interest earned, given by

$$D = \frac{r}{100 \times p} \times V_0$$

Example 30 Calculating the payment from a perpetuity

Elizabeth invests her superannuation payout of \$500 000 into a perpetuity that will provide a monthly income.

If the interest rate for the perpetuity is 6% per annum, what monthly payment will Elizabeth receive?

Explanation	Solution
1 Find the monthly interest earned.	$D = \dfrac{r}{100 \times p} \times V_0$ $= \dfrac{6}{100 \times 12} \times 500000$ $= 2500$
2 Write your answer, rounding as required.	Elizabeth will receive \$2500 every month from her investment.

Example 31 Calculating the investment required to establish a perpetuity

Calculate how much money will need to be invested in a perpetuity account, earning interest of 4.8% per annum compounding monthly, if \$300 will be withdrawn every month.

Explanation

1 Use the rule $D = \frac{r}{100 \times p} \times V_0$ to write down an equation that can be solved for V_0.

Solution

$$300 = \frac{4.8}{100 \times 12} \times V_0$$
$$V_0 = \frac{300}{0.004}$$
$$= 75000$$

2 Write your answer.

$75 000 will need to be invested to establish the perpetuity investment.

Problems involving perpetuities can also be solved using a financial calculator.

Example 32 Calculating the interest rate of a perpetuity

A university mathematics faculty has $30 000 to invest. It intends to award an annual mathematics prize of $1500 with the interest earned from investing this money in a perpetuity.

What is the minimum interest rate that will allow this prize to be awarded indefinitely?

Explanation

We will consider just one compounding period because all compounding periods will be identical.

Calculation method

1 Use the rule $D = \frac{r}{100 \times p} \times V_0$ and solve the equation for r.

Solution

$$1500 = \frac{r}{100 \times 1} \times 30\,000$$
$$1500 = r \times 300$$
$$r = \frac{1500}{300} = 5$$

2 Write your answer.

The minimum annual interest rate to award this prize indefinitely is 5%.

Financial solver

1 Open Finance Solver and enter the following.

- **N**: 1 (one payment)
- **PV**: −30 000
- **Pmt or PMT**: 1500 (prize is $1500 each year)

N:	1
I%:	
PV:	-30000
Pmt or PMT:	1500
FV:	30000
Pp/y or P/Y:	1
Cp/Y or C/Y:	1

- **FV**: 30 000 (the balance will be the same after each payment)
- **Pp/Y**: 1 (yearly payment)
- **Cp/Y**: 1 (interest compounds yearly)

2 Solve for the unknown interest rate (**I%**). On the:

- *TI-Nspire:* Move the cursor to the **I%** entry box and press enter to solve.
- *ClassPad:* Tap on the **I%** entry box and tap **Solve** The amount **5** now appears in the **I%** entry box.

N:	1
I%:	5
PV:	-30000
Pmt or PMT:	1500
FV:	30000
Pp/y or P/Y:	1
Cp/Y or C/Y:	1

3 Write your answer, rounding as required.

The minimum annual interest rate to award this prize indefinitely is 5%.

Exercise 8I

Calculating the payment from a perpetuity

Example 30

1 Suzie invests her inheritance of \$642 000 in a perpetuity that pays 6.1% per annum compounding quarterly.

a What quarterly payment does she receive?

b After five quarterly payments, how much money remains invested in the perpetuity?

c After 10 quarterly payments, how much money remains invested in the perpetuity?

2 Craig wins \$1 000 000 in a lottery and decides to place it in a perpetuity that pays 5.76% per annum interest, compounding monthly.

a What monthly payment does he receive?

b How much interest does he earn in the first year?

3 Donna sold her cafe business for \$720 000 and invested this amount in a perpetuity. The perpetuity earns interest at a rate of 3.6% per annum. Interest is calculated and paid monthly.

a What monthly payment will Donna receive from this investment?

b After three years, the interest rate for the perpetuity increases. Describe whether Donna's monthly payment will increase, decrease or stay the same.

Calculating the investment required to establish a perpetuity

Example 31

4 Geoff wishes to set up a fund so that every year \$2500 is donated to the RSPCA in his name.
If the interest on his initial investment averages 2.5% per annum, compounding annually, how much should he invest?

5 Barbara wishes to start a scholarship that will reward the top mathematics student each quarter with a \$600 prize.
If the interest on the initial investment averages 4.8% per annum, compounding quarterly, how much should be invested?

6 Omar inherits \$920 000 and splits the money between a perpetuity and an annuity investment.
The perpetuity pays \$2340 each month based on an interest rate of 5.2% per annum that compounds monthly.
The annuity investment has an interest rate of 4.8% per annum that compounds monthly.

a Calculate how much Omar invested in the perpetuity.
b State how much is initially invested in the annuity investment.
c Omar realises that he only needs \$2000 from his perpetuity each month and so he adds \$340 as an aditional payment into the annuity investment each month. Find the value of the annuity investment after three years.

Calculating the interest rate of a perpetuity

Example 32

7 If Sandra has \$80 000 to invest, what is the minimum interest rate she requires to provide an annual donation of \$2400 indefinitely into the future if interest is compounding annually?

8 Benjamin has \$12 000 to invest in a perpetuity to provide a prize of \$750 each year. What is the minimum interest rate that he requires in order to pay the prize in perpetuity if interest compounds annually?

9 On retiring from work, Tyson received a superannuation payout of \$694 400.
If Tyson invests the money in a perpetuity, he would then receive \$3645.60 each month for the rest of his life. At what annual percentage rate is interest earned by this perpetuity?

Analysis of perpetuities

10 Marco invests \$350 000 in a perpetuity from which he will receive a regular monthly payment of \$1487.50.
The perpetuity earns interest at the rate of 5.1% per annum.

a Determine the total amount, in dollars, that Marco will receive after one year of monthly payments.

b Write down the value of the perpetuity after Marco has received one year of monthly payments.

c Let M_n be the value of Marco's perpetuity after n months. Write down a recurrence relation in terms of M_0, M_{n+1} and M_n, that would model the value of this perpetuity over time.

11 Zihan invests \$200 000 in a perpetuity from which he will receive a regular payment. There are two options available:

- Option A: Earn interest at a rate of 3.6% per annum, compounding monthly with a regular monthly payment.
- Option B: Earn interest at a rate of 3.8% per annum, compounding annually with a regular annual payment.

a Calculate the monthly payment from Option A.

b Calculate the annual payment from Option B.

c Determine which option pays the most over one year.

d Let Z_n be the value of the perpetuity after n payments that pays the most over the course of a year. Write down a recurrence relation in terms of Z_0, Z_{n+1} and Z_n, that would model the value of this perpetuity over time.

Exam 1 style questions

12 Which of the following recurrence relations could be used to model the value of a perpetuity investment, P_n, after n months?

A $P_0 = 100\,000, \quad P_{n+1} = 1.005P_n + 500$

B $P_0 = 100\,000, \quad P_{n+1} = 1.005P_n - 500$

C $P_0 = 100\,000, \quad P_{n+1} = 0.005P_n - 500$

D $P_0 = 200\,000, \quad P_{n+1} = 1.003P_n + 600$

E $P_0 = 200\,000, \quad P_{n+1} = 1.103P_n - 600$

13 Aaliyah invests \$120 000 in a perpetuity from which she will receive a regular monthly payment. The perpetuity has a compound interest rate of 5.2% per annum and compounds monthly. The amount that Aaliyah will receive from the perpetuity in the first two years is closest to

A \$520 **B** \$624 **C** \$6240 **D** \$12 480 **E** \$149 760

Key ideas and chapter summary

Assign-ment

Reducing balance loan	A **reducing balance loan** is a loan that attracts compound interest but is reduced in value by making regular payments. Each payment partly pays the interest that has been added and partly reduces the value of the loan.
Annuity	An **annuity** is an investment that earns compound interest and from which regular payments are made.
Amortisation	An **amortising loan** is one that is paid back with periodic payments. An amortising investment is one that is exhausted by regular withdrawals. **Amortisation of reducing balance loans** tracks the distribution of each periodic payment, in terms of the interest paid and the reduction in the value of the loan. **Amortisation of an annuity** tracks the source of each withdrawal, in terms of the interest earned and the reduction in the value of the investment.
Amortisation table	An **amortisation table** shows the amortisation (payment) of all or part of a reducing balance loan or annuity. It has columns for the payment number, the payment amount, the interest paid or earned, the principal reduction or increase and the balance after the payment has been made.
Finance Solver	**Finance Solver** is a function on a CAS calculator that performs financial calculations. It can be used to determine any of the principal, interest rate, periodic payment, future value or number of payments given all of the other values.
Interest-only loan	An **interest-only loan** is a loan where the regular payments made are equal in value to the interest charged. Interest-only loans have the same value after each payment is made.
Perpetuity	A **perpetuity** is an annuity where the regular payments or withdrawals are the same as the interest earned. The value of a perpetuity remains constant.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

8A **1** I can generate a sequence from a recurrence relation that combines both geometric and linear growth or decay. ☐

See Example 1, and Exercise 8A Question 1

8A **2** I can model compound interest investments with additions to the principal. ☐

See Example 2 and 3, and Exercise 8A Question 2 and 4

8A **3** I can use a recurrence relation to analyse compound interest investments with additions to the principal. ☐

See Example 4, and Exercise 8A Question 8

8A **4** I can determine the annual interest rate from a recurrence relation. ☐

See Example 5, and Exercise 8A Question 10

8B **5** I can model a reducing balance loan with a recurrence relation. ☐

See Example 6 and 7, and Exercise 8B Question 1 and 2

8B **6** I can use a recurrence relation to analyse a reducing balance loan. ☐

See Example 8, and Exercise 8B Question 7

8B **7** I can model an annuity with a recurrence relation. ☐

See Example 9, and Exercise 8B Question 9

8C **8** I can apply the amortisation process. ☐

See Example 10, and Exercise 8C Question 1

8C **9** I can construct an amortisation table for a reducing balance loan. ☐

See Example 11, and Exercise 8C Question 1

8C **10** I can analyse an amortisation table for a reducing balance loan. ☐

See Example 12, and Exercise 8C Question 4

8C **11** I can analyse an amortisation table for an annuity to find the interest rate. ☐

See Example 13, and Exercise 8C Question 7

8C **12** I can interpret and construct an amortisation table for a compound interest investment with additional payments. ☐

See Example 14, and Exercise 8C Question 8

8D **13** I can find the final payment for a reducing balance loan or annuity. ☐

See Example 15, and Exercise 8D Question 1

8D **14** I can find the total payment made and the total interest paid. ☐

See Example 16, and Exercise 8D Question 5

8D **15** I can plot from an amortisation table. ☐

See Example 17, and Exercise 8D Question 9

8E **16** I can determine the value of an investment with regular additions made to the principal using a financial solver. ☐

See Example 18, and Exercise 8E Question 1

8E **17** I can determine the balance and final payment of a reducing balance loan after a given number of payments. ☐

See Example 19, and Exercise 8E Question 3

8E **18** I can determine the balance of an annuity using a finance solver. ☐

See Example 20, and Exercise 8E Question 7

8F **19** I can find the interest rate for an investment with additional payments. ☐

See Example 21, and Exercise 8F Question 1

8F **20** I can find the regular monthly payment and the time taken for an investment with additions to the principal. ☐

See Example 22, and Exercise 8F Question 2

8F **21** I can determine the payment amount, total cost and total amount of interest paid for a reducing balance loan. ☐

See Example 23, and Exercise 8F Question 5

8F **22** I can find the interest rate, time taken and regular payment for an annuity. ☐

See Example 24, and Exercise 8F Question 10

8G **23** I can find the value of an investment when the regular payment changes. ☐

See Example 25, and Exercise 8G Question 1

8G **24** I can analyse the impact of a change in the interest rate on a reducing balance loan, an annuity and an investment. ☐

See Example 26, and Exercise 8G Question 4, 8 and 10

Review

8H **25** I can find the repayment amount for an interest-only loan with and without finance solver. ☐

See Example 27, and Exercise 8H Question 1

8H **26** I can find the amount borrowed for an interest-only loan. ☐

See Example 28, and Exercise 8H Question 8

8H **27** I can find the interest rate for an interest-only loan. ☐

See Example 29, and Exercise 8H Question 11

8I **28** I can calculate the payment from a perpetuity. ☐

See Example 30, and Exercise 8H Question 1

8I **29** I can calculate the investment required to establish a perpetuity. ☐

See Example 31, and Exercise 8H Question 4

8I **30** I can calculate the interest rate of a perpetuity. ☐

See Example 32, and Exercise 8H Question 7

Multiple-choice questions

1 An investment of \$18 000, earning compound interest at the rate of 6.8% per annum, compounding yearly, and with regular additions of \$2500 every year can be modelled with a recurrence relation. If V_n is the value of the investment after n years, the recurrence relation is

A $V_0 = 18000, V_{n+1} = 1.006V_n - 2500$
B $V_0 = 2500, V_{n+1} = 1.068V_n - 18\,000$
C $V_0 = 18\,000, V_{n+1} = 1.068V_n + 2500$
D $V_0 = 18000, V_{n+1} = 1.068V_n - 2500$
E $V_0 = 2500, V_{n+1} = 1.006V_n - 18000$

2 Let V_n be the value of an investment after n months. The investment is modelled by the recurrence relation $V_0 = 25\,000,\ V_{n+1} = 1.007V_n - 400$. The annual interest rate for this investment is

A 0.084% **B** 0.7% **C** 2.8% **D** 8.4% **E** 36.4%

3 The value of an annuity investment, in dollars, after n months, V_n, can be modelled by the recurrence relation shown below

$$V_0 = 20\,000, \quad V_{n+1} = 1.045V_n + 500$$

The increase in the value of this investment in the second month is closest to

A \$500
B \$900
C \$1100
D \$1500
E \$2300

Questions 4 and 5 relate to the following information.

A loan of $28 000 is charged interest at the rate of 6.4% per annum, compounding monthly. It is repaid with regular monthly payments of $1200.

4 Correct to the nearest cent, the value of the loan after 5 months is

A $21 611.35 **B** $22 690.33 **C** $23 763.59 **D** $24 831.16 **E** $31 363.91

5 Following 24 regular payments of $1200, a final payment is made to fully repay the loan. The final payment on the loan, correct to the nearest cent, will be

A $1125.41 **B** $1131.41 **C** $1175.20 **D** $1181.47 **E** $1200

6 A loan of $6000 is to be repaid in full by 11 quarterly payments followed by a final payment that is as close to the regular payment as possible. Interest at 10% per annum is calculated on the remaining balance each quarter. The regular quarterly payment that is required to pay out the loan is closest to

A $527.50 **B** $573.10 **C** $584.92 **D** $600 **E** $630.64

7 Paula borrows $12 000 from a bank, to be repaid over 5 years. Interest of 12% per annum is charged monthly on the amount of money owed. If Paula makes regular monthly payments of $266.90, then the amount she owes at the end of the second year is closest to

A $2880 **B** $5590 **C** $6410 **D** $8040 **E** $9120

8 Ayush invests $12 000 in an annuity from which he receives a regular monthly payment of $239. The annuity earns interest of 7.2% per annum, compounding monthly. The balance of the annuity after three months is closest to

A $11 495 **B** $11 496 **C** $11 665 **D** $12 938 **E** $12 939

9 James invests $50 000 in an annuity from which he receives a regular monthly payment of $925.30.
The balance of the annuity, in dollars, after n months, J_n, can be modelled by a recurrence relation of the form

$$J_0 = 50\,000, \qquad J_{n+1} = 1.0035J_n - 925.30$$

The balance of the annuity after six months is closest to

A $45 289 **B** $45 458 **C** $45 459 **D** $56 659 **E** $56 660

Review

Questions 10–13 refer to the following amortisation table for a reducing balance loan.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	40000.00
1	400.00	160.00	240.00	39760.00
2	400.00	159.04	240.96	39519.04
3	400.00	158.08	241.92	39277.12

10 The principal of this loan is

A \$20 000 **B** \$30 000 **C** \$40 000 **D** \$50 000 **E** \$60 000

11 The periodic payment amount on this loan is

A \$80 **B** \$160 **C** \$240 **D** \$400 **E** \$560

12 The principal reduction of the loan from the third payment is

A \$158.08 **B** \$159.04 **C** \$240 **D** \$240.96 **E** \$241.92

13 Assuming that payments are made monthly and interest compounds monthly, the annual interest rate on the loan is

A 0.4% **B** 0.48% **C** 4% **D** 4.8% **E** 16%

14 Amir borrows \$1500 in a reducing balance loan at a rate of 3.6% per annum, compounding quarterly.
She makes regular repayments of \$383.45 each quarter for three quarters followed by a final payment.
To pay out her loan fully, her final payment is

A \$0.01 **B** \$0.10 **C** \$383.35 **D** \$383.45 **E** \$383.55

15 Monthly withdrawals of \$220 are made from an account that has an opening balance of \$35 300, invested at 7% per annum, compounding monthly. The balance of the account after 1 year is closest to

A \$32 660 **B** \$33 500 **C** \$35 125 **D** \$35 211 **E** \$40 578

16 Tilly invests \$5000 in an account that pays interest at the rate of 3.9% per annum, compounding annually.
She makes an additional payment of \$1200 each year.
The number of years that it will take the investment to first reach a balance of \$20 000 is

A 1 **B** 2 **C** 9 **D** 10 **E** 13

17 Twenty years ago, Oscar invested \$65 000 in an account earning interest at the rate of 2.8% per annum, compounding monthly.
After 10 years, he made a one-off payment of \$20 000 to the account.
For the next 10 years, the account earned interest at the rate of 3.2% per annum, compounding monthly. The balance of the account today is closest to

A \$65 000 **B** \$85 975.39 **C** \$105 975.39 **D** \$113 719.51 **E** \$145 879.51

18 The monthly payment on an interest-only loan of \$175 000, at an interest rate of 5.9% per annum, compounding monthly, is closest to

A \$198 **B** \$397 **C** \$860 **D** \$1117 **E** \$2581

19 A scholarship will be set up to provide an annual prize of \$400 to the best Mathematics student in a school. The scholarship is paid for by investing an amount of money into a perpetuity, paying interest of 3.4% per annum, compounding annually. The amount that needs to be invested to provide this scholarship is closest to

A \$400 **B** \$800 **C** \$1176 **D** \$11 764 **E** \$11 765

20 Pham invests \$74 000 from which he will receive a regular monthly payment.
The perpetuity has a compound interest rate of 4.8% per annum and compounds monthly. The amount that Pham will receive from the perpetuity in the first two years is closest to

A \$296 **B** \$3233 **C** \$3552 **D** \$7104 **E** \$77 756.64

Written response questions

1 Josie is considering borrowing \$250 000 to buy a house. A home loan at her bank will charge interest at the rate of 4.8% per annum, compounding monthly. Josie will make monthly payments of \$1800.
Let V_n be the value of Josie's loan after n months.

a Write down a recurrence model for the value of Josie's loan after n months.

b After 12 months, how much would Josie owe on this loan? Round your answer to the nearest cent.

c After how many months is the loan first below \$200,000?

d If Josie chose an interest-only loan for the first year:

i what would her monthly payments be?

ii how much interest in total would she pay in the first year?

iii how much would she owe after the 12th payment?

2 Samantha inherited \$150 000 from her aunt. She decides to invest this money into an account paying 6.25% per annum interest, compounding monthly.

a If Samantha deposited her money into a perpetuity, what monthly payment would she receive?

b If Samantha deposited her money into an annuity and withdrew \$1000 per month, how much would she have in the account after 1 year?

c If Samantha deposited her money into an annuity and withdrew \$2000 per month, after how many months is the investment first below \$100,000?

d If Samantha deposited her money into an annuity and withdrew \$4000 per month for each month until the final month:

  i How many regular payments of \$4000 would she receive?

  ii What would be the value of her last withdrawal?

3 A loan of \$10 000 is to be repaid over 5 years with 19 equal quarterly payments of \$656.72 followed by a final payment. Interest is charged at the rate of 11% per annum compounding quarterly.
Find:

a the final payment. Round your answer to the nearest cent.

b the sum of all repayments, to the nearest dollar.

c the total amount of interest paid, to the nearest dollar.

4 The Andersons were offered a \$24 800 loan to pay for a new car. Their loan is to be repaid in equal monthly payments of \$750, except for the last month when less than this will be required to fully pay out the loan. The interest rate is 10.8% per annum, compounding monthly.

a Find the number of months needed to repay this loan.

b Calculate the amount of the final payment. Round your answer to the nearest cent.

c Calculate the total interest that is paid on the loan.

5 Elsa borrowed \$100 000 at 9.6% per annum, compounding quarterly. The loan was to be repaid over 25 years with 99 equal quarterly payments followed by a final payment that is as close to the regular payment as possible.

a How much of the first quarterly payment went towards paying off the principal?

b Elsa inherits some money and decides to terminate the loan after 10 years by paying what is owing in a lump sum. How much will this lump sum be?

6 Helene won \$750 000 in a lottery. She decides to place the money in an investment account that pays 4.5% per annum interest, compounding monthly.

a How much will Helene have in the investment account after 10 years?

b After 10 years, Helene withdraws the money from the investment account and places it in an annuity. The annuity pays 3.5% per annum, compounding monthly. Helene receives \$6000 per month from the annuity. For how many months will she receive \$6000?

c Helene's accountant suggests that rather than purchase an annuity she places the money in a perpetuity so that she will be able to leave some money to her grandchildren. If she places \$1,100,000 into a perpetuity that pays 3.6% per annum compounding monthly, how much is the monthly payment that Helene will receive?

Chapter 9

Revision: Recursion and financial modelling

Revision

9A Exam 1 style questions

1 A sequence of numbers is generated by the recurrence relation shown below.

$$A_0 = 5, \qquad A_{n+1} = 2A_n - 1$$

What is the value of A_3?

A 2 **B** 5 **C** 11 **D** 17 **E** 33

2 Consider the recurrence relation below.

$$T_0 = 6000, \qquad T_{n+1} = T_n + 270$$

The recurrence relation could be used to model a

A simple interest investment of \$6000 with an annual interest rate of 0.045%.
B simple interest investment of \$6000 with an annual interest rate of 0.45%.
C simple interest investment of \$6000 with an annual interest rate of 4.5%.
D compound interest investment of \$6000 with an annual interest rate of 0.45%.
E compound interest investment of \$6000 with an annual interest rate of 4.5%.

3 The value of an investment, in dollars, after n months, V_n, can be modelled by the recurrence relation shown below.

$$V_0 = 12\,000, \qquad V_{n+1} = RV_n$$

This investment earns compound interest at the rate of 7.2% per annum, compounding monthly.
What is the value of R in this recurrence relation?

A 0.006 **B** 0.072 **C** 1.006 **D** 1.072 **E** 1.72

4 The value of a commercial fridge, purchased for \$9000, is depreciated by 10% per annum using a reducing balance method.
Recursive calculations can determine the value of the fridge after n years, V_n.
Which one of the following recursive calculations is **not** correct?

A $V_0 = 9000$
B $V_1 = 0.9 \times 9000$
C $V_2 = 0.9 \times 8100$
D $V_3 = 0.9 \times 7290$
E $V_4 = 0.9 \times 6560$

5 Lorraine has taken out a personal loan of \$14 000.
Interest for this loan compounds quarterly and Lorraine makes no repayments.
After one year she owes \$14 686.98.
The effective annual rate of interest for the first year of Lorraine's loan is closest to

A 4.7% **B** 4.8% **C** 4.9% **D** 5.0% **E** 5.1%

6 Luke has purchased a caravan for \$75 000.
He depreciates the value of the caravan using the reducing balance method.
For the first two years of reducing balance depreciation, the annual depreciation rate was 12%.
Luke then changed the annual depreciation to d per cent.
After three more years of reducing balance depreciation, the value of the caravan was \$41638.56.
The changed depreciation rate, d per cent, is closest to

A 9% **B** 9.5% **C** 10% **D** 10.5% **E** 11.5%

7 Tran invests \$375 000 in an account that pays interest at the rate of 3.6% per annum, compounding monthly.
He makes additional payments of \$400 each month into this account.
The value of Tran's account, in dollars, after n months, T_n, can be modelled by the recurrence relation shown below.

$$T_n = 375\,000, \qquad T_{n+1} = 1.003T_n + 400$$

The balance of Tran's account first exceeds \$650 000 at the end of month

A 15 **B** 16 **C** 143 **D** 144 **E** 145

8 Carlos has borrowed \$54 000 to go on a holiday.
Interest on this loan is charged at the rate of 7.9% per annum, compounding monthly.
Carlos intends to repay the loan over five years with 60 monthly repayments.
After 59 equal repayments of \$1092, he finds that a small adjustment to the final repayment is required to fully repay the loan to the nearest cent.
Compared to the 59 earlier repayments, the final repayment will be

A \$17.81 lower
B \$25.12 lower
C \$17.81 higher
D \$25.12 higher
E \$1109.81 higher

9 Consider the recurrence relation shown below

$$V_0 = 175\,000, \qquad V_{n+1} = 1.002V_n - 350$$

This recurrence relation could be used to determine the value of

A a perpetuity with an interest rate of 2.4% per annum, compounding monthly.
B an annuity with an interest rate of 2% per annum, compounding annually.
C an annuity investment with an interest rate of 2% per annum, compounding monthly.
D an item depreciating at a flat rate of 2.4% per annum of the purchase price.
E a compound interest investment earning interest at the rate of 0.24% per annum, of the purchase price.

10 Witter has a reducing balance loan.
Six lines of the amortisation table for Witter's loan are shown below.

Payment number	Payment	Interest	Principal reduction	Balance
13				402 148.19
14	9500.00	1742.64	7757.36	394 390.83
15	9500.00	1709.03	7790.97	386 599.86
16	9500.00	1675.27	7824.73	378 775.13
17	9500.00	1609.79	7890.21	370 844.92
18	9500.00	1576.26	7923.74	362 961.18

The interest rate for Witter's loan changed after one of these repayments had been made.
The first repayment with the lower interest rate was repayment number

A 14 **B** 15 **C** 16 **D** 17 **E** 18

11 Dina invested \$225 000 in an annuity.
This investment earned interest at the rate of 5.2% per annum, compounding quarterly.
Immediately after the interest has been added to the account each quarter, Dina withdraws a payment of \$3800.
A recurrence relation that can be used to determine the value of Dina's investment after n quarters, D_n, is

A $D_0 = 225\,000,\quad D_{n+1} = 0.948D_n - 3800$
B $D_0 = 225\,000,\quad D_{n+1} = 0.987D_n - 3800$
C $D_0 = 225\,000,\quad D_{n+1} = 1.0052D_n - 3800$
D $D_0 = 225\,000,\quad D_{n+1} = 1.013D_n - 3800$
E $D_0 = 225\,000,\quad D_{n+1} = 1.052D_n - 3800$

12 Alicia invested some money in a perpetuity from which she receives a payment of \$675.10 each quarter.
The perpetuity pays interest of 4.3% per annum, compounding quarterly.
How much money did Alicia invest in the perpetuity?

A \$6280 **B** \$15 700 **C** \$18 840 **D** \$62 800 **E** \$188 400

13 Geoff has a current balance of \$317 922.75 in his superannuation account.
Geoff's employer deposits \$420 into this account every fortnight.
This account earns interest at the rate of 3.1% per annum, compounding fortnightly.
The account will first exceed \$720 000 after

A 26 years
B 235 fortnights
C 236 fortnights
D 394 fortnights
E 395 fortnights

14 Lucy borrows \$180 000 in an interest-only loan.
Interest is calculated and paid monthly.
If Lucy pays \$1080 per month, then the annual interest rate is

A 0.06% **B** 0.6% **C** 0.72% **D** 6% **E** 7.2%

15 Maya borrowed \$32 000 to buy a car and was charged interest at the rate of 10.4% per annum, compounding monthly.
For the first year of the loan, Maya made monthly repayments of \$380.
For the second year of the loan, Maya made monthly repayments of \$510.
The total amount of interest that Maya paid over this two-year period was closest to

A \$3050 **B** \$3268 **C** \$4362 **D** \$6318 **E** \$10 680

9B Exam 2 style questions

1 Abigail invests some money into a savings account that earns interest compounding quarterly.
The interest is calculated and paid at the end of each quarter.
Let A_n be the balance of Abigail's savings account, in dollars, after n quarters.
The recursive calculations below show the balance of Abigail's savings account for the first two quarters.

$$A_0 = 8500$$
$$A_1 = 1.013 \times 8500 = 8610.50$$
$$A_2 = 1.013 \times 8610.50 = 8722.44$$

a How much money did Abigail initially invest?

b How much interest in total did Abigail earn by the end of the second quarter?

c Write down a recurrence relation, in terms of A_0, A_{n+1} and A_n that can be used to model the balance, in dollars, of Abigail's savings account after n quarters.

d What is the annual interest rate?

e Abigail plans to use her savings to buy a car that will cost \$10 000. At the end of which quarter will Abigail first have enough money to buy the car?

2 Tina spent \$25 000 on a new coffee machine for her cafe.
The value of the coffee machine will depreciate by \$0.50 per hour of use.
The recurrence relation below can be used to model the value of the coffee machine, V_n, after n years.

$$V_0 = 25\,000, \qquad V_{n+1} = V_n - 936$$

a Use recursion to show that the value of the coffee machine after three years is \$22 192.

b Tina uses the coffee machine all 52 weeks of the year for the same number of hours each week.
For how many hours each week is the coffee machine used?

3 Yash has a reducing balance loan where he makes monthly payments.
Five lines of the amortisation table for Yash's loan are shown below.

Payment number	Payment	Interest	Principal reduction	Balance
0	0	0.00	0.00	260 000.00
1	2150.00	1170.00	980.00	259 020.00
2	2150.00	1165.59	984.41	258 035.59
3	2150.00	1161.16	988.84	257 046.75
4	2150.00	A	B	C

a What amount did Yash originally borrow?

Interest is calculated monthly and Yash makes monthly payments.

b Show that the interest rate for this loan is 5.4% per annum.

c Write down the values of A, B and C.

4 Millie takes out a reducing balance loan of \$240 000. The interest rate for the loan is 3.6% per annum, compounding fortnightly.
Millie pays \$1350 per fortnight and decides to do so for all payments except the final payment which will be lower.

a How many of Millie's payments will be exactly \$1350?

b After seven years of repayments, Millie decides to pay the remaining balance of her loan.
How much will Millie need to pay?

c In which of the years will Millie pay the most interest?

5 Hugh bought a motorscooter to deliver food for a local restaurant.
He paid $4450 for the scooter.
The value of Hugh's scooter depreciated by a fixed amount for each evening shift that he completed.
After 20 evening shifts, the value of the scooter decreased by $50.

a What was the value of Hugh's scooter after 20 evening shifts?

b Write a calculation that shows that the value of Hugh's scooter depreciated by $2.50 per evening shift.

c The value of Hugh's scooter after n evening shifts can be determined using a rule.
Complete the rule below by writing the appropriate numbers in the boxes provided.

$$H_n = \boxed{} - \boxed{} \times n$$

d Using the rule, find the value of Hugh's scooter after 200 evening shifts.

e The value of the scooter continues to depreciate by $2.50 per evening delivery shift.
After how many shifts will the value of Hugh's scooter first fall below $3000?

6 A government grant of $65 000 was invested in an annuity to provide a monthly payment to a local artist.
The annuity pays interest at the rate of 3.54% per annum, compounding monthly.
After four months the annuity had a balance of $43592.50 remaining.

a What is the value, in dollars, of the monthly payment to the artist?

b After four months, the artist realises that she will not complete the artwork within the originally set twelve months.
To extend the time that the annuity will last, the artist will receive a reduced payment of $2000 per month.
The annuity will end with one final monthly payment that will be smaller than $2000.
Calculate the number of months that the artist will receive $2000 per month.

Chapter 10

Matrices

Chapter questions

- What is a matrix?
- How is the order of a matrix defined?
- How are the positions of the elements of a matrix specified?
- How do we use matrices to represent information and solve practical problems?
- How do we use matrices to represent a network diagram?
- What are the rules for adding and subtracting matrices?
- How do we multiply a matrix by a scalar?
- What is the method for multiplying a matrix by another matrix?
- How do we form and use permutation, communication and dominance matrices?
- How can your CAS calculator be used to do matrix operations?

Matrix algebra was first studied in England in the middle of the nineteenth century. Matrices are now used in many areas of science and business: for example, in physics, medical research, encryption and internet search engines.

In this chapter we will show how addition and multiplication of matrices can be defined and how matrices can be used to describe the relationship between people, businesses and sporting teams.

In Chapter 11 we will see how matrices can be used to represent networks.

10A What is a matrix?

Learning intentions

- To be able to state the order of a given matrix.
- To be able to describe the location of an element in a matrix.
- To be able to determine the transpose of a matrix.
- To be able to define and recognise diagonal, symmetric and triangular matrices.
- To be able to define and recognise identity matrices.

A **matrix** (plural **matrices**) is a rectangular array or table of numbers or symbols, arranged in rows and columns. We form a matrix from data in the following way.

The table of data shown below displays the heights, weights, ages and pulse rates of eight students.

Name	*Height*	*Weight*	*Age*	*Pulse rate*
Mahdi	173	57	18	86
Dave	179	58	19	82
Jodie	167	62	18	96
Simon	195	84	18	71
Kate	173	64	18	90
Pete	184	74	22	78
Mai	175	60	19	88
Tran	140	50	34	70

$$D = \begin{bmatrix} 173 & 57 & 18 & 86 \\ 179 & 58 & 19 & 82 \\ 167 & 62 & 18 & 96 \\ 195 & 84 & 18 & 71 \\ 173 & 64 & 18 & 90 \\ 184 & 74 & 22 & 78 \\ 175 & 60 & 19 & 88 \\ 140 & 50 & 34 & 70 \end{bmatrix}$$

If we extract the numbers from the table and enclose them in square brackets, we form a matrix. We might call this matrix D (for data matrix). We use capital letters A, B, C, etc. to name matrices.

Rows and columns

Rows and columns are the building blocks of matrices. We number rows from the top down: row 1, row 2, etc. Columns are numbered from the left across: column 1, column 2, etc.

$$\begin{array}{cc} & \begin{array}{cccc} & & \text{Col. 3} & \end{array} \\ \begin{array}{c} \\ \text{Row 2} \\ \\ \\ D = \\ \\ \\ \\ \end{array} & \begin{bmatrix} 173 & 57 & 18 & 86 \\ 179 & 58 & 19 & 82 \\ 167 & 62 & 18 & 96 \\ 195 & 84 & 18 & 71 \\ 173 & 64 & 18 & 90 \\ 184 & 74 & 22 & 78 \\ 175 & 60 & 19 & 88 \\ 140 & 50 & 34 & 70 \end{bmatrix} \end{array}$$

Order of a matrix

In its simplest form, a matrix is just a rectangular array (rows and columns) of numbers.

The **order** (or size) of matrix D is said to be 8×4, read '8 by 4' because it has *eight rows* and *four columns*.

Order of a matrix

Order of a matrix = number of rows × number of columns

The numbers, or entries, in the matrix are called **elements**.

The number of elements in a matrix is determined by its order. For example, the matrix D has order 8×4 and the the number of elements in matrix D is 32 ($8 \times 4 = 32$).

Row matrices

Matrices come in many shapes and sizes. For example, from this same set of data, we could have formed the matrix called K

$$K = \begin{bmatrix} 173 & 64 & 18 & 90 \end{bmatrix}$$

This matrix has been formed from just one row of the data: the data values for Kate. Because it only contains *one row* of numbers, it is called a **row matrix** (or **row vector**). It is a 1×4 matrix: one row by four columns. It contains $1 \times 4 = 4$ elements.

Column matrices

Equally, we could form a matrix called H (for height matrix). This matrix is formed from just one column of the data, the heights of the students.

Because it only contains *one column* of numbers, it is called a **column matrix** (or **column vector**). This is an 8×1 matrix: eight rows by one column. It contains $8 \times 1 = 8$ elements.

$$H = \begin{bmatrix} 173 \\ 179 \\ 167 \\ 195 \\ 173 \\ 184 \\ 175 \\ 140 \end{bmatrix}$$

Example 1

State the order of each of the following matrices.

a $\begin{bmatrix} 1 & 5 \\ 3 & 0 \\ 7 & 6 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 5 & 8 & 9 & 0 \end{bmatrix}$ **c** $\begin{bmatrix} 1 \\ 4 \\ 4 \\ 9 \end{bmatrix}$ **d** $\begin{bmatrix} 1 & 4 & 4 & 5 \\ 4 & 3 & 4 & 6 \\ 4 & 3 & 2 & 1 \\ 9 & 1 & 0 & 7 \end{bmatrix}$

Solution

a 3 rows and 2 columns.
Order is 3×2

b 1 row and 5 columns
Order is 1×5

c 4 rows and 1 column.
Order is 4×1

d 4 rows and 4 columns.
Order is 4×4

Switching rows and columns: the transpose of a matrix

If you switch the rows and columns in a matrix you have what is called the **transpose** of the matrix.

For example, the *transpose* of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

The transpose of a row matrix is a column matrix and vice versa.

For example, the *transpose* of the matrix $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$.

The symbol we use to indicate the transpose of a matrix is T.

Thus, $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Note: The transpose of a 3×2 matrix is a 2×3 matrix because the rows and columns are switched.

Example 2 The transpose of a matrix

a Write down the transpose of $\begin{bmatrix} 7 & 4 \\ 8 & 1 \end{bmatrix}$.

b Write down the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix}^T$.

c If $A = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}$, write down the matrix A^T.

Explanation

a The transpose of the matrix is obtained by switching (interchanging) its rows and columns.

Solution

$$\begin{bmatrix} 7 & 8 \\ 4 & 1 \end{bmatrix}$$

Explanation

b The symbol T is an instruction to transpose the matrix.

Solution

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix}^T = \begin{bmatrix} 0 & 3 \\ 1 & 2 \\ 2 & 5 \end{bmatrix}$$

Explanation

c The symbol T is an instruction to transpose matrix A.

Solution

$$A^T = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Square matrices

As a final example, we could form a matrix we call M (for males). This matrix contains only the data for the males. As this matrix has four rows and four columns, it is a 4×4 matrix. It contains $4 \times 4 = 16$ elements.

$$M = \begin{bmatrix} 173 & 57 & 18 & 86 \\ 179 & 58 & 19 & 82 \\ 195 & 84 & 18 & 71 \\ 184 & 74 & 22 & 78 \end{bmatrix}$$

A matrix with an *equal* number of *rows* and *columns* is called a **square matrix**.

Example 3 Matrix facts

For each of the matrices below, write down its type, order and the number of elements.

Solution

Matrix	*Type*	*Order*	*No. of elements*
$A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 4 \\ 2 & -1 & 6 \end{bmatrix}$	Square matrix rows = columns	3×3 3 rows, 3 cols.	9 $3 \times 3 = 9$
$B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$	Column matrix single column	3×1 3 rows, 1 col.	3 $3 \times 1 = 3$
$C = \begin{bmatrix} 3 & 1 & 0 & 5 & -3 & 1 \end{bmatrix}$	Row matrix single row	1×6 1 row, 6 cols.	6 $1 \times 6 = 6$

Diagonal, symmetric and triangular matrices

Some square matrices occur so often in practice that they have their own names.

Diagonal matrices

A square matrix has two diagonals: $\begin{bmatrix} \color{red}{1} & 2 & 3 & 4 \\ 5 & \color{red}{6} & 7 & 8 \\ 9 & 0 & \color{red}{1} & 2 \\ 3 & 4 & 5 & \color{red}{6} \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 & \color{blue}{4} \\ 5 & 6 & \color{blue}{7} & 8 \\ 9 & \color{blue}{0} & 1 & 2 \\ \color{blue}{3} & 4 & 5 & 6 \end{bmatrix}$

In practice, the diagonal going downwards from left to right in the matrix (coloured red) turns out to be more important than the other diagonal (coloured blue), so we give it a special name: the *leading diagonal*.

A square matrix is called a *diagonal matrix* if all of the elements off the leading diagonal are zero. The elements on the leading diagonal may or may not be zero.

The matrices opposite are all diagonal matrices: $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

Identity matrices

Diagonal matrices in which each element in the diagonal is 1 are of special importance. They are called identity or unit matrices and have their own name and symbol (I).

Every order of square matrix has its own **identity matrix**, three of which are shown opposite.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Symmetric matrices

A symmetric matrix is a square matrix that is unchanged by transposition (switching rows and columns). In a symmetric matrix, the elements above the leading diagonal are a mirror image of the elements below the diagonal. Three are shown.

$$\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 4 & 6 \\ 2 & 1 & 5 & 7 \\ 4 & 5 & 3 & 8 \\ 6 & 7 & 8 & 5 \end{bmatrix}$$

Triangular matrices

Triangular matrices come in two types:

1. An upper triangular matrix is a square matrix in which all elements below the leading diagonal are zeros.
2. A lower triangular matrix is a square matrix in which all elements above the leading diagonal are zeros.

Examples of triangular matrices are shown.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 6 & 5 & 4 & 0 \\ 0 & 9 & 8 & 7 \end{bmatrix}$$

upper triangular matrix   lower triangular matrix

Example 4 **Types of matrices**

Consider the following square matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 5 \\ 3 & 4 & 7 \\ 5 & 7 & 2 \end{bmatrix}$$

Write down:

a the upper triangular matrices

b the identity matrix

c the diagonal matrices

d the symmetric matrices.

Explanation	Solution
a All the elements below the leading diagonal are 0.	$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
b Elements in the leading diagonal are all 1 and the other elements 0.	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
c All elements other than those in the leading diagonal are zero.	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
d The matrix must be its own transpose.	$\begin{bmatrix} 1 & 3 & 5 \\ 3 & 4 & 7 \\ 5 & 7 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Some notation

In some situations, we talk about a matrix and its elements without having specific numbers in mind. We can do this as follows.

For the matrix A, which has n rows and m columns, we write:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nm} \end{bmatrix} \qquad a_{23}$$

row number column number

Thus:

- a_{21} represents the element in the second row and the first column
- a_{12} represents the element in the first row and the second column
- a_{22} represents the element in the second row and the second column
- a_{mn} represents the element in the mth row and the nth column.

Example 5 Identifying the elements in a matrix

For the matrices A and B, opposite, write down the values of:

a a_{12} **b** a_{21} **c** a_{33} **d** b_{31}.

$$A = \begin{bmatrix} 1 & 5 & 3 \\ -1 & 0 & 4 \\ 2 & -2 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Explanation	Solution
a a_{12} is the element in the first row and the second column of A.	$a_{12} = 5$
b a_{21} is the element in the second row and the first column of A.	$a_{21} = -1$
c a_{33} is the element in the third row and the third column of A.	$a_{33} = 6$
d b_{31} is the element in the third row and the first column of B.	$b_{31} = 1$

In some instances, there is a rule connecting the value of each matrix with its row and column number. In such circumstances, it is possible to construct this matrix knowing this rule and the order of the matrix.

Example 6 Constructing a matrix given a rule for its *ij*th term

A is a 3×2 matrix. The element in row i and column j is given by $a_{ij} = i + j$. Construct the matrix.

Explanation

1 The matrix is square and will have the form:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Use the rule

$$a_{ij} = i + j$$

to generate the elements one by one. For example $a_{32} = 3 + 2 = 5$

2 Write down the matrix.

Solution

$$\begin{array}{c|cc} & j=1 & j=2 \\ \hline \begin{matrix} i=1 \\ i=2 \\ i=3 \end{matrix} & \multicolumn{2}{c}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}} \end{array}$$

where

$a_{11} = 1 + 1 = 2$ $a_{12} = 1 + 2 = 3$

$a_{21} = 2 + 1 = 3$ $a_{22} = 2 + 2 = 4$

$a_{31} = 3 + 1 = 4$ $a_{32} = 3 + 2 = 5$

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix}$$

Entering a matrix into a CAS calculator

Later in this chapter, you will learn about matrix arithmetic: how to add, subtract and multiply matrices. While it is possible to carry out these tasks by hand, for all but the smallest matrices this is very tedious. Most matrix arithmetic is better done with the help of a CAS calculator. However, before you can perform matrix arithmetic, you need to know how to enter a matrix into your calculator.

CAS 1: How to enter a matrix on the TI-Nspire CAS

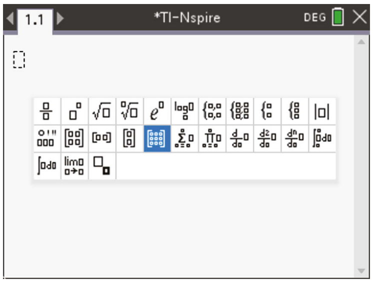
Enter the matrix $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ and determine its transpose (A^T).

Steps

1 Press ctrl + N. Select **Add Calculator**.

2 Press and use the cursor ▼ ▶ arrows to highlight the matrix template shown. Press enter.

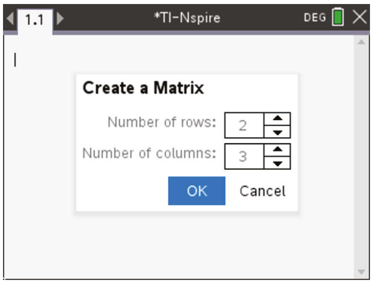
Note: **Math Templates** can also be accessed by pressing ctrl + menu **>Templates**.



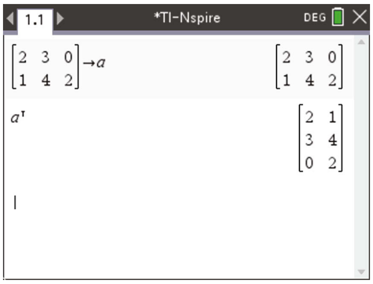
3 Use the ▼ arrow to select the **Number of rows** required (number of rows in this example is 2).

Press tab to move to the next entry and repeat for the **Number of columns** (the number of columns in this example is 3).

Press tab to highlight **OK**; press enter.



4 Type the values into the matrix template. Use tab to move to the required position in the matrix to enter each value. When the matrix has been completed, press tab to move outside the matrix, press ctrl var, followed by A. Press enter. This will store the matrix as the variable a.



5 When you type ***A*** (or ***a***) it will paste in the matrix $\begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$. Press enter to display.

6 To find a^T, type in **a** (for matrix A) and then menu **>Matrix & Vector>Transpose** **>** enter as shown.

Note: Superscript T can also be accessed from the symbols palette (ctrl).

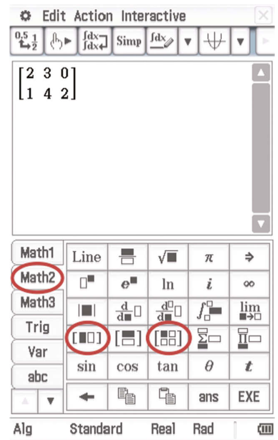
CAS 1: How to enter a matrix using the ClassPad

Enter the matrix $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ and determine its transpose (A^T).

Steps

1 **a** Open the **Main** ($\sqrt{\alpha}$) application Press Keyboard to display the soft keyboard.

b Select the Math2 keyboard.



2 Tap the 2×2 matrix icon, followed by the 1×2 matrix icon. This will add a third column and create a 2×3 matrix.

3 Type the values into the matrix template.

Note: Tap at each new position to enter the new value or use the black cursor key on the hard keyboard to navigate to the new position.

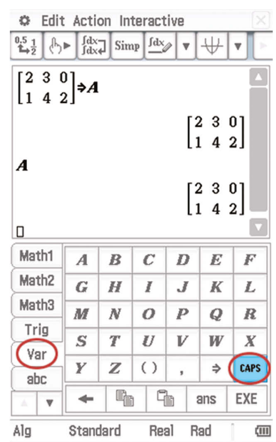
4 To assign the matrix the variable name ***A***:

a move the cursor to the very right-hand side of the matrix

b tap the variable assignment key ⇒ followed by var CAPS ***A***

c press EXE to confirm your choice.

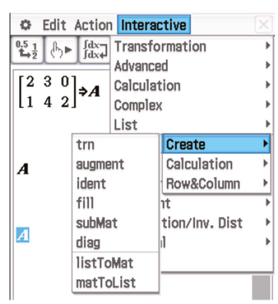
Note: Until it is reassigned, ***A*** will represent the matrix as defined above.

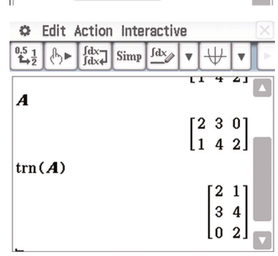


5 To calculate the transpose matrix $\boldsymbol{A}^T$:

a type and highlight ***A*** (by swiping with the stylus)

b select **Interactive** from the menu bar, tap **Matrix-Create** and then tap **trn**.





Skill-sheet

Exercise 10A

Order of a matrix

Example 1

1 State the order of each of the following matrices.

a $\begin{bmatrix} 1 & 5 & 9 \\ 3 & 0 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 7 & 6 & 12 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 6 \\ 3 & 4 \\ 11 & 8 \end{bmatrix}$ **d** $\begin{bmatrix} 18 \\ 7 \\ 1 \end{bmatrix}$ **e** $\begin{bmatrix} 3 & 4 & 4 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{bmatrix}$

2 Write down the order of the following matrices:

a $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 2 & 3 \end{bmatrix}$

3 How many elements are there in a

a 2×6 matrix? **b** 3×5 matrix? **c** 7×4 matrix?

4 A matrix has 12 elements. What are its possible orders? (There are six.)

The transpose of a matrix

Example 2

5 Determine the transpose of each matrix

a $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ **c** $\begin{bmatrix} 9 & 1 & 0 & 7 \\ 8 & 9 & 1 & 5 \end{bmatrix}$

Types of matrices and their elements

Example 3

6 For each of the following matrices, state whether it is a column, row or square matrix, and state the order and the number of elements.

a $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$ **c** $\begin{bmatrix} 7 & 11 & 2 & 1 \end{bmatrix}$

Example 4

7 Consider the following square matrices.

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 7 \\ 0 & 0 & 2 \end{bmatrix}$$

Identify:

a the upper triangular matrices **b** the identity matrix

c the diagonal matrices **d** the symmetric matrices.

Example 5

8 Complete the sentences below that relate to the following matrices.

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 6 \\ -1 & 0 \\ 1 & 3 \\ 4 & -4 \end{bmatrix} \quad E = \begin{bmatrix} 4 & 3 & 1 \\ 0 & -1 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

a The square matrices are ☐ and ☐.
b Matrix B has ☐ rows.
c The row matrix is ☐.
d The column matrix is ☐.
e Matrix D has ☐ rows and ☐ columns.
f The order of matrix E is ☐ × ☐.
g The order of matrix A is ☐ × ☐.
h The order of matrix B is ☐ × ☐.
i The order of matrix D is ☐ × ☐.
j There are ☐ elements in matrix E.
k There are ☐ elements in matrix A.

l $a_{14} =$ ☐ **m** $b_{31} =$ ☐ **n** $c_{11} =$ ☐ **o** $d_{41} =$ ☐
p $e_{22} =$ ☐ **q** $d_{32} =$ ☐ **r** $b_{11} =$ ☐ **s** $c_{12} =$ ☐

Constructing a matrix given a rule for its *ij*th term

Example 6

9 B is a 3×2 matrix. The element in row i and column j is given by $b_{ij} = i \times j$. Construct the matrix.

10 C is a 4×1 matrix. The element in row i and column j is given by $c_{ij} = i + 2j$. Construct the matrix.

11 D is a 3×2 matrix. The element in row i and column j is given by $d_{ij} = i - 3j$. Construct the matrix.

12 E is a 1×3 matrix. The element in row i and column j is given by $e_{ij} = i + j^2$. Construct the matrix.

13 F is a 2×2 matrix. The element in row i and column j is given by $f_{ij} = (i + j)^2$. Construct the matrix.

Entering a matrix into a CAS calculator and determining the transpose

14 Enter the following matrices into your calculator and determine the transpose.

a $B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}$ **b** $C = \begin{bmatrix} 4 & -4 \\ -2 & 6 \end{bmatrix}$ **c** $E = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$ **d** $F = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Exam 1 style questions

15 Consider the following four matrices.

■ $\begin{bmatrix} 5 & 0 & 5 & 0 \\ 0 & 5 & 7 & 5 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ ■ $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ ■ $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ ■ $\begin{bmatrix} 5 & 0 & 5 & 0 \\ 0 & 5 & 7 & 5 \\ 0 & 8 & 5 & 0 \\ 9 & 0 & 0 & 5 \end{bmatrix}$

How many of these matrices are diagonal matrices?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

16 The element in row i and column j of matrix A is a_{ij}. A is a 3×3 matrix. It is constructed using the rule $m_{ij} = 3i + 2j$. A is

A $\begin{bmatrix} 5 & 3 & 5 \\ 1 & 4 & 7 \\ 6 & 7 & 15 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 7 \\ 0 & 0 & 2 \end{bmatrix}$ **C** $\begin{bmatrix} 5 & 9 & 10 \\ 7 & 11 & 12 \\ 9 & 13 & 14 \end{bmatrix}$ **D** $\begin{bmatrix} 5 & 7 & 9 \\ 8 & 10 & 12 \\ 11 & 13 & 15 \end{bmatrix}$ **E** $\begin{bmatrix} 5 & 8 & 11 \\ 7 & 10 & 13 \\ 9 & 13 & 15 \end{bmatrix}$

17 Consider the matrix A, where $A = \begin{bmatrix} 5 & 8 & 11 \\ 7 & 10 & 13 \\ 9 & 12 & 15 \end{bmatrix}$. The element in row i and column j of matrix A is a_{ij}. The elements in matrix A are determined by the rule

A $a_{ij} = 4 + j$ **B** $a_{ij} = 2i + 3j$ **C** $a_{ij} = i + j + 3$

D $a_{ij} = 3i + 2j$ **E** $a_{ij} = 2i - j + 2$

18 The matrix $\begin{bmatrix} 1 & 3 & 5 & 0 \\ 3 & 4 & 1 & 0 \\ 5 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is an example of

A a diagonal matrix. **B** an identity matrix. **C** a symmetric matrix.

D an upper triangular matrix. **E** a lower triangular matrix.

19 The element in row i and column j of matrix A is a_{ij}. The elements in matrix A are determined using the rule $a_{ij} = 2i + j$. Matrix A could be

A $\begin{bmatrix} 3 & 4 \\ 5 & 4 \end{bmatrix}$ **B** $\begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$ **C** $\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$ **D** $[5]$ **E** $\begin{bmatrix} 3 & 4 & 5 \\ 5 & 6 & 6 \\ 7 & 8 & 7 \end{bmatrix}$

10B Using matrices to represent information

Learning intentions

- To be able to represent data given in a table by a matrix.
- To be able to represent a network diagram with a matrix.

At the start of this chapter we used a matrix to store numerical information in a data table. Matrices can also be used to carry codes that encrypt credit-card numbers for internet transmission or to carry all the information needed to solve sets of simultaneous equations. A less obvious application is using matrices to represent network diagrams.

Using a matrix to represent data tables

The numerical information in a data table is frequently presented in rows and columns. As such, it is a relatively straightforward process to convert this information into matrix form.

Example 7 Representing information in a table by a matrix

The table opposite shows the three types of membership of a local gym and the number of males and females enrolled in each. Construct a matrix to display the numerical information in the table.

	Gym membership		
Gender	*Weights*	*Aerobics*	*Fitness*
Males	16	104	86
Females	75	34	94

Explanation

1 Draw a blank (2×3) matrix.
Label the rows M for male and F for female.
Label the columns W for weights, A for aerobics and F for fitness.

2 Fill in the elements of the matrix row by row, starting at the top left-hand corner of the table.

Solution

$$\begin{array}{cc} & \begin{array}{ccc} W & A & F \end{array} \\ \begin{array}{c} M \\ F \end{array} & \begin{bmatrix} & & \\ & & \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccc} W & A & F \end{array} \\ \begin{array}{c} M \\ F \end{array} & \begin{bmatrix} 16 & 104 & 86 \\ 75 & 34 & 94 \end{bmatrix} \end{array}$$

Example 8 Entering a credit card number into a matrix

Convert the 16-digit credit card number: 4454 8178 1029 3161 into a 2×8 matrix, listing the digits in pairs, one under the other. Ignore the spaces.

Explanation	Solution
1 Write out the sequence of numbers. Note: Writing the number down in groups of four (as on the credit card) helps you keep track of the figures.	4454 8178 1029 3161
2 Fill in the elements of the matrix row by row, starting at the top left-hand corner of the table.	$\begin{bmatrix} 4 & 5 & 8 & 7 & 1 & 2 & 3 & 6 \\ 4 & 4 & 1 & 8 & 0 & 9 & 1 & 1 \end{bmatrix}$

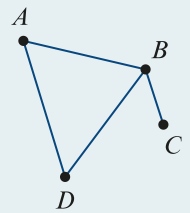
Using matrices to represent network diagrams

A less obvious use of matrices is to represent the information contained in network diagrams. Network diagrams consist of a series of numbered or labelled points joined in various ways. They are a powerful way of representing and studying things as different as friendship networks, airline routes, electrical circuits and road links between towns.

Example 9 Representing a network diagram by a matrix

Represent the network diagram shown opposite as a 4×4 matrix M, where the:

- matrix element = 1 if the two points are joined by a line
- matrix element = 0 if the two points are not connected.



Explanation

1 Draw a blank 4×4 matrix, labelling the rows and columns A, B, C, D to indicate the points.

Solution

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

2 Fill in the elements of the matrix row by row, starting at the top left-hand corner:

- $m_{11} = 0$ (no line joining point A to itself)
- $m_{12} = 1$ (a line joining points A and B)
- $m_{13} = 0$ (no line joining points A and C)
- $m_{14} = 1$ (a line joining points A and D)

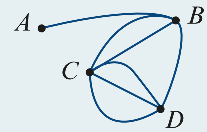
and so on until the matrix is complete.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Note: If a network contains no 'loops' (lines joining points to themselves) the elements in the leading diagonal will always be zero. Knowing this can save a lot of work.

Example 10 Interpreting a matrix representing a network diagram

The diagram opposite shows the roads connecting four towns: town A, town B, town C, and town D. This diagram has been represented by a 4×4 matrix, M. The elements show the number of roads between each pair of towns.



a In the matrix M, $m_{24} = 1$. What does this tell us?

b In the matrix M, $m_{34} = 3$. What does this tell us?

c In the matrix M, $m_{41} = 0$. What does this tell us?

d What is the sum of the elements in row 3 of matrix M and what does this tell us?

e What is the sum of all the elements of matrix M and what does this tell us?

$$M = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 3 \\ 0 & 1 & 3 & 0 \end{bmatrix} \end{array}$$

Explanation

a There is one road between town B and town D.

b There are three roads between town D and town C.

c There is no road between town D and town A.

d 5: the total number of roads between town C and the other towns in the network.

e 14: the total number of different ways you can travel between towns.

Note: For each road, there are two ways you can travel; for example, from town A to town B ($a_{12} = 1$) and from town B to town A ($a_{21} = 1$).

Exercise 10B

Representing a table of data in matrix form

Example 7

1 The table opposite gives the number of residents, TVs and computers in three households.

Household	*Residents*	*TVs*	*Computers*
A	4	2	1
B	6	2	3
C	2	1	0

Use the table to:

a construct a matrix to display the numbers in the table. What is its order?

b construct a row matrix to display the numbers in the table relating to household B. What is its order?

c construct a column matrix to display the numbers in the table relating to computers. What is its order? What does the sum of its elements tell you?

2 The table opposite gives the yearly car sales for two car dealers.

Car sales	*Small*	*Medium*	*Large*
Honest Joe's	24	32	11
Super Deals	32	34	9

Use the table to:

a construct a matrix to display the numbers in the table. What is its order?

b construct a row matrix to display the numbers in the table relating to Honest Joe's. What is its order?

c construct a column matrix to display the numerical information in the table relating to small cars. What is its order? What does the sum of its elements tell you?

3 Four exporters A, B, C and D sell refrigerators (R), dishwashers (D), microwave ovens (M) and televisions (T). The sales in a particular month can be represented by a matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} R & D & M & T \end{array} \\ \begin{bmatrix} 120 & 95 & 370 & 250 \\ 430 & 380 & 950 & 900 \\ 60 & 50 & 150 & 100 \\ 200 & 100 & 470 & 50 \end{bmatrix} \end{array}$$

a What is the order of the matrix?

b Construct a row matrix to display the exports of B. What is its order? What does the sum of the elements of this matrix tell you?

c Construct a column matrix to display the numerical information in the table relating to microwave ovens. What is its order?

4 At a certain school there are 200 girls and 110 boys in Year 7. The numbers of girls and boys in the other year levels are 180 and 117 in Year 8, 135 and 98 in Year 9, 110 and 89 in Year 10, 56 and 53 in Year 11, and 28 and 33 in Year 12. Summarise this information in a matrix.

Example 8

5 Convert the 16-digit credit card number 3452 8279 0020 3069 into a 2×8 matrix. List the digits in pairs, one under the other. Ignore any spaces.

6 The statistics for five members of a basketball team are recorded as follows:

Player A points 21, rebounds 5, assists 5
Player B points 8, rebounds 2, assists 3
Player C points 4, rebounds 1, assists 1
Player D points 14, rebounds 8, assists 6
Player E points 0, rebounds 1, assists 2

Express this information in a 5×3 matrix.

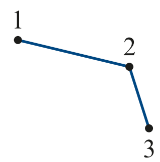
Representing the information in a network diagram in matrix form

Example 9

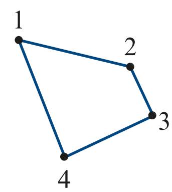
7 Represent each of the following network diagrams by a matrix A using the rules:

- matrix element = 1 if points are joined by a line
- matrix element = 0 if points are not joined by a line.

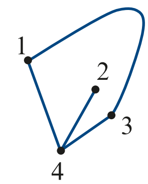
a



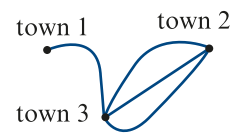
b



c



8 The diagram opposite shows the roads interconnecting three towns: town 1, town 2 and town 3. Represent this diagram with a 3×3 matrix where the elements represent the number of roads between each pair of towns.

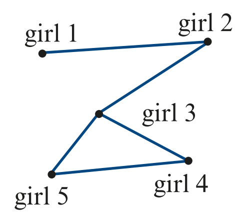


Example 10

9 The network diagram opposite shows a friendship network between five girls: girl 1 to girl 5.

This network has been represented by a 5×5 matrix, F, using the rule:

- element = 1 if the pair of girls are friends
- element = 0 if the pair of girls are not friends.



$$F = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

a In the matrix F, $f_{34} = 1$. What does this tell us?

b In the matrix F, $f_{25} = 0$. What does this tell us?

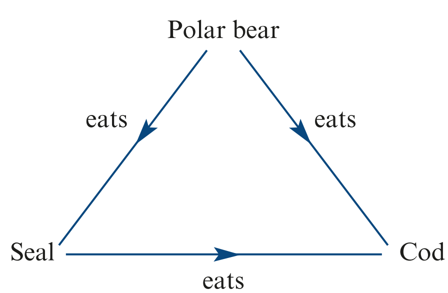
c What is the sum of the elements in row 3 of the matrix and what does this tell us?

d Which girl has the least friends? The most friends?

10 **a** The diagram below shows a 'food web' for polar bears (P), seals (S) and cod (C).

The matrix F below has been set up to represent the information in the diagram.

$$F = \begin{array}{c} \\ P \\ S \\ C \end{array}\!\!\begin{array}{c} \begin{array}{ccc} P & S & C \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

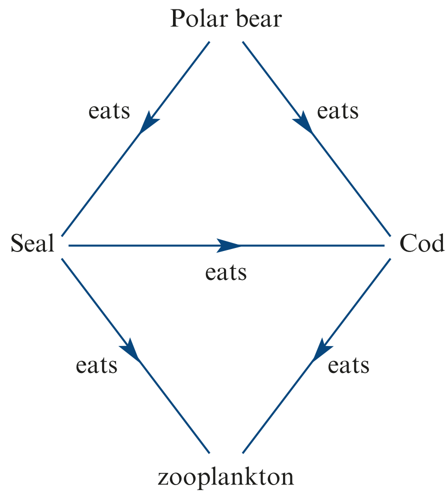


i What does the '1' in column C, row P, of matrix F represent?

ii What does the column of zeroes in matrix F represent?

b The diagram below shows a 'food web' for polar bears (P), seals (S) cod (C) and zooplankton (Z).
Complete the matrix W to represent the information in the diagram.

$$W = \begin{array}{c} \\ P \\ S \\ C \\ Z \end{array} \begin{array}{c} \begin{array}{cccc} P & S & C & Z \end{array} \\ \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \end{array}$$



Exam 1 style questions

11 The matrix below shows the number of boys and girls at a large school.

$$\begin{array}{r} \\ \textit{Year 10} \\ \textit{Year 11} \\ \textit{Year 12} \end{array} \begin{array}{c} \begin{array}{cc} \textit{Boys} & \textit{Girls} \end{array} \\ \begin{bmatrix} 157 & 163 \\ 148 & 154 \\ 139 & 145 \end{bmatrix} \end{array}$$

How many girls are there in Year 11?

A 157 **B** 148 **C** 154 **D** 139 **E** 145

12 A small chain of delicatessens with shopnames Allbright, Bunchof, Crisp, Delic and Elite (A, B, C, D, E) each sell the products Feta, Goatmilk, Haloumi, Insalata and Jam (F, G, H, I, J). The number of weekly sales of each product at each of the shops is shown in the matrix below.

$$P = \begin{array}{c} \\ F \\ G \\ H \\ I \\ J \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 34 & 40 & 52 & 106 & 27 \\ 42 & 154 & 38 & 55 & 68 \\ 136 & 145 & 11 & 44 & 77 \\ 136 & 147 & 43 & 72 & 111 \\ 139 & 140 & 66 & 56 & 145 \end{bmatrix} \end{array}$$

Find which product had the highest weekly sales at any single one of the shops. The name of this product and the shop is

A Feta at Allbright **B** Goatmilk at Bunchof **C** Jam at Elite
D Insalata at Bunchof **E** Haloumi at Allbright

10C Matrix arithmetic: addition, subtraction and scalar multiplication

Learning intentions

- To be able to establish when two matrices are equal.
- To be able to recognise when matrix addition and subtraction are defined and to peform these operations.
- To be able to undertake scalar multiplication.
- To be able to identify zero matrices.
- To be able to undertake addition, subtraction and multiplication by a scalar with a CAS calculator.
- Using the operations of addition, subtraction and scalar multiplication of matrices in practical situations.

Equality of two matrices

Equal matrices have the same order and each corresponding element is identical in value. It is not sufficient for the two matrices to contain an identical set of numbers; they must also be in the same positions.

For example:

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is equal to $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ because the corresponding elements are equal.

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is *not* equal to $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ because the numbers are in different positions.

Example 11

Given that

$$\begin{bmatrix} 2 & a \\ 8 & b+1 \end{bmatrix} = \begin{bmatrix} 2 & 10 \\ 20 & 7 \end{bmatrix}$$

find the value of a and the value of b.

Solution

$a = 10$ and $b + 1 = 7$ which implies $b = 6$.

Matrix addition and subtraction

Adding and subtracting matrices

If two matrices are of the same order (have the same number of rows and columns), they can be added (or subtracted) by adding (or subtracting) their corresponding elements.

Example 12 **Adding two matrices**

$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & 1 \end{bmatrix}$ $\quad$ Find $A + B$.

Explanation

1 As the two matrices have the same order, 2×3, they can be added.

2 Add corresponding elements.

Solution

$$A + B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+1 & 3+2 & 0+3 \\ 1+2 & 4+(-2) & 2+1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 & 3 \\ 3 & 2 & 3 \end{bmatrix}$$

Likewise, if we have two matrices of the same order (same number of rows and columns), we can subtract the two matrices by subtracting their corresponding elements.

Example 13 **Subtracting two matrices**

$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & 1 \end{bmatrix}$ $\quad$ Find $A - B$.

Explanation

1 As the two matrices have the same order, 2×3, they can be subtracted.

2 Subtract corresponding elements.

Solution

$$A - B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 3-2 & 0-3 \\ 1-2 & 4-(-2) & 2-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -3 \\ -1 & 6 & 1 \end{bmatrix}$$

Multiplying matrices by a number (scalar multiplication)

Mutiplying a matrix by a number has the effect of multiplying each element in the matrix by that number.

Multiplying a matrix by a number is called **scalar multiplication**, because it has the effect of scaling the matrix by that number. For example, multiplying a matrix by 2 doubles each element in the matrix.

Example 14 Scalar multiplication

$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 4 & -4 \\ -2 & 6 \end{bmatrix}$ Find $3A$ and $0.5C$.

Explanation

Multiplying a matrix by a number has the effect of multiplying each element by that number.

Solution

$$3\begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 0 \\ 3 & 12 & 6 \end{bmatrix}$$

$$0.5\begin{bmatrix} 4 & -4 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$

The zero matrix

If $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $X - Y = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$.

A matrix of any order with *all zeros* is known as a **zero matrix**. The symbol O is used to represent a zero matrix. The matrices below are all examples of zero matrices.

$$O = [0], \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example 15 The zero matrix

$G = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix} \quad H = \begin{bmatrix} 9 & 0 \\ -6 & 3 \end{bmatrix}$ Show that $3G - 2H = O$.

Solution

$$\begin{aligned} 3G - 2H &= 3 \times \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix} - 2 \times \begin{bmatrix} 9 & 0 \\ -6 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 0 \\ -12 & 6 \end{bmatrix} - \begin{bmatrix} 18 & 0 \\ -12 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 18 - 18 & 0 - 0 \\ -12 - (-12) & 6 - 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \therefore 3G - 2H &= O \end{aligned}$$

Using a CAS calculator to perform matrix addition, subtraction and scalar multiplication

For small matrices, it is usually quicker to add, subtract or multiply a matrix by a number (scalar multiplication) by hand. However, if dealing with larger matrices, it is best to use a CAS calculator.

CAS 2: How to add, subtract and scalar multiply matrices using the TI-Nspire CAS

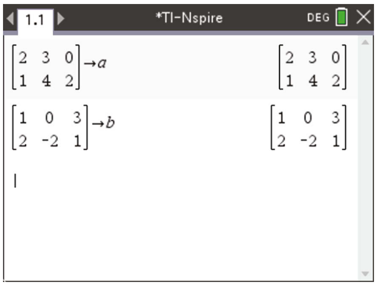
If $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}$, find:

a $A + B$ **b** $A - B$ **c** $3A - 2B$

Steps

1 Press ctrl + N. Select **Add Calculator**.

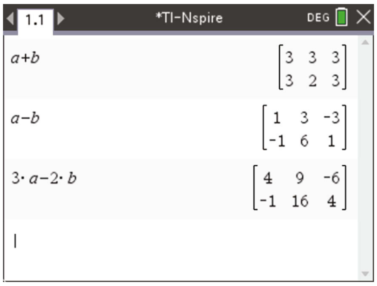
2 Enter and store the matrices A and B into your calculator.



a To determine $A + B$, type ***a* + *b***.
Press enter to evaluate.

b To determine $A - B$, type ***a* − *b***.
Press enter to evaluate.

c To determine $3A - 2B$, type **3*a* − 2*b***.
Press enter to evaluate.



CAS 2:How to add, subtract and scalar multiply matrices with the ClassPad

If $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}$, find:

a $A + B$ **b** $A - B$ **c** $3A - 2B$

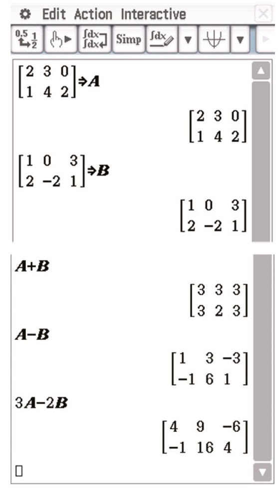
Steps

1 Enter the matrices A and B into your calculator using the var keyboard.

2 **a** To calculate $A + B$, type ***A*** + ***B*** and then press EXE to evaluate.

b To calculate $A - B$, type ***A*** – ***B*** and then press EXE to evaluate.

c To calculate $3A - 2B$, type **3*A*** – **2*B*** and then press EXE to evaluate.



Example 16 Processing data using addition, subtraction and scalar multiplication

The sales data for two used car dealers, Honest Joe's and Super Deals, are displayed below.

Car sales	*2014*			*2015*		
	Small	*Medium*	*Large*	*Small*	*Medium*	*Large*
Honest Joe's	24	32	11	26	38	16
Super Deals	32	34	9	35	41	12

Solution

a Construct two matrices, A and B, to represent the sales data for 2014 and 2015 separately.

$$A = \begin{bmatrix} 24 & 32 & 11 \\ 32 & 34 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 26 & 38 & 16 \\ 35 & 41 & 12 \end{bmatrix}$$

b Construct a new matrix $C = A + B$. What does this matrix represent?

$$C = A + B$$
$$= \begin{bmatrix} 24 & 32 & 11 \\ 32 & 34 & 9 \end{bmatrix} + \begin{bmatrix} 26 & 38 & 16 \\ 35 & 41 & 12 \end{bmatrix}$$
$$= \begin{bmatrix} 50 & 70 & 27 \\ 67 & 75 & 21 \end{bmatrix}$$

Matrix C represents the total sales for 2014 and 2015 for the two dealers.

c Construct a new matrix, $D = B - A$. What does this matrix represent?

$$D = B - A$$
$$= \begin{bmatrix} 2 & 6 & 5 \\ 3 & 7 & 3 \end{bmatrix}$$

Matrix D represents the increase in sales from 2014 and 2015 for the two dealers.

d Both dealers want to increase their 2015 sales by 50% by 2016. Construct a new matrix $E = 1.5B$. Explain why this matrix represents the planned sales figures for 2016.

$$E = 1.5B = 1.5 \times \begin{bmatrix} 26 & 38 & 16 \\ 35 & 41 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 \times 26 & 1.5 \times 38 & 1.5 \times 16 \\ 1.5 \times 35 & 1.5 \times 41 & 1.5 \times 12 \end{bmatrix}$$

$$= \begin{bmatrix} 39 & 57 & 24 \\ 52.5 & 61.5 & 18 \end{bmatrix}$$

Forming the scalar product 1.5B multiplies each element by 1.5. This has the effect of increasing each value by 50%.

Skill-sheet

Exercise 10C

Equality of two matrices

Example 11

1 Given that

$$\begin{bmatrix} 4 & a+2 \\ 6 & b-1 \end{bmatrix} = \begin{bmatrix} 3 & 11 \\ 10 & 6 \end{bmatrix}$$

find the value of a and the value of b.

Matrix addition, subtraction and scalar multiplication

Example 12
Example 13

2 The questions below relate to the following six matrices. Computations will be quicker if done by hand.

$$A = \begin{bmatrix} 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \quad F = \begin{bmatrix} 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

a Which matrices are equal?

b Which matrices have the same order?

c Which matrices can be added or subtracted?

d Compute each of the following, where possible.

i $A + B$ **ii** $D + E$ **iii** $C - F$ **iv** $A - B$ **v** $E - D$

Example 14

vi $3B$ **vii** $4F$ **viii** $3C + F$ **ix** $4A - 2B$ **x** $E + F$

Example 15

3 Let $G = \begin{bmatrix} 8 & 0 \\ -4 & 2 \end{bmatrix}$ and $H = \begin{bmatrix} 2 & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$. Show that $2G - 8H = O$.

4 Evaluate each of the folowing giving your answer as a single matrix.

a $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

b $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

c $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + 2 \times \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

d $\begin{bmatrix} 1 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \end{bmatrix}$

e $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

f $3\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

g $3\begin{bmatrix} 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

h $\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

5 Use a calculator to evaluate the following.

a $2.2 \times \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - 1.1 \times \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

b $\begin{bmatrix} 1.2 & 0.2 \\ 4.5 & 3.3 \end{bmatrix} - 3.5 \times \begin{bmatrix} 0.4 & 4 \\ 1 & 2 \end{bmatrix}$

c $5 \times \begin{bmatrix} 1 & 2 & 1 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - 2 \times \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 1 \\ 0.5 & 0 & -2 \end{bmatrix}$

d $0.8 \times \begin{bmatrix} 1 & 2 & 1 & 4 \\ 1 & 0 & -1 & 2 \end{bmatrix} + 0.2 \times \begin{bmatrix} -1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{bmatrix}$

6 Find the values of x, y, z and w in the following. $2\begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 12 & 6 \end{bmatrix}$

Applications of matrix addition, subtraction and scalar multiplication

Example 16

7 The number of DVDs sold in a company's city, suburban and country stores for each 3-month period in a year is shown in the table.

	DVD sales (thousands)			
Store location	*Jan–March*	*April–June*	*July–Sept*	*Oct–Dec*
City	2.4	2.8	2.5	3.4
Suburban	3.5	3.4	2.6	4.1
Country	1.6	1.8	1.7	2.1

a Construct four 3×1 matrices A, B, C, and D that show the sales in each of the three-month periods during the year.

b Evaluate $A + B + C + D$. What does the sum $A + B + C + D$ represent?

8 The numbers of females and males enrolled in three different gym programs for 2014 and 2015, *Weights*, *Aerobics* and *Fitness*, are shown in the table.

	2014			*2015*		
Gym membership	*Weights*	*Aerobics*	*Fitness*	*Weights*	*Aerobics*	*Fitness*
Females	16	104	86	24	124	100
Males	75	34	94	70	41	96

a Construct two matrices, A and B, which represent the gym memberships for 2014 and 2015 separately.

b Construct a new matrix $C = A + B$. What does this matrix represent?

c Construct a new matrix $D = B - A$. What does this matrix represent? What does the negative element in this matrix represent?

d The manager of the gym wants to double her 2015 membership by 2018. Construct a new matrix E that would show the membership in 2018 if she succeeds with her plan. Evaluate.

Exam 1 style questions

9 $\begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} - \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} =$

A $\begin{bmatrix} -2 & 2 \\ -3 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 4 \\ -3 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} -2 & 4 \\ -6 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} -2 & 0 \\ -6 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 2 \\ -6 & 0 \end{bmatrix}$

10 If $M = \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 4 \\ 3 & 0 \end{bmatrix}$, then $2M - 2N =$

A $\begin{bmatrix} 0 & 0 \\ -9 & 2 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & -2 \\ -6 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & -4 \\ -12 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 4 \\ 12 & -2 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 2 \\ 6 & -1 \end{bmatrix}$

11 The table below shows information about two matrices A and B.

	Order	Rule
A	3×3	$a_{ij} = ij$
B	3×3	$b_{ij} = i + j$

The sum $A + B$ is

A $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$ **C** $\begin{bmatrix} 3 & 5 & 7 \\ 3 & 4 & 5 \\ 3 & 6 & 19 \end{bmatrix}$

D $\begin{bmatrix} 3 & 2 & 7 \\ 3 & 4 & 5 \\ 14 & 15 & 16 \end{bmatrix}$ **E** $\begin{bmatrix} 3 & 5 & 7 \\ 5 & 8 & 11 \\ 7 & 11 & 15 \end{bmatrix}$

10D Matrix arithmetic: the product of two matrices

Learning intentions

- To be able to establish when multiplication of two matrices can be undertaken.
- To be able to undertake the multiplication of matrices.
- To be able to understand that the order of multiplication of matrices is important.
- To be able to apply multiplication of matrices to practical problems.
- To be able to use your CAS calculator to undertake multiplication of matrices.
- To be able to understand the role of 'summing matrices'.
- To be able to work with calculations involving powers of matrices.

The process of multiplying two matrices involves both multiplication and addition. The process can be illustrated using Australian Rules football scores.

An illustration of matrix multiplication

Two teams, the Ants and the Bulls, play each other in a game of Australian rules football. At the end of the game:

- the Ants had scored 11 goals 5 behinds
- the Bulls had scored 10 goals 9 behinds.

Now calculate each team's score in points:

- one goal = 6 points
- one behind = 1 point.

Thus we can write:

$$11 \times 6 + 5 \times 1 = 71 \text{ points}$$
$$10 \times 6 + 9 \times 1 = 69 \text{ points}$$

Matrix multiplication follows the same pattern.

$$\begin{array}{lcccccccc} & \text{Goals} \quad \text{Behinds} & & \text{Point values} & & & & \text{Final points} \\ \begin{array}{l}\text{Ants score:}\\ \text{Bulls score:}\end{array} & \begin{bmatrix} 11 & 5 \\ 10 & 9 \end{bmatrix} & \times & \begin{bmatrix} 6 \\ 1 \end{bmatrix} & = & \begin{bmatrix} 11 \times 6 + 5 \times 1 \\ 10 \times 6 + 9 \times 1 \end{bmatrix} & = & \begin{bmatrix} 71 \\ 69 \end{bmatrix} \end{array}$$

The order of matrices and matrix multiplication

Look at the **order** of each of the matrices involved in the **matrix multiplication** below.

$$\begin{array}{lcccccccc} & \text{Goals} \quad \text{Behinds} & & \text{Point values} & & & & \text{Final points} \\ \begin{array}{l}\text{Ants score:}\\ \text{Bulls score:}\end{array} & \begin{bmatrix} 11 & 5 \\ 10 & 9 \end{bmatrix} & \times & \begin{bmatrix} 6 \\ 1 \end{bmatrix} & = & \begin{bmatrix} 11 \times 6 + 5 \times 1 \\ 10 \times 6 + 9 \times 1 \end{bmatrix} & = & \begin{bmatrix} 71 \\ 69 \end{bmatrix} \\ \text{Order of matrices:} & 2 \times 2 & & 2 \times 1 & & & & 2 \times 1 \end{array}$$

Thus, multiplying a 2×2 matrix by a 2×1 matrix gives a 2×1 matrix.

Two observations can be made here:

1 To perform matrix multiplication, the *number of columns in the first matrix* (2) needs to be the same as the *number of rows in the second matrix* (2). For example, if there were three columns in the first matrix, there would not be enough elements in the second matrix to complete the multiplication. When this happens, we say that matrix multiplication is not defined.

2 The final result of multiplying the two matrices is a 2×1 matrix. For each row in the first matrix, there will be a row in the product matrix (there are two rows). For each column in the second matrix, there will be a column in the product matrix (there is one column).

These observations can be generalised to give two important rules for matrix multiplication.

Rule 1: Condition for matrix multiplication to be defined

Matrix multiplication of two matrices requires the *number of columns*in the *first matrix* to *equal* the *number of rows* in the *second matrix*.

That is, if A is of order $m \times n$ and B is of order $r \times s$, then the product AB is only defined if $n = r$.

For example, if $A = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, then:

- $AB = \underset{2 \times 3}{\begin{bmatrix} 3 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}} \times \underset{3 \times 1}{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}}$ is *defined*: columns in $A(3)$ = rows in $B(3)$
- $BC = \underset{3 \times 1}{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}} \times \underset{2 \times 2}{\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}}$ is *not defined*: columns in $B(1) \neq$ rows in $C(2)$.

Example 17 **Is a matrix product defined?**

$A = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Which of the following matrix products are defined?

a AB **b** BC **c** AC

Explanation	Solution
a Write down the matrix product. Under each matrix, write down its order (columns × rows)	$AB = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \end{bmatrix}$; not defined order: $2 \times 2 \quad 1 \times 2$
b The matrix product is defined if the number of columns in matrix 1 = the number of rows in matrix 2.	$BC = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; defined order: $1 \times 2 \quad 2 \times 1$
c Write down your conclusion.	$AC = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; defined order: $2 \times 2 \quad 2 \times 1$

Once we know that two matrices can be multiplied, we can use the order of the two matrices to determine the order of the resulting matrix.

Rule 2: Determining the order of the product matrix

If two matrices can be multiplied, then the *product matrix* will have the same *number of rows* as the *first matrix* and the same *number of columns* as the *second matrix*.

That is, if A is of order $m \times n$ and B is of order $n \times s$, then AB will be of order $m \times s$.

For example, if $A = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, then:

- $AB = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ is *defined* and will be of order 2×1.

  $$2 \times \underbrace{3 \qquad 3}_{\text{equal}} \times 1 \qquad\qquad 2 \times 1$$

- $AB = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$ is *defined* and will be of order 2×3.

  $$2 \times \underbrace{2 \qquad 2}_{\text{equal}} \times 3 \qquad\qquad 2 \times 3$$

Example 18 **Determining the order of a matrix product**

$$A = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 3 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The following matrix products are defined. What is their order?

a BA **b** BC **c** AC

Explanation

1 Write down the matrix product. Under each matrix, write down its order rows × columns.

2 The order of the product matrix is given by rows in matrix 1 × columns in matrix 2.

3 Write down the order.

Solution

a $BA = \begin{bmatrix} 3 & 1 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix}$; order of BA 1×2

order: (1×2) (2×2)

b $BC = \begin{bmatrix} 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$; order of BC 1×1

order: (1×2) (2×1)

c $AC = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$; order of AC 2×1

order: (2×2) (2×1)

Order of multiplication is important when multiplying matrices

You might have noticed in Example 17 that while the matrix product BA was defined, the matrix product AB in Example 18 was not defined. Order is important in matrix multiplication. For example, if we have two matrices, M and N, and form the products MN and NM, frequently the products will be different. We will return to this point when in the next section.

Determining matrix products

For large matrices, the process of matrix multiplication is complex and can be error prone and tedious to do by hand. Fortunately, CAS calculators will do it for us, and that is perfectly acceptable.

However, before we show you how to use a CAS calculator to multiply matrices, we will illustrate the process by multiplying a row matrix by a column matrix and a rectangular matrix by a column matrix *by hand*.

In terms of understanding matrix multiplication, and using this knowledge to solve problems later in this module, these are the two most important worked examples in the chapter.

Example 19 **Multiplying a row matrix by a column matrix (by hand)**

Evaluate the matrix product AB, where $A = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$.

Explanation

1 Write down the matrix product and, above each matrix, write down its order. Use this information to determine whether the matrix product is defined and its order.

2 To determine the matrix product:

a multiply each element in the row matrix by the corresponding element in the column matrix

b add the results

c write down your answer.

Solution

$$AB = \overset{1\times 3}{\begin{bmatrix} 1 & 3 & 2 \end{bmatrix}} \overset{3\times 1}{\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}}$$

Defined: the number of columns in A equals the number of rows in B.

The order of AB is 1×1.

$$\begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

$$= [1 \times 2 + 3 \times 4 + 2 \times 1]$$

$$= [16]$$

$\therefore AB = [16]$

Example 20 **Multiplying a rectangular matrix by a column matrix (by hand)**

Evaluate the matrix product AB, where $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Explanation

1 Write down the matrix product and, above each matrix, write down its order. Use this information to determine whether the matrix product is defined and its order.

Solution

$$AB = \overset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}} \overset{2\times 1}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

Defined: the number of columns in A equals the number of rows in B.

The order of AB is 2×1.

2 To determine the matrix product:

a multiply each element in the row matrix by the corresponding element in the column matrix

b add the results

c write down your answer.

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1\times 2 + 0 \times 3 \\ 2 \times 2 + 3 \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 13 \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} 2 \\ 13 \end{bmatrix}$$

Using a CAS calculator to multiply two matrices

In principle, if you can multiply a row matrix by a column matrix, you can work out the product between any two matrices, provided it is defined. However, because you have to do it for every possible row/column combination, it soon gets beyond even the most patient and careful person. For that reason, in practice we use technology to do the calculation for us.

We will illustrate how to use a calculator to multiply matrices by evaluating the matrix product in the football score example given earlier.

CAS 3: How to multiply two matrices using the TI-Nspire CAS

If $C = \begin{bmatrix} 11 & 5 \\ 10 & 9 \end{bmatrix}$ and $D = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, find the matrix CD.

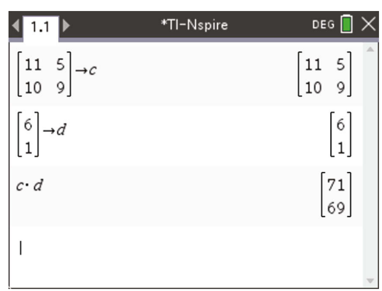
Steps

1 Press ctrl + N. Select **Add Calculator**.

2 Enter and store the matrices C and D into your calculator.

3 To calculate matrix CD, type $\boldsymbol{c} \times \boldsymbol{d}$. Press enter to evaluate.

Note: You must put a multiplication sign between the c and d.



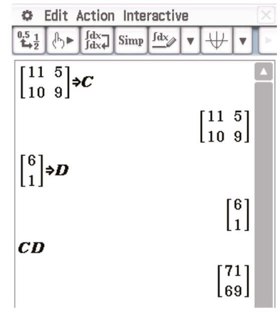
CAS 3:How to multiply two matrices using the ClassPad

If $C = \begin{bmatrix} 11 & 5 \\ 10 & 9 \end{bmatrix}$ and $D = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$ find the matrix product CD.

Steps

1 Enter the matrices C and D into your calculator.

2 To calculate $C \times D$, type ***CD*** and then press EXE to evaluate.



Applications of the product of two matrices

The summing matrix

A row or column matrix in which all the elements are 1 is called a *summing matrix*.

The matrices opposite are all examples of summing matrices.

The rules for using a summing matrix to sum the rows and columns of a matrix follow.

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \quad \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} \quad \begin{bmatrix}1 & 1\end{bmatrix} \quad \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

Using matrix multiplication to sum the rows and columns of a matrix

- To *sum* the *rows* of an $m \times n$ matrix, *post-multiply* the matrix by an $n \times 1$ summing matrix.
- To *sum the columns* of an $m \times n$ matrix, *pre-multiply* the matrix by a $1 \times m$ summing matrix.

Example 21 Using matrix multiplication to sum the rows and columns of a matrix

Use matrix multiplication to generate a matrix that:

a displays

i row sums of the matrix $\begin{bmatrix}1 & 3 & 0\\2 & 6 & 7\\3 & 0 & 1\end{bmatrix}$ **ii** column sums of the matrix $\begin{bmatrix}1 & 3 & 0\\2 & 6 & 7\\3 & 0 & 1\end{bmatrix}$

b displays the column sums of the matrix $\begin{bmatrix}2 & 5 & -1 & -3 & 4\\0 & 6 & 2 & -2 & 3\end{bmatrix}$.

Explanation

a i To sum the rows of a 3×3 matrix, *post*-multiply a 3×1 summing matrix.

Solution

$$\begin{bmatrix}1 & 3 & 0\\2 & 6 & 7\\3 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}4\\15\\4\end{bmatrix}$$

ii To sum the columns of a 3×3 matrix, *pre*-multiply a 1×3 summing matrix.

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 7 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 8 \end{bmatrix}$$

b To sum the columns of a 2×5 matrix, *pre*-multiply a 1×2 summing matrix.

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & -1 & -3 & 4 \\ 0 & 6 & 2 & -2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 11 & 1 & -5 & 7 \end{bmatrix}$$

Example 22 shows a practical application of matrix multiplication.

Example 22 A practical application of matrix multiplication

$E = \begin{array}{c} \begin{bmatrix} 25 \\ 40 \end{bmatrix} \end{array} \begin{array}{l} \textit{Walk} \\ \textit{Run} \end{array}$ Matrix E gives the energy in kilojoules consumed per minute when walking and running.

$T = \begin{array}{cc} [20 & 40] \\ \textit{Walk} & \textit{Run} \end{array}$ Matrix T gives the times (in minutes) a person spent walking and running in a training session.

Compute the matrix product TE and show that it gives the total energy consumed during the training session.

Solution

$$T \times E = \begin{bmatrix} 20 & 40 \end{bmatrix} \begin{bmatrix} 25 \\ 40 \end{bmatrix} = [20 \times 25 + 40 \times 40] = [2100]$$

The total energy consumed is:

20 minutes × 25 kJ/minute + 40 minutes × 40 kJ/minute = 2100 kJ

This is the value given by the matrix product TE.

You could work out the energy consumed on the training run for one person just as quickly without using matrices. However, the advantage of using a matrix formulation is that, with the aid of a calculator, you could almost as quickly have worked out the energy consumed by 10 or more runners, all with different times spent walking and running.

Matrix powers

Now that we can multiply matrices, we can also determine the **power of a matrix**. This is an important tool when we meet communication and dominance matrices in the next section and transition matrices in the next chapter.

The power of a matrix

Just as we define

2^2 as 2×2,

2^3 as $2 \times 2 \times 2$,

2^4 as $2 \times 2 \times 2 \times 2$ and so on,

we define the various powers of matrices as

A^2 as $A \times A$,

A^3 as $A \times A \times A$,

A^4 as $A \times A \times A \times A$ and so on.

Only square matrices can be raised to a power.

Example 23 Evaluating matrix expressions involving powers

If $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, determine:

a $2A + B^2 - 2C$ **b** $(2A - B)^2 - C^2$ **c** $AB^2 - 3C^2$

Explanation

1 Write down the matrices.

2 Enter the matrices A, B and C into your calculator.

Solution

$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \rightarrow a \qquad \begin{bmatrix} 1 & 0 \\ 2. & -1. \end{bmatrix}$

$\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \rightarrow b \qquad \begin{bmatrix} -1. & 1. \\ 2. & 1. \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow c \qquad \begin{bmatrix} 0. & 1. \\ 1. & 1. \end{bmatrix}$

$2 \cdot a + b^2 - 2 \cdot c \qquad \begin{bmatrix} 5. & -2. \\ 2. & -1. \end{bmatrix}$

$(2 \cdot a - b)^2 - c^2 \qquad \begin{bmatrix} 6. & -1. \\ -1. & 5. \end{bmatrix}$

$a \cdot b^2 - 3 \cdot c^2 \qquad \begin{bmatrix} 0. & -3. \\ 3. & -9. \end{bmatrix}$

3 Type in each of the expressions as written, and press to evaluate. Write down your answer.

a $2A + B^2 - 2C = \begin{bmatrix} 5 & -2 \\ 2 & -1 \end{bmatrix}$

b $(2A + B)^2 - C^2 = \begin{bmatrix} 6 & -1 \\ -1 & 5 \end{bmatrix}$

c $AB^2 - 3C^2 = \begin{bmatrix} 0 & -3 \\ 3 & -9 \end{bmatrix}$

Note: For CAS calculators you must use a multiplication sign between a and b^2 in the last example, otherwise it will be read as variable $(ab)^2$.

Skill-sheet

Exercise 10D

Matrix multiplication

Example 17

1 The questions below relate to the following five matrices.

$$A = \begin{bmatrix} 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad E = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

a Which of the following matrix products are defined?

i AB **ii** BA **iii** AC **iv** CE

v EC **vi** EA **vii** DB **viii** CD

Example 19

Example 20

b Compute the following products by hand.

i AB **ii** CE **iii** DB **iv** AD

c Enter the five matrices into your calculator and compute the following matrix expressions.

i AB **ii** EC **iii** $AB - 3CE$ **iv** $2AD + 3B$

2 Evaluate each of the following matrix products *by hand*.

a $\begin{bmatrix} 0 & 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \end{bmatrix} =$ **b** $\begin{bmatrix} 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$ **c** $\begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} =$

d $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$ **e** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} =$ **f** $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} =$

3 Evaluate each of the following matrix products using a CAS calculator.

a $\begin{bmatrix} 0.5 \\ -1.5 \\ 2.5 \end{bmatrix} \times \begin{bmatrix} 2 & 4 & 6 \end{bmatrix} =$

b $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \times \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} =$

c $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ -1 & 2 \end{bmatrix}$

d $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 0 & 2 & 1 \\ -1 & 4 & 2 \\ -2 & 1 & 2 \end{bmatrix} =$

Using summing matrices to sum the rows and columns of matrices

Example 21

4 For the matrix opposite, write down a matrix that can be used to: $\begin{bmatrix} 2 & 5 \\ -1 & 1 \\ 9 & 3 \end{bmatrix}$.

a sum its rows **b** sum its columns

5 Show how the matrix $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ can be used to sum the rows of $\begin{bmatrix} 7 & 1 & 2 \\ 1 & 2 & 2 \\ 8 & 1 & 4 \end{bmatrix}$.

6 Show how the matrix $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ can be used to sum the columns of $\begin{bmatrix} 9 & 0 & 2 \\ 1 & 7 & 3 \\ 8 & 3 & 4 \end{bmatrix}$.

7 Use matrix multiplication to construct a matrix that:

a displays row sums of the matrix $\begin{bmatrix} 2 & 4 & 1 & 7 & 8 \\ 1 & 9 & 0 & 0 & 2 \\ 3 & 4 & 3 & 3 & 5 \\ 2 & 1 & 1 & 1 & 7 \\ 5 & 3 & 6 & 7 & 9 \end{bmatrix}$

b displays the column sums of the matrix $\begin{bmatrix} 4 & 5 & 1 & 2 & 1 \\ 0 & 3 & 4 & 5 & 1 \\ 4 & 2 & 1 & 7 & 9 \end{bmatrix}$.

Practical applications of matrix multiplication

Example 22

8 Six teams play an indoor soccer competition.
If a team:

- wins, it scores two points
- draws, it scores one point
- loses, it scores zero points.

This is summarised in the points matrix opposite.

$$P = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{array}{l} \textit{Win} \\ \textit{Draw} \\ \textit{Lose} \end{array}$$

The results of the competition are summarised in the results matrix. Work out the final points score for each team by forming the matrix product RP.

$$R = \begin{array}{c} \begin{array}{ccc} W & D & L \end{array} \\ \begin{bmatrix} 4 & 1 & 0 \\ 3 & 1 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix} \end{array} \begin{array}{l} \text{Team 1} \\ \text{Team 2} \\ \text{Team 3} \\ \text{Team 4} \\ \text{Team 5} \\ \text{Team 6} \end{array}$$

9 Four people complete a training session in which they walked, jogged and ran at various times.
The energy consumed in kJ/minute when walking, jogging or running is listed in the energy matrix opposite.

$$E = \begin{bmatrix} 25 \\ 40 \\ 65 \end{bmatrix} \begin{array}{l} \text{Walk} \\ \text{Jog} \\ \text{Run} \end{array}$$

The time spent in each activity (in minutes) by four people is summarised in the time matrix opposite. Work out the total energy consumed by each person, by forming the matrix product TE.

$$T = \begin{array}{c} \begin{array}{ccc} W & J & R \end{array} \\ \begin{bmatrix} 10 & 20 & 30 \\ 15 & 20 & 25 \\ 20 & 20 & 20 \\ 30 & 20 & 10 \end{bmatrix} \end{array} \begin{array}{l} \text{Person 1} \\ \text{Person 2} \\ \text{Person 3} \\ \text{Person 4} \end{array}$$

10 A manufacturer sells three types of fruit straps, A, B and C, through outlets at two shops, Energy (E) and Nourishing (N).
The number of fruit straps sold per month at each shop is given by the matrix Q.

$$Q = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 25 & 34 & 19 \\ 30 & 45 & 25 \end{bmatrix} \end{array} \begin{array}{l} E \\ N \end{array}$$

a Write down the order of matrix Q.

The matrix P, shown opposite, gives the selling price, in dollars, of each type of fruit strap A, B and C.

$$P = \begin{bmatrix} 2.50 \\ 1.80 \\ 3.20 \end{bmatrix} \begin{array}{l} A \\ B \\ C \end{array}$$

b **i** Evaluate the matrix M, where $M = QP$.
ii What information do the elements of matrix M provide?

c Explain why the matrix PQ is not defined.

11 Matrix X shows the number of cars of models a and b bought by four dealers A, B, C, D. Matrix Y shows the cost in dollars of cars a and b. Find XY and explain what it represents.

$$X = \begin{array}{l} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \end{array} \qquad Y = \begin{bmatrix} 26\,000 \\ 32\,000 \end{bmatrix} \begin{array}{l} a \\ b \end{array}$$

12 It takes John 5 minutes to drink a milk shake which costs \$2.50, and 12 minutes to eat a banana split which costs \$3.00.

a Find the product $\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and interpret the result in fast-food economics.

b Two friends join John. Find $\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ and interpret the result.

13 The final grades for Physics and Chemistry are made up of three components: tests, practical work and exams. Each semester, a mark out of 100 is awarded for each component. Wendy scored the following marks in the three components for Physics:

Semester 1 tests 79, practical work 78, exam 80

Semester 2 tests 80, practical work 78, exam 82

a Represent this information in a 2×3 matrix.

To calculate the final grade for each semester, the three components are weighted: tests are worth 20%, practical work is worth 30% and the exam is worth 50%.

b Represent this information in a 3×1 matrix.

c Calculate Wendy's final grade for Physics in each semester.

Wendy also scored the following marks in the three components for Chemistry:

Semester 1 tests 86, practical work 82, exam 84

Semester 2 tests 81, practical work 80, exam 70

d Calculate Wendy's final grade for Chemistry in each semester.

Students who gain a total score of 320 or more for Physics and Chemistry over the two semesters are awarded a Certificate of Merit in Science.

e Will Wendy be awarded a Certificate of Merit in Science?

She asks her teacher to re-mark her Semester 2 Chemistry exam, hoping that she will gain the necessary marks to be awarded a Certificate of Merit.

f How many extra marks on the exam does she need?

Powers of matrices

Example 23

14 If $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, determine A^2, A^3, A^4 and A^7.

15 If $A = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$, determine A^4, A^5, A^6 and A^7.

16 If $B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix}$ use your calculator to find B^3.

17 If $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$, evaluate:

a $A + 2B - C^2$ **b** $AB - 2C^2$ **c** $(A + B + 2C)^2$

d $4A + 3B^2 - C^3$ **e** $(A - B)^3 - C^3$

Exam 1 style questions

18 The matrix product $\begin{bmatrix} 6 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ is equal to

A $[30]$ **B** $\begin{bmatrix} 18 \\ 12 \\ 0 \end{bmatrix}$ **C** $[39]$ **D** $3 \times \begin{bmatrix} 2 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ **E** $2 \times \begin{bmatrix} 3 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

19 Matrix $P = \begin{bmatrix} 1 & 4 \\ 2 & 6 \\ 1 & 2 \end{bmatrix}$ and matrix $Q = \begin{bmatrix} 1 & 5 & 2 \\ 2 & 1 & 6 \end{bmatrix}$

Matrix $R = P \times Q$. Element r_{31} is determined by the calculation

A $1 \times 1 + 4 \times 2$ **B** $1 \times 1 + 4 \times 0$ **C** $1 \times 1 + 2 \times 2$

D $4 \times 1 + 5 \times 0$ **E** $4 \times 2 + 5 \times 1$

20 There are two types of chocolate boxes, Minty Chews and Orange Delight, available in a shop. The cost, in dollars, to purchase a box is shown in the table below.

Chocolate	Cost($)
Minty Chews	6
Orange Delight	8

Liam is doing all his Christmas shopping by buying chocolate boxes. He buys 7 boxes of Minty Chews and 9 boxes of Orange Delight. The total cost in dollars of these chocolates can be determined by which one of the following calculations?

A $[7] \times \begin{bmatrix} 6 & 8 \end{bmatrix}$ **B** $\begin{bmatrix} 7 & 9 \end{bmatrix} \times \begin{bmatrix} 6 & 8 \end{bmatrix}$ **C** $\begin{bmatrix} 7 \\ 9 \end{bmatrix} \times \begin{bmatrix} 6 & 8 \end{bmatrix}$

D $\begin{bmatrix} 7 & 9 \end{bmatrix} \times \begin{bmatrix} 6 \\ 8 \end{bmatrix}$ **E** $\begin{bmatrix} 9 & 7 \end{bmatrix} \times \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

10E Matrix inverse, the determinant and matrix equations

Learning intentions

- To be able to detemine by multiplication when two matrices are inverses.
- To be able to determine the determinant of a 2×2 matrix.
- To be able to determine the inverse of a 2×2 matrix.
- To be able to use a CAS calculator to determine the inverse and determinant of an $n \times n$ matrix.
- To be able to solve simple matrix equations.

The inverse matrix A^{-1}

So far, you have been shown how to add, subtract and multiply matrices, but what about dividing them? As you might expect matrix division, like matrix multiplication, is a more complicated process than its equivalent process for dividing numbers.

The starting point for matrix division is the **inverse matrix**. You will see why as we proceed.

The inverse matrix A^{-1}

The inverse of a square matrix A is called A^{-1}.

The inverse matrix has the property $AA^{-1} = A^{-1}A = I$.

Having defined the inverse matrix, two questions immediately come to mind. Does the inverse of a matrix actually exist? If so, how can we calculate it?

First we will demonstrate that at least some matrices have inverses. We can do this by showing that two matrices, which we will call A and B, have the property $AB = I$ and $BA = I$, where I is the **identity matrix**. If this is the case, we can then say that $B = A^{-1}$.

Example 24 Demonstrating that two matrices are inverses

Show that the matrices $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ are inverses.

Explanation	Solution
1 Write down A and B.	$A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$

2 Form the product AB and evaluate. You can use your calculator to speed things up if you wish.	$AB = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ $= \begin{bmatrix} 2\times 5 + 3\times(-3) & 2\times(-3)+3\times 2 \\ 3\times 5 + (-5)\times 3 & 3\times(-3)+5\times 2 \end{bmatrix}$ $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\therefore AB = I$
3 Form the product BA and evaluate. You can use your calculator here to speed things up if you wish.	$BA = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ $= \begin{bmatrix} 5\times 2 + (-3)\times 3 & 5\times 3+(-3)\times 5 \\ (-3)\times 2 + 2\times 3 & (-3)\times 3+2\times 5 \end{bmatrix}$ $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\therefore BA = I$
4 Write down your conclusion.	Because $AB = I$ and $BA = I$, we conclude that A and B are inverses.

While Example 24 clearly demonstrates that the matrices $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ both have an inverse, many square matrices do *not* have inverses. To see why, we need to introduce another new matrix concept, the **determinant**, and see how it relates to finding the inverse of a square matrix. To keep things manageable, we will restrict ourselves initially to 2×2 matrices.

The determinant of a matrix

The determinant of a 2 × 2 matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the determinant of matrix A is given by:

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \times d - b \times c$$

Example 25 Finding the determinant of a 2×2 matrix

Find the determinant of the matrices:

a $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ **b** $B = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$ **c** $C = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$

Solution

1 Write down the matrix and use the rule $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \times d - b \times c$.

2 Evaluate.

a $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ $\therefore \det(A) = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 2 \times 5 - 3 \times 3 = 1$

b $B = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$ $\therefore \det(B) = \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} = 2 \times 3 - 2 \times 3 = 0$

c $C = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$ $\therefore \det(C) = \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} = 2 \times 3 - 2 \times 4 = -2$

From Example 25, we can see that the determinant of a matrix is a number that can take on both positive and negative values as well as being zero. For a matrix to have an inverse, its determinant must be non-zero.

How to determine the inverse of a 2×2 matrix

Normally you will use a calculator to determine the inverse of a matrix, but we need to do the following example by hand to show you why some square matrices do not have an inverse. To do this we first need to consider the rule for finding the determinant of a 2×2 matrix.

The rule for finding the inverse of a 2×2 matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its inverse, A^{-1}, is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

provided $ad - bc \neq 0$; that is, provided $\det(A) \neq 0$.

The most important thing about this rule is that it shows immediately why you cannot calculate an inverse for some matrices. These are the matrices whose determinant is zero.

Example 26 Using the rule to find the inverse of a 2×2 matrix

Find the inverse of the following matrices.

a $A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$ **b** $B = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$

Explanation

1 Write down the matrix and use the rule

$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \times d - b \times c$

to evaluate the determinant.

Use the rule

$A^{-1} = \dfrac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

to evaluate A^{-1}.

2 Write down the matrix and use the rule $\det(B) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} =$ $a \times d - b \times c$.

to evaluate the determinant.

Solution

a

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = 2 \times 4 - 3 \times 2 = 2$$

$$\therefore A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -1.5 & 1 \end{bmatrix}$$

b

$$B = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

$$\therefore \det(B) = \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} = 2 \times 3 - 2 \times 3 = 0$$

$\det(B) = 0$

$\therefore B$ does not have an inverse.

Using a CAS calculator to determine the determinant and inverse of an $n \times n$ matrix

There are rules for finding the inverse of a square matrix of any size, but in practice we tend to use a calculator. The same goes for calculating determinants, although the inverse and determinant of a 2×2 matrix are often computed more quickly by hand.

CAS 4: How to find the determinant and inverse of a matrix using the TI-Nspire CAS

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$, find $\det(A)$ and A^{-1}.

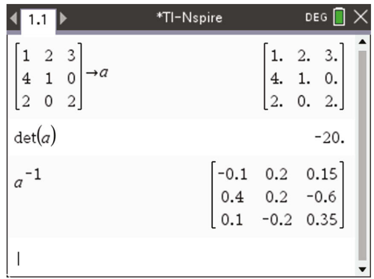
Steps

2 Press ctrl + N. Select **Add Calculator**.

3 Enter the matrix A into your calculator.

4 To calculate det(A), type **det(*a*)** and press enter to evaluate.

Note: det() can also be accessed using menu>**Matrix & Vector>Determinant**.

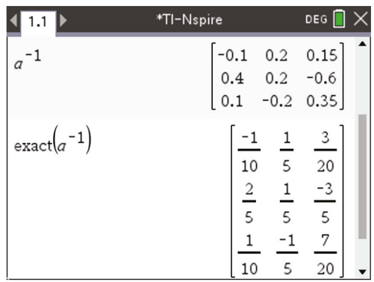


4 To calculate the inverse matrix A^{-1} type **a ∧ – 1** and press enter to evaluate. If you want to see the answer in fractional form, enter as **exact (a∧ –1)** and press enter to evaluate.

Note:

1 Long strings of decimals can be avoided by asking for an exact inverse. Type in **exact(a^{-1})**.

2 If the matrix has no inverse, the calculator will respond with the error message **Singular matrix**.



CAS 4: How to find the determinant and inverse of a matrix using the ClassPad

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$, find det($A$) and A^{-1}.

Steps

1 Enter the matrix A into your calculator.

Note: Change the status of the calculator to **Standard** for fractions to be displayed. Tapping on **Decimal** will change the calculator to **Standard**.

2 To calculate det(A):

a type and highlight ***A*** (by swiping with the stylus)

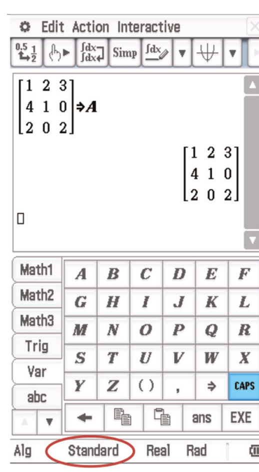
b select **Interactive** from the menu bar, tap **Matrix-Calculation**, then tap **det**.

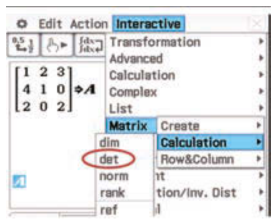
3 To calculate the inverse matrix A^{-1}:

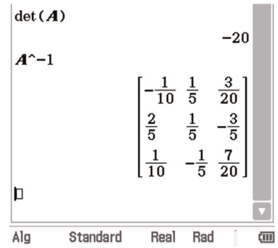
a type **A^–1**

b press EXE to evaluate.

Note: If the matrix has no inverse, the calculator will respond with the message **Undefined**.







Solving matrix equations

In the following example the solution of simple matrix equations is illustrated. The techniques are analagous to those we use in solving linear equations while noting that there are some important differences such as multiplication of matrices is not commutative and to take care that operations are defined on the matrices involved.

Example 27

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 8 & -1 \\ 4 & 6 \end{bmatrix}$ $D = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $E = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

Solve each of the following matrix equations for X

a $B + X = C$ **b** $BX = C$ **c** $XB = C$

d $BX = D$ **e** $AX = E$ **f** $BX + \begin{bmatrix} 4 \\ 5 \end{bmatrix} = D$

Explanation

a **1** Write down B and C.

2 Form the equation $B + X = C$ and note that X must be a 2×2 matrix.

3 Calculate $X = C - B$.

Solution

$B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 8 & -1 \\ 4 & 6 \end{bmatrix}$

$B + X = C$

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + X = \begin{bmatrix} 8 & -1 \\ 4 & 6 \end{bmatrix}$$

$$X = \begin{bmatrix} 8 & -1 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 \\ 1 & 4 \end{bmatrix}$$

b **1** Form the equation $BX = C$ and note that X must be a 2×2 matrix.

2 Note that $B^{-1}B = I$, the 2×2 identity matrix. Calculate $X = B^{-1}C$.

$$BX = C$$
$$B^{-1}BX = B^{-1}C$$
$$X = B^{-1}C$$
$$X = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 8 & -1 \\ 4 & 6 \end{bmatrix}$$
$$\therefore X = \begin{bmatrix} 12 & -8 \\ -16 & 15 \end{bmatrix}$$

c **1** Form the equation $XB = C$ and note that X must be a 2×2 matrix.

2 Note that $BB^{-1} = I$, the 2×2 identity matrix. Calculate $X = CB^{-1}$.

$$XB = C$$
$$XBB^{-1} = CB^{-1}$$
$$X = CB^{-1}$$
$$X = \begin{bmatrix} 8 & -1 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$
$$\therefore X = \begin{bmatrix} 12 & -8 \\ -16 & 15 \end{bmatrix}$$

d **1** Form the equation $BX = D$ and note that X must be a 2×1 matrix.

2 Calculate $X = B^{-1}D$.

$$BX = D$$
$$B^{-1}BX = B^{-1}D$$
$$X = B^{-1}D$$
$$X = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$
$$\therefore X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

e **1** Form the equation $AX = E$ and note that X must be a 3×1 matrix.

2 Calculate $X = A^{-1}E$.

$$AX = E$$
$$A^{-1}AX = A^{-1}E$$
$$X = A^{-1}E$$
$$X = \begin{bmatrix} -2 & -1 & 5 \\ 1 & 0 & -1 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$
$$\therefore X = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

f **1** Form the equation $BX + \begin{bmatrix} 4 \\ 5 \end{bmatrix} = D$ and note that X must be a 2×1 matrix.	$BX + \begin{bmatrix} 4 \\ 5 \end{bmatrix} = D$
	$BX = D - \begin{bmatrix} 4 \\ 5 \end{bmatrix}$
2 Calculate $X = B^{-1}\left(D - \begin{bmatrix} 4 \\ 5 \end{bmatrix}\right)$.	$X = B^{-1}\left(D - \begin{bmatrix} 4 \\ 5 \end{bmatrix}\right)$
	$X = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ -2 \end{bmatrix}$
	$\therefore X = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

Skill-sheet

Exercise 10E

Review of the properties of the identity matrix

1 **a** Write down the:

i 2×2 identity matrix **ii** 3×3 identity matrix **iii** 4×4 identity matrix.

b If $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, show that $AI = IA = A$.

c If $C = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$, show that $CI = IC = C$.

Demonstrating that one matrix is the inverse of the other

Example 24

2 Show that each of the following pairs of matrices are inverses by multiplying one by the other. Use a calculator if you wish.

a $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & -1 \\ -1.5 & 1 \end{bmatrix}$

c $\begin{bmatrix} 9 & 7 \\ 4 & 3 \end{bmatrix}$ and $\begin{bmatrix} -3 & 7 \\ 4 & -9 \end{bmatrix}$ **d** $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}$

Calculating the determinant of a matrix

Example 25

3 Determine (by hand) the value of the determinant for each of the following matrices.

a $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ **b** $B = \begin{bmatrix} 0 & 3 \\ 1 & 4 \end{bmatrix}$ **c** $C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ **d** $D = \begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix}$

Calculating the inverse of a matrix

Example 26

4 Use a calculator to determine the inverse of each of the following matrices.

a $A = \begin{bmatrix} 1.1 & 2.2 \\ 0 & 3.0 \end{bmatrix}$ **b** $B = \begin{bmatrix} 0.2 & -0.1 \\ 10 & 4 \end{bmatrix}$ **c** $D = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ **d** $E = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Solving matrix equations

Example 27

5 Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$ $C = \begin{bmatrix} -6 & -1 \\ 5 & 6 \end{bmatrix}$ $D = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $E = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$

Solve each of the following matrix equations for X

a $B + X = C$ **b** $BX = C$ **c** $XB = C$

d $BX = D$ **e** $AX = E$ **f** $BX + \begin{bmatrix} 7 \\ 6 \end{bmatrix} = D$

6 Find the 2×2 matrix A such that $A\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 12 & 14 \end{bmatrix}$.

7 Suppose that A, B, C and X are 2×2 matrices and that both A and B have inverses. Solve the following for X:

a $AX = C$ **b** $ABX = C$ **c** $AXB = C$

d $A(X + B) = C$ **e** $AX + B = C$ **f** $XA + B = A$

8 If $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15\,000 \\ 20\,000 \\ 10\,000 \end{bmatrix}$ find x, y and z

9 Three crop sprays are manufactured by combining chemicals A, B and C as follows:

Spray *P* One barrel of spray P contains 1 unit of A, 3 units of B and 4 units of C.

Spray *Q* One barrel of spray Q contains 3 units each of A, B and C.

Spray *R* One barrel of spray R contains 2 units of A and 5 units of B.

To control a certain crop disease, a farmer requires 6 units of chemical A, 10 units of chemical B and 6 units of chemical C. How much of each type of spray should the farmer use?

10 A factory makes and assembles three products, P, Q and R, each requiring different quantities of three components, a, b and c. The following matrix A represents the required quantities of components for each product, and the matrix K represents the daily production of components at the factory.

$$A = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} P & Q & R \end{array} \\ \begin{bmatrix} 5 & 3 & 2 \\ 2 & 2 & 4 \\ 0 & 2 & 3 \end{bmatrix} \end{array} \quad \text{and} \quad K = \begin{array}{c} a \\ b \\ c \end{array}\begin{bmatrix} 95 \\ 80 \\ 40 \end{bmatrix}$$

a Find the inverse of A.

b Assume that the factory uses all components that are produced. Find the rate of assembly of P, Q and R at the factory, expressed as number of products per day.

11 Bronwyn and Noel have a clothing warehouse in Summerville. They are supplied by three contractors: Brad, Flynn and Lina.
The matrix shows the number of dresses, pants and shirts that one worker, for each of the contractors, can produce in a week.

$$\begin{array}{r} \\ \text{Dresses} \\ \text{Pants} \\ \text{Shirts} \end{array}\begin{array}{c} \begin{array}{ccc} \text{Brad} & \text{Flynn} & \text{Lina} \end{array} \\ \begin{bmatrix} 5 & 6 & 10 \\ 3 & 4 & 5 \\ 2 & 6 & 5 \end{bmatrix} \end{array}$$

The number produced varies because of the different equipment used by the contractors.
The warehouse requires 310 dresses, 175 pants and 175 shirts in a week. How many workers should each contractor employ to meet the requirement exactly?

Exam 1 style questions

12 The determinant of the matrix $\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$ is

A 4 **B** 0 **C** −4 **D** 1 **E** 2

13 The inverse of the matrix $\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$ is

A −1 **B** $\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$

14 Consider the matrix equation

$$3 \times \begin{bmatrix} 4 & 2 \\ 6 & 7 \end{bmatrix} + X = \begin{bmatrix} 14 & 12 \\ 18 & 22 \end{bmatrix}$$

Which one of the following is the matrix X?

A $\begin{bmatrix} 4 & 2 \\ 6 & 7 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & 5 \\ 6 & 7 \end{bmatrix}$ **C** $\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & 6 \\ 0 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 4 & 2 \\ 6 & 7 \end{bmatrix}$

10F Binary, permutation and communication matrices

Learning intentions

- To be able to identify and work with binary matrices, permutation matrices and communication matrices.

Binary matrices

The following matrices are examples of binary matrices.

$$\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Binary matrices are at the heart of many practical matrix applications, including analysing communication systems and using the concept of dominance to rank players in sporting competitions.

Permutation matrices

A **permutation**[1] **matrix** is a square binary matrix in which there is only one '1' in each row and column.

The following matrices are examples of permutation matrices.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

An identity matrix is a special permutation matrix. A permutation matrix can be used to rearrange the elements in another matrix.

Example 28 **Applying a permutation matrix**

X is the column matrix $X = \begin{bmatrix} T \\ A \\ R \end{bmatrix}$. P is the permutation matrix $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

a Show that:

i pre-multiplying[2] X by P changes the matrix X to the matrix $Y = \begin{bmatrix} R \\ A \\ T \end{bmatrix}$

ii pre-multiplying X by P^2 leaves the matrix X unchanged.

b What can be deduced about P^2 from the result in **a ii**?

---

[1] The word 'permutation' means a rearrangement a group of objects, in this case the elements of a matrix, into a different order.

[2] When we form the matrix product $A \times B$, we say that we are pre-multiplying by A.

Explanation

a i Form the matrix product PX. To find the first entry in the resulting column matrix note that the 1 of the permutation matrix is in the third column. of the first row. We obtain

$$0 \times T + 0 \times A + 1 \times R = R$$

To find the second entry in the resulting column matrix note that the 1 of the permutation matrix is in the second column of the second row and so on.

ii Form the matrix product P^2X.

b To leave the matrix X unchanged, P^2 must be an identity matrix.

Solution

$$PX = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} T \\ A \\ R \end{bmatrix} = \begin{bmatrix} R \\ A \\ T \end{bmatrix}$$

$$P^2X = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^2\begin{bmatrix} T \\ A \\ R \end{bmatrix} = \begin{bmatrix} T \\ A \\ R \end{bmatrix}$$

P^2 is the identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Example 29

Find a permutation matrix that takes the column matrix $\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$ to the column matrix $\begin{bmatrix} D \\ C \\ B \\ A \end{bmatrix}$

Explanation

- We want to move D from the fourth place to the first place. Place the 1 in the fourth column of the first row.
- We want to move C from the third place to the second place. Place the 1 in the third column of the second row.
- We want to move B from the second place to the third place. Place the 1 in the second column of the third row.
- We want to move A from the first place to the fourth place. Place the 1 in the first column of the fourth row.

Solution

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} D \\ C \\ B \\ A \end{bmatrix}.$$

Inverses of permutation matrices

Every permutation matrix has an inverse and this inverse is the adjoint of the matrix. That is if P is a permutation matrix then P^{-1} exists and

$$P^{-1} = P^T$$

Example 30

A 4×4 permutation matrix $P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ applied to a 4×1 column matrix A gives the column matrix $\begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix}$. Determine the matrix A.

Solution

$$PA = \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix}$$

Using your calculator.

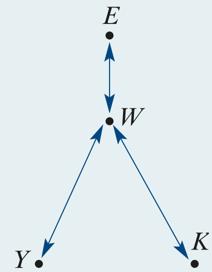
$$\therefore A = P^{-1}\begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = P^T\begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = \begin{bmatrix} W \\ Z \\ X \\ Y \end{bmatrix}$$

Communication matrices

A **communication matrix** is a square binary matrix in which the 1s represent the links in a communication system.

Example 31 Constructing a communication matrix

Eva, Wong, Yumi and Kim are students who are staying in a backpacker's hostel. Because they speak different languages they can have problems communicating. The situation they have to deal with is that:



- Eva speaks English only
- Yumi speaks Japanese only
- Kim speaks Korean only
- Wong speaks English, Japanese and Korean.

Conveniently, this information can be summarised in a network diagram, as shown above. In this diagram, the arrow linking Eva and Wong indicates that they can communicate directly because they both speak English.

The task is to construct a communication matrix.

Explanation

1 There are four people so a 4×4 matrix is needed. Label the columns and rows E, W, Y and K.

2 Label the rows 'Speaker' and the columns 'Receiver'.

3 Designate each element as a '1' or '0' according to the following rules:

- the element $= 1$ if two people can communicate directly because they speak the same language
- the element $= 0$ if two people *cannot* communicate directly because they do not have a common language.

The completed matrix is shown opposite.

Solution

$$\begin{array}{cc} & \textit{Receiver} \\ & \begin{array}{cccc} E & W & Y & K \end{array} \\ \textit{Speaker} \begin{array}{c} E \\ W \\ Y \\ K \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

There is little point in having a matrix representation of a communication system if we already have a network diagram. However, several questions that are not so easily solved with a network diagram can be answered using a communication matrix.

For example, we can see from the network diagram that Eva, who speaks only English, cannot communicate directly with Yumi, who speaks only Japanese. We call this a *one-step communication link*.

However, Eva can communicate with Yumi by sending a message via Wong, who speaks both English and Japanese. In the language of communication systems we call this a *two-step communication link*.

The power of the matrix representation is that *squaring* the communication matrix generates a matrix that identifies all possible *two-step communication links* in a communication network. For example, if we call the communication matrix C, we have:

$$C^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} E & W & Y & K \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ E \\ W \\ Y \\ K \end{array}$$

We have the *two-step link*: Eva → Wong → Yumi.

That is, Eva can communicate with Yumi by using Wong as a go-between.

However, if we use the same process to help us interpret the '1' in row E, column E, we will find that it represents the two-step link Eva → Wong → Eva. This is not a very useful thing to know. Two-step links that have the same sender and receiver are said to be **redundant communication links** because they do not contribute to the communication between different people.

Redundant communication links

A communication link is said to be redundant if the sender and the receiver are the same person.

All of the non-zero elements in the leading diagonal of a communication matrix, or its powers, represent redundant links in the matrix.

However, all of the remaining non-zero elements represent meaningful two-step communication links.

For example, the 1 in row Y, column K represents the two-step communication link that enables Yumi to send a message to Kim.

$$C^2 = \begin{array}{c} \begin{array}{cccc} E & W & Y & K \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ E \\ W \\ Y \\ K \end{array}$$

Finally, the total number of one and two-step links in a communication system, T, can be found by evaluating $T = C + C^2$.

$$T = C + C^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} E & W & Y & K \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ E \\ W \\ Y \\ K \end{array}$$

Analysing communication matrices

- A communication matrix (C) is a square binary matrix in which the 1s are used to identify the direct (one-step) links in the communication system.
- The number of two-step links in a communication system can be identified by squaring its communication matrix.
- The total number of one and two-step links in a communication system can be found by evaluating the matrix sum $T = C + C^2$.

These statements can be readily generalised to include the determination of three (or more) step links by evaluating the matrices C^3, C^4, etc. However, unless the communication networks are extremely large, most of the multi-step links identified will be redundant.

Note: In all cases, the diagonal elements of a communication matrix (or its power) represent redundant communication links.

Skill-sheet

Exercise 10F

Permutation matrices

1 Which of the following binary matrices are permutation matrices?

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 28

2 X is the row matrix: $X = \begin{bmatrix} \text{S} & \text{H} & \text{U} & \text{T} \end{bmatrix}$

P is the permutation matrix: $P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

a What does matrix X change to if it is post-multiplied by P?

b For what value of n does XP^n first equal X?

Example 29

3 Find a permutation matrix that takes the column matrix $\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$ to the column matrix $\begin{bmatrix} C \\ D \\ A \\ B \end{bmatrix}$

Inverses of permutation matrices

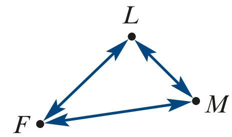
Example 30

4 A 4×4 permutation matrix $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ applied to a 4×1 column matrix A gives the column matrix $\begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix}$. Determine the matrix A.

Communication matrices

Example 31

5 Freya (F), Lani (L) and Mei (M) are close friends who regularly send each other messages. The direct (one-step) communication links between the friends are shown in the diagram opposite.

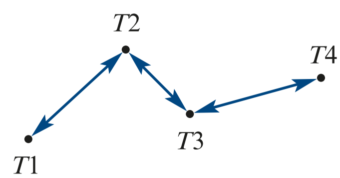


a Construct a communication matrix C from this diagram.

b Calculate C^2.

c How many different ways can Mei send a message to Freya?

6 Four fire towers $T1$, $T2$, $T3$ and $T4$, can communicate with one another as shown in the diagram opposite. In this diagram an arrow indicates that a direct channel of communication exists between a pair of fire towers.



For example, a person at tower 1 can directly communicate with a person in tower 2 and vice versa.

The communication matrix C can also be used to represent this information.

$$C = \begin{array}{c} \begin{array}{cccc} T1 & T2 & T3 & T4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \square & 0 \\ 0 & 1 & 0 & 1 \\ 0 & \square & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ T1 \\ T2 \\ T3 \\ T4 \end{array}$$

a Explain the meaning of a zero in the communication matrix.

b Which two towers can communicate directly with $T2$?

c Write down the values of the two missing elements in the matrix.

The matrix C^2 is shown opposite.

$$C^2 = \begin{array}{c} \begin{array}{cccc} T1 & T2 & T3 & T4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ T1 \\ T2 \\ T3 \\ T4 \end{array}$$

d Explain the meaning of the 1 in row $T3$, column $T1$.

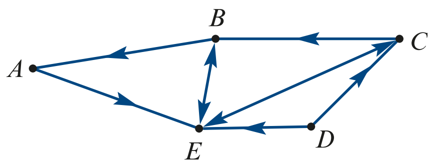
e How many of the two-step communication links shown in the matrix C^2 are redundant?

f Construct a matrix that shows the total number of one and two-step communication links between each pair of towers.

g Which of the four towers need a three-step link to communicate with each other?

Representing a large network diagram by a matrix

7 Construct a 5×5 matrix to represent the communication network diagram opposite.



Exam 1 style questions

8 The matrix below shows how five people, Adam(A), Bertie(B), Catherine(C), David(D) and Evan (E), can communicate with each other.

$$\text{Sender}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \text{Receiver} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

A '1' in the matrix shows that the person named in that row can send a message directly to the person named in that column. Adam wants to send a message to David. This can be done through a sequence of communications formed from the five people. Which of the following is a possible sequence of communications to get the message from Adam to David?

A A, D **B** A, B, D **C** A, B, C, D **D** A, B, E, D **E** A, E, C, B, D

9 Matrix P is a 3×3 permutation matrix. Matrix Z is another matrix such that the matrix product $P \times Z$ is defined. This matrix product results in the third row becoming the first row, the second row becoming the third and the first row becoming the second row of matrix Z. The permutation matrix P is

A $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

D $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

10G Dominance matrices

Learning intentions

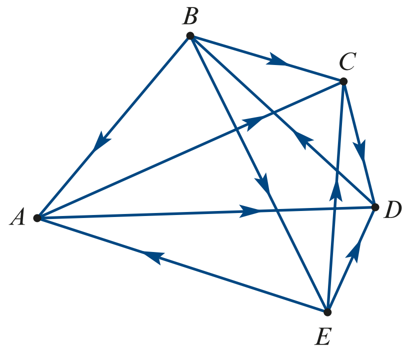
- To be able to construct and interpret dominance matrices.

In many group situations, certain individuals are said to be dominant. This is particularly true in sporting competitions. Problems of identifying dominant individuals in a group can be analysed using the same approach we used to analyse communication networks.

For example, five players – Anna, Birgit, Cas, Di and Emma – played in a **round-robin tournament**[3] of tennis to see who was the dominant (best) player.

The results were as follows:

- Anna defeated Cas and Di
- Birgit defeated Anna, Cas and Emma
- Cas defeated Di
- Di defeated Birgit
- Emma defeated Anna, Cas and Di.



We can use a network diagram to display the results graphically, as shown opposite. In this diagram, the arrow from *B to A* tells us that, when they played, Birgit defeated Anna.

Both Birgit and Emma had three wins each so there is a tie. How can we resolve this situation and see who is the best player? One way of doing this is to calculate a dominance score for each player. We do this by a constructing a series of dominance matrices.

One-step dominances

The first **dominance matrix**, D, records the number of one-step dominances between the players.

For example, Anna has a one-step dominance over Cas because, when they played, Anna *beat* Cas.

$$D = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \textit{Dominance} \\ 2 \\ 3 \\ 1 \\ 1 \\ 3 \end{array}$$

This matrix can be used to calculate a one-step dominance score for each player, by summing each of the rows of the matrix. According to this analysis, B and E are equally dominant with a dominance score of 3.

[3] A round-robin tournament is one in which each of the participants play each other once.

Now let us take into account two-step dominances between players.

Two-step dominances

A two-step dominance occurs when a player beats another player who has beaten someone else. For example, Birgit has a two-step dominance over Di because Birgit defeated Cas who defeated Di.

Two-step dominances can be determined using the same technique used to obtain two-step links in a communication network. We simply square the one-step dominance matrix.

$$D^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix} \end{array} \begin{array}{c} \textit{Dominance} \\ 2 \\ 6 \\ 1 \\ 3 \\ 4 \end{array}$$

The two-step dominances for these players are shown in matrix D^2. Reading across the 'B row'.

- The 1 in column A represents the two-step dominance 'Birgit defeated Emma who defeated Anna'.
- The 2 in column C represents the two-step dominances 'Birgit defeated Emma who defeated Cas' and 'Birgit defeated Anna who defeated Cas'
- The 3 in column D represents the three two-step dominances 'Birgit defeated Emma who defeated Di', 'Birgit defeated Anna who defeated Di' and 'Birgit defeated Cas who defeated Di'.
- In column E the 0 tells us that there are no two-step dominances for Birgit over Emma even though there was a one-step dominance.

We can combine the information contained in both D and D^2 by calculating a new matrix $T = D + D^2$.

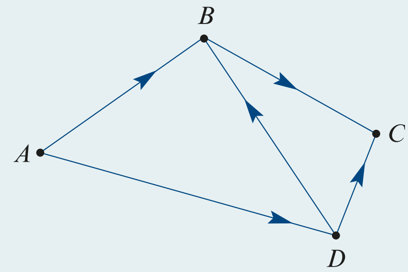
$$T = D + D^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 2 & 0 & 3 & 3 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 3 & 0 \end{bmatrix} \end{array} \begin{array}{c} \textit{Total} \\ 4 \\ 9 \\ 2 \\ 4 \\ 7 \end{array}$$

Using these total dominance scores:

- Birgit is the top-ranked player with a total dominance score of 9
- Eva is second with a total score of 7
- Anne and Di are equal third with a total score of 4
- Cas is the bottom-ranked player with a total score of 2.

Example 32 Determining dominance

Four people, A, B, C and D, have been asked to form a committee to decide on the location of a new toxic waste dump.



From previous experience, it is known that:

- A influences the decisions of B and D
- B influences the decisions of C
- C influences the decisions of no one
- D influences the decisions of C and B.

a Use the graph to construct a dominance matrix that takes into account both one-step and two-step dominances.

b From this matrix, determine who is the most influential person on the committee.

Explanation

a Construct the one-step dominance matrix D.

Solution

$$D = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \textit{One-step} \\ 2 \\ 1 \\ 0 \\ 2 \end{array}$$

Construct the two-step dominance matrix D^2.

$$D^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \textit{Two-step} \\ 3 \\ 0 \\ 0 \\ 1 \end{array}$$

Form the sum $T = D + D^2$.

$$T = D + D^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array} \begin{array}{c} \textit{Total} \\ 5 \\ 1 \\ 0 \\ 3 \end{array}$$

b The person with the highest total dominance score is the most influential.

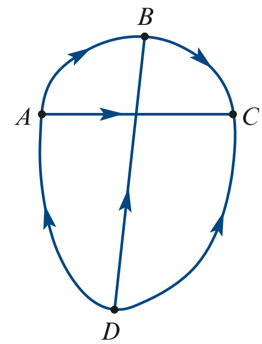
Person A is the most influential person with a total dominance score of 5.

Exercise 10G

Dominance matrices

Example 32

1 The results of a competition between teams A, B, C and D are displayed opposite. An arrow from D to C indicates that team D defeated team C.



a Construct a dominance matrix showing one-step dominance between the teams. Rank the teams according to one-step dominances.

b Construct a dominance matrix showing two-step dominances between the teams. Rank the teams, taking into account both one-step and two-step dominances.

2 Five students play each other at chess. The dominance matrix shows the winner of each game with a '1' and the loser or no match with a '0'. For example, row 2 indicates that B loses to A, D and E but beats C.

a Find the one-step dominance score for each student and use these to rank them.

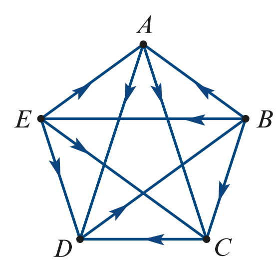
b Calculate the two-step dominance matrix.

c Determine the matrix $T = D + D^2$ and use this matrix to rank the players.

$$\textit{Losers}$$

$$\textit{Winners} \quad \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

3 Five friends – Ann, Bea, Cat, Deb and Eve – competed in a round-robin tennis tournament. The results were as follows:



- Ann defeated Cat and Deb
- Bea defeated Ann, Cat and Eve
- Cat defeated Deb
- Deb defeated Bea
- Eve defeated Ann, Cat and Deb.

Using this information:

a Construct a one-step dominance matrix, D.

b Construct a two-step dominance matrix, D^2.

c Use the dominance scores from the matrix $D + D^2$ to rank the five players.

4 The following dominance matrix, M, gives the results of a series of squash matches between five friends, where $m_{ij} = 1$ if player i beat player j.

$$M = \begin{array}{r} \\ \text{Ash} \\ \text{Ben} \\ \text{Carl} \\ \text{Dot} \\ \text{Elle} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Ash} & \text{Ben} & \text{Carl} & \text{Dot} & \text{Elle} \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

a How many matches were there?

b Describe the outcomes of the matches.

c Use the dominance scores from the matrix M to give a ranking of the players.

5 Five chess players – A, B, C, D and E – competed in a round-robin chess tournament. The results were as follows:

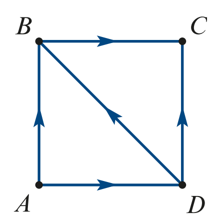
- A defeated B and D
- B defeated C and E
- C defeated A and D
- D defeated B
- E defeated A, C and D.

Using this information:

a Construct a one-step dominance matrix, M.

b Use the dominance scores from the matrix $M + M^2$ to rank the players.

6 A committee of four people – A, B, C and D – will decide on the location of a new toxic waste dump. From previous experience, it is known that:



- A influences the decisions of B and D
- B influences the decisions of C
- C influences the decisions of no one
- D influences the decisions of B and C.

Using this information:

a Construct a matrix that takes into account both one-step and two-step dominance.

b From this matrix, determine who is the most influential person on the committee.

7 The following table gives the results of the first round of games at a chess club.

Game	A vs B	C vs D	A vs D	B vs C	B vs D	A vs C
Winner	A	C	D	B	B	A

a Create a one-step dominance matrix, D.

b Create a ranking of the four players using D and D^2.

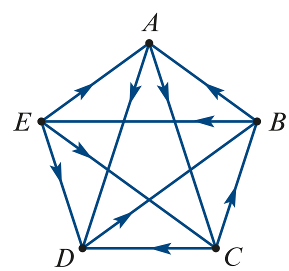
8 Four schools – A, B, C and D – compete in a round-robin hockey tournament. The results are summarised by the following dominance matrix:

$$M = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Create an ordering of the four schools using M, M^2 and M^3.

9 Five teams competed in a football tournament. Each team played every other team once. The results are shown in the diagram. (For example, the arrow pointing from A to C indicates that A defeated C.)

Create a rank order for the five teams by taking into consideration both one-step and two-step dominance.



Exam 1 style questions

10 Four teams, A, B, C and D, competed in a round-robin competition where each team played each of the other teams once. There were no draws. The results are shown in the matrix below.

$$\text{winner} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \textit{loser} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & x & y & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & z & 1 & 0 \end{bmatrix} \end{array}$$

The values of x, y and z are

A $x = 0, y = 0, z = 0$ **B** $x = 0, y = 1, z = q$ **C** $x = 1, y = 0, z = 0$
D $x = 1, y = 0, z = 1$ **E** $x = 1, y = 1, z = 1$

11 There are five hens in a coup. Their owner calls them Alpha, Beta, Gamma, Delta and Epsilon. There is a pecking order in the coop, and the following dominance matrix, M, was formed by the owner:

$$M = \begin{array}{c} \\ \text{Alpha} \\ \text{Beta} \\ \text{Gamma} \\ \text{Delta} \\ \text{Epsilon} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Alpha} & \text{Beta} & \text{Gamma} & \text{Delta} & \text{Epsilon} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Based on the matrix $M + M^2$, which of the following best describes the pecking order in the coop?

A Beta, Epsilon, Delta, Alpha, Gamma

B Beta, Epsilon, Gamma, Delta, Alpha

C Beta, Epsilon, Delta, Gamma, Alpha

D Epsilon, Beta, Delta, Alpha, Gamma

E Epsilon, Beta, Delta, Gamma, Alpha

12 Four soccer teams, X, Y, Z and W, compete in a round-robin competition. In each game, there is a winner and a loser. To decide the winner of the tournament, the sum of the one-step dominance matrix, D, and the two-step dominance matrix, D^2, is found. This sum is

$$D + D^2 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 \end{bmatrix}$$

Which one of the following is the correct one-step dominance for this tournament?

A $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

B $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

C $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

D $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

E $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

Review

Key ideas and chapter summary

Assign-ment

Matrix	A **matrix** is a rectangular array of numbers or symbols (elements) enclosed in brackets (plural: matrices).
Row matrix	A **row matrix** contains a *single row* of elements.
Column matrix	A **column matrix** contains a *single column* of elements.
Transpose	The **transpose** of a matrix is obtained by interchanging its rows and columns.
Square matrix	A **square matrix** has an **equal number of rows and columns**.
Zero matrix	A **zero (null) matrix**, O, contains only zeros.
Order	The **order** (or size) of a matrix is given by the number of rows and columns in that order.
Locating an element	The location of each element in the matrix is specified by its row and column number in that order.
Equal matrices	Matrices are *equal* when they have the *same order* and *corresponding elements* are *equal* in value.
Adding and subtracting matrices	Two matrices of the same order can be added or subtracted, by adding or subtracting corresponding elements.
Scalar multiplication	Multiplying a matrix by a number (**scalar multiplication**) multiplies every element in the matrix by that number.
Matrix multiplication	**Matrix multiplication** is a process of multiplying rows by columns. To multiply a row matrix by a column matrix, each element in the row matrix is multiplied by each element in the column matrix and the results added. For example: $\begin{bmatrix}1 & 0 & 3\end{bmatrix}\begin{bmatrix}4\\2\\5\end{bmatrix} = [1\times 4 + 0\times 2 + 3\times 5] = [19]$
Power of a matrix	The **power of a matrix** is defined in the same way as the powers of numbers: $A^2 = A\times A$, $A^3 = A\times A\times A$, and so on. Only **square matrices** can be raised to a power. A^0 is defined to be I, the **identity matrix**.

Identity matrix An **identity matrix**, I, is a square matrix with 1s down the leading diagonal and zeros elsewhere.

Determinant The **determinant** of a *matrix*, A, is written as det(A).
Only **square matrices** have determinants.
If $A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$ then $\det(A) = \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 3 \times 3 = -5$
For higher order matrices, a calculator is used to calculate the determinant.

Inverse The **inverse** of a matrix, A, is written as A^{-1} and has the property that $AA^{-1} = A^{-1}A = I$.
Only **square matrices** have inverses.
The *inverse* of a matrix is *not defined* if det(A) = 0.
A calculator is used to determine the inverse of a matrix.

Binary matrix A **binary matrix** is a matrix whose elements are either zeros or ones.

Permutation matrix A **permutation matrix** is a square binary matrix in which there is only a single 1 in each row and column.

Communication matrix A **communication matrix** is a square binary matrix in which the 1s represent direct (one-step) communication links.

Redundant communication link A communication link is said to be redundant if the sender and the receiver are the same people.

Round-robin tournament A **round-robin tournament** is one in which each of the participants plays every other competitor once.

Dominance matrix A **dominance matrix** is a square binary matrix in which the 1s represent one-step dominances between the members of a group.

Review

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

10A **1** I can state the order of a given matrix. ☐

See Example 1, and Exercise 10A Question 1

10A **2** I can find the transpose of a matrix. ☐

See Example 2, and Exercise 10A Question 4

10A **3** I can identify a matrix as being a square, column or row matrix. ☐

See Example 3, and Exercise 10A Question 5

10A **4** I can identify a square matrix as being an identity or diagonal or symmetric or upper/lower triangular matrix. ☐

See Example 4, and Exercise 10A Question 6

10A **5** I can identify the a_{ij} term in a matrix A. ☐

See Example 5, and Exercise 10A Question 7

10A **6** I can construct a matrix given a rule for a_{ij}. ☐

See Example 6, and Exercise 10A Question 8

10A **7** I can enter a matrix into a CAS calculator. ☐

See CAS 1, and Exercise 10A Question 13

10B **8** I can enter information from a table into a matrix. ☐

See Example 7, and Exercise 10B Question 1

10B **9** I can represent a network diagram by a matrix. ☐

See Example 9, and Exercise 10B Question 4

10B **10** I can interpret a matrix representing a network diagram. ☐

See Example 10, and Exercise 10B Question 6

10C **11** I can recognise when two matrices are equal and use this to solve problems. ☐

See Example 11, and Exercise 10C Question 1

10C **12** I can add two matrices of the same order together. ☐

See Example 12, and Exercise 10C Question 2

10C **13** I can subtract one matrix from another when they have the same order. ☐

See Example 13, and Exercise 10C Question 2

10C **14** I can multiply a matrix by a scalar. ☐

See Example 14, and Exercise 10C Question 2

10C **15** I recognise the role of the zero matrix and can undertake operations using the zero matrix. ☐

See Example 15, and Exercise 10C Question 3

10C **16** I can use addition, subtraction and scalar multiplication to process data. ☐

See Example 16, and Exercise 10C Question 5

10D **17** I can determine if the product of two given matrices is defined. ☐

See Example 17, and Exercise 10D Question 1

10D **18** II can determine the order of a matrix product. ☐

See Example 18, and Exercise 10D Question 1

10D **19** I can multiply a row matrix by a column matrix by hand. ☐

See Example 19, and Exercise 10D Question 2

10D **20** I can multiply a rectangular matrix by a column matrix by hand. ☐

See Example 20, and Exercise 10D Question 2

10D **21** I can use summing matrices to sum the rows or columns of a matrix. ☐

See Example 21, and Exercise 10D Question 4

10D **22** I can undertake multiplications of matrices to solve practical problems. ☐

See Example 22, and Exercise 10D Question 8

10D **23** I can evaluate matrix expressions involving powers. ☐

See Example 23, and Exercise 10D Question 11

10E **24** I can recognise that two matrices are inverses if their product is the identity matrix. ☐

See Example 24, and Exercise 10E Question 2

10E **25** I can evaluate the determinant of a 2×2 matrix by hand. ☐

See Example 25, and Exercise 10E Question 3

10E **26** I can calculate the inverse of a 2×2 matrix by hand. ☐

See Example 26, and Exercise 10E Question 4

Review

10E **27** I can calculate the inverse and determinant of a $n \times n$ matrix using CAS. ☐

See CAS 4, and Exercise 10E Question 4

10E **28** I can solve simple matrix equations. ☐

See Example 27, and Exercise 10E Question 5

10F **29** I can use a permutation matrix to rearrange the elements of a column or row matrix. ☐

See Example 28, and Exercise 10F Question 2

10F **30** I can construct a permutation matrix to rearrange the elements of a column or row matrix in a given order. ☐

See Example 29, and Exercise 10F Question 3

10F **31** I can find the inverse of a permutation matrix. ☐

See Example 30, and Exercise 10F Question 4

10F **32** I can construct a communication matrix from information given in written form or a diagram. ☐

See Example 31, and Exercise 10F Question 5

10F **33** I can construct a dominance matrix from information given in written form or a diagram. ☐

See Example 30, and Exercise 10F Question 1

Multiple choice questions

The following matrices are needed for Questions 1 to 8.

$$U = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \quad V = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad W = \begin{bmatrix} 1 & -1 \end{bmatrix} \quad X = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad Z = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

1 The row matrix is:

A U **B** V **C** W **D** X **E** Z

2 The square matrices are:

A U and V **B** X and Y **C** Y and W **D** U and Y **E** U, V and X

3 The order of matrix X is:

A 2×2 **B** 2×3 **C** 3×2 **D** 3×3 **E** 6

4 The following matrices can be added:

A U and V **B** V and W **C** X and Y **D** U and Y **E** none of the above

5 The following matrix product is *not* defined:

A WV **B** XZ **C** YV **D** XY **E** UY

6 $-2Y =$

A $\begin{bmatrix} 0 & -2 \\ 2 & -4 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 2 \\ -2 & 4 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & -2 \\ -2 & 4 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} -2 & -1 \\ -4 & 2 \end{bmatrix}$

7 The order of matrix product XZ is:

A 1×3 **B** 2×1 **C** 3×1 **D** 3×2 **E** 3×3

8 $U^T =$

A $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

9 In the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 3 \\ -5 & -4 & 7 \end{bmatrix}$, the element $a_{23} =$

A -4 **B** -1 **C** 0 **D** 3 **E** 4

10 $2\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} =$

A $\begin{bmatrix} 5 & 0 \\ -4 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 5 & 0 \\ 4 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 3 & 0 \\ -2 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 6 & 0 \\ 1 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 5 & 0 \\ -3 & 1 \end{bmatrix}$

11 $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} =$

A $\begin{bmatrix} 10 \end{bmatrix}$ **B** $\begin{bmatrix} 12 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}$ **E** not defined

The following matrices are needed for Questions 12 to 16.

$$U = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \quad V = \begin{bmatrix} 43 \\ 45 \end{bmatrix} \quad W = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \quad X = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad Y = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

12 The matrix that cannot be raised to a power is:

A U **B** V **C** W **D** X **E** Y

13 $\det(U) =$

A -2 **B** 0 **C** 1 **D** 2 **E** 4

14 $Y^{-1} =$

A $\begin{bmatrix} 0.5 & 0 \\ -0.5 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **D** $\frac{1}{8}\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ **E** not defined

15 $U^{-1} =$

A $\begin{bmatrix} 0.5 & 0 \\ -0.5 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **D** $\frac{1}{8}\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ **E** not defined

16 The matrix product that is defined is:

A UX **B** XY **C** VW **D** UW **E** WX

17 $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} =$

A $\begin{bmatrix} 18 \end{bmatrix}$ **B** $\begin{bmatrix} 12 \end{bmatrix}$ **C** $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ **D** $\begin{bmatrix} 4 \\ 10 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & 2 \\ 6 & 4 \end{bmatrix}$

18 X is a 3×2 matrix. Y is a 2×3 matrix. Z is a 2×2 matrix. Which of the following matrix expressions is *not* defined?

A XY **B** YX **C** $XZ - 2X$ **D** $YX + 2Z$ **E** $XY - YX$

19 $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 4 \end{bmatrix}$.

The matrix expression that displays the mean of the numbers 3, 5, 2, 4 is:

A $\frac{1}{4}(A + B)$ **B** $\frac{1}{2}(A + B)$ **C** $\frac{1}{4}B$ **D** $\frac{1}{4}AB$ **E** $\frac{1}{4}BA$

20 Consider the following four matrix expression.

$$\begin{bmatrix} 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 \\ 5 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 6 & 7 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 6 & 7 \end{bmatrix}$$

How many of these four matrix expressions are defined?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

21 If both A and B are $m \times n$ matrices, where $m \neq n$, then $A + B$ is

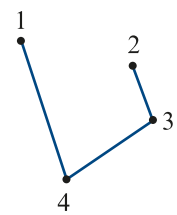
A an $m \times n$ matrix **B** an $m \times m$ matrix **C** an $n \times n$ matrix
D a $2m \times 2n$ matrix **E** not defined

22 The matrix expression $\begin{bmatrix} 4 & 6 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} -6 & -12 \\ 4 & -1 \end{bmatrix}$ is equal to

A $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 0 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
D $\begin{bmatrix} 13 & 24 \\ 0 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 24 \\ 0 & 3 \end{bmatrix}$

23 The diagram opposite is to be represented by a matrix, A, where:

- element = 1 if the two points are joined by a line
- element = 0 if the two points are not connected.



The matrix A is:

A $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ **B** $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

24 If matrix $M = \begin{bmatrix} 3 & 0 \\ 4 & 1 \\ 7 & 2 \\ 9 & 6 \end{bmatrix}$ then the transpose matrix $M^T =$

A $\begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 7 \\ 6 & 9 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 6 & 9 \\ 2 & 7 \end{bmatrix}$ **C** $\begin{bmatrix} 3 & 4 & 7 & 9 \\ 0 & 1 & 2 & 6 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 1 & 2 & 6 \\ 3 & 4 & 7 & 9 \end{bmatrix}$ **E** $\begin{bmatrix} 3 & 4 & 0 & 1 \\ 2 & 6 & 7 & 9 \end{bmatrix}$

25 M is a 6×6 matrix. N is a 5×6 matrix. Which one of the following matrix expressions is defined?

A $NM - 2N$ **B** $M(MN)^{-1}$ **C** M^2N **D** N^TM **E** M^TN

Review

26 Matrix A_1 is the 4×1 column matrix $\begin{bmatrix} C \\ B \\ A \\ D \end{bmatrix}$

A second 4×1 column matrix, A_2, contains the same elements as A_1, but the elements are ordered from top to bottom in alphabetical order. Matrix $A_2 = P \times A_1$, where P is a permutation matrix. Matrix P is

A $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

27 Four teams, X, Y, Z and W, compete in a round-robin competition. In each game, there is a winner and a loser. The sum of the one-step dominance matrix, D, and the two-step dominance matrix, D^2, is found. This sum is

$$D + D^2 = \begin{array}{c} \\ X \\ Y \\ Z \\ W \end{array} \begin{array}{c} \begin{array}{cccc} X & Y & Z & W \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & 2 & 0 \end{bmatrix} \end{array}$$

In the first two games:

- team W defeated team X
- team Z defeated team W.

Which one of the following is the correct one-step dominance for this tournament?

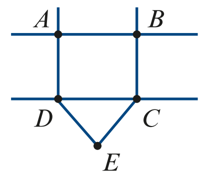
A $\begin{array}{c} \\ X \\ Y \\ Z \\ W \end{array} \begin{array}{c} \begin{array}{cccc} X & Y & Z & W \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$ **B** $\begin{array}{c} \\ X \\ Y \\ Z \\ W \end{array} \begin{array}{c} \begin{array}{cccc} X & Y & Z & W \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \end{array}$ **C** $\begin{array}{c} \\ X \\ Y \\ Z \\ W \end{array} \begin{array}{c} \begin{array}{cccc} X & Y & Z & W \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$

D $\begin{array}{c} \\ X \\ Y \\ Z \\ W \end{array} \begin{array}{c} \begin{array}{cccc} X & Y & Z & W \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$ **E** $\begin{array}{c} \\ X \\ Y \\ Z \\ W \end{array} \begin{array}{c} \begin{array}{cccc} X & Y & Z & W \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$

28 The matrix $\begin{bmatrix} 1 & 5 & 3 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is an example of a

A symmetric matrix. **B** unit matrix. **C** triangular matrix.
D diagonal matrix. **E** communication matrix.

29 A, B, C, D and E are five intersections joined by roads, as shown in the diagram opposite. Some of these roads are one-way only.



The matrix opposite indicates the direction that cars can travel along each of these roads.

In this matrix:

- the 1 in column A and row B indicates that cars can travel directly from A to B
- the 0 in column B and row A indicates that cars cannot travel directly from B to A (either it is a one-way road or no road exists).

$$\begin{array}{c} \textit{From intersection} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \\ A \\ B \\ C \ \ \textit{To intersection} \\ D \\ E \end{array}$$

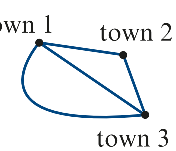
Cars can travel in both directions between intersections:

A A and D **B** B and C **C** C and D **D** D and E **E** C and E

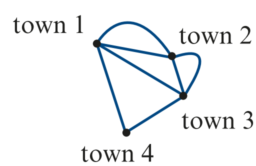
Written response questions

1 The following diagrams represent the road network joining several towns. Represent each road by a matrix.

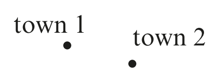
a



b



c



2 Heights in feet and inches can be converted into centimetres using matrix multiplication. The matrix $C = \begin{bmatrix} 30.45 \\ 2.54 \end{bmatrix}$ can be used as a conversion matrix (1 foot = 30.45 cm and 1 inch equals 2.54 cm).

a What is the order of matrix C?

Jodie tells us that her height is 5 feet 4 inches. We can write her height as a matrix $J = \begin{bmatrix} 5 & 4 \end{bmatrix}$.

b What is the order of matrix J?

Review

c Is the matrix product JC defined? Why?

d Evaluate the matrix product JC. Explain why it gives Jodie's height in centimetres.

e Matrix $H = \begin{bmatrix} 5 & 8 \\ 6 & 1 \end{bmatrix}$ gives the heights in feet and inches of two other people.

Use the conversion matrix C and matrix multiplication to generate a matrix that displays the heights of these two people in centimetres.

3 Books can be classified as *fiction* or *non-fiction* and come in either *hardback* or *paperback* form. The table shows the number of book titles carried by two bookshops in each of the categories.

	Bookshop 1		*Bookshop 2*	
Number of titles	*Hardback*	*Paperback*	*Hardback*	*Paperback*
Fiction	334	876	354	987
Non-fiction	213	456	314	586

a How many non-fiction paperback titles does bookshop 1 carry?

b The matrix $A = \begin{bmatrix} 334 & 876 \\ 213 & 456 \end{bmatrix}$ displays the number of book titles available at bookshop 1 in all categories. What is the order of this matrix?

c Write down a matrix equivalent to matrix A that displays the number of book titles available at bookshop 2. Call this matrix B.

d Construct a new matrix, $C = A + B$. What does this matrix represent?

e The average cost of books is \$45 for a hardback and \$18.50 for a paperback. These values are summarised in the matrix $E = \begin{bmatrix} 45.00 \\ 18.50 \end{bmatrix}$.

i What is the order of matrix E?

ii Construct the matrix product AE and evaluate.

iii What does the product AE represent?

f Bookshop 1 plans to double the number of titles it carries in every category. Write down a matrix expression that represents this situation and evaluate.

4 Mathematics and Physics are offered in a first year university science course.

The matrix $N = \begin{array}{c} \begin{bmatrix} 600 \\ 320 \end{bmatrix} \end{array}\begin{array}{l} \textit{Mathematics} \\ \textit{Physics} \end{array}$ lists the number of students enrolled in each subject.

The matrix $P = \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0.15 & 0.225 & 0.275 & 0.25 & 0.10 \end{bmatrix} \end{array}$ lists the proportion of these students expected to be awarded an A, B, C, D or E grade in each subject.

a Write down the order of matrix P.

b Let the matrix $R = NP$.

i Evaluate the matrix R.

ii Explain what the matrix element R_{13} represents.

c Students enrolled in Mathematics have to pay an extra fee of \$220, while students enrolled in Physics pay an extra fee of \$197.

i Write down a clearly labelled row matrix, called F, that lists these fees.

ii Show a matrix calculation that will give the total fees fees, L, paid in dollars by the students enrolled in Mathematics and Physics. Find this amount.

5 In a simplified game of darts, the possible scores are 25, 50 or 75.
G is a column matrix that lists the possible scores
In one game of 15 throws, Daniel achieved

$$G = \begin{bmatrix} 25 \\ 50 \\ 75 \end{bmatrix}$$

- Eight '25 scores'
- Six '50 scores'
- One '75 score'

a Write a row matrix, N, that shows the number of each score that Daniel had.

b Matrix P is found by multiplying matrix N with matrix G so that $P = N \times G$. Evaluate matrix P.

c In this context, what does the information in matrix P provide?

6 A mining company operates three mines, A, B and C. Each of the mines produces three types of minerals, p, q and r. Consider the following two matrices:

$$X = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} p & q & r \end{array} \\ \begin{bmatrix} 20 & 20 & 40 \\ 0 & 40 & 20 \\ 60 & 40 & 60 \end{bmatrix} \end{array} \quad \text{and} \quad Y = \begin{array}{c} A \\ B \\ C \end{array} \begin{bmatrix} 46\,000 \\ 34\,000 \\ 106\,000 \end{bmatrix}$$

Matrix X gives the number of tonnes of each of the minerals extracted per day from each of the mines, and matrix Y gives the total revenue (in dollars) from selling the minerals extracted from each of the mines on one day.

a Calculate the total number of tonnes of minerals produced by mine A.

b Calculate the total number of tonnes of mineral q produced.

c Calculate the total revenue of the three mines.

d In the matrix equation $XA = Y$

i What is the order of matrix A?

ii What do the elements of matrix A represent?

iii We know that $A = X^{-1}Y$. Find A.

Chapter 11

Transition matrices and Leslie matrices

Chapter objectives

- ▶ How do you construct a transition matrix from a transition diagram and vice versa?
- ▶ How do you construct a transition matrix to model the transitions in a population?
- ▶ How do you use a matrix recurrence relation, S_0 = initial state matrix, $S_{n+1} = TS_n$, to generate a sequence of state matrices?
- ▶ How do you informally identify the equilibrium state or steady-state matrix in the case of regular state matrices?
- ▶ How do you use a matrix recurrence relation S_0 = initial state matrix, $S_{n+1} = TS_n + B$ to model systems that include external additions or reductions at each step of the process?
- ▶ How do you use and interpret Leslie matrices to analyse population growth?

In this chapter we use matrices to model proportional change of the numbers in a particular state to itself and other states from one time to the next.

For example, in the first example of this chapter, there are two states which in this case are the towns in which rental cars finish up each day.

In sections A-E of this chapter we look at transition matrices which satisfy certain conditions and describe the proportional change.

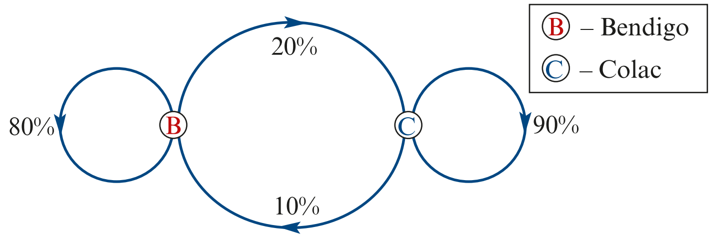
In section F of the chapter Leslie matrices are used to model population change in a particular way.

11A Transition matrices - setting up a transition matrix

Learning intentions

- To be able to set up a transition matrix from a diagram or written information.

A car rental firm has two branches: one in Bendigo and one in Colac. Cars are usually rented and returned in the same town. However, a small percentage of cars rented in Bendigo each week are returned in Colac, and vice versa. The diagram below describes what happens on a weekly basis.



What does this diagram tell us?

From week to week:

- 0.8 (or 80%) of cars rented each week in Bendigo are returned to Bendigo
- 0.2 (or 20%) of cars rented each week in Bendigo are returned to Colac
- 0.1 (or 10%) of cars rented each week in Colac are returned to Bendigo
- 0.9 (or 90%) of cars rented each week in Colac are returned to Colac.

The percentages (written as proportions) are summarised in the form of the matrix below.

$$\text{Returned to} \quad \begin{array}{c} \\ \text{Bendigo} \\ \text{Colac} \end{array} \begin{array}{c} \text{Rented in} \\ \begin{array}{cc} \text{Bendigo} & \text{Colac} \end{array} \\ \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \end{array}$$

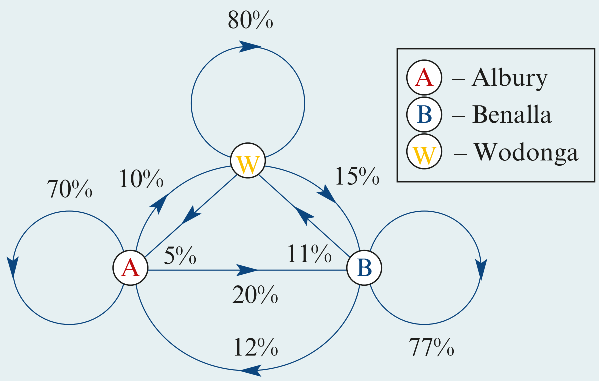
This matrix is an example of a **transition matrix (T)**. It describes the way in which transitions are made between two *states*:

- state 1: the rental car is based in Bendigo.
- state 2: the rental car is based in Colac.

Note: In this situation, where the total number of cars remains constant, the columns in a transitional matrix will always add to one (100%). For example, if 80% of cars are returned to Bendigo, then 20% must be returned to Colac.

Example 1 Setting up a transition matrix

The diagram gives the weekly return rates of rental cars at three locations: Albury, Wodonga and Benalla. Construct a transition matrix that describes the week-by-week return rates at each of the three locations. Convert the percentages to proportions.



Explanation

1 There are three locations from which the cars can be rented and returned: Albury (A), Wodonga (W) and Benalla (B). To account for all the possibilities, a 3×3 matrix is needed. Construct a blank matrix labelling the rows and columns A, W and B, respectively. Column labels indicate where the car was rented. The row labels indicate where the cars were returned to.

2 Complete the matrix by writing each of the percentages (converted to proportions) into the appropriate locations. Start with column A and write in values for each row: 0.7 (70%), 0.1 (10%) and 0.2 (20%).

3 Mentally check your answer by summing columns; they should sum to 1.

Solution

Rented in

$$\begin{array}{rc} & \begin{array}{ccc} \text{A} & \text{W} & \text{B} \end{array} \\ \text{Returned to} \begin{array}{c} \text{A} \\ \text{w} \\ \text{B} \end{array} & \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \end{array}$$

$$\begin{array}{rc} & \begin{array}{ccc} \text{A} & \text{W} & \text{B} \end{array} \\ \begin{array}{c} \text{A} \\ \text{w} \\ \text{B} \end{array} & \begin{bmatrix} 0.7 & \quad & \quad \\ 0.1 & & \\ 0.2 & & \end{bmatrix} \end{array}$$

$$\begin{array}{rc} & \begin{array}{ccc} \text{A} & \text{W} & \text{B} \end{array} \\ \begin{array}{c} \text{A} \\ \text{w} \\ \text{B} \end{array} & \begin{bmatrix} 0.7 & 0.05 & 0.12 \\ 0.1 & 0.8 & 0.11 \\ 0.2 & 0.15 & 0.77 \end{bmatrix} \end{array}$$

Example 2 Setting up a transition matrix

A factory has a large number of machines. Machines can be in one of two states: operating or broken. Broken machines are repaired and come back into operation, and vice versa. On a given day:

- 85% of machines that are operational stay operating
- 15% of machines that are operating break down

- 5% of machines that are broken are repaired and start operating again
- 95% of machines that are broken stay broken.

Construct a transition matrix to describe this situation. Use the columns to define the situation at the 'Start' of the day and the rows to describe the situation at the 'End' of the day.

Explanation

1 There are two machine states: operating (O) or broken (B). To account for all the possibilities, a 2×2 transition matrix is needed. Construct a blank matrix, labelling the rows and columns O and B, respectively.

Solution

$$\begin{array}{ccc} & & \text{Start} \\ & & \begin{array}{cc} O & \quad B \end{array} \\ \text{End} & \begin{array}{c} \text{O} \\ \text{B} \end{array} & \begin{bmatrix} & \\ & \end{bmatrix} \end{array}$$

2 Complete the matrix by writing each of the percentages (converted to proportions) into the appropriate locations. Start with column O and write in the values for each row: 0.85 (85%) and 0.15 (15%).

$$\begin{array}{cc} & \begin{array}{cc} O & \quad B \end{array} \\ \begin{array}{c} \text{O} \\ \text{B} \end{array} & \begin{bmatrix} 0.85 & \\ 0.15 & \end{bmatrix} \end{array}$$

3 Mentally check your answer by summing the columns; they should sum to 1.

$$\begin{array}{cc} & \begin{array}{cc} O & \quad B \end{array} \\ \begin{array}{c} \text{O} \\ \text{B} \end{array} & \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \end{array}$$

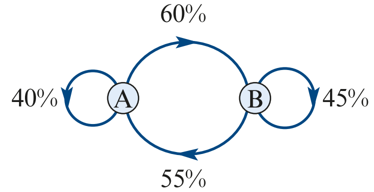
Exercise 11A

Setting up a transition matrix from a transition diagram

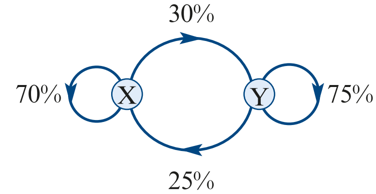
Example 1

1 The diagrams below describe a series of transitions between the states indicated. Construct a transition matrix that can be used to represent each of these diagrams. Use columns to define the starting points. Convert the percentages to proportions.

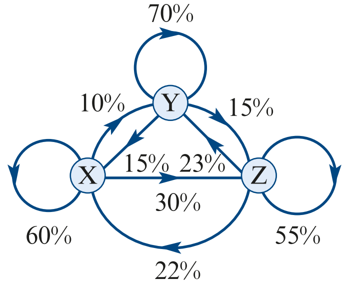
a



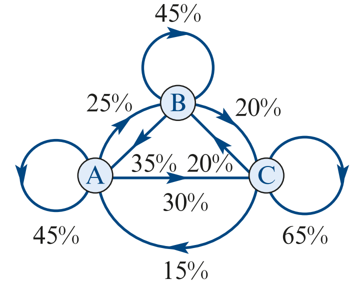
b



c



d



Example 2

2 A factory has a large number of machines which can be in one of two states, *operating* (O) or *broken down* (B). It is known that that an operating machine breaks down by the end of the day on 4% of the days, and that 98% of machines which have broken down are repaired by the end of the day.

Complete the 2×2 transition matrix T to describe this.

$$\begin{array}{ccc} & & \text{Today} \\ & & \begin{array}{cc} O & \quad B \end{array} \\ T = & \begin{array}{c} O \\ B \end{array} & \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \end{array} \text{Tomorrow}$$

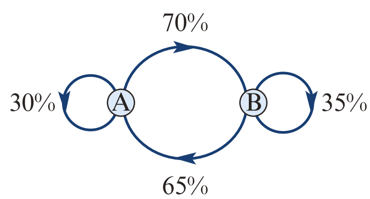
3 A large company has 1640 employees, 60% of whom currently work full-time (F) and 40% of whom currently work part-time (P). Every year 20% of full-time workers move to part-time work, and 14% of part-time workers move to full-time work.

Complete the 2×2 transition matrix T to describe this.

$$\begin{array}{ccc} & & \text{This year} \\ & & \begin{array}{cc} F & \quad P \end{array} \\ T = & \begin{array}{c} F \\ P \end{array} & \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \end{array} \text{Next year}$$

Exam 1 style questions

4 In a particular newsagent, the two top-selling newspapers are the Argus and the Bastion. The transition diagram below shows the way shoppers at this newsagent change their newspaper choice from today to tomorrow.



A transition matrix that provides the same information as the transition diagram is

A
$$\begin{array}{ccc} & & \text{Today} \\ & & \begin{array}{cc} A & \quad B \end{array} \\ \text{Tomorrow} & \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 65\% & 30\% \\ 70\% & 35\% \end{bmatrix} \end{array}$$

B
$$\begin{array}{ccc} & & \text{Today} \\ & & \begin{array}{cc} A & \quad B \end{array} \\ \text{Tomorrow} & \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 30\% & 65\% \\ 70\% & 35\% \end{bmatrix} \end{array}$$

C
$$\begin{array}{ccc} & & \text{Today} \\ & & \begin{array}{cc} A & \quad B \end{array} \\ \text{Tomorrow} & \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 30\% & 65\% \\ 35\% & 70\% \end{bmatrix} \end{array}$$

D
$$\begin{array}{ccc} & & \text{Today} \\ & & \begin{array}{cc} A & \quad B \end{array} \\ \text{Tomorrow} & \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 30\% & 35\% \\ 65\% & 70\% \end{bmatrix} \end{array}$$

E
$$\begin{array}{ccc} & & \text{Today} \\ & & \begin{array}{cc} A & \quad B \end{array} \\ \text{Tomorrow} & \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 65\% & 70\% \\ 35\% & 30\% \end{bmatrix} \end{array}$$

11B Interpreting transition matrices

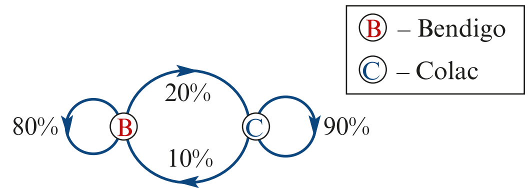
Learning intentions

- To be able to interpret a transition matrix and a transition diagram.

Let us return to the car rental problem at the start of this section. As we saw then, the following transition matrix, T, and its transition diagram can be used to describe the weekly pattern of rental car returns in Bendigo and Colac.

Rented in

$$T = \begin{array}{c c} \begin{matrix} B & \quad C \end{matrix} & \\ \begin{bmatrix} 0.80 & 0.10 \\ 0.20 & 0.90 \end{bmatrix} & \begin{matrix} B \\ C \end{matrix} \end{array} \text{ Returned to}$$



Using this information alone, a number of predictions can be made.

For example, if 50 cars are rented in Bendigo this week, the transition matrix predicts that:

- 80% or 40 of these cars will be returned to Bendigo next week ($0.80 \times 50 = 40$)
- 20% or 10 of these cars will be returned to Colac next week ($0.20 \times 50 = 10$).

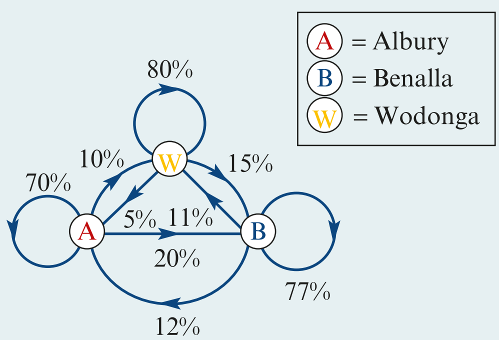
Further, if 40 cars are rented in Colac this week, the transition matrix predicts that:

- 10% or 4 of these cars will be returned to Bendigo next week ($0.10 \times 40 = 4$)
- 90% or 36 of these cars will be returned to Colac next week ($0.90 \times 40 = 36$).

Example 3 **Interpreting a transition matrix**

The following transition matrix, T, and its transition diagram can be used to describe the weekly pattern of rental car returns in three locations: Albury, Wodonga and Benalla.

$$T = \begin{array}{c c} \begin{matrix} A & \quad W & \quad B \end{matrix} & \\ \begin{bmatrix} 0.7 & 0.05 & 0.12 \\ 0.1 & 0.8 & 0.11 \\ 0.2 & 0.15 & 0.77 \end{bmatrix} & \begin{matrix} A \\ W \\ B \end{matrix} \end{array} \text{ Returned to}$$



Use the transition matrix T and its transition diagram to answer the following questions.

a What percentage of cars rented in Wodonga each week are predicted to be returned to:

i Albury? **ii** Benalla? **iii** Wodonga?

b Two hundred cars were rented in Albury this week. How many of these cars do we expect to be returned to:

i Albury? **ii** Benalla? **iii** Wodonga?

c What percentage of cars rented in Benalla each week are *not* expected to be returned to Benalla?

d One hundred and sixty cars were rented in Albury this week. How many of these cars are expected to be returned to either Benalla or Wodonga?

Solution

a **i** 0.5 or 5% **ii** 0.15 or 15% **iii** 0.80 or 80%

b **i** $0.70 \times 200 = 140$ cars **ii** $0.20 \times 200 = 40$ cars **iii** $0.10 \times 200 = 20$ cars

c $11 + 12 = 23\%$ or $100 - 77 = 23\%$

d 20% of 160 + 10% of 160 = 48 cars

Exercise 11B

Interpreting transition matrices

Example 3

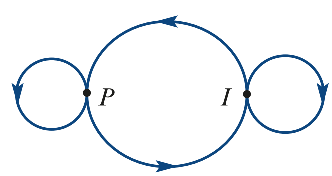
1 When people go to the movies they buy either a bag of popcorn (P) or an ice cream (I). Experience has shown that:

- 85% of people who buy popcorn this time will buy popcorn next time
- 15% of people who buy popcorn this time will buy an ice cream next time
- 75% of people who buy an ice cream this time will buy an ice cream next time
- 25% of people who buy ice cream this time will buy popcorn next time.

a Construct a transition matrix and transition diagram that can be used to describe this situation. Use the models below.

This time

$$T = \begin{array}{c} \begin{array}{cc} P & I \end{array} \\ \left[\begin{array}{cc} \quad & \quad \\ \quad & \quad \end{array} \right] \end{array} \begin{array}{l} P \\ I \end{array} \text{ Next time}$$



b Eighty people are seen buying popcorn at the movies. How many of these are expected to buy popcorn next time they go to the movies?

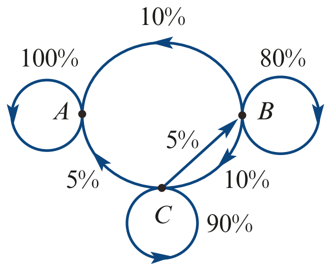
c Sixty people are seen buying an ice cream at the movies. How many of these are expected to buy popcorn next time they go to the movies?

d On another occasion, 120 people are seen buying popcorn and 40 are seen buying an ice cream. How many of these are expected to buy an ice cream next time they attend the movies?

2 On Windy Island, sea birds are observed nesting at three sites: A, B and C. The following transition matrix and accompanying transition diagram can be used to predict the movement of these sea birds between these sites from year to year.

This year

$$T = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1.0 & 0.10 & 0.05 \\ 0 & 0.80 & 0.05 \\ 0 & 0.10 & 0.90 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \end{array} \text{ Next year}$$



a What percentage of sea birds nesting at site B this year were expected to nest at:

i site A next year? **ii** site B next year? **iii** site C next year?

b This year, 850 sea birds were observed nesting at site B. How many of these are expected to:

i still nest at site B next year? **ii** move to site A to nest next year?

c This year, 1150 sea birds were observed nesting at site A. How many of these birds are expected to nest at:

i site A next year? **ii** site B next year? **iii** site C next year?

d What does the '1' in column A, row A of the transition matrix indicate?

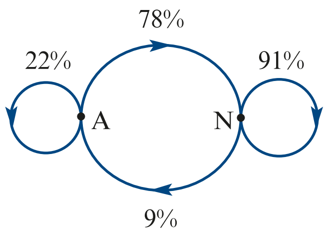
3 A car insurance company finds that:

- 22% of car drivers involved in an accident this year (A) are also expected to be involved in an accident next year
- 9% of drivers who are *not* involved in an accident this year (N) are expected to be involved in an accident next year.

The transition diagram that can be used to describe this situation is shown below.

This year

$$T = \begin{array}{c} \begin{array}{cc} A & N \end{array} \\ \begin{bmatrix} 0.22 & 0.09 \\ 0.78 & 0.91 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ N \end{array} \text{ Next year}$$



a In 2015, 84 000 drivers insured with the company were *not* involved in an accident.

i How many of these drivers were *not* expected to be involved in an accident in 2016?

ii How many of these drivers were expected to be involved in an accident in 2016?

b In 2015, 25 000 drivers insured with the company were involved in an accident.

i How many of these drivers were expected to be involved in an accident in 2016?

ii How many of these drivers were expected to be involved in an accident in 2017?

iii How many of these drivers were expected to be involved in an accident in 2018?

4 Fleas can move beteen three locations A, B and C. The way a flea moves after 5 seconds in a location can be exactly described by the transition matrix.

$$\begin{array}{cc} & \text{Now} \\ & \begin{array}{ccc} A & B & C \end{array} \\ T = & \begin{bmatrix} 0.60 & 0.10 & 0.70 \\ 0.20 & 0.80 & 0.10 \\ 0.20 & 0.10 & 0.20 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \end{array} \text{ After 5 seconds}$$

The move is not dependent on any previous move.

a If there are 30 fleas at location A at the beginning of the 5-second period, how many fleas would you expect to

i stay at A **ii** go to B **iii** go to C

at the end of the 5-second period?

b If there were 60 fleas at each of the locations how many fleas would you expect to have at

i A **ii** B **iii** C

after one 5-second period?

c Find the product $T\begin{bmatrix} 60 \\ 60 \\ 60 \end{bmatrix}$ and comment.

d At the conclusion of the first 5-second period there are 30 fleas at C.

i How many of these go to A in the next 5-second period?

ii How many of these go to B in the next 5-second period?

iii How many of these go to C in the next 5-second period?

e Evaluate the product $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} T \begin{bmatrix} 60 \\ 60 \\ 60 \end{bmatrix}$ and comment.

Exam 1 style questions

5 There are 120 students in a Year 12 class. Each week every student participates in one of three activities: Sport (S), Outdoor Activities (O) or First Aid (F).

$$\begin{array}{cc} & \text{This week} \\ & \begin{array}{ccc} S & O & F \end{array} \\ T = & \begin{bmatrix} 0.6 & 0.2 & 0.4 \\ 0.1 & 0.7 & 0.1 \\ 0.3 & 0.1 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} S \\ O \\ F \end{array} \text{ Next week}$$

The activities that the children select each week change according to the transition matrix opposite.

From the transition matrix it can be concluded that:

A in the first week of the program, eighty students do Sport, twenty students children do Outdoor activities and twenty Students do First Aid.

B at least 50% of the students do not change their activities from the first week to the second week.

C in the long term, all of the children will choose the same activity.

D Sport is the most popular activity in the first week

E 40% of the students will do First Aid each week.

6 Warren text messages a friend each week day of this week. His friends are Arthur (A), Belinda (B), Connie (C), Danielle (D) and Eleanor (E). On Monday, Warren will send a text message to Connie. Based on the transition matrix, the order in which Warren will text message each of his friends for the next four days is:

$$\begin{array}{c} \text{Today} \\ T = \begin{array}{cc} \begin{array}{ccccc} A & B & C & D & E \end{array} & \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \end{array} \text{Tomorrow} \end{array}$$

A Arthur, Belinda, Connie, Danielle

B Danielle, Belinda , Arthur, Connie

C Danielle, Belinda, Arthur, Eleanor

D Eleanor, Arthur, Danielle, Belinda

E Eleanor, Danielle, Belinda, Arthur

11C Transition matrices – using recursion

Learning intentions

- To be able to use a matrix recurrence relation: S_0 = initial state matrix, $S_{n+1} = TS_n$, to generate a sequence of state matrices.
- To be able to informally identify the equilibrium state or steady-state matrix in the case of regular state matrices.

We return again to the car rental problem. The car rental firm now plans to buy 90 new cars. Fifty will be based in Bendigo and 40 in Colac.

Given this pattern of rental car returns, the first question the manager would like answered is:

> ‘If we start with 50 cars in Bendigo, and 40 cars in Colac, how many cars will be available for rent at both towns after 1 week, 2 weeks, etc?’

You have met this type of problem earlier when doing financial modelling (Chapter 8). For example, if we invest \$1000 at an interest rate of 5% per annum, how much will we have after 1 year, 2 years, 3 years, etc?

We solved this type of problem by using a **recurrence relation** to model the growth in our investment year-by-year. We do the same with the car rental problem, the only difference being that we are now working with matrices.

Constructing a matrix recurrence relation

A recurrence relation must have a *starting point*.

In this case it is the **initial state matrix**: $S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix}$

Generating S_1

To find out the number of cars in Bendigo and Colac after 1 week, we use the transition matrix $T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$ to generate the next **state matrix** in the sequence, S_1, as follows:

$$S_1 = T\,S_0$$

$$= \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}\begin{bmatrix} 50 \\ 40 \end{bmatrix} = \begin{bmatrix} 0.8 \times 50 + 0.1 \times 40 \\ 0.2 \times 50 + 0.9 \times 40 \end{bmatrix}$$

$$\text{or } S_1 = \begin{bmatrix} 44 \\ 46 \end{bmatrix}$$

Thus, after 1 week we predict that there will be 44 cars in Bendigo and 46 in Colac.

Generating S_2

Following the same pattern, after 2 weeks;

$$S_2 = T\ S_1 = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}\begin{bmatrix} 44 \\ 46 \end{bmatrix} = \begin{bmatrix} 39.8 \\ 50.2 \end{bmatrix}$$

Thus, after 2 weeks we predict that there will be 39.8 cars in Bendigo and 50.2 in Colac.

Generating S_3

After 3 weeks:

$$S_3 = T\ S_2 = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}\begin{bmatrix} 39.8 \\ 50.2 \end{bmatrix} = \begin{bmatrix} 36.9 \\ 53.1 \end{bmatrix}$$

Thus, after 3 weeks we predict that there will be 36.9 cars in Bendigo and 53.1 in Colac.

A pattern is now emerging. So far we have seen that:

$$S_1 = TS_0$$
$$S_2 = TS_1$$
$$S_3 = TS_2$$

If we continue this pattern we have:

$$S_4 = TS_3$$
$$S_5 = TS_4$$

or, more generally, $S_{n+1} = TS_n$.

With this rule as a starting point, we now have a recurrence relation that will enable us to model and analyse the car rental problem on a step-by-step basis.

Recurrence relation

$$S_0 = \text{intial value}, \quad S_{n+1} = TS_n$$

Let us return to the factory problem in Example 2.

Example 4 Using a recursion relation to calculate state matrices step-by-step

The factory has a large number of machines. The machines can be in one of two states: operating (O) or broken (B). Broken machines are repaired and come back into operation and vice versa.

At the start, 80 machines are operating and 20 are broken.

Use the recursion relation

$$S_0 = \text{intial value}, \; S_{n+1} = TS_n$$

where

$$S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$$

to determine the number of operational and broken machines after 1 day and after 3 days.

Explanation	Solution
1 Write down a column matrix with S_0 representing the initial operational state of the machines, and the transition matrix.	$S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix} \quad T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$
2 Use the rule $S_{n+1} = TS_n$ to determine the operational state of the machines after one day by forming the product $S_1 = TS_0$ and evaluate.	$S_1 = TS_0 = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}\begin{bmatrix} 80 \\ 20 \end{bmatrix} = \begin{bmatrix} 69 \\ 31 \end{bmatrix}$ After 1 day, 69 machines are operational and 31 are broken.
3 To find the operational state of the machines after 3 days, we must first find the operating state of the machines after 2 days (S_2) and use this matrix to find S_3 using $S_3 = TS_2$.	$S_2 = TS_1 = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}\begin{bmatrix} 69 \\ 31 \end{bmatrix} = \begin{bmatrix} 60.2 \\ 39.8 \end{bmatrix}$ $S_3 = TS_2 = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}\begin{bmatrix} 60.2 \\ 39.8 \end{bmatrix} = \begin{bmatrix} 53.16 \\ 46.84 \end{bmatrix}$ After 3 days, 53 machines are operating and 47 are broken.

Calculator hint: In practice, generating matrices recursively is performed on your CAS calculator as shown opposite for the calculations performed in Example 11.

$\begin{bmatrix} 80 \\ 20 \end{bmatrix} \rightarrow s0$	$\begin{bmatrix} 80. \\ 20. \end{bmatrix}$
$\begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \rightarrow t$	$\begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$
t.s0	$\begin{bmatrix} 69. \\ 31. \end{bmatrix}$
$\begin{bmatrix} 69 \\ 31 \end{bmatrix}$	$\begin{bmatrix} 60 & 2 \\ 39 & 8 \end{bmatrix}$

and so on.

A rule for determining the state matrix of a system after *n* steps

While we can use the recurrence relation:

$$S_0 = \text{intial value}, \; S_{n+1} = TS_n$$

to generate state matrices step-by-step, there is a more efficient method when we need to determine the state matrix after a large number of steps.
If we follow through the process step-by-step we have:

$$S_1 = TS_0$$
$$S_2 = TS_1 = T(TS_0) = T^2S_0$$
$$S_3 = TS_2 = T(TS_1) = T^2S_1 = T^2(TS_0) = T^3S_0$$

Continuing the process

$$S_4 = T^4S_0$$
$$S_5 = T^5S_0$$

or more generally, $S_{\text{n}} = T^nS_0$.

We now have a simple rule for finding the value, S_n, of the state matrix after n steps.

A rule for finding the state matrix after *n* steps

If the recurrence rule for determining state matrices is

$$S_0 = \text{initial state matrix}, \; S_{n+1} = TS_n,$$

the state matrix after n steps (or transitions) is given by $S_n = T^nS_0$.

Let us return to the factory problem we analysed in Example 2.

Example 5 **Determining the nth state of a system using the rule $S_n = T^n S_0$**

The factory has a large number of machines. The machines can be in one of two states: operating (O) or broken (B). Broken machines are repaired and come back into operation and vice versa.

Initially, 80 machines are operating and 20 are broken, so:

$$S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$$

Determine the number of operational and broken machines after 10 days.

Explanation

1 Write down the transition matrix, T, and initial state matrix, S_0. Enter the matrices into your calculator. Use T and S.

2 To find out how many machines are in operation and how many are broken after 10 days, write down the rule $S_n = T^n S_0$ and substitute $n = 10$ to give $S_{10} = T^{10} S_0$.

3 Enter the expression $T^{10} S$ into your calculator and evaluate.

4 Write down your answer in matrix form and then in words.

Solution

$$T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \quad S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix}$$

$$S_n = T^n S_0$$
$$\therefore S_{10} = T^{10} S_0$$

$$\begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \rightarrow t \qquad \begin{bmatrix} 0.85 & 0.05. \\ 0.15 & 0.95. \end{bmatrix}$$

$$\begin{bmatrix} 80 \\ 20 \end{bmatrix} \rightarrow s \qquad \begin{bmatrix} 80. \\ 20. \end{bmatrix}$$

$$t^{10} \cdot s \qquad \begin{bmatrix} 30.9056 \\ 69.0944 \end{bmatrix}$$

$$S_{10} = \begin{bmatrix} 30.9 \\ 69.1 \end{bmatrix}$$

After 10 days, 31 machines will be operational and 69 broken.

Using the inverse matrix of a transition matrix

In the above we have seen how to move from left to right in the sequence of state matrices by multiplying by the transition matrix.

$$S_0, S_1, S_2, \ldots S_n, S_{n+1} \ldots$$

We can move from right to left through the transition states by using the inverse of the transition matrix. In general, the inverse is not a transition matrix.

$$S_{n+1} = TS_n \qquad \text{and} \qquad S_n = T^{-1} S_{n+1}$$

Example 6

We have a transition matrix $T = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$

and we know that the state matrix $S_4 = \begin{bmatrix} 25\,587 \\ 34\,413 \end{bmatrix}$.

Determine S_3 and S_2.

Solution

We know that $S_4 = TS_3$. Hence $S_3 = T^{-1}S_4$. First $T^{-1} = \begin{bmatrix} \frac{7}{3} & -1 \\ -\frac{4}{3} & 2 \end{bmatrix}$.

You should hold this in your calculator and then

$$S_3 = T^{-1}S_4 = \begin{bmatrix} \frac{7}{3} & -1 \\ -\frac{4}{3} & 2 \end{bmatrix}\begin{bmatrix} 25\,587 \\ 34\,413 \end{bmatrix} = \begin{bmatrix} 25\,290 \\ 34\,710 \end{bmatrix}$$

and

$$S_2 = T^{-1}S_3 = \begin{bmatrix} \frac{7}{3} & -1 \\ -\frac{4}{3} & 2 \end{bmatrix}\begin{bmatrix} 25\,290 \\ 34\,710 \end{bmatrix} = \begin{bmatrix} 24\,300 \\ 35\,700 \end{bmatrix}$$

Note: To calculate S_2 given S_3 we could have used:

$$S_2 = \left(T^{-1}\right)^2 S_4$$

The steady-state solution

A second question a manager might like answered about the car rental is as follows.

> 'Will the number of rental cars available from each location vary from week to week or will they settle down to some fixed value?'

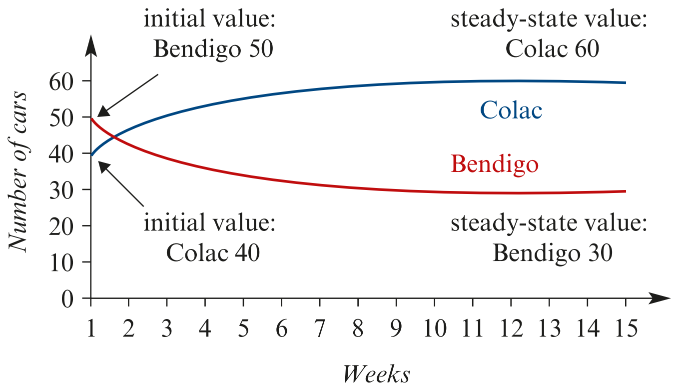
To investigate this question, we start by listing the state matrices from week 0 to week 15.

Week	0	1	2	3	4–11	12	13	14	15
State matrix	$\begin{bmatrix} 50 \\ 40 \end{bmatrix}$	$\begin{bmatrix} 44 \\ 46 \end{bmatrix}$	$\begin{bmatrix} 39.8 \\ 50.2 \end{bmatrix}$	$\begin{bmatrix} 36.9 \\ 53.1 \end{bmatrix}$	...	$\begin{bmatrix} 30.3 \\ 59.7 \end{bmatrix}$	$\begin{bmatrix} 30.2 \\ 59.8 \end{bmatrix}$	$\begin{bmatrix} 30.1 \\ 59.9 \end{bmatrix}$	$\begin{bmatrix} 30.1 \\ 59.9 \end{bmatrix}$

What you should notice is that, as the weeks go by, the number of cars at each of the locations starts to settle down. We call this the *steady- or equilibrium- state solution*.

For the rental car problem, the *steady-state solution* is 30.1 (in practice, 30) cars at the Bendigo branch and 59.9 (in practice, 60) cars at the Colac branch, which means the numbers of cars at each location will *not* change from then on.

This can be seen more clearly in the graph below (the points have been joined to guide the eye).



In summary, even though the number of cars returned to each location varied from day to day, the numbers at each location eventually settled down to an equilibrium or steady-state solution. In the steady state, the number of cars at each location remained the same.

Important

1. In the *steady state*, cars are still moving between Bendigo and Colac, but the number of cars rented in Bendigo and returned to Colac is balanced by the number of cars rented in Colac and returned to Bendigo. Because of this balance, the steady state is also called the *equilibrium state*.
2. For a system to have a steady state, the transition matrix must be *regular* and the columns must add up to 1. A *regular matrix* is one whose powers never contain any zero elements. In practical terms, this means that every state represented in the transition matrix is accessible, either directly or indirectly from every other state.

A strategy for estimating the steady-state solution

In the car rental problem we found that, even though the number of cars returned to each location initially varied from day to day, it eventually settled down so the number of cars at each location remained the same.

Although we arrived at this conclusion by repeated calculations, we can arrive at the solution much faster by using the rule $S_n = T^n S_0$ to find the nth state.

Estimating the steady state solution

If S_0 is the initial state matrix, then the **steady-state matrix**, S, is given by

$$S = T^n S_0$$

as n tends to infinity (∞).

Note: While in practice we cannot evaluate T^n for $n = \infty$, we find that, depending on the circumstances, large values of n can often give a very close approximation to the steady-state solution.

Example 7 Estimating the steady-state solution

For the car rental problem:

$$S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

Estimate the steady-state solution by calculating S_n for $n = 10, 15, 17$ and 18.

Explanation

1 Write down the transition matrix T and initial state matrix S_0. Enter the matrices into your calculator. Use T and S.

2 Use the rule $S_n = T^n S_0$ to write down the expression for the nth state for $n = 10$.

3 Enter the expression $T^{10}S$ into your calculator and evaluate.

4 Repeat the process for $n = 15, 17$ and 18.

5 Write down your answer in matrix form and then in words. This result agrees with the graphical result arrived at earlier.

Solution

$$T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}, \ S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix}$$

$$S_n = T^n S_0$$

$$\therefore S_{10} = T^{10} S_0 = \begin{bmatrix} 30.6 \\ 59.4 \end{bmatrix}$$

$t^{10} \cdot s$ $\begin{bmatrix} 30.565 \\ 59.435 \end{bmatrix}$

$t^{15} \cdot s$ $\begin{bmatrix} 30.095 \\ 59.905 \end{bmatrix}$

$t^{17} \cdot s$ $\begin{bmatrix} 30.047 \\ 59.953 \end{bmatrix}$

$t^{18} \cdot s$ $\begin{bmatrix} 30.033 \\ 59.967 \end{bmatrix}$

$$S_{15} = \begin{bmatrix} 30.1 \\ 59.9 \end{bmatrix}, \quad S_{17} = \begin{bmatrix} 30.0 \\ 60.0 \end{bmatrix}, \quad S_{18} = \begin{bmatrix} 30.0 \\ 60.0 \end{bmatrix}$$

The estimated steady-state solution is 30 cars based in Bendigo and 60 cars based in Colac.

Note: To establish a steady state to a given degree of accuracy, in this case one decimal place, at least two successive state matrices must agree to this degree of accuracy.

Skill-sheet

Exercise 11C

Calculating state matrices step-by-step and by rule

Example 4

1 For the initial state matrix $S_0 = \begin{bmatrix} 200 \\ 400 \end{bmatrix}$ and the transition matrix $T = \begin{bmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{bmatrix}$: use the recursion relation: $S_0 =$ initial state matrix, $S_{n+1} = TS_n$, to determine:

a S_1 **b** S_2 **c** S_3

Example 5

2 For the initial state matrix $S_0 = \begin{bmatrix} 200 \\ 400 \end{bmatrix}$ and the transition matrix $T = \begin{bmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{bmatrix}$: use the recursion relation: $S_0 =$ initial state matrix, $S_{n+1} = T^n S_0$, to determine:

a S_5 **b** S_7 **c** S_{12}

Example 6

3 We have a transition matrix $T = \begin{bmatrix} 0.65 & 0.4 \\ 0.35 & 0.6 \end{bmatrix}$

and we know that the state matrix $S_5 = \begin{bmatrix} 5461 \\ 4779 \end{bmatrix}$.

Determine S_4 and S_3.

Example 7

4 For the initial state matrix $S_0 = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$ and the transition matrix $T = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$:

a use the recursion relation: $S_0 =$ initial state matrix, $S_{n+1} = TS_n$, to determine:

i S_1 **ii** S_2 **iii** S_3

b determine the value of T^5

c use the rule $S_n = T^n S_0$ to determine:

i S_2 **ii** S_3 **iii** S_7

d by calculating $S_n = T^n S_0$ for $n = 10, 15, 21$ and 22, show that the steady-state matrix is close to $\begin{bmatrix} 200 \\ 100 \end{bmatrix}$.

5 For the initial state matrix $S_0 = \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$ and the transition matrix $T = \begin{bmatrix} 0.7 & 0.4 & 0.1 \\ 0.2 & 0.1 & 0.3 \\ 0.1 & 0.5 & 0.6 \end{bmatrix}$:

a use a recurrence relation to determine:

i S_1 **ii** S_2 **iii** S_3

b use the relationship $S_n = T^n S_0$ to determine:

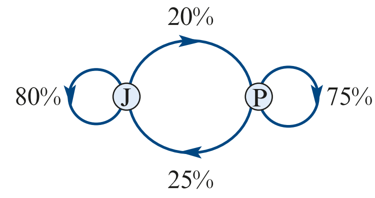
i S_2 **ii** S_3 **iii** S_7

c by calculating $S_n = T^n S_0$ for $n = 10, 15, 17$ and 18, show that the steady-state matrix is close to $\begin{bmatrix} 247.1 \\ 129.4 \\ 223.5 \end{bmatrix}$.

Practical applications of transition matrices

6 Two fast-food outlets, Jill's and Pete's, are located in a small town.
In a given week:

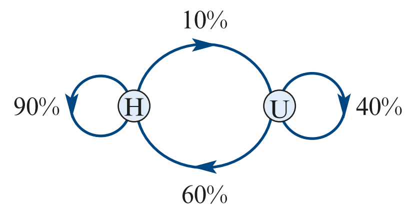
- 80% of people who go to Jill's return the next week
- 20% of people who go to Jill's go to Pete's the next week
- 25% of people who go to Pete's go to Jill's the next week
- 75% of people who go to Pete's return the next week.



a Construct a transition matrix to describe this situation. Call the matrix T.
b Initially, 400 people eat at Jill's and 400 eat at Pete's. Write down a column matrix S_0 that describes this situation.
c How many of these people do we expect to go to Jill's the next week? How many to Pete's?
d How many do we expect to go to Jill's after 5 weeks? How many to Pete's?
e In the long term, how many do we expect to eat at Jill's each week? How many of these people do we expect to eat at Pete's?

7 Imagine that we live in a world in which people are either 'happy' or 'unhappy', but the way people feel can change from day to day.
In this world:

- 90% of people who are happy today will be happy tomorrow
- 10% of people who are happy today will be unhappy tomorrow
- 40% of people who are unhappy today will be unhappy tomorrow
- 60% of people who are unhappy today will be happy tomorrow.



a Construct a transition matrix to describe this situation. Call the matrix T.
b On a given day, out of 2000 people, 1500 are happy and 500 are unhappy. Write down a column matrix, S_0, that describes this situation.
c The next day, how many of these people do we expect to be 'happy' and how many 'unhappy'?
d After 4 days, how many of these people do we expect to be 'happy' and how many 'unhappy'?
e In the long term, how many people do we expect to be 'happy' and how many 'unhappy'?

8 In another model of this world, people can be 'happy', 'neither happy nor sad', or 'sad', but the way people feel can change from day to day.

The transition matrix opposite shows how people's feelings may vary from day to day in this world, and the proportions of people involved.

$$T = \begin{array}{c c} & \begin{array}{ccc} H & N & S \end{array} \\ \begin{array}{c} H \\ N \\ S \end{array} & \begin{bmatrix} 0.80 & 0.40 & 0.35 \\ 0.15 & 0.30 & 0.40 \\ 0.05 & 0.30 & 0.25 \end{bmatrix} \end{array}$$

In the transition matrix, the columns define the situation today and the rows define the situation tomorrow.

a On a given day, out of 2000 people, 1200 are 'happy', 600 are 'neither happy nor sad' and 200 are 'sad'. Write down a matrix, S_0, that describes this situation.

b The next day, how many people do we expect to be happy?

c After 5 days, how many people do we expect to be happy?

d In the long term, how many of the 2000 people do we expect to be happy?

Exam 1 style questions

9 Students at a boarding school have a choices of two breakfast cereals, Crispies (C) and Krunchies (K). The change in the percentage of students who have each cereal on consecutive days is descibed by the transition matrix T shown below.

$$T = \begin{array}{c c} & \begin{array}{cc} \multicolumn{2}{c}{\text{Today}} \\ C & K \end{array} \\ \begin{array}{c} C \\ K \end{array} & \begin{bmatrix} 0.42 & 0.56 \\ 0.58 & 0.44 \end{bmatrix} \end{array} \text{ Tomorrow}$$

On Monday 25% of the students ate Crispies. What percentage of the students ate Krunchies on Tuesday?

A 47.5% **B** 48% **C** 50.25% **D** 52.5% **E** 62%

10 A factory employs the same number of workers each day. The workers are allocated to work with either machine A or machine B. The workers may be allocated to work on a different machine from day to day, as shown in the transition matrix below.

$$T = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.32 & 0.16 \\ 0.68 & 0.84 \end{bmatrix} \end{array}$$

Machine A has 72 workers each day on it. Each day, the number of workers machine B will be

A 24 **B** 36 **C** 72 **D** 288 **E** 306

11 Consider the matrix recurrence relation below.

$$S_0 = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}, S_{n+1} = TS_n \text{ where } T = \begin{bmatrix} x & 0.4 & y \\ 0.6 & z & 0.4 \\ 0.1 & 0.2 & w \end{bmatrix}$$

Matrix T is a regular transition matrix. Given that $S_1 = \begin{bmatrix} 14 \\ 26 \\ 20 \end{bmatrix}$ which of the following is true?

A $x = 0.3, y = 0.1, z = 0.4, w = 0.5$

B $x = 0.3, y = 0.3, z = 0.4, w = 0.7$

C $x = 0.2, y = 0.7, z = 0.3, w = 0.3$

D $x = 0.2, y = 0.8, z = 0.3, w = 0.2$

E $x = 0.3, y = 0.6, z = 0.4, w = 0.4$

11D Transition matrices – using the rule $S_{n+1} = TS_n + B$

Learning intentions

- To be able to use a matrix recurrence relation S_0 = initial state matrix, $S_{n+1} = TS_n + B$ to model systems that include external additions or reductions at each step of the process.

To date, we have only considered matrix recurrence models of the form

$$S_0 = \text{ initial state matrix, } S_{n+1} = TS_n$$

This recurrence model can be used to model situations where the total number of objects in the system, like cars, machines, people or birds, remains unchanged. For example, in the car rental problem 90 cars are available for rental. But what happens if management wants to increase the total number of cars available for rent by adding, say, an extra car at each location each week?

To allow for this situation we need to use the matrix recurrence relation:

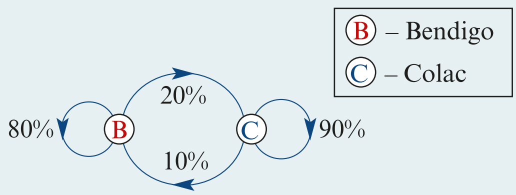
$$S_0 = \text{initial state matrix, } S_{n+1} = TS_n + B$$

where B is a column matrix.
The next example applies this model to the rental car problem.

Example 8 **Determining the nth state of a system using the rule $S_n = T^n S_0 + B$**

A rental starts with 90 cars, 50 located at Bendigo and 40 located at Colac.

Cars are usually rented and returned in the same town. However, a small percentage of cars rented in Bendigo are returned in Colac and vice versa. The transition diagram opposite gives these percentages.



To increase the number of cars, two extra cars are added to the rental fleet at each location each week. The recurrence relation that can be used to model this situation is:

$$S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix}, \; S_{n+1} = TS_n + B \quad \text{where} \quad T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Determine the number of cars at Bendigo and Colac after:

a 1 week **b** 2 weeks.

Explanation

a Use the rule $S_1 = TS_o + B$ to determine the state matrix after 1 week and write your conclusion.

Solution

$$S_1 = TS_0 + B$$
$$= \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 50 \\ 40 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 44 \\ 46 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 46 \\ 48 \end{bmatrix}$$

Thus, we predict that there will be 46 cars in Bendigo and 48 cars in Colac.

b Use the rule $S_2 = TS_1 + B$ to determine the state matrix after 2 weeks and write your conclusion.

$$S_2 = TS_1 + B$$
$$= \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 46 \\ 48 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 41.6 \\ 52.4 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 43.6 \\ 54.4 \end{bmatrix}$$

Thus, we predict that there will be 43.6 cars in Bendigo and 54.4 cars in Colac.

Unfortunately, the recurrence rule $S_{n+1} = TS_n + B$ does not lead to a simple rule for the state matrix after n steps, so we need to work our way through this sort of problem step-by-step.

Using the inverse matrix of a transition matrix

In the above we have seen how to move from left to right in the sequence of state matrices by applying $S_{n+1} = TS_n + B$

$$S_0, S_1, S_2, \ldots S_n, S_{n+1} \ldots$$

We can move from right to left through the transition states by the observation

$$S_{n+1} = TS_n + B \qquad \text{and} \qquad S_n = T^{-1}(S_{n+1} - B)$$

Question 4 in the Exercise can be completed in this way.

Skill-sheet

Exercise 11D

Using a recurrence rule to calculate state matrices

Example 7

1 For the transition matrix $T = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$ and the state matrix $S_0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$:

a use the recurrence rule $S_{n+1} = TS_n$ to determine:

i S_1 **ii** S_3

b use the recurrence rule $S_{n+1} = TS_n + R$, where $R = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, to determine:

i S_1 **ii** S_2

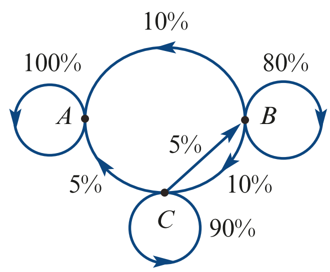
c use the recurrence rule $S_{n+1} = TS_n - B$, where $B = \begin{bmatrix} -20 \\ 20 \end{bmatrix}$, to determine:

i S_1 **ii** S_2

Practical application

2 On Windy Island, sea birds are observed nesting at three sites: A, B and C. The following transition matrix and accompanying transition diagram can be used to predict the movement of sea birds between these sites from year to year.

$$\begin{array}{cc} & \text{This year} \\ & \begin{array}{ccc} A & B & C \end{array} \\ T = & \begin{bmatrix} 1.0 & 0.10 & 0.05 \\ 0 & 0.80 & 0.05 \\ 0 & 0.10 & 0.90 \end{bmatrix} \end{array} \begin{array}{l} A \\ B \\ C \end{array} \text{ Next year}$$



Initially, 10 000 sea birds were observed nesting at each site, so $S_0 = \begin{bmatrix} 10\,000 \\ 10\,000 \\ 10\,000 \end{bmatrix}$.

a Use the recurrence rule $S_{n+1} = TS_n$ to:

i determine S_1, the state matrix after 1 year

ii predict the number of sea birds nesting at site B after 2 years.

b Without calculation, write down the number of sea birds predicted to nest at each of the three sites in the long term. Explain why this can be done without calculation.

c To help solve the problem of having all the birds eventually nesting at site A, the ranger suggests that 2000 sea birds could be removed from site A each year and relocated in equal numbers to sites B and C.

The state matrix, S_2, is now given by

$$S_2 = TS_1 + N$$

where $S_1 = \begin{bmatrix} 10000 \\ 10000 \\ 10000 \end{bmatrix}$, $T = \begin{bmatrix} 1.0 & 0.10 & 0.05 \\ 0 & 0.80 & 0.05 \\ 0 & 0.10 & 0.90 \end{bmatrix}$ and $N = \begin{bmatrix} -2000 \\ 1000 \\ 1000 \end{bmatrix}$.

Evaluate:

i S_2 **ii** S_3 (assuming that $S_3 = TS_2 + N$) **iii** S_4 (assuming that $S_4 = TS_3 + N$).

Exam 1 style questions

3 The matrix S_{n+1} is determined from the matrix S_n using the recurrence relation $S_{n+1} = T \times S_n - C$, where

$$T = \begin{bmatrix} 0.6 & 0.75 & 0.1 \\ 0.2 & 0.2 & 0.1 \\ 0.2 & 0.05 & 0.8 \end{bmatrix}, S_0 = \begin{bmatrix} 2000 \\ 1000 \\ 1000 \end{bmatrix}, S_1 = \begin{bmatrix} 1975 \\ 650 \\ 1125 \end{bmatrix}.$$

and C is a column matrix. Matrix S_2 =

A $\begin{bmatrix} 1860 \\ 687.5 \\ 1452.5 \end{bmatrix}$ **B** $\begin{bmatrix} 1700 \\ 560.5 \\ 1022.5 \end{bmatrix}$ **C** $\begin{bmatrix} 1710 \\ 587.5 \\ 1202.5 \end{bmatrix}$ **D** $\begin{bmatrix} 1600 \\ 1725.5 \\ 1200.5 \end{bmatrix}$ **E** $\begin{bmatrix} 1650 \\ 550.5 \\ 1032.5 \end{bmatrix}$

4 Supporters of a football team attend home games. There are 3 areas, bays A, B and C, where they sit. There is considerable moving of position from game to game and the numbers attending the home games gradually decline as the year progresses. Let X_n be the state matrix that shows the number of supporters in each bay n weeks into the the season. The number of supporters in each location can be determined by the matrix recurrence relation

$$X_{n+1} = TX_n - D$$

where

$$T = \begin{array}{c} \text{This game} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.5 \\ 0.3 & 0.7 & 0.2 \\ 0.6 & 0.1 & 0.3 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \end{array} \text{ Next game} \quad \text{and } D = \begin{bmatrix} 70 \\ 70 \\ 70 \end{bmatrix}$$

If $X_3 = \begin{bmatrix} 9830 \\ 11\,130 \\ 7830 \end{bmatrix}$ then X_2 =

A $\begin{bmatrix} 12370 \\ 9510 \\ 4650 \end{bmatrix}$ **B** $\begin{bmatrix} 4000 \\ 10\,000 \\ 15\,000 \end{bmatrix}$ **C** $\begin{bmatrix} 12\,000 \\ 11\,000 \\ 5000 \end{bmatrix}$ **D** $\begin{bmatrix} 7500 \\ 8600 \\ 12\,000 \end{bmatrix}$ **E** $\begin{bmatrix} 79300 \\ 245232 \\ 231011 \end{bmatrix}$

11E Leslie matrices

Learning intentions

- ▶ To use and interpret Leslie matrices to analyse changes in population over time.

Leslie matrices are used to construct discrete models of population growth. In particular, they are used to model changes in the sizes of different age groups within a population.

The general setting

Leslie matrices were developed by Patrick Holt Leslie (1900–1972) while he was working in the Bureau of Animal Population at the University of Oxford. They are used by biologists and ecologists to model changes over time in various animal populations.

Age groups First the population is divided into age groups. The time period for each age group is the same length. Together they cover the life span of the population. For example, in a study of a human population, we could use a time period of 10 years and consider eleven age groups as follows:

Age group(i)	1	2	3	4	$\cdots$	10	11
Age range (years)	0–10	10–20	20–30	30–40	$\cdots$	90–100	100–110

Note: Only the females of the species are counted in the population, as they are the ones who give birth to the new members of the population.

A **Leslie matrix** is a transition matrix that can be used to describe the way population changes over time. It takes into account two factors for the females in each age group: the *birth rate*, b_i, and *survival rate*, s_i, where i is the number of the age group.

Birth rates We ignore migration, and so the population growth is entirely due to new female births. The birth rate, b_i, for age group i is the average number of female offspring from a mother in age group i during one time period. For example, average birth rate of women in age group 4 (20 – 30 years) might be 1.7 female children for the 10 year period.

Survival rates The survival rate, s_i, for age group i is the proportion of the population in age group i that progress to age group $i + 1$. Note that $0 \leq s_i \leq 1$.

For example, the survival rate for age group 2 might be 0.95, that is 95% of females in this 10 – 20 year age group would survive to progress to age group 3, 20 – 30 years.

Note: The survival rate of the last age group (100 – 110) is taken to be 0.

A simple example

We start with a simple example where the life span of the species is 9 years. We will divide the population into three age groups. This means we use a time period of 3 years.

Age group(i)	1	2	3
Age range (years)	0–3	3–6	6–9

A Leslie matrix for three age groups is a 3×3 matrix of the form

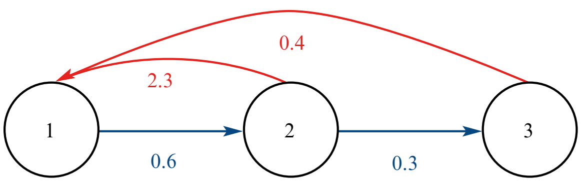
$$L = \begin{bmatrix} b_1 & b_2 & b_3 \\ s_1 & 0 & 0 \\ 0 & s_2 & 0 \end{bmatrix}$$

Suppose that the survival rates are $s_1 = 0.6$ and $s_2 = 0.3$, and that the birth rates are $b_1 = 0$, $b_2 = 2.3$ and $b_3 = 0.4$. Then the Leslie matrix is

$$\begin{array}{cc} & \text{From age group } i \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ L = & \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ 1 \text{ birth rate} \\ 2 \text{ survival rate} \\ 3 \text{ survival rate} \end{array} \quad \text{To age group } i+1$$

Life cycle transition diagram

The above Leslie matrix can be represented by a diagram which we will refer to as a **Life cycle transition diagram**.



The population state matrix

The **population state matrix** is a column matrix that lists the number in each age group at a given time.

The **initial population state matrix** is denoted by S_0. Suppose that for our example, initially the population has 400 females in each age group. We represent the initial population state matrix S_0, as a 3×1 column matrix as shown below.

$$S_0 = \begin{bmatrix} 400 \\ 400 \\ 400 \end{bmatrix} \begin{array}{c} \text{Age group} \\ 1 \\ 2 \\ 3 \end{array}$$

We can now use the Leslie matrix, L, in combination with the initial state matrix S_0 to generate the state matrix after one time period, S_1, to find the size of each age group after one time period (3 years) as follows:

$$S_1 = LS_0 = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}\begin{bmatrix} 400 \\ 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 1080 \\ 240 \\ 120 \end{bmatrix}$$

Thus after one time period, there are 1080 females in age group 1, 240 in age group 2 and 120 in age group 3 and the total population size has increased from 1200 (= 400 + 400 + 400) to 1440 (= 1080 + 240 + 120). Similarly, to find the number in each age group after two time periods we calculate S_2 from S_1 as follows:

$$S_2 = LS_1 = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}\begin{bmatrix} 1080 \\ 240 \\ 120 \end{bmatrix} = \begin{bmatrix} 600 \\ 648 \\ 72 \end{bmatrix}$$

Thus, after two time periods, there are 600 females in age group 1648 in age group 2 and 72 in age group 3 and the over-all population size has decreased to 1320.

Finding the population matrix S_n after n time periods.

To speed up the process we can make use of the explicit formula for the state matrix S_n after n time periods. Notice that there is a pattern when calculating the population state matrices:

$$\begin{aligned} S_1 &= LS_0 \\ S_2 &= LS_1 = L^2S_0 \\ S_3 &= LS_2 = L^3S_0 \\ &\vdots \\ S_{n+1} &= LS_n = L^nS_0 \end{aligned}$$

In general, we can find the population matrix S_n using the rule

$$S_n = L^nS_0$$

Using this rule, to find S_3, we have

$$S_3 = L^3S_0 = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}^3\begin{bmatrix} 400 \\ 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 1519.2 \\ 360 \\ 194.4 \end{bmatrix}$$

Continuing in this way, we can see the change over time in the total population and in the distribution of the age groups.

Change in the population over time

Time period	0	1	2	3	4	5
Age 0–3 years	400	1080	600	1519.2	905.76	2139.70
Age 3–6 years	400	240	648	360.0	911.52	543.46
Age 6–9 years	400	120	72	194.4	108.00	273.46
Total	1200	1440	1320	2073.6	1925.28	2956.61

Leslie matrices

An $m \times m$ **Leslie matrix** has the form

$$L = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{m-1} & b_m \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{m-1} & 0 \end{bmatrix}$$

where:

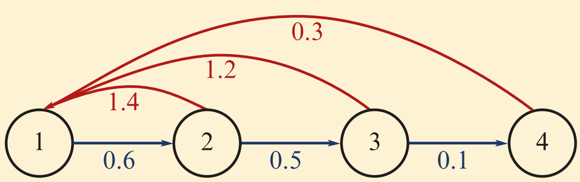
- m is the number of age groups being considered
- s_i, the survival rate, is the proportion of the population in age group i that progress to age group $i + 1$
- b_i, the birth rate, is the average number of female offspring from a mother in age group i during one time period.

Leslie matrix and its interpretation

From age group

$$L = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1.4 & 1.2 & 0.3 \\ 0.6 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \quad \text{To age group}$$

This is a Leslie matrix with 4 age groups. The corresponding life-cycle transition diagram is shown here.



Recursive rules

The population matrix S_n is an $m \times 1$ matrix representing the size of each age group after n time periods. This is calculated using a recursive formula

$$S_0 \text{ is the initial state matrix, } S_{n+1} = LS_n$$

or the explicit rule

$$S_n = L^n S_0$$

Example 9 Determining state matrices and life cycle diagrams

Use the Leslie matrix and initial state matrix below to answer the following questions.

$$\begin{array}{ccl} & \text{From age group} & \\ & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ L = & \begin{bmatrix} 0 & 1.8 & 2.6 & 0.1 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \ \text{To age group} \end{array} \qquad S_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

a Write down

i the birth rate for age group 2 **ii** the survival rate for age group 3

b Complete a life cycle diagram for this Leslie matrix.

c Evaluate the following population state matrices.

S_1, S_5 and S_{20}

d Given that $S_{16} = \begin{bmatrix} 9.53 \\ 2.42 \\ 1.22 \\ 0.16 \end{bmatrix}$, determine S_{17}

Explanation

a **i** The birth rate for age group 2 is given in the matrix position, row 1, column 2.

ii The survival rate for age group 3 is given in the matrix position, row 4, column 3.

b **Survival rates**

$s_1 = 0.2, s_2 = 0.4, s_3 = 0.3$

Birth rates

$b_2 = 1.8, b_3 = 2.6, b_4 = 0.1$

c $S_1 = LS_0$

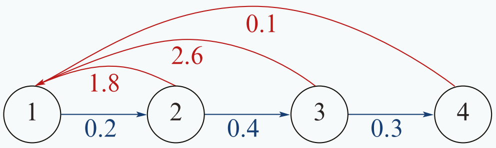
$S_5 = L^5 S_0$

$S_{20} = L^{20} S_0$

Solution

Birth rate for age group 2 = 1.8

Survival rate for age group 3 = 0.3



Using a calculator.

$$S_1 = \begin{bmatrix} 0 \\ 200 \\ 0 \\ 0 \end{bmatrix}, \ S_5 = \begin{bmatrix} 149.76 \\ 26.4 \\ 16.64 \\ 8.64 \end{bmatrix}, \ S_{20} = \begin{bmatrix} 3.84 \\ 0.97 \\ 0.49 \\ 0.19 \end{bmatrix}$$

d $S_{17} = LS_{16}$

(Further investigation would reveal that the population continues to decrease over time.)

$$S_{17} = \begin{bmatrix} 7.54 \\ 1.91 \\ 0.97 \\ 0.37 \end{bmatrix}$$

Example 10 Entering information in a Leslie matrix and state matrix

Information about a population of female goats is given in the following table.

Age group (years)	0 – 1	1 – 2	2 – 3	3 – 4	4 – 5
Initial population	10	25	40	20	15
Birth rates	0	0.2	0.9	0	0
Survival rates	0.6	0.7	0.5	0.2	0

a Write down the initial population state matrix, S_0.

b Using the information in the table above, write down a Leslie matrix to describe the change in population of female goats.

c Construct a life cycle transition diagram for this Leslie matrix.

d Determine the number of 3 – 4 year old female goats in the population after 3 years. Round your answer to the nearest whole number.

Explanation

a Enter the initial population numbers into a 5×1 matrix.

Solution

$$S_0 = \begin{bmatrix} 10 \\ 25 \\ 40 \\ 20 \\ 15 \end{bmatrix}$$

b Enter the birthrates and survival rates into a 5×5 matrix.

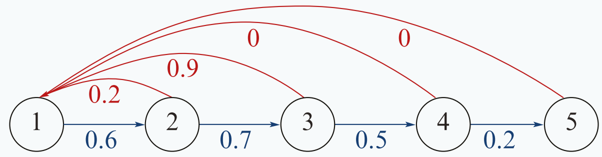
$$L = \begin{bmatrix} 0 & 0.2 & 0.9 & 0 & 0 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 \end{bmatrix}$$

c **Survival rates**

$s_1 = 0.6, s_2 = 0.7, s_3 = 0.5,$
$s_4 = 0.2$

Birth rates

$b_2 = 0, 2, b_3 = 0.9, b_4 = 0,$
$b_5 = 0$



d $S_3 = L^3 S_0$

$$\begin{bmatrix} 0 & 0.2 & 0.9 & 0 & 0 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 \end{bmatrix}^3 \begin{bmatrix} 10 \\ 25 \\ 40 \\ 20 \\ 15 \end{bmatrix} = \begin{bmatrix} 8.7 \\ 10.17 \\ 17.22 \\ 2.1 \\ 1.75 \end{bmatrix}$$

There are two goats in the 3-4 year old age group in this population.

Long term (limiting) behaviour of population numbers

The following examples demonstrate numerical techniques for modelling the use of Leslie matrices.

Example 11 Limiting behaviour for Leslie matrices

Consider the following Leslie matrix L and initial population matrix S_0:

$$L = \begin{bmatrix} 0 & 4 & 4 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \quad \text{and} \quad S_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix}$$

a Find

i S_5 **ii** S_{10} **iii** S_{50}

Premultiply each of these state matrices by $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ to calculate the total populations at each of these stages and comment.

b Determine S_{25} and S_{26}. Divide each age group population for S_{26} by the corresponding age group population for S_{25} and show that $S_{26} \approx 1.1915 S_{25}$ and comment.

Solution

a

i $S_5 = \begin{bmatrix} 1000 \\ 250 \\ 62.5 \end{bmatrix}$ **ii** $S_{10} = \begin{bmatrix} 2500 \\ 531.25 \\ 218.75 \end{bmatrix}$ **iii** $S_{50} = \begin{bmatrix} 2\,777\,063 \\ 582\,688.05 \\ 244521.18 \end{bmatrix}$

Finding the total populations according to this model

i $\begin{bmatrix}1 & 1 & 1\end{bmatrix} S_5 = [1312.5]$. The population is approximately 1312 after 5 years.

ii $\begin{bmatrix}1 & 1 & 1\end{bmatrix} S_{10} = [3250]$. The population is approximately 3250 after 10 years.

iii $\begin{bmatrix}1 & 1 & 1\end{bmatrix} S_{50} = [3\,604\,272.2]$. The population is approximately 3 604 272 after 50 years.

A numerical investigation reveals that the population increases without bound.

b We calculate S_{25} and S_{26}:

$$S_{25} = L^{25} S_0 = \begin{bmatrix} 34\,781.25 \\ 7297.85 \\ 3062.50 \end{bmatrix}, \quad S_{26} = L^{26} S_0 = \begin{bmatrix} 41\,441.41 \\ 8695.31 \\ 3648.93 \end{bmatrix}$$

Then we can find the rate of increase in each age group during the 26th time period:

$$\frac{41\,441.41}{34\,781.25} \approx \frac{8695.31}{7297.85} \approx \frac{3648.93}{3062.50} \approx 1.1915$$

This suggests that the age-group proportions have stabilised after the 25 time periods.

The long-term growth rate is approximately 1.19. (That is, after a certain stage, the population is increasing by 19% each time period.)

Note: Try different entries in S_0 to see if you get the same behaviour. The long-term growth rate is largely dependent on the Leslie matrix L.

Limiting behaviour of Leslie matrices

Often we will find that, after a long enough time, the proportion of the population in each age group does not change from one time period to the next. This happens if we can find a real number k such that $LS_{n+1} = kS_n$ for some sufficiently large n. This does not happen with every Leslie matrix as we see in Example 13.

Example 12 A Leslie matrix and state matrix with constant rate of increase

Consider the following Leslie matrix L and initial population matrix S_0:

$$L = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \quad \text{and} \quad S_0 = \begin{bmatrix} 1600 \\ 800 \\ 200 \end{bmatrix}.$$

a Determine

i S_{10} **ii** S_{14} **iii** S_{15}

b The rate of increase of the population is a constant and each of the age group populations increase in the same way. Find this rate by comparing S_{14} and S_{15}.

c Confirm the ratio of the age group populations stays constant for S_0, S_1 and S_{10} at $8 : 4 : 1$

Solution

a

i $S_{10} = \begin{bmatrix} 9906.78 \\ 4953.39 \\ 1238.35 \end{bmatrix}$ **ii** $S_{14} = \begin{bmatrix} 20542.70 \\ 10271.35 \\ 2567.84 \end{bmatrix}$ **iii** $S_{15} = \begin{bmatrix} 24651.24 \\ 12325.62 \\ 3081.4043 \end{bmatrix}$

By comparing S_{14} and S_{15},

$$\frac{24651.24}{20542.70} = \frac{12325.62}{10271.35} = \frac{3081.4043}{2567.84} \approx 1.2$$

we find that the growth rate is 1.2.

b $8 : 4 : 1 = 1600 : 800 : 200 \approx 9906.78 : 4953.39 : 1238.35$

Example 13 Periodic, increasing and decreasing rates of change

Consider the following Leslie matrix L and initial population matrix S_0:

$$L = \begin{bmatrix} 0 & 0 & b_3 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \quad \text{and} \quad S_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix}$$

Investigate the long-term behaviour of the population if:

a $b_3 = 8$ **b** $b_3 = 4$ **c** $b_3 = 10$

Solution

a Let $b_3 = 8$. Use your calculator to store the matrices L and S_0. Then compute:

$$S_1 = LS_0 = \begin{bmatrix} 0 \\ 250 \\ 0 \end{bmatrix}, \quad S_2 = L^2S_0 = \begin{bmatrix} 0 \\ 0 \\ 125 \end{bmatrix}, \quad S_3 = L^3S_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix}$$

The population will continue to cycle through these three states; this is because $L^3 = I$.

b Let $b_3 = 4$. Then a numerical investigation suggests that the population decreases over the long term:

$$S_1 = LS_0 = \begin{bmatrix} 0 \\ 250 \\ 0 \end{bmatrix}, \quad S_5 = L^5S_0 = \begin{bmatrix} 0 \\ 0 \\ 62.5 \end{bmatrix}, \quad S_{50} = L^{50}S_0 = \begin{bmatrix} 0 \\ 0 \\ 0.0019 \end{bmatrix}$$

c Let $b_3 = 10$. Then a numerical investigation suggests that the population increases over the long term:

$$S_1 = LS_0 = \begin{bmatrix} 0 \\ 250 \\ 0 \end{bmatrix}, \quad S_5 = L^5S_0 = \begin{bmatrix} 0 \\ 0 \\ 156.25 \end{bmatrix}, \quad S_{50} = L^{50}S_0 = \begin{bmatrix} 0 \\ 0 \\ 4440.89 \end{bmatrix}$$

Note: A population can increase, decrease, become constant or oscillate.

Skill-sheet

Exercise 11E

Determining state matrices and life cycle diagrams

Example 8

1 Use the Leslie matrix and initial state matrix below to answer the following questions.

$$L = \begin{array}{cccc|c} \multicolumn{4}{c}{\text{From age group}} & \\ 1 & 2 & 3 & 4 & \\ \hline 0 & 1.9 & 2.1 & 1.1 & 1 \\ 0.7 & 0 & 0 & 0 & 2 \\ 0 & 0.5 & 0 & 0 & 3 \\ 0 & 0 & 0.6 & 0 & 4 \end{array} \ \text{To age group} \qquad S_0 = \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}$$

a Write down

i The birth rate for age group 2 **ii** The survival rate for age group 3

b Complete the life cycle diagram for this Leslie matrix.

c Evaluate the following population state matrices.

i S_1 **ii** S_3 **iii** S_{20}

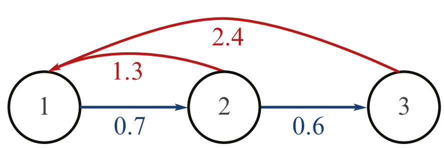
d Given that $S_7 = \begin{bmatrix} 2613 \\ 1200 \\ 485 \\ 168 \end{bmatrix}$ determine S_8. Give your values correct to the nearest whole number.

2 Complete the life cycle diagram corresponding to each of the following Leslie matrices:

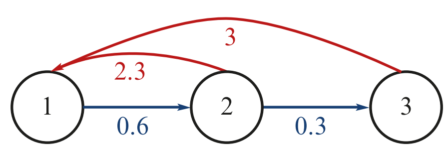
a $\begin{bmatrix} 0 & 2.9 & 3.1 & 2.1 \\ 0.8 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 0 & 0.42 \\ 0.6 & 0 & 0 \\ 0 & 0.75 & 0 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 0 & 3 & 8 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \end{bmatrix}$

3 Construct the Leslie matrix corresponding to each life cycle diagram.

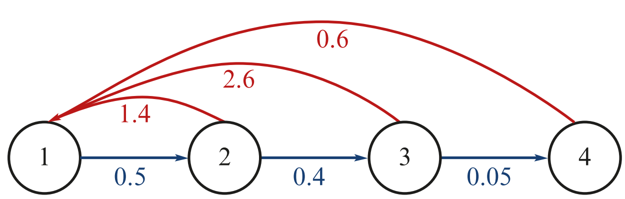
a



b



c



Entering information in a Leslie matrix and state matrix

Example 9

4 Information about a population of female kangaroos in a particular area is given in the following table.

Age group (years)	0 – 4	4 – 8	8 – 12	12 – 16	4 – 5
Initial population	15	20	30	15	10
Birth rates	0	0.2	0.9	1.1	0
Survival rates	0.8	0.9	0.7	0.8	0

a Write down the initial population state matrix, S_0.

b Write down the Leslie matrix.

c Complete the life cycle diagram for this Leslie matrix.

d Determine the population state matrix after

i one year, (S_1) **ii** after 5 years, (S_5).

e Determine the number of 4 – 8 year old female kangaroos in the population after 5 years.

f State the initial total population.

g Use multiplication of state matrices by the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ to find the total population after

i one year, **ii** after 5 years **iii** after 10 years.

h It is conjectured that the population is increasing by about 10% per annum. Calculate each of the following and comment

i 1.1×90 **ii** $1.1^5 \times 90$ **iii** $1.1^{10} \times 90$.

5 Information about a population of female locusts is given in the following table.

Stage	Eggs	Nymphs	Adults
Initial population	0	0	50
Birth rates	0	0	1000
Survival rates	0.02	0.05	0

a Write down the initial population state matrix, S_0.

b Write down the Leslie matrix.

c Construct a time life-cycle transition diagram for this Leslie matrix.

d Determine the population state matrix after

i one year, (S_1) **ii** after 3 years, (S_3) **iii** after 4 years, (S_4).

e If the Initial population is now:

Stage	Eggs	Nymphs	Adults
Initial population	50	100	50

find the populations of each after

i one year (S_1) **ii** after 3 years (S_3). **iii** after 4 years (S_4).

Limiting behaviour for Leslie matrices

Example 10

6 Consider the following Leslie matrix L and initial population state matrix S_0:

$$L = \begin{bmatrix} 0 & 2 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \quad \text{and} \quad S_0 = \begin{bmatrix} 204 \\ 96 \\ 23 \end{bmatrix}$$

a Find

i S_5 **ii** S_{10} **iii** S_{20}

Premultiply each of these state matrices by $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ to calculate the total populations at each of these stages and comment.

b Determine S_{20} and S_{21}. Divide each age group population for S_{21} by the corresponding age group population for S_{20} and show that $S_{21} \approx 1.057 S_{20}$ and comment.

Example 11

7 A Leslie matrix that models a certain population of female animals is

$$\mathbf{L} = \begin{bmatrix} 0 & 2.5 & 1 \\ 0.6 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}$$

where the animals have a maximum life span of 9 years, and the population has been divided into three age groups of 3 years each.

a Assume that each age group initially consists of 400 females. What is the number of females in each age group after:

i 3 years **ii** 6 years **iii** 9 years?

b Now assume that the initial population is 1200 and $S_0 = \begin{bmatrix} 767 \\ 362 \\ 71 \end{bmatrix}$. Find S_n after

i 3 years **ii** 6 years **iii** 9 years.

c Calculate

i $1.27S_0$ **ii** 1.27^2S_0 **iii** 1.27^3S_0

Compare these answers to the answers of part **b**

Example 12

8 Consider the following Leslie matrix L and initial population matrix S_0:

$$L = \begin{bmatrix} 0 & 0 & 12 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix} \quad \text{and} \quad S_0 = \begin{bmatrix} 1200 \\ 0 \\ 0 \end{bmatrix}$$

a Find:

i LS_0 **ii** L^2S_0 **iii** L^3S_0

b Comment on these results in terms of the population behaviour. Try using a different initial population matrix S_0.

c Now investigate for each of the following Leslie matrices. Comment on population increase or decrease.

i $L = \begin{bmatrix} 0 & 0 & 6 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$ **ii** $L = \begin{bmatrix} 0 & 0 & 15 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$

9 A Leslie matrix that models a certain population of female insects is

$$\mathbf{L} = \begin{bmatrix} 0 & 3 & 2 & 2 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$$

where the insects have a maximum life span of 4 months, and the population has been divided into four age groups of 1 month each.

Assume that each age group initially consists of 400 females. What is the number of females in each age group after:

a 1 month **b** 2 months **c** 3 months?

10 For a certain species of fish, we consider three age groups each of one year in length. These fish reproduce only during their third year and then die. Assume that 20% of fish survive their first year and that 50% of these survivors make it to reproduction age. The initial population consists of 1000 newborns.

a Investigate what happens for each of the following values of b_3:

i $b_3 = 10$ **ii** $b_3 = 15$ **iii** $b_3 = 6$

b For $b_3 = 20$, determine the long-term growth rate and the proportion of fish in each age group.

Exam 1 style questions

11 The Leslie matrix for a certain endangered species is:

$$L = \begin{bmatrix} 0.9 & 2.5 & 0.4 \\ 0.3 & 0 & 0 \\ 0 & 0.45 & 0 \end{bmatrix}$$

Some of the species were moved into a sanctuary. The initial female population in the sanctuary is given by

$$S_0 = \begin{bmatrix} 130 \\ 40 \\ 20 \end{bmatrix}$$

The best estimate of the total female population after 7 years is

A 1000 **B** 1500 **C** 2000 **D** 2500 **E** 3000

12 The Leslie matrix $L = \begin{bmatrix} 0 & 2 & b \\ c & 0 & 0 \\ 0 & d & 0 \end{bmatrix}$ satisfies the matrix equation

$$L\begin{bmatrix} 16 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ 2 \end{bmatrix}$$

The values of b, c and d are

A $b = 2, d = \frac{1}{4}, c = \frac{1}{2}$ **B** $b = 2, d = \frac{1}{2}, c = \frac{1}{4}$ **C** $b = 4, d = \frac{1}{4}, c = \frac{1}{2}$

D $b = 4, d = \frac{1}{2}, c = \frac{1}{4}$ **E** $b = 2, d = \frac{1}{4}, c = \frac{1}{2}$

13 A population of birds is modelled by using the Leslie matrix

$$L = \begin{bmatrix} 0 & 2 & 1.5 \\ 0.44 & 0 & 0 \\ 0 & 0.55 & 0 \end{bmatrix}$$

The growth has reached the point where the rates of growth of the different age groups of the population are constant and the state matrix at this point is $S_k = \begin{bmatrix} 1000 \\ 400 \\ 200 \end{bmatrix}$. The rate of growth per time period is

A 10% **B** 11% **C** 12%

D 13% **E** 14%

Review

Key ideas and chapter summary

Assign-ment

State matrix Transition matrixes

A **state matrix** S_n is a column matrix whose elements represent the nth state of a dynamic system defined by a recurrence relation of the form: S_0 = initial state, $S_{n+1} = TS_n$. Here T is a square matrix called a **transition matrix**.

Steady-state matrix

The **steady-state matrix**, S, represents the equilibrium state of a system. For regular matrices, this equilibrium state of a system can be estimated by calculating $T^n S_0$ for a large value of n.

Leslie matrices

- An $m \times m$ **Leslie matrix** has the form

$$L = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{m-1} & b_m \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{m-1} & 0 \end{bmatrix}$$

  where:
  - m is the number of age groups being considered
  - s_i, the survival rate, is the proportion of the population in age group i that progress to age group $i + 1$
  - b_i, the birth rate, is the average number of female offspring from a mother in age group i during one time period.
- The population matrix S_n is an $m \times 1$ matrix representing the size of each age group after n time periods. This is calculated using a recursive formula

  S_0 is the initial state matrix, $S_{n+1} = LS_n$

  or the explicit rule

  $S_n = L^n S_0$

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

11A **1** I can set up a transition matrix from a diagram. ☐

See Example 1, and Exercise 11A Question 1

11A **2** I can set up a transition matrix from a written information. ☐

See Example 2, and Exercise 11A Question 2

11B **3** I can interpret a transition matrix and a transition diagram. ☐

See Example 3, and Exercise 11B Question 1

11C **4** I can use a recurrence relation to calculate state matrices step by step ☐

See Example 4, and Exercise 11C Question 1

11C **5** I can use a recurrence relation $S_{n+1} = T^n S_0$ to determine the nth state. ☐

See Example 5, and Exercise 11C Question 2

11C **6** I can use the inverse of a transition matrix. ☐

See Example 6, and Exercise 11C Question 4

11C **7** I can estimate steady state solution for suitable transition matrices. ☐

See Example 7, and Exercise 11C Question 4

11D **8** I can use the matrix recurrence relation S_0 = initial state matrix, $S_{n+1} = TS_n + B$. ☐

See Example 8, and Exercise 11D Question 1

11E **9** I can determine state matrices and construct life cycle diagrams in situations modelled by Leslie matrices. ☐

See Example 9, and Exercise 11E Question 1

11E **10** I can enter information into a Leslie matrix from written information. ☐

See Example 10, and Exercise 11E Question 2

11E **11** I can use numerical techniques to consider the limiting behaviour of Leslie matrices. ☐

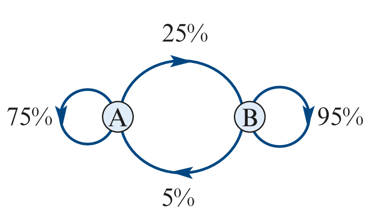
See Example 12, and Exercise 11E Question 3

11E **12** I can identify the properties of a Leslie matrix and the state matrices when there is a constant rate. ☐

See Example 13, and Exercise 11E Question 4

Multiple choice questions

1 The transition matrix that can be used to represent the information in the diagram shown is:



A $\begin{array}{cc} & \begin{array}{cc} & \text{From} \\ A & B \end{array} \\ \text{To} \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.75 & 0.25 \\ 0.05 & 0.95 \end{bmatrix} \end{array}$

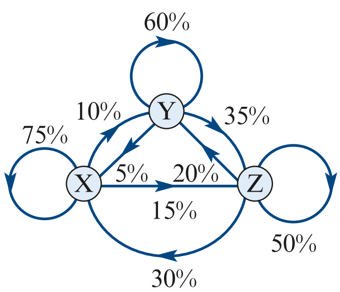
B $\begin{array}{cc} & \begin{array}{cc} & \text{From} \\ A & B \end{array} \\ \text{To} \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.75 & 0.05 \\ 0.25 & 0.95 \end{bmatrix} \end{array}$

C $\begin{array}{cc} & \begin{array}{cc} & \text{From} \\ A & B \end{array} \\ \text{To} \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.75 & 0.25 \\ 0.95 & 0.05 \end{bmatrix} \end{array}$

D $\begin{array}{cc} & \begin{array}{cc} & \text{From} \\ A & B \end{array} \\ \text{To} \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.75 & 0.95 \\ 0.25 & 0.05 \end{bmatrix} \end{array}$

E $\begin{array}{cc} & \begin{array}{cc} & \text{From} \\ A & B \end{array} \\ \text{To} \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.25 & 0.05 \\ 0.75 & 0.95 \end{bmatrix} \end{array}$

2 The transition matrix that can be used to represent the information in the diagram shown is:



A $\begin{array}{cc} & \begin{array}{ccc} X & Y & Z \end{array} \\ \text{To} \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} 0.75 & 0.05 & 0.30 \\ 0.10 & 0.60 & 0.20 \\ 0.15 & 0.35 & 0.50 \end{bmatrix} \end{array}$

B $\begin{array}{cc} & \begin{array}{ccc} X & Y & Z \end{array} \\ \text{To} \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} 0.75 & 0.10 & 0.15 \\ 0.60 & 0.05 & 0.35 \\ 0.50 & 0.30 & 0.20 \end{bmatrix} \end{array}$

C $\begin{array}{cc} & \begin{array}{ccc} X & Y & Z \end{array} \\ \text{To} \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} 0.75 & 0.10 & 0.15 \\ 0.10 & 0.05 & 0.35 \\ 0.50 & 0.30 & 0.20 \end{bmatrix} \end{array}$

D $\begin{array}{cc} & \begin{array}{ccc} X & Y & Z \end{array} \\ \text{To} \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} 0.75 & 0.05 & 0.15 \\ 0.10 & 0.60 & 0.20 \\ 0.15 & 0.35 & 0.50 \end{bmatrix} \end{array}$

E $\begin{array}{cc} & \begin{array}{ccc} X & Y & Z \end{array} \\ \text{To} \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} 0.75 & 0.05 & 0.15 \\ 0.15 & 0.35 & 0.50 \\ 0.10 & 0.60 & 0.20 \end{bmatrix} \end{array}$

The following information is needed for Questions 3 to 8.

3 A system is defined by a transition matrix $T = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}$ with $S_0 = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$.

For this system, $S_1 =$

A $\begin{bmatrix} 60 \\ 200 \end{bmatrix}$ **B** $\begin{bmatrix} 140 \\ 160 \end{bmatrix}$ **C** $\begin{bmatrix} 160 \\ 140 \end{bmatrix}$ **D** $\begin{bmatrix} 166 \\ 144 \end{bmatrix}$ **E** $\begin{bmatrix} 200 \\ 100 \end{bmatrix}$

4 For this system, T^2 is:

A $\begin{bmatrix} 0.36 & 0.25 \\ 0.16 & 0.25 \end{bmatrix}$ B $\begin{bmatrix} 0.56 & 0.55 \\ 0.44 & 0.45 \end{bmatrix}$ C $\begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}$ D $\begin{bmatrix} 1.2 & 1.0 \\ 0.8 & 1.0 \end{bmatrix}$

E not defined

5 For this system, S_3 is closest to:

A $\begin{bmatrix} 160 \\ 140 \end{bmatrix}$ B $\begin{bmatrix} 166.6 \\ 133.4 \end{bmatrix}$ C $\begin{bmatrix} 166.7 \\ 133.3 \end{bmatrix}$ D $\begin{bmatrix} 640 \\ 560 \end{bmatrix}$ E $\begin{bmatrix} 400 \\ 800 \end{bmatrix}$

6 For this system, the steady-state matrix is closest to:

A $\begin{bmatrix} 166.5 \\ 133.5 \end{bmatrix}$ B $\begin{bmatrix} 166.6 \\ 133.4 \end{bmatrix}$ C $\begin{bmatrix} 166.7 \\ 133.3 \end{bmatrix}$ D $\begin{bmatrix} 166.8 \\ 133.2 \end{bmatrix}$ E $\begin{bmatrix} 166.9 \\ 133.1 \end{bmatrix}$

7 If $L_1 = TS_0 + B$, where $B = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$, then L_1 equals:

A $\begin{bmatrix} 70 \\ 220 \end{bmatrix}$ B $\begin{bmatrix} 150 \\ 180 \end{bmatrix}$ C $\begin{bmatrix} 170 \\ 160 \end{bmatrix}$ D $\begin{bmatrix} 176 \\ 164 \end{bmatrix}$ E $\begin{bmatrix} 210 \\ 120 \end{bmatrix}$

8 If $P_1 = TS_0 - 2B$, where $B = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$, then P_1 equals:

A $\begin{bmatrix} 140 \\ 100 \end{bmatrix}$ B $\begin{bmatrix} 170 \\ 100 \end{bmatrix}$ C $\begin{bmatrix} 180 \\ 100 \end{bmatrix}$ D $\begin{bmatrix} 170 \\ 160 \end{bmatrix}$ E $\begin{bmatrix} 180 \\ 180 \end{bmatrix}$

9 A system of state matrices S_n is defined by the matrix equation $S_{n+1} = GS_n$ where $G = \begin{bmatrix} 0 & -0.5 \\ 1.5 & 0.5 \end{bmatrix}$.

If $S_1 = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$, then S_2 equals:

A $\begin{bmatrix} -12.5 \\ -2.5 \end{bmatrix}$ B $\begin{bmatrix} -10 \\ 25 \end{bmatrix}$ C $\begin{bmatrix} 10 \\ 20 \end{bmatrix}$ D $\begin{bmatrix} 10 \\ 25 \end{bmatrix}$ E $\begin{bmatrix} 15 \\ 30 \end{bmatrix}$

10 $T = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}$ is a transition matrix. $S_5 = \begin{bmatrix} 22 \\ 18 \end{bmatrix}$ is a state matrix.

If $S_5 = TS_4$, then S_4 equals:

A $\begin{bmatrix} 18 \\ 22 \end{bmatrix}$ B $\begin{bmatrix} 20 \\ 20 \end{bmatrix}$ C $\begin{bmatrix} 21.8 \\ 18.2 \end{bmatrix}$ D $\begin{bmatrix} 22 \\ 18 \end{bmatrix}$ E $\begin{bmatrix} 18.2 \\ 21.2 \end{bmatrix}$

Review

11 A large population of birds lives on a remote island. Every night each bird settles at either location A or location B.

On the first night the number of birds at each location was the same. On each subsequent night, a percentage of birds changed the location at which they settled.

The movement of birds between the two locations is described by the transition matrix T shown opposite. Assume this pattern of movement continues.

$$T = \begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.8 & 0 \\ 0.2 & 1 \end{bmatrix} \end{array}$$

In the long term, the number of birds that settle at location A will:

A not change **B** gradually decrease to zero **C** gradually increase

D eventually settle at around 20% of the island's bird population

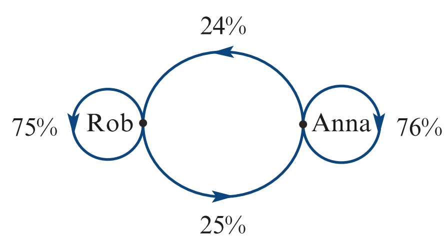
E eventually settle at around 80% of the island's bird population

Use the following information to answer Questions 12 and 13.

Two politicians, Rob and Anna, are the only candidates for a forthcoming election. At the beginning of the election campaign, people were asked for whom they planned to vote. The numbers were as per the table.

Candidate	*Number of people planing to vote for candidate*
Rob	5692
Anna	3450

During the election campaign, it is expected that people may change the candidate that they plan to vote for each week according to the transition diagram shown.



12 The total number of people who are expected to change the candidate that they plan to vote for 1 week after the election campaign begins is:

A 828 **B** 1423 **C** 2251 **D** 4269 **E** 6891

13 The election campaign will run for 10 weeks. If people continue to follow this pattern of changing the candidate they plan to vote for, the expected winner after 10 weeks will be:

A Rob by about 50 votes **B** Rob by about 100 votes

C Rob by fewer than 10 votes **D** Anna by about 100 votes

E Anna by about 200 votes

Written response questions

1 The Diisco (D) and the Spin (S) are two large music venues in the same city. They both open on the same Saturday night and will open on every Saturday night.

The matrix A_1 opposite is the attendance matrix for the first Saturday. This matrix shows the number of people who attended the first Diisco and the number of people who attended the Spin.

$$A_1 = \begin{array}{c} \begin{bmatrix} 500 \\ 240 \end{bmatrix} \begin{array}{c} D \\ S \end{array} \end{array}$$

The number of people expected to attend the second Saturday for each venue can be determined using the matrix equation

$$A_2 = GA_1$$

where G is the matrix

$$G = \begin{array}{c} \text{This Saturday} \\ \begin{array}{cc} D & S \end{array} \\ \begin{bmatrix} 1.2 & -0.4 \\ 0.2 & 0.6 \end{bmatrix} \begin{array}{c} D \\ S \end{array} \end{array} \text{ Next Saturday}$$

a **i** Determine A_2, the attendance matrix for the second Saturday.

ii What was the total attendance on the second Saturday.

Assume that the attendance matrices for successive Saturdays can be determined as follows:

$$A_3 = GA_2, \quad A_4 = GA_3, \text{ and so on such that } A_{n+1} = GA_n$$

b Determine the attendance matrix (with the elements written correct to the nearest whole number) for the eighth Saturday.

c Describe the way in which the number of people attending the Diiscoo is expected to change over the next 80 or so Saturdays.

Suppose instead that 500 people attend the first Diiscoo, and 490 people attend the Spin.

d Describe the way in which the attendance at both venues changes if attendance follows this prediction.

2 Suppose that the trees in a forest are classified into three age groups: young trees (0–15 years), middle-aged trees (16–30 years) and old trees (more than 30 years). A time period is 15 years, and it is assumed that in each time period:

- 10% of young trees, 20% of middle-aged trees and 40% of old trees die
- surviving trees enter into the next age group; old trees remain old
- dead trees are replaced by young trees.

Review

Complete the 3×3 transition matrix T to describe this.

$$\begin{array}{c c} & \begin{array}{ccc} Y & M & O \end{array} \\ \begin{array}{c} Y \\ M \\ O \end{array} & \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \end{array}$$

3 The following table represents a study of a particular population of marsupials, which has been divided into eight age groups. The table gives the initial population, birth rate and survival rate for each age group.

Age group	1	2	3	4	5	6	7	8
Initial population	0	100	100	50	0	0	0	0
Birth rate	0	0.1	0.9	0.2	0	0	0	0
Survival rate	0.98	0.95	0.95	0.9	0.7	0.5	0.1	0

a Write down the Leslie matrix for this population.

b Calculate S_2 and S_3.

c Estimate the long-term growth rate of the population.

4 The growth of algae in a particular lake is being studied to protect the ecology from a disastrous algal bloom. The algae can live for up to four days. So the population is divided into four age groups of one day each. The fertility rates and survival rates are being monitored so that the population can be modelled using a Leslie matrix.

At the beginning of the study in late winter (day 0), it was observed that the algae concentration in the lake was 3200 cells per millilitre of water, with equal numbers in each age group. The fertility rates on the four days of life were 0.2, 0.5, 0.6 and 0.4 respectively. The survival rate for each of the first three days of life was 0.7.

a Write down a Leslie matrix to represent this particular model.

b Find the population matrix for cells per millilitre of water on day 20, correct to three significant figures.

c Find the population matrix for cells per millilitre of water on day 21. Hence find the rate of change in the algae concentration per day at this stage.

d With the coming of spring on day 21, the fertility rates increased to 0.3, 0.6, 0.7 and 0.5; the survival rate remained unchanged. Find the population matrix after a further three weeks (i.e. on day 42).

e With the arrival of warmer weather on day 42, the fertility rates increased to 0.3, 0.7, 0.8 and 0.5; the survival rate increased to 0.85. Suppose that an algal bloom is declared if the concentration of algae reaches 100 000 cells per millilitre of water. Using trial and error, find the day of the study on which an algal bloom was declared.

Chapter 12

Revision: Matrices

Revision

12A Exam 1 style questions: Matrices

1 Matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \\ 6 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 6 \end{bmatrix}$ Matrix $Q = A \times B$. The element in row i and column j of matrix Q is q_{ij}. Element q_{33} is determined by the calculation

A $6 \times 2 + 2 \times 1$ **B** $2 \times 4 + 3 \times 2$ **C** $1 \times 4 + 4 \times 6$

D $6 \times 3 + 2 \times 0$ **E** $6 \times 4 + 2 \times 6$

2 Matrices P and W are defined below.

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad W = \begin{bmatrix} D \\ O \\ G \\ S \end{bmatrix}$$

$P^n W = W$ for $n =$

A 1 **B** 2 **C** 3 **D** 4 **E** 5

3 The element in row i and column j of matrix A is a_{ij}. A is a 3×3 matrix. It is constructed using the rule $a_{ij} = 2i + 4j$. A is

A $\begin{bmatrix} 6 & 10 & 14 \\ 8 & 12 & 16 \\ 10 & 14 & 18 \end{bmatrix}$ **B** $\begin{bmatrix} 6 & 10 & 14 \\ 10 & 14 & 18 \\ 8 & 12 & 16 \end{bmatrix}$ **C** $\begin{bmatrix} 6 & 10 & 8 \\ 10 & 12 & 14 \\ 14 & 16 & 18 \end{bmatrix}$

D $\begin{bmatrix} 14 & 16 & 18 \\ 6 & 10 & 8 \\ 10 & 12 & 14 \end{bmatrix}$ **E** $\begin{bmatrix} 6 & 10 & 14 \\ 10 & 12 & 18 \\ 14 & 14 & 20 \end{bmatrix}$

4 $\begin{bmatrix} \mathrm{O} & \mathrm{T} & \mathrm{S} & \mathrm{P} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$

A $\begin{bmatrix} \mathrm{S} & \mathrm{T} & \mathrm{O} & \mathrm{P} \end{bmatrix}$ **B** $\begin{bmatrix} \mathrm{P} & \mathrm{O} & \mathrm{S} & \mathrm{T} \end{bmatrix}$ **C** $\begin{bmatrix} \mathrm{T} & \mathrm{O} & \mathrm{P} & \mathrm{S} \end{bmatrix}$

D $\begin{bmatrix} \mathrm{O} & \mathrm{P} & \mathrm{S} & \mathrm{T} \end{bmatrix}$ **E** $\begin{bmatrix} \mathrm{T} & \mathrm{O} & \mathrm{S} & \mathrm{P} \end{bmatrix}$

5 $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ equals

A [0] **B** [1] **C** [2] **D** [3] **E** [5]

6 Matrix A has four rows and three columns.
Matrix B has three rows and four columns.
Matrix $C = B \times A$ has:

A two rows and three columns
B three rows and two columns
C three rows and three columns
D four rows and two columns
E four rows and three columns

7 Matrix $N = \begin{bmatrix} 14 \\ 3 \\ 12 \\ 2 \end{bmatrix}$. The matrix P is the permutation matrix such that the 4×1 matrix $M = P \times N$ has the smallest value at the top and the elements are in increasing order as you go down the matrix. The matrix P is

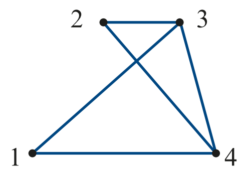
A $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
B $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
C $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

D $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$
E $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

8 Matrix X is of order $p \times q$ and matrix Y is of order $q \times r$. The matrix products XY^{-1} and $X^{-1}Y$ are both defined:

A for no values of p, q or r
B when $p = r$
C when $p = q = r$ only
D when $p = q$ only
E for all values of p, q and r

9 The diagram opposite is to be represented by a matrix A, where:



- element = 1 if the two points are joined by a line
- element = 0 if the two points are not connected.

The matrix A is:

A $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$
B $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$
C $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$
D $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$
E $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Revision

10 The order of three matrices: A, B and C are respectively

$$A : 3\times 5 \qquad B : 3\times 4 \qquad C : 4\times 5$$

The transpose of matrix A, for example, is written as A^T. What is the order of the product $(C^T \times B^T)\times A$?

A 4×3 **B** 3×4 **C** 4×5 **D** 5×5 **E** 3×3

11 A and B are $n\times n$ matrices. Which of the following is not always true?

A $(AB)^T = B^TA^T$
B $(A-B)(A+B) = A^2 - B^2$
C $(A^T)^T = A$
D $A + B = B + A$
E $A\times kB = kAB$ where k is a real number.

12 A is a 5×5 matrix.
B is a 8×5 matrix.
Which one of the following matrix expressions is defined?

A AB^T **B** A^3B **C** $BA + 3A$
D $A(BA)^{-1}$ **E** $A^2 - BA$

The following information is needed for Questions 13 to 15.

The recurrence relation $S_0 = \begin{bmatrix}100\\100\end{bmatrix}$, $S_{n+1} = TS_n$, where $T = \begin{bmatrix}0.1 & 0.8\\0.9 & 0.2\end{bmatrix}$, can be used to generate a sequence of matrices. In this sequence:

13 $S_1 =$

A $\begin{bmatrix}90\\110\end{bmatrix}$ **B** $\begin{bmatrix}100\\100\end{bmatrix}$ **C** $\begin{bmatrix}110\\90\end{bmatrix}$ **D** $\begin{bmatrix}120\\80\end{bmatrix}$ **E** $\begin{bmatrix}140\\60\end{bmatrix}$

14 S_5 is closest to:

A $\begin{bmatrix}90\\110\end{bmatrix}$ **B** $\begin{bmatrix}93.1\\106.9\end{bmatrix}$ **C** $\begin{bmatrix}95.5\\104.5\end{bmatrix}$ **D** $\begin{bmatrix}107.9\\92.1\end{bmatrix}$ **E** $\begin{bmatrix}106.9\\93.1\end{bmatrix}$

15 The steady-state matrix is closest to:

A $\begin{bmatrix}93.0\\107.0\end{bmatrix}$ **B** $\begin{bmatrix}93.6\\106.4\end{bmatrix}$ **C** $\begin{bmatrix}94.1\\105.9\end{bmatrix}$ **D** $\begin{bmatrix}106.4\\93.6\end{bmatrix}$ **E** $\begin{bmatrix}107\\93\end{bmatrix}$

16 Consider the permutation matrix $P = \begin{bmatrix}0 & 0 & 0 & 1\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}$ The row matrix $X =$ $\begin{bmatrix}B & O & A & R\end{bmatrix}$ is permuted by finding the product XP^2. The result is

A $\begin{bmatrix} O & A & R & B \end{bmatrix}$ **B** $\begin{bmatrix} A & O & R & B \end{bmatrix}$ **C** $\begin{bmatrix} A & R & B & O \end{bmatrix}$

D $\begin{bmatrix} A & B & R & O \end{bmatrix}$ **E** $\begin{bmatrix} B & O & A & R \end{bmatrix}$

The following information relates to Questions 17 to 19.

People in a suburb are very fitness conscious but get sick of going to the same gymnasium. It is found that they go to four gymnasiums A, B, C and D but there are changes in attendance every month. It is found that following transition matrix can be used to predict the number of people at each of the four gymnasiums each month.

$$\begin{array}{c} \text{This month} \\ \begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ T = & \begin{bmatrix} 0.30 & 0.25 & 0.1 & 0.1 \\ 0.45 & 0.5 & 0.15 & 0.1 \\ 0.13 & 0.2 & 0.45 & 0.2 \\ 0.12 & 0.05 & 0.3 & 0.6 \end{bmatrix} \begin{array}{c} A \\ B \\ C \\ D \end{array} \end{array} \end{array} \quad \text{Next month} \qquad S_0 = \begin{bmatrix} 300 \\ 300 \\ 300 \\ 300 \end{bmatrix}$$

17 Three hundred people go to gym C in September. How many of these people are in gym A in October.

A 280 **B** 290 **C** 30 **D** 320 **E** 330

18 This transition matrix predicts that, in the long term, the people

A go only to A
B go only to B
C go to only A and C
D go only to B and D
E continue to visit all four gyms

19 After 12 months the total number of customers at gymnasiums B and C is closest to

A 500 **B** 600 **C** 620 **D** 650 **E** 700

20 A town has two hardware shops: Fairtrade (F) and Bungles (B). The percentage of shoppers at each shop changes from day to day, as shown in the transition matrix T.

$$\begin{array}{cc} & \text{Today} \\ & \begin{array}{cc} F & B \end{array} \\ \text{Tomorrow} \begin{array}{c} F \\ B \end{array} & \begin{bmatrix} 65\% & 70\% \\ 35\% & 30\% \end{bmatrix} \end{array}$$

On a particular Monday, 35% of shoppers went to Fairtrade. The matrix recursion relation $S_{n+1} = TS_n$ is used to model this situation. The percentage of shoppers who go to Fairtrade on Wednesday of the same week is closest to

A 65% **B** 66% **C** 67% **D** 68% **E** 69%

Revision

21 The matrix gives the results of a table tennis round robin competition between five players: A, B, C, D and E. A '1' indicates a win of 'row' over 'column'.

$$\text{Winner}\quad \begin{array}{c} \text{Loser} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \\ D \\ E \end{array}$$

When the sum of the one-step and two-step dominances is used to rank the players in this competition, the ranking is:

A B, E, D, A, C
B B, E, C, D, A
C B, E, D, C, A
D E, B, D, A, C
E E, B, D, C, A

22 A taxi company has two depots at A and B. They always keep x taxis at A and z taxis at B. The transition matrix T shows how the taxis change their nightly location.

$$\begin{array}{cc} & \text{Today} \\ & \begin{array}{cc} A & B \end{array} \\ \text{Tomorrow} \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 65\% & 70\% \\ 35\% & 30\% \end{bmatrix} \end{array}$$

On a particular night 14 taxis came from depot B to depot A. How many taxis in the fleet?

A 24 **B** 36 **C** 42 **D** 60 **E** 80

23 The matrix S_{n+1} is determined from the matrix S_n using the recurrence relation $S_{n+1} = T \times S_n - C$, where

$$T = \begin{bmatrix} 0.3 & 0.8 & 0.4 \\ 0.6 & 0.1 & 0.3 \\ 0.1 & 0.1 & 0.3 \end{bmatrix}, \qquad S_0 = \begin{bmatrix} 80 \\ 60 \\ 20 \end{bmatrix}, S_1 = \begin{bmatrix} 58 \\ 41 \\ 17 \end{bmatrix}$$

and C is a column matrix. Matrix S_2 is equal to

A $\begin{bmatrix} 58 \\ 41 \\ 17 \end{bmatrix}$ **B** $\begin{bmatrix} 35 \\ 25 \\ 12 \end{bmatrix}$ **C** $\begin{bmatrix} 14.85 \\ 8.95 \\ 4.2 \end{bmatrix}$ **D** $\begin{bmatrix} 30.5 \\ 26.5 \\ 15.5 \end{bmatrix}$ **E** $\begin{bmatrix} 59.5 \\ 41.5 \\ 15 \end{bmatrix}$

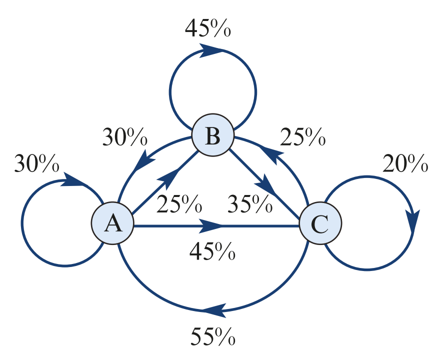
24 A group of people travel every weekday but they have a choice of the type of transport they use: train (T), bus (B) or Car (C). They change from day to day according to the transition matrix.

$$\begin{array}{cc} & \text{Today} \\ & \begin{array}{ccc} T & B & C \end{array} \\ \text{Tomorrow} \begin{array}{c} T \\ B \\ C \end{array} & \begin{bmatrix} 65\% & 70\% & 50\% \\ 20\% & 10\% & 25\% \\ 15\% & 20\% & 25\% \end{bmatrix} \end{array}$$

On Monday 30% take a car and 70% take the train and no one takes the bus. What is the percentage of people who are not expected to change their mode of transport from Tuesday to Wednesday?

A 50.45% **B** 46% **C** 48.45% **D** 45.975% **E** 100%

25 The transition matrix that can be used to represent the information in the diagram shown is:



A $$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ To \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0.30 & 0.30 & 0.55 \\ 0.25 & 0.45 & 0.25 \\ 0.45 & 0.35 & 0.20 \end{bmatrix} \end{array}$$

B $$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ To \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0.30 & 0.30 & 0.55 \\ 0.60 & 0.05 & 0.35 \\ 0.1 & 0.30 & 0.20 \end{bmatrix} \end{array}$$

C $$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \text{To} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0.75 & 0.10 & 0.15 \\ 0.60 & 0.05 & 0.35 \\ 0.50 & 0.30 & 0.20 \end{bmatrix} \end{array}$$

D $$\begin{array}{cc} & \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \text{To} \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} & \begin{bmatrix} 0.75 & 0.05 & 0.15 \\ 0.10 & 0.60 & 0.20 \\ 0.45 & 0.35 & 0.20 \end{bmatrix} \end{array}$$

E $$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \text{To} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0.30 & 0.30 & 0.55 \\ 0.15 & 0.35 & 0.50 \\ 0.45 & 0.35 & 0.20 \end{bmatrix} \end{array}$$

Revision

12B Exam 2 style questions: Matrices

1 At some stage, the matrix $C = \begin{array}{c} \begin{bmatrix} 1.316 \\ 1.818 \\ 0.167 \end{bmatrix} \end{array} \begin{array}{l} \textit{US dollar rate} \\ \textit{Euro rate} \\ \textit{HK dollar rate} \end{array}$

could be used to convert US dollars (US\$), European euro (€) and Hong Kong dollars (HK\$) into Australian dollars (A\$).

a What is the order of matrix C?

You return from an overseas trip with US\$102, €262 and HK\$516.

We can write this information as the matrix $H = \begin{bmatrix} 102 & 262 & 516 \end{bmatrix}$.

b What is the order of matrix H?

c Is the matrix product HC defined? Why?

d **i** Evaluate the matrix product HC.

ii What does the matrix product represent and why?

e Matrix $M = \begin{bmatrix} 125 & 216 & 54 \\ 0 & 34 & 453 \\ 0 & 356 & 0 \end{bmatrix}$ gives the amounts in US dollars, euros and HK dollars that three other people want to change into Australian dollars. The rows represent people. The columns indicate the amounts of each currency they have. Use the conversion matrix C and matrix multiplication to generate a matrix that displays the amounts of Australian currency that each person will receive.

2 Lake Blue and Lake Green are two small lakes connected by a channel. This enables fish to move between the two lakes on a daily basis. Research has shown that each day:

- 67% of fish in Lake Blue stay in Lake Blue
- 33% of fish in Lake Blue move to Lake Green
- 72% of fish in Lake Green stay in Lake Green
- 28% of fish in Lake Green move to Lake Blue.

a Construct a transition matrix, T, of the form:

$$\begin{array}{ccc} & & \textit{From} \\ & & \textit{Blue} \quad \textit{Green} \\ \textit{To} & \begin{array}{r} \textit{Blue} \\ \textit{Green} \end{array} & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \end{array}$$

to describe this situation.

b Today there are currently 4000 fish in Lake Blue and 6000 fish in Lake Green. Write down a column matrix, S_0, that describes this situation.

c How many fish do you expect to be in each lake tomorrow?

d How many fish do you expect to be in each lake in 3 days' time?

e In the long term, how many fish do you expect to be in each lake?

3 The life cycle of a type of insect can be descibed by a Leslie matrix. The stages of life which are used are

- Egg E
- Juvenile J
- Young adult Y
- adult A

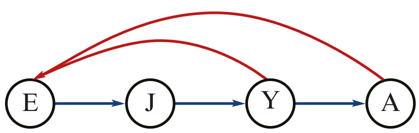
From stage of life

$$\begin{array}{cc} & \begin{array}{cccc} E & J & Y & A \end{array} \\ L = & \begin{bmatrix} 0 & 0 & 20 & 30 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \end{bmatrix} \begin{array}{c} E \\ J \\ Y \\ A \end{array} \end{array} \text{ To next stage of life}$$

The initial population is described by the 4×1 state matrix $S_0 = \begin{bmatrix} 897 \\ 438 \\ 43 \\ 2 \end{bmatrix} \begin{array}{c} E \\ J \\ Y \\ A \end{array}$

The time period in this model is one week and the state matrix after n weeks can be determined by $S_{n+1} = LS_n$.

a From the Leslie matrix complete the life-cycle diagram.



b How many eggs are produced in total each week?

c How many insect eggs are there after one week?

d How many weeks pass before there are more than 1000 eggs?

e What percentage of young adult insects become adult insects each week?

f How many insects of every type (including eggs) are there after?

i 8 weeks **ii** 9 weeks

Give answers correct to the nearest whole number.

g It is known that after some weeks the rate of increase per week of the entire population (including eggs) is very close to being a constant. Use the results of **f** to give an estimate of this rate per week as a percentage correct to the nearest percent. (That is, in the form $a\%$, where a is a whole number.)

4 The following transition matrix, T, is used to help predict attendance at a weekly club meeting

$$\begin{array}{cc} & \textit{This meeting} \\ & \begin{array}{cc} \textit{attend} & \textit{not attend} \end{array} \\ \mathrm{T} = & \begin{bmatrix} 0.80 & 0.30 \\ 0.20 & 0.70 \end{bmatrix} \begin{array}{l} \textit{attend} \\ \textit{not attend} \end{array} \end{array} \textit{ Next meeting}$$

S_1 is the attendance matrix for the first club meeting of the year.

$$S_1 = \begin{bmatrix} 110 \\ 40 \end{bmatrix} \begin{array}{l} \textit{attend} \\ \textit{not attend} \end{array}$$

S_1 indicates that 110 club members attended the first meeting and 40 club members did not attend the first meeting.

a Use T and S_1 to:

i determine S_2, the attendance matrix for the second meeting.

ii predict the number of club members attending the third meeting.

b Write down a matrix equation for S_n in terms of T, n and S_1.

c How many weeks does it take for the attendance to fall below 91?

d In the long term, how many club members are predicted to attend meetings?

5 A chemist wholesaler stocks three brands of hand sanitizer Cleanup (C), Loveneasy (L) and Orama (O). The number of half litre bottles of these sanitizers sold in March 2022 is shown in matrix A below.

$$A = \begin{bmatrix} 3000 \\ 1500 \\ 2500 \end{bmatrix} \begin{array}{l} C \\ L \\ O \end{array}$$

a **i** What is the order of matrix A?

ii The wholesaler expected that in April 2022 the sales of all three brands of sanitizer would increase by 20%. She multiplied matrix A by a real number, k, to determine the expected volume of sales for April. What is the value of k?

b A small chain of 4 chemists operate through this wholesaler and communicate with each other rather inefficiently through an online facility. The communication links are shown in this communication matrix receiver

$$M = \text{sender} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

- the '1' in row A, column B indicates that A can send information to B.
- the '0' in row D, column C indicates that D cannot send information to C.

i Which pairs of chemists can send information directly to each other?

ii D needs to send documents to C. What is the sequence of communication links that will successfully get the information from D to C?

iii Matrix M^2 below shows the number of two-step communication links between each pair of chemists

$$M^2 = \text{sender} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \text{receiver} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Communication from C to B in two different two-step communication links is possible. List each two-step communication link for this pair.

c The total purchases of the three hand sanitizers, Cleanup (C), Loveneasy (L) and Orama (O), change from month. Let T denote the transition matrix and S_n represent the state matrix describing the number of shoppers buying each brand n months after June 2022.

$$T = \begin{array}{c} \text{this month} \\ \begin{array}{ccc} C & L & O \end{array} \\ \begin{bmatrix} 0.7 & 0.8 & 0.5 \\ 0.05 & 0.1 & 0.2 \\ 0.25 & 0.1 & 0.3 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ C \\ L \\ O \end{array} \text{next month}$$

The initial state matrix S_0 below shows the number of shoppers at the four chemists who bought each brand of hand sanitizer in June 2022.

$$S_0 = \begin{bmatrix} 200 \\ 150 \\ 250 \end{bmatrix} \begin{array}{c} C \\ L \\ O \end{array}$$

i Calculate S_1.

ii How many of the shoppers bought the same sanitizer in July as they did in June 2022?

iii Consider the shoppers who were expected to buy Cleanup in September 2022. What percentage of these shoppers also bought Cleanup in August 2022?

d The shopping habits changed over the months. A rule to model this situation is $S_{n+1} = T \times S_n + B$, where S_n represents the state matrix describing the number of shoppers n months after June 2021.

Here T is as above and $S_0 = \begin{bmatrix} 200 \\ 150 \\ 250 \end{bmatrix} \begin{array}{c} C \\ L \\ O \end{array}$

If $S_1 = \begin{bmatrix} 405 \\ 90 \\ 165 \end{bmatrix} \begin{array}{c} C \\ L \\ O \end{array}$ find B.

6 In a cinema complex there are four cinemas A, B, C and D. They vary in the number of seats, the standard of seating and other amenities. The number of empty (E) and occupied seats (O) on a Friday afternoon is shown in the matrix below.

$$M = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cc} E & O \end{array} \\ \begin{bmatrix} 30 & 60 \\ 60 & 120 \\ 20 & 50 \\ 10 & 85 \end{bmatrix} \end{array}$$

a What is the order of matrix M?

b What is the

i total number of seats

ii percentage of seats which are occupied on the Friday afternoon.

The cost of admission to each of the cinemas is given in the following matrix

$$Q = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} \$8.50 & \$12.60 & \$18.00 & \$25.80 \end{bmatrix} \end{array}$$

c The total payments for this Friday afternoon can be determined by the calculation $Total = Q \times L$, where L is a 4×1 matrix. Write the matrix L with its entries and calculate the matrix $Total$.

7 A commercial art school offers classes in pottery (P), sculpture (S), drawing (D) and weaving (W). Students are allowed to change activities every month. In January 2022 the number of students in each class can be described by the state matrix and the movement from one month to the next is described by the transition matrix T.

$$S_0 = \begin{bmatrix} 30 \\ 30 \\ 30 \\ 30 \end{bmatrix} \qquad T = \begin{array}{c} \\ P \\ S \\ D \\ W \end{array} \begin{array}{c} \begin{array}{cccc} P & S & D & W \end{array} \\ \begin{bmatrix} 0.6 & 0.2 & 0.4 & 0.1 \\ 0.2 & 0.3 & 0.1 & 0.6 \\ 0.1 & 0.2 & 0.2 & 0.2 \\ 0.1 & 0.3 & 0.3 & 0.1 \end{bmatrix} \end{array}$$

Let S_n be the state matrix n months after January 2022. Use $S_n = T^n S_0$

a Determine S_1 and S_2.

b The number in each class stabilises quite quickly according to this model. Find S_{10} and give the numbers in each class (rounded to the nearest whole number) after 10 months.

c How many of those doing pottery moved to weaving from January to February?

d How many people stayed in the same activity from January to February?

e How many of those doing pottery moved to weaving from February to March?

f How many people stayed in the same activity from February to March?

8 A local take-away shop popular with students sells hamburgers (H), fish and chips (F) and sandwiches (S). The number of each item sold over three weeks is shown in matrix M.

$$M = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{l} \begin{array}{ccc} H & F & S \end{array} \\ \begin{bmatrix} 160 & 200 & 50 \\ 180 & 210 & 55 \\ 210 & 240 & 80 \end{bmatrix} \end{array}\begin{array}{l} \\ \textit{week } 1 \\ \textit{week } 2 \\ \textit{week } 3 \end{array}$$

a How many hamburgers were sold in these three weeks?

b What does the element m_{23} indicate?

c the total sales in dollars for three weeks for each of these items is given in the matrix below.

$$C = \begin{array}{c} H \\ F \\ S \end{array}\begin{bmatrix} 8250 \\ 9100 \\ 2220 \end{bmatrix}$$

Determine the unit cost of each of these items.

d The matrix expression shown gives the total cost of all hamburgers and sandwiches in these three weeks.

$$L \times C$$

Matrix L is a 1×3 matrix. Write down matrix L.

Chapter 13

Graphs, networks and trees: travelling and connecting problems

Chapter objectives

- ▶ What is a graph?
- ▶ How do we identify the features of a graph?
- ▶ How do we draw a graph?
- ▶ How do we apply graphs in practical situations?
- ▶ How do we construct an adjacency matrix from a graph?
- ▶ How do we define and draw a planar graph?
- ▶ How do we identify the type of walk on a graph?
- ▶ How do we find the shortest path between two vertices of a graph?
- ▶ How do we find the minimum distance required to connect all vertices of a graph?

In this chapter graphs and their use as networks representing connections between objects will be introduced, in addition to exploring their properties and applications.

Problems involving networks will be investigated and you will learn unique algorithms such as Djikstra's and Prim's to solve such problems.

13A Graphs and networks

Learning intentions

- To be able to define and identify a graph, vertex, edge and loop.
- To be able to find the degree of a vertex and the sum of degrees.
- To be able to describe the features of a graph.
- To be able to define and identify a planar graph and its faces.
- To be able to apply Euler's formula and use it to verify if a graph is planar.

Representing connections with graphs

There are many situations in everyday life that involve connections between people or objects. Towns are connected by roads, computers are connected to the internet and people connect to each other through being friends on social media. A diagram that shows these connections is called a **graph**.

Edges and vertices

Six people – Anna, Brett, Cora, Dario, Ethan and Frances – have connections on a social media website. The graph shows these connections.

Anna is a friend of Brett, Ethan and Frances.

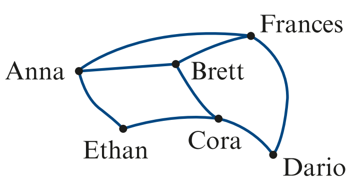
Brett is a friend of Anna, Cora and Frances.

Cora is a friend of Brett, Dario and Ethan.

Dario is a friend of Cora and Frances.

Ethan is a friend of Anna and Cora.

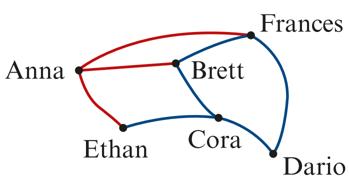
Frances is a friend of Anna, Brett and Dario.



The graph shows each of the people as a dot called a **vertex**. The *vertices* (plural of vertex) are joined together by a line that indicates the social media friendship between the people. The lines that join the vertices in the graph are called **edges**.

Degree of a vertex

Anna has three friends. The vertex representing Anna has three edges attached to it, connecting Anna to one of her friends. The number of edges attached to a vertex is called the **degree** of that vertex.



The degree of the vertex representing Anna is *odd*, because there is an odd number of edges connected to it. The degree of the vertex representing Dario is *even* because there is an even number of edges connected to it.

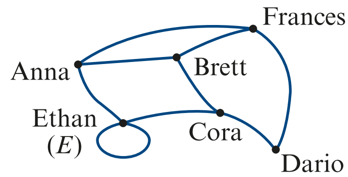
In symbolic form, we can let the letter A represent the vertex for Anna. The degree of this vertex can be written as $\deg(A)$. In this graph, $\deg(A) = 3$.

Loops

Imagine that Ethan is able to add himself as a friend on the social media website.

The edge representing this connection would connect the vertex representing Ethan, E, back to itself. This type of edge is called a **loop**.

A loop is attached twice to a vertex and so it will contribute two degrees. So $\deg(E) = 4$.



Edges, vertices and loops

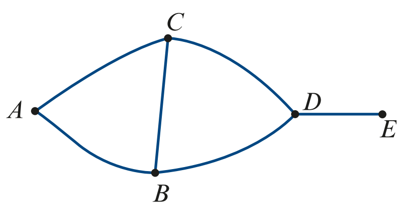
- A graph consists of **vertices** joined by **edges**.
- The number of edges attached to a vertex is called the degree of the vertex. The symbolic form for the degree of vertex A is $\deg(A)$.
- A loop connects a vertex to itself. Loops contribute two degrees to a vertex.

Describing graphs

Graphs that represent connections between objects can take different forms and have different features. This means that there is a variety of ways to describe these graphs.

Simple graphs

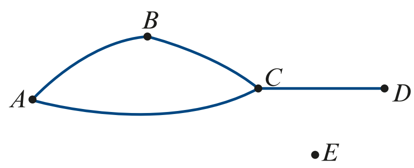
Simple graphs do not have any loops. There are no duplicate or **multiple edges** either.



Isolated vertex

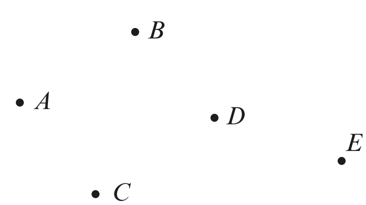
A graph has an **isolated vertex** if there is a vertex that is not connected to another vertex by an edge.

The isolated vertex in this graph is E, because it is not connected to any other vertex by an edge.

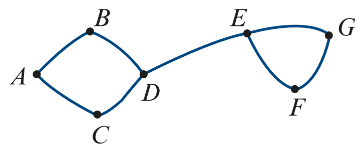


Degenerate graphs

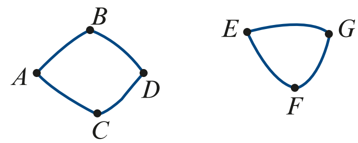
Degenerate graphs have all vertices isolated. This means that there are no edges in the graph at all.



Connected graphs and bridges

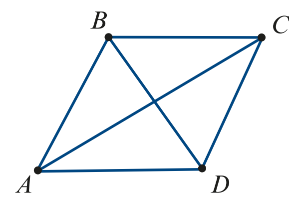


A **connected graph** has every vertex connected to every other vertex, either directly or indirectly via other vertices. The graph on the right is connected. A **bridge** is an edge in a connected graph that, if removed, will cause the graph to be disconnected. The graph on the right has a bridge connecting vertex D to vertex E.



The graph on the right shows the bridge from vertex D to vertex E removed. There are now two separate sections of the graph that are not connected to each other.

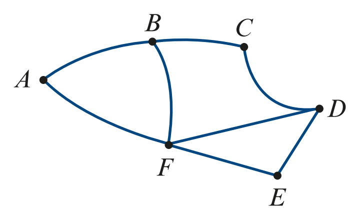
Complete graphs



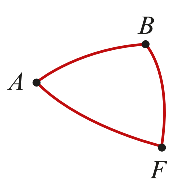
If there is an edge between every pair of vertices, the graph is called a **complete graph**. Every vertex in the graph is connected directly by an edge to every other vertex in the graph.

Subgraphs

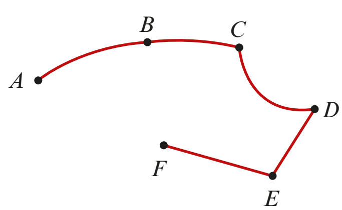
A **subgraph** is a part of a larger graph. All of the edges and vertices in the subgraph must exist in the original graph. If there are extra edges or vertices, the graph will not be a subgraph of the larger graph.



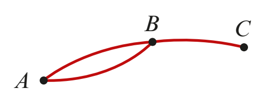
Graph 1



Graph 2



Graph 3



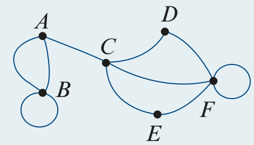
Graph 4

Graphs 2 and 3 above are subgraphs of graph 1. All of the vertices and edges in graphs 2 and 3 exist in graph 1.

Graph 4 above is not a subgraph of graph 1. There are two edges connecting vertex A to vertex B, but in graph 1 there is only one.

Example 1 **Graphs**

A connected graph is shown on the right.



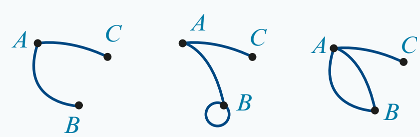
a What is the degree of vertex C?

b Which vertices have a loop?

c What is the degree of vertex F?

d A bridge exists between two vertices. Which vertices are they?

e Draw a subgraph of this graph that involves only vertices A, B and C.

Explanation	Solution
a Count the number of times an edge connects to vertex C. There are four connections.	The degree of vertex C is 4. deg(C) = 4.
b A vertex has a loop if an edge connects it to itself.	Vertex B and vertex F have loops.
c Count the number of times an edge connects to vertex F. Remember that a loop contributes two degrees.	The degree of vertex F is 5. deg(F) = 5
d Look for an edge that, if removed, would disconnect the graph.	A bridge exists between vertex A and vertex C.
e There are a few possible answers for this question. Some are shown on the right.	

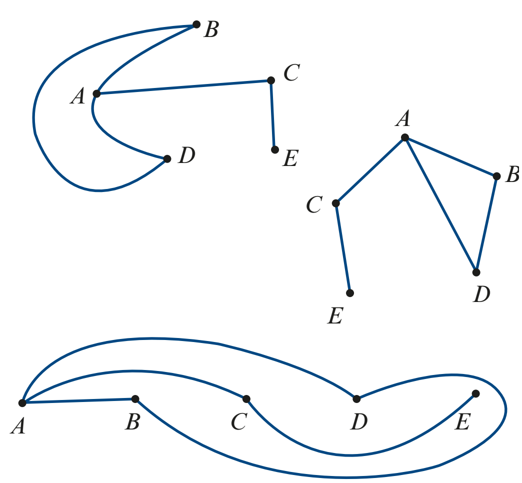
Equivalent (Isomorphic) graphs

All of the graphs shown in the diagram below contain exactly the same information. For example, the edge between vertex E and C exists in all of them. The vertex A is connected to B, D and C as well.

The location of the vertices and edges in the diagram are unimportant. As long as the connections are all represented accurately, the graph can be drawn in any way that you prefer.

The first of the graphs has some curved edges and the second has all straight edges. The third has the vertices arranged in a straight line.

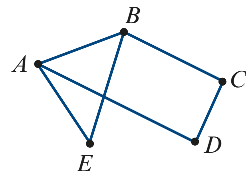
The style that they are drawn in and the position of the vertices relative to each other is unimportant. It *is* important that the information contained in the graph – the connections between the vertices – is correct.



All of these graphs are considered to be *equivalent* to each other because they all contain identical information. Each has edges connecting the same vertices. Graphs that contain identical information like this are called **equivalent graphs** or **isomorphic graphs**.

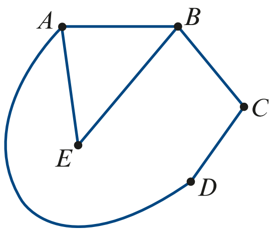
Planar graphs

The graph opposite has two edges that overlap. It is important to note that there is *no vertex at the point of overlap of the edges.*



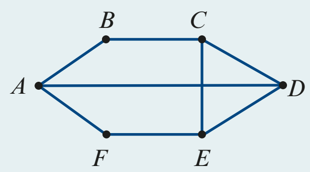
It can help to think of an edge as an insulated electrical wire. It is quite safe to cross two such electrical wires because the wires themselves never touch and never interfere with each other. The edges that cross over in this diagram are similar, in that they do not intersect and do not interfere with each other.

If a graph has edges that cross, it may be possible to redraw the graph so that the edges no longer cross. The edge between vertices A and D has been moved, but none of the information in the graph has changed. Graphs where this is possible are called **planar graphs**. If it is impossible to draw an equivalent graph without crossing edges, the graph is called a *non-planar graph*.



Example 2 Redrawing a graph in planar form

Show that this graph is planar by redrawing it so that no edges cross.



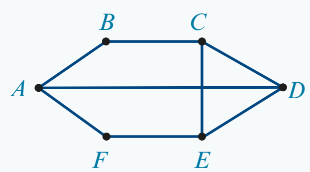
Explanation

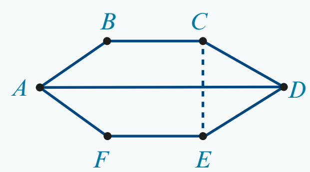
1 Choose one of the edges that crosses over another edge.

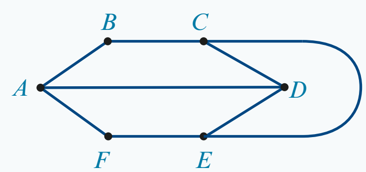
2 Remove it temporarily from the graph.

3 Redraw the edge between the same vertices but without crossing over another edge.

Solution

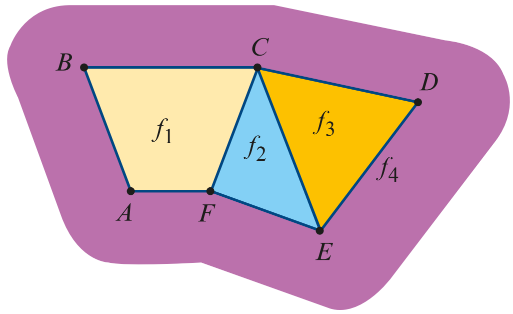






Euler's formula

Leonard Euler (pronounced 'oiler') was one of the most prolific mathematicians of all time. He contributed to many areas of mathematics and his proof of the rule named after him is considered to be the beginning of the branch of mathematics called topology.



Faces

A planar graph defines separate regions of the paper it is drawn on. These regions are enclosed spaces that you could colour in and these regions are called **faces**. An often-forgotten face of a graph is the space outside of the graph itself, covering the infinite space around it. This face is labelled f_4 in the graph above.

The number of faces for a graph can be counted. In the graph shown above, there are four faces, labelled f_1, f_2, f_3 and f_4.

Euler's formula

There is a relationship between the number of vertices, v, the number of edges, e, and the number of faces, f, in a connected planar graph.

In words: number of vertices + number of faces = number of edges +2

In symbols: $v + f = e + 2$

Euler's formula

For any planar graph:

$$v + f = e + 2$$

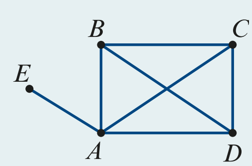
where v is the number of vertices, e is the number of edges and f is the number of faces in the graph.

Example 3 Verifying Euler's formula

For the graph shown on the right:

a redraw the graph into planar form

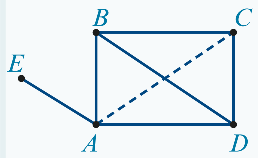
b verfiy Euler's formula for this graph.

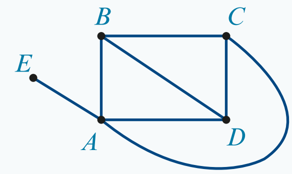


Explanation

a Temporarily remove an edge that crosses another edge and redraw it so that it does not cross another edge.

Solution





b Count the number of vertices, edges and faces.

In the planar graph there are five vertices, seven edges and four faces.

$v + f = e + 2$

$5 + 4 = 7 + 2$

$9 = 9$

Euler's formula is verified.

Example 4 Using Euler's formula

A connected planar graph has six vertices and nine edges. How many faces does the graph have? Draw a connected planar graph with six vertices and nine edges.

Explanation

a Write down the known values.

Solution

$v = 6 \quad e = 9$

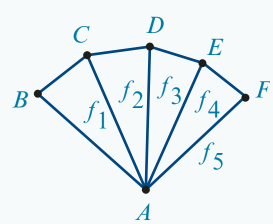
b Substitute into Euler's formula and solve for the unknown value.

$$\begin{aligned} v + f &= e + 2 \\ 6 + f &= 9 + 2 \\ 6 + f &= 11 \\ f &= 11 - 6 \\ f &= 5 \end{aligned}$$

This graph has five faces, labelled f_1, f_2, f_3, f_4 and f_5.

c Sketch the graph.

Note: There are other possible graphs.



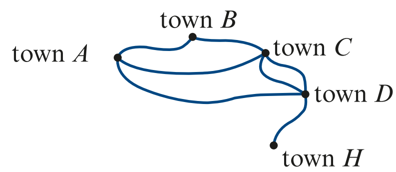
Skillsheet

Exercise 13A

Drawing and describing graphs

Example 1

1 This section of a road map can be considered a graph, with towns as vertices and the roads connecting the towns as edges.



a Give the degree of:

i town A **ii** town B **iii** town H.

b What is the sum of the degrees of all the vertices of this graph?

c A bridge exists between two towns. Which towns are they?

d Draw a subgraph of this road map that contains only towns H, D and C.

2 Draw a graph that:

a has three vertices, two of which are odd

b has four vertices and five edges, one of which is a loop

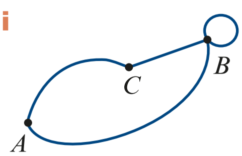
c has six vertices, eight edges and one bridge

d has six vertices, two of which are odd, and contains a subgraph that is a triangle.

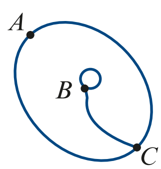
Equivalent graphs

3 In each question below, three graphs are isomorphic and the fourth is not. Identify the graph which is not isomorphic to the others.

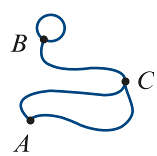
a **i**



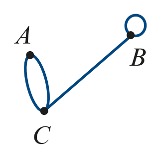
ii



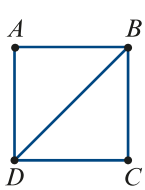
iii



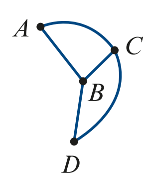
iv



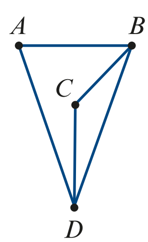
b **i**



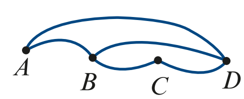
ii



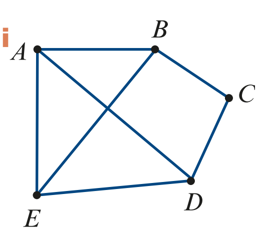
iii



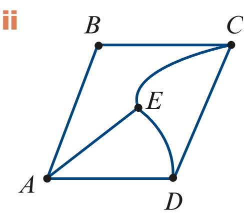
iv



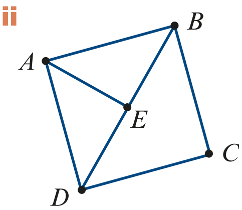
c **i**



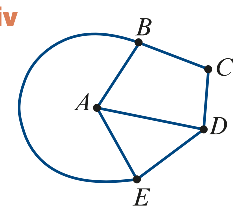
ii



iii



iv

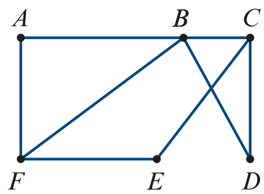


Drawing planar graphs

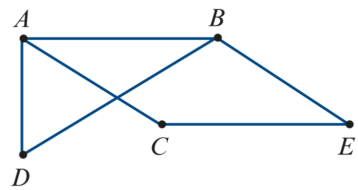
Example 2

4 Where possible, show that the following graphs are planar by redrawing them in a suitable planar form.

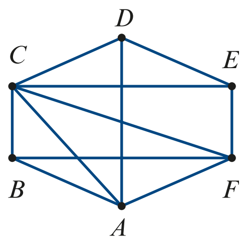
a



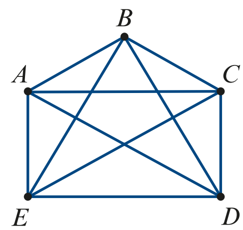
b



c



d



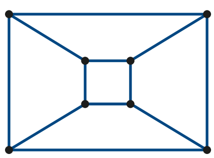
Euler's formula

Example 3

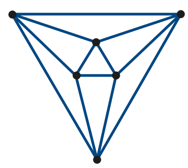
5 For each of the following graphs:

i state the values of v, e and f **ii** verify Euler's formula.

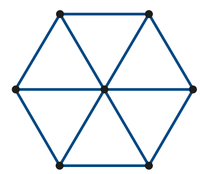
a



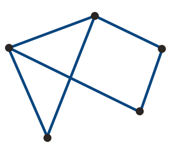
b



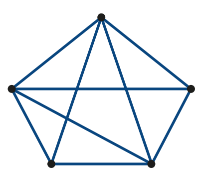
c



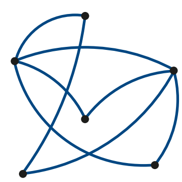
d



e



f



6 For a planar connected graph, find:

a f, if $v = 8$ and $e = 10$ **b** v, if $e = 14$ and $f = 4$ **c** e, if $v = 10$ and $f = 11$.

Properties of graphs

Example 4

7 A connected planar graph has eight vertices and thirteen edges. Find the number of faces of this graph.

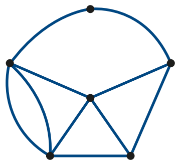
8 A connected graph has five vertices and seven edges. Find the sum of the degrees of the vertices.

9 Find the number of edges needed to make a complete graph with six vertices.

Exam 1 style questions

10 Consider the graph opposite.

The number of vertices with a degree of 4 is

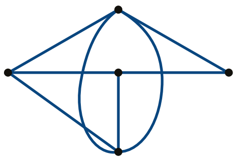


A 1 **B** 2 **C** 3
D 4 **E** 5

11 A planar graph has four faces. The graph could have

A Seven vertices and seven edges
B Seven vertices and four edges
C Seven vertices and five edges
D Four vertices and seven edges
E Five vertices and seven edges

Use the following information for questions 12 and 13.



12 The number of faces in the graph above is

A 3 **B** 4 **C** 5 **D** 6 **E** 7

13 Consider the following five statements about the graph above:

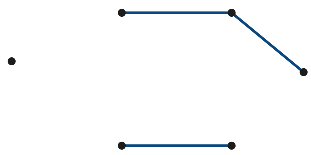
- The graph is planar.
- It is a simple graph.
- It is a complete graph.
- The graph contains a bridge.
- The sum of degrees of the vertices is 16.

How many of these statements are true?

A 1 **B** 2 **C** 3 **D** 4 **E** 5

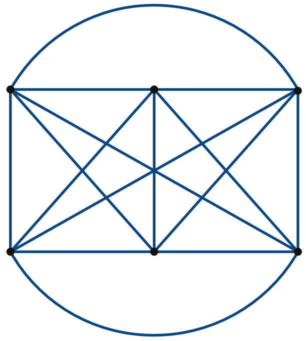
14 Consider the graph shown opposite.

The minimum number of edges that must be added to make this a complete graph is



A 1 **B** 2 **C** 6
D 8 **E** 12

15 The following graph with six vertices is a complete graph. Edges are removed so that the graph will have the minimum number of edges to remain connected. The number of edges that are removed is



A 6 **B** 8 **C** 10 **D** 12 **E** 14

13B Adjacency matrices

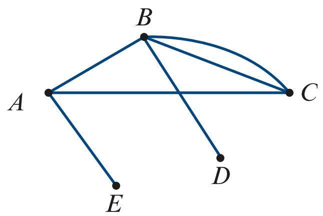
Learning intentions

- To be able to use an adjacency matrix to represent a graph.

Summarising the connections in a graph

A matrix can be used to summarise the information in a graph. A matrix that records the number of connections between vertices of a graph is called an **adjacency matrix**.

A graph and the adjacency matrix for that graph are shown here.



$$\begin{array}{c|c} & \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The adjacency matrix has:

- five rows and five columns, one for each vertex in the graph
- row and column labels that match the vertices in the graph, A, B, C, D, E
- a '0' in the intersection of row A and column D because there is no edge connecting A to D
- a '0' in the intersection of row A and column A because there is no edge connecting A to itself; that is, there is no loop at vertex A
- a '1' in the intersection of row A column B because there is one edge connecting A to B
- a '2' in the intersection of row C and column B because there are two edges connecting C to B.

The number of edges between every other pair of vertices in the graph is recorded in the adjacency matrix in the same way.

Example 5 Drawing a graph from an adjacency matrix

Draw the graph that is represented by the following adjacency matrix.

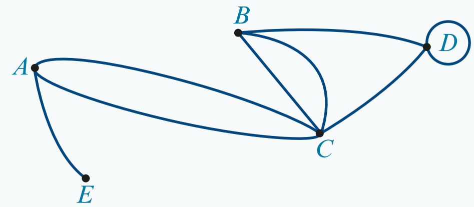
$$\begin{array}{c|c} & \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Explanation

1 Draw a dot for each vertex and label A to E.

2 There is a '2' in the intersection of row A and column C. This means there are two edges connecting vertex A and vertex C. Add these to the graph.

Solution



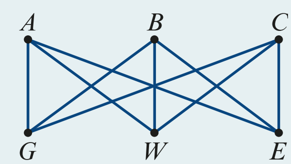
3 Note the '1' in the intersection of row D and column D. This shows that there is a loop at vertex D.

4 Look at every intersection of row and column and add edges to the graph, if they do not already exist.

Note: This graph is drawn as a planar graph, but this is not strictly necessary unless required by the question.

Example 6

Construct an adjacency matrix that can be used to represent the graph opposite. This graph represents the ways that three houses A, B and C are connected to three utility outlets, gas (G), water (W) and electricity (E).



Explanation

The convention used to enter the values is the same as discussed above.

Solution

$$\begin{array}{c} \\ A \\ B \\ C \\ G \\ W \\ E \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & G & W & E \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The graph in Example 6 is called a bipartite graph as the set of vertices is separated into two sets of objects Houses (A, B, C) and Utility outlets (G, W, E) with each edge connecting a vertex in each set. You will meet bipartite graphs again in Chapter 14 when studying allocation problems.

Adjacency matrices

The adjacency matrix A of a graph is an $n \times n$ matrix in which, for example, the entry in row C and column F is the number of edges joining vertices C and F.

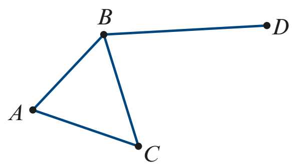
A loop is a single edge connecting a vertex to itself.

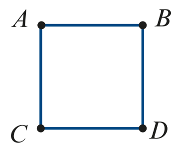
Loops are counted as one edge.

Exercise 13B

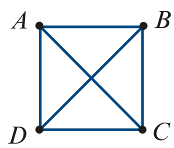
Writing adjacency matrices

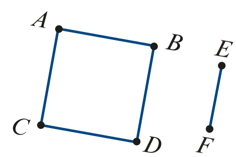
1 For each of the following graphs, write down the adjacency matrix.

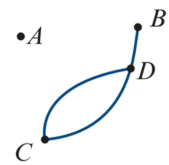
a


b


c


d


e


f


Drawing graphs from adjacency matrices

Example 5

2 Draw a graph from each of the following adjacency matrices.

a
$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

b
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

c
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Properties of graphs

3 The adjacency matrix on the right has a row and column for vertex C that contains all zeros. What does this tell you about vertex C?

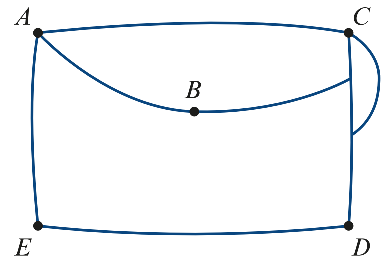
$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

4 Every vertex in a graph has one loop. What feature of the adjacency matrix would tell you this information?

5 A graph has five vertices: A, B, C, D and E. It has no duplicate edges and no loops. If this graph is complete, write down the adjacency matrix for the graph.

Exam 1 style questions

6 The map opposite shows the pathways between five buildings: A, B, C, D and E. An adjacency matrix for the graph that represents this map is formed. The number of zeros in this matrix is

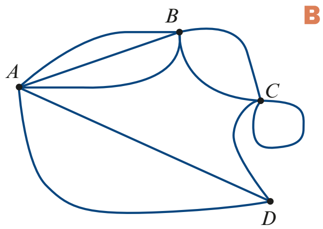


A 9 **B** 10 **C** 11
D 12 **E** 13

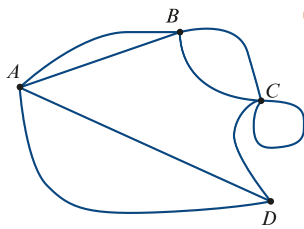
7 The adjacency matrix opposite shows the number of pathways between four points A, B, C and D. A graph that could be represented by the adjacency matrix is

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

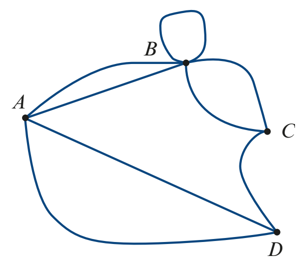
A



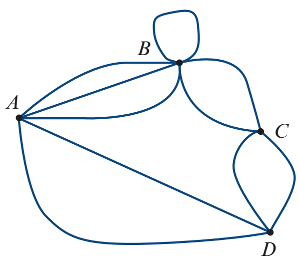
B



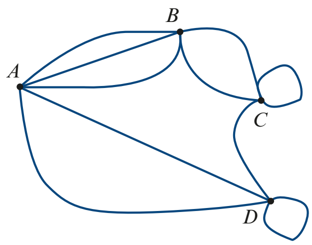
C



D



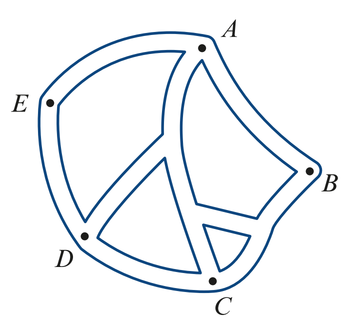
E



Use the following information to answer questions 8 and 9.

The map opposite shows the pathways between five schools: A, B, C, D and E.

An adjacency matrix for the graph that represents this map is formed.



8 Of the 25 elements in the adjacency matrix, the number ‘1’ appears

A 7 times **B** 8 times **C** 9 times
D 10 times **E** 11 times

9 Of the 25 elements in the adjacency matrix, the numbers ‘2’ or ‘3’ appear

A 6 times **B** 7 times **C** 8 times **D** 9 times **E** 10 times

Use the following information to answer questions 10 and 11.

10 A graph has four vertices A, B, C, and D. The adjacency matrix for this graph is shown opposite. Which one of the following statements about this graph is **not** true?

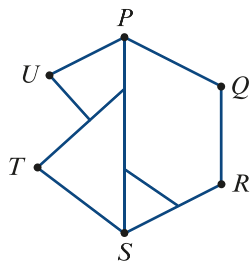
$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

A The graph is connected.
B The graph contains a loop.
C The graph contains multiple edges.
D The graph is planar.
E The graph contains a bridge

11 The number of faces of the graph represented by the adjacency matrix above is

A 2 **B** 3 **C** 4 **D** 5 **E** 6

12 The map opposite shows all the road connections between six towns, P, Q, R, S, T and U. The road connections could be represented by the adjacency matrix



A
$$\begin{array}{cc} & \begin{array}{cccccc} P & Q & R & S & T & U \end{array} \\ \begin{array}{c} P \\ Q \\ R \\ S \\ T \\ U \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

B
$$\begin{array}{cc} & \begin{array}{cccccc} P & Q & R & S & T & U \end{array} \\ \begin{array}{c} P \\ Q \\ R \\ S \\ T \\ U \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

C
$$\begin{array}{cc} & \begin{array}{cccccc} P & Q & R & S & T & U \end{array} \\ \begin{array}{c} P \\ Q \\ R \\ S \\ T \\ U \end{array} & \begin{bmatrix} 0 & 1 & 1 & 2 & 1 & 2 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ 2 & 0 & 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

D
$$\begin{array}{cc} & \begin{array}{cccccc} P & Q & R & S & T & U \end{array} \\ \begin{array}{c} P \\ Q \\ R \\ S \\ T \\ U \end{array} & \begin{bmatrix} 0 & 1 & 1 & 2 & 1 & 2 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ 2 & 0 & 2 & 0 & 3 & 2 \\ 1 & 0 & 1 & 3 & 0 & 1 \\ 2 & 0 & 1 & 2 & 0 & 0 \end{bmatrix} \end{array}$$

E
$$\begin{array}{cc} & \begin{array}{cccccc} P & Q & R & S & T & U \end{array} \\ \begin{array}{c} P \\ Q \\ R \\ S \\ T \\ U \end{array} & \begin{bmatrix} 0 & 1 & 1 & 2 & 1 & 2 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ 2 & 0 & 2 & 1 & 3 & 2 \\ 1 & 0 & 1 & 3 & 0 & 1 \\ 2 & 0 & 1 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

13C Exploring and travelling

Learning intentions

- ▶ To be able to identify a walk as a trail, path, circuit or cycle.
- ▶ To be able to identify a walk as an Eulerian trail, Eulerian circuit, Hamiltonian path or Hamiltonian cycle.
- ▶ To be able to use the degrees of the vertices to identify if an Eulerian trail or circuit is possible.

Travelling

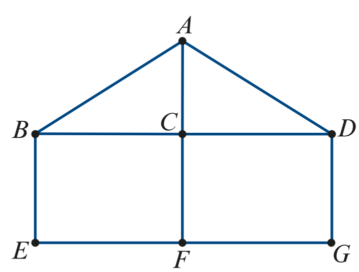
Graphs can be used to model and analyse problems involving exploring and **travelling**. These problems include minimising the distance travelled or time taken between different locations using different routes. For example, a courier driver would like to know the shortest route to use for deliveries, and a tour guide would like to know the quickest route that allows tourists to see a number of sights without retracing their steps.

To solve these types of problems, you will need to learn the language we use to describe the different ways of navigating through a graph, from one vertex to another.

Walks, trails, paths, circuits and cycles

The different ways of navigating through graphs, from one vertex to another, are described as *walks, trails, paths, circuits* and *cycles*.

The graph opposite will be used to explain and define each of these terms.

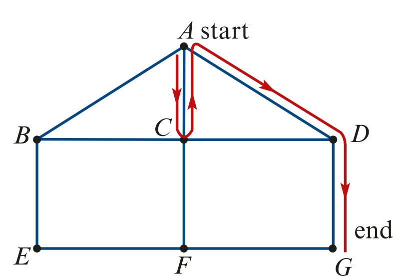


Walks

A **walk** is a sequence of edges, linking successive vertices in a graph.

A walk starts at one vertex and follows any route to finish at another vertex.

The red line in the graph opposite traces out a walk. This walk can be written down by listing the vertices in the order they are visited: $A–C–A–D–G$.

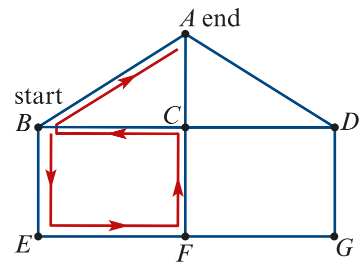


Trails

A **trail** is a walk with no repeated edges.

The red line in the graph opposite traces out a trail. This trail can be written down by listing the vertices in the order they are visited: $B–E–F–C–B–A$.

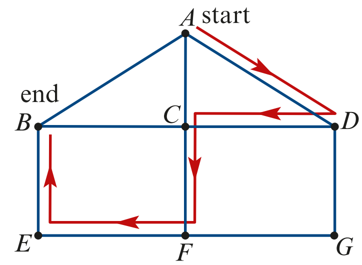
Note: There are *no repeated edges* in this trail, but one vertex (B) is repeated.



Paths

A **path** is a walk with no repeated edges and no repeated vertices.

The red line in the graph opposite traces out a path. This path can be written down by listing the vertices in the order they are visited: $A–D–C–F–E–B$.

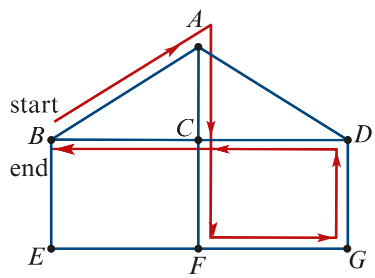


Circuits

A **circuit** is a trail (no repeated edges) that starts and ends at the same vertex. Circuits are also called *closed trails*.

The red line in the graph opposite traces out a circuit. This circuit can be written down by listing the vertices in the order they are visited: $A–C–F–G–D–C–B–A$.

Note: There are *no repeated edges* in this circuit, but one vertex, C, is repeated. The start and end vertices are also repeated because of the definition of a circuit.

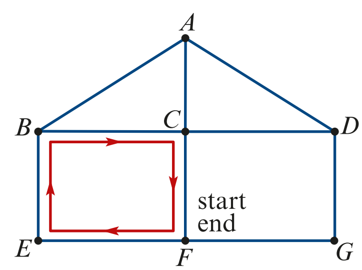


Cycles

A **cycle** is a path (no repeated edges, no repeated vertices) that starts and ends at the same vertex. The start and end vertex is an exception to repeated vertices. Cycles are also called *closed paths*.

The red line in the graph opposite traces out a cycle. This cycle can be written down by listing the vertices in the order they are visited: $F–E–B–C–F$.

Note: There are *no repeated edges* and *no repeated vertices* in this cycle, except for the start and end vertices.



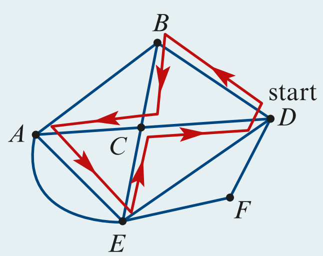
Walks, trails, paths, circuits and cycles

- A *walk* is a sequence of edges, linking successive vertices in a graph.
- A *trail* is a walk with no repeated edges.
- A *path* is a walk with no repeated edges and no repeated vertices.
- A *circuit* is a trail (no repeated edges) that starts and ends at the same vertex.
- A *cycle* is a path (no repeated edges and no repeated vertices) that starts and ends at the same vertex.

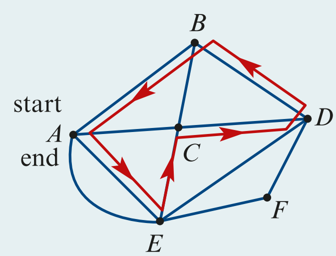
Example 7 **Identifying types of walks**

Identify the walk in each of graphs below as a trail, path, circuit, cycle or walk only.

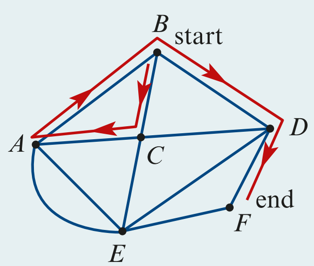
a



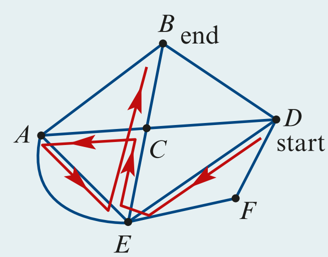
b



c



d



Solution

a This walk starts and ends at the same vertex so it is either a circuit or a cycle. The walk passes through vertex C twice without repeated edges, so it must be a circuit.

b This walk starts and ends at the same vertex so it is either a circuit or a cycle. The walk has no repeated vertex or edge so it is a cycle.

c This walk starts at one vertex and ends at a different vertex, so it is not a circuit or a cycle. There is one repeated vertex (B) and no repeated edge, so it must be a trail.

d This walk starts at one vertex and ends at a different vertex so it is not a circuit or a cycle. There are repeated vertices (C and E) and repeated edges (the edge between C and E), so it must be a walk only.

Eulerian trails and circuits

Trails and circuits that follow every edge, without duplicating any edge, of a graph are called **Eulerian trails** and **Eulerian circuits**. Eulerian trails and circuits are important for some real-life applications. If, for example, a graph shows towns as vertices and roads as edges, then being able to identify a route through the graph that follows every road can be important for mail delivery, or for checking the condition of the roads.

Eulerian trails and circuits exist under easily identified conditions.

Eulerian trails and circuits

Eulerian trails

An Eulerian trail follows every edge of a graph.

An Eulerian trail will exist if the graph:

- is connected
- has exactly *zero* or *two* vertices that have an *odd degree*.
- If there are no odd vertices, the Eulerian trail can start at any vertex in the graph.
- if there are two odd vertices, the Eulerian trail will *start* at one of the odd vertices and *finish* at the other.

Eulerian circuits

An Eulerian circuit is an Eulerian trail (follows every edge) that starts and ends at the same vertex.

An Eulerian circuit will exist if the graph:

- is connected
- has vertices that *all* have an *even degree.*

An Eulerian circuit can start at *any* of the vertices.

Note: If a graph has more than two odd-degree vertices, neither an eulerian trail nor an eulerian circuit exists.

Hamiltonian paths and cycles

Paths and cycles that pass through every vertex of a graph only once are called **Hamiltonian paths** and **Hamiltonian cycles**, named after the mathematician William Rowan Hamilton.

Hamiltonian paths and cycles have real-life applications to situations where every vertex of a graph needs to be visited, but the route taken is not important. If, for example, the vertices of a graph represent people and the edges of the graph represent email connections between those people, a hamiltonian path would ensure that every person in the graph received a message intended for everyone.

Unlike Eulerian trails and circuits, Hamiltonian paths and cycles do not have a convenient rule or feature that identifies them. Inspection is the only way to identify them.

Hamiltonian paths and cycles

Hamiltonian paths

A Hamiltonian path visits every vertex of a graph.

Hamiltonian cycles

A Hamiltonian cycle is a Hamiltonian path (every vertex) that starts and ends at the same vertex.

Note: Inspection is the only way to identify Hamilton paths and cycles.

Remember: Eulerian trails and circuits do not repeat edges. Hamiltonian paths and cycles do not repeat vertices.

Hint: To remember the difference between eulerian and hamiltonian travels, remember that eulerian refers to edges, and both start with ‘e’.

Example 8 Eulerian and Hamiltonian travel

A map showing the towns of St Andrews, Kinglake, Yarra Glen, Toolangi and Healesville is shown on the right. Consider only the yellow routes in your answer. St Andrews and Yarra Glen are considered connected, although this route is not fully shown in the image.

a Draw a graph with a vertex representing each of these towns and edges representing the direct road connections between the towns.

b Explain why an Eulerian trail, but not an Eulerian circuit, is possible through this graph.

c Write down an Eulerian trail that begins at Toolangi.

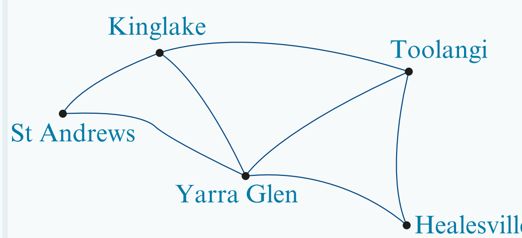
d Write down a Hamiltonian cycle that begins at Healesville.

Explanation

a A road connection exists between:

- St Andrews and Kinglake
- St Andrews and Yarra Glen
- Kinglake and Yarra Glen
- Kinglake and Toolangi
- Yarra Glen and Toolangi
- Yarra Glen and Healesville
- Healesville and Toolangi.

Solution



b The graph has two odd-degree vertices (Toolangi and Kinglake).

There are exactly two odd-degree vertices in this graph. An Eulerian trail will exist, but an Eulerian circuit does not.

c There are a few different answers to this question. One of these is shown.

An Eulerian trail, starting at Toolangi is: Toolangi–Healesville–Yarra Glen–Toolangi–Kinglake–Yarra Glen–St Andrews–Kinglake

d There are two different answers to this question. One of these is shown.

A Hamiltonian cycle that begins at Healesville is: Healesville–Yarra Glen–St Andrews–Kinglake–Toolangi–Healesville

Skill-sheet

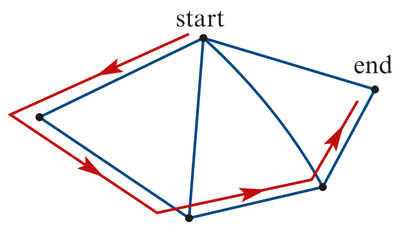
Exercise 13C

Describing travels

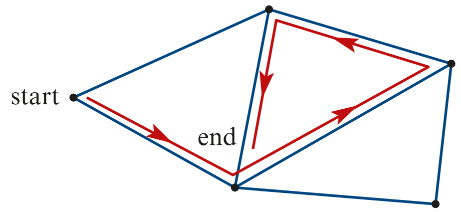
Example 7

1 Identify the walk in each of the graphs below as a trail, path, circuit or walk only.

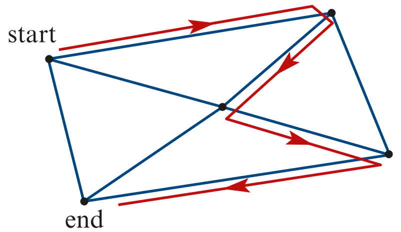
a



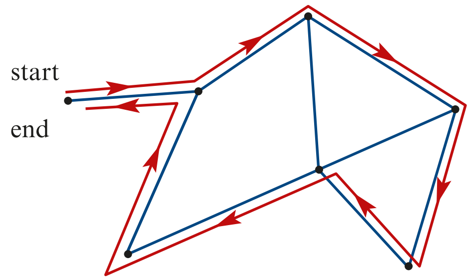
b



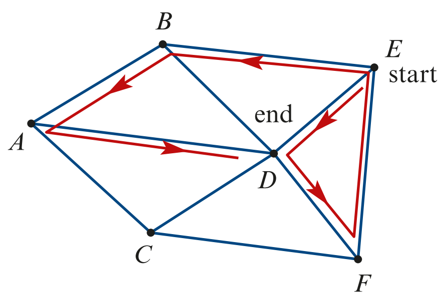
c



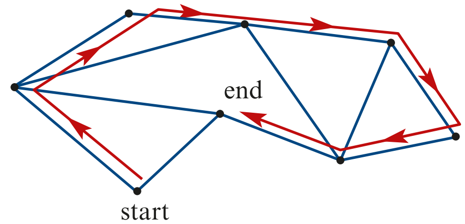
d



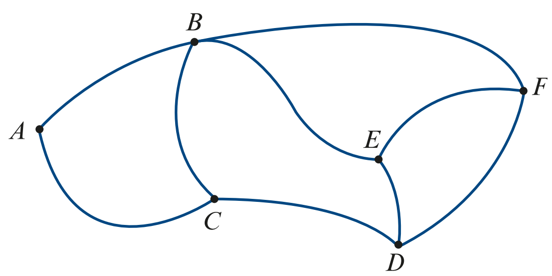
e



f



2 Using the graph opposite, identify the walks below as a trail, path, circuit, cycle or walk only.



a $A–B–E–B–F$

b $B–C–D–E–B$

c $C–D–E–F–B–A$

d $A–B–E–F–B–E–D$

e $E–F–D–C–B$

f $C–B–E–F–D–E–B–C–A$

g $F–E–B–C–A–B–F$

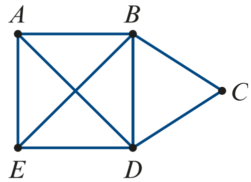
h $A–C–D–E–B–A$

Eulerian trails and circuits

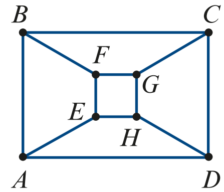
3 **i** Identify whether each graph below has an Eulerian circuit, an Eulerian trail, both or neither.

ii Name the Eulerian circuits or trails found.

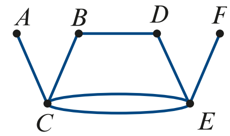
a



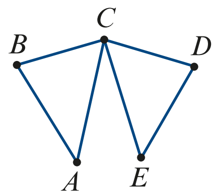
b



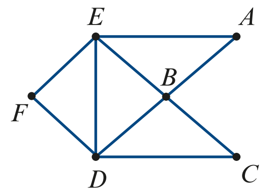
c



d



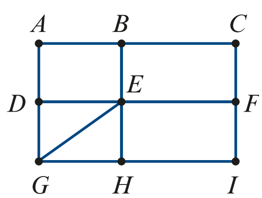
e



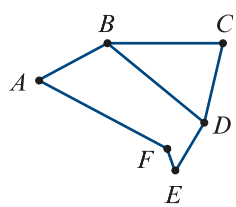
Hamiltonian paths and cycles

4 List a Hamiltonian cycle for each of the following.

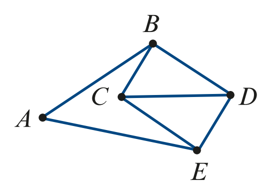
a



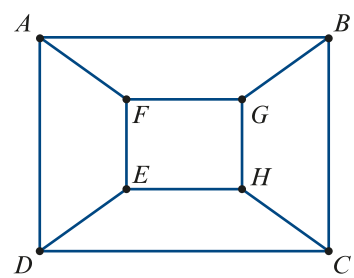
b



c



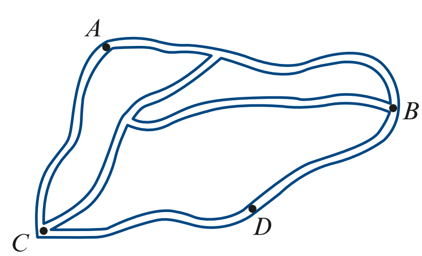
5 List a Hamiltonian path for this graph, starting at F and finishing at G.



Applications

Example 8

6 Four children each live in a different town. The diagram opposite is a map of the roads that link the four towns, A, B, C and D.



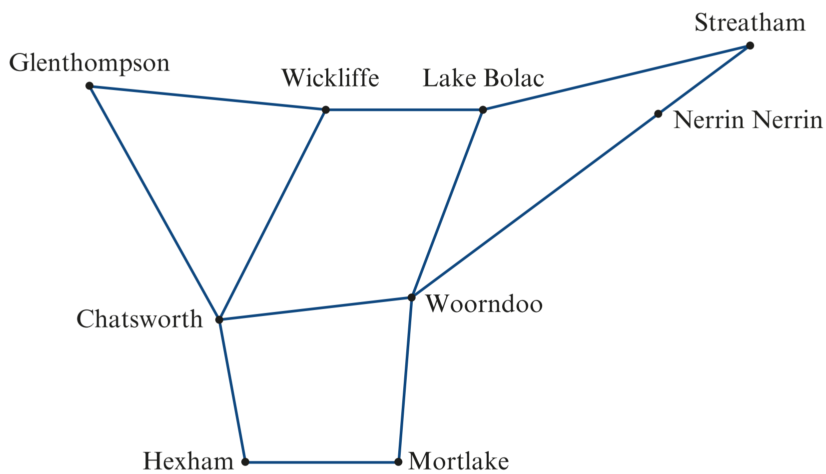
a How many different ways can a vehicle travel between town A and town B without visiting any other town?

b How many different trails are there from town A to town D?

c Draw this map as a graph by representing towns as vertices and each different route between two towns as an edge.

d Explain why a vehicle at A could not follow an Eulerian circuit through this graph.

7 The following graph shows the roads linking nine Victorian country towns.



a Verify Euler's formula for this graph.

Cycling enthusiasts from the nine towns are planning a race that uses the roads linking the towns.

b The organisers who live in Lake Bolac want to visit each town to gain support for the race. They plan to visit each town on the same day but not pass through any town more than once. They will start and finish at Lake Bolac.

  i What name is given for the route they plan to take?

  ii Identify the two routes they can follow.

c In planning the race route, the organisers would like the cyclists to travel along each road linking the towns, but only once, and start and finish at Lake Bolac.

  i What name is given for the type of route they would like the cyclists to take?

  ii Explain why this cannot be done.

d The race planning in part **c** above can start and finish at Lake Bolac with only one road being travelled along twice.

  i Which road is this?

  ii Identify one possible route the race can follow starting and finishing at Lake Bolac, with only one road being travelled along twice.

Properties of graphs

8 A graph has six vertices, A, B, C, D, E and F. The adjacency matrix for this graph is shown opposite.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

a Is the graph connected?

b Is the graph planar?

c Does the graph contain a bridge?

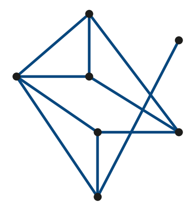
d Does the graph contain an Eulerian trail?

e Does the graph contain an Eulerian circuit?

f Does the graph contain a Hamiltonian path?

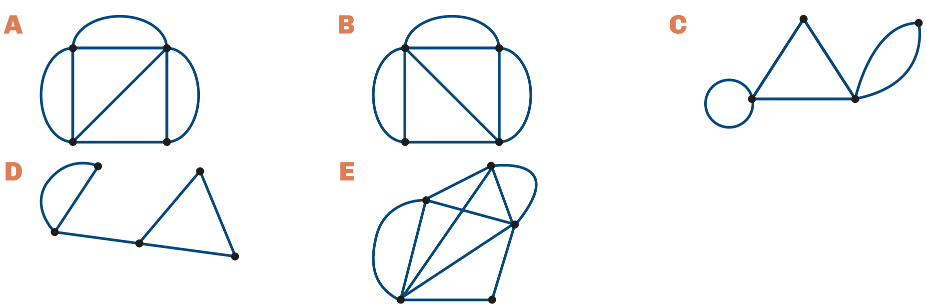
g Does the graph contain a Hamiltonian cycle?

9 Consider the graph opposite.
What is the minimum number of edges that must be added for an Eulerian circuit to exist?



Exam 1 style questions

10 Which one of the following graphs has an Eulerian circuit?



11 Consider the graph opposite. Which one of the following is **not** a Hamiltonian cycle for this graph?

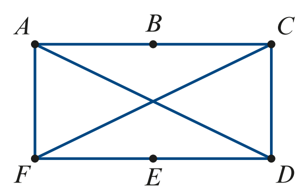
A $ABCFEDA$

B $BADEFCB$

C $CDEFABC$

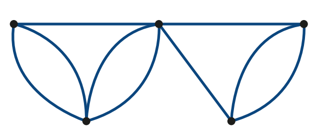
D $DEFACBD$

E $EFCBADE$

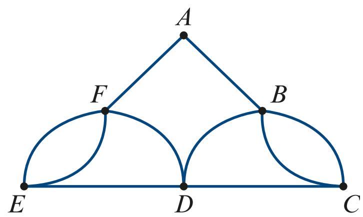


12 An Eulerian trail for the graph opposite will be possible if only one edge is removed. In how many different ways could this be done?

A 1 **B** 2 **C** 3 **D** 4 **E** 5



Use the following graph to answer questions 13 and 14.



13 Which one of the following is a Hamiltonian path for the graph above?

A $AFEDBA$ **B** $ABCDEF$ **C** $AFEDBCBA$
D $ABDFEDCBA$ **E** $FDEFABC$

14 The graph above will have a Eulerian circuit if an edge could be added between the vertices

A E and C **B** A and B **C** A and F
D A and D **E** F and B

13D Weighted graphs and networks

Learning intentions

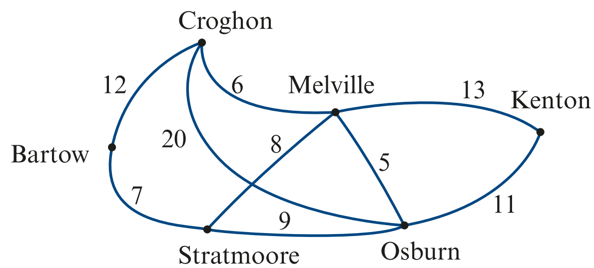
- To be able to analyse a weighted graph.
- To be able to find the shortest path between two vertices for a network.

Weighted graphs

The edges of graphs represent connections between the vertices. Sometimes there is more information known about that connection. If the edge of a graph represents a road between two towns, we might also know the length of this road, or the time it takes to travel this road.

Extra numerical information about the edge that connects vertices can be added to a graph by writing the number next to the edge. Graphs that have a number associated with each edge are called **weighted graphs**.

The weighted graph in the diagram on the right shows towns, represented by vertices, and the roads between those towns, represented by edges. The numbers, or *weights*, on the edges are the distances along the roads.



Weighted graphs in which the weights are physical quantities, for example distance, time or cost, are called **networks**.

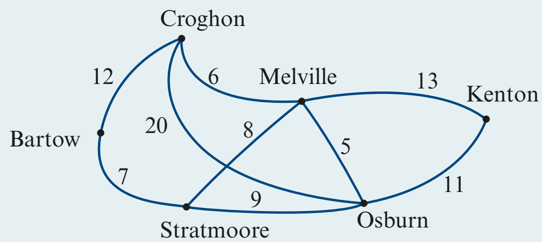
Shortest path problems

When we know numerical information about the connections, travelling through a graph will have extra considerations. If the weights of a network represent time, we can choose a route that will allow us to travel in the shortest time. If the weights represent distance, we can determine a route that will allow us to travel the shortest distance.

These types of problems involve finding the **shortest path** from one vertex to another. In networks that have only a few vertices, it is often easy to find the shortest path between two vertices by inspection. All of the possible route options should be listed, but it is sometimes obvious that certain routes are going to be much longer than others.

Example 9 Finding the shortest path from one vertex to another

Find the shortest path from Bartow to Kenton in the network shown on the right.



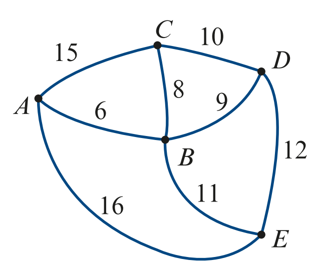
Explanation	Solution
1 List options for travelling from Bartow to Kenton.	B–S–M–O–K B–S–M–K B–S–O–K
2 Add the weights for each route.	$B–S–M–O–K$ $7 + 8 + 5 + 11 = 31$ km $B–S–M–K$ $7 + 8 + 13 = 28$ km $B–S–O–K$ $7 + 9 + 11 = 27$ km
3 Write your answer.	The shortest path from Bartow to Kenton is 27 km with route $B–S–O–K$.

Skill-sheet

Exercise 13D

Weighted graphs and networks

1 The network on the right shows towns A, B, C, D and E represented by vertices. The edges represent road connections between the towns. The weights on the edges are the average times, in minutes, it takes to travel along each road.



a Which two towns are 12 minutes apart by road?

b How long will it take to drive from C to D via B?

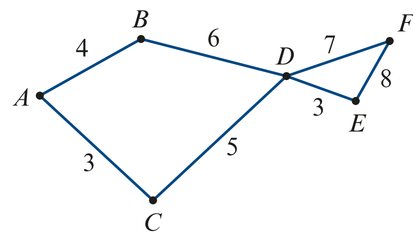
c A motorist intends to drive from D to E via B. How much time will they save if they travel directly from D to E?

d Find the shortest time it would take to start at A, finish at E and visit every town exactly once.

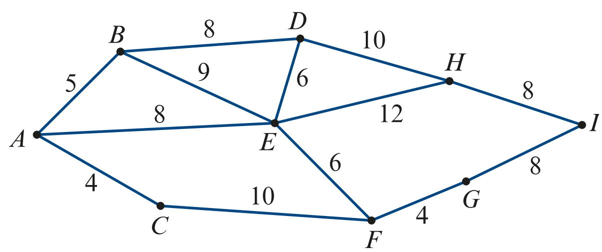
Shortest path by inspection

Example 9

2 By inspection, find the length of the shortest path from A to E.



3 The network on the right shows the distance, in kilometres, along walkways that connect landmarks A, B, C, D, E, F, G, H and I in a national park.



a What distance is travelled on the path A–B–E–H–I?

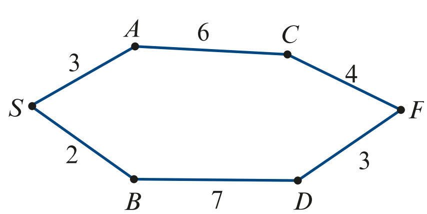
b What distance is travelled on the circuit F–E–D–H–E–A–C–F?

c What is the distance travelled on the shortest cycle starting and finishing at E?

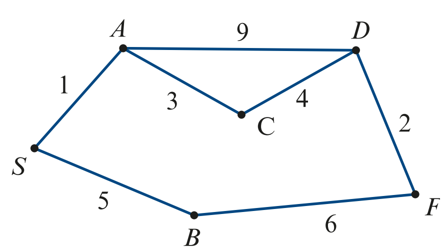
d Find the shortest path from A to I.

4 Determine the shortest path from S to F in the following weighted graphs. Write down the length of the shortest path.

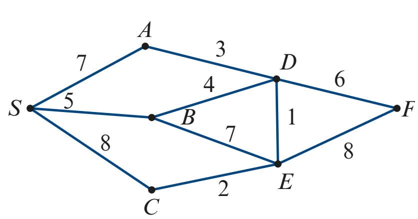
a



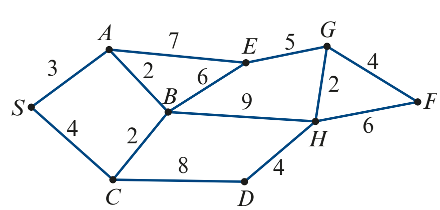
b



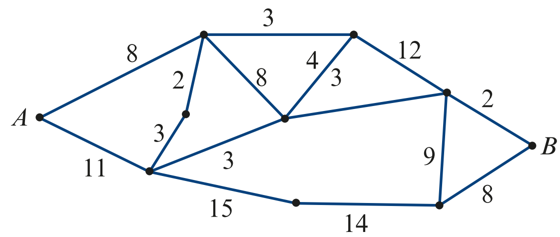
c



d

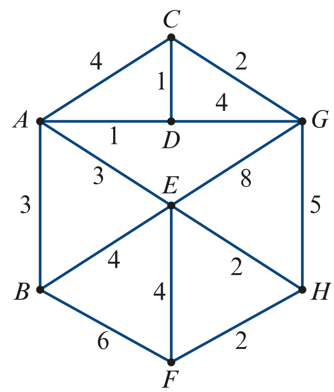


5 In the network opposite, the vertices represent small towns and the edges represent roads. The numbers on each edge indicate the distances (in kilometres) between towns.
Determine the length of the shortest path between the towns labelled A and B.



Exam 1 style questions

Use the following information to answer questions 6 and 7.



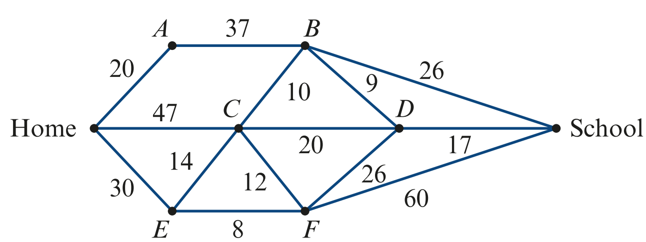
6 What is the shortest path between C and E?

A $C - A - E$ **B** $C - D - A - E$ **C** $C - D - G - E$
D $C - G - E$ **E** $C - G - H - E$

7 What is the shortest path between B and G?

A $B - E - G$ **B** $B - E - H - G$ **C** $B - A - C - G$
D $B - A - D - G$ **E** $B - A - D - C - G$

Use the following information to answer questions 8 and 9.



8 Victoria rides her bike to school each day. The edges of the network on the previous page represent the roads that Victoria can use to ride to school from her home. The numbers on the edges give the time taken, in minutes, to travel along each road. The fastest Victoria can ride from home to school is

A 80 minutes
B 81 minutes
C 83 minutes
D 84 minutes
E 98 minutes

9 Which of the following represent the fastest route for Victoria's journey from home to school.

A *Home – A – B – School*
B *Home – A – B – D – School*
C *Home – C – D – School*
D *Home – C – B – D – School*
E *Home – E – C – B – D – School*

13E Dijkstra's algorithm

Finding the shortest path from one vertex of a graph to another is easy to determine if the graph is small and does not have too many vertices and edges. When there are many vertices and many edges, a systematic method, called an **algorithm**, can be used to find the shortest path.

Dutch computer scientist, Edsger Wybe Dijkstra (pronounced 'Dyke-stra') developed an algorithm for determining the shortest path through a graph. This algorithm, and others like it, have important applications to computerised routing and scheduling programs, such as GPS navigation devices.

Note: An alternative version is available online. Either method can be used. There is no curriculum requirement for you to know both methods.

The algorithm

You may choose to read through the example first to see a detailed implementation. Here we write the algorithm to emphasise its repetitive aspect.

Step 1: Assign the starting vertex a label of value zero and circle the vertex and the zero together.

Once a vertex and its label have been circled it cannot be changed

Step 2: Consider the vertex which has been most recently circled. Suppose this vertex to be X and the label of value d assigned to it. Then, in turn, consider each vertex directly joined to X but not yet permanently circled. For each such vertex, Y say, temporarily assign it with the value d + (the weight of edge XY) if Y does not have a temporary value or if it does, assign the lesser of d + (the weight of edge XY) and the existing temporary value.

Step 3: Choose the least of all temporary value labels on the network. Make this value label permanent by circling it.

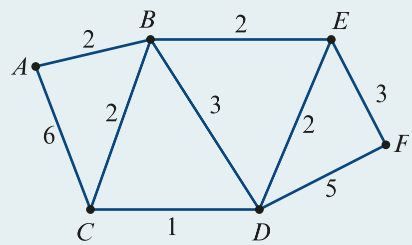
Step 4: Repeat Steps 2 and 3 until the destination node has a permanent label.

Step 5: Go backwards through the network, retracing the path of shortest length from the destination vertex to the starting vertex by

- starting at destination and go to the circled vertex with value = destination value − edge value.
- continuing to move back to the start vertex folowing this procedure.

Example 10 Using Dijkstra's algorithm to find the shortest path in a network: graphical method

Find the shortest path from A to F in the weighted graph shown on the right.

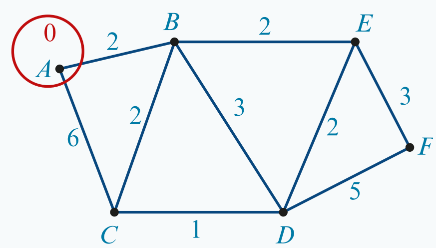


Explanation

Step 1

- Assign the starting vertex a zero and circle the vertex and its new value of zero.

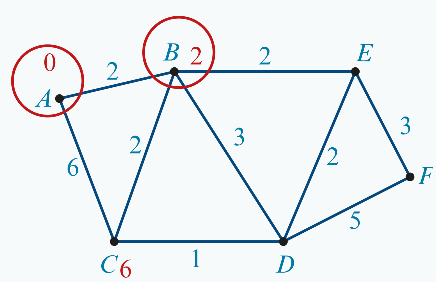
Solution



- A is the starting vertex, it is assigned zero and it is circled.

Step 2

- Assign a value to each vertex connected to the starting vertex. The value assigned is the length of the edge connecting it to the starting vertex.
- Circle the vertex with the lowest assigned value.



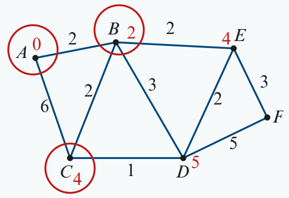
- The starting vertex A is connected to vertices B and C.
- The vertex B is assigned 2 and the vertex C is assigned 6.
- Vertex B is circled because it has the lowest value.

Explanation

Step 3

- From the newly circled vertex, assign a value to each vertex connected to it by adding the value of each connecting edge to the newly circled vertex's value.
- If a connecting vertex already has a value assigned and the new value is less than it, replace it with the new value.
- If a vertex is circled, it cannot have its value changed.
- Consider all uncircled vertices and circle the one with the lowest value.

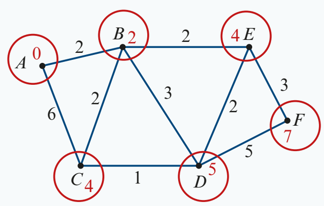
Solution



- The newly circled vertex B is connected to three vertices; C, D and E. Starting with vertex B's value of 2, E is assigned 4 (adding 2 from the connecting edge) and D is assigned 5 (adding 3 from the connecting edge).
- The vertex C will be re-assigned 4 (adding 2 from the connecting edge) because it is lower than 6.
- Now there are two uncircled vertices with the lowest assigned value 4, vertices C and E; it **does not** matter which one is circled. C is circled.

Step 4

- Repeat **Step 3** until the destination vertex and its assigned value are circled.
- The length of the shortest path will be the assigned value of the destination vertex.



- Vertex F is the destination vertex, assigned a value of 7. Therefore the shortest path from A to F has a length of 7.

Step 5

- The shortest path is found by backtracking. Starting at the destination vertex, move to the circled vertex whose value is equal to the destination vertex's assigned value, subtract the connecting edge value. Continue to subtract the connecting edge value from one circled vertex to the next until you reach the starting vertex.

Note: Once a vertex is assigned a value, it cannot be assigned a larger value, even if it has not been circled yet. You do not need to circle all vertices. Stop when the destination vertex is circled.

- To find the shortest path, start at F and consider the two connecting edges to it. The edge of length 3 is correct, because 7 minus 3 equals 4, the value of vertex E.
- Likewise from E, subtract the connecting edge of 2 to vertex B to equal 2, then subtract the connecting edge of 2 to A to equal zero.

Therefore the shortest path from A to F is: $A - B - E - F$ with a length of 7.

Skill-sheet

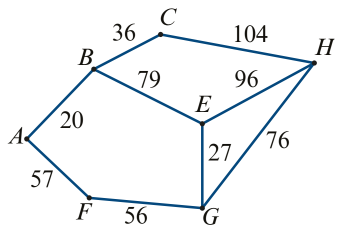
Exercise 13E

Calculations within Dijkstra's algorithm

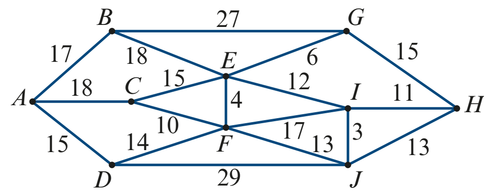
Example 10

1 Using Dijkstra's algorithm, determine the shortest path and write down the length of the shortest path from A to H in the following weighted graphs.

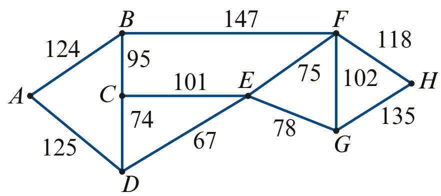
a



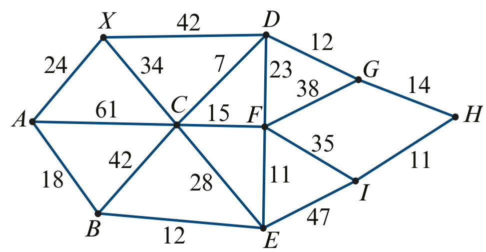
b



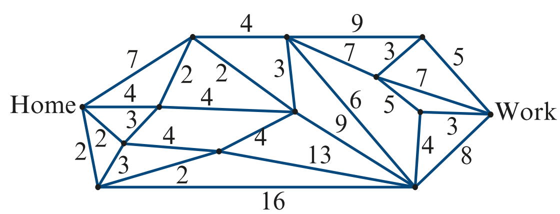
c



d



2 Renee drives to work each day. The edges of the network opposite represent the roads that Renee can use to drive to work. The numbers on the edges give the time, in minutes, to travel along each road. What is the shortest time that Renee can drive between home and work?



Note: See Chapter review, written-response question 1 for more problems using Dijkstra's algorithm.

13F Trees and minimum connector problems

Learning intentions

- ▶ To be able to identify a tree.
- ▶ To be able to find a spanning tree for a graph.
- ▶ To be able to find the minimum spanning tree for a network using Prim's algorithm.

In the previous applications of networks, the weights on the edges of the graph were used to determine a minimum pathway through the graph. In other applications, it is more important to minimise the number and weights of the edges in order to keep all vertices connected to the graph. For example, a number of towns might need to be connected to a water supply. The cost of connecting the towns can be minimised by connecting each town into a network or water pipes only once, rather than connecting each town to every other town.

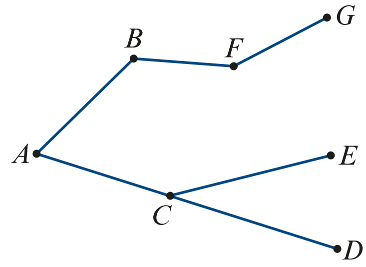
Problems of this type are called connector problems. In order to solve connector problems, you need to learn the language of networks that have as few edges as possible.

Trees

A **tree** is a connected graph that has no loops, multiple edges or cycles.

This tree has seven vertices and six edges.

The number of edges is always one less than the number of vertices.



Spanning trees

Every connected graph will have at least one subgraph that is a tree. A subgraph is a tree, and if that tree connects all of the vertices in the graph, then it is called a **spanning tree**.

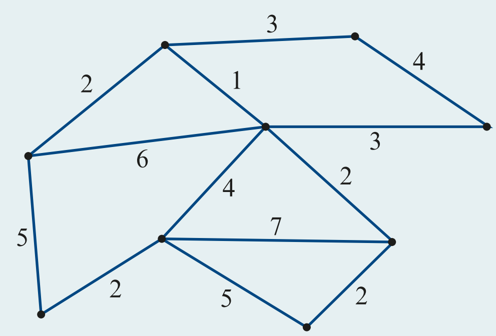
> **Trees**
>
> A *tree* has no loops, multiple edges or cycles.
>
> If a tree has n vertices, it will have $n - 1$ edges.
>
> A *spanning tree* is a tree that connects all of the vertices of a graph.

There can be more than one spanning tree for any connected graph. The *total weight* of a spanning tree is the total of all the weights on the edges that make up the tree.

Example 11 **Finding the weight of a spanning tree**

a Draw one spanning tree for the graph shown.

b Calculate the weight of this spanning tree.

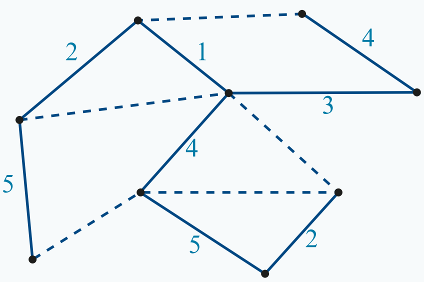


Explanation	Solution
a **1** Count the number of vertices and edges in the graph.	There are 9 vertices and 13 edges.
2 Calculate the number of edges in the spanning tree.	The spanning tree will have 8 edges.

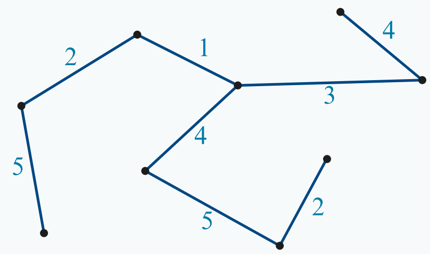
3 Calculate how many edges must be removed.

Remove 13 – 8 = 5 edges.

4 Choose edges to remove.



b Add the weights of the remaining edges.



Weight = 5 + 2 + 1 + 4 + 5 + 2 + 3 + 4
= 26

Minimum spanning trees

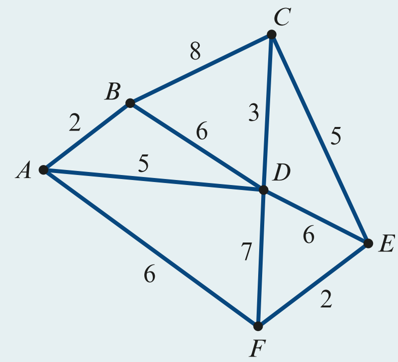
One of the spanning trees from a particular connected graph will have the *smallest* total weight. This tree is called the **minimum spanning tree**. Minimum spanning trees can be found using an algorithm called **Prim's algorithm**.

Prim's algorithm for finding a minimum spanning tree

- Choose a starting vertex (any will do).
- Inspect the edges starting from the starting vertex and choose the one with the smallest weight. (If there are two edges that have the same weight, it does not matter which one you choose). The starting vertex, the edge and the vertex it connects to form the beginning of the minimum spanning tree.
- Now inspect all of the edges starting from both of the vertices you have in the tree so far. Choose the edge with the smallest weight, ignoring edges that would connect the tree back to itself. The vertices and edges you already have, plus the extra edge and vertex it connects form the minimum spanning tree so far.
- Keep repeating this process until all of the vertices are connected.

Example 12 Finding the minimum spanning tree

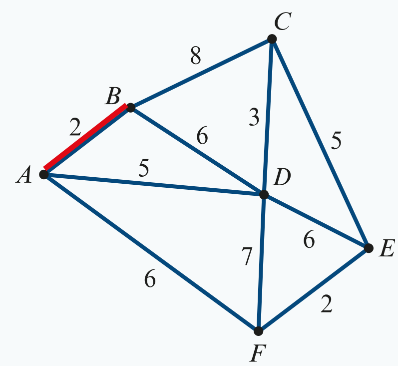
Apply Prim's algorithm to find the minimum spanning tree for the graph shown on the right. Write down the total weight of the minimum spanning tree.



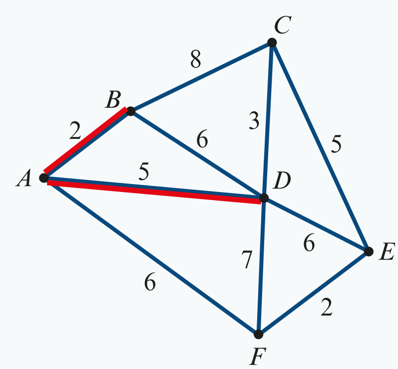
Explanation

Start with vertex A.
The smallest weighted edge from vertex A is to B with weight 2.

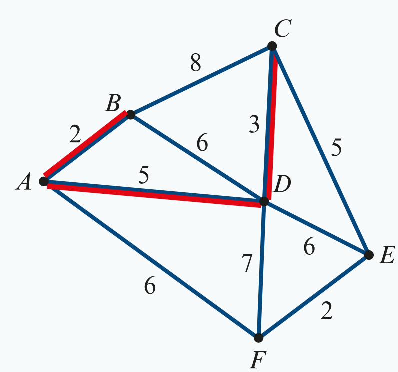
Solution



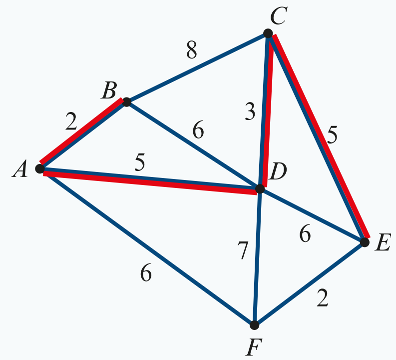
Look at vertices A and B. The smallest weighted edge from either vertex A or vertex B is from A to D with weight 5.



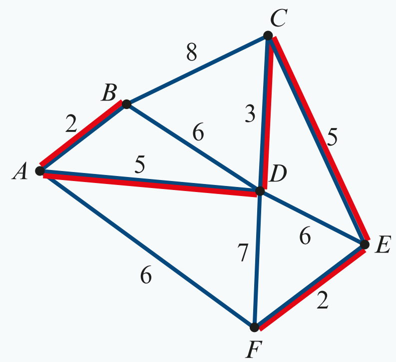
Look at vertices A, B and D. The smallest weighted edge from vertex A, B or D is from D to C with weight 3.



Look at vertices A, B, D and C. The smallest weighted edge from vertex A, B, D or C is from C to E with weight 5.



Look at vertices A, B, D, C and E. The smallest weighted edge from vertex A, B, D, C or E is from E to F with weight 2.
All vertices have been included in the graph. This is the minimum spanning tree.



Add the weights to find the total weight of the minimum spanning tree.

The total weight of the minimum spanning tree is $2 + 5 + 3 + 5 + 2 = 17$.

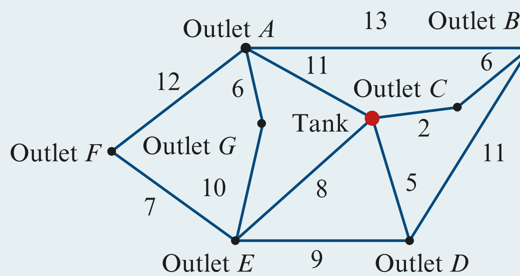
Minimum Connector problems

Minimum spanning trees represent the least weight required to keep all of the vertices connected in the graph. If the edges of a graph represent the cost of connecting towns to a gas pipeline, then the total weight of the minimum spanning tree would represent the minimum cost of connecting the towns to the gas. This is an example of a *connector problem*, where the cost of keeping towns or other objects connected together is important to make as low as possible.

Example 13 Solving a connector problem

Water is to be piped from a water tank to seven outlets on a property. The distances (in metres) of the outlets from the tank and from each other are shown in the network below.

Starting at the tank, the aim is to find the minimum length of pipe, in metres, which will be needed to have water piped to all outlets in the property.



a On the diagram, show where the water pipes will be placed in order to minimise the length required.

b Calculate the total length, in metres, of the water pipe that is required to obtain this minimum length.

Explanation

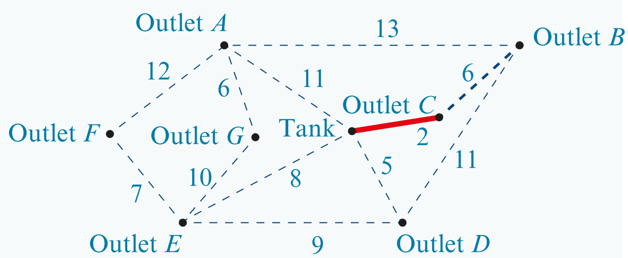
a **1** The water pipes will be a minimum length if they are placed on the edges of the minimum spanning tree for the network.

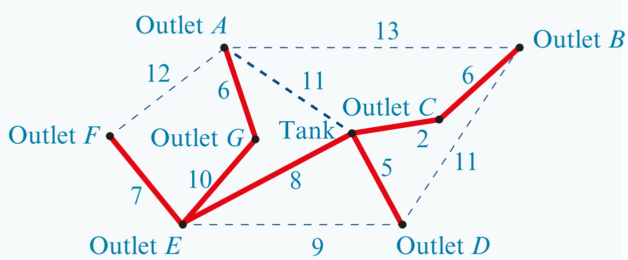
A starting point for Prim's algorithm is the vertex that is connected to the tank by the edge with the smallest weight. The starting vertex (Tank), the edge and the vertex it connects to form the beginning of the minimum spanning tree.

2 Follow Prim's algorithm to find the minimum spanning tree.

b Add the weights of the minimum spanning tree. Write your answer.

Solution





The length of water pipe required is $2 + 6 + 5 + 8 + 7 + 10 + 6 = 44$ metres

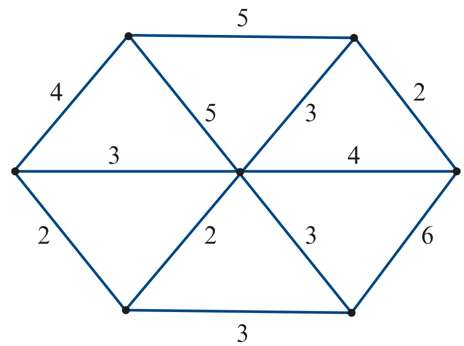
Skill-sheet

Exercise 13F

Spanning trees

Example 11

1 A weighted graph is shown on the right.



a How many edges must be removed in order to leave a spanning tree?

b Remove some edges to form three different trees.

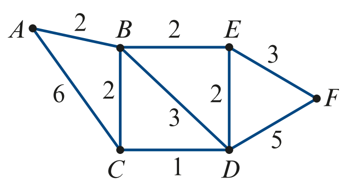
c For each tree in part **b**, find the total weight.

Minimum spanning trees and Prim's algorithm

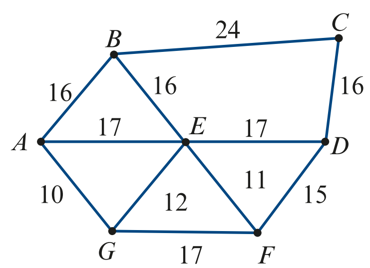
Example 12

2 Find a minimum spanning tree for each of the following graphs and give the total weight.

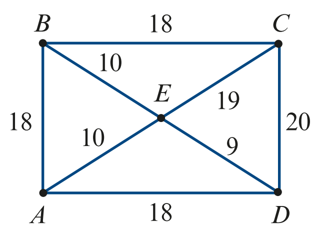
a



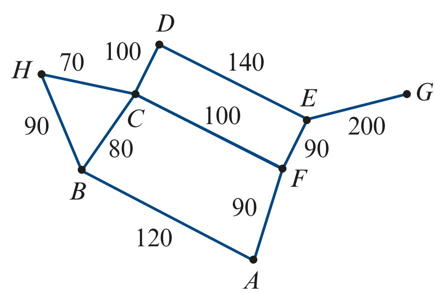
b



c



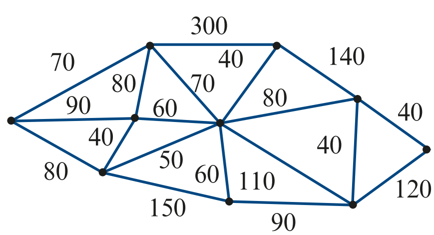
d



Connector problems

Example 13

3 In the network opposite, the vertices represent water tanks on a large property and the edges represent pipes used to move water between these tanks. The numbers on each edge indicate the lengths of pipes (in m) connecting different tanks.



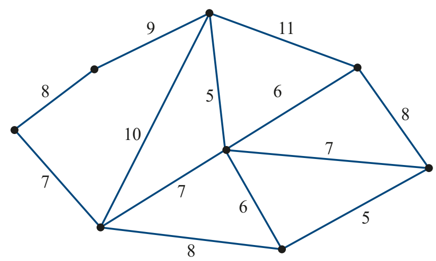
Determine the shortest length of pipe needed to connect all water storages.

Exam 1 style questions

4 For the graph opposite, the length of the minimum spanning tree is

A 44 **B** 45 **C** 46

D 47 **E** 48



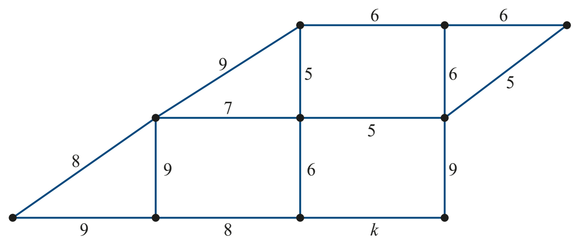
5 The minimum spanning tree for the graph below includes the edge with weight labelled k.

The total weight of all edges for the minimum spanning tree is 58.

The value of k is

A 6 **B** 7 **C** 8

D 9 **E** 10



6 The minimum spanning tree for the graph opposite includes two edges with weights a and b. The length of the minimum spanning tree is 124.

The values of a and b could be

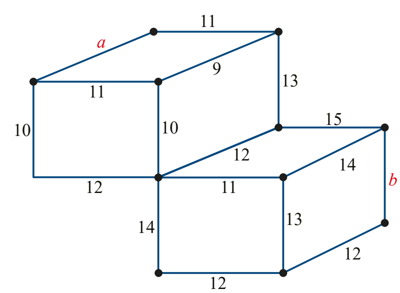
A $a = 6$ and $b = 17$

B $a = 12$ and $b = 12$

C $a = 10$ and $b = 14$

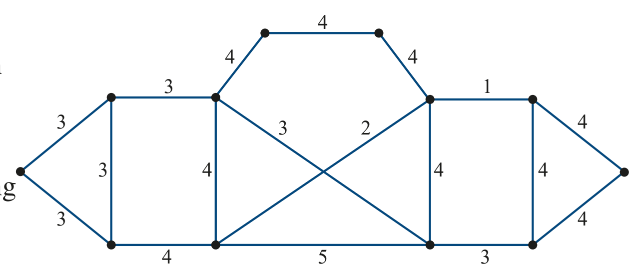
D $a = 10$ and $b = 15$

E $a = 13$ and $b = 12$



7 A minimum spanning tree is to be drawn for the weighted graph opposite. How many edges with weight 4 will **not** be included in any particular minimum spanning tree?

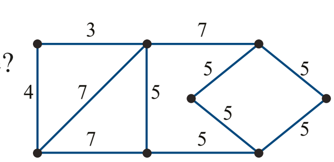
A 1 **B** 2 **C** 3 **D** 4 **E** 5



8 Consider the weighted graph opposite.

How many different minimum spanning trees are possible?

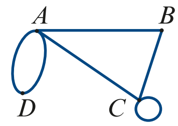
A 1 **B** 2 **C** 3 **D** 4 **E** 5



Review

Key ideas and chapter summary

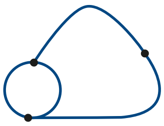
Assign-ment

Graph	A **graph** is a diagram that consists of a set of points called **vertices** and a set of lines called **edges**. Each edge joins two vertices. 
Edge	In the graph above, the lines joining A, B, C and D are edges.
Vertex	In the graph above, the points A, B, C, D are vertices.
Loop	A **loop** is an edge that connects a vertex to itself. In the graph above there is a loop at vertex C.
Degree of a vertex	The **degree of a vertex** is the number times edges attach to a vertex. The degree of vertex A is written as $\deg(A)$. In the graph above, $\deg(A) = 4$, $\deg(B) = 2$ and $\deg(D) = 2$. A loop has degree of 2. In the graph above, $\deg(C) = 4$.
Multiple edge	Sometimes a graph has two or more identical edges. These are called **multiple edges**. In the graph above, there are multiple edges between vertex A and vertex D.
Simple graph	**Simple graphs** are graphs that do not have loops and do not have multiple edges.
Isolated vertex	An **isolated vertex** is one that is not connected to any other vertex. Isolated vertices have degree of zero.
Degenerate graph	A **degenerate graph** has no edges. All of the vertices are isolated.
Connected graph	A **connected graph** has no isolated vertex. There is a path between each pair of vertices.
Bridge	A **bridge** is a single edge in a connected graph that, if it were removed, leaves the graph disconnected.
Complete graph	A **complete graph** has every vertex connected to every other vertex by an edge.
Subgraph	A **subgraph** is a graph that is part of a larger graph and has some of the same vertices and edges as that larger graph. A subgraph does not have any extra vertices or edges that do not appear in the larger graph.
Equivalent graph (isomorphic graph)	Graphs that contain identical information (connections between vertices) to each other are **equivalent graphs** or **isomorphic graphs**.

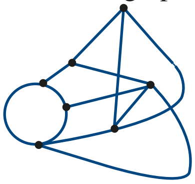
Face An area in a graph or network that can only be reached by crossing an edge. One such area is always the area surrounding a graph.

Planar graph A **planar graph** can be drawn so that no two edges overlap or intersect, except at the vertices.

This graph is planar.



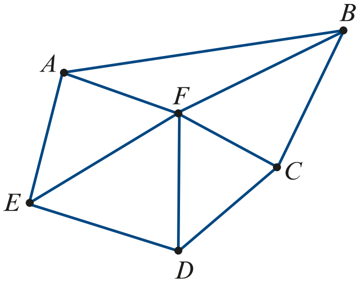
This graph is **not** planar.



Euler's formula **Euler's formula** applies to planar graphs.
If v = the number of vertices, e = the number of edges and f = the number of faces then $v + f = e + 2$

Adjacency matrix An **adjacency matrix** is a square matrix that uses a zero or an integer to record the number of edges connecting each pair of vertices in the graph.

Travelling Movement through a graph from one vertex to another along the edges is called **travelling** through the graph.



Walk A **walk** is a sequence of edges, linking successive vertices, in a graph.
In the graph above, E–A–F–D–C–F–E–A is an example of a walk.

Trail A **trail** is a walk with no repeated edges.
In the graph above, A–F–D–E–F–C is an example of a trail.

Path A **path** is a walk with no repeated vertices and no repeated edges.
In the graph above, F–A–B–C–D is an example of a path.

Circuit A **circuit** is a trail (no repeated edges) that starts and ends at the same vertex.
In the graph above, A–F–D–E–F–B–A is an example of a circuit.

Cycle	A **cycle** is a path (no repeated edges nor vertices) that starts and ends at the same vertex. The start and end vertex is an exception to repeated vertices. In the graph above, $B–F–D–C–B$ is an example of a cycle.
Eulerian trail	An **Eulerian trail** is a trail (no repeated edges) that includes all of the edges of a graph. Eulerian trails exist if the graph is connected and has exactly zero or two vertices of odd degree. The remaining vertices have even degree.
Eulerian circuit	An **Eulerian circuit** is a trail (no repeated edges) that includes all of the edges of a graph and that starts and ends at the same vertex. Eulerian circuits exist if the graph is connected and has all of the vertices with an even degree.
Hamiltonian path	A **Hamiltonian path** is a path (no repeated edges or vertices) that includes all of the vertices of a graph.
Hamiltonian cycle	A **Hamiltonian cycle** is a path (no repeated edges or vertices) that starts and ends at the same vertex. The starting vertex is an exception to repeated vertices.
Weighted graph	A **weighted graph** has numbers, called weights, associated with the edges of a graph. The weights often represent physical quantities as additional information to the edge, such as time, distance or cost.
Network	A **network** is a weighted graph where the weights represent physical quantities such as time, distance or cost.
Shortest path	The **shortest path** through a network is the path along edges so that the total of the weights of that path is the minimum for that network. Shortest path problems involve finding minimum distances, costs or times through a network. Shortest paths can be determined by inspection or by using Dijkstra's algorithm.
Dijkstra's algorithm	**Dijkstra's algorithm** is an algorithm for determining the shortest path through a network from one vertex to another.
Tree	A **tree** is a connected graph that contains no cycles, multiple edges or loops. A tree with n vertices has $n - 1$ edges.
Spanning tree	A **spanning tree** is a tree that connects every vertex of a graph. A spanning tree is found by counting the number of vertices (n) and removing enough edges so that there are $n - 1$ edges left that connect all vertices.

Minimum spanning tree	A **minimum spanning tree** is a spanning tree for which the sum of the weights of the edges is as small as possible.
Prim's algorithm	**Prim's algorithm** is an algorithm for determining the minimum spanning tree of a network.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

13A **1** I can identify edges, vertices and loops in a graph. ☐

See Example 1, and Exercise 13A Question 1

13A **2** I can determine the degree of a vertex in a graph. ☐

See Example 1, and Exercise 13A Question 1

13A **3** I can define and identify simple graphs, isolated vertices, degenerate graphs, connected graphs, bridges and subgraphs. ☐

See Example 1, and Exercise 13A Question 1

13A **4** I can recognise isomorphic graphs. ☐

See Exercise 13A Question 3

13A **5** I can define planar graphs. ☐

See Example 2, and Exercise 13A Question 5

13A **6** I can redraw graphs in planar form. ☐

See Example 3, and Exercise 13A Question 4

13A **7** I can use Euler's formula. ☐

See Example 4, and Exercise 13A Question 6

13B **8** I can write an adjacency matrix from a graph. ☐

See Exercise 13B Question 1

13B **9** I can construct a graph from an adjacency matrix. ☐

See Example 5, and Exercise 13B Question 2

13C **10** I can define walks, trails, paths, circuits and cycles through a graph. ☐

See Example 6, and Exercise 13C Question 1

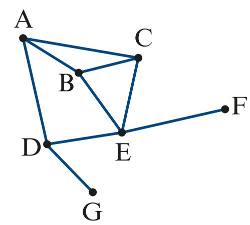
13C	**11**	I can identify Eulerian trails and circuits through graphs.	☐
		See Example 7, and Exercise 13C Question 3	
13C	**12**	I can determine whether an Eulerian trail or circuit exists in a graph.	☐
		See Example 7, and Exercise 13C Question 3	
13C	**13**	I can identify Hamiltonian paths and cycles through graphs.	☐
		See Example 7, and Exercise 13C Question 4	
13D	**14**	I can define a weighted graph.	☐
		See Exercise 13D Question 1	
13D	**15**	I can calculate the shortest path from one vertex to another by inspection.	☐
		See Example 8, and Exercise 13D Question 3	
13D	**16**	I can calculate the shortest path from one vertex to another using Dijkstra's algorithm.	☐
		See Example 9, and Exercise 13D Question 9	
13E	**17**	I can define tree, spanning tree, minimum spanning tree.	☐
		See Example 11, and Exercise 13E Question 1	
13E	**18**	I can draw a minimum spanning tree using Prim's algorithm.	☐
		See Example 12, and Exercise 13E Question 2	

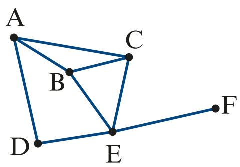
Multiple-choice questions

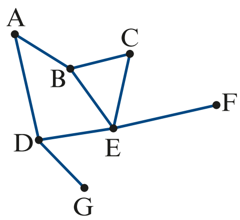
1 The minimum number of edges for a graph with seven vertices to be connected is:

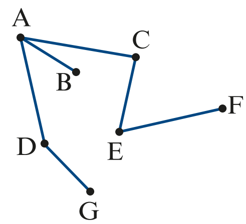
A 4 **B** 5 **C** 6 **D** 7 **E** 21

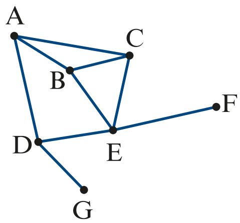
2 Which of the following graphs is a spanning tree for the network shown?

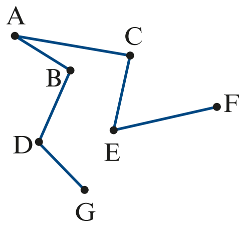


A


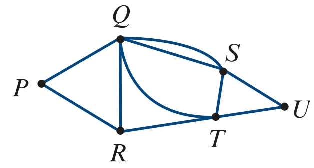
B


C


D 

E 

3 For the graph shown, which vertex has degree 5?

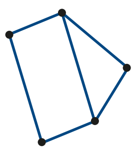


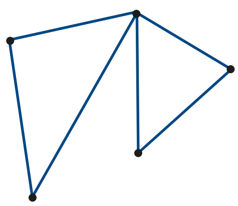
A Q **B** T **C** S **D** R **E** U

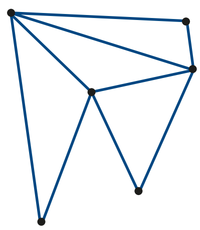
4 A connected graph with 15 vertices divides the plane into 12 regions. The number of edges connecting the vertices in this graph will be:

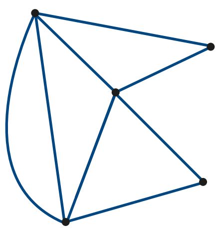
A 15 **B** 23 **C** 24 **D** 25 **E** 27

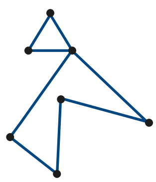
5 Which of the following graphs does *not* have an Eulerian circuit?

A 

B 

C 

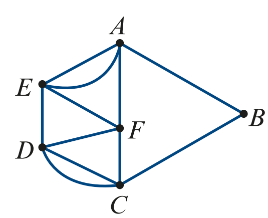
D 

E 

6 A connected planar graph divides the plane into a number of regions. If the graph has eight vertices and these are linked by 13 edges, then the number of regions is:

A 5 **B** 6 **C** 7 **D** 8 **E** 10

7 For the graph shown, which of the following paths is a Hamiltonian cycle?



A $A–B–C–D–C–F–D–E–F–A–E–A$

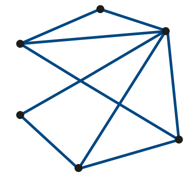
B $A–E–F–D–C–B–A$

C $A–F–C–D–E–A–B–A$

D $A–B–C–D–E–A$

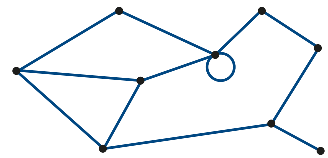
E $A–E–D–C–B–A–F$

8 The graph opposite has:



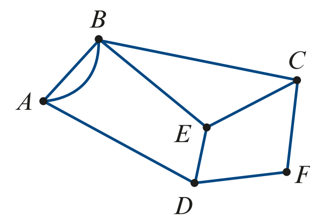
A four faces **B** five faces
C six faces **D** seven faces
E eight faces

9 The sum of the degrees of the vertices on the graph shown here is:



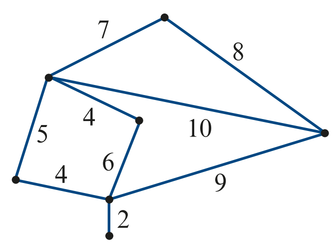
A 20 **B** 21 **C** 22
D 23 **E** 24

10 For the graph shown, which additional edge could be added to the network so that the graph formed would contain an Eulerian trail?



A $A–F$ **B** $D–E$ **C** $A–B$
D $C–F$ **E** $B–F$

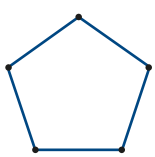
11 For the graph shown here, a minimum spanning tree has length:



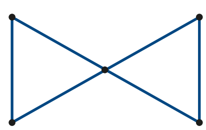
A 30 **B** 31 **C** 33
D 34 **E** 26

12 Of the following graphs, which one has both Eulerian circuit and Hamiltonian cycles?

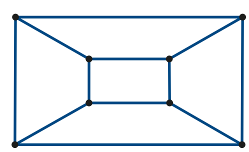
A



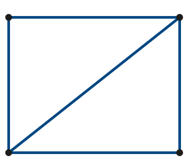
B



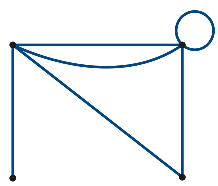
C



D



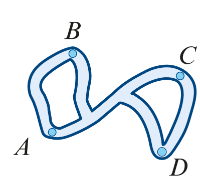
E



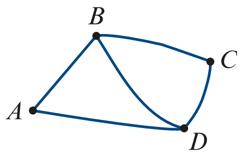
13 A graph with six vertices is connected with the minimum number of edges. The minimum number of extra edges needed to make this a complete graph is

A 5 **B** 6 **C** 10 **D** 14 **E** 16

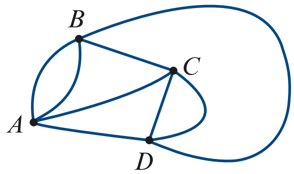
14 Four towns, A, B, C and D, are linked by roads as shown. Which of the following graphs could be used to represent the network of roads? Each edge represents a route between two towns.



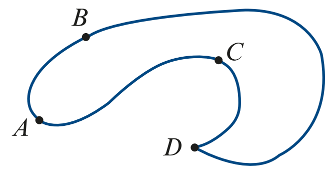
A



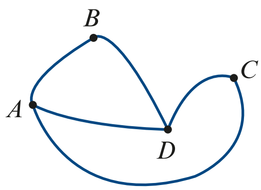
B



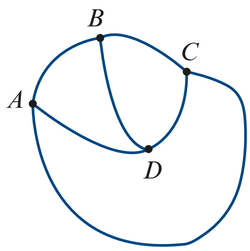
C



D

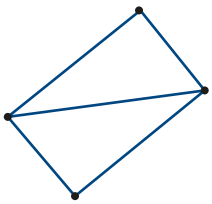


E

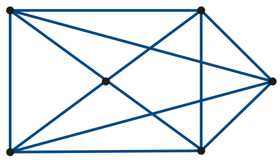


15 Which *one* of the following graphs has an Eulerian circuit?

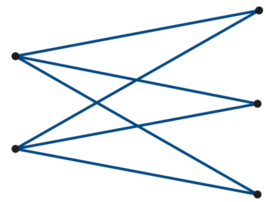
A



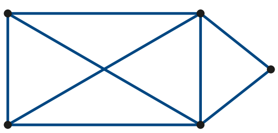
B



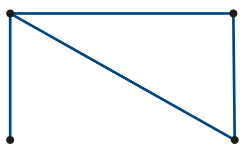
C



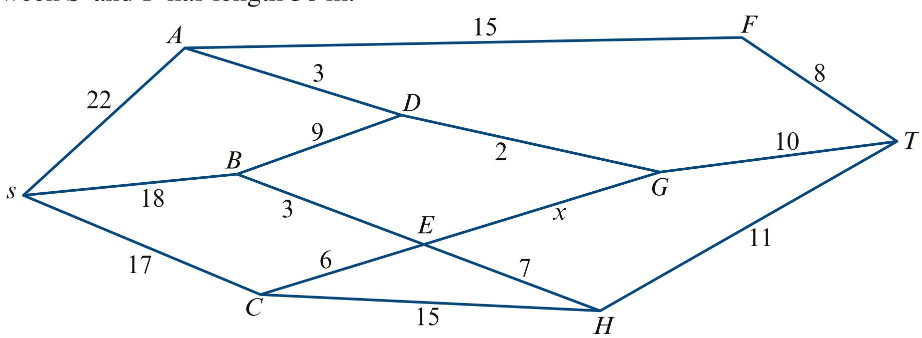
D



E



16 The network below shows the distance, in metres, between points. The shortest path between S and T has length 36 m.



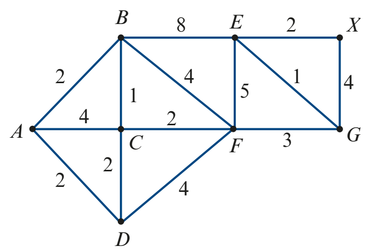
The value of x is.

A 4 **B** 5 **C** 6 **D** 7 **E** 8

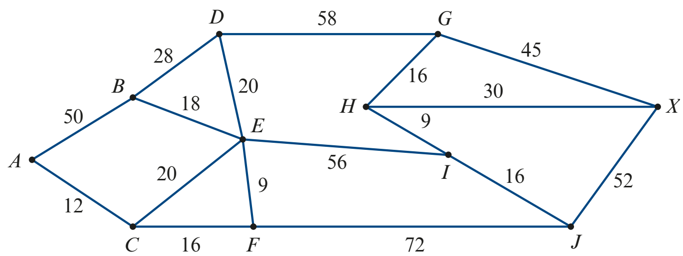
Written response questions

1 Determine the shortest path and write down the length of the shortest path from vertex A to vertex X in the following weighted graphs.

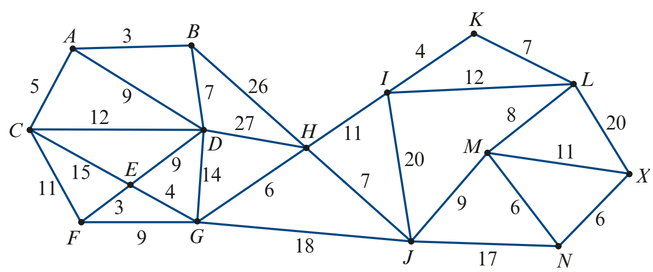
a



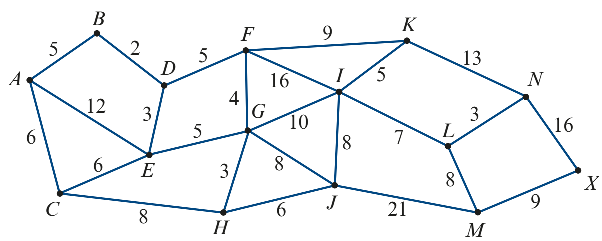
b



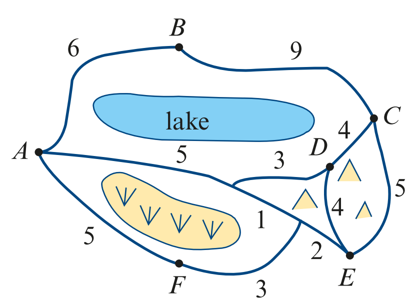
c



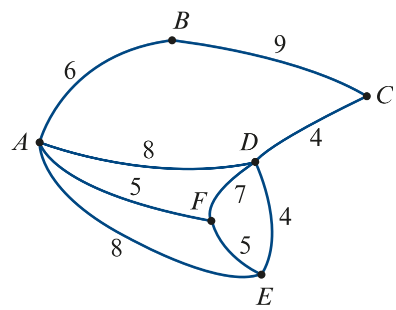
d



2 The map shows six campsites, A, B, C, D, E and F, which are joined by tracks. The numbers by the paths show lengths, in kilometres, of that section of track.



a **i** Complete the graph opposite, which shows the shortest direct distances between campsites. (The campsites are represented by vertices and tracks are represented by edges.)



ii A telephone cable is to be laid to enable each campsite to phone each other campsite. For environmental reasons, cables can only be laid along the tracks and cables can only connect to one another at the campsites. What is the minimum length of cable necessary to complete this task?

iii Fill in the missing entries for the adjacency matrix shown for the completed graph formed above.

$$
\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}
\begin{array}{c}
\begin{array}{cccccc} A & B & C & D & E & F \end{array} \\
\left[\begin{array}{cccccc}
0 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & - & - & - & - \\
1 & 0 & - & - & - & - \\
1 & 0 & - & - & - & -
\end{array}\right]
\end{array}
$$

b A walker follows the route $A–B–A–F–E–D–C–E–F–A$.

i How far does this person walk?

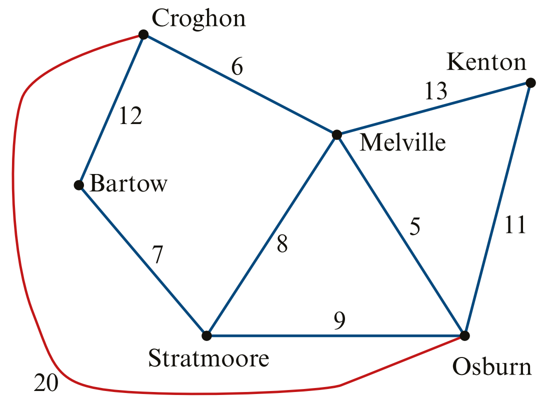
ii Why is the route *not* a Hamiltonian cycle?

iii Write down a route that a walker could follow that is a Hamiltonian cycle.

iv Find the distance walked in following this Hamiltonian cycle.

c It is impossible to start at A and return to A by going along each track exactly once. An extra track joining two campsites can be constructed so that this is possible. Which two campsites need to be joined by a track to make this possible? (Two tracks are only considered the same if they share their entire length. For example, A–E and F–E are considered different tracks even though they share a common stretch near E.)

3 The network on the right shows six villages represented as vertices of the graph. The edges represent the roads connecting the villages. The weights on the edges are the distances, in kilometres, alog each of the roads.



a What is the degree of the vertex representing Melville?

b Determine the sum of the degrees of the vertices of this graph.

c Verify Euler's formula for this graph.

A salesperson might need to travel to every village in this network to conduct business.

d If the salesperson follows the path Stratmoore – Melville – Kenton– Osburn – Melville– Croghon – Bartow – Stratmoore, has the salesperson followed a Hamiltonian cycle? Give a reason to justify your answer.

e If the salesperson follows the path Croghorn– Bartow – Stratmoore – Melville– Kenton – Osburn, what is the mathematical term for this path?

It would make sense for the salesperson to avoid visiting a certain village more than once, and it would also make sense for them to return 'home' after travelling the shortest distance possible.

f If the salesperson starts and ends in Bartow, find the shortest route and state the shortest distance the salesperson would have to travel.

g If the salesperson can start and end at any village in the network, what is the shortest route possible?

A road inspector must travel along every road connecting the six villages.

h Explain why the inspector could not follow an Eulerian circuit through this road network.

i The inspector may start and end their route at different villages, but would like to travel along each road once only. Which villages can the inspector start their route from? Write down a path the inspector could take to complete their work.

j The speed limit for each of these roads is 60 km/hr. If the inspector must complete their work by 5 p.m, what is the latest time that the inspector can begin their work?

New electrical cables connecting the villages are required. They will be installed along some of the roads listed in the graph above. These cables will form a connected graph and the shortest total length of cable will be used.

k Give a mathematical term to describe a graph that represents these cables.

l Draw the graph that represents these cables and find the total length of cable required.

Chapter 14

Flow, matching and scheduling problems

Chapter objectives

- How do we define a directed graph?
- How do we define flow?
- How do we calculate the maximum flow through a network?
- How do we draw and use a bipartite graph to solve allocation problems?
- How do we find the optimal allocation of multiple groups of objects?
- How do we identify predecessors of an activity?
- How do we draw an activity network and use it to plan for a project?
- How do we account for float times in our project?
- How do we find the earliest starting time and latest finishing time for an activity in a project?
- How do we identify the critical path of an activity network?

In the previous chapter, undirected graphs were used to define and represent situations. In this chapter, directed graphs will be used to model networks and solve problems involving travel, connection, flow, matching, allocation and scheduling.

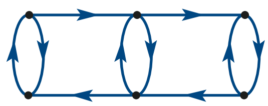
14A Flow problems

Learning intentions

- To be able to define and describe flow.
- To be able to calculate the maximum flow through a flow network by observation.
- To be able to identify cuts and calculate cut capacities.
- To be able to determine the maximum flow through a flow network by finding the capacity of the minimum cut.
- To be able to solve flow problems by finding minimum cut capacities.

Directed graphs

In the previous chapter, graphs were used to represent connections between people, places or objects. The vertices of a graph represented objects, such as towns, and edges represented the connections between them, such as roads. **Weighted graphs** included extra numerical information about the connections, such as distance, time or cost. When a graph has this numerical information we call it a **network**.



A **directed graph**, or **digraph**, records directional information on networks using arrows on the edges. The network on the right shows roads around a city. The vertices are the intersections of the roads and the edges are the possible road connections between the intersections. The arrows show that some of the roads only allow traffic in one direction, while others allow traffic in both directions.

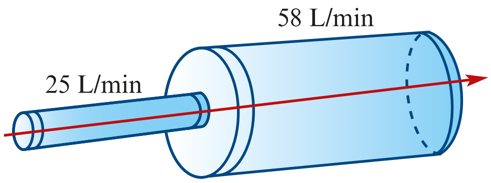
Understanding minimum flow

One of the applications of directed graphs to real-life situations is flow problems. Flow problems involve the transfer or **flow** of material from one point, called the **source**, to another point called the **sink**. Examples of this include water flowing through pipes, or traffic flowing along roads.

source → flow through network → sink

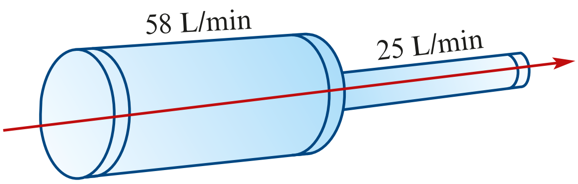
Water flows through pipes in only one direction. In a digraph representing water flow, the vertices are the origin and destination of the water and the edges represent the pipes connecting them. The weights on the edges would be the amount of water that can flow through the pipe in a given time. The weights of flow problem directed graphs are called **capacities**.

The diagram on the right shows two pipes that are joined together, connecting the source of water to the sink. There is a small pipe with capacity 25 litres per minute joined to a large pipe with capacity 58 litres per minute.



Even though the large pipe has a capacity greater than 25 litres per minute, the small pipe will only allow 25 litres of water through each minute. The flow through the large pipe will never be more than 25 litres per minute. The large pipe will experience flow below its capacity.

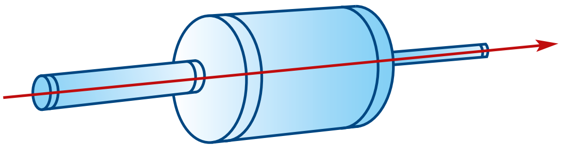
If we reverse the connection and direct water through the large capacity pipe into the smaller capacity pipe, there will be a 'bottleneck' of flow at the junction.



The large capacity pipe is capable of delivering 58 litres of water every minute to the small pipe, but the small pipe will only allow 25 litres per minute to pass.

In both of these situations, the flow through the entire pipe system (both pipes from source to sink) is restricted to a maximum of 25 litres per minute. This is the capacity of the smallest pipe in the connection.

If we connect more pipes together, one after the other, we can calculate the overall capacity or **maximum flow** of the pipe system by looking for the *smallest capacity pipe* in that system.

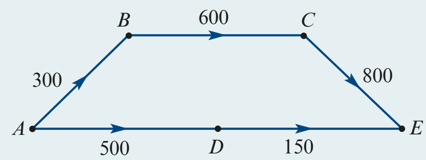


Maximum flow

If pipes of different capacities are connected one after the other, the *maximum flow* through the pipes is equal to the *minimum* capacity of the individual pipes.

Example 1 Calculating the maximum flow

In the digraph shown on the right, the vertices A, B, C, D and E represent towns. The edges of the graph represent roads and the weights of those edges are the maximum number of cars that can travel on the road each hour. The roads allow only one-way travel.



a Find the maximum traffic flow from A to E through town C.

b Find the maximum traffic flow from A to E overall.

c A new road is being built to allow traffic from town D to town C. This road can carry 500 cars per hour.

i Add this road to the digraph.

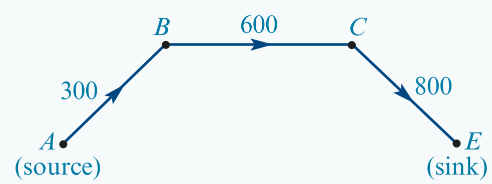
ii Find the maximum traffic flow from A to E overall after this road is built.

Explanation

a Look at the subgraph that includes town C. The smallest capacity of the individual roads is 300 cars per hour. This will be the maximum flow through town C.

The maximum flow from A to E through town C is equal to the smallest capacity road along that route.

Solution



The maximum flow is 300 cars per hour.

Explanation

b Look at the two subgraphs from A to E. The maximum flow through D will be 150 cars per hour (minimum capacity). Add the maximum flow through C to the maximum flow through D.

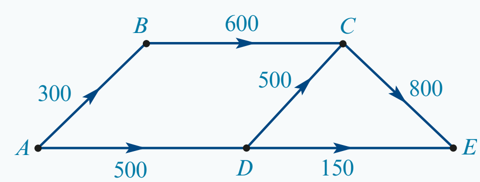
Solution

The maximum flow from A to E overall is:
$300 + 150 = 450$ cars per hour

Explanation

c **i** Add the edge to the diagram.

Solution



Explanation

ii The maximum flow through A–B–C–E is 300. But C–E has capacity 800. If another 500 cars per hour come through A–D–C, they will also be able to travel from C to E.

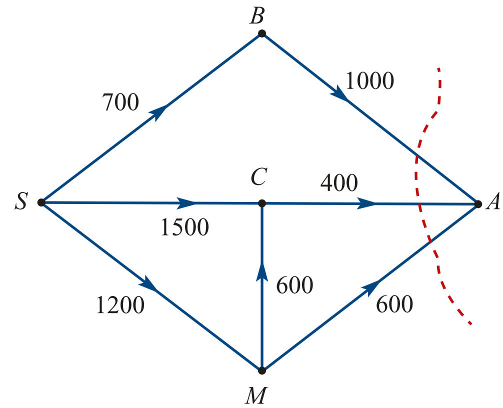
Solution

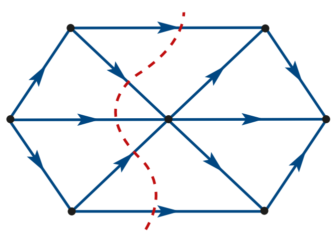
The new maximum flow is now 800 cars per hour.

Cuts

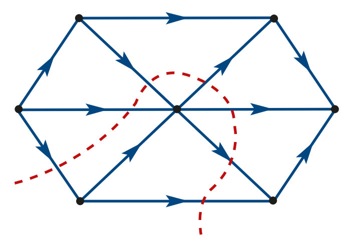
It is difficult to determine the maximum flow by inspection for directed networks that involve many vertices and edges. We can simplify the search for maximum flow by searching for **cuts** within the digraph.

A cut divides the network into two parts, completely separating the source from the sink. It is helpful to think of cuts as imaginary breaks within the network that completely block the flow through that network. For the network or water pipes shown in this diagram, the dotted line is a cut. This cut completely blocks the flow of water from the source (S) to the sink (A).





The dotted line on the graph above is a valid cut because it separates the source and the sink completely. No material can flow from the source to the sink.

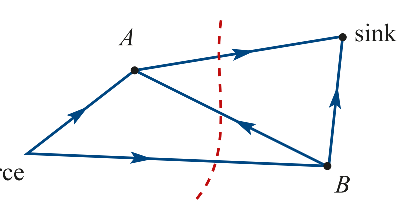


The dotted line on the graph above is *not* a valid cut because material can still flow from the source to the sink. Not all of the pathways from source to sink have been blocked by the cut.

Capacity of a cut

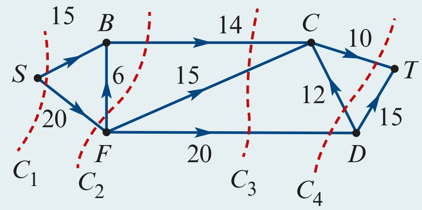
The **cut capacity** is the sum of all the capacities of the edges that the cut passes through, taking into account the direction of flow. The capacity of an edge is only counted if it flows from the source side to the sink side of the cut.

In the simple network shown, the cut passes through three edges. The edge B to A is not counted in the capacity of the cut because the flow for that edge is from the sink side to the source side of the cut.



Example 2 Calculating cut capacity

Calculate the capacity of the four cuts shown in the network on the right. The source is vertex S and the sink is vertex T.



Explanation	Solution
All edges in C_1 are counted.	The capacity of $C_1 = 15 + 20 = 35$
Note that the edge from F to B is not counted in C_2.	The capacity of $C_2 = 14 + 20 = 34$
All edges in C_3 are counted.	The capacity of $C_3 = 14 + 15 + 20 = 49$
Note that the edge from D to C is not counted in C_4.	The capacity of $C_4 = 20 + 10 = 30$

The capacity of a cut is important to help determine the maximum flow through any digraph. Look for the smallest, or minimum, cut capacity that exists in the graph. This will be the same as the maximum flow that is possible through that graph. This is known as the *maximum-flow minimum-cut theorem*.

Cut, cut capacity and minimum cut capacity

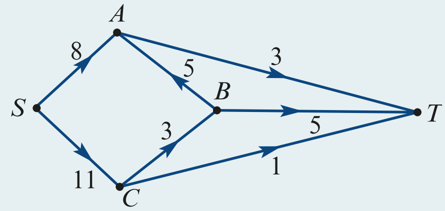
A *cut* is an imaginary line across a directed graph that completely separates the *source* (start of the flow) from the *sink* (destination of the flow).

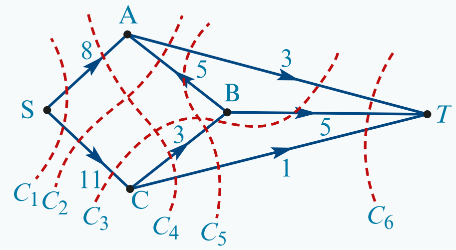
The *cut capacity* is the sum of the capacities of the edges that are cut. Only edges that flow from the source side of the cut to the sink side of the cut are included in a cut capacity calculation.

The ***minimum cut*** *capacity* possible for a graph equals the *maximum flow* through the graph.

Example 3 Calculating maximum flow using cuts

Determine the maximum flow from S to T for the digraph shown on the right.



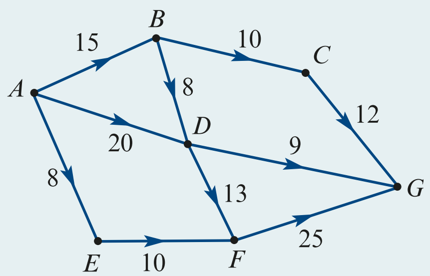
Explanation	Solution
1 Mark in all possible cuts on the network.	
2 Calculate the capacity of all the cuts.	The capacity of $C_1 = 8 + 11 = 19$ The capacity of $C_2 = 3 + 11 = 14$ The capacity of $C_3 = 3 + 5 + 11 = 19$ The capacity of $C_4 = 8 + 3 + 1 = 12$ The capacity of $C_5 = 3 + 3 + 1 = 7$ The capacity of $C_6 = 3 + 5 + 1 = 9$
3 Identify the minimum cut capacity and write your answer.	The minimum cut capacity is 7 so the maximum flow from S to T is 7.

Calculating maximum flow by tracking flow through a network (Optional)

There is another method we can use to calculate the maximum flow through a network. This method relies on tracking the flow through every edge.

Example 4 Calculating maximum flow

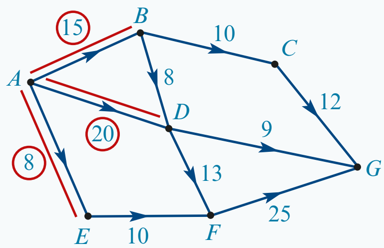
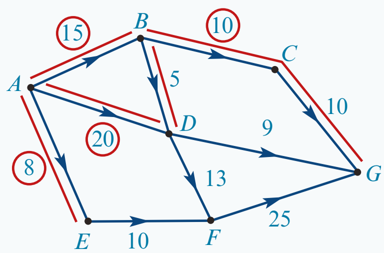
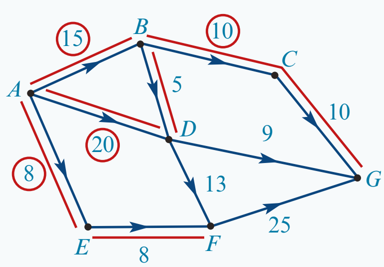
The koala sanctuary in Cowes allows visitors to walk through their park. The park is represented by a network below, where each edge represents one-way tracks for visitors through the park. The direction of travel on each track is shown by an arrow. The numbers on the edges indicate the maximum number of people who are permitted to walk along each track each hour.



a Starting at A, how many people are permitted to walk to G each hour?

b Given that one group of nine people would like to walk from A to G together as a group, list all the different routes they could take so that the entire group of nine will stay together for the duration of their walk.

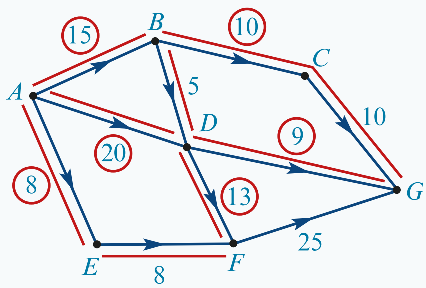
c What is the largest group of people that could walk through the koala sanctuary if they must stay together in a group for the entire duration of the walk?

Explanation	Solution
a Firstly, consider the edges coming from the source. When calculating the maximum flow through a network, always assume the initial edges from the source are flowing at their maximum capacity.	The vertex A is the source in this network. The three edges connected to A, flowing towards B, D and E will be flowing at maximum capacity because they are coming from the source. Draw lines along these edges and circle their capacities as they are flowing at maximum capacity. 
Now, from the vertices accepting flow from the source, start observing the flow through the network and if the maximum capacity of each edge can be achieved via direct flow from another edge or by splitting the flow of one edge into two.	At vertex B, although there is a flow of 15 coming in from the source, the edges taking flow towards the sink at G are of different capacities. From B to C only a capacity of 10 can flow, therefore the same 10 can also only flow from C to G as no other edges are connected in this route. From B, as 10 flows to C, the leftover 5 can flow to D. Even though the edge from B to D has a maximum capacity of 8, there is only 5 available to flow through this edge; cross off 8 and write a 5 next to it.  At vertex E, a flow of 8 is coming from the source. Although the edge from E to F has a capacity of 10, only a maximum of 8 can flow through it. Just as before, cross out the 10 and write an 8 next to it to signify the flow passing through the edge. 

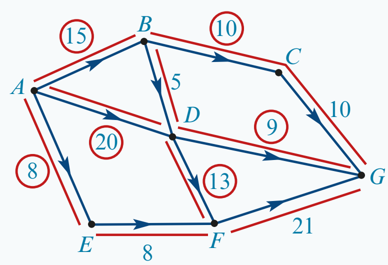
Explanation

Solution

At vertex D there is a total of 25 flowing into it, coming from vertices B and A. This flow of 25 can be redistributed to the two edges coming from D towards the sink at G. Of the 25, 9 can flow directly to G and 13 can flow from D to F. The maximum capacities of these edges can be achieved, so circle these numbers along the edges.



Finally, the edge from F to G has a maximum capacity of 25, however only 21 ($8 + 13 = 21$) is coming through. Cross out 25, replace with 21 to indicate the correct flow through the edge.



The maximum flow through the network is the total amount incoming to the *sink* vertex G, which is $10 + 9 + 21 = 40$. Therefore a maximum of 40 people can walk through the koala sanctuary each hour.

b A group of 9 must begin at vertex A, only pass through edges with a capacity greater than or equal to 9 and end at vertex G. Take note of the direction of the arrow heads.

Starting from vertex A there are only two edges the group of 9 people can walk along; the edges going to vertices B and D. Walking through vertex B there is one walk possible: $A - B - C - G$. Walking through vertex D there are two walks possible: $A - D - G$ and $A - D - F - G$.

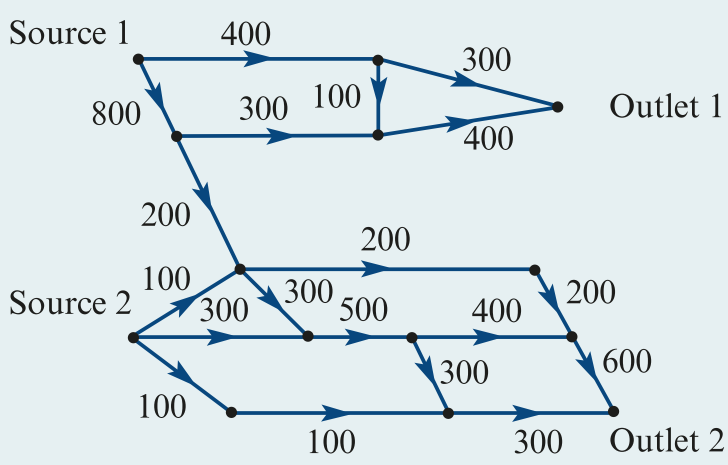
Explanation	Solution
c From the starting vertex A, consider the largest possible group that could start a walk through the sanctuary and then analyse how many of that group could then walk to vertex G given the capacities of each of the edges throughout the network.	Starting at vertex A the largest possible group of people that can enter the sanctuary is 20, however after reaching vertex D the group of 20 cannot stay together as they move towards vertex G because the capacity of the edges from D can only take a maximum of 13 people. From D, moving towards G a group of 13 could pass through with no other restrictions. This is the largest group of people that can enter the sanctuary at vertex A and pass through to vertex G, together as one group, given the restrictions of the edge capacities.

Calculating maximum flow from more than one source

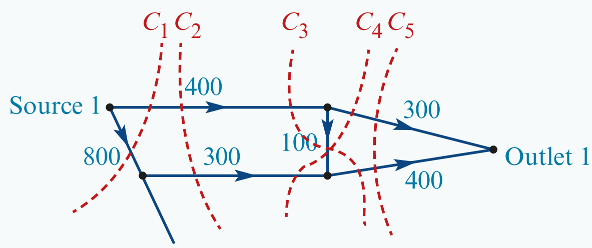
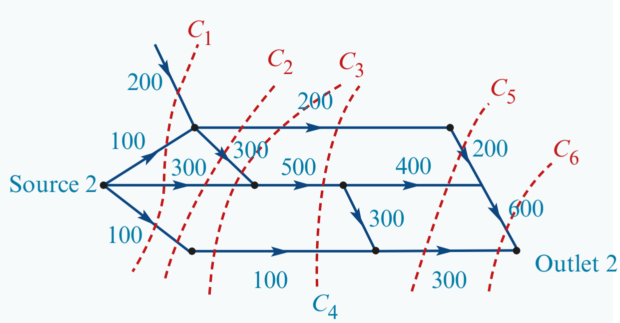
Example 5 **Calculating maximum flow from more than one source**

Water enters a network of pipes at either Source 1 or Source 2 and flows out at either Outlet 1 or Outlet 2.

The numbers next to the arrows represent the maximum rate, in kilolitres per minute, at which water can flow through each pipe.



Determine the maximum rate, in kilolitres per minute, at which water can flow from these pipes into the ocean at Outlet 1 and Outlet 2.
Note that although this method gives us the maximum flow for each outlet, we cannot always add these values up to find the total maximum flow through the system, because we might not be able to achieve maximum flow for every outlet at the same time.

Explanation	Solution
The outlets need to be considered separately.	
Outlet 1	
Look for the minimum cut that prevents water reaching Outlet 1.	The capacity of C_1 is: $400 + 800 = 1200$
	The capacity of C_2 is: $400 + 300 = 700$
	The capacity of C_3 is: $400 + 400 = 800$
Note: The pipe with capacity 200 leading towards Outlet 2 does not need to be considered in any cut because this pipe *always* prevents water from reaching Outlet 1.	The capacity of C_4 is: $300 + 100 + 300 = 700$
	The capacity of C_5 is: $300 + 400 = 700$
	The minimum cut/maximum flow is 700 kilolitres per minute.
	
Outlet 2	
Look for the minimum cut that prevents water reaching Outlet 2.	The capacity of C_1 is: $200 + 100 + 300 + 100 = 700$
	The capacity of C_2 is: $200 + 300 + 300 + 100 = 900$
Note: The pipe with capacity 200 leading towards Outlet 2 will need to be considered in any cut because this pipe delivers water towards Outlet 2 and must be 'cut' like all the others. Other cuts are possible, but have not been included in the diagram.	The capacity of C_3 is: $200 + 300 + 300 + 100 = 900$
	The capacity of C_4 is: $200 + 500 + 100 = 800$
	The capacity of C_5 is: $200 + 400 + 300 = 900$
	The capacity of C_6 is: $600 + 300 = 900$
	The minimum cut/maximum flow is 700 kilolitres per minute.
	

Note: The maximum flow through a network with two sources can also be determined by tracking the flow as outlined in Example 4.

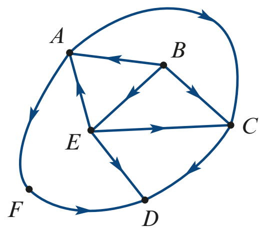
Skill-sheet

Exercise 14A

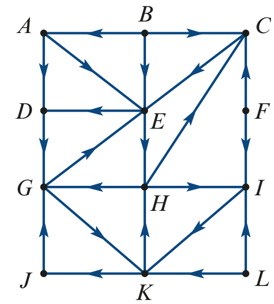
Directed graphs

1 Find the number of vertices that can be reached from vertex A in each of the directed graphs below.

a



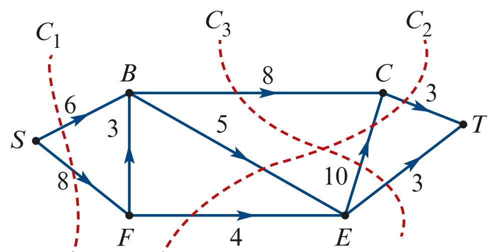
b



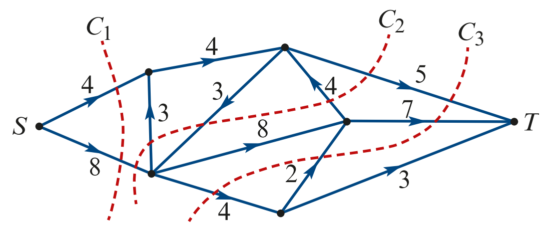
Cuts

Example 2

2 Determine the capacity of each of the cuts in the digraph opposite. The source is vertex S and the sink is vertex T.



3 Determine the capacity of each of the cuts in the digraph opposite. The source is vertex S and the sink is vertex T.

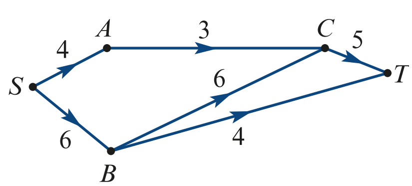


Maximum-flow

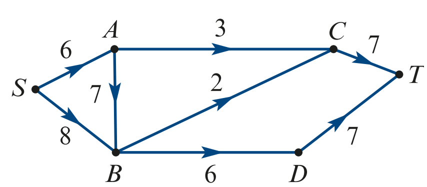
Example 3
Example 4

4 Find the maximum flow for each of the following graphs. The source is vertex S and the sink is vertex T.

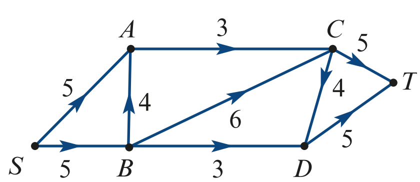
a



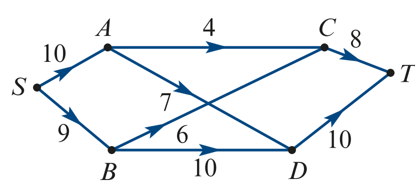
b



c

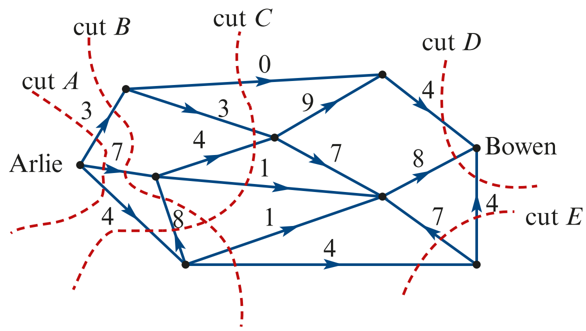


d



Minimum-cut maximum-flow

5 A train journey consists of a connected sequence of stages formed by edges on the directed network opposite from Arlie to Bowen. The number of available seats for each stage is indicated beside the corresponding edge, as shown on the diagram on the right.



The five cuts, A, B, C, D and E, shown on the network are attempts to find the maximum number of available seats that can be booked for a journey from Arlie to Bowen.

a Write down the capacity of cut A, cut B, cut C and cut D.

b Explain why cut E is not a valid cut when trying to find the minimum cut between Arlie and Bowen.

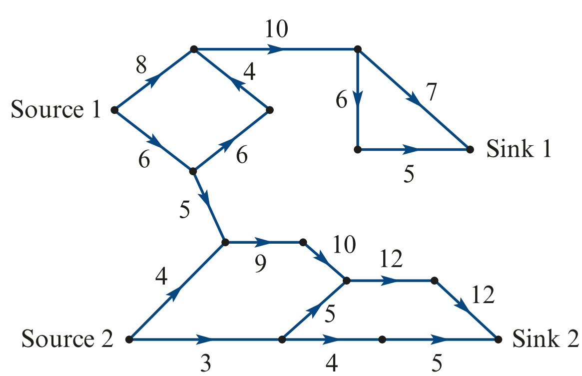
c Find the maximum number of available seats for a train journey from Arlie to Bowen.

Networks with more than one source

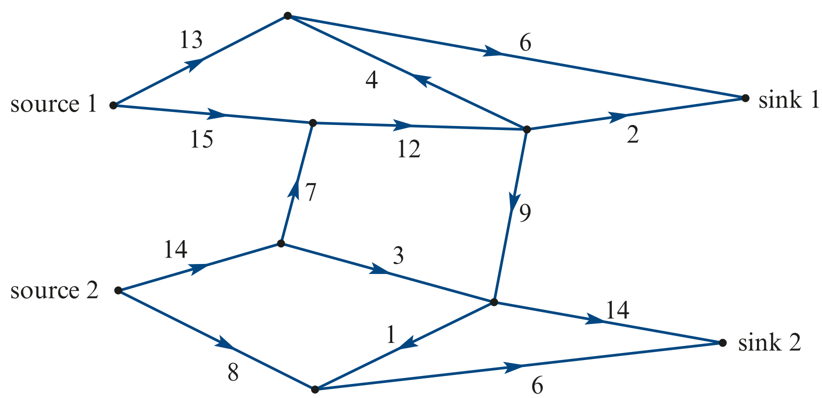
Example 5

6 In each of the following, water pipes of different capacities are connected to two water sources and two sinks. Networks of water pipes are shown in the diagrams below. The numbers on the edges represent the capacities, in kilolitres per minute, of the pipes. For each of the following, find the maximum flow, in kilolitres per minute, to each of the sinks in this network.

a

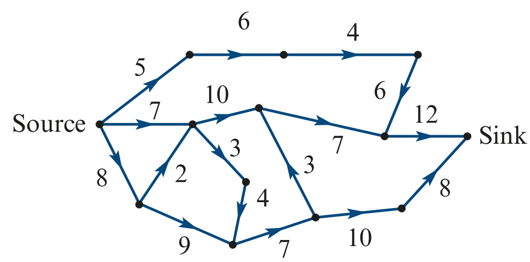


b



Analysis of maximum-flow problems

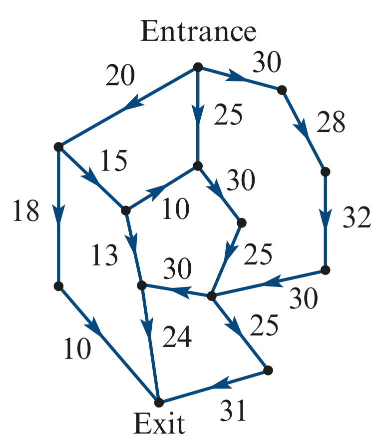
7 The flow of water through a series of pipes, in litres per minute, is shown in the directed network below.



a How many different routes from the *source* to the *sink* are possible?

b Determine the maximum flow from the *source* to the *sink*.

8 The corridors people can walk through to visit different exhibits in a museum are given as a directed network opposite. To avoid congestion around every exhibit, the museum imposes a maximum capacity policy throughout each corridor between exhibits. The numbers on the edges represent the maximum number of people that can walk through each corridor of the museum every 30 minutes.



a On the network opposite, identify a cut that has a capacity of 80.

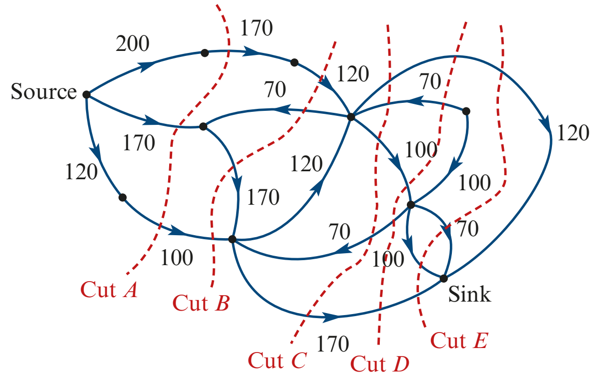
b Determine the maximum flow of people from the entrance to the exit of the museum.

c One group of primary school students would like to walk through the museum. The teacher explains that this can happen unsupervised if all students in the group remain together, not separating to explore different routes. Given that a group of students must stay together from the entrance to the exit, what is the largest group of students possible that can pass through the museum every 30 minutes?

Exam 1 style questions

Questions 9 and 10 refer to the diagram opposite

The flow of liquid through a series of pipes, in litres per minute, is shown in the directed network opposite. Five cuts labelled A to E are shown on the network.



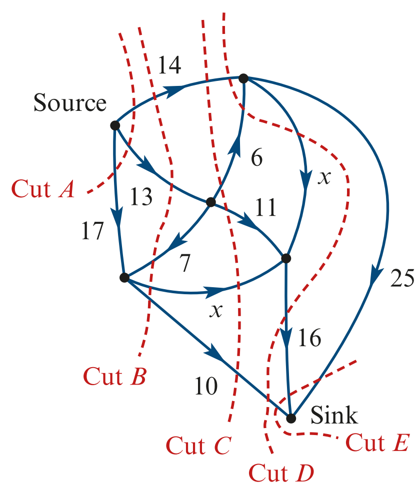
9 The capacity of Cut E is

A 70 **B** 170 **C** 290 **D** 390 **E** 460

10 The number of these cuts with a capacity equal to the maximum flow of liquid from the source to the sink, in litres per minute, is

A 0 **B** 1 **C** 2 **D** 3 **E** 4

11 The flow of water through a series of pipes, in litres per minute, is shown in the network opposite. The weightings of two edges are labelled x. Five cuts labelled A to E are shown on the network. The maximum flow of water from the source to the sink, in litres per minute, is given by the capacity of



A Cut A if $x = 4$ **B** Cut B if $x = 6$ **C** Cut C if $x = 8$

D Cut D if $x = 6$ **E** Cut E if $x = 8$

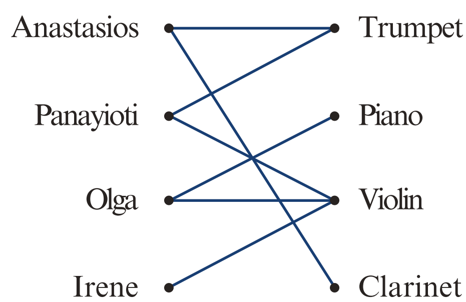
14B Matching and allocation problems

Learning intentions

- To be able to define and describe bipartite graphs.
- To be able to solve assignment problems using the Hungarian Algorithm.

Bipartite graphs

In some situations, the vertices of a graph belong in two separate sets. Consider a music school that has four teachers: Anastasios, Panayioti, Olga and Irene. These teachers, between them can teach four different instruments: trumpet, piano, violin and clarinet. The teachers and instruments are represented by vertices, arranged vertically as shown in the diagram opposite.



The edges of the diagram connect the teachers to the instruments they can teach.

This type of graph is called a **bipartite graph.** Each edge in a bipartite graph joins a vertex from one group to a vertex in the other group.

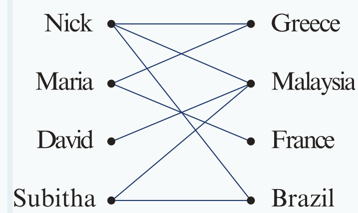
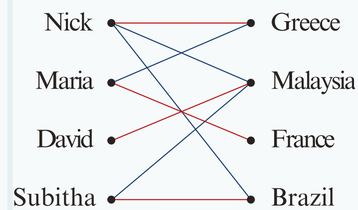
In the situation described above, the school would need to match each teacher to one instrumental class; this is an example of an **allocation problem**. The bipartite graph above graphically shows the instrument(s) that each teacher can teach and can help the school assign each teacher to an instrument. Anastasios is the only teacher who can teach clarinet, Irene can only teach violin, therefore Olga must teach piano and Panayioti must teach trumpet.

Example 6

Nick, Maria, David and Subitha are presenters on a TV travel show. Each presenter will be assigned a story to film about one country that they have visited before.

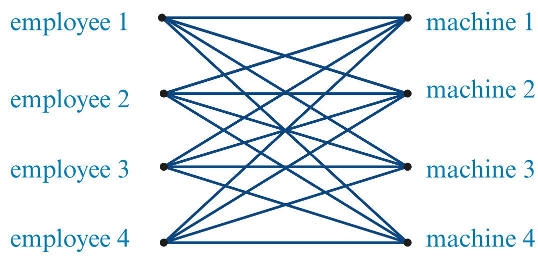
- Nick has visited Greece, Malaysia and Brazil
- Maria has visited Greece and France
- Subitha has visited Malaysia and Brazil
- David has visited Malaysia

Construct a bipartite graph of the information above and use it to decide on the assignment of each presenter to one country.

Explanation	Solution
The two groups of items are: Presenters and Countries. Draw a vertex for each presenter in one column and each country in another. Join the vertex of each presenter to the vertex of each country they have visited.	
To allocate a presenter to each country, begin by identifying the vertices with only one edge connected to them. ■ David is the only presenter who has visited Malaysia, so he must visit that country. ■ France has only been visited by one presenter: Maria. She must be allocated to France, therefore she cannot be allocated to Greece, the other country she has visited before.	Highlight definite allocations on the bipartite graph 
Given that each presenter can only visit one country, the final allocations can be deduced by eliminating impossible allocations. ■ Maria is allocated to France, therefore she cannot visit Greece. Nick is the only other presenter who has visited Greece, therefore he must be allocated that country. ■ As David is allocated to Malaysia, Subitha has only one other country available to visit. ■ Subitha must be allocated to Brazil and this is also supported by the fact Nick must be allocated to Greece. Write the assignments.	■ Nick will be allocated Greece ■ Maria will be allocated France ■ David will be allocated Malaysia ■ Subitha will be allocated Brazil

The Hungarian algorithm

The graph on the right shows four employees in a factory. There are four different machines that are used in the production of an item. Every employee can use every machine and so this is a **complete bipartite graph**. Employees and machines can be matched in many different ways.



Rather than just assign an employee to a machine randomly, the factory could use information about how well each employee uses each machine, perhaps in terms of how quickly each performs the task. The times taken would be the weights on the edges of the bipartite graph. Rather than writing all of the weights on a complete bipartite graph (which would be a very complicated diagram), we can summarise the time information in a table and then use an algorithm, called the **Hungarian algorithm**, to allocate employees to machines in order to minimise the time taken to finish the tasks.

The table on the right shows the four employees: Wendy, Xenefon, Yolanda and Zelda. The machines in a factory are represented by the letters A, B, C and D.

The numbers in the table are the times, in minutes, it takes each employee to finish the task on each machine.

Employee	*A*	*B*	*C*	*D*
Wendy	30	40	50	60
Xenefon	70	30	40	70
Yolanda	60	50	60	30
Zelda	20	80	50	70

The table is called a **cost matrix**. Even though the numbers do not represent money value, this table contains information about the cost, in terms of time, of employees using each machine. The cost matrix can be used to determine the best way to allocate an employee to a machine so that the overall cost, in terms of the time taken to finish the work, is minimised. The Hungarian algorithm is used to do this.

Performing the Hungarian algorithm

Step 1: Subtract the lowest value in each row, from every value in that row.

- 30 has been subtracted from every value in the row for Wendy.
- 30 has been subtracted from every value in the row for Xenefon.
- 30 has been subtracted from every value in the row for Yolanda.
- 20 has been subtracted from every value in the row for Zelda.

Employee	*A*	*B*	*C*	*D*
Wendy	0	10	20	30
Xenefon	40	0	10	40
Yolanda	30	20	30	0
Zelda	0	60	30	50

Step 2: If the minimum number of lines required to cover all the zeros in the table is equal to the number of allocations to be made, jump to step 6. Otherwise, continue to step 3.

- The zeros can be covered with three lines. This is less than the number of allocations to be made (4).
- Continue to step 3.

Employee	*A*	*B*	*C*	*D*
Wendy	0	10	20	30
Xenefon	40	0	10	40
Yolanda	30	20	30	0
Zelda	0	60	30	50

Step 3: If a column does not contain a zero, subtract the lowest value in that column from every value in that column.

- Column *C* does not have a zero.
- 10 has been subtracted from every value in column *C*.

Employee	*A*	*B*	*C*	*D*
Wendy	0	10	10	30
Xenefon	40	0	0	40
Yolanda	30	20	20	0
Zelda	0	60	20	50

Step 4: If the minimum number of lines required to cover all the zeros in the table is equal to the number of allocations to be made, jump to step 6. Otherwise, continue to step 5a.

- The zeros can be covered with three lines. This is less than the number of allocations to be made (4).
- Continue to step 5a.

Employee	*A*	*B*	*C*	*D*
Wendy	0	10	10	30
Xenefon	40	0	0	40
Yolanda	30	20	20	0
Zelda	0	60	20	50

Step 5a: Add the smallest uncovered value to any value that is covered by two lines. Subtract the smallest uncovered value from all the uncovered values.

- The smallest uncovered element is 10.
- 10 has been *added* to Xenefon–*A* and Xenefon–*D* because these values are covered by two lines.
- 10 has been *subtracted* from all the uncovered values.

Employee	*A*	*B*	*C*	*D*
Wendy	0	0	0	30
Xenefon	50	0	0	50
Yolanda	30	10	10	0
Zelda	0	50	10	50

Step 5b: Repeat from step 4.

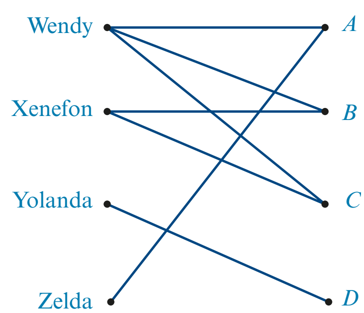
- The zeros can be covered with a minimum of four lines. This is the same as the number or allocations to make.
- Continue to step 6.

Employee	*A*	*B*	*C*	*D*
Wendy	0	0	0	30
Xenefon	50	0	0	50
Yolanda	30	10	10	0
Zelda	0	50	10	50

Step 6: Draw a bipartite graph with an edge for every zero value in the table.

In the bipartite graph:

- Wendy will be connected to *A*, *B* and *C*
- Xenefon will be connected to *B* and *C*
- Yolanda will be connected to *D*
- Zelda will be connected to *A*.



Step 7: Make the allocation and calculate minimum cost

- Zelda must operate machine *A* (20 minutes).
- Yolanda must operate machine *D* (30 minutes).
- Wendy can operate either machine *B* (40 minutes) or *C* (50 minutes).
- Xenefon can operate either machine *B* (30 minutes) or *C* (40 minutes).

Note: Because Wendy and Xenefon can operate either B or C, there are two possible allocations. Both allocations will have the same minimum cost.

The minimum time taken to finish the work = 20 + 30 + 50 + 30 = 130 minutes.

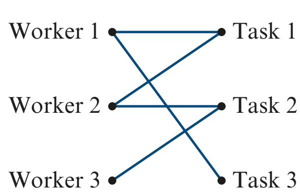
Skill-sheet

Exercise 14B

Bipartite graphs

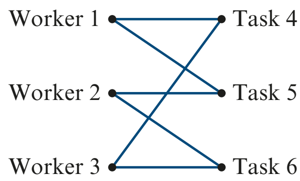
Example 6

1 **a** On Monday, three workers are each to be allocated one task at work. The bipartite graph below shows which task(s) each person is able to complete.



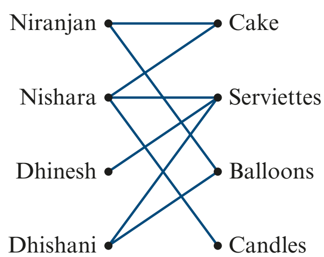
If each person completes a different task, write down the task each worker must be allocated to on Monday.

b On Tuesday, the same three workers will be allocated to a new set of tasks. The bipartite graph below shows which task(s) each person is able to complete.



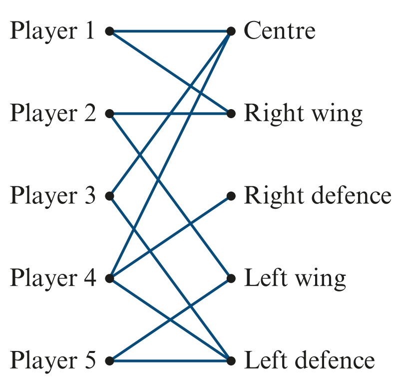
Given that Worker 2 must complete Task 6, write down the new task each worker must be allocated to on Tuesday.

2 It is Miko's birthday and his sister Aria has asked some of his friends to assist with the celebrations by purchasing some items for a party. The bipartite graph below shows which item(s) each person is able to purchase on their way to the party.



Each friend must purchase an item. Write down which item each friend must purchase.

3 The sport of ice hockey has six player positions: goalie, left defence, right defence, right wing, left wing and centre. A group of six have decided to play. Only one person is happy to play goalie. The other five people must be allocated to the other five positions.
The bipartite graph below shows which positions each of the five players can play.



Each player plays a different position. Write down two possible allocations, describing which position each player must play.

4 Gloria, Minh, Carlos and Trevor are buying ice-cream. They have a choice of five flavours: chocolate, vanilla, peppermint, butterscotch and strawberry. Gloria likes vanilla and butterscotch, but not the others. Minh only likes strawberry. Carlos likes chocolate, peppermint and butterscotch. Trevor likes all flavours.

a Explain why a bipartite graph can be used to display this information.

b Draw a bipartite graph with the people on the left and flavours on the right.

c What is the degree of the vertex representing Trevor?

The Hungarian algorithm

5 **a** A cost matrix is shown. Find the allocation(s) by the Hungarian algorithm that will give the minimum cost.

	A	*B*	*C*	*D*
W	110	95	140	80
X	105	82	145	80
Y	125	78	140	75
Z	115	90	135	85

b Find the minimum cost for the given cost matrix and give a possible allocation.

	A	*B*	*C*	*D*
W	2	4	3	5
X	3	5	3	4
Y	2	3	4	2
Z	2	4	2	3

6 A school is to enter four students in four track events: 100 m, 400 m, 800 m and 1500 m. The four students' times (in seconds) are given in the table. The rules permit each student to enter only one event. The aim is to obtain the minimum total time.

Student	*100 m*	*400 m*	*800 m*	*1500 m*
Dimitri	11	62	144	379
John	13	60	146	359
Carol	12	61	149	369
Elizabeth	13	63	142	349

Use the Hungarian algorithm to select the 'best' student for each event.

7 Three volunteer workers, Joe, Meg and Ali, are available to help with three jobs. The time (in minutes) in which each worker is able to complete each task is given in the table opposite. Which allocation of workers to jobs will enable the jobs to be completed in the minimum time?

	Job		
Student	*A*	*B*	*C*
Joe	20	20	36
Meg	16	20	44
Ali	26	26	44

8 A company has four machine operators and four different machines that they can operate. The table shows the hourly cost in dollars of running each machine for each operator. How should the machinists be allocated to the machines to minimise the hourly cost from each of the machines with the staff available?

	Machine			
Operator	*W*	*X*	*Y*	*Z*
A	38	35	26	54
B	32	29	32	26
C	44	26	23	35
D	20	26	32	29

9 A football association is scheduling football games to be played by three teams (the Champs, the Stars and the Wests) on a public holiday. On this day, one team must play at their Home ground, one will play Away and one will play at a Neutral ground.

The costs (in $'000s) for each team to play at each of the grounds are given in the table below.

Determine a schedule that will minimise the total cost of playing the three games and determine this cost.

Note: There are two different ways of scheduling the games to achieve the same minimum cost. Identify both of these.

Team	*Home*	*Away*	*Neutral*
Champs	10	9	8
Stars	7	4	5
Wests	8	7	6

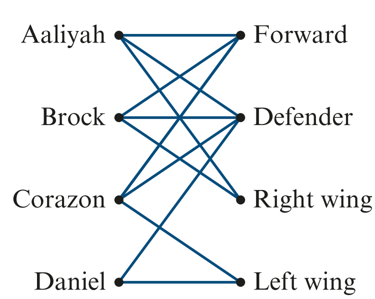
10 A roadside vehicle assistance organisation has four service vehicles located in four different places. The table below shows the distance (in kilometres) of each of these service vehicles from four motorists in need of roadside assistance.

	Motorist			
Service vehicle	*Jess*	*Mark*	*Raj*	*Karla*
A	18	15	15	16
B	7	17	11	13
C	25	19	18	21
D	9	22	19	23

Determine a service vehicle assignment that will ensure that the total distance travelled by the service vehicles is minimised. Determine this distance.

Exam 1 style questions

11 The sport of futsal has five player positions: goalkeeper, forward, defender, right wing and left wing. In a group of five friends, Ezekiel will always play goalkeeper, but the other four friends Aaliyah, Brock, Corazon and Daniel will rotate their responsibilities and are able to play a number of positions each.
The bipartite graph below shows which positions each of the four friends can play.



Based on the bipartite graph, which one of the following allocations is **not** possible?

A

Friend	Position
Aaliyah	Right wing
Brock	Forward
Corazon	Defender
Daniel	Left wing

B

Friend	Position
Aaliyah	Right wing
Brock	Defender
Corazon	Forward
Daniel	Left wing

C

Friend	Position
Aaliyah	Forward
Brock	Right wing
Corazon	Left wing
Daniel	Defender

D

Friend	Position
Aaliyah	Forward
Brock	Defender
Corazon	Left wing
Daniel	Right wing

E

Friend	Position
Aaliyah	Forward
Brock	Right wing
Corazon	Defender
Daniel	Left wing

Use the following information to answer questions 12 and 13

Five people work at a bank. Each person will perform one task. The time taken for each person to complete tasks 1, 2, 3, 4 and 5, in hours, is shown in the table below.

	Anita	**Brad**	**Carmen**	**Dexter**	**Electra**
Task 1	1	2	2	5	4
Task 2	4	9	7	11	6
Task 3	5	3	3	9	4
Task 4	8	5	6	6	7
Task 5	5	8	4	6	9

12 The manager of the bank wants to allocate the tasks so as to minimise the total time taken to complete the five tasks. If each person starts their allocated task at the same time, then the first person to finish could be either

A Anita or Brad **B** Anita or Elektra **C** Brad or Carmen
D Brad or Dexter **E** Brad or Elektra

13 Before the tasks are performed, it is found that Elektra will only require 4 hours to complete Task 5 rather than 9 hours. If the tasks are allocated based on this new information, the minimum total time for all tasks will

A increase by 4 days. **B** decrease by 4 days. **C** decrease by 3 days.
D decrease by 2 days. **E** decrease by 1 day.

14 Four people, Xena, Wilson, Yasmine, Zachary, are each assigned a different job by their manager. The table below shows the time, in hours, that each person would take to complete each of the four jobs.

	Job 1	Job 2	Job 3	Job 4
Xena	5	3	7	p
Wilson	1	2	5	6
Yasmine	1	7	1	5
Zachary	4	7	6	p

Wilson takes 6 minutes to complete Job 4, while Yasmine only takes 5 minutes to complete Job 4. Both Xena and Zachary take p minutes to complete Job 4.
The manager will allocate the jobs as follows:

- Job 1 to Wilson
- Job 2 to Xena
- Job 3 to Yasmine
- Job 4 to Zachary

This allocation will achieve the minimum total completion time if the value of p is not greater than

A 6 **B** 7 **C** 8 **D** 9 **E** 10

14C Precedence tables and activity networks

Learning intentions

- ▶ To be able to identify activities in a project.
- ▶ To be able to understand the precedence that some activities have over others in a project.
- ▶ To be able to identify immediate predecessors of activities from an activity network.
- ▶ To be able to draw an activity network from a precedence table.
- ▶ To be able to understand and explain the need for dummy activities in projects.
- ▶ To be able to include dummy activities in activity networks as required.

Drawing activity networks from precedence tables

Building a house, manufacturing a product, organising a wedding and other similar projects all require many individual **activities** to be completed before the project is finished. The individual activities often rely upon each other and some can't be performed until other activities are complete.

In the organisation of a wedding, invitations would be sent out to guests, but a plan for seating people at the tables during the reception can't be completed until the invitations are accepted. When building a house, the plastering of the walls can't begin until the house is sealed from the weather.

For any project, if activity A must be completed before activity B can begin then activity A is said to be an **immediate predecessor** of activity B. The activities within a project can have multiple immediate predecessors and these are usually recorded in a table called a **precedence table**.

This precedence table shows some of the activities involved in a project and their immediate predecessors.

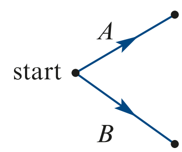
Activity	*Immediate predecessors*
A	–
B	–
C	A
D	B
E	B
F	C, D
G	E, F

The information in the precedence table can be used to draw a network diagram called an **activity network**.

Activity networks do not have labelled vertices, other than the *start* and *finish* of the project. The activities in the project are represented by the edges of the diagram and *so it is the edges that must be labelled*, not the vertices.

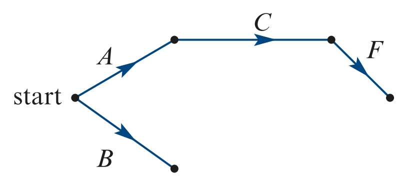
Activities A and B have no immediate predecessors.

These activities can start immediately and can be completed at the same time.



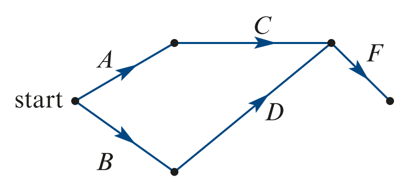
Activity A is an immediate predecessor of activity C, so activity C must follow immediately after activity A.

Activity C is an immediate predecessor of activity F, so activity F must follow immediately after activity C.



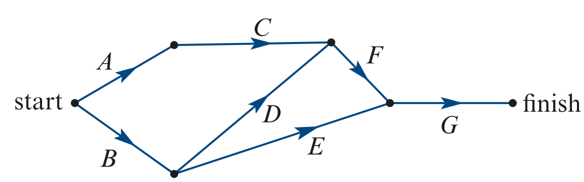
Activity D has immediate predecessor activity B so it follows immediately after activity B.

Activity D is also an immediate predecessor of activity F so activity F must follow immediately after activity D.



Activity E has immediate predecessor activity B so it will follow immediately after activity B.

Activity G has immediate predecessor activity F and activity E and so it must follow immediately after both of these activities.



Activity G is not an immediate predecessor for any activity and so the project is finished after this activity is complete.

Activity networks

When activity A must be completed before activity B can begin, activity A is called an immediate *predecessor* of activity B.

A table containing the activities of a project, and their immediate predecessors, is called a *precedence table*.

An *activity network* can be drawn from a precedence table. Activity networks have edges representing activities. The vertices are not labelled, other than the start and finish vertices.

Example 7 Drawing an activity network from a precedence table.

Draw an activity network from the precedence table shown on the right.

In this solution, the activity network will be drawn from the finish back to the start.

Activity	*Immediate predecessors*
A	–
B	A
C	A
D	A
E	B
F	C
G	D
H	E, F, G

Explanation

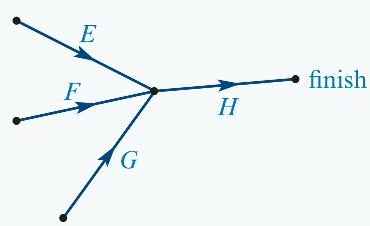
H is not an immediate predecessor for any other activity so it will lead to the finish of the project.

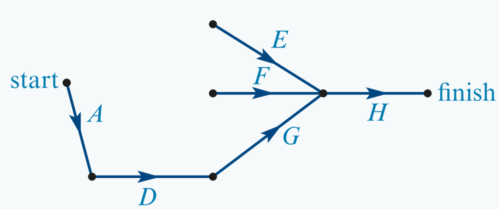
H has immediate predecessors E, F and G and so these three activities will lead into activity H.

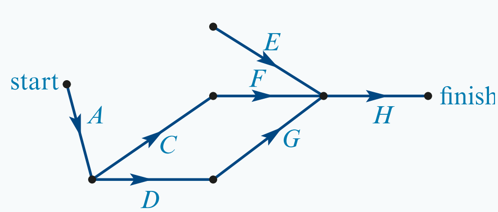
Activity D is an immediate predecessor of activity G *and* has immediate predecessor activity A. There will be a path through activity A, activity D and then activity G.

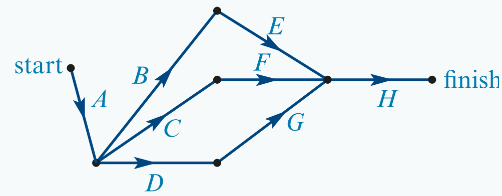
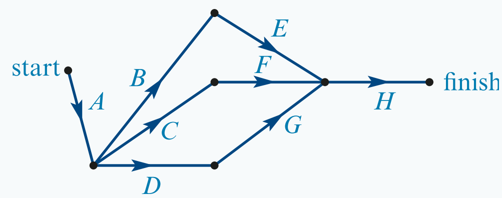
Activity C is an immediate predecessor of activity F *and* has immediate predecessor activity A. There will be a path through activity A, activity C and then activity F.

Solution







Explanation	**Solution**
Activity B is an immediate predecessor of activity E *and* has immediate predecessor activity A. There will be a path through activity A, activity B and then activity E.	
Activity A has no immediate predecessors, so it is the start of the project.	

Sketching activity networks

Activities that have no immediate predecessors follow from the start vertex.

Activities that are not immediate predecessors for other activities lead to the finish vertex.

For every other activity, look for:

- which activities it has as immediate predecessors
- which activities it is an immediate predecessor for.

Construct the activity network from this information.

Dummy activities

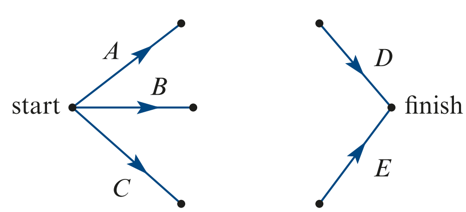
Sometimes two activities will have some of the same immediate predecessors, but not all of them. In this very simple precedence table, activity D and activity E share the immediate predecessor activity B, but they both have an immediate predecessor activity that the other does not.

Activity	*Immediate predecessors*
A	–
B	–
C	–
D	A, B
E	B, C

This overlap of predecessors presents some difficulty when constructing the activity network, but this difficulty is easily overcome.

Activity D and activity E are not immediate predecessors for any other activity, so they will lead directly to the finish vertex of the project.

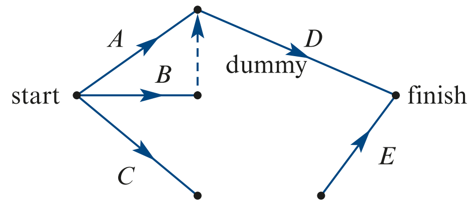
Activites A, B and C have no immediate predecessors, so they will follow directly from the start vertex of the project.



The start and finish of the activity network are shown in the diagram above. We need to use the precedence information for activity D and activity E to join these two parts together.

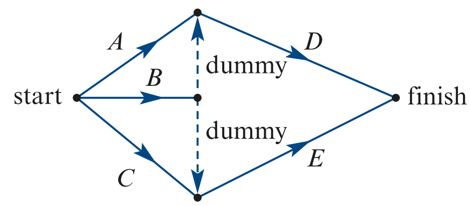
Activity D needs to follow directly from activity A and activity B, but we can only draw one edge for activity D. Activity E needs to follow directly from both activity B and activity C, but again we only have one edge for activity E, not two.

The solution is to draw the diagram with activity D starting after one of its immediate predecessors, and using a **dummy activity** for the other. The dummy activities are represented by dotted edges and are, in effect, imaginary. They are not real activities, but they allow all of the predecessors from the table to be correctly represented.



The dummy activity for D allows activity D to directly follow both activity A and B.

A dummy activity is also needed for activity E because it, too, has to start after two different activities, activity B and C.



Dummy activities

A dummy activity is required if two activities share some, but not all, of their immediate predecessors.

A dummy activity will be required *from the end* of each shared immediate predecessor *to the start* of the activity that has additional immediate predecessors.

Dummy activities are represented in the activity network using *dotted lines*.

Example 8 Using a dummy activity in an activity network

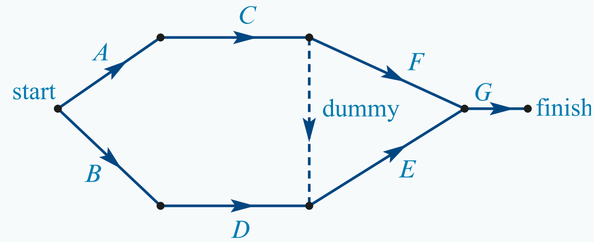
Draw an activity network from the precedence table shown on the right.

Activity	*Immediate predecessors*
A	–
B	–
C	A
D	B
E	C, D
F	C
G	E, F

Explanation

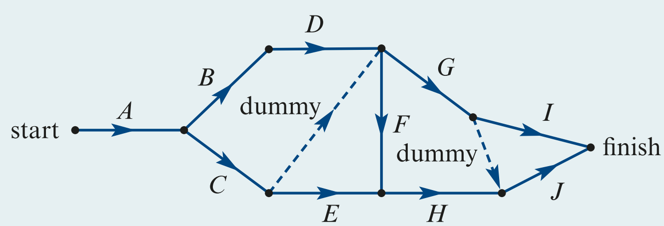
- A and B will lead from the start vertex.
- G will lead to the end vertex.
- A dummy will be required from the end of activity C (shared immediate predecessor) to the start of activity E (the activity with an additional immediate predecessor.)

Solution



Example 9 Creating a precedence table from an activity network

Write down a precedence table for the activity network shown on the right.



Explanation

1. Create a table with a row for each activity.
2. Look at the start of an activity. Write down all of the activities that lead directly to this activity in the immediate predecessor column.
3. Activity C is a predecessor of activity E, and the dummy activity makes it also a predecessor of F and G.
4. Activity G is a predecessor of activity I, and the dummy activity makes it also a predecessor of J.

Solution

Activity	*Immediate predecessors*
A	–
B	A
C	A
D	B
E	C
F	D, C
G	D, C
H	E, F
I	G
J	G, H

Skill-sheet

Exercise 14C

Constructing activity networks from precedence tables

Example 7

1 Draw an activity network for each of the precedence tables below.

a

Activity	Immediate predecessors
A	–
B	A
C	A
D	B
E	C

b

Activity	Immediate predecessors
P	–
Q	–
R	P
S	Q
T	R, S

c

Activity	Immediate predecessors
T	–
U	–
V	T
W	U
X	V, W
Y	X
Z	Y

d

Activity	Immediate predecessors
F	–
G	–
H	–
I	F
J	G, I
K	H, J
L	K

e

Activity	Immediate predecessors
K	–
L	–
M	K
N	M
O	N, L
P	O
Q	P
R	M
S	R, Q

f

Activity	Immediate predecessors
A	–
B	–
C	–
D	B
E	A, D
F	E, C
G	F
H	G
I	E, C
J	G
K	H, I

Constructing activity networks requiring dummy activities from precedence tables

Example 8

2 Draw an activity network for the following precedence tables. Dummy activities will need to be used.

a

Activity	*Immediate predecessors*
F	–
G	–
H	F
I	H, G
J	G

b

Activity	*Immediate predecessors*
A	–
B	A
C	A
D	B
E	B, C

c

Activity	*Immediate predecessors*
P	–
Q	–
R	P
S	Q
T	Q
U	R, S
V	R, S, T

d

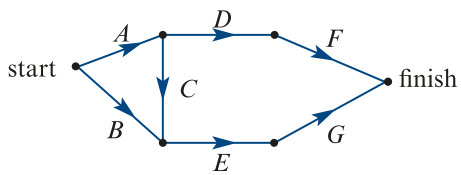
Activity	*Immediate predecessors*
A	–
B	A
C	A
D	B, C
E	C
F	E
G	D
H	F, G

Constructing precedence tables from activity networks

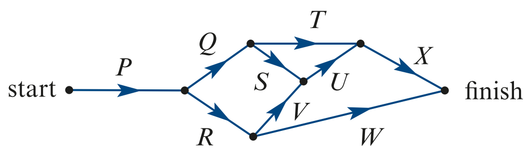
Example 9

3 Write down a precedence table for the activity networks shown below.

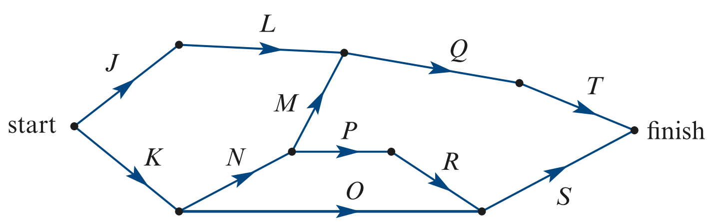
a



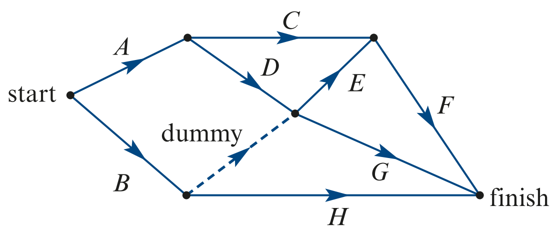
b



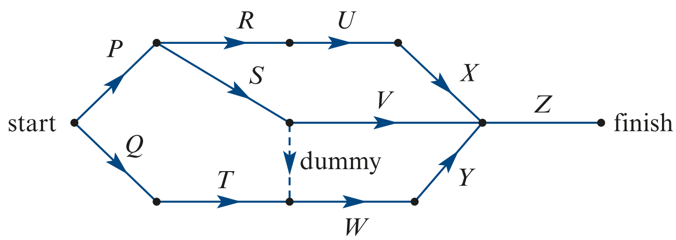
c



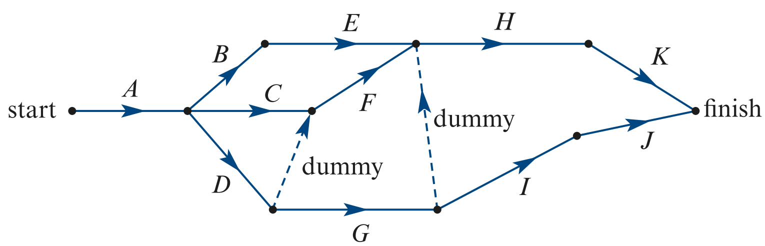
d



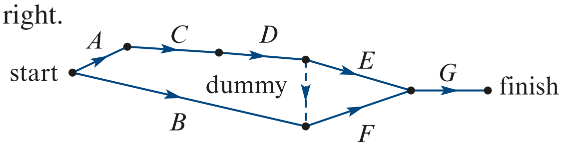
e



f



4 The following activity network shows the activities in a project to repair a dent in a car panel. The activities are listed in the table on the right.



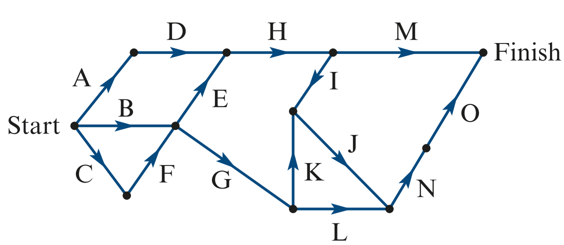
a Which activity or activities are the immediate predecessors of the event 'remove broken component'?

b Which activities are the immediate predecessors of the activity 'install new component'?

Activity	*Description*
A	Remove panel
B	Order component
C	Remove broken component
D	Pound out dent
E	Repaint
F	Install new component
G	Replace panel

Analysis of activity networks and precedence tables

5 Consider the following activity network for a project.



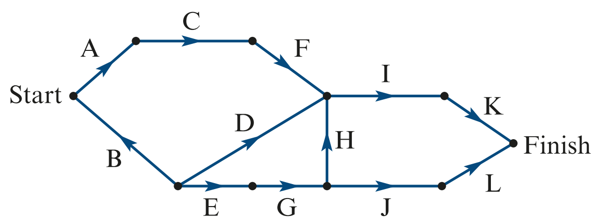
a Write down a precedence table for the network above.

b Write down the two paths from *start* to *finish* that begin with activity A.

c Write down the four paths from *start* to *finish* that begin with activity B.

d Write down the four paths from *start* to *finish* that begin with activity C.

6 Consider the following activity network.



a Which activities are immediate predecessors of activity I?

b Which activities must be completed before activity I can commence?

7 The information in the table opposite can be used to complete a directed network. This network will require a dummy activity.

a Which activity will the dummy activity be drawn from the end of?

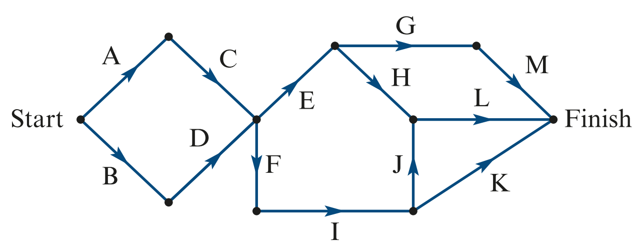
b Which activity will the dummy activity be drawn to the start of?

c Why is it necessary to include a dummy activity in this network?

Activity	*Immediate predecessors*
A	–
B	–
C	A
D	B
E	A
F	C, D
G	B
H	F
I	G
J	I
K	G, H, J
L	E

Exam 1 style questions

8 The activity network below shows the sequence of activities required to complete a project.



Beginning with activity B, the number of paths from start to finish is

A 1 **B** 2 **C** 3 **D** 4 **E** 5

Use the following information to answer Questions 9 and 10

A project involves eight activities, A to H.

The immediate predecessor(s) of each activity is shown in the table opposite. A directed network for this project will require a dummy activity.

Activity	*Immediate predecessors*
A	–
B	–
C	–
D	A
E	C
F	B, E
G	B
H	G

9 The dummy activity will be drawn from the end of

A activity B to the start of activity F.

B activity B to the start of activity E.

C activity G to the start of activity F.

D activity G to the start of activity E.

E activity E to the start of activity F.

10 The number of paths from start to finish is

A 4 **B** 5 **C** 6 **D** 7 **E** 8

14D Scheduling problems

Learning intentions

- To be able to understand activity networks that include weights (durations) of each activity.
- To be able to determine the EST for activities using forward scanning.
- To be able to determine the LST for activities using backward scanning.
- To be able to calculate and understand the existence of float times for some activities.
- To be able to identify the critical path and completion time of a project.

Scheduling

Projects that involve multiple activities are usually completed against a time schedule. Knowing how long individual activities within a project are likely to take allows managers of such projects to hire staff, book equipment and also to estimate overall costs of the project. Allocating time to the completion of activities in a project is called *scheduling*. Scheduling problems involve analysis to determine the minimum overall time it would take to complete a project.

Weighted precedence tables

The estimated time to complete activities within a project can be recorded in a precedence table, alongside the immediate predecessor information.

A precedence table that contains the estimated duration, in days, of each activity is shown on the right.

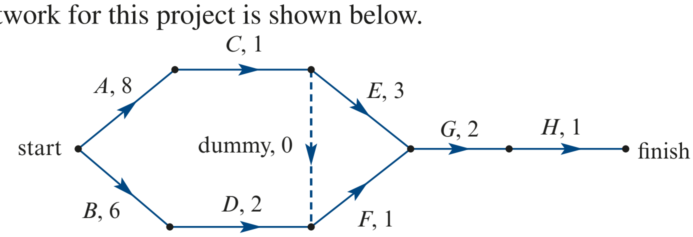
These estimated times are the weights for the edges of the activity network and need to be recorded alongside the name of the activity on the graph.

A dummy activity is required from the end of activity C (C is repeated in immediate predecessors) to the start of activity F (F is the activity that has an extra immediate predecessor).

Activity	*Estimated completion time (days)*	*Immediate predecessors*
A	8	–
B	6	–
C	1	A
D	2	B
E	3	C
F	1	C, D
G	2	E, F
H	1	G

The weight (duration) of dummy activities is always zero.

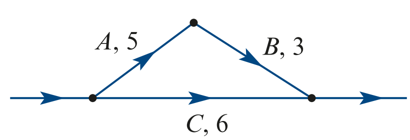
The activity network for this project is shown below.



Float times

The diagram below shows a small section of a different activity network. There are three activities shown, with their individual durations, in hours.

Activity B and activity C are both immediate predecessors to the next activity, so the project cannot continue until both of these tasks are finished. Activity B cannot begin until activity A is finished.



Activity C can be completed at the same time as activity A and activity B.

Activity A and B will take a total of $5 + 3 = 8$ hours, while activity C only requires 6 hours. There is some flexibility around when activity C needs to start. There are $8 - 6 = 2$ hours spare for the completion of activity C. This value is called the **float time** for activity C.

The flexibility around the starting time for activity C can be demonstrated with the following diagram.

	A	A	A	A	A	B	B	B
Start at same time	C	C	C	C	C	C	Slack	Slack
Delay C by 1 hour	Slack	C	C	C	C	C	C	Slack
Delay C by 2 hour	Slack	Slack	C	C	C	C	C	C

The five red squares represent the 5 hours it takes to complete activity A. The three green squares represent the 3 hours it takes to complete activity B.

The six yellow squares represent the 6 hours it takes to complete activity C. Activity C does not have to start at the same time as activity A because it has some slack time available (2 hours).

Activity C should not be delayed by more than 2 hours because this would cause delays to the project. The next activity requires B and C to be complete before it can begin.

Calculating and recording earliest starting times (EST)

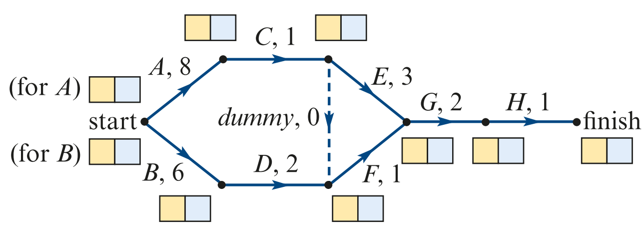
In order for a project to be completed in the shortest time possible, it is important that certain key activities start at the earliest possible time. The **earliest starting time**, or **EST**, for each activity is the earliest time after the start of the entire project that the individual activity can start. An EST of 8 means an activity can start 8 hours (or whatever time period is given) after the start of the project.

The EST for each activity is found by a process called **forward scanning**.

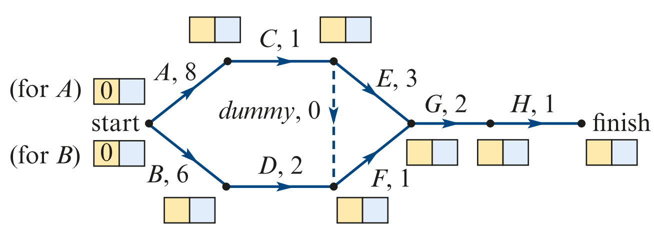
Forward scanning

Forward scanning will be demonstrated using the activity network below. The durations of each are in days.

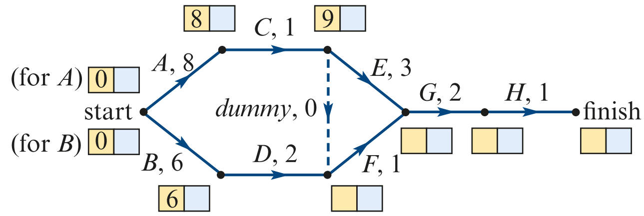
1 Draw a box, split into two cells, next to each vertex of the activity network, as shown in the diagram opposite. If more than one activity begins at a vertex, draw a box for each of these activities.



2 Activities that begin at the start of the project have an EST of zero (0). Write this in the left box, shown shaded yellow in the diagram.

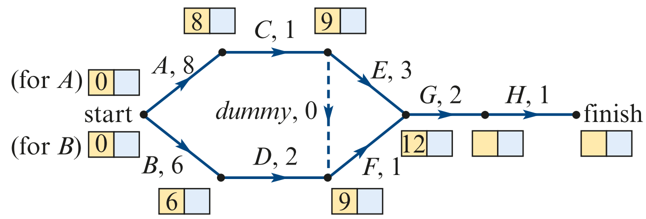


3 Calculate the EST of each activity of the project by adding the EST of the immediate predecessor to the duration of the immediate predecessor.



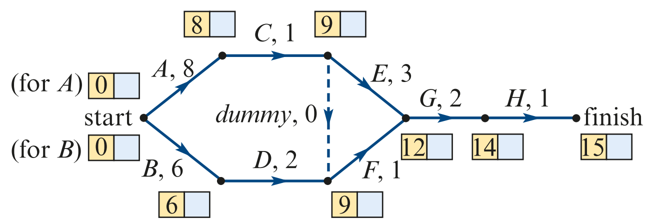
Notes: EST of C = EST of A + duration of A (EST of $C = 0 + 8 = 8$)
EST of D = EST of B + duration of B (EST of $D = 0 + 6 = 6$)
EST of E = EST of C + duration of C (EST of $C = 8 + 1 = 9$)

4 If an activity has more than one predecessor, calculate the EST using each of the predecessors and choose the *largest* value.



Notes: EST of $F = 6 + 2 = 8$ or EST of $F = 9 + 0 = 9$. Use 9.
EST of $G = 9 + 1 = 10$ or EST of $G = 9 + 3 = 12$. Use 12.

5 The EST value at the finish of the project is the minimum time it takes to complete the project.



Notes: The minimum time to complete this project is 15 days.

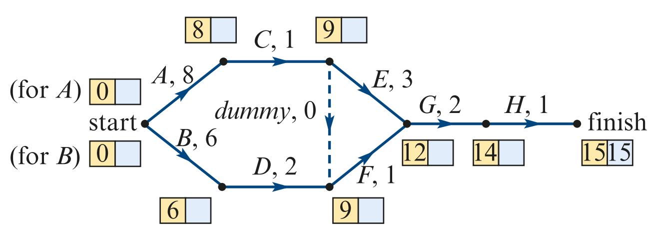
Calculating and recording latest starting times (LST)

Some activities, as we saw earlier, have some flexibility around the time that they can start. The **latest start time**, or LST, for each activity is the latest time after the start of the entire project that the individual activity can start. LSTs for each activity are calculated using the reverse of the process used to calculate the ESTs. This process is called **backward scanning**.

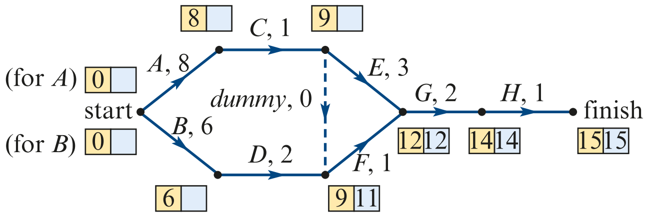
Backward scanning

Backward scanning will be demonstrated using the activity network with completed forward scanning from above.

1 Copy the minimum time to complete the project into the right cell shown shaded blue in the diagram.

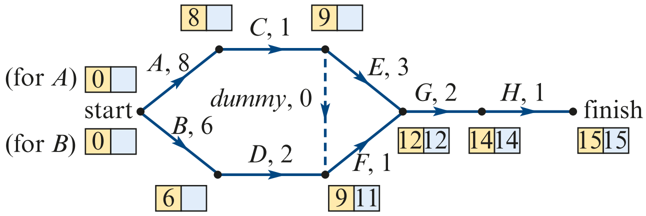


2 Calculate the LST for each activity by subtracting the duration of the activity from the LST of the following activity.



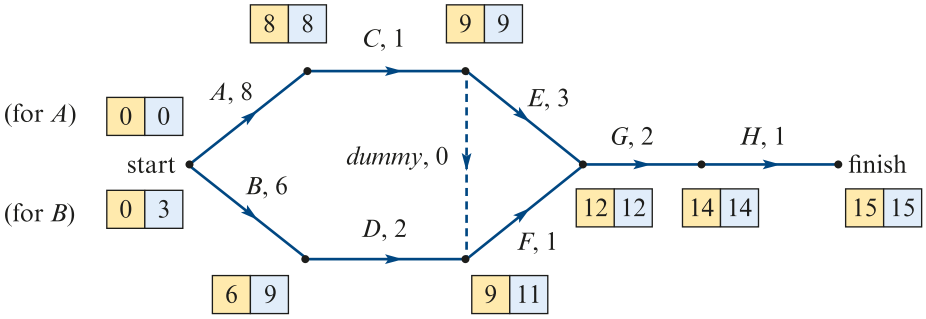
Notes: **1** LST of H = LST of finish – duration of H (LST of $H = 15 - 1 = 14$).
2 LST of G = LST of H – duration of G (LST of $G = 14 - 2 = 12$).
3 LST of F = LST of G – duration of F (LST of $F = 12 - 1 = 11$).
4 LST of E = LST of G – duration of E (LST of $E = 12 - 3 = 9$).

3 If more than one activity have the same predecessor, calculate the LST using each of the activities that follow and choose the *smallest* value.



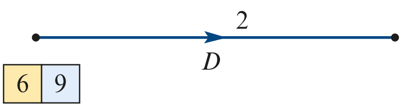
Note: LST of $E = 12 - 3 = 9$ (from duration of E) or LST of $E = 11 - 0 = 11$ (from duration of *dummy*). Use 9.

The completed activity network with all EST and LST is shown below.



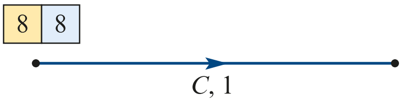
Identifying float times and the critical path

The boxes at the vertices in the activity network above give the EST and LST for the activity that begins at that vertex.



The EST for activity D is 6 and the LST for activity D is 9. This means activity D has a float time of $9 - 6 = 3$ hours.

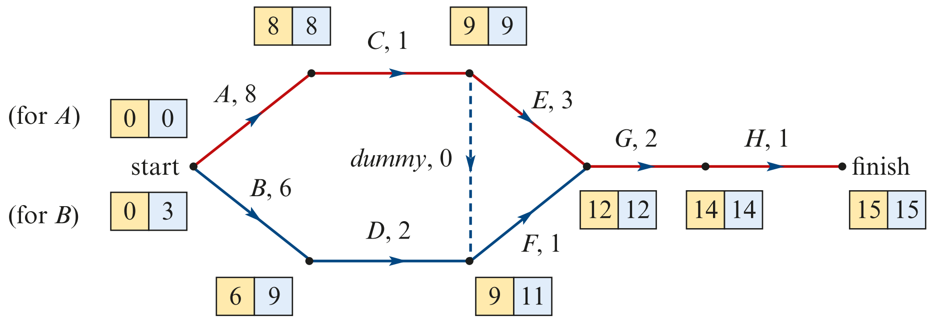
Activity D can be delayed by 3 hours without delaying the rest of the project.



The EST for activity C is 8 and the LST for activity C is 8. This means activity C has a float time of $8 - 8 = 0$ hours.

Activity C has no flexibility around its starting time at all. Any delay to the start of this activity will delay the whole project and extend the minimum time for completion.

Activities that have no float time are critical ones for completion of the project. Tracking through the activity network along the edges of critical activities gives the **critical path** for the project. The critical path for this project is highlighted in red on the diagram below.



Critical path

- A critical path is the longest or equal longest path in an activity network.
- There can be more than one critical path in an activity network.
- The critical path is the sequence of activities that cannot be delayed without affecting the overall completion time of the project.

The process for determining the critical path is called **critical path analysis**.

Critical path analysis

- Draw a box with two cells next to each vertex of the activity network.
- Calculate the EST for each activity by forward scanning:

  EST = EST of predecessor + duration of predecessor
- If an activity has more than one predecessor, the EST is the *largest* of the alternatives.
- The minimum overall completion time of the project is the EST value at the end vertex.
- Calculate the LST for each activity by backward scanning:

  LST = LST of following activity − duration of activity
- If an activity has more than one following activity, the LST is the *smallest* of the alternatives.
- Float = LST − EST
- If float time = 0, the activity is on the critical path.

Example 10 Finding the critical path from a precedence table

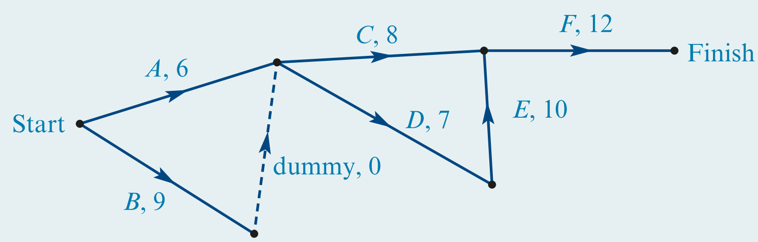
A project has six activities as shown in the precedence table opposite and the associated activity network is shown below.

a Complete the critical path analysis to calculate the EST and LST for each activity.

b Write down the critical path of this project.

c What is the minimum time required to complete the project?

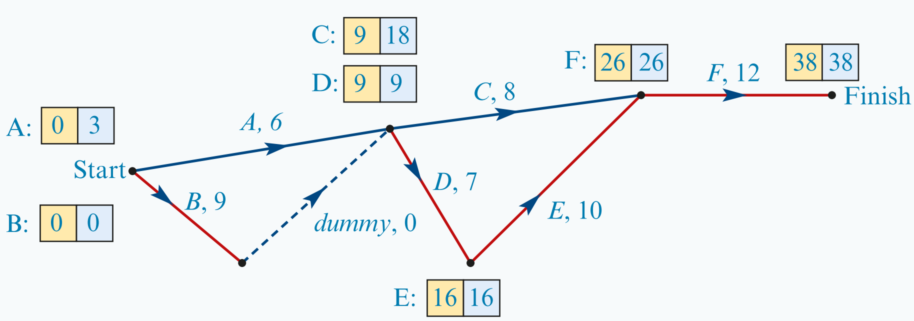
Activity	*Duration (days)*	*Immediate predecessors*
A	6	–
B	9	–
C	8	A, B
D	7	A, B
E	10	D
F	12	C, E



Explanation

Solution

a



b The critical path is highlighted in red.

Note: The dummy is not included in the critical path.

The critical path of this project is $B \rightarrow D \rightarrow E \rightarrow F$.

c The minimum completion time is in EST of the end box.

The minimum completion time of this project is 38 days.

Example 11 Finding the critical path

A project has eight activities as shown in the precedence table opposite.

Activity	*Duration (weeks)*	*Immediate predecessors*
A	3	–
B	5	–
C	12	–
D	4	A
E	7	A
F	5	B, D
G	6	E, F
H	2	C

a Draw an activity network for this project.

b Complete the critical path analysis to calculate the EST and LST for each activity.

c What is the earliest starting time for activity H?

d What is the latest starting time for activity H?

e What is the float time of activity H?

f Write down the critical path of this project.

g What is the minimum time required to complete the project?

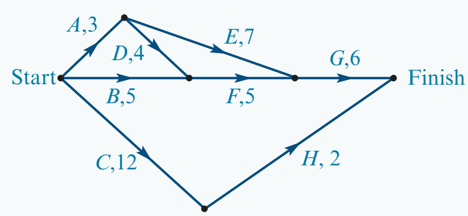
h The person responsible for completing activity E falls sick three weeks into the project. If he will be away from work for two weeks, will this cause the entire project to be delayed?

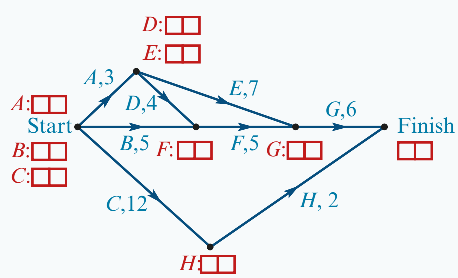
Explanation

a A, B and C have no predecessors and so can begin at the same time. Continue drawing the network as outlined by the table above, including arrowheads, activity labels, duration labels and correct immediate predecessors.

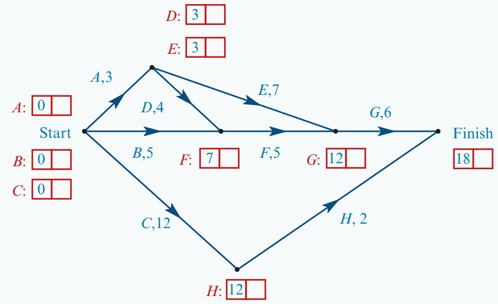
b Begin by drawing boxes, split into two cells, at the beginning of each activity. Label them with the name of each activity. You must also include a box, split into two cells, at the final vertex where the project finishes.

Solution

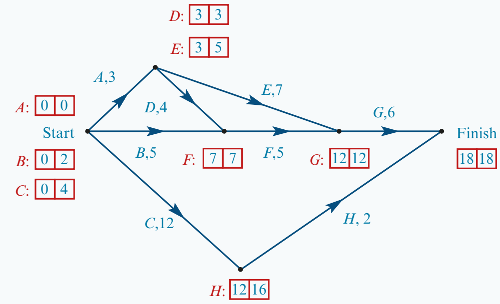




Use forward scanning to identify the EST for each activity. Activities with no immediate predecessors always have an EST of zero. Add the left cell value at the start of the activity to the duration and write the result in the left cell at the end of the activity. Use the largest of the possibilities if there is more than one activity ending at the same vertex. Identify the minimum project completion time as the left cell value at the finish vertex



Use backward scanning to identify the LST for each activity. Subtract the duration from the right cell value at the end of the activity and write the result in the right cell at the start of the activity. Use the smallest of the possibilities if there is more than one activity beginning at the same vertex.



c EST values are in the left cell at the start of each activity.

The EST for activity H is 12 weeks

d LST values are in the right cell at the end of each activity.

The EST for activity H is 16 weeks

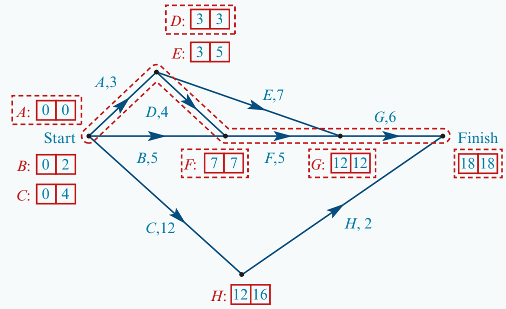
e Float = $LST - EST$.

Float $H = LST - EST$

$= 16 - 12$

$= 4$ weeks

f The critical path joins all of the activities that have the same EST and LST, and therefore which have zero float time.



The critical path for this project is

$A - D - F - G$

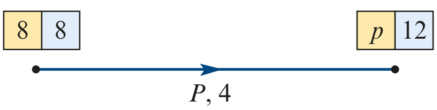
g The minimum time required to complete the project is the EST (also, always equal to the LST) at the finish vertex.	18 weeks
h If the float time is more or equal to the delay in the start of activity E, the project will not be affected.	The person will be away for two weeks, starting 3 weeks into the project. This is equal to the float time for activity E, and so delaying the start of activity E until the person comes back to work will not affect the overall completion time of the project.

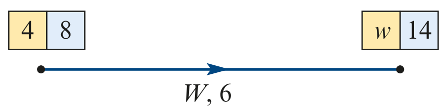
Skill-sheet

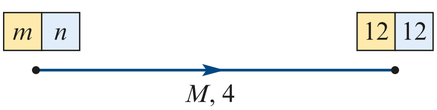
Exercise 14D

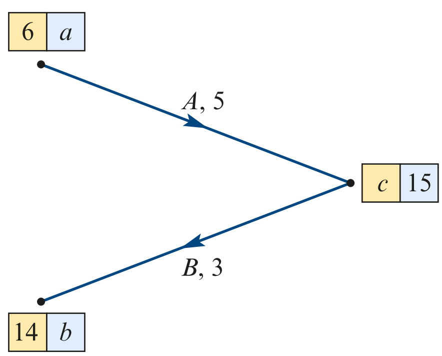
Calculations from elements of an activity network

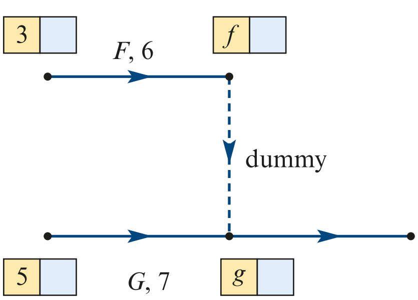
1 Write down the value of each pronumeral in the sections of activity networks below.

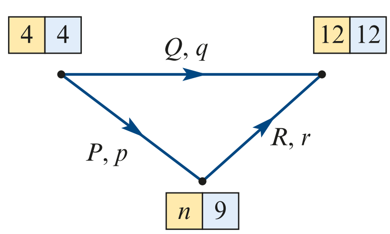
a


b


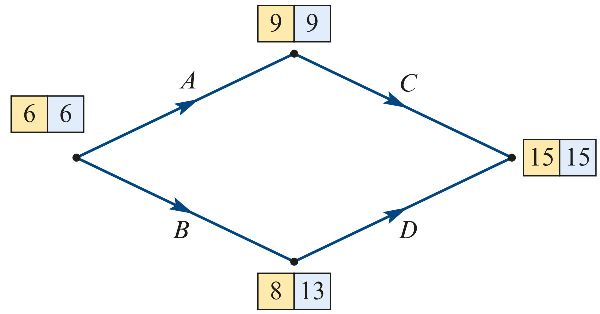
c


d


e


f


2 Consider the section of an activity network shown in the diagram opposite.



a What is the duration of activity A?

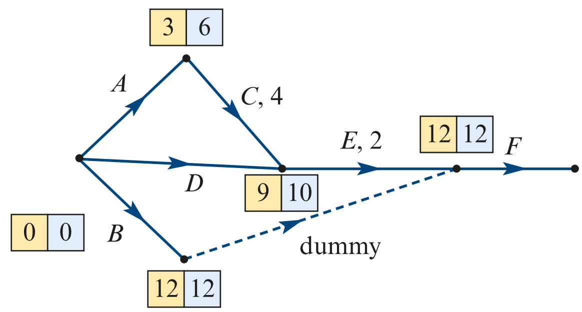
b What is the critical path through this section of the activity network?

c What is the float time of activity B?

d What is the latest time that activity D can start?

e What is the duration of activity D?

3 Consider the section of an activity network shown in the diagram below.



a What is the duration of activity B?

b What is the latest start time for activity E?

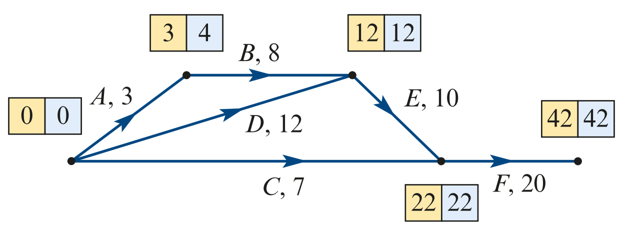
c What is the earliest time that activity E can start?

d What is the float time for activity E?

e What is the duration of activity A?

f What is the duration of activity D?

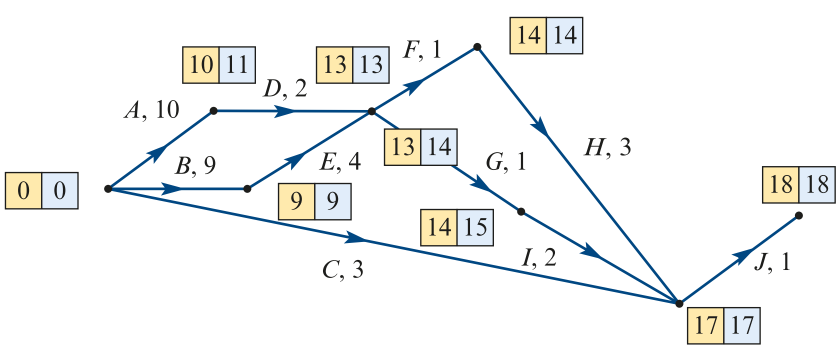
4 An activity network is shown in the diagram opposite.



a Write down the critical path for this project.

b Calculate the float times for non-critical activities.

5 An activity network is shown in the diagram below.

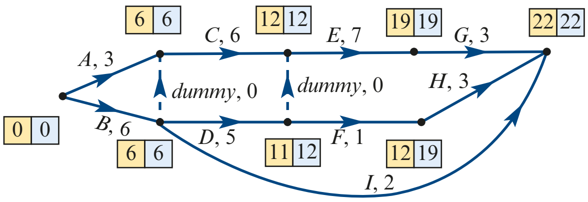


a Write down the critical path.

b Write down the float times for all non-critical activities.

6 A precedence table and activity network for a project are shown below. The precedence table is incomplete.

Activity	*Duration (weeks)*	*Immediate predecessors*
A	3	–
B	6	–
C	6	
D		*B*
E	7	
F	1	*D*
G		*E*
H	3	
I	2	*B*



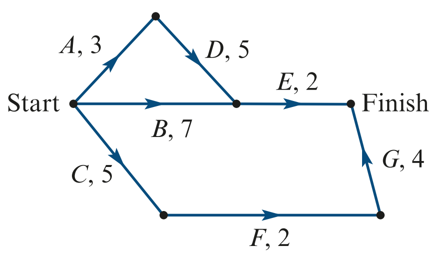
a Complete the table above.

b Write down the critical path for this project.

Critical path analysis from a given activity network

Example 10

7 Consider the following activity network for a project. The duration of each activity is given in the network, in days.



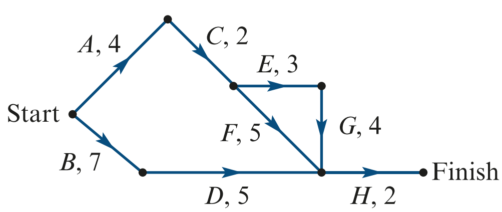
a Determine the earliest start time for activity E.

b Find the minimum completion time for this project.

c Write down the critical path for this project.

d Which activity has a float time of two days?

8 Consider the following activity network for a project.



a Write down the three activities that are immediate predecessors of activity H.

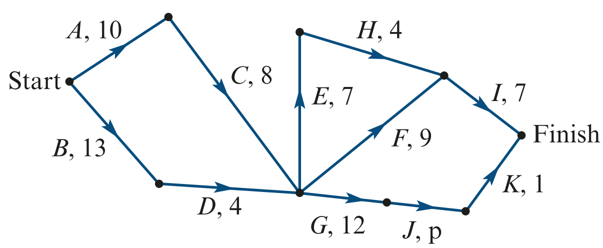
b Determine the earliest start time of activity H.

c For activity H the earliest start time and the latest start time are the same. What does this tell us about activity H?

d Determine the minimum completion time, in hours, for this project.

e Which activity could be delayed for the longest time without affecting the minimum completion time of the project?

9 Consider the following activity network for a project.



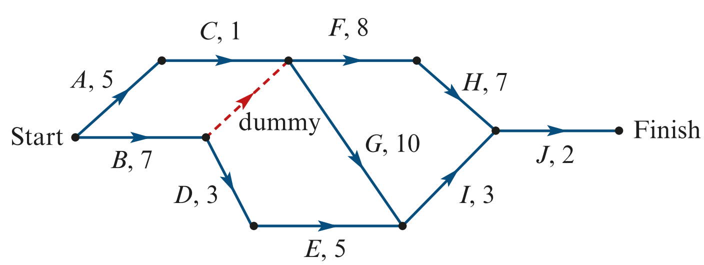
a Determine the earliest start time for the following activities:

i H **ii** I **iii** J

b Determine the value of p, in weeks, that would create more than one critical path.

c If the value of p is 3 weeks, what will be the float time, in weeks, of activity G.

10 Consider the following activity network for a project.



a Write down the two immediate predecessors of activity G.

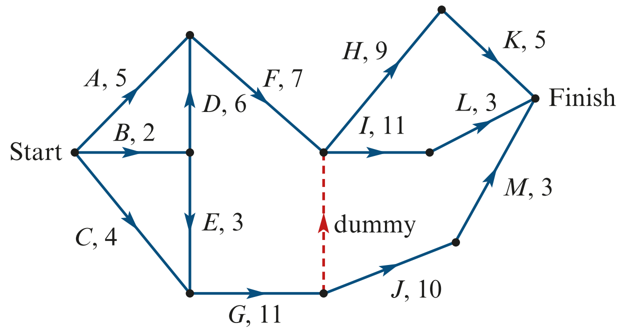
b Which of the ten activities must be completed before activity I can begin?

c Write down the critical path of for this project.

d Determine the float time of activity E.

e Which three activities could have their completion times increased by two days without altering the minimum completion time?

11 Consider the following activity network for a project.



a Complete a precedence table for this network, using two columns, one column for the activities and a second column for the Immediate predecessors.

b How many activities have an earliest start time of 16 hours?

c Find the latest start time of activity F.

d There are two critical paths. Write down both critical paths.

e How many activities can be delayed by 1 hour without increasing the minimum completion time of the project?

Critical path analysis from precedence table only

Example 11

12 The precedence table for a project is shown opposite.

a Draw an activity network for this project.

b Complete the critical path analysis to calculate the EST and LST for each activity.

c Write down the critical path of this project.

d What is the minimum time required to complete the project?

Activity	*Duration (weeks)*	*Immediate predecessors*
P	4	–
Q	5	–
R	12	–
S	3	P
T	6	Q
U	3	S
V	4	R
W	8	R, T, U
X	13	V
Y	6	W, X

13 The precedence table for a project is shown opposite.

a Draw an activity network for this project.

b Complete the critical path analysis to calculate the EST and LST for each activity.

c Write down the critical path of this project

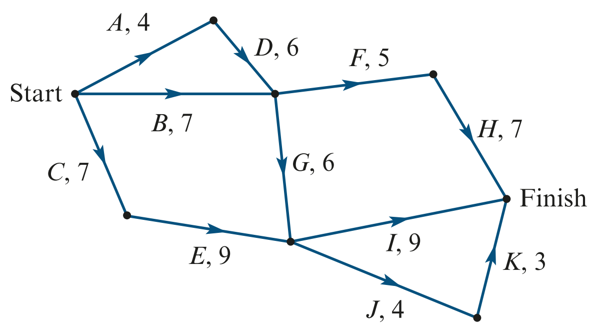
d What is the minimum time required to complete the project?

Activity	*Duration (weeks)*	*Immediate predecessors*
I	2	–
J	3	–
K	5	–
L	4	I
M	8	J, N
N	1	K
O	6	L, M
P	6	J, N
Q	7	J, N
R	5	K
S	1	O
T	9	Q, R

Exam 1 style questions

Use the following information to answer questions 14, 15 and 16

The directed network opposite shows the sequence of eleven activities that are needed to complete a project. The time, in days, that it takes to complete each activity is also shown.



14 The earliest starting time, in days, for activity J is

A 12 **B** 13 **C** 14 **D** 15 **E** 16

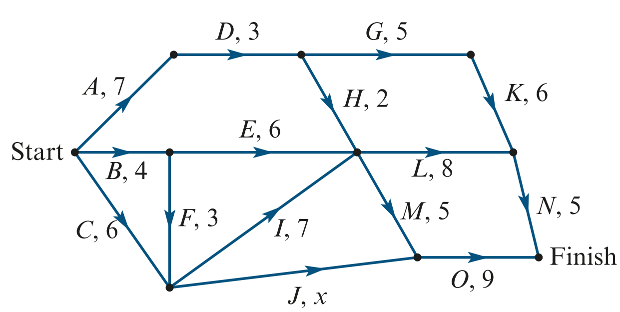
15 The number of activities that have exactly two immediate predecessors is

A 1 **B** 2 **C** 3 **D** 4 **E** 5

16 How many of these activities could be delayed without affecting the minimum completion time of the project?

A 3 **B** 4 **C** 5 **D** 6 **E** 7

17 The directed graph opposite shows the sequence of activities required to complete a project. The time taken to complete each activity, in weeks, is also shown.



The minimum completion time for this project is 28 weeks. The time taken to complete activity J is labelled x. The maximum value of x is

A 12 **B** 10 **C** 8 **D** 4 **E** 2

18 A project consists of ten activities, A to J. The table below shows the immediate predecessor(s) and earliest start time, in days, of each activity.

Activity	*Immediate predecessors*	*Earliest starting time*
A	–	0
B	–	0
C	–	0
D	A	6
E	B	5
F	B	5
G	C	4
H	D, E	13
I	F, G	14
J	H, I	25

It is known that activity H has a completion time of ten days. The project can still be completed in minimum time if activity D is delayed. The maximum length of the delay for activity D is

A one day **B** two days **C** seven days **D** eight days **E** nine days

14E Crashing

Learning intentions

- To be able to use crashing to reduce the completion time of a project.
- To be able to minimise the cost of crashing activities to achieve the maximum reduction in completion time of a project.

Altering completion times

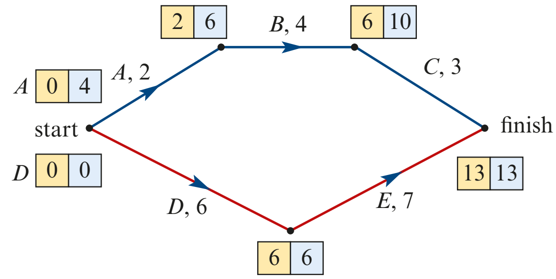
The minimum time it takes to complete a project depends upon the time it takes to complete the individual activities of the project, and upon the predecessors each of the activities have. Critical path analysis can be completed to find the overall minimum completion time.

Sometimes, the managers of a project might arrange for one or more activities within the project to be completed in a shorter time than originally planned. Changing the conditions of an activity within a project, and recalculating the minimum completion time for the project, is called **crashing**.

An individual activity could be crashed by employing more staff, sourcing alternate materials or simply because weather or other factors allow the activity to be completed in a shorter time than usual.

A simple crashing example

A simple activity network is shown in the diagram on the right. The forwards and backwards scanning processes have been completed and the critical path has been determined. The critical path is shown in red on the diagram.



The minimum time for completion is currently 13 hours. In order to reduce this overall time, the manager of the project should try to complete one, or more, of the activities in a shorter time than normal. Reducing the time taken to complete activity A, B or C would not achieve this goal however. These activities are not on the critical path and so they already have slack time. Reducing their completion time will not shorten the overall time taken to complete the project.

Activity D and E, on the other hand, lie on the critical path. Reducing the duration of these activities will reduce the overall time for the project. If activity D was reduced in time to 4 hours instead, the project will be completed in 11, not 13, hours.

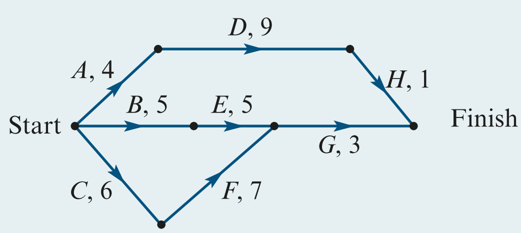
Crashing with cost

Shortening the completion time for any individual activity could result in an extra cost for the project. In the simple example above, the cost of reducing the completion time of activity D by 1 hour is \$150, while the cost of reducing the completion time of activity E by 1 hour is \$18.

Clearly it is best to reduce the completion time, or crash, the activity that will cost the least.

Example 12 Crashing one activity with cost

The directed network below shows the sequence of 8 activities that are needed to complete a project. The time, in days, that it takes to complete each activity is also shown.



a Write down the critical path for this project.

b What is the minimum completion time for the project?

Activity F can be reduced by a maximum of 3 days at a cost of $100 per day.

c What is the new minimum completion time for the project?

d What is the minimum cost that will achieve the greatest reduction in time taken to complete the project?

Explanation

a In crashing problems, we first need to identify the critical path, or paths. We will do this by remembering that a critical path is the longest or equal longest path in the activity network. Using this method, set up a table, list all possible paths from *Start* to *Finish* for the directed network and calculate the length of each path. The critical path is the path with the longest time from *Start* to *Finish*.

Solution

Path	*Duration(days)*
$A-D-H$	14
$B-E-G$	13
$C-F-G$	16

The critical path is $C-F-G$

Explanation

b Write the duration of the critical path identified in the previous part.

c Crash all possible activities by the maximum reduction. Add a new column to the summary table to get an overview of the new duration of each path. This may result in a new critical path. Consider the cost of crashing and whether it is worth applying the maximum reduction.

Solution

16 days

Path	*Duration* (days)	*New duration with maximum reduction* (F by 3)
$A-D-H$	14	14
$B-E-G$	13	13
$C-F-G$	16	13

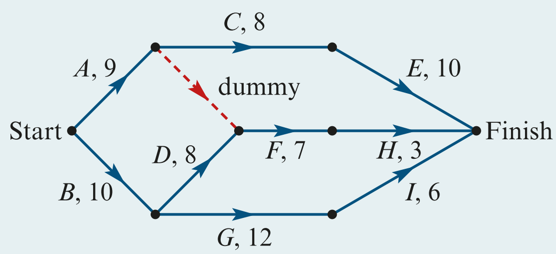
The new minimum completion time for the project is 14 days.

Activity F originally took 7 days to complete. It can be crashed, which means activity F may be reduced by a maximum of 3 days, to result in a completion time of 4 days. It is possible to choose to reduce acitvity F by 0, 1, 2 or 3 days. Reducing activity F by the maximum 3 days would result in the original critical path to be reduced from a total of 16 days, down to 13 days. Considering there is a cost of $100 per day, this is **not** a desirable outcome; crashing activity F by 3 days results in a new critical path $A-D-H$ with a total completion time of 14 days. If we crash activity F by 2 days only, we create 2 equal critical paths requiring 14 days to complete the project.

d Reducing activity F by 2 days allows us to reduce the overall completion time of the project at minimum cost.	\$200.

Example 13 Crashing multiple activities with cost

The directed network below shows the sequence of 9 activities that are needed to complete a project. The time, in days, that it takes to complete each activity is also shown.



The minimum completion time for the project is 28 days. It is possible to reduce the completion time for activities B, E, G, H and I. The completion time for each of these five activities can be reduced by a maximum of two days.

a What is the new minimum completion time, in days, that the project could take?

The reduction in completion time for each of these five activities will incur an additional cost. The table opposite shows the five activities that can have their completion times reduced and the associated daily cost, in dollars.

Activity	Daily cost($)
B	1500
E	2000
G	700
H	900
I	800

b What is the minimum cost that will achieve the greatest reduction in time taken to complete the project?

Explanation

a List all possible paths from *Start* to *Finish*, including the completion time of each. Crash all activities by their maximum reduction. Identify the new critical path (path with the longest completion time) after the maximum reductions are applied.

Solution

Path	Duration (days)	New duration after maximum reduction (B,E,G,H,I by 2)
$A-C-E$	27	25
$A-F-H$	19	17
$B-D-F-H$	28	24
$B-G-I$	28	22

$A-C-E$ is the new critical path with a duration of 25 days.

Explanation

b **1** Begin with the new critical path $A - C - E$. The reduction of activity E must occur to achieve the new minimal completion time, therefore reducing activity E by 2 days is essential.

2 Ignore the path $A - F - H$ because its completion time of 19 days is already lower than the critical path.

3 Consider the path $B - D - F - H$. It has a completion time of 28 days and must be reduced to 25 days to equal the critical path of $A - C - E$. There are two options; reduce B by 2 and H by 1 or reduce B by 1 and H by 2. From the table, it is more expensive per day to reduce B than H, however by choosing to reduce B this will also reduce the completion time of the final path $B - G - I$, which is more cost effective; reducing activity B reduces the completion time of *two* different paths. So reduce activity B by 2 and H by 1 day to reduce the overall completion time of $B - D - F - H$ down to 25 days.

4 The final path $B - G - I$ has already been reduced by 2 days due to the reduction of activity B previously chosen. One more activity must be reduced for this path to equal the critical path. Activity G has a lower cost of reduction than activity I per day, so include this in your calculation.

5 Calculate your total cost of crashing.

Solution

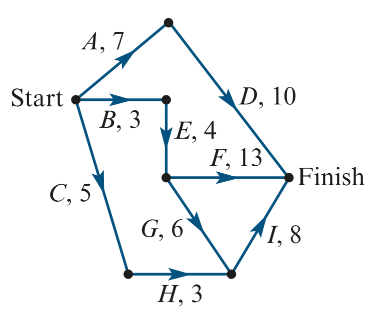
b Cost of crashing $= E$ by 2 days $+ B$ by 2 days $+ H$ by 1 day $+ G$ by 1 day

$$= 2000 \times 2 + 1500 \times 2 + 900 + 700$$

$$= \$8600$$

Exercise 14E

1 The activity network for a project is shown in the diagram below. The duration for each activity is in hours.



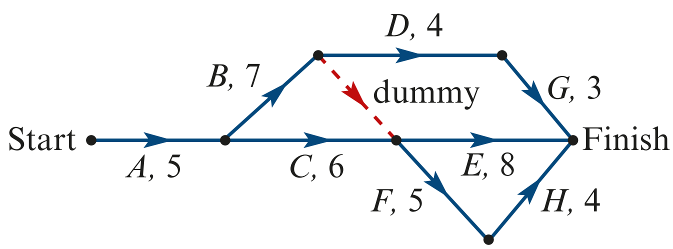
a List all four paths from the Start to the Finish of the project, with their respective completion times.

b Identify the critical path and the minimum completion time for the project.

c If Activity E is reduced by 3 hours, identify the new minimum completion time for this project.

Example 11

2 The directed network below shows the sequence of 8 activities that are needed to complete a project. The time, in days, that is takes to complete each activity is also shown.



a Write down the critical path for this project.

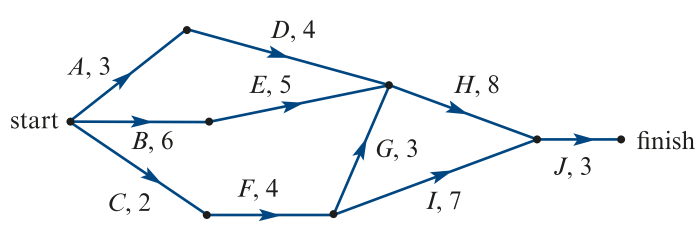
b What is the minimum completion time for the project?

Activity B can be reduced by a maximum of 3 days at a cost of $100 per day.

c What is the new minimum completion time for the project?

d What is the minimum cost that will achieve the greatest reduction in time taken to complete the project?

3 The activity network for a project is shown in the diagram on the right. The duration for each activity is in hours.



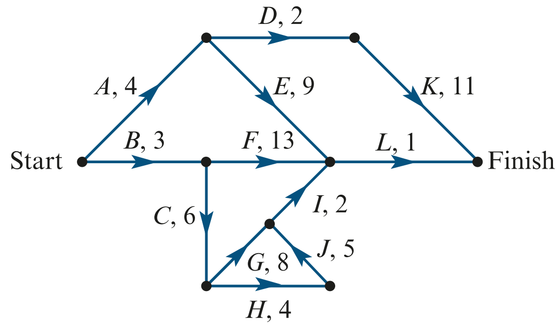
a Identify the critical path for this project.

b What is the maximum number of hours that the completion time for activity *E* can be reduced by without changing the minimum completion time of the project?

c What is the maximum number of hours that the completion time for activity *H* can be reduced without changing the minimum completion time of the project?

d Every activity can be reduced in duration by a maximum of 2 hours. If every activity was reduced by the maximum amount possible, what is the new minimum completion time for the project?

4 The activity network for a project is shown in the diagram below. The duration for each activity is in hours.



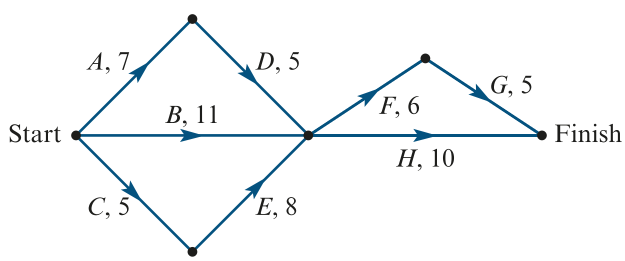
a How many activities could be delayed by 4 hours without altering the minimum completion time for the project?

b If the project is to be crashed by reducing the completion time of one activity only, what is the minimum time, in hours, that the project can be completed in?

c Activity G can be reduced in time at a cost of \$200 per hour. Activity J can be reduced in time at a cost of \$150 per hour. What is the cost of reducing the completion time of this project as much as possible?

Example 13

5 The directed network below shows the sequence of 8 activities that are needed to complete a project. The time, in days, that it takes to complete each activity is also shown.



The minimum completion time for the project is 24 days. It is possible to reduce the completion time for activities D, E and H. The completion time for each of these three activities can be reduced by a maximum of two days.

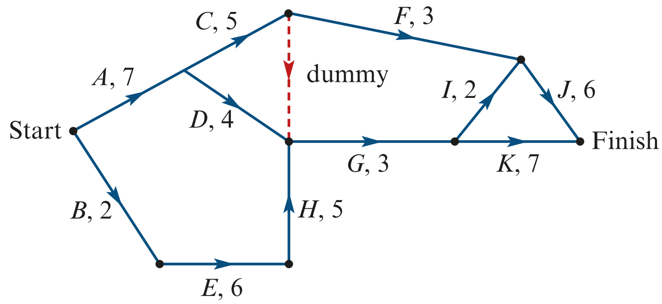
a What is the new minimum completion time, in days, of the project?

The reduction in completion time for each of these three activities will incur an additional cost. The table opposite shows the three activities that can have their completion times reduced and the associated daily cost, in dollars.

Activity	Daily cost(\$)
D	170
E	350
H	200

b What is the minimum cost that will achieve the greatest reduction in time taken to complete the project?

6 The directed network below shows the sequence of 11 activities that are needed to complete a project. The time, in days, that it takes to complete each activity is also shown.



a Which activities are immediate predecessors to activity G?

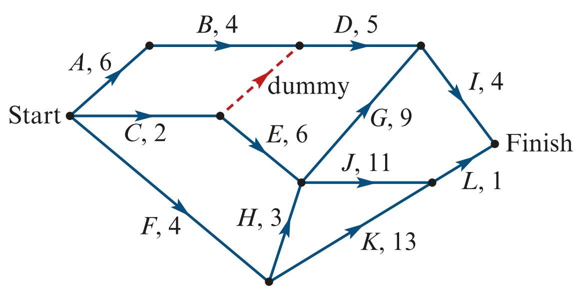
b Which activities, if crashed, would create more than one critical path?

c The project could finish earlier if some activities were crashed. Five activities, B, E, G, H and I, can all be reduced by one hour. The cost of this crashing is \$150 per hour.

i What is the minimum number of hours in which the project could now be completed?

ii What is the minimum cost of completing the project in this time?

7 The directed network below shows the sequence of 12 activities that are needed to complete a project. The time, in weeks, that it takes to complete each activity is also shown.



a Determine the shortest time in which this project can be completed.

b Determine the earliest start time for activity D.

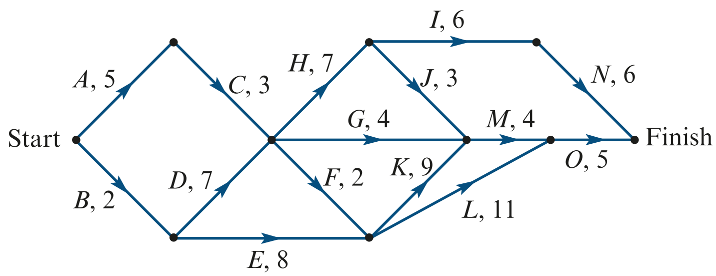
c Determine the latest start time for activity H.

d Which activity has a float time of more than two weeks?

e The completion times for activities D, E, G, H and J can each be reduced by a maximum of two weeks. The table opposite shows the five activities than can have their completion time reduced and the associated weekly cost, in dollars.
What is the minimum cost to complete the project in the shortest time possible?

Activity	Weekly cost($)
D	2000
E	1000
G	500
H	1500
J	3000

8 The directed network below shows the sequence of 15 activities that are needed to complete a maintenance project at the MCG. The time, in days, that it takes to complete each activity is also shown.



a What is the minimum completion time?

b How many activities are on the critical path?

c How many paths have a completion time of 28 days?

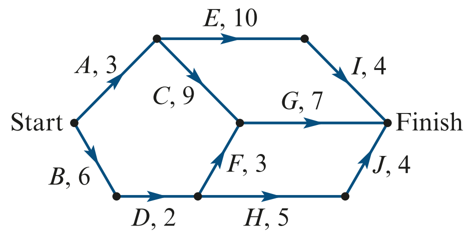
d The completion times for activities H, J, K, L and M can each be reduced by a maximum of two days. The cost of reducing the time of each activity is \$500 per day. The MCG requires the overall completion time for the maintenance project to be reduced by three days at minimum cost. Complete the table below, showing the reductions in individual activity completion times that would achieve this.

Activity	*Reduction in completion time* (0, 1 or 2 days)
H	
J	
K	
L	
M	

Exam 1 style questions

Questions 9 and 10 refer to the diagram opposite.

The directed graph opposite shows the sequence of activities required to complete a project. All times are in hours.



9 There is one critical path for this project.
The critical path is

A $A - E - I$

B $A - C - G$

C $B - D - F - G$

D $B - D - H - J$

E $A - C - F - H - J$

10 Four critical paths would exist if the duration of activities

A A and B were reduced by one hour.

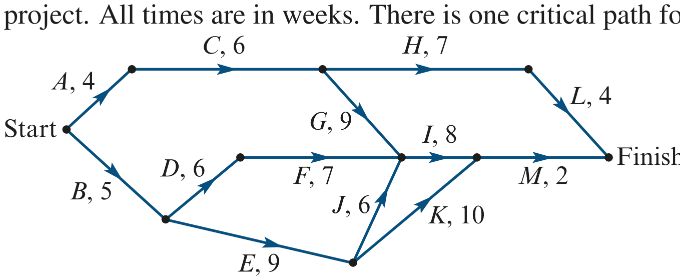
B C and G were reduced by one hour.

C A and C were reduced by two hours.

D B and G were reduced by two hours.

E D and H were reduced by three hours.

11 The directed graph below shows the sequence of activities required to complete a project. All times are in weeks. There is one critical path for this project.



The total completion time of the project can be reduced by four weeks by reducing

A activity B by four weeks

B activity F by four weeks.

C activity J by four weeks.

D activity I by three weeks and activity J by one week.

E activity D by three weeks and activity E by one week.

Review

Key ideas and chapter summary

Assign-
ment

Weighted graph A **weighted graph** is a graph in which a number representing the size of same quantity is associated with each edge. These numbers are called weights.

Network A **network** is a weighted graph in which the weights are physical quantities, for example distance, time or cost.

Directed graph (digraph) A **directed graph** is a graph where direction is indicated for every edge. This is often abbreviated to **digraph**.

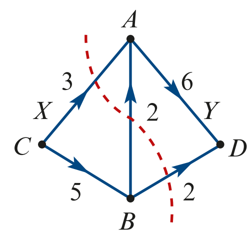
Flow The transfer of material through a directed network. **Flow** can refer to the movement of water or traffic.

Capacity The maximum flow of substance that an edge of a directed graph can allow during a particular time interval. The **capacity** of water pipes is the amount of water (usually in litres) that the pipe will allow through per time period (minutes, hours, etc.). Other examples of capacity are number of cars per minute or number of people per hour.

Source The **source** is the origin of the material flowing through a network.

Sink The **sink** is the final destination of the material flowing through a network.

Cut A **cut** is a line dividing a directed graph into two parts (shown as a broken line dividing the graph below into two sections, labelled X and Y).



Cut capacity The sum of the capacities (weights) of the edges directed from X to Y that the cut passes through. For the weighted digraph shown, the capacity of the cut is 7.

Minimum cut The **minimum cut** is the cut with the minimum capacity. The cut must separate the source from the sink.

Maximum flow The **maximum flow** through a directed graph is equal to the capacity of the minimum cut.

Bipartite graph A **bipartite graph** has two distinct groups or categories for the vertices. Connections exist between a vertex or vertices from one group with a vertex or vertices from the other group. There are no connections between the vertices within a group.

Allocation An **allocation** is made when each of the vertices in one group from a bipartite graph are matched with one of the vertices in the other group from that graph. An allocation is possible when both groups have exactly the same number of vertices. The vertices in each group are matched to only one vertex from the other group.

Cost matrix A table that contains the costs of allocating objects from one group (such as people) to another (such as tasks). The 'cost' can be money, or other factors such as the time taken.

Hungarian algorithm The **Hungarian algorithm** is an algorithm that is used to determine the best allocation to minimise the overall cost.

Activity network An **activity network** is a directed graph that shows the required order of completing individual activities that make up a project.

Immediate predecessor If activity A is an **immediate predecessor** to activity B, activity A must be completed before activity B can begin.

Precedence table A **precedence table** is a table that records the activities of a project and their immediate predecessors. Precedence tables can also contain the duration of each activity.

Dummy activity A **dummy activity** has zero cost. It is required if two activities share some, but not all, of the same immediate predecessors. It allows the network to show all precedence relationships in a project correctly.

Earliest starting time (EST) **EST** is the earliest time an activity in a project can begin.

Latest starting time (LST) **LST** is the latest time an activity in a project can begin, without affecting the overall completion time for the project.

Float (slack) time **Float (slack) time** is the difference between the latest starting time and the earliest starting time.

Float = LST – EST

The float time is sometimes called the slack time. It is the largest amount of time that an activity can be delayed without affecting the overall completion time for the project.

Forward scanning **Forward scanning** is a process of determining the EST for each activity in an activity network. The EST of an activity is added to the duration of that activity to determine the EST of the next activity. The EST of any activity is equal to the largest forward scanning value determined from all immediate predecessors.

Backward scanning — **Backward scanning** is a process of determining the LST for each activity in an activity network. The LST of an activity is equal to the LST of the activity that follows, minus the duration of the activity.

Critical path — The **critical path** is the series of activities that cannot be delayed without affecting the overall completion time of the project. Activities on the critical path have no slack time. Their EST and LST are equal.

Critical path analysis — **Critical path analysis** is a project planning method in which activity durations are known with certainty.

Crashing — **Crashing** is the process of shortening the length of time in which a project can be completed by reducing the time required to complete individual activities. Reducing the individual activity completion times often costs money; this increases the overall cost of a project.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

14A **1** I can define and describe a directed graph. ☐

See Example 1, and Exercise 14A Question 1

14A **2** I can determine the maximum flow for any section of sequential edges of a directed graph. ☐

See Example 4, and Exercise 14A Question 4

14A **3** I can determine cut capacities. ☐

See Example 2, and Exercise 14A Question 2

14A **4** I can determine the maximum flow as equal to the minimum cut capacity. ☐

See Example 3, and Exercise 14A Question 5

14B **5** I can draw directed and weighted bipartite graphs. ☐

See Exercise 14B Question 4

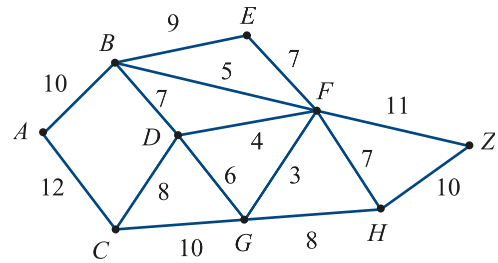
14B **6** I can use the Hungarian algorithm to determine an optimum allocation in order to minimise cost. ☐

See Exercise 14B Question 2

14C	7	I can create an activity network from a precedence table.	☐
		See Example 5, and Exercise 14C Question 5	
14C	8	I can write down a precedence table from an activity network.	☐
		See Exercise 14C Question 1	
14C	9	I can decide when to use dummy activities in an activity network.	☐
		See Example 6, and Exercise 14C Question 3	
14D	10	I can use forward scanning to determine the earliest starting time of activities in an activity network.	☐
		See Example 8, and Exercise 14D Question 12	
14D	11	I can use backward scanning to determine the latest starting time of activities in an activity network.	☐
		See Example 8, and Exercise 14D Question 12	
14D	12	I can determine the float time for activities in an activity network.	☐
		See Example 8, and Exercise 14D Question 7	
14D	13	I can determine the overall minimum completion time for a project using critical path analysis.	☐
		See Example 8, and Exercise 14D Question 8	
14D	14	I can determine the critical path for an activity network.	☐
		See Example 8, and Exercise 14D Question 9	
14E	15	I can use crashing to reduce the completion time of a project.	☐
		See Example 9, and Exercise 14E Question 1	

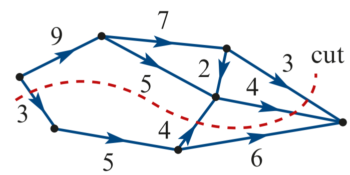
Multiple-choice questions

1 The shortest path from A to Z in the network on the right has length:



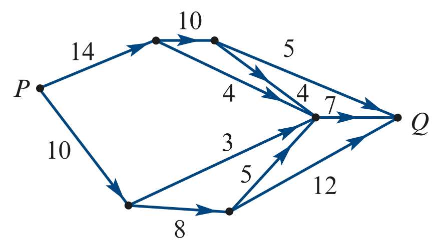
A 10 **B** 15
C 22 **D** 26
E 28

2 For the network shown on the right, the capacity of the cut is:



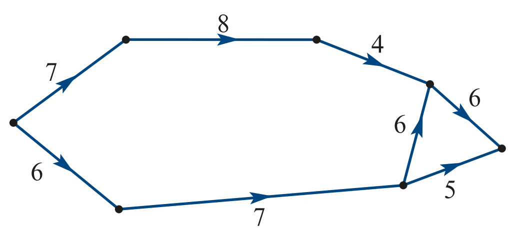
A 3 **B** 6 **C** 9
D 10 **E** 14

3 In the communications network shown, the numbers represent transmission capacities for information (data) in scaled units. What is the maximum flow of information from station P to station Q?



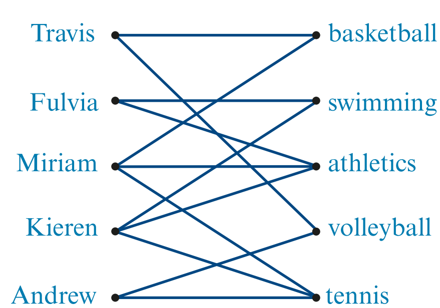
A 20 **B** 22
C 23 **D** 24
E 30

4 The maximum flow in the network opposite, from source to sink, is:



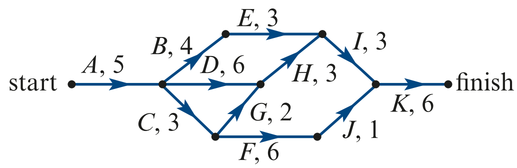
A 10 **B** 11
C 12 **D** 13
E 14

5 A group of five students represent their school in five different sports. The information is displayed in a bipartite graph. From this graph we can conclude that:



A Travis and Miriam played all the sports between them.
B In total, Miriam and Fulvia played fewer sports than Andrew and Travis.
C Kieren and Miriam each played the same number of sports.
D In total, Kieren and Travis played fewer different sports than Miriam and Fulvia.
E Andrew played fewer sports than any of the others.

6 This activity network is for a project where the component times in days are shown. The critical path for the network of this project is given by:



A $A–B–E–I–K$

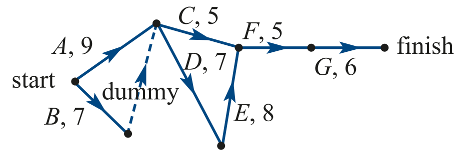
B $A–D–H–I–K$

C $A–C–G–H–I–K$

D $A–C–F–J–K$

E $A–D–G–F–J–K$

7 The activity network shown represents a project development with activities and their durations (in days) listed on the edges of the graph. Note that the dummy activity takes zero time.



The earliest time (in days) that activity *F* can begin is:

A 0 **B** 12 **C** 14 **D** 22 **E** 24

8 The table opposite lists the seven activities in a project and the earliest start time, in hours, and the predecessor(s) of each task. The time taken for activity *F* is five hours. Without affecting the time taken for the entire project, the time taken for activity *D* could be increased by:

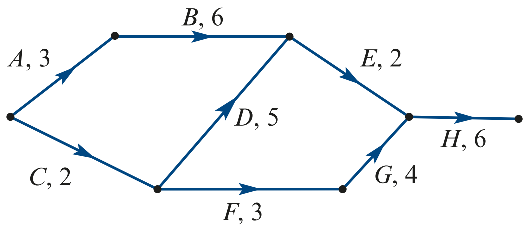
A 0 hours **B** 2 hours

C 3 hours **D** 4 hours

E 12 hours

Task	*Immediate predecessor*	*EST*
A	–	0
B	–	0
C	*A*	24
D	*B*	29
E	*C*	39
F	*D*	41
G	*E*, *F*	50

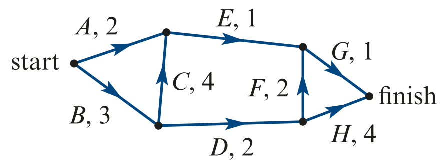
9 The edges in this activity network correspond to the tasks involved in the preparation of an examination. The numbers indicate the time, in weeks, needed for each task. The total number of weeks needed for the preparation of the examination is:



A 14 **B** 15 **C** 16 **D** 17 **E** 27

Review

10 The activity network represents a manufacturing process with activities and their duration (in hours) listed on the edges of the graph. The earliest time (in hours) after the start that activity G can begin is:



A 3 **B** 5 **C** 6 **D** 7 **E** 8

Written-response questions

1 An English class recently performed poorly in their essay writing assessment. To help them improve, the teacher separated the class into groups of five and assigned one of the following tasks to each person: Introduction, Body paragraph 1, Body paragraph 2, Body paragraph 3 and Conclusion. Each task will be completed by one person. The table below shows the time, in minutes, that each person would take to complete each of the five tasks.

	Intro	Par 1	Par 2	Par 3	Con
Alvin	16	14	19	9	9
Billy	17	18	10	9	9
Chloe	9	8	6	15	8
Danielle	11	12	11	16	6
Elena	10	10	8	15	8

The tasks will be allocated so that the total time of completing the five tasks is a minimum

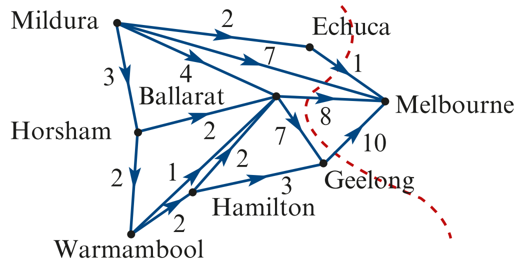
a Complete the sentences below by clearly stating which task each student should write in order for the essay to be completed in the minimum time possible.

- Alvin should write the...
- Billy should write the...
- Chloe should write the...
- Danielle should write the...
- Elena should write the...

b What is the minimum total time the group will dedicate to completing the essay?

Review

2 WestAir Company flies routes in western Victoria. The network shows the layout of connecting flight paths for WestAir, which originate in Mildura and terminate in either Melbourne or on the way to Melbourne. On this network, the available spaces for passengers flying out of various locations on one morning are shown.



The network has one cut shown.

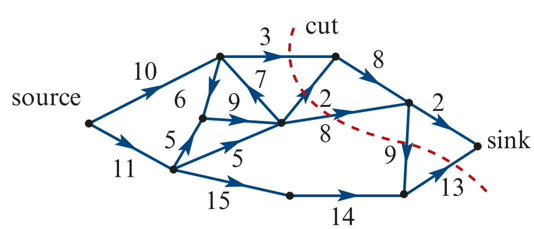
a What is the capacity of this cut?

b What is the maximum number of passengers who could travel from Mildura to Melbourne for the morning?

3 A school swimming team wants to select a 4×200 metre relay team. The fastest times of its four best swimmers in each of the strokes are shown in the table below. Which swimmer should swim which stroke to give the team the best chance of winning, and what would be their time to swim the relay?

Swimmer	*Backstroke*	*Breaststroke*	*Butterfly*	*Freestyle*
Rob	76	78	70	62
Joel	74	80	66	62
Henk	72	76	68	58
Sav	78	80	66	60

4 In the network opposite, the values on the edges give the maximum flow possible between each pair of vertices. The arrows show the direction of flow in the network. Also shown is a cut that separates the source from the sink.



a Determine the capacity of the cut shown.

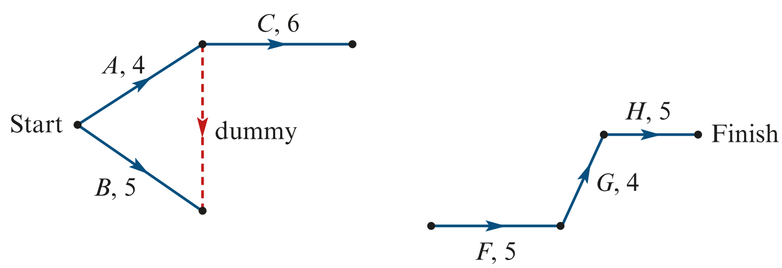
b Determine the maximum flow through this network.

Review

5 A project requires eight activities (A – H) to be completed. The duration, in hours, and the immediate predecessor(s) of each activity are shown in the table below.

Activity	*Duration (hours)*	*Immediate predecessor(s)*
A	4	–
B	5	–
C	6	A
D	7	$A,\ B$
E	10	C, D
F	5	D
G	4	F
H	5	$E,\ G$

a The directed network that shows these activities is shown below. Add the three missing features to the network.



b Determine the earliest start time for activity E.

c What is the float time of activity G?

d How many of these activities have a non-zero float time?

e Write down the critical path for this project.

f What is the minimum completion time for this project?

g The project could finish earlier if some activities were crashed.

i Activity E can be crashed by two hours. If this occurs, what will be the new critical path for the project?

ii In addition to activity E crashing by two hours, activities, A, B and D can also be crashed by one hour each. What is the minimum number of days in which the project can now be completed?

h After careful deliberation, it is decided that crashing the original directed network is not possible. Alternatively, an upgrade will be made to this project, where one extra activity will be added. This activity has a duration of five hours, an earliest starting time of twelve hours and a latest starting time of seventeen hours. Complete the following sentence by filling in the boxes provided.

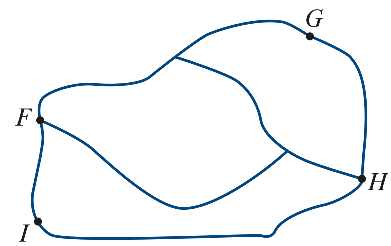
The extra activity could be represented on the network above by a directed edge from the end of activity ☐ to the start of activity ☐

Chapter 15

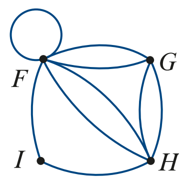
Revision: Networks and decision mathematics

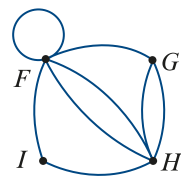
15A Exam 1 style questions

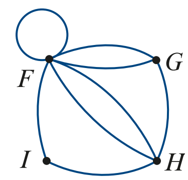
1 The diagram opposite shows a map of the roads between four towns: F, G, H and I.

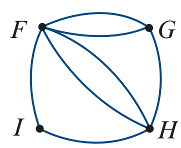


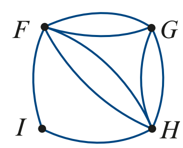
A graph that represents the connections between the towns on the map is:

A


B


C


D


E


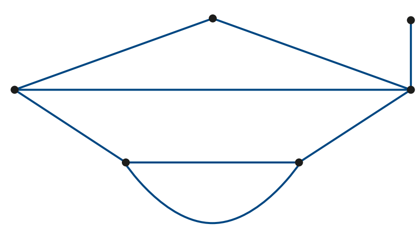
2 Five people are to be each allocated one of five tasks (A, B, C, D, E). The table shows the time, in hours, that each person takes to complete the tasks. The tasks must be completed in the least possible total amount of time. If no person can help another, Francis should be allocated task:

Name	*A*	*B*	*C*	*D*	*E*
Francis	12	15	99	10	14
David	10	9	10	7	12
Herman	99	10	11	6	12
Indira	8	8	12	9	99
Natalie	8	99	9	8	11

A A **B** B **C** C **D** D **E** E

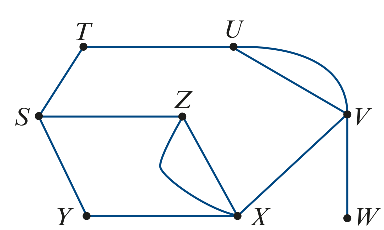
3 The sum of the degrees of all the vertices in the graph opposite is:

A 6 **B** 7 **C** 8
D 15 **E** 16



4 Adding which one of the following edges to the graph opposite makes an Eulerian trail possible?

A ST **B** SU **C** SX
D XW **E** ZY



5 A connected planar graph has an equal number of vertices and faces. If there are 20 edges in this graph, the number of vertices must be:

A 9 **B** 10 **C** 11 **D** 20 **E** 22

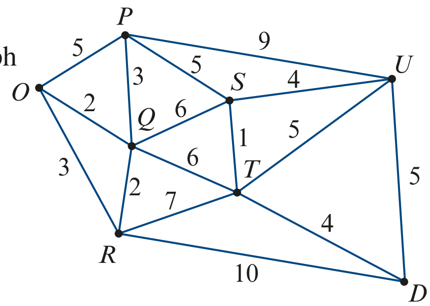
6 Underground water pipes are needed to water a new golf course. Water will be pumped from the dam in the back corner of the course. To find the smallest total length of water pipe needed, we must find:

A a critical path
B a minimal spanning tree
C the shortest Eulerian circuit
D the shortest Hamiltonian cycle
E the perimeter of the golf course

7 Which one of the following is a true statement about a critical path in a project?

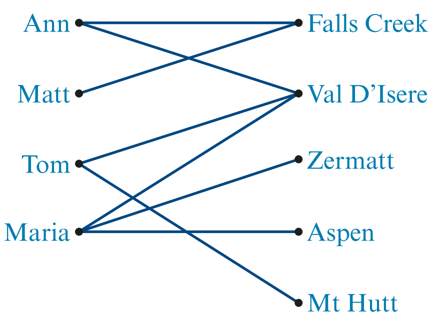
A Knowledge of the critical path can be used to decide if any tasks in a project can be delayed without extending the length of time of the project.
B All tasks on the critical path must be completed before any other task in the same project can be started.
C Decreasing the times of tasks not on the critical path will decrease the length of time of the project.
D The critical path must always include at least two tasks in a project.
E There is only one critical path in any project.

8 The length of the shortest path between the origin, O, and destination, D, in the weighted graph shown here is:



A 11 **B** 12 **C** 13
D 14 **E** 15

9 Four students, talking about five ski resorts they have visited, represented their information on the bipartite graph shown here. Which one of the following statements is implied by this bipartite graph?



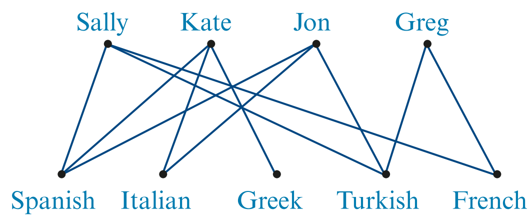
A Ann and Maria between them have visited fewer ski resorts than Matt and Tom between them.
B Matt and Tom have been to four ski resorts between them.
C Maria has visited fewer ski resorts than any of the others.
D Ann and Maria between them have visited all five ski resorts discussed.
E Ann and Tom between them have visited fewer resorts than Matt and Maria between them.

10 A gas pipeline is to be constructed to link several towns in the country. Assuming the pipeline construction costs are the same everywhere in the region, the cheapest network formed by the pipelines and the towns as vertices would form:

A a Hamiltonian cycle **B** an Eulerian circuit **C** a minimum spanning tree

D a critical path **E** a complete graph

11 Which one of the following statements is not implied by this bipartite graph?



A There are more translators of French than Greek.

B Sally and Kate can translate five languages between them.

C Jon and Greg can translate four languages between them.

D Kate and Jon can translate more languages between them than can Sally and Greg.

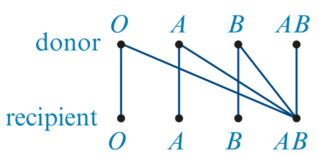
E Sally and Jon can translate more languages between them than can Kate and Greg.

12 There are four different human blood types: O, A, B and AB. The relationships between donor and recipients for these blood types are as follows:

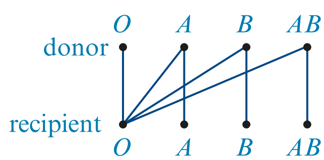
- Type O can donate blood to any type.
- Type AB can receive blood from any type.
- Each type can donate blood to its own type.
- Each type can receive blood from its own type.

Which one of the following donor–recipient bipartite graphs correctly represents this information?

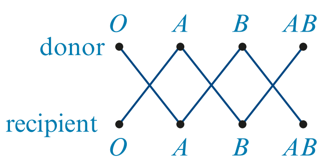
A



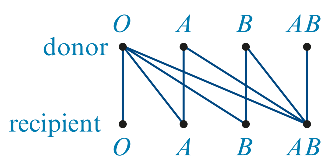
B



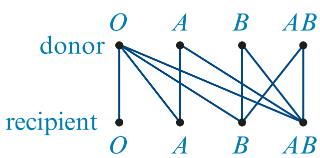
C



D

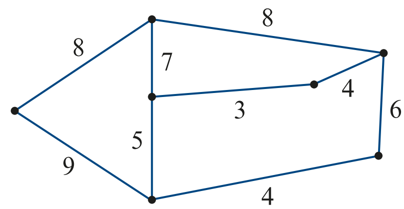


E



13 For the weighted graph shown, the length (total weight) of the minimum spanning tree is:

A 28 **B** 29 **C** 30
D 31 **E** 32



14 A connected graph with 12 edges divides a plane into four faces. The number of vertices in this graph will be:

A 6 **B** 10 **C** 12 **D** 13 **E** 14

15 The number of edges for a complete graph with twenty vertices is:

A 10 **B** 20 **C** 21 **D** 180 **E** 190

16 The number of vertices for a complete graph with twenty-one edges is:

A 7 **B** 8 **C** 14 **D** 42 **E** 43

17 The number of edges for a tree with four vertices must be:

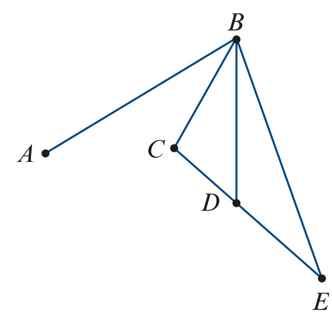
A 1 **B** 2 **C** 3 **D** 4 **E** 5

18 The number of vertices for a tree with 13 edges must be:

A 6 **B** 7 **C** 14 **D** 15 **E** 26

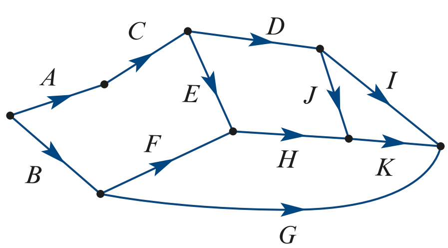
19 What additional edge could be added to the graph opposite to ensure that the resulting graph would contain an Eulerian circuit?

A AB **B** AC **C** AD
D AE **E** BC

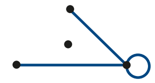


20 The graph opposite represents a project with activities listed on the edges of the graph. Which of the following statements must be true?

A A must be completed before B can start.
B A must be completed before F can start.
C E and F must start at the same time.
D E and F must finish at the same time.
E E cannot start until A is finished.



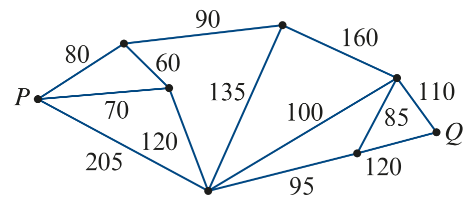
21 An adjacency matrix for the graph opposite could be:



A $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

D $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

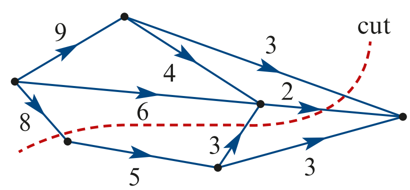
22 A vehicle is travelling from town P to town Q. The journey requires the vehicle to travel along a network linking suitable fuel stops. The cost, in dollars, of travel between these is shown on the network opposite, where the vertices represent fuel stops.



What is the minimum cost, in dollars, for the trip?

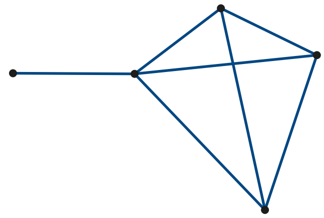
A 400 **B** 405 **C** 410 **D** 420 **E** 440

23 The capacity of the cut in the network flow diagram shown is:



A 0 **B** 2 **C** 10
D 13 **E** 16

24 The sum of the degrees of the vertices on the graph shown is:



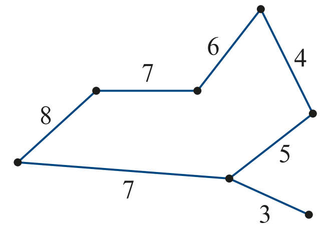
A 12 **B** 13 **C** 14
D 15 **E** 16

25 A connected planar graph divides the plane into a number of faces. If the graph has nine vertices and these are linked by 20 edges, then the number of faces is:

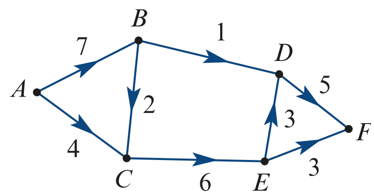
A 11 **B** 13 **C** 21 **D** 27 **E** 31

26 The sum of the weights of the minimum spanning tree of the weighted graph is:

A 2 **B** 30 **C** 32
D 33 **E** 35



The following graph relates to Questions 27 and 28.



27 The maximum flow in the network linking vertex A to vertex F is:

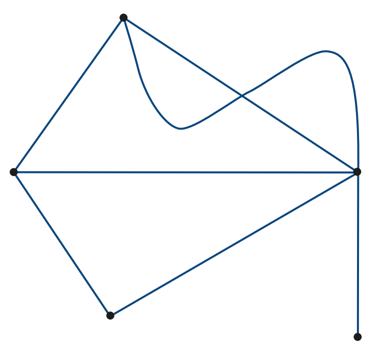
A 5 **B** 6 **C** 7 **D** 8 **E** 9

28 The number of ways that vertex F can be reached from vertex A is:

A 1 **B** 2 **C** 3 **D** 4 **E** 5

15B Exam 2 style questions

1 Consider the graph opposite.



a What is the sum of the degrees of all the vertices in the graph?

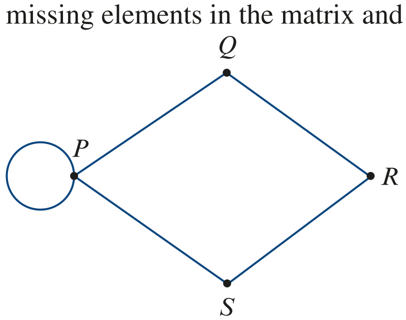
b What is the maximum number of edges that can be removed so that the graph remains connected?

c What is the maximum number of edges that can be removed for an Eulerian trail to exist?

d Circle the bridge in the graph.

e Verify Euler's formula for the graph opposite.

2 The adjacency matrix and graph below represent the same information. Some elements are missing from the adjacency matrix and some edges from the graph. Write down the missing elements in the matrix and add the missing edges to the graph.



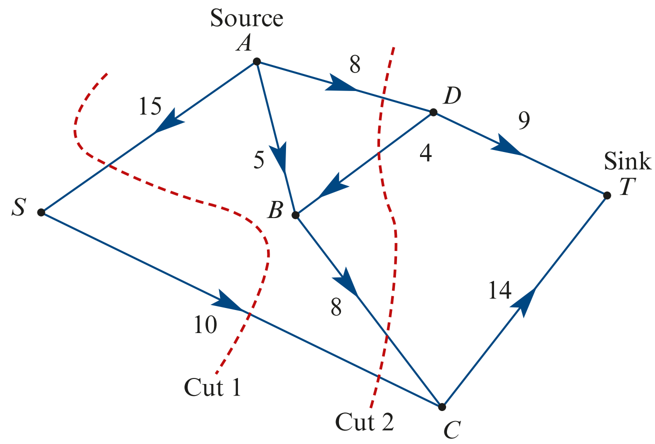
$$\begin{array}{c} \\ P \\ Q \\ R \\ S \end{array}\begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{bmatrix} \boxed{} & 1 & 0 & 1 \\ 1 & 0 & \boxed{} & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

3 Consider the diagram opposite, which represents the number of cars per hour over a suburban road system.

a Give a reason why Cut 1 is not a valid cut.

b What is the capacity of Cut 2?

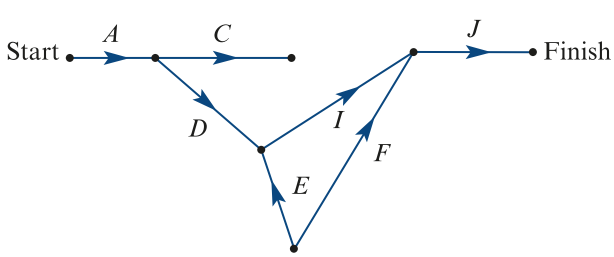
c What is the maximum flow for this network?



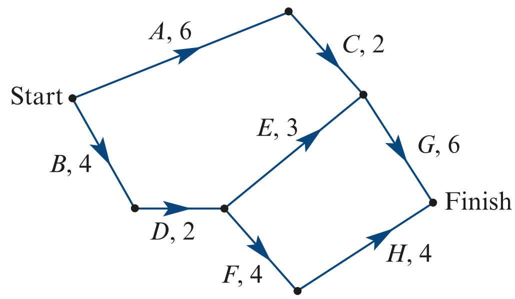
4 A project requires nine activities ($A - J$) to be completed. The duration, in hours, and the immediate predecessor(s) of each activity are shown in the table below.

Activity	*Immediate predecessors*
A	–
B	–
C	A
D	A
E	B
F	B
G	C
H	C
I	D, E, G
J	H, I, F

The directed network that shows these activities is shown below. Three features are missing. Using the table above, complete the network diagram below.



5 Consider the activity network below.



a Determine the earliest start time of activity G.

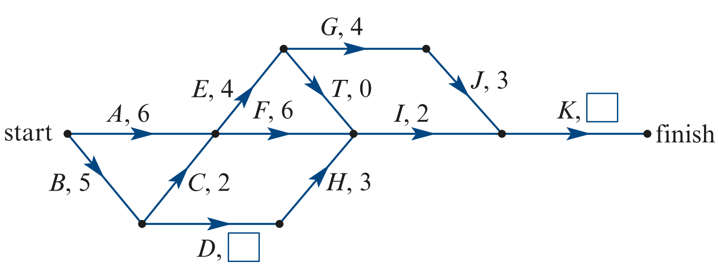
b Determine the latest start time of activity F.

c Determine the float time for activity C.

d Write down the critical path.

e Determine the minimum completion time for this project.

6 All the activities and their durations (in hours) in a project at a quarry are shown in the network diagram below. The least time required for completing this entire project is 30 hours.



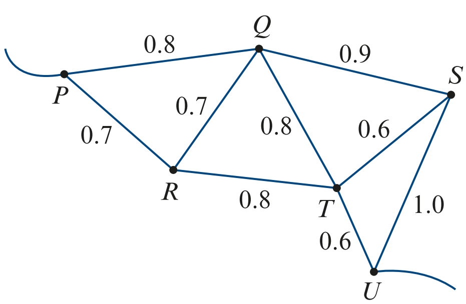
For each activity in this project, the table on the next page shows the completion time, the earliest starting time and the latest starting time.

Activity	*Completion time (hours)*	*Earliest starting time (hours)*	*Latest starting time (hours)*
A	6	0	
B	5	0	0
C	2	5	5
D		5	9
E	4	7	7
F	6	7	
G	4	11	11
H	3	9	13
I	2	13	16
J	3	15	15
K		18	18

a Complete the missing times in the table above.

b Write down the critical path for this project.

7 A rural town, built on hills, contains a set of roads represented by edges in the network shown here. The numbers on the network refer to distances along the roads (in kilometres) and the letters refer to intersections of the roads. The edges without endpoints refer to the two roads in and out of town.



a **i** What is the length of the shortest route through the town from P to U?

ii A safety officer who enters the town at P needs to examine all intersections in the town before leaving from U to travel to the next town. To save time, she wants to pass through each intersection only once. State a path through the network of roads that would enable her to do this.

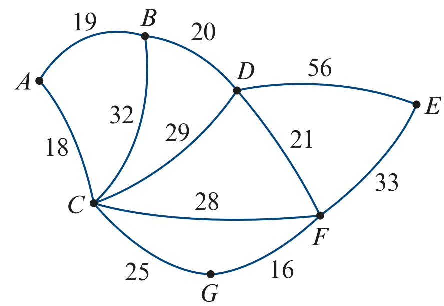
b A technician from the electricity company is checking the overhead cables along each street. The technician elects to follow an Eulerian path through the network streets (ignoring the roads in and out of town) starting at R and finishing at S.

i Complete the following Eulerian trail:

R–Q–P–R–____–____–____–T–U–S

ii How would the technician benefit from choosing an Eulerian path?

8 Seven towns on an island have been surveyed for transport and communications needs. The towns (labelled A, B, C, D, E, F, G) form the network shown here. The road distances between the towns are marked in kilometres. (All towns may be treated as points being of no size compared to the network lengths.)



a Explain what is meant by the description of the graph as 'planar'.

b Verify Euler's formula for the graph above.

An inspector of roads is stationed at B. Starting from B, she must travel the complete network of roads to examine them.

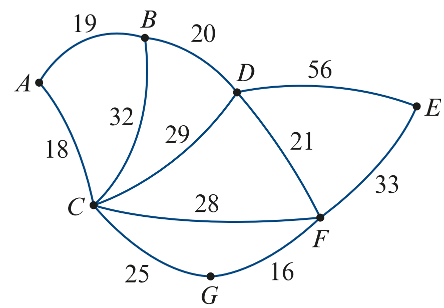
c If she wishes to travel the least distance where will she end up in the network?

d What will that distance be?

e Is the route unique? Briefly justify your answer.

f Determine the shortest distance that a fire truck stationed at E must travel to assist at an emergency at A.

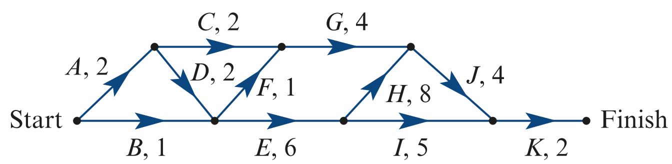
g To establish a cable network for telecommunications on the island, it is proposed to put the cable underground beside the existing roads. What is the minimal length of cable required here if back-up links are not considered necessary; that is, there are no loops in the cable network?



The Island Bank has outlets in each of the towns. The regional assistant manager stationed at C must visit each outlet every second Friday and then return to the office at C.

h Treating the towns as vertices and roads as edges in a graph, what is the distance of a journey that forms a Hamiltonian cycle in the graph?

9 The assembly of machined parts in a manufacturing process can be represented by the following network. The activities are represented by the letters on the edges and the numbers represent the time taken (in hours) for the activities scheduled.



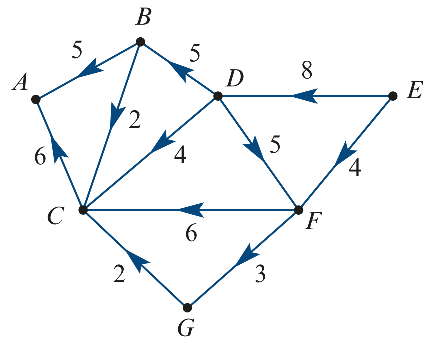
Activity	A	B	C	D	E	F	G	H	I	J	K
EST	0	0	2	2	4	4		10	10	18	22

a The earliest start times (EST) for each activity except G are given in the table. Complete the table by finding the EST for G.

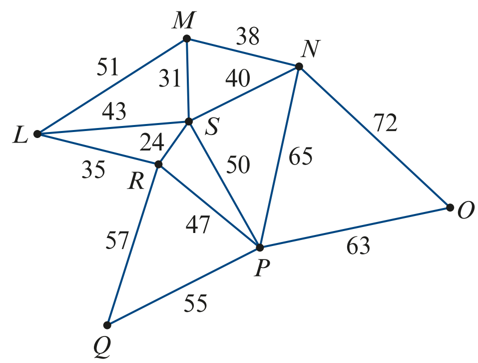
b What is the shortest time required to assemble the product?

c What is the float (slack time) for activity I?

10 A reservoir at E pumps water through pipes along the network routes shown. The capacities of the flow are given in the digraph shown here in megalitres per day.
Occasionally, there are fire emergencies in the forest beside A and additional flow of water is used. What is the maximum flow that can reach A from E?



11 A number of towns need to be linked by pipelines to a natural gas supply. In the network shown, the existing road links between towns L, M, N, O, P, Q and R and to the supply point, S, are shown as edges. The towns and the gas supply are shown as vertices. The distances along roads are given in kilometres.



a What is the shortest distance along roads from the gas supply point S to the town O?

b The gas company decides to run the gas lines along the existing roads. To ensure that all nodes on the network are linked, the company does not need to place pipes along all the roads in the network.

i What is the usual name given to the network within a graph (here, the road system) which links all nodes (towns and supply) and which gives the shortest total length?

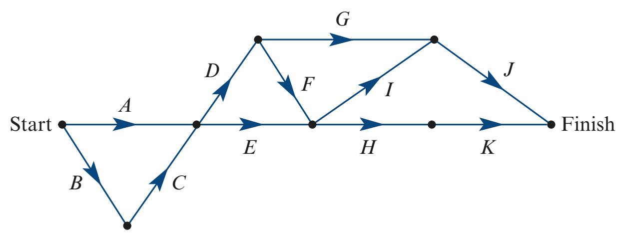
ii Sketch this network.

iii What is the minimum length of gas pipeline the company can use to supply all the towns by running the pipes along the existing roads?

c The gas company decides it wants to run the pipeline directly to any town which is linked by road to its supply at S. Towns not directly connected to S by road will be linked via other towns in the network.

What is the minimum length of pipeline that will enable all towns to be connected to the gas supply under these circumstances?

12 In laying a pipeline, the various jobs involved have been grouped into a set of specific tasks A–K, which are performed in the precedence described in the network below.



a List all the task(s) that must be completed before task E is started. The durations of the tasks are given in Table 1.

b Use the information in Table 1 to complete Table 2.

Table 1 Task durations

Task	*Normal completion time (months)*
A	10
B	6
C	3
D	4
E	7
F	4
G	5
H	4
I	5
J	4
K	3

Table 2 Starting times for tasks

Task	*EST*	*LST*
A	0	0
B	0	
C	6	7
D	10	10
E		11
F	14	14
G	14	18
H	18	20
I	18	
J	23	23
K	22	24

c For this project:

i write down the critical path

ii determine the length of the critical path (that is, the earliest time the project can be completed).

d If the project managers are prepared to pay more for additional labour and machinery, the time taken to complete task A can be reduced to 8 months, task E can be reduced to 5 months and task I can be reduced to 4 months.

Under these circumstances:

i what would be the critical path(s)?

ii how long would it take to complete the project?

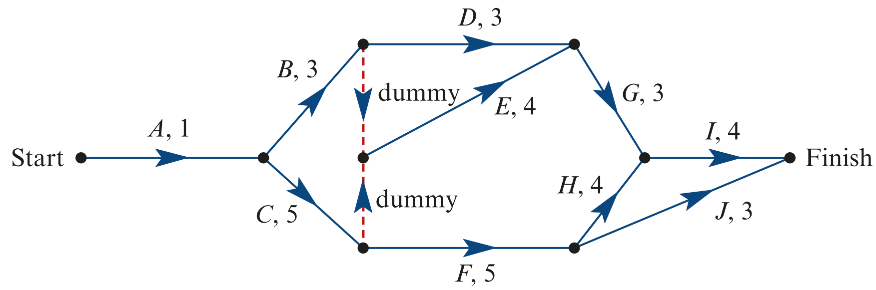
13 Camp sites A, B, C and D are to be supplied with food. Four local residents, W, X, Y and Z, offer to supply one campsite each. The cost in dollars of supplying one load of food from each resident to each campsite is tabulated.

Camp site	W	X	Y	Z
A	30	70	60	20
B	40	30	50	80
C	50	40	60	50
D	60	70	30	70

a Find the two possible matchings between campsites and residents so that the total cost is a minimum.

b State this minimum cost.

14 Consider the activity network below.



By employing more workers it is possible to reduce the time of some activities, however this will incur extra costs. The activities which can be reduced in time, the associated costs and maximum reduction in time are shown in the table below.

Activity	Cost (dollars per week)	Maximum reduction (weeks)
E	1000	2
F	1500	3
H	2000	3
J	200	2

a What is the new minimum completion time now possible for the project?

b What is the minimum cost of completing the project in this time?

c How many activities will be reduced in time to achieve the new minimum completion time at minimal cost?

Revision

Chapter 16

Revision of Chapters 1–15

Revision

16A Exam 1 questions

Data analysis, probability and statistics

1 In a study of cats, data relating to the following five variables were collected:

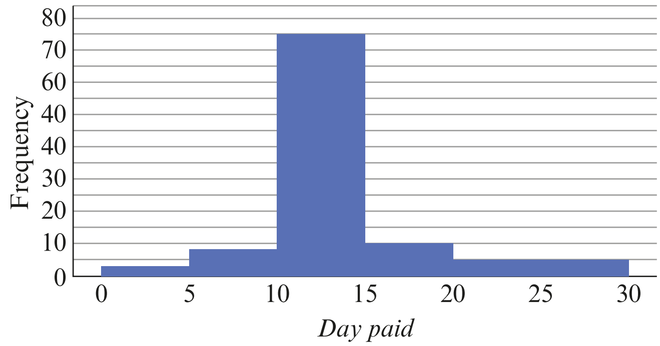
- *name*
- *breed*
- *age* (1 = less than 1 year, 2 = 1–5 years, 3 = more than 5 years)
- *weight in kg*
- *length in cm*

The number of these variables that are discrete numerical variables is:

A 0 **B** 1 **C** 2 **D** 3 **E** 4

Use the following information to answer questions 2 and 3

The invoices from Janelle's company request payment from her customers within 14 days. She records the number of days between the customers receiving the invoice, and the invoice being paid (*day paid*), for 105 invoices. Her data is shown in the histogram.



2 Janelle decides to consider invoices which were paid in 15 days or more as late payments. The percentage of the invoices classified as late payments is closest to:

A 9.5% **B** 10.0% **C** 19.0% **D** 20.0% **E** 85.7%

3 The third quartile (Q_3) for this distribution could be:

A 8 days **B** 14 days **C** 15 days **D** 20 days **E** 22 days

4 A class of 28 students in Year 11 (16 girls and 12 boys) sat for a French test. The mean score on the test for the 16 girls in the class was 42. The mean score on the test for the 12 boys in the class was 28. The mean score on the test for the whole class was

A 30 **B** 32 **C** 35 **D** 36 **E** 37

5 The amount of flour in a box, in grams, is approximately normally distributed with a mean of 1002 g and a standard deviation of 4 g. The percentage of these boxes which contain from 998 g to 1010 g of flour is closest to

A 32% **B** 68% **C** 81.5% **D** 95% **E** 97.35%

Use the following information to answer questions 6–8

The *kilometres driven* by secondhand cars were recorded to the nearest 100 km for a sample of 300 cars. The mean and the five number summary for these data are shown in the table opposite.

mean	36 947
minimum	500
first quartile (Q_1)	15 000
median (M)	32 000
third quartile (Q_3)	48 500
maximum	213 000

6 The difference between the mean *kilometres driven* and the median *kilometres driven*, to the nearest 100 km, is closest to:

A 500 km **B** 4900 km **C** 5000 km
D 16 500 km **E** 33 500 km

7 Of these 300 cars, the number that have been driven less than 15 000 km is closest to:

A 50 **B** 75 **C** 100 **D** 125 **E** 150

8 The shape of the distribution of the the *kilometres driven* is best described as

A approximately symmetric
B positively skewed
C positively skewed with one or more outliers
D negatively skewed
E negatively skewed with one or more outliers

9 Suppose that the weights of adults wombats are approximately normally distributed. If 16% of wombats weigh less than 3.5 kg, and 0.15% of wombats weigh more than 5.5 kg, the mean and standard deviation of the weight of wombats, in kg, is closest to

A mean = 4.0, standard deviation = 0.5 **B** mean = 4.3, standard deviation = 0.4
C mean = 5.5, standard deviation = 2.0 **D** mean = 4.5, standard deviation = 1.0
E mean = 4.5, standard deviation = 0.5

Use the following information to answer Questions 10 and 11

The data in the following table were collected when a group of 250 students from Year 9 and Year 10 were asked where they would like to go for their school camp.

	Year level		
Preferred camp	Year 9	Year 10	Total
Beach	78	43	121
Snow	40	66	106
Other	12	11	23
Total	130	120	250

10 The percentage of students surveyed who chose to go to the beach is closest to:

A 48.4% **B** 60.0% **C** 35.8% **D** 65.0% **E** 35.5%

11 The data in the table supports the contention that there is an association between *preferred camp* and *year level* because:

A more students preferred to go the beach than go to the snow.

B 35.8% of students in Year 10 preferred to go to the beach, compared to only 55.0% of Year 10 students who preferred to go the snow.

C 60.0% of students in Year 9 preferred to go to the beach, compared to only 30.8% of Year 9 students who preferred to go the snow.

D 48.2% of students preferred the beach compared to 42.4% who preferred the snow.

E 60.0% of students in Year 9 preferred to go to the beach, compared to only 35.8% of Year 10 students who preferred to go the beach.

Use following information to answer Questions 12 and 13

In a study of the effect of sunlight on the growth of seedlings, a number of seedlings were planted in various locations in a garden bed. The average number of hours of sunlight each plant received each day for fourteen days (*sunlight hours*), as well as the amount they grew, in cm, over that period of time (*growth*) were recorded. *Growth* can be predicted from *sunlight hours* from the regression line:

$$growth = 4.28 + 0.586 \times sunlight\ hours, \quad \text{with } r = 0.690$$

12 The percentage of variation in *growth* NOT explained by the variation in the *sunlight hours* is closest to:

A 16.9% **B** 31.0% **C** 34.3% **D** 47.6% **E** 52.4%

13 This regression line predicts that, on average, *growth*:

A increases by 4.28 cm for each additional hour of sunlight

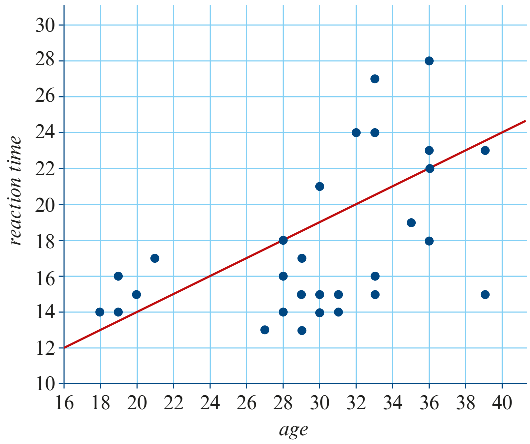
B increases by 0.586 cm for each additional hour of sunlight

C decreases by 0.586 cm for each additional hour of sunlight

D decreases by 4.28 cm for each additional hour of sunlight

E increases by 0.690 cm for each additional hour of sunlight

14 The scatterplot shows the reaction time, measured in hundredths of seconds, for a group of 20 people, together with their age. A least squares line had been fitted to the scatterplot with *age* as the explanatory variable, and *reaction time* as the response variable. The equation of the least squares line is closest to:



A $reaction\ time = 26 + 2.0 \times age$
B $reaction\ time = 4.0 + 0.5 \times age$
C $reaction\ time = 12.0 + 0.5 \times age$
D $reaction\ time = 12.0 + 0.3 \times age$
E $reaction\ time = 4.0 - 0.5 \times age$

15 The table below shows the *life expectancy* in years and the percentage of government expenditure which is spent on health (*health*) in 8 countries.

Health (%)	17.3	10.3	4.7	6.0	20.1	6.4	13.2	7.7
Life expectancy (years)	82	76	68	69	83	75	76	76

A least squares line which enables a country's *life expectancy* to be predicted from their expenditure on *health* is fitted to the data. The value of the residual (to the nearest year) when the actual percentage of government expenditure which is spent on health is 6% is closest to:

A -3 **B** -2 **C** 1 **D** 2 **E** 3

16 The price of shares in a company has increased non-linearly in the last 12 months. A log transformation was applied to the maximum share price each month (*share price*), and a least squares line fitted to the transformed data, with *month* as the explanatory variable. The equation of the least squares line is:

$$\log(share price) = 0.855 + 0.154 \times month$$

Using this equation, the maximum monthly share price in month 15 is closest to:

A \$0.50 **B** \$1.04 **C** \$3.16 **D** \$1462.18 **E** \$94.42

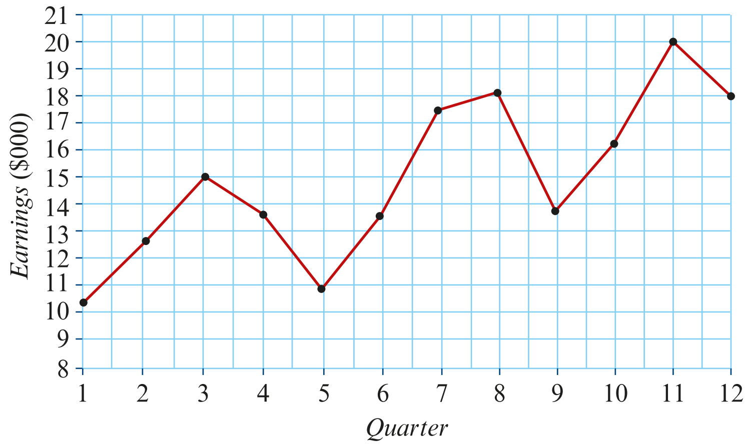
17 The table below records the monthly electricity cost (in dollars) for an apartment over one calendar year.

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
123	90	153	136	101	129	153	143	95	61	85	107

The six-mean smoothed with centring cost of electricity in July is closest to:

A \$129 **B** \$120 **C** \$128 **D** \$131 **E** \$143

18 The time series plot below shows shows earnings per quarter (\$000) for a certain salesperson over a 3 year period.



The nine-median smoothed earnings for the salesperson in Quarter 8, in \$000s, is closest to:

A 14 **B** 16 **C** 17.5 **D** 18 **E** 20

19 The table below shows the long term mean monthly sales figures (in \$'000s) for a company, and the associated seasonal indices for the sales. The long-term mean sales figure for January is missing.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Sales		80.0	70.3	62.6	54.6	55.0	52.1	54.2	56.5	52.8	61.8	99.7
SI	0.727	1.289	1.132	1.008	0.880	0.886	0.840	0.874	0.911	0.850	0.996	1.607

The long-term mean sales figure for January is closest to:

A 45.1 **B** 58.9 **C** 62.1 **D** 73.4 **E** 85.4

20 The number of job applications received by a large supermarket chain is seasonal. Data has been collected, and a least squares regression line fitted to the deseasonalised data. The equation of the line is

$$\textit{deseasonalised job applications} = 457.8 + 12.27 \times \textit{month number}$$

where *month number* 1 is January 2022.

The monthly seasonal indices for job applications are shown in the following table:

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
1.14	1.06	1.22	1.03	0.95	0.95	0.83	0.70	0.78	0.88	1.23	1.23

The actual number of job applications predicted for January 2023 was closest to

A 605 **B** 617 **C** 542 **D** 690 **E** 704

Recursion and financial modelling

21 Consider the recurrence relation below.

$$A_0 = 20, \qquad A_{n+1} = 2A_n - 26$$

Which term of the sequence generated by this recurrence relation is the first to be negative?

A A_1 **B** A_2 **C** A_3 **D** A_4 **E** A_5

22 The following recurrence relation can generate a sequence of numbers

$$M_0 = 48 \qquad M_{n+1} = M_n + B$$

The value of M_3 is 30. The value of B is

A -18 **B** -6 **C** -3 **D** 3 **E** 6

23 An amount of money is deposited into an account that earns compound interest. Which combination of interest rate and compounding period has the largest effective interest rate?

A 4.0% per annum, compounding quarterly

B 4.1% per annum, compounding weekly

C 4.1% per annum, compounding monthly

D 4.2% per annum, compounding fortnightly

E 4.2% per annum, compounding quarterly

24 A tractor was purchased for \$189 000.
Using the reducing balance method, the value of the tractor depreciates by 8% each year.
Which one of the following recurrence relations could be used to determine the value of the tractor after n years, V_n?

A $V_0 = 189\,000, \qquad V_{n+1} = 0.92V_n$

B $V_0 = 189\,000, \qquad V_{n+1} = 1.08V_n$

C $V_0 = 189\,000, \qquad V_{n+1} = V_n - 15120$

D $V_0 = 189\,000, \qquad V_{n+1} = 0.92V_n - 15120$

E $V_0 = 189\,000, \qquad V_{n+1} = 1.08V_n - 15120$

25 Nicos borrowed \$15 000 to pay for a holiday.
He was charged interest at the rate of 5.9% per annum, compounding monthly.
The loan was repaid with monthly repayments of \$600.
After four months, the total interest that Nicos had paid was closest to

A \$74 **B** \$280 **C** \$2100 **D** \$2400 **E** \$13 000

26 Reuben has purchased a new oven for his restaurant for \$37 000.
He depreciates the value of the oven using the unit cost method at the rate of \$3 per hour of use.
The recurrence relation below can be used to model the value of the oven, V_n, after n years.

$$V_0 = 37\ 000, \qquad V_{n+1} = V_n - 5460$$

Reuben uses the oven all 52 weeks of the year for the same number of hours each week.
The number of hours each week that the oven is used for is closest to

A 15 **B** 21 **C** 35 **D** 105 **E** 5460

27 The first three lines of an amortisation table for a reducing balance loan are shown below.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	140 000.00
1	755.00	560.00	195.00	139 805.00
2	755.00	559.22		

What is the principal reduction from repayment number 2?

A \$195.00 **B** \$195.78 **C** \$560.00 **D** \$559.22 **E** \$755.00

28 An annuity investment earns interest at the rate of 5.3% per annum, compounding monthly.
Tim initially invested \$60 000 and will add monthly payments of \$1600.
The value of this investment will first exceed \$74 000 after

A five months
B six months
C seven months
D eight months
E nine months

29 Raj borrowed \$54 000 to buy a car and was charged interest at the rate of 9.6% per annum, compounding monthly.
For the first year of the loan, Raj made monthly repayments of \$1080.
For the second year of the loan, Raj made monthly repayments of \$1200.
The total amount of interest that Raj paid over this two-year period is closest to

A \$8785 **B** \$12 960 **C** \$14 400 **D** \$18 575 **E** \$27 360

30 Victoria has invested an amount of money in a perpetuity.
The perpetuity earns interest at the rate of 4.8% per annum.
Interest is calculated and paid quarterly.
If Victoria receives \$1020 per quarter from the perpetuity, then the amount that she has invested is

A \$4896 **B** \$21 250 **C** \$85 000 **D** \$255 000 **E** \$489 600

Matrices

31 If matrix $M = \begin{bmatrix} 3 & 4 & 2 \\ 5 & 6 & 8 \end{bmatrix}$ then its transpose M^T is

A $\begin{bmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$ **B** $\begin{bmatrix} 3 & 5 \\ 4 & 6 \\ 2 & 8 \end{bmatrix}$ **C** $\begin{bmatrix} 5 & 6 & 8 \\ 3 & 4 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 5 & 4 & 8 \\ 3 & 6 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 4 & 3 \\ 6 & 5 \\ 8 & 7 \end{bmatrix}$

32 Matrix $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 0 \end{bmatrix}$ and matrix $B = \begin{bmatrix} 1 & 9 & 11 \\ 0 & 12 & 14 \end{bmatrix}$. Matrix $R = A \times B$. Element r_{32} is calculated by

A $3 \times 9 + 4 \times 12$ **B** $7 \times 9 + 0 \times 12$ **C** $1 \times 11 + 6 \times 14$
D $1 \times 11 + 6 \times 14$ **E** $7 \times 11 + 0 \times 12$

33 A gymnasium has the same number of customers every day. They can do either pilates (P) or yoga (Y). The customers may change activities every day. Their change in involvement is shown in the transition matrix below.

$$\begin{array}{cc} & \text{Today} \\ & \begin{array}{cc} P & Y \end{array} \\ \text{Tomorrow} \begin{array}{c} P \\ Y \end{array} & \begin{bmatrix} 65\% & 70\% \\ 35\% & 30\% \end{bmatrix} \end{array}$$

There must be 20 customers in the pilates class. Each day the number of customers in the yoga class is

A 10 **B** 12 **C** 15 **D** 20 **E** 25

34 Consider the following communication matrix

$$\begin{array}{ccc} & & \text{receiver} \\ & & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \text{sender} & \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

How many 4-step communications are there from C to B?

A 1 **B** 2 **C** 3 **D** 4 **E** 5

35 Matrices A, B and C have orders 2×5, 2×4 and 5×4 respectively. What is the order of the matrix $C^T \times (B^T \times A)^T$?

A 5×5 **B** 4×5 **C** 2×4 **D** 4×4 **E** 2×5

36 Matrix A is a 5×5 matrix.

- Matrix B is a row matrix.
- Matrix C is a column matrix.

Which one of the matrix products below could result in a 1×1 matrix?

A ACB **B** ABC **C** CAB **D** BAC **E** BCA

37 Matrix $P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ and matrix $Z = \begin{bmatrix} a \\ b \\ c \\ a \\ b \end{bmatrix}$. The smallest value of n such that $P^n Z = Z$ is

A 1 **B** 2 **C** 3 **D** 4 **E** 5

Questions 38 and 39 are based on the following information.

A survey company has selected 1000 people to rate a news program. The rating given by a survey participant can be low (L), medium (M) or good (G). By observing for a long while it is observed that the participants change their rating from week to week as shown in the transition matrix T below.

$$T = \begin{array}{c} \text{This week} \\ \begin{array}{ccc} L & M & G \end{array} \\ \begin{bmatrix} 0.8 & 0.2 & 0.3 \\ 0.1 & 0.7 & 0.1 \\ 0.1 & 0.1 & 0.6 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ L \\ M \\ G \end{array} \text{ Next week}$$

The expected number of each rating received after n weeks can be determined by the recurrence relation

$$S_0 = \begin{bmatrix} 300 \\ 600 \\ 100 \end{bmatrix}, S_{n+1} = TS_n$$

where S_0 is the state matrix for the first week of the survey with 300 low ratings, 600 medium ratings and 100 high ratings.

38 What percentage of these 1000 participants are not expected to change their rating in the second week from the first?

A 60% **B** 43% **C** 56% **D** 72% **E** 82 %

39 In the long term (say after more than 20 weeks of the survey), how many of the participants will give a low rating?

A 150 **B** 250 **C** 350 **D** 450 **E** 550

40 Consider the matrix recurrence relation below.

$$S_0 = \begin{bmatrix} 25 \\ 16 \\ 56 \end{bmatrix}, \; S_{n+1} = TS_n \quad \text{where } T = \begin{bmatrix} x & 0.4 & 0.6 \\ y & 0.2 & 0.4 \\ z & w & v \end{bmatrix}$$

(Matrix T is a regular transition matrix, The columns of the matrix T sum to one).
Which one of the following statements is not necessarily true?

A $z + w + v = 1$ **B** $x + y + z = 1$ **C** $w = 0.4$
D $v = 0$ **E** $z + w + v \geq 0.4$

41 Four teams, X, Y, Z and W, competed in a round-robin competition where each team played each of the other teams once. There were no draws. The results are shown in the matrix below.

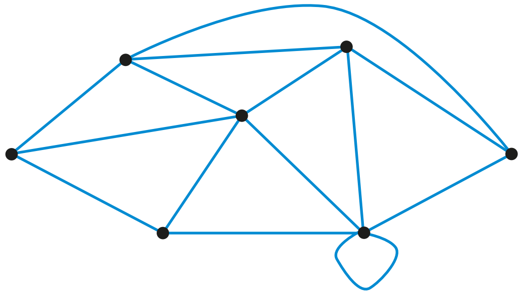
$$\begin{array}{cc} & \text{loser} \\ & \begin{array}{cccc} X & Y & Z & W \end{array} \\ \text{winner} \begin{array}{c} X \\ Y \\ Z \\ W \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ a & 0 & 0 & 1 \\ 0 & 1 & 0 & c \\ b & 0 & 1 & 0 \end{bmatrix} \end{array}$$

A '1' in the matrix shows that the team named in that row defeated the team named in that column. In this matrix, the values of a, b and c are

A $a = 1, b = 0, c = 0$ **B** $a = 0, b = 1, c = 0$ **C** $a = 1, b = 0, c = 1$
D $a = 0, b = 1, c = 1$ **E** $a = 1, b = 1, c = 0$

Networks and decision mathematics

42 Consider the graph below.



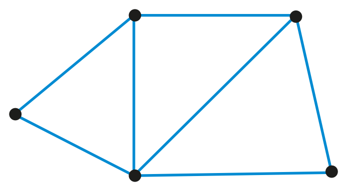
The number of vertices with a degree greater than or equal to 5 is

A 0 **B** 1 **C** 2 **D** 3 **E** 4

43 What is the maximum number of edges a bipartite graph with 30 vertices can have?

A 60 **B** 125 **C** 200 **D** 225 **E** 250

44 Consider the graph opposite. Euler's formula can be verifted for this graph. What values of e, v and f can be used in this verification?



A $e = 5, v = 5, f = 3$

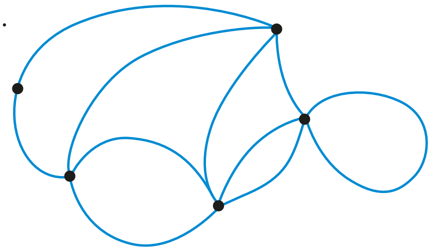
B $e = 7, v = 5, f = 4$

C $e = 6, v = 5, f = 3$

D $e = 6, v = 5, f = 4$

E $e = 8, v = 4, f = 4$

45 The graph opposite has five vertices and ten edges. How many of the vertices in this graph have an even degree?

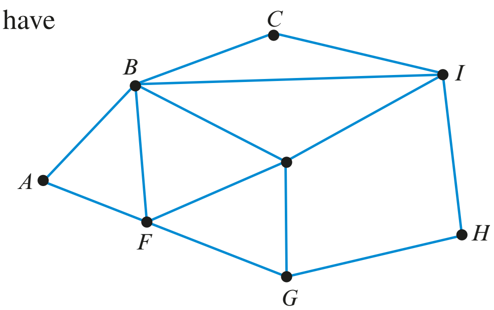


A 0 **B** 1 **C** 2 **D** 3 **E** 4

46 For a connected graph with five vertices and five edges, the sum of the degrees of the vertices is

A 4 **B** 6 **C** 8 **D** 9 **E** 10

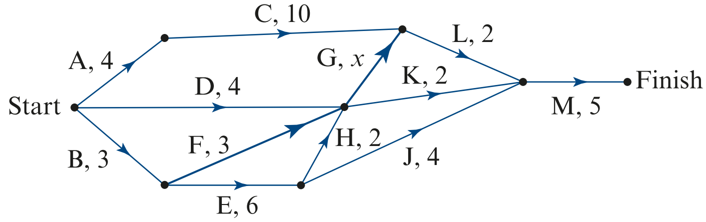
47 If an extra edge is added to this graph it will have an Eulerian circuit. The edge is



A AB **B** BG **C** QG

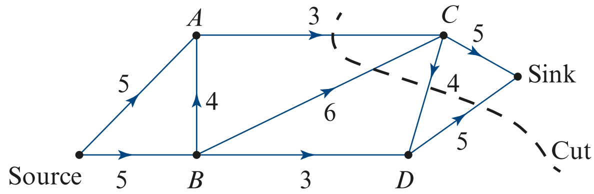
D IB **E** AG

48 The directed graph below shows the sequence of activities required to complete a project. The time taken to complete each activity, in hours, is also shown. The minimum completion time for this project is 21 hours. The time taken to complete activity G is labelled x. The maximum value of x is



A 1 **B** 2 **C** 3 **D** 5 **E** 7

49 The flow of water through a series of pipes is shown in the network below. The numbers on the edges show the maximum flow through each pipe in litres per minute.



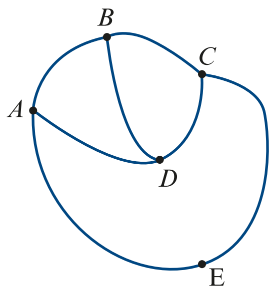
The capacity of the cut in litres per minute is

A 10 **B** 12 **C** 14 **D** 18 **E** 20

50 The network opposite shows the cabling between five locations A, B, C, D and E.

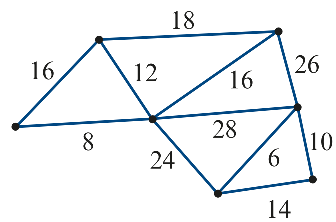
An adjacency matrix for this network is formed. The number of zeros in this matrix is

A 8 **B** 9 **C** 10 **D** 11 **E** 12



51 The minimum spanning tree for the graph opposite has a weight of

A 52 **B** 72 **C** 76
D 80 **E** 86



16B Exam 2 questions

Data analysis, probability and statistics

1 The weights (in kg) carried by the horses in a handicap race are given below.

60 57 57 55 54 53 53 53 52 52 51.5 51

a Calculate the mean and the standard deviation of the weights carried by the horses in this race, rounding your answers to three decimal places.

b One horse in the race carried a weight of 51 kg. Use your answer to part **a.** to calculate the standardised score (z) for the weight carried by that horse, rounding your answer to one decimal place.

c Suppose that the weights carried by horses in handicap races are approximately normally distributed. If the weight carried by one horse, Silver, has a standardised score of $z = -2$:

i Using the mean and standard deviation determined in part **a.**, how much weight was carried by Silver? Round your answer to one decimal place.

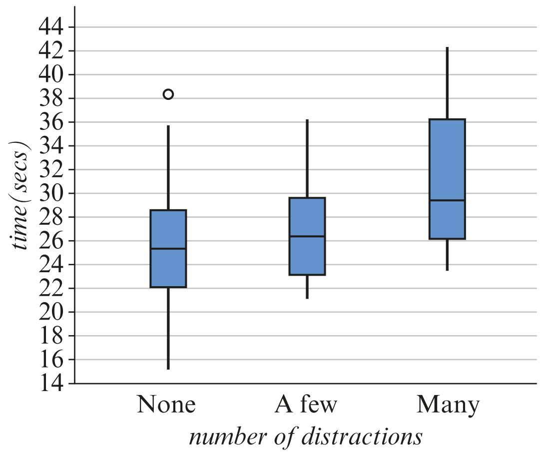
ii What percentage of horses would be expected to carry weight less than Silver?

2 In a study on the effect of distractions on concentration, three groups of subjects were timed completing a complex task. One group completed the task with no distractions, a second group with a few distractions, and the third group with many distractions. The two variables in this study are *number of distractions* and *time* in seconds.

Five number summaries describing the distribution of *time* for each group are displayed below, along with the size of each group.

The associated boxplots are shown following the table.

Number of distractions	Group size	Min	Q_1	Median	Q_3	Max
none	32	15.2	22.0	25.0	28.2	38.0
a few	36	21.0	23.0	26.2	29.4	36.0
many	36	23.4	26.0	29.2	36.0	42.1



a Which variable is the explanatory variable and which is the response variable in this study?

b Which variables in this study are numerical?

c For the group who completed the task with no distractions

i What is the interquartile range?

ii Show why the value of 38.0 seconds is an outlier for this group.

d People who took more than 36 seconds to complete the task were classified as slow. How many people in the study would be classified as slow?

e Do the boxplots support the contention that there is an association between *number of distractions* and *time*? Refer to the values of an appropriate statistic in your response.

3 The equation of the least squares line that relates the fuel consumption of a certain car, in litres/100 km, to the speed at which the car is travelling, in km/hr is:

$$\textit{fuel consumption} = 6.827 + 0.0218 \times \textit{speed}$$

a Use the summary statistics shown to determine the coefficient of determination as a percentage, rounded to one decimal place.

	fuel consumption	*speed*
mean	8.7556	88.444
standard deviation	0.52941	22.367

b Interpret the value of the coefficient of determination in terms of *fuel consumption* and *speed*.

c Use the equation to predict the *fuel consumption* of the car if it is travelling at 100 km/hr. Round the answer to one decimal place.

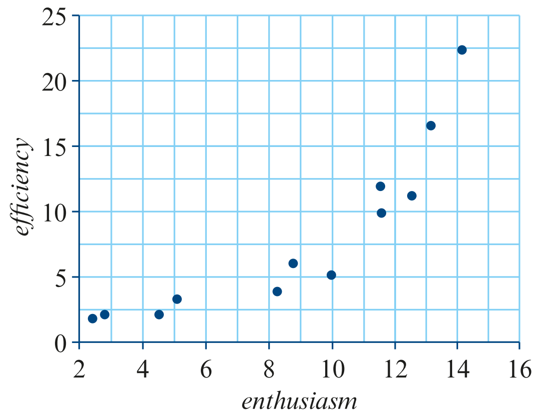
d Write down the slope of the regression line and interpret in terms of *fuel consumtion* and the *speed*.

e When the *speed* was 72 km/hr, the actual *fuel consumption* was 8.3 litre/100 km. Show that, when the least squares line is used to predict the fuel consumption at 72 km/hr, the residual is −0.10 rounded to two decimal places.

4 In a study of the association between a person's *enthusiasm* for their job (a numerical variable measured on a scale from 0 to 15), and their *efficiency* when performing their job (a numerical variable measured on a scale from 0 to 25), the following data was collected from a group of 12 employees.

enthusiasm	14.2	13.2	11.6	12.6	11.6	10.0	8.8	8.3	5.1	4.5	2.8	2.4
efficiency	22.5	16.7	12.0	11.3	10.0	5.2	6.1	4.0	3.4	2.2	2.1	1.9

The following scatterplot was constructed, with *enthusiasm* as the explanatory variable, and *efficiency* as the response variable.



a Describe the association between *efficiency* and *enthusiasm* in terms of form and strength.

b Which transformations could be used in order to linearise the association?

c Apply a log transformation to the variable *efficiency* to linearise the association. Fit a least squares line to the transformed data, and write down its equation. Round the values of the intercept and slope to three significant figures.

d Use the equation from part **c.** to predict the *efficiency* score for a person who scores 9.3 on *enthusiasm*.

Revision

5 The table below shows the quarterly house sales achieved by a real estate company in the years 2020-2021.

Year	Q1	Q2	Q3	Q4
2020	52	59	68	27
2021	57	65	75	29

a Use the data in the table to find seasonal indices. Give your answers rounded to two decimal places.

b The number of houses sold in each of the four quarters in 2022 is shown in the table below.

Year	Q1	Q2	Q3	Q4
2022	63	69	80	33

Use the seasonal indices from part **a** to deseasonalise the data. Round your answers to the nearest whole number.

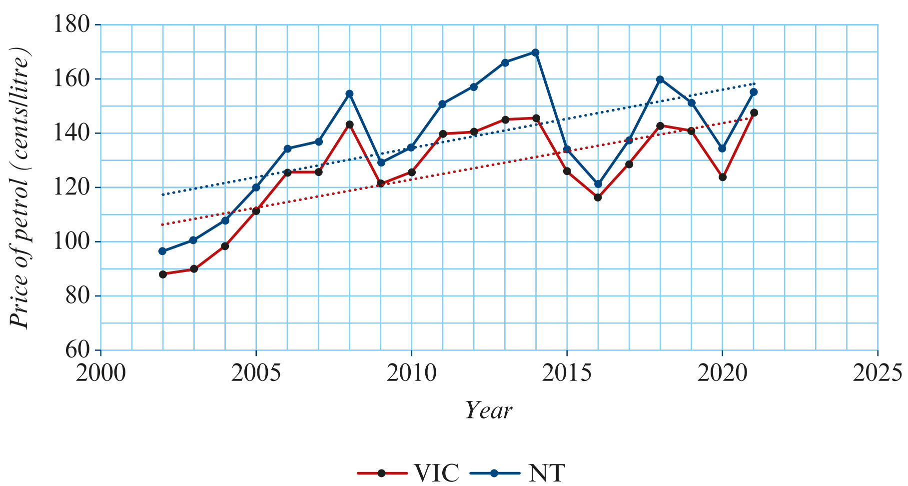
6 The following table gives the value of average price of unleaded petrol in Victoria each year from 2015–2021.

Year	2015	2016	2017	2018	2019	2020	2021
Price (cents/litre)	126.3	116.4	128.7	143.4	141.1	123.9	147.6

a Find the centred four-mean smoothed value of petrol in Victoria for the year 2017 in cents/litre, rounding your answer to one decimal place.

b Find the five-median smoothed value of petrol in Victoria for the year 2019 in cents/litre, rounding your answer to one decimal place

The following time series plot shows the price of petrol in Victoria in cents/litre, and the price of petrol in the Northern Territory (NT) in cents/litre, over the years 2002–2021.



Least squares regression lines (shown on the plot) have been fitted to both sets of time series data, and the following equations determined:

Victoria: $\quad petrol\ price = -4071.53 + 2.08699 \times year$

NT: $\quad petrol\ price = -4290.07 + 2.20128 \times year$

c **i** Write down the slope of the least squares line for Victoria rounded to two decimal places, and interpret.

ii Write down the slope of the least squares line for the NT rounded to two decimal places, and interpret.

d Use the least squares regression lines to predict the price of petrol in 2026:

i in Victoria **ii** in the NT

e Do the equations predict that the difference in petrol prices between Victoria and the NT will decrease, stay the same, or increase? Explain your answer, quoting appropriate statistics.

Recursion and financial modelling

7 Joslyn originally paid $9500 for office furniture that she now wishes to sell. Joslyn will sell the furniture at a depreciated value.

a Joslyn could use a reducing balance method, with an annual depreciation rate of 7.4%.
Using this depreciation method, what is the value of the furniture five years after it was purchased? Round your answer to the nearest cent.

b If Joslyn used a reducing balance depreciation method and the furniture was sold for $6890 after five years, what annual percentage rate of depreciation did this represent? Give your answer correct to one decimal place.

c Joslyn could use a flat rate depreciation method.
Let J_n be the value, in dollars, of Joslyn's furniture n years after it was purchased.
The value of the furniture, J_n can be modelled by the recurrence relation below.

$$J_0 = 9500, \qquad J_{n+1} = J_n - 855$$

i Using this depreciation method, what is the value of the furniture five years after it was purchased?

ii What annual flat rate of depreciation is represented?

8 Marina would like to buy a new bicycle and has saved $2600.

a Marina could invest this money in an account that pays interest which compounds monthly.
The balance of this investment after n months, M_n, could be determined using the recurrence relation below.

$$M_0 = 2600, \qquad M_{n+1} = 1.003 \times M_n$$

Calculate the total interest that would be earned by Marina's investment in the first five months. Round your answer to the nearest cent.

Marina could invest the \$2600 in a different account that allows her to make an additional payment of \$140 each month.

b Marina would like to have a balance of \$5500, to the nearest dollar, after 18 months. What annual interest rate would Marina require? Round your answer to two decimal places.

c The interest rate is 3% per annum, compounding monthly. Let V_n be the value of Marina's investment after n months.
Write down a recurrence relation, in terms of V_0, V_{n+1} and V_n, that would model the change in the value of this investment.

9 Alessandro borrows \$12 000 with interest on the loan charged at the rate of 7.9% per annum, compounding monthly.
Immediately after the interest has been calculated and charged each month, Alessandro will make a repayment.

a Alessandro considers making interest only repayments. What would be the value of each interest only repayment?

b Alessandro makes equal monthly repayments for four years. After these four years, the balance of his loan will be \$2946.24 correct to two decimal places. What amount, in dollars, will Alessandro repay each month during the four years?

c If Alessandro instead decides to fully repay the loan in three years with 35 equal monthly payments followed by a final payment that is as close to the regular payment as possible. Find both the regular payment and the final payment. Round your answers to the nearest cent.

Matrices

10 A regional city has three supermarkets HSL (H), Radcliffs (R) and Cottonworths (C)
The total number of shoppers at each of the stores on a weekday is shown in matrix W.

$$W = \begin{array}{c} \begin{array}{ccc} H & R & C \end{array} \\ \begin{bmatrix} 1500 & 2500 & 3200 \end{bmatrix} \end{array}$$

a Write down the order of matrix W.

Each of the supermarkets has a meat counter and a delicatessen The proportion of daily shoppers who only purchase from the meat counter (M), only from the delicatessan (D) and those that do shopping from several sections (G) is the same for each supermarket and is described in the matrix shown here.

$$P = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.7 \end{bmatrix} \begin{array}{c} M \\ D \\ G \end{array}$$

b Find the matrix $Q = P \times W$ and describe what the value element q_{32} is.

c On a particular day at Cottonworths the amount spent, in dollars, is described by the following matrix. The amounts indicate the typical amount spent by a customer.

$$A = \begin{array}{c} \begin{bmatrix} 20 \\ 20 \\ 60 \end{bmatrix} \end{array} \begin{array}{l} M \\ D \\ G \end{array}$$

The numbers of each type of of customer that spent exactly these amounts is descibed by the matrix

$$T = \begin{array}{c} \begin{array}{ccc} M & D & G \end{array} \\ \begin{bmatrix} 150 & 250 & 600 \end{bmatrix} \end{array}$$

Show a matrix multiplication that will give the total amount spent on that day.

d There can be a lot of change in the daily shopping numbers of customers between the three supermarkets. We use a 3×3 transition matrix and a recurrence relation to describe the changed shopping locations of customers in the city month by month.

$$\text{Let } S_0 = \begin{bmatrix} 1500 \\ 2500 \\ 3200 \end{bmatrix} \begin{array}{l} H \\ R \\ C \end{array} \text{ and } T = \begin{array}{c} \text{This month} \\ \begin{array}{ccc} H & R & C \end{array} \\ \begin{bmatrix} 0.13 & 0.8 & 0.2 \\ 0.7 & 0.1 & 0.2 \\ 0.17 & 0.1 & 0.6 \end{bmatrix} \end{array} \begin{array}{l} H \\ R \\ C \end{array} \text{ Next month}$$

The recurrence relation is $S_{n+1} = T \times S_n$ where S_n is the state matrix n months after our starting time.

i Find S_1

ii Find S_{50}

iii Describe the long term situation for the number of customers at each of the supermarkets.

11 When a new infectious disease was first noticed in a particular country, there were 100 people already infected. As the disease spread, the country's health authority collected the following information:

- The duration of the disease is at most 3 weeks. People who contract the disease either die during these 3 weeks or else recover during the third week.
- The survival rate of people who have the disease is 90% in the first week and 80% in the second week.
- People who contract the disease are not infectious during the first week. In their second week of the disease they have a 90% probability of infecting one other person, and in their third week a 70% probability of infecting one other person.

Using this information:

a Construct a Leslie matrix L for the disease based on three one-week stages.

b Assume that, when the disease is first noticed, the 100 infected people are all in the second week of the disease. Write down the initial population matrix S_0.

For each of the following, give answers to the nearest whole number.

c Determine how the disease is spreading by using the Leslie matrix L to find S_1, S_2, S_3 and S_4. Comment.

d Find S_{40}, and then use S_{40} to find S_{41}. Verify that, in these two population matrices, the sizes of the three groups are in nearly the same ratio. Hence estimate the growth rate of the disease at this stage.

e Suppose that the health authority had taken immediate action to reduce the rate of infection in the third week to 35%. Repeat parts **a**–**d**. Would this action have been sufficient to eradicate the disease? Give evidence for your answer.

f Now suppose that the health authority had taken more drastic action and reduced the rate of infection in the third week to 10%. Repeat parts **a**–**d**.

12 A certain population of female marsupials is divided into six age groups, each spanning 3 months. The population can then be modelled by the following Leslie matrix, L, and initial population matrix, S_0:

$$L = \begin{bmatrix} 0 & 0.3 & 0.8 & 0.7 & 0.4 & 0 \\ 0.6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0 \end{bmatrix} \quad \text{and} \quad S_0 = \begin{bmatrix} 24 \\ 16 \\ 24 \\ 8 \\ 0 \\ 0 \end{bmatrix}$$

a Determine the population matrices S_1, S_{40} and S_{41}.

b Using S_{40} and S_{41}, show that the long-term growth rate of the population is approximately 1.03.

Now assume that the initial population is 2202 and the sizes of the six age groups (from youngest to oldest) are 715, 416, 363, 317, 247 and 144.

c Write down S_0 and determine S_1.

d Using S_0 and S_1, show that the growth rate of the population over the first 3 months is approximately 1.03.

13 The choice of seats and their costs at the Leslie theatre are

- Stalls (S) $34.00
- Balcony (B) $42.00
- Dress circle (D) $60.00

The number of seats available in each class are

- Stalls (S) 200
- Balcony (B) 150
- Dress circle (D) 80

a The column matrix A contains the number of seats in each class.

$$A = \begin{array}{c} \begin{bmatrix} 200 \\ 150 \\ 80 \end{bmatrix} \end{array} \begin{array}{c} S \\ B \\ D \end{array}$$

State the order of A.

b Matrix C gives the cost of each type of seat

$$\begin{array}{cc} & \begin{array}{ccc} S & B & D \end{array} \\ C = & \begin{bmatrix} 34 & 42 & 60 \end{bmatrix} \end{array}$$

Determine the matrix $C \times A$ and explain what it represents.

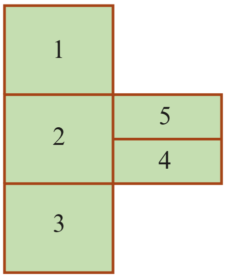
c All seats are sold. Let $M = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$.
What does the matrix product

$$M \begin{bmatrix} 200 & 0 & 0 \\ 0 & 150 & 0 \\ 0 & 0 & 80 \end{bmatrix} C^T$$

represent?

Networks and decision mathematics

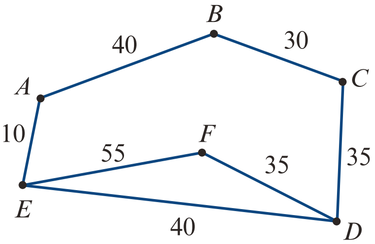
14 **a** A very large country home garden is divided into five regions labelled 1 to 5 on the diagram opposite. The red lines represent the boundary stonewalls between two regions.



i Draw a graph, where the five regions of the garden are represented as vertices and the edges of the graph represent the boundary stonewalls between areas.

ii What is the sum of the degrees of the vertices of this graph?

b Region 1 of of the garden contains 6 circular beds that are labelled A to F, as shown in the graph opposite. The owner wants to have a walk around region 1 visiting each bed on the way. The numbers on the edges joining the vertices give the shortest distance, in metres, between beds.

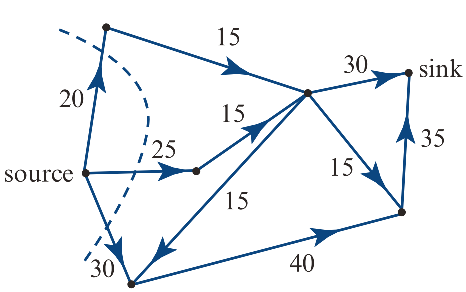


i Explain why the owner could not follow an Eulerian circuit through this network.

ii If the owner follows the shortest Hamiltonian path, name a garden bed at which the owner could start and a garden bed at which the owner could finish.

iii List an Eulerian trail for the graph.

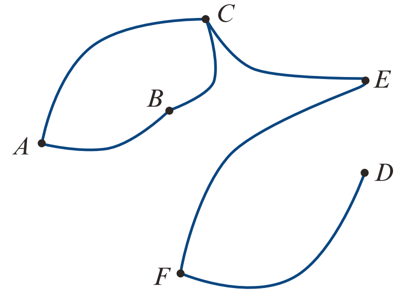
c All areas of the garden require a constant supply of water. The directed graph opposite shows an irrigation system for the garden with capacity of each section shown in litres per minute. The beginning of the system is labelled source and the end of the system labelled sink.



i What is the capacity of the marked cut in litres per minute?

ii Determine the maximum flow of water, in litres per minute, from the source to the sink.

15 **a** The Penvale swimming club has six new members A, B, C, D, E and F. The graph below shows the members who have competed together before joining the club. For example, the edge between A and B shows that they have previously competed together.



i How many of these swimmers had E competed with before joining the club?

ii Who had competed with both A and B before joining the club?

b The swimming club has a medley relay team. Three of the new club members, A, B and C can can complete the following sectors of the medley race: backstroke, butterfly and breastroke. The table below shows the average times in seconds for 100 m for these sectors for each of the three swimmers. The freestyle swimmer has been chosen and has much better freestyle times than the three new members. How should the swimmers be allocated to minimise the team's time?

Swimmer	Backstroke	Breaststroke	Butterfly
A	72	74	66
B	68	72	62
C	70	76	62

c The Penvale swimming club rooms are to undergo renovation. This project involves eight activities: A to I. The table opposite shows the earliest start time (EST) and duration, in months, for each activity. The immediate predecessor(s) is also shown. The duration for activity D is missing. The information in the table opposite can be used to complete an activity network. This network will require a dummy activity.

Activity	EST	Duration	Immediate predecessor(s)
A	0	3	–
B	0	6	–
C	6	2	A, B
D	8	. . .	C
E	8	9	C
F	6	3	B
G	15	4	D
H	9	9	F
I	19	2	E, G, H

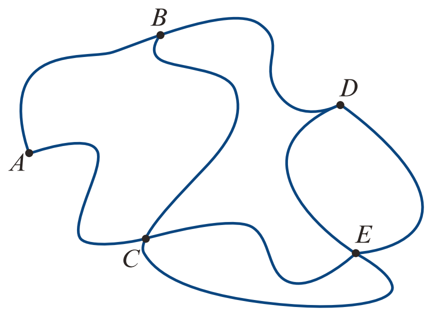
i What is the duration, in months, of activity D?

ii Draw the associated activity network for this renovation.

iii Name the four activities that have a float time.

iv The project is to be crashed by reducing the completion time of one activity only. What is the minimum time, in months, that the project can be completed in?

16 A family is visiting a theme park and will visit five rides. A map of the theme park is shown opposite with the vertices representing the rides and the edges representing the paths connecting the rides.

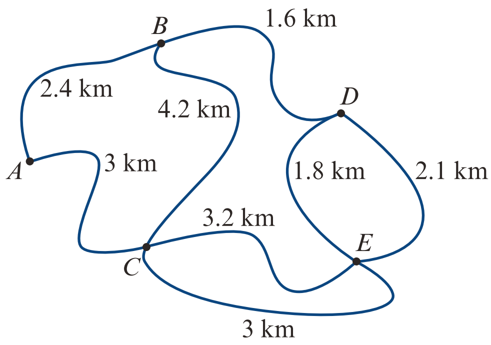


a Determine a Hamiltonian cycle beginning and ending at A that the family can follow.

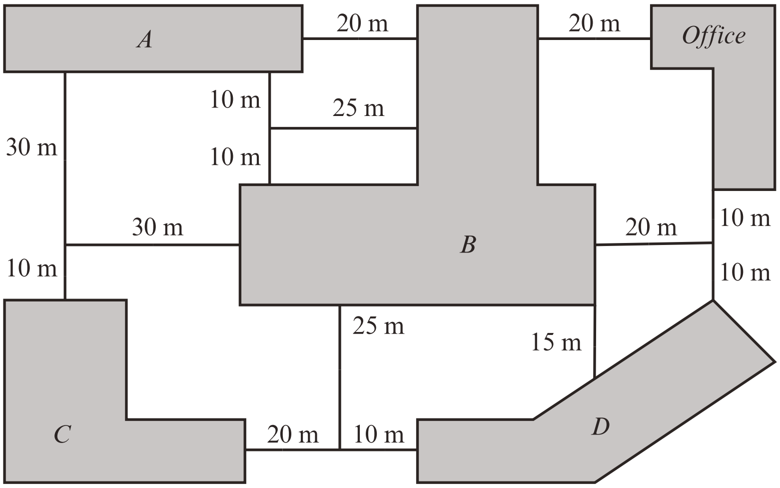
b Determine an Eulerian trail that the family can follow. Explain how you determined this trail.

The network opposite shows the lengths of the paths that join the rides. The family decides to visit the rides by following an Eulerian trail. Assume that the family can walk at a speed of 3 km/hr. The theme park will close at 5 p.m.

c What is the latest time that the family can enter the theme park?



17 The diagram below shows the buildings of a new university. The lines on the diagram show the location of the pathways between the buildings.



a **i** How many different ways can a student walk directly from building A to building B?

ii Represent this diagram as a weighted graph in planar form.

iii Which buildings are immediately adjacent to building C?

Some of the pathways will be covered to protect students from the rain as they move between buildings. The covering structure will cost $240 per metre to make and install.

b **i** Modify your planar weighted graph from question **aii** above to show only the shortest direct pathway between adjacent buildings.

ii How much will the covered walkways on these pathways cost to build?

c It has been decided that covering all of these walkways is too expensive. Only the minimum number of pathways that are necessary to allow students to walk from one building to any other while remaining under cover will be built.

i Draw a graph that shows the pathways that should be covered so that the overall cost of making and installing the covering structure is a minimum.

ii Calculate the cost of the covering structure in part **i**.

18 Anthony is creating a robot for a university project. The activities required to design and build the robot are shown in the table below, along with their duration in weeks and the immediate predecessor for each activity.

Activity	Description	Duration (weeks)	Immediate predecessors
A	Research robot design and control	5	-
B	Design the internal electronics	8	A
C	Design the remote control	3	A
D	Construct and assemble the robot	15	B
E	Write the code to control the robot	10	B
F	Construct and program the remote control	6	C
G	Debug the code to control the robot	4	E
H	Install the software	1	D, F
I	Test the robot	3	G, H

a Construct an activity network for this project.

b Determine the shortest time in which Anthony can expect to create his robot.

c **i** Construct a table that shows the EST, LST and float for each activity.

ii Write down the critical path of this project.

d Use the information in the table from **ci** to describe and explain what would happen if Anthony took:

i 3 weeks to research robot design and control instead of 5

ii 10 weeks to construct and program the remote control instead of 6

iii 20 weeks to construct and assemble the robot instead of 15

Glossary

68–95–99.7% rule: [p. 77] A rule for determining the percentage of values that lie within one, two and three deviations of the mean in a normal distribution.

A

Activity: [p. 683] A task to be completed as part of a project. Activities are represented by the edges in the project diagram.

Activity network: [p. 684] An **activity network** is a weighted directed graph that shows the required order of completion of the activities that make up a project. The weights indicate the durations of the activities they represent.

Adding to the principal: see **annuity investment**.

Adjacency matrix: [p. 619] A square matrix showing the number of edges joining each pair of vertices in a graph.

Algorithm: [p. 637] A step-by-step procedure for solving a particular problem that involves applying the same process repeatedly. Examples include Prim's algorithm and the Hungarian algorithm.

Allocation: [p. 74] **Allocation** is the process of assigning a series of tasks to different members of a group in a way that enables the tasks to be completed for the minimum time or cost.

Amortisation: [p. 407] **Amortisation** is the repayment of a loan or an investment with regular payments made over a period of time.

Amortisation table: [p. 408] An **amortisation table** charts the amortisation (repayment) of a reducing balance loan or annuity on a step-by-step (payment-by payment) basis or the payment of a compound interest investment with additional payments.

Annuity: [p. 402] An **annuity** is a compound interest investment from which regular payments are made.

B

Backward scanning: [p. 696] **Backward scanning** is the process of determining the LST for each activity in a project activity network.

Balance: [p. 342] The **balance** of a loan or investment is the amount owed or accrued after a period of time.

Bar chart: [p. 8] A statistical graph used to display the frequency distribution of categorical data.

Binary matrix: [p. 523] A matrix whose elements are either zero or one.

Bipartite graph (bigraph): [p. 674] A graph whose set of vertices can be split into two subsets, X and Y, in such a way that each edge of the graph joins a vertex in X and a vertex in Y.

Bivariate data: [p. 104] Data in which each observation involves recording information about two variables for the same person or thing. An example would be data recording the height and weight of the children in a preschool.

Boxplot: [p. 61] A graphical display of the five-number summary of a data set showing outliers if present. *See* **outliers**.

Bridge: [p. 611] A single edge in a connected graph that, if removed, leaves the graph disconnected.

C

Capacities (flow network) [p. 660] The weights of the directed edges in a flow network are called **capacities**. They give the maximum amount that can move between the two points in the flow network represented by these vertices in a particular time interval. This could be, for example, the maximum amount of water in litres per minute or the maximum number of cars per hour.

Categorical variable: [p. 2] Categorical variables are used to represent characteristics of individuals, for example place of birth, house number. Categorical variables come in types, nominal and ordinal.

Centre of a distribution: [p. 22] A measure of location of a distribution. Measures of centre include the median and the mean.

Centring: [p. 270] If smoothing takes place over an even number of data values, the smoothed values do not align with an original data value. A second stage of smoothing is carried out to centre the smoothed values at an original data value.

Circle of transformations: [p. 239] Provides guidance in choosing the transformations that can be used to linearise various forms of scatterplots.

Circuit: [p. 625] A **walk** with no repeated edges that starts and ends at the same vertex. *See also* **cycle (graphs)**.

Coefficient of determination (r^2): [p. 146] A coefficient which gives a measure of the predictive power of a regression line. It gives the percentage of variation in the RV that can be explained by the variation in the EV.

Column matrix: [p. 474] A matrix with only one column.

Column vector: [p. 474] *see* **column matrix**.

Communication matrix [p. 525] A square binary matrix in which the 1s represent direct (one-step) communication links.

Complete graph: [p. 611] A graph with edges connecting all pairs of vertices.

Compound interest: [p. 360] Where the interest paid on a loan or investment is added to the principal and subsequent interest is calculated on the total.

Compound interest investments with additional payments: [p. 393] A **compound interest investments with periodic payments** is an investment to which additions are made to the principal on a regular basis. Also known as 'adding to the principal'.

Compounding period: [p. 393] The **compounding period** is the time period for the calculation of interest for an investment or loan. Typical compounding periods are yearly, quarterly, monthly or daily.

Connected graph: [p. 611] A graph in which there is a path between every pair of vertices.

Continuous variable: [p. 3] A numerical variable that represents a quantity that is measured rather than counted, for example the weights of people in kilograms.

Correlation coefficient r: [p. 141] A statistical measure of the strength of the linear association between two numerical variables.

Cost matrix: [p. 676] A **cost matrix** is a table that contains the cost of allocating objects from one group, such as people, to objects from another group, such as tasks. The cost can be money, or other factors such as the time taken to complete the project.

Crashing: [p. 708] **Crashing** is the process of shortening the length of time taken to complete a project by reducing the time required to complete individual activities.

Critical path: [p. 698] The project path that has the longest completion time.

Critical path analysis: [p. 698] A project planning method in which activity durations are known with certainty.

Cut: [p. 663] A line dividing a directed (flow) graph into two parts in a way that separates all 'sinks' from their 'sources'.

Cut capacity: [p. 663] The **capacity of a cut** is the sum of the capacities of the cuts passing through the cut that represents flow from the source to the sink. Edges that represent flow from the sink to the source do not contribute to the capacity of the cut.

Cycle (graphs): [p. 625] A **walk** with no repeated vertices that starts and ends at the same vertex. *See also* **circuit**.

Cycle (time series): [p. 258] Periodic movement in a time series but over a period greater than a year.

D

Data transformation: [p. 211] Using a mathematical rule to change the scale on either the x- or y-axis in order to linearise a non-linear scatterplot.

Degenerate graph: [p. 610] A graph in which no vertex is connected to any other vertex. All the vertices are isolated.

Degree of a vertex (deg(A)): [p. 609] The number of edges attached to the vertex. The degree of vertex A is written as deg(A).

Depreciation: [p. 343] The reduction in value of an item over time.

Deseasonalise: [p. 285] The process of removing seasonality in time series data.

Determinant: [p. 514] A number associated with square matrices. The determinant of a matrix A, written det(A), is used to decide if the matrix has an inverse. If det(A) = 0, the matrix has no inverse; it is singular.

Dijkstra's algorithm: [p. 637] An algorithm for finding the shortest path between two vertices in a weighted graph. Pronounced 'Di-stra': 'Di' as in 'die' and 'stra' as in 'car'.

Directed graph (digraph): [p. 660] A graph or network in which directions are associated with each of the edges.

Discrete variable: [p. 3] A **numerical variable** that represents a quantity that is determined by counting; for example, the number of people waiting in a queue is a discrete variable.

Dominance matrix: [p. 531] A square binary matrix in which the 1s represent one-step dominances between the members of a graph.

Dot plot: [p. 28] A statistical graph that uses dots to display individual data values on a number line; suitable for small sets of data only.

Dummy activity: [p. 687] An artificial activity of zero time duration added to a project diagram to ensure that all predecessor activities are properly accounted for.

E

Earliest starting time (EST): [p. 695] The earliest time an activity in a project can be started.

Edge: [p. 609] A line joining one vertex in a graph or network to another vertex or itself (a loop).

Effective interest rate: [p. 375] Used to compare the interest paid on loans (or investments) with the same annual nominal interest rate r but with different compounding periods (daily, monthly, quarterly, annually, other).

Elements: [p. 474] The numbers or symbols displayed in a matrix.

Equal matrices: [p. 491] Matrices that have the same order and identical elements in identical positions.

Equivalent graph: [p. 613] *see* **isomorphic graphs**.

Eulerian circuit: [p. 627] An Eulerian walk that starts and finishes at the same vertex. To have an Eulerian circuit, a network must be connected and all vertices must be of even degree.

Eulerian trail: [p. 627] A walk in a graph or network that includes every edge just once (but does not start and finish at the same vertex).

To have an Eulerian walk (but not an eulerian circuit), a network must be connected and have exactly two vertices of odd degree, with the remaining vertices having even degree.

Euler's formula: [p. 614] The formula $v - e + f = 2$, which relates the number of vertices, edges and faces in a connected graph.

Explanatory variable: [p. 105] When investigating associations in **bivariate data**, the explanatory variable (EV) is the variable used to explain or predict the value of the **response variable** (RV).

Extrapolation: [p. 182] Using a mathematical model to make a prediction *outside* the range of data used to construct the model.

F

Face: [p. 614] An area in a graph or network that can only be reached by crossing an edge. One such area is always the area surrounding a graph.

Finance Solver: [p. 422] A **finance solver** is a computer/calculator application that automates the computations associated with analysing a reducing balance loan, an annuity or an annuity investment.

Five-number summary [p. 61] A list of the five key points in a data distribution: the minimum value (min), the first quartile (Q_1), the median (M), the third quartile (Q_3) and the maximum value (max).

Flat-rate depreciation: [p. 344] Depreciation where the value of an item is reduced by the same amount each year. Flat-rate depreciation is equivalent, but opposite, to simple interest.

Float (slack) time: [p. 694] The amount of time available to complete a particular activity that does not increase the total time taken to complete the project.

Flow: [p. 660] **Flow** is the movement of something from a source to a sink.

Forward scanning: [p. 695] **Forward scanning** is the process of determining the EST for each activity in a project activity network.

Frequency table: [p. 7] A listing of the values a variable takes in a data set along with how often (frequently) each value occurs. Frequency can be recorded as a count or as a percentage.

G

Geometric decay [p. 359] When a recurrence rule involves multiplying by a factor less than one, the terms in the resulting sequence are said to decay geometrically.

Geometric growth [p. 359] When a recurrence rule involves multiplying by a factor greater than one, the terms in the resulting sequence are said to grow geometrically.

Graph: [p. 609] A collection of points called vertices and a set of connecting lines called edges.

H

Hamiltonian cycle: [p. 627] A Hamiltonian path that starts and finishes at the same vertex.

Hamiltonian path: [p. 627] A path through a graph or network that passes through each vertex exactly once. It may or may not start and finish at the same vertex.

Histogram: [p. 15] A statistical graph used to display the frequency distribution of a numerical variable; most suitable for medium to large sized data sets.

Hungarian algorithm: [p. 676] An algorithm for solving allocation (assignment) problems.

I

Identity matrix (I): [p. 477] A matrix that behaves like the number one in arithmetic. Any matrix multiplied by an identity matrix remains unchanged. An identity matrix is represented by the symbol I.

Immediate predecessor: [p. 683] An activity that must be completed immediately before another one can start.

Initial state matrix: [p. 560] A column matrix used to represent the starting state of a dynamic system.

Interest: [p. 342] The amount of money paid (earned) for borrowing (lending) money over a period of time.

Interest-only loans: [p. 446] A loan on which only the interest is paid. At the end of the loan, the principal must be repaid in full.

Interest rate: [p. 342] The rate at which interest is charged or paid. Usually expressed as a percentage of the money owed or lent.

Interpolation: [p. 182] Using a regression line to make a prediction *within* the range of values of the explanatory variable.

Interquartile range (IQR): [p. 47] The interquartile range is defined as $IQR = Q_3 - Q_1$. IQR gives the spread of the middle 50% of data values.

Inverse matrix: [p. 513] A matrix which, when multiplied by the original matrix, gives the identity matrix (I). For a matrix A, the inverse is written as A^{-1} and has the property that $A^{-1}A = AA^{-1} = I$.

Irregular (random) fluctuations [p. 262] Unpredictable fluctuations in a time series. Always present in any real world time series plot.

Isolated vertex: [p. 610] A vertex that is not connected to any other vertex. Its degree is zero.

Isomorphic graphs: [p. 613] Equivalent graphs. Graphs that have the same number of edges and vertices that are identically connected.

Iteration [p. 336] Each application of a recurrence rule to calculate a new term in a sequence is called an iteration.

L

Latest start time (LST): [p. 696] The latest time an activity in a project can begin, without affecting the overall completion time for the project.

Least squares method: [p. 170] One way of finding the equation of a regression line. It minimises the sum of the squares of the residuals. It works best when there are no outliers.

Linear decay [p. 340] When a recurrence rule involves subtracting a fixed amount, the terms in the resulting sequence are said to decay linearly.

Linear growth [p. 340] When a recurrence rule involves adding a fixed amount, the terms in the resulting sequence are said to grow linearly.

Linear regression: [p. 169] The process of fitting a straight line to bivariate data.

Log scale [p. 34] A scale used to transform a strongly skewed histogram to symmetry or linearise a scatterplot.

Logarithmic transformations (log *x* or log *y*): [p. 221] Transformations that linearise a scatterplot by compressing the upper end of the scale on an axis.

Loop: [p. 610] An edge in a graph or network that joins a vertex to itself.

Lower fence: [p. 63] *See* **outliers**.

M

Matrix: [p. 473] A rectangular array of numbers or symbols set out in rows and columns within square brackets (plural: matrices).

Matrix multiplication: [p. 499] The process of multiplying a matrix by a matrix.

Maximum flow (graph): [p. 661] The capacity of the 'minimum' cut.

Maximum or minimum value of the objective function: [p. 759] The value found by evaluating the objective function's value at the vertices or along the boundaries of the **feasible region**.

Mean ($\bar{x}$): [p. 50] The balance point of a data distribution. The mean is given by $\bar{x} = \dfrac{\Sigma x}{n}$, where Σx is the sum of the data values and n is the number of data values. Best used for symmetric distributions.

Median: [p. 44] The median (M) is the middle value in a data distribution. It is the midpoint of a distribution dividing an ordered data set into two equal parts. Can be used for skewed or symmetric distributions.

Minimum cut (graph): [p. 664] The cut through a graph or network with the minimum capacity.

Minimum spanning tree: [p. 642] The spanning tree of minimum length. For a given connected graph, there may be more than one minimum spanning tree.

Modal category or modal interval: [p. 9] The category or data interval that occurs most frequently in a data set.

Mode: [p. 9] The most frequently occurring value in a data set. There may be more than one.

Modelling: [ch. 7] Mathematical **modelling** is the use of a mathematical rule or formula to represent real-life situations.

Moving mean smoothing: [p. 268] In three-moving mean smoothing, each original data value is replaced by the mean of itself and the value on either side. In five-moving mean smoothing, each original data value is replaced by the mean of itself and the two values on either side.

Moving median smoothing: [p. 277] **Moving median smoothing** is a graphical technique for smoothing a time series plot using moving medians rather than moving means.

Multiple edges: [p. 610] Where more than one edge connects the same two vertices in a graph.

N

Negatively skewed distribution: [p. 21] A data distribution with a long tail to the left.

Network: [p. 633] A weighted graph in which the weights are physical quantities, for example distance, time or cost.

Nominal interest rate: [p. 373] The annual interest rate for a loan or investment that assumes the compounding period is 1 year. If the compounding period is less than a year, for example monthly, the actual or **effective interest rate** will be greater than r.

Nominal variable: [p. 3] A categorical variable that generates data values that can be used by name only – for example, eye colour: blue, green, brown.

Normal distribution: [p. 76] A data distribution that has a bell shape. For normal distributions, the 68–95–99.7% rule can be used to relate the mean and standard deviation to percentages in the distribution.

Numerical variable: [p. 3] A variable used to represent quantities that are counted or measured. For example, the number of people in a queue, the heights of these people in cm. Numerical variables come in types: **discrete** and **continuous**.

O

Order: [p. 474] Used to indicate the size and shape of a matrix. For a matrix with n rows and m columns, the order of a matrix is written as $(n \times m)$.

Ordinal variable: [p. 3] **A categorical variable** that generates data values that can be used to both name and order, for example house number.

Outliers: [p. 23] Data values that appear to stand out from the main body of a data set. In a boxplot possible outliers are defined as data values greater than the upper fence $(Q_3 + 1.5 \times IQR)$ or less than the lower fence $(Q_1 - 1.5 \times IQR)$.

P

Parallel box plots: [p. 125] A statistical graph in which two or more box plots are drawn side-by-side so that the distributions can be compared.

Path [p. 625] A **walk** with no repeated vertices. *See also* **trail**.

Percentage frequency: [p. 7] Frequency expressed as a percentage.

Permutation matrix: [p. 523] A square binary matrix in which there is only a single one in each row and column.

Perpetuity: [p. 452] An investment where an equal amount is paid out on a regular basis forever.

Planar graph: [p. 613] A graph that can be drawn in such a way that no two edges intersect, except at the vertices.

Positively skewed distribution: [p. 21] A data distribution that has a long tail to the right.

Power of a matrix: [p. 507] Defined in the same way as the powers of numbers: $A^2 = A \times AA^3 = A \times A \times A$, etc. Only square matrices can be raised to a power. A^0 is defined to be I, the identity matrix.

Precedence table: [p. 684] A table that records the activities of a project, their immediate predecessors and often the duration of each activity.

Prim's algorithm: [p. 642] An algorithm for determining a minimum spanning tree in a connected graph.

Principal (P): [pp. 342] The initial amount borrowed, lent or invested.

Q

Quartiles (Q_1, Q_2, Q_3): [p. 47] Summary statistics that divide an ordered data set into four equal sized groups.

R

Range (R): [p. 46] The difference between the smallest and the largest observations in a data set; a measure of spread.

Reciprocal transformations ($1/x$ or $1/y$): [p. 230] Transformations that linearise a scatterplot by compressing the upper end of the scale on an axis to a greater extent than the log transformation.

Recurrence relation: [p. 336] A relation that enables the value of the next term in a sequence to be obtained by one or more current terms. Examples include 'to find the next term, add two to the current term' and 'to find the next term, multiply the current term by three and subtract five'.

Reducing-balance depreciation: [p. 361] When the value of an item is reduced by the same percentage each year. Reducing-balance depreciation is equivalent to, but opposite to, compound interest.

Reducing-balance loan: [p. 400] A loan that attracts compound interest, but where regular repayments are also made. In most instances the repayments are calculated so that the amount of the loan and the interest are eventually repaid in full.

Redundant communication link: [p. 527] A communication link is said to be **redundant** if the sender and the receiver are the same people.

Reseasonalise [p. 285] The process of converting seasonal data back into its original form.

Residual: [p. 170] The vertical distance from a data point to a straight line fitted to a scatterplot is called a residual:

residual = actual value − predicted value

Residuals are sometimes called *errors of prediction*.

Residual plot: [p. 184] A plot of the residuals against the explanatory variable. Residual plots can be used to investigate the linearity assumption.

Response variable [p. 105] The variable of primary interest in a statistical investigation.

Round-robin tournament: [p. 531] A tournament in which each participant plays each other participant once.

Row matrix: [p. 474] A matrix with only one row. A row matrix is also called a row vector.

Row vector: [p. 474] *See* **row matrix**.

S

Scalar multiplication: [p. 492] The multiplication of a matrix by a number.

Scatterplot: [p. 130] A statistical graph used for displaying bivariate data. Data pairs are represented by points on a coordinate plane, the EV is plotted on the horizontal axis and the RV is plotted on the vertical axis.

Scrap value: [p. 344] The value at which an item is no longer of use to a business.

Seasonal indices: [p. 284] Indices calculated when the data shows seasonal variation. Seasonal indices quantify seasonal variation. A seasonal index is defined by the formula:

$$\text{seasonal index} = \frac{\text{value for season}}{\text{seasonal average}}$$

For seasonal indices, the average is 1 (or 100%).

Seasonality: [p. 259] The tendency for values in the time series to follow a seasonal pattern, increasing or decreasing predictably according to time periods such as time of day, day of the week, month, or quarter.

Segmented bar chart: [p. 9] A statistical graph used to display the information contained in a two-way frequency table. It is a useful tool for identifying associations between two categorical variables.

Sequence: [p. 334] A list of numbers or symbols written down in succession, for example 5, 15, 25, . . .

Shape of a distribution: [p. 21] The general form of a data distribution described as symmetric, positively skewed or negatively skewed.

Shortest path: [p. 634] The path through a graph or network with minimum length.

Simple graph: [p. 610] A graph with no loops or multiple edges.

Simple interest: [p. 342] Interest that is calculated for an agreed period and paid only on the original amount invested or borrowed.

Singular matrix: [p. 517] A matrix that does not have an inverse; its determinant is zero.

Sink: *See* **sink and source**.

Sink and source: [p. 660] In a flow network, a **source** generates flow while a **sink** absorbs the flow.

Slope (of a straight line): [p. 169] The slope of a straight line is defined to be: slope $= \frac{\text{rise}}{\text{run}}$. The slope is also known as the **gradient**.

Smoothing: [p. 268] A technique used to eliminate some of the variation in a time series plot so that features such as seasonality or trend are more easily identified.

Source *See* **sink and source**.

Spanning tree: [p. 641] A subgraph of a connected graph that contains all the vertices of the original graph, but without any multiple edges, circuits or loops.

Spread of a distribution: [p. 46 & p. 52] A measure of the degree to which data values are clustered around some central point in the distribution. Measures of spread include the standard deviation (s), the interquartile range (IQR) and the range (R).

Square matrix: [p. 476] A matrix with the same number of rows as columns.

Squared transformations (x^2 or y^2): [p. 212] Transformations that linearise a scatterplot by stretching out the upper end of the scale on an axis.

Standard deviation (s): [p. 52] A summary statistic that measures the spread of the data values around the mean. The standard deviation is given by $s = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}}$.

Standardised (z) scores: [p. 80] The value of the standard score gives the distance and direction of a data value from the mean in terms of standard deviations.
The rule for calculating a standardised score is:

$$\text{standardised score} = \frac{\text{actual score} - \text{mean}}{\text{standard deviation}}$$

State matrix: [p. 559] A column matrix that represents the starting state of a dynamic system.

Statistical question: [p. 10] A question that depends on data for its answer.

Steady-state matrix: [p. 565] A column matrix that represents the final state of a dynamic system. Also called the equilibrium state.

Stem plot (stem-and-leaf plot): [p. 29] A method for displaying data in which each observation is split into two parts, a 'stem' and a 'leaf'. A stem plot is an alternative display to a histogram, suitable for small to medium sized data sets. When data values are tightly clustered, stems can be split to give finer detail.

Strength of a linear relationship: [p. 143] Classified as weak, moderate or strong. Determined by observing the degree of scatter in a scatterplot or calculating a correlation coefficient.

Structural change (time series) [p. 260] A sudden change in the established pattern of a time series plot.

Subgraph: [p. 611] Part of a graph that is also a graph in its own right.

Summary statistics [p. 43] **Statistics** that give numerical values to special features of a data distribution, such as centre and spread. Summary statistics include the mean, median, range, standard deviation and IQR.

Symmetric distribution: [p. 21] A data distribution in which the data values are evenly spread out around the mean. In a symmetric distribution, the mean and the median are the same.

T

Time series data: [p. 253] A collection of data values along with the times (in order) at which they were recorded.

Time series plot: [p. 253] A line graph where the values of the response variable are plotted in time order.

Trail [p. 625] A walk with no repeated edges. *See also* **path**.

Transition matrix (T): [p. 551] A square matrix that describes the transitions made between the states of a system.

Transpose: [p. 475] The transpose of a matrix is obtained by interchanging its rows and columns.

Tree: [p. 641] A connected graph with no circuits, multiple edges or loops.

Trend: [p. 257] The tendency for values in the time series to generally increase or decrease over a significant period of time.

Trend line forecasting: [p. 296] Using a line fitted to an increasing or decreasing time series to predict future values.

Triangular matrix: [p. 477] An **upper triangular matrix** is a square matrix in which all elements below the leading diagonal are zeros. A **lower triangular matrix** is a square matrix in which all elements above the leading diagonal are zeros.

Two-way frequency table: [p. 110] A frequency table in which subjects are classified according to two categorical variables. Two-way frequency tables are commonly used to investigate the associations between two categorical variables.

U

Unit-cost depreciation: [p. 345] Depreciation based on how many units have been produced or consumed by the object being depreciated. For example, a machine filling bottles of drink may be depreciated by 0.001 cents per bottle it fills.

Univariate data: [p. 1] Data associated with a single variable.

Upper fence: [p. 63] *See* **outliers**.

V

Vertex (graph): [p. 609] The points in a graph or network (*pl* vertices).

W

Walk [p. 624] Any continuous sequence of edges, linking successive vertices, that connects two different vertices in a graph. *See also* **trail** and **path**.

Weighted graph: [p. 633] A graph in which a number representing the size of some quantity is associated with each edge. These numbers are called weights.

Z

Zero matrix (O): [p. 493] A matrix that behaves like zero in arithmetic. Represented by the symbol O. Any matrix with zeros in every position is a zero matrix.

Answers

Chapter 1

Exercise 1A

1 **a** numerical **b** numerical **c** categorical
d categorical **e** numerical
f numerical **g** categorical
h categorical

2 **a** nominal **b** nominal **c** ordinal
d ordinal **e** ordinal **f** nominal

3 **a** discrete **b** discrete **c** continuous
d continuous **e** discrete
f continuous

4 B **5** D **6** B

Exercise 1B

1 **a**

	Frequency	
Grades	Count	%
A	3	27.3
B	5	45.5
C	3	27.3
Total	11	100.1

b

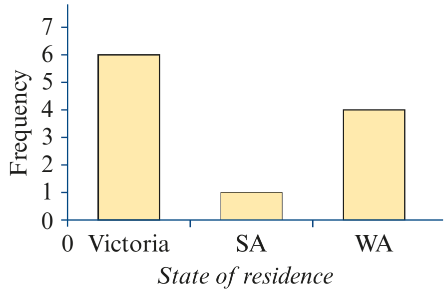
	Frequency	
Shoe size	Count	%
8	6	50.0
9	3	25.0
10	2	16.7
11	0	0
12	1	8.3
Total	12	100.0

2 **a** categorical

b

	Frequency	
State of residence	Count	%
Victoria	6	54.5
SA	1	9.1
WA	4	36.4
Total	11	100.0

c

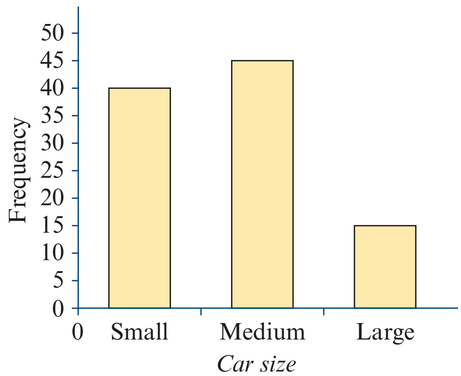


3 **a** categorical

b

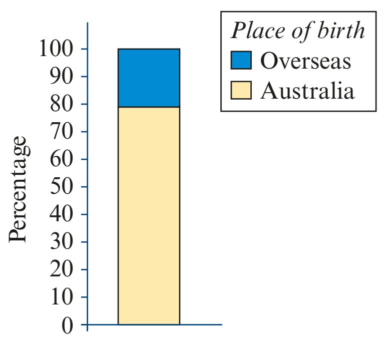
Car size	Frequency	
	Count	%
Small	8	40
Medium	9	45
Large	3	15
Total	20	100

c



4 **a** nominal

b

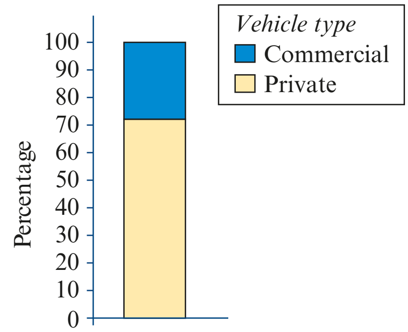


5 **a** nominal

b

Type of vehicle	Frequency	
	Count	%
Private	132 736	73
Commercial	49 109	27
Total	181 845	100

c



6 **a** 20, 55 **b** 5 **c** 20 **d** 55%

e Report: 20 schools were classified according to school type. The majority of these schools, 55%, were found to be government schools. Of the remaining schools, 25% were independent while 20% were Catholic schools.

7 **a** 7, 45.5, 100.0

b Report: When 22 students were asked the question, ‘How often do you play sport’, the most frequent response was ‘sometimes’, given by 45.5% of the students. Of the remaining students, 31.8% of the students responded that they played sport ‘rarely’ while 22.7% said that they played sport ‘regularly’.

8 Report: The eye colours of 11 children were recorded. The majority, 54.5%, had brown eyes. Of the remaining children, 27.3% had blue eyes and 18.2% had hazel eyes.

9 B

Exercise 1C

1 a

Number	Frequency Count	%
0	6	30
1	4	20
2	3	15
3	3	15
4	2	10
5	2	10
Total	20	100

b 20%

c 0

2 a

Number	Frequency Count	%
2	1	2.5
3	0	0
4	17	42.5
5	13	32.5
6	9	22.5
Total	40	100.0

b 2.5%

c 4

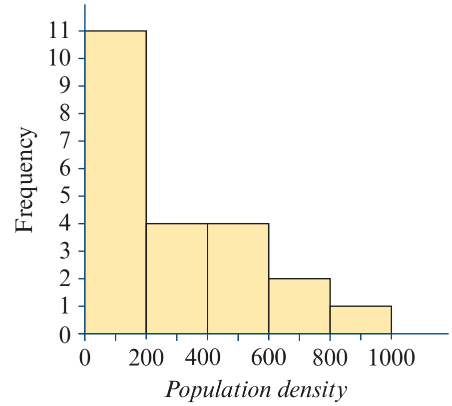
3 a

Height (cm)	Frequency
160–164	5
165–169	5
170–174	5
175–179	6
180–184	3
185–189	1
Total	25

b 175–179

c 16%

4



5 a **i** 17% **ii** 13% **iii** 46% **iv** 33%

b **i** 6 **ii** 4

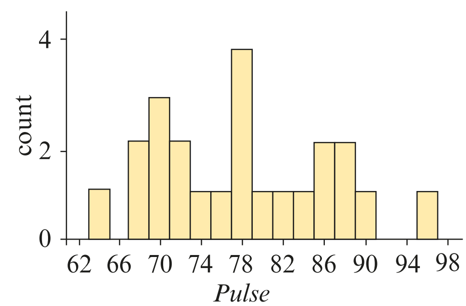
c 15–19 words/sentence

6 a 21

b **i** 13 **ii** 8 **iii** 5 **iv** 0

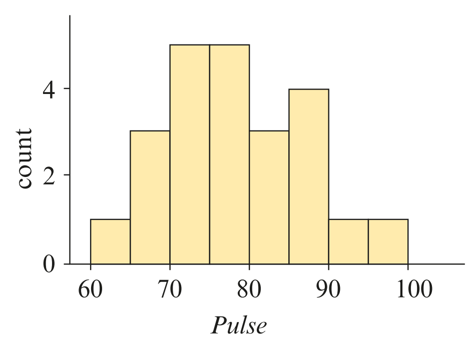
c **i** 4.8% **ii** 57.1%

7 a



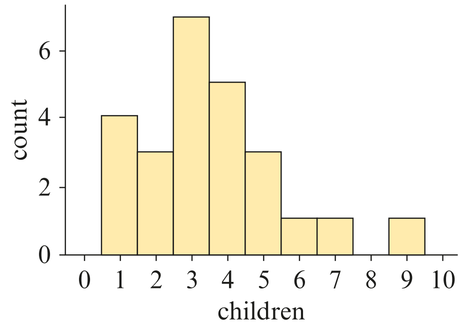
b **i** 69 **ii** 3; 69, 70, 70

c



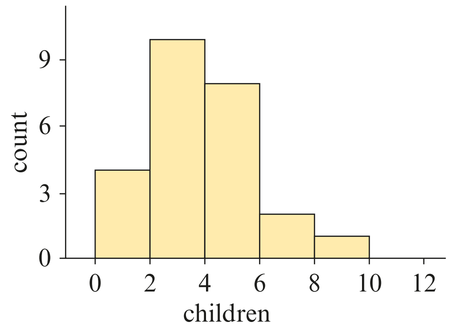
d 3

8 a



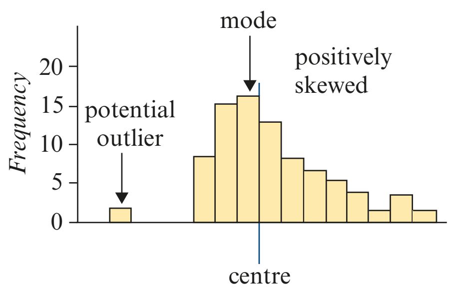
b 3.5, 5

c



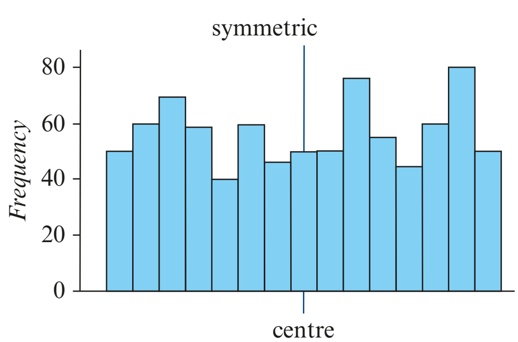
d **i** 2 **ii** 6 and 7

9 a



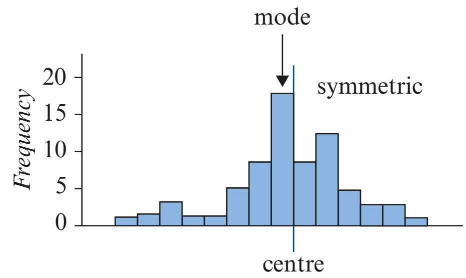
Histogram A

b



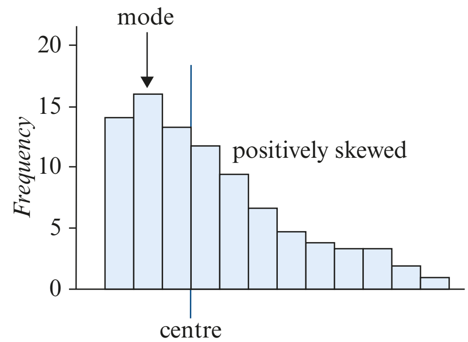
Histogram B

c



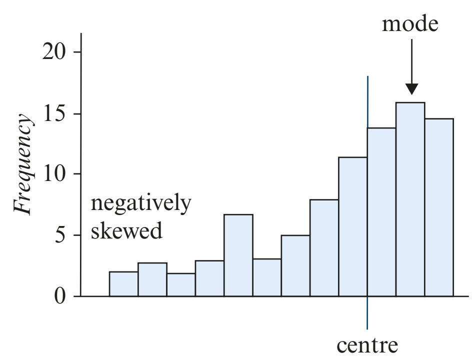
Histogram C

d



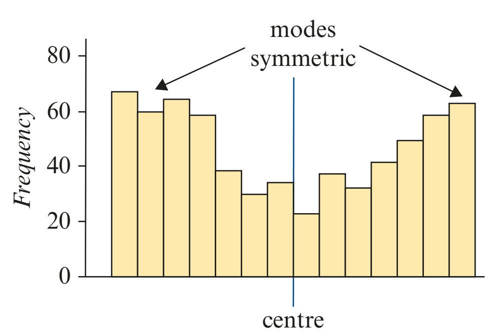
Histogram D

e



Histogram E

f



Histogram F

10 a All of the distributions are approximately symmetric about their respective centres.

b There are no clear outliers in any of the distributions.

c In A the centre lies in the interval 8-10, in B it lies in the interval 24-26, and in C it lies in the interval 40-42.

d The spread is the lowest in B, since the range is only 8, compared to 14 for A, and 18 for C.

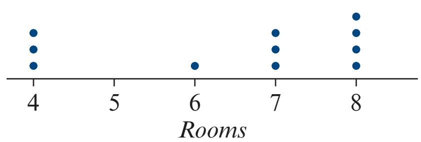
11 B

12 A

Exercise 1D

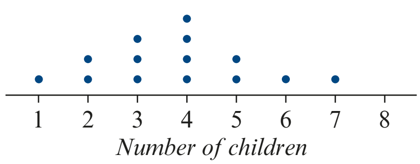
1 a discrete

b



2 a discrete

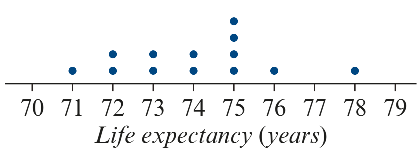
b



c 4; mode is the most frequently occurring number of children for these families

3 a continuous

b



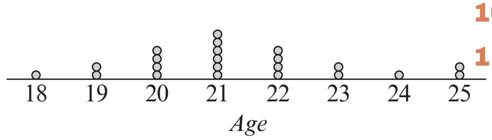
c 75; mode is the most common life expectancy for these countries

4 a negatively skewed

b positively skewed

5

a



b 21 years

c approximately symmetric **d** 14%

6 a continuous

b

Key: 1|6 = 16

0	3 3 6 9 9
1	2 2 6 7
2	0 2 2 5 7 8 9
3	1 5
4	
5	4 6
6	
7	
8	
9	9 9
10	0

7 a continuous

b i

Key:16|5 = 16.5

16	579
17	0 1 2 3 6 6 7
18	2 4 5
19	3 9

ii

Key:16|5 = 16.5

16	5 7 9
17	0 1 2 3
17	6 6 7
18	2 4
18	5
19	3
19	9

8 a positively skewed

b negatively skewed

9 a 40 **b** approx symmetric

c 11

10 C

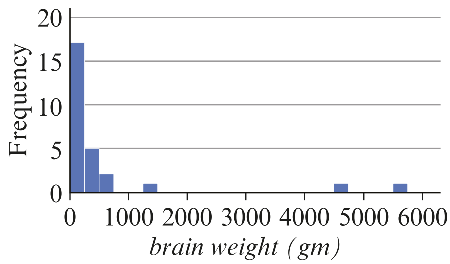
11 B

Exercise 1E

1 **a** 0.4 **b** 1.4 **c** 2.4 **d** 3.4
e −0.3 **f** −1.3 **g** −2.3 **h** −3.3

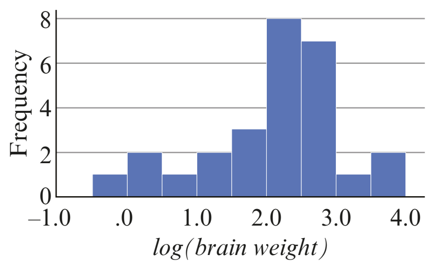
2 **a** 0.0032 **b** 0.032 **c** 0.32 **d** 1.0

3 **a**



The shape is positively skewed with outliers.

b



The shape is slightly negatively skewed but closer to symmetric.

4 **a** −0.4 **b** 3.8 **c** 100 g **d** 0.1 g
e **i** 5 **ii** 12 **iii** 24

5 B **6** D

Exercise 1F

1 **a** 5 **b** 12

2 $850

3 M=1

4 **a** M=7.3 **b** R=6.4

5 **a** $M = 2$ **b** $Q_1 = 1,\ Q_3 = 3$ **c** $IQR = 2$
d $R = 7$

6 **a** $M = 11$ **b** $Q_1 = 10,\ Q_3 = 15$
c $IQR = 5,\ R = 18$

7 **a** approximately symmetric with no outliers.
b $M = 26$
c $Q_1 = 17.5, Q_3 = 30.5$
d $IQR = 13,\ R = 29$

8 **a** positively skewed with a possible outlier at 6.
b $M = 0$
c $IQR = 1$ **d** $R = 6$

9 **a** $M = 21$ **b** $Q_1 = 10.5,\ Q_3 = 28$
c $IQR = 17.5,\ R = 54$

10 Median from 65 to less than 70, Q_1 from 60 to less than 65, Q_3 from 75 to less than 80.

11 **a** Median in the interval 5.0-9.9.
b max $IQR = 19.9$

12 **a** $n = 4,\ \Sigma x = 12,\ \bar{x} = 3$
b $n = 5,\ \Sigma x = 104,\ \bar{x} = 20.8$
c $n = 7,\ \Sigma x = 21,\ \bar{x} = 3$

13 **a** $\bar{x} = 3,\ M = 3,$ Mode = 2
b $\bar{x} = 5,\ M = 5,$ Mode = 5

14 **a** **i** mean = 36.1 **ii** median = 36.0
b The mean and median almost co-incide because the distribution is approximately symmetric.

15 **a** **i** mean = \$3.65 **ii** median = \$1.70
b The median. The mean is inflated because of the one large sale and not representative of the sales in general.

16 **a** strongly positively skewed distribution
b positively skewed distribution with outliers

17 **a** symmetric; either
b mean = 82.55 median= 82.5

18 **a** IQR **b** range **c** standard deviation

19 7.1, 0 **20** b, d, f

21 **a** 20.1, 1.8 **b** symmetric

22

	TVs	*Cars*	*Alcohol*
mean	450	376	14.9
SD	100	107	5.1

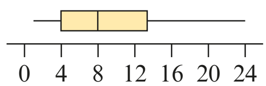
23 C **24** C **25** C **26** B

Exercise 1G

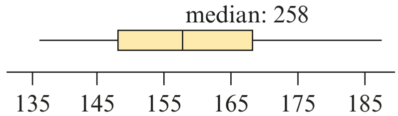
1 4, 5, 6, 7, 9

2 136, 148, 158, 169, 189

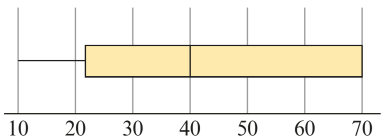
3 **a**



b



4 **a**

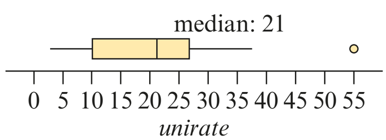


b Q_3 and the maximum are equal.

5 **b** Upper fence = 26.5 + 1.5 × 26 = 50.5

c 55 is larger than the upper fence.

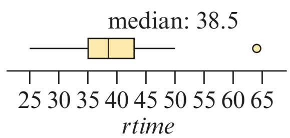
d



6 **a** lower fence = 7, upper fence = 95.

b 6 and 99

7 **a**



b 25, 35, 38.5, 43, 64; 64 is a possible outlier.

8 **a** **i** 10 **ii** 5, 21 **iii** 16 **iv** 0, 45 **v** none

b **i** 27 **ii** 12, 42 **iii** 30 **iv** 5, 50 **v** none

c **i** 38 **ii** 32, 42 **iii** 10 **iv** 5, 50 **v** 5

d **i** 16 **ii** 14, 21 **iii** 7 **iv** 1.5, 50 **v** 1.5, 3, 36, 40, 50

e **i** 34 **ii** 30, 39 **iii** 9 **iv** 30, 55 **v** 55

9 **a** **i** 55 **ii** 15

b **i** 100 **ii** 20

10 **a** 30;

b no, 31 > 30, so inside the lower fence

11 **a** 25% **b** 75% **c** 25% **d** 50% **e** 75%

12 **a** 25% **b** 25% **c** 50% **d** 25% **e** 75% **f** 50%

13 Boxplot 1 matches histogram B, Boxplot 2 matches histogram D, Boxplot 3 matches histogram C, Boxplot 4 matches histogram A.

14 **a** The distribution is negatively skewed with no outliers. The distribution is centred at about 42, the median value. The spread of the distribution, as measured by the IQR, is 15 and, as measured by the range, 47.

b The distribution is postively skewed with no outliers. The distribution is centred at 800, the median value. The spread of the distribution, as measured by the IQR, is 1200 and, as measured by the range, 3200.

15 **a** The distribution is negatively skewed with an outlier. The distribution is centred at 39, the median value. The spread of the distribution, as measured by the *IQR*, is 10 and, as measured by the range, 45. There is an outlier at 5.

b The distribution is positively skewed with outliers. The distribution is centred at 16, the median value. The spread of the distribution, as measured by the *IQR*, is 6 and, as measured by the range, 35. The outliers are at 5, 8, 36 and 40.

c The distribution is approximately symmetric but with outliers. The distribution is centred at 41, the median value. The spread of the distribution, as measured by the *IQR*, is 7 and, as measured by the range, 36. The outliers are at 10, 15, 20 and 25.

16 The median time it takes Taj to travel to university is 70 minutes. The range is of the distribution of travel time is 60 minutes, but the interquartile range is only 15 minutes. The distribution of travel times is positively skewed with two outliers, unusually long travel times of 110 minutes and 120 minute respectively.

17 B **18** A **19** B **20** D

Exercise 1H

1 **a** 114 and 154 **b** 94 and 174 **c** 74 and 194 **d** 154 **e** 94 **f** 74 **g** 134

2 **a** 68% **b** 99.7% **c** 16% **d** 2.5% **e** 0.15% **f** 50%

3 **a** **i** 84% **ii** 50% **iii** 47.5% **b** 25

4 **a** **i** 99.7% **ii** 2.5% **iii** 81.5% **b** 800

5 **a** **i** 50% **ii** 34% **iii** 81.5% **b** 1994

6 **a** $z = 1$ **b** $z = 2$ **c** $z = -1$ **d** $z = 0$ **e** $z = -3$ **f** $z = 0.5$

7 **a** 1.0 **b** −1.0 **c** 1.4 **d** −1.4

8

Subject	*z-score*	*Rating*
English	2.25	Top 2.5%
Biology	3	Top 0.15%
Chemistry	0	Exactly average
Further Maths	1.1	Top 16%
Psychology	−2.25	Bottom 2.5%

9 **a** 2.5% **b** 15.85%

10 **a** 2.5% **b** 13.5%

11 **a** 120 **b** 116 **c** 142 **d** 100 **e** 72 **f** 50

12 $1.50

13 101 g

14 mean = 3.5 kg, st dev = 0.5 kg

15 mean = 66.0 marks, st dev = 7.7 marks

16 **a** 0.2 **b** 46.5 **c** 2.5% **d** 34% **e** 16% **f** 97.5%

17 **a** **i** 16% **ii** 2.5%
b 130
c 133

18 **a** **i** 84% **ii** 97.5%
b 184 cm
c 144 cm
d 150.4 cm

19 A **20** D **21** C **22** C **23** C

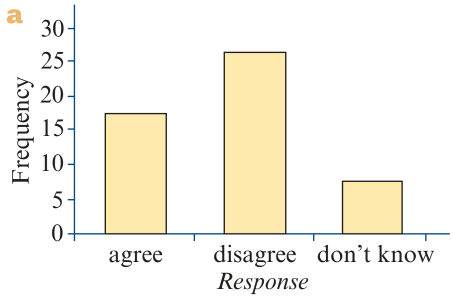
Chapter 1 review

Multiple-choice questions

1 A	**2** B	**3** D	**4** C
5 D	**6** B	**7** C	**8** C
9 C	**10** D	**11** E	**12** B
13 C	**14** B	**15** A	**16** B
17 A	**18** D	**19** D	**20** A
21 B	**22** B	**23** A	**24** D
25 B	**26** D	**27** C	**28** B
29 C	**30** E	**31** A	**32** A
33 E	**34** B	**35** E	

Written-response questions

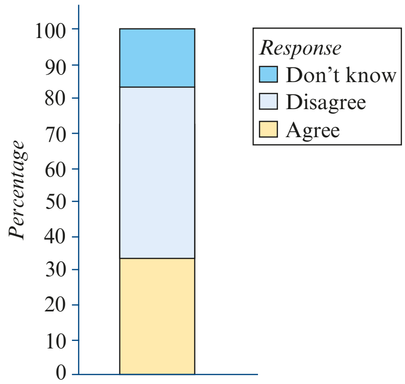
1 **a**



b

Legalised?	Frequency Count	%
Agree	18	34.6
Disagree	26	50.0
Don't know	8	15.4
Total	52	100.0

c



d Report: In response to the question, 'Do you agree that the use of marijuana should be legalised?', 50% of the 52 students disagreed. Of the remaining students, 34.6% agreed, while 15.4% said that they didn't know.

2 a **i** 50 **ii** 5

b \$105 – < \$110 **c** 28 **d** 16%

e **i** approximately symmetric

ii \$110 – < \$115 **iii** \$120 – < \$125

3 a positively skewed.

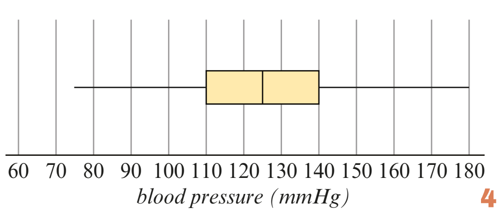
b $M = 2.65$ kg

c $IQR = 1.25$ kg

d 15.6%

e No, it is less than the upper fence (5.33 kg).

4 a



b lower fence = 65 , upper fence = 185, no outliers.

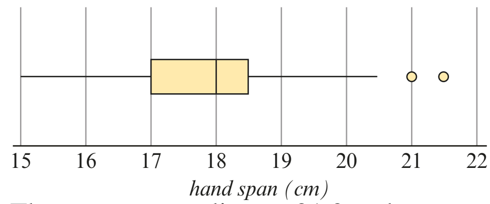
c **i** 68% **ii** 68 mmHg **iii** 3

iv 0

5 a 18 cm

b 5.5%

c



There are two outliers at 21.0 and one at 21.5.

Chapter 2

Exercise 2A

1 a Two categorical

b One categorical and one numerical

c Two numerical **d** Two categorical

2 a EV: *colour*; RV: *toxicity*

b EV: *diet*; RV: *weight loss*

c EV: *age*; RV: *price*

d EV: *fuel*; RV: *cost*

e EV: *location*; RV: *house price*

3 a EV: *age* (numerical), RV: *exercise level* (categorical)

b EV: *years of education* (numerical), RV: *salary level* (numerical)

c EV: *temperature* (numerical), RV: *comfort level* (categorical)

d EV: *time of year* (categorical), RV: *incidence of hay fever* (categorical)

e EV: *age group* (categorical), RV: *musical taste* (categorical)

f EV: *state of residence* (categorical), RV: *AFL team* (categorical)

4 B **5** B **6** C

Exercise 2B

1 a EV: *gender*, RV: *intends to go to university*

b

Intends to go to university	*Gender* Male	Female
Yes	4	8
No	4	4
Total	8	12

2 a EV: *age group*, RV: *reduce university fees?*

b

Reduce university fees?	*Age group* 17-18	19-25	26 or more
Yes	8	6	6
No	3	3	4
Total	11	9	10

c

Reduce university fees?	*Age group (%)* 17-18	19-25	26 or more
Yes	72.7	66.7	60.0
No	27.3	33.3	40.0
Total	100.0	100.0	100.0

3 a *enrolment status*

b No. The percentage of full-time and part-time students who drank alcohol is similar: 80.5% to 81.8%. This indicates that drinking behaviour is not related to enrolment status.

4 a *handedness*

b

Handedness	*Gender (%)* Male	Female
Left	9.0	9.8
Right	91.0	90.2
Total	100.0	100.0

c No, there is little difference in the percentage of males and females who are left handed, 9.0% compared to 9.8%.

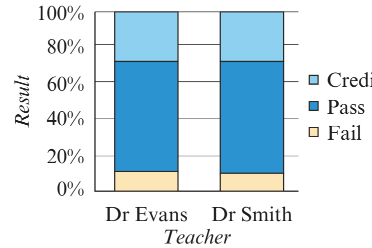
5 a *course* **b** ordinal **c** 54.9%

d Yes; the percentage of Business students who exercise regularly (18.6%) was much higher that the percentage of Arts who exercise regularly (5.9%).

6 a

Result	*Teacher (%)* Dr Evans	Dr Smith
Fail	11.1	9.4
Pass	61.1	62.5
Credit	27.8	28.1
Total	100.0	100.0

b



c There is no evidence students of Dr Evans receive higher grades than students of Dr Smith. The percentage of students achieving each grade level is almost the same for both classes (eg. 61.1% compared 62.5% for students who received a Pass).

7 The data supports the contention that people who are satisfied with their job are more likely to be satisfied with their life, with 70% of people who are satisfied with their job reporting that they are satisfied with their life, compared to only 50% of people who are dissatisfied with their job.

8 a EV: *type of treatment*, RV: *treatment outcome*

b The data supports the contention that the special pillow is more effective at treating snoring than the drug treatment, with just over 30% of people who used the special pillow reporting they were cured, compared to only 10% of people who used the drug reported they were cured.

9 a 11.9% **b** 52.3% **c** marital status
d ordinal
e Yes. There are several ways that this can be seen. For example, by comparing the married and widowed groups, we can see that a smaller percentage of those widowed found life exciting (33.8%) compared to those who were married (47.6%). Or: a bigger percentage of widowed people found life pretty routine (54.3% to 48.7%) and dull (11.9% to 3.7%) compared to those who were married.

10 A **11** B **12** C

Exercise 2C

1 a EV: *country of origin*, RV: *number of days away*
b The number of days these tourists spend away from home was associated with their country of origin. The median number of days spent away from home for Japanese tourists (M = 17 days) is considerably higher than for Australian tourists (M = 7 days). The variability for the number of days away is also higher for Japanese tourists (IQR = 16.5) compared to that for Australian tourists (IQR = 10.5).

2 a *age*: numerical, *gender*: categorical
b From this information it can be concluded that the median age of the people admitted to the hospital during this week was associated with their age. The median age of the females (34 years) admitted to the hospital was considerably higher than the median age of males (25.5 years). The variability of the ages was also higher for the females (IQR = 28 years) compared that of the males (IQR = 13 years).

3 a *hours online*: numerical, *year level*: categorical
b From this information it can be concluded that the median number of hours spent online was associated with year level. The median time spent online by the Year 10 students (20 hours) was higher than the median number of hours by the Year 11 students (16.5 hours). The variability of the hours spent online was lower for the Year 10 students (IQR = 9.5 hours) compared that of the Year 11 students (IQR = 13 hours).

4 a *age at marriage*: numerical, *gender*: categorical
b For this data there is an association between age at marriage and gender. The age at marriage is higher for men (M = 23 years) than for women (M = 20.5 years). The variability is also greater for the men (IQR = 12 years) than for the women (IQR = 8.5 years). The distributions of age at marriage are positively skewed for both men and women. There are no outliers.

5 a *pulse rate*: numerical, *gender*: categorical
b For this data there is an association between pulse rate and gender. The pulse rates for males (M = 73 beats/min) are lower than the pulse

rates for women (M = 76 beats/min). The variability is also lower for the males (IQR = 8 beats/min) than for the women (IQR = 14 beats/min). Both distributions are approximately symmetric, with no outliers.

6 **a** *lifetime*: numerical; *price*: categorical

b For this data there is an association between the lifetime of a battery and its price. The lifetime of the high price batteries (M = 51 hours) is longer than that of the medium price batteries (M = 35 hours), which is in turn slightly longer than that of the low price batteries (M = 32 hours). The variability in lifetime increased as price decreased, from IQR = 7 hours for the high price batteries, to IQR = 12 hours for the medium price batteries, and IQR = 17 hours for the low price batteries. All three distributions are approximately symmetric, with no outliers.

7 A **8** D

Exercise 2D

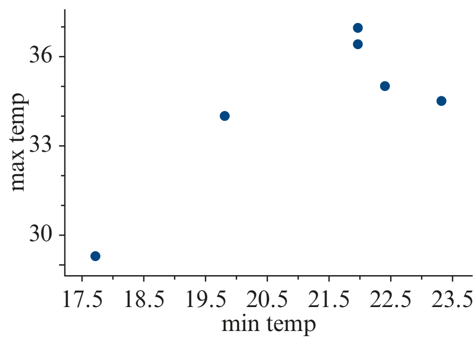
1 **a** number of seats

b numerical

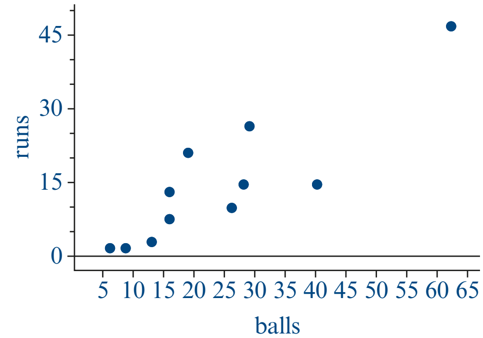
c 8 aircraft

d around 800 km/h

2



3



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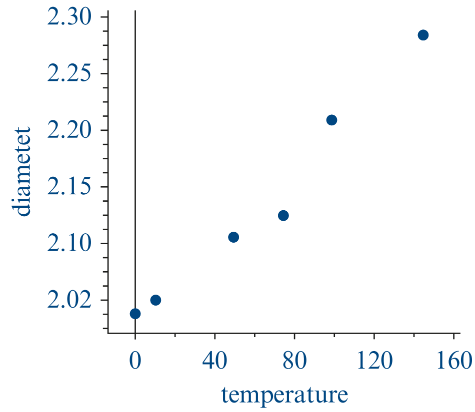
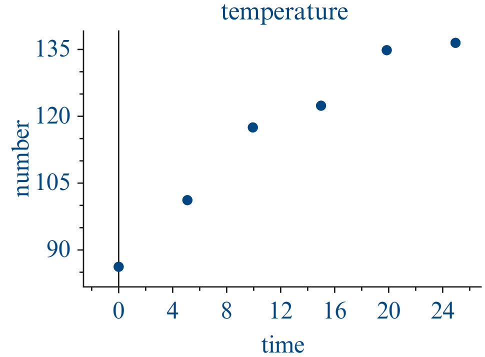


5



6 D

Exercise 2E

1 Note: There are no absolute right or wrong answers to these questions as answering them requires a degree of personal judgment.

i no association **ii** yes, positive

iii yes, positive **iv** yes, positive

v yes, negative **vi** yes, negative

2 **a** **i** moderate, positive, linear association

ii weak, negative, linear association

iii strong, positive, linear association

iv no association

b **i** Those people who smoke more tend to have a higher lung cancer mortality rate.

ii Older children tended to score lower on the aptitude test.

iii Intersections with higher levels of traffic volume also tended to have higher CO levels.

Exercise 2F

1 **a** A: strong, positive, non-linear relationship with no outliers

B: strong, negative, linear relationship with an outlier

C: weak, negative, linear relationship with no outliers

b A: non-linear

2 $r = 0.73$

3 **a–c** Answers given in question.

4 **a** strong positive

b strong positive

c moderate negative

Exercise 2G

1 **a** 45.6% **b** 11.9% **c** 32.1% **d** 45.3% **e** 1.5%

2 **a** The coefficient of determination is $r^2 = (-0.611)^2 = 0.373$ or 37.3%; that is, 37.3% of the variation observed in hearing test scores can be explained by variation in age.

b The coefficient of determination is $r^2 = (0.716)^2 = 0.513$ or 51.3%; that is, 51.3% of the variation observed in mortality rates can be explained by variation in smoking rates.

c The coefficient of determination is $r^2 = (-0.807)^2 = 0.651$ or 65.1%; that is, 65.1% of the variation observed in life expectancies can be explained by variation in birth rates.

d The coefficient of determination is $r^2 = (0.818)^2 = 0.669$ or 66.9%; that is, 66.9% of the variation observed in daily maximum temperature is explained by the variability in daily minimum temperatures.

e The coefficient of determination is $r^2 = (0.8782)^2 = 0.771$ or 77.1%; that is, 77.1% of the variation in the runs scored by a batsman is explained by the variability in the number of balls they face.

3 **a** $r = 0.906$ **b** $r = -0.353$

4 E **5** A **6** E **7** D

8 D

Exercise 2H

Note: These answers are for guidance only. Alternative explanations for the source of an association may be equally acceptable as the variables suggested.

1 Not necessarily. In general, older children are taller and have been learning mathematics longer. Therefore they tend to do better on mathematics tests. Age is the probable common cause for this association.

2 Not necessarily. While one possible explanation is that religion is encouraging people to drink, a better explanation might be that towns with large numbers of churches also have large populations, thus explaining the larger amount of alcohol consumed. Town size is the probable common cause for this association.

3 Probably not. The amount of ice-cream consumed and the number of drownings would both be affected by weather conditions. Weather conditions are the probable common cause.

4 Maybe but not necessarily. Bigger hospitals tend to treat more people with serious illnesses and these require longer hospital stays. A common cause could be the type of patients treated at the hospital.

5 Not necessarily. Possible confounding variables include age and diet.

6 There is no logical link between eating cheese and becoming tangled in bed sheets and dying. The correlation is probably spurious and the result of coincidence.

7 Not necessarily. For example, the more serious the fire, the more fire trucks in attendance and the greater the fire damage. A possible common cause is the severity of the fire.

8 E

Exercise 2I

1 a segmented bar chart
b scatterplot **c** parallel box plots
d scatterplot **e** scatterplot
f segmented bar chart
g segmented bar chart
h parallel box plots or back-to-back stem plots

2 E

3 D

Chapter 2 review

Multiple-choice questions

1 A	**2** D	**3** B	**4** D
5 E	**6** B	**7** D	**8** A
9 C	**10** E	**11** D	**12** E
13 C	**14** C	**15** C	**16** D
17 A	**18** E	**19** B	**20** C
21 E	**22** C		

Written-response questions

1 a *Number of accidents* and *age*; both categorical variables

b RV: *Number of accidents*; EV: *age*

c 470

d

No of accidents	< 30	≥ 30
At most one	21.7%	42.5%
More than one	78.3%	57.5%

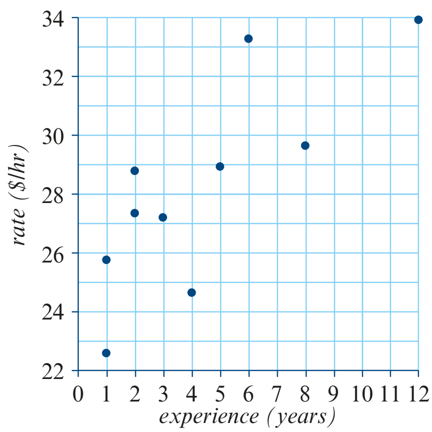
e The statement is correct. Of drivers aged less than 30, 78.3% had more than one accident compared to only 57.5% of drivers in the older category.

2 a Numerical: *conversation test score*. Categorical: *completed weeks of course*

b There is an association between the students' scores on the conversation test, and the number of weeks of the course they have completed. The median score at the beginning of the course (M=38) showed a little improvement after six weeks (M =42), followed by a very large improvement by the end of the 12 week course (M =72). The variability of the scores changed little over the course (IQR=12 at the beginning, IQR = 12 at 6 weeks, IQR = 14 at 12 weeks). The distributions of scores at 0 weeks is approximately symmetric with an outlier at 66, positively skewed with an outlier at 76 at 6 weeks, and approximately symmetric with no outliers at 12 weeks.

3 a *rate* is the response variable, *experience* is the explanatory variable.

b



c There is a moderate positive linear relationship; that is, people with more experience are generally being paid a higher hourly pay rate.

d 0.786. Note that this value is on the borderline between a moderate and a strong linear relationship, but with such a small data set it is difficult to assess the strength from the scatterplot.

e Coefficient of determination = 0.618; that is, 61.8% of variation in pay rate is explained by the variation in experience.

4 a 60%

b For these people there is an association between the person's quality of sleep and their participation in the course, with 85% of people rating their sleep quality as good after the course, compared to only 60% of people rating their sleep quality as good before the course.

Chapter 3

Exercise 3A

1 A residual is the difference between a data value and its value predicted by a regression line.

2 C

3 The data is numerical; the association is linear; there are no clear outliers.

4 a x

b $y = 9.23 + 1.00x$

5 a RV: *pollution level*; EV: *traffic volume*

b *pollution level* $= -330 + 49 \times$ *traffic volume*

6 a RV: *life expectancy*; EV: *birth rate*

b *life expectancy* $= 110 - 1.5 \times$ *birth rate*

7 a y **b** $r = -0.450$

8 a RV: *distance travelled*; EV: *age of car*

b $r = 0.947$

9 a r is also negative.

b Slope is zero: regression line is horizontal.

c intercept $= \bar{y}$ (mean of RV)

10–11 Answers given in question.

12 a Answers given in question.

b $runs = -2.6 + 0.73 \times balls\ faced$

13 a RV: *number of TVs*

b Answer given in question.

c $number\ of\ TVs =$
$61.2 + 0.930 \times number\ of\ cars$

14 C **15** A **16** C

Exercise 3B

1 $mark = 80 - 4.3 \times days\ absent$

2 a 2.9: On average a person who is 0 cm in height has a hand span of 2.9 cm - not sensible!

b 0.33: On average a person's hand span increases by 0.33 cm for each 1 cm increase in height.

3 a 575: On average, the company will have $575 in sales when their online advertising expenditure is $0.

b 4.85: On average sales will increase by $4.85 for each additional $1.00 spent on online advertising.

4 **a** 80 cm, extrapolating
b 92 cm, interpolating
c 98 cm, extrapolating

5 **a** \$487.50, extrapolating
b \$1023.50, interpolating
c \$1224.50, extrapolating

6 **a** 173 cm, reliable, interpolating
b 189 cm, unreliable, extrapolating
c 165 cm, reliable, interpolating

7 **a** 20.3%
b 42.3%
c The number of *hours* is more important as it explains 42.3% of the variation in *exam score*, much more than *IQ* which explains only 20.3% of the variation in *exam score*.

8 Answers given in question.

9 **a** 9.7 **b** −0.8

10 **a** 2 **b** −1 **c** 2

11 A: clear curved pattern in the residuals (not random), C: curved pattern in the residuals (not random).

12 **a** 27.8: On average a packet of chips with 0 gm of fat contains 27.8 calories.
b 14.7: On average, the calorie content increases by 14.7 for each one additional gram of fat included.
c 75.7% of the variation in calorie content of the chips is explained by the variation in fat content.
d 145.4 **e** −13.4

13 **a** −0.278: On average, for each additional one metre the golfer is from the hole the success rate decreases by 27.8%.
b 73.5
c 3.54 m
d −0.705
e 49.7%: 49.7% of the variation in success rate in putting is explained by the variation in the distance the golfer is from the hole.

14 **a** yes, linear relationship
b 0.9351 or 93.5%
c 93.5%
d $\textit{pay rate} = 18.56 + 0.289 \times \textit{experience}$
e Intercept = \$18.56. On average, a worker with no experience will earn \$18.56 per hour.
f On average, the pay rate increases by 29 cents per hour for each additional one year of experience.
g **i** \$20.87 **ii** \$0.33
h yes; no clear pattern in the residual plot

15 **a** $r = -0.608$
b 37% of the variation in the hearing test score is explained by the variation in age.
c $\textit{score} = 4.9 - 0.043 \times \textit{age}$
d −0.043; the hearing test score, on average, decreases by 0.043 for each one additional year of age.
e **i** 4.04 **ii** −2.04
f **i** 0.3 **ii** −0.4
g yes; no clear pattern in the residual plot

16 negative, drug dose, −0.9492; 55.9; −9.31; decreases, 9.31; 55.9; 90.1, response time, drug dose; clear pattern

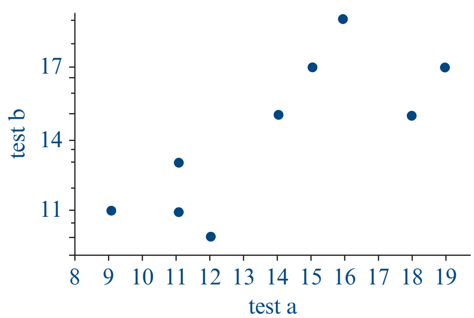
17 The scatterplot shows that there is a strong positive linear relationship between radial length and femur length: $r = 0.9876$. There are no outliers. The equation of the least squares regression line is:
$\textit{radial length} = -7.2 + 0.74 \times \textit{femur length}$

The slope of the regression line predicts that, on average, radial length increased by 0.74 cm for each one centimetre increase in femur length.
The coefficient of determination indicates that 97.5% of the variation in radial lengths can be explained by the variation in femur lengths.
The residual plot shows no clear pattern, supporting the assumption that the relationship between radial and femur length is linear.

18 E **19** B **20** A

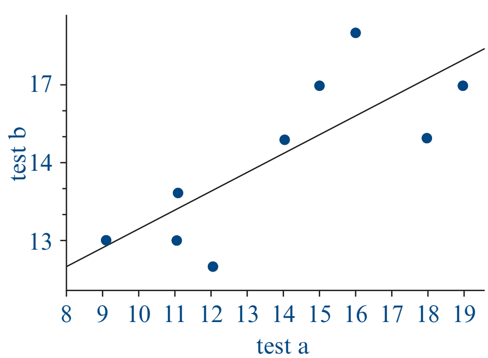
Exercise 3C

1 a

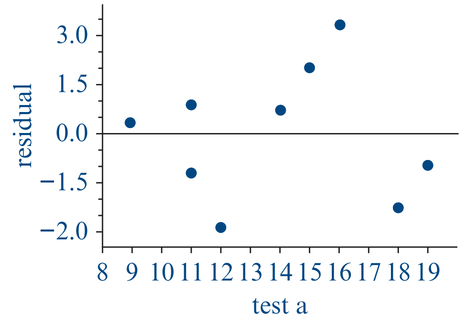


b *test B score* $= 4.2 + 0.72 \times$ *test A score*, $r = 0.78$, $r^2 = 0.61$

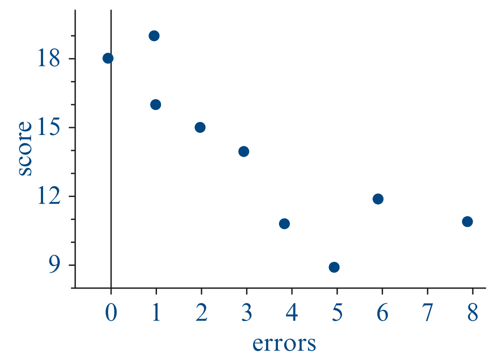
c



d

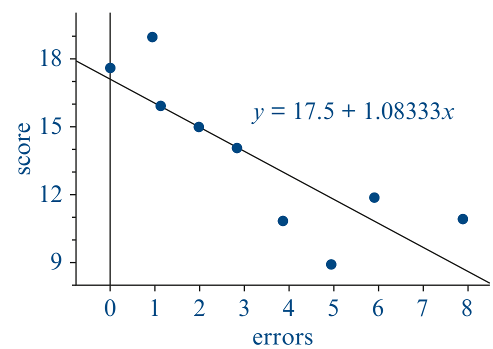


2 a

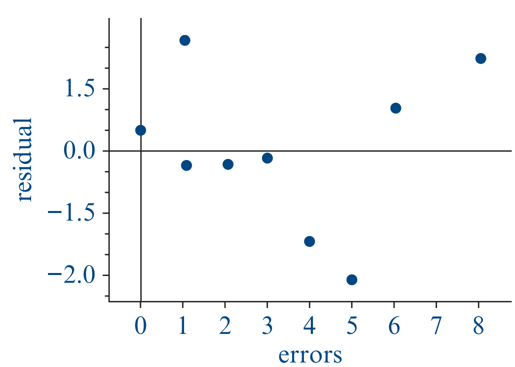


b $score = 17.5 - 1.08 \times error$, $r = -0.841$, $r^2 = 0.707$

c

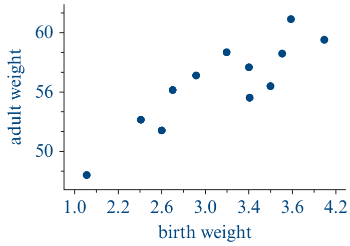


d



3 a RV: *adult weight*; EV: *birth weight*

b



c i strong positive linear association with no outliers

ii approximately 0.9

d *adult weight* = 38.4 + 5.87 × *birth weight*, $r^2 = 0.765$, $r = 0.875$

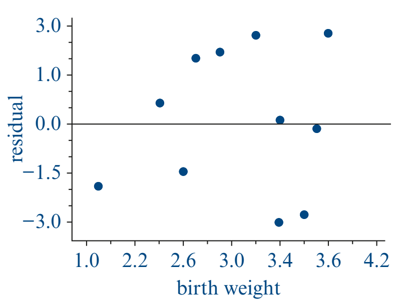
e 76.5% of the variation in the adult weight is explained by the variation in birth weight.

f On average, adult weight increases by 5.9 kg for each additional kilogram of birth weight.

g i 56.0 **ii** 53.1 **iii** 61.3

h Yes. 76.5% of the variation in the adult weight is explained by the variation in birth weight.

i



The lack of a clear pattern in the residual plot supports the assumption that the association between adult weight and birth weight is linear.

Chapter 3 review

Multiple-choice questions

1 C	**2** D	**3** A	**4** C
5 E	**6** C	**7** B	**8** B
9 D	**10** A	**11** A	**12** A
13 D	**14** E	**15** A	**16** C
17 C	**18** A	**19** A	**20** D
21 C	**22** C	**23** D	

Written-response questions

1 a i 5 years

ii mean= 767, st dev = 35

b *airspeed* = 673 + 0.372 × *number of seats*

c 74.1%

2 a days of rain **b** −6.88, 2850 **c** 2024

d decrease, 6.88 **e** −0.696

f 48.4, days of rain **g i** 1873 **ii** −483

h interpolation

3 a *cost*

b There is a strong, positive, linear association between the cost of the meals and the number of meals prepared.

c i $307.30 **ii** extrapolating

d i 222.48: the fixed costs of preparing meals is $222.48.

ii $4.039: The slope of the regression line predicts that, on average, meal preparation costs increase by $4.039 for each additional meal produced.

e Answer given in question.

4 a RV: *height*; EV: *femur length*

b *height* = 36.3 + 5.35 × *femur length*

c On average, height increases by 5.35 cm for each cm increase in femur length.

d $r^2 = 0.988$; that is, 98.8% of the variation in height is explained by the variation in femur length.

e 97.6%

5 a RV: *height*; EV: *age*

b strong positive association with no outliers.

c Answer given in question.

d i Answer given in question

ii Extrapolating

e On average, for each additional one year of age, height increases by 6.4 cm.

f $r^2 = 0.995$; that is, 99.5% of the variation in height is explained by the variation in age.

g i 140.3 cm ii −0.7 cm

h i Answer given in question.

ii Residuals show a clear curved pattern.

6 a moderate, positive linear association with no outliers

b i 142

ii extrapolating

c −6.3

d i linearity

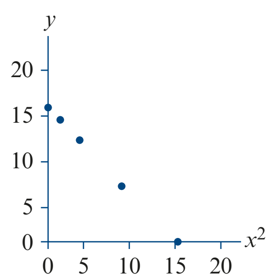
ii the lack of a clear pattern in the residual plot supports the linearity assumption.

Chapter 4

Exercise 4A

1 a 19.5 b 11.7 c 23.8 d 126.7

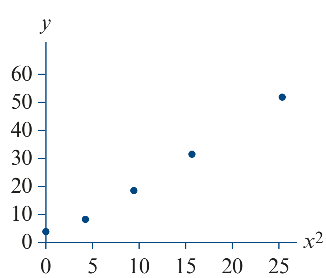
2 a



b $y = 16 - x^2$

c when $x = -2, y = 12$

3 a

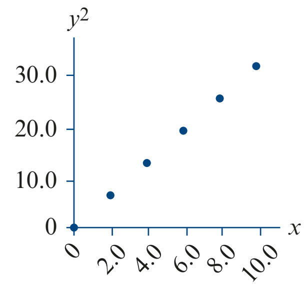


b $y = 1 + 2x^2$

c when $x = 6, \; y = 73$

4 a ±4.7 b ±1.3 c 6 d −8

5 a



b $y^2 = 1.5 + 3.1x$

c $y = \pm 5.4$, but only the positive solution applies here because the model is only defined for $y > 0$.

6 a $\textit{number of people} = 0.0 + 4.1 \times \textit{diameter}^2$

b 7

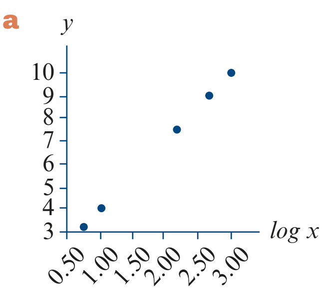
7 a $\textit{time}^2 = 18 - 9.3 \times \textit{amount}$ b 3.8 min

8 D 9 A 10 B

Exercise 4B

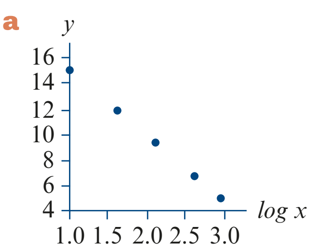
1 a 6.6 b 1.1 c −3.1 d 138.5

2 a



b $y = 1 + 3\log x$ c 7

3 a

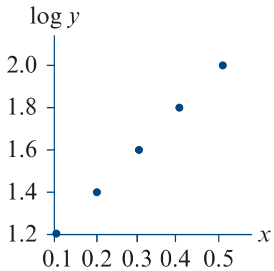


b $y = 20 - 5\log x$ c 5

4 a 100 b 218.8 c 1 000 000

d 0.8

5 a



b $\log y = 1 + 2x$ **c** 158.5

6 a $level = 1.8 + 2.6 \log(time)$ to 2 sig. figs

b 2.8 to 1 d.p.

7 a $\log(number) = 1.314 + 0.08301 \times month$ to 4 sig. figs

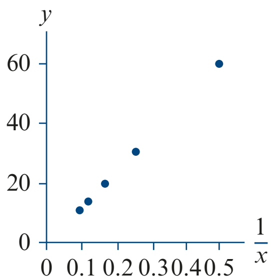
b 139 to nearest whole number

8 C **9** E **10** A

Exercise 4C

1 a 13.3 **b** 2.8 **c** 4.9 **d** 309.5

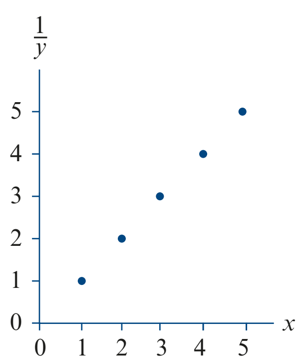
2 a



b $y = \frac{120}{x}$ **c** 24

3 a 0.17 **b** 0.07 **c** 0.16 **d** 0.06

4 a



linear

b $\frac{1}{y} = x$

c 4

5 a $horsepower = 22.1 + \frac{690}{consumption}$

b 99 to nearest whole number

6 a $\frac{1}{errors} = -0.00024 + 0.050 \times times$ to 2 sig. figs

b 3 to nearest whole number

7 A **8** E **9** C

Exercise 4D

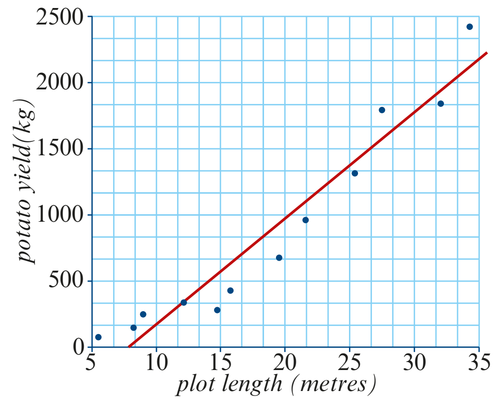
1 a $\log y, \frac{1}{y}, \log x, \frac{1}{x}$

b None; trend needs to be consistently increasing or decreasing.

c $\log y, \frac{1}{y}, x^2$

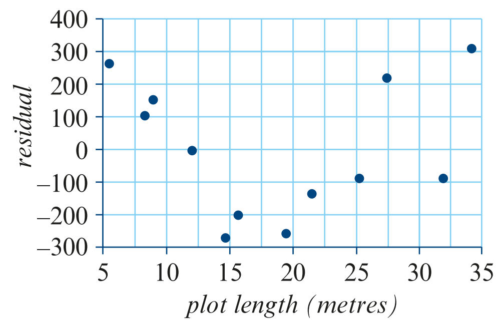
d x^2, y^2

2 a



b $yield = -620.0 + 80.23 \times length$

c



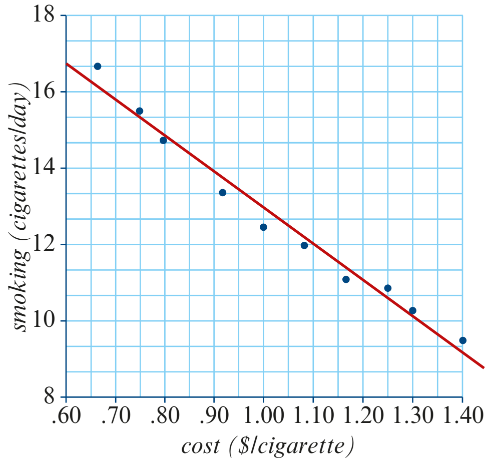
No, the residuals show a clear curved pattern.

d $\log y, \frac{1}{y}, x^2$

e $yield = 3.983 + 2.030 \times (length)^2$

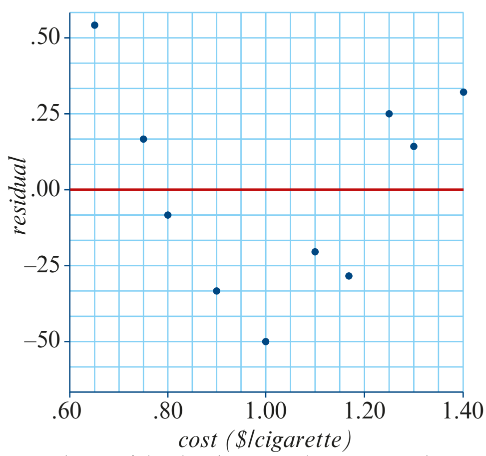
f $r^2 = 97.5\%$

3 a



b $smoking = 22.49 - 9.501 \times cost$

c



No, the residuals show a clear curved pattern.

d $\log x$, $\log y$, $\frac{1}{y}$, $\frac{1}{x}$

e Either the $\log x$ and $\frac{1}{x}$ could be recommended as both transformations give very good results. That is

$smoking = 3.420 + \frac{9.045}{cost}$ or

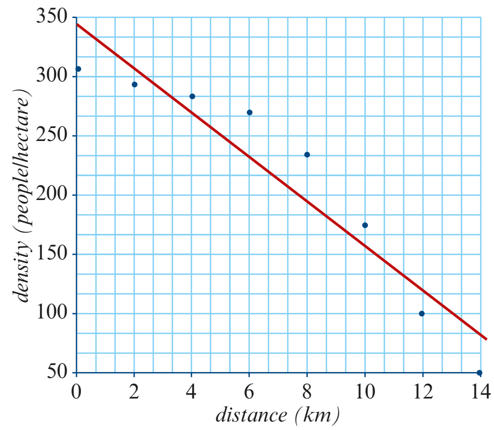
$smoking = 12.73 - 21.90 \times \log(cost)$

The $\frac{1}{x}$ transformation is more intuitive and easier to interpret.

f $\frac{1}{x}$: $r^2 = 99.3\%$

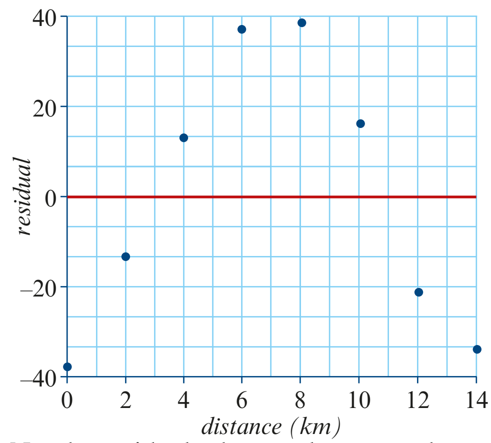
$\log x$: $r^2 = 99.6\%$

4 a



b $density = 345.3 - 18.65 \times distance$

c



No, the residuals show a clear curved pattern.

d x^2, y^2

e $density = 308.9 - 1.345 \times (distance)^2$

f $r^2 = 99.1\%$

Chapter 4 review

Multiple-choice questions

1 A	**2** D	**3** D	**4** B
5 A	**6** B	**7** E	**8** D
9 D	**10** D	**11** D	

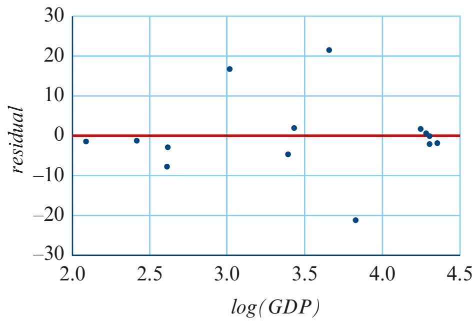
Written-response questions

1 a $\frac{1}{age} = 2.606 - 1.053 \times length$

b 2.6 years

2 a $literacy\ rate = -44.2 + 33.3 \log(GDP)$

b

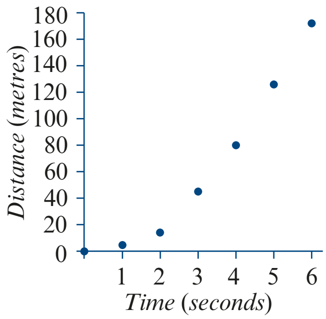


Residual plot shows no clear pattern

c 89%

d 0.077

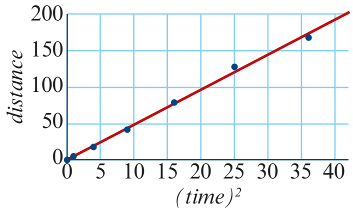
3 a



b

Time	0	1	2	3	4	5	6
Distance	0	5.2	18	42	79	128	168
Time^2	0	1	4	9	16	25	36

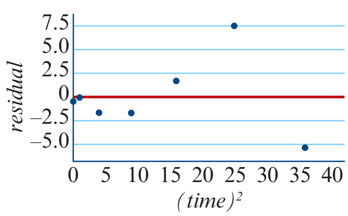
c



d $distance = 0.45 + 4.8 \times time^2$

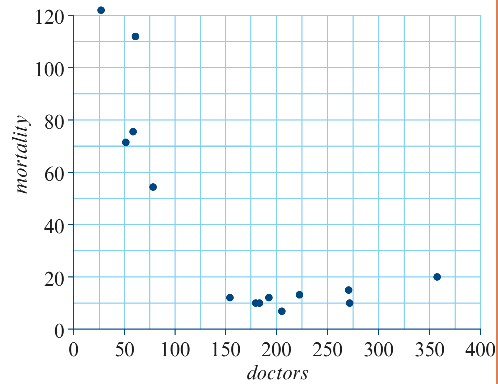
e 236 metres

f

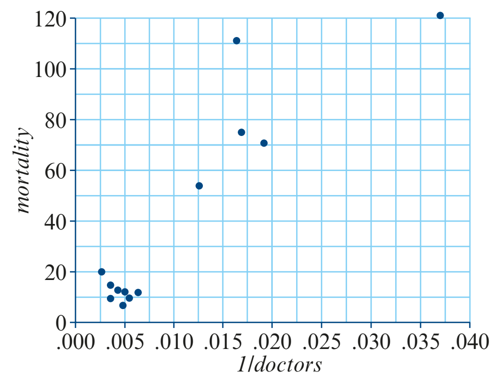


The residual plot shows no clear structure indicating that the assumption of linearity is justified.

4 a

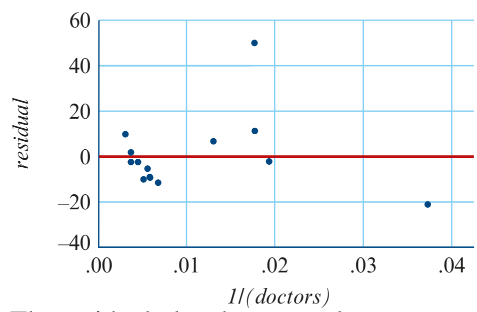


b



c $mortality = -1.194 + 3856 \times \dfrac{1}{doctors}$

d



The residual plot shows no clear structure indicating that the assumption of linearity is justified.

e r^2=82.8%

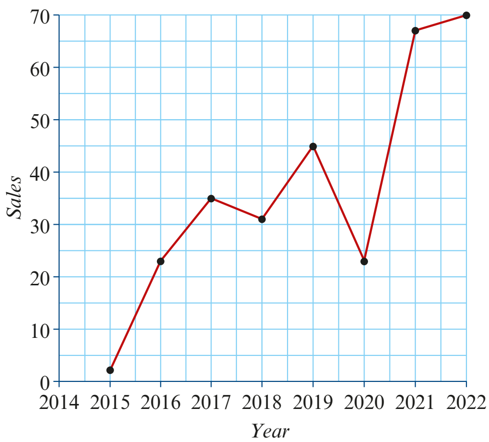
f 37

g Since 100 is within the range of the data we are interpolating, and the prediction is reliable.

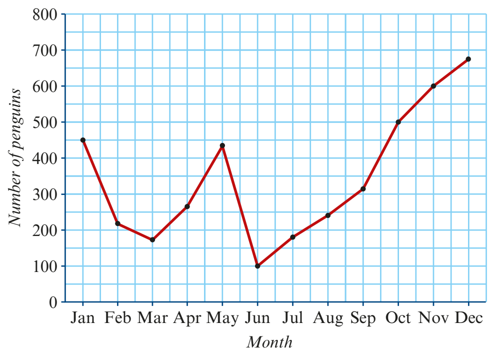
Chapter 5

Exercise 5A

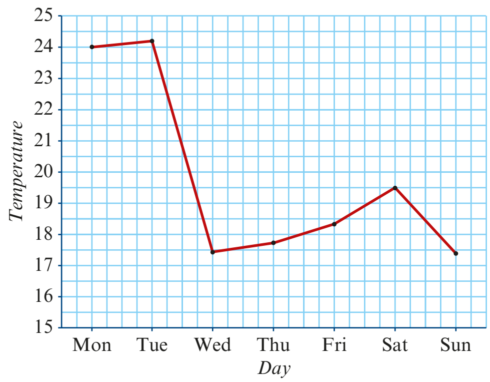
1



2



3



4

Feature	A	B	C
Irreg fluct	✓	✓	✓
Inc			✓
Dec	✓		

5

Feature	A	B	C
Irreg fluct	✓	✓	✓
Inc trend			✓
Dec trend	✓		
Cycles		✓	
Seasonality	✓		✓

6

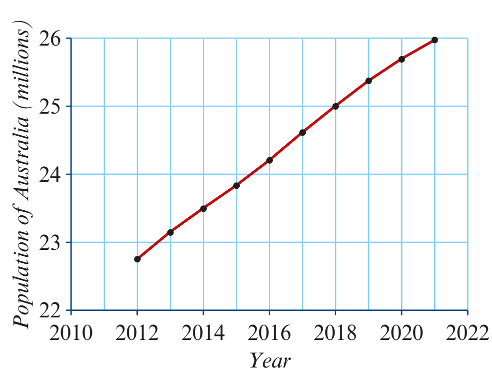
Feature	A	B	C
Irreg fluct	✓		✓
Struct change	✓		
Inc trend	✓		
Dec trend	✓		
Seasonality			✓

7

Feature	A	B	C
Irreg fluct	✓	✓	✓
Struct change		✓	
Inc trend	✓		
Dec trend			
Outlier			✓

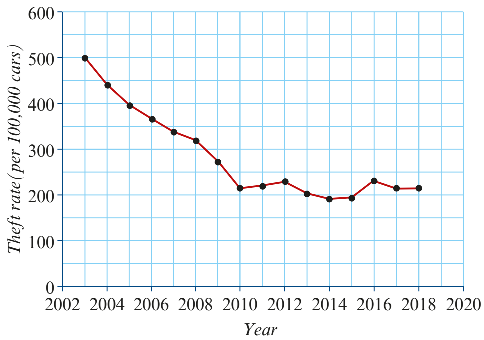
8 The number of mobile phones per 100 people increases rapidly over the years 2000-2008. The number continues to increase from 2009 until 2019, but the increase in the number of phones is at much lower rate than in the preceding years.

9 **a**



b The plot shows a steady increase in the population of Australia over the years 2012 - 2021.

10 a



b The plot shows a steady decline in the number of vehicle thefts over the years 2003 -2010, after which the number of vehicle thefts has remained reasonably steady, showing only irregular variation.

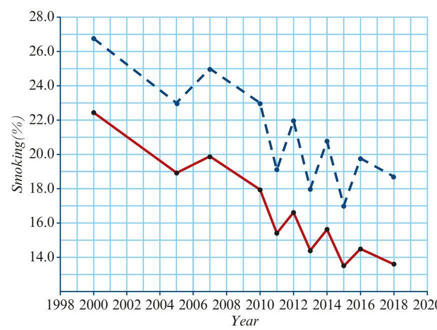
11 The number of cases of measles show an increasing trend between 1989 and 1992. In 1993-1994 there is a rapid increase in the number of measles cases, followed be a rapid decrease in 1994-1995. The number of cases continued to decrease until 2000, since then have remained low, showing only irregular variation over the years 2001-2019.

12 The number of overseas arrivals (millions people per month) in Australia increased steadily from November 2011 until April 2020. The number of arrivals is clearly seasonal, with the peak time for arrivals in the January quarter each year. The number of arrivals dropped suddenly to almost zero in April 2020, and remained at this level until October 2021.

13 a i The percentage of males who smoke has consistently decreased since 1945, while the percentage of females who smoke increased from 1945 to 1975 but then decreased at a similar rate to males over the period 1975–1992.

ii The difference in smoking rates between males and females has decreased over these years.

b i



ii Whilst both plots show irregular fluctuation, overall the percentages of male and females who smoke have declined substantially over the years 2000-2018.

iii The difference in smoking rates between males and females has remained almost the same over these years.

14 E

15 D

Exercise 5B

1 a i 3 **ii** 1 **iii** 4

b i 3.2 **ii** 1.2 **iii** 2.2

c i 2.6 **ii** 2.0

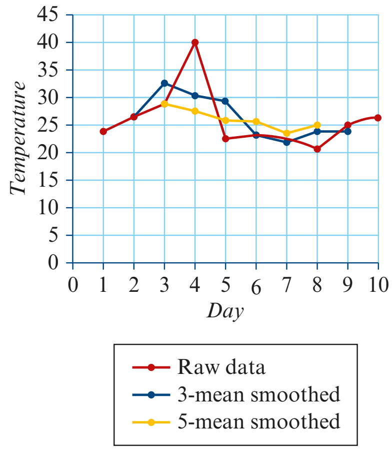
d 2.3

2 a 24.4 **b** 20.0 **c** 23.2

3

t	1	2	3	4	5	6	7	8	9
y	10	12	8	4	12	8	10	18	2
3-mean	–	10	8	8	8	10	12	10	–
5-mean	–	–	9.2	8.8	8.4	10.4	10	–	–

4 a, c

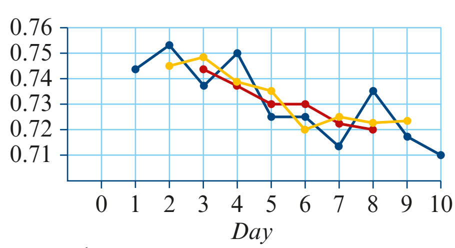


The smoothed plots show that the 'average' maximum temperature changes relatively slowly over the 10-day period (the 5-day average varies by only 5°) when compared to the daily maximum, which can vary quite widely (for example, nearly 20° between the fourth and fifth day) over the same period of time.

b

Day	Temp. (°C)	3-moving mean	5-moving mean
1	24	–	–
2	27	26.3	–
3	28	31.7	28.2
4	40	30.0	28.0
5	22	28.3	27.0
6	23	22.3	25.6
7	22	22.0	22.6
8	21	22.7	23.4
9	25	24.0	–
10	26	–	–

5 a, c



- exchange rate
- 3-moving mean exchange rate
- 5-moving mean exchange rate

The exchange rate has a downward trend over the 10-day period. This is most obvious from the smoothed plots, particularly the 5-moving mean plot.

b

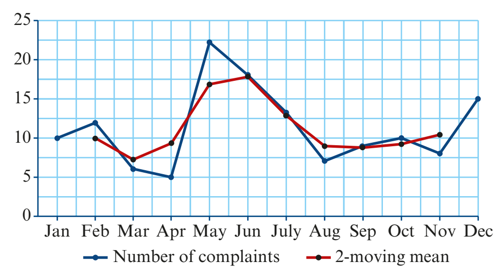
Day	Exchange rate	3-moving mean	5-moving mean
1	0.743	–	–
2	0.754	0.745	–
3	0.737	0.747	0.742
4	0.751	0.737	0.738
5	0.724	0.733	0.730
6	0.724	0.720	0.729
7	0.712	0.724	0.722
8	0.735	0.721	0.720
9	0.716	0.721	–
10	0.711	–	–

6 a 3.8 **b** 2.0

7 a 3.3 **b** 1.5 **c** 2.4 **d** 1.9

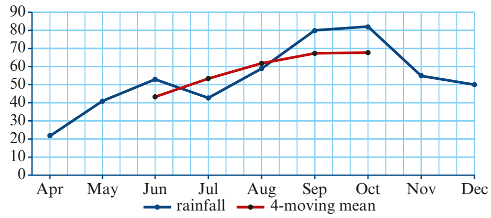
8 a 13.1 **b** 12.2 **c** 10.7

9 a, c



Two-mean smoothing of the plot has not had a big effect, with both plots showing that the that the number of complaints between April and July is considerably higher than the number in the rest of the year.

10 a, c



Four-mean smoothing of the plot shows a steady increase in rainfall from June to October.

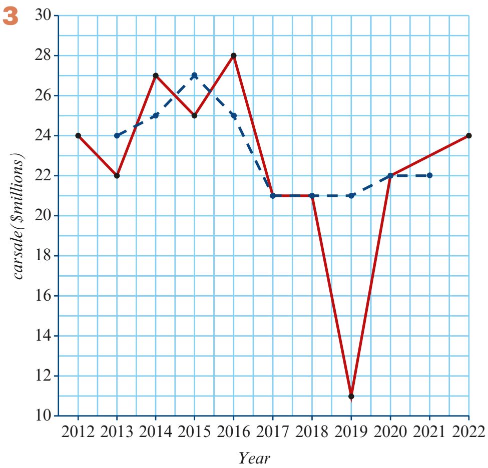
11 A **12** C **13** E **14** D

Exercise 5C

1 **a** (3, 3) **b** (2, 2) **c** (3, 2) **d** (3, 3)

2 **a** 30°C **b** 25°C

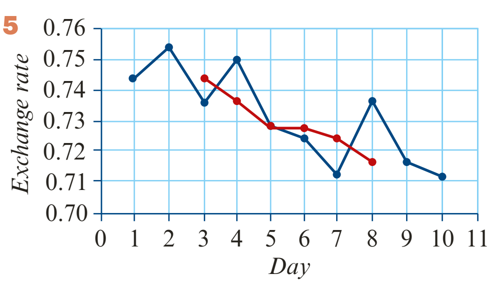
3



The smoothed plot shows that sales were quite consistent up to 2017 when they dropped, and have remained at this lower level from 2017-2022.

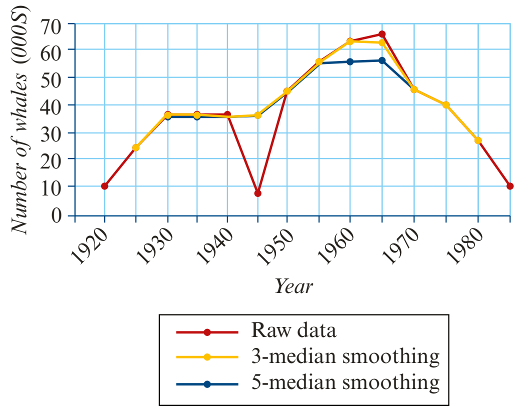
4 **a** 25°C **b** 25°C

5

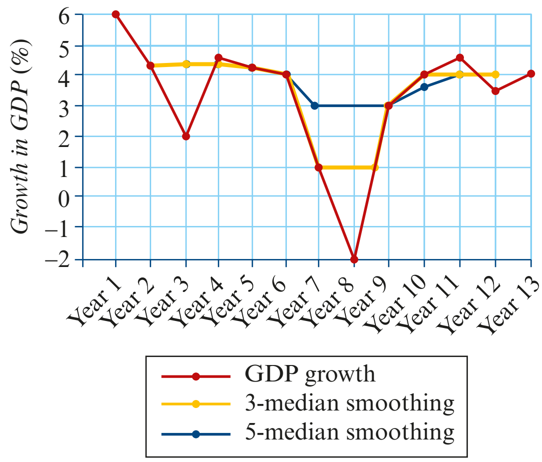


The smoothed plot shows that there was a general decreasing trend in the exchange rate over this period.

6



7 **a** 4 **b** **i, ii**



c The plot of GDP growth over 1 year, shows a great deal of variability, with no clear trend apparent. When smoothed over a 3-year period, GDP growth is still variable but to a lesser extent. No clear trend is apparent, but GDP appears to be going through a period of below average growth during the time period from Year 7 to Year 9. When smoothed over a 5-year period, GDP growth is much less variable but clearly shows the period of below average growth during the from period from Year 7 to Year 9.

8 D **9** C **10** D

Exercise 5D

1 a 1.0

b Sales in February are typically 30% higher than sales in the average month.

c Sales in September are typically 10% lower than sales in the average month.

2 a 7.8 **b** 6.7

3 a 3.9 **b** 6.9

4 a Increase by 42.9%.

b Decrease by 23.1%

5 a 1.2 **b** 1514 **c** 1437 **d** 1005

6

	Sum	Aut	Win	Spr
Number of students:	56	125	126	96
Deseasonalised:	112	125	97	80
Seasonal index	0.5	1.0	1.3	1.2

7 a, c

	Q1	Q2	Q3	Q4
Deseasonalised:	152	142	148	153
Seasonal index	1.30	1.02	0.58	1.10

b In quarter 1 the restaurant chain employs 30% more waiters than the number employed in an average quarter.

8

Q1	Q2	Q3	Q4
0.89	0.83	1.12	1.16

9

Jan	Feb	Mar	April	May	June
0.89	0.96	1.04	1.26	1.33	1.11

July	Aug	Sept	Oct	Nov	Dec
0.67	0.74	0.59	0.81	1.11	1.48

10

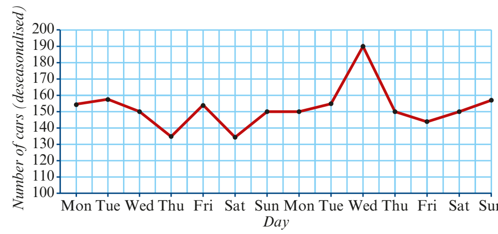
Jan	Feb	Mar	April	May	June
1.06	0.96	1.18	1.18	1.03	0.86

July	Aug	Sept	Oct	Nov	Dec
0.96	0.79	0.74	0.54	1.18	1.50

11 a

Mon	Tues	Wed	Thur	Fri	Sat	Sun
155	157	150	134	153	134	150
150	154	190	148	143	150	157

11 b



12 a, d

b

Q1	Q2	Q3	Q4
1.03	0.93	0.93	1.11

c

Year	Q1	Q2	Q3	Q4
1	206	209	211	205
2	214	212	211	215
3	224	220	218	221

13 E **14** D **15** E **16** B

17 B

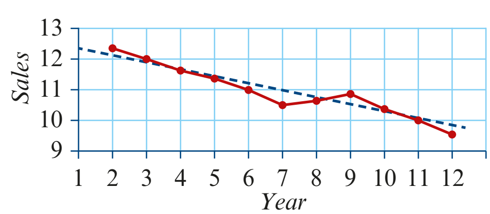
Exercise 5E

1 a There was a general increasing trend in the number of university students in Australia during the period 2010–2019

b *students*(000*s*) = −37563 + 18.927 × *year*. On average, the number of university students in Australia has increased by 18 927 per year.

c 859 000 (to nearest thousand)

2 a



b There was a general decreasing trend in the percentage of retail sales made in department stores.

c $sales = 12.5 - 0.258 \times year$

The percentage of total retail sales that are made in department stores is decreasing by 0.258% per year.

d 8.6%

3 a $age = -147 + 0.0882 \times year$; On average, the average age of mothers increased by 0.0882 years (equivalent to 1 month) each year between 2010 and 2020.

b 32.0 years; Unreliable as we are extrapolating 10 years beyond the period in which the data were collected.

4 a $earnings = -83\,280 + 42.07 \times year$; On average, average weekly earnings increased by \$42.07 each year between 2014 and 2021.

b \$2122.10; Unreliable as we are extrapolating 9 years beyond the period in which the data were collected.

5 a deseasonalised number $= 50.9 + 1.59 \times$ quarter number

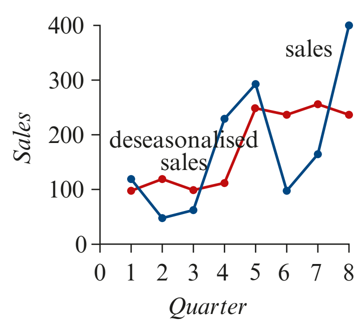
b deseasonalised number = 76.34

reseasonalised (actual) number = 90 (to the nearest whole number)

6 a

Year	Q1	Q2	Q3	Q4
1	122	128	118	130
2	250	245	263	236

b



The deseasonalised sales appear to show an increasing trend over time.

c deseasonalised sales $= 80.8 + 23.5 \times quarter$

d forecasted actual sales $= 386.3 \times 1.13 = 437$

7 C **8** E

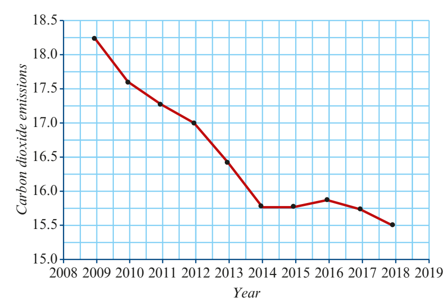
Chapter 5 review

Multiple-choice questions

1 E **2** E **3** A **4** B **5** E **6** C
7 D **8** D **9** C **10** D **11** B **12** C
13 B **14** A **15** E **16** D **17** D **18** A
19 B **20** E

Written-response questions

1 a



b Carbon dioxide emissions decreased between 2009 and 2014, then remained reasonable steady over the years 2014–2017, showing only irregular fluctuations. Between 2017 and 2018 there was a small decrease in carbon dioxide emissions.

c $CO_2\ emissions = 612.0 - 0.2958 \times year$

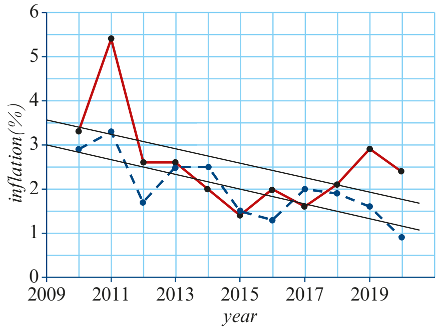
d 12.7

e Unreliable as we are extrapolating 8 years beyond the period in which the data were collected.

2 a i $inflation = 332 - 0.164 \times year$

b i $inflation = 339 - 0.167 \times year$

a ii, b ii



c The trend lines are parallel. As such, they will never cross, so the inflation rate for China will remain higher than the inflation rate for Australia.

d 1.7

3 a

	Sum	Aut	Win	Spr
SI	0.29	0.36	1.37	1.98

b

	Sum	Aut	Win	Spr
Deseas	269	239	255	273

Chapter 6

Exercise 6A

1 A **2** B **3** B **4** B
5 D **6** B **7** D **8** E
9 E **10** E **11** D **12** A
13 D **14** A **15** A **16** E
17 C **18** B **19** B **20** B
21 C **22** E **23** D **24** C
25 A **26** D

Exercise 6B

1 E **2** A **3** E **4** C
5 C **6** B **7** E **8** B
9 B **10** E **11** B **12** E
13 D **14** B **15** E **16** D
17 A **18** D

Exercise 6C

1 C **2** C **3** A **4** D
5 B **6** B **7** E **8** B
9 D **10** A **11** E **12** E
13 B **14** A **15** E **16** C
17 D

Exercise 6D

1 A **2** A **3** A **4** B
5 E **6** B **7** A **8** B
9 D **10** E **11** D **12** E
13 E **14** B **15** B **16** D
17 A

Exercise 6E

1 a *age*, *distance*

b mean = 7.17 km, sd = 3.46 km

c $z = 1.7$

d

	Gender	
Study mode	Female	Male
On campus	3	3
Online	4	2
Total	7	5

e i 60%

e ii Yes, there is an association between study mode preference and course. A higher percentage of students business chose to study online (60%), compared to only 36% for both students of Health and Social Science.

2 a The distribution of *distance* is positively skewed, with outliers at 17 km, 18 km, and 19 km.

b 30

c i Lower fence = -2, Upper fence = 14.

ii A distance of 1 km is within the fences.

d i 1 km **ii** 1.5 km

3 a On average, height increases by 0.815 cm for each additional 1 cm increase in arm span.

b **i** Females: $r^2 = 64.6\%$

ii Males: $r^2 = 69.9\%$

iii Since the value of the coefficient of determination for males (69.9%) is higher than the value for females (64.6%), then we can say that arm span is a better predictor of height for males than for females.

c **i** The models predict that when both have arm span measurements of 160 cm, a male will be 1.8 cm taller than a female.

ii The models predict that when both have arm span measurements of 190 cm, a female will be 4.6 cm taller than a male.

iii The differences predicted are not reliable for a height of 160 cm as this value is outside the range of height data for males. The prediction is not reliable for a height of 190 cm as this value is outside the range of height data for females.

4 **a** There is a moderate strength, non-linear association between *expenditure* and *score*.

b y^2, $\log x$, $\frac{1}{x}$

c **i** The linearity assumption.

ii No, there is a clear structure in the residual plot. If the linearity assumption had been met the residuals would have been randomly scattered around a horizontal line at $y = 0$.

d **i** $score = 12.99 + 120.6 \times \log(expenditure)$

ii 495

5 **a** **i** \$12 000

ii \$11 000

b \$52 208.29

c **i** Slope = \$1525.80. On average the price of bitcoin is increasing by \$1525.80 each month.

ii \$92 837

iii Unreliable as we are extrapolating several years beyond the period in which the data were collected.

Chapter 7

Exercise 7A

1 **a** 2, 8, 14, 20, 26 **b** 5, 2, −1, −4, −7
c 1, 4, 16, 64, 256 **d** 64, 32, 16, 8, 4

2 **a** 6, 14, 30, 62, 126
b 24, 16, 12, 10, 9 **c** 1, 2, 5, 14, 41
d 124, 60, 28, 12, 4

3 **a** 4, 6, 8, 10, 12 **b** 24, 20, 16, 12, 8
c 2, 6, 18, 54, 162 **d** 50, 10, 2, 0.4, 0.08
e 5, 13, 29, 61, 125
f 18, 16.4, 15.12, 14.096, 13.2768

4 **a** 2, 5, 8, 11, 14 **b** 50, 45, 40, 35, 30
c 1, 3, 9, 27, 81 **d** 3, −6, 12, −24, 48
e 5, 9, 17, 33, 65 **f** 2, 7, 17, 37, 77
g −2, −1, 2, 11, 38
h −10, 35, −100, 305, −910

5 **a** 12, 57, 327, 1947, 11667
b 20, 85, 280, 865, 2620
c 2, 11, 47, 191, 764
d 64, 15, 2.75, −0.3125, −1.078125
e 48000, 45000, 42000, 39000, 36000
f 25000, 21950, 19205, 16734.50, 14511.05

6 **a** $A_2 = 6$ **b** $B_4 = 2$ **c** $C_3 = 27$
d $D_5 = 95$

7 **a** $V_0 = 4,\quad V_{n+1} = V_n + 2$
b $V_0 = 24,\quad V_{n+1} = V_n - 4$
c $V_0 = 2,\quad V_{n+1} = 3V_n$

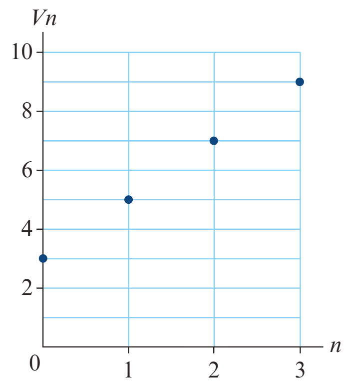
8 **a** $V_0 = 5,\quad V_{n+1} = V_n + 5$
b $V_0 = 13,\quad V_{n+1} = V_n - 4$
c $V_0 = 1,\quad V_{n+1} = 4V_n$
d $V_0 = 64,\quad V_{n+1} = 0.5V_n$

9 5

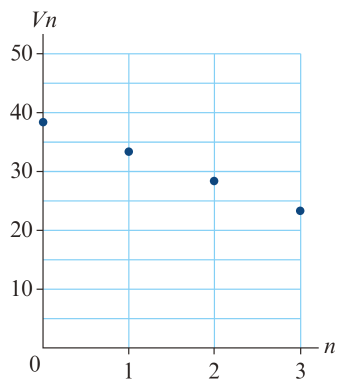
10 0 **11** E **12** D **13** E

Exercise 7B

1 a



b



2 a $V_0 = 8000$ **b** \$320

c $V_0 = 8000, \quad V_{n+1} = V_n + 320$

3 a $H_0 = 41\ 000$ **b** \$2542

c $H_0 = 41\ 000, \quad H_{n+1} = H_n + 2542$

4 a $V_0 = 2000$ **b** 14 years

$V_1 = 2000 + 76 = 2076$

$V_2 = 2076 + 76 = 2152$

$V_3 = 2152 + 76 = 2228$

5 a \$7518, \$8036, \$8554 **b** 6 years

6 a i \$15 000 **ii** \$525 **iii** 3.5%

b 29 years

7 a \$12 300

b $C_0 = 82\ 000, \quad C_{n+1} = C_n - 12\ 300$

8 a $\frac{8}{100} \times 2800 = 244$ **b** 2800

c $W_0 = 2800, \quad W_{n+1} = W_n - 224$

9 a \$2100, \$1700, \$1300 **b** 4

10 a \$22 195, \$21 390, \$20 585 **b** 17

11 a i \$1500 **ii** \$102 **iii** 6.8%

b \$684

c 14 years

12 a \$32 600

b $M_0 = 32\ 600, \quad M_{n+1} = M_n - 10$

13 a 450, 449.95, 449.90, 449.85, 449.80

b \$449

14 a \$47 800, \$47 600, \$47 400

b \$45 000 **c** 25 000 km

15 a 7200 **b** 72 000 **c** \$720 **d** 10%

16 C **17** C **18** B

Exercise 7C

1 a $A_n = 4 + 2n, A_{20} = 44$

b $A_n = 10 - 3n, A_{20} = -50$

c $A_n = 5 + 8n, A_{20} = 165$

d $A_n = 300 - 18n, A_{20} = -60$

2 a 5000 **b** \$270 **c** $V_n = 5000 - 270n$

d \$7430

3 a 12 000 **b** \$864

c $V_n = 12\ 000 + 864n$ **d** \$19 776

4 a \$8000 **b** \$512

c i \$14 144 **ii** 16 years

5 a \$2000 **b** \$70

c i \$2420 **ii** 29 years

6 a \$5600 **b** \$1260 **c** $V_n = 5600 - 1260n$

d \$1820

7 a \$7000 **b** \$1225 **c** $V_n = 7000 - 1225n$

d 5 years

8 a \$1700 **b** \$212.50 **c** \$850

d \$212.50 **e** 8 years

9 a \$65 000 **b** \$3250 **c** 5%

d \$42 250 **e** 11 years

10 a \$29 000 **b** \$0.25 (25 cents)

c \$24 000 **d** 96 000 km

11 **a** \$9700 **b** \$0.388 per km

c $V_n = 35\,400 - 0.388n$

d 74 000 km

12 **a** **i** \$0.026875 **ii** \$69 687.50

iii \$20 156.25

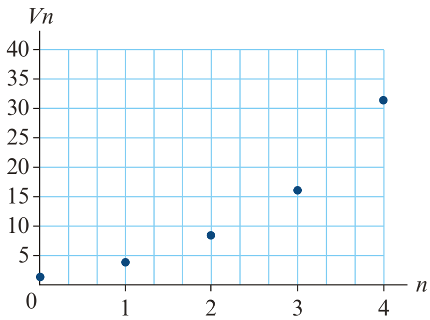
b \$9218.75

c 1 486 400

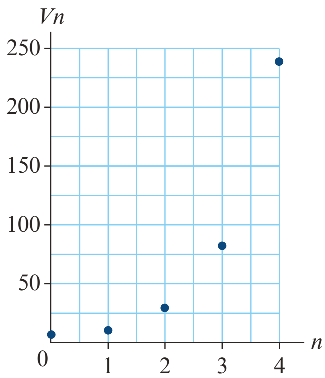
13 D **14** C

Exercise 7D

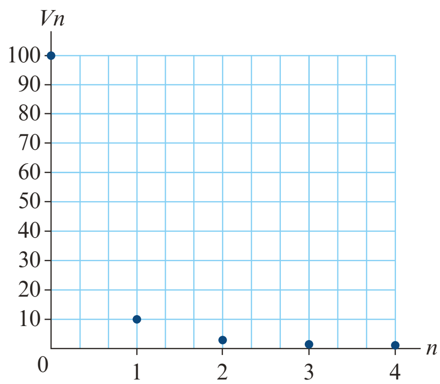
1 **a** 2, 4, 8, 16, 32



b 3, 9, 27, 81, 243



c 100, 10, 1, 0.1, 0.01



2 **a** $V_0 = 6000$

$V_1 = 1.042 \times 6000 = 6252$

$V_1 = 1.042 \times 6252 = 6514.58$

$V_2 = 1.042 \times 6514.58 = 6788.20$

b 7 years

3 **a** $V_0 = 20\,000$

$V_1 = 1.063 \times 20\,000 = 21\,260$

$V_1 = 1.063 \times 21\,260 = 22\,599.38$

$V_2 = 1.063 \times 22\,599.38 = 24\,023.14$

b 7 years

4 **a** \$5000 **b** 1.068

c $V_0 = 5000,\quad V_{n+1} = 1.068 \times V_n$

d \$6947.46 **e** \$1947.46

5 **a** \$18 000 **b** 1.094

c $V_0 = 18\,000,\quad V_{n+1} = 1.094V_n$

d \$25 783.50 **e** 4 years

6 $V_0 = 9800,\quad V_{n+1} = 0.965V_n$

7 $M_0 = 28\,600,\quad M_{n+1} = 0.926M_n$

8 **a** $V_0 = 18\,000,\quad V_{n+1} = 0.955V_n$

b \$17 190, \$16 416.45, \$15 677.71, \$14 972.21, \$14 298.46

c \$15 677.71 **d** \$3701.54

9 **a** $W_0 = 4000,\quad W_{n+1} = 0.959W_n$

b \$3527.90 **c** 755.46

10 **a** $S_0 = 13\,420,\quad S_{n+1} = 0.888S_n$

b \$11 916.96, \$10 582.26, \$9397.05, \$8344.58, \$7409.99

c \$7409.99 **d** \$1185.21

11 C **12** E **13** E **14** C

Exercise 7E

1 **a** $V_n = 2^n \times 6, V_4 = 96$

b $V_n = 3^n \times 10, V_4 = 810$

c $V_n = 0.5^n \times 1, V_4 = 0.0625$

d $V_n = 0.25^n \times 80, V_4 = 0.3125$

2 **a** **i** 3000 **ii** 10%

b $V_n = 1.1^n \times 3000$

c \$4831.53

3 **a** **i** \$2000 **ii** 6%
b $V_n = 1.06^n \times 2000$
c \$2524.95 **d** \$837.04

4 **a** $V_n = 1.125^n \times 8000$
b \$11 390.63 **c** \$3390.63
d \$1265.63

5 **a** **i** \$1200 **ii** 12%
b $V_n = 0.88^n \times 1200$ **c** \$490.41

6 **a** $V_n = 0.905^n \times 38\,500$
b \$23 372.42 **c** \$15 127.58

7 6 years **8** 100 years **9** 4

10 6% **11** \$9223.52 **12** \$32 397.18

13 C **14** E

Exercise 7F

1 **a** 0.40% **b** 2.08% **c** 0.40% **d** 0.14%
e 0.03%

2 **a** 6.48% **b** 5.8% **c** 14.82%
d 9.88% **e** 8.03%

3 **a** $V_0 = 8000, \quad V_{n+1} = 1.048\,V_n$
b $V_0 = 8000, \quad V_{n+1} = 1.012\,V_n$
c $V_0 = 8000, \quad V_{n+1} = 1.004\,V_n$

4 **a** $V_0 = 20\,000, \quad V_{n+1} = 1.005 \times V_n$
b $V_n = 1.005^n \times 20\,000$ **c** \$26 977

5 **a** $V_0 = 8000, \quad V_{n+1} = 1.012 \times V_n$
b $V_n = 1.012^n \times 8000$
c \$9231.16

6 **a** $V_0 = 7600, \quad V_{n+1} = 1.005 \times V_n$
b $V_n = 1.005^n V_n \times 7600$
c \$7791.91 **d** 139 months

7 **a** $V_0 = 3500, \quad V_{n+1} = 1.02 \times V_n$
b \$3788.51

8 **a** 4.68% **b** 4.70% **c** Monthly

9 **a** 8.25% **b** 8.24% **c** Monthly

10 **a** A – 8.62%, B – 8.11%
b A – \$3018.10, B – \$2837.08
c B – this loan will be charged less interest

11 **a** A – 5.43%, B – 5.61%
b A – \$7603, B – \$7860
c B – this investment will earn more interest

12 **a** 6.38% **b** 8.76% **c** 4.91% **d** 13.10%

13 6.78%

14 E **15** C **16** E **17** A

Chapter 7 review

Multiple-choice questions

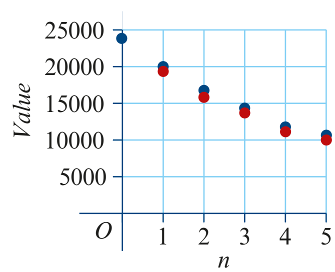
1 C	**2** E	**3** D	**4** C	**5** A
6 A	**7** D	**8** A	**9** B	**10** A
11 D	**12** C	**13** C	**14** B	**15** A
16 B	**17** C	**18** D	**19** B	**20** C

Written-response questions

1 **a** $V_0 = 20\,000, \; V_{n+1} = V_n + 1880$
b \$29 400
c $W_0 = 20\,000, \; W_{n+1} = 1.094W_n$
d $W_n = 1.094^n \times 20\,000$
e \$31 341.27

2 \$328.03

3 **a** **i** $V_0 = 22\,500, \; V_{n+1} = V_n - 2700$
ii \$9000
b **i** $V_0 = 22\,500, \; V_{n+1} = 0.84V_n$
ii \$9409.77
c



4 **a** \$0.20
b Let V_n be the value of the vacuum cleaner after cleaning n offices.
$V_0 = 650, \; V_{n+1} = V_n - 0.20$
c \$250

5 **a** \$6575 **b** \$6722.75 **c** 6.9%

6 **a** $V_0 = 30\,000, \; V_{n+1} = 1.0075\,V_n$
b \$31 142.00
c \$32 814.21 **d** \$34 318.81

7 $234.57

8 3.5

Chapter 8

Exercise 8A

1 a 2, 5, 11, 23, 47
b 50, 90, 170, 330, 650
c 128, 96, 80, 72, 68

2 a $500 **b** $100 **c** 1.03
d $V_0 = 500, V_{n+1} = 1.03V_n + 100$

3 a $300 000 **b** $50 000
c 1.052
d $V_0 = 300\,000, V_{n+1} = 1.052V_n + 50\,000$

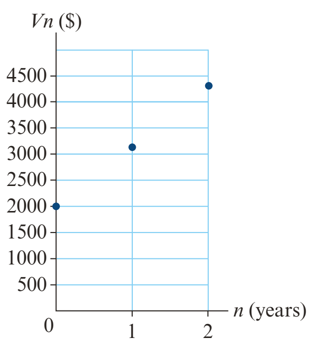
4 a 1.003
b $V_0 = 3500, V_{n+1} = 1.003V_n + 150$
c $3821.48

5 a $V_0 = 1700, V_{n+1} = 1.008V_n + 100$
b $2395.38

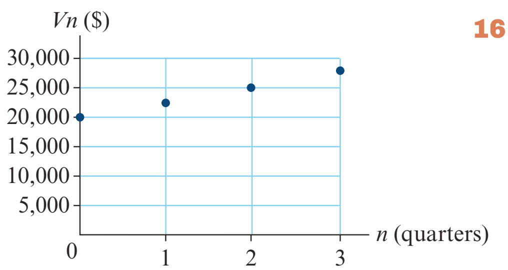
6 $V_0 = 1500, V_{n+1} = 1.0002V_n + 4$

7 $V_0 = 24000, V_{n+1} = 1.005V_n + 500$, $27 766.81

8 a $2000 **b** $1000 **c** $4412.80
d



9 a $20 000 **b** $2000
c $27 689.06
d



10 8%

11 10%

12 C 13 B 14 C

Exercise 8B

1 $V_0 = 5000, V_{n+1} = 1.054V_n - 1400$

2 a $V_0 = 2000, D = 339$
b $R = 1.005$
c $V_0 = 2000, V_{n+1} = 1.005V_n - 339$

3 a $B_0 = 10\,000, B_{n+1} = 1.03B_n - 2600$
b $5331

4 $V_0 = 3500, V_{n+1} = 1.004V_n - 280$

5 $V_0 = 150\,000, V_{n+1} = 1.0014V_n - 650$

6 a $V_0 = 235\,000, V_{n+1} = 1.0001V_n - 150$
b $234 620.46

7 a $2500 **b** $626 **c** 8%
d $1117.03

8 a $5000 **b** $865 **c** $r = 12\%$
d $3361.85

9 a $V_0 = 20\,000, D = 3375$
b $R = 1.072$
c $V_0 = 20\,000, V_{n+1} = 1.072V_n - 3375$

10 a $V_0 = 750\,000, D = 4100$
b $R = 1.0045$
c $V_0 = 750\,000, V_{n+1} = 1.0045V_n - 4100$

11 a $V_0 = 40\,000, V_{n+1} = 1.015V_n - 10\,380$
b $10 217.70

12 a $5000 **b** $1030 **c** 12%
d $2030.50 **e** $3090

13 a $3052.65 **b** $6000

14 a $1 000 000 **b** $400 **c** 2.88%
d $996 796.16

15 a $18 400 **b** 6.6%
c $9762.84

16 D **17** E **18** A

Exercise 8C

1 a \$14 000 **b** \$1540 **c** \$260

d \$13 740

e

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	14 000.00
1	1800.00	1540.00	260.00	13 740

f

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	14 000.00
1	1800.00	1540.00	260.00	13 740.00
2	1800.00	1511.40	288.60	13 451.40
3	1800.00	1479.65	320.35	13 131.05

2 a \$12 000 **b** 0.005

c \$60 **d** \$240 **e** \$11 760

f

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	12 000.00
1	300.00	60.00	240.00	11 760.00

g

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	12 000.00
1	300.00	60.00	240.00	11 760.00
2	300.00	58.80	241.20	11 518.80
3	300.00	57.59	242.41	11 276.39

3 a \$36 000 **b** 0.02

c

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	36 000.00
1	1000.00	720.00	280.00	35 720.00
2	1000	714.40	285.60	35 434.40
3	1000	708.69	291.31	35 143.09

4 a 1%

b $A = 16.75$, $B = 334.85$, $C = 342.17$

5 a \$4000, \$100 **b** 2.5%

c $A = 64.81$, $B = 505.16$, $C = 1076.65$, $D = 26.92$, $E = 530.73$

6 $A = 421.26$, $B = 458\,736.22$

7 a i \$15.00 **ii** 0.25%

b $A = 12.53$, $B = 495.47$, $C = 4517.30$

8 a 1%

b $A = 53.02$, $B = 153.02$, $C = 5454.52$

9 D **10** D **11** D

Exercise 8D

1 $554.16

2 $1262.60

3 $692.58

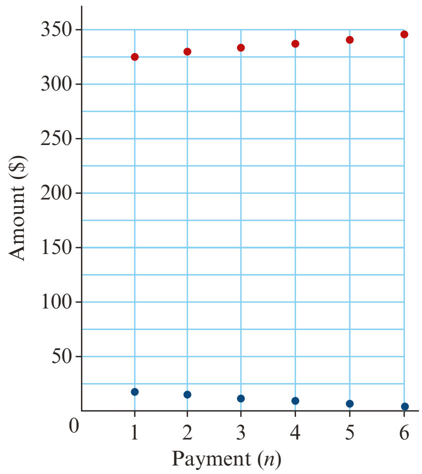
4 $771.27

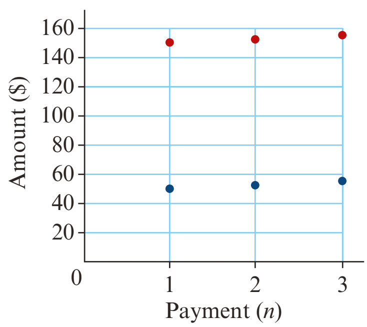
5 a 180.53 b 1380.53 c $624.67

6 a $A = 345.69, B = 1.72, C = 343.97$
b $4131.23 c $131.23

7 a $A = 3903.19, B = 34.82, C = 3868.37$
b $31 227.69 c $1227.69

8 $12 165.50, $165.50

9


10


11 E 12 C

Exercise 8E

1 a Negative b Negative
c $123 003.55 d $733 636.83

2 a $22 327.26 b $37 864.50

3 a Positive b Negative
c $6061.91

4 a $21 867.22 b $12 095.13

5 a $225 788.13 b $5452.89

6 a $34 093.96 b $344.64

7 a Negative b $28 674
c $6825.74

8 a $15 133.81 b $1732.49

9 a $7627.37 b $298.51

10 E 11 E 12 B

Exercise 8F

1 8.39%

2 a 2.7% b $741.19
c 60 months

3 a $500
b i $6000 ii 6491.15
iii $491.15
c 23 months

4 55 payments

5 a $197 793.85
b i $2158.06 ii $774 237.82
iii $534 237.82

6 a $617.80 b $617.72
c $18 533.92 d $1533.92

7 a i $5253.40 ii $210 135.61
b i $4175.11 ii $4174.54
iii $250 506.03

8 $446 972.41

9 $2591.94

10 a 9.24% b $2348.50
c 57 months

11 96 quarters

12 a $4000 b $22 965.28

13 E

14 C

15 B

Exercise 8G

1 $46 615.21
2 $178 558.60
3 a $81 939.67 b $67 141.09
4 $416.37
5 $338 807.90
6 a $5312.50 b $6500.67 c $6495.55 d $2 268 945.88 e $1 518 945.88
7 a 153 months b $229.96
8 a $V_0 = 125\,000, V_{n+1} = 1.0045V_n - 850$ b $11 966
9 a $S_0 = 150\,000, S_{n+1} = 1.005 \times S_n$ b $272 909.51 c 2.72%
10 a $29 697.05 b $34 378
11 a $3504.76 b $104.76 c $18929.68 d 58 payments
12 B
13 B
14 E

Exercise 8H

1 $600
2 $350
3 $462
4 $44 550
5 a $2775 b $32 775
6 a $2555 b $306 600 c $1 006 600
7 a $182 b $35 000 c $272.48 d $64 892.48
8 $40 000
9 $16 000
10 $28 000
11 2.9%
12 3.6%
13 4.8%
14 a $1312 b $78 720 c $86 400 d $1440 e 5.4%
15 C 16 B 17 D

Exercise 8I

1 a $9790.50 b $642 000 c $642 000
2 a $4800 b $57 600
3 a $2160 b It will increase
4 $100 000
5 $50 000
6 a $540 000 b $380 000 c $451 866.88
7 3%
8 6.25%
9 6.3%
10 a $17 850 b $350 000 c $M_0 = 350\,000, M_{n+1} = 1.00425M_n - 1487.50$
11 a $600 b $7600 c Option B d $Z_0 = 200\,000, Z_{n+1} = 1.038Z_n - 7600$
12 E
13 D

Chapter 8 review

Multiple-choice questions

1 C	2 D	3 D	4 B
5 D	6 C	7 D	8 B
9 C	10 C	11 D	12 E
13 D	14 E	15 C	16 D
17 E	18 C	19 E	20 D

Written-response questions

1 a $V_0 = 250\,000, V_{n+1} = 1.004V_n - 1800$ b $240 185.96 c 56 months

d **i** \$1000 **ii** \$12 000
iii \$250 000

2 **a** \$781.25 **b** \$147 298.48
c 38 months
d **i** 41 payments **ii** \$3323.07

3 **a** \$656.65 **b** \$13 134
c \$3134

4 **a** 40 months **b** \$320.78
c \$4770.78

5 **a** \$247.04 **b** \$83 713.37

6 **a** \$1 175 244.58 **b** 290 months
c \$3300

Chapter 9

Exercise 9A

1 E **2** C **3** C **4** E **5** C **6** D
7 E **8** D **9** A **10** D **11** D **12** D
13 E **14** E **15** D

Exercise 9B

1 **a** \$8500 **b** \$222.44
c $A_0 = 8500, A_{n+1} = 1.013 \times A_n$
d 5.2% **e** 13 quarters

2 **a** $V_0 = 25\,000,$
$V_1 = 25\,000 - 936 = 24\,064,$
$V_2 = 24\,064 - 936 = 23\,128,$
$V_3 = 23\,128 - 936 = 22\,192$
b 36

3 **a** \$260 000
b $\dfrac{1170}{260\,000} \times 100 = 5.4\%$
c $A = 1156.71, B = 993.29, C = 256\,053.46$

4 **a** 204 **b** \$29516.73 **c** The first

5 **a** \$4400 **b** $\dfrac{50}{20} = \$2.50$
c $H_n = 4450 - 2.50n$ **d** \$3950
e 581

6 **a** \$5520 **b** 22 months

Chapter 10

Exercise 10A

1 **a** 2×3 **b** 1×3 **c** 3×2 **d** 3×1
e 3×3

2 **a** 2×3 **b** 4×1 **c** 1×3

3 **a** 12 **b** 15 **c** 28

4 $1 \times 12,\ 12 \times 1,\ 6 \times 2,\ 2 \times 6,\ 4 \times 3,\ 3 \times 4$

5 **a** $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 5 \end{bmatrix}$ **c** $\begin{bmatrix} 9 & 8 \\ 1 & 9 \\ 0 & 1 \\ 7 & 5 \end{bmatrix}$

6 **a** square; 2×2; 4 **b** column 3×1; 3
c row; 1×4; 4

7 **a** $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 7 \\ 0 & 0 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
c $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$
d First 3 are symmetric.

8 **a** C, E **b** 3 **c** A **d** B
e 4, 2 **f** 3, 3 **g** 1, 5 **h** 3, 1
i 4, 2 **j** 9 **k** 5 **l** 0
m 1 **n** 0 **o** 4 **p** −1
q 3 **r** 3 **s** 1

9 $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$ **10** $\begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$ **11** $\begin{bmatrix} -2 & -5 \\ -1 & -4 \\ 0 & -3 \end{bmatrix}$

12 $\begin{bmatrix} 2 & 5 & 10 \end{bmatrix}$ **13** $\begin{bmatrix} 4 & 9 \\ 9 & 16 \end{bmatrix}$

14 **a** $\begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 4 & -2 \\ -4 & 6 \end{bmatrix}$ **c** $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$
d $\begin{bmatrix} 2 & 3 \end{bmatrix}$

15 C **16** D **17** B **18** C
19 B

Exercise 10B

1 a $\begin{bmatrix} 4 & 2 & 1 \\ 6 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix}$, 3×3

b $\begin{bmatrix} 6 & 2 & 3 \end{bmatrix}$, 1×3

c $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$, 3×1; the total number of computers owned by the three households

2 a $\begin{bmatrix} 24 & 32 & 11 \\ 32 & 34 & 9 \end{bmatrix}$, 2×3;

b $\begin{bmatrix} 24 & 32 & 11 \end{bmatrix}$, 1×3

c $\begin{bmatrix} 24 \\ 32 \end{bmatrix}$, 2×1; the total number of small cars sold by the car dealers

3 a 4×4 **b** $\begin{bmatrix} 430 & 380 & 950 & 900 \end{bmatrix}$; 1×4;The total exports of B

c $\begin{bmatrix} 370 \\ 950 \\ 150 \\ 470 \end{bmatrix}$; 4×1

4 $\begin{bmatrix} 200 & 110 \\ 180 & 117 \\ 135 & 98 \\ 110 & 89 \\ 56 & 53 \\ 28 & 33 \end{bmatrix}$

5 $\begin{bmatrix} 3 & 5 & 8 & 7 & 0 & 2 & 3 & 6 \\ 4 & 2 & 2 & 9 & 0 & 0 & 0 & 9 \end{bmatrix}$

6 $\begin{bmatrix} 21 & 5 & 5 \\ 8 & 2 & 3 \\ 4 & 1 & 1 \\ 14 & 8 & 6 \\ 0 & 1 & 2 \end{bmatrix}$

7 a $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

c $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

8 $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix}$

9 a Girls 3 and 4 are friends.
b Girls 2 and 5 are not friends.
c 3: girl 3 has three friends.
d girl 1, girl 3

10 a i Polar bears eat cod.
ii Nothing eats polar bears.

b
$$W = \begin{array}{c} \begin{array}{cccc} P & S & C & Z \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ P \\ S \\ C \\ Z \end{array}$$

11 C **12** B

Exercise 10C

1 $a = 9, b = 7$

2 a C, F
b A and B, C and F, D and E
c A and B, C and F, D and E

d **i** $\begin{bmatrix} 4 & 4 \end{bmatrix}$ **ii** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

iii $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **iv** $\begin{bmatrix} -2 & 2 \end{bmatrix}$

v $\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$ **vi** $\begin{bmatrix} 9 & 3 \end{bmatrix}$

vii $\begin{bmatrix} 0 & 4 & 16 \\ 12 & 8 & 4 \end{bmatrix}$ **viii** $\begin{bmatrix} 0 & 4 & 16 \\ 12 & 8 & 4 \end{bmatrix}$

ix $\begin{bmatrix} -2 & 10 \end{bmatrix}$ **x** not defined

4 a $\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$ **b** $\begin{bmatrix} -3 & -1 \\ 3 & 1 \end{bmatrix}$

c $\begin{bmatrix} 9 & 8 \\ 6 & 7 \end{bmatrix}$ **d** $\begin{bmatrix} 0 & 0 \end{bmatrix}$ **e** $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

f $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ **g** $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ **h** not defined

5 a $\begin{bmatrix} -2.2 & 1.1 \\ 7.7 & 4.4 \end{bmatrix}$ **b** $\begin{bmatrix} -0.2 & -13.8 \\ 1 & -3.7 \end{bmatrix}$

c $\begin{bmatrix} 5 & 8 & 7 \\ 16 & 0 & 3 \\ -1 & 5 & 4 \end{bmatrix}$ **d** $\begin{bmatrix} 0.6 & 2 & 1 & 3.2 \\ 1 & 0 & -0.6 & 2 \end{bmatrix}$

6 $x = 2, y = 4, z = 6, w = -4$

7 a

$$A = \begin{bmatrix} 2.4 \\ 3.5 \\ 1.6 \end{bmatrix} \; B = \begin{bmatrix} 2.8 \\ 3.4 \\ 1.8 \end{bmatrix} \; C = \begin{bmatrix} 2.5 \\ 2.6 \\ 1.7 \end{bmatrix} \; D = \begin{bmatrix} 3.4 \\ 4.1 \\ 2.1 \end{bmatrix}$$

b $\begin{bmatrix} 11.1 \\ 13.6 \\ 7.2 \end{bmatrix}$; the total (yearly) DVD sales for each store

8 a

$$A = \begin{bmatrix} 16 & 104 & 86 \\ 75 & 34 & 94 \end{bmatrix} \; B = \begin{bmatrix} 24 & 124 & 100 \\ 70 & 41 & 96 \end{bmatrix}$$

b $C = \begin{bmatrix} 40 & 228 & 186 \\ 145 & 75 & 190 \end{bmatrix}$; the total number of females and males enrolled in each of the three programs for the two years

c $D = \begin{bmatrix} 8 & 20 & 14 \\ -5 & 7 & 2 \end{bmatrix}$; the increase in the number of females and males enrolled in each of the three programs for the two years; a decrease in the number of men enrolled in weights classes

d $E = \begin{bmatrix} 48 & 248 & 200 \\ 140 & 82 & 192 \end{bmatrix}$

9 C **10** C **11** E

Exercise 10D

1 a i, ii, iv, v, vi, vii

b **i** [6] **ii** [2] **iii** $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ **iv** $\begin{bmatrix} -3 & 7 \end{bmatrix}$

c **i** [6] **ii** $\begin{bmatrix} 2 & 0 & -2 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ **iii** [0]

iv undefined

2 a [0] **b** [1] **c** [3] **d** $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

e $\begin{bmatrix} 5 & 5 \\ 1 & 2 \end{bmatrix}$ **f** $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$

3 a $\begin{bmatrix} 1 & 2 & 3 \\ -3 & -6 & -9 \\ 5 & 10 & 15 \end{bmatrix}$ **b** $\begin{bmatrix} 6 & 7 \\ 19 & 18 \end{bmatrix}$

c $\begin{bmatrix} 4 & 0 \\ 2 & -2 \\ 1 & 4 \end{bmatrix}$ **d** $\begin{bmatrix} -5 & 15 & 9 \\ -2 & 8 & 4 \\ -5 & 8 & 7 \end{bmatrix}$

4 a $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ post-multiply by this matrix.

b $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ pre-multiply by this matrix.

5 $\begin{bmatrix} 7 & 1 & 2 \\ 1 & 2 & 2 \\ 8 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 13 \end{bmatrix}$.

6 $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 9 & 0 & 2 \\ 1 & 7 & 3 \\ 8 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 18 & 10 & 9 \end{bmatrix}$

7 a $\begin{bmatrix} 22 \\ 12 \\ 18 \\ 12 \\ 30 \end{bmatrix}$ **b** $\begin{bmatrix} 8 & 10 & 6 & 14 & 11 \end{bmatrix}$

8 $\begin{bmatrix} 9 \\ 7 \\ 6 \\ 4 \\ 3 \\ 1 \end{bmatrix}$ **9** $\begin{bmatrix} 3000 \\ 2800 \\ 2600 \\ 2200 \end{bmatrix}$

10 a 2×3

b i $\begin{bmatrix} 184.50 \\ 236 \end{bmatrix}$

ii the total revenue from selling products A, B and C at Energy and Nourishing respectively

c number of columns in $P \neq$ number of rows in Q

11 $XY = \begin{bmatrix} 110\,000 \\ 116\,000 \\ 154\,000 \\ 58\,000 \end{bmatrix}$ It gives the total sales of each of the dealers.

12 a $\begin{bmatrix} 29 \\ 8.50 \end{bmatrix}$, John took 29 minutes to eat food costing \$8.50

b $\begin{bmatrix} 29 & 22 & 12 \\ 8.50 & 8.00 & 3.00 \end{bmatrix}$, John's friends took 22 and 12 minutes to eat food costing \$8.00 and \$3.00 respectively

13 a $\begin{bmatrix} 79 & 78 & 80 \\ 80 & 78 & 82 \end{bmatrix}$ **b** $\begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$

c Semester 1: 79.2; Semester 2: 80.4

d Semester 1: 83.8; Semester 2: 75.2

e No, total score is 318.6

f 3 marks

14 $\begin{bmatrix} 5 & 5 \\ 5 & 10 \end{bmatrix}, \begin{bmatrix} 15 & 20 \\ 20 & 35 \end{bmatrix}, \begin{bmatrix} 50 & 75 \\ 75 & 125 \end{bmatrix}, \begin{bmatrix} 2250 & 3625 \\ 3625 & 5875 \end{bmatrix}$

15 $\begin{bmatrix} 5 & 7 \\ 7 & 26 \end{bmatrix}, \begin{bmatrix} 2 & 19 \\ 19 & 59 \end{bmatrix}, \begin{bmatrix} 17 & 40 \\ 40 & 137 \end{bmatrix}, \begin{bmatrix} 23 & 97 \\ 97 & 314 \end{bmatrix}$

16 $\begin{bmatrix} 24 & 30 & 36 \\ 38 & 59 & 64 \\ 33 & 54 & 51 \end{bmatrix}$

17 a $\begin{bmatrix} -1 & 5 \\ 5 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} -3 & 8 \\ 6 & -3 \end{bmatrix}$

c $\begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$ **d** $\begin{bmatrix} 16 & 2 \\ 2 & 39 \end{bmatrix}$

e $\begin{bmatrix} 29 & -5 \\ -5 & 13 \end{bmatrix}$

18 A **19** C **20** D

Exercise 10E

1 a i $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **ii** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

iii $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

b $AI = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix};$

$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$\therefore AI = IA = A$

c

$CI = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix};$

$IC = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

$CI = IC = C$

3 a 3 **b** −3 **c** 0 **d** −8

4 a $\begin{bmatrix} \frac{10}{11} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{20}{9} & \frac{1}{18} \\ \frac{-50}{9} & \frac{1}{9} \end{bmatrix}$

c Matrix has no inverse, det $(D) = 0$

d $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

5 a $\begin{bmatrix} -9 & -3 \\ 1 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} -28 & -15 \\ 39 & 22 \end{bmatrix}$

c $\begin{bmatrix} -14 & 9 \\ -9 & 8 \end{bmatrix}$ **d** $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **e** $\begin{bmatrix} 3.5 \\ 2.5 \\ -1.5 \end{bmatrix}$

f $\begin{bmatrix} -8 \\ 10 \end{bmatrix}$

6 $\begin{bmatrix} -1 & 2 \\ -3 & 5 \end{bmatrix}$

7 a $X = A^{-1}C$

b $X = (AB)^{-1}C = B^{-1}A^{-1}C$

c $X = A^{-1}CB^{-1}$ **d** $X = A^{-1}C - B$

e $X = A^{-1}(C - B)$

f $X = (A - B)A^{-1} = I - BA^{-1}$

8 $x = -5000, y = 15\,000, z = 0$

9

Spray	P	Q	R
Barrels	$\frac{8}{13}$	$\frac{46}{39}$	$\frac{12}{13}$

10 a $\begin{bmatrix} 0.1 & 0.25 & -0.4 \\ 0.3 & -0.75 & 0.8 \\ -0.2 & 0.5 & -0.2 \end{bmatrix}$

b

Product	P	Q	R
Number per day	13.5	0.5	13

11 Brad 20; Flynn 10; Lina 15

12 A **13** E **14** D

Exercise 10F

1 B only

2 a $X = \begin{bmatrix} \text{H} & \text{U} & \text{T} & \text{S} \end{bmatrix}$ **b** $n = 4$

3 $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ **4** $\begin{bmatrix} X \\ W \\ Z \\ Y \end{bmatrix}$

5 a $C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ **b** $C^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

c 2

6 a There is no direct communication link between the towers.

b $T1$ and $T3$

c 1, 0

d There is a 2-step communication link between $T3$ and $T1$.

e 6

f

$$T = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc} T1 & T2 & T3 & T4 \\ \end{array}$$

$$\begin{array}{cc} T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} & \begin{array}{c} T1 \\ T2 \\ T3 \\ T4 \end{array} \end{array}$$

g $T1$ and $T4$

7

$$\begin{array}{c|c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

8 D **9** D

Exercise 10G

1 a

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

D, A, B, C

b

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

D, A, B, C

2 a

$$\begin{array}{ccc} & & \text{losers} \\ & & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \text{winners} & \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$A:4,\ B:1,\ C:0,\ D:2,\ E:2$

A; D and E equal; B; C.

b

$$D^2 = \begin{bmatrix} 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{bmatrix}$$

c

$$T = \begin{bmatrix} 0 & 3 & 3 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 \end{bmatrix} \begin{array}{c} 9 \\ 1 \\ 0 \\ 3 \\ 5 \end{array}$$

The tie can be broken using two-step dominances to give the ranking $A,\ E,\ D,\ B,\ C$.

3 a

$$\begin{array}{ccc} & \begin{array}{ccccc} A & B & C & D & E \end{array} & \text{Score} \\ D = \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \\ 1 \\ 1 \\ 3 \end{array} \end{array}$$

b

$$D^2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix}$$

c
$$D + D^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 2 & 0 & 3 & 3 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 3 & 0 \end{bmatrix}\end{array}\begin{array}{c} \text{Score} \\ 4 \\ 9 \\ 2 \\ 4 \\ 7 \end{array}$$

The matrix $D + D^2$ gives the following ranking:

Rank	Player	Score
First	Bea	9
Second	Eve	7
Equal third	Ann and Deb	4
Fifth	Cat	2

4 a 10

b
- Ash defeats Carl and Dot
- Ben defeats Ash, Carl and Dot
- Carl defeats Dot
- Dot defeats Elle
- Elle defeats Ash, Ben and Carl

c Ben = Elle, Ash, Carl = Dot

5 a $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$

b E, B, $A = C$, D

6 a $\begin{bmatrix} 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$ **b** A

7 a $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ **b** A, B, D, C

8 A, B, D, C

9 E, B, $A = C$, D

10 C **11** A **12** D

Chapter 10 review

Multiple-choice questions

1 C	**2** D	**3** B	**4** D
5 D	**6** A	**7** B	**8** C
9 D	**10** E	**11** A	**12** B
13 D	**14** E	**15** A	**16** D
17 D	**18** E	**19** D	**20** C
21 A	**22** C	**23** C	**24** C
25 A	**26** C	**27** C	**28** C
29 E			

Written-response questions

1 a $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

c $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2 a 2×1 **b** 1×2

c Yes; number of columns in C equals the number of rows in J.

d $[162.41]$; $5 \times 30.45 + 4 \times 2.54 = 162.41$

e $\begin{bmatrix} 172.57 \\ 185.24 \end{bmatrix}$

3 a 456 **b** 2×2

c $B = \begin{bmatrix} 354 & 987 \\ 314 & 586 \end{bmatrix}$

d $C = \begin{bmatrix} 688 & 1863 \\ 527 & 1042 \end{bmatrix}$; the total number of books of each type in the two stores

e **i** 2×1

ii $\begin{bmatrix} 31\,236 \\ 18\,021 \end{bmatrix}$

iii total value of fiction and non-fiction books at bookshop 1

f $2A = \begin{bmatrix} 668 & 1752 \\ 426 & 912 \end{bmatrix}$

Answers 10 review

4 a 1×5

b i $R = \begin{bmatrix} 90 & 135 & 165 & 150 & 60 \\ 48 & 72 & 63 & 88 & 32 \end{bmatrix}$

ii the number of students expected to get a C in Mathematics .

c i $F = \begin{array}{c} \begin{matrix} M & P \end{matrix} \\ \begin{bmatrix} 220 & 197 \end{bmatrix} \end{array}$

ii $FN = \begin{bmatrix} 220 & 197 \end{bmatrix} \begin{bmatrix} 600 \\ 320 \end{bmatrix} = [195\,040]$

The total fees paid are $195 040.

5 a $N = \begin{bmatrix} 8 & 6 & 1 \end{bmatrix}$ **b** $NG = [575]$

c total number of points scored by Daniel

6 a 80 tonnes **b** 100 tonnes

c $186, 000

d i 3×1

ii The price per tonne of each of the minerals

iii $\begin{bmatrix} 1000 \\ 700 \\ 300 \end{bmatrix}$

d $$\begin{array}{cc} & \begin{matrix} A & B & C \end{matrix} \\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 0.45 & 0.35 & 0.15 \\ 0.25 & 0.45 & 0.20 \\ 0.30 & 0.20 & 0.65 \end{bmatrix} \end{array}$$

2 $$\begin{array}{cc} & \begin{matrix} O & B \end{matrix} \\ \begin{matrix} O \\ B \end{matrix} & \begin{bmatrix} 0.96 & 0.98 \\ 0.04 & 0.02 \end{bmatrix} \end{array}$$

3 $$\begin{array}{cc} & \begin{matrix} F & P \end{matrix} \\ \begin{matrix} F \\ P \end{matrix} & \begin{bmatrix} 0.80 & 0.14 \\ 0.20 & 0.86 \end{bmatrix} \end{array}$$

4 B

Chapter 11

Exercise 11A

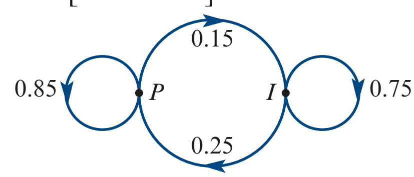
1 a $$\begin{array}{cc} & \begin{matrix} A & B \end{matrix} \\ \begin{matrix} A \\ B \end{matrix} & \begin{bmatrix} 0.40 & 0.55 \\ 0.60 & 0.45 \end{bmatrix} \end{array}$$

b $$\begin{array}{cc} & \begin{matrix} X & Y \end{matrix} \\ \begin{matrix} X \\ Y \end{matrix} & \begin{bmatrix} 0.70 & 0.25 \\ 0.30 & 0.75 \end{bmatrix} \end{array}$$

c $$\begin{array}{cc} & \begin{matrix} X & Y & Z \end{matrix} \\ \begin{matrix} X \\ Y \\ Z \end{matrix} & \begin{bmatrix} 0.6 & 0.15 & 0.22 \\ 0.1 & 0.7 & 0.23 \\ 0.3 & 0.15 & 0.55 \end{bmatrix} \end{array}$$

Exercise 11B

1 a $T = \begin{bmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{bmatrix}$



b $0.85 \times 80 = 68$ **c** $0.25 \times 60 = 15$

d $0.15 \times 120 + 0.75 \times 40 = 48$

2 a i 10% **ii** 80% **iii** 10%

b i 680 **ii** 85

c i 1150 **ii** 0 **iii** 0

d All (100%) of the sea birds who nest at site A this year will nest at site A next year.

3 a i 76440 **ii** 7560

b i 5500 **ii** 1210 **iii** 266

4 a i 18 **ii** 6 **iii** 6

b i 84 **ii** 66 **iii** 30

c $\begin{bmatrix} 84 \\ 66 \\ 30 \end{bmatrix}$

d **i** 21 **ii** 3 **iii** 6

e $\begin{bmatrix}180\end{bmatrix}$

5 B **6** E

Exercise 11C

1 **a** $S_1 = \begin{bmatrix}380\\220\end{bmatrix}$ **b** $S_2 = \begin{bmatrix}398\\202\end{bmatrix}$

c $S_3 = \begin{bmatrix}399.8\\200.2\end{bmatrix}$

2 **a** $S_5 = \begin{bmatrix}399.998\\200.02\end{bmatrix}$ **b** $S_7 = \begin{bmatrix}400\\200\end{bmatrix}$

c $S_{12} = \begin{bmatrix}400\\200\end{bmatrix}$

3 **a** $S_4 = \begin{bmatrix}5460\\4780\end{bmatrix}$ **b** $S_3 = \begin{bmatrix}5456\\4784\end{bmatrix}$

4 **a** **i** $S_1 = \begin{bmatrix}130\\170\end{bmatrix}$ **ii** $S_2 = \begin{bmatrix}151\\149\end{bmatrix}$

iii $S_3 = \begin{bmatrix}165.7\\134.3\end{bmatrix}$

b $T^5 = \begin{bmatrix}0.72269 & 0.55462\\0.27731 & 0.44538\end{bmatrix}$

c **i** $S_2 = \begin{bmatrix}151\\149\end{bmatrix}$ **ii** $S_3 = \begin{bmatrix}165.7\\134.3\end{bmatrix}$

iii $S_7 = \begin{bmatrix}191.8\\108.2\end{bmatrix}$

d See solutions

5 **a** **i** $S_1 = \begin{bmatrix}180\\130\\290\end{bmatrix}$ **ii** $S_2 = \begin{bmatrix}207\\136\\257\end{bmatrix}$

iii $S_3 \begin{bmatrix}225\\132.1\\242.9\end{bmatrix}$

b **i** $S_2 \begin{bmatrix}207\\136\\257\end{bmatrix}$ **ii** $S_3 = \begin{bmatrix}225\\132.1\\242.9\end{bmatrix}$

iii $S_7 = \begin{bmatrix}224.9\\129.7\\225.4\end{bmatrix}$

c See solutions

6 **a** $T = \begin{bmatrix}0.80 & 0.25\\0.20 & 0.75\end{bmatrix}$

b $S_0 = \begin{bmatrix}400\\400\end{bmatrix}$

c $S_1 = \begin{bmatrix}420\\380\end{bmatrix}$, 420 to Jill's and 380 to Pete's

d $S_5 = \begin{bmatrix}442.2\\357.8\end{bmatrix}$, 442 to Jill's and 358 to Pete's

e steady state solution: $S_s = \begin{bmatrix}444.4\\355.6\end{bmatrix}$, 444 to Jill's and 356 to Pete's

7 **a** $T = \begin{bmatrix}0.90 & 0.60\\0.10 & 0.40\end{bmatrix}$

b $S_0 = \begin{bmatrix}1500\\500\end{bmatrix}$

c $S_1 = \begin{bmatrix}1650\\350\end{bmatrix}$, 1650 are happy and 350 are unhappy

d $S_4 = \begin{bmatrix}1712.55\\287.45\end{bmatrix}$, 1713 are happy and 287 are unhappy

e steady state solution: $S = \begin{bmatrix}1714.3\\285.7\end{bmatrix}$, 1714 are happy and 286 are unhappy

8 **a** $S_0 = \begin{bmatrix}1200\\600\\200\end{bmatrix}$

b $S_1 = \begin{bmatrix} 1270 \\ 440 \\ 290 \end{bmatrix}$, 1270 are happy

c $S_5 = \begin{bmatrix} 1310.33 \\ 429.82 \\ 259.85 \end{bmatrix}$, 1310 are happy

d steady state solution: $\begin{bmatrix} 1311.7 \\ 429.1 \\ 259.1 \end{bmatrix}$, 1312 are happy

9 A **10** E **11** A

Exercise 11D

1 a i $\begin{bmatrix} 80 \\ 120 \end{bmatrix}$ **ii** $\begin{bmatrix} 68.8 \\ 131.2 \end{bmatrix}$

b i

$$S_1 = TS_0 + R$$

$$= \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 100 \\ 100 \end{bmatrix} + \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 80 \\ 120 \end{bmatrix} + \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 90 \\ 125 \end{bmatrix}$$

ii

$$S_2 = TS_1 + R$$

$$= \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 90 \\ 125 \end{bmatrix} + \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 79 \\ 136 \end{bmatrix} + \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 89 \\ 141 \end{bmatrix}$$

c i

$$S_1 = TS_0 - B$$

$$= \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 100 \\ 100 \end{bmatrix} - \begin{bmatrix} -20 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 80 \\ 120 \end{bmatrix} - \begin{bmatrix} -20 \\ 20 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

ii

$$S_2 = TS_1 - B$$

$$= \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 100 \\ 100 \end{bmatrix} - \begin{bmatrix} -20 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 80 \\ 120 \end{bmatrix} - \begin{bmatrix} -20 \\ 20 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

2 a i $S_1 = \begin{bmatrix} 11500 \\ 8500 \\ 10000 \end{bmatrix}$ **ii** 7300

b *A: 30 000, B: 0, C: 0*; While the sea birds move between nesting sites each year, the '1' in the transition matrix indicates that, once a sea bird nests at site *A*, it continues to nest at this site. Meanwhile, some of the birds who nest at sites *B* and *C* each year will move to site *A* until, in the long term, all birds are nesting at site *A*.

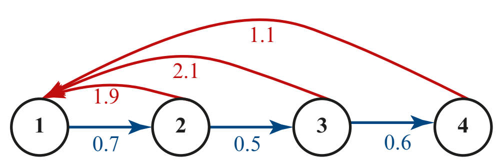
c i $\begin{bmatrix} 9500 \\ 9500 \\ 11000 \end{bmatrix}$ **ii** $\begin{bmatrix} 9000 \\ 9150 \\ 11850 \end{bmatrix}$ **iii** $\begin{bmatrix} 8507.5 \\ 8912.5 \\ 12\,580 \end{bmatrix}$

3 C **4** B

Exercise 11E

1 a i 1.9 **ii** 0.6

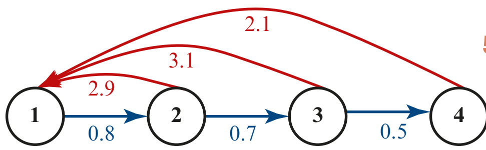
b



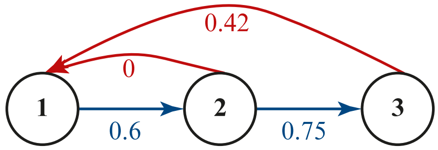
c i $\begin{bmatrix} 510 \\ 70 \\ 50 \\ 60 \end{bmatrix}$ **ii** $\begin{bmatrix} 784.8 \\ 212.8 \\ 178.5 \\ 21 \end{bmatrix}$ **iii** $\begin{bmatrix} 208\,276 \\ 103\,876 \\ 36\,984 \\ 15\,815.8 \end{bmatrix}$

d $\begin{bmatrix} 3483 \\ 1829 \\ 600 \\ 291 \end{bmatrix}$

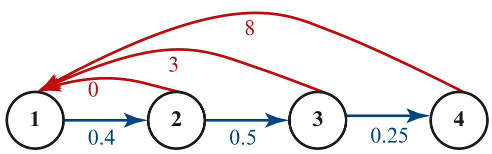
2 a



b



c

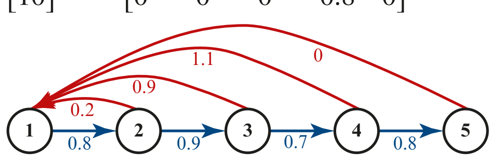


3 a $\begin{bmatrix} 0 & 1.3 & 2.4 \\ 0.7 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 2.3 & 3 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}$

c $\begin{bmatrix} 0 & 1.4 & 2.6 & 0.6 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \end{bmatrix}$

4 a $\begin{bmatrix} 15 \\ 20 \\ 30 \\ 15 \\ 10 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 0.2 & 0.9 & 1.1 & 0 \\ 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 \end{bmatrix}$

c



d i $\begin{bmatrix} 48 \\ 12 \\ 18 \\ 21 \\ 12 \end{bmatrix}$ **ii** $\begin{bmatrix} 58 \\ 37 \\ 22 \\ 21 \\ 19 \end{bmatrix}$

e 37 **f** 90

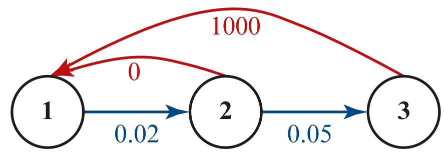
g i 111 **ii** 158 **iii** 235

h i 99 **ii** 145 **iii** 233

It appear that the population rate of increase approaches 10% per year. Further investigation confirms this.

5 a $\begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 0 & 1000 \\ 0.02 & 0 & 0 \\ 0 & 0.05 & 0 \end{bmatrix}$

c



d i $\begin{bmatrix} 50\,000 \\ 0 \\ 0 \end{bmatrix}$ **ii** $\begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix}$

iii $\begin{bmatrix} 50\,000 \\ 0 \\ 0 \end{bmatrix}$

e i $\begin{bmatrix} 50\,000 \\ 1 \\ 5 \end{bmatrix}$ **ii** $\begin{bmatrix} 50 \\ 100 \\ 50 \end{bmatrix}$

iii $\begin{bmatrix} 50\,000 \\ 1 \\ 5 \end{bmatrix}$

6 a i $\begin{bmatrix} 269 \\ 127 \\ 30 \end{bmatrix}$ **ii** $\begin{bmatrix} 356 \\ 168 \\ 40 \end{bmatrix}$ **iii** $\begin{bmatrix} 622 \\ 294 \\ 70 \end{bmatrix}$

i 427 **ii** 565 **iii** 986

b At this stage the rate of increase is approximately 5.7%

7 a i $\begin{bmatrix} 1400 \\ 240 \\ 100 \end{bmatrix}$ **ii** $\begin{bmatrix} 700 \\ 840 \\ 60 \end{bmatrix}$ **iii** $\begin{bmatrix} 2160 \\ 420 \\ 210 \end{bmatrix}$

b **i** $\begin{bmatrix} 976 \\ 460 \\ 91 \end{bmatrix}$ **ii** $\begin{bmatrix} 1241 \\ 586 \\ 115 \end{bmatrix}$ **iii** $\begin{bmatrix} 1579 \\ 745 \\ 146 \end{bmatrix}$

c **i** $\begin{bmatrix} 974 \\ 460 \\ 90 \end{bmatrix}$ **ii** $\begin{bmatrix} 1237 \\ 584 \\ 115 \end{bmatrix}$ **iii** $\begin{bmatrix} 1571 \\ 741 \\ 145 \end{bmatrix}$

The population appears to be iincreasing at a rate of 27%

d

8 **a** **i** $\begin{bmatrix} 0 \\ 300 \\ 0 \end{bmatrix}$ **ii** $\begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix}$ **iii** $\begin{bmatrix} 200 \\ 0 \\ 0 \end{bmatrix}$

b Population cycles through three states

c **i** Population decreases by 50% every three time periods

ii Population increases by 25% every three time periods

9 **a** $\begin{bmatrix} 2800 \\ 200 \\ 200 \\ 40 \end{bmatrix}$ **b** $\begin{bmatrix} 1080 \\ 1400 \\ 100 \\ 20 \end{bmatrix}$ **c** $\begin{bmatrix} 4440 \\ 540 \\ 700 \\ 10 \end{bmatrix}$

10 **a** **i** Every 3 years, the population returns to 1000 newborns

ii Every 3 years, the population increases by 50% and returns to only newborns

iii Every 3 years, the population decreases by 40% and returns to only newborns

b Long-term growth rate 1.37; long-term ratio of age groups 818 : 130 : 52

11 D **12** D **13** A

Chapter 11 review

Multiple-choice questions

1 B **2** A **3** C **4** B **5** B
6 C **7** C **8** A **9** B **10** B
11 B **12** C **13** E

Written-response questions

1 **a** **i** $\begin{bmatrix} 504 \\ 244 \end{bmatrix}$ **ii** 748

b $\begin{bmatrix} 517 \\ 257 \end{bmatrix}$

c Diiscoo attendance is expected to increase to around 520 and stay at that level.

d Diiscoo attendance is expected to decrease to around 20 and stay at that level.

2 $\begin{array}{c} \\ Y \\ M \\ O \end{array} \begin{array}{c} \begin{array}{ccc} Y & M & O \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.4 \\ 0.9 & 0 & 0 \\ 0 & 0.8 & 0.6 \end{bmatrix} \end{array}$

3 **a**

$$\begin{bmatrix} 0 & 0.1 & 0.9 & 0.2 & 0 & 0 & 0 & 0 \\ 0.98 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.95 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.95 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0 \end{bmatrix}$$

b $S_2 = \begin{bmatrix} 104.5 \\ 107.8 \\ 0 \\ 90.25 \\ 85.5 \\ 31.5 \\ 0 \\ 0 \end{bmatrix}$, $S_3 = \begin{bmatrix} 28.83 \\ 102.41 \\ 102.41 \\ 0 \\ 81.225 \\ 59.85 \\ 15.75 \\ 0 \end{bmatrix}$

c 1.035

4 a $\begin{bmatrix} 0.2 & 0.5 & 0.6 & 0.4 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.7 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 980 \\ 692 \\ 488 \\ 344 \end{bmatrix}$

c $\begin{bmatrix} 972 \\ 680 \\ 484 \\ 342 \end{bmatrix}$, 0.992 **d** $\begin{bmatrix} 7467 \\ 4762 \\ 3038 \\ 1938 \end{bmatrix}$

e Day 56

Chapter 12

Exercise 12A

1 E **2** D **3** A **4** A **5** B **6** C
7 D **8** C **9** C **10** D **11** B **12** A
13 A **14** B **15** C **16** C **17** C **18** E
19 D **20** C **21** A **22** D **23** B **24** D
25 A

Exercise 12B

1 a 3×1 **b** 1×3

c HC; because the number of columns in H equals the number of rows in C

d i [696.72]

ii the number of Australian dollars ($696.72) that you would receive by converting your foreign currency into Australian dollars; $HC = 102 \times 1.316 + 262 \times 1.818 + 516 \times 0.167$

e $\begin{bmatrix} 566.21 \\ 137.46 \\ 647.21 \end{bmatrix}$

2 a $\begin{bmatrix} 0.67 & 0.28 \\ 0.33 & 0.72 \end{bmatrix}$

b $S_0 = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}$

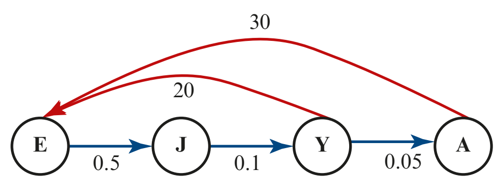
c $S_1 = \begin{bmatrix} 4360 \\ 5640 \end{bmatrix}$; 4360 fish in Lake Blue and 5640 fish in Lake Green

d $S_3 = \begin{bmatrix} 4555.156 \\ 5444.844 \end{bmatrix}$; 4555 fish in Lake Blue and 5445 fish in Lake Green

e $S_s = \begin{bmatrix} 4590.2 \\ 5409.8 \end{bmatrix}$; 4590 fish in Lake Blue and 5410 fish in Lake Green

3

a



b 50 **c** 920 **d** 5 weeks

e 5%

f i 1667 **ii** 1706

g 2%

4a i $\begin{bmatrix} 100 \\ 50 \end{bmatrix}$ **ii** 95

b $S_n = T^n S_1$ **c** 5 weeks

d 90

5a i 3×1 **ii** $k = 1.2$

b i A and B and A and C

ii $D \to B \to A \to C$

iii $C \to A \to B$ and $C \to D \to B$

c i $S_1 = \begin{bmatrix} 385 \\ 75 \\ 140 \end{bmatrix}$ **ii** 230

iii 70.56%

d $B = \begin{bmatrix} 20 \\ 15 \\ 25 \end{bmatrix}$

6a 4×2

b **i** 435 **ii** 72.4%

c $L = \begin{bmatrix} 60 \\ 120 \\ 50 \\ 85 \end{bmatrix}$ $Q \times L = \begin{bmatrix} 5115 \end{bmatrix}$

7a $S_1 = \begin{bmatrix} 39 \\ 36 \\ 21 \\ 24 \end{bmatrix}$ $S_2 = \begin{bmatrix} 41.4 \\ 35.1 \\ 20.1 \\ 23.4 \end{bmatrix}$

b $S_{10} = \begin{bmatrix} 42.76 \\ 34.64 \\ 19.72 \\ 22.87 \end{bmatrix}$

(P) 43, (S) 35, (D) 20, (W) 23

c 3

d 36

e 3.9

f 47.4

8a 550

b The number of sandwiches sold in week 3.

c Hamburgers \$15, fish and chips \$14 sandwiches \$12

d $L = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$

Chapter 13

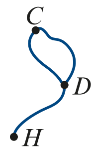
Exercise 13A

1 a **i** 3 **ii** 2 **iii** 1

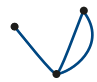
b 14

c town D and town H

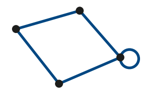
d



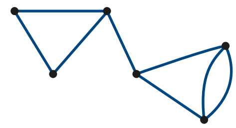
2 a



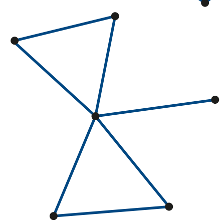
b



c

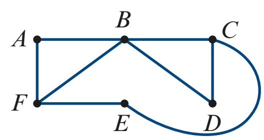


d

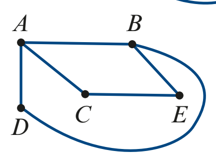


3 a i **b** ii **c** ii

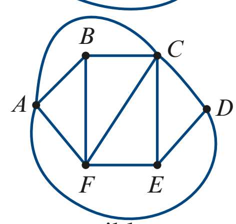
4 a



b



c



d not possible

5 a **i** $v = 8,\ f = 6,\ e = 12$

b **i** $v = 6,\ f = 8,\ e = 12$

c **i** $v = 7,\ f = 7,\ e = 12$

d **i** $v = 5,\ f = 3,\ e = 6$

e **i** $v = 5,\ f = 6,\ e = 9$

f **i** $v = 6,\ f = 4,\ e = 6$

6 a 4 **b** 12 **c** 19

7 7 **8** 14 **9** 15

10 C **11** E **12** C **13** B

14 E **15** C

Exercise 13B

1 a

$$\begin{array}{cc} & \begin{matrix}A & B & C & D\end{matrix} \\ \begin{matrix}A\\B\\C\\D\end{matrix} & \begin{bmatrix}0&1&1&0\\1&0&1&1\\1&1&0&0\\0&1&0&0\end{bmatrix}\end{array}$$

b

$$\begin{array}{cc} & \begin{matrix}A & B & C & D\end{matrix} \\ \begin{matrix}A\\B\\C\\D\end{matrix} & \begin{bmatrix}0&1&1&0\\1&0&0&1\\1&0&0&1\\0&1&1&0\end{bmatrix}\end{array}$$

c

$$\begin{array}{cc} & \begin{matrix}A & B & C & D\end{matrix} \\ \begin{matrix}A\\B\\C\\D\end{matrix} & \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\end{array}$$

d

$$\begin{array}{cc} & \begin{matrix}A & B & C & D\end{matrix} \\ \begin{matrix}A\\B\\C\\D\end{matrix} & \begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix}\end{array}$$

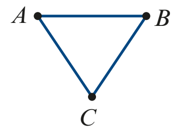
e

$$\begin{array}{cc} & \begin{matrix}A & B & C & D & E & F\end{matrix} \\ \begin{matrix}A\\B\\C\\D\\E\\F\end{matrix} & \begin{bmatrix}0&1&1&0&0&0\\1&0&0&1&0&0\\1&0&0&1&0&0\\0&1&1&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\end{bmatrix}\end{array}$$

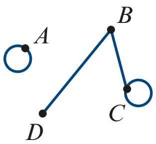
f

$$\begin{array}{cc} & \begin{matrix}A & B & C & D\end{matrix} \\ \begin{matrix}A\\B\\C\\D\end{matrix} & \begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&2\\0&1&2&0\end{bmatrix}\end{array}$$

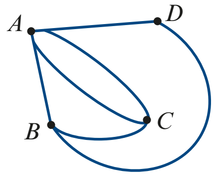
2 a



b



c



3 C is an isolated vertex.

4 Leading diagonals will all be '1'.

5

$$\begin{array}{cc} & \begin{matrix}A & B & C & D & E\end{matrix} \\ \begin{matrix}A\\B\\C\\D\\E\end{matrix} & \begin{bmatrix}0&1&1&1&1\\1&0&1&1&1\\1&1&0&1&1\\1&1&1&0&1\\1&1&1&1&0\end{bmatrix}\end{array}$$

6 E **7** A **8** E **9** A
10 C **11** B **12** E

Exercise 13C

1 a path **b** trail **c** path **d** walk
e trail **f** path

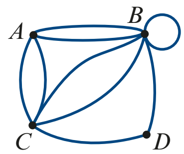
2 a walk **b** cycle **c** path **d** walk
e path **f** walk **g** **h**

3 a i Euler trail
ii $A–B–E–D–B–C–D–A–E$
b neither
c i Euler trail
ii $A–C–E–C–B–D–E–F$
d i Euler circuit
ii $A–B–C–E–D–C–A$
e i Euler circuit
ii $E–F–D–E–A–B–D–C–B–E$

4 a $A–B–C–F–I–H–E–G–D–A$
b $A–B–C–D–E–F–A$
c $A–B–D–C–E–A$

5 $F–A–B–C–D–E–H–G$

6 a 2 **b** 7 **c**



d Vertices are not all even.

7 a $v = 9, e = 12, f = 5$

$v - e + f = 2$

b i Hamiltonian cycle

ii Lake Bolac – Streatham – Nerrin Nerrin–Woorndoo–Mortlake –Hexham–Chatworth–Glenthompson–Wickliffe–Lake Bolac.

and the reverse of this

c i Eulerian circuit

ii Not all vertices have an even degree

d i Lake Bolac - Wickliffe

ii LWMHCGWCWNSLWL

8 a Yes **b** Yes **c** No **d** Yes **e** No **f** Yes **g** Yes

9 3

10 C **11** D **12** E **13** B **14** A

Exercise 13D

1 a D–E **b** 17 minutes **c** 8 minutes **d** 36 minutes

2 11

3 a 34 **b** 56 **c** E–B–A–E, 22

d A–E–F–G–I or A–C–F–G–I

4 a S–B–D–F, 12

b $S - A - C - D - F$, 10

c $S - B - D - F$, 15

d $S - A - E - G - F$, 19

5 19 km

6 B **7** C **8** A **9** E

Exercise 13E

1 a $A - B - C - H$, 160

b $A - C - F - E - G - H$, 53

c $A - D - E - F - H$, 385

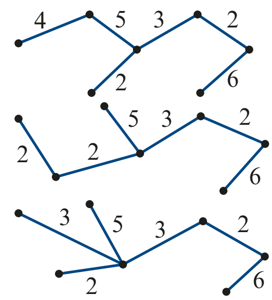
d $A - B - E - F - I - H$, 87

2 23 minutes

Exercise 13F

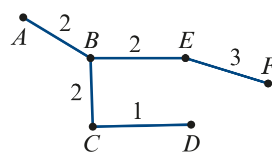
1 a 6

b



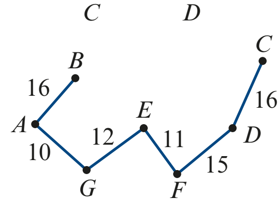
c 22, 20, 21 (Answers will vary)

2 a



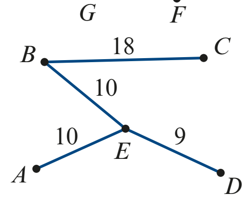
10

b



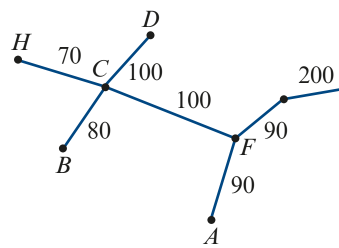
80

c



47

d



730

3 490 m

4 A **5** B **6** C **7** E
8 B

Chapter 13 review

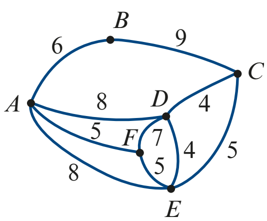
Multiple-choice questions

1 C	**2** C	**3** A	**4** D
5 A	**6** C	**7** B	**8** B
9 E	**10** B	**11** A	**12** A
13 C	**14** B	**15** B	**16** B

Written-response questions

1 a $A–B–C–F–G–E–X$, 11
b $A–C–E–I–H–X$, 127
c $A–D–E–G–H–J–M–X$, 55
d $A–B–D–E–G–I–L–M–X$, 49

2 a i



ii 24

iii

$$\begin{array}{c|c} & \begin{matrix} A & B & C & D & E & F \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

b i 45 km
ii Some vertices are visited more than once.
iii $F–E–D–C–B–A–F$
iv 33 km (for route above; other answers possible)
c C and F

3 a 4 **b** 18
c $v = 6, e = 9, f = 5; v - e + f = 2$
d-l See solutions

Chapter 14

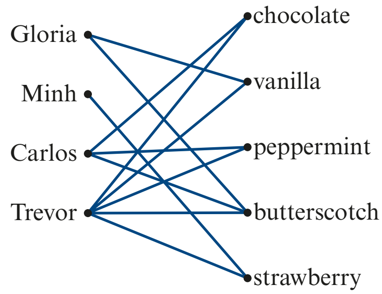
Exercise 14A

1 a 3 **b** 8
2 $C_1 = 14,\ C_2 = 12,\ C_3 = 21$
3 $C_1 = 12,\ C_2 = 16,\ C_3 = 16$
4 a 9 **b** 11 **c** 8 **d** 18
5 a A, 14; B, 23; C, 12; D, 16; E, not a cut that can be used to determine the maximum number of available seats
b It does not prevent flow from source (Arlie) to sink (Bowen).
c 12
6 a sink 1=10,sink 2=11
b sink 1 = 8, sink 2 = 18
7 a 9 **b** 18
8 a Cut passes through edges with weights 20,10,30,30
b 59 **c** 25
9 D **10** C **11** B

Exercise 14B

1 a Worker 1–Task 3; Worker 2–Task 1; Worker 3–Task 2
b Worker 1–Task 5; Worker 2–Task 6; Worker 3–Task 4
2 Niranjan – Cake; Nishara – Candles; Dinesh – Serviettes; Dhishani – Balloons
3 Two answers possible
Player 1 –Right Wing; Player 2 – Left wing; Player 3 –Centre; Player 4 – Right Defence; Player 5 – Left Defence
Or
Player 1 – Centre Player 2 – Right wing Player 3 – Left Defence Player 4 – Right Defence Player 5 – Left Wing
4 a two distinct groups of vertices (people and flavours)

b



c 5

5 **a** W–D, X–A, Y–B, Z–C

b e.g., minimum cost is 11; W–A, X–B, Y–D, Z–C

6 Dimitri 800 m, John 400 m, Carol 100 m, Elizabeth 1500 m

7 Joe C, Meg A, Ali B

8 A–Y, B–Z, C–X, D–W

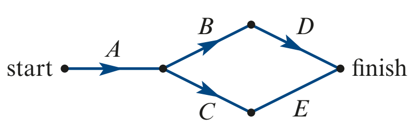
9 Champs Home, Stars Away, Wests Neutral; or Champs Neutral, Stars Away, Wests Home.
Cost = \$20 000

10 A Mark, B Karla, C Raj, D Jess; or A Karla, B Raj, C Mark, D Jess; 55 km

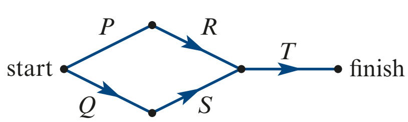
11 D **12** A **13** E **14** D

Exercise 14C

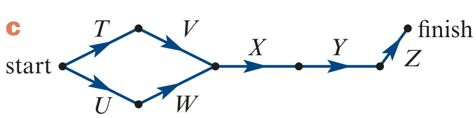
1 **a**



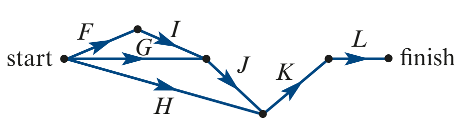
b



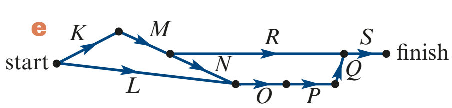
c



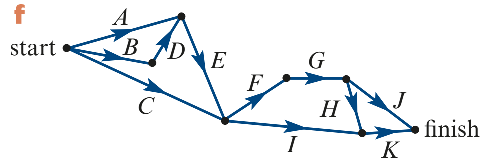
d



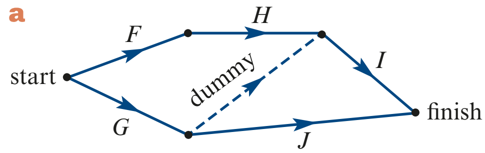
e



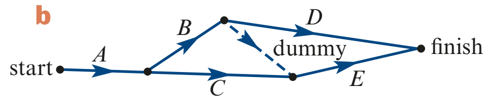
f



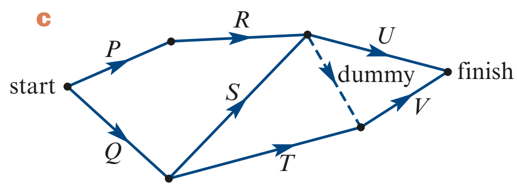
2 **a**



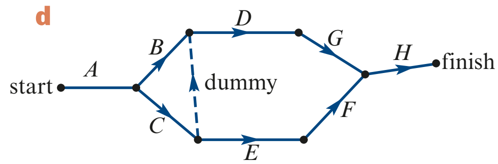
b



c



d



3

a

Activity	Immediate predecessors
A	–
B	–
C	A
D	A
E	B, C
F	D
G	E

b

Activity	Immediate predecessors
P	–
Q	P
R	P
S	Q
T	Q
U	S, V
V	R
W	R
X	T, U

c

Activity	*Immediate predecessors*
J	–
K	–
L	J
M	N
N	K
O	K
P	N
Q	L, M
R	P
S	O, R
T	Q

d

Activity	*Immediate predecessors*
A	–
B	–
C	A
D	A
E	D, B
F	C, E
G	D, B
H	B

e

Activity	*Immediate predecessors*
P	–
Q	–
R	P
S	P
T	Q
U	R
V	S
W	S, T
X	U
Y	W
Z	V, X, Y

f

Activity	*Immediate predecessors*
A	–
B	A
C	A
D	A
E	B
F	C, D
G	D
H	E, F, G
I	G
J	I
K	H

4 **a** Remove panel.

b ‘Order component’ and ‘Pound out dent’

5 **a**

Activity	*Immediate predecessors*
A	–
B	–
C	–
D	A
E	B, F
F	C
G	B, F
H	D, E
I	H
J	I, K
K	G
L	G
M	H
N	J, L
O	N

b $A - D - H - M$

$A - D - H - I - J - N - O$

c $B - E - H - M$

$B - E - H - I - J - N - O$

$B - G - K - J - N - O$

$B - G - L - N - O$

d $C - F - E - H - M$

$C-F-E-H-I-J-N-O$
$C-F-G-K-J-N-O$
$C-F-G-L-N-O$

6 **a** D, F, H
b A, B, C, D, E, F, G, H

7 **a** G **b** K
c G is the immediate predecessor of both I and K, however activity K has other immediates predecesors not common to activity I

8 D **9** A **10** A

Exercise 14D

1 **a** $p = 12$ **b** $w = 10$
c $m = 8,\ n = 8$
d $a = 10,\ b = 18,\ c = 11$
e $f = 9,\ g = 12$
f $q = 8,\ r = 3,\ p = 5,\ n = 9$

2 **a** 3 **b** A–C **c** 5 **d** 13
e 2

3 **a** 12 **b** 10 **c** 9 **d** 1
e 3 **f** 9

4 **a** D–E–F **b** A: 1, B: 1, C: 15

5 **a** B–E–F–H–J
b A: 1, C: 14, D: 1, G: 1, I: 1

6 **a**

Activity	*Duration (weeks)*	*Immediate predecessors*
A	3	–
B	6	–
C	6	A, B
D	5	B
E	7	C, D
F	1	D
G	3	E
H	3	F
I	2	B

b B–C–E–G

7 **a** 8 **b** 11 **c** $C-F-G$
d B

8 **a** D, F, G **b** 13
c Activity H lies on the critical path and if delayed, the completion time of the project will be extended.
d 15 **e** F

9 **a** **i** 25 **ii** 29 **iii** 30
b 5 **c** 2

10 **a** B, C **b** A, B, C, D, E, G
c B, F, H, J **d** 4 **e** E, G, I

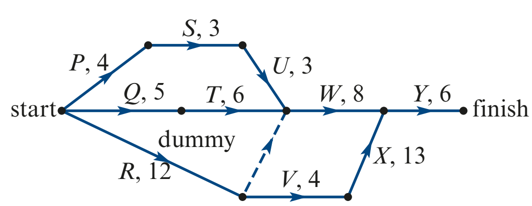
11

a

Activity	*Immediate predecessors*
A	–
B	–
C	–
D	B
E	B
F	A, D
G	C, E
H	F, G
I	F, G
J	G
K	H
L	I
M	J

b 3 **c** 9
d $B-E-G-H-K$
$B-E-G-I-L$ **e** 6

12 **a**

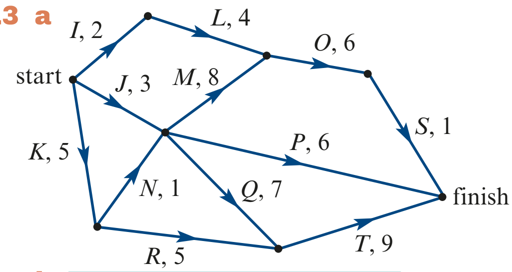


b

Activity	*EST*	*LST*
P	0	11
Q	0	10
R	0	0
S	4	15
T	5	15
U	7	18
V	12	12
W	12	21
X	16	16
Y	29	29

c $R–V–X–Y$ **d** 35 weeks

13 **a**



b

Activity	*EST*	*LST*
I	0	9
J	0	3
K	0	0
L	2	11
M	6	7
N	5	5
O	14	15
P	6	16
Q	6	6
R	5	8
S	20	21
T	13	13

c $K–N–Q–T$ **d** 22 weeks

14 E **15** D **16** C **17** A
18 D

Exercise 14E

1 **a** $A-D$; $B-E-F$; $B-E-G-I$; $C-H-I$
b $B-E-G-I, 21$ hours
c 18 hours

2 **a** $A-B-F-H$ **b** 21 days
c 20 days **d** \$100

3 **a** $B–E–H–J$ **b** 2 hours
c 6 hours **d** 14 hours

4 **a** 4 **b** 17 hours **c** \$1200

5 **a** 22 days **b** \$870

6 **a** C, D, H
b B, E, H, I, J
c **i** 21 days **ii** \$450

7 **a** 21 days **b** 10 days **c** 5 days **d** K
e \$6000

8 **a** 29 days **b** 6 **c** 4
d Two answers possible:
$H, 2$ $J, 0$ $K, 2$ $L, 1$ $M, 1$
$H, 2$ $J, 0$ $K, 1$ $L, 1$ $M, 2$

9 B **10** B **11** D

Chapter 14 review

Multiple-choice questions

1 D **2** D **3** A **4** A
5 C **6** B **7** E **8** D
9 D **10** E

Written-response questions

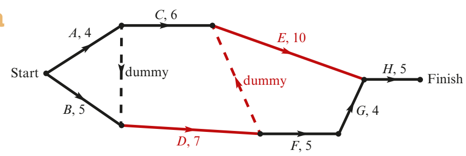
1 **a** Alvin should write the Body par 3
Billy should write the Body par 2
Chloe should write the Body par 1
Danielle should write the Conclusion
Elena should write the Introduction
b 43 minutes

2 **a** 26 **b** 15

3 Rob – breaststroke, Joel – backstroke, Henk – freestyle, Sav – butterfly or Rob – breaststroke, Joel – butterfly, Henk – backstroke, Sav – freestyle. Time = 276

4 **a** 26 **b** 15

5 a



b 12 **c** 1 hour **d** 4

e $B-D-E-H$ **f** 27 hours

g **i** $B-D-F-G-H$

ii 22 hours

h D, H must be in that order

Chapter 15

Exercise 15A

1 A	**2** E	**3** E	**4** B
5 C	**6** B	**7** A	**8** B
9 E	**10** C	**11** E	**12** D
13 D	**14** B	**15** E	**16** A
17 C	**18** C	**19** C	**20** E
21 B	**22** A	**23** D	**24** C
25 B	**26** C	**27** C	**28** E

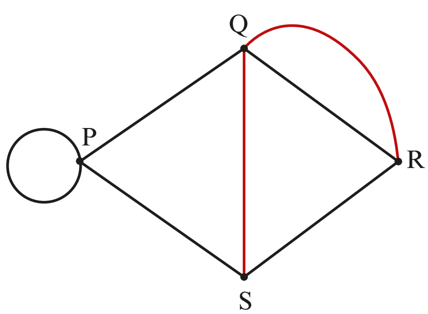
Exercise 15B

1 a 14 **b** 3 **c** 3

e $5+5=7+2, 9=9$

2

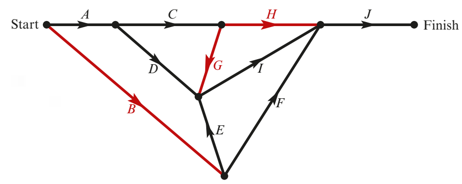
$$\begin{array}{c|cccc} & P & Q & R & S \\ \hline P & 1 & 1 & 0 & 1 \\ Q & 1 & 0 & 2 & 1 \\ R & 0 & 2 & 0 & 1 \\ S & 1 & 1 & 1 & 0 \end{array}$$



3 a Cut 1 does not isolate the source from the sink.

b 26 **c** 22

4



5 a 9 **b** 7 **c** 1

d $B-D-E-G$ **e** 15

6 a $A \to 1, D \to 4, F \to 10, K \to 12,$

b $B–C–E–G–J–K$

7 a **i** 2.1 km

ii $PQRTSU$ or $PRQSTU$ or $PRQTSU$ or $PRTQSU$

b **i** $R–Q–P–R–T–Q–S–T–U–S$ or $R–Q–P–R–T–S–Q–T–U–S$

ii travel each road only once

8 a None of the edges overlap.

b $7+6-11=2$ **c** C **d** 297 km

e no **f** 79 km **g** 127 km

h 187 km

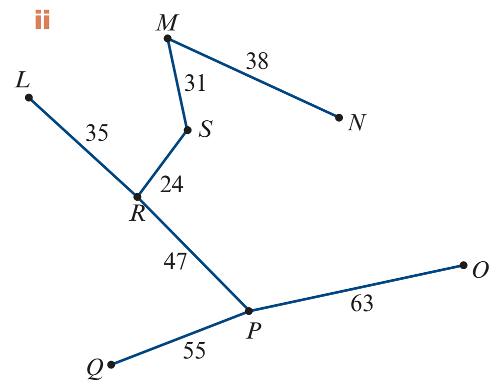
9 a 5 **b** 24 hours **c** 7 hours

10 11 megalitres/day

11 a 112 km

b **i** minimum spanning tree

ii



iii 293 km

c 306 km

12 a A, B, C

b LST for B is 1, EST for E is 10, LST for I is 18

c **i** $A–D–F–I–J$ **ii** 27 months

d **i** $B–C–D–F–I–J$ **ii** 25 months

13 **a** $A–Z$, $B–W$, $C–X$, $D–Y$, or $A–Z$, $B–X$, $C–W$, $D–Y$

b \$130

14 **a** 15 weeks **b** \$8500 **c** 3

Chapter 16

Exercise 16A

Data analysis, probability and statistics

1 A	**2** C	**3** B	**4** D	**5** C
6 B	**7** B	**8** C	**9** A	**10** A
11 E	**12** E	**13** B	**14** B	**15** A
16 D	**17** B	**18** B	**19** A	**20** E

Recusion and financial modelling

21 C	**22** B	**23** D	**24** A	**25** B
26 C	**27** B	**28** D	**29** A	**30** C

Matrices

31 B	**32** B	**33** A	**34** E	**35** D
36 D	**37** E	**38** D	**39** E	**40** A
41 B				

Networks

42 C	**43** D	**44** B	**45** D	**46** E
47 B	**48** C	**49** C	**50** D	**51** C

Exercise 16B

Data analysis, probability and statistics

1 **a** mean = 54.042, stand dev = 2.717

b $z = -1.1$

c **i** 48.6 kg **ii** 2.5%

2 **a** EV: *number of distractions*, RV: *time*

b *time*

c **i** IQR = 6.2 seconds

ii Upper fence = $28.2 + 1.5 \times 6.2 = 37.5$

d 10 people

e From this information it can be concluded that the time taken to complete the task is associated with the number of distractions. The median time taken by the group who completed the task with no distractions was 25.0 seconds, faster than the group with a few distractions which has a median time of 26.2 seconds, which was in turn faster than the group with many distractions which took a median time of 29.2 seconds to complete the task.

3 **a** $r^2 = 84.8\%$

b 84.8% of the variation in *fuel consumption* can be explained by the variation in *speed*.

c 9.0 litres/100 km

d slope = 0.0218. On average, for each additional 1 km/hr increase in the *speed* of the car, the *fuel consumption* increases by 0.0218 litres/100 km.

e predicted value = 8.40, actual value= 8.30. Thus residual= −0.10.

4 **a** There is a strong, non-linear relationship between *efficiency* and *enthusiasm*.

b $\log y$, $\frac{1}{y}$, x^2

c log(*efficiency*) = 0.0205 + 0.0860 × *enthusiasm*

d 6.6

5 **a**

	Q1	Q2	Q3	Q4
SI	1.01	1.15	1.32	0.52

b

Year	Q1	Q2	Q3	Q4
2022	62	60	61	63

6 **a** 130.6 cents/litre

b 141.1 cents/litre

c **i** Victoria, slope = 2.09. On average, the price of fuel in Victoria is increasing by 2.09 cents/litre each year.

ii NT, slope = 2.20. On average, the price of fuel in the NT is increasing by 2.20 cents/litre each year.

d **i** 156.7 cents/litre.

ii 169.7 cents/litre.

e The difference is predicted to increase over time. The cost of petrol in the NT is already higher than the cost of petrol in Victoria, and the cost is increasing at a higher rate in the NT (on average 2.20 cents/litre each year) than it is increasing in Victoria (on average 2.09 cents/litre each year).

Recusion and financial modelling

7 **a** \$6468.13 **b** 6.2%

c **i** \$5225 **ii** 9%

8 **a** \$39.23 **b** 6.42%

c $V_0 = 2600, V_{n+1} = 1.0025_{vn} + 140$

9 **a** \$79 **b** \$344.78

c \$375.48, \$375.60

Matrices

10 **a** 1×3

b $Q = \begin{bmatrix} 150 & 250 & 320 \\ 300 & 500 & 640 \\ 1050 & 1750 & 2240 \end{bmatrix}$

The number of shoppers who shopped from several sections at Radcliffs.

c $T \times A$

d **i** $S_1 = \begin{bmatrix} 2835 \\ 1940 \\ 2425 \end{bmatrix}$

ii $S_{50} = \begin{bmatrix} 2791.33 \\ 2577.88 \\ 1830.79 \end{bmatrix}$

iii The long term customer numbers are: HSL (2791), Radcliffs (2578) Cottonworths (1831)

11 **a** $L = \begin{bmatrix} 0 & 0.9 & 0.7 \\ 0.9 & 0 & 0 \\ 0 & 0.8 & 0 \end{bmatrix}$ **b** $S_0 = \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix}$

c $S_1 = \begin{bmatrix} 90 \\ 0 \\ 80 \end{bmatrix}, S_2 = \begin{bmatrix} 56 \\ 81 \\ 0 \end{bmatrix}, S_3 = \begin{bmatrix} 73 \\ 50 \\ 65 \end{bmatrix},$

$S_4 = \begin{bmatrix} 91 \\ 66 \\ 40 \end{bmatrix};$

number of cases is increasing and spreading through the three stages

d $S_{40} = \begin{bmatrix} 5309 \\ 4258 \\ 3036 \end{bmatrix}, S_{41} = \begin{bmatrix} 5957 \\ 4778 \\ 3406 \end{bmatrix},$ 1.122

e Not sufficient to eradicate disease; growth rate after 40 weeks is approximately 1.027

f Sufficient to eradicate disease; growth rate after 40 weeks is approximately 0.94

12 **a** $S_1 = \begin{bmatrix} 29.6 \\ 14.4 \\ 14.4 \\ 21.6 \\ 6.4 \\ 0.0 \end{bmatrix}, S_{40} = \begin{bmatrix} 96.11 \\ 55.97 \\ 48.88 \\ 42.69 \\ 33.15 \\ 19.30 \end{bmatrix},$

$S_{41} = \begin{bmatrix} 99.04 \\ 57.67 \\ 50.37 \\ 43.99 \\ 34.16 \\ 19.89 \end{bmatrix}$

c $S_0 = \begin{bmatrix} 715 \\ 416 \\ 363 \\ 317 \\ 247 \\ 144 \end{bmatrix}, S_1 = \begin{bmatrix} 735.9 \\ 429.0 \\ 374.4 \\ 326.7 \\ 253.6 \\ 148.2 \end{bmatrix}$

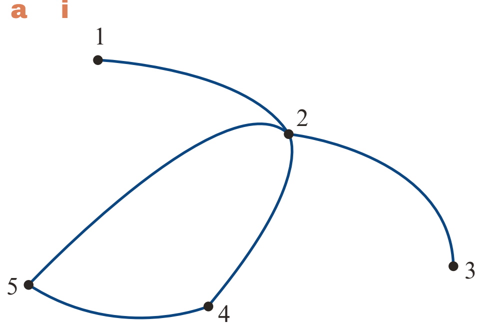
13 a 3×1

b $C \times A = [17900]$; the total cost of the seats.

c The product $= [11600]$; the total cost of the stalls and dress circle.

Networks

14 a i



ii $1 + 4 + 1 + 2 + 2 = 10$

b i Vertices D and E are odd.

ii E and F

iii $E - F - D - E - A - B - C - D$

c i Capacity $= 20 + 25 + 30 = 75$

ii Maximum flow = minimum cut
$= 15 + 15 + 30 = 60$

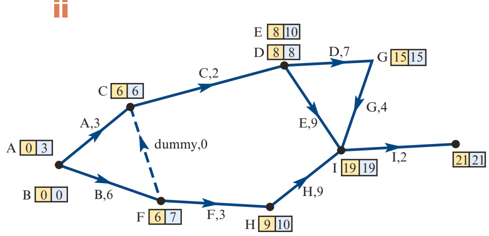
15 a i 2

ii C

b A on breastroke,
B on backstroke,
C on butterfly

c i 7 months

ii



iii A, E, F, H

iv 18 months

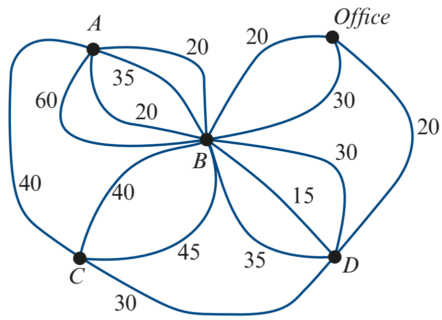
16 a $A - B - D - E - C - A$

b $B - C - E - D - E - C - A - B - D$;
Must start and end at a vertex of odd degree.

c 9:54 am

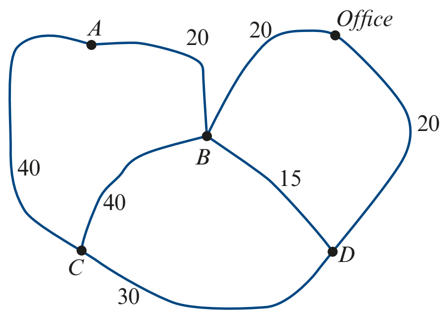
17 a i 4

ii



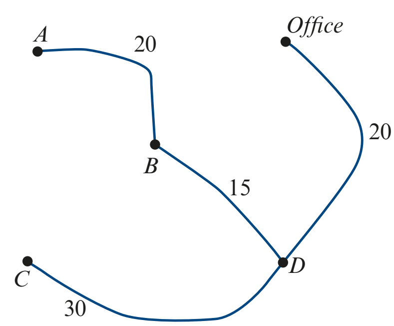
iii A, B, D

b i



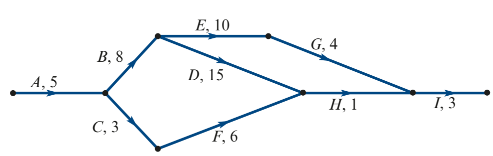
ii \$44 400

c i



ii \$20 400

18 a



b 32 weeks

c **i**

Activity	Duration	EST	LST	Float
A	5	0	0	0
B	8	5	5	0
C	3	5	19	14
D	15	13	13	0
E	10	13	15	2
F	6	8	22	14
G	4	23	25	2
H	1	28	28	0
I	3	29	29	0

ii $A - B - D - H - I$

d **i** The project would be completed in a minimum of 30 weeks.

ii Nothing. This activity has a float of 14 and so it could be extended in duration by 14 weeks.

iii The project would be completed in a minimum of 37 weeks.