ATAR Notes

Maths Methods Units 3&4

Edition 1

COMPLETE COURSE NOTES

- Provides a perfectly condensed summary of the study design and allows you to focus on exactly what you need to know.
- Specifically designed to help you in your SACs and exams, with tips and insights from a past student who knows how to do well.
- Written in a way that's easy to understand, so you can immediately grasp key concepts.
- Graduating with a 48 raw in Methods, Sam has since delivered Methods lectures and tutorials to thousands of students across the state.

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Mathematical Methods Units 3&4
Complete Course Notes

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Title: VCE Mathematical Methods Units 3&4 Complete Course Notes

ISBN: 978-1-925534-04-7

Preface

Welcome to Methods Units 3&4. This is a challenging but rewarding subject, and these summary notes aim to guide you through the Areas of Study and to provide you with a greater understanding of the course content.

The most important thing that you can do is to develop your problem solving skills by completing lots of questions! Methods is a very practical subject so you should make the most out of these notes by completing questions from your textbook as well.

The Methods course is broken up into four areas of study.

1. **Functions and Graphs:** involves sketching elementary functions and showing key features as well as applying transformations on to the functions.
2. **Algebra:** involving algebra of functions, including composite functions, and finding solutions of equations.
3. **Calculus:** involving differentiation, integration and their applications.
4. **Probability and Statistics:** involving probability tables, functions and their representations, as well as statistical inference for samples.

Throughout the year you will complete a range of assessment which will be used to calculate your study score.

- Graded Assessment 1 - SACs (School Asessed Coursework): 34%
- Graded Assessment 2 - Exam 1: 22%
- Graded Assessment 3 - Exam 2: 44%

The two exams at the end of the year will make up the majority of your study score, and influence the scaling of your SAC scores.

Exam 1: is a 1 hour exam which will contain short-answer and some extended-answer questions covering all the areas of study.

Exam 2: is a 2 hour exam which will contain multiple-choice question and extended-answer questions. You are allowed to use calculators and a bound reference.

More information regarding the Methods course can be found in the 2016 Mathematics Study Design published by VCAA here:
www.vcaa.vic.edu.au/Documents/vce/mathematics/MathematicsSD-2016.pdf
(relevant information for Methods begins on Page 70).

Best of luck with your studies!

— Samuel Goh

Contents

Area of Study 1

Functions and Graphs

1.1 Introduction to Functions

1.1.1 Set Notation

Before we look at sketching functions, it is important to get our notation correct.

Sets

A set is a collection of things - when we are sketching graphs, our sets are a collection of numbers. To denote a set, you list each element and surround this list by braces. For example, {1, 2, 3, 4, 5} is a set of numbers.

To say that a variable x is an element of some set A, we use the symbol $\in$, and can express this as $x \in A$. This means that our x variable is made up of elements from the set A.

The intersection of sets is expressed using the $\cap$ symbol while the union of sets is expressed using the $\cup$ symbol. If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cap B = \{3\}$while $A \cup B = \{1, 2, 3, 4, 5\}$.

Interval Notation

What if we want to include all numbers between 3 and 4? We could use: $\{x : 3 \leq x \leq 4\}$. However, interval notation allows us to simplify this to $[3, 4]$. In interval notation, *closed* brackets mean that the interval is inclusive of endpoints. *Open* brackets indicate that an interval is exclusive of endpoints. The interval $(3, 4)$ would include any number between 3 and 4 excluding 3 and 4. Closed brackets are sketched using closed circles (filled), while open brackets are sketched using open circles (hollow).

Pre-determined Sets

There are some commonly used sets which you can use in Methods.

- R for real numbers: all numbers except complex numbers.
- Z: integers.
- Q: rational numbers.
- Q': irrational numbers.
- N: natural numbers which are all postiive integers.

1.1.2 Functions

A function is a relation between ordered pair and coordinates, and is specified by its rule and domain. In order to define a function, we use the following notation:

$f : R \rightarrow R, f(x) = x^2$

- The f is used to state that we are defining a function.
- The first R is the domain. We can restrict a function to a specific domain here, if need be.
- The second R is the codomain. For the purposes of VCE, this will always be R.
- Finally, the rule of the function is stated as $f(x) = ...$

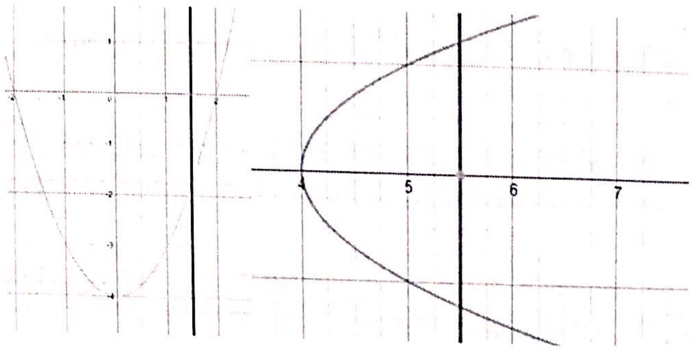
A function can be imagined as a machine or a factory. Whatever comes out of our function $f(x)$, is dependent on what we put into the function (x). Putting in a different input will usually yield a different output. $f(x)$ is essentially a description of the processes inside the machine or factory.

If we know how the machine works, we know the output to expect when we put in a certain input. Likewise, if we have a certain output, we can work backwards to find out what the input into the machine was.

Requirements

Although every function is a relation of ordered pairs and coordinates, not every relation is a function. For a relation to be considered a function, it must be one-to-one. This means that there must be one value of $f(x)$ for a specific x. In order to test for this, you can use the vertical line test through a graph. If the graph touches the vertical line more than once, it fails the vertical line test and is not a function.

The first relation below is a function, as the vertical line does not touch the curve more than once. The second relation is *not* a function, as the vertical line touches the curve more than once.

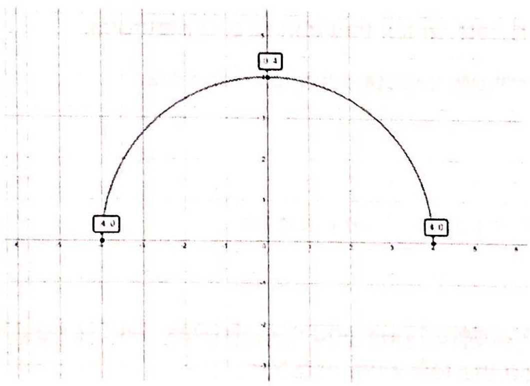


1.1.3 Domain

The domain is the set of x values which you place into a function. Another way of thinking about the domain, is as the set of x values in which the graph *exists* over.

Implied or Maximal Domain

In some circumstances, there are certain rules which *imply* the domain of a function, or which *limit* the x values in which the function can exist over. For example, take the graph of $y = \sqrt{16 - x^2}$. This graph is the top half of a semicircle as shown below.



From the image, it is clear that the maximal domain is $x \in [-4, 4]$. We cannot take the square root of a negative number, so we can only input x values between -4 and 4 if we want to get an output.

1.1.4 Range

The range is the set of numbers which are outputted from a function if you were to input every number in the domain. Another way of thinking about the range is as the set of y value in which the graph exists over.

For the semicircle shown above, the range is $y \in [0, 4]$ as the function only exists over those y values.

1.1.5 Key Features

When sketching functions, it is very important to show key features of your graph. Every time you do a sketch, run through this list in your head.

- Axis: x axis, y axis
- Origin: $(0, 0)$
- Asymptote
- Turning Points: maximums, minimums
- Stationary Points of Inflection
- Cusps
- Points of Discontinuity
- Endpoints

1.2 Polynomials

In this section, we will be looking at polynomials - you have most likely seen these before as part of your Year 11 course, so this will serve as a recap. A polynomial is a combination of terms, variables and exponents. A polynomial cannot have a variable in the denominator, and the exponents must be natural numbers. The degree of a polynomial is the value of the highest exponent.

1.2.1 Linear Functions

A linear function is a first degree polynomial and forms a straight line when sketched. As a linear function forms a straight line, only two points are required to sketch the graph. You may see a linear function in a few forms, the most common being gradient-intercept form: $y = mx + c$. In this form, m is the gradient of the line and c is the vertical intercept. Linear functions can also be seen in the form: $ax + by = c$.

Sketching a linear function is as easy as finding two points on the line, then drawing a line which passes through both of them.

Example 1.1

Sketch the graph of $2y + 4x = 8$.

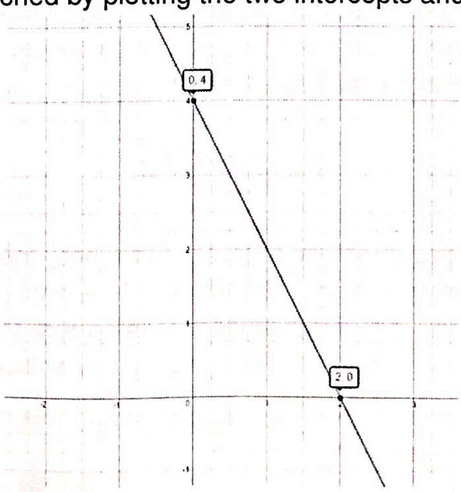
In this form, it is very easy to find the two intercepts. Let $x = 0$ and solve for y:

$$\begin{aligned} 2y + 4(0) &= 8 \\ y &= 4 \end{aligned}$$

Let $y = 0$ and solve for x:

$$\begin{aligned} 2(0) + 4x &= 8 \\ x &= 2 \end{aligned}$$

The graph can then easily be sketched by plotting the two intercepts and drawing a line through them:



An alternative way of sketching the graph would be to rearrange the graph into gradient-intercept form. The rearranged form is:

$$y = 4 - 2x$$

From this form, the y-intercept can be read as (0,4), and you can set $y = 0$ in order to solve for the x-intercept.

Finding the Equation

Given one point on a straight line, (x_1, y_1) and the gradient m, the equation of a linear function is given as:

$$y - y_1 = m(x - x_1)$$

Given two points on a straight line, (x_1, y_1) and (x_2, y_2), the gradient can be calculated as:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

This gradient can then be substituted into the equation above:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

1.2.2 Quadratic Functions

A quadratic function is a second degree polynomial which contains an x^2 term and no higher power. A quadratic function has a turning point and up to 2 x intercepts. There are three forms which you will commonly see a quadratic in:

$$\text{Turning Point: } y = a(x - b)^2 + c$$

$$\text{Factor: } y = a(x - b)(x - c)$$

$$\text{Expanded: } y = ax^2 + bx + c$$

When a quadratic is in turning point form, it is very easy to read off the coordinates of the vertex (b, c), while factor form makes it very easy to find the x intercepts which in this case are $x = b$ and $x = c$. In the expanded form, the y intercept is very easy to find as setting $x = 0$ will leave $y = c$ as the y intercept.

How do I sketch it?

Sketching a quadratic function can be done with a combination of factorisation, completing the square and solving for intercepts. However, sometimes it isn't possible to factorise, so the quadratic formula is useful. The quadratic formula states that if $ax^2 + bx + c = 0$, then:

$$x = \frac{-b \pm \sqrt{b^2 - 4ax}}{2a}$$

It's a very good idea to memorise the quadratic formula. Now, note the $b^2 - 4ac$ in the square root. You know you can't square root a negative number. Therefore, if $b^2 - 4ac < 0$, this means that x is undefined (over real numbers) and therefore the quadratic equation has no real solutions. If $b^2 - 4ac > 0$, there will be two solutions (note the $\pm$) and if $b^2 - 4ac = 0$, there is only one solution.

Since this $b^2 - 4ac$ value determines the number of solutions in this quadratic equation, it is called the discriminant and is denoted by the symbol Δ.

Example 1.2

Sketch the graph of $y = x^2 + 4x - 12$.

Let's factorise this equation so we can easily find the intercepts:

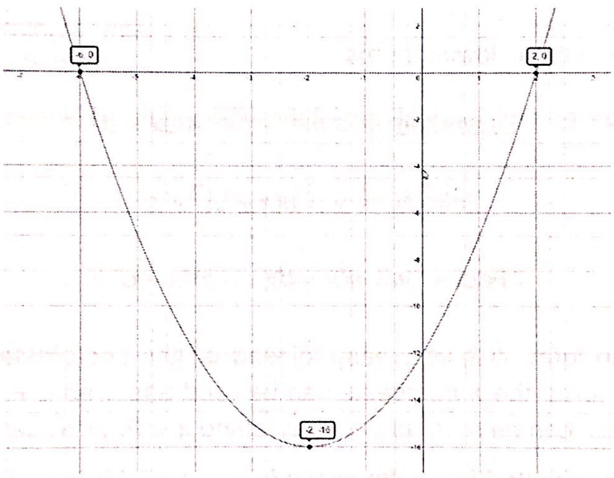
$$x^2 + 4x - 12 = (x - 2)(x + 6)$$

Our intercepts are (2, 0) and (–6, 0). However, we stil need to find the turning point before sketching the graph accurately. What you will notice is that a quadratic is symmetrical about its turning point, which means that the x coordinate of the turning point is half way between the two intercepts. This means the turning point is located at $x = \frac{2-6}{2} = -2$. We can then substitute this into the equation to find the y coordinate of the turning point:

Let $x = -2$:

$$y = (-2)^2 + 4(-2) - 12 = -16$$

Using the coordinates of the two intercepts and the coordinates of the turning point, we can sketch the graph:



An alternate method of sketching this graph would be to complete the square:

$$\begin{aligned} y &= x^2 + 4x - 12 \\ &= x^2 + 4x + 4 - 4 - 12 \\ &= (x + 2)^2 - 16 \end{aligned}$$

In this form, the turning point of (–2, 16) is evident, and solving for the x intercepts is a matter of some straightforward algebra.

Let $y = 0$:

$$\begin{aligned} 0 &= (x + 2)^2 - 16 \\ 16 &= (x + 2)^2 \\ \pm 4 &= x + 2 \\ x &= \pm 4 - 2 \end{aligned}$$

Therefore, the x intercepts are (–6, 0) and (2, 0).

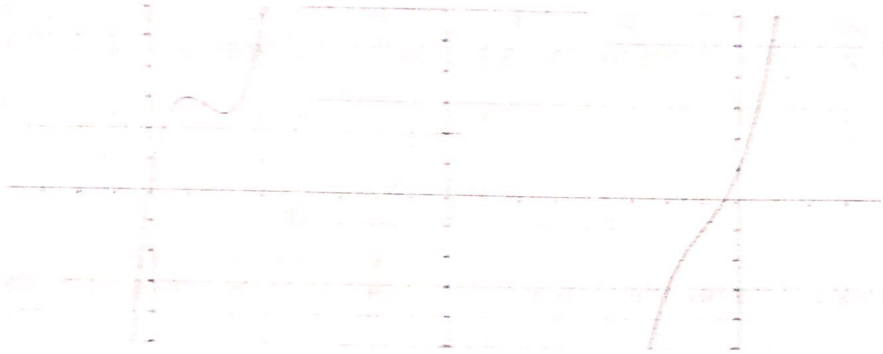
1.2.3 Cubic Functions

A cubic function is a third degree polynomial. They can have two turning points, one stationary point of inflexion or no stationary points. Below is an example of each scenario. The first graph has two stationary points, the second has a stationary point of inflexion and the third has no stationary points.

$$f : R \rightarrow R, f(x) = 2x^3 - 9x^2 + 12x + 1$$

$$g : R \rightarrow R, g(x) = x^3 + 4$$

$$h : R \rightarrow R, h(x) = x^3 + 3x^2 + 6x + 2$$



Cubic functions can be seen in the following forms:

Expanded: $y = ax^3 + bx^2 + cx + d$

Inflection: $y = a(x - b)^3 + c$

Factor: $y = a(x - b)(x - c)(x - d)$

When a cubic is in inflection form, it is very easy to read off the coordinates of the stationary point of inflection as (b, c). In factor form, the x intercepts can be read easily as $x = b$, $x = c$ and $x = d$. In the expanded form, the y intercept can be read as $y = d$ as letting $x = 0$ will leave $y = d$.

If a cubic in factor form has a repeated factor, for example $y = a(x - b)^2(x - c)$, there will be a turning point at $x = b$.

How do I sketch it?

Sketching cubic functions isn't as simple as sketching a linear or quadratic function. A good way of sketching cubics is to find x-intercepts, then use calculus to find turning points. You can then plot the intercepts and the turning points and connect them with a smooth continuous curve. We will cover finding the turning points in the calculus portion of these notes.

Example 1.3

Find the x-intercepts of the graph: $y = (x - 1)^3 - 27$

This can be solved quite easily as there is only a single term which is cubed, so we can work backwards to find x. Let $y = 0$:

$$\begin{aligned} (x-1)^3 - 27 &= 0 \\ (x-1)^3 &= 27 \\ x - 1 &= 3 \\ x &= 4 \end{aligned}$$

Example 1.4

Find the x-intercepts of the graph: $y = x^3 - 3x^2 + 6x - 4$

In your technology exam, it is safest to use the solve function on your calculator. By hand, finding intercepts is a little more difficult. We can use the factor theorem and test different x values to see if $y = 0$. In this expression, setting $x = 1$, will result in $y = 1^3 - 3(1)^2 + 6(1) - 4 = 0$. Therefore $(x - 1)$ is a factor of the polynomial. We can now use long division to factorise the expression to:

$$\begin{aligned} x^3 - 3x^2 + 6x - 4 &= 0 \\ (x-1)(x^2 - 2x + 4) &= 0 \end{aligned}$$

The x intercepts are $x = 1$ or $x^2 - 2x + 4 = 0$. Looking at our second possible solution, the discriminant is less than zero $((-2)^2 - 4(1)(4) = -12 < 0)$, so there will not be any solution for our second factor. Therefore $x = 1$ is our only solution. You can sketch this graph on your calculator to verify this.

1.2.4 Quartic Functions

Quartic functions are fourth order polynomials which involve an x^4term. Here are some properties:

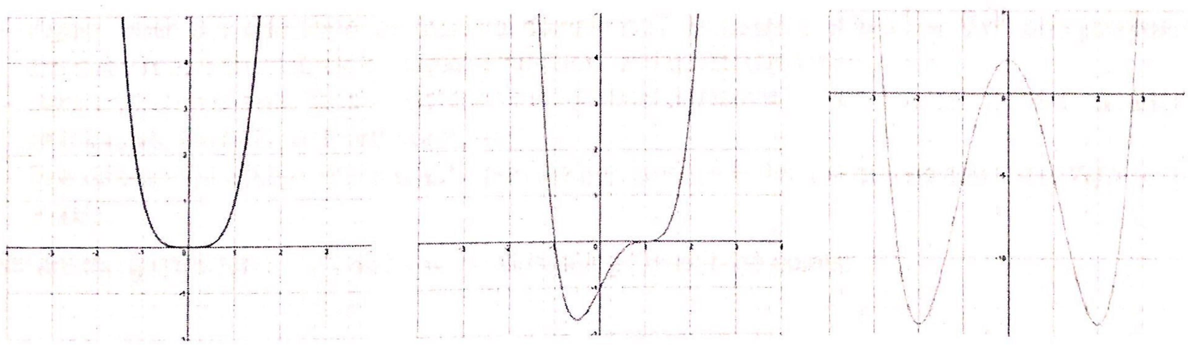
- They may have 1-3 turning points.
- They may have 0-4 x-intercepts.

Quartic functions can be expressed in many different forms. Here are some examples:

$$f : R \to R, f(x) = x^4$$

$$g : R \to R, g(x) = x^4 - 2x^3 + 2x - 1$$

$$h : R \to R, h(x) = x^4 - 8x^2 + 2$$



Quartic functions can be seen in the following forms:

Expanded: $y = ax^4 + bx^3 + cx^2 + dx + e$

Turning Point: $y = a(x - b)^4 + c$

Factor: $y = a(x - b)(x - c)(x - d)(x - e)$

When a quartic is in turning point form, it will have a turning point at (b, c). In factor form, the x intercepts can be read easily as $x = b$, $x = c$, $x = d$ and $x = e$. In th expanded form, the y intercept can be read as $y = e$ as letting $x = 0$ will leave $y = e$.

If a quartic in factor form has a repeated factor, for example $y = a(x-b)^2(x-c)^2$, there will be turning points at $x = b$ and $x = c$.

How do I sketch it?

Sketching quartic functions or solving quartic equations will require a variety of skills. The factor theorem, which was used in the cubic functions example is extremely useful, as well as long division and other factorisation skills. The goal should be to find x and y intercepts, then connect them together with a smooth curve. Finding the turning points (covered in Chapter 4) and also substituting other x values in to find other points can help improve the accuracy.

If you have a calculator available though, use that to avoid making any errors.

1.3 Additional Functions

1.3.1 Extended Power Functions

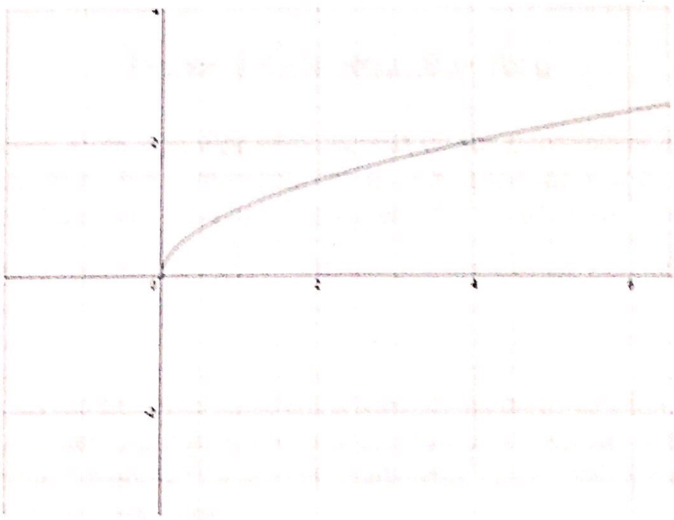
Square Root Functions

The most basic square root function is: $y = \sqrt{x}$.

Some properties to note:

- The domain is $x \in [0, \infty)$.
- The range is $y \in [0, \infty)$
- The graph is increasing over its maximal domain.

Here is a sketch of the most basic function:



Transformations (covered later) can be used to modify its shape, which will modify its domain and range.

How do I sketch it?

Sketching a square root function is quite simple. Firstly, find the endpoint where the square root finishes. In our most basic example, this point is $(0, 0)$. The endpoint will occur when the inside of the square root equals zero.

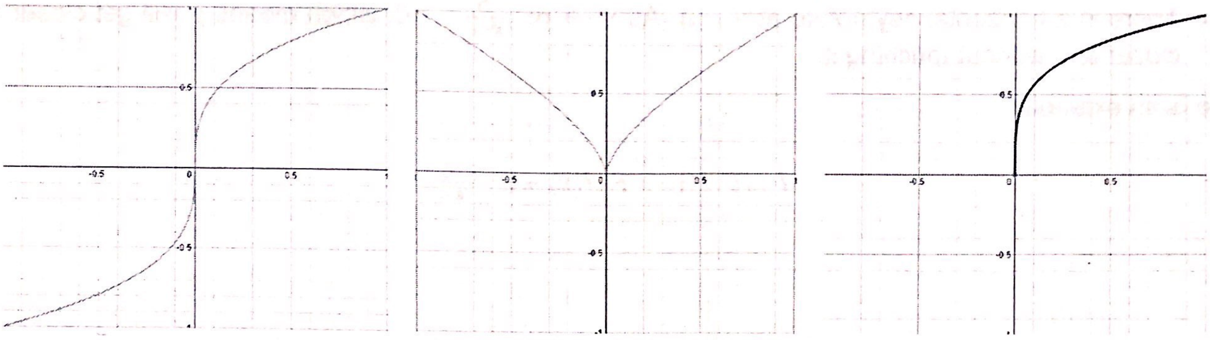
Then, one or two extra points on the curve is usually enough to draw an accurate square root function. Substituting $x = 4$ into $\sqrt{x}$ will give a point $(4, 2)$ and substituting $x = 9$ into $\sqrt{x}$ will give a point $(9, 3)$. Using the three points $(0, 0)$, $(4, 2)$ and $(9, 3)$ and connecting them with a smooth line should result in a fairly accurate square root function.

Functions of the form $y = x^{\frac{a}{b}}$, where $\frac{a}{b} < 1$

These functions are less common but quite interesting. A few properties you should know:

- When b is odd, then the maximal domain is R. Remember that $x^{\frac{a}{b}} = \sqrt[b]{x^a}$, and you can take the cube root, fifth root, etc of negative numbers and positive numbers.
- When b is even (and a odd) then the maximal domain is $[0, \infty)$. You cannot take square roots of negative numbers, or fourth roots, etc.
- These graphs concave towards the x-axis.

Here is an illustration of the first and second points with the graph $y = x^{\frac{1}{3}}$, $y = x^{\frac{2}{3}}$ $y = x^{\frac{1}{4}}$:



See how the first graph spans reals and the third one doesn't? Also, note how both graphs concave towards the x-axis.

Take a look at how $y = x^{\frac{2}{3}}$ looks like a bird. The reason it comes across like this is because:

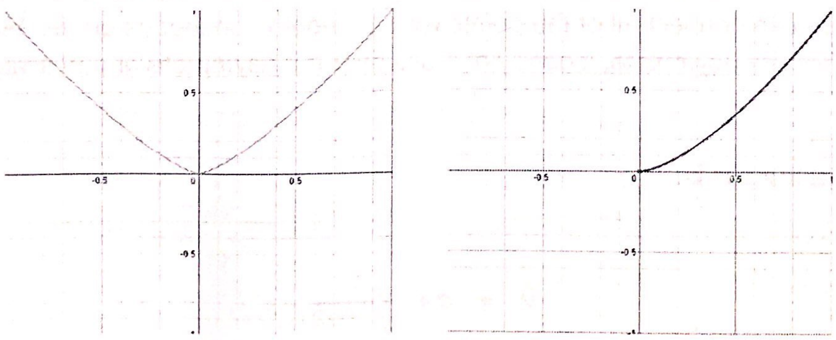
- $\frac{2}{3} < 1$, so the graph always concaves towards the x-axis.
- $x^{\frac{2}{3}} = (x^{\frac{1}{3}})^2$, so if you took the y-coordinates of the first graph, and squared them, you'd get the corresponding y-coordinate in the second graph. Also, $x^{\frac{2}{3}}$ is positive when $x < 0$ because x^2 is positive.

Functions of the form $y = x^{\frac{a}{b}}$, where $\frac{a}{b} > 1$

Some more interesting functions. Stuff you should know about these:

- Again, when b is odd, then the maximal domain is R. Remember that $x^{\frac{a}{b}} = \sqrt[b]{x^a}$, and you can take the cube root, fifth root, etc of negative numbers and positive numbers.
- Again, when b is even (and a odd) then the maximal domain is $[0, \infty)$. You cannot take square roots of negative numbers, or fourth roots, etc.
- The difference between these graphs and the previous set is that the graphs concave AWAY from the x-axis.

Below are the graphs for $y = x^{\frac{4}{3}}$ and $y = x^{\frac{3}{2}}$, illustrating these three points:



Area of Study 1 – Functions and Graphs

1.3.2 Hyperbolas

The general form of a hyperbola is:

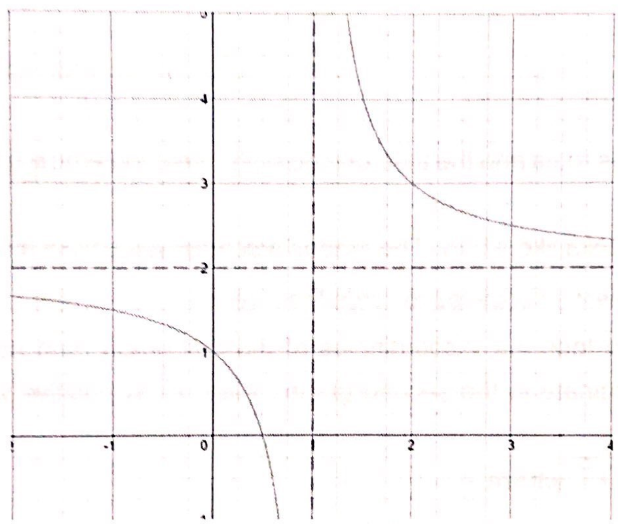
$$y = a + \frac{b}{x - c}$$

Some properties to note:

- There is a vertical asymptote at $x = c$. This is because setting $x = c$ will result in $c - c = 0$ on the denominator which will make y undefined.
- There is a horizontal asymptote at $y = a$. As $x \to \infty$, $\frac{b}{x-c} \to 0$, which means y will get closer and closer to a without touching it.

Here is an example:

$$f : R\backslash\{1\} \to R, f(x) = 2 + \frac{1}{x-1}$$



There are asymptotes at $x = 1$ and $y = 2$, which leads to a domain of $x \in R\backslash\{1\}$ and a range of $y \in R\backslash\{2\}$.

How do I sketch it?

Sketching a hyperbola can be done by looking at a few key points. First of all, draw in the two asymptotes with a dotted line. This will help you make sure that the shape is correct. The two axes intercepts are also useful, so set $x = 0$ and $y = 0$ to find out their position.

Once the asymptotes and the intercepts are found, one or two points on each side of the asymptotes is also useful. An easy point can be found by substituting $x = 2$ into $f(x)$ above. $f(2) = 2 + \frac{1}{2-1} = 3$, which gives$(2, 3)$. Then, you can connect all of the points with a smooth continuous curve. Make sure that when your curve approaches the asymptote, it does not cross over the asymptote or curl away.

1.3.3 Truncus

The general form of a truncus is:

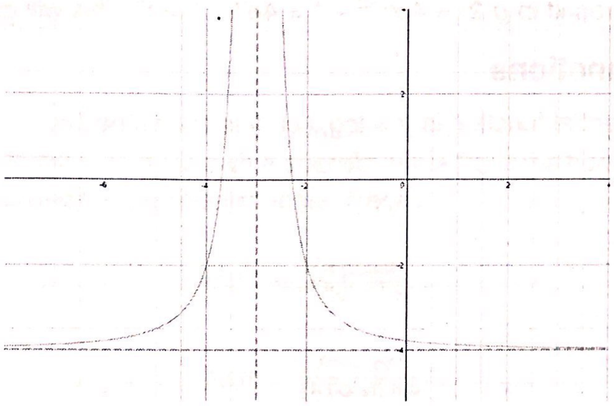
$$y = a + \frac{b}{(x - c)^2}$$

Some properties to note:

- There is a vertical asymptote at $x = c$. This is similar to the hyperbola as letting $x = c$ will make y undefined.
- There is a horizontal asymptote at $y = a$. This is similar to the hyperbola, because as x approaches positive or negative infinity, $\frac{b}{(x-c)^2}$ will approach zero.
- The range of the graph is $y \in (a, \infty)$ or $y \in (-\infty, a)$, depending on the reflection of the graph.

Here is an example:

$$f : R \setminus \{-3\} \rightarrow R, f(x) = -4 + \frac{2}{(x+3)^2}$$



There are asymptote at $x = -3$ and $y = -4$, which results in a domain of $x \in R \setminus \{-3\}$ and a range of $y \in (-2, \infty)$.

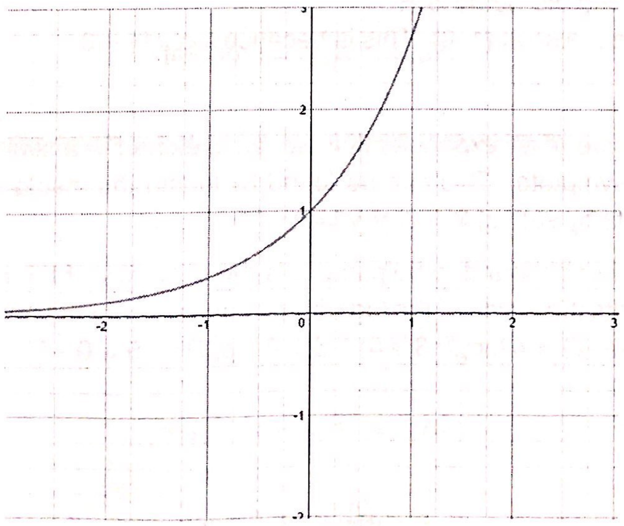
How do I sketch it?

A truncus can be sketched in a similar way to a hyperbola. First, sketch the asymptotes with a dotted line. Then find the axis intercepts. Add a few extra points, a continuous curve and you've got your truncus!

1.4 Exponentials and Logarithms

1.4.1 Exponential Functions

The basic form of an exponential function is $y = e^x$. It is shown below.



Some properties to note:

- There is an asymptote at $y = 0$. The curve approaches this asymptote when x approaches negative infinity.
- The domain is R and the range is $(0, \infty)$.
- The graph crosses the y-axis at $(0, 1)$. This is because $e^0 = 1$.

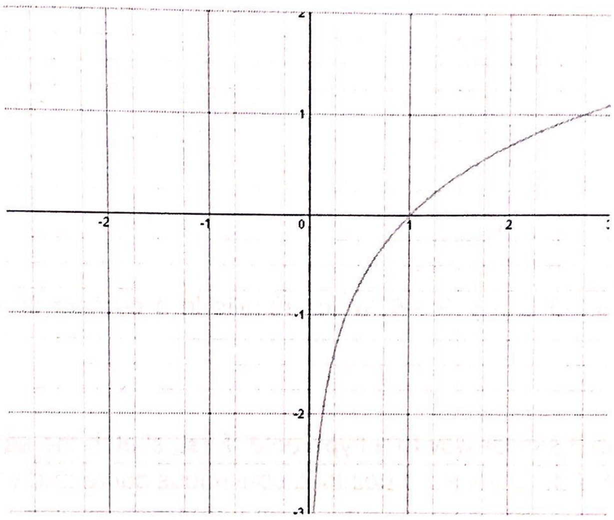
How do I sketch it?

An exponential function can be sketched easily by firstly drawing in the asymptote. The next step would be to find any x and y intercepts by letting $x = 0$ and $y = 0$.

Finally, one additional point can be found to help improve the accuracy of the curve. An easy point can be found at the x value which sets the exponent of the exponential function to zero. For example, if $g(x) = 4e^{x-2} + 1$, setting $x = 2$ will result in $g(2) = 4e^{2-2} + 1 = 4e^0 + 1 = 5$. This will give us a point of $(2, 5)$.

1.4.2 Logarithmic Functions

The basic form of an exponential function is $y = \log_e(x)$. It is shown below.



Some properties to note:

- The $y = \log_e(x)$ graph is the inverse of $y = e^x$. That is, it is the reflection across the line $y = x$.
- There is an asymptote at $x = 0$.
- The domain is $(0, \infty)$ and the range is R.
- The graph crosses the x-axis at $(1, 0)$. This is because $\log_e(1) = 0$.

How do I sketch it?

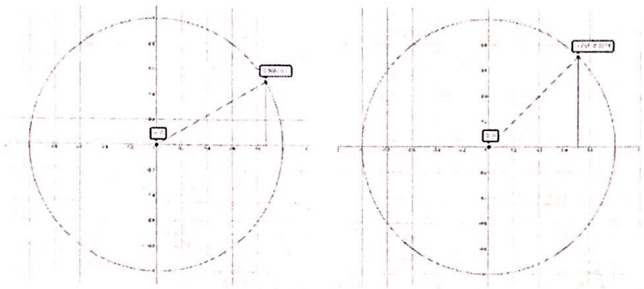
Since a logarithm is the inverse of an exponential, it can be sketched in a similar way. The first step would be to draw in the vertical asymptote. This can be found by letting the inside of the logarithm equal zero. Then, find any axis intercepts by setting $x = 0$ or $y = 0$.

Again, one additional point can be found to help improve the accuracy of the curve. An easy point can be found at the x value which sets the inside of the logarithm to equal 1. For example, if $g(x) = 4\log_e(2x+4)-5$, setting $x = -\frac{3}{2}$ will result in $g(-2) = 4\log_e(-3 + 4) - 5 = 4\log_e(1) - 5 = 0 - 5 = -5$. This will give us a point of $(-\frac{3}{2}, -5)$.

1.5 Circular Functions

1.5.1 Unit Circle

The unit circle is the foundation behind the circular functions covered in Methods. The unit circle is simply a circle with radius of 1 centred at the origin (0, 0).



If we place a triangle inside of the unit circle with a specific angle from the horizontal, we can calculate the values of the trigonometric functions at a specific angle. Recall:

$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$$

In the first image above, the angle is set at 30 degrees. In this position:

$$O = 0.5, H = 1, A = 0.866$$

Therefore:

$$\sin(30°) = \frac{0.5}{1} = 0.5$$

$$\cos(30°) = \frac{0.866}{1} = 0.866$$

$$\tan(30°) = \frac{0.5}{0.866} = 0.577$$

In the second image above, the angle is set at 45 degrees. In this position:

$$O = 0.707, H = 1, A = 0.707$$

Therefore:

$$\sin(45°) = \frac{0.707}{1} = 0.707$$

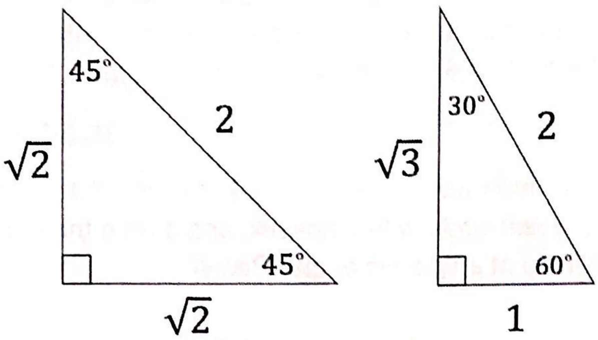
$$\cos(45°) = \frac{0.707}{1} = 0.707$$

$$\tan(45°) = \frac{0.707}{0.707} = 1$$

The unit circle is used to determine the values of trigonometric ratios because the radius, or the hypotenuse is always going to be equal to 1. Sine and cosine are both calculated by dividing by the hypotenuse, so the value of sine is simply the vertical distance in the triangle, and the value of cosine is simply the horizontal distance in the triangle.

When trigonometric ratios are graphed, the x-axis becomes the angle, and the value of the trigonometric ratio at specific x-values is plotted on the y-axis. This results in the trigonometric ratio being evaluated at a range of different angles.

Special Triangles



The two special triangles above can be used to find the exact values of the trigonometric ratios at 30, 45, and 60 degrees.

In the previous section we used the approximation from the unit circle to find the values of the trigonometric ratios. Let's use these special triangles to find the exact value of these ratios.

$$\sin(30°) = \frac{1}{2}$$
$$\cos(30°) = \frac{\sqrt{3}}{2}$$
$$\tan(30°) = \frac{1}{\sqrt{3}}$$

$$\sin(45°) = \frac{\sqrt{2}}{2}$$
$$\cos(45°) = \frac{\sqrt{2}}{2}$$
$$\tan(45°) = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

Units

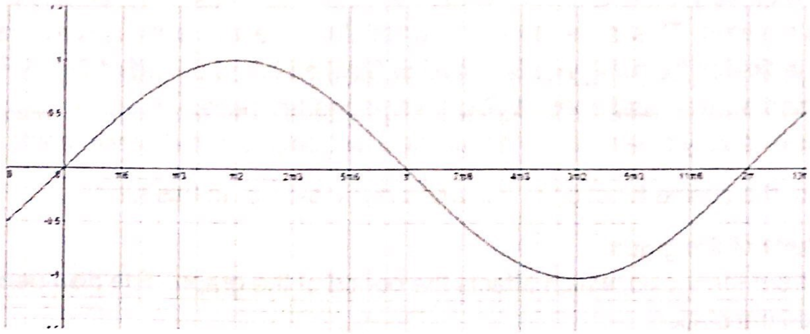
Both radians and degrees can be used for sketching circular functions. However, radians are used more often.

$$\pi \text{rad} = 180 \text{ degrees}$$
$$1 \text{ rad} = \frac{180}{\pi} \text{ degrees}$$

- To convert from radians to degrees, multiply by $\frac{180}{\pi}$.
- To convert from degrees to radians, divide by $\frac{180}{\pi}$ or multiply by $\frac{\pi}{180}$.

1.5.2 Sine Function

The basic form of the sine function is $y = \sin(x)$. It is shown below.



Some properties to note:

- The sine curve has an amplitude. This is the distance from midway up the graph to the top of the graph. A sine curve without a dilation away from the x-axis has an amplitude of 1.
- The curve has a period. This is the amount of units along the x-axis the curve covers before it repeats itself. Without a dilation from the y-axis, a sine function has a period of 2π.
- The graph begins at $(0, 0)$ and ends at $(2\pi, 0)$ after one full cycle.

A transformed form of the sine function is $y = a\sin(n(x + b)) + c$. In this form:

- a is the amplitude of the graph.
- n is a dilation from the y-axis, which affects the period of the graph. The new period is $\frac{2\pi}{n}$.
- b is a horizontal translation.
- c is a vertical translation.

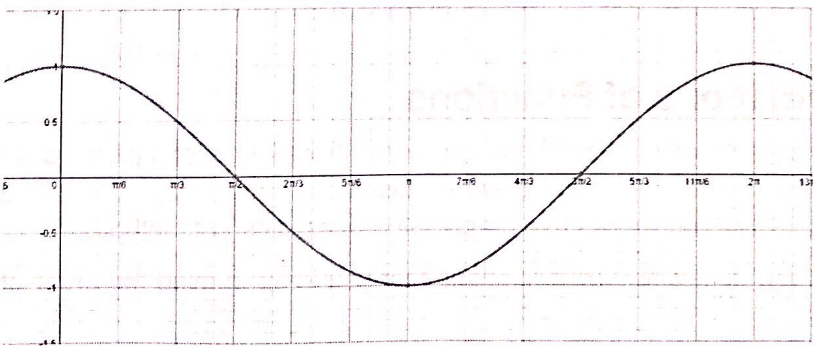
How do I sketch it?

The sine function can be sketched by starting at the most basic form; the normal $y = \sin(x)$. In its usual form, it starts at $(0, 0)$, moves up to $(\frac{\pi}{2}, 1)$, down and through $(\pi, 0)$ to $(\frac{3\pi}{2}, -1)$ and back up to $(2\pi, 0)$ where the function finishes its cycle and then repeats. Using that information, we can determine the position of the curve after transformations.

If the curve is $f(x) = 4\sin(x) + 5$, after all of the values have been outputted from $\sin(x)$, they are multiplied by 4 and then 5 is added to them. This means the new points on the transformed curve would be: $(0, 5)$, $(\frac{\pi}{2}, 9)$, $(\pi, 5)$ and so on.

1.5.3 Cosine Function

The basic form of the cosine function is $y = \cos(x)$. It is shown below.



Some properties to note:

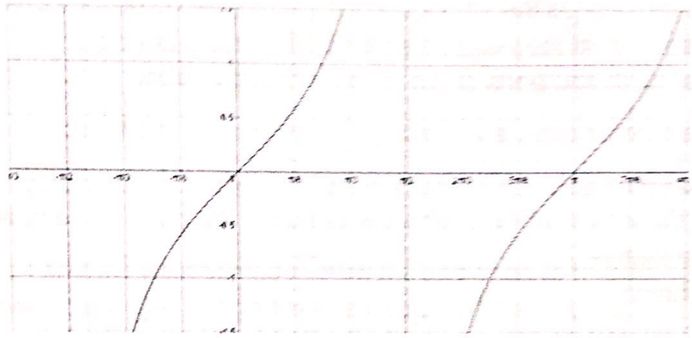
- The cosine curve has an amplitude. This is the distance from midway up the graph to the top of the graph. A cosine curve without a dilation away from the x-axis has an amplitude of 1.
- The curve has a period. This is the amount of units along the x-axis the curve covers before it repeats itself. Without a dilation from the y-axis, a cosine function has a period of 2π.
- The graph begins at $(0, 1)$ and ends at $(2\pi, 1)$ after one full cycle.
- The cosine curve is very similar to a sine function, but begins at a different point.

A transformed form of the cosine function is $y = a\cos(n(x + b)) + c$. In this form:

- a is the amplitude of the graph.
- n is a dilation from the y-axis, which affects the period of the graph. The new period is $\frac{2\pi}{n}$.
- b is a horizontal translation.
- c is a vertical translation.

1.5.4 Tangent Function

The basic form of the tangent function is $y = \tan(x)$. It is shown below.



Some properties to note:

- The curve has a period. This is the amount of units along the x-axis the curve covers before it repeats itself. Without a dilation from the y-axis, a tangent function has a period of π.
- The graph has vertical aymptotes which represent the areas where $\tan(x)$ is undefined. The asymptotes are at $\pm\frac{\pi}{2}$ and occur every π units after that and before that.

A transformed form of the tangent function is $y = a\tan(n(x + b)) + c$. In this form:

- a is the amplitude of the graph.
- n is a dilation from y-axis, which affects the period of the graph. The new period is $\frac{\pi}{n}$.
- b is a horizontal translation.
- c is a vertical translation.

1.6 Transformations of Functions

Transformations are a major topic in the Methods course. Transformations allow us to change the appearance of a function. The three transformations we will discuss are dilations, reflections and translations. We will then cover applying transformations to a graph using three different methods.

It is important when you deal with transformations that you keep in mind the order of transformations. It does matter!

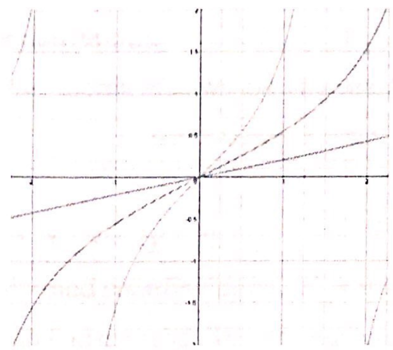
The best way to describe transformation is using DRT. Dilations, Reflections, and finally Translations. Attempting to describe transformations in a different way can be very messy.

1.6.1 Dilations

Dilations allow us to *stretch* graphs away or towards from the axes by certain factors.

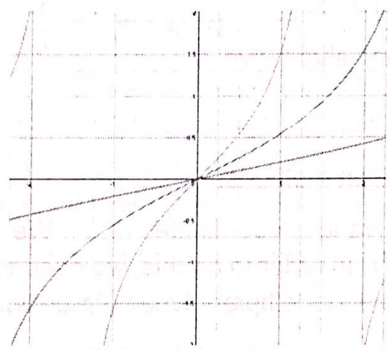
x-axis

The graph of $y = x^2$ after a dilation of factor 2 away from the x-axis and a dilation of factor 4 away from the x-axis are shown below. Notice how the dilated graphs appear taller when dilated from the x-axis.



y-axis

The graph of $y = \tan(x)$ is shown below after a dilation of factor 2 from the y-axis and a dilation of factor 5 from the y-axis is applied. Notice how he dilated graphs appear wider when dilated from the y-axis.

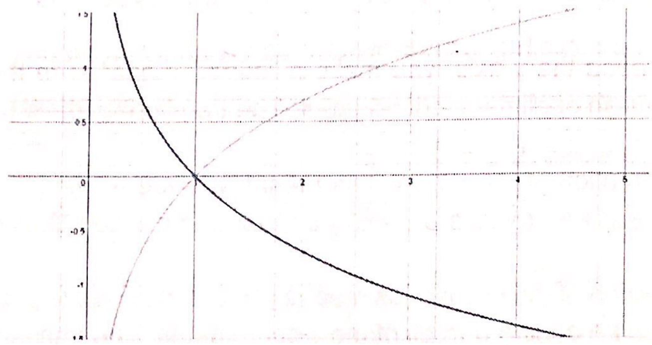


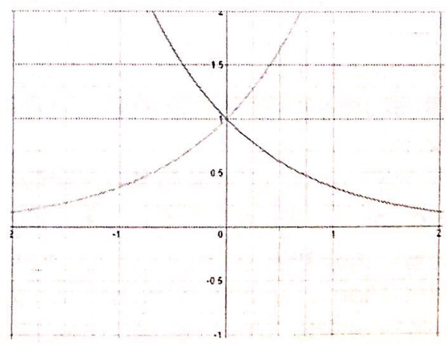
Be very careful when reading the wording of a question. A dilation *away* from an axis is very different form a dilation *towards* an axis. One will pull the graph apart, while one will squish the graph together.

1.6.2 Reflections

Reflections allow us to *flip* graphs about the axis. The x-axis and the y-axis become a "mirror" which reflects what was originally above, below, or on each side of it.

The reflection of $y = \log_e(x)$ across the x-axis and the reflection of $y = e^x$ across the y-axis are below:

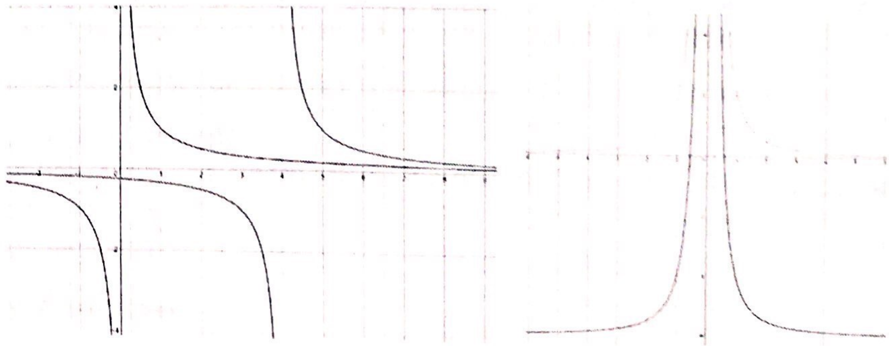




1.6.3 Translations

Translations allow us to *move* graphs horizontally and vertically. The size of the graph does not change.

The graph of $y = \frac{1}{x}$ translated 4 units in the positive x direction is shown below, as well as the graph of $y = \frac{2}{x^2}$ translated 6 units in the negative y direction.



The wording is also important with translations. A translation to the right can be expressed as:

- Translation h units in the positive direction of the x axis.
- Translation h units right.
- Translation h units parallel to the x axis.

It is important you are familiar with each alternative.

1.6.4 Method 1: Dash

Let's look at the first of the transformation methods!

Say we have a graph of $y = f(x)$, with various coordinates (x, y). We can transform it into another graph of $y' = f(x')$ with new coordinates (x', y'). In order to do this, we need to look at how the original coordinates are manipulated to get a new function under different types of transformations.

- Dilations by factor a from the x-axis: you are stretching the graph away from the x-axis. What you are basically doing is taking all the y-coordinates of the points in the graph and multiplying them by a. If $a > 1$, the graph is stretched away from the x-axis. If $0 < a < 1$, we end up "constricting" the graph and pushing it down towards the axis. The transformation is $(x, y) \longrightarrow (x, ay)$.
- Dilations by factor b from the y-axis: you are stretching the graph away from the y-axis. What you are basically doing is taking all of the x-coordinates of the points on the graph and multiplying them by b. The transformation is $(x, y) \longrightarrow (bx, y)$.
- Reflections in the x-axis: everything above the x-axis, you make a "mirror image" of below the x-axis. Therefore, you are changing the sign of all of the y-coordinates. The transformation is $(x, y) \longrightarrow (x, -y)$.
- Reflections in the y-axis: everything around the y-axis, you make a "mirror image" of to the other side of the y-axis. You are changing the sign of all the x-coordinates. The transformation is $(x, y) \longrightarrow (-x, y)$.
- Translations by c units in the positive direction of the x-axis: this means you are moving the graph by c units to the right. This basically means that you are adding c to all of the x-coordinates. The transformation is $(x, y) \longrightarrow (x + c, y)$.
- Translation by d units in the positive direction of the y-axis: this means you are moving the graph by d units upwards. This basically means you are adding d to all of the y-coordinates. The transformation is $(x, y) \longrightarrow (x, y + d)$.

Example 1.5

Apply the following transformations to the graph of $y = x^2$:

Dilation by factor 2 from the x-axis.

Reflection across the x-axis.

Translation of 4 units right and 2 units downwards.

Step 1: Let's start by looking at what the coordinates will be after each individual transformation.

Dilation: $(x, y) \rightarrow (x, 2y)$

Reflection: $(x, 2y) \rightarrow (x, -2y)$

Translation: $(x, -2y) \rightarrow (x+4, -2y) \rightarrow (x+4, -2y-2)$

This means we can express our transformed coordinates as: $x' = x + 4$ and $y' = -2y - 2$.

Step 2: Rearrange for x and y so we can find out what they are equivalent to with respect to the transformed coordinates.

$$x = x' - 4 \text{ and } y = -\tfrac{1}{2}(y' + 2)$$

Step 3: Substitute in x and y into your original equation to find the new curve after transformations.

$$\begin{aligned} -\frac{1}{2}(y' + 2) &= (x' - 4)^2 \\ y' + 2 &= -2(x' - 4)^2 \\ y' &= -2(x' - 4)^2 - 2 \end{aligned}$$

Therefore, our transformed graph is $y = -2(x-4)^2 - 2$.

Example 1.6

Apply the following transformations to the graph of $y = \log_e(x)$:

Dilation by factor 4 from the x-axis, dilation by factor 3 from the y-axis.

Reflection across the y-axis.

Translation of 2 units left and 1 units downwards.

Step 1: Let's start by looking at what the coordinates will be after each individual transformation.

Dilation: $(x, y) \rightarrow (3x, 4y)$

Reflection: $(3x, 4y) \rightarrow (-3x, 4y)$

Translation: $(-3x, 4y) \rightarrow (-3x - 2, 4y - 1)$

This means we can express our transformed coordinates as: $x' = -3x - 2$ and $y' = 4y - 1$.

Step 2: Rearrange for x and y so we can find out what they are equivalent to with respect to the transformed coordinates.

$$x = -\tfrac{1}{3}(x' + 2) \text{ and } y = \tfrac{1}{4}(y' + 1)$$

Step 3: Substitute in x and y into your original equation to find the new curve after transformations.

$$\begin{aligned} y &= \log_e(x) \\ \frac{1}{4}(y' + 1) &= \log_e(-\frac{1}{3}(x' + 2)) \\ y' + 1 &= 4\log_e(-\frac{1}{3}(x' + 2)) \\ y' &= 4\log_e(-\frac{1}{3}(x' + 2)) - 1 \end{aligned}$$

Therefore, our transformed graph is $y = 4\log_e(-\tfrac{1}{3}(x + 2)) - 1$

1.6.5 Method 2: Matrices

The matrix method is very similar to the dash method, but uses a matrix to define the transformations instead of coordinates. This is often assessed in exam 2. The following notation is used.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$$

This statement is essentially saying that we are going to transform x and y into x' and y' and this transformation will be based upon the matrices $\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$.

- Dilations from the x-axis: Set a as your dilation factor. If there is no dilation from the x-axis, $a = 1$.
- Dilations from the y-axis: Set b as your dilation factor. If there is no dilation from the y-axis, $b = 1$.
- Reflections in the x-axis: Multiply b by -1.
- Reflections in the y-axis: Multiply a by -1.
- Translation left and right: Let c equal your translation horizontally. If $c > 0$, the graph will be translated to the right.
- Translation up or down: Let d equal your translation vertically. If $d > 0$, the graph will be translated upwards.

Example 1.7

Apply the following transformations to the graph of $y = x^2$:

Dilation by factor 2 from the x-axis.

Reflection across the x-axis.

Translation of 4 units right and 2 units downwards.

Step 1: Let's start by filling in the matrix after each transformation.

Since there is a dilation by factor 2 from the x-axis, $b = 2$. There is then a reflection across the x-axis so $b = 2 \times -1 = -2$. Finally, there is a translation 4 units right and 2 units downwards, so $c = 2$ and $d = -4$. The matrices will look as follows.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

Step 2: Expand the matrix.

$x' = x + 4$ and $y' = -2y - 2$

Notice that we have the same x' and y' from the dash method? We can then finish off this method using the exact same approach from that section.

Step 3: Rearrange for x and y.

$$x = x' - 4 \text{ and } y = -\tfrac{1}{2}(y' + 2)$$

Step 4: Substitute in x and y into your original equation to find the new curve after transformations.

$$\begin{aligned} -\frac{1}{2}(y' + 2) &= (x' - 4)^2 \\ y' + 2 &= -2(x' - 4)^2 \\ y' &= -2(x' - 4)^2 - 2 \end{aligned}$$

Therefore, our transformed graph is $y = -2(x - 4)^2 - 2$.

Example 1.8

Apply the following transformations to the graph of $y = \log_e(x)$:

Dilation by factor 4 from the x-axis, dilation by factor 3 from the y-axis.

Reflection across the y-axis.

Translation of 2 units left and 1 unit downwards.

Step 1: Let's start by filling in the matrix after each transformation.

Since there is a dilation by factor 4 from the x-axis, $b = 4$. There is also a dilation by factor 3 from the y-axis, so $a = 3$. There is then a reflection across the y-axis so $a = 3 \times -1 = -3$. Finally, there is a translation 2 units left and 1 unit downwards, so $c = -2$ and $d = -1$. The matrices will look as follows.

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Step 2: Expand the matrix.

$$x' = -3x - 2 \text{ and } y' = 4y - 1$$

Notice that we have the same x' and y' from the dash method? We can then finish off this method using the exact same approach from that section.

Step 3: Rearrange for x and y.

$$x = -\tfrac{1}{3}(x' + 2) \text{ and } y = \tfrac{1}{4}(y' + 1)$$

Step 4: Substitute in x and y into your original equation to find the new curve after transformations.

$$\begin{aligned} y &= \log_e(x) \\ \frac{1}{4}(y' + 1) &= \log_e(-\frac{1}{3}(x' + 2)) \\ y' + 1 &= 4\log_e(-\frac{1}{3}(x' + 2)) \\ y' &= 4\log_e(-\frac{1}{3}(x' + 2)) \end{aligned}$$

Therefore, our transformed graph is $y = 4\log_e(-\frac{1}{3}(x + 2)) - 1$.

1.6.6 Method 3: Function

The function method is another method of applying transformations. It can be extremely quick once you have had some practice with it. It all starts at $f(x)$ and transformations are applied by modifying the function definition directly.

- Dilations by factor a from the x-axis: Multiply the entire $f(x)$ by a, or $af(x)$.
- Dilations by factor b from the y-axis: Multiply all x values by $\frac{1}{b}$ or $f(\frac{x}{b})$.
- Reflections in the x-axis: Multiply the entire function by -1 or $-f(x)$.
- Reflections in the y-axis: Multiply all x values by -1 or $f(-x)$.
- Translations by c units in the positive direction of the x-axis: Subtract c from all x values or $f(x - c)$.
- Translation by d units in the positive direction of the y-axis: Add d to the output of $f(x)$ or $f(x) + d$.

Example 1.9

Apply the following transformations to the graph of $y = x^2$:

Dilation by factor 2 from the x-axis.

Reflection across the x-axis.

Translation of of 4 units right and 2 units downwards.

Step 1: Let $f(x) = x^2$. Determine the form of the function after each transformation.

Dilation: $f(x) \rightarrow 2f(x)$

Reflection: $2f(x) \rightarrow -2f(x)$

Translation: $-2f(x) \rightarrow -2f(x-4) \rightarrow -2f(x-4)-2$

Step 2: Apply the transformations:

$f(x) = x^2 \rightarrow -2f(x-4)-2 = -2(x-4)^2-2$

Example 1.10

Apply the following transformations to the graph of $y = \log_e(x)$:

Dilation by factor 4 from the x-axis, dilation by factor 3 from the y-axis.

Reflection across the y-axis.

Translation of of 2 units left and 1 units downwards.

Step 1: Let $f(x) = \log_e(x)$. Determine the form of the function after each transformation.

Dilation: $f(x) \rightarrow 4f(x) \rightarrow 4f(\frac{x}{3})$

Reflection: $4f(\frac{x}{3}) \rightarrow 4f(-\frac{x}{3})$

Translation: $4f(-\frac{x}{3}) \rightarrow 4f(-\frac{(x+2)}{3}) \rightarrow 4f(-\frac{(x+2)}{3})-1$

Step 2: Apply the transformations:

$f(x) = \log_e(x) \rightarrow 4f(-\frac{(x+2)}{3})-1 = 4\log_e(-\frac{1}{3}(x+2))-1$

1.6.7 Identifying Transformations

Now that we've learnt how to apply transformations onto a graph or function, how can we identify them? The first thing is to remember DRT which stands for dilations, reflections and translations. When identifying transformations, always write them in this order. Secondly, it is useful to take out coefficients of x, which can make it much easier to read the transformations.

The quickest way to identify transformations is to use the function method to read the changes between $f(x)$, the original function, and the function given which has been transformed. Let's work through some examples.

Example 1.11

Determine the sequence of transformations to transform the graph of $f(x) = e^{2x}$ to $g(x) = 4e^{2x-4}+5$.

First of all, let's take out coefficients from $g(x)$: $4e^{2x-4}+5 \rightarrow 4e^{2(x-2)}+5$.

Now let's see what has happened to the $f(x)$ to result in $g(x)$. With a bit of analysis, we can come to this conclusion: $g(x) = 4f(x-2)+5$. This means there has been a dilation by factor 4 from the x-axis, a translation 2 units to the right, and a translation of 5 units up.

Take note that I've written these transformations in DRT order, with the dilations coming first, then the translations after.

Example 1.12

Determine the sequence of transformations required to transform the graph of $f(x) = x^3 + 2x^2 - 5$ *to* $g(x) = -2(x-4)^3 - 4(x-4)^2 - 2$.

We have to be a little careful here, since we are dealing with quite a few negatives. Let's see what has happened to $f(x)$ to result in $g(x)$. Again, with a little bit of analysis, you will find that $g(x) = -2f(x-4) - 12$. If that was a little fast in one step, let's break it up:

$$f(x) \longrightarrow -2f(x) : -2x^3 - 4x^2 + 10$$

$$-2f(x) \longrightarrow -2f(x-4) : -2(x-4)^3 - 4(x-4)^2 + 10$$

$$-2f(x-4) \longrightarrow -2f(x-4) - 12 : -2(x-4)^3 - 4(x-4)^2 - 12 \implies g(x)$$

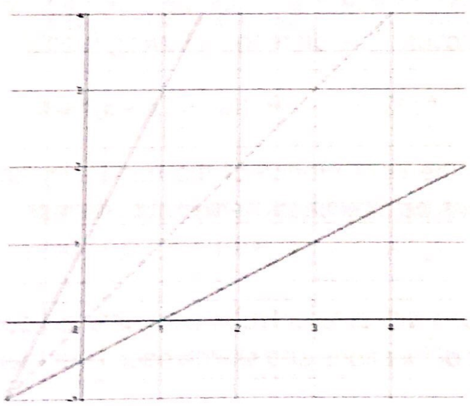
So in order to transform our graph of $f(x)$ into $g(x)$, $f(x)$ becomes $-2f(x-4) - 12$. This corresponds to a:

- Dilation by factor 2 from the x-axis
- Reflection across the x-axis
- Translation 4 units up
- Translation 12 units down

1.7 Other Functions

1.7.1 Inverse Functions

The inverse of a function is a function with all of its x and y coordinates swapped. For example, if we had the function with rule $y = 2x + 1$ and then made a second function with the x and y coordinates swapped, we'll get the graphs below.



If we took three points on the first graph - $(0,1)$, $(1,3)$ and $(2,5)$ - the corresponding points on the inverse function would be $(1,0)$, $(3,1)$ and $(5,2)$. The inverse function is also the reflection of the original function about the line $y = x$ (dotted).

Finding The Rule

In order to find the rule of an inverse function, you can simply swap x and y. So instead of $y = 2x + 1$ the rule of the inverse will be $x = 2y + 1$. Then, make y the subject:

$$\begin{aligned} x &= 2y + 1 \\ x - 1 &= 2y \\ y &= \frac{x-1}{2} \end{aligned}$$

Domain and Range

Now, let's see how the domain and range get affected when you take an inverse of a function. Say we have a function with rule $y = f(x)$. The rule of the inverse is denoted by $y = f^{-1}(x)$. Given that the x and y coordinates get swapped, it makes sense to say that the possible x-values (dom f) become the possible y-values in the inverse function (ran f^{-1}) and the possible y-values in the original function (ran f) become the possible x-values in the inverse (dom f^{-1}). Therefore, we can categorically say that:

$$\text{dom } f^{-1} = \text{ran } f$$

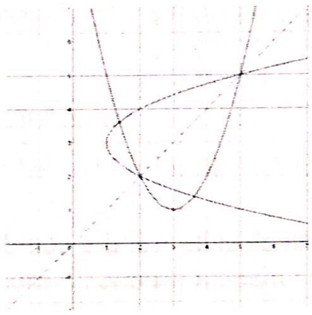
$$\text{ran } f^{-1} = \text{dom } f$$

Restrictions

Remember that for a function to exist, it must pass the vertical line test. This means that if we take a vertical line and move it across the graph, it will only intersect the line once. When we take the inverse of a function, we have to make sure it passes the vertical line test too, which means we can only take the inverse function of a one-to-one function. This is illustrated with the function:

$$f : R \rightarrow R, f(x) = (x-3)^2 + 1$$

This is a simple quadratic with a turning point at (3,1). It passes the vertical line test. However, if we flip the quadratic across the line y=x, the inverse will fail the vertical line test.



Therefore, we can restrict a function so that it will have an inverse function. If we restrict the function to $x \in [3, \infty)$, the inverse will be a function. Therefore the following function would have an inverse:

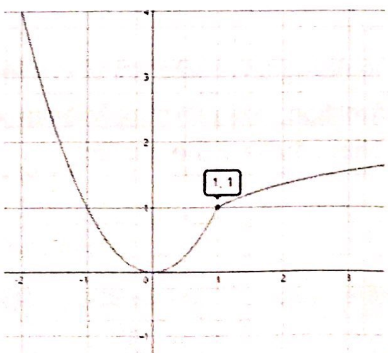
$$f : [3, \infty) \rightarrow R, f(x) = (x-3)^2 + 1$$

When sketched, this would leave the right branch of the quadratic, and the top half of the square root function, which will both be functions as they pass the vertical line test.

1.7.2 Hybrid Functions

Hybrid functions are simply a function with different rules over different domains. To sketch a hybrid function, simply work your way down the list of functions and sketch each over their respective domain. For example:

$$f(x) = \begin{cases} x^2 & x \leq 1 \\ \frac{1}{2}\log_e(x) + 1 & x > 1 \end{cases}$$



Be careful of:

- Endpoints: make sure you show open circles and closed circles in the correct places.
- Range: using hybrid functions may significantly change the range of a function.

Evaluating Hybrid Functions

You may be required to evaluate the value of $f(x)$ for different values of x. How is this done when $f(x)$ is made up of multiple curves? Simply substitute in x into the function which is sketched for that x value.

For example, if we wanted to evaluate $f(4)$, we simply substitute $x = 4$ into $\frac{1}{2}\log_e(x) + 1$ as that particular function is defined for $x > 1$. Similarly, if we wanted to evaluate $f(-2)$, we can substitute $x = -2$ into x^2 as x^2 is defined over the region $x \leq 1$.

1.7.3 Composite Functions

A composite function is a function inside of another function. For example, let's let $f(x) = \sin(3x)$ and $g(x) = \sqrt{x}$.

$f(g(x))$ is $g(x)$ inside of $f(x)$. We are inputting a $g(x)$ into all of the x's in $f(x)$ which results in:

$$f(g(x)) = \sin(3\sqrt{x})$$

$g(f(x))$ is $f(x)$ inside of $g(x)$. We are inputting a $f(x)$ into all of the x's in $g(x)$ which results in:

$$g(f(x)) = \sqrt{\sin(3x)}$$

Restrictions

There is an important restriction when dealing with composite functions. The function $f(g(x))$ only exists if the range of $g(x)$ is a subset of the domain of $f(x)$. The reason of this is because we are taking the output of $g(x)$ then putting this into $f(x)$. It is not possible to calculate $f(g(x))$ if $g(x)$ outputs a range which cannot be inputted into the domain of $f(x)$, this means that range of $g(x)$ must fit into the domain of $f(x)$ for the composite function to exist.

In mathematical notation, ran $g(x) \subseteq$ dom $f(x)$.

Domain

The domain of $f(g(x))$ is equivalent to the domain of $g(x)$. This is because whatever you put into $g(x)$ can be put into $f(g(x))$. Similarly, the domain of $g(f(x))$ is the domain of $f(x)$.

1.7.4 Sum/Difference Functions

If $y = f(x)$ and $y = g(x)$ are the rules of two functions, $y = f(x) + g(x)$ would be the rule of the sum function. We also express it as $y = (f+g)(x)$. The rule of the difference function is simply $y = f(x) - g(x) = (f-g)(x)$.

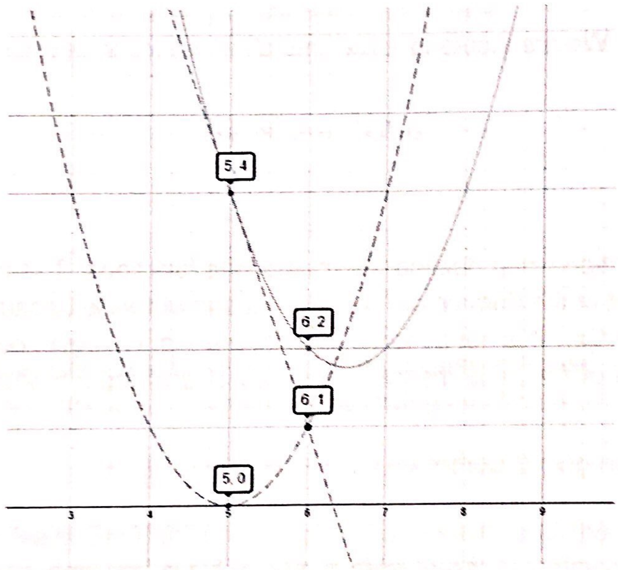
The domain of $(f+g)(x)$ and $(f-g)(x)$ would be dom $f \cap$ dom g. The reason for this is because for $f+g$ and $f-g$ to be defined at a given value of x, both $f(x)$ and$g(x)$ have to be defined at that value of x.

When sketching a sum/difference function, the original functions $f(x)$ and $g(x)$ can be first drawn lightly, then various points from each function can be added/subtracted and \ then connected to show the resulting function.

The following points on $f(x)$ and $g(x)$ are useful when sketching sum and difference functions:

- Intersection points: If $f(x)$ and $g(x)$ intersect at the point $(0,4)$, the equivalent point in their sum function would be $(0, 4+4) = (0,8)$ and the equivalent point in a difference function would be $(0, 4-4) = (0,0)$
- Intercepts (of either graph). If $f(x)$ has an x-intercept of $(5,0)$ and $g(x)$ has the point $(5,4)$, the sum of the functions at $x = 5$ is simply the value of $g(x)$ at $x = 5$ since $f(x)$ has no value.

For example, the graph of $y = f(x) + g(x)$ is shown below where $f(x) = (x-5)^2$ and $g(x) = 19-3x$. $f(x)$ and $g(x)$ are in dotted lines.



Notice the intersection point at $(6,1)$? When the points are added together, the point on the sum function is simply double to result in $(6,2)$. Notice how the quadratic has an x-intercept at $(5,0)$ and the linear line has a point at $(5,4)$? The sum function at $x = 5$ is just $(5,4)$.

1.7.5 Product Functions

If $y = f(x)$ and $y = g(x)$ are the rules of two functions, rule of the product function would be $y = f(x)g(x) = (fg)(x)$. The domain of $(fg)(x)$ would also be dom $f \cap$ dom g. The reason for this is because for fg to be defined at a given value of x, both $f(x)$ and $g(x)$ have to be defined at that value of x.

When sketching product functions, intersection points and intercepts of $f(x)$ and $g(x)$ are again useful when calculating the values of the resulting function $fg(x)$. The only big difference is that when one graph has an x-intercept ($y = 0$), the entire product function will be zero as multiplying anything by zero results in zero.

Area of Study 2

Algebra

2.1 Solutions to Trigonometric Equations

2.1.1 Solving Trigonometric Equations

Before solving trigonometric equations, let's look at some exact values of each of these ratios.

x	$\sin(x)$
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1

x	$\cos(x)$
0	1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	0

x	$\tan(x)$
0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{3}$	$\sqrt{3}$
$\frac{\pi}{2}$	undef.

Make sure you memorise these values or have a good way of deriving them.

Example 2.1

Solve for x: $\sin(\frac{x}{2}) = \frac{1}{2}, x \in [0, 4\pi]$

The restriction after the equation states that our solutions for x can only be between 0 and 4π. This means that our solutions for $\frac{x}{2}$ can only be between 0 and 2π or:

$$\begin{aligned} x &\in [0, 4\pi] \\ \frac{x}{2} &\in [0, 2\pi] \end{aligned}$$

Now you know that the stuff in the brackets has to be between 0 and 2π. Are there any solutions to this equation?

From exact values, you know that $\sin(\frac{\pi}{6}) = \frac{1}{2}$. You also can use symmetry properties to determine that $\sin(\frac{5\pi}{6}) = \frac{1}{2}$. There are no other solutions for $\sin(\frac{x}{2}) = \frac{1}{2}$ which fit inside of the domain $[0, 2\pi]$. Therefore you can say:

$$\begin{aligned} \sin(\frac{x}{2}) &= \frac{1}{2} \\ \frac{x}{2} &= \frac{\pi}{6}, \frac{5\pi}{6} \\ x &= \frac{\pi}{3}, \frac{5\pi}{3} \end{aligned}$$

Example 2.2

Solve for x: $\cos(x)^2 - \frac{3}{2}\cos(x) + \frac{1}{2} = 0, x \in [0, 2\pi]$

This equation is a little more difficult to solve, but notice how it resembles a quadratic equation? There is stuff^2 + more stuff $\times$ stuff + constant = 0.

So let's let $u = \cos(x)$:

$$\begin{aligned} u^2 - \frac{3}{2}u + \frac{1}{2} &= 0 \\ (u - \frac{1}{2})(u-1) &= 0 \end{aligned}$$

Therefore $u = \frac{1}{2}$ or $\underline{u = 1}$. Let's deal with each scenario separately.

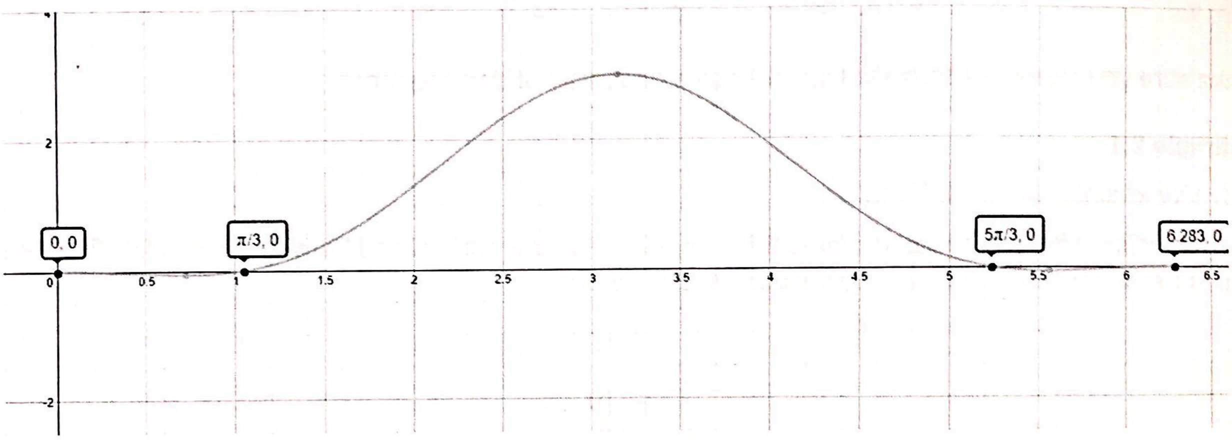
Let $u = \frac{1}{2} \implies \cos(x) = \frac{1}{2}$

From exact values, we know that $\cos(\frac{\pi}{3}) = \frac{1}{2}$. By symmetry, cosine is also positive in the fourth quadrant, therefore $\cos(2\pi - \frac{\pi}{3}) = \cos(\frac{5\pi}{3}) = \frac{1}{2}$. So our first two solutions are $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$. Let's consider the other scenario.

Let $u = 1 \implies \cos(x) = 1$.

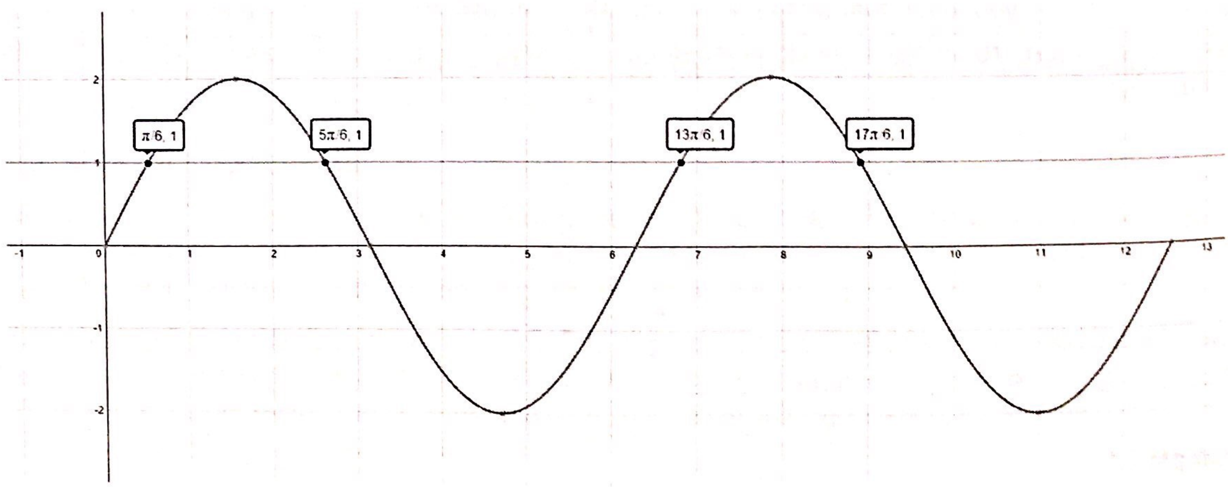
From exact values or a quick sketch, it is clear that $\cos(0) = 1$ and $\cos(2\pi) = 1$. Therefore our other two solutions are $x = 0$ or $x = 2\pi$.Combining these solutions from both scenarios gives $x = 0, \frac{\pi}{3}, \frac{5\pi}{3}, 2\pi$.

This is illustrated on the graph below.



2.1.2 General Solutions

Trigonometric equations are usually solved within a set domain. If we were to solve $2\sin(x) = 1$ over the interval $x \in [0, 4\pi]$, it would be clear that there were four solutions, as illustrated in the image below.

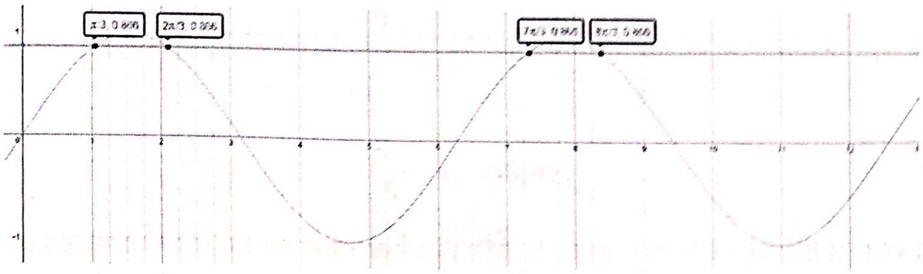


However, if no domain restriction is specified, there will be an infinite amount of solutions to this equation. We can use a general solution to summarise all of the possible solutions to this equation over an infinite domain.

Example 2.3

Write a general solution to the equation: $\sin(x) = \frac{\sqrt{3}}{2}$

Here is a quick sketch of the graph. Note that $\frac{\sqrt{3}}{2} \approx 0.866$.



We are essentially trying to find all of the times that the $y = \sin(x)$ graph touches $y = \frac{\sqrt{3}}{2}$. Firstly, remember that $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$, so $x = \frac{\pi}{3}$ is a solution. Now, if you did a full revolution and added 2π, you'd get $x = \frac{7\pi}{3}$ which is another solution in the next cycle. Add another 2π and you'll get $x = \frac{13\pi}{3}$. So if you took $\frac{\pi}{3}$ and added $2\pi n$ which is the period of the graph, where n is an integer (positive or negative!), you would get all the solutions. So, one general solution is $x = 2\pi n + \frac{\pi}{3}, n \in Z$. Make sure you specify that n is an integer by writing that $n \in Z$ (remember that Z is the set of integers).

Now, for the other general solution. With a bit of symmetry, we can find that $\cos(\frac{2\pi}{3}) = \frac{\sqrt{3}}{2}$. In other words, $\frac{2\pi}{3}$ is a solution. Now, by the same logic as earlier, the other general solution here is $x = 2\pi n + \frac{2\pi}{3}, n \in Z$. Hence, the overall general solution is:

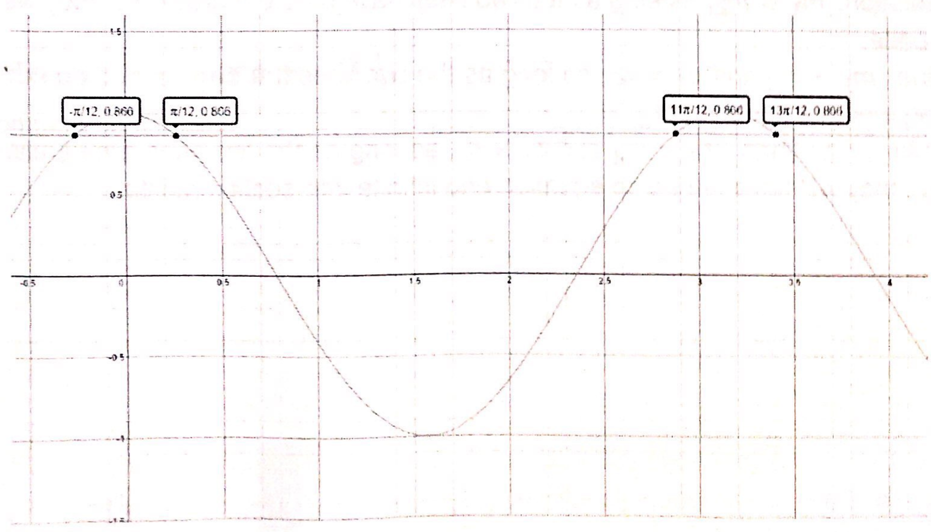
$$x = \begin{cases} 2\pi n + \frac{\pi}{3} & n \in Z \\ 2\pi n + \frac{2\pi}{3} & n \in Z \end{cases}$$

To recap the process, we found the first two solutions in the first period of the graph ($\frac{\pi}{3}$ and $\frac{2\pi}{3}$), and then added on the period multiplied by n, which allows us to summarise all of the solutions to the equations over an infinite domain.

Example 2.4

Write a general solution to the equation $\cos(2x) = \frac{\sqrt{3}}{2}$.

Here is a sketch of the graph. Note that $\frac{\sqrt{3}}{2} \approx 0.866$.



Similar to our previous example, let's find two solutions. $\cos(x) = \frac{\sqrt{3}}{2}$ when $x = \frac{\pi}{6}$ and $x = -\frac{\pi}{6}$. This means that $x = \pm\frac{\pi}{6}$.

If:

$$\begin{aligned} \cos(x) &= \frac{\sqrt{3}}{2} \\ x &= \pm\frac{\pi}{6} \end{aligned}$$

Then:

$$\begin{aligned} \cos(2x) &= \frac{\sqrt{3}}{2} \\ 2x &= \pm\frac{\pi}{6} \\ x &= \pm\frac{\pi}{12} \end{aligned}$$

So our first two solutions are $x = \pm\frac{\pi}{12}$. If we add multiples of the period to these solutions, we will reveal all of our solutions over an infinite domain.

Therefore our general solution is: $x = \pi n \pm \frac{\pi}{12}$, $n \in Z$

When writing general solutions for cosine equations, you can often use one equation with a plus or minus ($\pm$) sign instead of two separate equations. This is what makes it different from a sine general solution.

Example 2.5

Write the general solution to the equation $\tan(x) = 1$.

From our exact values, $\tan(\frac{\pi}{4}) = 1$, so our first solution is $x = \frac{\pi}{4}$. If you sketch this equation on your CAS, you will see that there is actually only one solution to the equation every period. This means we can just add our period multiplied by n onto the solution to write our general solution. Our solutions are basically $x = \frac{\pi}{4}$ with π (period) added on again and again.

The general solution is: $x = \pi n + \frac{\pi}{4}$, $n \in Z$.

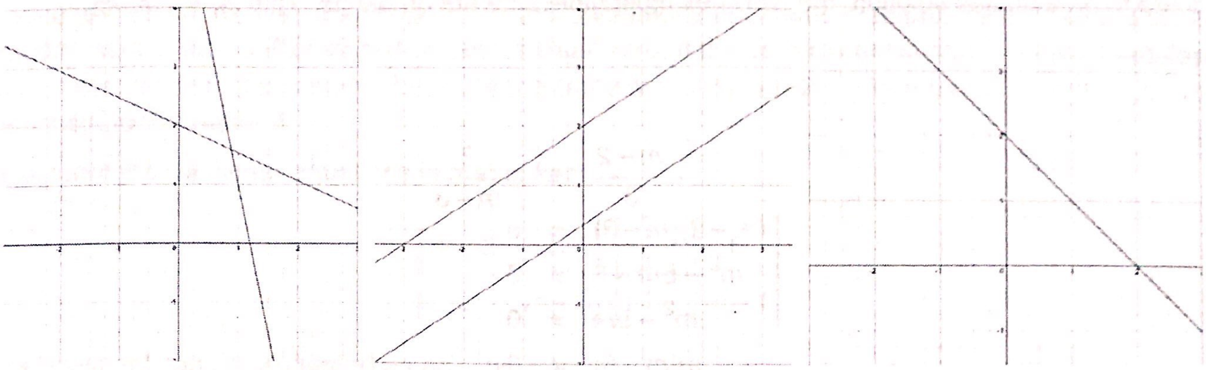
2.2 Systems of Linear Equations

2.2.1 Introduction

If we were to take two linear equations, there are three possible scenarios for the intersection between the two lines. There can be a unique solution (one solution), no solution, or infinitely many solutions. The requirements for each scenario are below where m is the gradient of the line and c is the y-intercept.

- Unique Solution: $m_1 \neq m_2$, as long as the two lines have different gradients, they will intersect once and only once.
- No Solution: $m_1 = m_2$ and $c_1 \neq c_2$, as long as the two lines are parallel and do not have the same y-intercept, they will never meet.
- Infinitely Many Solutions: $m_1 = m_2$ and $c_1 = c_2$, as long as the two lines are parallel and share the y-intercept, they will have the same equation and infinite intersection points.

These three scenarios are shown below.



If we are given the equation for two lines with one unknown, such as $(m-2)x+3y = 6$ and $2x+(m-3)y = m-1$, we can solve for the value of m which will give us one of the three scenarios.

2.2.2 Method 1: Coefficients

The first method involves the comparison of coefficients. We can compare the gradient and the y-intercept of the two lines and set m to be a value such that they create one of the three scenarios.

Let's use the two lines $(m-2)x + 3y = 6$ and $2x + (m-3)y = m-1$.

Start by rearranging the two lines into gradient intercept form.

[1]

$$\begin{aligned} (m-2)x + 3y &= 6 \\ 3y &= 6-(m-2)x \\ y &= 2-\frac{(m-2)}{3}x \end{aligned}$$

For our first line, $m_1 = -\frac{(m-2)}{3}$ (coefficient of x) and $c_1 = 2$.

[2]

$$\begin{aligned} 2x + (m-3)y &= m-1 \\ (m-3)y &= m-1-2x \\ y &= \frac{m-1-2x}{m-3} \\ y &= \frac{m-1}{m-3}-\frac{2}{m-3}x \end{aligned}$$

For our second line, $m_1 = -\frac{2}{m-3}$ and $c_2 = \frac{m-1}{m-3}$.

Now we can compare coefficients.

Unique Solution

For there to be a unique solution, the two lines must have different gradients. That is, $m_1 \neq m_2$.

Therefore:

$$\begin{aligned} -\frac{m-2}{3} &\neq -\frac{2}{m-3} \\ (m-2)(m-3) &\neq 6 \\ m^2-5m+6 &\neq 6 \\ m^2-5m &\neq 0 \\ m(m-5) &\neq 0 \end{aligned}$$

Hence, $m \neq 0$ or $m \neq 5$. As long as m is not equal to 0 or 5, there will be a unique solution as the two lines have different gradients.

No Solution

For there to be no solution, the two lines must have the same gradient but different y-intercepts. $m_1 = m_2$ and $c_1 \neq c_2$

We can now set up equation involving our gradients. The working out is exactly the same as the last, except there is an equals sign.

$$\begin{aligned} -\frac{m-2}{3} &= -\frac{2}{m-3} \\ m(m-5) &= 0 \end{aligned}$$

We now have $m = 0$ or $m = 5$ which will make the two lines have the same gradient. But we now need to check if they will have the same y-intercept under each method.

Let $m = 0$:

$$c_1 = 2$$

$$c_2 = \frac{0-1}{0-3} = \frac{1}{3}$$

Let $m = 5$:

$$c_1 = 2$$

$$c_2 = \frac{5-1}{5-3} = 2$$

Therefore, if $m = 0$, the lines will have two different y-intercepts, and if $m = 5$, the two lines will have the same y-intercept. For there to be no solutions $m = 0$.

Infinitely Many Solutions

For there to be infinitely many solutions $m_1 = m_2$ and $c_1 = c_2$. This means that the equation of the two lines are exactly the same. In our working out for no solution, we deduced that $m = 0$ or $m = 5$ will lead the lines to have the same solution. When substituting the different m values into c_1 and c_2, we found that $m = 5$ will give the two lines the same y-intercept.

Therefore, $m = 5$ will give the two lines infinitely many solutions. Sketch these two lines on your CAS calculator and test out different m values to see this in action!

2.2.3 Method 2: Matrices

The second method involves matrices. For two simultaneous equations in matrix form to have a solution, the determinant of the coefficient matrix cannot equal zero. If the determinant is equal to zero, then the two lines will not have a unique solution. Let's use the same two lines, $(m-2)x+3y = 6$ and $2x+(m-3)y = m-1$, to demonstrate this method.

First, present the two linear equations in matrix form:

$$\begin{bmatrix} m-2 & 3 \\ 2 & m-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ m-1 \end{bmatrix}$$

The determinant can be calculated as $ad - bc$ or:

$$(m-2)(m-3)-(3)(2) \implies m^2-5m+6-6 \implies m^2-5m$$

Unique Solution

If we want the two lines to have a unique solution, we don't want the determinant to equal zero. Therefore:

$$\begin{aligned} m^2-5m &\neq 0 \\ m(m-5) &\neq 0 \end{aligned}$$

Hence, $m \neq 0$ or $m \neq 5$ for there to be a unique solution. Notice that this is the same conclusion we reached under the coefficients method.

No Solution

If the determinant of the matrix equals zero, we can have no solutions, infinitely many solutions, but no unique solutions. Therefore, we want the determinant to equal zero:

$$\begin{aligned} m^2-5m &= 0 \\ m(m-5) &= 0 \end{aligned}$$

Letting $m = 0$ or $m = 5$ will provide no solutions or infinitely many solutions. We can now substitute each m value back into our linear equations to check which value leads to each scenario.

Let $m = 0$:

$$(0-2)x+3y = 6 \implies -2x+3y = 6$$

$$2x+(0-3)y = 0-1 \implies 2x-3y = -1 \implies -2x+3y = 1$$

Let $m = 5$:

$$5-2)x+3y = 6 \implies 3x+3y = 6$$
$$2x+(5-3)y = 5-1 \implies 2x+2y = 4$$

When $m = 0$, the two lines have different equations. Simply multiplying the second equation by -1 will lead to $-2x+3y = -1$ which is a different equation to the first equation. Therefore, if $m = 0$, the graphs will have no solutions at all.

Infinitely Many Solutions

Notice that if we let $m = 5$, divide our first equation by 3 and our second equation by 2, we will end up with the same line $x+y = 2$. This means that $m = 5$ will provide two lines with an infinite amount of solutions as they are the exact same line.

2.3 Solving Exponents and Logarithms

2.3.1 Exponent Laws

The following exponent laws are useful when solving exponential equations:

$$
\begin{aligned}
x^m \times x^n &= x^{m+n} \\
\frac{x^m}{x^n} &= x^{m-n} \\
(x^m)^n &= x^{m \times n} \\
(\frac{x}{y})^n &= \frac{x^n}{y^n} \\
x^{-n} &= \frac{1}{x^n} \\
(\frac{x}{y})^{-n} &= (\frac{y}{x})^n \\
x^0 &= 1
\end{aligned}
$$

2.3.2 Logarithm Laws

The following logarithm laws are useful when solving logarithmic equations:

$$
\begin{aligned}
\log_e(x \times y) &= \log_e(x) + \log_e(y) \\
\log_e(\frac{x}{y}) &= \log_e(x) - \log_e(y) \\
\log_e(x^n) &= n \times \log_e(x) \\
\log_e(e) &= 1 \\
\log_e(1) &= 0
\end{aligned}
$$

2.3.3 Solving Exponential Equations

Exponential equations involve an a^x term - in Methods, you'll most likely see e^x in equations. Solving exponential equations is best shown using examples.

Example 2.6

Solve the following equation for x: $3^{x+2} = 27^x$.

The first thing we need to do is get rid of the x as an exponent so that the algebra becomes easier. To do this, we need to get the LHS and RHS on the same base number. Right now, there is a 3 as a base on the LHS and a 27 as a base on the RHS. Recognise that $27 = 3^3$ and therefore we can manipulate the equation:

$$
\begin{aligned}
3^{x+2} &= 27^x \\
&= (3^3)^x \\
&= 3^{3x}
\end{aligned}
$$

Now that we have the same base on each side, we can equate the exponents.

$$
\begin{aligned}
x + 2 &= 3x \\
2x &= 2 \\
x &= 1
\end{aligned}
$$

Area of Study 2 – Algebra

Example 2.7

Solve the following equation for x: $9^x - 2 \times 3^x + 1 = 0$.

This looks like quite a mouthful and VCAA love giving this kind of question. Exponentials equations are much easier to solve when everything has the same base. Notice that $9 = 3^2$, so we can convert each term to have a base of 3:

$$\begin{aligned} 3^{2x} - 2 \times 3^x + 1 &= 0 \\ (3^x)^2 - 2 \times (3^x) + 1 &= 0 \end{aligned}$$

This now looks like a quadratic equation. It's in the form of stuff^2 + more stuff $\times$ stuff + constant = 0. Just to illustrate this, let $u = 3^x$ and we can solve for u:

$$\begin{aligned} u^2 - 2u + 1 &= 0 \\ (u-1)^2 &= 0 \\ u &= 1 \end{aligned}$$

Then, you can solve for x:

$$\begin{aligned} u &= 3^x = 1 \\ 3^x &= 1 \\ x &= 0 \end{aligned}$$

That makes for some neat factorisation there. Now, if you can't be bothered letting $u = 3^x$ you can directly factorise:

$$\begin{aligned} (3^x - 1)^2 &= 0 \\ 3^x &= 1 \\ x &= 0 \end{aligned}$$

2.3.4 Solving Logarithmic Equations

Example 2.8

Solve, for x, the equation $log_e(2x + 5) = 0$.

Isolating the $2x + 5$ and using the definition of log, we can get:

$$\begin{aligned} 2x + 5 &= e^0 = 1 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

Example 2.9

Solve, for x, the following equation: $log_e(x-1) + log_e(x+2) = log_e(6x-8)$.

Just like in the previous example, you want to get rid of the 'logs' in this question which make it very difficult to solve. If you see sums of logs, you can 'lump' them together using the log laws, as such:

$$\begin{aligned} \log_e((x-1)(x+2)) &= \log_e(6x-8) \\ (x-1)(x+2) &= 6x-8 \\ x^2 + x - 2 &= 6x - 8 \\ x^2 - 5x + 6 &= 0 \\ (x-2)(x-3) &= 0 \\ x &= 2, 3 \end{aligned}$$

We just have to do a final check because, since the stuff inside the log *must* be positive, we have to check if these solutions are valid. Since the equation is $\log_e(x-1) + \log_e(x+2) = \log_e(6x-8)$, $x - 1 > 0$, $x + 2 > 0$ and $6x - 8 > 0$. Both of the solutions satisfy these conditions, so are both valid.

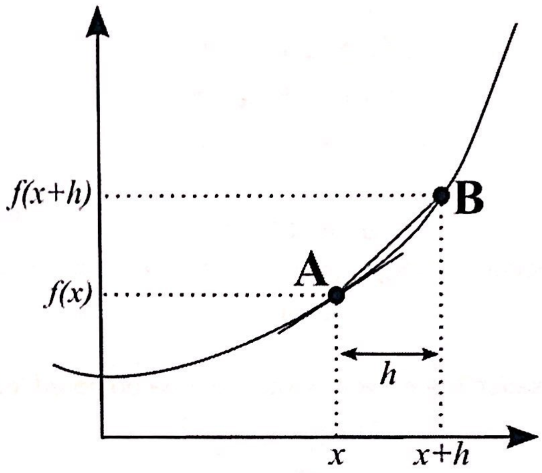
Area of Study 3

Calculus

3.1 Differentiation

3.1.1 Differentiation and Limits

Differentiation is a key concept in Mathematical Methods. It allows you to find the gradient of a curve at any point, to determine rates of change and solve for stationary points. You should have already had some experience with Units 1 and 2, and here we seek to build on those skills.



First Principles

In Year 11, you would have covered differentiation by first principles. When applying this approach, we are essential finding the $\frac{rise}{run}$ over an extremely small interval h which gives us the value of the gradient at a specific point x.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Applying this to $f(x) = x^2 + 4x$, the working is as follows.

$$\begin{aligned}
f(x) &= x^2 + 4x \\
f(x+h) &= (x+h)^2 + 4(x+h) \\
&= x^2 + 2xh + h^2 + 4x + 4h \\
\Rightarrow f'(x) &= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2 + 4x + 4h) - (x^2 + 4x)}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2 + 4h}{h} \\
&= \lim_{h \to 0} \frac{(2x + h + 4)\cancel{h}}{\cancel{h}} \\
\therefore f'(x) &= 2x + 4
\end{aligned}$$

However, you will most likely be using other methods to find the derivative (covered later) which will allow you to save time in your SACs and exams.

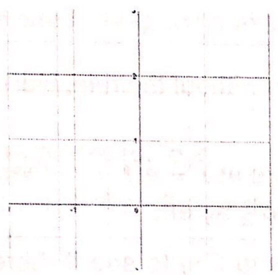
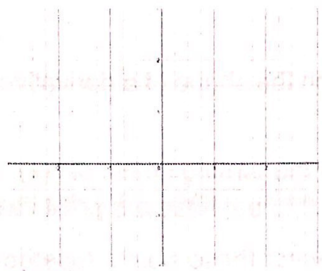
Special Points

Even if a graph exists at a specific point, the derivative may not. More specifically, the derivative does not exist at:

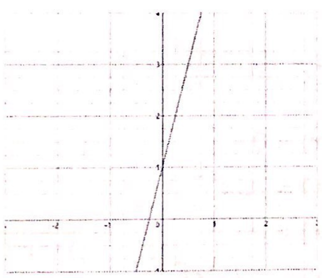
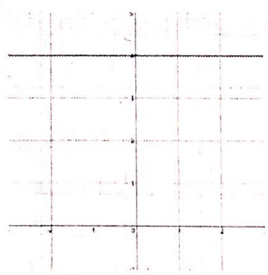
- Sharp Points: You cannot draw a tangent at a sharp point as it will not match each side of the point.
- Points of Discontinuity: Any point where the graph "jumps" will lead to the derivative not existing at that point.
- End Points

3.1.2 Graphs of Derivatives

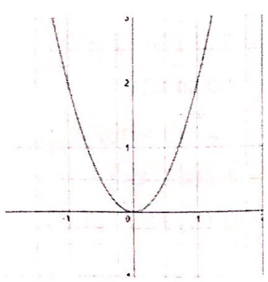
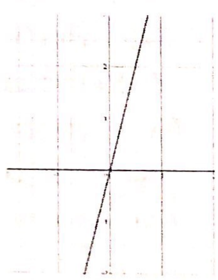
It is important that you understand the relationship between the original function and its derivative function. Below are graphs of an original function and their derivative, followed by some general comments.

$f(x) = 2$	$f'(x) = 0$
	

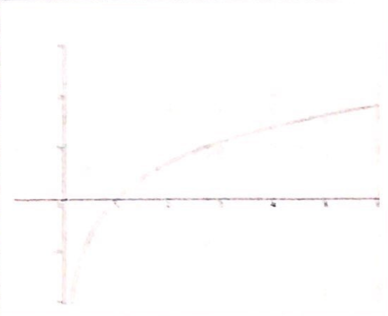
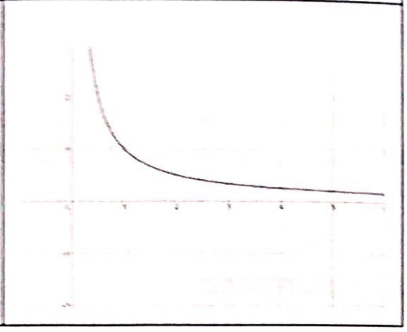
$f(x)$ is a horizontal line, which means that it has no gradient. Therefore $f'(x) = 0$.

$f(x) = 4x + 1$	$f'(x) = 4$
	

$f(x)$ is a line with a gradient of 4, therefore $f'(x) = 4$ over the entire domain.

$f(x) = 2x^2$	$f'(x) = 4x$
	

$f(x)$ is a parabola with a turning point at $x = 0$. For $x \in (-\infty, 0)$, the parabola has a negative gradient, and for $x \in (0, \infty)$, the parabola has a positive gradient. This is evident in the value of the deriative function $f'(x) = 4x$. The turning point at $x = 0$ has a zero gradient, so the derivative function $f'(x) = 0$ at $x = 0$.

$f(x) = \log_e(x)$	$f'(x) = \frac{1}{x}$
	

$f(x)$ is a logarithm. The gradient is aways positive but is decreasing as x approaches infinity. This is shown in the derivative function $f'(x)$ as it is always positive, and approaches zero. The gradient is always positive, but it will never reach zero.

Sketching Derivatives

If you are asked to sketch the shape of a derivative function without the rule, start with turning points if there are any.

Find the x-coordinate of the turning points of $f(x)$. If they are at $x = a, b, c...$, then sketch on your derivative function $(a, 0), (b, 0), (c, 0)...$ as these are points with a zero gradient.

The next step is to connect these points together. Observe $f(x)$ to see if there is a positive gradient or negative gradient around the turning points and sketch it in appropriately.

If there are no stationary points, the derivative function will always be positive or always be negative.

3.1.3 Standard Derivatives

Below is a list of standard derivatives - it's a good idea to use these in your bound reference book. Your textbook should also have a similar version of these.

$$
\begin{array}{rcl}
f(x) = c & \rightarrow & f'(x) = 0 \\
f(x) = ax^n & \rightarrow & f'(x) = nax^{n-1} \\
f(x) = e^{ax} & \rightarrow & f'(x) = ae^{ax} \\
f(x) = e^{g(x)} & \rightarrow & f'(x) = g'(x)e^{g(x)} \\
f(x) = \log_e(ax) & \rightarrow & f'(x) = \dfrac{1}{x} \\
f(x) = \log_e(g(x)) & \rightarrow & f'(x) = \dfrac{g'(x)}{g(x)} \\
f(x) = \sin(ax) & \rightarrow & f'(x) = a\cos(ax) \\
f(x) = \sin(g(x)) & \rightarrow & f'(x) = g'(x)\sin(g(x)) \\
f(x) = \cos(ax) & \rightarrow & f'(x) = -a\sin(ax) \\
f(x) = \cos(g(x)) & \rightarrow & f'(x) = -g'(x)\sin(g(x)) \\
f(x) = \tan(ax) & \rightarrow & f'(x) = a\sec^2(ax) \\
f(x) = \tan(g(x)) & \rightarrow & f'(x) = g'(x)\sec^2(g(x))
\end{array}
$$

3.2 Tools for Differentiation

In this section we will be covering the chain rule, product rule and quotient rule. All three rules are extremely useful and can be used in combination with each other.

The best way to become familiar with them is to practise!

3.2.1 Chain Rule

The chain rule is a useful rule which allows us to find the derivative of functions which are inside of other functions.

The chain rule is most commonly seen in the form: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

u is defined as a function of x which is inside of an outer function y.

However, the chain rule can be extended to include any amount of functions. The chain rule with three terms can be expressed as: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$

The chain rule is best explained using some examples.

Example 3.1

Differentiate $y = \sin(8x^2)$ *with respect to* x.

Firstly, notice that there is a function $(8x^2)$ sitting inside of another function (sine function). Let's use the chain rule to differentiate it:

$$\text{Let } u = 8x^2 \implies \frac{du}{dx} = 16x$$

$$\text{Let } y = \sin(u) \implies \frac{dy}{du} = \cos(u) = \cos(8x^2)$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 16x\cos(8x^2)$$

Example 3.2

Differentiate $y = \log_e(\sqrt{\cos(x)})$.

This example is a little tougher than the previous one. We can identify a cosine function sitting inside a square root function, sitting inside of a logarithm. Let's use the chain rule to differentiate it with more terms:

$$\text{Let } v = \cos(x) \implies \frac{dv}{dx} = -\sin(x)$$

$$\text{Let } u = \sqrt{v} \implies \frac{du}{dv} = \frac{1}{2\sqrt{v}} = \frac{1}{2\sqrt{\cos(x)}}$$

$$\text{Let } y = \log_e(u) \implies \frac{dy}{du} = \frac{1}{u} = \frac{1}{\sqrt{v}} = \frac{1}{\sqrt{\cos(x)}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx} = \frac{1}{\sqrt{\cos(x)}} \times \frac{1}{2\sqrt{\cos(x)}} \times -\sin(x) = \frac{-\sin(x)}{2\cos(x)} = -\frac{1}{2}\tan(x)$$

A quicker way of thinking about the chain rule is that the derivative of a composite function is "the derivative of the outside function multiplied by the derivative of the inside function".

Area of Study 3 – Calculus

3.2.2 Product Rule

The product rule allows us to find the derivative of functions which are multiplied together.

The product rule states if $f(x)$ is the product of two functions $u(x)$ and $v(x)$ then:

$$f(x) = u(x) \times v(x) \quad \longrightarrow \quad f'(x) = u(x) \cdot v'(x) + u'(x) \cdot v(x)$$

Alternatively:

$$y = u \times v \quad \longrightarrow \frac{dy}{dx} = u \times \frac{dv}{dx} + \frac{du}{dx} \times v$$

Example 3.3

Differentiate $f(x) = x^2 \sin(8x)$.

$$\text{Let } u(x) = x^2 \implies u'(x) = 2x$$

$$\text{Let } v(x) = \sin(8x) \implies v'(x) = 8\cos(8x)$$

$$f(x) = u(x) \times v'(x) + u'(x) \times v(x)$$

$$f(x) = 8x^2 \cos(8x) + 2x \sin(8x)$$

3.2.3 Quotient Rule

The quotient rule allows us to find the derivative of functions which are divided by another function.

The quotient rule states that if $f(x)$ is the quotient of two functions $u(x)$ and $v(x)$, then:

$$f(x) = \frac{u(x)}{v(x)} \quad \longrightarrow \quad f'(x) = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{(v(x))^2}$$

Alternatively:

$$y = \frac{u}{v} \quad \longrightarrow \frac{dy}{dx} = \frac{\frac{du}{dx} \times v - u \times \frac{dv}{dx}}{v^2}$$

Example 3.4

Differentiate $f(x) = \frac{x^2}{\sin(8x)}$

$$\begin{aligned} \textit{Let } u(x) &= x^2 & \textit{Let } v(x) &= \sin(8x) \\ u'(x) &= 2x & v'(x) &= 8\cos(8x) \end{aligned}$$

$$\begin{aligned} f'(x) &= \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{(v(x))^2} \\ \Rightarrow f'(x) &= \frac{2x \times \sin(8x) - x^2 \times 8\cos(8x)}{(\sin(8x))^2} \\ &= \frac{2x\sin(8x) - 8x^2\cos(8x)}{\sin^2(8x)} \end{aligned}$$

3.3 Applications of Differentiation

3.3.1 Stationary Points

Finding stationary points can be very easy! At *all* stationary points, the gradient of a function is zero. That means it has no gradient and the tangent to the curve is horizontal.

Since the derivative of a function is equal to the gradient of the curve at a point, the derivative of a function must be zero at a stationary point.

- **Step 1:** find the derivative of the function and let it equal zero. Solve for the x value(s) for which this occurs. The x values are the values at which the stationary points occur.
- **Step 2:** a lot of questions will ask for the y value of the stationary point. We know the x value(s) of the stationary point, so finding the corresponding y value is as easy as substituting it back into the original equation.
- Make sure to substitute your x value(s) into the original function, and not the derivative function!
- **Step 3:** sometimes you will need to classify the type of stationary point. A stationary point can be a local minimum, local maximum, or a stationary point of inflection. In order to classify the stationary point, a gradient table can be used. A gradient table for different types of stationary points at $x = a$ is shown below.

Local Maximum

x	$x < a$	$x = a$	$x > a$
$\frac{dy}{dx}$	+	0	−
slope	/	-	\

Local Minimum

x	$x < a$	$x = a$	$x > a$
$\frac{dy}{dx}$	−	0	+
slope	\	-	/

Stationary Point of Inflection (+)

x	$x < a$	$x = a$	$x > a$
$\frac{dy}{dx}$	+	0	+
slope	/	-	/

Stationary Point of Inflection (-)

x	$x < a$	$x = a$	$x > a$
$\frac{dy}{dx}$	−	0	−
slope	\	-	\

Example 3.5

Find the coordinates of the stationary points of the graph $y = 3x^3 + 6x^2 + 4$ *and determine their nature.*

First of all, let's differentiate the graph:

$$\begin{aligned} y &= 3x^3 + 6x^2 + 4 \\ \frac{dy}{dx} &= 9x^2 + 12x \end{aligned}$$

Now let's let $\frac{dy}{dx} = 0$ and solve for the x values when this occurs.

$$\begin{aligned} 0 &= 9x^2 + 12x \\ 0 &= x(9x + 12) \end{aligned}$$

Applying the null factor law, $x = 0$ or $x = -\frac{12}{9} = -\frac{4}{3}$.

These are the x coordinates of the stationary points. Let's substitute them back into y to find their y coordinate.

Let $x = 0$, $y = 4$. Let $x = -\frac{4}{3}$, $y = \frac{68}{9}$. The coordinates of the stationary points are $(0, 4)$ and $(-\frac{4}{3}, \frac{68}{9})$.

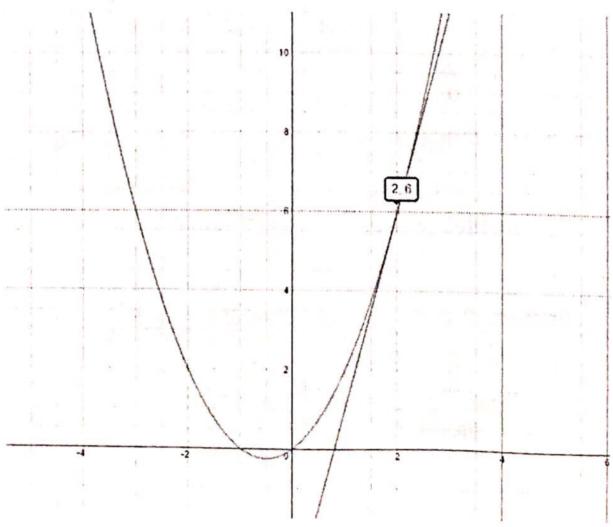
Time to find out their nature. Let's draw up a gradient table, and substitue values into the derivative on either side of each stationary point.

x	-2	$-\frac{4}{3}$	-1	0	1
$\frac{dy}{dx}$	12 (+)	0	−3 (-)	0	21(+)
slope	/	-	\	-	/

Based on the gradient table, it is clear that we have a local maximum at $x = -\frac{4}{3}$ and a local minimum at $x = 0$.

3.3.2 Tangent Lines

A tangent is a line that *just* touches a curve at a point. The tangent line touching the curve $y = x^2 + x$ at $x = 2$ is shown below.



To find the equation of a tangent, we need two pieces of information: the gradient and a point. The point we use can be the same point as where the tangent touches the curve because they both pass through this point. To find the gradient of the tangent, we use the gradient of the curve (the value of $\frac{dy}{dx}$ at that point) because they have the same gradient. Once we have these two pieces of information we can substitute them into the equation $y - y_1 = m(x - x_1)$ to find the equation of a tangent.

Example 3.6

Find the equation of the tangent to the graph $y = x^2 + x$ at $x = 2$.

First, find the coordinates of the point at which the tangent touchest the graph.

$$\begin{aligned} y &= x^2 + x \\ \text{Let } x = 2\ y &= 2^2 + 2 = 6 \end{aligned}$$

So our coordinate is $(2, 6)$. Now let's find the gradient at that point.

$$\begin{aligned} \frac{dy}{dx} &= 2x + 1 \\ \text{When } x = 2\ \frac{dy}{dx} &= 2(2) + 1 = 5 \end{aligned}$$

Now we can substitute it into $y - y_1 = m(x - x_1)$.

$$\begin{aligned} y - 6 &= 5(x - 2) \\ y &= 5x - 4 \end{aligned}$$

3.3.3 Optimisation Problems

Okay, so now we know how to find derivatives, stationary points and the nature of stationary points. What are some real life applications then? None. But VCAA really like to ask application style questions a lot on the exams.

Example 3.7

A farmer has $120m$ of fencing and needs to make a rectangular enclosure for his sheep. In order to save fencing, he decides to use a river on one side as the length of the enclosure and fencing on the other three sides. What dimensions should he make his enclosure in order to make sure he gets the largest possible area for his enclosure?

Step 1: Draw a diagram! This helps a lot as it lets you visualise what you are about to do.

Step 2: Define your variables, and write an equation that connects them.

Let l be the length and w be the width.

Since the total fencing is $120m$ and one of the length sides is a river, $l + 2w = 120$

Step 3: Write the equation that describes the quantity you want to maximise/minimise.

Maximise the area. Area is given as $A = l \times w$

Step 4: Before we can differentiate A, we need to get it in terms of one variable. Rearrange the equation from Step 2 to find l in terms of w, then substitute this into the equation in Step 3.

$l = 120 - 2w$. Substituting, $A = (120 - 2w) \times w$

Step 5: Now optimise! This is done by finding the stationary point.

$$\frac{dA}{dw} = 120 - 4w \quad \frac{dA}{dw} = 0 \text{ for a stationary point, } 120 - 4w = 0$$

So when solved, $w = 30m$

Step 6: If required, prove this is a maximum/minimum.

Draw up a gradient table!

w	25	30	35
$\frac{dy}{dx}$	20 (+)	0	–20(-)
slope	/	-	\

So, when $w = 30$ there is a local maximum.

Step 7: Lastly, solve for what the question is asking for!

Don't go and find the area when $w = 30$, because it doesn't ask for it.

Solve for l because we want the dimensions! $l = 120 - 2w \longrightarrow l = 60m$

Step 8: State your answer and get those marks!

So, the dimensions for maximum area are $30m \times 60m$ (river opposite the $60m$ side).

Area of Study 3 – Calculus

3.4 Integration

3.4.1 Standard Integrals

Below is a list of standard integrals - just like the derivative table, make sure to put these in your bound reference if you haven't committed them to memory.

$$f(x) = a \longrightarrow \int f(x)dx = ax + c$$

$$f(x) = ax^n \longrightarrow \int f(x)dx = \frac{ax^{n+1}}{n+1} + c$$

$$f(x) = (ax+b)^n \longrightarrow \int f(x)dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

$$f(x) = \frac{1}{x} \longrightarrow \int f(x)dx = \log_e |x| + c$$

$$f(x) = e^{ax} \longrightarrow \int f(x)dx = \frac{1}{a}e^{ax} + c$$

$$f(x) = \sin(ax) \longrightarrow \int f(x)dx = -\frac{1}{a}\cos(ax) + c$$

$$f(x) = \cos(ax) \longrightarrow \int f(x)dx = \frac{1}{a}\sin(ax) + c$$

3.4.2 Integral Properties

There are several properties of integrals which can help speed up your working out.

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b k\,f(x)\,d(x) = k\int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

3.4.3 Integration by Recognition

There are some integrals which we cannot integrate straight away. Sometimes, if we differentiate a similar function first, we can use that derivative to integrate a more complex function. The integral properties are extremely useful for this.

Example 3.8

Differentiate $x\cos(x)$ and use the result to find the integral of $x\sin(x)$.

Let's find the derivative first. $x\cos(x)$ is the product of x and $\cos(x)$ so we can use the product rule to differentiate it. The result is $\frac{d}{dx}(x\cos(x)) = \cos(x) - x\sin(x)$. Or, we can write this as $\int(\cos(x) - x\sin(x))dx = x\cos(x)$.

Do you notice how there is an $x\sin(x)$ term in the integral? This is the integral we want to find and we can isolate it with a bit of algebra. Start by splitting up the integral using the properties covered in the previous section.

$$\begin{aligned}
\int(\cos(x) - x\sin(x))dx &= x\cos(x) \\
\int\cos(x)dx - \int x\sin(x)dx &= x\cos(x) \\
-\int x\sin(x)dx &= x\cos(x) - \int\cos(x)dx \\
\int x\sin(x)dx &= \int\cos(x)dx - x\cos(x) \\
\int x\sin(x)dx &= \sin(x) - x\cos(x) \\
\therefore \int x\sin(x)dx &= \sin(x) - x\cos(x) + C
\end{aligned}$$

3.5 Applications of Integration

3.5.1 Definite and Indefinite Integrals

Before looking at applications of integrals, let's first make clear the difference between a definite and indefinite integral.

Indefinite Integral

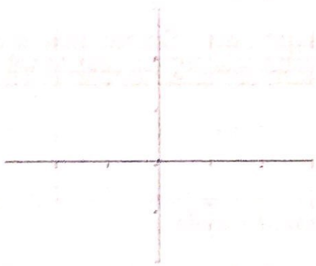
Indefinite integrals are integrals without any bounds. An indefinite integral allows us to find a family of functions which have a derivative $f(x)$. In mathematical notation: $\int f(x) = F(x) + C$. $f(x)$ is a derivative function, and when we do not add anything to the terminals of the integral sign, we simply find a family of functions which satisfy the above statement.

Definite Integrals

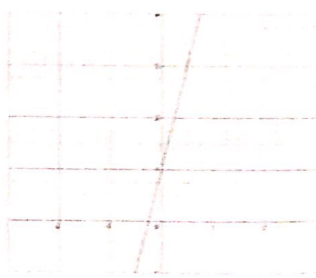
Definite integrals are integrals with boundaries. It is an integral evaluated over a specific domain. A definite integral will have terminals such as $\int_a^b f(x)dx$.

3.5.2 Graphs of Antiderivatives

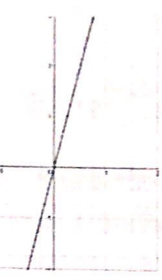
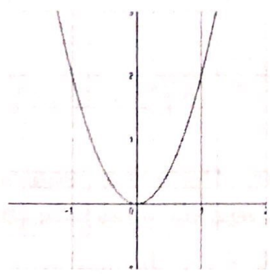
To show the relationship between a function and its antiderivative, we are going to use the same images from the "Graphs of Derivatives" section, but look at them in the opposite direction. Be aware that the integrals could be translated vertically to any position, as dictated by the constant of integration C.

$f(x) = 0$	$\int f(x)dx = C$
	

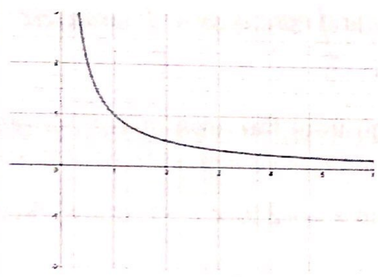
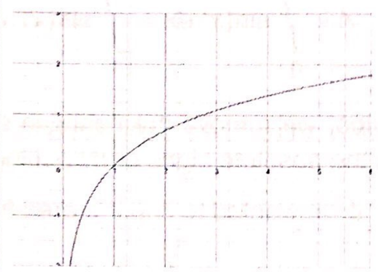
$f(x) = 0$ for all $x \in R$. This means that the antiderivative will always have a zero gradient and can be characterised by a horizontal line.

$f(x) = 4$	$\int f(x)dx = 4x + C$
	

$f(x)$ is a horizontal line at $y = 4$. This means that the antiderivative will always have a gradient of 4, which is represented by the line $4x + C$.

$f(x) = 4x$	$\int f(x)dx = 2x^2 + C$
	

$f(x)$ is a straight line that is negative for $x \in (-\infty, 0)$, $f(x) = 0$ for $x = 0$ and is positive for $(0, \infty)$. Therefore, the antiderivative will be decreasing (negative slope) for negative x values, will have a turning point at $x = 0$ and will be increasing (positive slope) for positive x values.

$f(x) = \frac{1}{x}$	$\int f(x)dx = \log_e(x) + C$
	

$f(x)$ is always positive, which means the antiderivative will always be increasing. However, $f(x)$ is positive but gets closer to zero as x approaches infinity. This means that the antiderivative function is going to start with a very steep positive gradient, and this gradient will become less steep as x approaches infinity.

3.5.3 Fundamental Theorem of Calculus

Before looking at areas or average values, we need to know about the fundamental theorem of calculus.

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

This theorem states that:

$\int_a^b f(x)dx$ is called the definite integral.

$f(x)$ is called the integrand.

$F(x)$ is the antiderivative of f(x) without +c.

a and b are called the terminals.

For this to be true, f must be a continuous function over the interval $[a, b]$.

3.5.4 Finding Areas Under Curves

The fundamental theorem of calculus, covered in the previous section, can be used to find the area under a graph. The area under a graph is simply the integral of the function with specified terminals. The area under $f(x)$ from $x = a$ to $x = b$ can be calculated as:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

Signed Areas

Let's now try find the area under the graph of $y = x$ from $x = -2$ to $x = -1$.

Great! Let's jump into it.

$$\int_{-2}^{-1} x\, dx = [\tfrac{x^2}{2}]_{-2}^{-1} = \tfrac{1}{2} - 2 = -\tfrac{3}{2} \text{ sq units}$$

But there is a problem here. There is a negative answer for our area! Area is a unit of measurement and we should only get a positive area. This is because the area from $x = -2$ to $x = -1$ of $y = x$ is underneath the x-axis.

In fact, $\int_a^b f(x)\, dx$ gives us what is termed 'the net signed area' of the regions contained between the curve and the x-axis over that interval. This means that if we were to evaluate integral of $\sin(x)\, dx$ from 0 to 2π we would get zero. The area above the x-axis from 0 to π would just cancel out with the area below the x-axis from π to 2π. If we want to find the area contained between y=sin(x) and the x -axis from 0 to 2π we need to use the expression:

$$A = \int_0^{\pi} \sin(x)\, dx + \left| \int_{\pi}^{2\pi} \sin(x)\, dx \right| = \int_0^{\pi} \sin(x)\, dx - \int_{\pi}^{2\pi} \sin(x)\, dx$$

By using this expression, we remove the negative sign from the area under the graph from π to 2π and make it positive using the absolute signs or by putting a negative sign in front of the integral.

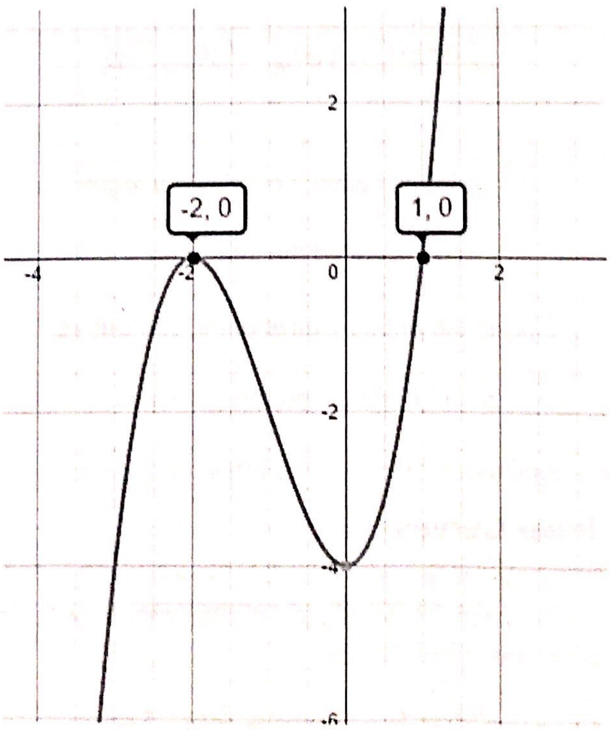
Applying this to $y = x$, if we wanted to find the area from $x = -2$ to $x = -1$, our working out should look like:

$$A = -\int_{-2}^{-1} x\, dx = -[\tfrac{x^2}{2}]_{-2}^{-1} = -(\tfrac{1}{2} - 2) = \tfrac{3}{2} \text{ square units}$$

Example 3.9

Find the area under the graph of $y = x^3 + 3x^2 - 4$ *from* $x = -2$ *to* $x = 2$.

The first thing we should do is a very quick sketch of the graph.



If we want to find the area from $x = -2$ to $x = 2$, we need to split up the integral into two sections. This is because the positive area from $x = 1$ to $x = 2$ will be affected by the negative area from $x = -2$ to $x = -1$. Let's express the area as the sum of two smaller areas and make sure to multiply the negative area by -1.

$$A = -\int_{-2}^{1}(x^3 + 3x^2 - 4)dx + \int_{1}^{2}(x^3 + 3x^2 - 4)dx = -(\tfrac{-27}{4}) + \tfrac{27}{4} = \tfrac{27}{2}$$

3.5.5 Areas Between Two Curves

Now that we are able to find the area under one curve, we can apply this knowledge to find the area between two curves. If there is one function above another one, such as $f(x)$ and $g(x)$, we can find the area underneath $f(x)$, subtract the area under $g(x)$ and the result will be the area between the two curves.

In other words, the area of the region bounded by two curves from $x = a$ to $x = b$ can be calculated as:

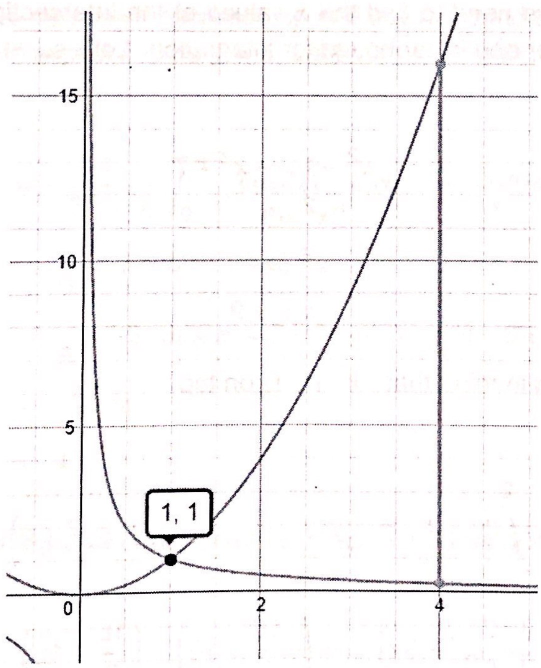
$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b \left[f(x) - g(x)\right]\,dx$$

Make sure that you do the top curve minus the bottom curve.

Example 3.10

Find the area between the curves of $y = x^2$ and $y = \frac{1}{x}$, and the lines $x = 1$ and $x = 4$.

A graph of the two curves is shown below:



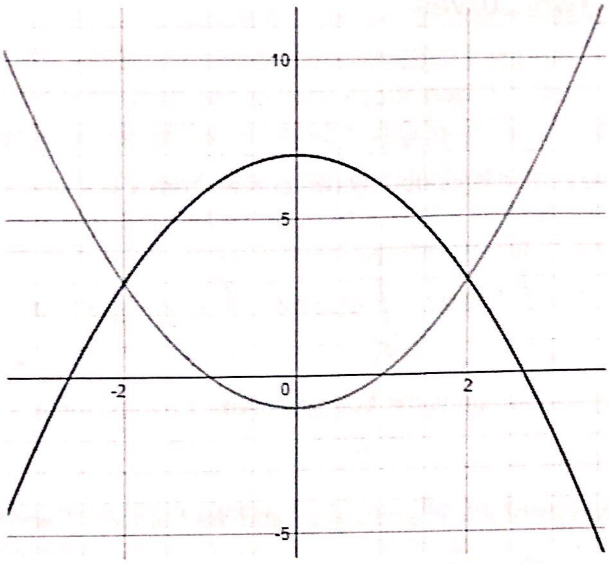
The area we want to find is the area between the quadratic and the $y = \frac{1}{x}$, up to the vertical line at $x = 4$. To do this, we can directly apply the formula, keeping in mind that the curve on top is $y = x^2$

$$A = \int_1^4 (x^2 - \tfrac{1}{x})dx = [\tfrac{x^3}{3} - \log_e(x)]_1^4 = (\tfrac{64}{3} - \log_e(4)) - (\tfrac{1}{3} - 0) = 21 - \log_e(4)$$

Example 3.11

Find the area enclosed by the curves with equations $y = x^2 - 1$ and $y = -x^2 + 7$.

Draw a quick sketch of these two curves by hand or on your CAS calculator so you can visualise the area you are trying to find. A sketch is shown below.



Before applying the formula, we need to find the x values of the intersection points, as these will be the values that we use as our upper and lower bound for integration. Let's solve for the intersection points.

$$\begin{aligned} x^2 - 1 &= -x^2 + 7 \\ 2x^2 &= 8 \\ x^2 &= 4 \\ \therefore x &= \pm 2 \end{aligned}$$

Now let's find the area, keeping in mind that $-x^2 + 7$ is on top.

$$\begin{aligned} A &= \int_{-2}^{2} \left[(-x^2 + 7) - (x^2 - 1)\right] dx = \int_{-2}^{2} \left(-2x^2 + 8\right) dx \\ &= \left[-\frac{2}{3}x^3 + 8x\right]_{-2}^{2} = \left(-\frac{16}{3} + 16\right) - \left(\frac{16}{3} - 16\right) \\ &= \frac{64}{3} \text{ square units} \end{aligned}$$

3.5.6 Average Value of a Function

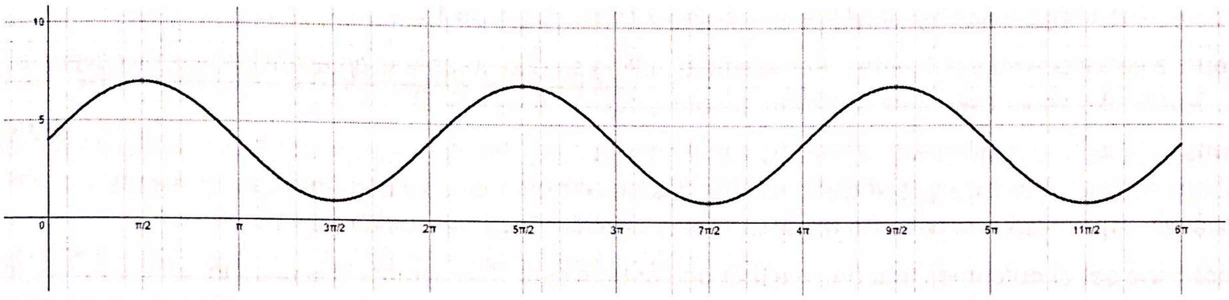
The final application of integration we will be looking at is the average value of a function. The formula for the average value of a function $f(x)$ over the interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$. The formula essentially adds up all of the area under a function, then divides it by how many units in the x direction that corresponds to, which leads to the average value of a function.

The formula is very similar to the formula for calculating the mean of data in statistics. To find the mean of a sample, you add up the value of each item in the sample (the area) , then divide it by the amount of items in the sample (the x units).

Example 3.12

Find the average value of the graph of $y = 3\sin(x) + 4$ *over the interval* $x \in [0, 6\pi]$.

Before actually doing the calculation, here is a sketch of the graph over the interval.



Just by looking at the graph, we can see it oscillates up and down around the line $y = 4$. So if we were to find the average value, we could expect it to be $y = 4$. Let's verify this with the formula.

$$
\begin{aligned}
y_{AVG} &= \frac{1}{(6\pi - 0)}\int_0^{6\pi}(3\sin(x) + 4)dx === \frac{1}{6\pi}(24\pi) = 4 \\
&= \frac{1}{6\pi}[-3\cos(x) + 4x]_0^{6\pi} \\
&= \frac{1}{6\pi}[(-3\cos(6\pi) + 24\pi) - (-3\cos(0) + 0)] \\
&= \frac{1}{6\pi}(24\pi) \\
&= 4
\end{aligned}
$$

Therefore, we have verified that the average value of the graph of $y = 3\sin(x)+4$ over the interval $x \in [0, 6\pi]$ is $y = 4$.

Area of Study 4

Probability and Statistics

4.1 Introduction to Probability

4.1.1 Definitions

Let's do a very quick recap of terms and definitions that you should be familiar with.

Set Notation: we use curly brackets to describe a set of events. For example, when rolling a die the set of possible events would be written formally as $\{1,2,3,4,5,6\}$.

Sample Space: this is the set of all possible outcomes and is represented by ε. For rolling a die, $\varepsilon = \{1,2,3,4,5,6\}$. Rolling an odd number would not be called the sample space as that set only contains $\{1,3,5\}$. The total probability of all the events in a sample space is 1.

Event: a set of possible outcomes. For example, rolling an odd number is an event because it covers rolling a 1, 3 or 5. We could even give this event a name, say A , $A = \{1,3,5\}$.

Element: if we are dealing with a specific set, then we use $\in$, the element symbol, to show that an outcome belongs to a set. For example, if we rolled a 3, it is an odd number so it belongs in the previous set, $3 \in A$. However, if we rolled a 4, which is not in the set of odd numbers we would write $4 \notin A$.

Union: the set of outcomes that are in either one or both sets. The symbol $\cup$ is used to denote union (think of u in union to remember this symbol). If B is the event a number less than or equal to 4 is rolled, then $A \cup B = \{1,2,3,4\}$.

Intersection: the set of outcomes that are common to both sets. The symbol $\cap$ is used to denote intersection (think the n in and to remember this symbol). Using the same sets A and B , $A \cap B = \{1,3\}$.

Null Set: if an event is impossible, such as rolling a 7 or the intersection of rolling an odd number and rolling a number divisible by 4, it is called a null set. The symbol $\emptyset$ or empty curly brackets {} are used.

Complement: the complement is the other matching side of a set. For rolling an odd number, A , the complement would be rolling an even number, A' (note the dash). The key is that, combined, A and A' give the sample space ε . So, to find $Pr(A')$, we would do $1 - Pr(A)$.

Mutually Exclusive: events are mutually exclusive if the intersection of the two events is a null set. This means that for two mutually exclusive events A and B, $Pr(A \cap B) = 0$.

Independent Events: events that are independent have no influence on one another. This means that: $Pr(A \cap B) = Pr(A) \times Pr(B)$.

4.1.2 Conditional Probability

Let's look into conditional probability. Consider two events A and B. If the chance of B occurring influences the chance of A occurring, then the two events are conditional. For example, if we where choosing students from the audience and if we knew which students had stayed up until 4am in the morning (event B), it would increase our chances of choosing a student who had failed the test (event A).

Conditional probability is written as $Pr(A|B)$. This is read as "The chance that A happens given that I already know that B has happened" or, in a shorter form "The chance of A given B".

The formula for conditional probability is written as:

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

Tree diagrams are often used to solve conditional probability questions, but are not restricted to conditional probability questions.

Example 4.1

What is the probability of rolling a 5 on a dice, given that you roll an odd number?

Let's let event A be rolling a 5 on a dice, which has a probability of $\frac{1}{6}$ then let's let event B be rolling an odd number, which has a probability of $\frac{1}{2}$. In order to complete the formula, we need to know $Pr(A \cap B)$. What is the intersection of rolling a 5 and rolling an odd number? Well, the only way they can both happen is if you roll a 5, which is an odd number. Therefore $Pr(A \cap B) = \frac{1}{6}$.

Now we can apply the formula:

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

Because there is a condition that the roll is an odd number, the probability of rolling a 5 has been doubled, since there's essentially half the amount of possible outcomes (1, 3, 5) given the condition.

4.1.3 Discrete and Continuous Variables

Discrete Variables

A discrete variable is a variable which can take on specific values. Most of the time in Methods you'll be dealing with integers. Examples of discrete variables are number of people in a car, number of likes on a Facebook post or the amount of 5's you roll in game of Monopoly.

Continuous Variables

A continuous distribution is one where data can take on any value within a range. Examples of continuous variables are how heavy a person is, the height of a person or the distance required to get to work.

4.1.4 Measures

Expected Value

The expected value goes by many names. It is also referred to as the mean or the average and is denoted by either $E(X)$ or μ.

Median

The median of a function is the middle value. Since it is the middle value, it means that there is 0.5 chance of a value above the median occurring, and there is a 0.5 chance of a value below the median occurring.

Mode

The mode can be thought of as the "most". That's how most people remember it. The mode is the event which is the most likely to occur. For a distribution X, the mode is the x value with the highest probability.

Variance and Standard Deviation

The variance and standard deviations are measures of spread. No, not how well you spread jam on your toast this morning, but how spread out data is. For example, if we were looking at Tomic's aces over an entire tennis tournament, we might find that the number greatly varies from 0 to 20 per game (large spread), or that he consistently gets 10-12 aces per match (small spread).

The variance is often expressed as σ^2. This is because the standard deviation is the square root of the variance. The standard deviation is the root mean square of the distance from the mean.

Some additional useful properties are:

$$Var(aX + B) = a^2 Var(X)$$

$$SD(aX + B) = a\ SD(X)$$

4.2 Discrete Probability

4.2.1 Definition

A discrete random variable is just an event of integer values. For example, the result of rolling a dice is a discrete random variable as it is random, a specific event and it only has integer values. Other discrete random variables could be shirt sizes, number of heads obtained after 3 coin tosses and the number of aces hit in a tennis match.

For each possible value of X, a discrete random variable, the probability of each outcome is written as: $p(x) = Pr(X = x)$. There is a difference between X and x. The former refers to a whole set, whereas the latter only refers to some element value. To state the probability of rolling a 1, we would write $Pr(X = 1) = \frac{1}{6}$.

There are two rules for making sure that X is a probability distribution function.

- Each probability $p(x)$ must be between 0 and 1 (inclusive).

$$0 \leq p(x) \leq 1$$

- The sum of all probabilities is 1.

$$\sum_{x} p(x) = 1$$

4.2.2 Calculations

There are not many calculations of probabilities required as discrete probability is most commonly expressed in probability tables. For example, the probability of rolling a specific number of a loaded die may be be as follows:

x	1	2	3	4	5	6
$Pr(X = x)$	$\frac{1}{24}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{7}{12}$	$\frac{1}{12}$	$\frac{1}{24}$

In this die, the chance of rolling a 4 is much greater than rolling any of the other values.

However, if you are required to fill in a probability table or calculate values, you can use basic probability skills from Year 11.

4.2.3 Measures

Mean

The mean of a discrete distribution is the sum of the x values multiplied by their respective $Pr(X = x)$.

For the probability table in 5.2.2, the mean is $1 \times \frac{1}{24} + 2 \times \frac{1}{6} + 3 \times \frac{1}{12} + 4 \times \frac{7}{12} + 5 \times \frac{1}{12} + 6 \times \frac{1}{24} = \frac{29}{8} = 3.625$.

Median

The median is the x value with 50% of the probability above and below it. This can be found by adding up probabilities from the left side or the right side until the value is greater than 0.5

For the probability table in 5.2.2, $\frac{1}{24} + \frac{1}{6} + \frac{1}{12} \approx 0.29$, $\frac{1}{24} + \frac{1}{6} + \frac{1}{12} + \frac{7}{12} \approx 0.86$. *As adding* $\frac{7}{12}$ *which is the probability that* $x = 4$ *pushes the sum over 0.5, the median is 4.*

Mode

The mode is the most frequently occurring x value or the x value with the maximum probability.

For the probability table in 5.2.2, the mode is $x = 4$ *again, as this is the* x *value with the maximum chance of occurring.*

Variance and Standard Deviation

To find the variance, we take the average of the squared distance from the mean:

$$Var(X) = E[(X-\mu)^2]$$

Because this is quite difficult to calculate, a rearranged formula is preferred:

$$Var(X) = E(X^2) - [E(X)]^2$$

For the probability table in the previous section, the variance is:

$$\begin{aligned} E(X^2) &= (1 \times \frac{1}{24} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{12} + 4^2 \times \frac{7}{12} + 5^2 \times \frac{1}{12} + 6^2 \times \frac{1}{24}) \\ &= 14.375 \\ E(X) &= 3.625 \\ Var(X) &= 14.375 - 3.625^2 \\ &= 1.2344 \end{aligned}$$

The standard deviation is the square root of the variance. For the probability table, the square root is:

$$\begin{aligned} SD(X) &= \sqrt{Var(X)} \\ &= \sqrt{1.2344} = 1.1110 \end{aligned}$$

4.3 Binomial Distributions

A binomial distribution is part of discrete probability and looks at the probabilities associated with an event over multiple trials. It is also referred to as a Bernoulli sequence.

4.3.1 Definition

Binomial trials must have the following properties:

- There are n independent identical trials. This means that they do not change or have an influence on one another.
- Each trial has only 2 possible outcomes, a success (S) or failure (F).
- The probability of success is always p and the failure is $(1-p)$ for each trial.

The number of successes, x, from n trials where p is the probability of success is written as $X \sim Bi(n,p)$. This notation is read as: "X is distributed binomially with n trials and a probability of success of p. When making reference to a binomial distribution, the same Pr notation is used. For example, $Pr(X=1)$ is the probability that a particular binomial trial X will succeed exactly 1 time out of n times given its probability p. $Pr(X>1)$ is the probability that a binomial trial X will succeed more than 1 time out of n times given its probability p.

4.3.2 Calculations

Given $X \sim Bi(n,p)$, the probability of x amount of successes is given as:

$$Pr(X=x) = ^{n}C_r . p^x (1-p)^{n-x}$$

You may have to calculate a particular probability by hand, so as a quick refresher: $^nC_r = \frac{n!}{r!(n-r)!}$ where the ! denotes a factorial ($4! = 4 \times 3 \times 2 \times 1 = 24$).

You may also be required to calculate the probability of multiple successes. If a question asks to find $Pr(X \leq 1)$, this can be calculated as $Pr(X=0) + Pr(X=1)$. Another question may ask to find $Pr(X \geq 1)$, one method would be to calculate $Pr(X=1) + Pr(X=2) + Pr(X=3) ... + Pr(X=n)$, but a much quicker method is to calculate $Pr(X \geq 1) = 1 - Pr(X=0)$ as all the probability in a sample space must add up to 1.

Area of Study 4 – Probability and Statistics

Example 4.2

Joe is aiming to score a strike in bowling. He bowls 10 times and has a probability of 0.4 that he scores a strike. What is the probability that he scores more than 8 strikes?

Start with our definition:

$$
\begin{aligned}
X &\sim Bi(10, 0.4) \\
Pr(X > 8) &= Pr(X = 9) + Pr(X = 10) \\
&= {}^{10}C_9(0.4)^9(1-0.4)^{10-9} + {}^{10}C_{10}(0.4)^{10}(1-0.4)^{10-10} \\
&= 0.001678
\end{aligned}
$$

4.3.3 Measures

Mean

The mean of a binomial distribution is given by $E(X) = np$. It is simply the number of trials multiplied by the probability of success of one trial.

For the example given in 5.3.2, the expected amount of strikes by Joe is calculated as $E(X) = 10 \times 0.4 = 4$. *Joe can be expected to make 4 strikes out of 10.*

Variance and Standard Deviation

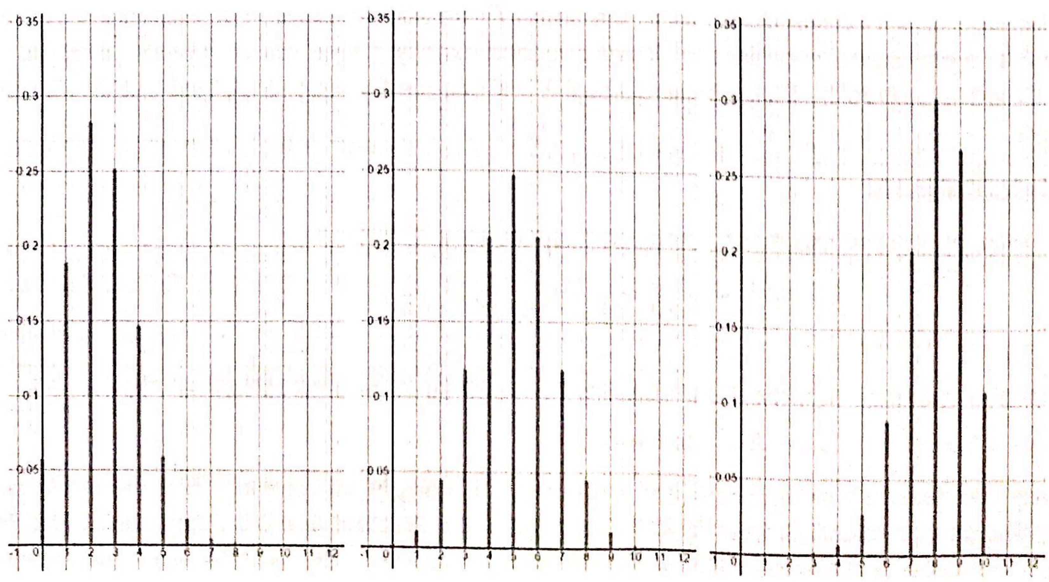
The variance and standard deviation are given by:

$$
\begin{aligned}
Var(X) &= np(1-p) \\
SD(X) &= \sqrt{np(1-p)}
\end{aligned}
$$

For the example given in 5.3.2, the variance is $Var(X) = 10(0.4)(1-0.4) = 2.4$ *and the standard deviation is* $SD(X) = \sqrt{2.4} = 1.55$.

4.3.4 Graphing a Binomial Distribution

A binomial distribution can be graphed, with the number of successes on the x axis, and the probability of that specific number of successes on the y-axis, $Pr(X = x)$. The graphs for $p = 0.25$, $p = 0.5$ and $p = 0.8$ are shown below. $n = 10$ for all the graphs.



When $p < 0.5$, the graph is positively skewed. This means that the probability that less than half of the trials are successful is greater than the probability that more than half of your trials are successful.

When $p = 0.5$, the graph is symmetrical. This means you are most likely going to succeed in half your trials, since the probability of one specific trial is half.

When $p > 0.5$, the graph is negatively skewed. This means that the probability that more than half of the trials are successful is greater than the probability that less than half the trials are successful.

When n is a large number, the step size between columns is smaller, which gives it a smoother appearance. When n is large enough, the binomial distribution can resemble a normal distribution (covered later).

When n is a smaller number, the step size between columns is larger, giving a more blocky appearance.

4.4 Probability Density Functions

Now that we've looked at discrete distributions, let's look at some continuous distributions. The first we will look at are probability density functions.

4.4.1 Definition

A continuous random variable, X, is defined if it can take any value in an interval. The plot of X is known as a probability density function (also known as PDF). The area under the probability function from a to b is the probability that X is between a and b.

For a function $f(x)$ to be a probability density function, it must:

- Be positive or zero for any x value.
- Have the total area enclosed by $f(x)$ equalling 1 (as the sum of all probabilities must be 1).

Mathematically, these two conditions are written as:

$$\text{For all } x \in R,\ f(x) \geq 0$$

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

If either one of these conditions are not met, then $f(x)$ is not a probability density function.

4.4.2 Calculations

The probability is the area under the curve $f(x)$, not the value of $f(x)$. Integration is back! .

To find the probability from a to b, we can calculate the area under the curve from $x = a$ to $x = b$:

$$Pr(A \leq X \leq B) = \int_a^b f(x)\, dx$$

It is important to note that the probability of obtaining an exact value of a or b is zero. This is because we are dealing with a continuous variable which can take on any value in the interval, so the probability of one specific value is zero, and we can only find the probability over a range.

Therefore: $Pr(a \leq X \leq b) = Pr(a < X \leq b) = Pr(a \leq X < b) = Pr(a < X < b)$. So finding the probability of x greater than 1 and less than 5 is the exact same as finding the probability that x is greater or equal to 1 and less than or equal to 5.

Example 4.3

It was found that the amount of time watching a short video on a video sharing site could be modelled using the probability function $f(x) = \frac{1}{25e^{\frac{x}{25}}}$ *for* $x > 0$, *where* x *is the time spent watching a video in seconds. What is the probability that a specific person will spend between 10 seconds and 40 seconds watching a video?*

Simply integrate the function with the correct terminals.

$$Pr(10 < x < 40) = \int_{10}^{40} \frac{1}{25e^{\frac{x}{25}}} dx = 0.7981$$

4.4.3 Measures

Mean

The mean of a continuous random variable is given by:

$$E(X) = \mu = \int_{-\infty}^{\infty} xf(x)\, dx$$

Think of it as multiplying each x value with its associated probability and summing it all up (although this is not strictly true).

For the example provided in 5.4.2, the mean can be calculated as $\int_0^{\infty} xf(x)dx = \int_0^{\infty} x(\frac{1}{25e^{\frac{x}{25}}})dx = \lim_{t\to\infty}[-(x+25)e^{-\frac{x}{25}}]_0^t = 25$. *Therefore, those on the video sharing site can be expected to spend a mean time of 25 seconds.*

Median

The median of X distributed across $f(x)$ is found by solving for m where $Pr(X \leq m) = Pr(X \geq m) = 0.5$. By solving this, we are essentially finding the x value with half the probability above it, and half of the probability below it.

In mathematical notation:

$$\int_{-\infty}^{m} f(x)\, dx = 0.5 \text{ and solve for } m$$

$$\text{or } \int_{m}^{\infty} f(x)\, dx = 0.5 \text{ and solve for } m$$

For the example provided in 5.4.2, the median can be calculated by solving for m in $\int_0^m \frac{1}{25e^{\frac{x}{25}}}dx = 0.5 \implies 17.33$. *This means that half of the data lies below 17.33 seconds, and half of the data lies above 17.33 seconds.*

Mode

The mode is just the absolute maximum of $f(x)$. This could be a turning point, an end point or even a sharp point. To get the best idea of what the mode is, it is always helpful to draw a graph.

Keep in mind that the mode is the x value with the maximum probability, not the maximum probability itself ($f(x)$).

Variance and Standard Deviation

The variance of X is calculated by:

$$Var(X) = \sigma^2 = E(X-\mu)^2 = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)\, dx = (\int_{-\infty}^{\infty} x^2 f(x)\, dx) - (\int_{-\infty}^{\infty} x\, f(x)\, dx)^2$$

The standard deviation is the square root of the variance.

$$SD(X) = \sigma = \sqrt{Var(X)}$$

For the example provided in 5.4.2, the variance is $(\int_0^{\infty} x^2(\frac{1}{25e^{\frac{x}{25}}})dx) - (\int_0^{\infty} x(\frac{1}{25e^{\frac{x}{25}}})dx)^2 = 625$, *and the standard deviation is* $\sqrt{625} = 25$.

4.5 Normal Distributions

The normal distribution is a very special type of distribution that is characterised by a symmetrical bell shape. The function is given as: $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$. However, there is no need to memorise this as your CAS has a built in normal distribution function!

The curve of a normal distribution always has a mean of μ and a variance of σ^2. It can be written as $X \sim N(\mu, \sigma^2)$ which is read as: X is normally distributed around a mean of μ with a variance of σ^2.

4.5.1 Calculations

Your calculator will have its own syntax which can be used to find the probability within a certain region in the bell curve.

On the Ti-Nspire the notation is as follows:

$$normCdf(lower, upper, \mu, \sigma)$$

It can be also found in Menu, Probability, Distributions, Normal Cdf.

On a CASIO Classpad, the notation is as follows:

$$normCDf(lower, upper, \mu, \sigma)$$

It can also be found in Action, Distribution, Continuous, normCDf.

Example 4.4

A particular school in Melbourne took a survey to find out the average amount of text messages their students sent each day. The results were normally distributed with a mean of 35 and a standard deviation of 8.

Find the probability that a randomly chosen student from the school sends more than 25 texts a day.

Simply input the data into your calculator using the appropriate syntax:

For the Ti-Nspire CAS: $normCdf(25, \infty, 35, 8) = 0.8944$

This means that there is an 89% chance that a randomly chosen student sends more than 25 texts a day.

Be aware that this is not the working out you should show on an exam. You should not use calculator syntax. The appropriate working out is as follows:

$$\begin{aligned} X &\sim N(35, 8^2) \\ Pr(X > 25) &= 0.8944 \end{aligned}$$

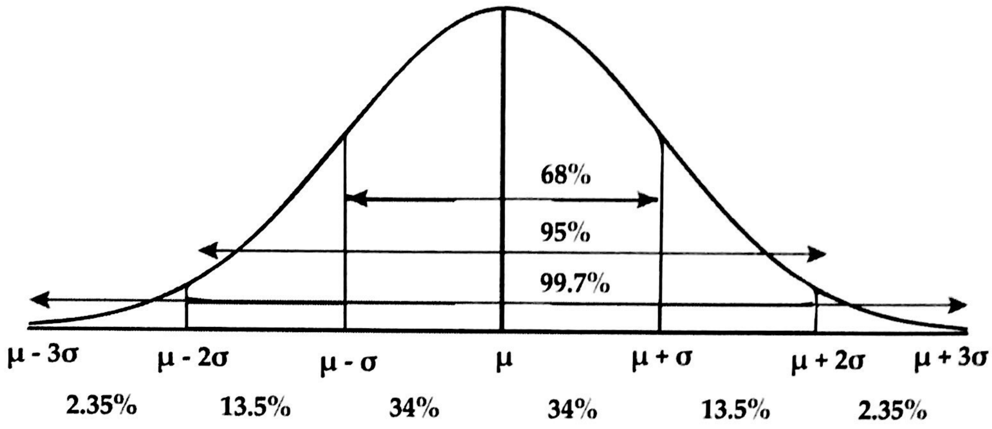
This working out uses proper mathematical notation to define the normal distribution, then calculates the probability required.

4.5.2 68-95-99.7% Rule

There are times when we can't perform an exact calculation on our CAS calculator (Exam 1), so the 68-95-99.7% rule is useful for finding approximations as to the amount of data within a certain region.

The 68-95-99.7% rule states that:

- Approximately 68% of values lie within one standard deviation of the mean.
- Approximately 95% of values lie within two standard deviations of the mean.
- Approximately 99.7% of values lie within three standard deviations of the mean.



For example, VCE study scores are normally distributed with a mean of 30 and a standard deviation of 7. This means that:

- 68% of students will receive a study score between 23 and 37.
- 95% of students will receive a study score between 16 and 44.
- 99.7% of students will receive a study score between 9 and 51. (Note: This is rounded back to 50 when VCAA actually does the calculation.)

4.5.3 Measures

The amazing thing about the normal distribution is that all of the measures are the same. That is, the mean, the median and the mode all occur at the same value. This is because it is symmetrical, so there is 50% of data on each side of the curve, and the maximum occurs in the middle of the curve, so the mode is centred as well.

4.5.4 Standard Normal Distribution

The standard normal distribution is a normal distribution with a mean of $\mu = 0$ and a standard deviation of $\sigma = 1$. Instead of X, Z is used to denote the standard normal distribution. Another way of expressing it is as $Z \sim N(0, 1)$.

The standard normal distribution allows us to standardise different bell curves using z-scores.

We can convert to a standard normal distribution using the following formula:

$$z = \frac{x-\mu}{\sigma}$$

The z-score formula can also be used in reverse to find specific values (x, μ, σ) given a z-score.

Example 4.5

Suppose a student at school A took a test and achieved 85%. Those test results for school A were normally distributed with a mean of 75% and a standard deviation of 9%. Suppose another student at school B took a much harder test and achieved 55% and the test results for that test had a mean of 40% and a standard deviation of 14%. Which student performed better?

We can simply convert each of the test scores into a z-score and see which student performed better relative to the spread of their own cohort.

Student A: $z = \frac{85-75}{9} = 1.11$

Student B: $z = \frac{55-40}{14} = 1.07$

Student A scored 1.11 standard deviations away from the mean, while Student B scored 1.07 standard deviations away from their average, which means that Student A performed better relative to their cohort.

Example 4.6

Mr Normal is a Physical Education teacher at a school. His students have a height which is normally distributed with a mean of 149cm and a standard deviation of 6.7cm. He is putting together a basketball team and wants the tallest students. If Mr Normal only accepts students who are 2 standard deviations above the mean, what height must you be to be part of his basketball team?

Let's use the z-score formula in reverse:

$$z = \frac{x-\mu}{\sigma} \rightarrow 2 = \frac{x-149}{6.7} \rightarrow x = 162.4cm$$

Therefore, to be part of his team, you must have a minimum height of 162.4cm.

4.5.5 Inverse Normal Distribution

Earlier in this section we looked at calculating probabilities of normal distributions. We looked at using an area to calculate the probability associated with that area. The inverse normal distribution allows us to find the area that a particular probability covers in a bell curve. The syntax for the Ti-Nspire is: *invNorm(area,* μ, σ*)*. This can also be found in Menu, Probability, Distributions, Inverse Normal. Be aware that the area is calculated from left to right. We will discuss the effects of this later.

The syntax for the Casio Classpad is: *invNormCDf(tail setting, area,* μ, σ*)*. This can also be found in Action, Distribution, Inverse, invNormCDf. The Casio Classpad allows you to specify where you want the area to be measured from.

Example 4.7

Mr Normal decides that instead of accepting students with a minimum height of 162.4cm for his basketball team, he wants the top 5% of students in terms of their height. Recall that his students have heights which are normally distributed with a mean of 149cm and a standard deviation of 6.7cm. Calculate the new minimum height of a student to join the team if they need to be in the top 5% of students.

Let's use the inverse normal distribution to figure out the area in the z-score which will have 5% of the data above it.

Ti-Nspire: Notice that we want an area of 5% measured from the right side of the bell curve, but the calculator uses its area from left to right. Since the bell curve has a total area of 1, we can just say that 5% from the right hand side of the curve is the same as 95% from the left hand side of the curve. Let's use the inverse normal function:

$$invNorm(0.95, 149, 6.7) = 160.02$$

Casio Classpad: We want to find out which point of our bell curve has 5% of the data above it, so we are measuring from the right side of the curve. We can input it directly as:

$$invNormCDf("R", 0.05, 149, 6.7) = 160.02$$

Notice we have used "R" inside of the calculator syntax. This means that we are measuring out 5% of data from the right side of the curve. "L" can be used to measure data from the left hand side, and "C" can be used to measure data from the centre outwards.

Therefore, a student needs to be over 160cm in order to be in the top 5% of students and accepted into the basketball team.

The working out you should show on the exam is as follows:

$$\begin{aligned} X &\sim N(149, 6.7^2) \\ Pr(X > a) &= 0.05 \\ a &= 160.02 \end{aligned}$$

4.6 Introduction to Statistics

4.6.1 Parameters and Statistical Inference

Statistical inference allows us to use random samples in order to make judgements about a larger population. Before we get into some calculations, we first need to understand the difference between a population parameter and a sample statistic.

Parameters

In a population, parameters describe the population as a whole. They give us information about the entire set. In large populations, parameters can be quite difficult to measure, so samples are drawn from the population in order to make estimations.

Statistics

Sample statistics characterise a sample and give us information about a smaller subset of a larger population. Statistics are easier to gather than parameters as they deal with smaller amounts of data.

4.6.2 Sample Proportions

The samples you will be dealing with in this portion of the course will be based off the binomial distribution. That is, they will either have a success or failure. A sample proportion is used to describe the proportion of successes within a sample.

The sample proportion is calculated as:

$$\hat{P} = \frac{X}{n}$$

Where X is the binomial variables with a certain amount of successes and n is the sample size. The sample proportion can also be thought of as:

$$\hat{P} = \frac{\text{Successes}}{\text{Sample Size}}$$

In this form, the sample proportion resembles the formula used to calculate basic probability from Year 11. Not too hard to remember!

Variation in Sample Proportion

The sample proportion will vary between samples. This is because choosing a different sample out of your population may will result in a different proportion of successes and hence different sample proportions.

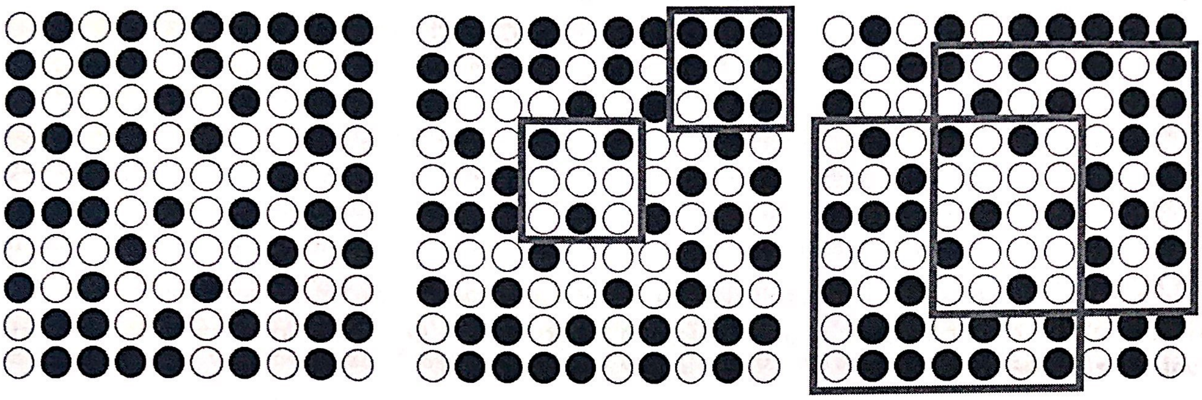
For example, say you took a sample of 15 Year 7 students and determined how many of them had gone overseas. In this scenario, we defined our success characteristic as a student who had gone overseas. If 6 had gone overseas, we would have 6 successes out of 15 and $\hat{P} = \frac{6}{15} = 0.4$.

If we decided to take a bigger sample of 25 Year 7 students and had 13 students who had gone overseas, our sample proportion for this new sample would be $\hat{P} = \frac{13}{25} = 0.52$.

4.6.3 Normality of Sample Proportions

For large sample sizes with a large value of n, a sample proportion becomes normalised. How exactly does this work?

Imagine a population of 100 balls, made up of 50 white balls, and 50 black balls. Let's say that having a white ball is considered a binomial success. This is illustrated in the first image below.

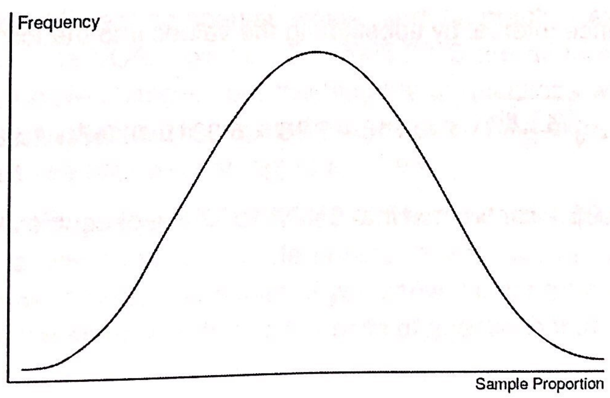


The second image shows a sample size of 9 being taken from the population, while the third image shows a sample size of 49 being taken from the population.

When $n = 9$, we can determine the sample proportions of the samples inside of the two blue squares. The top right square has a sample proportion of $\hat{P} = \frac{2}{9} \approx 0.22$, while the middle square has a sample proportion of $\hat{P} = \frac{6}{9} \approx 0.67$. There seems to be quite a lot of variation for the sample proportion we just took.

When $n = 49$, we can determine the sample proportions using the samples inside of the blue squares again. The bottom left sample has a sample proportion of $\hat{P} = \frac{27}{49} \approx 0.55$, while the middle right sample has a sample proportion of $\hat{P} = \frac{28}{49} \approx 0.57$. There is a lot less variation between the two sample proportions in this scenario.

As $n \longrightarrow \infty$, the binomial distribution becomes normalised. When taking samples from the population, you are extremely likely to achieve a sample proportion equivalent to the mean p, and less likely to achieve sample proportions which are further away from the mean on either side.



For our population of balls, the peak of the bell curve would occur at a sample proportion of $p = 0.5$, and moving further away from the mean will result in a decrease in the likelihood of achieving those sample proportions.

Mean and Standard Deviation

The mean of the distribution $\hat{P}$ is given as p, while the standard deviation is $\sqrt{\frac{p(1-p)}{n}}$. The approximate normality of a sample proportion can then be defined as $N(p, \frac{p(1-p)}{n})$.

4.6.4 Approximate Confidence Interval

Confidence intervals can be used to determine the interval in a population which should have a specific probability or certainty of success. For example, if we can say with 95% certainty that the proportion of students who have travelled overseas is between 0.3 and 0.5, then the 95% confidence interval for that population is [0.3, 0.5].

The formula for the approximate confidence interval is:

$$(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}})$$

This can be thought of as the mean plus or minus the standard deviation multiplied by a particular value z. This z value is based upon the z scores in a standard normal distribution. For a 95% confidence interval, 2.5% of the sample proportions will remain above and below the interval. We can then use the inverse normal function on our calculator to determine the z-score (we use the positive value) where this occurs.

The Ti-Nspire CAS calculates z-scores from the left of the curve, so we are looking for the z-score with 95%+2.5% of the data below it. Using *invNorm*(0.975, 0, 1) = 1.96. Therefore, for a 95% confidence interval, let $z = 1.96$.

Example 4.8

A tomato sauce manufacturer launched a special new recipe of tomato sauce to their thousands of customers. They surveyed 200 customers to see if they liked the improved recipe. Out of these 200 customers, 143 said they liked the new recipe. Calculate the sample proportion, mean, and approximate standard deviation of this sample. Then, find the 95% confidence interval and 50% confidence interval.

The calculations for the sample proportion, mean and standard deviation aren't difficult.

$$\begin{aligned} \hat{P} &= \frac{143}{200} = 0.715 \\ p &= \hat{P} = 0.715 \\ \sigma &= \sqrt{\frac{0.715(1-0.715)}{200}} = 0.0319 \end{aligned}$$

Now let's find a 95% confidence interval by substituting the values into the formula above with $z = 1.96$.

$$\text{Interval: } (\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}) \rightarrow (0.715 - 1.96 \times 0.0319, 0.715 + 1.96 \times 0.0319) = (0.652, 0.778)$$

Therefore, we can state with 95% confidence that 65.2% to 77.8% of people liked the new recipe of tomato sauce.

Section 5

Exam Preparation

This section is dedicated to providing you with tips and tricks which will help you smash the exams so you can produce the highest study score possible. Whether you have just started the year, or you are preparing for exams, there should be something here for you!

5.1 Practice Exams

There's really no better way to prepare for exams in mathematics than doing practice exams. But there's no need to go overboard or crazy. Here are some general tips for approaching practice exams.

Study Design Changes

There's a good chance that you will be looking at practice exams from before 2016. As the study design has changed, not all questions will be relevant, and the old exams will not include everything which can be assessed as part of the new study design. The major changes are addressed below:

- Modulus/Absolute Functions have been removed.
- Related Rates (Calculus) has been removed.
- Markov Chains (Probability) has been removed.
- Statistical Inference has been added.

Start small

First of all, start small. There's no need to jump into a full 120 minute exam straight away. Start with just one section of the exam. So if you are starting with an exam 1, maybe just do the first 5 questions or first 20 marks, then leave the rest of the 20 marks for later and review the first half. If you're doing an exam 2, you can start by doing just the multiple choice section or just the extended response section.

Feel free to use as much material as you want in your first few exams as well. If you are having difficulties starting off, feel free to use textbooks, study guides and other notes to help you in your first few papers. Remember that practice exams are for your benefit, so there's no need to simulate exam conditions straight away if you don't think you can handle them.

Prioritise VCAA exams!

Although there are countless different companies which produce practice exams, ensure that you get a good taste of VCAA exams. The VCAA exams from 2006-2015 are all from the previous study design. Some portions of the course have changed, but the majority of questions will still be helpful. There are usually questions aimed at the highest achieving students in each VCAA exam, and it's in your best interest to give them a shot or at least see how you can approach them.

Prioritising VCAA exams will also give you the best idea of what the exam questions will be like on the day. You may encounter questions with an unusual style in other exam company's papers. Sometimes it may not be worth spending the time to understand them if you know it is far from what VCAA could assess on the real exam. The time is better spent working on the style of questions that you know VCAA like to use.

Use assessment reports

The VCAA assessment reports are extremely useful when practicing for exams. VCAA provides a list of strength and weakness on the front page of assessment reports. You can work through the list of strengths and weaknesses to identify certain areas which you can work on.

Assessment reports can also provide hints as to what the next year's exam may involve. VCAA provide the average mark for every single question in the exam paper, and often poorly completed questions are repeated in later years.

For example, Question 4 from the VCAA 2013 exam 2 had an extremely low percentage of completion. The final question in the exam (diii) had only 2% scoring full marks, 1% scoring ½ marks and 97% of students scoring zero marks. What is notable is that the VCAA 2014 exam 1 had a question which resembled the VCAA 2013 and needed the same type of thinking to complete.

4diii.

Marks	0	1	2	Average
%	97	1	2	0.1

What are the best companies?

Students often ask what the best companies are to complete. Methods is one of the subjects with the most companies actively producing practice exams each year.

If you are tight for time due to other subjects, the exams which offer difficulties closest to VCAA's are Neap, Heffernan, MAV and Insight. These should be your priority unless you have more time than usual.

If you are feeling confident and want to try something more difficult, Kilbaha are known for their tough algebra questions and IARTV can also have some difficult exam 1 questions.

Take notes always

One thing that students often do is to complete the exam, put it down then leave it. Spend as much time as you need to writing down all of the errors or mistakes you made in the exam. It is not advisable to complete an exam, then move on to another one without properly spending time working on what you did wrong in the previous one. You can take notes using an Excel/Word document, or even a notepad if you prefer that.

If you are using Excel, a suggestion for the information you take note of is:

- Exam Company: This is useful as some companies are more difficult than others - it can explain some discrepancies in results.
- Exam Year: Always take note of the year as some years are also harder than others, especially with VCAA exams.
- Time Taken: You can use the time taken to finish to compare your exams over time to see if you are giving yourself more or less time to check answers at the end of the exam.
- Mark: Your result for the exam.
- Issues/Problems: Any major problems you had with the exam, and any areas you need to work on.

Doing exams digitally

Nowadays, a lot of practice exams are available digitally in PDF format. If you want to save on the paper, you can display the exams digitally on a laptop and tablet. However, keep in mind that you will have to provide your own writing paper and space for graphing as you will be copying off a screen.

If you decide to take this option though, remember to do your last few exam papers closer to the exam day on good old paper. This will ensure that you have a feel for how much working space is provided and you won't have to draw up axes every time you are required to sketch a graph.

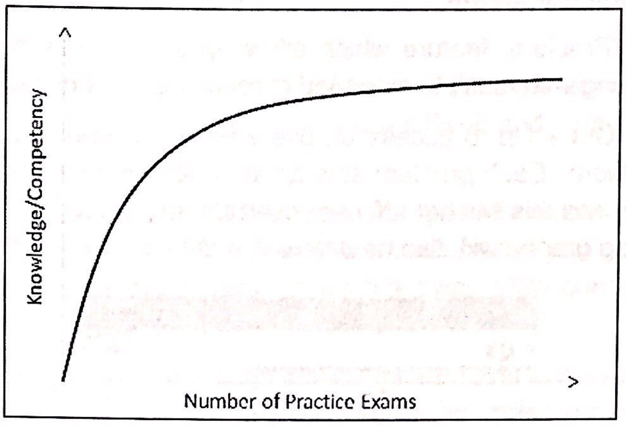
How many exams should I do?

I want to end this section by discussing quantity. First of all, do not do the exams for the numbers. It is really nice to quantify the amount of exams that you complete. However, make sure that you are actually benefiting from doing extra exams, and you are not just adding one more exam to a growing list.

If you have seen a question many times over and over again, don't feel pressured to doing them. It might be better off to skip the question, and focus on more difficult questions from practice exams. You may even finish the entire practice exam earlier which can leave more time for other subjects.

In short, do the amount that will make you comfortable going into the exams. If you are completing the subject in Year 11, you may have time to do over 30 sets of exams (60+ individual papers), but for most of you that's not possible - simply do enough to feel confident.

It's also important to remember that there's no significant advantage to practice exams once you have exceeded 20-25 sets. I personally think skill and competency have a logarithmic relationship with the amount of practice exams you complete. You will improve the most in your first few exams, but soon after that you will learn less from each new exam, so it's your decision when it's time to stop and focus on another subject.



5.2 Bound Reference

VCAA are generous enough to allow you to bring in one bound reference into exam 2 which can contain as much as you want. However, it is important that you do not go overboard and that you do use it wisely.

Bigger is not better

Having more quantatively does not mean you have more qualitatively. The VCAA exam 2 is long and difficult. There is a lot to do and you have roughly 90 seconds per mark in the exam. The majority of students do not complete the entire exam. I have seen too many students carrying in bound references which have their textbook, sticky taped to their exercise book and then sticky taped to a bundle of VCAA exams to create a behemoth of a reference. However, there is absolutely no need for that in the exam, and you'll find that some of the best bound references are only a handful of pages long.

Make it personal

Make your bound references yours. Make sure that it is yours. There is no point purchasing notes for practice exams if you don't know how to navigate them or they are written in a way which you don't understand.

One way of making your bound reference personal is to list any notes from practice exams into the bound reference. What do I mean by that? If you have kept track of all of your mistakes or areas you can improve in from practice exams, you can summarise that list and use it on the front of your bound reference. When you look at your bound reference, the things that you struggle with will be listed right in front so you can ensure you don't struggle with them on the real exam.

Practise exams with it

If you think you're going to rely on your bound reference a lot in the real exam, make sure you know where things are. Your goal should be to get as much information from the bound reference as quickly as possible. Therefore, use your real bound reference when you are completing practice exams. This will get the position of pages into muscle memory so you can turn to them quickly under pressure.

Use the study design

Finally, if you feel you are unsure what exactly to put in your bound reference, use the study design. The study design lists all of the expected outcomes from both units of methods, so use that list to check that you have covered all of the outcomes. If you haven't got a particular outcome, it might be something worth adding to your bound reference.

5.3 CAS Calculator Tips

These tips are written based on the TI-Nspire CAS calculators. However, the syntax and functions on other brands should be quite similar.

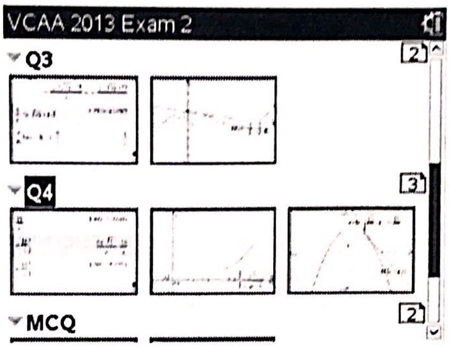
Radians and degrees

Firstly, always make sure that you set your calculator in the appropriate mode. It is safest to use radians at all times, then convert to degrees if a question specifically requires it. This is extremely important as trigonometric functions make up a large portion of the coursework and more importantly the exams.

Use problems for organising questions

The Ti-NSpire CAS has a "Problem" feature which allows you to access different workspaces. This is extremely helpful when dealing with multiple extended response questions as part of exam 2.

To insert a problem, Press Ctrl + Up to access an overview of all problems, then use Menu + 6 (Insert Problem) to add a new problem. Each problem acts as an independent workspace. This means that you define an $f(x)$ in Problem 1, and this will not affect or override any other $f(x)$ in any of the other problems. It also means that your saved graphs will also be different in different workspaces.



Take advantage of inbuilt features

You can speed through Exam 2 multiple choice if you know where certain functions are on your calculator.

For example, a tangent line and normal line can be calculated at a specific 'x' value automatically using Menu, 4: Calculus, 9: Tangent Line. This feature takes a function as its first argument, and an x value as its second argument. The CAS can also compute determinants, which are useful for finding solutions, and can also solve systems of equations.

As there is no working out necessary for Exam 2 multiple choice questions, the inbuilt functions on a CAS can help save time during multiple choice so you can tackle the other more demanding questions.

Know how to navigate

Spend plenty of time with your CAS so you can navigate the menus quickly. Want to access a binomial continuous distribution? You should be able to use Menu, 5, 5, E and that will instantly bring up the continuous distribution menu.

There are also some other keyboard shortcuts which may be useful:

- Integrals can be inserted using Shift + Plus
- Derivatives can be inserted using Shift + Minus
- The keyboard shortcuts that we're all used to on a computer can also be used. Ctrl + C and Ctrl + V can be used to copy maths from other areas of your workspace.

For those doing Further or Specialist Maths

Create template documents for all of your subjects with the correct settings enabled. For example, if you are doing Further Maths at the same time as Methods, it's a good idea to create a document called "Further Template" which has an empty Lists and Spreadsheet pages as well as the settings set in degrees with approximate values. You can save this empty document then open it up any time you do Further.

Then, you can create an empty document for a Methods template which uses radians as its angle measurement and uses exact values in calculations. You might want to add empty graphing pages onto that document as well. You can then save this empty document as "Methods Template" and open it up whenever you are doing Methods questions.

When you begin a practice exam, or enter the real exam for the subject, you can simply open up the template, and all the settings will be correct. You will also have graphing pages as well as List and Spreadsheet pages already created to work on.

5.4 General Exam Advice

Examination 1 Tips

The exam 1 for Methods comprises of 40 marks of short answer style questions. This exam is the most similar to the textbook questions, but there are often more difficult and time consuming questions towards the end. Exam 1 is calculator free.

You should aim to make the most of your reading time. You get 15 minutes reading time for both exam 1 and exam 2. 15 minutes reading time is a quarter of the writing time you get for exam 1. You may have enough time to plan your approach for the exam or even answer a few questions in your head before the writing begins.

The majority of exam 1 questions will feel quite similar to what you are used to seeing in a textbook or on a SAC. Therefore, I would prioritise smashing out the easiest questions first to get as many marks as possible, then tackling the hardest questions later. If you find yourself spending too much time on one question, stop and move on – there are more marks to be earned elsewhere.

Examination 2 Tips

The exam 2 for methods contains 80 available marks made up of 22 multiple choice questions worth one mark each, and 58 extended response style questions.

One major piece of advice I have is to avoid spending too much time on multiple choice. I'd recommend spending between 20 and 30 minutes on multiple choice. 30 minutes should be a maximum as this is one quarter of the time you have to solve questions. Although 22/80 marks is around one quarter of the marks, the questions later on in extended response are much more demanding.

Students often ask whether to begin exam 2 with the multiple choice section, or with the extended answer section. It is completely up to you. I personally chose to start with multiple choice, but I heard good arguments why you should start with extended response.

A common argument for starting with multiple choice is that it eases you into the exam and allows you to gain the easy marks earlier on. However, on the other hand, starting with extended response gives you more time and a less-stressful chance work on the more difficult questions – and you can guess multiple choice questions (and still get marks) if you run out of time.

General

- Read the questions very carefully. Take time in understanding what is required to solve the question, and how much working out you need to show.
- It's better to show more working out than less. Although you may understand how to answer a question in your head, you need to communicate this understanding to examiners in order to get the marks.
- If you are given a worded question, use a short sentence or statement to make your answer clear.
- Remember to show units if they are required in the question. Convert degrees to radians if necessary as well.
- Some answers only require a magnitude as an answer. For example, if you are asked to give the maximum depth, you can express your depth as a positive number. If a question asks for speed, you can remove a negative sign as speed is scalar.

Functions and Graphs

- Always label important points: axial intercepts, the axes, origin, sharp points, inflection points and turning points.
- Beware of open and closed circles. If you are sketching a graph with restrictions, make sure you double check to make sure your end points are included or not.
- When taking the inverse of a function, always write: "To find inverse swap x, and y"
- In exam 2, use a plot on your calculator before sketching graphs. This can give you a much better idea of what the shape of a graph looks like.
- When drawing addition of ordinates, use intercepts and intersection points as "easy" points to find out where the addition graph will lie.
- Solve for asymptotes on your calculator by letting the function equal $\frac{1}{0}$. This is extremely important as asymptotes do not show up on the graph screen.
- When solving for the intersection between a function and its inverse, it can often be quicker to find the intersection with $y = x$, as the inverse is simply a reflection over that line.
- Make sure that your inverse is a function if the question requires it. Use the horizontal line test on the normal function, and if it fails the horizontal line test then the inverse will fail the vertical line test.

Algebra

- The x-intercepts of a graph in factorised form are the opposite of what is in the brackets. The x intercepts of $p(x) = (x + a)(x + b)(x + c)$ are $x = -a$, $x = -b$ and $x = -c$.
- When dealing with composite functions, double check to make sure that it is defined.
- Remember that you cannot log or square a negative number. This will often eliminate solutions to equations.
- When asked for the inverse function, make sure to provide the rule as well as the domain.

Calculus

- A derivative will not be defined at endpoints, sharp points, cusps and points of discontinuity.
- Never forgot the $+C$ when asked to anti-differentiate. The only time you can omit this is if the question asks for "an antiderivative".
- Ensure you don't forget the dx or dy at the end of your integrals.
- Use brackets inside of your derivatives if there is addition and subtraction.
- A quick sketch when finding the area is always useful – this can ensure you're finding the positive area and breaking your integral where needed.
- A "use calculus" question means to show full working by hand. You cannot use your CAS to skip steps.
- A strictly increasing or decreasing function includes the endpoints or stationary points in a graph.
- Be creative when finding areas. It might be quicker to find the area under a line by using a trapezium formula or to double one side of a symmetrical region.

Probability

- Do not use calculator syntax as your working out. Calculator syntax varies between different calculator brands and companies.
- When sketching a probability function, you can draw a $y = 0$ line slightly above the axis so it is clear that you have drawn it.
- Beware of open and closed circles when sketching probability functions.
- Inequalities make a huge difference in probability. Be very careful with greater than, less than, at least etc.
- Sometimes approaching a probability question from the opposite direction can simplify working. For example, instead of finding $Pr(X \geq 1)$, you could find $Pr(X = 0)$, and then calculate $Pr(X \geq 1) = 1 - Pr(X = 0)$.

ATAR Notes

Maths Methods Units 3&4

COMPLETE COURSE NOTES

AMY, EMMANUEL COLLEGE
"the notes provided a concise overview of the course content, and going through them is a much faster and more efficient way of revising as opposed to rereading the textbook... it is almost as though a real person is explaining it to you – complex ideas are explained in simple terms, and this makes it much more inviting to read through."

LOUISE, HOPPERS CROSSING SECONDARY COLLEGE
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Published by InStudent Media Pty Ltd
ABN 26 601 049 852
91A Orrong Crescent
Caulfield North, Victoria, 3161
support@atarnotes.com
(03) 9916 7760

ISBN 978-1-925534-04-7