CAMBRIDGE
UNIVERSITY PRESS

10 & 10A

ESSENTIAL MATHEMATICS

FOR THE VICTORIAN CURRICULUM
SECOND EDITION

DAVID GREENWOOD
SARA WOOLLEY
JENNY GOODMAN
JENNIFER VAUGHAN
STUART PALMER

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org

First published 2016
Second Edition 2019
20 19 18 17 16 15 14 13 12 11 10 9 8 7 6

Cover and text designed by Loupe Studio
Typeset by diacriTech
Printed in China by C & C Offset Printing Co., Ltd.

A catalogue record for this book is available from the National Library of Australia at www.nla.gov.au

ISBN 978-1-108-77290-7 Paperback

Additional resources for this publication at www.cambridge.edu.au/GO

Cambridge University Press acknowledges the Australian Aboriginal and Torres Strait Islander peoples of this nation.
We acknowledge the traditional custodians of the lands on which our company is located and where we conduct our business. We pay our respects to ancestors and Elders, past and present. Cambridge University Press is committed to honouring Australian Aboriginal and Torres Strait Islander peoples' unique cultural and spiritual relationships to the land, waters and seas and their rich contribution to society.

Contents

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

About the authors

David Greenwood is the Head of Mathematics at Trinity Grammar School in Melbourne and has 25+ years' experience teaching mathematics from Years 7 to 12. He has run numerous workshops within Australia and overseas regarding the implementation of the Australian Curriculum and the use of technology for the teaching of mathematics. He has written more than 30 mathematics titles and has a particular interest in the sequencing of curriculum content and working with the Australian Curriculum proficiency strands.

Sara Woolley was born and educated in Tasmania. She completed an Honours degree in Mathematics at the University of Tasmania before completing her education training at the University of Melbourne. She has taught mathematics in Victoria from Years 7 to 12 since 2006 and is currently a Head of Mathematics. She has written more than 15 mathematics titles and specialises in lesson design and differentiation.

Jennifer Vaughan has taught secondary mathematics for over 30 years in New South Wales, Western Australia, Queensland and New Zealand and has tutored and lectured in mathematics at Queensland University of Technology. She is passionate about providing students of all ability levels with opportunities to understand and to have success in using mathematics. She has had extensive experience in developing resources that make mathematical concepts more accessible; hence, facilitating student confidence, achievement and an enjoyment of maths.

Jenny Goodman has taught in schools for over 25 years and is currently teaching at a selective high school in Sydney. Jenny has an interest in the importance of literacy in mathematics education, and in teaching students of differing ability levels. She was awarded the Jones Medal for education at Sydney University and the Bourke Prize for Mathematics. She has written for *CambridgeMATHS NSW* and was involved in the *Spectrum* and *Spectrum Gold* series.

Stuart Palmer has been a head of department in two schools and is now an educational consultant who conducts professional development workshops for teachers all over New South Wales and beyond. He is a Project Officer with the Mathematical Association of New South Wales, and also works with pre-service teachers at The University of Sydney and The University of Western Sydney.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Introduction

This second edition of *Essential Mathematics for the Victorian Curriculum* includes some substantial new features in the print and digital versions of the textbook, as well as in the Online Teaching Suite. The main new features are listed below.

Now you try

Every worked example now contains additional questions, without solutions, called 'Now you try'. We anticipate many uses of these questions, including giving students immediate practice at what they've seen in a worked example (instead of expecting students to simply 'absorb' the worked example by reading through it). We also anticipate these questions will be useful for the teacher to do in front of the class, given that students will not have seen the solution or answer beforehand.

Building understanding and changes to the exercise structure

To improve the flow of ideas from the beginning of each lesson through to the end of the exercise, a few structural changes have been made in each lesson. First, the Understanding questions have been taken out of the exercise, simplified into discussion-style questions, and placed immediately after the Key ideas. These questions are now called 'Building understanding' and are intended to consolidate the skills and concepts covered by the Key ideas, which students will then encounter in the worked examples. Each exercise now starts at Fluency, and the first question in each exercise has been revised to ensure that it links directly to the first worked example in the lesson. The exercise then continues as before through Problem-solving, Reasoning and Enrichment.

Learning intentions and Success criteria checklist

At the beginning of every lesson is a set of Learning intentions that describe what the student can expect to learn in the lesson. At the end of the chapter, these appear again in the form of a Success criteria checklist; students can use this to check their progress through the chapter. Every criterion is listed with an example question to remind students of what the mathematics mentioned looks like. These checklists can also be downloaded and printed off so that students can physically check them off as they accomplish their goals.

Applications and problem-solving

Included in each chapter is a set of three applications and problem-solving questions. These extended-response style problems apply the mathematics of the chapter to realistic contexts and provide important practice at this type of extended-response work before any final test is completed.

Workspaces and self-assessment

In the Interactive Textbook, students can complete almost any question from the textbook inside the platform via **workspaces**. Questions can be answered with full worked solutions using three input tools: 'handwriting' using a stylus, inputting text via a keyboard and in-built symbol palette, or uploading an image of work completed elsewhere. Then students can critically engage with their own work using the **self-assessment** tools, which allow them to rate their confidence with their work and also red-flag to the teacher any questions they have not understood. All work is saved, and teachers will be able to see both students' working-out and how they've assed their own work via the Online Teaching Suite.

Note that the workspaces and self-assessment feature is intended to be used as much or as little as the teacher wishes, including not at all. However, the ease with which useful data can be collected will make this feature a powerful teaching and learning tool when used creatively and strategically.

ISBN 978-1-108-77290-7

Guide to the working programs

As with the first edition, *Essential Mathematics for the Victorian Curriculum Second Edition* contains working programs that are subtly embedded in every exercise. The suggested working programs provide three pathways through each book to allow differentiation for Foundation, Standard and Advanced students.

Each exercise is structured in subsections that match the Australian Curriculum proficiency strands of Fluency, Problem-solving and Reasoning, as well as Enrichment (Challenge). (Note that Understanding is now covered by 'Building understanding' in each lesson.) In the exercises, the questions suggested for each pathway are listed in three columns at the top of each subsection:

- The left column (lightest shaded colour) is the Foundation pathway
- The middle column (medium shaded colour) is the Standard pathway
- The right column (darkest shaded colour) is the Advanced pathway.

	Foundation	Standard	Advanced
FLUENCY	1, 2–4(½)	2–5(½)	2–5(½)
PROBLEM-SOLVING	6, 7	6–8	7–9
REASONING	10	10–12	12–14
ENRICHMENT	–	–	15

The working program for Exercise 3A in Year 7

Gradients within exercises and proficiency strands

The working programs make use of the gradients that have been seamlessly integrated into the exercises. A gradient runs through the overall structure of each exercise – where there is an increasing level of mathematical sophistication required from Fluency through to Reasoning and Enrichment – but also within each proficiency strand; the first few questions in Fluency, for example, are easier than the last few, and the last Problem-solving question is more challenging than the first Problem-solving question.

The right mix of questions

Questions in the working programs are selected to give the most appropriate mix of *types* of questions for each learning pathway. Students going through the Foundation pathway should use the left tab, which includes all but the hardest Fluency questions as well as the easiest Problem-solving and Reasoning questions. An Advanced student can use the right tab, proceed through the Fluency questions (often half of each question), and have their main focus be on the Problem-solving and Reasoning questions, as well as the Enrichment questions. A Standard student would do a mix of everything using the middle tab.

Choosing a pathway

There are a variety of ways to determine the appropriate pathway for students through the course. Schools and individual teachers should follow the method that works for them. If required, there are two types of chapter pre-tests (now found online) that can be used as a tool for helping students select a pathway. For the prior-knowledge pre-test, the following are recommended guidelines:

- A student who gets 40% or lower should complete the Foundation questions
- A student who gets between 40% and 85% should complete the Standard questions
- A student who gets 85% or higher should complete the Advanced questions.

For schools that have classes grouped according to ability, teachers may wish to set one of the Foundation, Standard or Advanced pathways as their default setting for their entire class and then make individual alterations depending on student need. For schools that have mixed-ability classes, teachers may wish to set a number of pathways within the one class, depending on previous performance and other factors.

* The nomenclature used to list questions is as follows:

- 3, 4: complete all parts of questions 3 and 4
- 1-4: complete all parts of questions 1, 2, 3 and 4
- 10(½): complete half of the parts from question 10 (a, c, e, ... or b, d, f, ...)
- 2-4(½): complete half of the parts of questions 2, 3 and 4
- 4(½), 5: complete half of the parts of question 4 and all parts of question 5
- — : do not complete any of the questions in this section.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Guide to this resource

PRINT TEXTBOOK FEATURES

1 **Victorian Curriculum:** content strands, sub-strands and content descriptions are listed at the beginning of the chapter (see the teaching program for more detailed curriculum documents)

2 **In this chapter:** an overview of the chapter contents

3 **Working with unfamiliar problems:** a set of problem-solving questions not tied to a specific topic

4 **Chapter introduction:** sets context for students about how the topic connects with the real world and the history of mathematics

5 NEW **Learning intentions:** sets out what a student will be expected to learn in the lesson

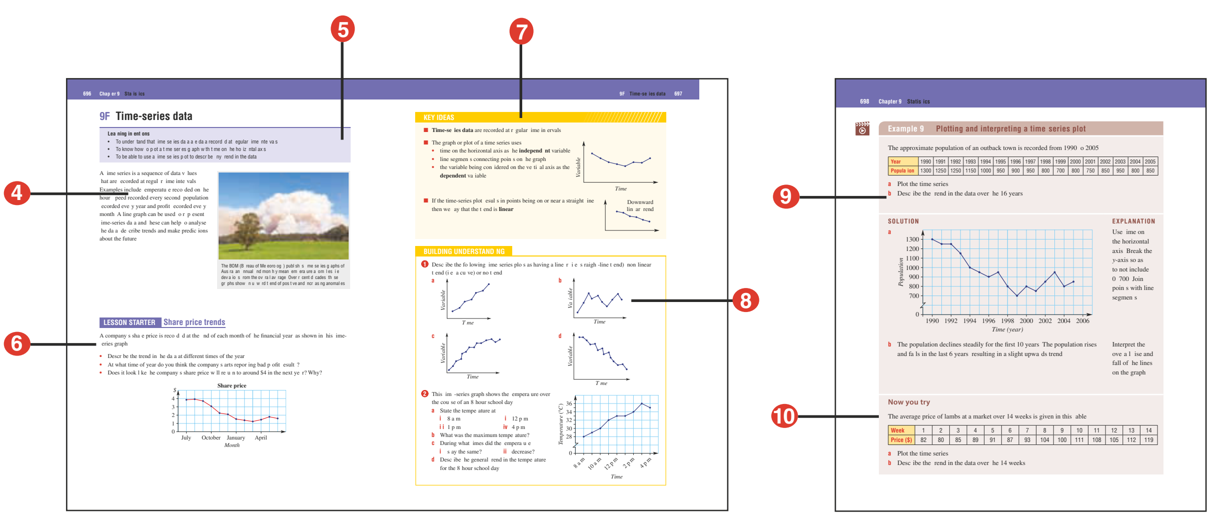
6 **Lesson starter:** an activity, which can often be done in groups, to start the lesson

7 **Key ideas:** summarises the knowledge and skills for the lesson

8 NEW **Building understanding:** a small set of discussion questions to consolidate understanding of the Key ideas (replaces Understanding questions formerly inside the exercises)

9 **Worked examples:** solutions and explanations of each line of working, along with a description that clearly describes the mathematics covered by the example

10 NEW **Now you try:** try-it-yourself questions provided after every worked example in exactly the same style as the worked example to give immediate practice



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

11 Revised exercise structure: the exercise now begins at Fluency, with the first question always linked to the first worked example in the lesson.

12 Working programs: differentiated question sets for three ability levels in exercises

13 Example references: show where a question links to a relevant worked example – the first question is always linked to the first worked example in a lesson

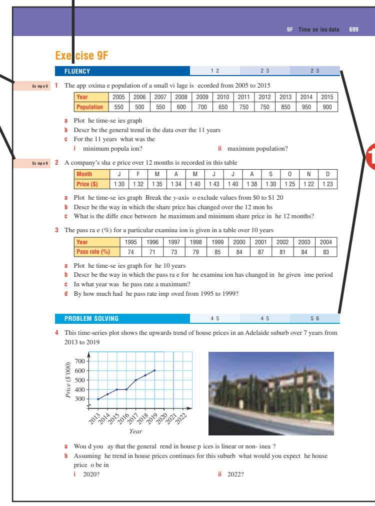
14 Problems and challenges: in each chapter provide practice with solving problems connected with the topic

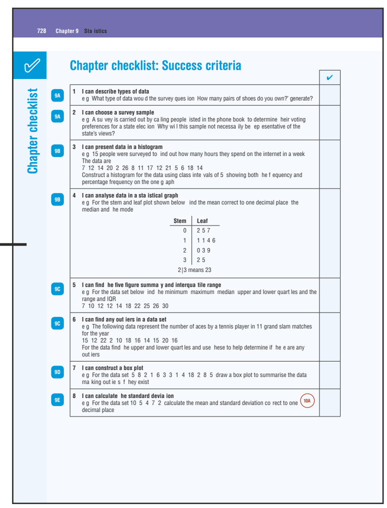
15 NEW **Success criteria checklist:** a checklist of the learning intentions for the chapter, with example questions

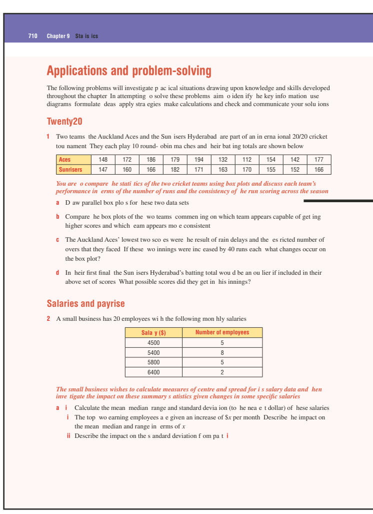
16 NEW **Applications and problem-solving:** a set of three extended-response questions across two pages that give practice at applying the mathematics of the chapter to real-life contexts

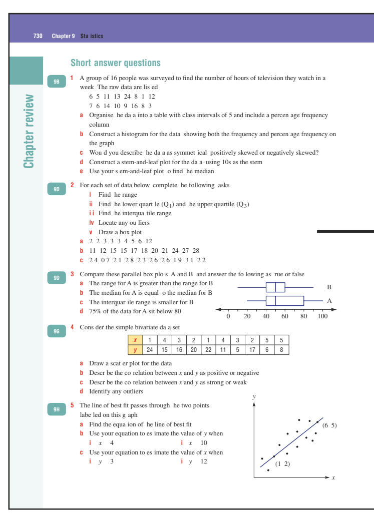
17 Chapter reviews: with short-answer, multiple-choice and extended-response questions; questions that are extension or 10A (at Year 10) are clearly signposted

18 Solving unfamiliar problems poster: at the back of the book outlines a strategy for solving any unfamiliar problem









ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

INTERACTIVE TEXTBOOK FEATURES

19 NEW **Workspaces:** almost every textbook question – including all working-out – can be completed inside the Interactive Textbook by using either a stylus, a keyboard and symbol palette, or uploading an image of the work.

20 NEW **Self-assessment:** students can then self-assess their own work and send alerts to the teacher. See the Introduction on page ix for more information

21 **Interactive question tabs** can be clicked on so that only questions included in that working program are shown on the screen

22 **HOTmaths resources:** a huge catered library of widgets, HOTsheets and walkthroughs seamlessly blended with the digital textbook

23 A revised set of **differentiated auto-marked practice quizzes** per lesson with saved scores

24 **Scorcher:** the popular competitive game

25 **Worked example videos:** every worked example is linked to a high-quality video demonstration, supporting both in-class learning and the flipped classroom

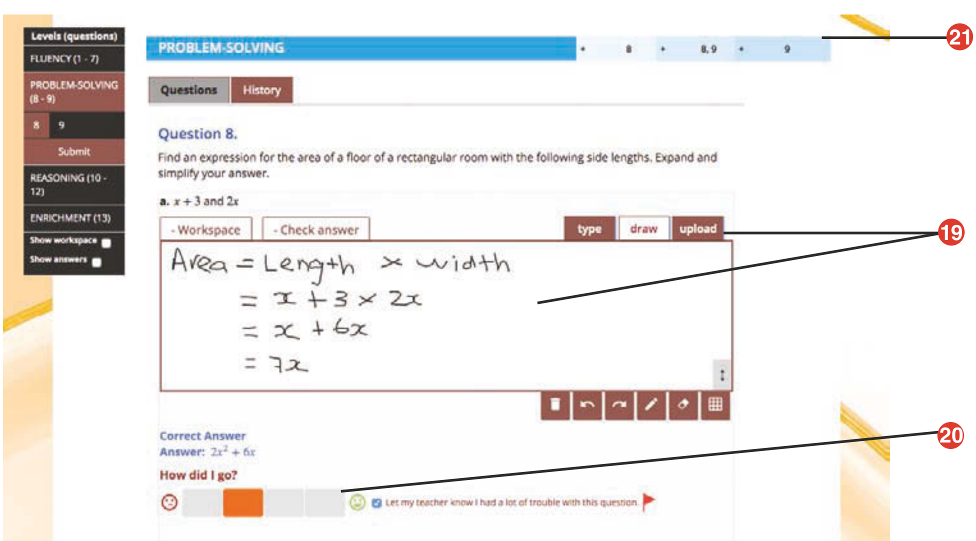
26 **Desmos graphing calculator**, scientific calculator and geometry tool are always available to open within every lesson

27 **Desmos interactives:** a set of Desmos activities written by the authors allow students to explore a key mathematical concept by using the Desmos graphing calculator or geometry tool

28 **Auto-marked prior knowledge pre-test** for testing the knowledge that students will need before starting the chapter

29 NEW **Auto-marked diagnostic pre-test** for setting a baseline of knowledge of chapter content

30 **Auto-marked progress quizzes and chapter review multiple-choice questions** in the chapter reviews can now be completed online

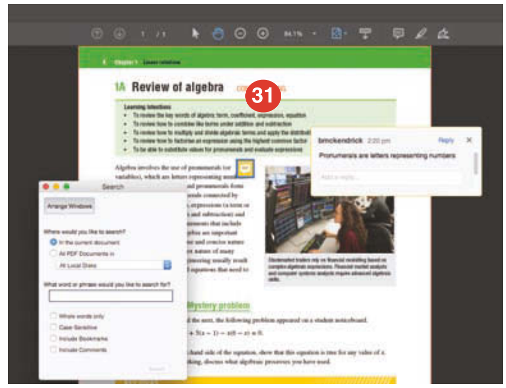


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

DOWNLOADABLE PDF TEXTBOOK

31 In addition to the Interactive Textbook, a **PDF version of the textbook** has been retained for times when users cannot go online. PDF search and commenting tools are enabled.



ONLINE TEACHING SUITE

32 **Learning Management System** with class and student analytics, including reports and communication tools

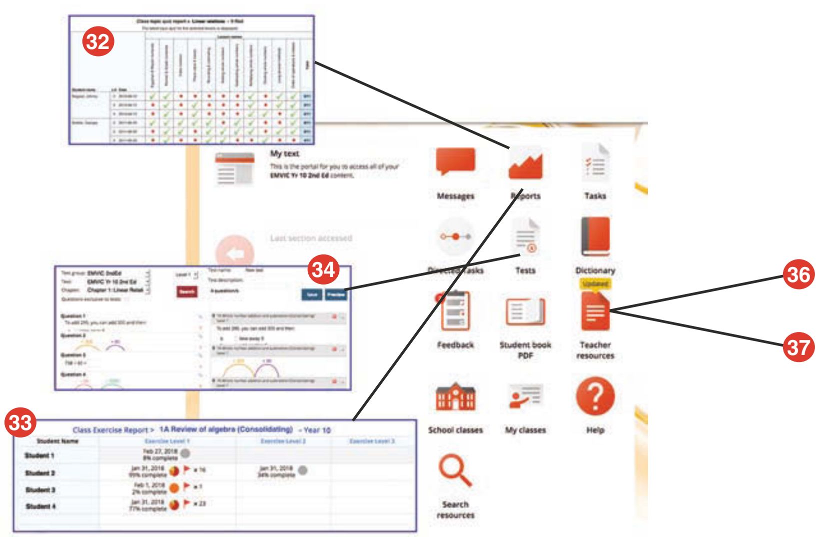
33 NEW **Teacher view of student's work and self-assessment** allows the teacher to see their class's workout, how students in the class assessed their own work, and any 'red flags' that the class has submitted to the teacher

34 **Powerful test generator** with a huge bank of levelled questions as well as ready-made tests

35 NEW **Revamped task manager** allows teachers to incorporate many of the activities and tools listed above into teacher-controlled learning pathways that can be built for individual students, groups of students and whole classes.

36 **Worksheets and four differentiated chapter tests in every chapter**, provided in editable Word documents

37 NEW **More printable resources:** all Pre-tests, Progress quizzes and Applications and problem-solving tasks are provided in printable worksheet versions



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Acknowledgements

The author and publisher wish to thank the following sources for permission to reproduce material:

Cover: © Getty Images / Mandy Disher Photography

Images: © Getty Images / Paul Gilham, Chapter 1 Opener / Tetra Images, 1A (1) / haydenbird, 1A (2) / fstop123, 1B (1) / aydinmutlu, 1C (1) / Petar Chernaev, 1D (1) / InkkStudios, 1D (2) / Terry Vine, 1E (1) / Ascent / PKS Media Inc., 1E (2) / JackF, 1F (1) / Robert Lang Photography, 1F (2) / Juan Napurí, 1F (3) / Mint Images, 1F (4) / Nullplus, 1G (1) / Paul Viant, 1G (2) / Ponsulak Kunsub, 1G (3) / Monty Rakusen, 1H (1) / Ulrike Schmitt-Hartmann, 1H (2) / ArtyFree, 1I (2) / Adam Hester, 1I (3) / Morten Falch Sortland, 1I (4) / John A. Beatty, 1I (5) / Wavebreakmedia Ltd, 1J (1) / Abraham López, 1J (2) / Monty Rakusen, 1K (1) / Stephanie Hager – HagerPhoto, 1K (2) / andresr, 1L (2) / egon69, 1L (3) / Hero Images, 1M (1) / Lkpgfoto, 1M (2) / Paul Bradbury, 1M (3) / Hero Images, 1M (4) / Jenny Dettrick, 1M (5) / Michael DeYoung, 1M (6) / Ryan Evans, Chapter 2 Opener / CLU, 2A (1) / Glowimages, 2A (2) / PIER, 2B (3) / Alexandra Ribeiro, 2B (2) / Jose A. Bernat Bacete, 2C (2) / Ana Maria Perez Leal, 2D (3) / Mahamood Razi, 2E (2) / Yuko Yamada, 2E (3) / W. Prasongsin, 2H (2) / Martin Barraud, 2H (3), 2D (1) / Clu E+, 2I (2) / hudiemm, 2J (1) / Ricardo Cortes – Cameron, 2B (1) / cdwheatley, 2C (1) / Zetter, 2D (1) / David De Lossy, 2E (1) / bernie_photo, 2F (2) / saoirse_2010, 2G (1) / Jmichi, 2H (1) / bjdlzx, 2I (1) / Peter Zelei Images, 2H (4) / Robert Cianflone, Chapter 3 Opener / Francesco Scatena, 3E (1) / Simply Creative Photography E+, 3E (2) / Geoff Dann, 3F (2) / Logan Campbell, 3J (2) / Astrakan Images, 3J (3) / Sunti Phuangphila, 3J (4) / Daniel Balogh, 3K (2) / Pedro Venncio, 3K (3) / Rosemary Calvert, 3K (4) / tomprout E+, 3M (2) / Gwenvidig, 3A (1) / Oxygen, 3A (2) / Lya_Cattel, 3B (1) / sturti, 3C (1) / aytugaskin, 3D (1) / Media photos, 3F (1) / Kateryna / SPL, 3G (1) / Stefanie Keller, 3H (1) / Ian Hitchcock, 3I (1) / Radius Images, 3J (1) / luoman, 3K (1) / gorodenkoff, 3M (1) / STR / Stringer / AFP, Chapter 4 Opener / 35007, 4A (1) / Robert Downer, 4B (1) / guvendemir, 4C (1) / Mark Stevenson/Stocktrek Images, 4C (2) / Vitali Kozhar, 4D (2) / Sylvain Bouzat, 4D (3) / mbbirdy, 4D (4) / pukrufus, 4E (2) / Matt Heal, 4E (4) / koto_feja, 4F (2) / Lori Andrews, 4G (2) / Jordan Siemens, 4G (3) / Alexander Spatari, 4F (3) / Image Source, 4F (4) / Bastian Sander, 4G (1) / THPStock, 4H (1) / Lizzie Roberts, 4I (1) / Roger Harris / SPL, 4K (3) / ilbusca, 4K (4) / d3sign, 4K (5) / Kieran Stone, 4K (6) / japatino, 4D (1) / Christian Lagerek / SPL, 4E (1) / Foto Bureau Nz Limited, 4E (4) / choja, 4F (1) / Julian Finney, 4J (1) / Petmal, 4K (1) / NurPhoto, Chapter 5 Opener / kurga, 5A (1) / RobertCrum, 5B (1) / Yuri_Arcurs, 5C (1) / Oksana Slepcov, 5G (3) / Image Studios UpperCut Images, 5G (4) / mseidelch, 5G (5) / People:Ama Agbeze, NurPhoto, 5G (6) / Lorianne Ende, 5G (2) / Construction Photography / Avalon, 5I (2), Chapter 6 Opener / gregobagel, 5D (1) / oticki, 5E (1) / Chris Kridler, 5G (1) / Leonid Andronov, 5H (1) / Andrew Merry, 5I (1) / Photo Talk, 6A (1) / I love Photo and Apple, 6A (2) / Cavan Images, 6A (3) / Hisham Ibrahim, 6B (2) / R. Piyasirisorost, 6B (3) / PASIEKA, 6B (1) / gwiggington, 6C (1) / Noppawat Tom Charoensinphon, 6D (1) / Monty Rakusen, 6D (2) / domin_domin, 6E (2) / Richard Drury, 6E (3) / Pornpimon Lekudom, 6F (2) / skaman306, 6G (2) / Tetra Images, 6G (3) / Krisana Antharith, 6H (2) / William Turner, 6H (3) / Jason Droege, 6I (2) / dem10, 6I (3) / Westend61, 6I (5) / Samere Fahim Photography, 6I (4) / Pbombaer, 6E (1) / Nordroden, 6F (1) / mikedabell, 6G (1) / HAYKIRDI, 6H (1) / Pete Saloutos, 6I (1) / Auscape, Chapter 7 Opener / Mark Williamson, 7A (2) / Thomas Barwick, 7B (2) / Jesse Chance, 7E (1) / technotr, 7B (1) / DenisTangneyJr, 7C (1) / Westend61, 7D (1) / Rich Jones, 7F (1) / Jochen Schlenker, 7F (2) / Paul Starns, 7F (3) / Paul Downing, 7G (1) / Alex Bamford, 7G (2) / Jesse Schwab, 7G (3) / Zvonimir Atletic, 7G (4) / anucha sirivisansuwan, 7I (2) / samvaltenbergs, 7K (3) / mphillips007, 7K (1) / Juanma Domínguez, 7K (2) / Saudi Desert Photos by TARIQ-M, 7K (4) / Flickr Moment Open, 7K (5) / rusm, 7H (1) / Ipopba, 7I (1) / Victor De Schwanberg, 7J (1) / Aytug Askin, 7J (2) / naulicreative, 7L (1) / Sean Gladwell, 7L (2) / Bletchley Park Trust, Chapter 8 Opener / TF-Images, 8A (1) / Tetra Images, 8A (2) / Monty Rakusen, 8A (3) / Tetra Images – fotog Brand X Pictures, 8B (2) / Witthaya Prasongsin, 8B (3) / Alistair Berg, 8B (4) / Cyndi Monaghan, 8B (5) / Hashaun Adderley, 8C (2) / MarkGillow, 8D (2) / duard Hupbutdinov, 8D (3) / track5, 8E (2) / Francesco Nacchia, 8E (4) / Martin Barraud, 8E (5) / Katarzyna Bialasiewicz, 8F (3) / Zero Creatives, 8E (3) / Ahmẽd HaẒẽm, 8F (1) / gilaxia, 8F (2) / simonkr, 8F (4) / Joni_R, 8G (1) / Tetra Images, 8G (2) / Hinterhaus Productions, 8G (3) / Busakorn Pongparnit, 8G (4) / efenzi, 8G (5) / Albasir Canizares, 8G (6) / Dmytro Aksonov, 8B (1) / Django, 8C (1) / Hero Images, 8D (1) / KenWiedemann, 8E (1) / Colin Hawkins, 8E (6) / Mark Kolbe, Chapter 9 Opener / Raylipscombe, 9A (1) / Clare Jackson, 9A (2) / kasayizgi, 9A (3) / sanjeri, 9B (1) / Jens Daffner, 9B (2) / ZekaG, 9B (3) / Jennifer A. Smith, 9C (3) / Yann Guichaoua, 9D (2) / Nora Carol, 9D (3) / JGI / Jamie Grill, 9E (2) / Amer Ghazzal, 9F (2) / Nazar Abbas, 9F (3) / NicoElNino, 9G (1) / d3sign, 9G (2) / Guido Mieth, 9G (3) / 10'000 Hours, 9G (4) / Barry Chan, 9G (5) / MB Photography, 9H (2) / Guido Mieth, 9H (3) / Suriyo Hmun Kaew, 9I (2) / dstellerm, 9I (3) / ismagilov, 9I (4) / South_agency, 9I (5) / Virginie Blanquart, 9I (6) / Sturti, 9C (1) / Supoj Buranaprapapong, 9C (2) / Tasmin Brown, 9D (1) / Wavebreakmedia, 9E (1) / ZambeziShark, 9F (1) / Matthias Tunger, 9H (1) / graytln, 9I (1) / Al Bello, Chapter 10 Opener / Adrian Wojcik, 10A (1) / jamesbenet, 10A (2) / Maskot, 10F (2) / Martin Koller, 10F (3) / Hero Images, Chapter 10, Semester review / Westend61, 10B (1) / ihoe, 10B (2) / andresr, 10C (1) / I love nature, 10C (2) / Rubberball, 10D (1) / Ariel Skelley, 10E (1) / yangong, 10F (1) / Derrek, 10G (1) / S. Greg Panosian, 10H (1) / Andrey Suslov, 10I (1) / Ray Lego, Chapter 11 Opener / Rafal Rodzoch, Chapter 11, Intro / Randrey, Chapter 11, 2.3; © Duk / Creative Commons Attribution-Share Alike 3.0 Unported license, 7A (1); © C. J. Mozzochi, Princeton N.J / Wikimedia Commons, 5B (2); © Phanie / Alamy, 1L (1); © Warner Bros. / Ronald Grant Archive / Alamy, 2F (1); U S. Department of Defense, Public Domain, 5F (1).

Every effort has been made to trace and acknowledge copyright. The publisher apologises for any accidental infringement and welcomes information that would redress this situation.

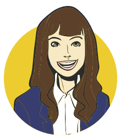
VCAA – The Victoria Curriculum F–10 content elements are © VCAA, reproduced by permission. Victoria Curriculum F–10 elements are accurate at the time of publication. The VCAA does not endorse or make any warranties regarding this resource. The Victorian Curriculum F–10 and related content can be accessed directly at the VCAA website. You can find the most up to date version of the Victoria Curriculum at https://victoriancurriculum.vcaa.vic.edu.au/.

Working with unfamiliar problems: Part 1

The questions on the next four pages are designed to provide practice in solving unfamiliar problems. Use the 'Working with unfamiliar problems' poster at the back of this book to help you if you get stuck.

In Part 1, apply the suggested strategy to solve these problems, which are in no particular order. Clearly communicate your solution and final answer.

For Question 1, try looking for number patterns and algebraic patterns.



1 Discover the link between Pascal's triangle and expanded binomial products and use this pattern to help you expand $(x + y)^6$.

Pascal's triangle

$(x + y)^0$				1			
$(x + y)^1$			1		1		
$(x + y)^2$		1		2		1	
$(x + y)^3$	1		3		3		1

For Questions 2 and 3, try making a list or table.

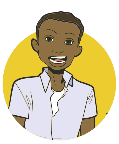
2 How many palindromic numbers are there between 10^1 and 10^3?

3 Find the smallest positive integer values for x so that $60x$ is:

a a perfect square

b a perfect cube

c divisible by both 8 and 9.



For Questions 4–8, try drawing a diagram to help you visualise the problem.

4 A Year 10 class raises money at a fete by charging players \$1 to flip their dollar coin onto a red and white checked tablecloth with 50 mm squares. If the dollar coin lands fully inside a red square the player keeps their \$1. What is the probability of keeping the \$1?
How much cash is likely to be raised from 64 players?

5 The shortest side of a 60° set square is 12 cm. What is the length of the longest side of this set square?

6 A Ferris wheel with diameter 24 metres rotates at a constant rate of 60 seconds per revolution.

a Calculate the time taken for a rider to travel:

i from the bottom of the wheel to 8 m vertically above the bottom

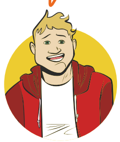
ii from 8 m to 16 m vertically above the bottom of the wheel.

b What fraction of the diameter is the vertical height increase after each one-third of the ride from the bottom to the top of the Ferris wheel?

ISBN 978-1-108-77290-7

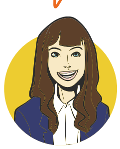
Cambridge University Press
Updated June 2022

For Question 9, try to break up the numbers to help simplify.



For Question 10, try to set up an equation

For Questions 11–13, try using algebra as a tool to work out the unknowns.



For Questions 14 and 15, try using concrete, everyday materials to help you understand the problem.

7 $ABCD$ is a rectangle with $AB = 16$ cm and $AD = 12$ cm. X and Y are points on BD such that AX and CY are each perpendicular to the diagonal BD. Find the length of the interval XY.

8 How many diagonal lines can be drawn inside a decagon (i.e. a 10-sided polygon)?

9 The symbol ! means factorial.
e.g. $4! = 4 \times 3 \times 2 \times 1 = 24$.
Simplify $9! \div 7!$ without the use of a calculator.

10 In 2017 Charlie's age is the sum of the digits of his birth year $19xy$ and Bob's age is one less than triple the sum of the digits of his birth year $19yx$. Find Charlie's age and Bob's age on their birthdays in 2017.

11 Let D be the difference between the squares of two consecutive positive integers. Find an expression for the average of the two integers in terms of D.

12 For what value of b is the expression $15ab + 6b - 20a - 8$ equal to zero for all values of a?

13 Find the value of k given $k > 0$ and that the area enclosed by the lines $y = x + 3, x + y + 5 = 0, x = k$ and the y-axis is 209 units^2.

14 The diagonal of a cube is $\sqrt{27}$ cm. Calculate the volume and surface area of this cube.

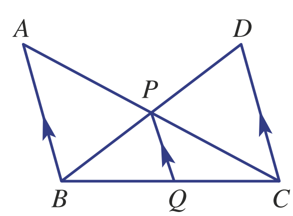
15 Two sides of a triangle have lengths 8 cm and 12 cm, respectively. Determine between which two values the length of the third side would fall. Give reasons for your answer.

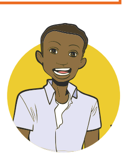
16 When $10^{89} - 89$ is expressed as a single number, what is the sum of its digits?

17 Determine the reciprocal of this product: $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) \dots \left(1 - \frac{1}{n+1}\right)$.

18 Find the value of $\dfrac{1002^2 - 998^2}{102^2 - 98^2}$, without using a calculator.

19 In the diagram at right, $AP = 9$ cm, $PC = 15$ cm, $BQ = 8.4$ cm and $QC = 14$ cm. Also, $CD \parallel QP \parallel BA$. Determine the ratio of the sides AB to DC.





For Questions 16–19, try using a mathematical procedure to find a shortcut to the answer.

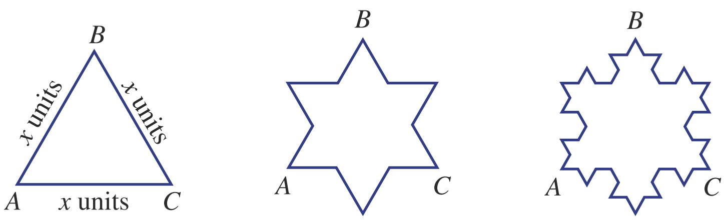
ISBN 978-1-108-77290-7

Part 2

Working with unfamiliar problems: Part 2

For the questions in Part 2, again use the 'Working with unfamiliar problems' poster at the back of this book, but this time choose your own strategy (or strategies) to solve each problem. Clearly communicate your solution and final answer.

1 The Koch snowflake design starts with an equilateral triangle. A smaller equilateral triangle is built onto the middle third of each side and its base is erased. This procedure can be repeated indefinitely.



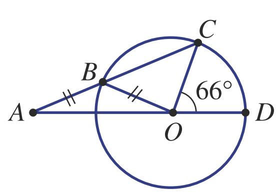
a For a Koch snowflake with initial triangle side length x units, determine expressions for the exact value of:
  i the perimeter after 5 procedure repeats and after n procedure repeats
  ii the sum of areas after 3 procedure repeats and the *change* in area after n procedure repeats.

b Comment on perimeter and area values as $n \to \infty$. Give reasons for your answers.

2 Two sides of a triangle have lengths in the ratio 3 : 5 and the third side has length 37 cm. If each side length has an integer value, find the smallest and largest possible perimeters, in cm.

3 The midpoints of each side of a regular hexagon are joined to form a smaller regular hexagon with side length k cm. Determine a simplified expression in terms of k for the exact difference in the perimeters of the two hexagons.

4 Angle COD is 66°. Find the size of angle CAD.



5 The graph of $y = ax^2 + 2x + 3$ has an axis of symmetry at $x = \frac{1}{4}$. Determine the maximum possible value of y.

6 Find the value of x and y given that $5^x = 125^{y-3}$ and $81^{x+1} = 9^y \times 3$.

7 A rectangular prism has a surface area of 96 cm^2 and the sum of the lengths of all its edges is 64 cm. Determine the exact sum of the lengths of all its internal diagonals (i.e. diagonals not on a face).

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

8 In a Year 10 maths test, six students gained 100%, all students scored at least 75% and the mean mark was 82.85%. If the results were all whole numbers, what is the smallest possible number of students in this class? List the set of results for this class size.

9 Determine the exact maximum vertical height of the line $y = 2x$ above the parabola $y = 2x^2 - 5x - 3$.

10 $A + B = 6$ and $AB = 4$. Without solving for A and B, determine the values of:

a $(A + 1)(B + 1)$ **b** $A^2 + B^2$ **c** $(A - B)^2$ **d** $\frac{1}{A} + \frac{1}{B}$

11 If $f(1) = 5$ and $f(x + 1) = 2f(x)$, determine the value of $f(8)$.

12 Four rogaining markers, $PQRS$, are in an area of bushland with level ground. Q is 1.4 km east of P, S is 1 km from P on a true bearing of 168° and R is 1.4 km from Q on a true bearing of 200°. To avoid swamps, Lucas runs the route $PRSQP$. Calculate the distances (in metres) and the true bearings from P to R, from R to S, from S to Q and from Q to P. Round your answers to the nearest whole number.

13 Consider all points (x, y) that are equidistant from the point $(4, 1)$ and the line $y = -3$. Find the rule relating x and y and then sketch its graph, labelling all significant features. (Note: Use the distance formula.)

14 A 'rule of thumb' useful for 4WD beach driving is that the proportion of total tide height change after either high or low tide is $\frac{1}{12}$ in the first hour, $\frac{2}{12}$ in the second hour, $\frac{3}{12}$ in the third hour, $\frac{3}{12}$ in the fourth hour, $\frac{2}{12}$ in the fifth hour and $\frac{1}{12}$ in the sixth hour.

a Determine the accuracy of this 'rule of thumb' using the following equation for tide height: $h = 0.7 \cos(30t) + 1$, where h is in metres and t is time in hours after high tide.

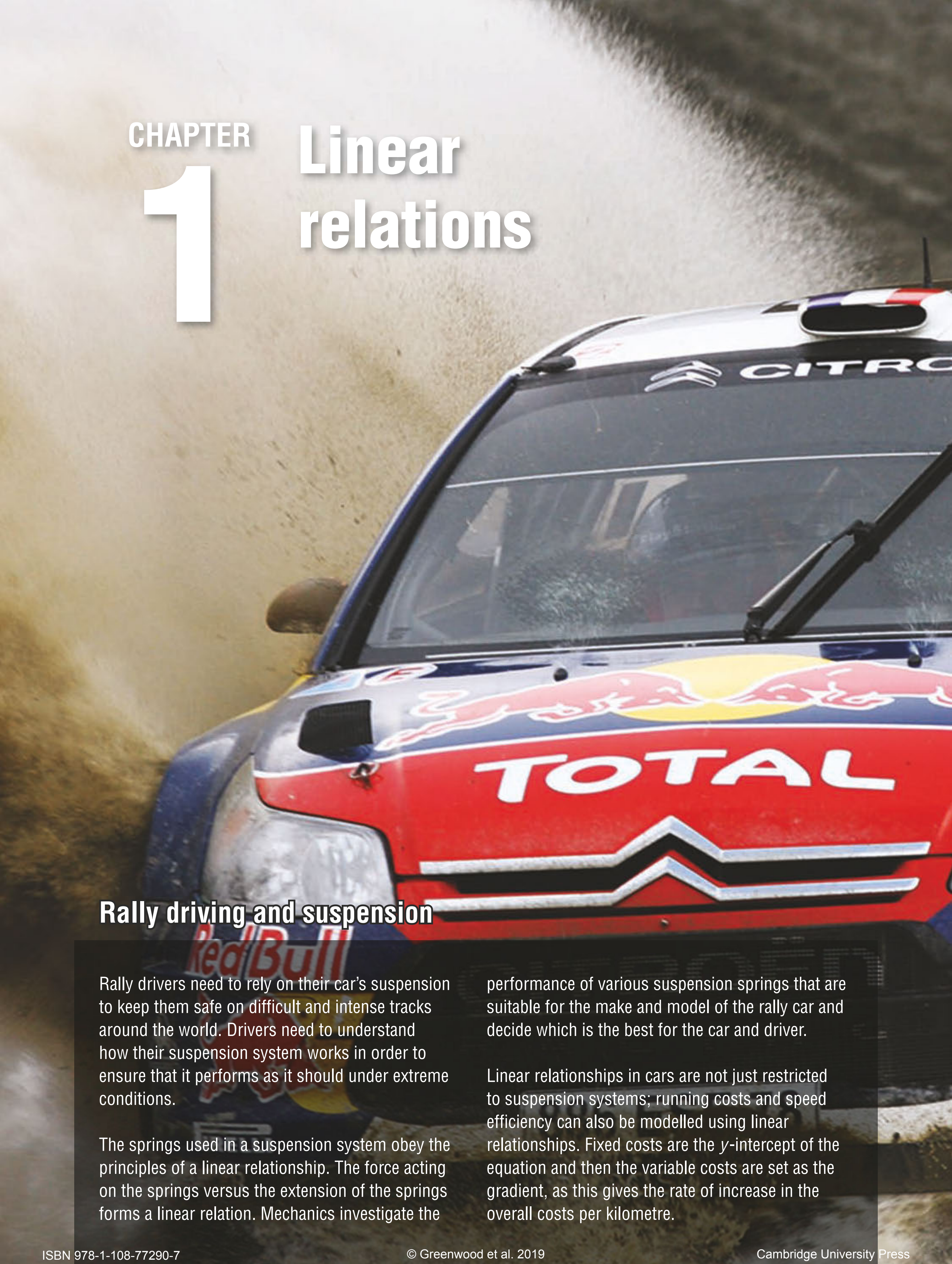
b Using $h = A \cos(30t) + D$, show that the proportion of total tide height change between any two given times, t_1 and t_2, is independent of the values of A and D.

15 All Golden Rectangles have the proportion $L : W = \Phi : 1$ where Φ (phi) is the Golden Number. Every Golden Rectangle can be subdivided into a square of side W and a smaller Golden Rectangle. Calculate phi as an exact number and also to six decimal places.

ISBN 978-1-108-77290-7 Cambridge University Press
 Updated June 2022

CHAPTER 1

Linear relations



Rally driving and suspension

Rally drivers need to rely on their car's suspension to keep them safe on difficult and intense tracks around the world. Drivers need to understand how their suspension system works in order to ensure that it performs as it should under extreme conditions.

The springs used in a suspension system obey the principles of a linear relationship. The force acting on the springs versus the extension of the springs forms a linear relation. Mechanics investigate the performance of various suspension springs that are suitable for the make and model of the rally car and decide which is the best for the car and driver.

Linear relationships in cars are not just restricted to suspension systems; running costs and speed efficiency can also be modelled using linear relationships. Fixed costs are the y-intercept of the equation and then the variable costs are set as the gradient, as this gives the rate of increase in the overall costs per kilometre.

ISBN 978-1-108-77290-7

Online resources

A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

In this chapter

Victorian Curriculum

NUMBER AND ALGEBRA
Patterns and algebra

Factorise algebraic expressions by taking out a common algebraic factor (VCMNA329)

Apply the four operations to simple algebraic fractions with numerical denominators (VCMNA331)

Substitute values into formulas to determine an unknown and re-arrange formulas to solve for a particular term (VCMNA333)

Linear and non-linear relationships

Solve problems involving linear equations, including those derived from formulas (VCMNA335)

Solve linear inequalities and graph their solutions on a number line (VCMNA336)

Solve simultaneous linear equations, using algebraic and graphical techniques including using digital technology (VCMNA337)

Solve problems involving gradients of parallel and perpendicular lines (VCMNA338)

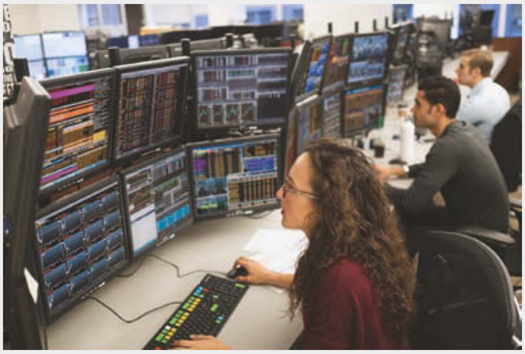
Solve linear equations involving simple algebraic fractions (VCMNA340)

1A Review of algebra CONSOLIDATING

Learning intentions

- To review the key words of algebra: term, coefficient, expression, equation
- To review how to combine like terms under addition and subtraction
- To review how to multiply and divide algebraic terms and apply the distributive law to expand brackets
- To review how to factorise an expression using the highest common factor
- To be able to substitute values for pronumerals and evaluate expressions

Algebra involves the use of pronumerals (or variables), which are letters representing numbers. Combinations of numbers and pronumerals form terms (numbers and pronumerals connected by multiplication and division), expressions (a term or terms connected by addition and subtraction) and equations (mathematical statements that include an equals sign). Skills in algebra are important when dealing with the precise and concise nature of mathematics. The complex nature of many problems in finance and engineering usually result in algebraic expressions and equations that need to be simplified and solved.



Stockmarket traders rely on financial modelling based on complex algebraic expressions. Financial market analysts and computer systems analysts require advanced algebraic skills.

LESSON STARTER Mystery problem

Between one school day and the next, the following problem appeared on a student noticeboard.

Prove that $8 - x^2 + \dfrac{3x - 9}{3} + 5(x - 1) - x(6 - x) = 0$.

- By working with the left-hand side of the equation, show that this equation is true for any value of x.
- At each step of your working, discuss what algebraic processes you have used.

KEY IDEAS

- Key words in algebra:
  - **term:** $5x$, $7x^2y$, $\dfrac{2a}{3}$, 7 (a constant term)
  - **coefficient:** -3 is the coefficient of x^2 in $7 - 3x^2$; 1 is the coefficient of y in $y + 7x$.
  - **expression:** $7x$, $3x + 2xy$, $\dfrac{x + 3}{2}$, $\sqrt{2a^2 - b}$
  - **equation:** $x = 5$, $7x - 1 = 2$, $x^2 + 2x = -4$
- Expressions can be evaluated by substituting a value for each pronumeral (variable).
  - Order of operations are followed: First brackets, then indices, then multiplication and division, then addition and subtraction, working then from left to right.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

- **Like terms** have the same pronumeral part and, using addition and subtraction, can be collected to form a single term.
  For example, $3x - 7x + x = -3x$
  $6a^2b - ba^2 = 5a^2b$
  Note that $a^2b = ba^2$

- The symbols for multiplication (×) and division (÷) are usually not shown.
  $7 \times x \div y = \frac{7x}{y}$
  $-6a^2b \div (ab) = \frac{-6a^2b}{ab}$
  $= -6a$

- The **distributive law** is used to expand brackets.
  - $a(b + c) = ab + ac$ $2(x + 7) = 2x + 14$
  - $a(b - c) = ab - ac$ $-x(3 - x) = -3x + x^2$

- **Factorisation** involves writing expressions as a product of factors.
  - Many expressions can be factorised by taking out the highest common factor (HCF).
    $15 = 3 \times 5$
    $3x - 12 = 3(x - 4)$
    $9x^2y - 6xy + 3x = 3x(3xy - 2y + 1)$

- Other general properties are:
  - **associative** $a \times (b \times c) = (a \times b) \times c$ or $a + (b + c) = (a + b) + c$
  - **commutative** $ab = ba$ or $a + b = b + a$ $\left(\text{Note: } \frac{a}{b} \neq \frac{b}{a} \text{ and } a - b \neq b - a.\right)$
  - **identity** $a \times 1 = a$ or $a + 0 = a$
  - **inverse** $a \times \frac{1}{a} = 1$ or $a + (-a) = 0$

BUILDING UNDERSTANDING

1 Which of the following is an equation?
A $3x - 1$ B $\frac{x + 1}{4}$ C $7x + 2 = 5$ D $3x^2y$

2 Which expression contains a term with a coefficient of -9?
A $8 + 9x$ B $2x + 9x^2y$ C $9a - 2ab$ D $b - 9a^2$

3 State the coefficient of a^2 in these expressions.
a $a + a^2$ b $\frac{3}{2} - 4a^2$ c $1 - \frac{a^2}{5}$ d $-\frac{7a^2}{3} - 1$

4 Decide whether the following pairs of terms are like terms.
a xy and $2yx$ b $7a^2b$ and $-7ba^2$ c $-4abc^2$ and $8ab^2c$

5 Evaluate:
a $(-3)^2$ b $(-2)^3$ c -2^3 d -3^2

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 1 Collecting like terms

Simplify by collecting like terms.

a $7a + 3a$ b $3a^2b - 2a^2b$ c $5xy + 2xy^2 - 2xy + 3y^2x$

SOLUTION

a $7a + 3a = 10a$

Keep the pronumeral and add the coefficients.

b $3a^2b - 2a^2b = a^2b$

$3a^2b$ and $2a^2b$ have the same pronumeral part, so they are like terms. Subtract coefficients and recall that $1a^2b = a^2b$.

c $5xy + 2xy^2 - 2xy + 3y^2x = 3xy + 5xy^2$

Collect like terms, noting that $3y^2x = 3xy^2$. The + or − sign belongs to the term that directly follows it.

EXPLANATION

Now you try

Simplify by collecting like terms.

a $4a + 13a$ b $5ab^2 - 2ab^2$ c $3xy + 4x^2y - xy + 2yx^2$

Example 2 Multiplying and dividing expressions

Simplify the following.

a $2h \times 7l$ b $-3p^2r \times 2pr$ c $-\frac{7xy}{14y}$

SOLUTION

a $2h \times 7l = 14hl$

Multiply the numbers and remove the × sign.

b $-3p^2r \times 2pr = -6p^3r^2$

Remember the basic index law: When you multiply terms with the same base you add the powers.

c $-\frac{7xy}{14y} = -\frac{x}{2}$

Cancel the highest common factor of 7 and 14 and cancel the y.

Now you try

Simplify the following.

a $3a \times 6b$ b $-2x^2y \times 5xy$ c $-\frac{4ab}{8a}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 3 Expanding the brackets

Expand the following using the distributive law. Simplify where possible.

a $2(x + 4)$ **b** $-3x(x - y)$ **c** $3(x + 2) - 4(2x - 4)$

SOLUTION

a $2(x + 4) = 2x + 8$

b $-3x(x - y) = -3x^2 + 3xy$

c $3(x + 2) - 4(2x - 4) = 3x + 6 - 8x + 16$
$= -5x + 22$

EXPLANATION

$2(x + 4) = 2 \times x + 2 \times 4$

Note that $x \times x = x^2$ and $-3 \times (-1) = 3$.

Expand each pair of brackets and simplify by collecting like terms.

Now you try

Expand the following using the distributive law. Simplify where possible.

a $3(x + 2)$ **b** $-2x(x - y)$ **c** $2(x + 3) - 3(2x - 1)$

Example 4 Factorising simple algebraic expressions

Factorise:

a $3x - 9$ **b** $2x^2 + 4x$

SOLUTION

a $3x - 9 = 3(x - 3)$

b $2x^2 + 4x = 2x(x + 2)$

EXPLANATION

HCF of $3x$ and 9 is 3.

Check that $3(x - 3) = 3x - 9$.

HCF of $2x^2$ and $4x$ is $2x$.

Check that $2x(x + 2) = 2x^2 + 4x$.

Now you try

Factorise:

a $2x - 10$ **b** $3x^2 + 9x$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 5 Evaluating expressions

Evaluate $a^2 - 2bc$ if $a = -3$, $b = 5$ and $c = -1$.

SOLUTION	EXPLANATION
$a^2 - 2bc = (-3)^2 - 2(5)(-1)$	Substitute for each pronumeral:
$= 9 - (-10)$	$(-3)^2 = -3 \times (-3)$ and $2 \times 5 \times (-1) = -10$
$= 19$	To subtract a negative number, add its opposite.

Now you try

Evaluate $b^2 - 3ac$ if $a = 1$, $b = -2$ and $c = -3$.

Exercise 1A

FLUENCY	1, 2–7(1/2)	2–7(1/2)	2–7(1/3)

1 Simplify by collecting like terms.

Example 1a
a **i** $5a + 9a$ **ii** $7a - 2a$

Example 1b
b **i** $4a^2b - 2a^2b$ **ii** $5x^2y - 4x^2y$

Example 1c
c **i** $4xy + 3xy^2 - 3xy + 2y^2x$ **ii** $6ab + 2ab^2 - 2ab + 4b^2a$

Example 1
2 Simplify by collecting like terms.

a $6a + 4a$ **b** $8d + 7d$ **c** $5y - 5y$
d $2xy + 3xy$ **e** $9ab - 5ab$ **f** $4t + 3t + 2t$
g $7b - b + 3b$ **h** $3st^2 - 4st^2$ **i** $4m^2n - 7nm^2$
j $0.3a^2b - ba^2$ **k** $4gh + 5 - 2gh$ **l** $7xy + 5xy - 3y$
m $4a + 5b - a + 2b$ **n** $3jk - 4j + 5jk - 3j$ **o** $2ab^2 + 5a^2b - ab^2 + 5ba^2$
p $3mn - 7m^2n + 6nm^2 - mn$ **q** $4st + 3ts^2 + st - 4s^2t$ **r** $7x^3y^4 - 3xy^2 - 4y^4x^3 + 5y^2x$

Example 2
3 Simplify the following.

a $4a \times 3b$ **b** $5a \times 5b$ **c** $-2a \times 3d$ **d** $5h \times (-2m)$
e $-6h \times (-5t)$ **f** $-5b \times (-6l)$ **g** $2s^2 \times 6t$ **h** $-3b^2 \times 7d^5$
i $4ab \times 2ab^3$ **j** $-6p^2 \times (-4pq)$ **k** $6hi^4 \times (-3h^4i)$ **l** $7mp \times 9mr$
m $\frac{7x}{7}$ **n** $\frac{6ab}{2}$ **o** $-\frac{3a}{9}$ **p** $-\frac{2ab}{8}$
q $\frac{4ab}{2a}$ **r** $-\frac{15xy}{5y}$ **s** $-\frac{4xy}{8x}$ **t** $-\frac{28ab}{56b}$

Example 3a,b
4 Expand the following, using the distributive law.

a $5(x + 1)$ **b** $2(x + 4)$ **c** $3(x - 5)$
d $-5(4 + b)$ **e** $-2(y - 3)$ **f** $-7(a + c)$
g $-6(-m - 3)$ **h** $4(m - 3n + 5)$ **i** $-2(p - 3q - 2)$
j $2x(x + 5)$ **k** $6a(a - 4)$ **l** $-4x(3x - 4y)$
m $3y(5y + z - 8)$ **n** $9g(4 - 2g - 5h)$ **o** $-2a(4b - 7a + 10)$
p $7y(2y - 2y^2 - 4)$ **q** $-3a(2a^2 - a - 1)$ **r** $-t(5t^3 + 6t^2 + 2)$
s $2m(3m^3 - m^2 + 5m)$ **t** $-x(1 - x^3)$ **u** $-3s(2t - s^3)$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 3c

5 Expand and simplify the following, using the distributive law.

a $2(x + 4) + 3(x + 5)$ **b** $4(a + 2) + 6(a + 3)$
c $6(3y + 2) + 3(y - 3)$ **d** $3(2m + 3) + 3(3m - 1)$
e $2(2 + 6b) - 3(4b - 2)$ **f** $3(2t + 3) - 5(2 - t)$
g $2x(x + 4) + x(x + 7)$ **h** $4(6z - 4) - 3(3z - 3)$
i $3d^2(2d^3 - d) - 2d(3d^4 + 4d^2)$ **j** $q^3(2q - 5) + q^2(7q^2 - 4q)$

Example 4

6 Factorise:

a $3x - 9$ **b** $4x - 8$ **c** $10y + 20$
d $6y + 30$ **e** $x^2 + 7x$ **f** $2a^2 + 8a$
g $5x^2 - 5x$ **h** $9y^2 - 63y$ **i** $xy - xy^2$
j $x^2y - 4x^2y^2$ **k** $8a^2b + 40a^2$ **l** $7a^2b + ab$
m $-5t^2 - 5t$ **n** $-6mn - 18mn^2$ **o** $-y^2 - 8yz$

Example 5

7 Evaluate these expressions if $a = -4$, $b = 3$ and $c = -5$.

a $-2a^2$ **b** $b - a$ **c** $abc + 1$ **d** $-ab$
e $\dfrac{a + b}{2}$ **f** $\dfrac{3b - a}{5}$ **g** $\dfrac{a^2 - b^2}{c}$ **h** $\dfrac{\sqrt{a^2 + b^2}}{\sqrt{c^2}}$

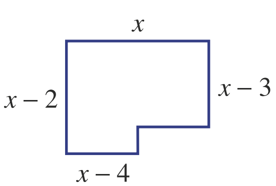
PROBLEM-SOLVING	8	8, 9	9

8 Find an expression for the area of a floor of a rectangular room with the following side lengths. Expand and simplify your answer.

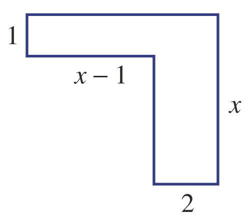
a $x + 3$ and $2x$ **b** x and $x - 5$

9 Find expressions in simplest form for the perimeter (P) and area (A) of these shapes. (Note: All angles are right angles.)

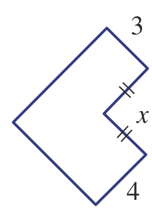
a



b



c



REASONING	10	10, 11	11, 12

10 When $a = -2$ give reasons why:

a $a^2 > 0$ **b** $-a^2 < 0$ **c** $a^3 < 0$

11 Decide whether the following are true or false for all values of a and b. If false, give an example to show that it is false.

a $a + b = b + a$ **b** $a - b = b - a$
c $ab = ba$ **d** $\dfrac{a}{b} = \dfrac{b}{a}$
e $a + (b + c) = (a + b) + c$ **f** $a - (b - c) = (a - b) - c$
g $a \times (b \times c) = (a \times b) \times c$ **h** $a \div (b \div c) = (a \div b) \div c$

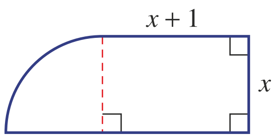
ISBN 978-1-108-77290-7

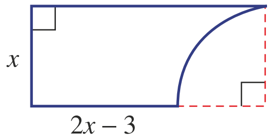
Cambridge University Press
Updated June 2022

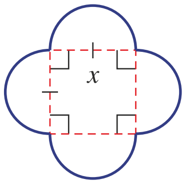
12 **a** Write an expression for the statement 'the sum of x and y divided by 2'.
b Explain why the statement above is ambiguous.
c Write an unambiguous statement describing $\frac{a+b}{2}$.

ENRICHMENT: Algebraic circular spaces	–	–	13

13 Find expressions in simplest form for the perimeter (P) and area (A) of these shapes. Your answers may contain π, for example 4π. Do not use decimals.

a


b


c


Architects, builders, carpenters and landscapers are among the many occupations that use algebraic formulas to calculate areas and perimeters in daily work.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

1B Multiplying and dividing algebraic fractions

Learning intentions

- To understand that expressions need to be in factorised form in order to cancel common factors
- To know that it is helpful to cancel common factors in fractions before multiplying or dividing
- To be able to multiply and divide fractions involving algebraic expressions

Since pronumerals represent numbers, the rules for algebraic fractions are the same as those for simple numerical fractions. This includes processes such as cancelling common factors, adding or subtracting with a lowest common denominator (LCD) and dividing, by multiplying by the reciprocal of the fraction that follows the division sign. In this section we focus on multiplying and dividing algebraic fractions.

The study of air-conditioning uses algebraic fractions to model airflow, air temperatures and humidity. The mechanical engineers who design ventilation systems, and the electricians who install and repair them, all require algebraic skills.

LESSON STARTER Describe the error

Here are three problems involving algebraic fractions. Each simplification contains one critical error. Find and describe the errors, then give the correct answer.

a $\dfrac{6x - \cancel{8}^{2}}{\cancel{4}_{1}} = \dfrac{6x - 2}{1} = 6x - 2$

b $\dfrac{2a}{9} \div \dfrac{2}{3} = \dfrac{2a}{9} \times \dfrac{2}{3} = \dfrac{4a}{27}$

c $\dfrac{3b}{7} \div \dfrac{2b}{3} = \dfrac{3b}{7} \times \dfrac{3b}{2} = \dfrac{9b^2}{14}$

KEY IDEAS

- Simplify **algebraic fractions** by factorising expressions where possible and cancelling common factors.
- For multiplication, cancel common factors and then multiply the numerators together and the denominators together.
- For division, multiply by the **reciprocal** of the fraction that follows the division sign. The reciprocal of a is $\dfrac{1}{a}$ and the reciprocal of $\dfrac{a}{b}$ is $\dfrac{b}{a}$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

BUILDING UNDERSTANDING

1 Simplify to find the answer in simplest form.

a $\frac{2}{3} \times \frac{6}{4}$ b $\frac{3}{4} \times \frac{10}{9}$ c $\frac{4}{7} \div \frac{2}{7}$ d $\frac{3}{6} \div \frac{6}{9}$

2 What is the reciprocal of each fraction?

a $\frac{3}{2}$ b $\frac{7a}{3}$ c $\frac{-4xy}{7t}$ d $\frac{-8x^2a}{b^2c}$

3 Simplify by cancelling common factors.

a $\frac{10x}{2}$ b $\frac{24x}{6}$ c $\frac{5a}{20}$ d $\frac{7}{21a}$

Example 6 Cancelling common factors

Simplify by cancelling common factors.

a $\frac{8a^2b}{2a}$ b $\frac{3-9x}{3}$

SOLUTION

a $\frac{8a^2b}{2a} = \frac{\cancel{8}^4 \times \cancel{a}^1 \times a \times b}{\cancel{2}_1 \times \cancel{a}_1}$
$= 4ab$

Cancel the common factors 2 and a.

b $\frac{3-9x}{3} = \frac{\cancel{3}^1(1-3x)}{\cancel{3}^1}$
$= 1 - 3x$

Factorise the numerator, then cancel the common factor of 3.

Now you try

Simplify by cancelling common factors.

a $\frac{9ab^2}{3b}$ b $\frac{5-10x}{5}$

Example 7 Multiplying and dividing algebraic fractions

Simplify the following.

a $\frac{2}{a} \times \frac{a+2}{4}$ b $\frac{2x-4}{3} \div \frac{x-2}{6}$

SOLUTION

a $\frac{\cancel{2}^1}{a} \times \frac{a+2}{\cancel{4}_2} = \frac{a+2}{2a}$

Cancel the common factor of 2 and then multiply the numerators and the denominators. a cannot be cancelled as it is not a common factor in $a + 2$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b $\frac{2x-4}{3} \div \frac{x-2}{6} = \frac{2x-4}{3} \times \frac{6}{x-2}$

$= \frac{2\cancel{(x-2)}^1}{\cancel{3}^1} \times \frac{\cancel{6}^2}{\cancel{(x-2)}^1}$

$= 4$

Multiply by the reciprocal of the second fraction.

Factorise $2x - 4$ and cancel the common factors.

Now you try

Simplify the following.

a $\frac{6}{a} \times \frac{a+1}{12}$

b $\frac{3x-12}{2} \div \frac{x-4}{4}$

Exercise 1B

FLUENCY	1, 2–5(1/2)	2–5(1/2)	2–5(1/3)

1 Simplify by cancelling common factors.

Example 6a

a **i** $\frac{6a^2b}{2a}$ **ii** $\frac{10xy^2}{5y}$

Example 6b

b **i** $\frac{4-8x}{4}$ **ii** $\frac{5-5x}{5}$

Example 6a

2 Simplify by cancelling common factors.

a $\frac{35x^2}{7x}$ **b** $\frac{-14x^2y}{7xy}$ **c** $\frac{-36ab^2}{4ab}$ **d** $\frac{8xy^3}{-4xy^2}$

e $\frac{-15pq^2}{30p^2q^2}$ **f** $\frac{-20s}{45s^2t}$ **g** $\frac{-48x^2}{16xy}$ **h** $\frac{120ab^2}{140ab}$

Example 6b

3 Simplify by cancelling common factors.

a $\frac{4x+8}{4}$ **b** $\frac{6a-30}{6}$ **c** $\frac{6x-18}{2}$ **d** $\frac{5-15y}{5}$

e $\frac{-2-12b}{-2}$ **f** $\frac{21x-7}{-7}$ **g** $\frac{9t-27}{-9}$ **h** $\frac{44-11x}{-11}$

i $\frac{x^2+2x}{x}$ **j** $\frac{6x-4x^2}{2x}$ **k** $\frac{a^2-a}{a}$ **l** $\frac{7a+14a^2}{21a}$

Example 7a

4 Simplify the following.

a $\frac{3}{x} \times \frac{x-1}{6}$ **b** $\frac{x+4}{10} \times \frac{2}{x}$ **c** $\frac{-8a}{7} \times \frac{7}{2a}$

d $\frac{x+3}{9} \times \frac{4}{x+3}$ **e** $\frac{y-7}{y} \times \frac{5y}{y-7}$ **f** $\frac{10a^2}{a+6} \times \frac{a+6}{4a}$

g $\frac{2m+4}{m} \times \frac{m}{m+2}$ **h** $\frac{6-18x}{2} \times \frac{5}{1-3x}$ **i** $\frac{b-1}{10} \times \frac{-5}{b-1}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 7b

5 Simplify the following.

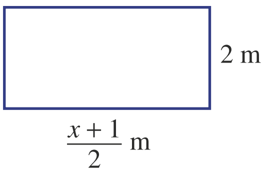
a $\dfrac{x}{5} \div \dfrac{x}{15}$ **b** $\dfrac{x+4}{2} \div \dfrac{x+4}{6}$ **c** $\dfrac{6x-12}{5} \div \dfrac{x-2}{3}$

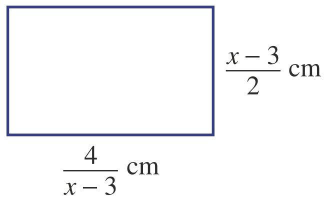
d $\dfrac{3-6y}{8} \div \dfrac{1-2y}{2}$ **e** $\dfrac{2}{a-1} \div \dfrac{3}{2a-2}$ **f** $\dfrac{2}{10x-5} \div \dfrac{10}{2x-1}$

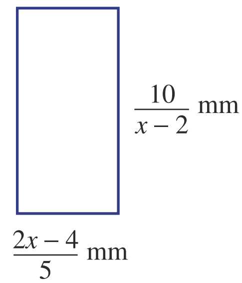
g $\dfrac{5}{3a+4} \div \dfrac{15}{-15a-20}$ **h** $\dfrac{2x-6}{5x-20} \div \dfrac{x-3}{x-4}$ **i** $\dfrac{t+1}{9} \div \dfrac{-t-1}{3}$

PROBLEM-SOLVING	6	6, 7(1/2)	7–8(1/2)

6 Find a simplified expression for the area of these rectangles.

a


b


c


7 Simplify these expressions.

a $\dfrac{x}{3} \times \dfrac{9x}{5} \times \dfrac{15}{3x}$ **b** $\dfrac{2}{a} \times \dfrac{a}{5} \times \dfrac{10}{3a}$

c $\dfrac{x-1}{2} \times \dfrac{4x}{2x-2} \times \dfrac{x+3}{5x}$ **d** $\dfrac{2x-1}{x} \div \dfrac{2x-1}{2} \div \dfrac{1}{2}$

e $\dfrac{2x-3}{5} \div \dfrac{14x-21}{10} \div \dfrac{x}{2}$ **f** $\dfrac{b^2-b}{b} \div \dfrac{b-1}{b^2} \times \dfrac{2}{b-1}$

8 Write the missing algebraic fraction.

a $\dfrac{x+3}{5} \times \square = 2$ **b** $\dfrac{1-x}{x} \times \square = 3$

c $\square \div \dfrac{x}{2} = \dfrac{3(x+2)}{x}$ **d** $\square \div \dfrac{2x-2}{3} = \dfrac{5x}{x-1}$

e $\dfrac{1}{x} \div \square \times \dfrac{x-1}{2} = 1$ **f** $\dfrac{2-x}{7} \times \square \div \dfrac{5x}{x-1} = x$

REASONING	9(1/2)	9(1/2), 10	9(1/2), 10, 11

9 Recall that $(x-1)^2 = (x-1)(x-1)$. Use this idea to simplify the following.

a $\dfrac{(x-1)^2}{x-1}$ **b** $\dfrac{3(x+2)^2}{x+2}$ **c** $\dfrac{4(x-3)^2}{2(x-3)}$

d $\dfrac{4(x+2)}{(x+2)^2}$ **e** $\dfrac{-5(1-x)}{(1-x)^2}$ **f** $\dfrac{(2x-2)^2}{x-1}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

10 Prove that the following all simplify to 1.

a $\frac{5x+5}{15} \times \frac{3}{x+1}$

b $\frac{3x-21}{2-x} \times \frac{4-2x}{6x-42}$

c $\frac{10-5x}{2x+6} \div \frac{20-10x}{4x+12}$

11 a Explain why $\frac{x-1}{2} \times \frac{4}{1-x} = \frac{x-1}{2} \times \frac{-4}{x-1}$.

b Use this idea to simplify these expressions.

i $\frac{2-a}{3} \times \frac{7}{a-2}$

ii $\frac{6x-3}{x} \div \frac{1-2x}{4}$

iii $\frac{18-x}{3x-1} \div \frac{2x-36}{7-21x}$

ENRICHMENT: Simplifying with quadratics	–	–	12(½)

12 You may recall that to factorise a monic quadratic of the form $x^2 + bx + c$ we look for factors of c which add to b. So for example: $x^2 - x - 6 = (x-3)(x+2)$.

So:

$$\frac{x^2-x-6}{6} \times \frac{3}{x-3} = \frac{{}^{1}\cancel{(x-3)}(x+2)}{\cancel{6}^{2}} \times \frac{\cancel{3}^{1}}{\cancel{(x-3)}_{1}}$$
$$= \frac{x+2}{2}$$

Now simplify these algebraic fractions which involve quadratics.

a $\frac{x^2-2x-8}{4} \times \frac{2}{x-4}$

b $\frac{x^2+5x+6}{x+2} \times \frac{x}{x+3}$

c $\frac{x+1}{x^2-4x-5} \times \frac{x-5}{3}$

d $\frac{3x-27}{4x} \times \frac{2x}{x^2-7x-18}$

e $\frac{4ab}{a^2+a} \div \frac{b}{a^2+2a+1}$

f $\frac{a+8}{a^2-5a-6} \div \frac{a^2+5a-24}{a-6}$

g $\frac{(x-y)^2}{xy} \div \frac{x^2-y^2}{x+y}$

h $\frac{y^2+4y+4}{x^2y} \div \frac{(y+2)^2}{xy^2+2xy}$

ISBN 978-1-108-77290-7

1C Adding and subtracting algebraic fractions

Learning intentions

- To know how to find the lowest common denominator of algebraic fractions
- To be able to combine numerators using expansion and addition of like terms
- To be able to add and subtract algebraic fractions

The sum or difference of two or more algebraic fractions can be simplified in a similar way to numerical fractions with the use of a common denominator.

Electricians, electrical and electronic engineers work with algebraic fractions when modelling the flow of electric energy in circuits. The application of algebra when using electrical formulas is essential in these professions.

LESSON STARTER Spot the difference

Here are two sets of simplification steps. One set has one critical error. Can you find and correct it?

$$\begin{aligned}\frac{2}{3} - \frac{5}{2} &= \frac{4}{6} - \frac{15}{6} \\ &= \frac{-11}{6}\end{aligned}$$

$$\begin{aligned}\frac{x}{3} - \frac{x+1}{2} &= \frac{2x}{6} - \frac{3(x+1)}{6} \\ &= \frac{2x - 3x + 3}{6} \\ &= \frac{-x+3}{6}\end{aligned}$$

KEY IDEAS

- Add and subtract algebraic fractions by firstly finding the lowest common denominator (LCD) and then combine the numerators.
- Expand numerators correctly by taking into account addition and subtraction signs. E.g. $-2(x + 1) = -2x - 2$ and $-5(2x - 3) = -10x + 15$.

BUILDING UNDERSTANDING

1 Expand the following.

a $2(x - 2)$ **b** $-(x + 6)$ **c** $-6(x - 2)$

2 Simplify these by firstly finding the lowest common denominator (LCD).

a $\frac{1}{2} + \frac{1}{3}$ **b** $\frac{4}{3} - \frac{1}{5}$ **c** $\frac{3}{7} - \frac{1}{14}$ **d** $\frac{5}{3} + \frac{7}{6}$

3 State the lowest common denominator for these pairs of fractions.

a $\frac{a}{3}, \frac{7a}{4}$ **b** $\frac{x}{2}, \frac{4xy}{6}$ **c** $\frac{3xy}{7}, \frac{-3x}{14}$ **d** $\frac{2}{x}, \frac{3}{2x}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 8 Adding and subtracting simple algebraic fractions

Simplify the following.

a $\frac{3}{4} - \frac{a}{2}$

b $\frac{2}{5} + \frac{3}{a}$

SOLUTION

a $\frac{3}{4} - \frac{a}{2} = \frac{3}{4} - \frac{2a}{4}$

$= \frac{3 - 2a}{4}$

b $\frac{2}{5} + \frac{3}{a} = \frac{2a}{5a} + \frac{15}{5a}$

$= \frac{2a + 15}{5a}$

EXPLANATION

The LCD of 2 and 4 is 4. Express each fraction as an equivalent fraction with a denominator of 4. Subtract the numerators.

The LCD of 5 and a is $5a$.

Add the numerators.

Now you try

Simplify the following.

a $\frac{5}{6} - \frac{a}{3}$

b $\frac{3}{4} + \frac{2}{a}$

Example 9 Adding and subtracting more complex algebraic fractions

Simplify the following algebraic expressions.

a $\frac{x + 3}{2} + \frac{x - 2}{5}$

b $\frac{2x - 1}{3} - \frac{x - 1}{4}$

SOLUTION

a $\frac{x + 3}{2} + \frac{x - 2}{5} = \frac{5(x + 3)}{10} + \frac{2(x - 2)}{10}$

$= \frac{5(x + 3) + 2(x - 2)}{10}$

$= \frac{5x + 15 + 2x - 4}{10}$

$= \frac{7x + 11}{10}$

EXPLANATION

LCD is 10.

Use brackets to ensure you retain equivalent fractions.

Combine the numerators, then expand the brackets and simplify.

Continued on next page

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b $\dfrac{2x-1}{3}-\dfrac{x-1}{4}=\dfrac{4(2x-1)}{12}-\dfrac{3(x-1)}{12}$

Express each fraction with the LCD of 12.

$=\dfrac{4(2x-1)-3(x-1)}{12}$

Combine the numberators.

$=\dfrac{8x-4-3x+3}{12}$

Expand the brackets: $4(2x-1)=8x-4$ and $-3(x-1)=-3x+3$.

$=\dfrac{5x-1}{12}$

Simplify by collecting like terms.

Now you try

Simplify the following algebraic expressions.

a $\dfrac{x+1}{3}+\dfrac{x-2}{2}$

b $\dfrac{3x-2}{2}-\dfrac{x-2}{5}$

Example 10 Adding and subtracting with algebraic denominators

Simplify the algebraic expression $\dfrac{3}{x-6}-\dfrac{2}{x+2}$.

SOLUTION

$$\frac{3}{x-6}-\frac{2}{x+2}=\frac{3(x+2)}{(x-6)(x+2)}-\frac{2(x-6)}{(x-6)(x+2)}$$
$$=\frac{3(x+2)-2(x-6)}{(x-6)(x+2)}$$
$$=\frac{3x+6-2x+12}{(x-6)(x+2)}$$
$$=\frac{x+18}{(x-6)(x+2)}$$

EXPLANATION

$(x-6)(x+2)$ is the lowest common multiple of $(x-6)$ and $(x+2)$.
Combine the numberators and then expand the brackets.

Recall that $-2\times(-6)=12$.

Collect like terms to simplify.

Now you try

Simplify the expression $\dfrac{4}{x-5}-\dfrac{3}{x+1}$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 1C

FLUENCY	1, 2–4(1/2)	2–5(1/2)	2–5(1/3)

Example 8a

1 Simplify the following.

a **i** $\frac{1}{4}-\frac{a}{2}$ **ii** $\frac{3}{10}-\frac{a}{5}$

Example 8b

b **i** $\frac{1}{3}+\frac{2}{a}$ **ii** $\frac{3}{7}+\frac{5}{a}$

Example 8a

2 Simplify the following.

a $\frac{2}{3}+\frac{a}{7}$ **b** $\frac{3}{8}+\frac{a}{2}$ **c** $\frac{3}{10}-\frac{3b}{2}$ **d** $\frac{2}{5}+\frac{4x}{15}$

e $\frac{1}{9}-\frac{2a}{3}$ **f** $\frac{a}{3}-\frac{a}{5}$ **g** $\frac{2x}{5}-\frac{x}{4}$ **h** $\frac{6b}{7}-\frac{b}{14}$

Example 8b

3 Simplify the following.

a $\frac{2}{3}+\frac{5}{a}$ **b** $\frac{3}{4}+\frac{2}{a}$ **c** $\frac{7}{9}-\frac{3}{a}$ **d** $\frac{4}{b}-\frac{3}{4}$

e $\frac{2}{7}-\frac{3}{2b}$ **f** $\frac{3}{2y}-\frac{7}{9}$ **g** $\frac{-4}{x}-\frac{2}{3}$ **h** $\frac{-9}{2x}-\frac{1}{3}$

Example 9a

4 Simplify the following algebraic expressions.

a $\frac{x+3}{4}+\frac{x+2}{5}$ **b** $\frac{x+2}{3}+\frac{x+1}{4}$ **c** $\frac{x-3}{4}+\frac{x+2}{2}$

d $\frac{x+4}{3}+\frac{x-3}{9}$ **e** $\frac{2x+1}{2}+\frac{x-2}{3}$ **f** $\frac{3x+1}{5}+\frac{2x+1}{10}$

g $\frac{x-2}{8}+\frac{2x+4}{12}$ **h** $\frac{5x+3}{10}+\frac{2x-2}{4}$ **i** $\frac{3-x}{14}+\frac{x-1}{7}$

Example 9b

5 Simplify these algebraic fractions.

a $\frac{2x+1}{3}-\frac{x-1}{2}$ **b** $\frac{3x-1}{3}-\frac{2x-3}{4}$ **c** $\frac{x+6}{5}-\frac{x-4}{3}$

d $\frac{x-3}{2}-\frac{2x+1}{7}$ **e** $\frac{7x+2}{7}-\frac{x+2}{3}$ **f** $\frac{10x-4}{3}-\frac{2x+1}{6}$

g $\frac{4-x}{6}-\frac{1-x}{5}$ **h** $\frac{1-3x}{5}-\frac{x+2}{3}$ **i** $\frac{6-5x}{2}-\frac{2-7x}{4}$

PROBLEM-SOLVING	6(1/2)	6(1/2)	6(1/2), 7

Example 10

6 Simplify the following algebraic expressions.

a $\frac{5}{x+1}+\frac{2}{x+4}$ **b** $\frac{4}{x-7}+\frac{3}{x+2}$ **c** $\frac{1}{x-3}+\frac{2}{x+5}$

d $\frac{3}{x+3}-\frac{2}{x-4}$ **e** $\frac{6}{2x-1}-\frac{3}{x-4}$ **f** $\frac{4}{x-5}+\frac{2}{3x-4}$

g $\frac{5}{2x-1}-\frac{6}{x+7}$ **h** $\frac{2}{x-3}-\frac{3}{3x+4}$ **i** $\frac{8}{3x-2}-\frac{3}{1-x}$

ISBN 978-1-108-77290-7

7 **a** Write the LCD for these pairs of fractions.

i $\frac{3}{a}, \frac{2}{a^2}$ **ii** $\frac{7}{x^2}, \frac{3+x}{x}$

b Now simplify these expressions.

i $\frac{2}{a} - \frac{3}{a^2}$ **ii** $\frac{a+1}{a} - \frac{4}{a^2}$ **iii** $\frac{7}{2x^2} + \frac{3}{4x}$

REASONING	8	8, 9	8, 9

8 Describe the error in this working, then fix the solution.

$$\begin{aligned}\frac{x}{2} - \frac{x+1}{3} &= \frac{3x}{6} - \frac{2(x+1)}{6}\\ &= \frac{3x}{6} - \frac{2x+2}{6}\\ &= \frac{x+2}{6}\end{aligned}$$

9 **a** Explain why $2x - 3 = -(3 - 2x)$.

b Use this idea to help simplify these expressions.

i $\frac{1}{x-1} - \frac{1}{1-x}$ **ii** $\frac{3x}{3-x} + \frac{x}{x-3}$ **iii** $\frac{x+1}{7-x} - \frac{2}{x-7}$

ENRICHMENT: Fraction challenges	–	–	10, 11

10 Simplify these expressions.

a $\frac{a-b}{b-a}$

b $\frac{5}{a} + \frac{2}{a^2}$

c $\frac{3}{x+1} + \frac{2}{(x+1)^2}$

d $\frac{x}{(x-2)^2} - \frac{x}{x-2}$

e $\frac{x}{2(3-x)} - \frac{x^2}{7(x-3)^2}$

f $\frac{1}{x} - \frac{1}{y} - \frac{1}{z}$

11 By first simplifying the left-hand side of these equations, find the value of a.

a $\frac{a}{x-1} - \frac{2}{x+1} = \frac{4}{(x-1)(x+1)}$

b $\frac{3}{2x-1} + \frac{a}{x+1} = \frac{5x+2}{(2x-1)(x+1)}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

1D Solving linear equations

Learning intentions

- To know the form of a linear equation
- To understand that an equivalent equation can be generated by applying the same operation to each side of the equation
- To be able to solve a linear equation involving two or more steps, including brackets and variables on both sides
- To be able to solve linear equations involving algebraic fractions
- To understand that solutions can be checked by substituting into both sides of an equation

A linear equation is a statement that contains an equals sign and includes constants and pronumerals with a power of 1 only. Here are some examples:

$$2x - 5 = 7$$
$$\frac{x+1}{3} = x + 4$$
$$-3(x+2) = \frac{1}{2}$$

We solve linear equations by operating on both sides of the equation until a solution is found.

A small business, such as a garden nursery, generates revenue from its sales. To calculate the number of employees (x) a business can afford, a linear revenue equation is solved for x: Revenue (y) = pay (m) × employees (x) + costs (c)

LESSON STARTER What's the best method?

Here are four linear equations.

- Discuss what you think is the best method to solve them using 'by hand' techniques.
- Discuss how it might be possible to check that a solution is correct.

a $\frac{7x-2}{3} = 4$ **b** $3(x-1) = 6$ **c** $4x + 1 = x - 2$ **d** $\frac{2x+1}{5} = \frac{x-1}{3}$

KEY IDEAS

- An equation is true for the given values of the pronumerals when the left-hand side equals the right-hand side.
  $2x - 4 = 6$ is true when $x = 5$ but false when $x \neq 5$.
- A **linear equation** contains pronumerals with a highest power of 1.
- Useful steps in solving linear equations are:
  - using inverse operations (backtracking)
  - collecting like terms
  - expanding brackets
  - multiplying by the lowest common denominator.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

BUILDING UNDERSTANDING

1. Decide whether the following are linear equations.
   a $x^2 - 1 = 0$ b $\sqrt{x} + x = 3$ c $\frac{x - 1}{2} = 5$ d $\frac{3x}{4} = 2x - 1$
2. Decide whether these equations are true when $x = 2$.
   a $3x - 1 = 5$ b $4 - x = 1$ c $\frac{2x + 1}{5} = x + 4$
3. Decide whether these equations are true when $x = -6$.
   a $-3x + 17 = x$ b $2(4 - x) = 20$ c $\frac{2 - 3x}{10} = \frac{-12}{x}$
4. Solve the following equations and check your solution using substitution.
   a $x + 8 = 13$ b $x - 5 = 3$ c $-x + 4 = 7$ d $-x - 5 = -9$

Example 11 Solving linear equations

Solve the following equations and check your solution using substitution.

a $4x + 5 = 17$ b $3(2x + 5) = 4x$

SOLUTION

a
$$\begin{aligned} 4x + 5 &= 17 \\ 4x &= 12 \\ x &= 3 \end{aligned}$$
Check: LHS $= 4 \times 3 + 5 = 17$, RHS $= 17$.

b
$$\begin{aligned} 3(2x + 5) &= 4x \\ 6x + 15 &= 4x \\ 2x + 15 &= 0 \\ 2x &= -15 \\ x &= -\frac{15}{2} \end{aligned}$$
Check:
$$\begin{aligned} \text{LHS} &= 3\left(2 \times \left(-\frac{15}{2}\right) + 5\right) \\ &= -30 \end{aligned}$$
$$\begin{aligned} \text{RHS} &= 4 \times \left(-\frac{15}{2}\right) \\ &= -30 \end{aligned}$$

EXPLANATION

a Subtract 5 from both sides and then divide both sides by 4.

Check by seeing if $x = 3$ makes the equation true.

b Expand the brackets.

Gather like terms by subtracting $4x$ from both sides.

Subtract 15 from both sides and then divide both sides by 2.

Check by seeing if $x = -\frac{15}{2}$ makes the equation true by substituting into the equation's left-hand side (LHS) and right-hand side (RHS) and confirming they are equal.

Now you try

Solve the following equations and check your solution using substitution.

a $2x + 7 = 13$ b $4(2x + 1) = 2x$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 12 Solving equations involving algebraic fractions

Solve the following equations and check your solution using substitution.

a $\frac{x+3}{4}=2$ **b** $\frac{4x-2}{3}=\frac{3x-1}{2}$ **c** $\frac{x+2}{3}-\frac{2x-1}{2}=4$

SOLUTION	EXPLANATION
a $\frac{x+3}{4}=2$	Multiply both sides by 4.
$x+3=8$	Subtract 3 from both sides.
$x=5$	
Check: LHS $=\frac{5+3}{4}=2$, RHS $=2$.	Check by seeing if $x=5$ makes the equation true.
b $\frac{4x-2}{3}=\frac{3x-1}{2}$	Multiply both sides by the LCD of 2 and 3, which is 6.
$\frac{\cancel{6}^{2}(4x-2)}{\cancel{3}_{1}}=\frac{\cancel{6}^{3}(3x-1)}{\cancel{2}_{1}}$	Cancel common factors.
$2(4x-2)=3(3x-1)$	
$8x-4=9x-3$	Expand the brackets and gather terms containing x by subtracting $8x$ from both sides.
$-4=x-3$	
$-1=x$	Rewrite with x as the subject. Check by seeing if $x=-1$ makes the equation true.
$\therefore x=-1$	
c $\frac{x+2}{3}-\frac{2x-1}{2}=4$	Multiply both sides by the LCD of 2 and 3, which is 6.
$\frac{6(x+2)}{3}-\frac{6(2x-1)}{2}=4\times 6$	
$2(x+2)-3(2x-1)=24$	Cancel common factors.
$2x+4-6x+3=24$	Expand, nothing that $-3\times(-1)=3$.
$-4x+7=24$	Simplify and solve for x.
$-4x=17$	
$x=-\frac{17}{4}$	Check your solution using substitution.

Now you try

Solve the following equations and check your solution using substitution.

a $\frac{x+1}{3}=4$ **b** $\frac{2x-1}{3}=\frac{x-3}{4}$ **c** $\frac{x+1}{2}-\frac{2x-1}{3}=2$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 1D

FLUENCY	1, 2–5(1/2)	2–5(1/3)	2–5(1/4)

1 Solve the following equations and check your solution using substitution.

Example 11a
a i $3x + 4 = 13$ ii $5x + 2 = 27$

Example 11b
b i $2(2x + 3) = 2x$ ii $5(x + 2) = 3x$

Example 11a
2 Solve the following equations and check your solution using substitution.

a $2x + 9 = 14$ b $4x + 3 = 14$ c $3x - 3 = -4$ d $6x + 5 = -6$
e $-3x + 5 = 17$ f $-2x + 7 = 4$ g $-4x - 9 = 9$ h $-3x - 7 = -3$
i $8 - x = 10$ j $5 - x = -2$ k $6 - 5x = 16$ l $4 - 9x = -7$

Example 11b
3 Solve the following equations and check your solution using substitution.

a $4(x + 3) = 16$ b $2(x - 3) = 12$ c $2(x - 4) = 15$
d $3(1 - 2x) = 8$ e $3(2x + 3) = -5x$ f $2(4x - 5) = -7x$
g $3(2x + 3) + 2(x + 4) = 25$ h $2(2x - 3) + 3(4x - 1) = 23$ i $2(3x - 2) - 3(x + 1) = 5$
j $5(2x + 1) - 3(x - 3) = 63$ k $5(x - 3) = 4(x - 6)$ l $4(2x + 5) = 3(x + 15)$
m $5(x + 2) = 3(2x - 3)$ n $3(4x - 1) = 7(2x - 7)$ o $7(2 - x) = 8 - x$

Example 12a
4 Solve the following equations and check your solution using substitution.

a $\frac{x - 4}{2} = 3$ b $\frac{x + 2}{3} = 5$ c $\frac{x + 4}{3} = -6$ d $\frac{2x + 7}{3} = 5$
e $\frac{2x + 1}{3} = -3$ f $\frac{3x - 2}{4} = 4$ g $\frac{x}{2} - 5 = 3$ h $\frac{3x}{2} + 2 = 8$
i $\frac{2x}{3} - 2 = -8$ j $5 - \frac{x}{2} = 1$ k $4 - \frac{2x}{3} = 0$ l $5 - \frac{4x}{7} = 9$
m $\frac{x + 1}{3} + 2 = 9$ n $\frac{x - 3}{2} - 4 = 2$ o $4 + \frac{x - 5}{2} = -3$ p $1 - \frac{2 - x}{3} = 2$

5 For each of the following statements, write an equation and solve it to find x.

a When 3 is added to x, the result is 7.
b When x is added to 8, the result is 5.
c When 4 is subtracted from x, the result is 5.
d When x is subtracted from 15, the result is 22.
e Twice the value of x is added to 5 and the result is 13.
f 5 less than x when doubled is -15.
g When 8 is added to 3 times x, the result is 23.
h 5 less than twice x is 3 less than x.

PROBLEM-SOLVING	7, 8, 11	6(1/2), 7, 9, 11, 13(1/2)	6–7(1/2), 9–12, 13(1/2)

Example 12b
6 Solve the following equations, which involve algebraic fractions.

a $\frac{2x + 12}{7} = \frac{3x + 5}{4}$ b $\frac{5x - 4}{4} = \frac{x - 5}{5}$ c $\frac{3x - 5}{4} = \frac{2x - 8}{3}$
d $\frac{1 - x}{5} = \frac{2 - x}{3}$ e $\frac{6 - 2x}{3} = \frac{5x - 1}{4}$ f $\frac{10 - x}{2} = \frac{x + 1}{3}$
g $\frac{2(x + 1)}{3} = \frac{3(2x - 1)}{2}$ h $\frac{-2(x - 1)}{3} = \frac{2 - x}{4}$ i $\frac{3(6 - x)}{2} = \frac{2(x + 1)}{5}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

7 Substitute the given values and then solve for the unknown in each of the following common formulas.

a $v = u + at$ Solve for a given $v = 6$, $u = 2$ and $t = 4$.

b $s = ut + \frac{1}{2}at^2$ Solve for u given $s = 20$, $t = 2$ and $a = 4$.

c $A = h\left(\frac{a+b}{2}\right)$ Solve for b given $A = 10$, $h = 4$ and $a = 3$.

d $A = P\left(1 + \frac{r}{100}\right)$ Solve for r given $A = 1000$ and $P = 800$.

8 The perimeter of a square is 68 cm. Determine its side length.

9 The sum of two consecutive numbers is 35. What are the numbers?

10 I ride four times faster than I jog. If a trip took me 45 minutes and I spent 15 of these minutes jogging 3 km, how far did I ride?

11 A service technician charges \$30 up front and \$46 for each hour that he works.

a What will a 4-hour job cost?

b If the technician works on a job for 2 days and averages 6 hours per day, what will be the overall cost?

c Find how many hours the technician worked if the cost is:

i \$76

ii \$513

iii \$1000 (round to the nearest half hour).

12 The capacity of a petrol tank is 80 litres. If it initially contains 5 litres and a petrol pump fills it at 3 litres per 10 seconds, find:

a the amount of fuel in the tank after 2 minutes

b how long it will take to fill the tank to 32 litres

c how long it will take to fill the tank.

Example 12c

13 Solve the following equations by multiplying both sides by the LCD.

a $\frac{x}{2} + \frac{2x}{3} = 7$

b $\frac{x}{4} + \frac{3x}{3} = 5$

c $\frac{3x}{5} - \frac{2x}{3} = 1$

d $\frac{2x}{5} - \frac{x}{4} = 3$

e $\frac{x-1}{2} + \frac{x+2}{5} = 2$

f $\frac{x+3}{3} + \frac{x-4}{2} = 4$

g $\frac{x+2}{3} - \frac{x-1}{2} = 1$

h $\frac{x-4}{5} - \frac{x+2}{3} = 2$

i $\frac{7-2x}{3} - \frac{6-x}{2} = 1$

ISBN 978-1-108-77290-7

REASONING	14	14	15

14 Solve $2(x - 5) = 8$ using the following two methods and then decide which method you prefer. Give a reason.

- **a** Method 1: First expand the brackets.
- **b** Method 2: First divide both sides by 2.

15 A family of equations can be represented using other pronumerals (sometimes called parameters). For example, the solution to the family of equations $2x - a = 4$ is $x = \dfrac{4 + a}{2}$.

Find the solution for x in these equation families.

- **a** $x + a = 5$
- **b** $6x + 2a = 3a$
- **c** $ax + 2 = 7$
- **d** $ax - 1 = 2a$
- **e** $\dfrac{ax - 1}{3} = a$
- **f** $ax + b = c$

ENRICHMENT: More complex equations	–	–	16, 17

16 Make a the subject in these equations.

- **a** $a(b + 1) = c$
- **b** $ab + a = b$
- **c** $\dfrac{1}{a} + b = c$
- **d** $a - \dfrac{a}{b} = 1$
- **e** $\dfrac{1}{a} + \dfrac{1}{b} = 0$
- **f** $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{c}$

17 Solve for x in terms of the other pronumerals.

- **a** $\dfrac{x}{2} - \dfrac{x}{3} = a$
- **b** $\dfrac{x}{a} + \dfrac{x}{b} = 1$
- **c** $\dfrac{x}{a} - \dfrac{x}{b} = c$

ISBN 978-1-108-77290-7

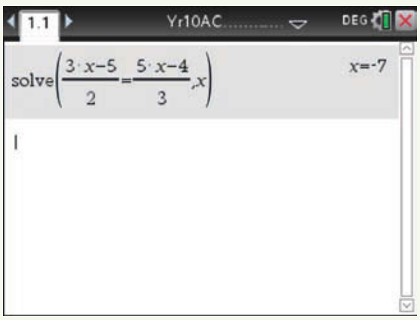
Using calculators to solve equations and inequalities

1 Solve the equation $\frac{3x-5}{2} = \frac{5x-4}{3}$.

2 Solve the inequality $5 < \frac{2x+3}{5}$.

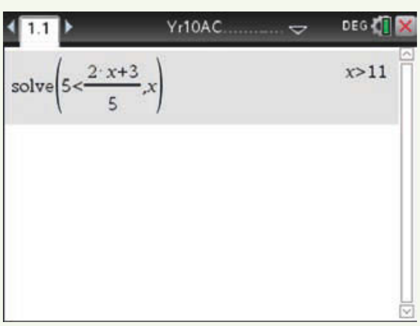
Using the TI-Nspire:

1 In a **Calculator** page use menu **>Algebra>Solve** and type the equation as shown.



Hint: use the fraction template (ctrl ÷)

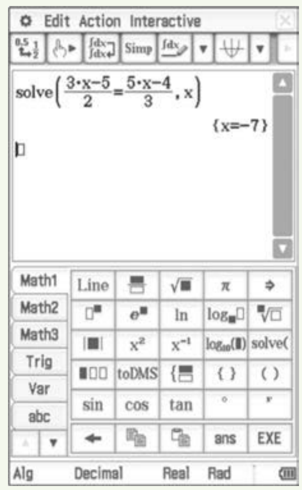
2 In a **Calculator** page use menu **>Algebra>Solve** and type the equation as shown.



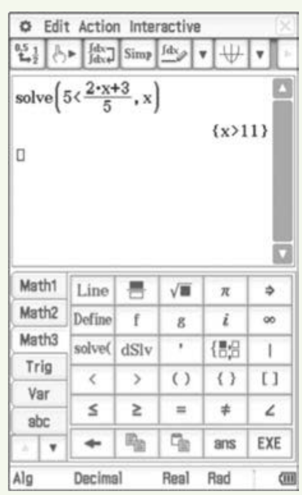
Hint: the inequality symbols (e.g. <) are accessed using ctrl =
Hint: use the fraction template (ctrl ÷)

Using the ClassPad:

1 Tap **solv(**, then and type the equation as shown.



2 Tap **solv(** and type the equation as shown.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

1E Linear inequalities

Learning intentions

- To know the meaning of the term inequality
- To be able to use and interpret the signs $>, \geqslant, \leqslant, <$
- To know how to interpret and represent an inequality on a number line
- To understand when to reverse the sign in an inequality
- To be able to solve a linear inequality

There are many situations in which a solution to the problem is best described using one of the symbols $<, \leqslant, >$ or $\geqslant$. For example, a pharmaceutical company might need to determine the possible number of packets of a particular drug that need to be sold so that the product is financially viable. This range of values may be expressed using inequality symbols.

Doctors, nurses and pharmacists can use an inequality to express the dosage range of a medication from the lowest effective level to the highest safe level.

An inequality is a mathematical statement that uses an *is less than* ($<$), an *is less than or equal to* ($\leqslant$), an *is greater than* ($>$) or an *is greater than or equal to* ($\geqslant$) symbol. Inequalities may result in an infinite number of solutions and these can be illustrated using a number line.

LESSON STARTER What does it mean for x?

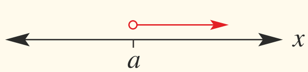
The following inequalities provide some information about the value of x.

a $2 \geqslant x$ **b** $-2x < 4$ **c** $3 - x \leqslant -1$

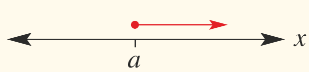
- Can you describe the possible values for x that satisfy each inequality?
- Test some values to check.
- How would you write the solution for x? Illustrate this on a number line.

KEY IDEAS

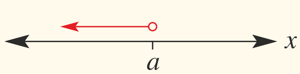
- The four **inequality signs** are $<, \leqslant, >$ and $\geqslant$.
  - $x > a$ means x is greater than a.



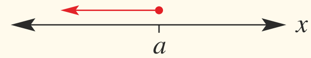
  - $x \geqslant a$ means x is greater than or equal to a.



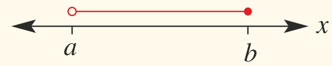
  - $x < a$ means x is less than a.



- $x \leqslant a$ means x is less than or equal to a.



- Also $a < x \leqslant b$ could be illustrated as shown.



- An open circle is used for $<$ or $>$ where the end point is not included.
- A closed circle is used for $\leqslant$ or $\geqslant$ where the end point is included.

■ Solving **linear inequalities** follows the same rules as solving linear equations, except:

- We reverse the inequality sign if we multiply or divide by a negative number.
  For example, $-5 < -3$ is equivalent to $5 > 3$ and if $-2x < 4$, then $x > -2$.
- We reverse the inequality sign if the sides are switched.
  For example, if $2 \geqslant x$, then $x \leqslant 2$.

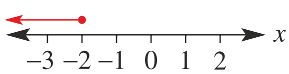
BUILDING UNDERSTANDING

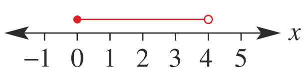
1 State three numbers that satisfy each of these inequalities.

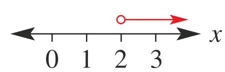
a $x \geqslant 3$ **b** $x < -1.5$ **c** $0 < x \leqslant 7$ **d** $-8.7 \leqslant x < -8.1$

2 Match the graph **a**–**c** with the inequality **A**–**C**.

A $x > 2$ **B** $x \leqslant -2$ **C** $0 \leqslant x < 4$

a


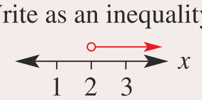
b


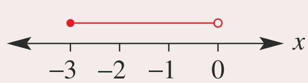
c


3 Phil houses x rabbits. If $10 < x \leqslant 13$, how many rabbits could Phil have?

Example 13 Writing inequalities from number lines

Write as an inequality.

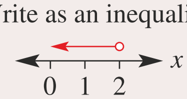
a


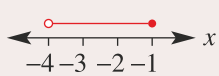
b


SOLUTION	EXPLANATION
a $x > 2$	An open circle means 2 is not included.
b $-3 \leqslant x < 0$	-3 is included but 0 is not.

Now you try

Write as an inequality.

a


b


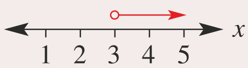
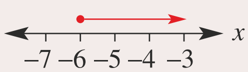
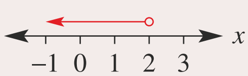
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 14 Solving linear inequalities

Solve the following inequalities and graph their solutions on a number line.

a $3x + 4 > 13$ **b** $4 - \frac{x}{3} \leqslant 6$ **c** $3x + 2 > 6x - 4$

SOLUTION	EXPLANATION
a $3x + 4 > 13$	Subtract 4 from both sides and then divide both sides by 3.
$3x > 9$	
$\therefore\ x > 3$	
	Use an open circle since x does not include 3.
b $4 - \frac{x}{3} \leqslant 6$	Subtract 4 from both sides.
$-\frac{x}{3} \leqslant 2$	Multiply both sides by -3 and reverse the inequality sign.
$\therefore x \geqslant -6$	
	Use a closed circle since x includes the number -6.
c $3x + 2 > 6x - 4$	Subtract $3x$ from both sides to gather the terms containing x.
$2 > 3x - 4$	Add 4 to both sides and then divide both sides by 3.
$6 > 3x$	
$2 > x$	
$\therefore\ x < 2$	Make x the subject. Switching sides means the inequality sign must be reversed.
	Use an open circle since x does not include 2.

Number line labels: (a) 1 2 3 4 5 x; (b) −7 −6 −5 −4 −3 x; (c) −1 0 1 2 3 x

Now you try

Solve the following inequalities and graph their solutions on a number line.

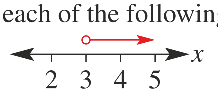
a $2x + 5 > 11$ **b** $2 - \frac{x}{3} \leqslant 4$ **c** $4x + 1 > 7x - 2$

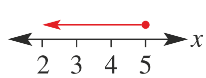
Exercise 1E

FLUENCY	1, 2–4(1/2)	2–5(1/2)	2–5(1/2)

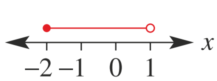
1 Write each of the following as an inequality.

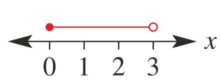
Example 13a

a **i** 

ii 

Example 13b

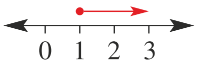
b **i** 

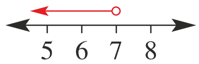
ii 

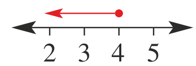
ISBN 978-1-108-77290-7

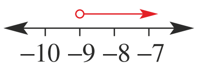
Example 13

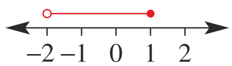
2 Write each of the following as an inequality.

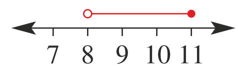
a 

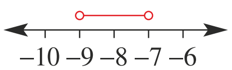
b 

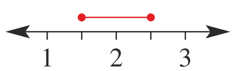
c 

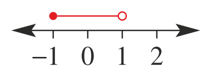
d 

e 

f 

g 

h 

i 

Example 14a

3 Solve the following inequalities and graph their solutions on a number line.

a $2x + 6 < 14$ **b** $3x + 5 \geqslant 20$ **c** $4x - 7 \geqslant 9$ **d** $\frac{x}{5} \leqslant 2$

e $\frac{x+4}{3} \leqslant 2$ **f** $\frac{5x-3}{2} > 6$ **g** $\frac{2x+3}{5} > 3$ **h** $\frac{x}{3} + 4 \leqslant 6$

i $\frac{x}{9} + 6 < 4$ **j** $-3 + \frac{x}{4} > 5$ **k** $3(3x - 1) \leqslant 7$ **l** $2(4x + 4) < 5$

Example 14b

4 Solve the following inequalities. Remember: if you multiply or divide by a negative number, you must reverse the inequality sign.

a $-5x + 7 \leqslant 9$ **b** $4 - 3x > -2$ **c** $-5x - 7 \geqslant 18$ **d** $\frac{3-x}{2} \geqslant 5$

e $\frac{5-2x}{3} > 7$ **f** $\frac{4-6x}{5} \leqslant -4$ **g** $3 - \frac{x}{2} \leqslant 8$ **h** $-\frac{x}{3} - 5 > 2$

Example 14c

5 Solve the following inequalities.

a $x + 1 < 2x - 5$ **b** $5x + 2 \geqslant 8x - 4$ **c** $7 - x > 2 + x$

d $3(x + 2) \leqslant 4(x - 1)$ **e** $7(1 - x) \geqslant 3(2 + 3x)$ **f** $-(2 - 3x) < 5(4 - x)$

PROBLEM-SOLVING	6, 7	6, 7, 8(½)	7, 8

6 For the following situations, write an inequality and solve it to find the possible values for x.

a 7 more than twice a number x is less than 12.

b Half of a number x subtracted from 4 is greater than or equal to −2.

c The product of 3 and one more than a number x is at least 2.

d The sum of two consecutive even integers, of which the smaller is x, is no more than 24.

e The sum of four consecutive even integers, of which x is the largest, is at most 148.

7 The cost of a satellite phone call is 30 cents plus 20 cents per minute.

a Find the possible cost of a call if it is:

  i shorter than 5 minutes

  ii longer than 10 minutes.

b For how many minutes can the phone be used if the cost per call is:

  i less than \$2.10?

  ii greater than or equal to \$3.50?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

8 Solve these inequalities by first multiplying by the LCD.

a $\frac{1+x}{2} < \frac{x-1}{3}$

b $\frac{2x-3}{2} \geqslant \frac{x+1}{3}$

c $\frac{3-2x}{5} \leqslant \frac{5x-1}{2}$

d $\frac{x}{2} \leqslant \frac{7-x}{3}$

e $\frac{5x}{3} \geqslant \frac{3(3-x)}{4}$

f $\frac{2(4-3x)}{5} > \frac{2(1+x)}{3}$

REASONING	9	9, 10	10, 11

9 How many whole numbers satisfy these inequalities? Give a reason.

a $x > 8$

b $2 < x \leqslant 3$

10 Solve these families of inequalities by writing x in terms of a. Consider cases where $a > 0$ and $a < 0$.

a $10x - 1 \geqslant a + 2$

b $\frac{2-x}{a} > 4$

c $a(1-x) > 7$

11 Describe the sets (in a form like $2 < x \leqslant 3$ or $-1 \leqslant x < 5$) that simultaneously satisfy these pairs of inequalities.

a $x < 5$
$x \geqslant -4$

b $x \leqslant -7$
$x > -9.5$

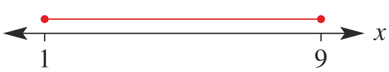
c $x \leqslant 10$
$x \geqslant 10$

ENRICHMENT: Mixed inequalities	–	–	12(½), 13

12 Solve the inequalities and graph their solutions on a number line. Consider this example first.

Solve $-2 \leqslant x - 3 \leqslant 6$.

$1 \leqslant x \leqslant 9$ (add 3 to both sides)



a $1 \leqslant x - 2 \leqslant 7$

b $-4 \leqslant x + 3 \leqslant 6$

c $-2 \leqslant x + 7 < 0$

d $0 \leqslant 2x + 3 \leqslant 7$

e $-5 \leqslant 3x + 4 \leqslant 11$

f $-16 \leqslant 3x - 4 \leqslant -10$

g $7 \leqslant 7x - 70 \leqslant 14$

h $-2.5 < 5 - 2x \leqslant 3$

13 Solve these inequalities for x.

a $\frac{3-x}{5} + \frac{1+x}{4} \geqslant 2$

b $\frac{2x-1}{3} - \frac{x+1}{4} < 1$

c $\frac{7x-1}{6} - \frac{2x-1}{2} \leqslant \frac{1}{2}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

1F Graphing straight lines CONSOLIDATING

Learning intentions

- To understand what it means for a point to lie on a line: graphically and algebraically
- To understand that straight lines have a constant gradient that can be positive, negative, zero or undefined
- To know how to determine the gradient of a line from its equation and use it and a point to sketch its graph
- To be able to find the axis intercepts of a linear graph and use them to sketch the graph
- To be able to sketch straight lines with only one intercept

In two dimensions, a straight-line graph can be described by a linear equation. Common forms of such equations are $y = mx + c$ and $ax + by = d$, where a, b, c, d and m are constants. From a linear equation a graph can be drawn by considering such features as x- and y-intercepts and the gradient.

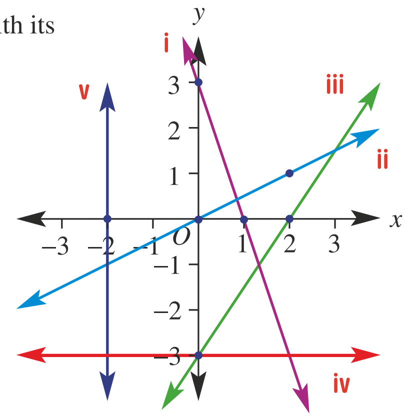
For any given straight-line graph the y-value changes by a fixed amount for each 1 unit change in the x-value. This change in y tells us the gradient of the line.

A financial analyst can use linear graphs to predict possible profit. The profit made by a lawn mower shop could be analysed with a straight-line graph of the equation:
Profit (y) = mower price (m) × sales (x) − costs (c)

LESSON STARTER Five graphs, five equations

Here are five equations and five graphs. Match each equation with its graph. Give reasons for each of your choices.

a $y = -3$
b $x = -2$
c $y = \frac{1}{2}x$
d $y = -3x + 3$
e $3x - 2y = 6$

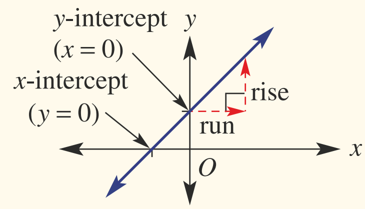


ISBN 978-1-108-77290-7

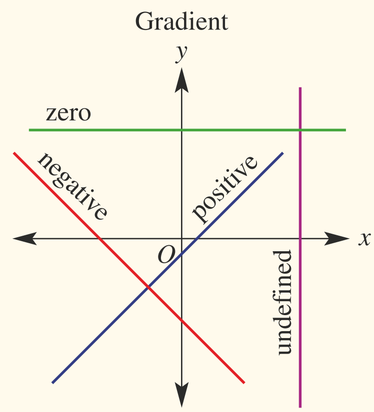
Cambridge University Press
Updated June 2022

KEY IDEAS

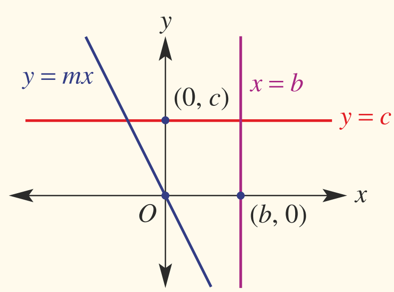
- The **gradient**, m, is a number that describes the slope of a line.
  - Gradient $= \dfrac{\text{rise}}{\text{run}}$
  - The gradient is the change in y per 1 unit change in x. Gradient is also referred to as the 'rate of change of y'.



- The **intercepts** are the points where the line crosses the x- and the y-axis.
  - The y-intercept is the y-value where $x = 0$.
  - The x-intercept is the x-value where $y = 0$.
- The gradient of a line can be positive, negative, zero (i.e. a horizontal line) or undefined (i.e. a vertical line).
- The gradient–intercept form of a straight line is $y = mx + c$, where m is the gradient and c is the y-intercept.



- Two points are needed to sketch most straight-line graphs.
- Special lines include those with only one axis intercept:
  - horizontal lines $y = c$
  - vertical lines $x = b$
  - lines passing through the origin $y = mx$.

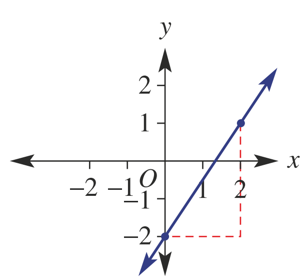


BUILDING UNDERSTANDING

1 Rearrange these equations into the form $y = mx + c$. Then state the gradient (m) and y-intercept (c).

a $y + 2x = 5$ **b** $2y = 4x - 6$ **c** $x - y = 7$ **d** $-2x - 5y = 3$

2 The graph of $y = \dfrac{3x}{2} - 2$ is shown.



a State the rise of the line if the run is:

i 2 **ii** 4 **iii** 7

b State the run in the line if the rise is:

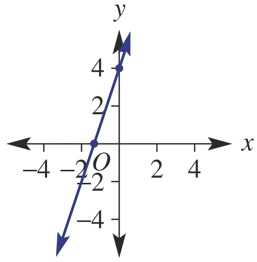
i 3 **ii** 9 **iii** 4

3 Match each of the following equations to one of the graphs shown.

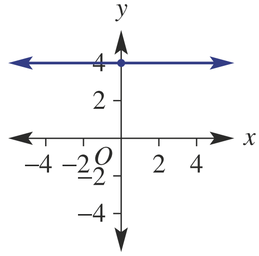
a $y = 3x + 4$ **b** $y = -2x - 4$ **c** $y = 4$

d $y = x$ **e** $y = -2x + 4$ **f** $x = -3$

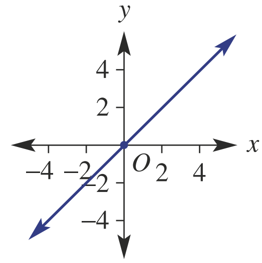
A



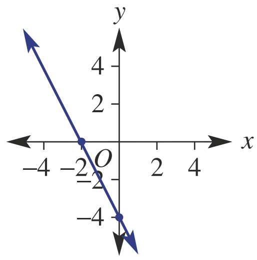
B



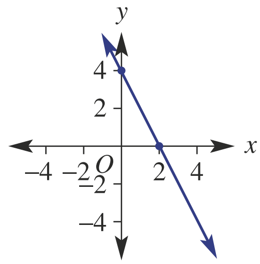
C



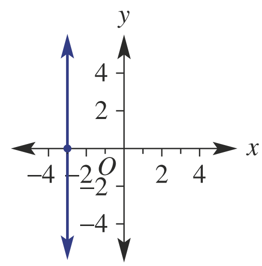
D



E



F



4 Find the value of the unknown in the following.

a $y = 2x - 4$ where $y = 0$ **b** $3x + 8y = 16$ where $x = 0$

Example 15 Deciding if a point is on a line

Decide if the point $(-2, 7)$ is on the line with the given equations.

a $y = -3x + 1$ **b** $2x + 2y = 1$

SOLUTION	EXPLANATION
a $y = -3x + 1$	Substitute $x = -2$ and $y = 7$ into the equation of the line.
$7 = -3(-2) + 1$	
$7 = 7$ True	If the equation is true, then the point is on the line.
$\therefore$ $(-2, 7)$ is on the line.	
b $2x + 2y = 1$	By substituting the point we find that the equation is false, so the point is not on the line.
$2(-2) + 2(7) = 1$	
$-4 + 14 = 1$	
$10 = 1$ False	
$\therefore$ $(-2, 7)$ is not on the line.	

Now you try

Decide if the point $(-1, 4)$ is on the line with the given equations.

a $y = -2x + 2$ **b** $3x + 3y = 2$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 16 Sketching linear graphs using the gradient–intercept method

Find the gradient and y-intercept for these linear relations and sketch each graph.

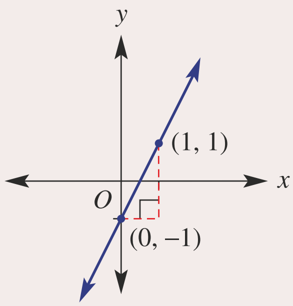
a $y = 2x - 1$ **b** $2x + 3y = 3$

SOLUTION

a $y = 2x - 1$

Gradient $= 2$

y-intercept $= -1$



EXPLANATION

In the form $y = mx + c$, the gradient is m (the coefficient of x) and c is the y-intercept.

Start by plotting the y-intercept at $(0, -1)$ on the graph.

Gradient $= 2 = \frac{2}{1}$, thus rise $= 2$ and run $= 1$.

From the y-intercept move 1 unit right (run) and 2 units up (rise) to the point $(1, 1)$. Join the two points with a line.

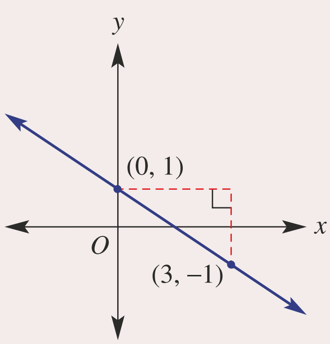
b $2x + 3y = 3$

$$3y = -2x + 3$$

$$y = -\frac{2}{3}x + 1$$

Gradient $= -\frac{2}{3}$

y-intercept $= 1$



Rewrite in the form $y = mx + c$ by subtracting $2x$ from both sides and then dividing both sides by 3.

Note: $\frac{-2x}{3}$ can also be written as $-\frac{2x}{3}$.

The gradient is the coefficient of x and the y-intercept is the constant term.

Start the graph by plotting the y-intercept at $(0, 1)$.

Gradient $= -\frac{2}{3}$ (run $= 3$ and fall $= 2$). From the point $(0, 1)$ move 3 units right (run) and 2 units down (fall) to $(3, -1)$.

Now you try

Find the gradient and y-intercept for these linear relations and sketch each graph.

a $y = 3x - 1$ **b** $3x + 4y = 4$

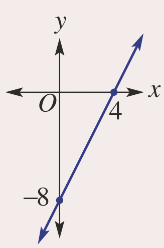
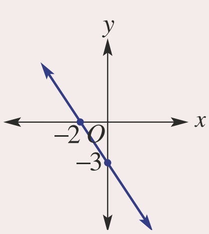
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 17 Sketching linear graphs using the x- and y-intercepts

Sketch the following by finding the x- and y-intercepts.

a $y = 2x - 8$ **b** $-3x - 2y = 6$

SOLUTION	EXPLANATION
a $y = 2x - 8$	
y-intercept ($x = 0$): $y = 2(0) - 8$, $y = -8$. The y-intercept is -8.	The y-intercept is at $x = 0$. For $y = mx + c$, c is the y-intercept.
x-intercept ($y = 0$): $0 = 2x - 8$, $8 = 2x$, $x = 4$. The x-intercept is 4.	The x-intercept is on the x-axis, so $y = 0$. Solve the equation for x.
	Plot and label the intercepts and join with a straight line.
b $-3x - 2y = 6$	
y-intercept ($x = 0$): $-3(0) - 2y = 6$, $-2y = 6$, $y = -3$. The y-intercept is -3	The y-intercept is on the y-axis so substitute $x = 0$. Simplify and solve for y.
x-intercept ($y = 0$): $-3x - 2(0) = 6$, $-3x = 6$, $x = -2$. The x-intercept is -2.	The x-intercept is on the x-axis so substitute $y = 0$. Simplify and solve for x.
	Sketch by drawing a line passing through the two axes intercepts. Label the intercepts.

Now you try

Sketch the following by finding the x- and y-intercepts.

a $y = 2x - 4$ **b** $-2x - 5y = 10$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

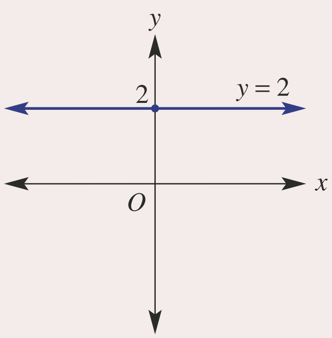
Example 18 Sketching lines with one intercept

Sketch these special lines.

a $y = 2$ **b** $x = -3$ **c** $y = -\frac{1}{2}x$

SOLUTION

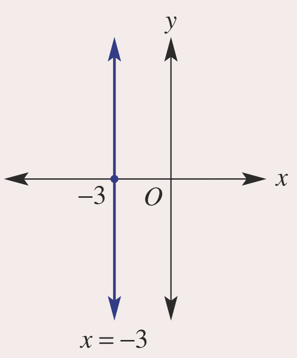
a



EXPLANATION

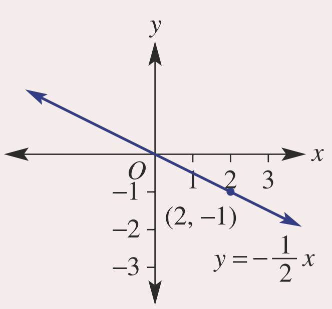
The y-coordinate of every point must be 2, hence $y = 2$ is a horizontal line passing through $(0, 2)$.

b



The x-coordinate of every point must be -3, hence $x = -3$ is a vertical line passing through $(-3, 0)$.

c



Both the x- and y-intercepts are $(0, 0)$, so the gradient can be used to find a second point.

The gradient $= -\frac{1}{2}$, hence use run $= 2$ and fall $= 1$.

Alternatively, substitute $x = 1$ to find a second point:

$$\begin{aligned} x = 1, y &= -\frac{1}{2} \times (1) \\ &= -\frac{1}{2} \end{aligned}$$

Now you try

Sketch these special lines.

a $y = -1$ **b** $x = 2$ **c** $y = -\frac{1}{3}x$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 1F

FLUENCY	1, 2–6(1/2)	2–6(1/2)	2–6(1/3)

Example 15a, Example 15b

1 Decide if the point $(-1, 5)$ is on the line with the given equations.
- a i $y = -3x + 2$ ii $y = -2x - 1$
- b i $2x + 2y = 5$ ii $3x + 3y = 12$

Example 15

2 Decide if the point $(3, -1)$ is on the line with the following equations.
- a $y = x - 4$
- b $y = -x + 2$
- c $y = -3x + 9$
- d $x + 2y = 6$
- e $-2x - y = -5$
- f $3y - 4x = -9$

Example 16a

3 Find the gradient and y-intercept for these linear relations and sketch a graph.
- a $y = 5x - 3$
- b $y = 2x + 3$
- c $y = -2x - 1$
- d $y = -x + 2$
- e $y = x - 4$
- f $y = -\frac{3}{2}x + 1$
- g $y = \frac{4}{3}x - 2$
- h $y = -\frac{7}{2}x + 6$
- i $y = 0.5x - 0.5$
- j $y = 1 - x$
- k $y = 3 + \frac{2}{3}x$
- l $y = 0.4 - 0.2x$

Example 16b

4 Find the gradient and y-intercept for these linear relations and sketch each graph.
- a $3x + y = 12$
- b $10x + 2y = 5$
- c $x - y = 7$
- d $3x - 3y = 6$
- e $4x - 3y = 9$
- f $-x - y = \frac{1}{3}$
- g $-y - 4x = 8$
- h $2y + x = \frac{1}{2}$

Example 17

5 Sketch the following by finding the x- and y-intercepts.
- a $y = 3x - 6$
- b $y = 2x + 4$
- c $y = 4x + 10$
- d $y = 3x - 4$
- e $y = 7 - 2x$
- f $y = 4 - \frac{x}{2}$
- g $3x + 2y = 12$
- h $2x + 5y = 10$
- i $4y - 3x = 24$
- j $x + 2y = 5$
- k $3x + 4y = 7$
- l $5y - 2x = 12$

Example 18

6 Sketch these special lines.
- a $y = -4$
- b $y = 1$
- c $x = 2$
- d $x = -\frac{5}{2}$
- e $y = 0$
- f $x = 0$
- g $y = 4x$
- h $y = -3x$
- i $y = -\frac{1}{3}x$
- j $y = \frac{5x}{2}$
- k $x + y = 0$
- l $4 - y = 0$

PROBLEM-SOLVING	7, 8	7, 8, 10	8–11

7 Sam is earning some money picking apples. She gets paid \$10 plus \$2 per kilogram of apples that she picks. If Sam earns \$$C$ for n kg of apples picked, complete the following.
- a Write a rule for C in terms of n.
- b Sketch a graph for $0 \leqslant n \leqslant 10$, labelling the endpoints.
- c Use your rule to find:
  - i the amount Sam earned after picking 9 kg of apples
  - ii the number of kilograms of apples Sam picked if she earned \$57.

ISBN 978-1-108-77290-7

8 A 90 L tank full of water begins to leak at a rate of 1.5 litres per hour. If V litres is the volume of water in the tank after t hours, complete the following.

a Write a rule for V in terms of t.
b Sketch a graph for $0 \leqslant t \leqslant 60$, labelling the endpoints.
c Use your rule to find:
  i the volume of water after 5 hours
  ii the time taken to completely empty the tank.

9 Alex earns \$84 for 12 hours of work.
a Write Alex's rate of pay per hour.
b Write the equation for Alex's total pay, \$$P$, after t hours of work.

10 It costs Jesse \$1600 to maintain and drive his car for 32 000 km.
a Find the cost in \$ per km.
b Write a formula for the cost, \$$C$, of driving Jesse's car for k kilometres.
c If Jesse also pays a total of \$1200 for registration and insurance, write the new formula for the cost to Jesse of owning and driving his car for k kilometres.

11 $D = 25t + 30$ is an equation for calculating the distance, D km, from home that a cyclist has travelled after t hours.
a What is the gradient of the graph of the given equation? What does it represent?
b What could the 30 represent?
c If a graph of D against t is drawn, what would be the intercept on the D-axis?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

REASONING	12	12, 13	13, 14

12 A student with a poor understanding of straight-line graphs writes down some incorrect information next to each equation. Decide how the error might have been made and then correct the information.

a $y = \dfrac{2x+1}{2}$ (gradient = 2)

b $y = 0.5(x + 3)$ (y-intercept = 3)

c $3x + y = 7$ (gradient = 3)

d $x - 2y = 4$ (gradient = 1)

13 Write expressions for the gradient and y-intercept of these equations.

a $ay = 3x + 7$

b $ax - y = b$

c $by = 3 - ax$

14 A straight line is written in the form $ax + by = d$. In terms of a, b and d, find:

a the x-intercept

b the y-intercept

c the gradient

ENRICHMENT: Graphical areas	–	–	15

15 Find the area enclosed by these lines.

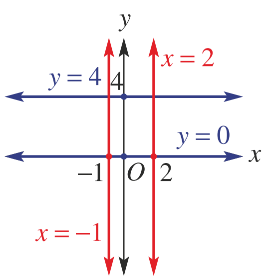
a $x = 2, x = -1, y = 0, y = 4$

b $x = 3, y = 2x, y = 0$

c $x = -3, y = -\dfrac{1}{2}x + 2, y = -2$

d $2x - 5y = -10, y = -2, x = 1$

e $y = 3x - 2, y = -3, y = 2 - x$



ISBN 978-1-108-77290-7

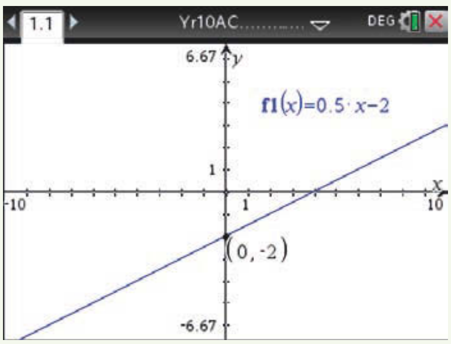
Cambridge University Press
Updated June 2022

Using calculators to sketch straight lines

1. Sketch a graph of $y = 0.5x - 2$ and locate the x- and y-intercepts.
2. Construct a table of values for $y = 0.5x - 2$.

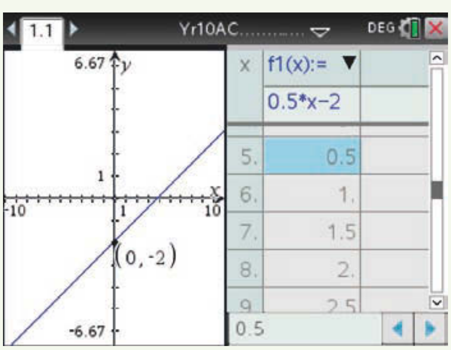
Using the TI-Nspire:

1. In a **Graphs** page, enter the rule $f1(x) = 0.5x - 2$. Use menu **>Trace>Graph Trace** and use the arrow keys to move left or right to observe intercepts. **Analyze Graph > Zero** can also be used for the x-intercept.



Hint: pressing enter will paste the intercept coordinates on the graph.

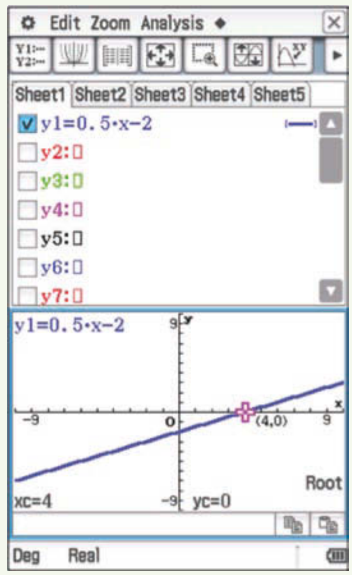
2. Press menu **>Table>Split-screen Table** to show the Table of Values.



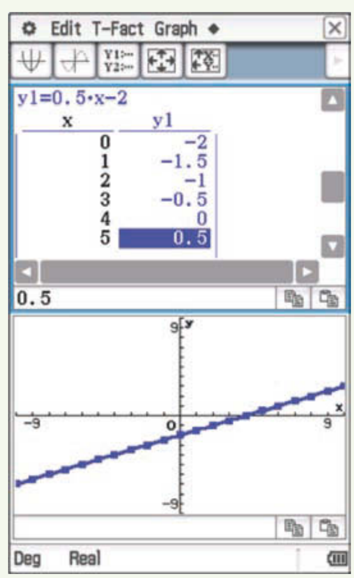
Hint: use ctrl + T as a shortcut to show the Table of Values.

Using the ClassPad:

1. In the **Graph&table** application enter the rule $y1 = 0.5x - 2$ followed by **EXE**. Tap the graph icon to see the graph. Tap **Zoom**, **Quick Standard**. Tap **Analysis**, **G-Solve**, **Root** to locate the x-intercept. Tap **Analysis**, **G-Solve**, ***y*-intercept** to locate the y-intercept.



2. Tap the table preferences icon and set the table preferences to start at –10 and end at 10 with steps 1. Tap the table icon to see the table.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

1G Finding an equation of a line

Learning intentions
- To understand that the gradient is the same between any two points on a straight line
- To know how to find the gradient of a line using two points
- To understand the gradient–intercept form, $y = mx + c$, of a straight line equation
- To be able to find the equation of a line given two points on the line
- To know the form of the equation of horizontal and vertical lines

It is a common procedure in mathematics to find the equation (or rule) of a straight line. Once the equation of a line is determined, it can be used to find the exact coordinates of other points on the line. Mathematics such as this can be used, for example, to predict a future company share price or the water level in a dam after a period of time.

Business equipment, such as a parcel courier's van, must eventually be replaced. For tax purposes, accountants calculate annual depreciation using the straight-line method. This reduces the equipment's value by an equal amount each year.

LESSON STARTER Fancy formula

Here is a proof of a rule for the equation of a straight line between any two given points.
Some of the steps are missing. See if you can fill them in.

$y = mx + c$

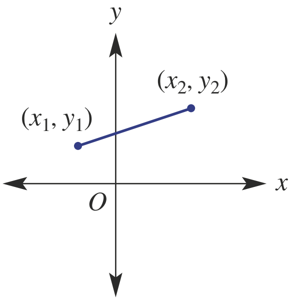
$y_1 = mx_1 + c$ (Substitute $(x, y) = (x_1, y_1)$.)

$\therefore c =$ ________

$\therefore y = mx +$ ________

$\therefore y - y_1 = m($______$)$

where $m = \dfrac{}{x_2 - x_1}$



KEY IDEAS

- Horizontal lines have the equation $y = c$, where c is the y-intercept.
- Vertical lines have the equation $x = k$, where k is the x-intercept.
- Given the gradient (m) and the y-intercept (c), use $y = mx + c$ to state the equation of the line.
- To find the equation of a line when given any two points, find the gradient (m), then:
  - substitute a point to find c in $y = mx + c$, or
  - use $y - y_1 = m(x - x_1)$, where $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and (x_1, y_1), (x_2, y_2) are points on the line.

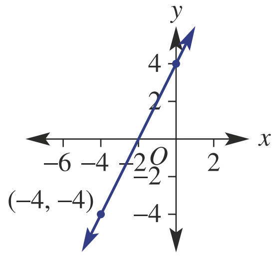
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

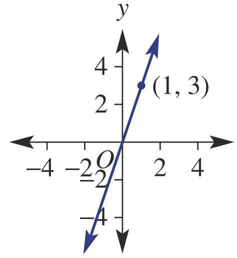
BUILDING UNDERSTANDING

1 State the gradient of the following lines.

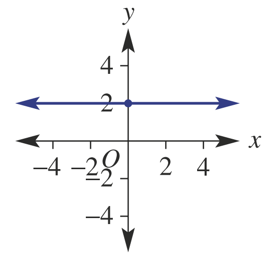
a



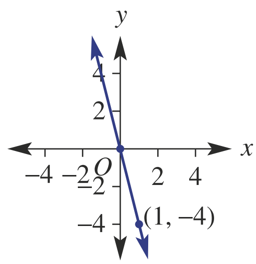
b



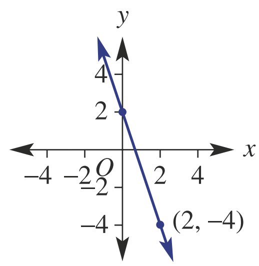
c



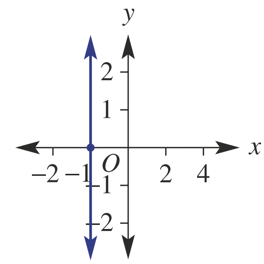
d



e



f



2 Find the value of c in $y = -2x + c$ when:

a $x = 3$ and $y = 2$

b $x = -1$ and $y = -4$

c $x = \frac{5}{2}$ and $y = 7$

Example 19 Finding the gradient of a line joining two points

Determine the gradient of the line joining the pair of points $(-3, 8)$ and $(5, -2)$.

SOLUTION

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

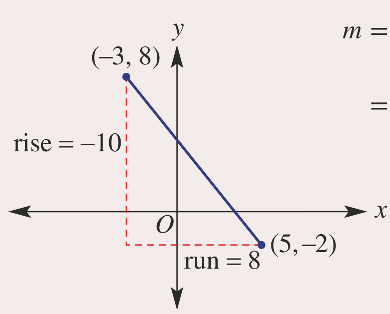
$$= \frac{-2 - 8}{5 - (-3)}$$

$$= -\frac{10}{8}$$

$$= -\frac{5}{4}$$

EXPLANATION

Use $(x_1, y_1) = (-3, 8)$
and $(x_2, y_2) = (5, -2)$.
Remember that $5 - (-3) = 5 + 3$.
Alternatively,



$$m = \frac{\text{rise}}{\text{run}}$$

$$= \frac{-10}{8} \text{ (from diagram)}$$

Now you try

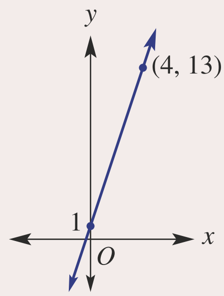
Determine the gradient of the line joining the pair of points $(-2, 6)$ and $(3, -1)$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 20 Finding the equation of a line given the y-intercept and a point

Find the equation of the straight line shown.



SOLUTION

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{13 - 1}{4 - 0}$$
$$= \frac{12}{4}$$
$$= 3$$

and $c = 1$

$\therefore y = 3x + 1$

EXPLANATION

The equation of a straight line is of the form $y = mx + c$.

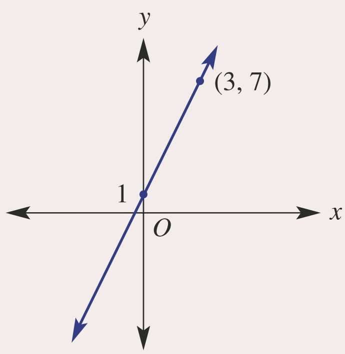
Find m using $(x_1, y_1) = (0, 1)$ and $(x_2, y_2) = (4, 13)$, or using $m = \frac{\text{rise}}{\text{run}}$ from the graph.

The y-intercept is 1.

Substitute $m = 3$ and $c = 1$.

Now you try

Find the equation of the line shown.

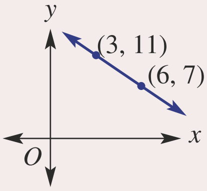


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 21 Finding the equation of a line given two points

Find the equation of the straight line shown.



SOLUTION

Method 1

$y = mx + c$

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{7 - 11}{6 - 3}$

$= -\dfrac{4}{3}$

Use $(x_1, y_1) = (3, 11)$ and $(x_2, y_2) = (6, 7)$ in gradient formula, or $m = \dfrac{\text{rise}}{\text{run}}$ from the graph where rise $= -4$ (fall).

$y = -\dfrac{4}{3}x + c$

Substitute $m = -\dfrac{4}{3}$ into $y = mx + c$.

$7 = -\dfrac{4}{3} \times (6) + c$

$7 = -8 + c$

$15 = c$

Substitute the point (6, 7) or (3, 11) to find the value of c.

$\therefore y = -\dfrac{4}{3}x + 15$

Write the rule with both m and c.

Method 2

$y - y_1 = m(x - x_1)$

$y - 11 = -\dfrac{4}{3}(x - 3)$

Choose $(x_1, y_1) = (3, 11)$ or alternatively choose (6, 7).

$m = -\dfrac{4}{3}$ was found using method 1.

$y = -\dfrac{4}{3}x + 4 + 11$

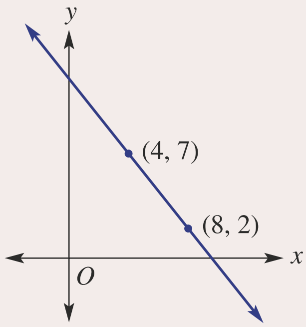
Expand brackets and make y the subject.

$= -\dfrac{4}{3}x + 15$

$-\dfrac{4}{3} \times (-3) = 4$

Now you try

Find the equation of the straight line shown.



ISBN 978-1-108-77290-7

Exercise 1G

FLUENCY	1, 2(1/2), 3, 4–5(1/2)	2–5(1/2)	2(1/3), 3–5(1/2)

Example 19

1 Determine the gradient of the line joining the following pairs of points.

a $(-2, 5)$ and $(2, -3)$
b $(-1, 4)$ and $(4, -2)$

Example 19

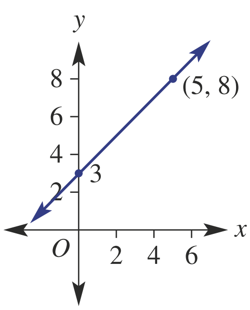
2 Determine the gradient of the line joining the following pairs of points.

a $(4, 2), (12, 4)$
b $(1, 4), (3, 8)$
c $(0, 2), (2, 7)$
d $(3, 4), (6, 13)$
e $(8, 4), (5, 4)$
f $(2, 7), (4, 7)$
g $(-1, 3), (2, 0)$
h $(-3, 2), (-1, 7)$
i $(-3, 4), (4, -1)$
j $(2, -3), (2, -5)$
k $(2, -3), (-4, -12)$
l $(-2, -5), (-4, -2)$

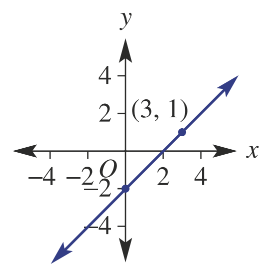
Example 20

3 Find the equation of the straight lines with the given y-intercepts.

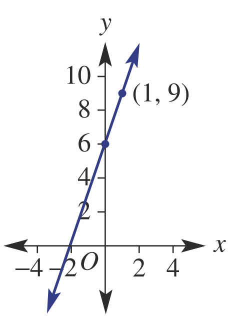
a



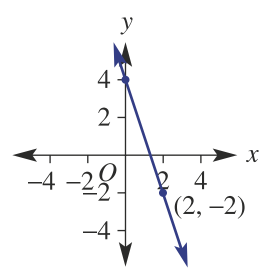
b



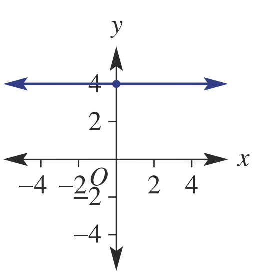
c



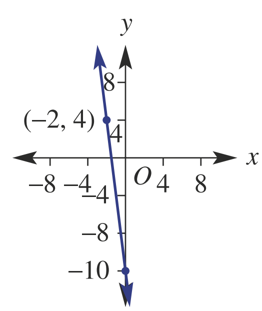
d



e



f

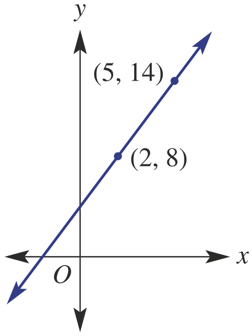


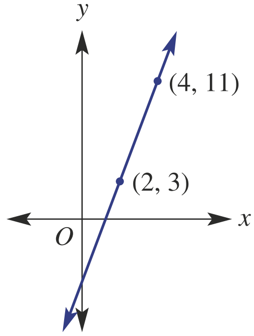
ISBN 978-1-108-77290-7

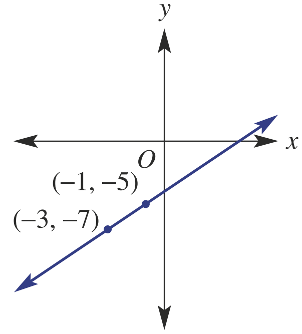
Cambridge University Press
Updated June 2022

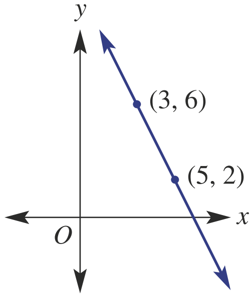
Example 21

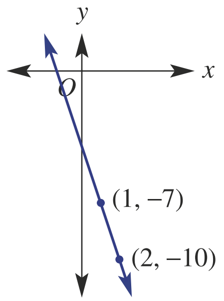
4 Find the equation of the straight lines with the given points.

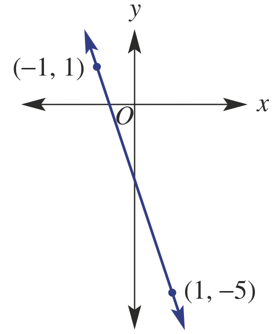
a


b


c


d


e


f


5 Given the following tables of values, determine the linear equation relating x and y in each case.

a

x	0	3
y	5	14

b

x	4	6
y	−4	−8

c

x	−1	3
y	−2	0

d

x	−2	1
y	2	−4

PROBLEM-SOLVING 6 | 6, 7 | 7, 8

6 Kyle invests some money in a simple savings fund and the amount increases at a constant rate over time. He hopes to buy a boat when the investment amount reaches \$20 000.
After 3 years the amount is \$16 500 and after 6 years the amount is \$18 000.

a Find a rule linking the investment amount (\$$A$) and time ($t$ years).
b How much did Kyle invest initially (i.e. when $t = 0$)?
c How long does Kyle have to wait before he can buy his boat?
d What would be the value of the investment after $12\frac{1}{2}$ years?

7 The cost of hiring a surfboard involves an up-front fee plus an hourly rate. Three hours of hire costs \$50 and 7 hours costs \$90.

a Sketch a graph of cost, \$$C$, for t hours of hire using the information given above.
b Find a rule linking \$$C$ in terms of t hours.
c **i** State the cost per hour.
ii State the up-front fee.

ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

8 **a** The following information applies to the filling of a flask with water, at a constant rate. In each case, find a rule for the volume, V litres, in terms of t minutes.

- **i** Initially (i.e. at $t = 0$) the flask is empty (i.e. $V = 0$) and after 1 minute it contains 4 litres of water.
- **ii** Initially (i.e. at $t = 0$) the flask is empty (i.e. $V = 0$) and after 3 minutes it contains 9 litres of water.
- **iii** After 1 and 2 minutes, the flask has 2 and 3 litres of water, respectively.
- **iv** After 1 and 2 minutes, the flask has 3.5 and 5 litres of water, respectively.

b For parts **iii** and **iv** above, find how much water was in the flask initially.

c Write your own information that would give the rule $V = -t + b$.

REASONING	9	9, 10	9, 10

9 A line joins the two points $(-1, 3)$ and $(4, -2)$.

a Calculate the gradient of the line using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ where $(x_1, y_1) = (-1, 3)$ and $(x_2, y_2) = (4, -2)$.

b Calculate the gradient of the line using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ where $(x_1, y_1) = (4, -2)$ and $(x_2, y_2) = (-1, 3)$.

c What conclusions can you draw from your results from parts **a** and **b** above? Give an explanation.

10 A line passes through the points $(1, 3)$ and $(4, -1)$.

a Calculate the gradient.

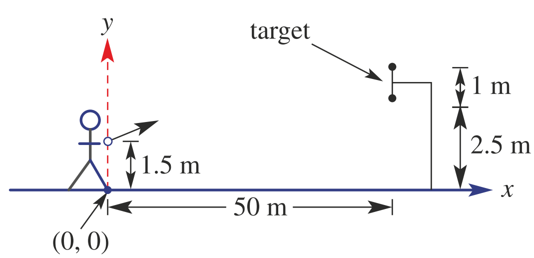
b Using $y - y_1 = m(x - x_1)$ and $(x_1, y_1) = (1, 3)$, find the rule for the line.

c Using $y - y_1 = m(x - x_1)$ and $(x_1, y_1) = (4, -1)$, find the rule for the line.

d What do you notice about your results from parts **b** and **c**? Can you explain why this is the case?

ENRICHMENT: Linear archery	–	–	11

11 An archer's target is 50 m away and is 2.5 m off the ground, as shown. Arrows are fired from a height of 1.5 m and the circular target has a diameter of 1 m.



a Find the gradient of the straight trajectory from the arrow (in firing position) to:

- **i** the bottom of the target
- **ii** the top of the target.

b If the position of the ground directly below the firing arrow is the point $(0, 0)$ on a Cartesian plane, find the equation of the straight trajectory to:

- **i** the bottom of the target
- **ii** the top of the target.

c If $y = mx + c$ is the equation of the arrow's trajectory, what are the possible values of m if the arrow is to hit the target?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Progress quiz

1A **1** Simplify the following.

a $15a^2b + 2ab - 6ba^2 + 8b$ **b** $-3xy \times 4x^2$ **c** $4(m + 5) + 3(3m - 2)$

1B **2** Simplify by cancelling common factors.

a $\dfrac{36mk^2}{9mk}$ **b** $\dfrac{3a - 12}{3}$ **c** $\dfrac{21x - 3x^2}{6x}$

d $\dfrac{4}{m} \times \dfrac{m + 3}{12}$ **e** $\dfrac{a + 4}{4a} \times \dfrac{18a^2}{a + 4}$ **f** $\dfrac{6h - 15}{6} \div \dfrac{2h - 5}{5}$

1C **3** Simplify the following algebraic fractions.

a $\dfrac{3}{4} + \dfrac{m}{8}$ **b** $\dfrac{2}{3} - \dfrac{5}{2x}$ **c** $\dfrac{a + 4}{8} + \dfrac{1 - 3a}{12}$ **d** $\dfrac{5}{m - 1} - \dfrac{2}{m - 3}$

1D **4** Solve the following equations and check your solution by substitution.

a $2x + 8 = 18$ **b** $2(3k - 4) = -17$ **c** $\dfrac{m + 5}{5} = 7$ **d** $\dfrac{2a - 3}{3} = \dfrac{4a + 2}{4}$

1E **5** Solve the following inequalities and graph their solutions on a number line.

a $2a + 3 > 9$ **b** $8 - \dfrac{x}{2} \leqslant 10$

c $5m + 2 > 7m - 6$ **d** $-(a - 3) \leqslant 5(a + 3)$

1F **6** Decide if the point $(-3, 2)$ is on the line with the given equations.

a $y = x + 2$ **b** $-2x + y = 8$

1F **7** Sketch the following linear relations. For parts **a** and **b**, use the method suggested.

a $y = -\dfrac{3}{2}x + 1$; Use the gradient and y-intercept.

b $-2x - 3y = 6$; Use the x- and y-intercepts.

c $y = 3$

d $x = -2$

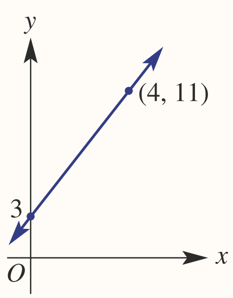
e $y = -\dfrac{3}{4}x$

1F **8** State the gradient and y-intercept of the following lines.

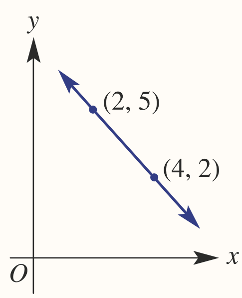
a $y = 3x - 2$ **b** $3x + 5y = 15$

1G **9** Find the equation of each straight line shown.

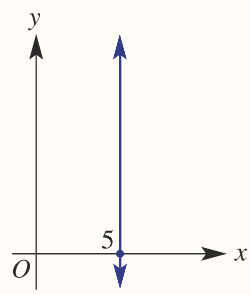
a



b



c



ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

1H Length and midpoint of a line segment

Learning intentions
- To know the meaning of the terms line segment and midpoint
- To understand that Pythagoras' theorem can be used to find the distance between two points
- To be able to find the length of a line segment (or distance between two points)
- To know how to find the midpoint of a line segment

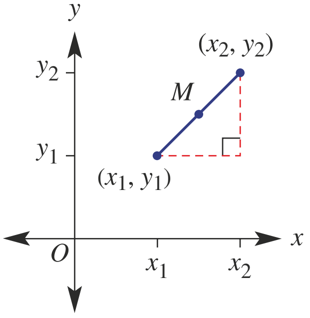
Two important features of a line segment (or line interval) are its length and midpoint. The length can be found using Pythagoras' theorem and the midpoint can be found by considering the midpoints of the horizontal and vertical components of the line segment.

Using coordinates to define locations and calculate distances are widely applied procedures, including by spatial engineers, geodetic engineers, surveyors, cartographers, navigators, geologists, archaeologists and biologists.

LESSON STARTER Developing the rules

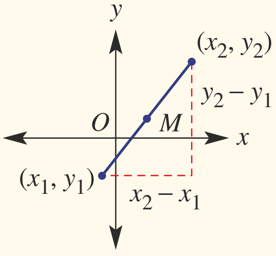
The line segment shown has endpoints (x_1, y_1) and (x_2, y_2).

- Length: Use your knowledge of Pythagoras' theorem to find the rule for the length of the segment.
- Midpoint: State the coordinates of M (the midpoint) in terms of x_1, y_1, x_2 and y_2. Give reasons for your answer.



KEY IDEAS

- The **length of a line segment** (or distance between two points (x_1, y_1) and (x_2, y_2)) d is given by the rule:
  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
  - This rule comes from Pythagoras' theorem where the distance d is the length of the hypotenuse of the right-angled triangle formed.



- The **midpoint** M of a line segment between (x_1, y_1) and (x_2, y_2) is given by:
  $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

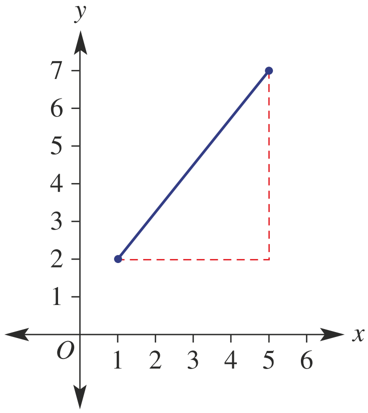
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

BUILDING UNDERSTANDING

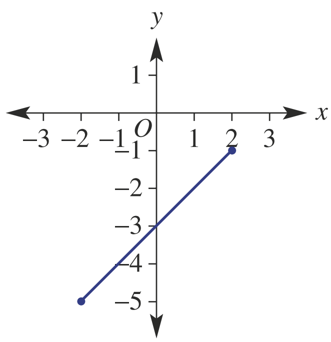
1 The endpoints for the given line segment are (1, 2) and (5, 7).

- **a** What is the horizontal distance between the two endpoints?
- **b** What is the vertical distance between the two endpoints?
- **c** Use Pythagoras' theorem ($c^2 = a^2 + b^2$) to find the exact length of the segment.
- **d** State the midpoint of the segment.



2 The endpoints for the given line segment are (−2, −5) and (2, −1).

- **a** What is the horizontal distance between the two endpoints?
- **b** What is the vertical distance between the two endpoints?
- **c** Use Pythagoras' theorem ($c^2 = a^2 + b^2$) to find the exact length of the segment.
- **d** State the midpoint of the segment.



3 Simplify the following.

- **a** $\dfrac{2+4}{2}$
- **b** $\dfrac{3+8}{2}$
- **c** $\dfrac{-4+10}{2}$
- **d** $\dfrac{-6+(-2)}{2}$

Example 22 Finding the distance between two points

Find the exact distance between each pair of points.

- **a** (0, 2) and (1, 7)
- **b** (−3, 8) and (4, −1)

SOLUTION

a
$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(1-0)^2 + (7-2)^2} \\ &= \sqrt{1^2 + 5^2} \\ &= \sqrt{26} \end{aligned}$$

b
$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(4-(-3))^2 + (-1-8)^2} \\ &= \sqrt{7^2 + (-9)^2} \\ &= \sqrt{49 + 81} \\ &= \sqrt{130} \end{aligned}$$

EXPLANATION

a Let $(x_1, y_1) = (0, 2)$ and $(x_2, y_2) = (1, 7)$.
Alternatively, sketch the points and use Pythagoras' theorem.
Simplify and express your answer exactly, using a surd.

b Let $(x_1, y_1) = (-3, 8)$ and $(x_2, y_2) = (4, -1)$.
Alternatively, let $(x_1, y_1) = (4, -1)$ and $(x_2, y_2) = (-3, 8)$.
Either way the answers will be the same.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Now you try

Find the exact distance between each pair of points.

a (0, 3) and (1, 5) **b** (−2, 7) and (3, −1)

Example 23 Finding the midpoint of a line segment joining two points

Find the midpoint of the line segment joining (−3, −5) and (2, 8).

SOLUTION

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{-3 + 2}{2}, \frac{-5 + 8}{2}\right)$$
$$= \left(-\frac{1}{2}, \frac{3}{2}\right)$$

EXPLANATION

Let $(x_1, y_1) = (-3, -5)$ and $(x_2, y_2) = (2, 8)$.

This is equivalent to finding the average of the x-coordinates and the average of the y-coordinates of the two points.

Now you try

Find the midpoint of the line segment joining (−2, −6) and (3, −2).

Example 24 Using a given distance to find coordinates

Find the values of a if the distance between $(2, a)$ and $(4, 9)$ is $\sqrt{5}$.

SOLUTION

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$\sqrt{5} = \sqrt{(4 - 2)^2 + (9 - a)^2}$$
$$\sqrt{5} = \sqrt{2^2 + (9 - a)^2}$$
$$5 = 4 + (9 - a)^2$$
$$1 = (9 - a)^2$$
$$\pm 1 = 9 - a$$

So $9 - a = 1$ or $9 - a = -1$.

$\therefore a = 8$ or $a = 10$

EXPLANATION

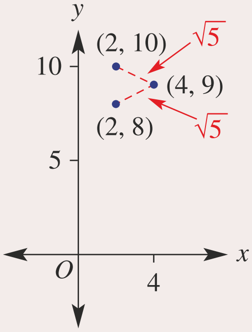
Substitute all the given information into the rule for the distance between two points.

Simplify and then square both sides to eliminate the square roots.

Subtract 4 from both sides and take the square root of each side.

Remember, if $x^2 = 1$ then $x = \pm 1$.

Solve for a. You can see there are two solutions.



Now you try

Find the values of a if the distance between $(1, a)$ and $(3, 7)$ is $\sqrt{13}$.

ISBN 978-1-108-77290-7

Exercise 1H

FLUENCY	1, 2–3(1/2)	2–3(1/2)	2–3(1/3)

Example 22a
Example 22b

1 Find the exact distance between:

a **i** $(0, 1)$ and $(1, 3)$ **ii** $(0, 3)$ and $(2, 7)$

b **i** $(-3, 7)$ and $(4, -2)$ **ii** $(-2, 8)$ and $(1, -1)$

Example 22

2 Find the exact distance between these pairs of points.

a $(0, 4)$ and $(2, 9)$ **b** $(0, -1)$ and $(3, 6)$ **c** $(-1, 4)$ and $(0, -2)$

d $(-3, 8)$ and $(1, 1)$ **e** $(-2, -1)$ and $(4, -2)$ **f** $(-8, 9)$ and $(1, -3)$

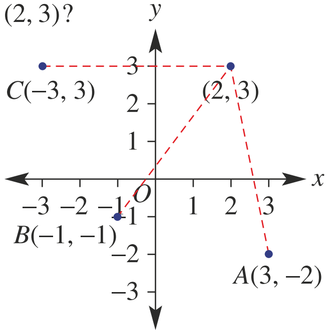
g $(-8, -1)$ and $(2, 0)$ **h** $(-4, 6)$ and $(8, -1)$ **i** $(-10, 11)$ and $(-4, 10)$

Example 23

3 Find the midpoint of the line segment joining the given points in Question **2**.

PROBLEM-SOLVING	4, 5–6(1/2)	4, 5–6(1/2)	5–6(1/2), 7

4 Which of the points A, B or C shown on these axes is closest to the point $(2, 3)$?



5 Find the value of a and b when:

a The midpoint of $(a, 3)$ and $(7, b)$ is $(5, 4)$.

b The midpoint of $(a, -1)$ and $(2, b)$ is $(-1, 2)$.

c The midpoint of $(-3, a)$ and $(b, 2)$ is $\left(-\frac{1}{2}, 0\right)$.

d The midpoint of $(-5, a)$ and $(b, -4)$ is $\left(-\frac{3}{2}, \frac{7}{2}\right)$.

Example 24

6 Find the values of a when:

a The distance between $(1, a)$ and $(3, 5)$ is $\sqrt{8}$.

b The distance between $(2, a)$ and $(5, 1)$ is $\sqrt{13}$.

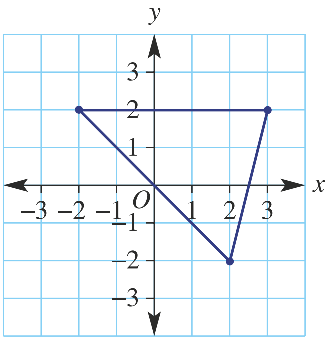
c The distance between $(a, -1)$ and $(4, -3)$ is $\sqrt{29}$.

d The distance between $(-3, -5)$ and $(a, -9)$ is 5.

7 A block of land is illustrated on this simple map, which uses the ratio 1 : 100 (i.e. 1 unit represents 100 m).

a Find the perimeter of the block, correct to the nearest metre.

b The block is to be split up into four triangular areas by building three fences that join the three midpoints of the sides of the block. Find the perimeter of the inside triangular area.



ISBN 978-1-108-77290-7

REASONING	8	8, 9	9, 10(½)

8 A line segment has endpoints $(-2, 3)$ and $(1, -1)$.
- a Find the midpoint using $(x_1, y_1) = (-2, 3)$ and $(x_2, y_2) = (1, -1)$.
- b Find the midpoint using $(x_1, y_1) = (1, -1)$ and $(x_2, y_2) = (-2, 3)$.
- c Give a reason why the answers to parts **a** and **b** are the same.
- d Find the length of the segment using $(x_1, y_1) = (-2, 3)$ and $(x_2, y_2) = (1, -1)$.
- e Find the length of the segment using $(x_1, y_1) = (1, -1)$ and $(x_2, y_2) = (-2, 3)$.
- f What do you notice about your answers to parts **d** and **e**? Give an explanation for this.

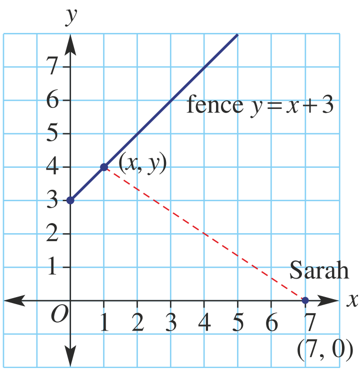
9 The distance between the points $(-2, -1)$ and $(a, 3)$ is $\sqrt{20}$. Find the values of a and use a Cartesian plane to illustrate why there are two solutions for a.

10 Find the coordinates of the point that divides the segment joining $(-2, 0)$ and $(3, 4)$ in the given ratio. Ratios are to be considered from left to right.

a 1 : 1 b 1 : 2 c 2 : 1 d 4 : 1 e 1 : 3 f 2 : 3

ENRICHMENT: Shortest distance	–	–	11

11 Sarah pinpoints her position on a map as $(7, 0)$ and wishes to hike towards a fence line that follows the path $y = x + 3$, as shown. (Note: 1 unit = 100 m).



- a Using the points $(7, 0)$ and (x, y), write a rule in terms of x and y for the distance between Sarah and the fence.
- b Use the equation of the fence line to write the rule in part **a** in terms of x only.
- c Use your rule from part **b** to find the distance between Sarah and the fence line to the nearest metre when:
  i $x = 1$ ii $x = 2$ iii $x = 3$ iv $x = 4$
- d Which x-value from part **c** gives the shortest distance?
- e Consider any point on the fence line and find the coordinates of the point such that the distance will be a minimum. Give reasons.

ISBN 978-1-108-77290-7

1I Perpendicular and parallel lines

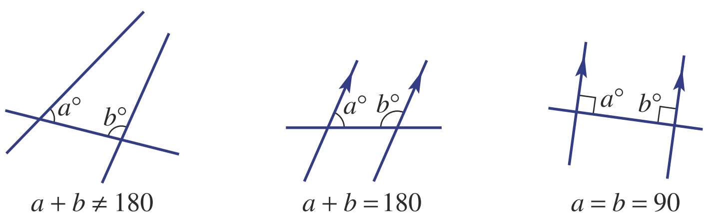
Learning intentions

- To know what it means for lines to be parallel or perpendicular
- To know that parallel lines have the same gradient
- To know that the gradients of perpendicular lines multiply to −1
- To be able to determine if lines are parallel or perpendicular using their gradients
- To be able to find the equation of a parallel or perpendicular line given a point on the line

Euclid of Alexandria (300 BC) was a Greek mathematician and is known as the 'father of geometry'. In his texts, known as *Euclid's Elements*, his work is based on five simple axioms. The fifth axiom, called the 'Parallel Postulate', states: 'It is true that if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, intersect on that side on which are the angles less than the two right angles.'

The mathematician Euclid of Alexandria

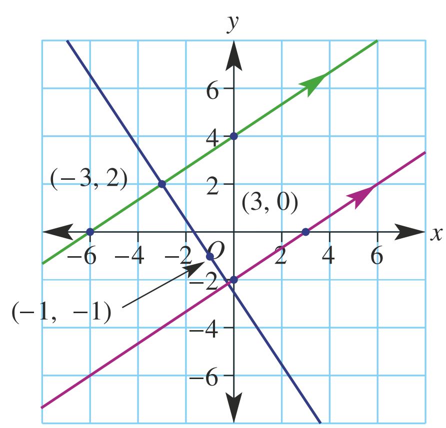
In simple terms, the Parallel Postulate says that if cointerior angles do not sum to 180°, then the two lines are not parallel. Furthermore, if the two interior angles are equal and also sum to 180°, then the third line must be perpendicular to the other two.



LESSON STARTER Gradient connection

Shown here is a pair of parallel lines and a third line that is perpendicular to the other two lines.

- Find the equation of each line, using the coordinates shown on the graph.
- What is common about the rules for the two parallel lines?
- Is there any connection between the rules of the parallel lines and the perpendicular line? Can you write down this connection as a formula?

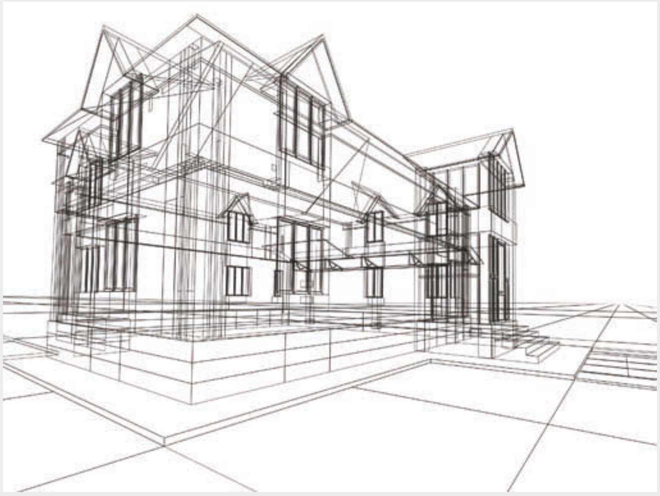


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

KEY IDEAS

- Two **parallel lines** have the same gradient.
  For example, $y = 3x - 1$ and $y = 3x + 8$ have the same gradient of 3.
- Two **perpendicular lines** with gradients m_1 and m_2 satisfy the following rule:
  $m_1 \times m_2 = -1$ or $m_2 = -\frac{1}{m_1}$ (i.e. m_2 is the negative reciprocal of m_1).
- Equations of parallel or perpendicular lines can be found by:
  - first finding the gradient (m)
  - then substituting a point to find c in $y = mx + c$.



Two planes, such as walls, intersect along a straight line. In a house design, linear equations in 3D can model the many parallel and perpendicular lines. Solving simultaneously finds the precise location of the intersection points.

BUILDING UNDERSTANDING

1 What is the gradient of the line that is parallel to the graph of these equations?

a $y = 4x - 6$ **b** $y = -7x - 1$ **c** $y = -\frac{3}{4}x + 2$ **d** $y = \frac{8}{7}x - \frac{1}{2}$

2 Use $m_2 = -\frac{1}{m_1}$ to find the gradient of the line that is perpendicular to the graphs of the following equations.

a $y = 3x - 1$ **b** $y = -2x + 6$ **c** $y = \frac{7}{8}x - \frac{2}{3}$ **d** $y = -\frac{4}{9}x - \frac{4}{7}$

3 A line is parallel to the graph of the rule $y = 5x - 2$ and its y-intercept is 4. The rule for the line is of the form $y = mx + c$.

a State the value of m. **b** State the value of c. **c** Find the rule.

4 Answer true or false.

a The lines $y = 2x$ and $y = 2x + 3$ are parallel.
b The lines $y = 3x$ and $y = -3x + 2$ are perpendicular.

ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

Example 25 Deciding if lines are parallel or perpendicular

Decide if the graph of each pair of rules will be parallel, perpendicular or neither.

a $y = -3x - 8$ and $y = \frac{1}{3}x + 1$

b $y = \frac{1}{2}x + 2$ and $2y - x = 5$

c $3x + 2y = -5$ and $x - y = 2$

SOLUTION	EXPLANATION
a $y = -3x - 8,\ m = -3$ (1)	Both equations are in the form $y = mx + c$.
$y = \frac{1}{3}x + 1,\ m = \frac{1}{3}$ (2)	
$-3 \times \frac{1}{3} = -1$	Test: $m_1 \times m_2 = -1$.
So the lines are perpendicular.	
b $y = \frac{1}{2}x + 2,\ m = \frac{1}{2}$ (1)	Write both equations in the form $y = mx + c$.
$2y - x = 5$	
$2y = x + 5$	
$y = \frac{1}{2}x + \frac{5}{2},\ m = \frac{1}{2}$ (2)	Both lines have a gradient of $\frac{1}{2}$, so the lines are parallel.
So the lines are parallel.	
c $3x + 2y = -5$	Write both equations in the form $y = mx + c$.
$2y = -3x - 5$	Note: $\frac{-3x}{2} = -\frac{3x}{2}$
$y = -\frac{3}{2}x - \frac{5}{2},\ m = -\frac{3}{2}$ (1)	
$x - y = 2$	Subtract x from both sides, then divide both sides by -1.
$-y = -x + 2$	
$y = x - 2,\ m = 1$ (2)	
$-\frac{3}{2} \times 1 \neq -1$	Test: $m_1 \times m_2 = -1$.
So the lines are neither parallel nor perpendicular.	The gradients are not equal and $m_1 \times m_2 \neq -1$.

Now you try

Decide if the graph of each pair of rules will be parallel, perpendicular or neither.

a $y = -4x - 1$ and $y = \frac{1}{4}x + 3$

b $y = \frac{1}{2}x + 1$ and $2y + x = 7$

c $5x - 3y = -10$ and $5x = 3y + 2$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 26 Finding the equation of a parallel or perpendicular line

Find the equation of the line that is:

a parallel to $y = -2x - 7$ and passes through $(1, 9)$

b perpendicular to $y = \frac{3}{4}x - 1$ and passes through $(3, -2)$

SOLUTION

a
$$y = mx + c$$
$$m = -2$$
$$y = -2x + c$$

Write the general equation of a line.

Since the line is parallel to $y = -2x - 7$, $m = -2$.

Substitute $(1, 9)$:
$$9 = -2(1) + c$$
$$11 = c$$
$$\therefore y = -2x + 11$$

Substitute the given point $(1, 9)$, where $x = 1$ and $y = 9$, and solve for c.

b
$$y = mx + c$$
$$m = \frac{-1}{\left(\frac{3}{4}\right)} = -\frac{4}{3}$$
$$y = -\frac{4}{3}x + c$$

The perpendicular gradient is the negative reciprocal of $\frac{3}{4}$.

Substitute $(3,-2)$:
$$-2 = -\frac{4}{3}(3) + c$$
$$-2 = -4 + c$$
$$c = 2$$
$$\therefore y = -\frac{4}{3}x + 2$$

Substitute $(3, -2)$ and solve for c.

Now you try

Find the equation of the line that is:

a parallel to $y = -3x - 5$ and passes through $(1, 5)$

b perpendicular to $y = \frac{2}{3}x - 3$ and passes through $(2, -1)$

Exercise 1I

FLUENCY	1, 2–3(1/2)	2–3(1/2)	2–3(1/2)

1 Decide if the graph of each pair of rules will be parallel, perpendicular or neither.

Example 25a **a** $y = -2x - 3$ and $y = \frac{1}{2}x + 2$

Example 25b **b** $y = \frac{1}{2}x + 1$ and $2y - x = 4$

Example 25c **c** $4x + 2y = -4$ and $x - y = 3$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 25

2 Decide if the line graphs of each pair of rules will be parallel, perpendicular or neither.

a $y = 3x - 1$ and $y = 3x + 7$

b $y = \frac{1}{2}x - 6$ and $y = \frac{1}{2}x - 4$

c $y = -\frac{2}{3}x + 1$ and $y = \frac{2}{3}x - 3$

d $y = -4x - 2$ and $y = x - 7$

e $y = -\frac{3}{7}x - \frac{1}{2}$ and $y = \frac{7}{3}x + 2$

f $y = -8x + 4$ and $y = \frac{1}{8}x - 2$

g $2y + x = 2$ and $y = -\frac{1}{2}x - 3$

h $x - y = 4$ and $y = x + \frac{1}{2}$

i $8y + 2x = 3$ and $y = 4x + 1$

j $3x - y = 2$ and $x + 3y = 5$

Example 26

3 Find the equation of the line that is:

a parallel to $y = x + 3$ and passes through $(1, 5)$

b parallel to $y = -x - 5$ and passes through $(1, -7)$

c parallel to $y = -4x - 1$ and passes through $(-1, 3)$

d parallel to $y = \frac{2}{3}x + 1$ and passes through $(3, -4)$

e parallel to $y = -\frac{4}{5}x + \frac{1}{2}$ and passes through $(5, 3)$

f perpendicular to $y = 2x + 3$ and passes through $(2, 5)$

g perpendicular to $y = -4x + 1$ and passes through $(-4, -3)$

h perpendicular to $y = \frac{2}{3}x - 4$ and passes through $(4, -1)$

i perpendicular to $y = \frac{4}{3}x + \frac{1}{2}$ and passes through $(-4, -2)$

j perpendicular to $y = -\frac{2}{7}x - \frac{3}{4}$ and passes through $(-8, 3)$.

PROBLEM-SOLVING	4(½)	4–6(½)	4–6(½), 7

4 This question involves vertical and horizontal lines. Find the equation of the line that is:

a parallel to $x = 3$ and passes through $(6, 1)$

b parallel to $x = -1$ and passes through $(0, 0)$

c parallel to $y = -3$ and passes through $(8, 11)$

d parallel to $y = 7.2$ and passes through $(1.5, 8.4)$

e perpendicular to $x = 7$ and passes through $(0, 3)$

f perpendicular to $x = -4.8$ and passes through $(2.7, -3)$

g perpendicular to $y = -\frac{3}{7}$ and passes through $\left(\frac{2}{3}, \frac{1}{2}\right)$

h perpendicular to $y = \frac{8}{13}$ and passes through $\left(-\frac{4}{11}, \frac{3}{7}\right)$.

5 Find the equation of the line that is parallel to these equations and passes through the given points.

a $y = \frac{2x - 1}{3}$, $(0, 5)$

b $y = \frac{3 - 5x}{7}$, $(1, 7)$

c $3y - 2x = 3$, $(-2, 4)$

d $7x - y = -1$, $(-3, -1)$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

6 Find the equation of the line that is perpendicular to the equations given in Question **5** and passes through the same given points.

7 A line with equation $3x - 2y = 12$ intersects a second line at the point where $x = 2$. If the second line is perpendicular to the first line, find where the second line cuts the x-axis.

REASONING	8	8, 9	8–10

8 Find an expression for the gradient of a line if it is:

a parallel to $y = mx + 8$

b parallel to $ax + by = 4$

c perpendicular to $y = mx - 1$

d perpendicular to $ax + by = -3$.

9 **a** Find the value of a if $y = \frac{a}{7}x + c$ is parallel to $y = 2x - 4$.

b Find the value of a if $y = \left(\frac{2a+1}{3}\right)x + c$ is parallel to $y = -x - 3$.

c Find the value of a if $y = \left(\frac{1-a}{2}\right)x + c$ is perpendicular to $y = \frac{1}{2}x - \frac{3}{5}$.

d Find the value of a if $ay = 3x + c$ is perpendicular to $y = -\frac{3}{7}x - 1$.

10 Find the equation of a line that is:

a parallel to $y = 2x + c$ and passes through (a, b)

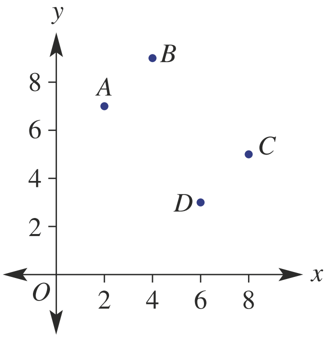
b parallel to $y = mx + c$ and passes through (a, b)

c perpendicular to $y = -x + c$ and passes through (a, b)

d perpendicular to $y = mx + c$ and passes through (a, b).

ENRICHMENT: Perpendicular and parallel geometry	–	–	11–13

11 A quadrilateral, $ABCD$, has vertex coordinates $A(2, 7)$, $B(4, 9)$, $C(8, 5)$ and $D(6, 3)$.



a Find the gradient of these line segments.

i AB **ii** BC **iii** CD **iv** DA

b What do you notice about the gradients of opposite and adjacent sides?

c What type of quadrilateral is $ABCD$?

12 The vertices of triangle ABC are $A(0, 0)$, $B(3, 4)$ and $C\left(\frac{25}{3}, 0\right)$.

a Find the gradient of these line segments.

i AB **ii** BC **iii** CA

b What type of triangle is $\triangle ABC$?

c Find the perimeter of $\triangle ABC$.

13 Find the equation of the perpendicular bisector of the line segment joining (1, 1) with (3, 5) and find where this bisector cuts the x-axis.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Applications and problem-solving

The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

Business profit

1 Abby runs an online business making and selling Christmas stockings. Over time she has worked out that the cost to make and deliver 3 Christmas stockings is \$125, while the cost to make and deliver 7 stockings is \$185. The cost to produce each Christmas stocking is the same. Costs involved also include the purchase of the tools to make the stockings.

Abby is interested in exploring the relationship between her profit, costs and selling price. She wants to determine the 'break-even' point and look at how this is impacted if the selling price is adjusted.

a Draw a graph of cost versus the number of stockings made using the information given and assuming a linear relationship.

b From your graph, determine a rule for the cost, C dollars, of making and delivering s stockings.

Abby sells each stocking for \$25.

c Write a rule for the amount she would receive (revenue), R dollars, from selling s stockings and sketch its graph on the same axes, as in part **a**.

d What are the coordinates of the point where the graphs intersect?

e Profit is defined as revenue – cost.

i Give a rule for the profit, P dollars, from selling s stockings.

ii Use your equation from part **i** to find how many stockings must be sold to break even.

f In the lead up to Christmas each year, Abby finds that she sells on average t stockings. She considers adjusting the selling price of the stockings at this time of year.

i Determine the minimum price, p dollars, she should sell her stockings for, in terms of t, to break even.

ii Use your result from part **i** to determine the minimum selling price if $t = 5$ or if $t = 25$.

Comparing speeds

2 To solve problems involving distance, speed and time we use the following well-known rule:

$$\text{distance} = \text{speed} \times \text{time}$$

We will explore how we can use this simple rule to solve common problems. These include problems where objects travel towards each other, follow behind or chase one another, and problems where speed is altered mid-journey.

a Two cars travel towards each other on a 100 km stretch of road. One car travels at 80 km/h and the other at 70 km/h.

i How far does each car travel in 1 hour?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

ii Complete the table below to determine how far each car has travelled after t hours.

	Speed (km/h)	Time (hours)	Distance (km)
Car A		t	
Car B		t	

iii Hence, if the cars set off at the same time, how long will it be before the cars meet (i.e. cover the 100 km between them)? Answer in minutes.

iv If two cars travel at x km/h and y km/h respectively and there is d km between them, determine after how long they will meet in terms of x, y and d.

b Ed's younger brother leaves the house on his bicycle and rides at 2 km/h. Ed sets out after his brother on his bike x hours later, travelling at 7 km/h.

i Use an approach like in part **a** to find a rule for the time taken for Ed to catch up to his younger brother in terms of x.

ii What is this time if $x = 1$?

c Meanwhile, Sam is driving from city A to city B. After 2 hours of driving she noticed that she covered 80 km and calculated that, if she continued driving at the same speed, she would end up being 15 minutes late. She therefore increased her speed by 10 km/h and she arrived at city B 36 minutes earlier than she planned. Find the distance between cities A and B.

Crossing the road

3 Coordinate geometry provides a connection between geometry and algebra where points and lines can be explored precisely using coordinates and equations.

We will investigate the shortest path between sets of points positioned on parallel lines to find the shortest distance to cross the road.

Two parallel lines with equations $y = 2x + 2$ and $y = 2x + 12$ form the sides of a road. A chicken is positioned at (2, 6) along one of the sides of the road. Three bags of grain are positioned on the other side of the road at (0, 12), (−2, 8) and (3, 18).

a What is the shortest distance the chicken would have to cover to get to one of the bags of grain?

b If the chicken crosses the road to get directly to the closest bag of grain, give the equation of the direct line the chicken walks along.

c By considering your equation in part **b**, explain why this is the shortest possible distance the chicken could walk to cross the road.

d Using the idea from part **c**, find the distance between the parallel lines with equations $y = 3x + 1$ and $y = 3x + 11$ using the point (3, 10) on the first line.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

1J Simultaneous equations using substitution

Learning intentions

- To understand that a single linear equation in two variables has an infinite number of solutions
- To know that two simultaneous linear equations (straight lines) can have 0 or 1 solution (points of intersection)
- To understand that the solution of two simultaneous equations satisfies both equations and lies on both straight line graphs
- To know how to substitute one algebraic expression for another to obtain an equation in one unknown
- To be able to solve simultaneous equations using the substitution method

When we try to find a solution to a set of equations rather than just a single equation, we say that we are solving simultaneous equations. For two linear simultaneous equations we are interested in finding the point where the graphs of the two equations meet. The point, for example, at the intersection of a company's cost equation and revenue equation is the 'break-even point'. This determines the point at which a company will start making a profit.

Simultaneous equations can solve personal finance questions such as: finding the best deal for renting a house or buying a car; the job where you will earn the most money over time; and the most profitable investment account.

LESSON STARTER Give up and do the algebra

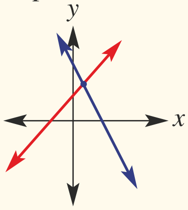
The two simultaneous equations $y = 2x - 3$ and $4x - y = 5\frac{1}{2}$ have a single solution.

- Use a guess and check (i.e. trial and error) technique to try to find the solution.
- Try a graphical technique to find the solution. Is this helpful?
- Now find the exact solution using the algebraic method of substitution.
- Which method is better? Discuss.

KEY IDEAS

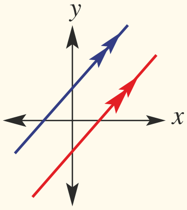
- Solving two **simultaneous equations** involves finding a solution that satisfies both equations.
  - When two straight lines are not parallel, there will be a single (**unique**) solution.

Non-parallel lines



1 point of intersection

Parallel lines (same gradient)



0 points of intersection

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

- The **substitution** method is usually used when at least one of the equations has one pronumeral as the subject. For example, $y = 3x + 2$ or $x = 3y - 1$.
  - By substituting one equation into the other, a single equation in terms of one pronumeral is formed and can then be solved.

    For example:

    $x + y = 8$ (1)

    $y = 3x + 4$ (2)

    Substitute (2) in (1): $x + (3x + 4) = 8$

    $4x + 4 = 8$

    $\therefore x = 1$

    Find y: $y = 3(1) + 4 = 7$

    So the solution is $x = 1$ and $y = 7$. The point $(1, 7)$ is the intersection point of the graphs of the two relations.

BUILDING UNDERSTANDING

1 By substituting the given values of x and y into both equations, decide whether it is the solution to these simultaneous equations.

a $x + y = 5$ and $x - y = -1$; $x = 2, y = 3$

b $3x - y = 2$ and $x + 2y = 10$; $x = 2, y = 4$

c $3x + y = -1$ and $x - y = 0$; $x = -1, y = 2$

d $2y = x + 2$ and $x - y = 4$; $x = -2, y = -6$

e $2(x + y) = -20$ and $3x - 2y = -20$; $x = -8, y = -2$

2 This graph represents the rental cost, $\$C$, after k kilometres of a new car from two car rental firms called Paul's Motor Mart and Joe's Car Rental.

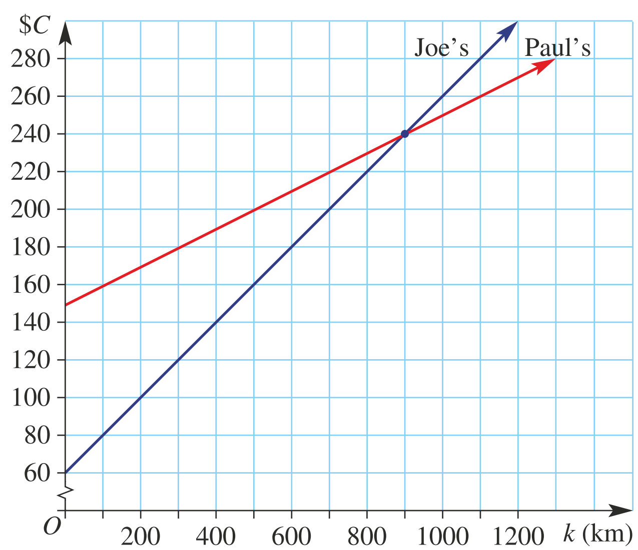
a **i** Determine the initial rental cost from each company.

ii Find the cost per kilometre when renting from each company.

iii Find the linear equations for the total rental cost from each company.

iv Determine the number of kilometres for which the cost is the same from both rental firms.

b If you had to travel 300 km, which company would you choose?



c If you had \$260 to spend on travel, which firm would give you the most kilometres?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 27 Solving simultaneous equations using substitution

Solve these pairs of simultaneous equations using the method of substitution.

a $2x + y = -7$ and $y = x + 2$

b $2x - 3y = -8$ and $y = x + 3$

SOLUTION

a $2x + y = -7 \quad (1)$

$y = x + 2 \quad (2)$

Substitute equation (2) into equation (1).

$$\begin{aligned} 2x + (x + 2) &= -7 \\ 3x + 2 &= -7 \\ 3x &= -9 \\ x &= -3 \end{aligned}$$

Substitute $x = -3$ into equation (2).

$$\begin{aligned} y &= -3 + 2 \\ &= -1 \end{aligned}$$

Solution is $x = -3, y = -1$.

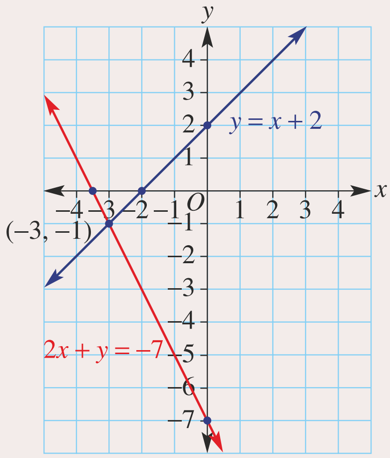
EXPLANATION

Label your equations.

Substitute equation (2) into equation (1) since equation (2) has a pronumeral as the subject.

Solve the resulting equation for x.

Substitute to find y.



b $2x - 3y = -8 \quad (1)$

$y = x + 3 \quad (2)$

Substitute equation (2) into equation (1).

$$\begin{aligned} 2x - 3(x + 3) &= -8 \\ 2x - 3x - 9 &= -8 \\ -x - 9 &= -8 \\ -x &= 1 \\ x &= -1 \end{aligned}$$

Substitute $x = -1$ into equation (2).

$$\begin{aligned} y &= -1 + 3 \\ &= 2 \end{aligned}$$

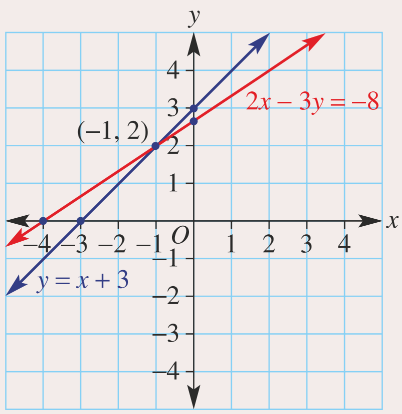
Solution is $x = -1, y = 2$.

Label your equations.

Substitute equation (2) into equation (1).

Expand and simplify then solve the equation for x.

Substitute $x = -1$ into either equation to find y.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Now you try

Solve these pairs of simultaneous equations using the method of substitution.

a $3x + y = 4$ and $y = x - 4$ **b** $x - 2y = -7$ and $y = x + 4$

Example 28 Solving with both equations in the form $y = mx + c$

Solve the pair of simultaneous equations: $y = -3x + 2$ and $y = 7x - 8$.

SOLUTION

$y = -3x + 2$ (1)

$y = 7x - 8$ (2)

Substitute equation (2) into equation (1).

$7x - 8 = -3x + 2$

$10x = 10$

$x = 1$

Substitute $x = 1$ into equation (1).

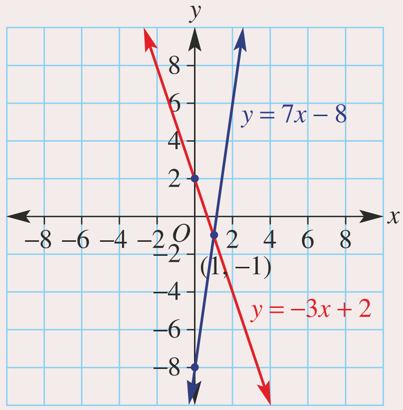
$y = -3(1) + 2$

$= -1$

Solution is $x = 1, y = -1$.

EXPLANATION

Write down and label each equation.



Now you try

Solve the pair of simultaneous equations: $y = -4x + 7$ and $y = 5x - 11$

Exercise 1J

FLUENCY	1, 2–3(1/2)	2–3(1/2)	2–3(1/3)

1 Solve these pairs of simultaneous equations using the method of substitution.

Example 27a **a** $2x + y = 1$ and $y = x - 5$

Example 27b **b** $3x - 2y = -9$ and $y = x + 4$

Example 27 **2** Solve the following pairs of simultaneous equations, using the method of substitution. You can check your solution graphically by sketching the pair of graphs and locating the intersection point.

a $y = x + 5$ and $3x + y = 13$ **b** $y = x + 3$ and $6x + y = 17$

c $y = x - 2$ and $3x - 2y = 7$ **d** $y = x - 1$ and $3x + 2y = 8$

e $y = x$ and $4x + 3y = 7$ **f** $y = x$ and $7x + 3y = 10$

g $x = 2y + 3$ and $11y - 5x = -14$ **h** $x = 3y - 2$ and $7y - 2x = 8$

i $x = 3y - 5$ and $3y + 5x = 11$ **j** $x = 4y + 1$ and $2y - 3x = -23$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 28

3 Solve the following pairs of simultaneous equations, using the method of substitution. Check your solution graphically if you wish.

- **a** $y = 4x + 2$ and $y = x + 8$
- **b** $y = -2x - 3$ and $y = -x - 4$
- **c** $x = y - 6$ and $x = -2y + 3$
- **d** $x = -7y - 1$ and $x = -y + 11$
- **e** $y = 4 - x$ and $y = x - 2$
- **f** $y = 5 - 2x$ and $y = \frac{3}{2}x - 2$
- **g** $y = 5x - 1$ and $y = \frac{11 - 3x}{2}$
- **h** $y = 8x - 5$ and $y = \frac{5x + 13}{6}$

PROBLEM-SOLVING	4, 6	4–6	5–7

4 The salary structures for companies A and B are given by:
Company A: \$20 per hour
Company B: \$45 plus \$15 per hour

- **a** Find a rule for $\$E$ earned for t hours for:
  - **i** company A
  - **ii** company B.
- **b** Solve your two simultaneous equations from part **a**.
- **c** **i** State the number of hours worked for which the earnings are the same for the two companies.
  - **ii** State the amount earned when the earnings are the same for the two companies.

5 The value of two cars is depreciating (i.e. decreasing) at a constant rate according to the information in this table.

Car	Initial value	Annual depreciation
Luxury sports coupe	\$62 000	\$5000
Family sedan	\$40 000	\$3000

- **a** Write rules for the value, $\$V$, after t years for:
  - **i** the luxury sports coupe
  - **ii** the family sedan.
- **b** Solve your two simultaneous equations from part **a**.
- **c** **i** State the time taken for the cars to have the same value.
  - **ii** State the value of the cars when they have the same value.

6 The sum of the ages of a boy and his mother is 48. If the mother is three more than twice the boy's age, find the difference in the ages of the boy and his mother.

7 The perimeter of a rectangular farm is 1800 m and its length is 140 m longer than its width. Find the area of the farm.

REASONING	8(1/2)	8(1/2), 9	8(1/2), 9, 10

8 When two lines have the same gradient, there will be no intersection point. Use this idea to decide if these pairs of simultaneous equations will have a solution.

- **a** $y = 3x - 1$ and $y = 3x + 2$
- **b** $y = 7x + 2$ and $y = 7x + 6$
- **c** $y = 2x - 6$ and $y = -2x + 1$
- **d** $y = -x + 7$ and $y = 2x - 1$
- **e** $2x - y = 4$ and $y = 2x + 1$
- **f** $7x - y = 1$ and $y = -7x + 2$
- **g** $2y - 3x = 0$ and $y = -\frac{3x}{2} - 1$
- **h** $\frac{x - 2y}{4} = 0$ and $y = \frac{x}{2} + 3$

ISBN 978-1-108-77290-7

9 For what value of k will these pairs of simultaneous equations have no solution?

a $y = -4x - 7$ and $y = kx + 2$

b $y = kx + 4$ and $3x - 2y = 5$

c $kx - 3y = k$ and $y = 4x + 1$

10 Solve these simple equations for x and y. Your solution should contain the pronumeral k.

a $x + y = k$ and $y = 2x$

b $x - y = k$ and $y = -x$

c $2x - y = -k$ and $y = x - 1$

d $y - 4x = 2k$ and $x = y + 1$

ENRICHMENT: Factorise to solve	–	–	11(1/2), 12

11 Factorisation can be used to help solve harder literal equations (i.e. equations including other pronumerals).

For example: $ax + y = b$ and $y = bx$

Substituting $y = bx$ into $ax + y = b$ gives:

$ax + bx = b$

$x(a + b) = b$ (Factor out x.)

$$x = \frac{b}{a + b}$$

Substitute to find y.

$$\therefore y = b \times \left(\frac{b}{a + b}\right) = \frac{b^2}{a + b}$$

Now solve these literal equations.

a $ax - y = b$ and $y = bx$

b $ax + by = 0$ and $y = x + 1$

c $x - by = a$ and $y = -x$

d $y = ax + b$ and $y = bx$

e $y = (a - b)x$ and $y = bx + 1$

f $ax + by = c$ and $y = ax + c$

g $\frac{x}{a} + \frac{y}{b} = 1$ and $y = ax$

h $ax + y = b$ and $y = \frac{x}{a}$

12 Make up your own literal equation like the ones in Question 11. Solve it and then test it on a classmate.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

1K Simultaneous equations using elimination

Learning intentions

- To be able to identify and form equations involving a matching pair of terms
- To be able to use the process of elimination to solve simultaneous equations

The elimination method for the solution of simultaneous equations is commonly used when the equations are written in the same form. One pronumeral is eliminated using addition or subtraction and the value of the other pronumeral can then be found using one of the original equations.

Using the initial cost of machinery and the production cost per item, financial analysts working for manufacturing companies, e.g. biscuit makers, can use simultaneous equations to determine the most profitable equipment to invest in.

LESSON STARTER Which operation?

Below are four sets of simultaneous equations.

- For each set discuss whether addition or subtraction would be used to eliminate one pronumeral.
- State which pronumeral might be eliminated first in each case.
- Describe how you would first deal with parts **c** and **d** so that elimination can be used.

a $\begin{aligned} x + y &= 5 \\ 2x + y &= 7 \end{aligned}$ **b** $\begin{aligned} -4x - 2y &= -8 \\ 4x + 3y &= 10 \end{aligned}$ **c** $\begin{aligned} 5x - y &= 1 \\ 3x - 2y &= -5 \end{aligned}$ **d** $\begin{aligned} 3x + 2y &= -5 \\ 4x - 3y &= -1 \end{aligned}$

KEY IDEAS

- The method of **elimination** is generally used when both equations are in the form $ax + by = d$.

  For example: $\begin{aligned} 2x - y &= 6 \\ 3x + y &= 10 \end{aligned}$ or $\begin{aligned} -5x + y &= -2 \\ 6x + 3y &= 5 \end{aligned}$

  - When there is no matching pair (as in the second example above) one or both of the equations can be multiplied by a chosen factor. This is shown in **Example 30a** and **b**.

BUILDING UNDERSTANDING

1 Find the answer.

a $2x$ subtract $2x$ **b** $5y$ add $-5y$ **c** $-2x$ add $2x$ **d** $-3y$ subtract $-3y$

2 Decide if you would add or subtract the two given terms to give a result of zero.

a $7x, 7x$ **b** $-5y, 5y$ **c** $2y, -2y$ **d** $-7y, -7y$

3 What is the resulting equation when $2x - 3y = 4$ is multiplied on both sides by the following?

a 2 **b** 3 **c** 4 **d** 10

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 29 Using the elimination method with a matching pair of terms

Solve the following pair of simultaneous equations using the elimination method.
$x + y = 6$ and $3x - y = 10$

SOLUTION

$$\begin{aligned} x + y &= 6 \quad (1) \\ 3x - y &= 10 \quad (2) \\ (1) + (2):\ 4x &= 16 \\ x &= 4 \end{aligned}$$

Substitute $x = 4$ into equation (1).

$$\begin{aligned} (4) + y &= 6 \\ \therefore\ y &= 2 \end{aligned}$$

Solution is $x = 4, y = 2$.

EXPLANATION

Label your equations to help you refer to them.

Add the two equations to eliminate y. Then solve for the remaining pronumeral x.

Substitute $x = 4$ into one of the equations to find y.

State the solution and check by substituting the solution into the original equations.

Now you try

Solve the following pair of simultaneous equations using the elimination method.
$x + y = 4$ and $5x - y = 14$

Example 30 Using the elimination method to solve simultaneous equations

Solve the following pairs of simultaneous equations using the elimination method.

a $y - 3x = 1$ and $2y + 5x = 13$ **b** $3x + 2y = 6$ and $5x + 3y = 11$

SOLUTION

a

$$\begin{aligned} y - 3x &= 1 \quad (1) \\ 2y + 5x &= 13 \quad (2) \\ (1) \times 2\quad 2y - 6x &= 2 \quad (3) \\ (2) - (3):\quad 11x &= 11 \\ x &= 1 \end{aligned}$$

Substitute $x = 1$ into equation (1).

$y - 3(1) = 1$

$\therefore\ y = 4$

Solution is $x = 1, y = 4$.

EXPLANATION

Label your equations.

Multiply equation (1) by 2 so that there is a matching pair ($2y$).

$2y - 2y = 0$ and $5x - (-6x) = 11x$.

Solve for x.

Substitute into one of the equations to find y.

Continued on next page

b

$$
\begin{array}{lrl}
 & 3x + 2y = 6 & (1) \\
 & 5x + 3y = 11 & (2) \\
(1) \times 3 & 9x + 6y = 18 & (3) \\
(2) \times 2 & \underline{10x + 6y = 22} & (4) \\
(4) - (3): & x \quad = 4 &
\end{array}
$$

Multiply equation (1) by 3 and equation (2) by 2 to generate $6y$ in each equation. (Alternatively, multiply (1) by 5 and (2) by 3 to obtain matching x coefficients.)

Subtract to eliminate y.

Substitute $x = 4$ into equation (1).

Substitute $x = 4$ into one of the equations to find y.

$3(4) + 2y = 6$

$2y = -6$

$y = -3$

State and check the solution.

Solution is $x = 4, y = -3$.

Now you try

Solve the following pairs of simultaneous equations using the elimination method.

a $y - 2x = 1$ and $4y + 3x = 15$

b $5x + 2y = 11$ and $3x + 5y = -1$

Exercise 1K

FLUENCY	1, 2–4(1/2)	2–4(1/2)	2–3(1/3), 4(1/2)

Example 29

1 Solve the pairs of simultaneous equations using the elimination method.

a $x + y = 5$ and $2x - y = 4$

b $x + y = 7$ and $5x - y = 17$

Example 29

2 Solve the following pairs of simultaneous equations using the elimination method.

a $x + y = 7$ and $5x - y = 5$

b $x + y = 5$ and $3x - y = 3$

c $x - y = 2$ and $2x + y = 10$

d $x - y = 0$ and $4x + y = 10$

e $3x + 4y = 7$ and $2x + 4y = 6$

f $x + 3y = 5$ and $4x + 3y = 11$

g $2x + 3y = 1$ and $2x + 5y = -1$

h $4x + y = 10$ and $4x + 4y = 16$

i $2x + 3y = 8$ and $2x - 4y = -6$

j $3x + 2y = 8$ and $3x - y = 5$

k $-3x + 2y = -4$ and $5x - 2y = 8$

l $-2x + 3y = 8$ and $-4x - 3y = -2$

Example 30a

3 Solve the following pairs of simultaneous equations using the elimination method.

a $3x + 5y = 8$ and $x - 2y = -1$

b $2x + y = 10$ and $3x - 2y = 8$

c $x + 2y = 4$ and $3x - y = 5$

d $3x - 4y = 24$ and $x - 2y = 10$

e $y - 3x = -\frac{1}{2}$ and $x + 2y = \frac{5}{2}$

f $7x - 2y = -\frac{5}{2}$ and $3x + y = -2$

Example 30b

4 Solve the following pairs of simultaneous equations using the elimination method.

a $3x + 2y = 6$ and $5x + 3y = 11$

b $3x + 2y = 5$ and $2x + 3y = 5$

c $4x - 3y = 0$ and $3x + 4y = 25$

d $2x + 3y = 10$ and $3x - 4y = -2$

e $-2y - 4x = 0$ and $3y + 2x = -2$

f $-7x + 3y = 22$ and $3x - 6y = -11$

PROBLEM-SOLVING	5	5, 6	6, 7

5 The sum of two numbers is 1633 and their difference is 35. Find the two numbers.

6 The cost of one apple and one banana at the school canteen is \$1 and the cost of 3 apples and 2 bananas is \$2.40. Find the cost of a single banana.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

7 A group of 5 adults and 3 children paid a total of \$108 for their concert tickets. Another group of 3 adults and 10 children paid \$155. Find the cost of an adult ticket and the cost of a child's ticket.

REASONING	8	8, 9(½)	9, 10

8 Describe the error made in this working and then correct the error to find the correct solution.

$$\begin{aligned} 3x - 2y &= 5 \quad (1) \\ \underline{-4x - 2y} &\underline{= -2} \quad (2) \\ (1) + (2):\quad -x &= 3 \\ \therefore\ x &= -3 \\ 3(-3) - 2y &= 5 \\ -9 - 2y &= 5 \\ -2y &= 14 \\ y &= -7 \end{aligned}$$

Solution is $x = -3$ and $y = -7$.

9 Solve these literal simultaneous equations for x and y.

a $ax + y = 0$ and $ax - y = 2$

b $x - by = 4$ and $2x + by = 9$

c $ax + by = 0$ and $ax - by = -4$

d $ax + by = a$ and $ax - by = b$

e $ax + by = c$ and $bx + ay = c$

10 Explain why there is no solution to the set of equations $3x - 7y = 5$ and $3x - 7y = -4$.

ENRICHMENT: Partial fractions	–	–	11(½)

11 Writing $\dfrac{6}{(x-1)(x+1)}$ as a sum of two 'smaller' fractions $\dfrac{a}{x-1} + \dfrac{b}{x+1}$, known as partial fractions, involves a process of finding the values of a and b for which the two expressions are equal. Here is the process.

$$\begin{aligned} \frac{6}{(x-1)(x+1)} &= \frac{a}{x-1} + \frac{b}{x+1} \\ &= \frac{a(x+1) + b(x-1)}{(x-1)(x+1)} \end{aligned}$$

$$\begin{aligned} \therefore\ a(x+1) + b(x-1) &= 6 \\ ax + a + bx - b &= 6 \\ ax + bx + a - b &= 6 \\ x(a+b) + (a-b) &= 0x + 6 \end{aligned}$$

By equating coefficients: $a + b = 0$ (1)

$a - b = 6$ (2)

(1) + (2) gives $2a = 6$

$\therefore a = 3$.

and so $b = -3$.

$$\therefore\ \frac{6}{(x-1)(x+1)} = \frac{3}{x-1} - \frac{3}{x+1}$$

Use this technique to write the following as the sum of two fractions.

a $\dfrac{4}{(x-1)(x+1)}$

b $\dfrac{7}{(x+2)(2x-3)}$

c $\dfrac{-5}{(2x-1)(3x+1)}$

d $\dfrac{9x+4}{(3x-1)(x+2)}$

e $\dfrac{2x-1}{(x+3)(x-4)}$

f $\dfrac{1-x}{(2x-1)(4-x)}$

ISBN 978-1-108-77290-7

1L Further applications of simultaneous equations

Learning intentions
- To know the steps involved in solving a word problem with two unknowns
- To be able to form linear equations in two unknowns from a word problem
- To be able to choose an appropriate method to solve two equations simultaneously

When a problem involves two unknown variables, simultaneous equations can be used to find the solution to the problem, provided that the two pronumerals can be identified and two equations can be written from the problem description.

Simultaneous equations can be used by farmers, home gardeners, nurses and pharmacists to accurately calculate required volumes when mixing solutions of different concentrations to get a desired final concentration.

LESSON STARTER 19 scores but how many goals?

Nathan heard on the news that his AFL team scored 19 times during a game and the total score was 79 points. He wondered how many goals (worth 6 points each) and how many behinds (worth 1 point each) were scored in the game. Nathan looked up simultaneous equations in his maths book and it said to follow these steps:

1 Define two variables.
2 Write two equations.
3 Solve the equations.
4 Answer the question in words.

Can you help Nathan with the four steps to find out what he wants to know?

KEY IDEAS

- When solving problems with two unknowns:
  - Define a variable for each unknown.
  - Write down two equations from the information given.
  - Solve the equations simultaneously to find the solution.
  - Interpret the solution and answer the question in words.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

BUILDING UNDERSTANDING

1 Let x and y be two numbers that satisfy the following statements. State two linear equations according to the information.
   a They sum to 16 but their difference is 2.
   b They sum to 30 but their difference is 10.
   c They sum to 7 and twice the larger number plus the smaller number is 12.
   d The sum of twice the first plus three times the second is 11 and the difference between four times the first and three times the second is 13.
2 The perimeter of a rectangle is 56 cm. If the length, l, of the rectangle is three times its width, w, state two simultaneous equations that would allow you to solve to determine the dimensions.
3 State expressions for the following.
   a the cost of 5 tickets at $$x$ each
   b the cost of y pizzas at $15 each
   c the cost of 3 drinks at $$d$ each and 4 pies at $$p$ each

Example 31 Setting up and solving simultaneous equations

The sum of the ages of two children is 17 and the difference in their ages is 5. If Kara is the older sister of Ben, determine their ages.

SOLUTION	EXPLANATION
Let k be Kara's age and b be Ben's age. $k + b = 17$ (1) $k - b = 5$ (2) (1) + (2): $2k = 22$ $\therefore k = 11$	Define the unknowns and use these to write two equations from the information in the question. • The sum of their ages is 17. • The difference in their ages is 5. Add the equations to eliminate b and then solve to find k.
Substitute $k = 11$ into equation (1). $11 + b = 17$ $b = 6$	Substitute $k = 11$ into one of the equations to find the value of b.
$\therefore$ Kara is 11 years old and Ben is 6 years old.	Answer the question in words.

Now you try

The sum of the ages of two children is 20 and the difference in their ages is 8. If Tim is the older brother of Tina, determine their ages.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 32 Solving further applications with two variables

John buys 3 daffodils and 5 petunias from the nursery and pays \$25. Julia buys 4 daffodils and 3 petunias for \$26. Determine the cost of each type of flower.

SOLUTION

Let $\$d$ be the cost of a daffodil and $\$p$ be the cost of a petunia.

$$3d + 5p = 25 \quad (1)$$
$$4d + 3p = 26 \quad (2)$$

$$(1) \times 4 \quad 12d + 20p = 100 \quad (3)$$
$$(2) \times 3 \quad 12d + 9p = 78 \quad (4)$$
$$(3) - (4): \quad 11p = 22$$
$$\therefore\ p = 2$$

Substitute $p = 2$ into equation (1).

$$3d + 5(2) = 25$$
$$3d + 10 = 25$$
$$3d = 15$$
$$\therefore\ d = 5$$

A petunia costs \$2 and a daffodil costs \$5.

EXPLANATION

Define the unknowns and set up two equations from the question.

If 1 daffodil costs d dollars then 3 will cost $3 \times d = 3d$.

- 3 daffodils and 5 petunias cost \$25.
- 4 daffodils and 3 petunias cost \$26.

Multiply equation (1) by 4 and equation (2) by 3 to generate $12d$ in each equation.

Subtract the equations to eliminate d and then solve for p.

Substitute $p = 2$ into one of the equations to find the value of d.

Answer the question in words.

Now you try

Georgie buys 2 coffees and 3 muffins for \$17 and Rick buys 4 coffees and 2 muffins for \$22 from the same shop. Determine the cost of each coffee and muffin.

Exercise 1L

FLUENCY	1–4	2–5	3–6

Example 31

1 The sum of the ages of two children is 24 and the difference between their ages is 8. If Nikki is the older sister of Travis, determine their ages by setting up and solving a pair of simultaneous equations.

Example 31

2 Cam is 3 years older than Lara. If their combined age is 63, determine their ages by solving an appropriate pair of equations.

Example 32

3 Luke buys 4 bolts and 6 washers for \$2.20 and Holly spends \$1.80 on 3 bolts and 5 washers at the same local hardware store. Determine the costs of a bolt and a washer.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

4 It costs \$3 for children and \$7 for adults to attend a school basketball game. If 5000 people attended the game and the total takings at the door was \$25 000, determine the number of children and the number of adults that attended the game.

5 A vanilla thickshake is \$2 more than a fruit juice. If 3 vanilla thickshakes and 5 fruit juices cost \$30, determine their individual prices.

6 A paddock contains ducks and sheep. There are a total of 42 heads and 96 feet in the paddock. How many ducks and how many sheep are in the paddock?

PROBLEM-SOLVING	7, 8	7, 8	8, 9

7 James has \$10 in 5-cent and 10-cent coins in his change jar and counts 157 coins in total. How many 10-cent coins does he have?

8 Connor the fruiterer sells two fruit packs.
Pack 1: 10 apples and 5 mangoes (\$12)
Pack 2: 15 apples and 4 mangoes (\$14.15)
Determine the cost of 1 apple and 5 mangoes.

9 Five years ago I was 5 times older than my son. In 8 years' time I will be 3 times older than my son. How old am I today?

REASONING	10	10, 11	11, 12

10 Erin goes off on a long bike ride, averaging 10 km/h. One hour later her brother Alistair starts chasing after her at 16 km/h. How long will it take Alistair to catch up to Erin? (*Hint*: Use the rule $d = s \times t$.)

11 Two ancient armies are 1 km apart and begin walking towards each other. The Vikons walk at a pace of 3 km/h and the Mohicas walk at a pace of 4 km/h. How long will they walk for before the battle begins?

12 A river is flowing downstream at a rate of 2 metres per second. Brendan, who has an average swimming speed of 3 metres per second, decides to go for a swim in the river. He dives into the river and swims downstream to a certain point, then swims back upstream to the starting point. The total time taken is 4 minutes. How far did Brendan swim downstream?

ENRICHMENT: Concentration time	–	–	13, 14

13 Molly has a bottle of 15% strength cordial and wants to make it stronger. She adds an amount of 100% strength cordial to her bottle to make a total volume of 2 litres of cordial drink. If the final strength of the drink is 25% cordial, find the amount of 100% strength cordial that Molly added. (*Hint*: Use Concentration = Volume (cordial) ÷ Total volume.)

14 A fruit grower accidentally made a 5% strength chemical mixture to spray his grape vines. The strength of spray should be 8%. He then adds pure chemical until the strength reaches 8% by which time the volume is 350 litres. How much pure chemical did he have to add?

ISBN 978-1-108-77290-7

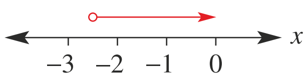
Cambridge University Press
Updated June 2022

1M Half planes EXTENDING

Learning intentions

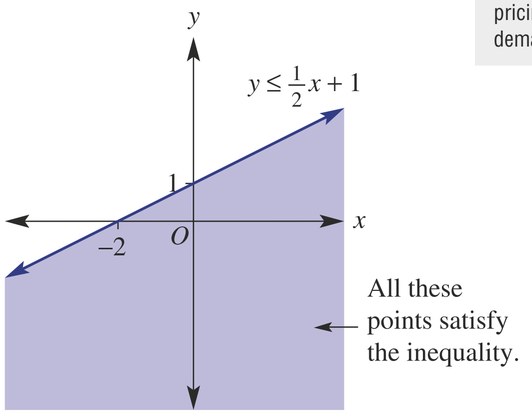
- To know the meaning of the term half plane
- To know how to determine which side of a line to shade to sketch a half plane
- To understand that if a pair of coordinates satisfy an inequality then the point is in the required region
- To be able to find the intersecting region of two or more half planes

You will remember that an inequality is a mathematical statement that contains one of these symbols: $<, \leqslant, >$ or $\geqslant$. The linear inequality with one pronumeral, for example $2x - 5 > -10$, has the solution $x > -2.5$.



Operations research analysts use half-plane graph calculations to optimise profit within certain limitations. For example, for airline companies to find the most economical combination of flight routes, seat pricing and pilot scheduling that aligns with customer demand.

Linear inequalities can also have two variables: $2x - 3y \geqslant 5$ and $y < 3 - x$ are two examples. The solutions to such inequalities will be an infinite set of points on a plane sitting on one side of a line. This region is called a half plane.



LESSON STARTER Which side do I shade?

You are asked to shade all the points on a graph that satisfy the inequality $4x - 3y \geqslant 12$.

- First, graph the equation $4x - 3y = 12$.
- Substitute the point $(-2, 3)$ into the inequality $4x - 3y \geqslant 12$. Does the point satisfy the inequality? Plot the point on your graph.
- Now test these points:

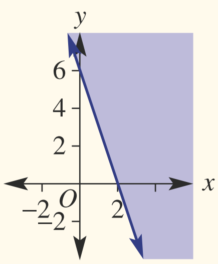
  a $(3, -2)$ **b** $(3, -1)$ **c** $(3, 0)$ **d** $(3, 1)$
- Can you now decide which side of the line is to be shaded to represent all the solutions to the inequality? Should the line itself be included in the solution?

ISBN 978-1-108-77290-7

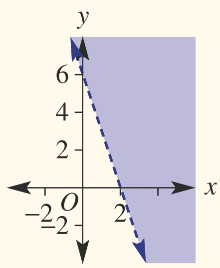
KEY IDEAS

- The solution to a linear inequality with two variables is illustrated using a shaded region called a **half plane**.
- When y is the subject of the inequality, follow these simple rules.
  - $y \geqslant mx + c$ Draw a solid line (as it is included in the region) and shade above.
  - $y > mx + c$ Draw a broken line (as it is not included in the region) and shade above.
  - $y \leqslant mx + c$ Draw a solid line (as it is included in the region) and shade below.
  - $y < mx + c$ Draw a broken line (as it is not included in the region) and shade below.

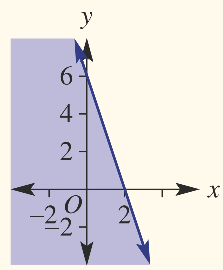
  Here are examples of each.



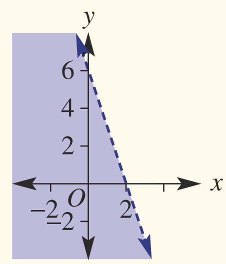
$y \geqslant -3x + 6$



$y > -3x + 6$

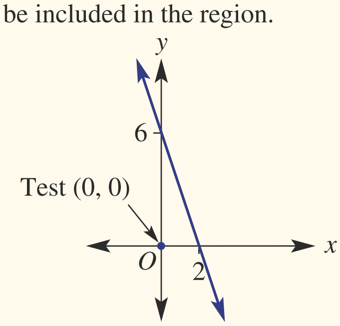


$y \leqslant -3x + 6$



$y < -3x + 6$

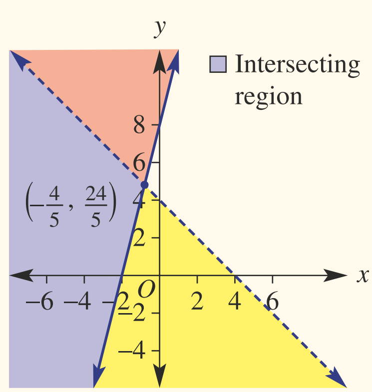
- If the equation is of the form $ax + by = d$, it is usually simpler to test a point, for example $(0, 0)$, to see which side of the line is to be included in the region.



- When two or more half planes are sketched on the same set of axes, the half planes overlap and form an **intersecting region**. The set of points inside the intersecting region will be the solution to the simultaneous inequalities.

  For example, $y \geqslant 4x + 8$

  $y < -x + 4$



  - To help define the intersecting region correctly, you should determine and label the point of intersection.

ISBN 978-1-108-77290-7

BUILDING UNDERSTANDING

1 Substitute the point (0, 0) into these inequalities to decide if the point satisfies the inequality; i.e. is the inequality true for $x = 0$ and $y = 0$?

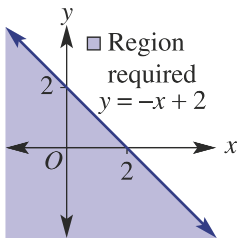
a $y < 3x - 1$ **b** $y > -\frac{x}{2} - 3$ **c** $y \geqslant 1 - 7x$

d $3x - 2y < -1$ **e** $x - y > 0$ **f** $2x - 3y \leqslant 0$

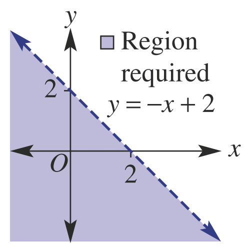
2 Match the rules with the graphs (**A**, **B** and **C**) below.

a $x + y < 2$ **b** $y \geqslant -x + 2$ **c** $y \leqslant -x + 2$

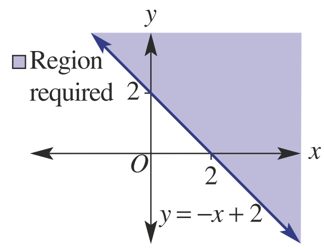
A



B



C



3 **a** Sketch the vertical line $x = -1$ and the horizontal line $y = 4$ on the same set of axes.

b Shade the region $x \geqslant -1$ (i.e. all points with an x-coordinate greater than or equal to -1).

c Shade the region $y \leqslant 4$ (i.e. all points with a y-coordinate less than or equal to 4).

d Now use a different colour to shade all the points that satisfy both $x \geqslant -1$ and $y \leqslant 4$ simultaneously.

Example 33 Sketching half planes

Sketch the half planes for the following linear inequalities.

a $y > 1.5x - 3$ **b** $y + 2x \leqslant 4$

SOLUTION

a $y = 1.5x - 3$

y-intercept ($x = 0$):

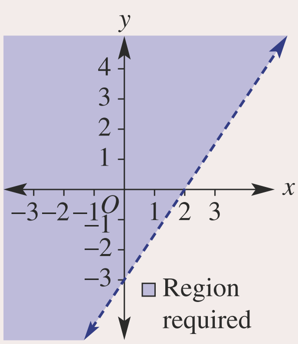
$y = -3$

x-intercept ($y = 0$):

$0 = 1.5x - 3$

$1.5x = 3$

$\therefore\ x = 2$



EXPLANATION

First, sketch $y = 1.5x - 3$ by finding the x- and y-intercepts.

Sketch a dotted line (since the sign is $>$ not $\geqslant$) joining the intercepts and shade above the line, since y is greater than $1.5x - 3$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b $y + 2x = 4$

y-intercept $(x = 0)$:

$y = 4$

x-intercept $(y = 0)$:

$2x = 4$

$x = 2$

First, sketch $y + 2x = 4$ by finding the x- and y-intercepts.

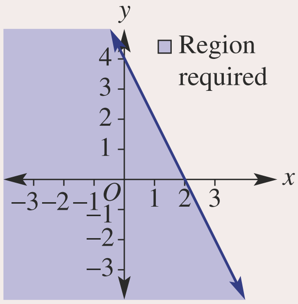
Shading: Test $(0, 0)$.

$0 + 2(0) \leqslant 4$

$0 \leqslant 4$ (True)

$\therefore$ $(0, 0)$ is included.

Decide which side to shade by testing the point $(0, 0)$; i.e. substitute $x = 0$ and $y = 0$. Since $0 \leqslant 4$, the point $(0, 0)$ should sit inside the shaded region.



Sketch a solid line since the inequality sign is $\leqslant$, and shade the region that includes $(0, 0)$.

Now you try

Sketch the half planes for the following linear inequalities.

a $y > 2.5x - 5$

b $y + 3x \leqslant 6$

Example 34 Finding the intersecting region

Sketch both the inequalities $4x + y \leqslant 12$ and $3x - 2y < -2$ on the same set of axes, show the region of intersection and find the point of intersection of the two lines.

SOLUTION

$4x + y = 12$

y-intercept $(x = 0)$:

$y = 12$

x-intercept $(y = 0)$:

$4x = 12$

$x = 3$

EXPLANATION

First sketch $4x + y = 12$ by finding the x- and y-intercepts.

Shading: Test $(0, 0)$.

$4(0) + 0 \leqslant 12$

$0 \leqslant 12$ (True)

So $(0, 0)$ is included.

Test $(0, 0)$ to see if it is in the included region.

Continued on next page

ISBN 978-1-108-77290-7

$3x - 2y = -2$
y-intercept ($x = 0$):
$-2y = -2$
$y = 1$
x-intercept ($y = 0$):
$3x = -2$
$x = -\frac{2}{3}$

Sketch $3x - 2y = -2$ by finding x- and y-intercepts.

Shading: Test (0, 0).
$3(0) + 2(0) < -2$
$0 < -2$ (False)
So (0, 0) is not included.

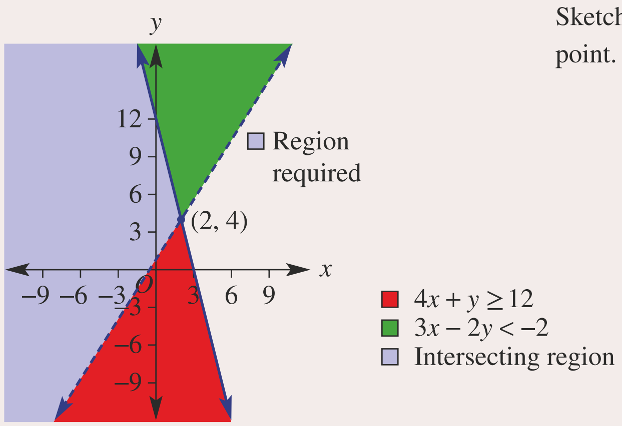
Test (0, 0) to see if it is in the included region.

Point of intersection:

$$\begin{aligned} & 4x + y = 12 \quad (1) \\ & 3x - 2y = -2 \quad (2) \\ (1) \times 2 \quad & \underline{8x + 2y = 24} \quad (3) \\ (2) + (3): \quad & 11x = 22 \\ & x = 2 \end{aligned}$$

Find the point of intersection by solving the equations simultaneously using the method of elimination.

Substitute $x = 2$ into equation (1).
$4(2) + y = 12$
$y = 4$
The point of intersection is (2, 4).



Sketch both regions and label the intersection point. Also label the intersecting region.

Now you try

Sketch both the inqualities $3x + y \leqslant 6$ and $2x - 3y < -7$ on the same set of axes. Show the region of intersection and find the point of intersection of the two lines.

ISBN 978-1-108-77290-7

Exercise 1M

FLUENCY	1, 2–3(1/2), 5(1/2)	2–5(1/2)	2–5(1/2)

1 Sketch the half planes for the following inequalities.

Example 33a
a **i** $y > 2x - 1$ **ii** $y < \frac{1}{2}x + 2$

Example 33b
b **i** $y + x \leqslant 3$ **ii** $y + 2x \geqslant 6$

Example 33a
2 Sketch the half planes for the following linear inequalities.

a $y \geqslant x + 4$ **b** $y < 3x - 6$ **c** $y > 2x - 8$ **d** $y \leqslant 3x - 5$
e $y < 2 - 4x$ **f** $y \leqslant 2x + 7$ **g** $y < 4x$ **h** $y > 6 - 3x$
i $y \leqslant -x$ **j** $x > 3$ **k** $x < -2$ **l** $y \geqslant 2$

3 Decide whether the following points are in the region defined by $2x - 3y > 8$.

a $(5, 0)$ **b** $(2.5, -1)$ **c** $(0, -1)$ **d** $(2, -5)$

4 Decide whether the following points are in the region defined by $4x + 3y \leqslant -2$.

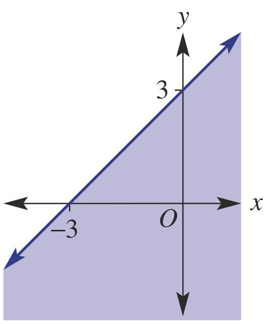
a $\left(-\frac{1}{2}, \frac{1}{2}\right)$ **b** $(-5, 6)$ **c** $(2, -3)$ **d** $(-3, 4)$

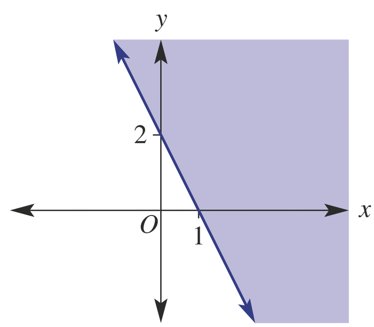
Example 33b
5 Sketch the half planes for the following linear inequalities.

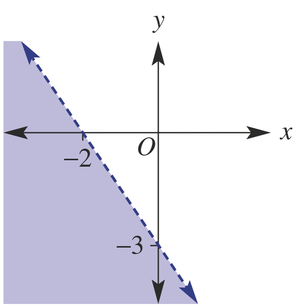
a $x + 3y < 9$ **b** $3x - y \geqslant 3$ **c** $4x + 2y \geqslant 8$ **d** $2x - 3y > 18$
e $-2x + y \leqslant 5$ **f** $-2x + 4y \leqslant 6$ **g** $2x + 5y > -10$ **h** $4x + 9y < -36$

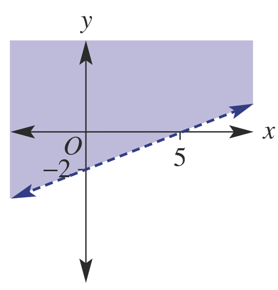
PROBLEM-SOLVING	6	6, 7(1/2)	6–7(1/2)

6 Write down the inequalities that give these half planes.

a


b


c


d


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 34

7 Sketch both inequalities on the same set of axes, shade the region of intersection and find the point of intersection of the two lines.

a $x + 2y \geqslant 4$
$2x + 2y < 8$

b $3x + 4y \leqslant 12$
$3x + y > 3$

c $2x - 3y > 6$
$y < x - 2$

d $3x - 5y \leqslant 15$
$y - 3x > -3$

e $y \geqslant -x + 4$
$2x + 3y \geqslant 6$

f $2y - x \leqslant 5$
$y < 10 - x$

g $3x + 2y \leqslant 18$
$4y - x < 8$

h $2y \geqslant 5 + x$
$y < 6 - 3x$

REASONING	8(1/2)	8(1/2)	8(1/2), 9

8 Sketch the following systems of inequalities on the same axes. Show the intersecting region and label the points of intersection. The result should be a triangle in each case.

a $x \geqslant 0$
$y \geqslant 0$
$3x + 6y \leqslant 6$

b $x \geqslant 0$
$y \leqslant 0$
$2x - y \leqslant 4$

c $x \geqslant 0$
$5x + 2y \leqslant 30$
$4y - x \geqslant 16$

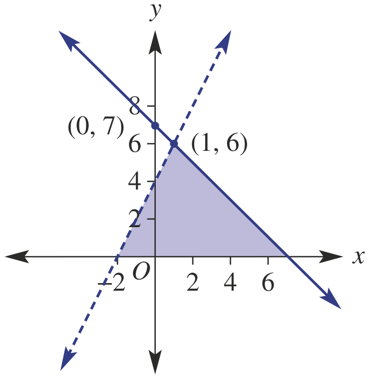
d $x < 2$
$y < 3$
$2x + 5y > 10$

e $x \leqslant 0$
$y < x + 7$
$2x + 3y \geqslant -6$

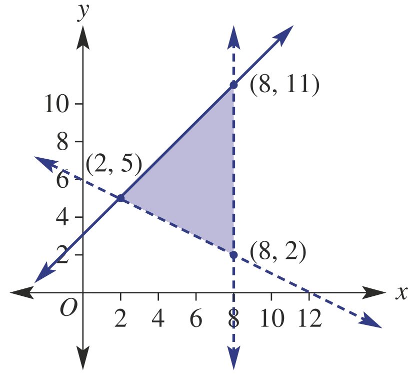
f $x + y \leqslant 9$
$2y - x \geqslant 6$
$3x + y \geqslant -2$

9 Determine the original inequalities that would give the following regions of intersection.

a



b



ENRICHMENT: Areas of regions	–	–	10, 11

10 Find the area of the triangles formed in Question **8** parts **a** to **d**.

11 **a** Find the exact area bound by:

i $x < 0$
$y > 0$
$x + 2y < 6$
$x - y > -7$

ii $y < 7$
$x + y > 5$
$3x - 2y < 14$

b Make up your own set of inequalities that gives an area of 6 square units.

ISBN 978-1-108-77290-7

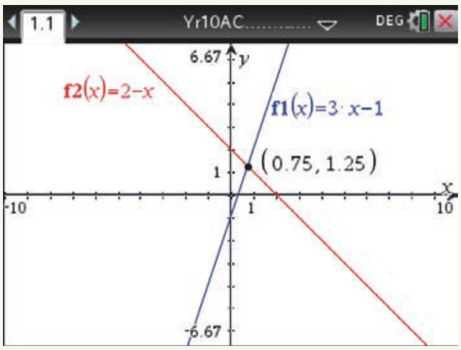
Cambridge University Press
Updated June 2022

Using calculators to find intersection points

1. Sketch a graph of $y = 3x - 1$ and $y = 2 - x$ and locate the intersection point.
2. Sketch the intersecting region of $y < 3x - 1$ and $y > 2 - x$.

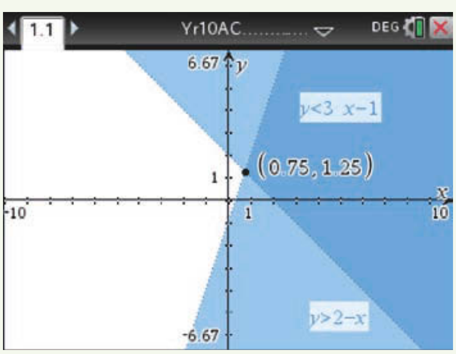
Using the TI-Nspire:

1. In a **Graphs** page enter the rules $f1(x) = 3x - 1$ and $f2(x) = 2 - x$. Select menu **>Analyze Graph>Intersection** and select the lower and upper bounds containing the intersection point. Press enter to paste the coordinates to the graph.



Hint: if multiple graphs are being entered use the down arrow to enter subsequent graphs.
Hint: if the graph entry line is not showing, press tab or double click in an open area.

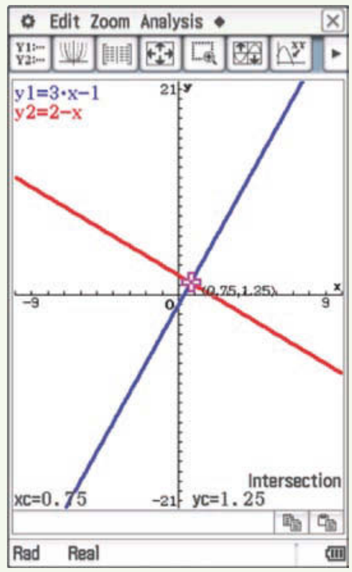
2. In a **Graphs** page press del and select the required inequality from the list and edit $f1(x)$ to $y < 3x - 1$ and $f2(x)$ to $y > 2 - x$.



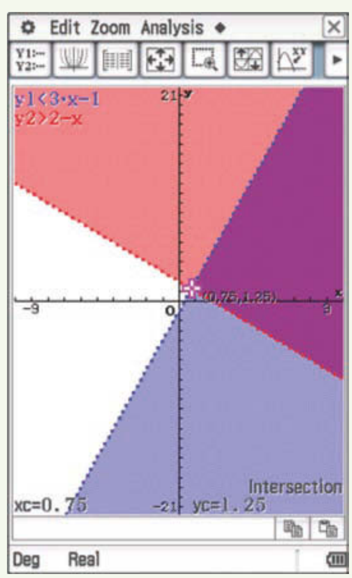
Find the intersection point as shown above.
Hint: if multiple graphs are being entered use the down arrow to enter subsequent graphs.
Hint: if the graph entry line is not showing, press tab or double click in an open area.

Using the ClassPad:

1. In the **Graph&Table** application enter the rules $y1 = 3x - 1$ and $y2 = 2 - x$ then tap . Tap **Zoom Quick**, **Quick Standard** to adjust the window. Tap **Analysis**, **G-Solve Intersect**.



2. Tap and clear all functions. With the cursor in $y1$ tap y=, select y<, enter the rule $3x - 1$ and press **EXE**. With the cursor in $y2$ tap y=, select y>, enter the rule $2 - x$ and press **EXE**. Tap .

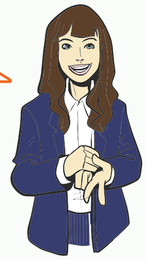


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Problems and challenges

Up for a challenge? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.



1 Tom walks at 4 km/h and runs at 6 km/h. He can save $3\frac{3}{4}$ minutes by running from his house to the train station instead of walking. How many kilometres is it from his house to the station?

2 A fraction is such that when its numerator is increased by 1 and its denominator is decreased by 1, it equals 1 and when its numerator is doubled and its denominator increased by 4 it is also equal to 1. What is the fraction?

3 Show that the following sets of points are collinear (i.e. in a straight line).

a $(2, 12)$, $(-2, 0)$ and $(-5, -9)$

b $(a, 2b)$, $(2a, b)$ and $(-a, 4b)$

4 Use two different methods from this chapter to prove that triangle ABC with vertices $A(1, 6)$, $B(4, 1)$ and $C(-4, 3)$ is a right-angled triangle.

5 Two missiles 2420 km apart are launched at the same time and are headed towards each other. They pass after 1.5 hours. The average speed of one missile is twice that of the other. What is the average speed of each missile?

6 Show that the points $(7, 5)$ and $(-1, 9)$ lie on a circle centred at $(2, 5)$ with radius 5 units.

7 A quadrilateral whose diagonals bisect each other at right angles will always be a rhombus. Prove that the points $A(0, 0)$, $B(4, 3)$, $C(0, 6)$ and $D(-4, 3)$ are the vertices of a rhombus. Is it also a square?

8 Solve this set of simultaneous equations:

$$\begin{aligned} x - 2y - z &= 9 \\ 2x - 3y + 3z &= 10 \\ 3x + y - z &= 4 \end{aligned}$$

9 A triangle, PQR, has $P(8, 0)$, $Q(0, -8)$ and point R is on the line $y = x - 2$. Find the area of the triangle PQR.

10 The average age of players at a ten pin bowling alley increases by 1 when either four 10-year olds leave or, alternatively, if four 22-year olds arrive. How many players were there originally and what was their average age?

ISBN 978-1-108-77290-7

Chapter summary

Linear relations

Midpoint and length of a line segment

Line segment joining (x_1, y_1) and (x_2, y_2) has:

Midpoint $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Length/distance

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Parallel and perpendicular lines

Parallel lines have the same gradient; e.g. $y = 2x + 7$ and $y = 2x - 4$

For perpendicular lines with gradients m_1 and m_2:

$m_1 \times m_2 = -1$ or $m_2 = -\frac{1}{m_1}$

e.g. $y = 3x + 4$ and $y = -\frac{1}{3}x + 2$

Finding a rule for a graph

For points (x_1, y_1) and (x_2, y_2)

$m = \frac{y_2 - y_1}{x_2 - x_1}$

To find line equation $y = mx + c$:

- find gradient, m
- substitute a point if c is unknown using $y = mx + c$ or $y - y_1 = m(x - x_1)$

Review of algebra

Add/subtract like terms only.

e.g. $2x + 4x + y = 6x + y$

Multiply: $3x \times 2y = 6xy$

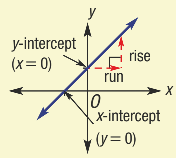
Divide: $4ab \div (2b) = \frac{\cancel{4ab}^{2}}{\cancel{2b}_{1}}$
$= 2a$

Expand: $a(b + c) = ab + ac$
$a(b - c) = ab - ac$

Graphs of linear relations

In the form $y = mx + c$, m is the gradient, c is the y-intercept.

$m = \frac{\text{rise}}{\text{run}}$



To graph:

a use gradient and y-intercept or

b locate x-intercept ($y = 0$) and y-intercept ($x = 0$)

Algebraic fractions

Add/subtract: find LCD.

e.g. $\frac{2}{a} + \frac{3}{5} = \frac{10}{5a} + \frac{3a}{5a} = \frac{10 + 3a}{5a}$

Multiply: cancel common factors first.

e.g. $\frac{\cancel{2}^{1}}{x + 2} \times \frac{3x + 6}{\cancel{8}_{4}} = \frac{1}{\cancel{x + 2}_{1}} \times \frac{3\cancel{(x + 2)}^{1}}{4} = \frac{3}{4}$

Divide: multiply by the reciprocal of the fraction following the ÷.

Solving linear equations

Some steps include:

- using inverse operations
- collecting like terms on one side
- multiplying by the LCD.

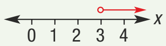
Linear inequalities

Equation contains '=' sign.

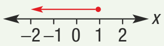
Inequalities have $>, \geq, <$ or $\leq$.

On a number line:

$x > 3$



x ° 1



Solve as per linear equations except when multiplying or dividing by a negative, reverse the inequality sign.

Simultaneous equations

Solve two equations in two pronumerals using:

- substitution when one pronumeral is the subject; e.g. $y = x + 4$.
- elimination when adding or subtracting multiples of equations eliminates one variable.

Applications of simultaneous equations

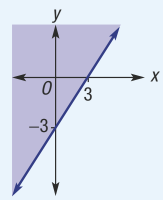
- Define unknowns using pronumerals.
- Form two equations from the information in the problem.
- Solve simultaneously.
- Answer the question in words.

Half planes (Ext)

Inequalities with two variables represent the region above or below the line.

For $y \geq x - 3$, shade above the line.

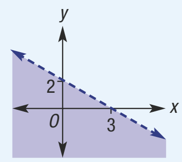
For $y > x - 3$, use a dashed line.



For $2x + 3y < 6$, test (0, 0) to decide which region to shade.

$2(0) + 3(0) < 6$ (true)

Include (0, 0); i.e. shade below dashed line.



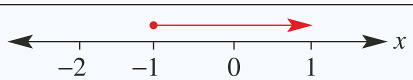
Two or more half planes form an intersecting region.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

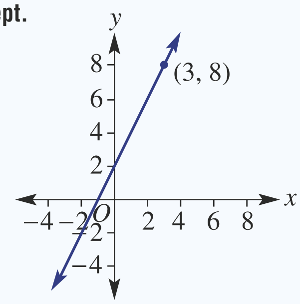
Chapter checklist

Chapter checklist: Success criteria

		✔
1A	**1. I can simplify expressions using the four operations $+, -, \times, \div$.** e.g. Simplify $3x \times 2y - 4x^2y + 2xy + 3x^2y$.	
1A	**2. I can expand brackets using the distributive law.** e.g. Expand and simplify $-4y(2y + 3)$.	
1A	**3. I can factorise simple algebraic expressions with a common factor.** e.g. Factorise $6x^2 - 15x$.	
1A	**4. I can substitute numbers for pronumerals and evaluate.** e.g. Given $a = -3$ and $b = 4$, evaluate $ab + a^2$.	
1B	**5. I can cancel common factors in algebraic fractions.** e.g. Simplify $\frac{4x - 2}{2}$.	
1B	**6. I can multiply and divide algebraic fractions.** e.g. Simplify $\frac{3x - 9}{20} \div \frac{2x - 6}{5}$.	
1C	**7. I can add and subtract simple algebraic fractions using the lowest common denominator.** e.g. Simplify $\frac{4}{x} + \frac{5}{6}$.	
1C	**8. I can add and subtract algebraic fractions with binomial numerators or denominators.** e.g. Simplify $\frac{x + 3}{3} - \frac{x - 2}{7}$ and $\frac{4}{x - 2} + \frac{3}{x + 4}$.	
1D	**9. I can solve linear equations with brackets and variables on both sides.** e.g. Solve $4(3x - 5) = 7x$.	
1D	**10. I can solve linear equations involving algebraic fractions.** e.g. Solve $\frac{x + 1}{3} + \frac{x - 2}{5} = 2$ and $\frac{1 - 2x}{3} = \frac{x + 1}{2}$.	
1E	**11. I can interpret number lines to write inequalities.** e.g. Write as an inequality. 	
1E	**12. I can represent a set of solutions on a number line.** e.g. Graph the inequality $x > 3$ on a number line.	
1F	**13. I can solve linear inequalities.** e.g. Solve $4 - \frac{x}{3} > 8$.	
1F	**14. I can determine if a point is on a straight line.** e.g. Decide if the point $(3, -1)$ is on the line $3x + 2y = 7$.	
1F	**15. I can find the gradient and *y*-intercept from a straight line equation.** e.g. State the gradient and y-intercept of $3y - 2x = 6$.	
1F	**16. I can use the gradient and *y*-intercept to sketch a graph.** e.g. Find the gradient and y-intercept of $y = -2x + 7$ and sketch its graph.	
1F	**17. I can find the *x*- and *y*-intercepts of a linear graph.** e.g. Find the x- and y-intercepts and sketch the graph of $3x + y = 9$	

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

		✔
1F	**18. I can sketch a horizontal or vertical line.** e.g. Sketch $y = 3$.	
1F	**19. I can sketch a line of the form $y = mx$.** e.g. Sketch $y = 2x$ labelling the axis intercept and one other point.	
1G	**20. I can find the gradient of a line joining two points.** e.g. Determine the gradient of the line joining the points $(-2, 4)$ and $(3, 1)$.	
1G	**21. I can find the equation of a line using a point and the y-intercept.** e.g. Find the equation of the straight line shown.  	
1G	**22. I can find the equation of a line given two points.** e.g. Find the equation of the straight line joining the points $(-2, -2)$ and $(2, 3)$.	
1H	**23. I can find the distance between two points.** e.g. Find the exact distance between the points $(2, 4)$ and $(5, 2)$.	
1H	**24. I can find the midpoint of a line segment joining two points.** e.g. Find the midpoint of the line segment joining $(-1, 5)$ and $(5, 2)$.	
1H	**25. I can use a given distance to find coordinates.** e.g. Find the values of a if the distance between $(3, a)$ and $(6, 10)$ is $\sqrt{34}$.	
1I	**26. I can decide if lines are parallel, perpendicular or neither.** e.g. Decide if the graph of the lines $y = 2x + 5$ and $2y + x = 3$ will be parallel, perpendicular or neither.	
1I	**27. I can find the equation of a parallel or perpendicular line.** e.g. Find the equation of the line that is parallel to $y = 3x - 4$ and passes through $(2, 4)$.	
1J	**28. I can solve simultaneous equations using substitution.** e.g. Solve the pair of simultaneous equations $x - 2y = 4$ and $y = x - 3$ using the method of substitution.	
1K	**29. I can solve simultaneous equations by adding or subtracting them.** e.g. Solve the simultaneous equations $x - 2y = 10$ and $x + y = 4$ using elimination.	
1K	**30. I can use the elimination method to solve simultaneous equations.** e.g. Solve the pair of simultaneous equations $2x + 3y = 5$ and $3x - 4y = -18$ using the elimination method.	
1L	**31. I can set up and solve simultaneous equations.** e.g. A teacher buys 5 of the same chocolate bars and 2 of the same ice-creams for \$18 while another teacher buys 4 of the same chocolate bar and 5 of the same ice-creams for \$28. Determine the individual costs of these chocolate bars and ice-creams.	
1M	**32. I can sketch a half plane.** e.g. Sketch the half plane $y > 2x - 5$. Ext	
1M	**33. I can find the intersecting region.** e.g. Sketch both the inequalities $2x + y \geqslant -2$ and $2x - 3y < 6$ on the same set of axes, showing the point of intersection of the two lines and the intersecting region. Ext	

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter review

Short-answer questions

1A **1** Simplify the following. You may need to expand the brackets first.

a $8xy + 5x - 3xy + x$ **b** $3a \times 4ab$

c $18xy \div (12y)$ **d** $3(b + 5) + 6$

e $-3m(2m - 4) + 4m^2$ **f** $3(2x + 4) - 5(x + 2)$

1B **2** Simplify these algebraic fractions by first looking to cancel common factors.

a $\dfrac{12x - 4}{4}$ **b** $\dfrac{5}{3x} \times \dfrac{6x}{5x + 10}$ **c** $\dfrac{3x - 3}{28} \div \dfrac{x - 1}{7}$

1C **3** Simplify the following algebraic fractions.

a $\dfrac{3}{7} - \dfrac{a}{2}$ **b** $\dfrac{5}{6} + \dfrac{3}{a}$

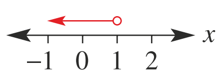
c $\dfrac{x + 4}{6} + \dfrac{x + 3}{15}$ **d** $\dfrac{2}{x - 3} - \dfrac{3}{x + 1}$

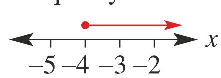
1D **4** Solve these linear equations.

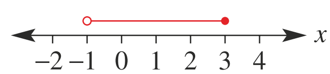
a $3 - 2x = 9$ **b** $3(2x + 1) = 7 - 2(x + 5)$

c $\dfrac{5x - 9}{4} = -2$ **d** $\dfrac{2x - 1}{3} = \dfrac{x + 2}{4}$

1E **5** Write each of the following as an inequality.

a 

b 

c 

1E **6** Solve the following inequalities.

a $4x - 3 > 17$ **b** $3x + 2 \leqslant 4(x - 2)$

c $1 - \dfrac{x}{3} < 2$ **d** $-2x \geqslant -4(1 - 3x)$

1D **7** Marie's watering can is initially filled with 2 litres of water. However, the watering can has a small hole in the base and is leaking at a rate of 0.4 litres per minute.

a Write a rule for the volume of water, V litres, in the can after t minutes.

b What volume of water remains after 90 seconds?

c How long would it take for all the water to leak out?

d If Marie fills the can with 2 litres of water at her kitchen sink, what is the maximum amount of time she can take to get to her garden if she needs at least 600 mL to water her roses?

1F **8** Sketch graphs of the following linear relations, labelling the points where the graph cuts the axes.

a $y = 3x - 9$ **b** $y = 5 - 2x$

c $y = 3$ **d** $x = 5$

e $y = 2x$ **f** $y = -5x$

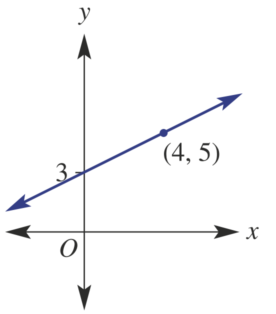
g $x + 2y = 8$ **h** $3x + 8y = 24$

ISBN 978-1-108-77290-7

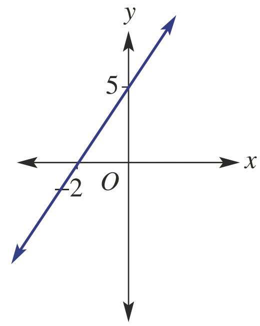
Cambridge University Press
Updated June 2022

1G **9** Find the equation of these straight lines.

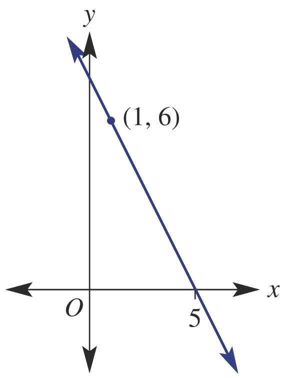
a



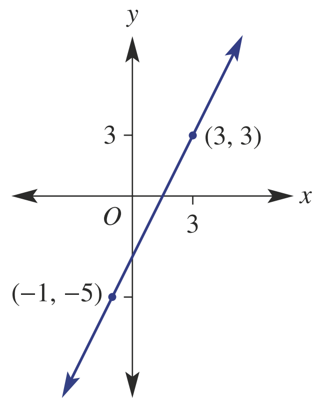
b



c



d



1G **10** For the line that passes through the points $(-2, 8)$ and $(3, 5)$, determine:

a the gradient of the line **b** the equation of the line.

1H **11** Find the midpoint and the exact length of the line segment joining these points.

a $(2, 5)$ and $(6, 11)$ **b** $(3, -2)$ and $(8, 4)$ **c** $(-1, -4)$ and $(2, -1)$

1I **12** Determine the equation of the line that is:

a parallel to the line $y = 3x + 8$ and passes through the point $(2, 4)$
b parallel to the line with equation $y = 4$ and passes through the point $(3, -1)$
c perpendicular to the line $y = 2x - 4$ and has a y-intercept with coordinates $(0, 5)$
d perpendicular to the line with equation $x + 3y = 5$ and passes through the point $(2, 5)$.

1H/I **13** Find the value(s) of the pronumeral in each situation below.

a The gradient of the line joining the points $(2, -5)$ and $(6, a)$ is 3.
b The line $bx + 2y = 7$ is parallel to the line $y = 4x + 3$.
c The distance between $(c, -1)$ and $(2, 2)$ is $\sqrt{13}$.

1J **14** Solve the following simultaneous equations, using the substitution method.

a $y = 5x + 14$
$y = 2x + 5$

b $3x - 2y = 18$
$y = 2x - 5$

1K **15** Solve these simultaneous equations by elimination.

a $3x + 2y = -11$
$2x - y = -5$

b $2y - 5x = 4$
$3y - 2x = 6$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

1L **16** At the movies Jodie buys three regular popcorns and five small drinks for her friends at a cost of \$24.50. Her friend Renee buys four regular popcorns and three small drinks for her friends at a cost of \$23.50. Find the individual costs of a regular popcorn and a small drink.

1M Ext **17** Sketch these half planes.

a $y \geqslant 3x - 4$

b $2x - 3y > -8$

1M Ext **18** Shade the intersecting region of the inequalities $x + 2y \geqslant 4$ and $3x - 2y < 12$ by sketching their half planes on the same axes and finding their point of intersection.

Multiple-choice questions

1A **1** The simplified form of $3x + 2x \times 7y - 3xy + 5x$ is:

A $10x + 4y$ **B** $35xy - 6x^2y$ **C** $\dfrac{35y - 6xy}{7}$ **D** $5xy - 6x^2$ **E** $8x + 11xy$

1B **2** $\dfrac{3x - 6}{2} \times \dfrac{8}{x - 2}$ simplifies to:

A 9 **B** -6 **C** $-4(x - 3)$ **D** 12 **E** $4(x - 2)$

1C **3** $\dfrac{4}{x - 1} - \dfrac{5}{2x - 3}$ simplifies to:

A $\dfrac{-1}{(x - 1)(2x - 3)}$ **B** $\dfrac{3x - 7}{(x - 1)(2x - 3)}$ **C** $\dfrac{3x - 17}{(2x - 3)(x - 1)}$

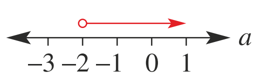
D $\dfrac{11 - 6x}{(x - 1)(2x - 3)}$ **E** $\dfrac{x - 7}{(2x - 3)(x - 1)}$

1D **4** The solution to $-4(2x - 6) = 10x$ is:

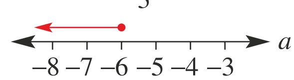
A $x = \dfrac{3}{2}$ **B** $x = 12$ **C** $x = \dfrac{4}{3}$ **D** $x = -12$ **E** $x = -\dfrac{4}{3}$

1E **5** The number line that represents the solution to the inequality $2 - \dfrac{a}{3} < 4$ is:

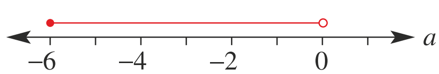
A



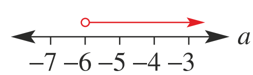
B



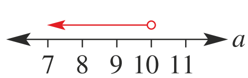
C



D



E



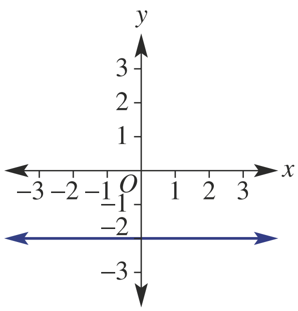
1F **6** If $(-1, 2)$ is a point on the line $ax - 4y + 11 = 0$, the value of a is:

A -19 **B** 3 **C** $-\dfrac{15}{2}$ **D** 5 **E** -1

ISBN 978-1-108-77290-7

1F **7** The graph shown has equation:

A $x = -2$
B $y = -2x$
C $y = -2$
D $x + y = -2$
E $y = x - 2$



1F **8** The gradient and the y-intercept, respectively, of the graph of $3x + 8y = 2$ are:

A $-\frac{3}{8}, \frac{1}{4}$ **B** $3, 2$ **C** $\frac{2}{3}, \frac{1}{4}$ **D** $-3, 2$ **E** $\frac{3}{8}, 2$

1G **9** The equation of the line joining the points $(-1, 3)$ and $(1, -1)$ is:

A $2y - x = 1$ **B** $y = 2x - 1$ **C** $y = -2x + 1$
D $y - 2x = 1$ **E** $y = \frac{1}{2}x + 1$

1H **10** The midpoint of the line segment joining the points $(a, -6)$ and $(7, b)$ is $(4.5, -1)$. The values of the pronumerals are:

A $a = 2, b = 8$ **B** $a = 3, b = -11$ **C** $a = 9, b = 5$
D $a = 2, b = 4$ **E** $a = 2.5, b = 5$

1I **11** The line that is perpendicular to the line with equation $y = -3x + 7$ is:

A $y = -3x + 2$ **B** $3x + y = -1$ **C** $y = 3x - 3$ **D** $3y = 4 - x$ **E** $3y - x = 4$

1I **12** The line that is parallel to the line with equation $y = 2x + 3$ and passes through the point $(-3, 2)$ has the equation:

A $2x + y = 5$ **B** $y = 2x + 8$ **C** $y = -\frac{1}{2}x + \frac{1}{2}$ **D** $y = 2x - 4$ **E** $y - 2x = -7$

1J **13** The solution to the simultaneous equations $2x - 3y = -1$ and $y = 2x + 3$ is:

A $x = -2, y = -1$ **B** $x = \frac{5}{2}, y = 8$ **C** $x = 2, y = 7$
D $x = -\frac{2}{3}, y = -\frac{1}{9}$ **E** $x = -3, y = 3$

1L **14** A community fundraising concert raises \$3540 from ticket sales to 250 people. Children's tickets were sold for \$12 and adult tickets sold for \$18. If x adults and y children attended the concert, the two equations that represent this problem are:

A $x + y = 250$; $18x + 12y = 3540$
B $x + y = 3540$; $216xy = 3540$
C $x + y = 250$; $12x + 18y = 3540$
D $x + y = 3540$; $18x + 12y = 250$
E $3x + 2y = 3540$; $x + y = 250$

1M Ext **15** The point that is *not* in the region defined by $2x - 3y \leqslant 5$ is:

A $(0, 0)$ **B** $(1, -1)$ **C** $(-3, 2)$
D $(2, -1)$ **E** $\left(\frac{5}{2}, 3\right)$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

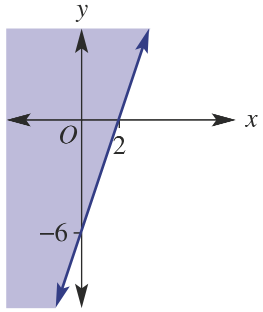
Chapter review

1M

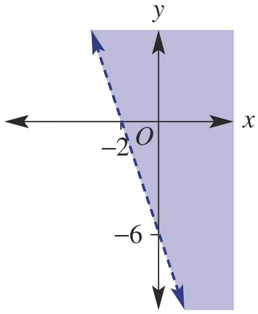
Ext

16 The half plane that represents the inequality $y < 3x - 6$ is:

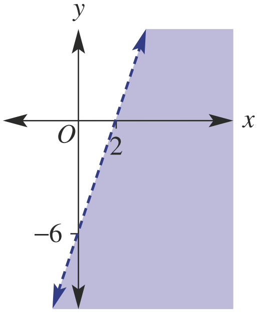
A



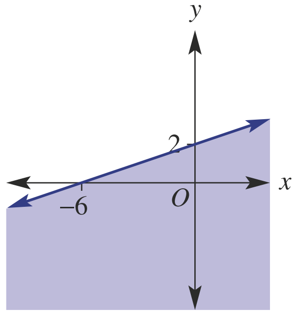
B



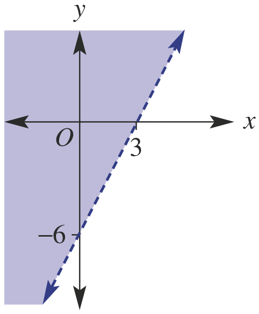
C



D



E



Extended-response questions

1 There are two shrubs in Chen's backyard that grow at a constant rate. Shrub A had an initial height of 25 cm and has grown to 33 cm after 2 months. Shrub B was 28 cm high after 2 months and 46 cm high after 5 months.

a Write a rule for the height, h cm, after t months for:
 i shrub A
 ii shrub B

b What was the initial height (i.e. at $t = 0$) of shrub B?

c Refer to your rules in part **a** to explain which shrub is growing at a faster rate.

d Graph each of your rules from part **a** on the same set of axes for $0 \leqslant t \leqslant 12$.

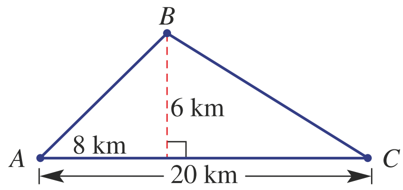
e Determine graphically after how many months the height of shrub B will overtake the height of shrub A.

f **i** Shrub B reaches its maximum height after 18 months. What is this height?
 ii Shrub A has a maximum expected height of 1.3 m. After how many months will it reach this height?
 iii Chen will prune shrub A when it is between 60 cm and 70 cm. Within what range of months after it is planted will Chen need to prune the shrub?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2 A triangular course has been roped off for a cross-country run. The run starts and ends at A and goes via checkpoints B and C, as shown.



a Draw the area of land onto a set of axes, taking point A to be the origin $(0, 0)$. Label the coordinates of B and C.

b Find the length of the course, to one decimal place, by calculating the distance of legs AB, BC and CA.

c A drink station is located at the midpoint of BC. Label the coordinates of the drink station on your axes.

d Find the equation of each leg of the course:

i AB ii BC iii CA

Ext **e** Write a set of three inequalities that would overlap to form an intersecting region equal to the area occupied by the course.

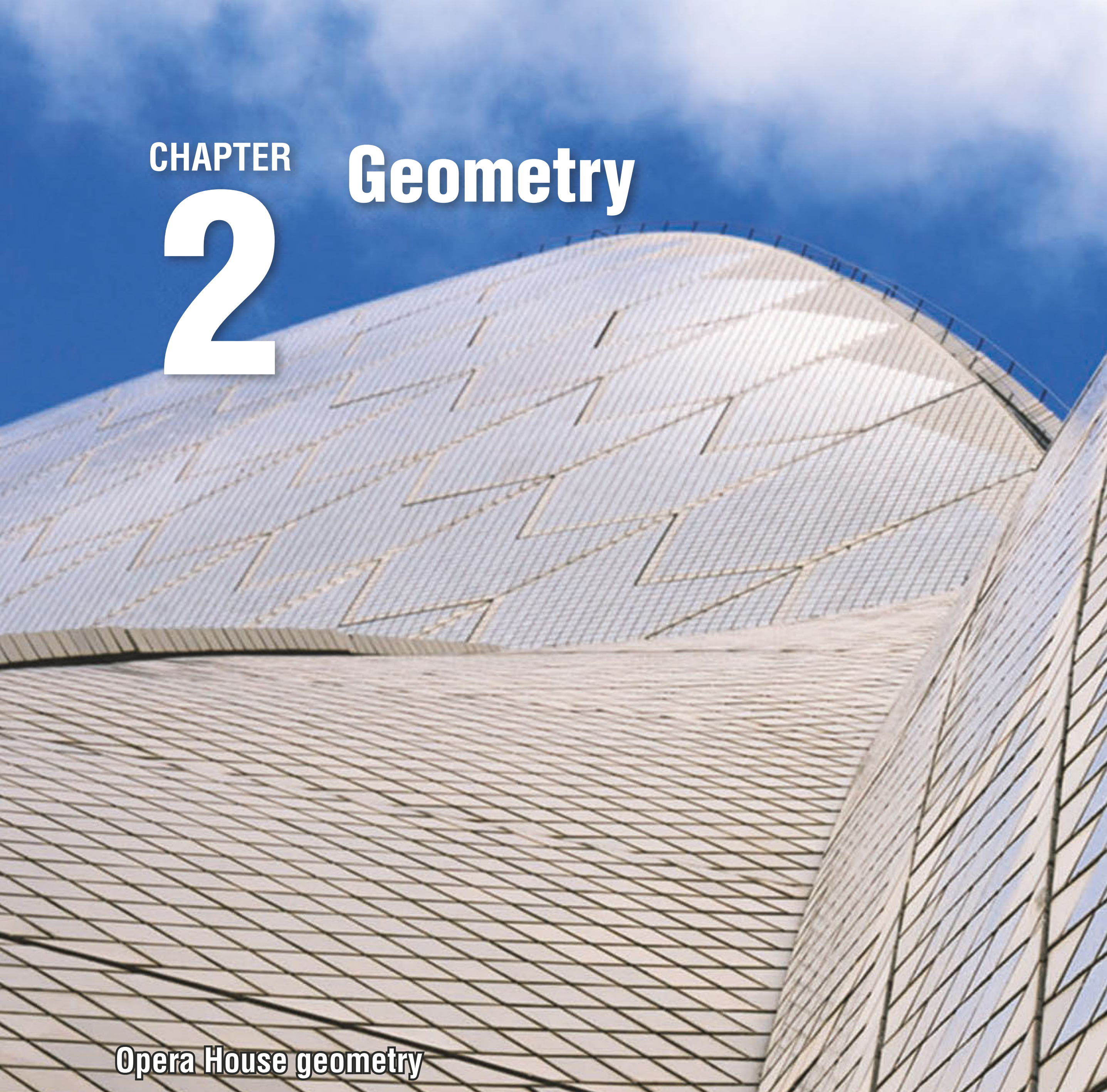
f A fence line runs beyond the course. The fence line passes through point C and would intersect AB at right angles if AB was extended to the fence line. Find the equation of the fence line.

Chapter review

ISBN 978-1-108-77290-7

CHAPTER

2 Geometry



Opera House geometry

The geometry of the Sydney Opera House is based on triangles drawn on a sphere. To understand how this was achieved, imagine an orange sliced into wedges and then each wedge cut into two pieces in a slanting line across the wedge. The orange skin of each wedge portion is a 3D triangle and illustrates the shape of one side of an Opera House sail. One full sail has two congruent curved triangular sides joined, each a reflection of the other. The edges of the sails form arcs of circles.

All the curved sides of the 14 Opera House sails could be joined to form one very large, whole sphere.

The Danish architect Jørn Utzon and engineer Ove Arup chose a radius of 75 m for the virtual sphere from which to design the curved triangular sails. The calculations required high level mathematical modelling, including the applications of circle and spherical geometry. Designed in the 1950s and

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

1960s, it was one of the first ever projects to use CAD (computer assisted drawing).

Overall the Opera House is 120 m wide, 85 m long and 67 m high (20 storeys above the water level). The sails are covered with about 1 million tiles and it is visited by over 8 million people each year.

Online resources

A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

In this chapter

Victorian Curriculum

MEASUREMENT AND GEOMETRY
Geometric reasoning

Formulate proofs involving congruent triangles and angle properties (VCMMG344)

Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes (VCMMG345)

(10A) Prove and apply angle and chord properties of circles (VCMMG366)

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2A Review of geometry CONSOLIDATING

Learning intentions

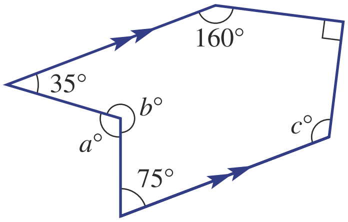
- To review the names of types of angles and the names and properties of angles in parallel lines
- To review the properties of triangles and quadrilaterals and the names and the angle sum rule of polygons
- To know the meaning of the term regular polygon
- To be able to work with polygon angle sums to find missing angles including exterior angles

Based on just five simple axioms (i.e. known or self-evident facts) the famous Greek mathematician Euclid (about 300 BC) was able to deduce many hundreds of propositions (theorems and constructions) systematically presented in the 13-volume book collection called the *Elements*. All the basic components of school geometry are outlined in these books, including the topics *angle sums of polygons* and *angles in parallel lines*, which are studied in this section.

Trade workers who regularly use geometry include: sheet metal workers building a commercial kitchen; plumbers joining parallel but separated water pipes; carpenters building house roof frames; and builders of steps and wheelchair ramps.

LESSON STARTER Three unknown angles

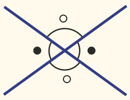
This hexagon contains a pair of parallel sides and three unknown angles, a, b and c.



- Find the value of a using the given angles in the hexagon. (*Hint*: Add a construction line parallel to the two parallel sides so that the angle of size $a°$ is divided into two smaller angles. Give reasons throughout your solution.)
- Find the value of b, giving a reason.
- What is the angle sum of a hexagon? Use this to find the value of c.
- Can you find a different method to find the value of c, using parallel lines?

KEY IDEAS

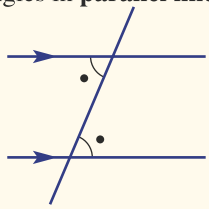
- Angles at a point
  - **Complementary** (sum to 90°)
  - **Supplementary** (sum to 180°)
  - **Revolution** (360°)
  - **Vertically opposite angles** (equal), as shown



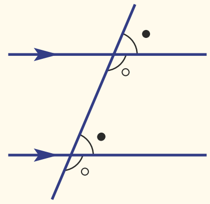
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

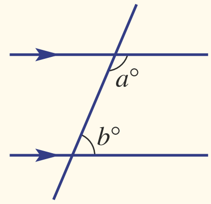
- Angles in **parallel lines**



Alternate angles are equal.



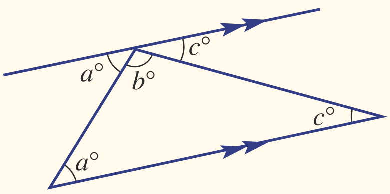
Corresponding angles are equal.



Cointerior angles are supplementary.
$a + b = 180$

- If two lines, AB and CD, are parallel, we write $AB \parallel CD$.

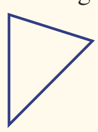
- **Triangles**
  - Angle sum is 180°.
    To prove this, draw a line parallel to a base and then mark the alternate angles in parallel lines.
    Note that angles on a straight line are supplementary.



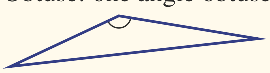
$a + b + c = 180$

  - Triangles classified by angles.

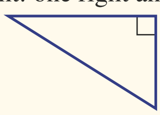
Acute: all angles acute



Obtuse: one angle obtuse

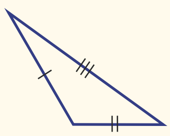


Right: one right angle

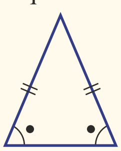


  - Triangles classified by side lengths.

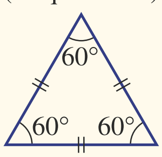
Scalene
(3 different sides)



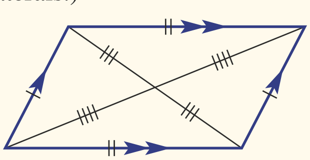
Isosceles
(2 equal sides)



Equilateral
(3 equal sides)



- **Quadrilaterals** (Refer to **Section 2C** for more details on quadrilaterals.)
  - **Parallelograms** are quadrilaterals with two pairs of parallel sides.
  - **Rectangles** are parallelograms with all angles 90°.
  - **Rhombuses** are parallelograms with sides of equal length.
  - **Squares** are parallelograms that are both rectangles and rhombuses.
  - **Kites** are quadrilaterals with two pairs of equal adjacent sides.
  - **Trapeziums** are quadrilaterals with at least one pair of parallel sides.



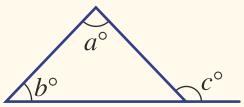
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

- **Polygons** have an angle sum given by $S = 180(n - 2)$, where n is the number of sides.
  - **Regular polygons** have equal sides and equal angles.

    A single interior angle $= \dfrac{180(n-2)}{n}$
- An **exterior angle** is supplementary to an interior angle.
  - For a triangle, the **exterior angle theorem** states that the exterior angle is equal to the sum of the two opposite interior angles.

    $c = a + b$



n	Name
3	triangle
4	quadrilateral
5	pentagon
6	hexagon
7	heptagon
8	octagon
9	nonagon
10	decagon

BUILDING UNDERSTANDING

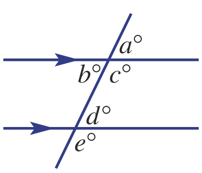
1 State the names of the polygons with 3 to 10 sides, inclusive.

2 Decide whether each of the following is true or false.

a The angle sum of a quadrilateral is 300°.
b A square has 4 lines of symmetry.
c An isosceles triangle has two equal sides.
d An exterior angle on an equilateral triangle is 120°.
e A kite has two pairs of equal opposite angles.
f A parallelogram is a rhombus.
g A square is a rectangle.
h Vertically opposite angles are supplementary.
i Cointerior angles in parallel lines are supplementary.

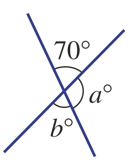
3 State the pronumeral in the diagram that matches the following descriptions.

a alternate to $d°$
b corresponding to $e°$
c cointerior to $c°$
d vertically opposite to $b°$

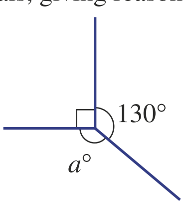


4 Find the values of the pronumerals, giving reasons.

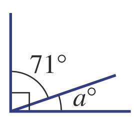
a



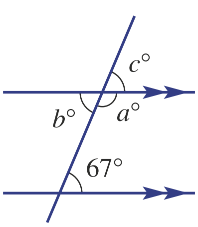
b



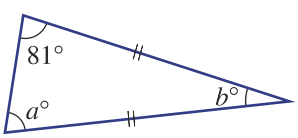
c



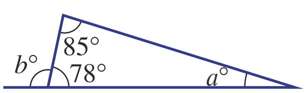
d



e



f



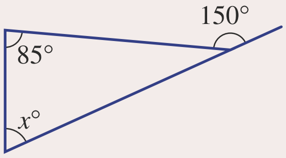
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

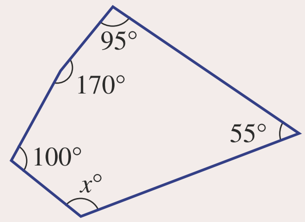
Example 1 Using exterior angles and angle sums

Find the value of x in the following, giving reasons.

a



b



SOLUTION

a $x + 85 = 150$ (exterior angle theorem)

$x = 65$

EXPLANATION

Use the exterior angle theorem for a triangle.

b
$$\begin{aligned} S &= 180(n - 2) \\ &= 180 \times (5 - 2) \\ &= 540 \end{aligned}$$

Use the rule for the angle sum of a polygon (5 sides, so $n = 5$).

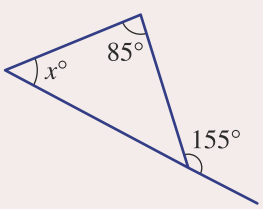
$$\begin{aligned} x + 100 + 170 + 95 + 55 &= 540 \\ x + 420 &= 540 \\ \therefore x &= 120 \end{aligned}$$

The sum of all the angles is 540°.

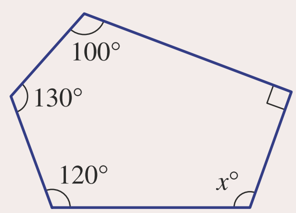
Now you try

Find the value of x in the following, giving reasons.

a



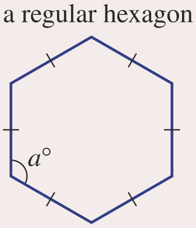
b



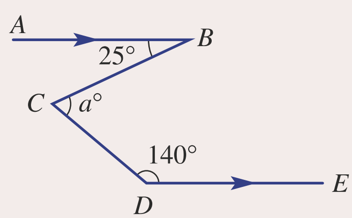
Example 2 Working with regular polygons and parallel lines

Find the value of the pronumeral, giving reasons.

a a regular hexagon



b



Continued on next page

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

SOLUTION

a $S = 180(n - 2)$
$\quad = 180 \times (6 - 2)$
$\quad = 720$
$a = 720 \div 6$
$\quad = 120$

EXPLANATION

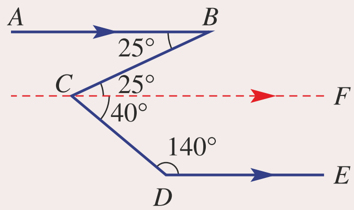
Use the angle sum rule for a polygon with $n = 6$.

In a regular hexagon there are 6 equal angles.

SOLUTION

b Construct a third parallel line, CF.
$\angle BCF = 25^\circ$ (alternate angles in $\parallel$ lines)
$\angle FCD = 180^\circ - 140^\circ$
$\quad = 40^\circ$ (cointerior angles in $\parallel$ lines)
$\therefore a = 25 + 40$
$\quad = 65$

EXPLANATION

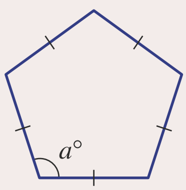


$\angle ABC$ and $\angle FCB$ are alternate angles in parallel lines. $\angle FCD$ and $\angle EDC$ are cointerior angles in parallel lines.

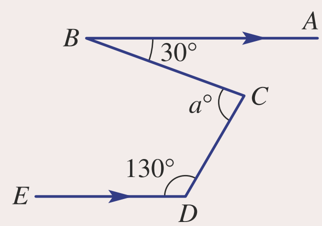
Now you try

Find the value of the pronumeral, giving reasons.

a a regular pentagon



b



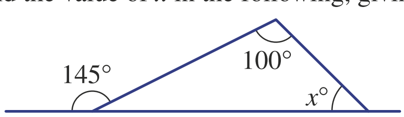
Exercise 2A

FLUENCY	1, 2, 3–4(½), 5	2, 3–4(½), 5	2–5(⅓)

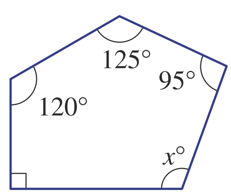
Example 1

1 Find the value of x in the following, giving reasons.

a



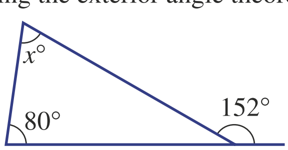
b



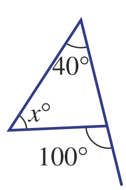
Example 1a

2 Using the exterior angle theorem, find the value of the pronumeral.

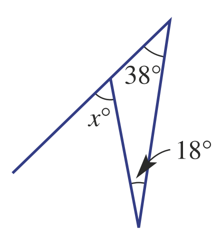
a



b



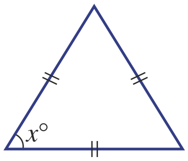
c

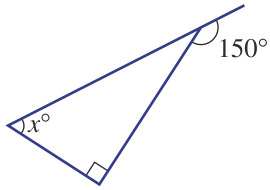


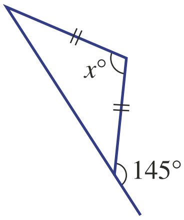
ISBN 978-1-108-77290-7

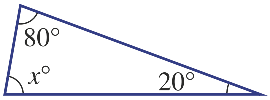
Cambridge University Press
Updated June 2022

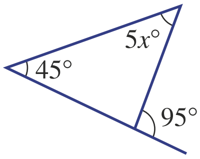
3 Find the value of the pronumeral, giving reasons.

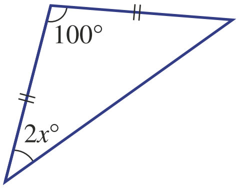
a


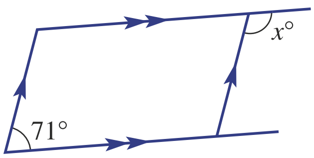
b


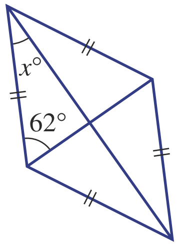
c


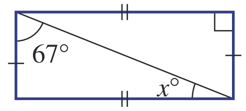
d


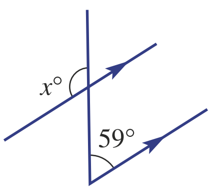
e


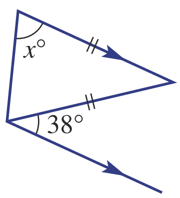
f


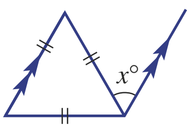
g


h


i


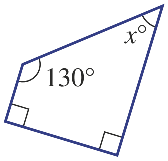
j


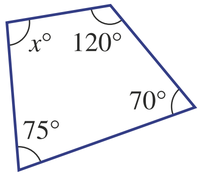
k


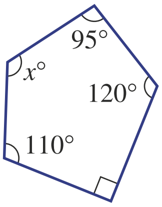
l


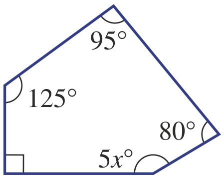
Example 1b

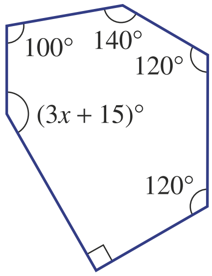
4 Find the value of x in the following, giving reasons.

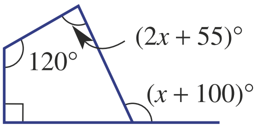
a


b


c


d


e


f


Example 2a

5 Find the size of an interior angle of these polygons if they are regular.

a pentagon **b** octagon **c** decagon

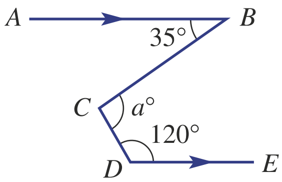
ISBN 978-1-108-77290-7

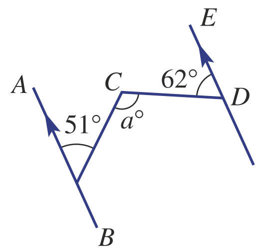
Cambridge University Press
Updated June 2022

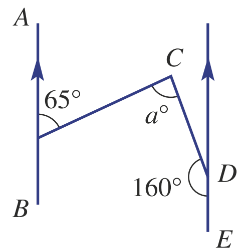
PROBLEM-SOLVING	6(1/2), 7	6(1/2), 7, 8	6(1/2), 8–10

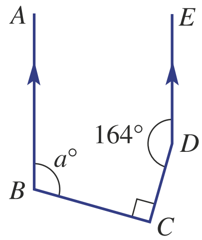
Example 2b

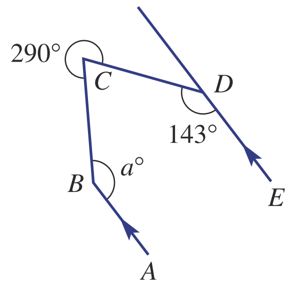
6 Find the value of the pronumeral a, giving reasons.

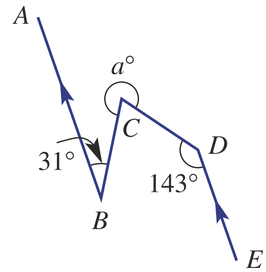
a


b


c


d


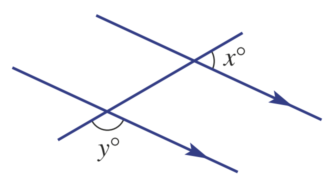
e


f


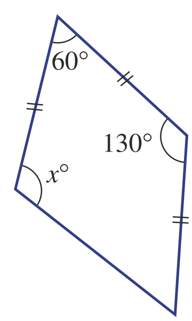
7 **a** Find the size of an interior angle of a regular polygon with 100 sides.
b What is the size of an exterior angle of a 100-sided regular polygon?

8 Find the number of sides of a polygon that has the following interior angles.
a 150° **b** 162° **c** 172.5°

9 In this diagram, $y = 4x$. Find the values of x and y.



10 Find the value of x in this diagram, giving reasons.
(*Hint*: Form isosceles and/or equilateral triangles.)



REASONING	11, 12	11–13	12–15

11 The rule for the sum of the interior angles of a polygon, $S°$, is given by $S = 180(n - 2)$.
a Show that $S = 180n - 360$.
b Find a rule for the number of sides n of a polygon with an angle sum S; i.e. write n in terms of S.
c Write the rule for the size of an interior angle $I°$ of a regular polygon with n sides.
d Write the rule for the size of an exterior angle $E°$ of a regular polygon with n sides.

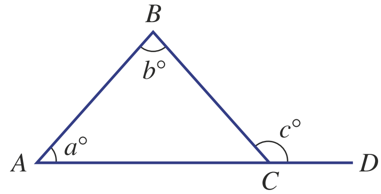
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

12 Prove that the exterior angle of a triangle is equal to the sum of the two opposite interior angles by following these steps.

a Write $\angle BCA$ in terms of a and b and give a reason.

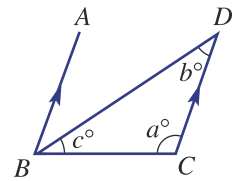
b Find c in terms of a and b using $\angle BCA$ and give a reason.



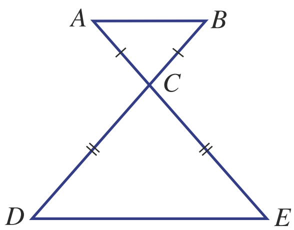
13 **a** Explain why in this diagram $\angle ABD$ is equal to $b°$.

b Using $\angle ABC$ and $\angle BCD$, what can be said about a, b and c?

c What does your answer to part **b** show?



14 Give reasons why AB and DE in this diagram are parallel; i.e. $AB \parallel DE$.

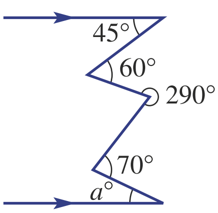


15 Each point on the Earth's surface can be described by a line of longitude (degrees east or west from Greenwich, England) and a line of latitude (degrees north or south from the equator). Investigate and write a brief report (providing examples) describing how places on the Earth can be located with the use of longitude and latitude.

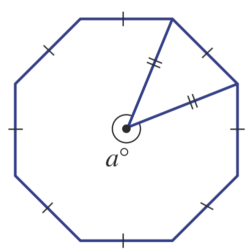
ENRICHMENT: Multilayered reasoning	–	–	16–18

16 Find the value of the pronumerals, giving reasons.

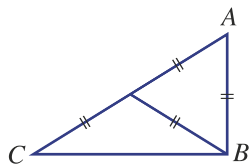
a



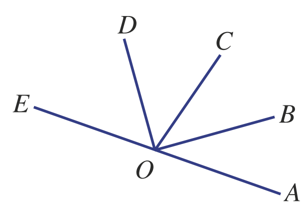
b



17 Give reasons why $\angle ABC = 90°$.



18 In this diagram $\angle AOB = \angle BOC$ and $\angle COD = \angle DOE$. Give reasons why $\angle BOD = 90°$.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exploring triangles with interactive geometry

Construct a triangle and illustrate the angle sum and the exterior angle theorem.

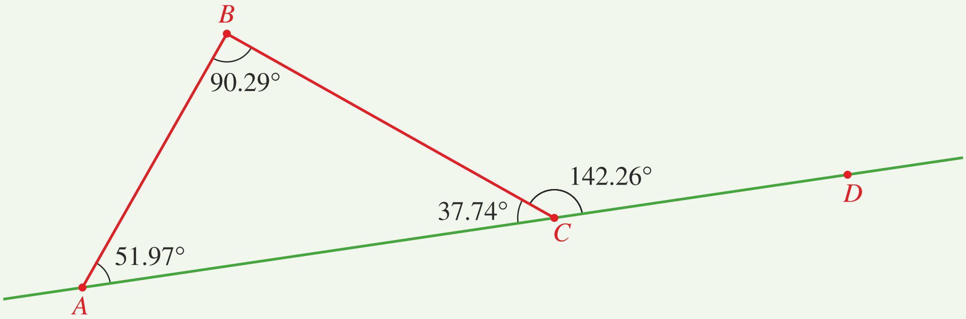
Interactive geometry instructions and screens

a Construct a line AD and triangle ABC, as shown.

b Measure all the interior angles and the exterior angle $\angle BCD$.

c Use the calculator to check that the angle sum is 180° and that $\angle BCD = \angle BAC + \angle ABC$.

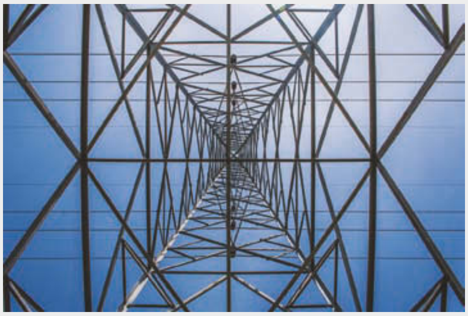
d Drag one of the points to check that these properties are retained for all triangles.



2B Congruent triangles

Learning intentions
- To know the meaning of the term congruent
- To be able to match corresponding sides and angles in congruent figures
- To know the four tests for congruence of triangles
- To know how to prove that two triangles are congruent using one of the tests
- To be able to use congruence of triangles to prove other properties

In geometry it is important to know whether or not two objects are in fact identical in shape and size. If two objects are identical, then we say they are congruent. Two shapes that are congruent will have corresponding (i.e. matching) sides equal in length and corresponding angles also equal. For two triangles it is not necessary to know every side and angle to determine if they are congruent. Instead, a set of minimum conditions is enough. There are four sets of minimum conditions for triangles and these are known as the tests for congruence of triangles.



Civil engineers apply congruent triangle geometry in the design and construction of buildings, bridges, cranes and electricity pylons. Triangles are the strongest form of support and congruent triangles evenly distribute the weight of the construction.

LESSON STARTER Which are congruent?

Consider these four triangles.

1 $\triangle ABC$ with $\angle A = 37^\circ$, $\angle B = 112^\circ$ and $AC = 5$ cm.
2 $\triangle DEF$ with $\angle D = 37^\circ$, $DF = 5$ cm and $\angle E = 112^\circ$.
3 $\triangle GHI$ with $\angle G = 45^\circ$, $GH = 7$ cm and $HI = 5$ cm.
4 $\triangle JKL$ with $\angle J = 45^\circ$, $JK = 7$ cm and $KL = 5$ cm.

Sarah says that only $\triangle ABC$ and $\triangle DEF$ are congruent. George says that only $\triangle GHI$ and $\triangle JKL$ are congruent and Tobias says that both pairs ($\triangle ABC$, $\triangle DEF$ and $\triangle GHI$, $\triangle JKL$) are congruent.

- Discuss which pairs of triangles might be congruent, giving reasons.
- What drawings can be made to support your argument?
- Who is correct: Sarah, George or Tobias? Explain why.

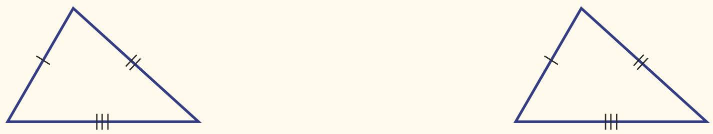
KEY IDEAS

- Two objects are said to be **congruent** when they are exactly the same size and shape. For two congruent triangles $\triangle ABC$ and $\triangle DEF$, we write $\triangle ABC \equiv \triangle DEF$.
  - When comparing two triangles, corresponding sides are equal in length and corresponding angles are equal.
  - When we prove congruence in triangles, we usually write vertices in matching order.

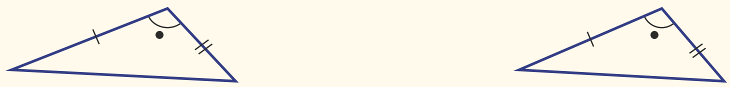
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

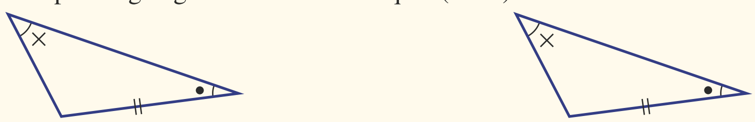
- Two triangles can be tested for **congruence** using the following conditions.
  - Corresponding sides are equal (SSS).



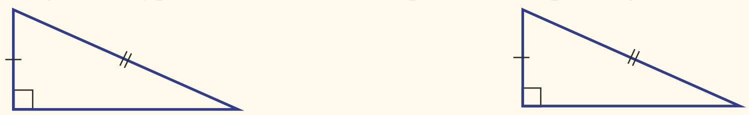
  - Two corresponding sides and the included angle are equal (SAS).



  - Two corresponding angles and a side are equal (AAS).



  - A right angle, the hypotenuse and one other pair of corresponding sides are equal (RHS).

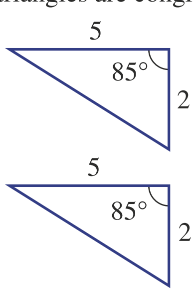


- $AB \parallel CD$ means AB is parallel to CD.
- $AB \perp CD$ means AB is perpendicular to CD.

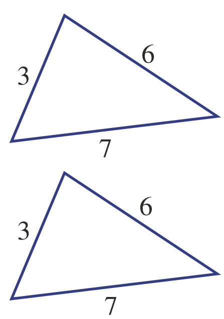
BUILDING UNDERSTANDING

1 Which of the tests (SSS, SAS, AAS or RHS) would be used to decide whether the following pairs of triangles are congruent?

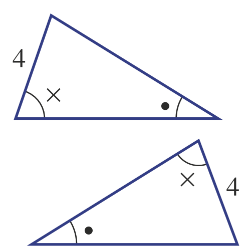
a



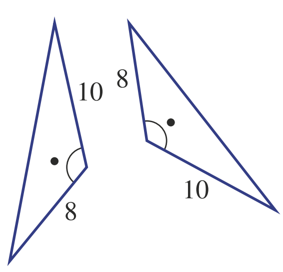
b



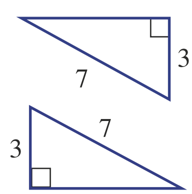
c



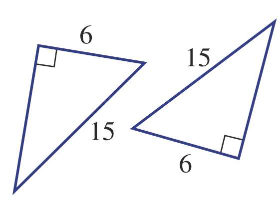
d



e



f

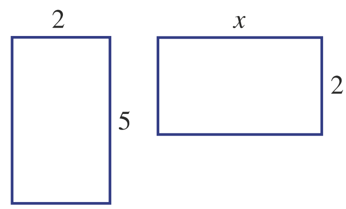


ISBN 978-1-108-77290-7

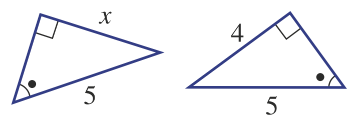
Cambridge University Press
Updated June 2022

2 Assume these pairs of figures are congruent and state the value of the pronumeral in each case.

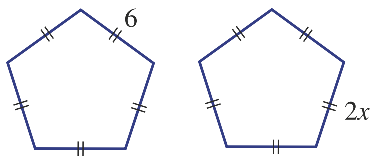
a



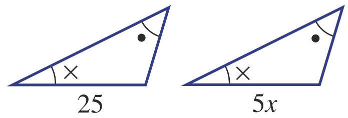
b



c



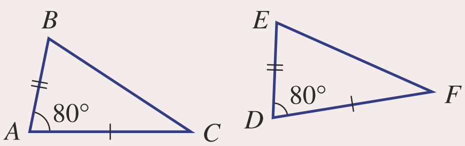
d



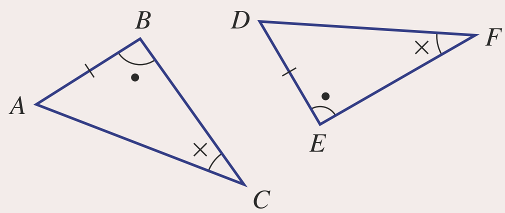
Example 3 Proving congruence in triangles

Prove that these pairs of triangles are congruent.

a



b

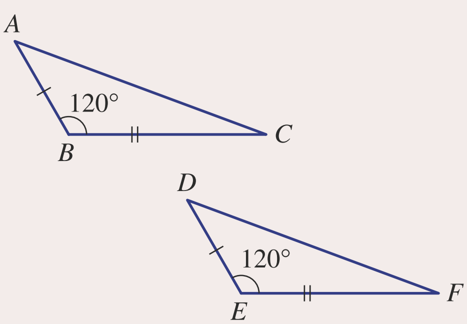


SOLUTION	EXPLANATION
a $AB = DE$ (given) S $\angle BAC = \angle EDF = 80°$ (given) A $AC = DF$ (given) S So, $\triangle ABC \equiv \triangle DEF$ (SAS)	List all pairs of corresponding sides and angles. The two triangles are therefore congruent, with two pairs of corresponding sides and the included angle equal.
b $\angle ABC = \angle DEF$ (given) A $\angle ACB = \angle DFE$ (given) A $AB = DE$ (given) S So, $\triangle ABC \equiv \triangle DEF$ (AAS)	List all pairs of corresponding sides and angles. The two triangles are therefore congruent with two pairs of corresponding angles and a corresponding side equal.

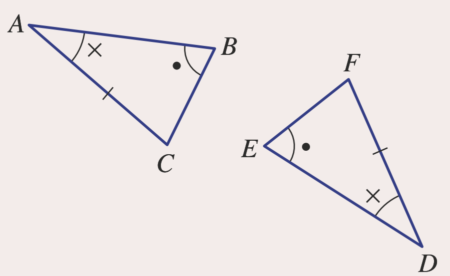
Now you try

Prove that these pairs of triangles are congruent.

a



b



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

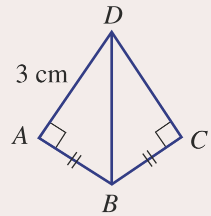
Example 4 Using congruence in proof

In this diagram, $\angle A = \angle C = 90°$ and $AB = CB$.

a Prove $\triangle ABD \equiv \triangle CBD$.

b Prove $AD = CD$.

c State the length of CD.



SOLUTION

a $\angle A = \angle C = 90°$ (given) R
BD is common H
$AB = CB$ (given) S
$\therefore \triangle ABD \equiv \triangle CBD$ (RHS)

b $\triangle ABD \equiv \triangle CBD$ so $AD = CD$
(corresponding sides in congruent triangles)

c $CD = 3$ cm

EXPLANATION

Systematically list corresponding pairs of equal angles and lengths.

Since $\triangle ABD$ and $\triangle CBD$ are congruent, the matching sides AD and CD are equal.

$AD = CD$ from part **b** above.

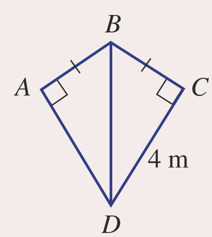
Now you try

In this diagram, $\angle A = \angle C = 90°$ and $AB = CB$.

a Prove $\triangle ABD \equiv \triangle CBD$.

b Prove $AD = CD$.

c State the length of CD.



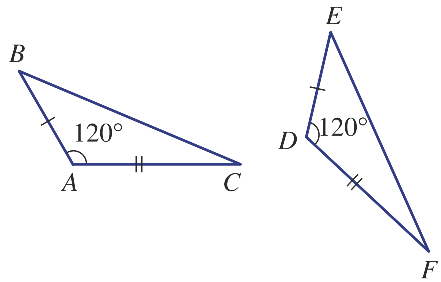
Exercise 2B

FLUENCY	1–3	2, 3–4(1/2)	2–4(1/2)

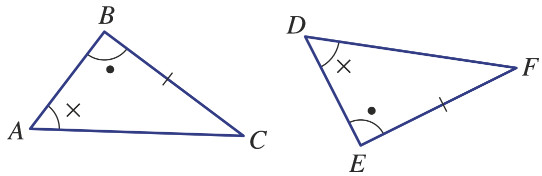
Example 3

1 Prove that these pairs of triangles are congruent.

a



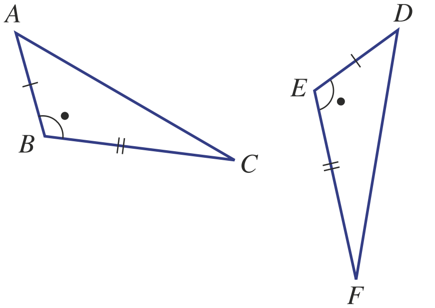
b



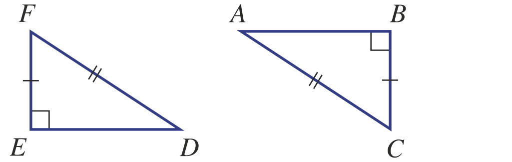
Example 3

2 Prove that these pairs of triangles are congruent.

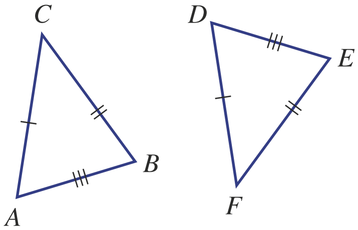
a



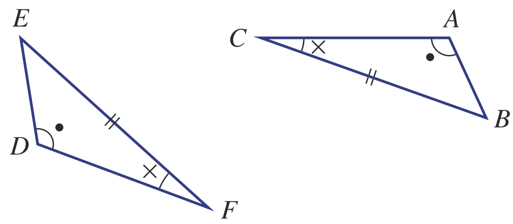
b



c

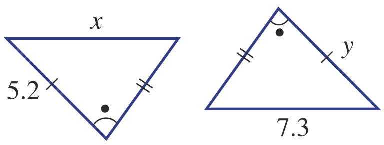


d

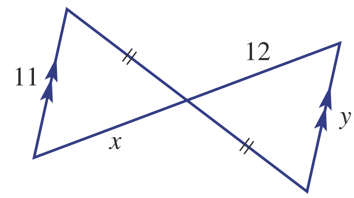


3 Find the value of the pronumerals in these diagrams, which include congruent triangles.

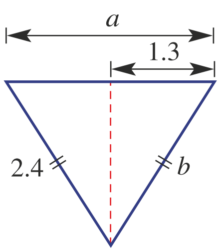
a



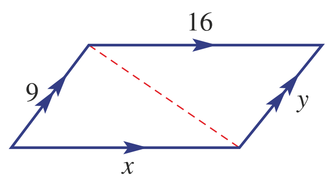
b

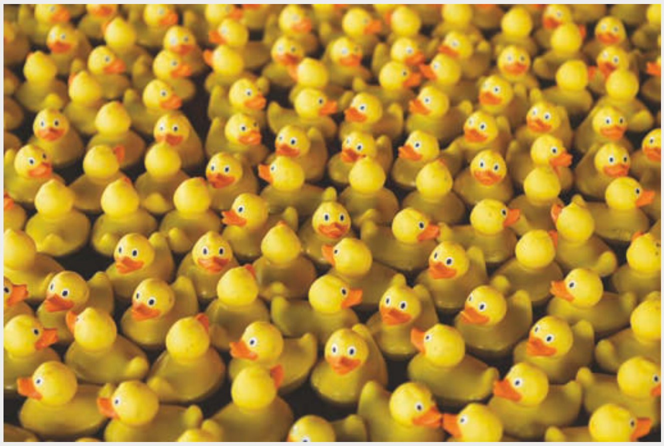


c



d



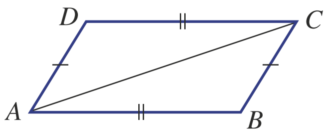


Congruent objects are identical: same size and shape.

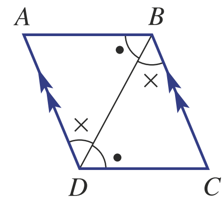
ISBN 978-1-108-77290-7

4 Prove that each pair of triangles is congruent, giving reasons. Write the vertices in matching order.

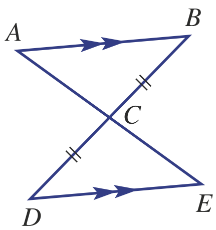
a



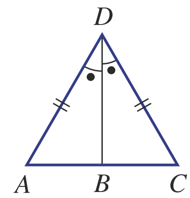
b



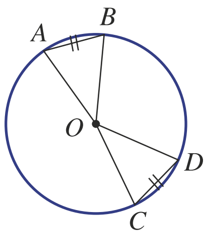
c



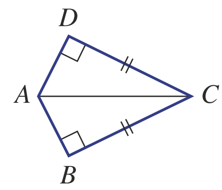
d



e



f



PROBLEM-SOLVING	5	5, 6	5–7

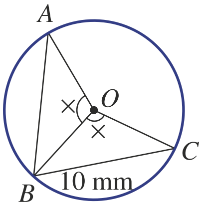
Example 4

5 In this diagram, O is the centre of the circle and $\angle AOB = \angle COB$.

a Prove $\triangle AOB \equiv \triangle COB$.

b Prove $AB = BC$.

c State the length of AB.



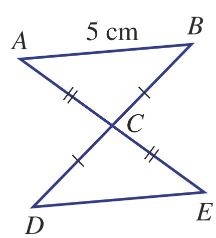
6 In this diagram, $BC = DC$ and $AC = EC$.

a Prove $\triangle ABC \equiv \triangle EDC$.

b Prove $AB = DE$.

c Prove $AB \parallel DE$.

d State the length of DE.

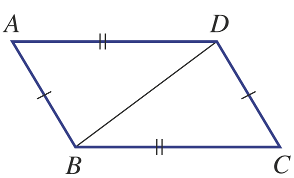


7 In this diagram, $AB = CD$ and $AD = CB$.

a Prove $\triangle ABD \equiv \triangle CDB$.

b Prove $\angle DBC = \angle BDA$.

c Prove $AD \parallel BC$.



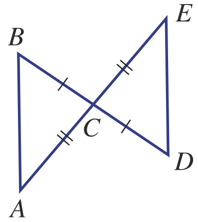
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

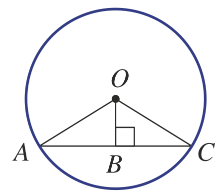
REASONING	8(1/2)	8(1/2)	8

8 Prove the following for the given diagrams. Give reasons.

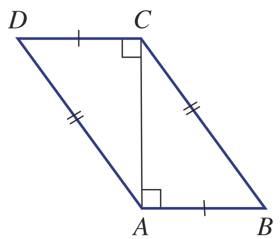
a $AB \parallel DE$



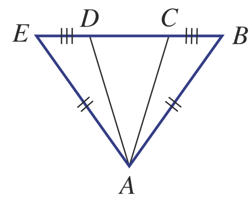
b OB bisects AC;
i.e. $AB = BC$



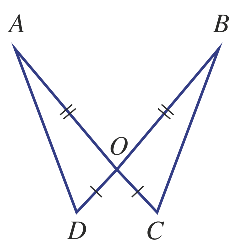
c $AD \parallel BC$



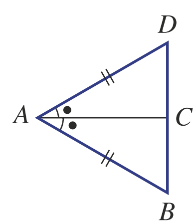
d $AD = AC$



e $\angle OAD = \angle OBC$



f $AC \perp BD$;
i.e. $\angle ACD = \angle ACB$



ENRICHMENT: Draw your own diagram	–	–	9

9 **a** A circle with centre O has a chord AB. M is the midpoint of the chord AB. Prove $OM \perp AB$.

b Two overlapping circles with centres O and C intersect at A and B. Prove $\angle AOC = \angle BOC$.

c $\triangle ABC$ is isosceles with $AC = BC$, D is a point on AC such that $\angle ABD = \angle CBD$, and E is a point on BC such that $\angle BAE = \angle CAE$. AE and BD intersect at F. Prove $AF = BF$.

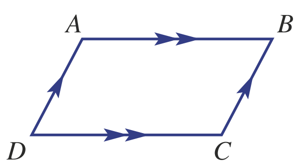
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2C Investigating parallelograms using congruence

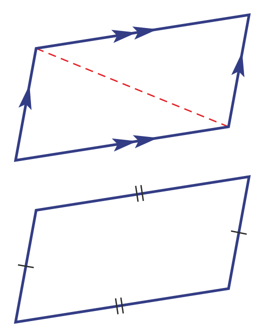
Learning intentions
- To review the properties of different types of parallelograms
- To know what is required to prove various properties of parallelograms
- To understand that showing two triangles are congruent can help prove that other angles and lengths are equal or that lines must be parallel or perpendicular
- To be able to use congruence to prove properties of quadrilaterals or test for types of quadrilaterals

Recall that parallelograms are quadrilaterals with two pairs of parallel sides. We therefore classify rectangles, squares and rhombuses as special types of parallelograms, the properties of which can be explored using congruence. By dividing a parallelogram into 'smaller' triangles, we can prove congruence for pairs of these triangles and use this to deduce the properties of the shape.



The parallelogram law of forces is widely applied in engineering, architecture and navigation. In a parallelogram *ABCD*, a boat aiming in direction *AB* against a tide or wind in direction *AD* results in the boat moving in the direction of the diagonal *AC*.

LESSON STARTER Aren't they the same proof?



Here are two statements involving the properties of a parallelogram.

1 A parallelogram (with parallel opposite sides) has opposite sides of equal length.
2 A quadrilateral with opposite sides of equal length is a parallelogram.
- Are the two statements saying the same thing?
- Discuss how congruence can be used to help prove each statement.
- Formulate a proof for each statement.

KEY IDEAS

- Some vocabulary and symbols:
  - If AB is parallel to BC, then we write $AB \parallel CD$.
  - If AB is perpendicular to CD, then we write $AB \perp CD$.
  - To bisect means to cut in half.

- **Parallelogram** properties and tests:
  - Parallelogram – a quadrilateral with opposite sides parallel.

Properties of a parallelogram	Tests for a parallelogram
• opposite sides are equal in length • opposite angles are equal • diagonals bisect each other	• if opposite sides are equal in length • if opposite angles are equal • if one pair of opposite sides are equal and parallel • if diagonals bisect each other

  - **Rhombus** – a parallelogram with all sides of equal length.

Properties of a rhombus	Tests for a rhombus
• all sides are equal length • opposite angles are equal • diagonals bisect each other at right angles • diagonals bisect the interior angles	• if all sides are equal length • if diagonals bisect each other at right angles

  - **Rectangle** – a parallelogram with all angles 90°.

Properties of a rectangle	Tests for a rectangle
• opposite sides are of equal length • all angles equal 90° • diagonals are equal in length and bisect each other	• if all angles are 90° • if diagonals are equal in length and bisect each other

  - **Square** – a parallelogram that is a rectangle and a rhombus.

Properties of a square	Tests for a square
• all sides are equal in length • all angles equal 90° • diagonals are equal in length and bisect each other at right angles • diagonals bisect the interior angles	• if all sides are equal in length and at least one angle is 90° • if diagonals are equal in length and bisect each other at right angles

BUILDING UNDERSTANDING

1. Name the special quadrilateral given by these descriptions.
   a. a parallelogram with all angles 90°
   b. a quadrilateral with opposite sides parallel
   c. a parallelogram that is a rhombus and a rectangle
   d. a parallelogram with all sides of equal length
2. Name all the special quadrilaterals that have these properties.
   a. All angles 90°.
   b. Diagonals are equal in length.
   c. Diagonals bisect each other.
   d. Diagonals bisect each other at 90°.
   e. Diagonals bisect the interior angles.
3. Give a reason why:
   a. A trapezium is not a parallelogram.
   b. A kite is not a parallelogram.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

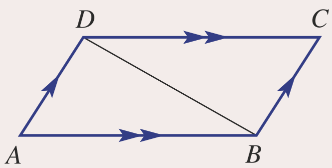
Example 5 Proving properties of quadrilaterals

a Prove that a parallelogram (with opposite sides parallel) has equal opposite angles.

b Use the property that opposite sides of a parallelogram are equal to prove that a rectangle (with all angles 90°) has diagonals of equal length.

SOLUTION

a



$\angle ABD = \angle CDB$ (alternate angles in $\parallel$ lines) A
$\angle ADB = \angle CBD$ (alternate angles in $\parallel$ lines) A
BD is common S
$\therefore \triangle ABD \equiv \triangle CDB$ (AAS)
$\therefore \angle DAB = \angle BCD$ (corresponding angles in congruent triangles)
Also since $\angle ADB = \angle CBD$ and $\angle ABD = \angle CDB$
then $\angle ADC = \angle CBA$
So opposite angles are equal.

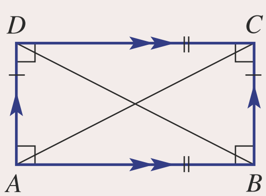
EXPLANATION

Draw a parallelogram with parallel sides and the segment BD.

Prove congruence of $\triangle ABD$ and $\triangle CDB$, noting alternate angles in parallel lines. Note also that BD is common to both triangles.
Corresponding angles in congruent triangles.

SOLUTION

b



Consider $\triangle ABC$ and $\triangle BAD$.
AB is common S
$\angle ABC = \angle BAD = 90°$ A
$BC = AD$ (opposite sides of a parallelogram are equal in length) S
$\therefore \triangle ABC \equiv \triangle BAD$ (SAS)
$\therefore AC = BD$, so diagonals are of equal length.

EXPLANATION

First, draw a rectangle with the given properties.

Choose $\triangle ABC$ and $\triangle BAD$, which each contain one diagonal.

Prove congruent triangles using SAS.

Corresponding sides in congruent triangles.

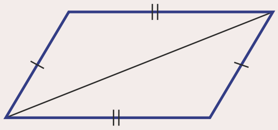
Now you try

Prove that a kite (with two pairs of equal adjacent sides) has one pair of equal angles.

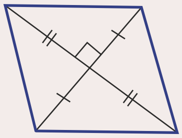
ISBN 978-1-108-77290-7

Example 6 Testing for a type of quadrilateral

a Prove that if opposite sides of a quadrilateral are equal in length, then it is a parallelogram.

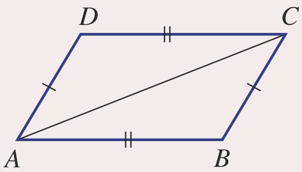


b Prove that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.



SOLUTION / EXPLANATION

a



First, label your quadrilateral and choose two triangles, $\triangle ABC$ and $\triangle CDA$.

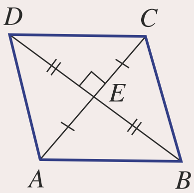
$AB = CD$ (given) S
$BC = DA$ (given) S
AC is common S
$\therefore \triangle ABC \equiv \triangle CDA$ (SSS)

Prove that they are congruent using SSS.

$\therefore \angle BAC = \angle DCA$
So $AB \parallel DC$ (since alternate angles are equal).
Also $\angle ACB = \angle CAD$.
$\therefore AD \parallel BC$ (since alternate angles are equal).
$\therefore ABCD$ is a parallelogram.

Choose corresponding angles in the congruent triangles to show that opposite sides are parallel. If alternate angles between lines are equal then the lines must be parallel.

b



All angles at the point E are 90°, so it is easy to prove that all four smaller triangles are congruent using SAS.

$\triangle ABE \equiv \triangle CBE \equiv \triangle ADE \equiv \triangle CDE$ by SAS

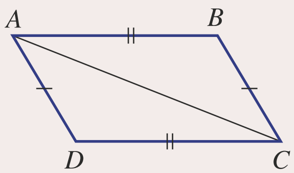
Corresponding sides in congruent triangles.

$\therefore AB = CB = CD = DA$
$\therefore ABCD$ is a rhombus.

Every quadrilateral with four equal sides is a rhombus.

Now you try

Prove, using this diagram, that if opposite sides of a quadrilateral are equal in length, then it is a parallelogram.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 2C

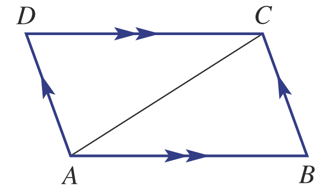
FLUENCY	1, 2	1–3	1–3

Example 5

1 Complete these steps to prove that a parallelogram (with opposite parallel sides) has equal opposite sides.

   a Prove $\triangle ABC \equiv \triangle CDA$.

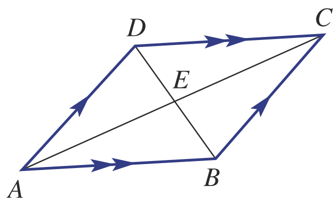
   b Hence, prove opposite sides are equal.



2 Complete these steps to prove that a parallelogram (with opposite equal parallel sides) has diagonals that bisect each other.

   a Prove $\triangle ABE \equiv \triangle CDE$.

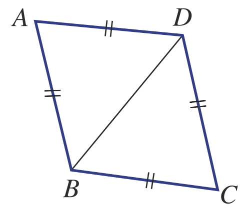
   b Hence, prove $AE = CE$ and $BE = DE$.



3 Complete these steps to prove that a rhombus (with sides of equal length) has diagonals that bisect the interior angles.

   a Prove $\triangle ABD \equiv \triangle CDB$.

   b Hence, prove BD bisects both $\angle ABC$ and $\angle CDA$.



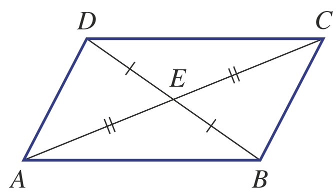
PROBLEM-SOLVING	4	4, 5	5, 6

Example 6

4 Complete these steps to prove that if the diagonals in a quadrilateral bisect each other, then it is a parallelogram.

   a Prove $\triangle ABE \equiv \triangle CDE$.

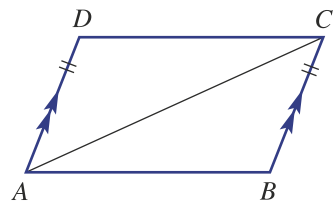
   b Hence, prove $AB \parallel DC$ and $AD \parallel BC$.



5 Complete these steps to prove that if one pair of opposite sides is equal and parallel in a quadrilateral, then it is a parallelogram.

   a Prove $\triangle ABC \equiv \triangle CDA$.

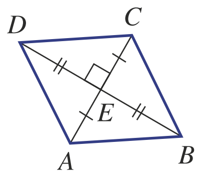
   b Hence, prove $AB \parallel DC$.



6 Complete these steps to prove that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.

   a Give a brief reason why $\triangle ABE \equiv \triangle CBE \equiv \triangle ADE \equiv \triangle CDE$.

   b Hence, prove $ABCD$ is a rhombus.



ISBN 978-1-108-77290-7

REASONING	7	7, 8	8, 9

7 Prove that the diagonals of a rhombus (i.e. a parallelogram with sides of equal length):

a intersect at right angles

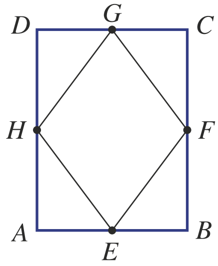
b bisect the interior angles.

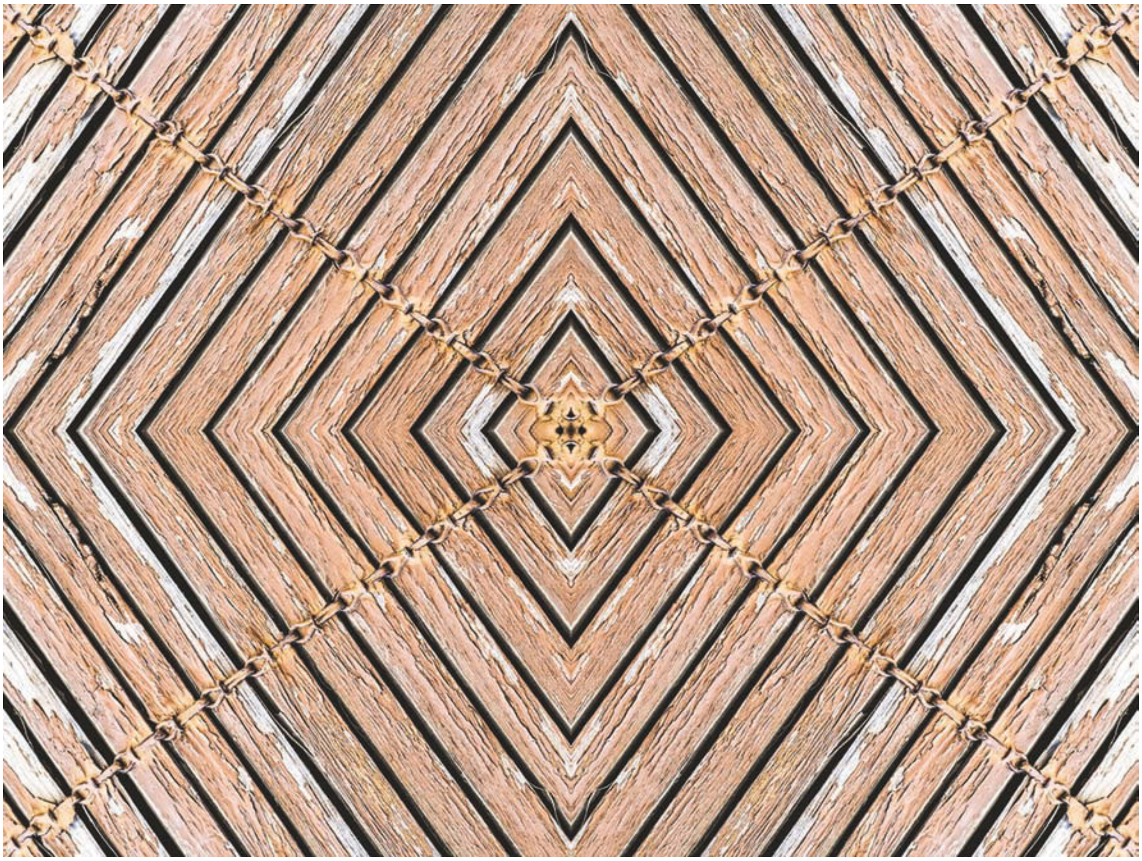
8 Prove that a parallelogram with one right angle is a rectangle.

9 Prove that if the diagonals of a quadrilateral bisect each other and are of equal length, then it is a rectangle.

ENRICHMENT: Rhombus in a rectangle	–	–	10

10 In this diagram, E, F, G and H are the midpoints of AB, BC, CD and DA, respectively, and $ABCD$ is a rectangle.
Prove that $EFGH$ is a rhombus.





ISBN 978-1-108-77290-7

2D Similar figures CONSOLIDATING

Learning intentions

- To know that similar figures have the same shape but a different size
- To understand that for figures to be similar their corresponding angles must be equal and their corresponding sides in the same ratio
- To know how to find the scale factor or ratio between two similar figures
- To be able to use the scale factor to determine side lengths on similar figures

You will recall that the transformation called enlargement involves reducing or enlarging the size of an object. The final image will be of the same shape but of different size. This means that matching pairs of angles will be equal and matching sides will be in the same ratio, just as in an accurate scale model.

Architects, engineers and governments use scale models to help find design errors and improvements. Businesses that specialise in making 3D scale models use physical construction, 3D printing and computer simulations.

LESSON STARTER The Q1 tower

The Q1 tower, pictured here, is located on the Gold Coast and was the world's tallest residential tower up until 2011. It is 245 m tall.

- Measure the height and width of the Q1 tower in this photograph.
- Can a scale factor for the photograph and the actual Q1 tower be calculated? How?
- How can you calculate the actual width of the Q1 tower using this photograph? Discuss.

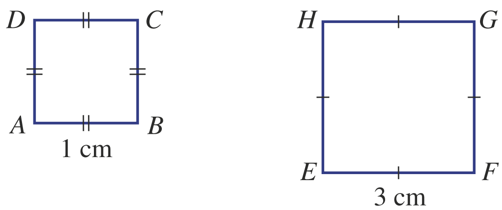
KEY IDEAS

- **Similar figures** have the same shape but are of different size.
  - Corresponding angles are equal.
  - Corresponding sides are in the same proportion (or ratio).
- **Scale factor** $= \dfrac{\text{image length}}{\text{original length}}$
- The symbols $|||$ or ~ are used to describe similarity and to write similarity statements. For example, $\triangle ABC\ |||\ \triangle DEF$ or $\triangle ABC \sim \triangle DEF$.

ISBN 978-1-108-77290-7

BUILDING UNDERSTANDING

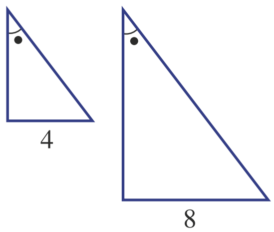
1 These two figures are squares.



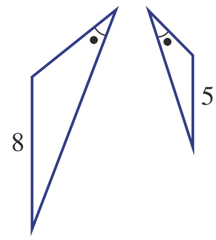
a Would you say that the two squares are similar? Why?
b What is the scale factor when the smaller square is enlarged to the size of the larger square?
c If the larger square is enlarged by a factor of 5, what would be the side length of the image?

2 Find the scale factor for each shape and its larger similar image.

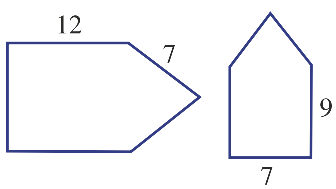
a



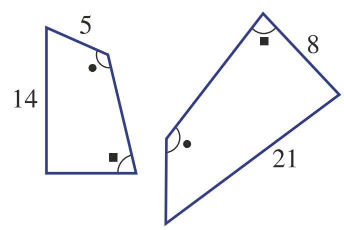
b



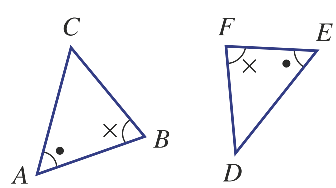
c



d

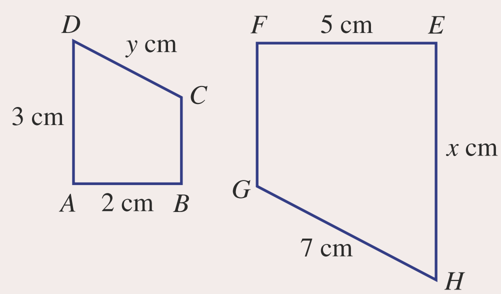


3 The two triangles shown opposite are similar.
a In $\triangle ABC$, which vertex corresponds to (matches) vertex E?
b In $\triangle ABC$, which angle corresponds to $\angle D$?
c In $\triangle DEF$, which side corresponds to BC?
d Give the similarity statement for the two triangles with matching vertices in the same order.



Example 7 Finding and using scale factors

These two shapes are similar.
a Write a similarity statement for the two shapes.
b Complete the following: $\frac{EH}{----} = \frac{FG}{----}$.
c Find the scale factor.
d Find the value of x.
e Find the value of y.



Continued on next page

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

SOLUTION

EXPLANATION

a A function since each x-value has only one y-value.

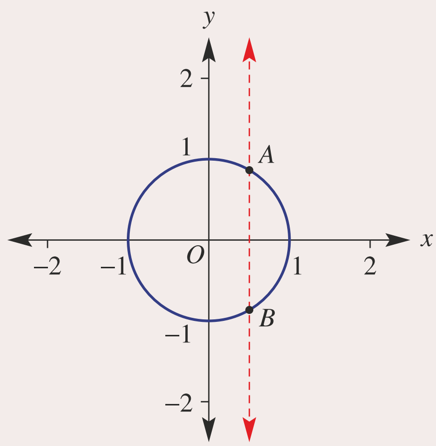
Each of the x-values; i.e. $x = 1, 2, 4$, and 5, occurs only once. So the coordinates represent a function.

b $y = x^2 + 3$ is a function since each x-value will produce only one y-value.

As the rule represents a parabola, each x-value will produce only one y-value and so it is a function.

c Not a function because a vertical line shows that one x-value links to more than one y-value.

A vertical line drawn anywhere through the graph will cross in more than one place, therefore it is not a function.



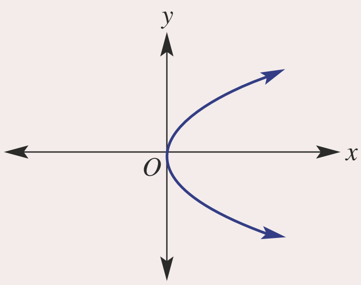
Now you try

From the following, identify which are functions.

a $\{(-1, 4), (0, 2), (1, 0), (2, -2)\}$

b $y = -x^2 + 1$

c



Example 20 Determining domain and range

Write down the allowable x-values (domain) and the resulting y-values (range) for each of these functions.

a $y = 4x - 1$

b $y = x^2 - 4$

SOLUTION

EXPLANATION

a Domain is the set of all real x-values.
Range is the set of all real y-values.

The function is a straight line. The input (i.e. x-values) can be any number and will produce any number as an output value.

Continued on next page

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

3 These two shapes are similar.

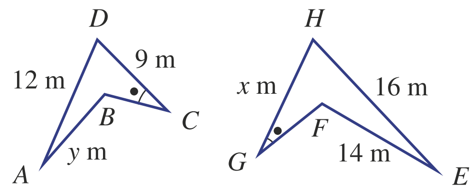
a Write a similarity statement for the two shapes.

b Complete the following: $\frac{EF}{----} = \frac{----}{CD}$.

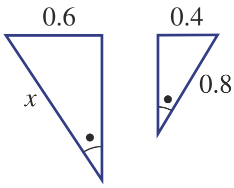
c Find the scale factor.

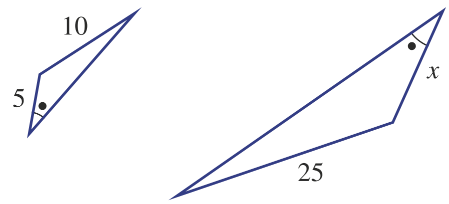
d Find the value of x.

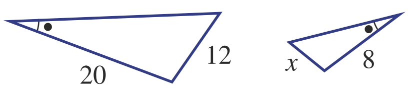
e Find the value of y.

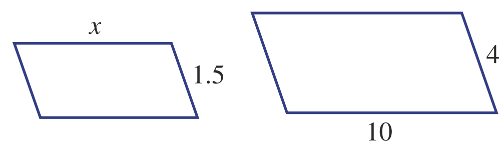


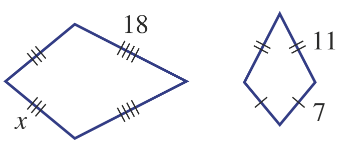
4 Find the value of the pronumeral in each pair of similar figures. Round to one decimal place where necessary.

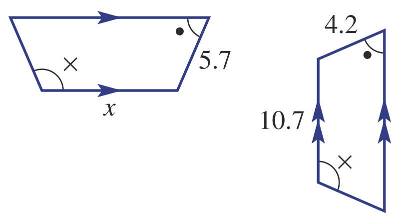
a


b


c


d


e


f


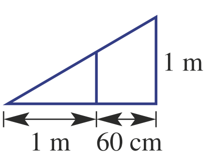
PROBLEM-SOLVING	5, 6	5, 6	6, 7

5 A 50 m tall structure casts a shadow 30 m in length. At the same time, a person casts a shadow of 1.02 m. Estimate the height of the person. (*Hint*: Draw a diagram of two triangles.)

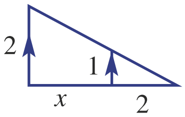
6 A BMX ramp has two vertical supports, as shown.

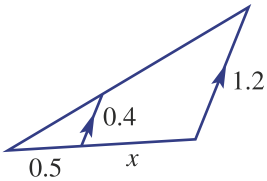
a Find the scale factor for the two triangles in the diagram.

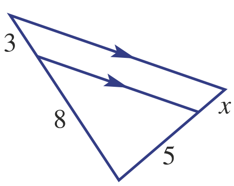
b Find the length of the inner support.

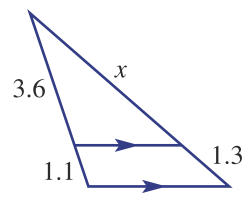


7 Find the value of the pronumeral if the pairs of triangles are similar. Round to one decimal place in part d.

a


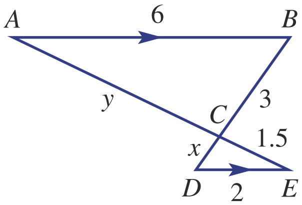
b


c


d


REASONING	8	8, 9	9, 10

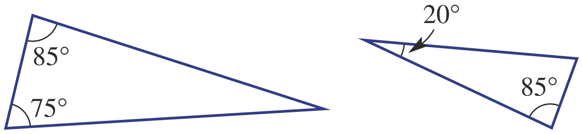
8 In this diagram, the two triangles are similar and $AB \parallel DE$.



a Which side in $\triangle ABC$ corresponds to DC? Give a reason.
b Write a similarity statement by matching the vertices.
c Find the value of x.
d Find the value of y.

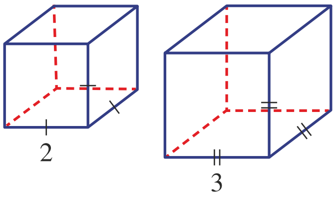
9 Decide whether each statement is true or false.
a All circles are similar.
b All squares are similar.
c All rectangles are similar.
d All rhombuses are similar.
e All parallelograms are similar.
f All trapeziums are similar.
g All kites are similar.
h All isosceles triangles are similar.
i All equilateral triangles are similar.
j All regular hexagons are similar.

10 These two triangles each have two given angles. Decide whether they are similar and give reasons.



ENRICHMENT: Length, area, volume and self-similarity	–	–	11, 12

11 Shown here is a cube of side length 2 and its image after enlargement.



a Write down the scale factor for the side lengths as an improper fraction.
b Find the area of one face of:
i the smaller cube **ii** the larger cube.
c Find the volume of:
i the smaller cube **ii** the larger cube.
d Complete this table.

Cube	Length	Area	Volume
Small	2		
Large	3		
Scale factor (fraction)			

e How do the scale factors for Area and Volume relate to the scale factor for side length?
f If the side length scale factor was $\frac{b}{a}$, write down expressions for:
i the area scale factor **ii** the volume scale factor.

12 An object that is similar to a part of itself is said to be self-similar. These are objects that, when magnified, reveal the same structural shape. A fern is a good example, as shown in this picture. Self-similarity is an important area of mathematics, having applications in geography, econometrics and biology.

Use the internet to explore the topic of self-similarity and write a few dot points to summarise your findings.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2E Proving and applying similar triangles

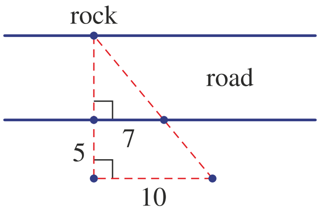
Learning intentions
- To know the four tests for similarity of triangles
- To understand that two corresponding pairs of sides must be known to identify a common ratio
- To know how to prove that two triangles are similar using the tests
- To be able to use similarity of triangles to find unknown values

As with congruent triangles, there are four tests for proving that two triangles are similar. When solving problems involving similar triangles, it is important to recognise and be able to prove that they are in fact similar.

Optical engineers and optometrists use similar triangle geometry to model the path of light rays through lenses and to calculate the size of virtual images. The designs of spectacles, cameras, microscopes, telescopes and projectors all use similar triangle analysis.

LESSON STARTER How far did the chicken travel?

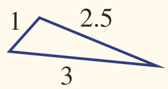
A chicken is considering how far it is across a road, so it places four pebbles in certain positions on one side of the road. Two of the pebbles are directly opposite a rock on the other side of the road. The number of chicken paces between three pairs of the pebbles is shown in the diagram.

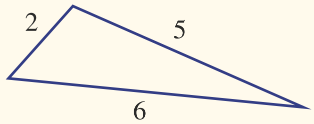


- Has the chicken constructed any similar triangles? If so, discuss why they are similar.
- What scale factor is associated with the two triangles?
- Is it possible to find how many paces the chicken must take to get across the road? If so, show a solution.
- Why did the chicken cross the road?
  Answer: To explore similar triangles.

KEY IDEAS

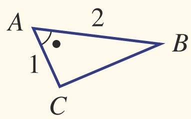
- Two objects are said to be **similar** if they are of the same shape but of different size.
  - For two similar triangles $\triangle ABC$ and $\triangle DEF$, we write $\triangle ABC \mathrel{|||} \triangle DEF$ or $\triangle ABC \sim \triangle DEF$.
  - When comparing two triangles, try to match up corresponding sides and angles, then look to see which similarity test can be used.
- Two triangles can be tested for **similarity** by considering the following conditions.
  - All pairs of corresponding sides are in the same ratio (or proportion) (SSS).

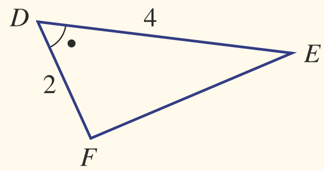




$$\frac{6}{3} = \frac{5}{2.5} = \frac{2}{1} = 2$$

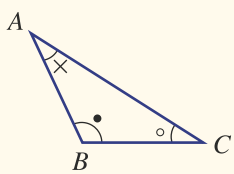
  - Two pairs of corresponding sides are in the same ratio and the included corresponding angles are equal (SAS).

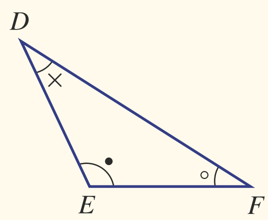




$$\frac{4}{2} = \frac{2}{1} = 2 \text{ and } \angle A = \angle D$$

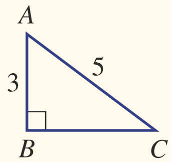
  - Three corresponding angles are equal (AAA). (Remember that two pairs of corresponding equal angles implies that all three pairs of corresponding angles are equal.)

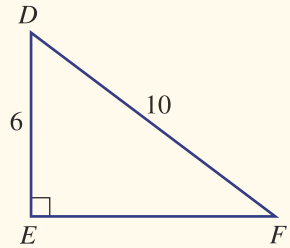




$$\angle A = \angle D$$
$$\angle B = \angle E$$
$$\angle C = \angle F$$

  - The hypotenuses of two right-angled triangles and another pair of corresponding sides are in the same ratio (RHS).





$$\angle B = \angle E = 90^\circ$$
$$\frac{10}{5} = \frac{6}{3} = 2$$

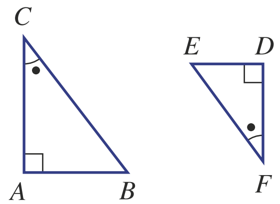
Note: If the test AAA is not used, then at least two pairs of corresponding sides in the same ratio are required for all the other three tests.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

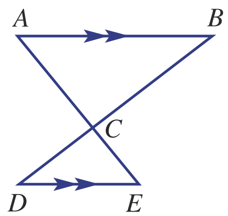
BUILDING UNDERSTANDING

1 This diagram includes two similar triangles.



a Which vertex in $\triangle DEF$ corresponds to vertex B?
b Which angle in $\triangle ABC$ corresponds to $\angle F$?
c Which side in $\triangle ABC$ corresponds to DE?
d Write a similarity statement for the two triangles.

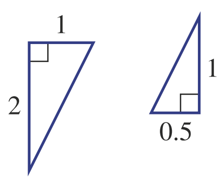
2 This diagram includes two similar triangles.



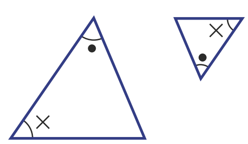
a Which angle in $\triangle CDE$ corresponds to $\angle B$ in $\triangle ABC$ and why?
b Which angle in $\triangle ABC$ corresponds to $\angle E$ in $\triangle CDE$ and why?
c Which angle is vertically opposite $\angle ACB$?
d Which side on $\triangle ABC$ corresponds to side CE on $\triangle CDE$?
e Write a similarity statement, being sure to write matching vertices in the same order.

3 Which similarity test (SSS, SAS, AAA or RHS) would be used to prove that these pairs of triangles are similar?

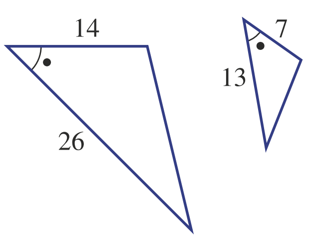
a



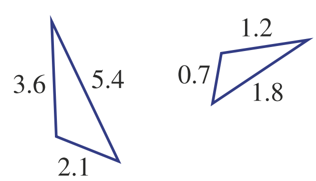
b



c



d



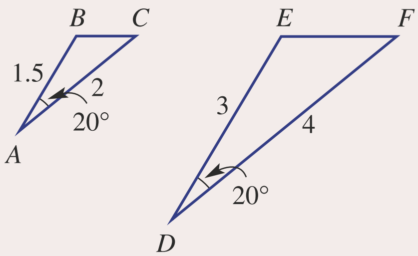
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

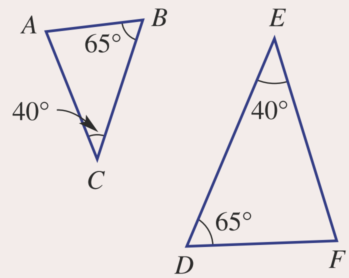
Example 8 Using similarity tests to prove similar triangles

Prove that the following pairs of triangles are similar.

a



b



SOLUTION

a $\dfrac{DF}{AC} = \dfrac{4}{2} = 2$ (ratio of corresponding sides) S

$\dfrac{DE}{AB} = \dfrac{3}{1.5} = 2$ (ratio of corresponding sides) S

$\angle BAC = \angle EDF = 20^\circ$ (given corresponding angles) A

$\therefore \triangle ABC \,|||\, \triangle DEF$ (SAS)

b $\angle ABC = \angle FDE = 65^\circ$ (given corresponding angles) A

$\angle ACB = \angle FED = 40^\circ$ (given corresponding angles) A

$\therefore \triangle ABC \,|||\, \triangle FDE$ (AAA)

EXPLANATION

DF and *AC* are corresponding sides and *DE* and *AB* are corresponding sides, and both pairs are in the same ratio.

The corresponding angle between the pair of corresponding sides in the same ratio is also equal.

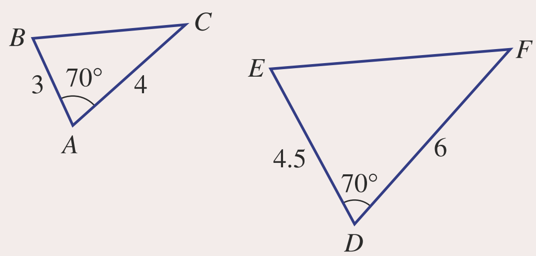
The two triangles are therefore similar.

There are two pairs of given corresponding angles. If two pairs of corresponding angles are equal, then the third pair must also be equal (due to angle sum).
The two triangles are therefore similar.

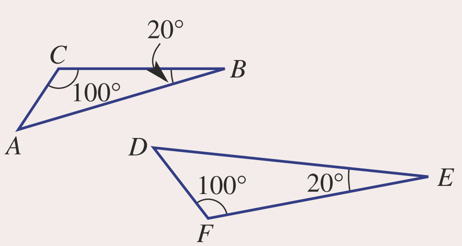
Now you try

Prove that the following pairs of triangles are similar.

a



b



ISBN 978-1-108-77290-7

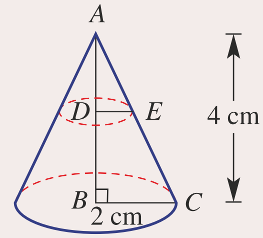
Cambridge University Press
Updated June 2022

Example 9 Establishing and using similarity

A cone has radius 2 cm and height 4 cm. The top of the cone is cut horizontally through D.

a Prove $\triangle ADE \mathbin{|||} \triangle ABC$.

b If $AD = 1$ cm, find the radius DE.



SOLUTION

a $\angle BAC$ is common A

$\angle ABC = \angle ADE$ (corresponding angles in parallel lines) A

$\therefore \triangle ADE \mathbin{|||} \triangle ABC$ (AAA)

EXPLANATION

All three pairs of corresponding angles are equal.

Therefore, the two triangles are similar.

SOLUTION

b $\dfrac{DE}{BC} = \dfrac{AD}{AB}$

$\dfrac{DE}{2} = \dfrac{1}{4}$

$\therefore DE = \dfrac{2}{4}$

$= 0.5$ cm

EXPLANATION

Given the triangles are similar, the ratio of corresponding sides must be equal.

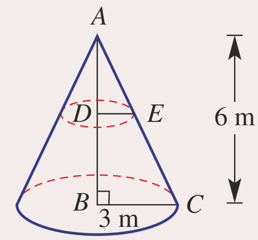
Solve for DE

Now you try

A cone has radius 3 m and height 6 m. The top of the cone is cut horizontally through D.

a Prove $\triangle ADE \mathbin{|||} \triangle ABC$.

b If $AD = 2$ m, find the radius DE.



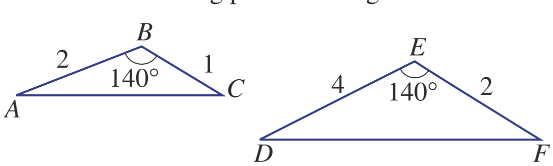
Exercise 2E

FLUENCY	1, 2, 3(1/2)	2–4(1/2)	2–4(1/2)

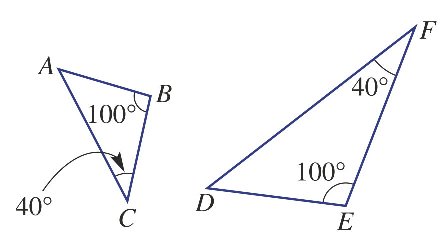
Example 8

1 Prove that the following pairs of triangles are similar.

a



b



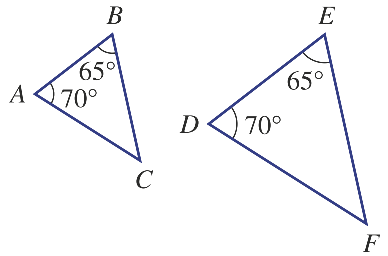
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

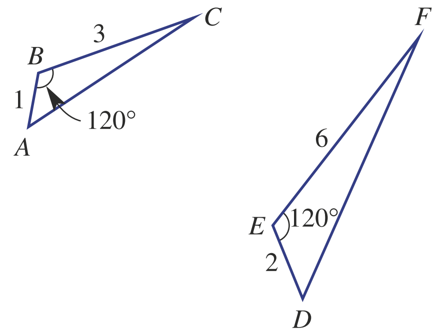
Example 8

2 Prove that the following pairs of triangles are similar.

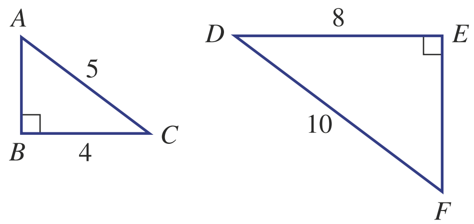
a



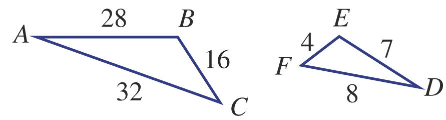
b



c

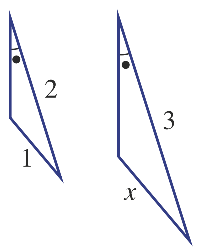


d

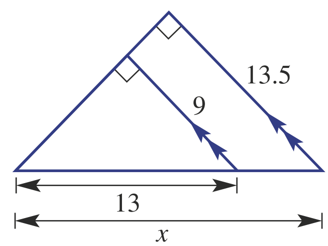


3 Find the value of the pronumerals in these pairs of similar triangles.

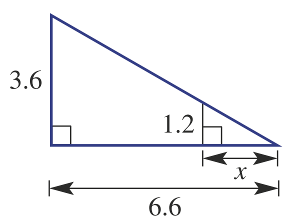
a



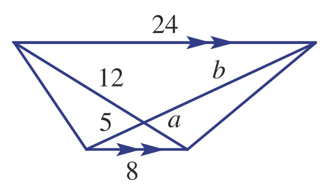
b



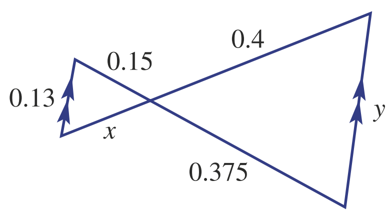
c



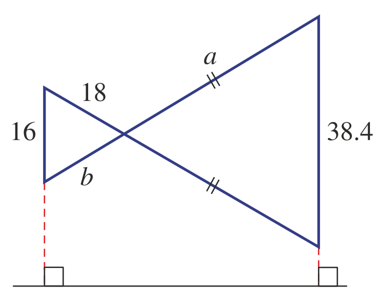
d



e



f

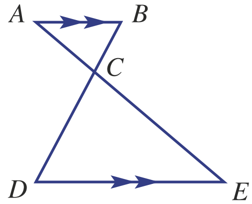


ISBN 978-1-108-77290-7

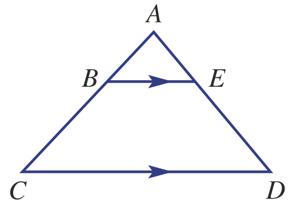
Cambridge University Press
Updated June 2022

4 For the following proofs, give reasons at each step.

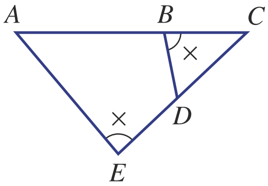
a Prove $\triangle ABC \mathrel{|||} \triangle EDC$.



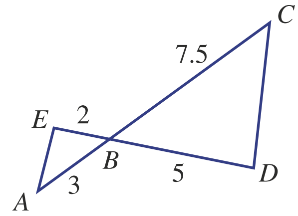
b Prove $\triangle ABE \mathrel{|||} \triangle ACD$.



c Prove $\triangle BCD \mathrel{|||} \triangle ECA$.



d Prove $\triangle AEB \mathrel{|||} \triangle CDB$.



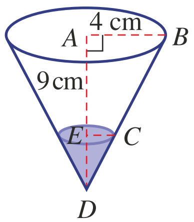
PROBLEM-SOLVING	5, 6	5, 7	7, 8

Example 9

5 A right cone with radius 4 cm has a total height of 9 cm. It contains an amount of water, as shown.

a Prove $\triangle EDC \mathrel{|||} \triangle ADB$.

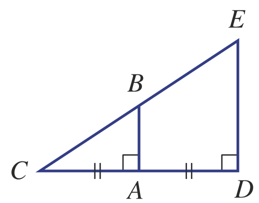
b If the depth of water in the cone is 3 cm, find the radius of the water surface in the cone.



6 A ramp is supported by a vertical stud AB, where A is the midpoint of CD. It is known that $CD = 4$ m and that the ramp is 2.5 m high; i.e. $DE = 2.5$ m.

a Prove $\triangle BAC \mathrel{|||} \triangle EDC$.

b Find the length of the stud AB.

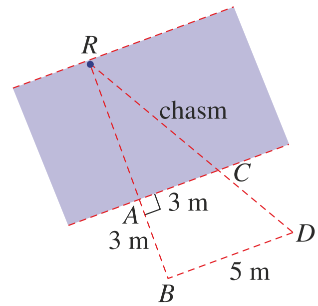


7 At a particular time in the day, Felix casts a shadow 1.3 m long and Curtis, who is 1.75 m tall, casts a shadow 1.2 m long. Find Felix's height, correct to two decimal places.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

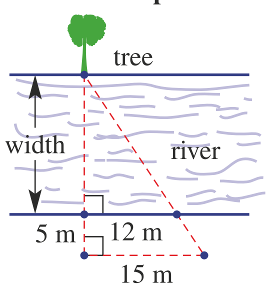
8 To determine the width of a chasm, a marker (A) is placed directly opposite a rock (R) on the other side. Point B is placed 3 m away from point A, as shown. Marker C is placed 3 m along the edge of the chasm, and marker D is placed so that BD is parallel to AC. Markers C and D and the rock are collinear (i.e. lie in a straight line). If BD measures 5 m, find the width of the chasm (AR).



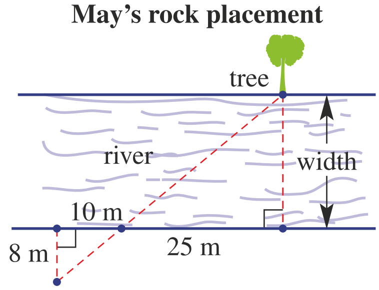
REASONING	9	9, 10	9–11

9 Aiden and May come to a river and notice a tree on the opposite bank. Separately they decide to place rocks (indicated with dots) on their side of the river to try to calculate the river's width. They then measure the distances between some pairs of rocks, as shown.

Aiden's rock placement

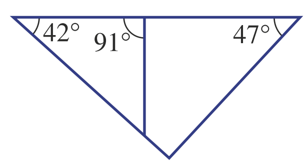


May's rock placement



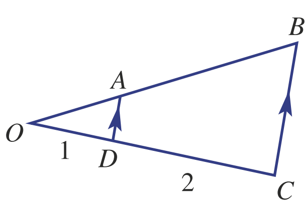
a Have both Aiden and May constructed a pair of similar triangles? Give reasons.
b Use May's triangles to calculate the width of the river.
c Use Aiden's triangles to calculate the width of the river.
d Which pair of triangles did you prefer to use? Give reasons.

10 There are two triangles in this diagram, each showing two given angles. Explain why they are similar.

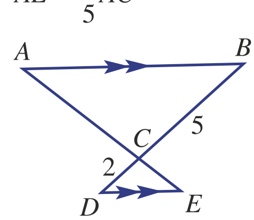


11 Prove the following, giving reasons.

a $OB = 3OA$



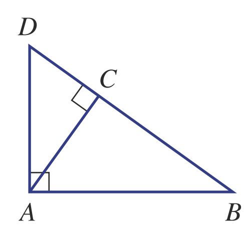
b $AE = \frac{7}{5}AC$



ENRICHMENT: Proving Pythagoras' theorem	–	–	12

12 In this figure $\triangle ABD$, $\triangle CBA$ and $\triangle CAD$ are right angled.

a Prove $\triangle ABD \;|||\; \triangle CBA$. Hence, prove $AB^2 = CB \times BD$.
b Prove $\triangle ABD \;|||\; \triangle CAD$. Hence, prove $AD^2 = CD \times BD$.
c Hence, prove Pythagoras' theorem $AB^2 + AD^2 = BD^2$.



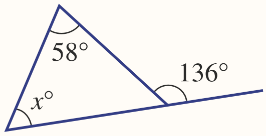
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

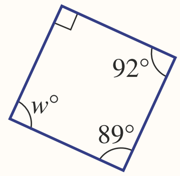
2A

1 Find the size of each pronumeral in the following polygons, giving reasons.

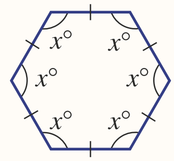
a



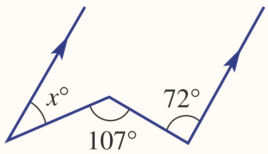
b



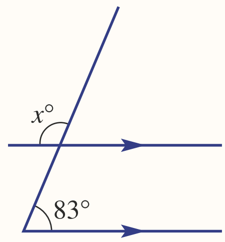
c



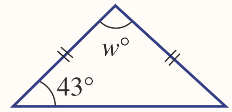
d



e

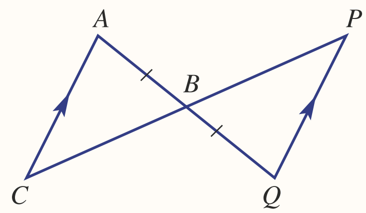


f



2B

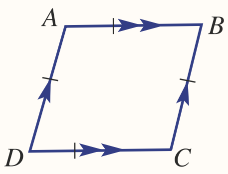
2 **a** Prove that $\triangle ABC$ is congruent to $\triangle QBP$.



b Prove that B is the midpoint of CP.

2C

3 Prove that a rhombus has its diagonals perpendicular to each other.



2C

4 Prove that if the opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram and the opposite sides are equal in length.

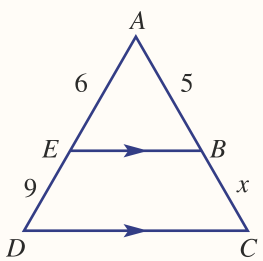
2D

5 For the two similar triangles shown:

a Write a similarity statement.

b Find the scale factor.

c Find the value of x.



ISBN 978-1-108-77290-7

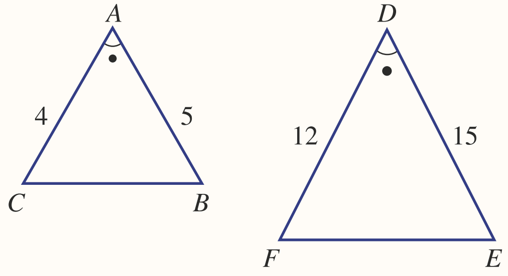
Cambridge University Press
Updated June 2022

Progress quiz

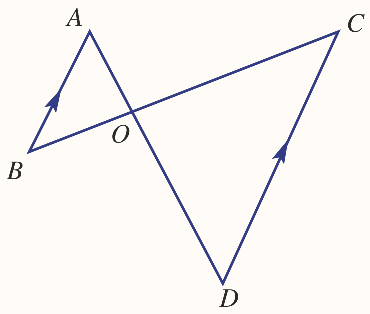
2E

6 Prove that the following pairs of triangles are similar.

a



b

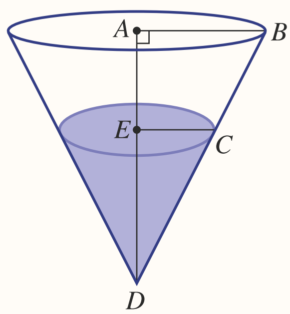


2E

7 A cone with radius 6 cm and height 10 cm is filled with water to a height of 5 cm.

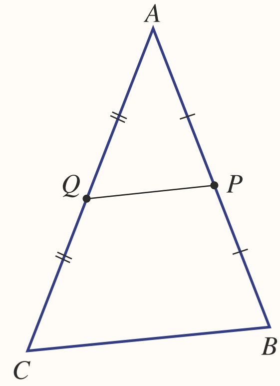
a Prove that $\triangle EDC$ is similar to $\triangle ADB$.

b Find the radius of the water's surface (EC).



2E

8 Prove that if the midpoints, Q and P, of two sides of a triangle ABC are joined as shown, then QP is $\frac{1}{2}$ that of CB. (First prove similarity.)



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2F Circles and chord properties 10A

Learning intentions
- To know the meaning of the terms circle, chord, sector, arc and segment
- To understand what is meant by an angle that is subtended by an arc or chord
- To know the chord theorems and how to apply them to find certain lengths and angles
- To be able to prove the chord theorems using congruent triangles

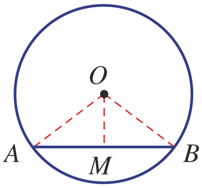
Although a circle appears to be a very simple object, it has many interesting geometrical properties. In this section we look at radii and chords in circles, and then explore and apply the properties of these objects. We use congruence to prove many of these properties.

Movie makers use circle geometry to help create the illusion of time passing slowly or of frozen motion, as in *The Matrix* (1999). Multiple cameras in a circle or arc sequentially or simultaneously photograph the actor and the images are stitched together.

LESSON STARTER Dynamic chords

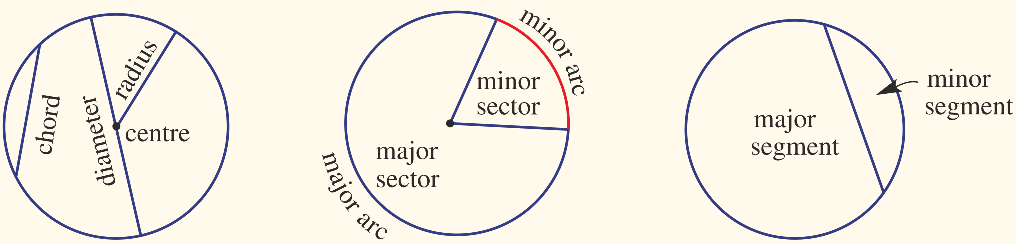
This activity would be enhanced with the use of interactive geometry.
Chord AB sits on a circle with centre O. M is the midpoint of chord AB.
Explore with interactive geometry software or discuss the following.

- Is $\triangle OAB$ isosceles and if so why?
- Is $\triangle OAM \equiv \triangle OBM$ and if so why?
- Is $AB \perp OM$ and if so why?
- Is $\angle AOM = \angle BOM$ and if so why?



KEY IDEAS

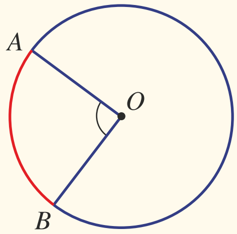
- Circle language



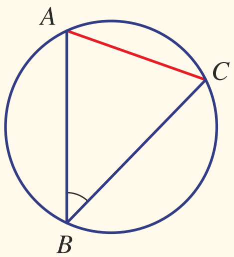
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

- An angle is **subtended** by an arc or chord if the arms of the angle meet the endpoints of the arc or chord.

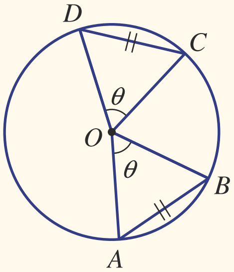


$\angle AOB$ is subtended at the centre by the minor arc AB.

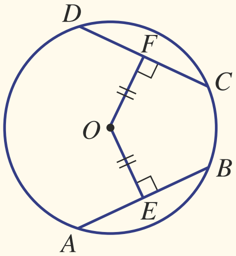


$\angle ABC$ is subtended at the circumference by the chord AC.

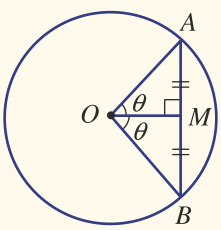
- **Chord theorem 1**: Chords of equal length subtend equal angles at the centre of the circle.
  - If $AB = CD$, then $\angle AOB = \angle COD$.
  - Conversely, if chords subtend equal angles at the centre of the circle, then the chords are of equal length.



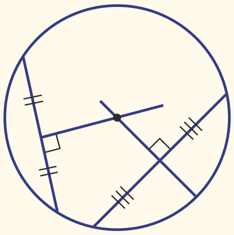
- **Chord theorem 2**: Chords of equal length are equidistant (i.e. of equal distance) from the centre of the circle.
  - If $AB = CD$, then $OE = OF$.
  - Conversely, if chords are equidistant from the centre of the circle, then the chords are of equal length.



- **Chord theorem 3**: The perpendicular from the centre of the circle to the chord bisects the chord and the angle at the centre subtended by the chord.
  - If $OM \perp AB$, then $AM = BM$ and $\angle AOM = \angle BOM$.
  - Conversely, if a radius bisects the chord (or angle at the centre subtended by the chord), then the radius is perpendicular to the chord.



- **Chord theorem 4**: The perpendicular bisectors of every chord of a circle intersect at the centre of the circle.
  - Constructing perpendicular bisectors of two chords will therefore locate the centre of a circle.



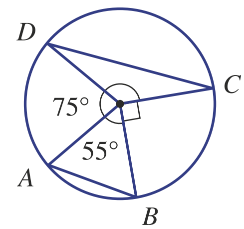
BUILDING UNDERSTANDING

1 Construct a large circle, then draw and label these features.

- **a** chord
- **b** radius
- **c** centre
- **d** minor sector
- **e** major sector

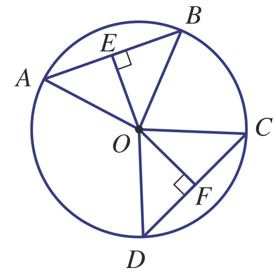
2 Give the size of the angle in this circle subtended by the following.

- **a** chord AB
- **b** minor arc BC
- **c** minor arc AD
- **d** chord DC



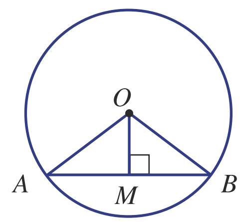
3 In this circle chords AB and CD are of equal length.

- **a** Measure $\angle AOB$ and $\angle COD$.
- **b** What do you notice about your answers from part **a**? Which chord theorem does this relate to?
- **c** Measure OE and OF.
- **d** What do you notice about your answers from part **c**? Which chord theorem does this relate to?



4 In this circle $OM \perp AB$.

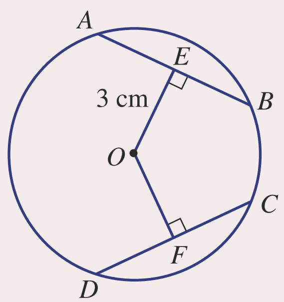
- **a** Measure AM and BM.
- **b** Measure $\angle AOM$ and $\angle BOM$.
- **c** What do you notice about your answers from parts **a** and **b**? Which chord theorem does this relate to?



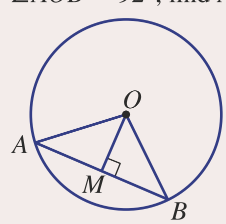
Example 10 Using chord theorems

For each part, use the given information and state which chord theorem is used.

a Given $AB = CD$ and $OE = 3$ cm, find OF.



b Given $OM \perp AB$, $AB = 10$ cm and $\angle AOB = 92°$, find AM and $\angle AOM$.



SOLUTION

a $OF = 3$ cm (using chord theorem 2)

EXPLANATION

Chords of equal length are equidistant from the centre.

Continued on next page

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

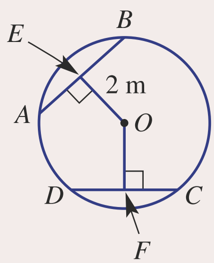
b Using chord theorem 3:
$AM = 5$ cm
$\angle AOM = 46°$

The perpendicular from the centre to the chord bisects the chord and the angle at the centre subtended by the chord.
$10 \div 2 = 5$ and $92 \div 2 = 46$.

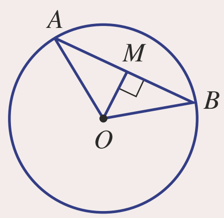
Now you try

For each part, use the given information and state which chord theorem is used.

a Given $AB = CD$ and $OE = 2$ m, find OF.



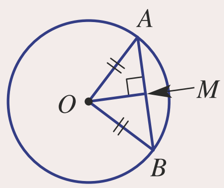
b Given $OM \perp AB$, $AB = 6$ m and $\angle AOB = 120°$, find AM and $\angle AOM$.



Example 11 Proving chord theorems

Prove chord theorem 3 in that the perpendicular from the centre of the circle to the chord bisects the chord and the angle at the centre subtended by the chord.

SOLUTION



$\angle OMA = \angle OMB = 90°$ (given) R
$OA = OB$ (both radii) H
OM is common S
$\therefore \triangle OMA \equiv \triangle OMB$ (RHS)
$\therefore AM = BM$ and $\angle AOM = \angle BOM$

EXPLANATION

First, draw a diagram to represent the situation. The perpendicular forms a pair of congruent triangles.

Corresponding sides and angles in congruent triangles are equal.

Now you try

Prove chord theorem 2 in that chords of equal length are equidistant (of equal distance) from the centre of the circle.

ISBN 978-1-108-77290-7

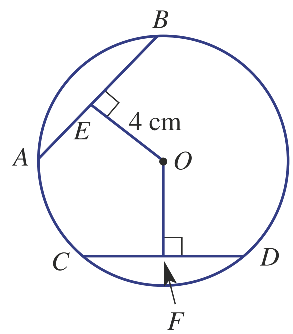
Exercise 2F

FLUENCY	1–3	2, 4(½)	2, 3, 4(½)

1 For each part, use the given information and state which chord theorem is used.

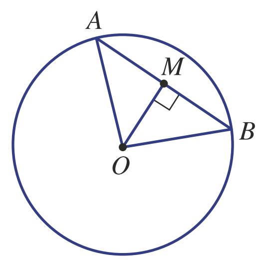
Example 10a

a Given $AB = CD$ and $OE = 4$ cm, find OF.



Example 10b

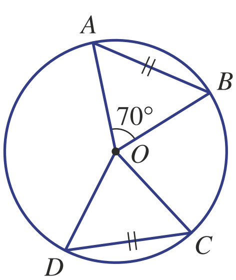
b Given $OM \perp AB$, $AB = 6$ m and $\angle AOB = 100°$, find AM and $\angle AOM$.



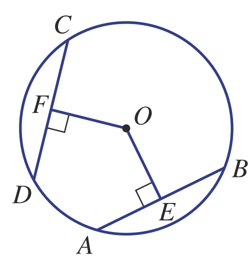
Example 10

2 For each part, use the information given and state which chord theorem is used.

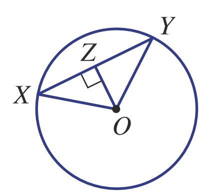
a Given $AB = CD$ and $\angle AOB = 70°$, find the value of $\angle DOC$.



b Given $AB = CD$ and $OF = 7.2$ cm, find the value of OE.



c Given $OZ \perp XY$, $XY = 8$ cm and $\angle XOY = 102°$, find the value of XZ and $\angle XOZ$.



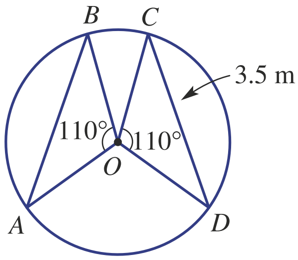
3 The perpendicular bisectors of two different chords of a circle are constructed. Describe where they intersect.

ISBN 978-1-108-77290-7

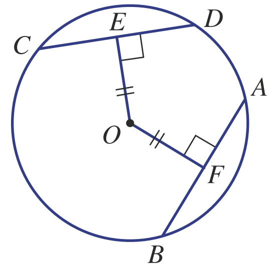
Cambridge University Press
Updated June 2022

4 Use the information given to complete the following.

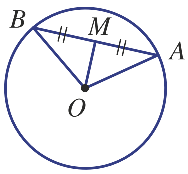
a Given $\angle AOB = \angle COD$ and $CD = 3.5$ m, find the value of AB.



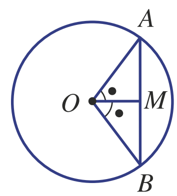
b Given $OE = OF$ and $AB = 9$ m, find the value of CD.



c Given M is the midpoint of AB, find the value of $\angle OMB$.



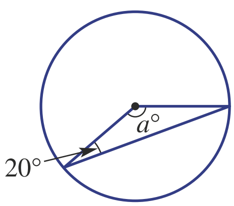
d Given $\angle AOM = \angle BOM$, find the value of $\angle OMB$.



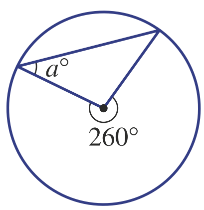
PROBLEM-SOLVING	5	5, 6	5(½), 6, 7

5 Find the size of each unknown angle $a°$.

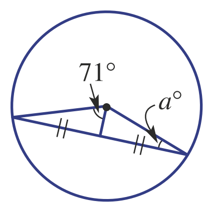
a



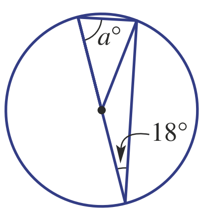
b



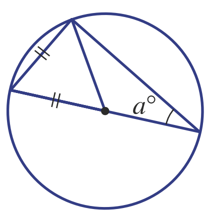
c



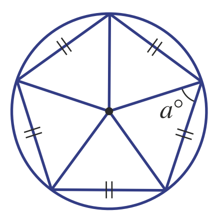
d



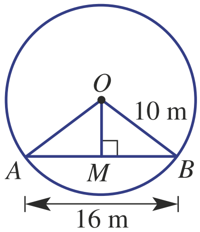
e



f



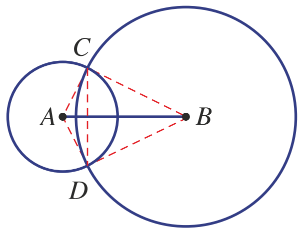
6 Find the length OM. (*Hint*: Use Pythagoras' theorem.)



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

7 In this diagram, radius $AD = 5$ mm, radius $BD = 12$ mm and chord $CD = 8$ mm. Find the exact length of AB, in surd form.



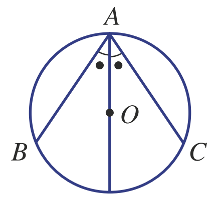
REASONING	8	8, 9	8–10

Example 11

8 a Prove chord theorem 1 in that chords of equal length subtend equal angles at the centre of the circle.
b Prove the converse of chord theorem 1 in that if chords subtend equal angles at the centre of the circle then the chords are of equal length.

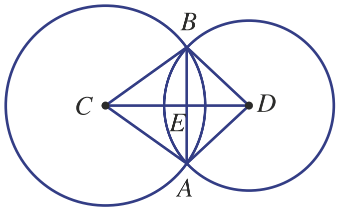
9 a Prove that if a radius bisects a chord of a circle then the radius is perpendicular to the chord.
b Prove that if a radius bisects the angle at the centre subtended by the chord, then the radius is perpendicular to the chord.

10 In this circle $\angle BAO = \angle CAO$. Prove $AB = AC$.
(*Hint*: Construct two triangles.)



ENRICHMENT: Common chord proof	–	–	11

11 For this diagram, prove $CD \perp AB$ by following these steps.
a Prove $\triangle ACD \equiv \triangle BCD$.
b Hence, prove $\triangle ACE \equiv \triangle BCE$.
c Hence, prove $CD \perp AB$.



ISBN 978-1-108-77290-7

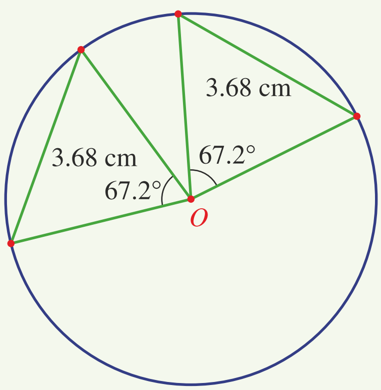
Cambridge University Press
Updated June 2022

Exploring chord theorems with interactive geometry

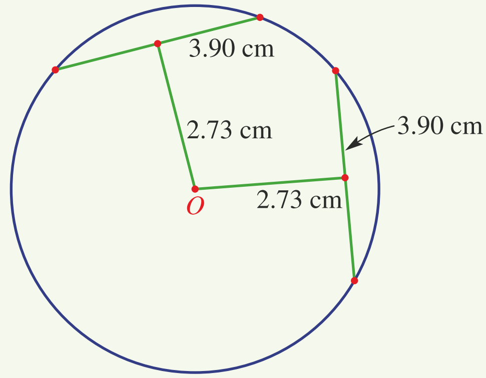
1. Construct a circle with centre O and any given radius.
2. Construct chords of equal length by rotating a chord about the centre by a given angle.
3. Illustrate the four chord properties by constructing line segments, as shown below.
   Measure corresponding angles and lengths to illustrate the chord properties.
   - Chord theorem 1 : Chords of equal length subtend equal angles at the centre of the circle.
   - Chord theorem 2 : Chords of equal length are equidistant from the centre of the circle.
   - Chord theorem 3 : The perpendicular from the centre of the circle to the chord bisects the chord and the angle at the centre subtended by the chord.
   - Chord theorem 4 : The perpendiculars of every chord of a circle intersect at the centre of the circle.
4. Drag the circle or one of the points on the circle to check that the properties are retained.

Chord theorem illustrations

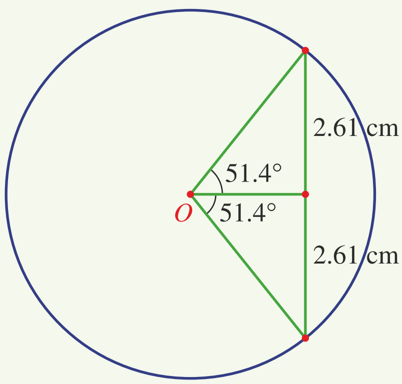
Chord theorem 1



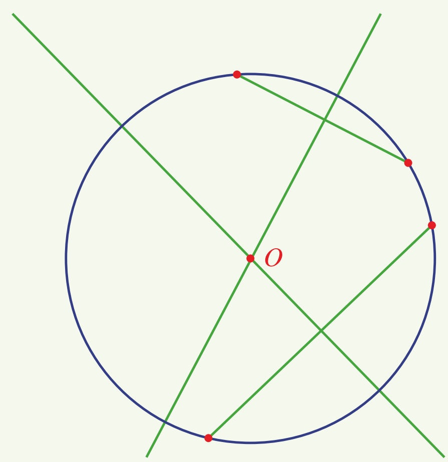
Chord theorem 2



Chord theorem 3



Chord theorem 4



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2G Angle properties of circles: Theorems 1 and 2 10A

Learning intentions

- To know the relationship between angles at the centre of a circle and at the circumference subtended by the same arc
- To know that a triangle in a semicircle creates a right angle at the circumference
- To be able to combine these theorems with other properties of circles

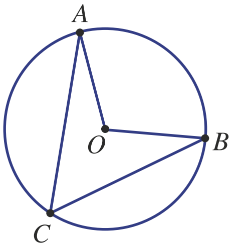
The special properties of circles extend to the pairs of angles formed by radii and chords intersecting at the circumference. In this section we explore the relationship between angles at the centre and at the circumference subtended by the same arc.

Road and railway tunnel design and construction are complex geological and technical processes. Civil engineers use geometry, including circle properties and theorems, to establish the geometrical requirements for structural stability.

LESSON STARTER Discover angle properties: Theorems 1 and 2

This activity can be completed with the use of a protractor and pair of compasses, but would be enhanced by using interactive geometry software.

- First, construct a circle and include two radii and two chords, as shown. The size of the circle and position of points A, B and C on the circumference can vary.
- Measure $\angle ACB$ and $\angle AOB$. What do you notice?
- Now construct a new circle with points A, B and C at different points on the circumference. (If dynamic software is used simply drag the points.) Measure $\angle ACB$ and $\angle AOB$ once again. What do you notice?
- Construct a new circle with $\angle AOB = 180°$ so AB is a diameter. What do you notice about $\angle ACB$?

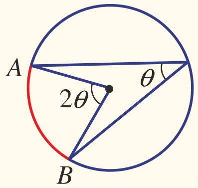


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

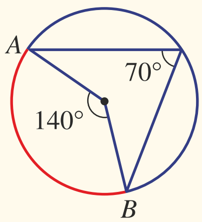
KEY IDEAS

- **Circle theorem 1:** Angles at the centre and circumference
  - The angle at the centre of a circle is twice the angle at a point on the circle subtended by the same arc.

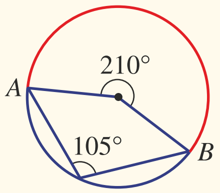


For example:

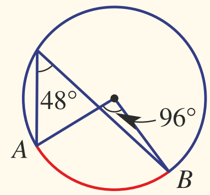
1



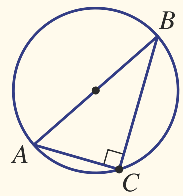
2



3



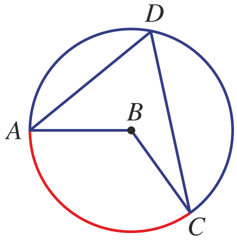
- **Circle theorem 2:** Angle in a semicircle
  - The angle in a semicircle is 90°.
  - This is a specific case of theorem 1, where $\angle ACB$ is known as the angle in a semicircle.



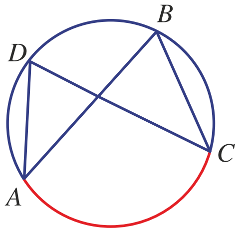
BUILDING UNDERSTANDING

1 Name another angle that is subtended by the same arc as $\angle ABC$ in these circles.

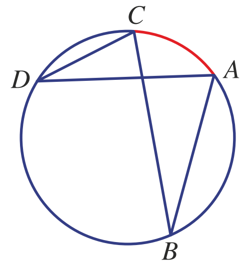
a



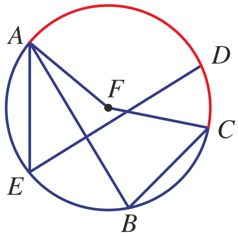
b



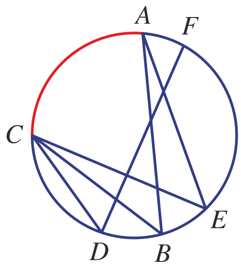
c



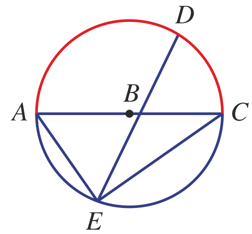
d



e



f



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

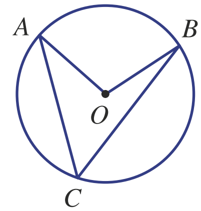
2 For this circle, O is the centre.

a Name the angle at the centre of the circle.

b Name the angle at the circumference of the circle.

c If $\angle ACB = 40°$, find $\angle AOB$ using circle theorem 1.

d If $\angle AOB = 122°$, find $\angle ACB$ using circle theorem 1.



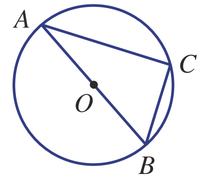
3 For this circle AB is a diameter.

a What is the size of $\angle AOB$?

b What is the size of $\angle ACB$ using circle theorem 2?

c If $\angle CAB = 30°$, find $\angle ABC$.

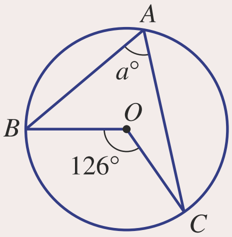
d If $\angle ABC = 83°$, find $\angle CAB$.



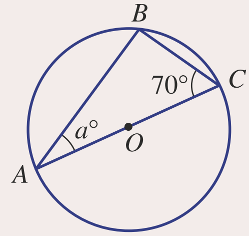
Example 12 Applying circle theorems 1 and 2

Find the value of the pronumerals in these circles.

a



b

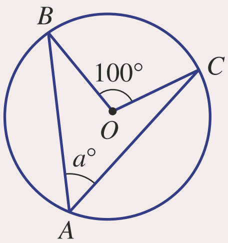


SOLUTION	EXPLANATION
a $2a = 126$ $\therefore a = 63$	From circle theorem 1, $\angle BOC = 2\angle BAC$.
b $\angle ABC$ is 90°. $\therefore a + 90 + 70 = 180$ $\therefore a = 20$	AC is a diameter, and from circle theorem 2 $\angle ABC = 90°$.

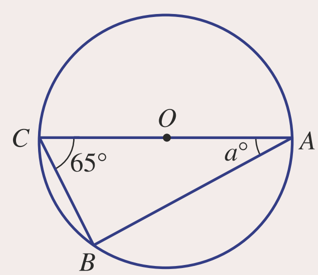
Now you try

Find the value of the pronumerals in these circles.

a



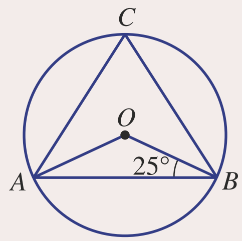
b



ISBN 978-1-108-77290-7

Example 13 Combining circle theorems with other circle properties

Find the size of $\angle ACB$.



SOLUTION

$\angle OAB = 25°$

$\angle AOB = 180° - 2 \times 25°$

$= 130°$

$\therefore \angle ACB = 65°$

EXPLANATION

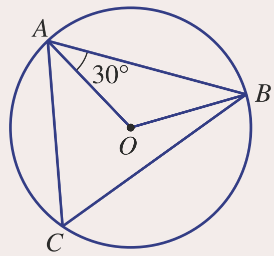
$\triangle AOB$ is isosceles.

Angle sum of a triangle is 180°.

The angle at the circumference is half the angle at the centre subtended by the same arc.

Now you try

Find the size of $\angle ACB$.



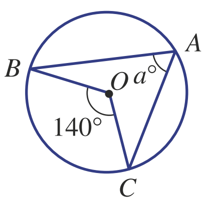
Exercise 2G

FLUENCY	1, 2(1/2), 3	2(1/2), 3, 4	2(1/2), 4, 5

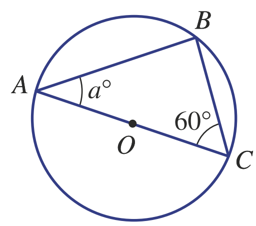
Example 12

1 Find the value of the pronumerals in these circles.

a



b



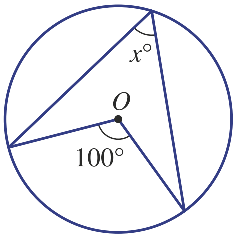
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

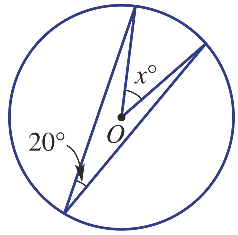
Example 12a

2 Find the value of x in these circles.

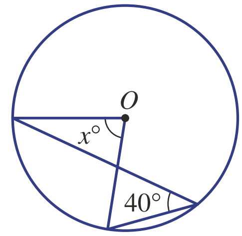
a



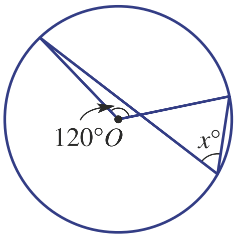
b



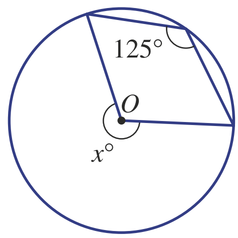
c



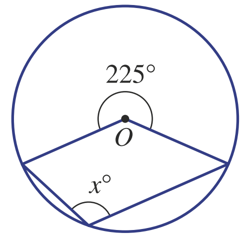
d



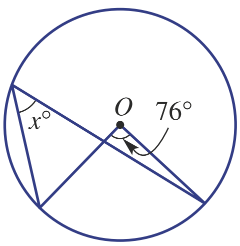
e



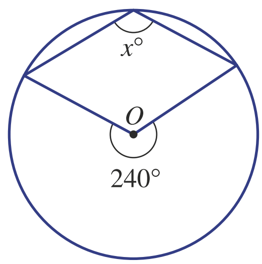
f



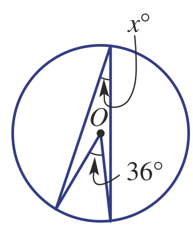
g



h



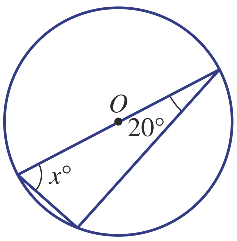
i



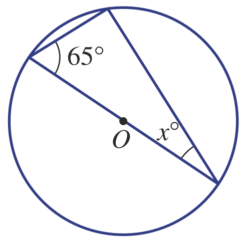
Example 12b

3 Find the value of x in these circles.

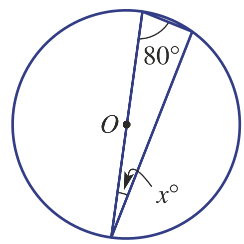
a



b

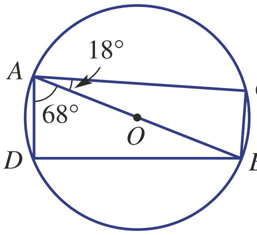


c

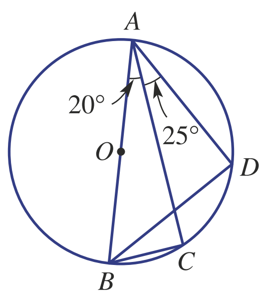


4 Find the size of both $\angle ABC$ and $\angle ABD$.

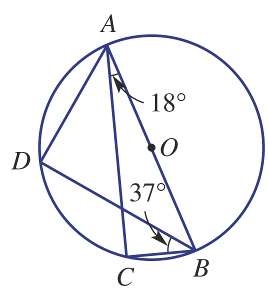
a



b



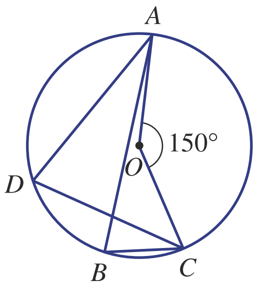
c



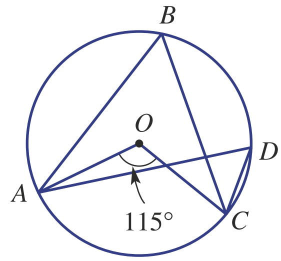
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

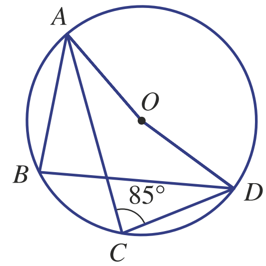
5 **a** Find the value of $\angle ADC$ and $\angle ABC$.



b Find the value of $\angle ABC$ and $\angle ADC$.



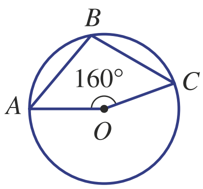
c Find the value of $\angle AOD$ and $\angle ABD$.



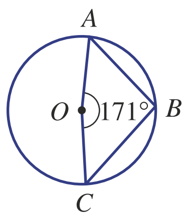
PROBLEM-SOLVING	6	6–7(1/2)	6–7(1/2)

6 Find the value of $\angle ABC$.

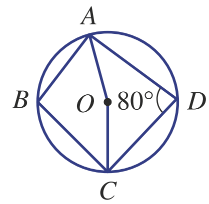
a



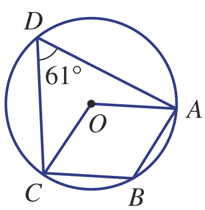
b



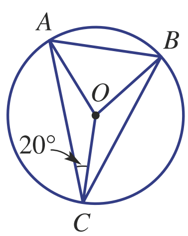
c



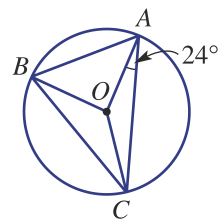
d



e



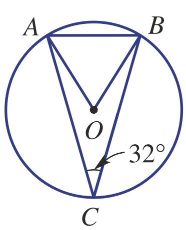
f



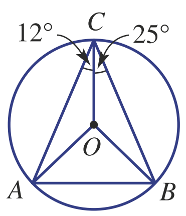
Example 13

7 Find the value of $\angle OAB$.

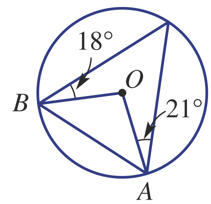
a



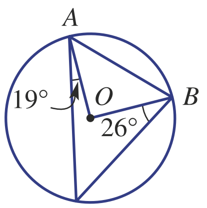
b



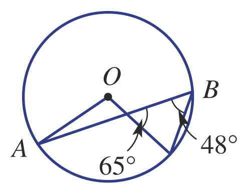
c



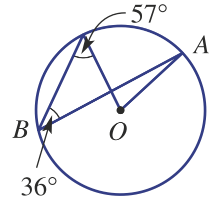
d



e



f

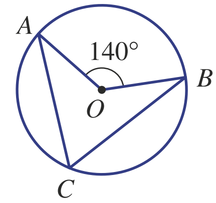


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

REASONING	8	8, 9	9–11

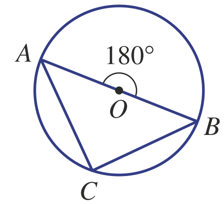
8 **a** For the first circle shown, use circle theorem 1 to find the value of $\angle ACB$.



b For the second circle shown, use circle theorem 1 to find the value of $\angle ACB$.

c For the second circle what does circle theorem 2 say about $\angle ACB$?

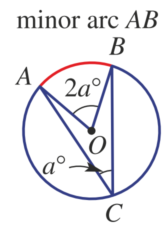
d Explain why circle theorem 2 can be thought of as a special case of circle theorem 1.



9 The two circles shown illustrate circle theorem 1 for both a minor arc and a major arc.

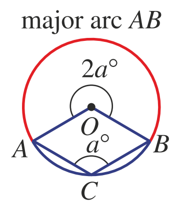
a When a minor arc is used, answer true or false.

- **i** $\angle AOB$ is always acute.
- **ii** $\angle AOB$ can be acute or obtuse.
- **iii** $\angle ACB$ is always acute.
- **iv** $\angle ACB$ can be acute or obtuse.



b When a major arc is used, answer true or false.

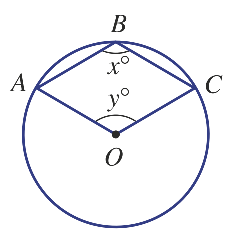
- **i** $\angle ACB$ can be acute.
- **ii** $\angle ACB$ is always obtuse.
- **iii** The angle at the centre $(2a°)$ is a reflex angle.
- **iv** The angle at the centre $(2a°)$ can be obtuse.



10 Consider this circle.

a Write reflex $\angle AOC$ in terms of x.

b Write y in terms of x.



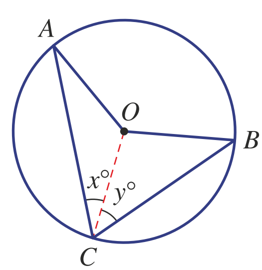
11 Prove circle theorem 1 for the case illustrated in this circle by following these steps and letting $\angle OCA = x°$ and $\angle OCB = y°$.

a Find $\angle AOC$ in terms of x, giving reasons.

b Find $\angle BOC$ in terms of y, giving reasons.

c Find $\angle AOB$ in terms of x and y.

d Explain why $\angle AOB = 2\angle ACB$.

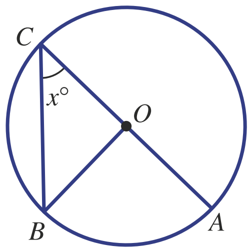


ISBN 978-1-108-77290-7

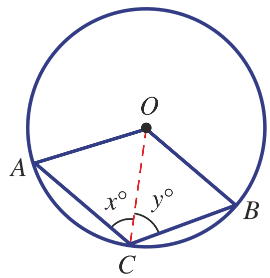
ENRICHMENT: Proving all cases	–	–	12, 13

12 Question **11** sets out a proof for circle theorem 1 using a given illustration. Now use a similar technique for these cases.

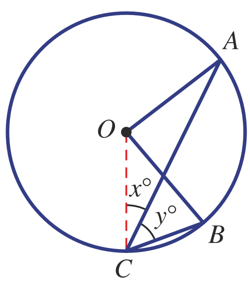
a Prove $\angle AOB = 2\angle ACB$; i.e. prove $\angle AOB = 2x°$.



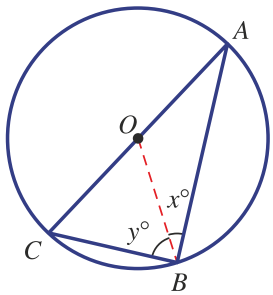
b Prove reflex $\angle AOB = 2\angle ACB$; i.e. prove reflex $\angle AOB = 2(x + y)°$.



c Prove $\angle AOB = 2\angle ACB$; i.e. prove $\angle AOB = 2y°$.



13 Prove circle theorem 2 by showing that $x + y = 90$.



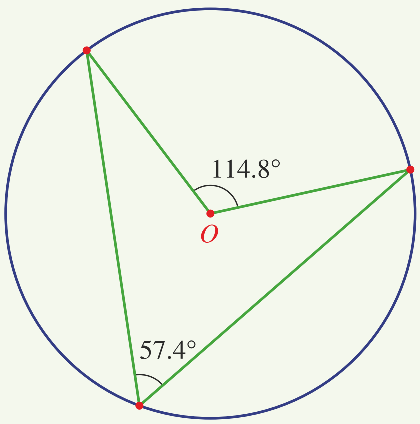
ISBN 978-1-108-77290-7

Exploring circle theorems with interactive geometry software

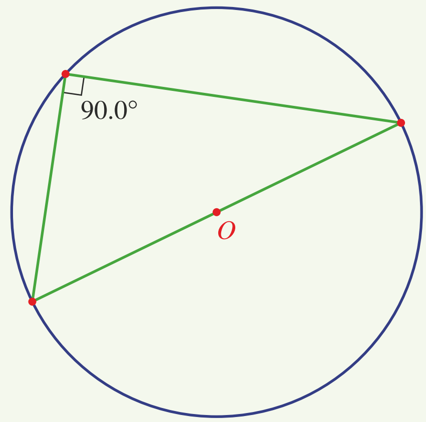
1. Construct a circle with centre O and any given radius.
2. Illustrate the four angle properties by constructing line segments, as shown.
   Measure corresponding angles to illustrate the angle properties.
   - Circle theorem 1: The angle at the centre of a circle is twice the angle at a point on the circle subtended by the same arc.
   - Circle theorem 2: The angle in a semicircle is 90°.
   - Circle theorem 3: Angles at the circumference of a circle subtended by the same arc are equal.
   - Circle theorem 4: Opposite angles in a cyclic quadrilateral are supplementary.
3. Drag the circle or one of the points on the circle to check that the properties are retained.

Circle theorem illustrations

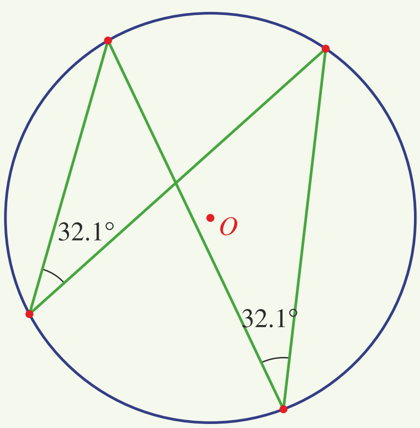
Circle theorem 1



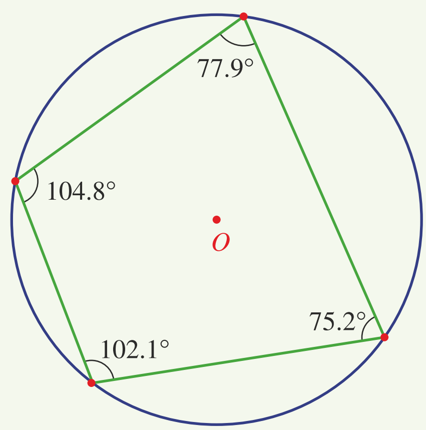
Circle theorem 2



Circle theorem 3



Circle theorem 4



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

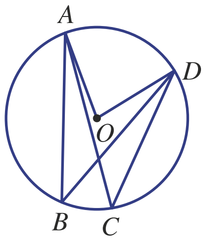
2H Angle properties of circles: Theorems 3 and 4 10A

Learning intentions

- To know that angles at the circumference subtended by the same arc or chord are equal
- To know that a cyclic quadrilateral is one that has all four vertices on the circumference of a circle
- To know that opposite angles in a cyclic quadrilateral are supplementary
- To be able to apply the circle theorems to find unknown angles

When both angles are at the circumference, there are two important properties of pairs of angles in a circle to consider.

You will recall from circle theorem 1 that in this circle $\angle AOD = 2\angle ABD$ and also $\angle AOD = 2\angle ACD$. This implies that $\angle ABD = \angle ACD$, which is an illustration of circle theorem 3 – angles at the circumference subtended by the same arc are equal.



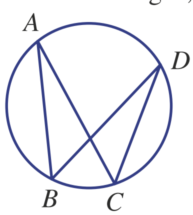
Agricultural engineers used circle geometry when designing the drive elements of the header, conveyer, separator, thresher and cutter units in the combine harvester.

The fourth theorem relates to cyclic quadrilaterals, which have all four vertices sitting on the same circle. This also will be explored in this section.

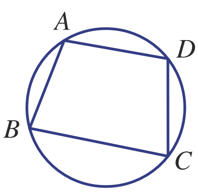
LESSON STARTER Discover angle properties: Theorems 3 and 4

Once again, use a protractor and a pair of compasses for this exercise or use interactive geometry software.

- Construct a circle with four points at the circumference, as shown.
- Measure $\angle ABD$ and $\angle ACD$. What do you notice? Drag A, B, C or D and compare the angles.



- Now construct this cyclic quadrilateral (or drag point C if using interactive geometry software).
- Measure $\angle ABC$, $\angle BCD$, $\angle CDA$ and $\angle DAB$. What do you notice? Drag A, B, C or D and compare angles.

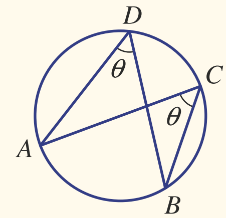


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

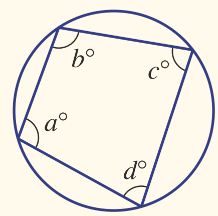
KEY IDEAS

- **Circle theorem 3**: Angles at the circumference
  - Angles at the circumference of a circle subtended by the same arc are equal.
    - As shown in the diagram, $\angle C = \angle D$ but note also that $\angle A = \angle B$.



- A **cyclic quadrilateral** has all four vertices sitting on the same circle.
- **Circle theorem 4**: Opposite angles in cyclic quadrilaterals
  - Opposite angles in a cyclic quadrilateral are supplementary (sum to 180°).

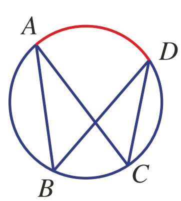
$a + c = 180$
$b + d = 180$



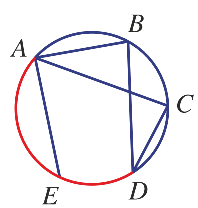
BUILDING UNDERSTANDING

1 Name another angle that is subtended by the same arc as $\angle ABD$.

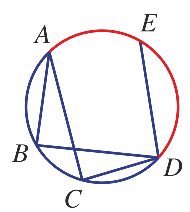
a



b



c



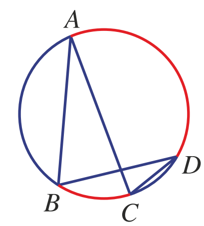
2 For this circle, answer the following.

a Name two angles subtended by arc AD.

b Using circle theorem 3, state the size of $\angle ACD$ if $\angle ABD = 85°$.

c Name two angles subtended by arc BC.

d Using circle theorem 3, state the size of $\angle BAC$ if $\angle BDC = 17°$.



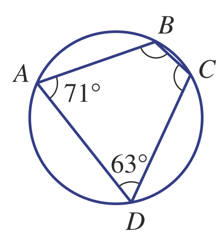
3 Circle theorem 4 states that opposite angles in a cyclic quadrilateral are supplementary.

a What does it mean when we say two angles are supplementary?

b Find the value of $\angle ABC$.

c Find the value of $\angle BCD$.

d Check that $\angle ABC + \angle BCD + \angle CDA + \angle DAB = 360°$.



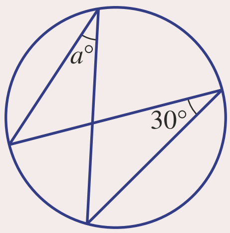
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

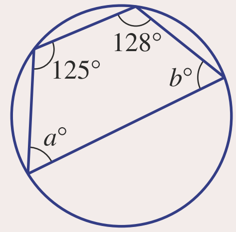
Example 14 Applying circle theorems 3 and 4

Find the value of the pronumerals in these circles.

a



b

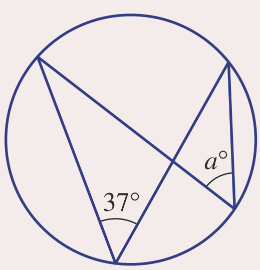


SOLUTION	EXPLANATION
a $a = 30$	The $a°$ and $30°$ angles are subtended by the same arc. This is an illustration of circle theorem 3.
b $a + 128 = 180$ $\therefore a = 52$ $b + 125 = 180$ $\therefore b = 55$	The quadrilateral is cyclic, so opposite angles sum to $180°$. This is an illustration of circle theorem 4.

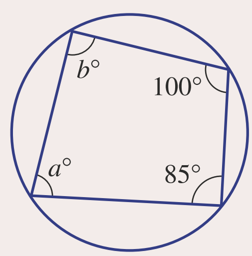
Now you try

Find the value of the pronumerals in these circles.

a



b



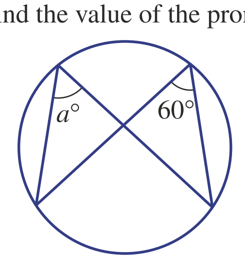
Exercise 2H

FLUENCY	1, 2–3(1/2)	2–3(1/2)	2–3(1/3)

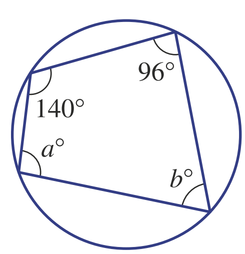
Example 14

1 Find the value of the pronumerals in these circles.

a



b



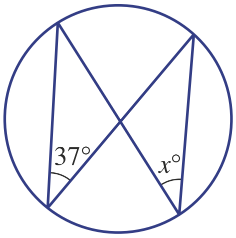
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

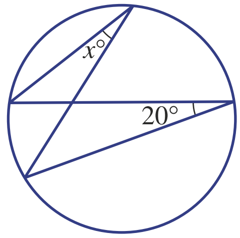
Example 14a

2 Find the value of x in these circles.

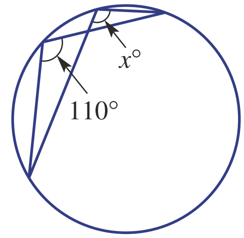
a



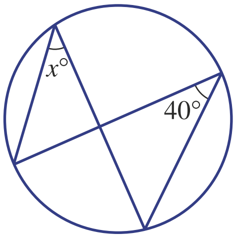
b



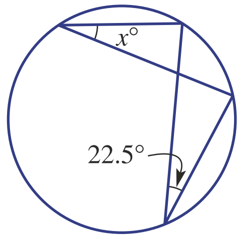
c



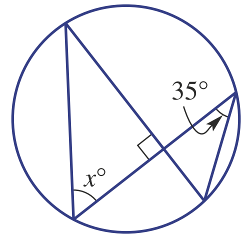
d



e



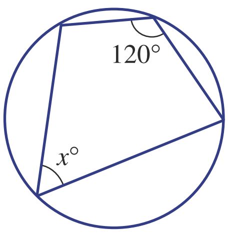
f



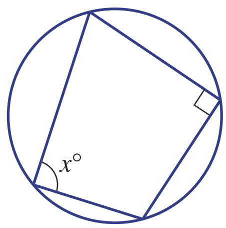
Example 14b

3 Find the value of the pronumerals in these circles.

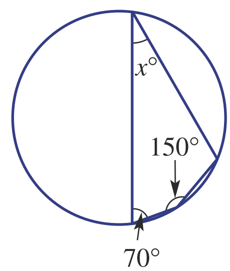
a



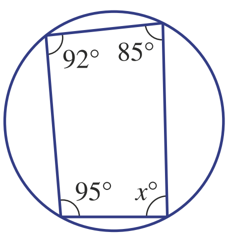
b



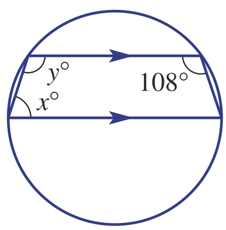
c



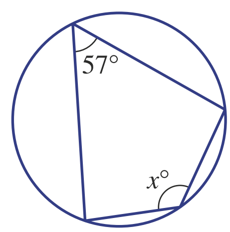
d



e



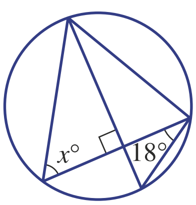
f



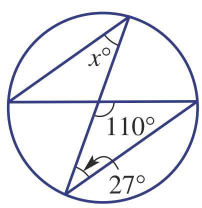
PROBLEM-SOLVING	4(1/2)	4–5(1/2)	4–5(1/3)

4 Find the value of x.

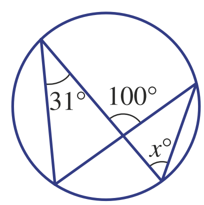
a



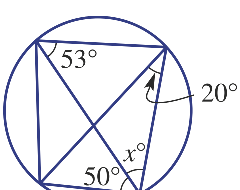
b



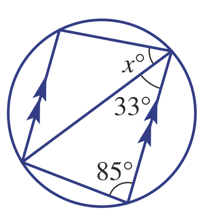
c



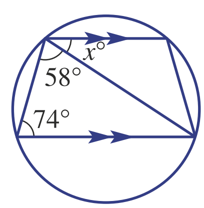
d



e



f

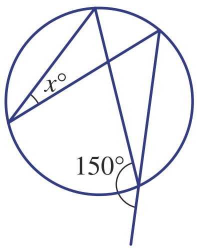


ISBN 978-1-108-77290-7

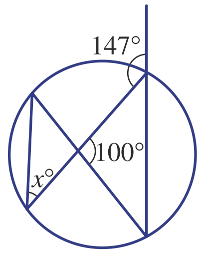
Cambridge University Press

Updated June 2022

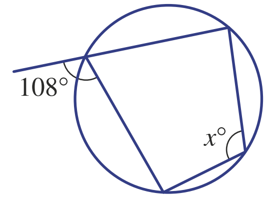
g



h

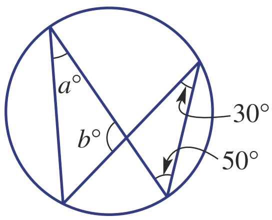


i

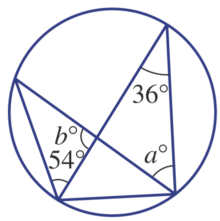


5 Find the values of the pronumerals in these circles.

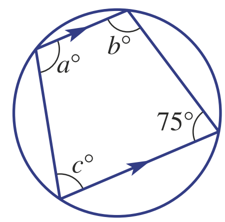
a



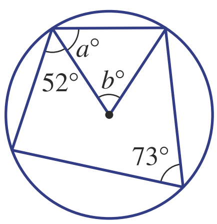
b



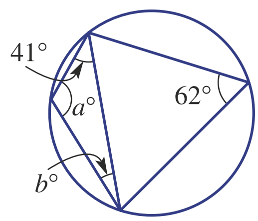
c



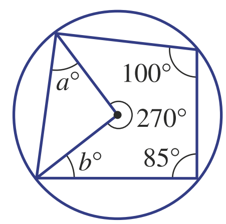
d



e



f



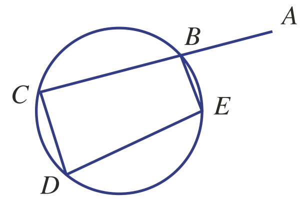
REASONING	6	6, 7	7, 8

6 $\angle ABE$ is an exterior angle to the cyclic quadrilateral $BCDE$.

a If $\angle ABE = 80°$, find $\angle CDE$.

b If $\angle ABE = 71°$, find $\angle CDE$.

c Prove that $\angle ABE = \angle CDE$ using circle theorem 4.

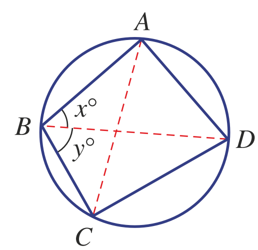


7 Prove that opposite angles in a cyclic quadrilateral are supplementary by following these steps.

a Explain why $\angle ACD = x°$ and $\angle DAC = y°$.

b Prove that $\angle ADC = 180° - (x + y)°$.

c What does this say about $\angle ABC$ and $\angle ADC$?

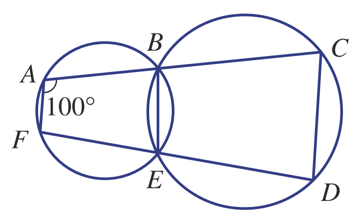


8 If $\angle BAF = 100°$, complete the following.

a Find:

i $\angle FEB$ **ii** $\angle BED$ **iii** $\angle DCB$

b Explain why $AF \parallel CD$.

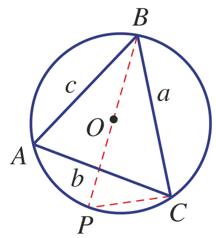


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

ENRICHMENT: A special property	–	–	9

9 Consider a triangle ABC inscribed in a circle. The construction line BP is a diameter and PC is a chord. If r is the radius, then $BP = 2r$.



a What can be said about $\angle PCB$? Give a reason.

b What can be said about $\angle A$ and $\angle P$? Give a reason.

c If $BP = 2r$, use trigonometry with $\angle P$ to write an equation linking r and a.

d Prove that $2r = \dfrac{a}{\sin A}$, giving reasons.See the poster on the inside back cover of your book for more information about solving unfamiliar problems.

ISBN 978-1-108-77290-7

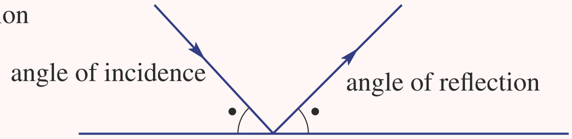
Cambridge University Press
Updated June 2022

Applications and problem-solving

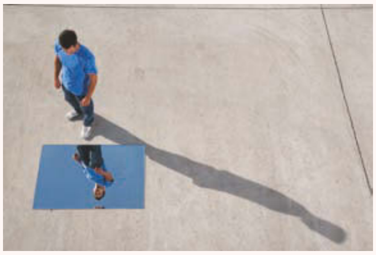
The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

Mirror, mirror

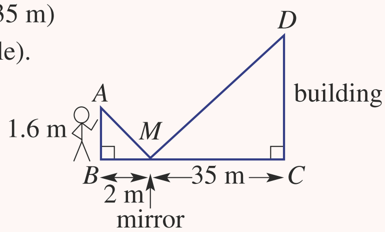
1 The law of reflection says that the angle of reflection is equal to the angle of incidence as shown.



George, who is 1.6 m tall, places a mirror (M) on the ground in front of a building and then moves backwards away from the mirror until he can sight the top of the building in the centre of the mirror.



George is interested in how the height of the building can be calculated using the mirror. He wants to only use measurements that can be recorded from ground level and combine these with the similar triangles that are generated after positioning the mirror on the ground.



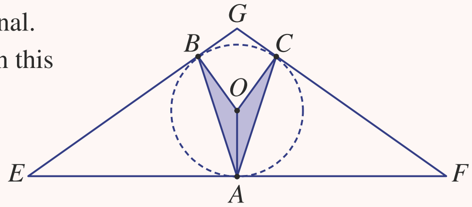
a The distance is measured between the mirror and the building (35 m) and the mirror and George's location (2 m) as shown (not to scale).
   i Prove that a pair of similar triangles has been formed.
   ii Use similarity to find the height of this building.

b Another building is 24 m high. If the mirror is placed 1.5 m from George such that he can see the top of the building in the centre of the mirror, how far is George from the base of the building?

c George moves to 20 m from the base of another building.
   i If the building is 11.2 m high, how far from George does the mirror need to be placed so that he can see the top of the building in the centre of the mirror?
   ii Repeat part **i** to find an expression for how far from George the mirror needs to be placed for a building of height y m. Answer in terms of y and use your expression to check your answer to part **i**.

Airport terminal

2 Engineers are working on the design of a new airport terminal. A cross-section of an airport terminal design is illustrated in this diagram where the roof is held up by a V-shaped support as shown. Points A, B and C sit on a circle with centre O and OA, OB and OC are perpendicular to EF, EG and FG respectively. Also, $EG = FG$ and $EA = AF$.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

The engineers are interested in the relationship between various angles within the design. Given the fixed geometric properties of the cross-section the engineers will explore how changing one angle affects the other angles so that they have a better understanding of the design limitations.

a If $\angle AEB$ is set at $40°$ find:
  i $\angle EGF$ **ii** $\angle BOC$ **iii** $\angle BAC$

b If $\angle OCA$ is set at $10°$ find:
  i $\angle BOC$ **ii** $\angle AEB$

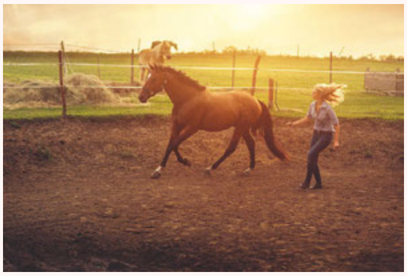
c If $\angle AEB = a°$ find $\angle BAC$ in terms of a.

d Use your rule in part **c** to verify your answer to part **a iii**.

e As $\angle AEB$ increases, describe what happens to $\angle BAC$ and make a drawing to help explain your answer.

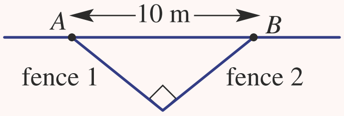
Circles on the farm

3 A circle is a locus defined as a set of points that are equidistant (the same distance) from a single point. While not always initially visible, such loci exist in many common situations.



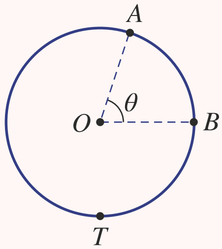
A farmer wishes to investigate the existence of such circle loci in everyday situations in a farm environment. These include fencing an area using two fixed posts and training a horse around a given point.

a A triangular region is being fenced on a farm. It connects to posts, A and B, 10 m apart on an existing fence. The two new fences are to meet at right angles as shown.



  i Describe and draw the location of all possible points where the two fence lines can meet.
  ii Which point from part **i** gives the maximum triangular area and what is this area?
  iii If the two fence posts are x m apart, give a rule for the maximum possible area of the triangular region in m^2.
  iv If fences 1 and 2 were replaced with a single fence in the shape of a semicircle, give an expression for the area gained in terms of x, where x metres is the distance between the fence posts.

b A horse is doing some training work around a circular paddock. The trainer stands on the edge of the paddock at T (as shown) watching the horse through binoculars as it moves from point A to point B.



  i Give an expression for $\angle ATB$.
  ii The horse is trotting at a constant rate of 1 lap (revolution) per minute. At what rate is the trainer moving his binoculars following the horse from A to B, in revolutions per minute?
  iii If the horse completes a lap (revolution) in x minutes, at what rate is the trainer moving his binoculars in terms of x?

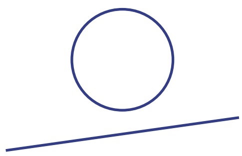
ISBN 978-1-108-77290-7

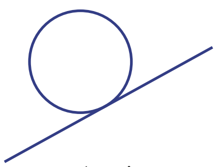
Cambridge University Press
Updated June 2022

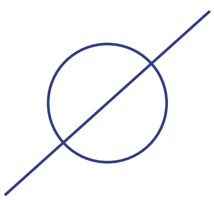
2I Tangents to a circle EXTENDING

Learning intentions
- To know that a tangent is a line that touches a circle or curve at one point
- To know that a tangent is perpendicular to the radius at the point of contact
- To be able to find angles involving tangents
- To know that two different tangents drawn from an external point to the circle create line segments of equal length
- To know that the angle between a tangent and a chord is equal to the angle in the alternate segment
- To be able to apply the alternate segment theorem

When a line and a circle are drawn, three possibilities arise: they could intersect 0, 1 or 2 times.


0 points


1 point


2 points

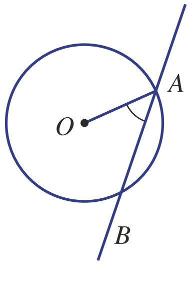
Where the 'pitch' circles meet, gears have a common tangent. Mechanical and auto engineers apply circle geometry when designing gears, including for vehicle engines, clocks, fuel pumps, automation machinery, printing presses and robots.

If the line intersects the circle once then it is called a tangent. If it intersects twice it is called a secant.

LESSON STARTER From secant to tangent

This activity is best completed using dynamic computer geometry software.

- Construct a circle with centre O and a secant line that intersects at A and B. Then measure $\angle BAO$.
- Drag B to alter $\angle BAO$. Can you place B so that line AB is a tangent? In this case what is $\angle BAO$?

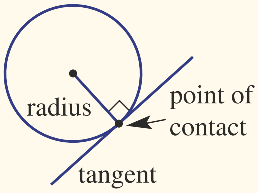


ISBN 978-1-108-77290-7

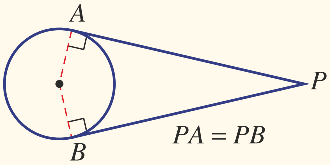
Cambridge University Press
Updated June 2022

KEY IDEAS

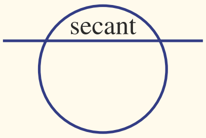
- A **tangent** is a line that touches a circle at a point called the **point of contact**.



  - A tangent intersects the circle exactly once.
  - A tangent is perpendicular to the radius at the point of contact.
  - Two different tangents drawn from an external point to the circle create line segments of equal length.



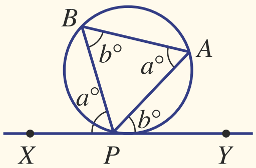
- A **secant** is a line that cuts a circle twice.



- **Alternate segment theorem**: The angle between a tangent and a chord is equal to the angle in the alternate segment.

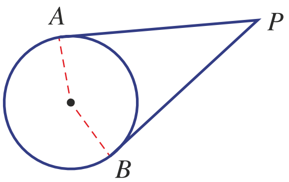
  $\angle APY = \angle ABP$ and

  $\angle BPX = \angle BAP$

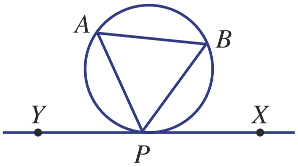


BUILDING UNDERSTANDING

1 a How many times does a tangent intersect a circle?
b At the point of contact, what angle does the tangent make with the radius?
c If AP is 5 cm, what is the length BP in this diagram?



2 For this diagram use the alternate segment theorem and name the angle that is:
a equal to $\angle BPX$
b equal to $\angle BAP$
c equal to $\angle APY$
d equal to $\angle ABP$



3 What is the interior angle sum for:
a a triangle?
b a quadrilateral?

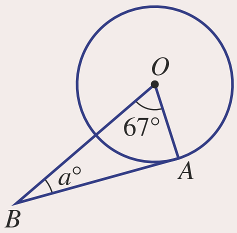
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

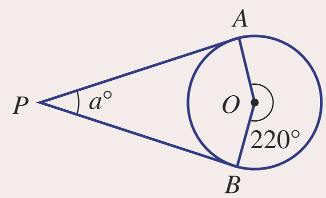
Example 15 Finding angles with tangents

Find the value of a in these diagrams that include tangents.

a



b



SOLUTION

a $\angle BAO = 90°$

$a + 90 + 67 = 180$

$\therefore a = 23$

b $\angle PAO = \angle PBO = 90°$

Obtuse $\angle AOB = 360° - 220° = 140°$

$a + 90 + 90 + 140 = 360$

$\therefore a = 40$

EXPLANATION

a BA is a tangent, so $OA \perp BA$.

The sum of the angles in a triangle is 180°.

b $PA \perp OA$ and $PB \perp OB$.

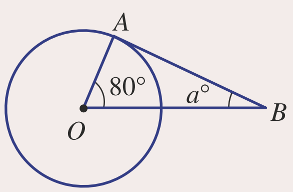
Angles in a revolution sum to 360°.

Angles in a quadrilateral sum to 360°.

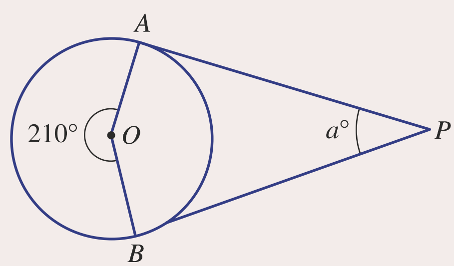
Now you try

Find the value of a in these diagrams that include tangents.

a



b

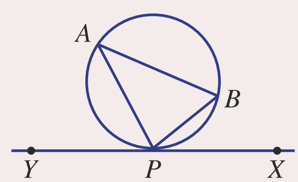


Example 16 Using the alternate segment theorem

In this diagram XY is a tangent to the circle.

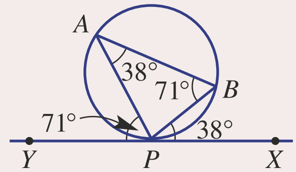
a Find $\angle BPX$ if $\angle BAP = 38°$.

b Find $\angle ABP$ if $\angle APY = 71°$.



ISBN 978-1-108-77290-7

SOLUTION	EXPLANATION
a $\angle BPX = 38^\circ$	The angle between a tangent and a chord is equal to the angle in the alternate segment.
b $\angle ABP = 71^\circ$	

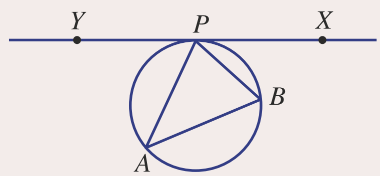


Now you try

In this diagram XY is a tangent to the circle.

a Find $\angle BPX$ if $\angle BAP = 50^\circ$.

b Find $\angle ABP$ if $\angle APY = 70^\circ$.



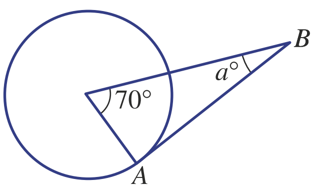
Exercise 2I

FLUENCY	1–5	2–6	2–6

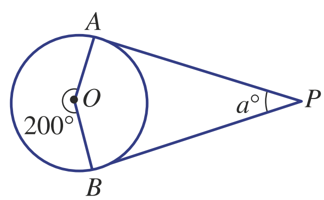
Example 15

1 Find the value of a in these diagrams that include tangents.

a



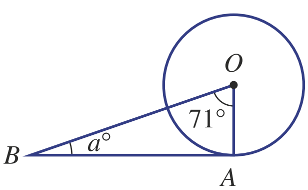
b



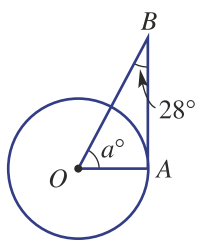
Example 15a

2 Find the value of a in these diagrams that include tangents.

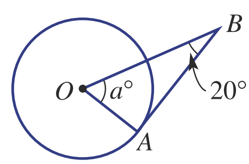
a



b



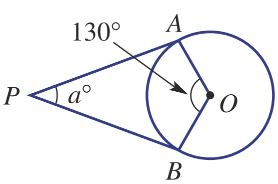
c



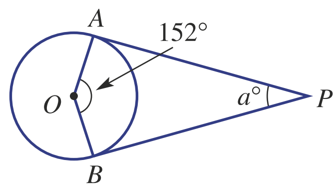
Example 15b

3 Find the value of a in these diagrams that include two tangents.

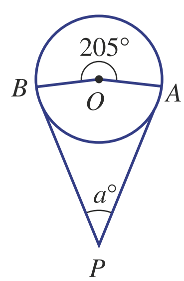
a



b



c



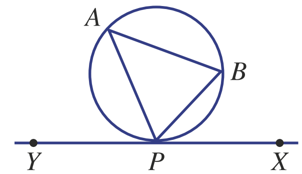
ISBN 978-1-108-77290-7

Example 16

4 In this diagram, XY is a tangent to the circle. Use the alternate segment theorem to find:

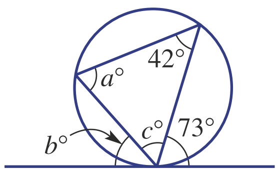
a $\angle PAB$ if $\angle BPX = 50°$

b $\angle APY$ if $\angle ABP = 59°$

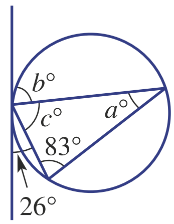


5 Find the value of a, b and c in these diagrams involving tangents.

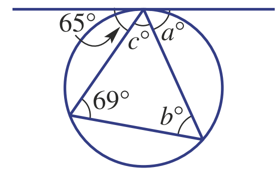
a



b

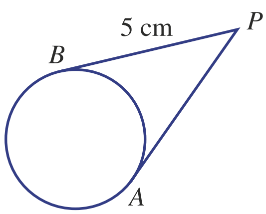


c

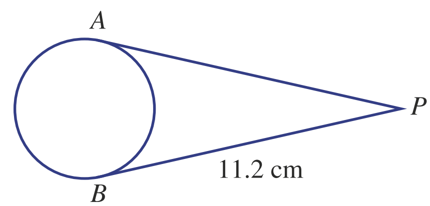


6 Find the length AP if AP and BP are both tangents.

a



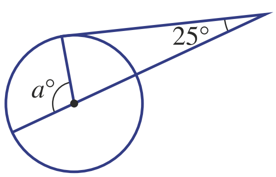
b



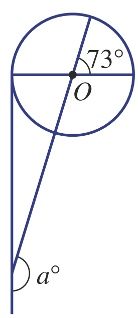
PROBLEM-SOLVING	7–8(1/2)	7–8(1/2)	7–8(1/3), 9

7 Find the value of a. All diagrams include one tangent line.

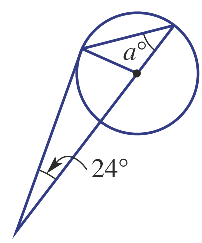
a



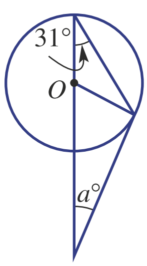
b



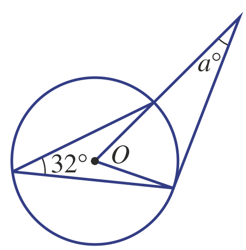
c



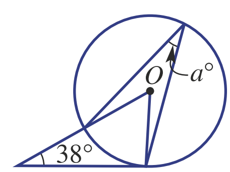
d



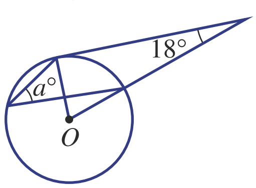
e



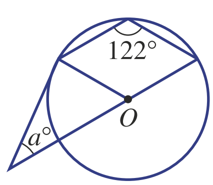
f



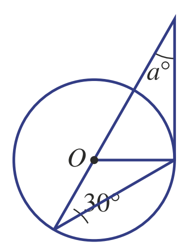
g



h



i

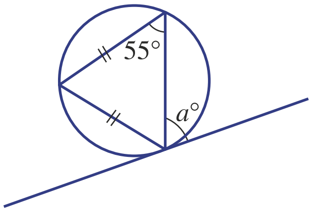


ISBN 978-1-108-77290-7

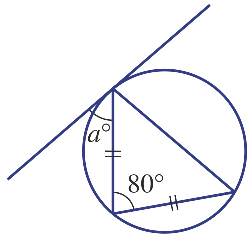
Cambridge University Press
Updated June 2022

8 Find the value of a in these diagrams involving tangents.

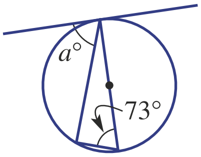
a



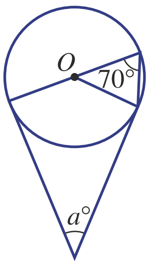
b



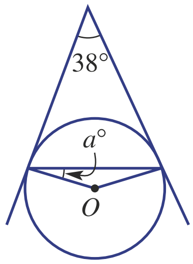
c



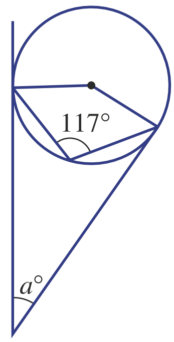
d



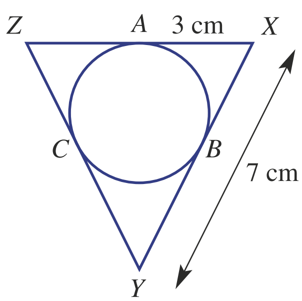
e



f



9 Find the length of CY in this diagram.



REASONING	10	10, 11	11–13

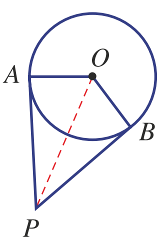
10 Prove that $AP = BP$ by following these steps.

a Explain why $OA = OB$.

b What is the size of $\angle OAP$ and $\angle OBP$?

c Hence, prove that $\triangle OAP \equiv \triangle OBP$.

d Explain why $AP = BP$.

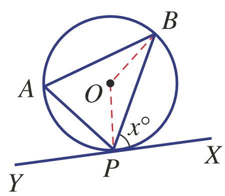


11 Prove the alternate angle theorem using these steps. First, let $\angle BPX = x°$, then give reasons at each step.

a Write $\angle OPB$ in terms of x.

b Write obtuse $\angle BOP$ in terms of x.

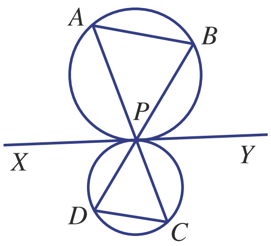
c Use circle theorem 1 from angle properties of a circle to write $\angle BAP$ in terms of x.



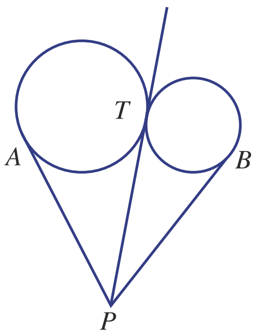
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

12 These two circles touch with a common tangent XY.
Prove that $AB \parallel DC$. You may use the alternate segment theorem.



13 PT is a common tangent. Explain why $AP = BP$.

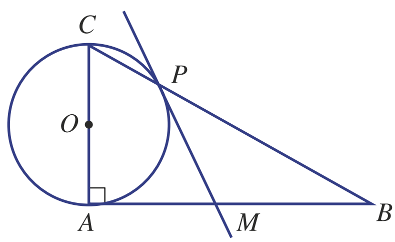


ENRICHMENT: Bisecting tangent	–	–	14

14 In this diagram, $\triangle ABC$ is right angled, AC is a diameter and PM is a tangent at P, where P is the point at which the circle intersects the hypotenuse.

a Prove that PM bisects AB; i.e. that $AM = MB$.

b Construct this figure using interactive geometry software and check the result. Drag A, B or C to check different cases.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2J Intersecting chords, secants and tangents EXTENDING

Learning intentions

- To know the difference between a chord, a tangent and a secant
- To know the relationship between the lengths of intersecting chords
- To know the relationship between the lengths of secants that intersect at an external point
- To know the relationship between the lengths of an intersecting secant and tangent
- To be able to apply these relationships to find unknown lengths

In circle geometry, the lengths of the line segments (or intervals) formed by intersecting chords, secants or tangents are connected by special rules. There are three situations in which this occurs:

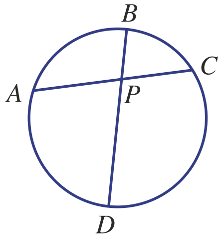
1. intersecting chords
2. intersecting secant and tangent
3. intersecting secants.

Architects use circle and chord geometry to calculate the dimensions of constructions, such as this glass structure.

LESSON STARTER Equal products

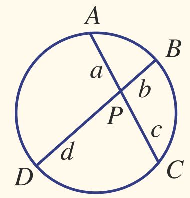
Use interactive geometry software to construct this figure and then measure AP, BP, CP and DP.

- Calculate $AP \times CP$ and $BP \times DP$. What do you notice?
- Drag A, B, C or D. What can be said about $AP \times CP$ and $BP \times DP$ for any pair of intersecting chords?

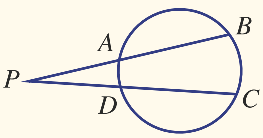


KEY IDEAS

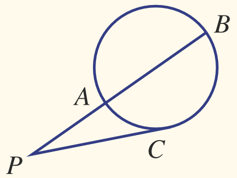
- When two chords intersect as shown, then $AP \times CP = BP \times DP$ or $ac = bd$.



- When two secants intersect at an external point P as shown, then $AP \times BP = DP \times CP$.



- When a secant intersects a tangent at an external point as shown, then $AP \times BP = CP^2$.

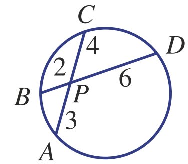


ISBN 978-1-108-77290-7

BUILDING UNDERSTANDING

1 State these lengths for the given diagram.

a AP **b** DP **c** AC **d** BD



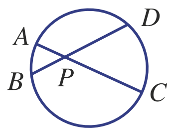
2 Solve these equations for x.

a $x \times 2 = 7 \times 3$ **b** $4x = 5 \times 2$

c $(x + 3) \times 7 = 6 \times 9$ **d** $7(x + 4) = 5 \times 11$

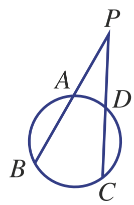
3 State the missing parts for the rules for each diagram.

a



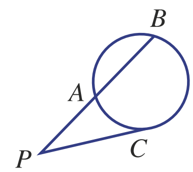
$AP \times CP =$ _____ $\times$ _____

b



$AP \times$ _____ $=$ _____ $\times$ _____

c

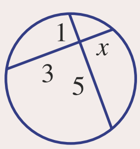


$AP \times$ _____ $=$ _____^{2}

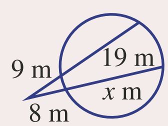
Example 17 Finding lengths using intersecting chords, secants and tangents

Find the value of x in each figure.

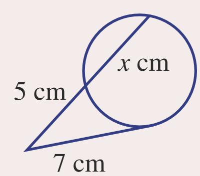
a



b



c



SOLUTION

a

$$x \times 3 = 1 \times 5$$
$$3x = 5$$
$$x = \frac{5}{3}$$

EXPLANATION

Equate the products of each pair of line segments on each chord.

b

$$8 \times (x + 8) = 9 \times 28$$
$$8x + 64 = 252$$
$$8x = 188$$
$$x = \frac{188}{8}$$
$$= \frac{47}{2}$$

Multiply the entire length of the secant ($19 + 9 = 28$ and $x + 8$) by the length from the external point to the first intersection point with the circle. Then equate both products.

Expand brackets and solve for x.

ISBN 978-1-108-77290-7

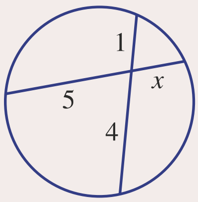
Cambridge University Press
Updated June 2022

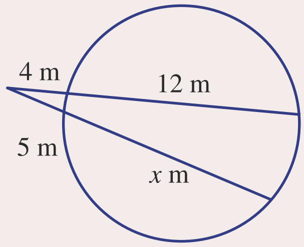
c $5 \times (x + 5) = 7^2$
$5x + 25 = 49$
$5x = 24$
$x = \dfrac{24}{5}$

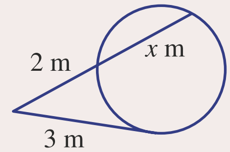
Square the length of the tangent and then equate with the product from the other secant.

Now you try

Find the value of x in each figure.

a


b


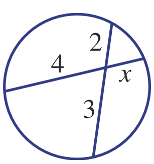
c


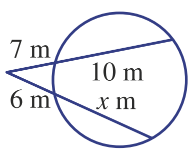
Exercise 2J

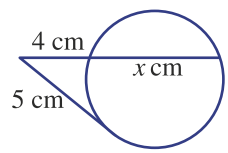
FLUENCY	1–4	2–4	2–4(1/3), 5

Example 17

1 Find the value of x in each figure.

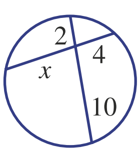
a


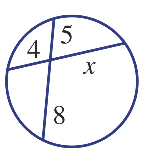
b


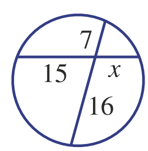
c


Example 17a

2 Find the value of x in each figure.

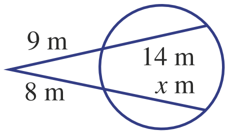
a


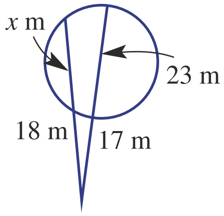
b


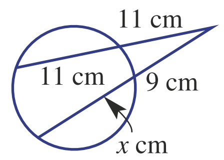
c


Example 17b

3 Find the value of x in each figure.

a


b


c


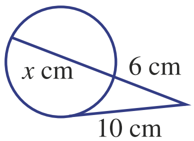
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

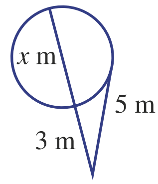
Example 17c

4 Find the value of x in each figure.

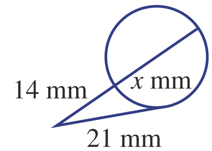
a



b

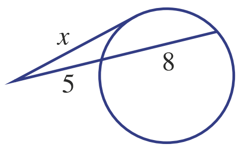


c

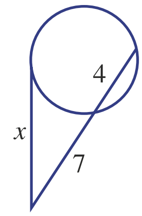


5 Find the exact value of x, in surd form. For example, $\sqrt{7}$.

a



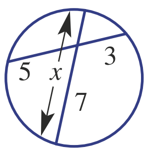
b



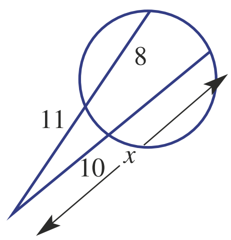
PROBLEM-SOLVING	6(1/2)	6(1/2)	6(1/2), 7

6 Find the exact value of x.

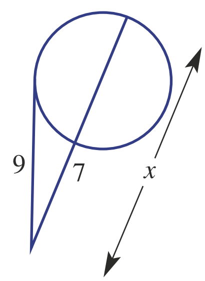
a



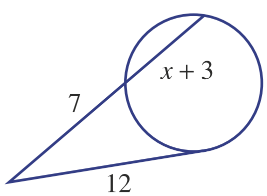
b



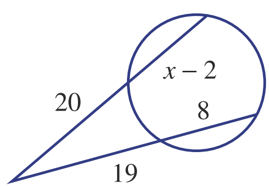
c



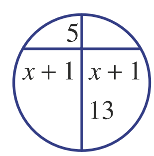
d



e

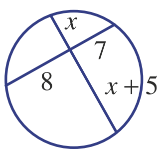


f

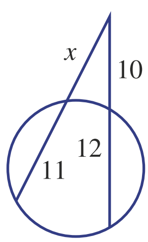


7 For each diagram, derive the given equations.

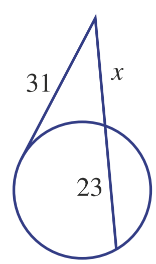
a $x^2 + 5x - 56 = 0$



b $x^2 + 11x - 220 = 0$



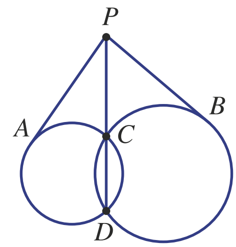
c $x^2 + 23x - 961 = 0$



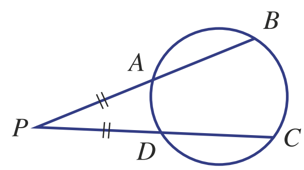
ISBN 978-1-108-77290-7

REASONING	8, 9	8–10	10–12

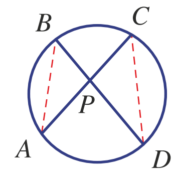
8 Explain why $AP = BP$ in this diagram, using your knowledge from this section.



9 In this diagram $AP = DP$. Explain why $AB = DC$.



10 Prove that $AP \times CP = BP \times DP$ by following these steps.



a What can be said about the pair of angles $\angle A$ and $\angle D$ and also about the pair of angles $\angle B$ and $\angle C$? Give a reason.

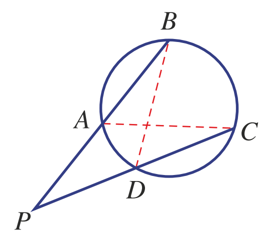
b Prove $\triangle ABP \parallel\!\mid \triangle DCP$.

c Complete:

$$\frac{AP}{\dots} = \frac{\dots}{CP}$$

d Prove $AP \times CP = BP \times DP$.

11 Prove that $AP \times BP = DP \times CP$ by following these steps.

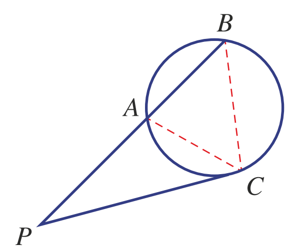


a Consider $\triangle PBD$ and $\triangle PCA$. What can be said about $\angle B$ and $\angle C$? Give a reason.

b Prove $\triangle PBD \parallel\!\mid \triangle PCA$.

c Prove $AP \times BP = DP \times CP$.

12 Prove that $AP \times BP = CP^2$ by following these steps.



a Consider $\triangle BPC$ and $\triangle CPA$. Is $\angle P$ common to both triangles?

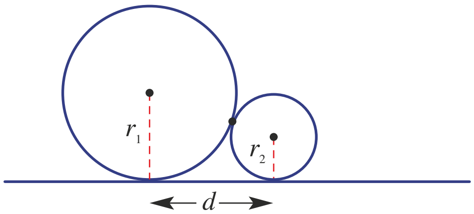
b Explain why $\angle ACP = \angle ABC$.

c Prove $\triangle BPC \parallel\!\mid \triangle CPA$.

d Prove $AP \times BP = CP^2$.

ENRICHMENT: Horizontal wheel distance	–	–	13

13 Two touching circles have radii r_1 and r_2. The horizontal distance between their centres is d. Find a rule for d in terms of r_1 and r_2.

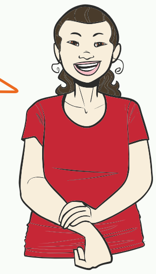


ISBN 978-1-108-77290-7

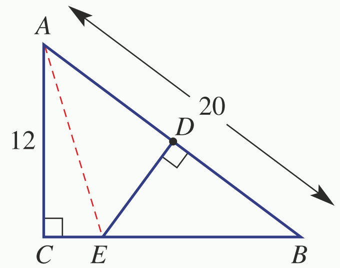
Cambridge University Press
Updated June 2022

Problems and challenges

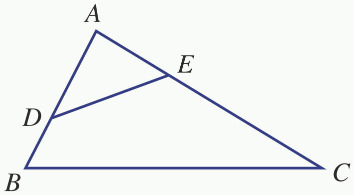
Up for a challenge? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.



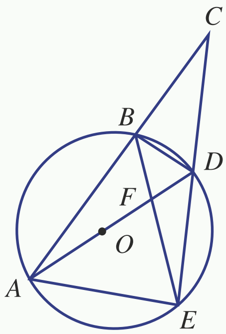
1 In a triangle ABC, angle C is a right angle, D is the midpoint of AB and DE is perpendicular to AB. The length of AB is 20 and the length of AC is 12. What is the area of triangle ACE?



2 In this diagram, $AB = 15$ cm, $AC = 25$ cm, $BC = 30$ cm and $\angle AED = \angle ABC$. If the perimeter of $\triangle ADE$ is 28 cm, find the lengths of BD and CE.



3 Other than straight angles, name all the pairs of equal angles in the diagram shown.



4 A person stands in front of a cylindrical water tank and has a viewing angle of 27° to the sides of the tank. What percentage of the circumference of the tank can they see?

5 An isosceles triangle ABC is such that its vertices lie on the circumference of a circle. $AB = AC$ and the chord from A to the point D on the circle intersects BC at E. Prove that $AB^2 - AE^2 = BE \times CE$.

6 D, E and F are the midpoints of the three sides of $\triangle ABC$. The straight line formed by joining two midpoints is parallel to the third side and half its length.

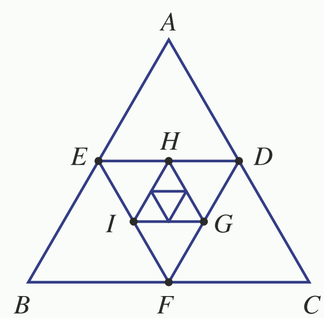
a Prove $\triangle ABC \mathbin{|||} \triangle FDE$.

$\triangle GHI$ is drawn in the same way such that G, H and I are the midpoints of the sides of $\triangle DEF$.

b Find the ratio of the area of:

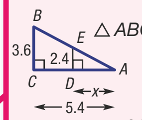
i $\triangle ABC$ to $\triangle FDE$ ii $\triangle ABC$ to $\triangle HGI$

c Hence, if $\triangle ABC$ is the first triangle drawn, what is the ratio of the area of $\triangle ABC$ to the area of the nth triangle drawn in this way?



ISBN 978-1-108-77290-7

Similar triangles application



$\triangle ABC \;|||\; \triangle AED$ (AAA)

$$\text{Scale factor} = \frac{3.6}{2.4} = 1.5$$

$\therefore\ x \times 1.5 = 5.4$

$x = 5.4 \div 1.5$

$x = 3.6$

Similar figures

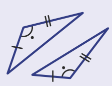
All corresponding angles are equal, corresponding sides are in the same ratio; i.e. same shape but different in size.
Tests for similar triangles:
SSS, SAS, AAA, RHS.
Written as $\triangle ABC \;|||\; \triangle DEF$ or $\triangle ABC \sim \triangle DEF$.

Polygons

Angle sum of polygon
$S = 180(n - 2)$, where n is the number of sides.

Congruent triangles

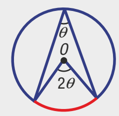
Such triangles are identical, written $\triangle ABC \equiv \triangle DEF$.
Tests for congruence are
SSS, SAS, AAS and RHS.



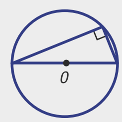
Congruent by SAS

Geometry

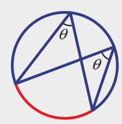
Angle properties of circles (10A)



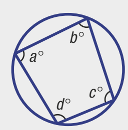
Angle at centre is twice angle at the circumference subtended by the same arc.



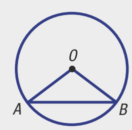
The angle in a semicircle is 90°.



Angles at circumference subtended by the same arc are equal.

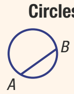


Opposite angles in cyclic quadrilaterals are supplementary.
$a + c = 180$
$b + d = 180$



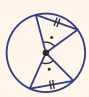
$\triangle OAB$ is isosceles given OA and OB are radii.

Circles and chords (10A)

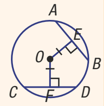


AB is a chord.

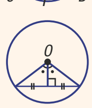
Chord theorem 1
Chords of equal length subtend equal angles.



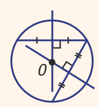
Chord theorem 2
If $AB = CD$,
then $OE = OF$.



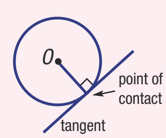
Chord theorem 3
Perpendicular from centre to chord bisects chord and angle at O.



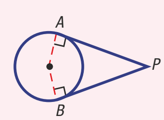
Chord theorem 4
Perpendicular bisectors of every chord of a circle intersect at the centre.



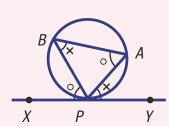
Tangents (Ext)



A tangent touches a circle once and is perpendicular to the radius at the point of contact.

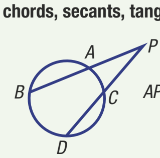


Tangents PA and PB have equal length; i.e. $PA = PB$.

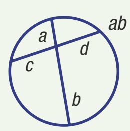


Alternate segment theorem: angle between tangent and chord is equal to the angle in the alternate segment.

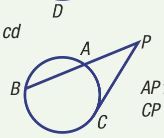
Intersecting chords, secants, tangents (Ext)



$AP \times BP = CP \times DP$



$ab = cd$



$AP \times BP = CP^2$
CP is a tangent.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

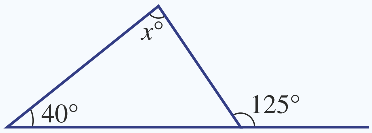
Chapter checklist

Chapter checklist: Success criteria

✔

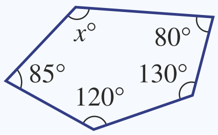
2A

1. I can apply the exterior angle theorem.
e.g. Find the value of x, giving reasons.



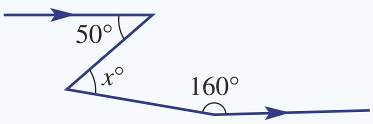
2A

2. I can find an unknown angle in a polygon.
e.g. Find the value of x in the polygon shown.



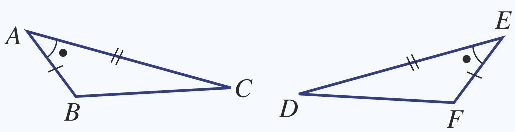
2A

3. I can work with angles in parallel lines.
e.g. Find the value of x, giving reasons.



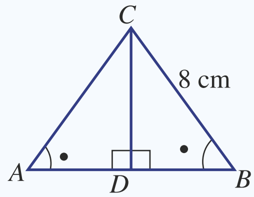
2B

4. I can prove congruence of triangles.
e.g. Prove that this pair of triangles are congruent.



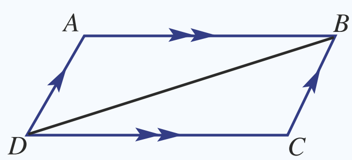
2B

5. I can use congruence in proof.
e.g. For the diagram shown, prove $\triangle ADC \equiv \triangle BDC$ and hence state the length of AC, giving a reason.



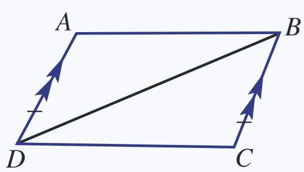
2C

6. I can prove properties of quadrilaterals.
e.g. Prove that a parallelogram (with opposite parallel sides) has equal opposite sides.



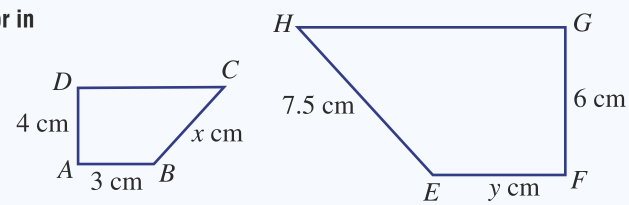
2C

7. I can test for a type of quadrilateral.
e.g. Prove that if one pair of opposite sides is equal and parallel in a quadrilateral then it is a parallelogram.



2D

8. I can find and use a scale factor in similar figures.
e.g. The two shapes shown are similar. Find the scale factor and use this to find the values of x and y.

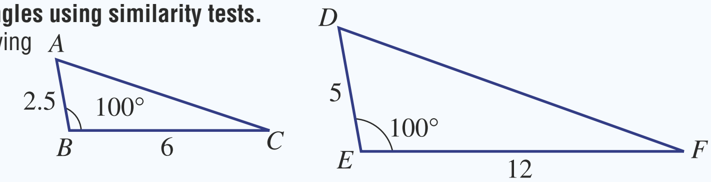


ISBN 978-1-108-77290-7

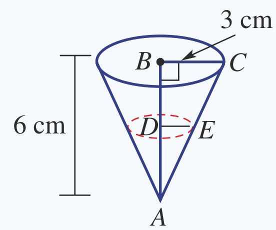
Cambridge University Press
Updated June 2022

Chapter checklist ✔

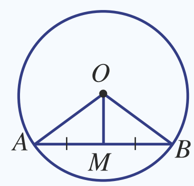
2E **9. I can prove similar triangles using similarity tests.**
e.g. Prove that the following triangles are similar.



2E **10. I can establish and use similarity.**
e.g. A cone has radius 3 cm and height 6 cm. Prove that $\triangle ADE \,|||\, \triangle ABC$ and find the radius DE if $AD = 2$ cm.

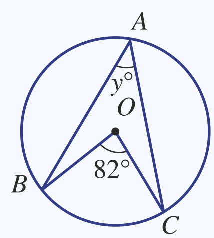


2F **11. I can use chord theorems.** 10A
e.g. Given $AM = BM$ and $\angle AOB = 100°$, find $\angle AOM$ and $\angle OMB$.

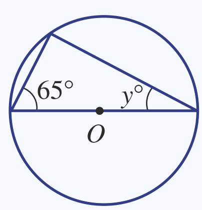


2F **12. I can prove chord theorems.** 10A
e.g. Prove chord theorem 1 in that chords of equal length subtend equal angles at the centre of a circle.

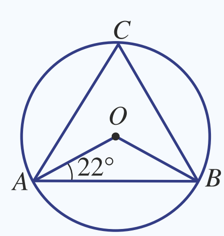
2G **13. I can apply circle theorem 1.** 10A
e.g. Find the value of y in the circle shown.



2G **14. I can apply circle theorem 2.** 10A
e.g. Find the value of y in the circle shown.



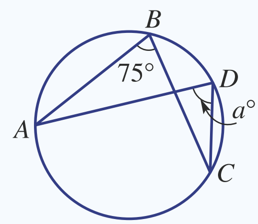
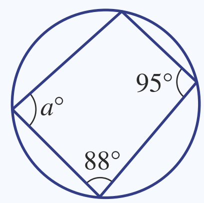
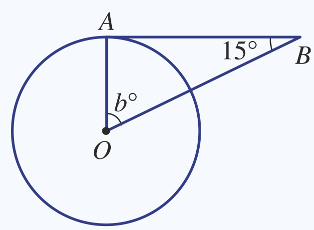
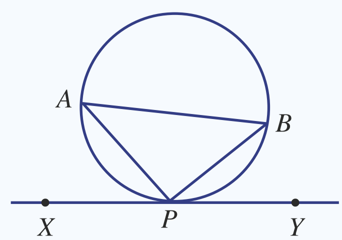
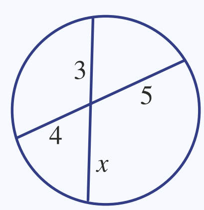
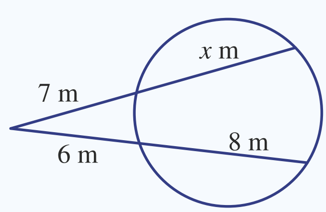
2G **15. I can apply circle theorems with other circle properties.** 10A
e.g. Find the size of $\angle ACB$.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter checklist

			✔
2H	**16. I can apply circle theorem 3.** e.g. Find the value of a.  	10A	
2H	**17. I can apply circle theorem 4.** e.g. Find the value of a.  	10A	
2I	**18. I can find angles involving tangents.** e.g. Find the value of b in this diagram involving a tangent.  	Ext	
2I	**19. I can use the alternate segment theorem.** e.g. In the diagram XY is a tangent to the circle. Find $\angle ABP$ if $\angle BAP = 23°$ and $\angle APX = 65°$.  	Ext	
2J	**20. I can find lengths using intersecting chords.** e.g. Find the value of x in the diagram.  	Ext	
2J	**21. I can find lengths using intersecting secants and tangents.** e.g. Find the value of x in the diagram.  	Ext	

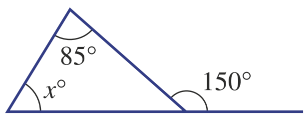
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

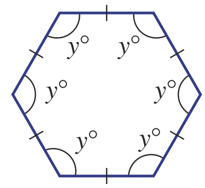
Short-answer questions

2A **1** Determine the value of each pronumeral.

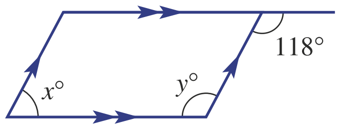
a



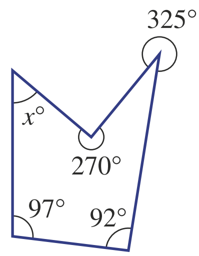
b



c

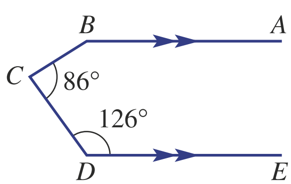


d

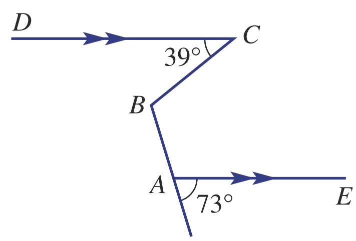


2A **2** Find the value of $\angle ABC$ by adding a third parallel line.

a

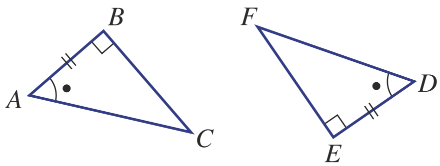


b

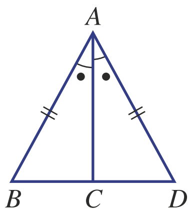


2B **3** Prove that each pair of triangles is congruent, giving reasons.

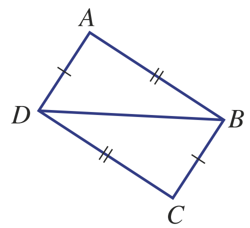
a



b



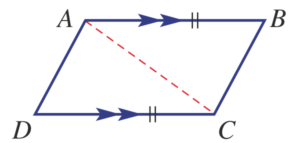
c



2C **4** Complete these steps to prove that if one pair of opposite sides is equal and parallel in a quadrilateral, then it is a parallelogram.

a Prove $\triangle ABC \equiv \triangle CDA$, giving reasons.

b Hence, prove $AD \parallel BC$.



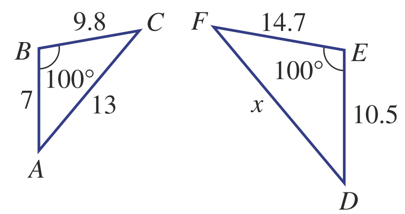
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

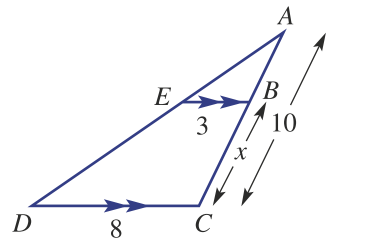
2E

5 In each of the following, identify pairs of similar triangles by proving similarity, giving reasons, and then use this to find the value of x.

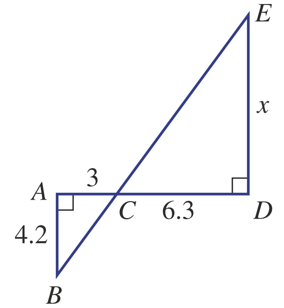
a



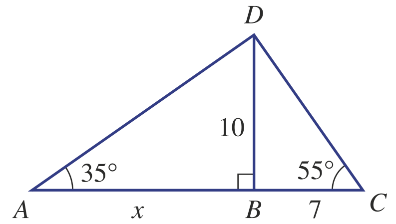
b



c



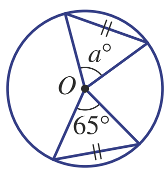
d



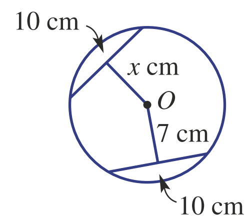
2F 10A

6 Find the value of each pronumeral and state the chord theorem used.

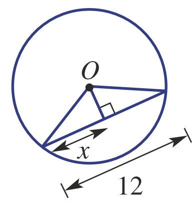
a



b



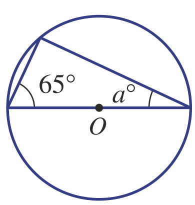
c



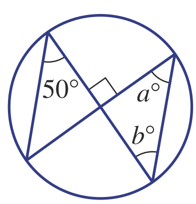
2G/H 10A

7 Use the circle theorems to help find the values of the pronumerals.

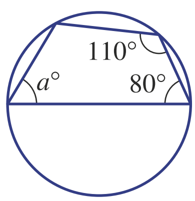
a



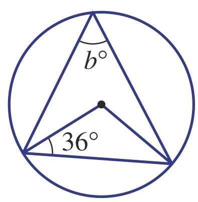
b



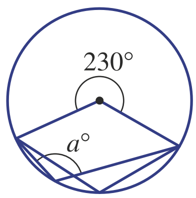
c



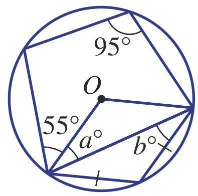
d



e



f



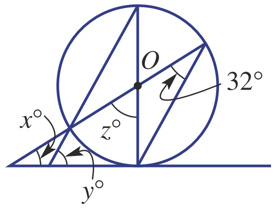
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

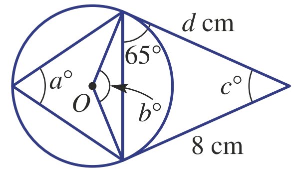
2I Ext

8 Find the value of the pronumerals in these diagrams involving tangents and circles.

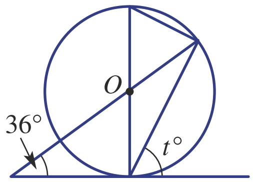
a



b



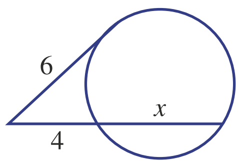
c



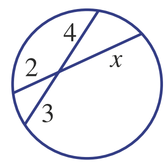
2J Ext

9 Find the value of x in each figure.

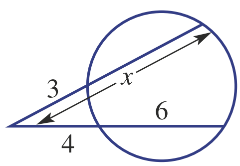
a



b



c

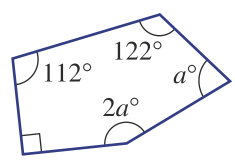


Multiple-choice questions

2A

1 The value of a in the polygon shown is:

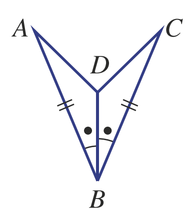
A 46 **B** 64 **C** 72
D 85 **E** 102



2B

2 The test that proves that $\triangle ABD \equiv \triangle CBD$ is:

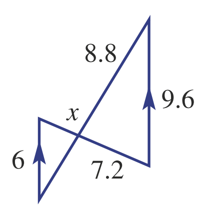
A RHS **B** SAS **C** SSS
D AAA **E** AAS



2D

3 The value of x in the diagram shown is:

A 4.32 **B** 4.5 **C** 3.6
D 5.5 **E** 5.2



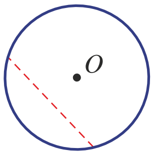
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2F 10A

4 The name given to the dashed line in the circle with centre O is:

A a diameter **B** a minor arc **C** a chord
D a tangent **E** a secant



2F 10A

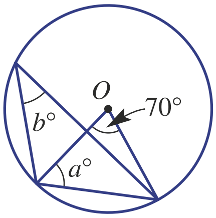
5 A circle of radius 5 cm has a chord 4 cm from the centre of the circle. The length of the chord is:

A 4.5 cm **B** 6 cm **C** 3 cm
D 8 cm **E** 7.2 cm

2G 10A

6 The values of the pronumerals in the diagram are:

A $a = 55, b = 35$
B $a = 30, b = 70$
C $a = 70, b = 35$
D $a = 55, b = 70$
E $a = 40, b = 55$



2H 10A

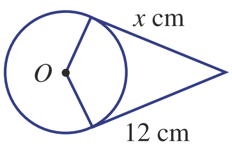
7 A cyclic quadrilateral has one angle measuring 63° and another angle measuring 108°. Another angle in the cyclic quadrilateral is:

A 63° **B** 108° **C** 122° **D** 75° **E** 117°

2I Ext

8 For the circle shown at right with radius 5 cm, the value of x is:

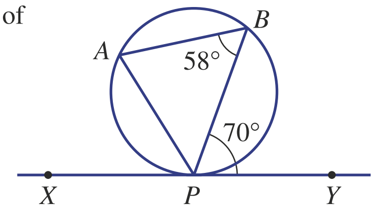
A 13 **B** 10.9 **C** 12
D 17 **E** 15.6



2I Ext

9 By making use of the alternate segment theorem, the value of $\angle APB$ is:

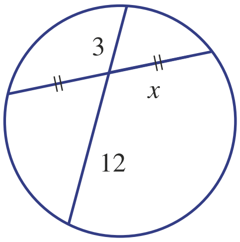
A 50° **B** 45° **C** 10°
D 52° **E** 25°



2J Ext

10 The value of x in the diagram is:

A 7.5 **B** 6 **C** 3.8
D 4 **E** 5

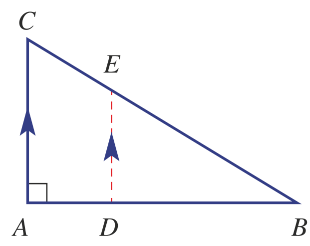


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Extended-response questions

1 The triangular area of land shown is to be divided into two areas such that $AC \parallel DE$. The land is to be divided so that $AC : DE = 3 : 2$.



a Prove that $\triangle ABC \,|||\, \triangle DBE$.

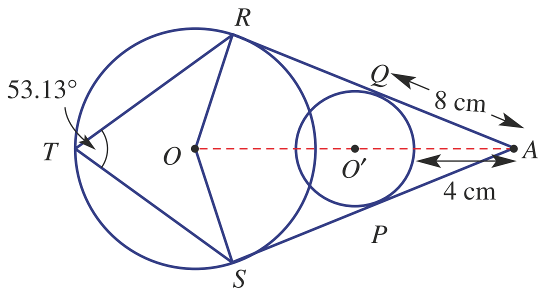
b If $AC = 1.8$ km, find DE.

c If $AD = 1$ km and $DB = x$ km:

i Show that $2(x + 1) = 3x$. ii Solve for x.

d For the given ratio, what percentage of the land area does $\triangle DBE$ occupy? Answer to one decimal place.

10A 2 The diagram below shows two intersecting circles sharing common tangents AR and AS. The distance between the centres O and O' of the two circles is 15 cm. Other measurements are as shown. Given that the two centres O and O' and the point A are in a straight line, complete the following.



a Find the values of these angles.

i $\angle ROS$ ii $\angle RAS$

b Use the rule for intersecting secants and tangents to help find the diameter of the smaller circle.

c Hence, what is the distance from A to O?

d By first finding AR, determine the perimeter of $AROS$. Round to one decimal place.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

CHAPTER

3 Indices and surds

Environmental science

Environmental scientists investigate and measure the growth rates of bacteria and their effects on the waterways of countries around the world. The Yarra River in Melbourne, one of Victoria's most important rivers, has been in the spotlight for the past decade or so with high levels of pollutants and bacteria, such as E. coli, making it unsafe in certain locations for swimming, fishing and even rowing.

Scientists look at the rate at which bacteria doubles. This process follows an exponential growth pattern of $2^0, 2^1, 2^2, 2^3, 2^4, 2^5, \ldots 2^n$ where n represents the number of generations. In a laboratory using favourable conditions, E. coli doubles approximately every 20 minutes. In summer where the temperature along the Yarra River can be over 30°C, the growth of this bacteria and others needs to be monitored. The Environmental Protection Agency (EPA) and Melbourne Water take samples along the waterway and analyse it for levels of bacteria and pollutants.

The decay rate of other pollutants is also examined by investigating their half-life, as this also follows an exponential pattern: $2^0, 2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}, \ldots$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Online resources

A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

In this chapter

Victorian Curriculum

NUMBER AND ALGEBRA
Patterns and algebra

Simplify algebraic products and quotients using index laws (VCMNA330)

Money and financial mathematics

Connect the compound interest formula to repeated applications of simple interest using appropriate digital technologies (VCMNA328)

Real numbers

(10A) Define rational and irrational numbers and perform operations with surds and fractional indices (VCMNA355)

Linear and non-linear relationships

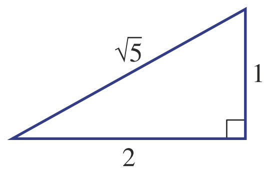
(10A) Solve simple exponential equations (VCMNA360)

3A Irrational numbers including surds 10A

Learning intentions

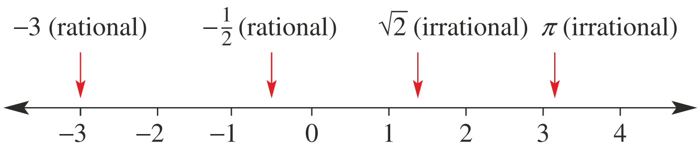
- To know the meaning of the terms rational number, irrational number and surd
- To know how to identify a number as rational or irrational
- To know simple rules related to surds
- To be able to simplify surds using the highest square number factor

You will recall that when using Pythagoras' theorem to find unknown lengths in right-angled triangles, many answers expressed in exact form are surds. The length of the hypotenuse in this triangle, for example, is $\sqrt{5}$, which is a surd.



Many formulas contain numbers that are surds. The formulas for the speed of a rising weather balloon and the speed of its falling measuring device both include the surd $\sqrt{2}$.

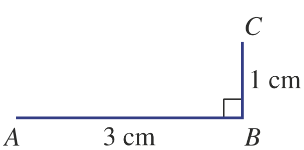
A surd is a number that uses a root sign ($\sqrt{\ }$), sometimes called a radical sign. They are irrational numbers, meaning that they cannot be expressed as a fraction in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Surds, together with other irrational numbers such as pi, and all rational numbers (fractions) make up the entire set of real numbers, which can be illustrated as points on a number line.



LESSON STARTER Constructing surds

Someone asks you: 'How do you construct a line that is $\sqrt{10}$ cm long?'
Use these steps to answer this question.

- First, draw a line segment AB that is 3 cm in length.
- Construct segment BC so that $BC = 1$ cm and $AB \perp BC$. You may wish to use a set square or pair of compasses.



- Now connect point A and point C and measure the length of the segment.
- Use Pythagoras' theorem to check the length of AC.

Use this idea to construct line segments with the following lengths. You may need more than one triangle for parts **d** to **f**.

a $\sqrt{2}$ **b** $\sqrt{17}$ **c** $\sqrt{20}$
d $\sqrt{3}$ **e** $\sqrt{6}$ **f** $\sqrt{22}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

KEY IDEAS

- All **real** numbers can be located as a point on a number line. Real numbers include:
  - **rational numbers** (i.e. numbers that can be expressed as fractions)

    For example: $\frac{3}{7}, -\frac{4}{39}, -3, 1.6, 2.\dot{7}, 0.\overline{19}$

    The decimal representation of a rational number is either a **terminating** or **recurring decimal**.
  - **irrational numbers** (i.e. numbers that cannot be expressed as fractions)

    For example: $\sqrt{3}, -2\sqrt{7}, \sqrt{12}-1, \pi, 2\pi-3$

    The decimal representation of an irrational number is an **infinite non-recurring decimal**.
- **Surds** are irrational numbers that use a root sign ($\sqrt{\ }$).
  - For example: $\sqrt{2}, 5\sqrt{11}, -\sqrt{200}, 1+\sqrt{5}$
  - These numbers are not surds: $\sqrt{4}(=2), \sqrt[3]{125}(=5), -\sqrt[4]{16}(=-2)$.
- The nth root of a number x is written $\sqrt[n]{x}$.
  - If $\sqrt[n]{x}=y$ then $y^n=x$. For example: $\sqrt[5]{32}=2$ since $2^5=32$.
- The following rules apply to surds.
  - $(\sqrt{x})^2=x$ and $\sqrt{x^2}=x$ when $x \geqslant 0$.
  - $\sqrt{xy}=\sqrt{x}\times\sqrt{y}$ when $x \geqslant 0$ and $y \geqslant 0$.
  - $\sqrt{\frac{x}{y}}=\frac{\sqrt{x}}{\sqrt{y}}$ when $x \geqslant 0$ and $y > 0$.
- When a factor of a number is a perfect square we call that factor a square factor. Examples of perfect squares are: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, …
- When simplifying surds, look for square factors of the number under the root sign and then use $\sqrt{a\times b}=\sqrt{a}\times\sqrt{b}$.

BUILDING UNDERSTANDING

1. Choose the correct word(s) from the words given in red to make the sentence true.
   - **a** A number that cannot be expressed as a fraction is a rational/irrational number.
   - **b** A surd is an irrational number that uses a root/square symbol.
   - **c** The decimal representation of a surd is a terminating/recurring/non-recurring decimal.
   - **d** $\sqrt{25}$ is a surd/rational number.
2. State the highest square factor of these numbers. For example, the highest square factor of 45 is 9.

   a 20 **b** 125 **c** 48 **d** 72

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 1 Defining and locating surds

Express each number as a decimal and decide if it is rational or irrational. Then locate all the numbers on the same number line.

a $-\sqrt{3}$ **b** 137% **c** $\frac{3}{7}$

SOLUTION

a $-\sqrt{3} = -1.732050807\ldots$

$-\sqrt{3}$ is irrational.

b $137\% = \frac{137}{100} = 1.37$

137% is rational.

c $\frac{3}{7} = 0.\overline{428571}$

$\frac{3}{7}$ is rational.

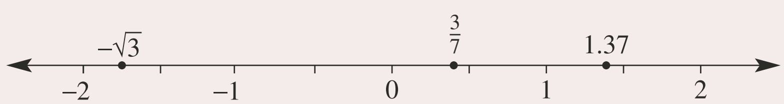
EXPLANATION

Use a calculator to express as a decimal.
The decimal does not terminate and there is no recurring pattern.

137% is a fraction and can be expressed using a terminating decimal.

$\frac{3}{7}$ is an infinitely recurring decimal.

Use the decimal equivalents to locate each number on the real number line.



Now you try

Express each number as a decimal and decide if they are rational or irrational. Then locate all the numbers on the same number line.

a $-\sqrt{5}$ **b** −40% **c** $\frac{2}{7}$

Example 2 Simplifying surds

Simplify the following.

a $\sqrt{32}$ **b** $3\sqrt{200}$ **c** $\frac{5\sqrt{40}}{6}$ **d** $\sqrt{\frac{75}{9}}$

SOLUTION

a
$$\begin{aligned}\sqrt{32} &= \sqrt{16 \times 2}\\ &= \sqrt{16} \times \sqrt{2}\\ &= 4\sqrt{2}\end{aligned}$$

EXPLANATION

When simplifying, choose the highest square factor of 32 (i.e. 16 rather than 4) as there is less work to do to arrive at the same answer.
Compare with
$\sqrt{32} = \sqrt{4 \times 8} = 2\sqrt{8} = 2\sqrt{4 \times 2} = 2 \times 2\sqrt{2} = 4\sqrt{2}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b $3\sqrt{200} = 3\sqrt{100 \times 2}$
$= 3 \times \sqrt{100} \times \sqrt{2}$
$= 3 \times 10 \times \sqrt{2}$
$= 30\sqrt{2}$

Select the apporopriate factors of 200 by finding its highest sqaure factor: 100.
Use $\sqrt{x \times y} = \sqrt{x} \times \sqrt{y}$ and simplify.

c $\dfrac{5\sqrt{40}}{6} = \dfrac{5\sqrt{4 \times 10}}{6}$
$= \dfrac{5 \times \sqrt{4} \times \sqrt{10}}{6}$
$= \dfrac{\cancel{10}^{5}\sqrt{10}}{\cancel{6}^{3}}$
$= \dfrac{5\sqrt{10}}{3}$

Select the appropriate factors of 40. The highest square factor is 4.

Cancel and simplify.

d $\sqrt{\dfrac{75}{9}} = \dfrac{\sqrt{75}}{\sqrt{9}}$
$= \dfrac{\sqrt{25 \times 3}}{\sqrt{9}} = \dfrac{5\sqrt{3}}{3}$

Use $\sqrt{\dfrac{x}{y}} = \dfrac{\sqrt{x}}{\sqrt{y}}$.

Then select the factors of 75 that include a square number and simplify.

Now you try

Simplify the following.

a $\sqrt{20}$ **b** $2\sqrt{300}$ **c** $\dfrac{2\sqrt{27}}{3}$ **d** $\sqrt{\dfrac{125}{16}}$

Example 3 Expressing as a single square root of a positive integer

Express these surds as a square root of a positive integer.

a $2\sqrt{5}$ **b** $7\sqrt{2}$

SOLUTION

a $2\sqrt{5} = \sqrt{4} \times \sqrt{5}$
$= \sqrt{20}$

b $7\sqrt{2} = \sqrt{49} \times \sqrt{2}$
$= \sqrt{98}$

EXPLANATION

Write 2 as $\sqrt{4}$ and then combine the two surds using $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$.

Write 7 as $\sqrt{49}$ and combine.

Now you try

Express these surds as a square root of a positive integer.

a $3\sqrt{2}$ **b** $5\sqrt{3}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 3A

FLUENCY	1, 2–6 (1/2)	2–7(1/3)	2–7(1/4)

Example 1

1 Express each number as a decimal and decide if they are rational or irrational. Then locate all the numbers on the same number line.

a $-\sqrt{2}$ b 150% c $\frac{6}{7}$

Example 1

2 Express each number as a decimal and decide if it is rational or irrational. Then locate all the numbers on the same number line.

a $\sqrt{5}$ b 18% c $\frac{2}{5}$ d -124%
e $1\frac{5}{7}$ f $-\sqrt{2}$ g $2\sqrt{3}$ h π

3 Decide if these numbers are surds.

a $\sqrt{7}$ b $2\sqrt{11}$ c $2\sqrt{25}$ d $-5\sqrt{144}$
e $\frac{3\sqrt{9}}{2}$ f $\frac{-5\sqrt{3}}{2}$ g $1-\sqrt{3}$ h $2\sqrt{1}+\sqrt{4}$

Example 2a

4 Simplify the following surds.

a $\sqrt{12}$ b $\sqrt{45}$ c $\sqrt{24}$ d $\sqrt{48}$
e $\sqrt{75}$ f $\sqrt{500}$ g $\sqrt{98}$ h $\sqrt{90}$
i $\sqrt{128}$ j $\sqrt{360}$ k $\sqrt{162}$ l $\sqrt{80}$

Example 2b,c

5 Simplify the following.

a $2\sqrt{18}$ b $3\sqrt{20}$ c $4\sqrt{48}$ d $2\sqrt{63}$
e $3\sqrt{98}$ f $4\sqrt{125}$ g $\frac{\sqrt{45}}{3}$ h $\frac{\sqrt{28}}{2}$
i $\frac{\sqrt{24}}{4}$ j $\frac{\sqrt{54}}{12}$ k $\frac{\sqrt{80}}{20}$ l $\frac{\sqrt{99}}{18}$
m $\frac{3\sqrt{44}}{2}$ n $\frac{5\sqrt{200}}{25}$ o $\frac{2\sqrt{98}}{7}$ p $\frac{3\sqrt{68}}{21}$
q $\frac{6\sqrt{75}}{20}$ r $\frac{4\sqrt{150}}{5}$ s $\frac{2\sqrt{108}}{18}$ t $\frac{3\sqrt{147}}{14}$

Example 2d

6 Simplify the following.

a $\sqrt{\frac{8}{9}}$ b $\sqrt{\frac{12}{49}}$ c $\sqrt{\frac{18}{25}}$ d $\sqrt{\frac{11}{25}}$
e $\sqrt{\frac{10}{9}}$ f $\sqrt{\frac{12}{144}}$ g $\sqrt{\frac{26}{32}}$ h $\sqrt{\frac{25}{50}}$
i $\sqrt{\frac{15}{27}}$ j $\sqrt{\frac{27}{4}}$ k $\sqrt{\frac{45}{72}}$ l $\sqrt{\frac{56}{76}}$

Example 3

7 Express these surds as a square root of a positive integer.

a $2\sqrt{3}$ b $4\sqrt{2}$ c $5\sqrt{2}$ d $3\sqrt{3}$
e $3\sqrt{5}$ f $6\sqrt{3}$ g $8\sqrt{2}$ h $10\sqrt{7}$
i $9\sqrt{10}$ j $5\sqrt{5}$ k $7\sqrt{5}$ l $11\sqrt{3}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

PROBLEM-SOLVING	8, 9	8, 9, 11(½)	10, 11(½)

8 Simplify by searching for the highest square factor.

a $\sqrt{675}$
b $\sqrt{1183}$
c $\sqrt{1805}$
d $\sqrt{2883}$

9 Determine the exact side length, in simplest form, of a square with the given area.

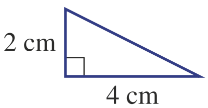
a 32 m^2 **b** 120 cm^2 **c** 240 mm^2

10 Determine the exact radius and diameter of a circle, in simplest form, with the given area.

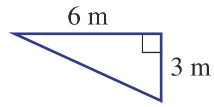
a $24\pi\text{ cm}^2$ **b** $54\pi\text{ m}^2$ **c** $128\pi\text{ m}^2$

11 Use Pythagoras' theorem to find the unknown length in these triangles, in simplest form.

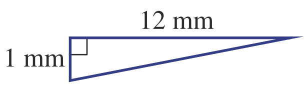
a



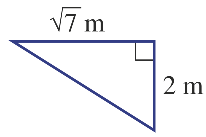
b



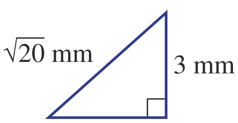
c



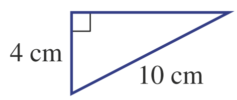
d



e



f



REASONING	12	12, 13	13, 14

12 Ricky uses the following working to simplify $\sqrt{72}$. Show how Ricky could have simplified $\sqrt{72}$ using fewer steps.

$$\begin{aligned}\sqrt{72} &= \sqrt{9 \times 8}\\ &= 3\sqrt{8}\\ &= 3\sqrt{4 \times 2}\\ &= 3 \times 2 \times \sqrt{2}\\ &= 6\sqrt{2}\end{aligned}$$

13 **a** List all the factors of 450 that are perfect squares.
b Now simplify $\sqrt{450}$ using the highest of these factors.

14 Use Pythagoras' theorem to construct a line segment with the given lengths. You can use only a ruler and a set square or compasses. Do not use a calculator.

a $\sqrt{10}$ cm
b $\sqrt{29}$ cm
c $\sqrt{6}$ cm
d $\sqrt{22}$ cm

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

ENRICHMENT: Proving that $\sqrt{2}$ is irrational	–	–	15

15 We will prove that $\sqrt{2}$ is irrational by the method called 'proof by contradiction'. Your job is to follow and understand the proof, then copy it out and try explaining it to a friend or teacher.

a Before we start, we first need to show that if a perfect square a^2 is even then a is even. We do this by showing that if a is even then a^2 is even and if a is odd then a^2 is odd.

If a is even then $a = 2k$, where k is an integer.

$$\begin{aligned} \text{So } a^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2 \times 2k^2, \text{ which must be even.} \end{aligned}$$

If a is odd then $a = 2k + 1$, where k is an integer.

$$\begin{aligned} \text{So } a^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2 \times (2k^2 + 2k) + 1, \text{ which must be odd.} \end{aligned}$$

$\therefore$ If a^2 is even then a is even.

b Now, to prove $\sqrt{2}$ is irrational let's suppose that $\sqrt{2}$ is instead rational and can be written in the form $\frac{a}{b}$ in simplest form, where a and b are integers ($b \neq 0$) and at least one of a or b is odd.

$$\therefore \sqrt{2} = \frac{a}{b}$$

$$\text{So } 2 = \frac{a^2}{b^2} \text{ (squaring both sides)}$$

$$a^2 = 2b^2$$

$\therefore a^2$ is even and, from part **a** above, a must be even.

If a is even, then $a = 2k$, where k is an integer.

$$\begin{aligned} \therefore \text{ If } a^2 &= 2b^2 \\ \text{Then } (2k)^2 &= 2b^2 \\ 4k^2 &= 2b^2 \\ 2k^2 &= b^2 \end{aligned}$$

$\therefore b^2$ is even and therefore b is even.

This is a contradiction because at least one of a or b must be odd. (Recall that $\frac{a}{b}$ in simplest form will have at least one of a or b being odd.) Therefore, the assumption that $\sqrt{2}$ can be written in the form $\frac{a}{b}$ must be incorrect and so $\sqrt{2}$ is irrational.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

3B Adding and subtracting surds 10A

Learning intentions

- To understand that only like surds can be combined under addition and subtraction
- To know how to add and subtract like surds
- To know that it is helpful to simplify all surds before determining if they can be added or subtracted

We can apply our knowledge of like terms in algebra to help simplify expressions involving the addition and subtraction of surds. Recall that $7x$ and $3x$ are like terms, so $7x + 3x = 10x$. The pronumeral x represents any number. When $x = 5$ then $7 \times 5 + 3 \times 5 = 10 \times 5$, and when $x = \sqrt{2}$ then $7\sqrt{2} + 3\sqrt{2} = 10\sqrt{2}$. Multiples of the same surd are called 'like surds' and can be collected (i.e. counted) in the same way as we collect like terms in algebra.

To design the Hearst Tower in New York, architects solved many equations, such a linear, quadratic and trigonometrical. Where possible, architects use surds in mathematical solutions to achieve precise results.

LESSON STARTER Can $3\sqrt{2} + \sqrt{8}$ be simplified?

To answer this question, first discuss these points.

- Are $3\sqrt{2}$ and $\sqrt{8}$ like surds?
- How can $\sqrt{8}$ be simplified?
- Now decide whether $3\sqrt{2} + \sqrt{8}$ can be simplified. Discuss why $3\sqrt{2} - \sqrt{7}$ cannot be simplified.

KEY IDEAS

- **Like surds** are multiples of the same surd.
  For example: $\sqrt{3}, -5\sqrt{3}, \sqrt{12} = 2\sqrt{3}, 2\sqrt{75} = 10\sqrt{3}$
- Like surds can be added and subtracted.
- Simplify all surds before attempting to add or subtract them.

BUILDING UNDERSTANDING

1 Decide if the following pairs of numbers are like surds.
a $\sqrt{3}, 2\sqrt{3}$
b $5, \sqrt{5}$
c $2\sqrt{2}, 2$
d $4\sqrt{6}, \sqrt{6}$
e $2\sqrt{3}, 5\sqrt{3}$
f $3\sqrt{7}, 3\sqrt{5}$
g $-2\sqrt{5}, 3\sqrt{5}$
h $-\sqrt{7}, -2\sqrt{7}$

2 Recall your basic skills in algebra to simplify these expressions.
a $11x - 5x$
b $2x - 7x$
c $-4a + 21a$
d $4t - 5t + 2t$

3 a Simplify the surd $\sqrt{48}$.
b Hence, simplify the following.
i $\sqrt{3} + \sqrt{48}$
ii $\sqrt{48} - 7\sqrt{3}$
iii $5\sqrt{48} - 3\sqrt{3}$

Example 4 Adding and subtracting surds

Simplify the following.
a $2\sqrt{3} + 4\sqrt{3}$
b $4\sqrt{6} + 3\sqrt{2} - 3\sqrt{6} + 2\sqrt{2}$

SOLUTION

a $2\sqrt{3} + 4\sqrt{3} = 6\sqrt{3}$

EXPLANATION

Collect the like surds by adding the coefficients: $2 + 4 = 6$.

SOLUTION

b $4\sqrt{6} + 3\sqrt{2} - 3\sqrt{6} + 2\sqrt{2} = \sqrt{6} + 5\sqrt{2}$

EXPLANATION

Collect like surds involving $\sqrt{6}$:
$4\sqrt{6} - 3\sqrt{6} = 1\sqrt{6} = \sqrt{6}$
Then collect those terms with $\sqrt{2}$.

Now you try

Simplify the following.
a $2\sqrt{5} + 3\sqrt{5}$
b $3\sqrt{7} + 2\sqrt{3} - 2\sqrt{7} + 5\sqrt{3}$

Example 5 Simplifying surds to add or subtract

Simplify these surds.
a $5\sqrt{2} - \sqrt{8}$
b $2\sqrt{5} - 3\sqrt{20} + 6\sqrt{45}$

SOLUTION

a
$$\begin{aligned} 5\sqrt{2} - \sqrt{8} &= 5\sqrt{2} - \sqrt{4 \times 2} \\ &= 5\sqrt{2} - 2\sqrt{2} \\ &= 3\sqrt{2} \end{aligned}$$

EXPLANATION

First, look to simplify surds: $\sqrt{8}$ has a highest square factor of 4 and can be simplified to $2\sqrt{2}$.
Then subtract like surds.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b $2\sqrt{5} - 3\sqrt{20} + 6\sqrt{45} = 2\sqrt{5} - 3\sqrt{4 \times 5} + 6\sqrt{9 \times 5}$
$= 2\sqrt{5} - 6\sqrt{5} + 18\sqrt{5}$
$= 14\sqrt{5}$

Simplify the surds and then collect like surds. Note that $3\sqrt{4 \times 5} = 3 \times \sqrt{4} \times \sqrt{5} = 6\sqrt{5}$.

Now you try

Simplify these surds.

a $7\sqrt{2} - \sqrt{8}$

b $2\sqrt{3} - 2\sqrt{27} + 3\sqrt{12}$

Exercise 3B

FLUENCY	1, 2–4(1/2)	2–4(1/2)	2–4(1/3)

1 Simplify the following.

Example 4a
a **i** $3\sqrt{7} + 2\sqrt{7}$ **ii** $7\sqrt{3} + 2\sqrt{3}$

Example 4b
b **i** $7\sqrt{5} + 2\sqrt{3} - 5\sqrt{5} + 3\sqrt{3}$ **ii** $4\sqrt{5} + 6\sqrt{11} - 2\sqrt{5} + 2\sqrt{11}$

Example 4a
2 Simplify the following.

a $2\sqrt{5} + 4\sqrt{5}$
b $5\sqrt{3} - 2\sqrt{3}$
c $7\sqrt{2} - 3\sqrt{2}$
d $8\sqrt{2} - 5\sqrt{2}$
e $7\sqrt{5} + 4\sqrt{5}$
f $6\sqrt{3} - 5\sqrt{3}$
g $4\sqrt{10} + 3\sqrt{10} - \sqrt{10}$
h $6\sqrt{2} - 4\sqrt{2} + 3\sqrt{2}$
i $\sqrt{21} - 5\sqrt{21} + 2\sqrt{21}$
j $3\sqrt{11} - 8\sqrt{11} - \sqrt{11}$
k $-2\sqrt{13} + 5\sqrt{13} - 4\sqrt{13}$
l $10\sqrt{30} - 15\sqrt{30} - 2\sqrt{30}$

Example 4b
3 Simplify the following.

a $2\sqrt{3} + 3\sqrt{2} - \sqrt{3} + 2\sqrt{2}$
b $5\sqrt{6} + 4\sqrt{11} - 2\sqrt{6} + 3\sqrt{11}$
c $3\sqrt{5} - 4\sqrt{2} + \sqrt{5} - 3\sqrt{2}$
d $5\sqrt{2} + 2\sqrt{5} - 7\sqrt{2} - \sqrt{5}$
e $2\sqrt{3} + 2\sqrt{7} + 2\sqrt{3} - 2\sqrt{7}$
f $5\sqrt{11} + 3\sqrt{6} - 3\sqrt{6} - 5\sqrt{11}$
g $2\sqrt{2} - 4\sqrt{10} - 5\sqrt{2} + \sqrt{10}$
h $-4\sqrt{5} - 2\sqrt{15} + 5\sqrt{15} + 2\sqrt{5}$

Example 5a
4 Simplify the following.

a $\sqrt{8} - \sqrt{2}$
b $\sqrt{8} + 3\sqrt{2}$
c $\sqrt{27} + \sqrt{3}$
d $\sqrt{20} - \sqrt{5}$
e $4\sqrt{18} - 5\sqrt{2}$
f $2\sqrt{75} + 2\sqrt{3}$
g $3\sqrt{44} + 2\sqrt{11}$
h $3\sqrt{8} - \sqrt{18}$
i $\sqrt{24} + \sqrt{54}$
j $2\sqrt{125} - 3\sqrt{45}$
k $3\sqrt{72} + 2\sqrt{98}$
l $3\sqrt{800} - 4\sqrt{200}$

PROBLEM-SOLVING	5(1/2)	5(1/2), 7(1/2)	5–7(1/3)

Example 5b
5 Simplify the following.

a $\sqrt{2} + \sqrt{50} + \sqrt{98}$
b $\sqrt{6} - 2\sqrt{24} + 3\sqrt{96}$
c $5\sqrt{7} + 2\sqrt{5} - 3\sqrt{28}$
d $2\sqrt{80} - \sqrt{45} + 2\sqrt{63}$
e $\sqrt{150} - \sqrt{96} - \sqrt{162} + \sqrt{72}$
f $\sqrt{12} + \sqrt{125} - \sqrt{50} + \sqrt{180}$
g $7\sqrt{3} - 2\sqrt{8} + \sqrt{12} + 3\sqrt{8}$
h $\sqrt{36} - \sqrt{108} + \sqrt{25} - 3\sqrt{3}$
i $3\sqrt{49} + 2\sqrt{288} - \sqrt{144} - 2\sqrt{18}$
j $2\sqrt{200} + 3\sqrt{125} + \sqrt{32} - 3\sqrt{242}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

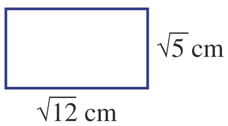
6 Simplify these surds that involve fractions. Remember to use the LCD (lowest common denominator).

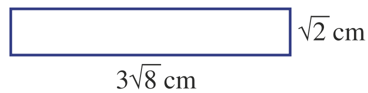
a $\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3}$ b $\frac{\sqrt{5}}{4} + \frac{\sqrt{5}}{3}$ c $\frac{\sqrt{2}}{5} - \frac{\sqrt{2}}{6}$

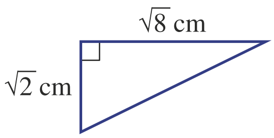
d $\frac{\sqrt{7}}{4} - \frac{\sqrt{7}}{12}$ e $\frac{2\sqrt{2}}{5} - \frac{\sqrt{2}}{2}$ f $\frac{3\sqrt{3}}{7} + \frac{\sqrt{3}}{2}$

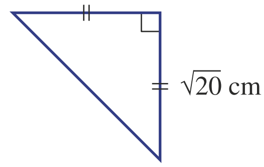
g $\frac{7\sqrt{5}}{6} - \frac{4\sqrt{5}}{9}$ h $\frac{3\sqrt{3}}{10} - \frac{8\sqrt{3}}{15}$ i $\frac{-5\sqrt{10}}{6} + \frac{3\sqrt{10}}{8}$

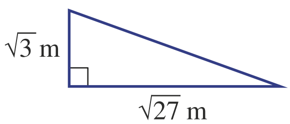
7 Find the perimeter of these rectangles and triangles, in simplest form.

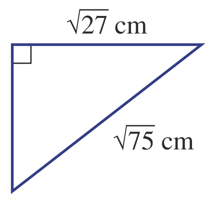
a


b


c


d


e


f


REASONING	8	8, 9(1/2)	9(1/2), 10

8 a Explain why $\sqrt{5}$ and $\sqrt{20}$ can be thought of as like surds.

b Explain why $3\sqrt{72}$ and $\sqrt{338}$ can be thought of as like surds.

9 Prove that each of the following simplifies to zero by showing all steps.

a $5\sqrt{3} - \sqrt{108} + \sqrt{3}$ b $\sqrt{6} + \sqrt{24} - 3\sqrt{6}$

c $6\sqrt{2} - 2\sqrt{32} + 2\sqrt{2}$ d $\sqrt{8} - \sqrt{18} + \sqrt{2}$

e $2\sqrt{20} - 7\sqrt{5} + \sqrt{45}$ f $3\sqrt{2} - 2\sqrt{27} - \sqrt{50} + 6\sqrt{3} + \sqrt{8}$

10 Prove that the surds in these expressions cannot be added or subtracted.

a $3\sqrt{12} - \sqrt{18}$ b $4\sqrt{8} + \sqrt{20}$ c $\sqrt{50} - 2\sqrt{45}$

d $5\sqrt{40} + 2\sqrt{75}$ e $2\sqrt{200} + 3\sqrt{300}$ f $\sqrt{80} - 2\sqrt{54}$

ENRICHMENT: Simplifying both surds and fractions	–	–	11(1/2)

11 To simplify the following, you will need to simplify surds and combine using a common denominator.

a $\frac{\sqrt{8}}{3} - \frac{\sqrt{2}}{5}$ b $\frac{\sqrt{12}}{4} + \frac{\sqrt{3}}{6}$ c $\frac{3\sqrt{5}}{4} - \frac{\sqrt{20}}{3}$ d $\frac{\sqrt{98}}{4} - \frac{5\sqrt{2}}{2}$

e $\frac{2\sqrt{75}}{5} - \frac{3\sqrt{3}}{2}$ f $\frac{\sqrt{63}}{9} - \frac{4\sqrt{7}}{5}$ g $\frac{2\sqrt{18}}{3} - \frac{\sqrt{72}}{2}$ h $\frac{\sqrt{54}}{4} + \frac{\sqrt{24}}{7}$

i $\frac{\sqrt{27}}{5} - \frac{\sqrt{108}}{10}$ j $\frac{5\sqrt{48}}{6} + \frac{2\sqrt{147}}{3}$ k $\frac{2\sqrt{96}}{5} - \frac{\sqrt{600}}{7}$ l $\frac{3\sqrt{125}}{14} - \frac{2\sqrt{80}}{21}$

ISBN 978-1-108-77290-7

3C Multiplying and dividing surds 10A

Learning intentions

- To know how to multiply and divide surds
- To understand that, by definition, $\sqrt{x} \times \sqrt{x}$ is equal to x and that this can be helpful in simplifying multiplications
- To be able to apply the distributive law to brackets involving surds

When simplifying surds such as $\sqrt{18}$, we write $\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$, where we use the fact that $\sqrt{xy} = \sqrt{x} \times \sqrt{y}$. This can be used in reverse to simplify the product of two surds. A similar process is used for division.

A surd represents an accurate value until approximated with a decimal. Surveyor training includes solving problems where the trigonometry ratios are expressed as surds because these are exact values and give accurate results.

LESSON STARTER Exploring products and quotients

When adding and subtracting surds we can combine like surds only. Do you think this is true for multiplying and dividing surds?

- Use a calculator to find a decimal approximation for $\sqrt{5} \times \sqrt{3}$ and for $\sqrt{15}$.
- Use a calculator to find a decimal approximation for $2\sqrt{10} \div \sqrt{5}$ and for $2\sqrt{2}$.
- What do you notice about the results from above? Try other pairs of surds to see if your observations are consistent.

KEY IDEAS

- When multiplying surds, use the following result.
  - $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$
  - More generally: $a\sqrt{x} \times b\sqrt{y} = ab\sqrt{xy}$
- When dividing surds, use the following result.
  - $\dfrac{\sqrt{x}}{\sqrt{y}} = \sqrt{\dfrac{x}{y}}$
  - More generally: $\dfrac{a\sqrt{x}}{b\sqrt{y}} = \dfrac{a}{b}\sqrt{\dfrac{x}{y}}$
- Use the distributive law to expand brackets.
  - $a(b + c) = ab + ac$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

BUILDING UNDERSTANDING

1 State the missing parts.

a $\sqrt{15} \div \sqrt{3} = \sqrt{\dfrac{15}{\_\_\_}}$
$= \sqrt{\_\_\_}$

b $\sqrt{42} \div \sqrt{7} = \sqrt{\dfrac{42}{\_\_\_}}$
$= \sqrt{\_\_\_}$

c $\sqrt{6} \times \sqrt{5} = \sqrt{6 \times \_\_\_}$
$= \sqrt{\_\_\_}$

d $\sqrt{11} \times \sqrt{2} = \sqrt{11 \times \_\_\_}$
$= \sqrt{\_\_\_}$

2 Use the definition of squares and square roots to simplify the following.

a $\sqrt{6} \times \sqrt{6}$ b $\sqrt{7^2}$ c $(\sqrt{5})^2$

3 Expand the brackets.

a $2(x + 3)$ b $5(2x - 1)$ c $6(5 - 4x)$

Example 6 Simplifying a product of two surds

Simplify the following.

a $\sqrt{2} \times \sqrt{3}$ b $2\sqrt{3} \times 3\sqrt{15}$ c $(2\sqrt{5})^2$

SOLUTION

a $\sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3}$
$= \sqrt{6}$

EXPLANATION

Use $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$.

b $2\sqrt{3} \times 3\sqrt{15} = 2 \times 3 \times \sqrt{3 \times 15}$
$= 6\sqrt{45}$
$= 6\sqrt{9 \times 5}$
$= 6 \times \sqrt{9} \times \sqrt{5}$
$= 18\sqrt{5}$

Use $a\sqrt{x} \times b\sqrt{y} = ab\sqrt{xy}$.
Then simplify the surd $\sqrt{45}$, which has a highest square factor of 9, using $\sqrt{9} = 3$.
Alternatively, using $\sqrt{15} = \sqrt{3} \times \sqrt{5}$:
$2\sqrt{3} \times 3\sqrt{15} = 2 \times 3 \times \sqrt{3} \times \sqrt{3} \times \sqrt{5}$
$= 2 \times 3 \times 3 \times \sqrt{5}$
$= 18\sqrt{5}$

c $(2\sqrt{5})^2 = 2\sqrt{5} \times 2\sqrt{5}$
$= 4 \times 5$
$= 20$

Recall that $a^2 = a \times a$.
Combine the whole numbers and surd components by multiplying $2 \times 2 = 4$ and $\sqrt{5} \times \sqrt{5} = 5$.

Now you try

Simplify the following.

a $\sqrt{5} \times \sqrt{3}$ b $3\sqrt{2} \times 4\sqrt{6}$ c $(3\sqrt{7})^2$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 7 Simplifying surds using division

Simplify these surds.

a $-\sqrt{10} \div \sqrt{2}$

b $\dfrac{12\sqrt{18}}{3\sqrt{3}}$

SOLUTION

a $-\sqrt{10} \div \sqrt{2} = -\sqrt{\dfrac{10}{2}}$
$= -\sqrt{5}$

Use $\sqrt{x} \div \sqrt{y} = \sqrt{\dfrac{x}{y}}$.

b $\dfrac{12\sqrt{18}}{3\sqrt{3}} = \dfrac{12}{3}\sqrt{\dfrac{18}{3}}$
$= 4\sqrt{6}$

Use $\dfrac{a\sqrt{x}}{b\sqrt{y}} = \dfrac{a}{b}\sqrt{\dfrac{x}{y}}$.

Now you try

Simplify these surds.

a $-\sqrt{15} \div \sqrt{5}$

b $\dfrac{14\sqrt{22}}{7\sqrt{11}}$

Example 8 Using the distributive law

Use the distributive law to expand the following and then simplify the surds where necessary.

a $\sqrt{3}(3\sqrt{5} - \sqrt{6})$

b $3\sqrt{6}(2\sqrt{10} - 4\sqrt{6})$

SOLUTION

a $\sqrt{3}(3\sqrt{5} - \sqrt{6}) = 3\sqrt{15} - \sqrt{18}$
$= 3\sqrt{15} - \sqrt{9 \times 2}$
$= 3\sqrt{15} - 3\sqrt{2}$

Expand the brackets $\sqrt{3} \times 3\sqrt{5} = 3\sqrt{15}$ and $\sqrt{3} \times \sqrt{6} = \sqrt{18}$.
Then simplify $\sqrt{18}$.

b $3\sqrt{6}(2\sqrt{10} - 4\sqrt{6}) = 6\sqrt{60} - 12 \times 6$
$= 6\sqrt{4 \times 15} - 72$
$= 12\sqrt{15} - 72$

Expand the brackets and simplify the surds.
Recall that $\sqrt{6} \times \sqrt{6} = 6$ and $\sqrt{4 \times 15} = 2\sqrt{15}$.

Now you try

Use the distributive law to expand the following and then simplify the surds where necessary.

a $\sqrt{2}(5\sqrt{3} - \sqrt{7})$

b $5\sqrt{3}(2\sqrt{6} - 3\sqrt{3})$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 3C

FLUENCY	1, 2–8(1/2)	2–8(1/2)	2–8(1/3)

1 Simplify the following.

Example 6a
a i $\sqrt{7}\times\sqrt{6}$ ii $\sqrt{11}\times\sqrt{5}$

Example 6b
b i $2\sqrt{3}\times 5\sqrt{7}$ ii $3\sqrt{7}\times 2\sqrt{5}$

Example 6c
c i $(2\sqrt{3})^2$ ii $(3\sqrt{5})^2$

Example 6a
2 Simplify the following.
a $\sqrt{3}\times\sqrt{5}$ b $\sqrt{7}\times\sqrt{3}$ c $\sqrt{2}\times\sqrt{13}$
d $\sqrt{5}\times\sqrt{7}$ e $\sqrt{2}\times(-\sqrt{15})$ f $-\sqrt{6}\times\sqrt{5}$
g $-\sqrt{6}\times(-\sqrt{11})$ h $-\sqrt{3}\times(-\sqrt{2})$ i $\sqrt{10}\times\sqrt{7}$

Example 7a
3 Simplify the following.
a $\sqrt{20}\div\sqrt{2}$ b $\sqrt{18}\div\sqrt{3}$ c $\sqrt{33}\div(-\sqrt{11})$
d $-\sqrt{30}\div(-\sqrt{6})$ e $\dfrac{\sqrt{15}}{\sqrt{5}}$ f $\dfrac{\sqrt{30}}{\sqrt{3}}$
g $\dfrac{\sqrt{40}}{\sqrt{8}}$ h $-\dfrac{\sqrt{26}}{\sqrt{2}}$ i $-\dfrac{\sqrt{50}}{\sqrt{10}}$

4 Simplify the following, making use of $\sqrt{x}\times\sqrt{x}=x, x\geqslant 0$, in each part.
a $\sqrt{3}\times\sqrt{3}$ b $\sqrt{5}\times\sqrt{5}$ c $\sqrt{9}\times\sqrt{9}$
d $\sqrt{14}\times\sqrt{7}$ e $\sqrt{2}\times\sqrt{22}$ f $\sqrt{3}\times\sqrt{18}$
g $\sqrt{10}\times\sqrt{5}$ h $\sqrt{12}\times\sqrt{8}$ i $\sqrt{5}\times\sqrt{20}$

Example 6b
5 Simplify the following.
a $2\sqrt{5}\times\sqrt{15}$ b $3\sqrt{7}\times\sqrt{14}$ c $4\sqrt{6}\times\sqrt{21}$
d $-5\sqrt{10}\times\sqrt{30}$ e $3\sqrt{6}\times(-\sqrt{18})$ f $5\sqrt{3}\times\sqrt{15}$
g $3\sqrt{14}\times 2\sqrt{21}$ h $-4\sqrt{6}\times 5\sqrt{15}$ i $2\sqrt{10}\times(-2\sqrt{25})$
j $-2\sqrt{7}\times(-3\sqrt{14})$ k $4\sqrt{15}\times 2\sqrt{18}$ l $9\sqrt{12}\times 4\sqrt{21}$

Example 6c
6 Simplify the following.
a $(\sqrt{11})^2$ b $(\sqrt{13})^2$ c $(2\sqrt{3})^2$
d $(5\sqrt{5})^2$ e $(7\sqrt{3})^2$ f $(9\sqrt{2})^2$

Example 7b
7 Simplify the following.
a $\dfrac{6\sqrt{14}}{3\sqrt{7}}$ b $\dfrac{15\sqrt{12}}{5\sqrt{2}}$ c $\dfrac{4\sqrt{30}}{8\sqrt{6}}$
d $-\dfrac{8\sqrt{2}}{2\sqrt{26}}$ e $-\dfrac{3\sqrt{3}}{9\sqrt{21}}$ f $\dfrac{12\sqrt{70}}{18\sqrt{14}}$

Example 8
8 Use the distributive law to expand the following and then simplify the surds where necessary.
a $\sqrt{3}(\sqrt{2}+\sqrt{5})$ b $\sqrt{2}(\sqrt{7}-\sqrt{5})$ c $-\sqrt{5}(\sqrt{11}+\sqrt{13})$
d $-2\sqrt{3}(\sqrt{5}+\sqrt{7})$ e $3\sqrt{2}(2\sqrt{13}-\sqrt{11})$ f $4\sqrt{5}(\sqrt{5}-\sqrt{10})$
g $5\sqrt{3}(2\sqrt{6}+3\sqrt{10})$ h $-2\sqrt{6}(3\sqrt{2}-2\sqrt{3})$ i $3\sqrt{7}(2\sqrt{7}+3\sqrt{14})$
j $6\sqrt{5}(3\sqrt{15}-2\sqrt{8})$ k $-2\sqrt{8}(2\sqrt{2}-3\sqrt{20})$ l $2\sqrt{3}(7\sqrt{6}+5\sqrt{3})$

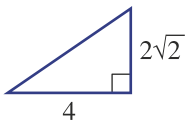
ISBN 978-1-108-77290-7

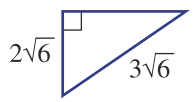
PROBLEM-SOLVING	10	9(½), 10	9(½), 10, 11

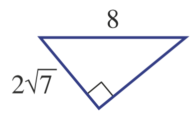
9 Simplify the following.

a $(2\sqrt{7})^2$
b $(-3\sqrt{2})^2$
c $-(5\sqrt{3})^2$
d $\sqrt{2}(3 - \sqrt{3}) - \sqrt{8}$
e $\sqrt{8}(\sqrt{6} + \sqrt{2}) - \sqrt{3}$
f $\sqrt{5}(\sqrt{2} + 1) - \sqrt{40}$
g $\sqrt{44} - 2(\sqrt{11} - 1)$
h $\sqrt{24} - 2\sqrt{2}(\sqrt{3} - 4)$
i $2\sqrt{3}(\sqrt{6} - \sqrt{3}) - \sqrt{50}$

10 Determine the unknown side of the following right-angled triangles. Recall that $a^2 + b^2 = c^2$ for right-angled triangles.

a


b


c


11 **a** The perimeter of a square is $2\sqrt{3}$ cm. Find its area.
b Find the length of a diagonal of a square that has an area of 12 cm^2.

REASONING	12	12, 13	13, 14

12 Use $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$ to prove the following results.

a $\sqrt{6} \times \sqrt{6} = 6$
b $-\sqrt{8} \times \sqrt{8} = -8$
c $-\sqrt{5} \times (-\sqrt{5}) = 5$

13 $\sqrt{8} \times \sqrt{27}$ could be simplified in two ways, as shown.

Method A

$$\begin{aligned}\sqrt{8} \times \sqrt{27} &= \sqrt{4 \times 2} \times \sqrt{9 \times 3}\\ &= 2\sqrt{2} \times 3\sqrt{3}\\ &= 2 \times 3 \times \sqrt{2 \times 3}\\ &= 6\sqrt{6}\end{aligned}$$

Method B

$$\begin{aligned}\sqrt{8} \times \sqrt{27} &= \sqrt{8 \times 27}\\ &= \sqrt{216}\\ &= \sqrt{36 \times 6}\\ &= 6\sqrt{6}\end{aligned}$$

a Describe the first step in method A.
b Why is it useful to simplify surds before multiplying, as in method A?
c Multiply by first simplifying each surd.

i $\sqrt{18} \times \sqrt{27}$
ii $\sqrt{24} \times \sqrt{20}$
iii $\sqrt{50} \times \sqrt{45}$
iv $\sqrt{54} \times \sqrt{75}$
v $2\sqrt{18} \times \sqrt{48}$
vi $\sqrt{108} \times (-2\sqrt{125})$
vii $-4\sqrt{27} \times (-\sqrt{28})$
viii $\sqrt{98} \times \sqrt{300}$
ix $2\sqrt{72} \times 3\sqrt{80}$

14 $\dfrac{\sqrt{12}}{\sqrt{3}}$ could be simplified in two ways.

Method A

$$\begin{aligned}\frac{\sqrt{12}}{\sqrt{3}} &= \sqrt{\frac{12}{3}}\\ &= \sqrt{4}\\ &= 2\end{aligned}$$

Method B

$$\begin{aligned}\frac{\sqrt{12}}{\sqrt{3}} &= \frac{2\cancel{\sqrt{3}}^{1}}{\cancel{\sqrt{3}}^{1}}\\ &= 2\end{aligned}$$

Choose a method to simplify these surds. Compare your method with that of another student.

a $\dfrac{\sqrt{27}}{\sqrt{3}}$
b $\dfrac{\sqrt{20}}{\sqrt{5}}$
c $-\dfrac{\sqrt{162}}{\sqrt{2}}$
d $-\dfrac{2\sqrt{2}}{5\sqrt{8}}$
e $\dfrac{2\sqrt{45}}{15\sqrt{5}}$
f $\dfrac{5\sqrt{27}}{\sqrt{75}}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

ENRICHMENT: Higher powers

	–	–	15–16(1/2)

15 Look at this example before simplifying the following.

$$\begin{aligned}(2\sqrt{3})^3 &= 2^3(\sqrt{3})^3\\ &= 2\times 2\times 2\times \sqrt{3}\times\sqrt{3}\times\sqrt{3}\\ &= 8\times 3\times\sqrt{3}\\ &= 24\sqrt{3}\end{aligned}$$

a $(3\sqrt{2})^3$
b $(5\sqrt{3})^3$
c $2(3\sqrt{3})^3$
d $(\sqrt{5})^4$
e $(-\sqrt{3})^4$
f $(2\sqrt{2})^5$
g $-3(2\sqrt{5})^3$
h $2(-3\sqrt{2})^3$
i $5(2\sqrt{3})^4$
j $\dfrac{(2\sqrt{7})^3}{4}$
k $\dfrac{(3\sqrt{2})^3}{4}$
l $\dfrac{(3\sqrt{2})^4}{4}$
m $\dfrac{(5\sqrt{2})^2}{4}\times\dfrac{(2\sqrt{3})^3}{3}$
n $\dfrac{(2\sqrt{3})^2}{9}\times\dfrac{(-3\sqrt{2})^4}{3}$
o $\dfrac{(2\sqrt{5})^3}{5}\times\dfrac{(-2\sqrt{3})^5}{24}$
p $\dfrac{(3\sqrt{3})^3}{2}\div\dfrac{(5\sqrt{2})^2}{4}$
q $\dfrac{(2\sqrt{5})^4}{50}\div\dfrac{(2\sqrt{3})^3}{5}$
r $\dfrac{(2\sqrt{2})^3}{9}\div\dfrac{(2\sqrt{8})^2}{(\sqrt{27})^3}$

16 Fully expand and simplify these surds.

a $(2\sqrt{3}-\sqrt{2})^2+(\sqrt{3}+\sqrt{2})^2$
b $(\sqrt{5}-\sqrt{3})^2+(\sqrt{5}+\sqrt{3})^2$
c $(\sqrt{3}-4\sqrt{5})(\sqrt{3}+4\sqrt{5})-(\sqrt{3}-\sqrt{5})^2$
d $-10\sqrt{3}-(2\sqrt{3}-5)^2$
e $(\sqrt{3}-2\sqrt{6})^2+(1+\sqrt{2})^2$
f $(2\sqrt{7}-3)^2-(3-2\sqrt{7})^2$
g $(2\sqrt{3}-3\sqrt{2})(2\sqrt{3}+3\sqrt{2})-(\sqrt{6}-\sqrt{2})^2$
h $\sqrt{2}(2\sqrt{5}-3\sqrt{3})^2+(\sqrt{6}+\sqrt{5})^2$

ISBN 978-1-108-77290-7

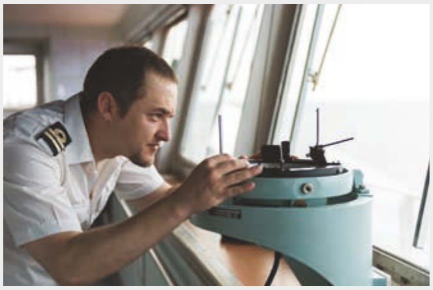
Cambridge University Press
Updated June 2022

3D Rationalising the denominator 10A

Learning intentions

- To understand that a surd multiplied by itself gives a whole number
- To know that rationalising the denominator refers to converting an irrational denominator to one that is rational
- To be able to rationalise the denominator

As you know, it is easier to add or subtract fractions when the fractions are expressed with the same denominator. In a similar way, it is easier to work with surds such as $\frac{1}{\sqrt{2}}$ and $\frac{\sqrt{3}-1}{\sqrt{5}}$ when they are expressed using a whole number in the denominator. The process that removes a surd from the denominator is called 'rationalising the denominator' because the denominator is being converted from an irrational number to a rational number.



Working through a problem using surds provides exact value solutions. Navigation training uses surd manipulation to solve problems of speed and direction, applying Pythagoras' theorem and trigonometry.

LESSON STARTER What do I multiply by?

When trying to rationalise the denominator in a surd like $\frac{1}{\sqrt{2}}$, you must multiply the surd by a chosen number so that the denominator is converted to a whole number.

- First, decide what each of the following is equivalent to.
  a $\frac{\sqrt{3}}{\sqrt{3}}$ **b** $\frac{\sqrt{2}}{\sqrt{2}}$ **c** $\frac{\sqrt{21}}{\sqrt{21}}$
- Recall that $\sqrt{x} \times \sqrt{x} = x$ and simplify the following.
  a $\sqrt{5} \times \sqrt{5}$ **b** $2\sqrt{3} \times \sqrt{3}$ **c** $4\sqrt{7} \times \sqrt{7}$
- Now, decide what you can multiply $\frac{1}{\sqrt{2}}$ by so that:
  – the value of $\frac{1}{\sqrt{2}}$ does not change, and
  – the denominator becomes a whole number.
- Repeat this for:
  a $\frac{1}{\sqrt{5}}$ **b** $\frac{3}{2\sqrt{3}}$

KEY IDEAS

- **Rationalising a denominator** involves multiplying by a number equivalent to 1, which changes the denominator to a whole number.
  $$\frac{x}{\sqrt{y}} = \frac{x}{\sqrt{y}} \times \frac{\sqrt{y}}{\sqrt{y}} = \frac{x\sqrt{y}}{y}$$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

BUILDING UNDERSTANDING

1 Simplify.

a $\frac{\sqrt{6}}{\sqrt{6}}$ **b** $\frac{2\sqrt{5}}{4\sqrt{5}}$ **c** $-\frac{\sqrt{8}}{\sqrt{2}}$ **d** $\frac{\sqrt{72}}{\sqrt{2}}$

2 State the missing number.

a $\sqrt{3} \times \_\_ = 3$ **b** $\sqrt{10} \times \sqrt{10} = \_\_$ **c** $2\sqrt{5} \times \_\_ = 10$ **d** $\_\_ \times 3\sqrt{7} = 21$

3 Use a calculator to find a decimal approximation to each number in the following pairs of numbers. What do you notice?

a $\frac{1}{\sqrt{7}}, \frac{\sqrt{7}}{7}$ **b** $\frac{5}{\sqrt{3}}, \frac{5\sqrt{3}}{3}$ **c** $\frac{11\sqrt{11}}{\sqrt{5}}, \frac{11\sqrt{55}}{5}$

Example 9 Rationalising the denominator

Rationalise the denominator in the following.

a $\frac{2}{\sqrt{3}}$ **b** $\frac{3\sqrt{2}}{\sqrt{5}}$ **c** $\frac{2\sqrt{7}}{5\sqrt{2}}$ **d** $\frac{1-\sqrt{3}}{\sqrt{3}}$

SOLUTION

a
$$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

b
$$\frac{3\sqrt{2}}{\sqrt{5}} = \frac{3\sqrt{2}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{10}}{5}$$

c
$$\frac{2\sqrt{7}}{5\sqrt{2}} = \frac{2\sqrt{7}}{5\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\cancel{2}^{1}\sqrt{14}}{\cancel{10}^{5}} = \frac{\sqrt{14}}{5}$$

d
$$\frac{1-\sqrt{3}}{\sqrt{3}} = \frac{1-\sqrt{3}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}-3}{3}$$

EXPLANATION

a Choose the appropriate fraction equivalent to 1 to multiply by. In this case, choose $\frac{\sqrt{3}}{\sqrt{3}}$ since $\sqrt{3} \times \sqrt{3} = 3$.

b Choose the appropriate fraction. In this case, use $\frac{\sqrt{5}}{\sqrt{5}}$ since $\sqrt{5} \times \sqrt{5} = 5$. Recall $\sqrt{2} \times \sqrt{5} = \sqrt{2 \times 5} = \sqrt{10}$.

c Choose the appropriate fraction; i.e. $\frac{\sqrt{2}}{\sqrt{2}}$.
$5 \times \sqrt{2} \times \sqrt{2} = 5 \times 2 = 10$
Cancel the common factor of 2.

d Expand using the distributive law:
$(1 - \sqrt{3}) \times \sqrt{3} = 1 \times \sqrt{3} - \sqrt{3} \times \sqrt{3} = \sqrt{3} - 3$

ISBN 978-1-108-77290-7

Now you try

Rationalise the denominator in the following.

a $\frac{3}{\sqrt{2}}$ b $\frac{4\sqrt{3}}{\sqrt{7}}$ c $\frac{2\sqrt{5}}{3\sqrt{2}}$ d $\frac{2-\sqrt{7}}{\sqrt{7}}$

Exercise 3D

FLUENCY	1, 2–5(1/2)	2–5(1/2)	2–5(1/3)

Example 9a
Example 9b
Example 9c
Example 9d

1 Rationalise the denominator in the following.

a i $\frac{3}{\sqrt{5}}$ ii $\frac{7}{\sqrt{6}}$

b i $\frac{2\sqrt{3}}{\sqrt{5}}$ ii $\frac{3\sqrt{5}}{\sqrt{2}}$

c i $\frac{3\sqrt{7}}{5\sqrt{3}}$ ii $\frac{2\sqrt{5}}{7\sqrt{6}}$

d i $\frac{1-\sqrt{2}}{\sqrt{2}}$ ii $\frac{4-\sqrt{5}}{\sqrt{5}}$

Example 9a

2 Rationalise the denominators.

a $\frac{1}{\sqrt{2}}$ b $\frac{1}{\sqrt{7}}$ c $\frac{3}{\sqrt{11}}$ d $\frac{4}{\sqrt{5}}$

e $\frac{5}{\sqrt{3}}$ f $\frac{8}{\sqrt{2}}$ g $\frac{\sqrt{5}}{\sqrt{3}}$ h $\frac{\sqrt{2}}{\sqrt{7}}$

3 Rewrite each of the following in the form $\frac{\sqrt{a}}{\sqrt{b}}$ and then rationalise the denominators.

a $\sqrt{\frac{2}{3}}$ b $\sqrt{\frac{5}{7}}$ c $\sqrt{\frac{6}{11}}$ d $\sqrt{\frac{2}{5}}$

e $\sqrt{\frac{7}{3}}$ f $\sqrt{\frac{6}{7}}$ g $\sqrt{\frac{10}{3}}$ h $\sqrt{\frac{17}{2}}$

Example 9b

4 Rationalise the denominators.

a $\frac{4\sqrt{2}}{\sqrt{7}}$ b $\frac{5\sqrt{2}}{\sqrt{3}}$ c $\frac{3\sqrt{5}}{\sqrt{2}}$

d $\frac{3\sqrt{6}}{\sqrt{7}}$ e $\frac{7\sqrt{3}}{\sqrt{10}}$ f $\frac{2\sqrt{7}}{\sqrt{15}}$

Example 9c

5 Rationalise the denominators.

a $\frac{4\sqrt{7}}{5\sqrt{3}}$ b $\frac{2\sqrt{3}}{3\sqrt{2}}$ c $\frac{5\sqrt{7}}{3\sqrt{5}}$ d $\frac{4\sqrt{5}}{5\sqrt{10}}$

e $\frac{2\sqrt{7}}{3\sqrt{35}}$ f $\frac{5\sqrt{12}}{3\sqrt{27}}$ g $\frac{9\sqrt{6}}{2\sqrt{3}}$ h $\frac{7\sqrt{90}}{2\sqrt{70}}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

PROBLEM-SOLVING	7	6(1/2), 7	6–8(1/3)

Example 9d

6 Rationalise the denominators.

a $\frac{1+\sqrt{2}}{\sqrt{3}}$ **b** $\frac{3+\sqrt{5}}{\sqrt{7}}$ **c** $\frac{2-\sqrt{3}}{\sqrt{5}}$

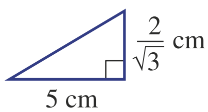
d $\frac{\sqrt{3}-\sqrt{5}}{\sqrt{2}}$ **e** $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{7}}$ **f** $\frac{\sqrt{10}-\sqrt{7}}{\sqrt{3}}$

g $\frac{\sqrt{2}+\sqrt{7}}{\sqrt{6}}$ **h** $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{10}}$ **i** $\frac{\sqrt{6}-\sqrt{10}}{\sqrt{5}}$

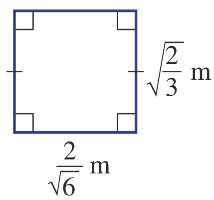
j $\frac{4\sqrt{2}-5\sqrt{3}}{\sqrt{6}}$ **k** $\frac{3\sqrt{5}+5\sqrt{2}}{\sqrt{10}}$ **l** $\frac{3\sqrt{10}+5\sqrt{3}}{\sqrt{2}}$

7 Determine the exact value of the area of the following shapes. Express your answers using a rational denominator.

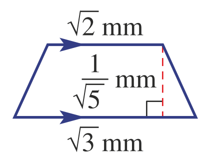
a



b



c



8 Simplify the following by first rationalising denominators and then using a common denominator.

a $\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{2}}$ **b** $\frac{3}{\sqrt{5}}+\frac{1}{\sqrt{2}}$ **c** $\frac{3}{\sqrt{7}}-\frac{2}{\sqrt{3}}$

d $\frac{5}{2\sqrt{3}}-\frac{2}{3\sqrt{2}}$ **e** $\frac{1}{3\sqrt{2}}+\frac{5}{4\sqrt{3}}$ **f** $\frac{3}{2\sqrt{5}}+\frac{2}{5\sqrt{3}}$

g $\frac{7\sqrt{2}}{5\sqrt{7}}-\frac{2\sqrt{7}}{3\sqrt{2}}$ **h** $\frac{10\sqrt{6}}{3\sqrt{5}}+\frac{4\sqrt{2}}{3\sqrt{3}}$ **i** $\frac{5\sqrt{2}}{3\sqrt{5}}-\frac{4\sqrt{7}}{3\sqrt{6}}$

REASONING	9	9, 10(1/2)	10(1/3), 11

9 Explain why multiplying a number by $\frac{\sqrt{x}}{\sqrt{x}}$ does not change its value.

10 Rationalise the denominators and simplify the following.

a $\frac{\sqrt{3}+a}{\sqrt{7}}$ **b** $\frac{\sqrt{6}+a}{\sqrt{5}}$ **c** $\frac{\sqrt{2}+a}{\sqrt{6}}$

d $\frac{\sqrt{3}-3a}{\sqrt{3}}$ **e** $\frac{\sqrt{5}-5a}{\sqrt{5}}$ **f** $\frac{\sqrt{7}-7a}{\sqrt{7}}$

g $\frac{4a+\sqrt{5}}{\sqrt{10}}$ **h** $\frac{3a+\sqrt{3}}{\sqrt{6}}$ **i** $\frac{2a+\sqrt{7}}{\sqrt{14}}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

11 To explore how to simplify a number such as $\frac{3}{4-\sqrt{2}}$, first answer these questions.

a Simplify.

i $(4-\sqrt{2})(4+\sqrt{2})$ **ii** $(3-\sqrt{7})(3+\sqrt{7})$ **iii** $(5\sqrt{2}-\sqrt{3})(5\sqrt{2}+\sqrt{3})$

b What do you notice about each question and answer in part **a** above?

c Now decide what to multiply $\frac{3}{4-\sqrt{2}}$ by to rationalise the denominator.

d Rationalise the denominator in these expressions.

i $\frac{3}{4-\sqrt{2}}$ **ii** $\frac{-3}{\sqrt{3}-1}$ **iii** $\frac{\sqrt{2}}{\sqrt{4}-\sqrt{3}}$ **iv** $\frac{2\sqrt{6}}{\sqrt{6}-2\sqrt{5}}$

ENRICHMENT: Binomial denominators	–	–	12(½)

12 Rationalise the denominators in the following by forming a 'difference of two perfect squares'.

For example:
$$\begin{aligned}\frac{2}{\sqrt{2}+1} &= \frac{2}{\sqrt{2}+1}\times\frac{\sqrt{2}-1}{\sqrt{2}-1}\\ &= \frac{2(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}\\ &= \frac{2\sqrt{2}-2}{2-1}\\ &= 2\sqrt{2}-2\end{aligned}$$

a $\frac{5}{\sqrt{3}+1}$ **b** $\frac{4}{\sqrt{3}-1}$ **c** $\frac{3}{\sqrt{5}-2}$ **d** $\frac{4}{1-\sqrt{2}}$

e $\frac{3}{1-\sqrt{3}}$ **f** $\frac{7}{6-\sqrt{7}}$ **g** $\frac{4}{3-\sqrt{10}}$ **h** $\frac{7}{2-\sqrt{5}}$

i $\frac{2}{\sqrt{11}-\sqrt{2}}$ **j** $\frac{6}{\sqrt{2}+\sqrt{5}}$ **k** $\frac{4}{\sqrt{3}+\sqrt{7}}$ **l** $\frac{\sqrt{2}}{\sqrt{7}+1}$

m $\frac{\sqrt{6}}{\sqrt{6}-1}$ **n** $\frac{3\sqrt{2}}{\sqrt{7}-2}$ **o** $\frac{2\sqrt{5}}{\sqrt{5}+2}$ **p** $\frac{b}{\sqrt{a}+\sqrt{b}}$

q $\frac{a}{\sqrt{a}-\sqrt{b}}$ **r** $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ **s** $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$ **t** $\frac{\sqrt{ab}}{\sqrt{a}-\sqrt{b}}$

ISBN 978-1-108-77290-7

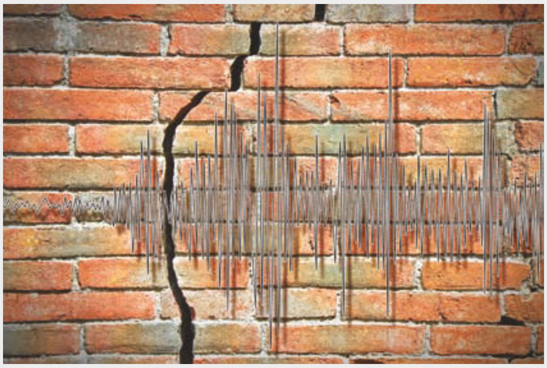
Cambridge University Press
Updated June 2022

3E Review of index laws CONSOLIDATING

Learning intentions

- To know that powers are used as a short hand way of writing repeated multiplications
- To understand that index laws for multiplication and division apply only to common bases
- To know how to combine powers with the same base under multiplication and division
- To know how to apply powers where brackets are involved
- To know that any number (except 0) to the power of zero is equal to 1
- To be able to combine a number of index laws to simplify an expression

From your work in Year 9 you will recall that powers (i.e numbers with indices) can be used to represent repeated multiplication of the same factor. For example, $2 \times 2 \times 2 = 2^3$ and $5 \times x \times x \times x \times x = 5x^4$. The five basic index laws and the zero power will be revised in this section.



Index laws efficiently simplify powers of a base. Powers of 2 calculate the size of digital data and bacterial populations, and powers of 10 are used when calculating earthquake and sound level intensities.

LESSON STARTER Recall the laws

Try to recall how to simplify each expression and use words to describe the index law used.

- $5^3 \times 5^7$
- $x^4 \div x^2$
- $(a^7)^2$
- $(2a)^3$
- $\left(\dfrac{x}{3}\right)^4$
- $(4x^2)^0$

KEY IDEAS

- Recall that $a = a^1$ and $5a = 5^1 \times a^1$.
- The index laws
  - Law 1: $a^m \times a^n = a^{m+n}$ — Retain the base and add the indices.
  - Law 2: $a^m \div a^n = \dfrac{a^m}{a^n} = a^{m-n}$ — Retain the base and subtract the indices.
  - Law 3: $(a^m)^n = a^{m \times n}$ — Retain the base and multiply the indices.
  - Law 4: $(a \times b)^m = a^m \times b^m$ — Distribute the index number across the bases.
  - Law 5: $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$ — Distribute the index number across the bases.
- The zero power: $a^0 = 1$ — Any number (except 0) to the power of zero is equal to 1.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

BUILDING UNDERSTANDING

1 Simplify, using index form.

a $3 \times 3 \times 3 \times 3$

b $7 \times 7 \times 7 \times 7 \times 7 \times 7$

c $2 \times x \times x \times 3 \times x$

d $2 \times b \times a \times 4 \times b \times a \times a$

2 State the missing parts to this table.

x	4	3	2	1	0
2^x		$2^3 = 8$			

3 State the missing components.

a $2^2 \times 2^3 = 2 \times 2 \times$ ______
$= 2^{--}$

b $\dfrac{x^5}{x^3} = \dfrac{x \times x \times x \times \_\_\_\_\_\_\_\_\_\_}{\_\_\_\_\_\_\_\_\_\_}$
$= x^{--}$

c $(a^2)^3 = a \times a$______ $\times$ ______
$= a^{--}$

d $(2x)^0 \times 2x^0 =$ ____ $\times$ ____
$= 2$

Example 10 Using index law 1

Simplify the following using the first index law.

a $x^5 \times x^4$

b $3a^2b \times 4ab^3$

SOLUTION

a $x^5 \times x^4 = x^9$

b $3a^2b \times 4ab^3 = 12a^3b^4$

EXPLANATION

There is a common base of x, so add the indices.

Multiply coefficients and add indices for each base a and b. Recall that $a = a^1$.

Now you try

Simplify the following using the first index law.

a $x^3 \times x^4$

b $2ab^2 \times 7a^2b^3$

Example 11 Using index law 2

Simplify the following using the second index law.

a $m^7 \div m^5$

b $4x^2y^5 \div (8xy^2)$

SOLUTION

a $m^7 \div m^5 = m^2$

EXPLANATION

Subtract the indices when dividing terms with the same base.

Continued on next page

ISBN 978-1-108-77290-7

b $4x^2y^5 \div (8xy^2) = \dfrac{4x^2y^5}{8xy^2}$
$= \dfrac{xy^3}{2}$
$= \dfrac{1}{2}xy^3$

First, express as a fraction.
Divide the coefficients and subtract the indices of x and y (i.e. $x^{2-1}\,y^{5-2}$).

Now you try

Simplify the following using the second index law.

a $m^5 \div m^3$
b $5x^2y^4 \div (10xy^2)$

Example 12 Combining index laws

Simplify the following using the index laws.

a $(a^3)^4$
b $(2y^5)^3$
c $\left(\dfrac{3x^2}{5y^2z}\right)^3$
d $\dfrac{3(xy^2)^3 \times 4x^4y^2}{8x^2y}$

SOLUTION

a $(a^3)^4 = a^{12}$

b $(2y^5)^3 = 2^3y^{15}$
$= 8y^{15}$

c $\left(\dfrac{3x^2}{5y^2z}\right)^3 = \dfrac{3^3x^6}{5^3y^6z^3}$
$= \dfrac{27x^6}{125y^6z^3}$

d $\dfrac{3(xy^2)^3 \times 4x^4y^2}{8x^2y} = \dfrac{3x^3y^6 \times 4x^4y^2}{8x^2y}$
$= \dfrac{12x^7y^8}{8x^2y}$
$= \dfrac{3x^5y^7}{2}$

EXPLANATION

a Use index law 3 and multiply the indices.

b Use index law 4 and multiply the indices for each base 2 and y. Note: $2 = 2^1$.

c Raise the coefficients to the power 3 and multiply the indices of each base by 3.

d Remove brackets first by multiplying the indices for each base.
Simplify the numerator using index law 1.

Simplify the fraction using index law 2, subtracting indices of the same base.

Now you try

Simplify the following using the index laws.

a $(a^2)^3$
b $(3y^3)^3$
c $\left(\dfrac{2x^3}{7yz^2}\right)^2$
d $\dfrac{4(x^2y)^3 \times 2xy^2}{2x^2y}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 13 Using the zero power

Evaluate, using the zero power.

a $4a^0$

b $2p^0 + (3p)^0$

SOLUTION	EXPLANATION
a $4a^0 = 4 \times 1$ $= 4$	Any number to the power of zero is equal to 1.
b $2p^0 + (3p)^0 = 2 \times 1 + 1$ $= 3$	Note: $(3p)^0$ is not the same as $3p^0$.

Now you try

Evaluate, using the zero power.

a $2a^0$

b $5p^0 + (7p)^0$

Exercise 3E

FLUENCY	1, 2–5(1/2)	2–5(1/3)	2–5(1/4)

1 Simplify the following using the first index law.

Example 10a

a **i** $x^2 \times x^5$ **ii** $x^6 \times x^2$

Example 10b

b **i** $5a^2b \times 2ab^2$ **ii** $4a^3b \times 3ab^4$

Example 10

2 Simplify, using the first index law.

a $a^5 \times a^4$
b $x^3 \times x^2$
c $b \times b^5$
d $7m^2 \times 2m^3$
e $2s^4 \times 3s^3$
f $t^8 \times 2t^8$
g $\frac{1}{5}p^2 \times p$
h $\frac{1}{4}c^4 \times \frac{2}{3}c^3$
i $\frac{3}{5}s \times \frac{3s}{5}$
j $2x^2y \times 3xy^2$
k $3a^2b \times 5ab^5$
l $3v^7w \times 6v^2w$
m $3x^4 \times 5xy^2 \times 10y^4$
n $2rs^3 \times 3r^4s \times 2r^2s^2$
o $4m^6n^7 \times mn \times 5mn^2$

Example 11

3 Simplify, using the second index law.

a $x^5 \div x^2$
b $a^7 \div a^6$
c $q^9 \div q^6$
d $b^5 \div b$
e $\frac{y^8}{y^3}$
f $\frac{d^8}{d^3}$
g $\frac{j^7}{j^6}$
h $\frac{m^{15}}{m^9}$
i $2x^2y^3 \div x$
j $3r^5s^2 \div (r^3s)$
k $6p^4q^2 \div (3q^2p^2)$
l $16m^7x^5 \div (8m^3x^4)$
m $\frac{5a^2b^4}{a^2b}$
n $\frac{8st^4}{2t^3}$
o $\frac{2v^5}{8v^3}$
p $\frac{7a^2b}{14ab}$
q $\frac{-3x^4y}{9x^3y}$
r $\frac{-8x^2y^3}{16x^2y}$

ISBN 978-1-108-77290-7
Cambridge University Press
Updated June 2022

Example 12a–c

4 Simplify using the third, fourth and fifth index laws.

a $(x^5)^2$ **b** $(t^3)^2$ **c** $4(a^2)^3$ **d** $5(y^5)^3$

e $(4t^2)^3$ **f** $(2u^2)^2$ **g** $(3r^3)^3$ **h** $(3p^4)^4$

i $\left(\frac{a^2}{b^3}\right)^2$ **j** $\left(\frac{x^3}{y^4}\right)^3$ **k** $\left(\frac{x^2y^3}{z^4}\right)^2$ **l** $\left(\frac{u^4w^2}{v^2}\right)^4$

m $\left(\frac{3f^2}{5g}\right)^3$ **n** $\left(\frac{3a^2b}{2pq^3}\right)^2$ **o** $\left(\frac{at^3}{3g^4}\right)^3$ **p** $\left(\frac{4p^2q^3}{3r}\right)^4$

Example 13

5 Evaluate the following using the zero power.

a $8x^0$ **b** $3t^0$ **c** $(5z)^0$ **d** $(10ab^2)^0$

e $5(g^3h^3)^0$ **f** $8x^0 - 5$ **g** $4b^0 - 9$ **h** $7x^0 - 4(2y)^0$

PROBLEM-SOLVING	6(1/2), 7	6(1/2), 7	6(1/3), 7, 8

Example 12d

6 Use appropriate index laws to simplify the following.

a $x^6 \times x^5 \div x^3$ **b** $x^2y \div (xy) \times xy^2$

c $x^4n^7 \times x^3n^2 \div (xn)$ **d** $\frac{x^2y^3 \times x^2y^4}{x^3y^5}$

e $\frac{m^2w \times m^3w^2}{m^4w^3}$ **f** $\frac{r^4s^7 \times r^4s^7}{r^4s^7}$

g $\frac{9x^2y^3 \times 6x^7y^5}{12xy^6}$ **h** $\frac{4x^2y^3 \times 12x^2y^2}{24x^4y}$

i $\frac{16a^8b \times 4ab^7}{32a^7b^6}$ **j** $(3m^2n^4)^3 \times mn^2$

k $-5(a^2b)^3 \times (3ab)^2$ **l** $(4f^2g)^2 \times f^2g^4 \div \left(3(fg^2)^3\right)$

m $\frac{4m^2n \times 3(m^2n)^3}{6m^2n}$ **n** $\frac{(7y^2z)^2 \times 3yz^2}{7(yz)^2}$

o $\frac{2(ab)^2 \times (2a^2b)^3}{4ab^2 \times 4a^7b^3}$ **p** $\frac{(2m^3)^2}{3(mn^4)^0} \times \frac{(6n^5)^2}{(-2n)^3m^4}$

7 Simplify.

a $(-3)^3$ **b** $-(3)^3$ **c** $(-3)^4$ **d** -3^4

8 Simplify.

a $\left((x^2)^3\right)^2$ **b** $\left((a^5)^3\right)^7$ **c** $\left(\left(\frac{a^2}{b}\right)^3\right)^5$

REASONING	9(1/2)	9(1/2), 10	9(1/3), 10, 11

9 Evaluate without the use of a calculator.

a $\frac{13^3}{13^2}$ **b** $\frac{18^7}{18^6}$ **c** $\frac{9^8}{9^6}$ **d** $\frac{4^{10}}{4^7}$

e $\frac{25^2}{5^4}$ **f** $\frac{36^2}{6^4}$ **g** $\frac{27^2}{3^4}$ **h** $\frac{32^2}{2^7}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

10 When Billy uses a calculator to raise -2 to the power 4 he gets -16 when the answer is actually 16. What has he done wrong?

11 Find the value of a in these equations in which the index is unknown.

a $2^a = 8$ **b** $3^a = 81$ **c** $2^{a+1} = 4$
d $(-3)^a = -27$ **e** $(-5)^a = 625$ **f** $(-4)^{a-1} = 1$

ENRICHMENT: Indices in equations	–	–	12–14(½)

12 If $x^4 = 3$, find the value of:

a x^8 **b** $x^4 - 1$ **c** $2x^{16}$ **d** $3x^4 - 3x^8$

13 Find the value(s) of x.

a $x^4 = 16$ **b** $2^{x-1} = 16$ **c** $2^{2x} = 16$ **d** $2^{2x-3} = 16$

14 Find the possible pairs of positive integers for x and y when:

a $x^y = 16$ **b** $x^y = 64$ **c** $x^y = 81$ **d** $x^y = 1$

ISBN 978-1-108-77290-7 Cambridge University Press
 Updated June 2022

3F Negative indices

Learning intentions
- To understand how a negative power relates to division
- To know how to rewrite expressions involving negative indices with positive indices
- To be able to apply index laws to expressions involving negative indices

The study of indices can be extended to include negative powers. Using the second index law and the fact that $a^0 = 1$, we can establish rules for negative powers.

$$a^0 \div a^n = a^{0-n} \text{ (index law 2)} \qquad \text{also} \qquad a^0 \div a^n = 1 \div a^n \text{ (as } a^0 = 1)$$
$$= a^{-n} \qquad\qquad\qquad\qquad = \frac{1}{a^n}$$

Therefore: $a^{-n} = \dfrac{1}{a^n}$

Also: $\dfrac{1}{a^{-n}} = 1 \div a^{-n}$
$$= 1 \div \frac{1}{a^n}$$
$$= 1 \times \frac{a^n}{1}$$
$$= a^n$$

Therefore: $\dfrac{1}{a^{-n}} = a^n$.

A half-life is the time taken for radioactive material to halve in size. Calculations of the quantity remaining after multiple halving use negative powers of 2. Applications include radioactive waste management and diagnostic medicine.

LESSON STARTER The disappearing bank balance

Due to fees, an initial bank balance of \$64 is halved every month.

Balance (\$)	64	32	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
Positive indices only	2^6	2^5							$\frac{1}{2^2}$	
Positive and negative indices	2^6		2^4					2^{-1}		

- Copy and complete the table and continue each pattern.
- Discuss the differences in the way indices are used at the end of the rows.
- What would be a way of writing $\dfrac{1}{16}$ using positive indices?
- What would be a way of writing $\dfrac{1}{16}$ using negative indices?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

KEY IDEAS

- $a^{-m} = \dfrac{1}{a^m}$ For example, $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$.
- $\dfrac{1}{a^{-m}} = a^m$ For example, $\dfrac{1}{2^{-3}} = 2^3 = 8$.

BUILDING UNDERSTANDING

1. State the next three terms in these patterns.
   a $2^3, 2^2, 2^1, 2^0, 2^{-1}$, ___, ___, ___
   b x^2, x^1, x^0, ___, ___, ___
2. Recall that $\dfrac{1}{4} = \dfrac{1}{2^2}$. Similarly, state these fractions using positive indices.
   a $\dfrac{1}{9}$ b $\dfrac{1}{25}$ c $\dfrac{5}{16}$ d $-\dfrac{2}{27}$
3. State each rule for negative indices by completing each statement.
   a $a^{-b} =$ _______ b $\dfrac{1}{a^{-b}} =$ _______
4. Express the following with positive indices using $a^{-m} = \dfrac{1}{a^m}$ and evaluate.
   a 5^{-2} b 3^{-3} c 4×7^{-2}

Example 14 Writing expressions using positive indices

Express each of the following using positive indices.

a b^{-4} b $3x^{-4}y^2$ c $\dfrac{5}{x^{-3}}$

SOLUTION	EXPLANATION
a $b^{-4} = \dfrac{1}{b^4}$	Use $a^{-n} = \dfrac{1}{a^n}$.
b $3x^{-4}y^2 = \dfrac{3y^2}{x^4}$	x is the only base with a negative power. $\dfrac{3}{1} \times \dfrac{1}{x^4} \times \dfrac{y^2}{1} = \dfrac{3y^2}{x^4}$.
c $\dfrac{5}{x^{-3}} = 5 \times x^3 = 5x^3$	Use $\dfrac{1}{a^{-n}} = a^n$ and note that $\dfrac{5}{x^{-3}} = 5 \times \dfrac{1}{x^{-3}}$.

Now you try

Express each of the following using positive indices.

a b^{-3} b $2x^{-2}y^3$ c $\dfrac{2}{x^{-4}}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 15 Using index laws with negative indices

Simplify the following expressing answers using positive indices.

a $\dfrac{2a^3b^2}{a^5b^3}$ **b** $\dfrac{4m^{-2}n^3}{8m^5n^{-4}}$

SOLUTION

a $\dfrac{2a^3b^2}{a^5b^3} = 2a^{-2}b^{-1}$

$= \dfrac{2}{a^2b}$

b $\dfrac{4m^{-2}n^3}{8m^5n^{-4}} = \dfrac{\cancel{4}^{1}m^{-7}n^7}{\cancel{8}^{2}}$

$= \dfrac{n^7}{2m^7}$

EXPLANATION

Use index law 2 to subtract powers with common base a^{3-5} and b^{2-3}.

Express with positive powers $\dfrac{2}{1} \times \dfrac{1}{a^2} \times \dfrac{1}{b}$.

Cancel common factor of 4 and subtract powers m^{-2-5} and $n^{3-(-4)}$.

Express with positive powers $\dfrac{1}{2} \times \dfrac{1}{m^7} \times \dfrac{n^7}{1}$.

Now you try

Simplify the following expressing answers using positive indices.

a $\dfrac{5a^2b^3}{a^4b^7}$ **b** $\dfrac{3m^{-3}n^6}{9m^4n^{-2}}$

Example 16 Simplifying more complex expressions

Simplify the following and express your answers using positive indices.

a $\dfrac{(p^{-2}q)^4}{5p^{-1}q^3} \times \left(\dfrac{p^{-2}}{q^3}\right)^{-3}$ **b** $\left(\dfrac{2m^3}{r^2n^{-4}}\right)^3 \div \left(\dfrac{5m^{-2}n^3}{r}\right)^2$

SOLUTION

a $\dfrac{(p^{-2}q)^4}{5p^{-1}q^3} \times \left(\dfrac{p^{-2}}{q^3}\right)^{-3} = \dfrac{p^{-8}q^4}{5p^{-1}q^3} \times \dfrac{p^6}{q^{-9}}$

$= \dfrac{p^{-2}q^4}{5p^{-1}q^{-6}}$

$= \dfrac{p^{-1}q^{10}}{5}$

$= \dfrac{q^{10}}{5p}$

EXPLANATION

Deal with brackets first by multiplying the power to each of the indices within the brackets.

Use index laws 1 and 2 to combine indices of like bases. Simplify each numerator and denominator first: $p^{-8+6} = p^{-2}$ and $q^{3+(-9)} = q^{-6}$.

Then $p^{-2-(-1)}q^{4-(-6)} = p^{-1}q^{10}$.

Use $a^{-n} = \dfrac{1}{a^n}$ to express p^{-1} with a positive index.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b $\left(\frac{2m^3}{r^2n^{-4}}\right)^3 \div \left(\frac{5m^{-2}n^3}{r}\right)^2 = \frac{2^3m^9}{r^6n^{-12}} \div \frac{5^2m^{-4}n^6}{r^2}$

Multiply the bracket power to each of the indices within the bracket.

$= \frac{8m^9}{r^6n^{-12}} \times \frac{r^2}{25m^{-4}n^6}$

Multiply by the reciprocal of the divisor.

$= \frac{8m^{13}r^{-4}}{25n^{-6}}$

Use index laws 1 and 2 to combine indices of like bases.

$= \frac{8m^{13}n^6}{25r^4}$

Write the answer with positive powers.

Now you try

Simplify the following and express your answers using positive indices.

a $\frac{(p^{-1}q)^3}{2p^{-2}q^2} \times \left(\frac{p^{-2}}{q^2}\right)^{-1}$

b $\left(\frac{2m^2}{r^4n^{-3}}\right)^2 \div \left(\frac{4m^{-1}n^2}{r}\right)^3$

Exercise 3F

FLUENCY	1, 2–5(1/2)	2–6(1/3)	2–6(1/4)

1 Express each of the following using positive indices.

Example 14a
a **i** b^{-2} **ii** a^{-5}

Example 14b
b **i** $4x^{-1}y^3$ **ii** $7x^{-3}y^2$

Example 14c
c **i** $\frac{2}{x^{-2}}$ **ii** $\frac{6}{x^{-7}}$

Example 14a,b
2 Express the following using positive indices.

a x^{-5} **b** a^{-4} **c** $2m^{-4}$ **d** $3y^{-7}$

e $3a^2b^{-3}$ **f** $4m^3n^{-3}$ **g** $10x^{-2}y^5z$ **h** $3x^{-4}y^{-2}z^3$

i $\frac{1}{3}p^{-2}q^3r$ **j** $\frac{1}{5}d^2e^{-4}f^5$ **k** $\frac{3}{8}u^2v^{-6}w^7$ **l** $\frac{2}{5}b^3c^{-5}d^{-2}$

Example 14c
3 Express the following using positive indices.

a $\frac{1}{x^{-2}}$ **b** $\frac{2}{y^{-3}}$ **c** $\frac{4}{m^{-7}}$ **d** $\frac{3}{b^{-5}}$

e $\frac{2b^4}{d^{-3}}$ **f** $\frac{3m^2}{n^{-4}}$ **g** $\frac{4b^4}{3a^{-3}}$ **h** $\frac{5h^3}{2g^{-3}}$

Example 15a
4 Use index laws 1 and 2 to simplify the following. Write your answers using positive indices.

a $x^3 \times x^{-2}$ **b** $a^7 \times a^{-4}$ **c** $2b^5 \times b^{-9}$ **d** $3y^{-6} \times y^3$

e $x^2y^3 \times x^{-3}y^{-4}$ **f** $4a^{-6}y^4 \times a^3y^{-2}$ **g** $2a^{-3}b \times 3a^{-2}b^{-3}$ **h** $6a^4b^3 \times 3a^{-6}b$

i $\frac{a^4b^3}{a^2b^5}$ **j** $\frac{m^3n^2}{mn^3}$ **k** $\frac{3x^2y}{6xy^2}$ **l** $\frac{4m^3n^4}{7m^2n^7}$

m $a^3b^4 \div (a^2b^7)$ **n** $p^2q^3 \div (p^7q^2)$ **o** $\frac{p^2q^2r^4}{pq^4r^5}$ **p** $\frac{12r^4s^6}{9rs^{-8}}$

ISBN 978-1-108-77290-7

Example 15b

5 Express the following in simplest form with positive indices.

a $\dfrac{2x^{-2}}{3x^{-3}}$ **b** $\dfrac{7d^{-3}}{10d^{-5}}$ **c** $\dfrac{5s^{-2}}{3s}$ **d** $\dfrac{4f^{-5}}{3f^{-3}}$

e $\dfrac{f^3g^{-2}}{f^{-2}g^3}$ **f** $\dfrac{r^{-3}s^{-4}}{r^3s^{-2}}$ **g** $\dfrac{3w^{-2}x^3}{6w^{-3}x^{-2}}$ **h** $\dfrac{15c^3d}{12c^{-2}d^{-3}}$

6 Express the following with positive indices.

a $\left(\dfrac{2x^2}{x^3}\right)^4$ **b** $\left(\dfrac{m^3}{4m^5}\right)^3$ **c** $2(x^{-7})^3$ **d** $4(d^{-2})^3$

e $(3t^{-4})^2$ **f** $5(x^2)^{-2}$ **g** $(3x^{-5})^4$ **h** $-8(x^5)^{-3}$

i $(4y^{-2})^{-2}$ **j** $(3h^{-3})^{-4}$ **k** $7(j^{-2})^{-4}$ **l** $2(t^{-3})^{-2}$

PROBLEM-SOLVING	7(1/2)	7–8(1/2)	7–8(1/3), 9

Example 16

7 Simplify the following and express your answers with positive indices.

a $(a^3b^2)^3 \times (a^2b^4)^{-1}$ **b** $(2p^2)^4 \times (3p^2q)^{-2}$ **c** $2(x^2y^{-1})^2 \times (3xy^4)^3$

d $\dfrac{2a^3b^2}{a^{-3}} \times \dfrac{2a^2b^5}{b^4}$ **e** $\dfrac{(3rs^2)^4}{r^{-3}s^4} \times \dfrac{(2r^2s)^2}{s^7}$ **f** $\dfrac{4(x^{-2}y^4)^2}{x^2y^{-3}} \times \dfrac{xy^4}{2x^{-2}y}$

g $\left(\dfrac{a^2b^3}{b^{-2}}\right)^2 \div \left(\dfrac{ab^4}{a^2}\right)^{-2}$ **h** $\left(\dfrac{m^4n^{-2}}{r^3}\right)^2 \div \left(\dfrac{m^{-3}n^2}{r^3}\right)^2$ **i** $\dfrac{3(x^2y^{-4})^2}{2(xy^2)^2} \div \dfrac{(xy)^{-3}}{(3x^{-2}y^4)^2}$

8 Evaluate without the use of a calculator.

a 5^{-2} **b** 4^{-3} **c** 2×7^{-2} **d** $5 \times (-3^{-4})$

e $3^{10} \times (3^2)^{-6}$ **f** $(4^2)^{-5} \times 4(4^{-3})^{-3}$ **g** $\dfrac{2}{7^{-2}}$ **h** $\dfrac{-3}{4^{-2}}$

i $\left(\dfrac{2}{3}\right)^{-2}$ **j** $\left(\dfrac{-5}{4}\right)^{-3}$ **k** $\dfrac{(4^{-2})^3}{4^{-4}}$ **l** $\dfrac{(10^{-4})^{-2}}{(10^{-2})^{-3}}$

9 The width of a hair on a spider is approximately 3^{-5} cm. How many centimetres is this, correct to four decimal places?

ISBN 978-1-108-77290-7

REASONING	11	10, 11	10, 12, 13

10 **a** Simplify these numbers.

i $\left(\frac{2}{3}\right)^{-1}$ **ii** $\left(\frac{5}{7}\right)^{-1}$ **iii** $\left(\frac{2x}{y}\right)^{-1}$

b What is $\left(\frac{a}{b}\right)^{-1}$ when expressed in simplest form?

11 A student simplifies $2x^{-2}$ and writes $2x^{-2} = \frac{1}{2x^2}$. Explain the error made.

12 Evaluate the following by combining fractions.

a $2^{-1} + 3^{-1}$ **b** $3^{-2} + 6^{-1}$ **c** $\left(\frac{3}{4}\right)^{-1} - \left(\frac{1}{2}\right)^{0}$

d $\left(\frac{3}{2}\right)^{-1} - 5(2^{-2})$ **e** $\left(\frac{4}{5}\right)^{-2} - \left(\frac{2^{-2}}{3}\right)$ **f** $\left(\frac{3}{2^{-2}}\right) - \left(\frac{2^{-1}}{3^{-2}}\right)^{-1}$

13 Prove that $\left(\frac{1}{2}\right)^{x} = 2^{-x}$ giving reasons.

ENRICHMENT: Simple equations with negative indices	–	–	14(½)

14 Find the value of x.

a $2^x = \frac{1}{4}$ **b** $2^x = \frac{1}{32}$ **c** $3^x = \frac{1}{27}$ **d** $\left(\frac{3}{4}\right)^x = \frac{4}{3}$

e $\left(\frac{2}{3}\right)^x = \frac{9}{4}$ **f** $\left(\frac{2}{5}\right)^x = \frac{125}{8}$ **g** $\frac{1}{2^x} = 8$ **h** $\frac{1}{3^x} = 81$

i $\frac{1}{2^x} = 1$ **j** $5^{x-2} = \frac{1}{25}$ **k** $3^{x-3} = \frac{1}{9}$ **l** $10^{x-5} = \frac{1}{1000}$

m $\left(\frac{3}{4}\right)^{2x+1} = \frac{64}{27}$ **n** $\left(\frac{2}{5}\right)^{3x-5} = \frac{25}{4}$ **o** $\left(\frac{3}{2}\right)^{3x+2} = \frac{16}{81}$ **p** $\left(\frac{7}{4}\right)^{1-x} = \frac{4}{7}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Progress quiz

3A 10A

1 Express each number as a decimal and decide if it is rational or irrational. Then locate all the numbers on the same number line.

a $\sqrt{10}$ b $\frac{22}{7}$ c π d 315%

3A 10A

2 Simplify the following.

a $\sqrt{98}$ b $2\sqrt{75}$ c $\frac{5\sqrt{32}}{8}$ d $\sqrt{\frac{125}{16}}$

3A 10A

3 Express $8\sqrt{3}$ as the square root of a positive integer.

3B 10A

4 Simplify the following.

a $7\sqrt{3} - 5\sqrt{3} + \sqrt{3}$ b $4\sqrt{2} - 3\sqrt{5} + 2\sqrt{2} + 5\sqrt{5}$
c $5\sqrt{48} - 2\sqrt{12}$ d $7\sqrt{45} - \sqrt{243} - 2\sqrt{20} + \sqrt{27}$

3C 10A

5 Simplify the following.

a $-\sqrt{3} \times \sqrt{5}$ b $-5\sqrt{21} \times (-\sqrt{14})$ c $14\sqrt{7} \div (21\sqrt{35})$
d $\sqrt{6} \times \sqrt{6}$ e $(2\sqrt{13})^2$

3C 10A

6 Use the distributive law to expand $2\sqrt{3}(\sqrt{6} + 5\sqrt{24})$ and simplify the surds where necessary.

3D 10A

7 Rationalise the denominators.

a $\frac{3}{\sqrt{7}}$ b $\frac{2\sqrt{3}}{\sqrt{5}}$ c $\frac{\sqrt{6} - 3\sqrt{5}}{\sqrt{2}}$

3E

8 Simplify, using index laws.

a $a^3 \times a^2$ b $4x^2y \times 3xy^3$ c $h^6 \div h^2$ d $5m^9n^4 \div (10m^3n)$
e $(a^2)^3$ f $(3m^5)^2$ g $\left(\frac{2p^4q^3}{7rt^2}\right)^2$ h $(2ab)^0 + 5m^0$

3F

9 Simplify the following where possible and express your answers using positive indices.

a x^{-3} b $2a^{-2}b^4c^{-3}$ c $\frac{7}{m^{-2}}$ d $\frac{4d^{-7}}{5d^{-5}}$
e $\left(\frac{4k^3}{k^7}\right)^2$ f $(2a^{-2})^{-3}$ g $6a^{-3}m^4 \times 2a^{-2}m^{-3}$ h $\frac{20c^{-3}d^2}{15c^{-1}d^{-3}}$

3F

10 Simplify the following and express your answers using positive indices.

a $\frac{(a^3b)^5}{5a^3b^{-2}} \times \left(\frac{a^{-4}}{b^6}\right)^2$ b $\left(\frac{3x^2}{c^2d^{-3}}\right)^2 \div \left(\frac{2x^{-2}c^3}{d}\right)^3$

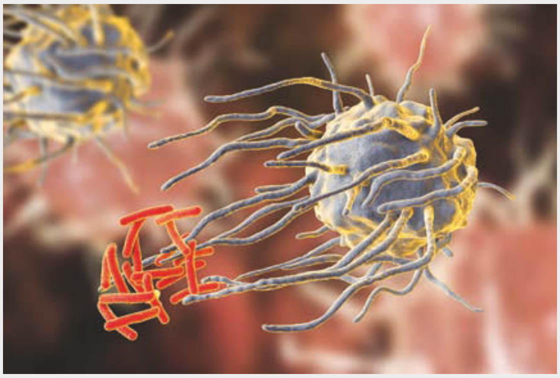
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

3G Scientific notation CONSOLIDATING

Learning intentions
- To understand that very large and very small numbers can be written in a shorthand form
- To know the general form of a number in scientific notation
- To be able to convert between scientific notation and basic numerals
- To know the meaning of the term significant figure
- To be able to round a number to a desired number of significant figures
- To know how to use technology in working with scientific notation

Scientific notation is useful when working with very large or very small numbers. Combined with the use of significant figures, numbers can be written down with an appropriate degree of accuracy and without the need to write all the zeros that define the position of the decimal place. The approximate distance between the Earth and the Sun is 150 million kilometres or 1.5×10^8 km when written in scientific notation using two significant figures. Negative indices can be used for very small numbers, such as $0.0000382\text{ g} = 3.82 \times 10^{-5}\text{ g}$.



Everyday users of scientific notation include astronomers, space scientists, chemists, engineers, environmental scientists, physicists, biologists, lab technicians and medical researchers. This image shows white blood cells engulfing tuberculosis bacteria.

LESSON STARTER Amazing facts large and small

Think of an object, place or living thing that is associated with a very large or small number.

- Give three examples using very large numbers.
- Give three examples using very small numbers.
- Can you remember how to write these numbers using scientific notation?
- How are significant figures used when writing numbers with scientific notation?

KEY IDEAS

- A number written using **scientific notation** is of the form $a \times 10^m$, where $1 \leqslant a < 10$ or $-10 < a \leqslant -1$ and m is an integer.
  - Large numbers: $24\,800\,000 = 2.48 \times 10^7$
    $9\,020\,000\,000 = 9.02 \times 10^9$
  - Small numbers: $0.00307 = 3.07 \times 10^{-3}$
    $-0.0000012 = -1.2 \times 10^{-6}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

- **Significant figures** are counted from left to right, starting at the first non-zero digit.
  - When using scientific notation the digit to the left of the decimal point is the first significant figure.
    For example: $20\,190\,000 = 2.019 \times 10^7$ shows four significant figures.
  - The $\boxed{\times 10^n}$, $\boxed{\textbf{EE}}$ or $\boxed{\textbf{Exp}}$ keys can be used on calculators to enter numbers using scientific notation; e.g. 2.3E–4 means 2.3×10^{-4}.

BUILDING UNDERSTANDING

1 How many significant figures are showing in these numbers?
 a 2.12×10^7 b 1.81×10^{-3} c 461 d 0.0000403

2 State these numbers as powers of 10.
 a 1000 b 10 000 000 c 0.000001 d $\dfrac{1}{1000}$

3 Convert to numbers using scientific notation by stating the missing power.
 a $43\,000 = 4.3 \times 10^{\square}$ b $712\,000 = 7.12 \times 10^{\square}$ c $9012 = 9.012 \times 10^{\square}$
 d $0.00078 = 7.8 \times 10^{\square}$ e $0.00101 = 1.01 \times 10^{\square}$ f $0.00003 = 3 \times 10^{\square}$

Example 17 Converting from scientific notation to a basic numeral

Write these numbers as a basic numeral.
a 5.016×10^5 b 3.2×10^{-7}

SOLUTION	EXPLANATION
a $5.016 \times 10^5 = 501\,600$	Move the decimal point 5 places to the right.
b $3.2 \times 10^{-7} = 0.00000032$	Move the decimal point 7 places to the left.

Now you try

Write these numbers as a basic numeral.
a 2.048×10^4 b 4.7×10^{-5}

Example 18 Converting to scientific notation using significant figures

Write these numbers in scientific notation, using three significant figures.
a 5 218 300 b 0.0042031

SOLUTION	EXPLANATION
a $5\,218\,300 = 5.22 \times 10^6$	Place the decimal point after the first non-zero digit. The digit following the third digit is at least 5, so round up.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b $0.0042031 = 4.20 \times 10^{-3}$

Round down in this case, but retain the zero to show the value of the third significant figure.

Now you try

Write these numbers in scientific notation, using three significant figures.

a 7 937 200 **b** 0.00027103

Exercise 3G

FLUENCY	1, 2–5(1/2)	2–5(1/2)	2–6(1/3)

Example 17a, Example 17b

1 Write these numbers as a basic numeral.

a **i** 3.126×10^3 **ii** 5.04×10^6

b **i** 2.8×10^{-3} **ii** 5.91×10^{-6}

Example 17a

2 Write these numbers as a basic numeral.

a 3.12×10^3 **b** 5.4293×10^4 **c** 7.105×10^5

d 8.213×10^6 **e** 5.95×10^4 **f** -8.002×10^5

g -1.012×10^4 **h** 9.99×10^6 **i** 2.105×10^8

j -5.5×10^4 **k** 2.35×10^9 **l** 1.237×10^{12}

Example 17b

3 Write these numbers as a basic numeral.

a 4.5×10^{-3} **b** 2.72×10^{-2} **c** 3.085×10^{-4}

d 7.83×10^{-3} **e** -9.2×10^{-5} **f** 2.65×10^{-1}

g 1.002×10^{-4} **h** -6.235×10^{-6} **i** 9.8×10^{-1}

j -5.45×10^{-10} **k** 3.285×10^{-12} **l** 8.75×10^{-7}

Example 18a

4 Write these numbers in scientific notation, using three significant figures.

a 6241 **b** −572 644 **c** 30 248

d 423 578 **e** −10 089 **f** 34 971 863

g 72 477 **h** 356 088 **i** 110 438 523

j 909 325 **k** −4 555 678 **l** 9 826 100 005

Example 18b

5 Write these numbers in scientific notation, using three significant figures.

a 0.002423 **b** −0.018754 **c** 0.000125

d −0.0078663 **e** 0.0007082 **f** 0.11396

g 0.000006403 **h** 0.00007892 **i** 0.000129983

j 0.00000070084 **k** 0.000000009886 **l** −0.0004998

6 Write in scientific notation, using the number of significant figures given in the brackets.

a −23 900 (2) **b** 5 707 159 (3) **c** 703 780 030 (2)

d 4875 (3) **e** 0.00192 (2) **f** −0.00070507 (3)

g 0.000009782 (2) **h** −0.35708 (4) **i** 0.000050034 (3)

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

PROBLEM-SOLVING	7–8(1/2)	7–8(1/2)	7–8(1/3), 9

7 Write the following numerical facts using scientific notation.

- **a** The area of Australia is about 7 700 000 km^2.
- **b** The number of stones used to build the Pyramid of Khufu is about 2 500 000.
- **c** The greatest distance of Pluto from the Sun is about 7 400 000 000 km.
- **d** A human hair is about 0.01 cm wide.
- **e** The mass of a neutron is about 0.000 000 000 000 000 000 000 000 001 675 kg.
- **f** The mass of a bacteria cell is about 0.000 000 000 000 95 g.

8 Use a calculator to evaluate the following, giving the answers in scientific notation using three significant figures.

- **a** $(2.31)^{-7}$
- **b** $(5.04)^{-4}$
- **c** $(2.83 \times 10^2)^{-3}$
- **d** $5.1 \div (8 \times 10^2)$
- **e** $(9.3 \times 10^{-2}) \times (8.6 \times 10^8)$
- **f** $(3.27 \times 10^4) \div (9 \times 10^{-5})$
- **g** $\sqrt{3.23 \times 10^{-6}}$
- **h** $\pi(3.3 \times 10^7)^2$
- **i** $\sqrt[3]{5.73 \times 10^{-4}}$

9 The speed of light is approximately 3×10^5 km/s and the average distance between Pluto and the Sun is about 5.9×10^9 km. How long does it take for light from the Sun to reach Pluto? Answer correct to the nearest minute.

REASONING	10	10, 11(1/2)	11–12(1/3), 13

10 Explain why 38×10^7 is not written using scientific notation.

11 Write the following using scientific notation.

- **a** 21×10^3
- **b** 394×10^7
- **c** 6004×10^{-2}
- **d** 179×10^{-6}
- **e** 0.2×10^4
- **f** 0.007×10^2
- **g** 0.01×10^9
- **h** 0.06×10^8
- **i** 0.4×10^{-2}
- **j** 0.0031×10^{-11}
- **k** 210.3×10^{-6}
- **l** 9164×10^{-24}

12 Combine your knowledge of index laws with scientific notation to evaluate the following and express using scientific notation.

- **a** $(3 \times 10^2)^2$
- **b** $(2 \times 10^3)^3$
- **c** $(8 \times 10^4)^2$
- **d** $(12 \times 10^{-5})^2$
- **e** $(5 \times 10^{-3})^{-2}$
- **f** $(4 \times 10^5)^{-2}$
- **g** $(1.5 \times 10^{-3})^2$
- **h** $(8 \times 10^{-8})^{-1}$
- **i** $(5 \times 10^{-2}) \times (2 \times 10^{-4})$
- **j** $(3 \times 10^{-7}) \times (4.25 \times 10^2)$
- **k** $(15 \times 10^8) \times (12 \times 10^{-11})$
- **l** $(18 \times 10^5) \div (9 \times 10^3)$
- **m** $(240 \times 10^{-4}) \div (3 \times 10^{-2})$
- **n** $(2 \times 10^{-8}) \div (50 \times 10^4)$
- **o** $(5 \times 10^2) \div (20 \times 10^{-3})$

13 Rewrite 3×10^{-4} with a positive index and use this to explain why, when expressing 3×10^{-4} as a basic numeral, the decimal point is moved four places to the left.

ENRICHMENT: $E = mc^2$	–	–	14

14 $E = mc^2$ is a formula derived by Albert Einstein (1879–1955). The formula relates the energy (E joules) of an object to its mass (m kg), where c is the speed of light (approximately 3×10^8 m/s).

Use $E = mc^2$ to answer these questions, using scientific notation.

- **a** Find the energy, in joules, contained inside an object with these given masses.
  - **i** 10 kg
  - **ii** 26 000 kg
  - **iii** 0.03 kg
  - **iv** 0.00001 kg
- **b** Find the mass, in kilograms, of an object that contains these given amounts of energy. Give your answer using three significant figures.
  - **i** 1×10^{25} J
  - **ii** 3.8×10^{16} J
  - **iii** 8.72×10^4 J
  - **iv** 1.7×10^{-2} J
- **c** The mass of the Earth is about 6×10^{24} kg. How much energy does this convert to?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

3H Rational indices 10A

Learning intentions

- To understand how a rational index relates to the root of a number
- To know how to convert between bases with rational indices and surd form
- To be able to evaluate some numbers with rational indices without a calculator
- To be able to apply index laws to expressions involving rational indices

The square and cube roots of numbers, such as $\sqrt{81} = 9$ and $\sqrt[3]{64} = 4$, can be written using fractional powers.

Fractional indices are used in finance, electrical engineering, architecture, carpentry and for solving packing problems. Volume to the power of one-third (i.e. the cube root) finds a cube's side length and helps find a sphere's radius.

The following shows that $\sqrt{9} = 9^{\frac{1}{2}}$ and $\sqrt[3]{8} = 8^{\frac{1}{3}}$.

Consider:

$$\begin{aligned}\sqrt{9} \times \sqrt{9} &= 3 \times 3 \\ &= 9\end{aligned} \quad \text{and} \quad \begin{aligned}9^{\frac{1}{2}} \times 9^{\frac{1}{2}} &= 9^{\frac{1}{2}+\frac{1}{2}} \\ &= 9\end{aligned}$$

$$\therefore \sqrt{9} = 9^{\frac{1}{2}}$$

Also:

$$\begin{aligned}\sqrt[3]{8} \times \sqrt[3]{8} \times \sqrt[3]{8} &= 2 \times 2 \times 2 \\ &= 8\end{aligned} \quad \text{and} \quad \begin{aligned}8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} &= 8^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} \\ &= 8\end{aligned}$$

$$\therefore \sqrt[3]{8} = 8^{\frac{1}{3}}$$

A rational index is an index that can be expressed as a fraction.

LESSON STARTER Making the connection

For each part below use your knowledge of index laws and basic surds to simplify the numbers. Then discuss the connection that can be made between numbers that have a $\sqrt{}$ sign and numbers that have fractional powers.

- $\sqrt{5} \times \sqrt{5}$ and $5^{\frac{1}{2}} \times 5^{\frac{1}{2}}$
- $\sqrt[3]{27} \times \sqrt[3]{27} \times \sqrt[3]{27}$ and $27^{\frac{1}{3}} \times 27^{\frac{1}{3}} \times 27^{\frac{1}{3}}$
- $(\sqrt{5})^2$ and $\left(5^{\frac{1}{2}}\right)^2$
- $(\sqrt[3]{64})^3$ and $\left(64^{\frac{1}{3}}\right)^3$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

KEY IDEAS

- $a^{\frac{1}{n}} = \sqrt[n]{a}$
  - $\sqrt[n]{a}$ is the nth root of a.

  For example: $3^{\frac{1}{2}} = \sqrt{3}$ or $\sqrt{3}$, $5^{\frac{1}{3}} = \sqrt[3]{5}$, $7^{\frac{1}{10}} = \sqrt[10]{7}$
- $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (\sqrt[n]{a})^m$ or $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$

  For example: $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = (\sqrt[3]{8})^2 = 2^2 = 4$ or $8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}} = (64)^{\frac{1}{3}} = \sqrt[3]{64} = 4$
- In most cases, the index laws apply to **rational indices** (i.e. fractional indices) just as they do for indices that are integers.

BUILDING UNDERSTANDING

1. State the missing components for each statement.

   a $2^{\square} = 8$ and $\sqrt[3]{8} =$ ____
   b $2^{\square} = 32$ and $\sqrt[\square]{32} = 2$
   c $3^{\square} = 81$ and $\sqrt[\square]{81} = 3$
   d $10^{\square} = 100000$ and $\sqrt[\square]{100000} = 10$
2. Evaluate:

   a $\sqrt{9}$
   b $\sqrt{121}$
   c $\sqrt[3]{27}$
   d $\sqrt[3]{64}$
   e $\sqrt[4]{16}$
   f $\sqrt[4]{81}$
   g $\sqrt[5]{32}$
   h $\sqrt[5]{100000}$
3. Using a calculator, enter and evaluate each pair of numbers in their given form. Round your answer to two decimal places.

   a $\sqrt[3]{7}, 7^{\frac{1}{3}}$
   b $\sqrt[5]{10}, 10^{\frac{1}{5}}$
   c $\sqrt{100}, 100^{\frac{1}{13}}$

Example 19 Writing in index form

Express the following in index form.

a $\sqrt{15}$
b $\sqrt{7x^5}$
c $3\sqrt[4]{x^7}$
d $10\sqrt{10}$

SOLUTION	EXPLANATION
a $\sqrt{15} = 15^{\frac{1}{2}}$	$\sqrt{\ }$ means the square root or $\sqrt[2]{\ }$. Note: $\sqrt[n]{a} = a^{\frac{1}{n}}$.
b $\sqrt{7x^5} = (7x^5)^{\frac{1}{2}}$ $= 7^{\frac{1}{2}}x^{\frac{5}{2}}$	Rewrite $\sqrt{\ }$ as power $\frac{1}{2}$, then apply index laws to simplify: $5 \times \frac{1}{2} = \frac{5}{2}$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

c $3\sqrt[4]{x^7} = 3(x^7)^{\frac{1}{4}}$
$= 3x^{\frac{7}{4}}$

$\sqrt[4]{\ }$ means to the power of $\frac{1}{4}$.
Apply index law 3 to multiply indices.

d $10\sqrt{10} = 10 \times 10^{\frac{1}{2}}$
$= 10^{\frac{3}{2}}$

Rewrite the square root as power $\frac{1}{2}$ and then add indices for the common base 10. Recall $10 = 10^1$, so $1 + \frac{1}{2} = \frac{3}{2}$.

An alternative answer is $\sqrt{100} \times \sqrt{10} = 1000^{\frac{1}{2}}$.

Now you try

Express the following in index form.

a $\sqrt{11}$ **b** $\sqrt{3x^7}$ **c** $2\sqrt[4]{x^9}$ **d** $7\sqrt{7}$

Example 20 Writing in surd form

Express the following in surd form.

a $3^{\frac{1}{5}}$ **b** $5^{\frac{2}{3}}$

SOLUTION

a $3^{\frac{1}{5}} = \sqrt[5]{3}$

b $5^{\frac{2}{3}} = (5^{\frac{1}{3}})^2$
$= (\sqrt[3]{5})^2$

Alternatively:
$5^{\frac{2}{3}} = (5^2)^{\frac{1}{3}}$
$= \sqrt[3]{25}$

EXPLANATION

$a^{\frac{1}{n}} = \sqrt[n]{a}$

Use index law 3 whereby $\frac{2}{3} = \frac{1}{3} \times 2$.
$5^{\frac{1}{3}} = \sqrt[3]{5}$

$\frac{1}{3} \times 2$ is the same as $2 \times \frac{1}{3}$.

Now you try

Express the following in surd form.

a $5^{\frac{1}{3}}$ **b** $11^{\frac{2}{3}}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 21 Evaluating numbers with fractional indices

Evaluate the following without a calculator.

a $16^{\frac{1}{2}}$ b $16^{\frac{1}{4}}$ c $27^{-\frac{1}{3}}$

SOLUTION

a $16^{\frac{1}{2}} = \sqrt{16}$
$= 4$

b $16^{\frac{1}{4}} = \sqrt[4]{16}$
$= 2$

c $27^{-\frac{1}{3}} = \dfrac{1}{27^{\frac{1}{3}}}$
$= \dfrac{1}{\sqrt[3]{27}}$
$= \dfrac{1}{3}$

EXPLANATION

$16^{\frac{1}{2}}$ means $\sqrt{16}$.

$16^{\frac{1}{4}}$ means $\sqrt[4]{16}$ and $2^4 = 16$.

Rewrite, using positive indices. Recall that $a^{-m} = \dfrac{1}{a^m}$.

$27^{\frac{1}{3}}$ means $\sqrt[3]{27}$ and $3^3 = 27$.

Now you try

Evaluate the following without a calculator.

a $25^{\frac{1}{2}}$ b $27^{\frac{1}{3}}$ c $16^{-\frac{1}{4}}$

Exercise 3H

FLUENCY	1, 2–5(1/2)	2–5(1/2)	2–5(1/3)

1 Express the following in index form.

Example 19a a i $\sqrt{13}$ ii $\sqrt{19}$

Example 19b b i $\sqrt{6x^3}$ ii $\sqrt{11x^7}$

Example 19c c i $4\sqrt[4]{x^5}$ ii $7\sqrt[3]{x^2}$

Example 19d d i $6\sqrt{6}$ ii $13\sqrt{13}$

Example 19a,b

2 Express the following in index form.

a $\sqrt{29}$ b $\sqrt[3]{35}$ c $\sqrt[5]{x^2}$ d $\sqrt[4]{b^3}$

e $\sqrt{2a}$ f $\sqrt[3]{4t^7}$ g $\sqrt[5]{10t^2}$ h $\sqrt[8]{8m^4}$

Example 19c,d

3 Express the following in index form.

a $7\sqrt{x^5}$ b $6\sqrt[3]{n^7}$ c $3\sqrt[4]{y^{12}}$

d $5\sqrt[3]{p^2r}$ e $2\sqrt[3]{a^4b^2}$ f $2\sqrt[4]{g^3h^5}$

g $5\sqrt{5}$ h $7\sqrt{7}$ i $4\sqrt[3]{4}$

Example 20

4 Express the following in surd form.

a $2^{\frac{1}{5}}$ **b** $8^{\frac{1}{7}}$ **c** $6^{\frac{1}{3}}$ **d** $11^{\frac{1}{10}}$

e $3^{\frac{2}{3}}$ **f** $7^{\frac{2}{3}}$ **g** $2^{\frac{3}{5}}$ **h** $3^{\frac{4}{7}}$

Example 21

5 Evaluate without using a calculator.

a $36^{\frac{1}{2}}$ **b** $27^{\frac{1}{3}}$ **c** $64^{\frac{1}{3}}$

d $49^{\frac{1}{2}}$ **e** $16^{\frac{1}{4}}$ **f** $125^{\frac{1}{3}}$

g $9^{-\frac{1}{2}}$ **h** $32^{-\frac{1}{5}}$ **i** $81^{-\frac{1}{4}}$

j $1000^{-\frac{1}{3}}$ **k** $400^{-\frac{1}{2}}$ **l** $10\,000^{-\frac{1}{4}}$

PROBLEM-SOLVING	6(1/2)	6–7(1/2)	6–8(1/3)

6 Evaluate without using a calculator.

a $8^{\frac{2}{3}}$ **b** $32^{\frac{3}{5}}$ **c** $36^{\frac{3}{2}}$

d $16^{\frac{5}{4}}$ **e** $16^{-\frac{3}{4}}$ **f** $27^{-\frac{2}{3}}$

g $64^{-\frac{2}{3}}$ **h** $25^{-\frac{3}{2}}$ **i** $\dfrac{1}{25^{-\frac{3}{2}}}$

j $\dfrac{2}{4^{\frac{5}{2}}}$ **k** $\dfrac{3}{9^{\frac{5}{2}}}$ **l** $\dfrac{10}{100^{\frac{3}{2}}}$

7 Use index laws to simplify the following.

a $a^{\frac{1}{2}} \times a^{\frac{3}{2}}$ **b** $m^{\frac{3}{2}} \times m^{\frac{3}{2}}$ **c** $x^{\frac{7}{3}} \div x^{\frac{4}{3}}$ **d** $b^{\frac{5}{4}} \div b^{\frac{3}{4}}$

e $\left(s^{\frac{3}{2}}\right)^{\frac{4}{7}}$ **f** $\left(y^{\frac{1}{3}}\right)^{\frac{1}{3}}$ **g** $\left(t^{\frac{2}{11}}\right)^{0}$ **h** $\left(\dfrac{a^{\frac{2}{3}}}{b^{\frac{4}{3}}}\right)^{\frac{3}{4}}$

8 Simplify the following.

a $\sqrt{25s^4}$ **b** $\sqrt[3]{27t^6}$ **c** $\sqrt[4]{16t^8}$ **d** $\sqrt[3]{125t^{12}}$

e $(x^3)^{\frac{1}{3}}$ **f** $(b^{12})^{\frac{1}{3}}$ **g** $\left(t^{\frac{1}{4}}\right)^{12}$ **h** $\left(m^{\frac{1}{5}}\right)^{10}$

i $(16a^2b^8)^{\frac{1}{2}}$ **j** $(216m^6n^3)^{\frac{1}{3}}$ **k** $(32x^{10}y^{15})^{\frac{1}{5}}$ **l** $(343r^9t^6)^{\frac{1}{3}}$

m $\sqrt{\dfrac{25}{49}}$ **n** $\sqrt[3]{\dfrac{8x^3}{27}}$ **o** $\left(\dfrac{32}{x^{10}}\right)^{\frac{1}{5}}$ **p** $\left(\dfrac{10^2x^4}{0.01}\right)^{\frac{1}{4}}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

REASONING	9	9	9, 10

9 As shown below, $16^{\frac{5}{4}}$ can be evaluated in two ways.

Method A

$$\begin{aligned} 16^{\frac{5}{4}} &= (16^5)^{\frac{1}{4}} \\ &= (1\,048\,576)^{\frac{1}{4}} \\ &= \sqrt[4]{1\,048\,576} \\ &= 32 \end{aligned}$$

Method B

$$\begin{aligned} 16^{\frac{5}{4}} &= \left(16^{\frac{1}{4}}\right)^5 \\ &= (\sqrt[4]{16})^5 \\ &= 2^5 \\ &= 32 \end{aligned}$$

a If $16^{\frac{5}{4}}$ is to be evaluated without a calculator, which method above would be preferable?

b Use your preferred method to evaluate the following without a calculator.

i $8^{\frac{5}{3}}$ **ii** $36^{\frac{3}{2}}$ **iii** $16^{\frac{7}{4}}$ **iv** $27^{\frac{4}{3}}$

v $125^{\frac{4}{3}}$ **vi** $\left(\frac{1}{9}\right)^{\frac{3}{2}}$ **vii** $\left(\frac{4}{25}\right)^{\frac{5}{2}}$ **viii** $\left(\frac{27}{1000}\right)^{\frac{4}{3}}$

10 Explain why $\sqrt[6]{64}$ is not a surd.

ENRICHMENT: Does it exist?	–	–	11

11 We know that when $y = \sqrt{x}$, where $x < 0$, y is not a real number. This is because the square of y cannot be negative; i.e. $y^2 \neq x$ since y^2 is positive and x is negative.

But we know that $(-2)^3 = -8$ so $\sqrt[3]{-8} = -2$.

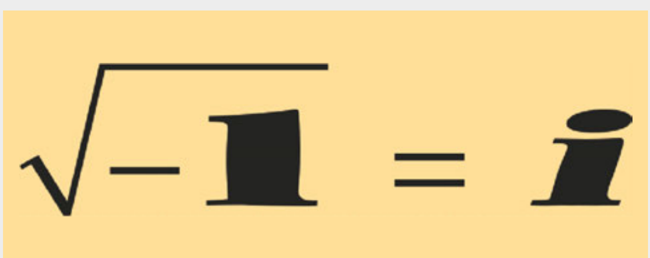
a Evaluate:

i $\sqrt[3]{-27}$ **ii** $\sqrt[3]{-1000}$ **iii** $\sqrt[5]{-32}$ **iv** $\sqrt[7]{-2187}$

b Decide if these are real numbers.

i $\sqrt{-5}$ **ii** $\sqrt[3]{-7}$ **iii** $\sqrt[5]{-16}$ **iv** $\sqrt[4]{-12}$

c If $y = \sqrt[n]{x}$ and $x < 0$, for what values of n is y a real number?



The square root of a negative value is not a real number. $\sqrt{-1} = i$ and is found in a special set of numbers called complex numbers.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

3I Exponential equations 10A

Learning intentions
- To know the form of an exponential equation
- To be able to rewrite an expression using its lowest base
- To be able to solve simple exponential equations using a common base

Equations can take many forms. For example, $2x - 1 = 5$ and $5(a - 3) = -3(3a + 7)$ are both linear equations; $x^2 = 9$ and $3x^2 - 4x - 9 = 0$ are quadratic equations; and $2^x = 8$ and $3^{2x} - 3^x - 6 = 0$ are exponential equations. Exponential equations contain a pronumeral within the index or indices of the terms in the equation. To solve for the unknown in exponential equations we use our knowledge of indices and surds and try to equate powers where possible.

Solving exponential equations can predict the timing of future outcomes. When will my new apartment double in value? When will Australia's population reach 30 million? How long before my coffee goes cold?

LESSON STARTER 2 to the power of what number is 5?

We know that 2 to the power of 2 is 4 and 2 to the power of 3 is 8, but 2 to the power of what number is 5? That is, what is x when $2^x = 5$?

- Use a calculator and trial and error to estimate the value of x when $2^x = 5$ by completing this table.

x	2	3	2.5	2.1	
2^x	4	8	5.65 …		
Result	too small	too big	too big		

- Continue trying values until you find the answer, correct to three decimal places.

KEY IDEAS

- A simple **exponential equation** is of the form $a^x = b$, where $a > 0$, $b > 0$ and $a \neq 1$.
  - There is only one solution to exponential equations of this form.
- Many exponential equations can be solved by expressing both sides of the equation using the same base.
  - We use this fact: if $a^x = a^y$ then $x = y$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

BUILDING UNDERSTANDING

1 a Evaluate the following.

i 2^2 ii 2^3 iii 2^4 iv 2^5

b Hence, state the value of x when:

i $2^x = 8$ ii $2^x = 32$ iii $2^x = 64$

2 Complete these patterns, which involve powers.

a 2, 4, 8, ___, ___, ___, ___, ___, ___

b 3, 9, 27, ___, ___, ___, ___, ___

c 4, 16, ___, ___, ___, ___

d 5, 25, ___, ___, ___

e 6, 36, ___, ___

3 State these numbers in index form. For example, $32 = 2^5$.

a 9 b 125 c 243 d 128 e 729

Example 22 Solving exponential equations

Solve for x in each of the following.

a $2^x = 16$ b $3^x = \frac{1}{9}$ c $25^x = 125$

SOLUTION	EXPLANATION
a $2^x = 16$ $2^x = 2^4$ $\therefore x = 4$	Rewrite 16 as a power, using the base 2. Equate powers using the result: if $a^x = a^y$ then $x = y$.
b $3^x = \frac{1}{9}$ $3^x = \frac{1}{3^2}$ $3^x = 3^{-2}$ $\therefore x = -2$	Rewrite 9 as a power of 3, then write using a negative index. Equate powers with the same base.
c $25^x = 125$ $(5^2)^x = 5^3$ $5^{2x} = 5^3$ $\therefore 2x = 3$ $x = \frac{3}{2}$	Since 25 and 125 are both powers of 5, rewrite both with a base of 5. Apply index law 3 to remove brackets and multiply indices, then equate powers and solve for x.

Now you try

Solve for x in each of the following.

a $3^x = 27$ b $2^x = \frac{1}{8}$ c $16^x = 64$

ISBN 978-1-108-77290-7

Example 23 Solving exponential equations with a variable on both sides

Solve $3^{2x-1} = 27^x$.

SOLUTION

$$\begin{aligned} 3^{2x-1} &= 27^x \\ 3^{2x-1} &= (3^3)^x \\ 3^{2x-1} &= 3^{3x} \\ \therefore 2x - 1 &= 3x \\ -1 &= x \\ \therefore x &= -1 \end{aligned}$$

EXPLANATION

Rewrite 27 as a power of 3.

Apply index law 3 to remove brackets and then equate powers.

Subtract $2x$ from both sides and answer with x as the subject.

Now you try

Solve $5^{3x-1} = 25^x$.

Exercise 3I

FLUENCY	1, 2–4(1/2)	2–4(1/2)	2–4(1/3)

1 Solve for x in each of the following.

Example 22a **a** i $5^x = 25$ ii $2^x = 8$

Example 22b **b** i $2^x = \frac{1}{4}$ ii $5^x = \frac{1}{125}$

Example 22c **c** i $9^x = 27$ ii $4^x = 8$

Example 22a **2** Solve for x in each of the following.

a $3^x = 27$ **b** $2^x = 8$ **c** $6^x = 36$
d $9^x = 81$ **e** $5^x = 125$ **f** $4^x = 64$
g $3^x = 81$ **h** $6^x = 216$ **i** $5^x = 625$
j $2^x = 32$ **k** $10^x = 10000$ **l** $7^x = 343$

Example 22b **3** Solve for x in each of the following.

a $7^x = \frac{1}{49}$ **b** $9^x = \frac{1}{81}$ **c** $11^x = \frac{1}{121}$
d $4^x = \frac{1}{256}$ **e** $3^x = \frac{1}{243}$ **f** $5^{-x} = \frac{1}{125}$
g $3^{-x} = \frac{1}{9}$ **h** $2^{-x} = \frac{1}{64}$ **i** $7^{-x} = \frac{1}{343}$

Example 22c **4** Solve for x in each of the following.

a $9^x = 27$ **b** $8^x = 16$ **c** $25^x = 125$
d $16^x = 64$ **e** $81^x = 9$ **f** $216^x = 6$
g $32^x = 2$ **h** $10000^x = 10$ **i** $7^{-x} = 49$
j $4^{-x} = 256$ **k** $16^{-x} = 64$ **l** $25^{-x} = 125$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

PROBLEM-SOLVING	5	5, 6(1/2)	6(1/3), 7

5 The population of bacteria in a dish is given by the rule $P = 2^t$, where P is the bacteria population and t is the time in minutes.

a What is the initial population of bacteria; i.e. when $t = 0$?

b What is the population of bacteria after:

i 1 minute? ii 5 minutes? iii 1 hour? iv 1 day?

c How long does it take for the population to reach:

i 8? ii 256? iii more than 1000?

Example 23

6 Solve for x in each of the following.

a $2^{x+1} = 8^x$ b $3^{2x+1} = 27^x$ c $7^{x+9} = 49^{2x}$

d $5^{x+3} = 25^{2x}$ e $6^{2x+3} = 216^{2x}$ f $9^{x+12} = 81^{x+5}$

g $27^{x+3} = 9^{2x}$ h $25^{x+3} = 125^{3x}$ i $32^{2x+3} = 128^{2x}$

j $27^{2x+3} = 9^{2x-1}$ k $9^{x-1} = 27^{2x-6}$ l $49^{2x-3} = 343^{2x-1}$

7 Would you prefer \$1 million now or 1 cent doubled every second for 30 seconds? Give reasons for your preference.

REASONING	8	8, 9(1/2)	9(1/2), 10, 11(1/2)

8 Consider a^x, where $a = 1$.

a Evaluate 1^x when:

i $x = 1$ ii $x = 3$ iii $x = 10\,000\,000$

b Are there any solutions to the equation $a^x = 2$ when $a = 1$? Give a reason.

9 Recall that $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[3]{x} = x^{\frac{1}{3}}$. Now solve the following.

a $3^x = \sqrt{81}$ b $5^x = \sqrt{25}$ c $6^x = \sqrt[3]{36}$ d $4^x = \sqrt[4]{64}$

e $2^x = \sqrt[4]{32}$ f $3^x = \sqrt[9]{27}$ g $25^x = \sqrt[5]{125}$ h $9^x = \dfrac{1}{\sqrt[3]{27}}$

10 a Write these numbers as decimals.

i $\dfrac{1}{2^2}$ ii 2^{-3} iii 10^{-3} iv $\left(\dfrac{1}{5}\right)^4$

b Write these decimal numbers as powers of prime numbers.

i 0.04 ii 0.0625 iii 0.5 iv 0.0016

11 Show how you can use techniques from this section to solve these equations involving decimals.

a $10^x = 0.0001$ b $2^x = 0.015625$ c $5^x = 0.00032$

d $(0.25)^x = 0.5$ e $(0.04)^x = 125$ f $(0.0625)^{x+1} = \dfrac{1}{2}$

ENRICHMENT: Mixing index laws with equations	–	–	12(1/2)

12 Solve for n in the following.

a $3^n \times 9^n = 27$ b $5^{3n} \times 25^{-2n+1} = 125$ c $2^{-3n} \times 4^{2n-2} = 16$

d $3^{2n-1} = \dfrac{1}{81}$ e $7^{2n+3} = \dfrac{1}{49}$ f $5^{3n+2} = \dfrac{1}{625}$

g $6^{2n-6} = 1$ h $11^{3n-1} = 11$ i $8^{5n-1} = 1$

j $\dfrac{3^{n-2}}{9^{1-n}} = 9$ k $\dfrac{5^{3n-3}}{25^{n-3}} = 125$ l $\dfrac{36^{3+2n}}{6^n} = 1$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

3J Graphs of exponentials

Learning intentions

- To know what defines an exponential relation
- To know the meaning of the term asymptote
- To know the basic features of an exponential graph
- To be able to sketch simple exponential graphs including those involving reflections
- To know how to find the point of intersection of an exponential graph and a horizontal line

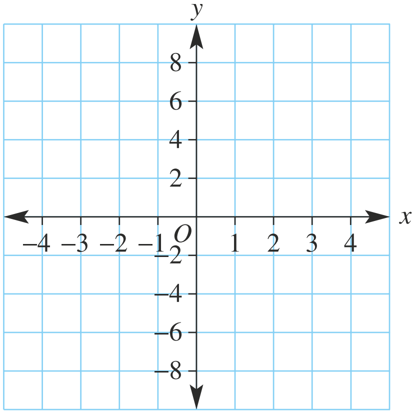
We saw earlier that indices can be used to describe some special relations. The population of the world, for example, or the balance of an investment account can be described using exponential rules that include indices. The rule $A = 100\,000(1.05)^t$ describes the account balance of \$100 000 invested at 5% p.a. compound interest for t years.

When a patient receives medication, the blood concentration decays exponentially as the body breaks it down. Exponential rules can determine the safe time between doses, from the highest safe level to the lowest effective level.

LESSON STARTER What do $y = 2^x$, $y = -2^x$ and $y = 2^{-x}$ all have in common?

Complete this table and graph all three relations on the same set of axes before discussing the points below.

x	–3	–2	–1	0	1	2	3
$y_1 = 2^x$	$\frac{1}{8}$			1		4	
$y_2 = -2^x$							
$y_3 = 2^{-x}$							



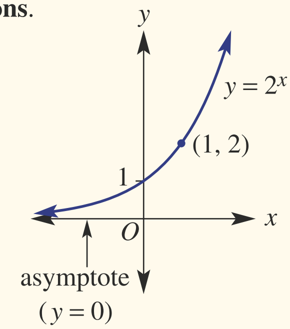
- Discuss the shape of each graph.
- Where does each graph cut the y-axis?
- Do the graphs have x-intercepts? Why not?
- What is the one feature they all have in common?

ISBN 978-1-108-77290-7

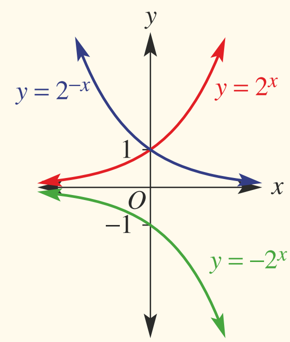
Cambridge University Press
Updated June 2022

KEY IDEAS

- $y = 2^x$, $y = (0.4)^x$, $y = 3 \times (1.1)^x$ are examples of **exponential relations**.
- An **asymptote** is a line that a curve approaches, by getting closer and closer to it, but never reaching.



- A simple **exponential** rule is of the form $y = a^x$, where $a > 0$ and $a \neq 1$.
  - y-intercept is 1.
  - $y = 0$ is the equation of the asymptote.
- The graph of $y = -a^x$ is the reflection of the graph of $y = a^x$ in the x-axis. (Note: $y = -a^x$ means $y = -1 \times a^x$.)
- The graph of $y = a^{-x}$ is the reflection of the graph of $y = a^x$ in the y-axis.



- To find the intersection points of a simple exponential and a horizontal line, use the method of substitution and equate powers after expressing both sides of the equation using the same base.
  For example, for $y = 2^x$ and $y = 16$, solve
  $$2^x = 16$$
  $$2^x = 2^4$$
  $$\therefore x = 4$$

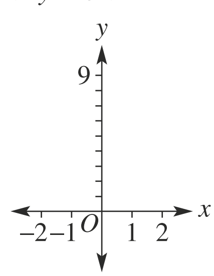
BUILDING UNDERSTANDING

1 Consider the exponential rule $y = 3^x$.

a Complete this table.

x	-2	-1	0	1	2
y		$\frac{1}{3}$	1		

b Plot the points in the table to form the graph of $y = 3^x$.



2 Complete the following.

a Graphs of the form $y = a^x$, $a > 0$ have an ______________ with equation $y = 0$ (the x-axis).

b The y-intercept of the graph $y = a^x$, $a > 0$ is ___________.

c The graph of $y = 4^{-x}$ is a reflection of the graph of $y = 4^x$ in the ______________.

d The graph of $y = -5^x$ is a reflection of the graph of $y = 5^x$ in the ______________.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

3 **a** Explain the difference between a^{-2} and $-a^2$.

b True or false: $-3^2 = \frac{1}{3^2}$? Explain why.

c Express with negative indices: $\frac{1}{5^3}, \frac{1}{3^2}, \frac{1}{2}$.

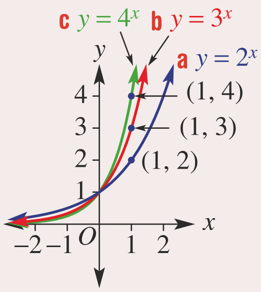
d Simplify: $-3^2, -5^3, -2^{-2}$.

Example 24 Sketching graphs of exponentials

Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = 2^x$ **b** $y = 3^x$ **c** $y = 4^x$

SOLUTION



EXPLANATION

$a^0 = 1$, so all y-intercepts are at 1.

$y = 4^x$ is steeper than $y = 3^x$, which is steeper than $y = 2^x$.

Substitute $x = 1$ into each rule to obtain a second point to indicate the steepness of each curve.

Now you try

Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

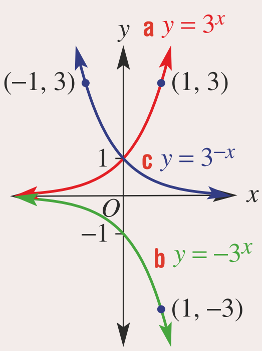
a $y = 2^x$ **b** $y = 5^x$

Example 25 Sketching with reflections

Sketch the graphs of these exponentials on the same set of axes.

a $y = 3^x$ **b** $y = -3^x$ **c** $y = 3^{-x}$

SOLUTION



EXPLANATION

The graph of $y = -3^x$ is a reflection of the graph of $y = 3^x$ in the x-axis.

Check: $x = 1, y = -3^1 = -3$

The graph of $y = 3^{-x}$ is a reflection of the graph of $y = 3^x$ in the y-axis.

Check: $x = 1, y = 3^{-1} = \frac{1}{3}$

$x = -1, y = 3^1 = 3$

Continued on next page

Now you try

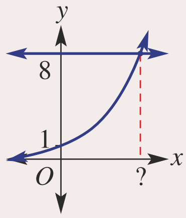
Sketch the graphs of these exponentials on the same set of axes.

a $y = 2^x$ **b** $y = -2^x$ **c** $y = 2^{-x}$

Example 26 Solving exponential equations

Find the intersection of the graphs of $y = 2^x$ and $y = 8$.

SOLUTION	EXPLANATION
$y = 2^x$	Set $y = 8$ and write 8 with base 2.
$8 = 2^x$	
$2^3 = 2^x$	Since the bases are the same, equate the powers.
$x = 3$	
$\therefore$ Intersection point is (3, 8).	



Now you try

Find the intersection of the graphs of $y = 3^x$ and $y = 27$.

Exercise 3J

FLUENCY	1–5	2–6	2–6

Example 24 **1** Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = 2^x$ **b** $y = 6^x$

Example 24 **2** Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = 2^x$ **b** $y = 4^x$ **c** $y = 5^x$

Example 25 **3** Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = 2^x$ **b** $y = -2^x$ **c** $y = 2^{-x}$

Example 25b **4** Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = -2^x$ **b** $y = -5^x$ **c** $y = -3^x$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 25c

5 Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = 2^{-x}$ **b** $y = 3^{-x}$ **c** $y = 6^{-x}$

6 **a** Find the coordinates on the graph of $y = 3^x$, where:

i $x = 0$ **ii** $x = -1$ **iii** $y = 1$ **iv** $y = 9$

b Find the coordinates on the graph of $y = -2^x$, where:

i $x = 4$ **ii** $x = -1$ **iii** $y = -1$ **iv** $y = -4$

c Find the coordinates on the graph of $y = 4^{-x}$, where:

i $x = 1$ **ii** $x = -3$ **iii** $y = 1$ **iv** $y = \frac{1}{4}$

PROBLEM-SOLVING	7, 8	7(1/2), 8, 9	7(1/2), 8–10

Example 26

7 **a** Find the intersection of the graphs of $y = 2^x$ and $y = 4$.

b Find the intersection of the graphs of $y = 3^x$ and $y = 9$.

c Find the intersection of the graphs of $y = -4^x$ and $y = -4$.

d Find the intersection of the graphs of $y = 2^{-x}$ and $y = 8$.

8 A study shows that the population of a town is modelled by the rule $P = 2^t$, where t is in years and P is in thousands of people.

a State the number of people in the town at the start of the study.

b State the number of people in the town after:

i 1 year **ii** 3 years

c When is the town's population expected to reach:

i 4000 people? **ii** 16 000 people?

9 A single bacterium divides into two every second, so one cell becomes 2 in the first second and in the next second two cells become 4 and so on.

a Write a rule for the number of bacteria, N, after t seconds.

b How many bacteria will there be after 10 seconds?

c How long does it take for the population to exceed 10 000? Round to the nearest second.

10 Use trial and error to find x when $2^x = 5$. Give the answer correct to three decimal places.

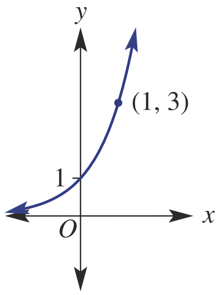
REASONING	11	11, 12	11–14

11 Match equations **a**–**f** with graphs **A**–**F** below.

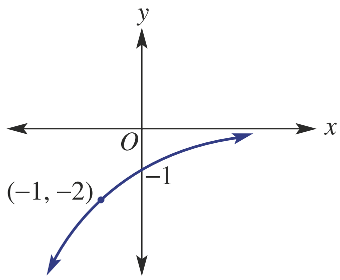
a $y = -x - 2$ **b** $y = 3^x$ **c** $y = 3^{-x}$

d $y = -2^x$ **e** $y = x$ **f** $y = -2^{-x}$

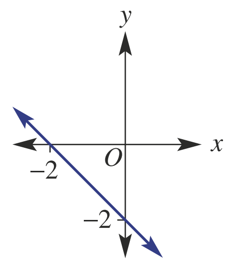
A



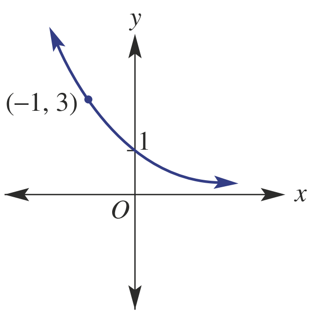
B



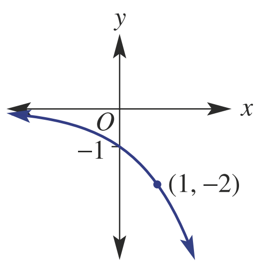
C



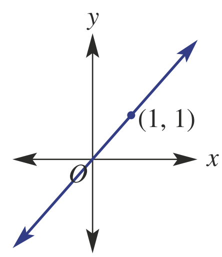
D



E



F



12 Explain why the point $(2, 5)$ does not lie on the curve with equation $y = 2^x$.

13 Describe and draw the graph of the line with equation $y = a^x$ when $a = 1$.

14 Explain why $2^x = 0$ is never true for any value of x.

ENRICHMENT: $y = 2^{-x}$ and $y = \left(\frac{1}{2}\right)^x$	–	–	15

15 Consider the exponential rules $y = 2^{-x}$ and $y = \left(\dfrac{1}{2}\right)^x$.

a Using $-3 \leqslant x \leqslant 3$, sketch graphs of the rules on the same set of axes. What do you notice?

b Write the following rules in the form $y = a^x$, where $0 < a < 1$.

i $y = 3^{-x}$ **ii** $y = 5^{-x}$ **iii** $y = 10^{-x}$

c Write the following rules in the form $y = a^{-x}$, where $a > 1$.

i $y = \left(\dfrac{1}{4}\right)^x$ **ii** $y = \left(\dfrac{1}{7}\right)^x$ **iii** $y = \left(\dfrac{1}{11}\right)^x$

d Prove that $\left(\dfrac{1}{a}\right)^x = a^{-x}$, for $a > 0$.

ISBN 978-1-108-77290-7

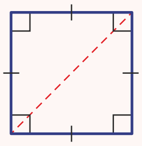
Cambridge University Press
Updated June 2022

Applications and problem-solving

The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

Square diagonals

1 Square sand boxes produced by a company for playgrounds are labelled on the packaging with their diagonal length.



A landscaper is interested in the relationship between this diagonal length and other properties of the sand box including perimeter and area.

a A square sand box has a diagonal length of $\sqrt{3}$ m. Give the area and perimeter of this sand box in simplified form.

b A second square sand box has diagonal length $(2 + 2\sqrt{2})$ m.

i Find the exact area occupied by this sand box in m^2, using $(a + b)(c + d) = ac + ad + bc + bd$ to expand.

ii Express the side length of the sand box in metres in the form $\sqrt{a + b\sqrt{c}}$ where a, b and c are integers.

c To determine the side length of the sand box in part **b** in simplified form, consider the following.

i Use expansion to show that $(\sqrt{x} + \sqrt{y})^2 = x + y + 2\sqrt{xy}$ where x and y are positive integers.

ii Make use of the result in part **i** to simplify $\sqrt{7 + 2\sqrt{10}}$ and $\sqrt{7 + 4\sqrt{3}}$.

iii Hence, simplify your answer to part **b ii** and give the perimeter of the sand box.

Air conditioner thermostat

2 An air conditioning unit inside a room has a thermostat that controls the temperature of the room. The temperature of the room, T°C, n hours after the air conditioning unit switches on is given by $T = 17 + \frac{8}{2^n}$.

The air conditioning unit is set to turn on when the room temperature reaches 25°C.

A technician wishes to investigate how exponential equations can model the change in air temperature and how thermostats can be used to control the use of air conditioners.

a If the air conditioning unit remains on for 1 hour after it switches on, what will be the temperature in the room?

b After how many hours of the unit being on would the temperature in the room reach 19°C?

The unit is programmed to switch off when the temperature in the room reaches 20°C.

c Find the longest consecutive period of time that the unit could be on for, correct to one decimal place.

d Sketch a graph of the temperature in the room, T, from when the unit switches on until when it switches off.

e Express the rule for the temperature T in the form $T = 17 + 2^{k-n}$ where k is an integer.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

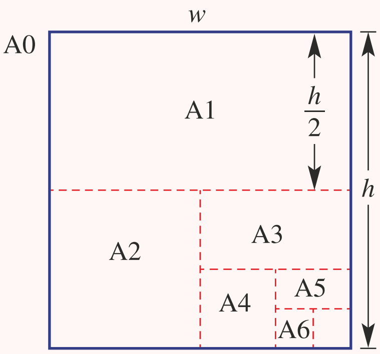
The thermostat is adjusted so that it turns on at 24°C and so that the fan strength is decreased. This unit switches off when the room is cooled to 21°C, which occurs after it has been on for 2 hours.

f Find the values of a and k, where a and k are integers, if the rule for the temperature, T°C, of the room n hours after this unit is turned on is given by $T = a + 2^{k-n}$.

International paper sizes

3 The A series of paper sizes, e.g. A4, are based on international standards. The paper sizes are such that the ratio between the height and width of each paper size is the same. The height is taken to be the longer side length of each rectangle.

Let an A0 piece of paper have width w mm and height h mm.



A paper company wants to explore the A series paper sizes and use ratios to connect the lengths and widths of successive sizes. It wishes to use these ratios to then determine various widths and heights and the rules that link these dimensions.

a Complete the table below for the corresponding height and width of the A series paper in terms of h and w.

An	Width	Height
A0	w	h
A1	$\frac{h}{2}$	w
A2		
A3		
A4		
A5		
A6		

b Determine the ratio of the height to the width of A series paper if it is the same for each paper size An.

c From your result in part **b**, write a rule for the height, h, of A series paper in terms of its width, w.

d A0 paper has an area of 1 square metre (1000 mm × 1000 mm). Determine the dimensions, w and h, of A0 paper in exact form in mm.

e Use your values from part **d** and your table from part **a** to determine the dimensions of an A4 sheet to the nearest millimetre. Measure a sheet of A4 paper to compare.

f Consider the table in part **a** and paper sizes An.

i Describe the changes to the values of the width and height as n increases when n is even and when n is odd.

ii Use your table and dimensions from part **d** to come up with rules for w and h when n is even and when n is odd.

iii Use your rule from part **ii** to find the length and width for A3 and A4 paper and check by measuring the paper.

ISBN 978-1-108-77290-7

3K Exponential growth and decay

Learning intentions

- To understand how percentage increase and decrease relate to exponential growth and decay
- To know the general form of the exponential growth and decay model
- To be able to write an exponential rule from a word problem and apply it

The population of a country increasing by 5% per year and an investment increasing, on average, by 12% per year are examples of exponential growth. When an investment grows exponentially, the increase per year is not constant. The annual increase is calculated on the value of the investment at that time, and this changes from year to year because of the added investment returns. The more money you have invested, the more interest you will make in a year.

Population growth can be modelled using exponential equations. Governments use projected population numbers when planning for future infrastructure, land use, and water, energy and food security.

In the same way, a population can grow exponentially. A growth of 5% in a large population represents many more babies born in a year than 5% of a small population.

Here we will focus on exponential growth and decay in general and compound interest will be studied in the next section.

LESSON STARTER A compound rule

Imagine you have an antique car valued at \$100 000 and you hope that it will increase in value at 10% p.a. The 10% increase is to be added to the value of the car each year.

- Discuss how to calculate the value of the car after 1 year.
- Discuss how to calculate the value of the car after 2 years.
- Complete this table.

Year	0	1	2	3
Value (\$)	100 000	$100\,000 \times 1.1$ $=$ ______	$100\,000 \times 1.1 \times$ ____ $=$ ______	______ $=$ ______

- Discuss how indices can be used to calculate the value of the car after the second year.
- Discuss how indices can be used to calculate the value of the car after the tenth year.
- What might be the rule connecting the value of the car (\$$A$) and the time n years?
- Repeat the steps above if the value of the car decreases by 10% p.a.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

KEY IDEAS

- **Per annum** (p.a.) means 'per year'.
- Exponential growth and decay can be modelled by the rule $A = ka^t$, where A is the amount, k is the initial amount and t is the time.
  - When $a > 1$, exponential growth occurs.
  - When $0 < a < 1$, exponential decay occurs.
- For a **growth** rate of $r\%$ p.a., the base 'a' is calculated using $a = 1 + \frac{r}{100}$.
- For a **decay** rate of $r\%$ p.a., the base 'a' is calculated using $a = 1 - \frac{r}{100}$.
- The basic **exponential formula** can be summarised as $A = A_0\left(1 \pm \frac{r}{100}\right)^n$.
  - The subscript zero is often used to indicate the initial value of a quantity (e.g. P_0 is initial population).

BUILDING UNDERSTANDING

1. An antique ring is purchased for \$1000 and is expected to grow in value by 5% per year. Round your answers to the nearest cent.
   a Find the increase in value in the first year.
   b Find the value of the ring at the end of the first year.
   c Find the increase in value in the second year.
   d Find the increase in value in the third year.
   e Find the value of the ring at the end of the fifth year.
2. The mass of a limestone 5 kg rock exposed to the weather is decreasing at a rate of 2% per annum.
   a Find the mass of the rock at the end of the first year.
   b State the missing numbers for the mass of the rock (M kg) after t years.
   $M = 5(1 - \_\_\_\_)^t$
   $= 5 \times \_\_\_\_^t$
   c Use your rule to calculate the mass of the rock after 5 years, correct to two decimal places.
3. Decide if the following represent exponential *growth* or exponential *decay*.
   a $A = 1000 \times 1.3^t$
   b $A = 350 \times 0.9^t$
   c $P = P_0\left(1 + \frac{3}{100}\right)^t$
   d $T = T_0\left(1 - \frac{7}{100}\right)^t$

Example 27 Writing exponential rules

Form exponential rules for the following situations.

a John has a painting that is valued at \$100 000 and it is expected to increase in value by 14% per annum.

b A city's initial population of 50 000 is decreasing by 12% per year.

ISBN 978-1-108-77290-7

SOLUTION

a Let A = the value in \$ of the painting at any time
n = the number of years the painting is kept
$r = 14$
$A_0 = 100\,000$

$$A = 100\,000\left(1 + \frac{14}{100}\right)^n$$

$\therefore A = 100\,000(1.14)^n$

EXPLANATION

Define your variables.

$$A = A_0\left(1 \pm \frac{r}{100}\right)^n$$

Substitute $r = 14$ and $A_0 = 100\,000$ and use '+' since we have growth.

b Let P = the population at any time
n = the number of years the population decreases
$r = 12$
$P_0 = 50\,000$

$$P = 50\,000\left(1 - \frac{12}{100}\right)^n$$

$\therefore P = 50\,000(0.88)^n$

Define your variables.

$$P = P_0\left(1 \pm \frac{r}{100}\right)^n$$

Substitute $r = 12$ and $P_0 = 50\,000$ and use '−' since we have decay.

Now you try

Form exponential rules for the following situations.

a Caz has a vase that is valued at \$50 000 and it is expected to increase in value by 16% per annum.

b A town's initial population of 10 000 is decreasing by 9% per year.

Example 28 Applying exponential rules

House prices are rising at 9% per year and Zoe's flat is currently valued at \$600 000.

a Determine a rule for the value of Zoe's flat (\$$V$) in n years' time.

b What will be the value of her flat:

i next year? **ii** in 3 years' time?

c Use trial and error to find when Zoe's flat will be valued at \$900 000, to one decimal place.

SOLUTION

a Let V = value of Zoe's flat at any time
V_0 = starting value \$600 000
n = number of years from now
$r = 9$
$V = V_0(1.09)^n$
$\therefore V = 600\,000(1.09)^n$

EXPLANATION

Define your variables.

$$V = V_0\left(1 \pm \frac{r}{100}\right)^n$$

Use '+' since we have growth.

Continued on next page

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b **i** When $n = 1$, $V = 600\,000(1.09)^1$
$= 654\,000$
Zoe's flat would be valued at \$654 000 next year.

Substitute $n = 1$ for next year.

ii When $n = 3$, $V = 600\,000(1.09)^3$
$= 777\,017.40$
In 3 years' time Zoe's flat will be valued at about \$777 017.

For 3 years, substitute $n = 3$.

c

n	4	5	4.6	4.8	4.7
V	846 949	923 174	891 894	907 399	899 613

Zoe's flat will be valued at \$900 000 in about 4.7 years' time.

Try a value of n in the rule. If V is too low, increase your n value. If V is too high, decrease your n value. Continue this process until you get close to 900 000.

Now you try

House prices are rising at 7% per year and Andrew's apartment is currently valued at \$400 000.

a Determine a rule for the value of Andrew's apartment (\$$V$) in n years' time.

b What will be the value of his apartment:

i next year? **ii** in 3 years' time?

c Use trial and error to find when Andrew's apartment will be valued at \$500 000, to one decimal place.

Exercise 3K

FLUENCY	1, 2–4	2(½), 3–5	2(½), 3, 5, 6

1 Form exponential rules for the following situations.

Example 27a

a Lara has a necklace that is valued at \$6000 and it is expected to increase in value by 12% per annum.

Example 27b

b A village's initial population of 2000 is decreasing by 8% per year.

Example 27

2 Define variables and form exponential rules for the following situations.

a A flat is purchased for \$200 000 and is expected to grow in value by 17% per annum.

b A house initially valued at \$530 000 is losing value at 5% per annum.

c The value of a car, bought for \$14 200, is decreasing at 3% per annum.

d An oil spill, initially covering an area of 2 square metres, is increasing at 5% per minute.

e A tank with 1200 litres of water is leaking at a rate of 10% of the water in the tank every hour.

f A human cell of area 0.01 cm^2 doubles its area every minute.

g A population, which is initially 172 500, is increasing at 15% per year.

h A substance of mass 30 g is decaying at a rate of 8% per hour.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 28

3 The value of a house purchased for \$500 000 is expected to grow by 10% per year. Let \$$A$ be the value of the house after t years.

a Write the missing number in the rule connecting A and t.

$A = 500\,000 \times \underline{\quad\quad}^{t}$

b Use your rule to find the expected value of the house after the following number of years. Round your answer to the nearest cent.

i 3 years **ii** 10 years **iii** 20 years

c Use trial and error to estimate when the house will be worth \$1 million. Round your answer to one decimal place.

4 A share portfolio, initially worth \$300 000, is reduced by 15% p.a. over a number of years. Let \$$A$ be the share portfolio value after t years.

a Write the missing number in the rule connecting A and t.

$A = \underline{\quad\quad} \times 0.85^{t}$

b Use your rule to find the value of the shares after the following number of years. Round your answer to the nearest cent.

i 2 years **ii** 7 years **iii** 12 years

c Use trial and error to estimate when the share portfolio will be valued at \$180 000. Round your answer to one decimal place.

5 A water tank containing 15 000 L has a small hole that reduces the amount of water by 6% per hour.

a Determine a rule for the volume of water (V) left after t hours.

b Calculate (to the nearest litre) the amount of water left in the tank after:

i 3 hours **ii** 7 hours

c How much water is left after two days? Round your answer to two decimal places.

d Using trial and error, determine when the tank holds less than 500 L of water, to one decimal place.

6 Megan invests \$50 000 in a superannuation scheme that has an annual return of 11%.

a Determine the rule for the value of her investment (V) after n years.

b How much will Megan's investment be worth in:

i 4 years? **ii** 20 years?

c Find the approximate time before her investment is worth \$100 000. Round your answer to two decimal places.

PROBLEM-SOLVING	7, 8	7, 8	8, 9

7 A certain type of bacteria grows according to the equation $N = 3000(2.6)^{t}$, where N is the number of cells present after t hours.

a How many bacteria are there at the start?

b Determine the number of cells (round to the whole number) present after:

i 1 hour **ii** 2 hours **iii** 4.6 hours

c If 50 000 000 bacteria are needed to make a drop of serum, determine how long you will have to wait to make a drop (to the nearest minute).

8 A car tyre has 10 mm of tread when new. It is considered unroadworthy when there is only 3 mm left. The rubber wears at 12.5% every 10 000 km.

a Write an equation relating the depth of tread (D) for every 10 000 km travelled.

b Using trial and error, determine when the tyre becomes unroadworthy, to the nearest 10 000 km.

c If a tyre lasts 80 000 km, it is considered to be of good quality. Is this a good quality tyre?

ISBN 978-1-108-77290-7

9 A cup of coffee has an initial temperature of 90°C and the surrounding temperature is 0°C.
   a If the temperature relative to surroundings reduces by 8% every minute, determine a rule for the temperature of the coffee (T°C) after t minutes.
   b What is the temperature of the coffee (to one decimal place) after:
      i 90 seconds?   ii 2 minutes?
   c When is the coffee suitable to drink if it is best consumed at a temperature of 68.8°C? Give your answer to the nearest second.

REASONING	10	10	10, 11

10 The monetary value of things can be calculated using different time periods. Consider a $1000 collector's item that is expected to grow in value by 10% p.a. over 5 years.
- If the increase in value is added annually then $r = 10$ and $t = 5$, so $A = 1000(1.1)^5$.
- If the increase in value is added monthly then $r = \frac{10}{12}$ and $t = 5 \times 12 = 60$, so $A = 1000\left(1 + \frac{10}{1200}\right)^{60}$.

   a If the increase in value is added annually, find the value of the collectors' item, to the nearest cent, after:
      i 5 years   ii 8 years   iii 15 years
   b If the increase in value is added monthly, find the value of the collectors' item, to the nearest cent, after:
      i 5 years   ii 8 years   iii 15 years

11 You inherit a $2000 necklace that is expected to grow in value by 7% p.a. What will the necklace be worth, to the nearest cent, after 5 years if the increase in value is added:
   a annually?   b monthly?   c weekly (assume 52 weeks in the year)?

ENRICHMENT: Half-life	–	–	12–14

Half-life is the period of time it takes for an object to decay by half. It is often used to compare the rate of decay for radioactive materials.

12 A 100 g mass of a radioactive material decays at a rate of 10% every 10 years.
   a Find the mass of the material after the following time periods. Round your answer to one decimal place, where necessary.
      i 10 years   ii 30 years   iii 60 years
   b Estimate the half-life of the radioactive material (i.e. find how long it takes for the material to decay to 50 g). Use trial and error and round your answer to the nearest year.

13 An ice sculpture, initially containing 150 L of water, melts at a rate of 3% per minute.
   a What will be the volume of the ice sculpture after half an hour? Round your answer to the nearest litre.
   b Estimate the half-life of the ice sculpture. Give your answer in minutes, correct to one decimal place.

14 The half-life of a substance is 100 years. Find the rate of decay per annum, expressed as a percentage correct to one decimal place.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

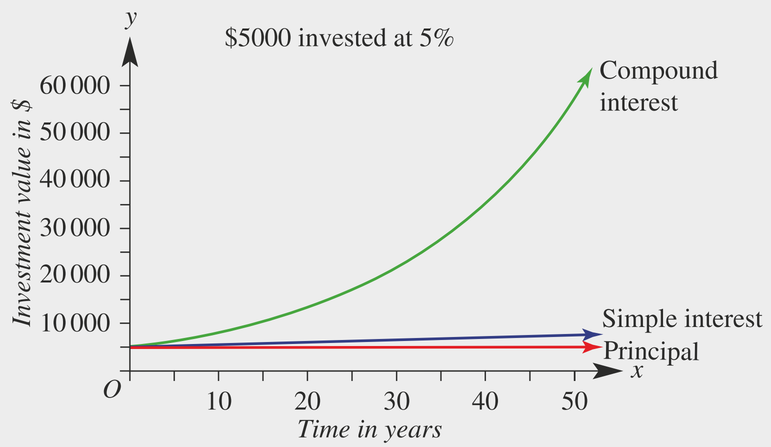
3L Compound interest

Learning intentions

- To know the meaning of the term compound interest
- To know how to apply the compound interest formula
- To know how compound interest is calculated for different time periods
- To be able to determine the total amount and the interest in a compound interest scenario

For simple interest, the interest is always calculated on the principal amount. Sometimes, however, interest is calculated on the actual amount present in an account at each time period that interest is calculated. This means that the interest is added to the amount, then the next lot of interest is calculated again using this new amount.

This process is called compound interest.



The 'magic' growth of compound interest comes from interest paid on previous interest. Retirement savings are especially suited to benefit from compound interest, as this type of investment grows at an increasingly faster rate over time, as you can see in the graph above.

Compound interest can be calculated using updated applications of the simple interest formula or by using the compound interest formula. It is a common example of exponential growth.

LESSON STARTER Investing using updated simple interest

Consider investing $400 at 12% per annum.

- Copy and complete the table below.

Time (n)	Amount (A)	Interest (I)	New amount
1st year	$400	$48	$448
2nd year	$448	$53.76	$501.76
3rd year	$501.76		
4th year			

- What is the balance at the end of 4 years if interest is added to the amount at the end of each year?
- Thinking about this as exponential growth, write a rule linking A with n.

ISBN 978-1-108-77290-7

KEY IDEAS

- **Compound interest** is calculated using updated applications of the simple interest formula.
  For example, $100 compounded at 10% p.a. for 2 years.
  Year 1: $100 + 10\%$ of $100 = \$110$
  Year 2: $110 + 10\%$ of $110 = \$121$, so compound interest $= \$21$
- The total amount in an account using compound interest for a given number of time periods is given by:
  $A = P\left(1 + \frac{r}{100}\right)^n$, where:
  - Principal (P) = the amount of money borrowed or invested.
  - Rate of interest (r) = the percentage applied to the principal per period of investment.
  - Periods (n) = the number of periods the principal is invested.
  - Amount (A) = the total amount of your investment.
- Interest = amount (A) − principal (P)

BUILDING UNDERSTANDING

1. Consider $500 invested at 10% p.a., compounded annually.
   a. How much interest is earned in the first year?
   b. What is the balance of the account once the first year's interest is added?
   c. How much interest is earned in the second year?
   d. What is the balance of the account at the end of the second year?
   e. Use your calculator to work out $500(1.1)^2$.
2. By considering an investment of $4000 at 5% p.a., compounded annually, calculate the missing values in the table below.

Year	Amount ($)	Interest ($)	New amount ($)
1	4000	200	4200
2	4200		
3			
4			
5			

3. Find the value of the following, correct to two decimal places.
   a. $\$1000 \times 1.05 \times 1.05$
   b. $\$1000 \times 1.05^2$
   c. $\$1000 \times 1.05 \times 1.05 \times 1.05$
   d. $\$1000 \times 1.05^3$
4. State the missing numbers.
   a. $700 invested at 8% p.a., compounded annually for 2 years.
   $A = \boxed{}(1.08)^{\boxed{}}$
   b. $1000 invested at 15% p.a., compounded annually for 6 years.
   $A = 1000\left(\boxed{}\right)^6$
   c. $850 invested at 6% p.a., compounded annually for 4 years.
   $A = 850\left(\boxed{}\right)^{\boxed{}}$

Example 29 Using the compound interest formula

Determine the amount after 5 years if \$4000 is compounded annually at 8%. Round to the nearest cent.

SOLUTION

$P = 4000, n = 5, r = 8$

$$A = P\left(1 + \frac{r}{100}\right)^n$$

$$= 4000\left(1 + \frac{8}{100}\right)^5$$

$$= 4000(1.08)^5$$

$$= \$5877.31$$

EXPLANATION

List the values for the terms

Write the formula and then substitute the known values.

Simplify and evaluate.

Write your answer to two decimal places (the nearest cent).

Now you try

Determine the amount after 6 years if \$3000 is compounded annually at 7%. Round to the nearest cent.

Example 30 Converting rates and time periods

Calculate the number of periods and the rates of interest offered per period for the following.

a 6% p.a. over 4 years, paid monthly

b 18% p.a. over 3 years, paid quarterly

SOLUTION

a $n = 4 \times 12 = 48$ $\quad r = 6 \div 12 = 0.5$

b $n = 3 \times 4 = 12$ $\quad r = 18 \div 4 = 4.5$

EXPLANATION

a 4 years is the same as 48 months, as 12 months = 1 year.
6% p.a. = 6% in 1 year.
Divide by 12 to find the monthly rate.

b There are 4 quarters in 1 year.

Now you try

Calculate the number of periods and the rates of interest offered per period for the following.

a 5% p.a. over 5 years, paid monthly

b 14% p.a. over 3 years, paid quarterly

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 31 Finding compounded amounts using months

Anthony's investment of \$4000 is compounded at 8.4% p.a. over 5 years. Determine the amount he will have after 5 years if the interest is paid monthly. Round to the nearest cent.

SOLUTION	EXPLANATION
$P = 4000$	List the values of the terms you know.
$n = 5 \times 12$ $= 60$	Convert the time in years to the number of periods (in this case, months); 60 months = 5 years.
$r = 8.4 \div 12$ $= 0.7$	Convert the rate per year to the rate per period (months) by dividing by 12.
$A = P\left(1 + \frac{r}{100}\right)^n$	Write the formula.
$= 4000(1 + 0.007)^{60}$	Substitute the values, $0.7 \div 100 = 0.007$.
$= 4000(1.007)^{60}$ $= \$6078.95$	Simplify and evaluate, rounding to the nearest cent.

Now you try

Wendy's investment of \$7000 is compounded at 6.2% p.a. over 4 years. Determine the amount she will have after 4 years if the interest is paid monthly. Round to the nearest cent.

Exercise 3L

FLUENCY	1, 2–5(½)	2–5(½)	2–5(½)

Example 29

1 Determine the amount after 4 years if \$5000 is compounded annually at 6%. Round to the nearest cent.

Example 29

2 Determine the amount after 5 years if:
- a \$4000 is compounded annually at 5%
- b \$8000 is compounded annually at 8.35%
- c \$6500 is compounded annually at 16%
- d \$6500 is compounded annually at 8%.

3 Determine the amount if \$100 000 is compounded annually at 6% for:
- a 1 year
- b 2 years
- c 3 years
- d 5 years
- e 10 years
- f 15 years.

Example 30

4 Calculate the number of periods (n) and the rates of interest (r) offered per period for the following. (Round the interest rate to three decimal places where necessary.)
- a 6% p.a. over 3 years, paid bi-annually
- b 12% p.a. over 5 years, paid monthly
- c 4.5% p.a. over 2 years, paid fortnightly
- d 10.5% p.a. over 3.5 years, paid quarterly
- e 15% p.a. over 8 years, paid quarterly
- f 9.6% p.a. over 10 years, paid monthly

ISBN 978-1-108-77290-7

Example 31

5 Calculate the value of the following investments if interest is compounded monthly.

a \$2000 at 6% p.a. for 2 years
b \$34 000 at 24% p.a. for 4 years
c \$350 at 18% p.a. for 8 years
d \$670 at 6.6% p.a. for $2\frac{1}{2}$ years
e \$250 at 7.2% p.a. for 12 years
f \$1200 at 4.8% p.a. for $3\frac{1}{3}$ years

PROBLEM-SOLVING	6	6, 7	6, 8

6 Darinia invests \$5000 compounded monthly at 18% p.a. Determine the value of the investment after:

a 1 month
b 3 months
c 5 months.

7 An investment of \$8000 is compounded at 12.6% over 3 years. Determine the amount the investor will have after 3 years if the interest is compounded monthly.

8 **a** For each rate below, calculate the amount of compound interest paid on \$8000 at the end of 3 years.

i 12% compounded annually
ii 12% compounded bi-annually (i.e. twice a year)
iii 12% compounded monthly
iv 12% compounded weekly
v 12% compounded daily

b What is the interest difference between annual and daily compounding in this case?

REASONING	9	9	9(½), 10

9 The following are expressions relating to compound interest calculations. Determine the principal (P), number of periods (n), rate of interest per period ($r\%$), annual rate of interest ($R\%$) and the overall time (t).

a $300(1.07)^{12}$, bi-annually
b $5000(1.025)^{24}$, monthly
c $1000(1.00036)^{65}$, fortnightly
d $3500(1.000053)^{30}$, daily
e $10\,000(1.078)^{10}$, annually
f $6000(1.0022)^{91}$, fortnightly

10 Paula must decide whether to invest her \$13 500 for 6 years at 4.2% p.a. compounded monthly or 5.3% compounded bi-annually. Decide which investment would be the best choice for Paula.

ENRICHMENT: Double your money	–	–	11

11 You have \$100 000 to invest and wish to double that amount. Use trial and error in the following.

a Determine, to the nearest whole number of years, the length of time it will take to do this using the compound interest formula at rates of:

i 12% p.a.
ii 6% p.a.
iii 8% p.a.
iv 16% p.a.
v 10% p.a.
vi 20% p.a.

b If the amount of investment is \$200 000 and you wish to double it, determine the time it will take using the same interest rates as above.

c Are the lengths of time to double your investment the same in part **a** and part **b**?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

3M Comparing simple and compound interest

Learning intentions

- To know the formulas for simple and compound interest
- To know how to use technology and the formulas to compare simple and compound interest

In the following exercise, we compare compound and simple interest and look at their applications to the banking world. You are expected to use technology to its best advantage when solving the problems in this section.

Finance industries employ highly trained mathematicians to model investment outcomes and analyse risk. In 2018, Australia's total pension funds (superannuation) invested in local and global markets exceeded $2600 billion.

LESSON STARTER Who earns the most?

- Ceanna invests $500 at 8% p.a., compounded monthly over 3 years.
- Huxley invests $500 at 10% p.a., compounded annually over 3 years.
- Loreli invests $500 at 15% p.a. simple interest over 3 years.
  - How much does each person have at the end of the 3 years?
  - Who earned the most?

KEY IDEAS

For either form of interest, you can calculate the total amount of your investment using technology.

- **CAS or graphics calculator**
  To create programs for the two types of interest, enter the following data. This will allow you to calculate both types of interest for a given time period. If you invest $100 000 at 8% p.a. paid monthly for 2 years, you will be asked for P, r, t or n and the calculator will do the work for you. (Note: Some modifications may be needed for other calculators or languages.)

```
Define simple()=
Prgm
Request "Enter Principal:",p
Request "Enter interest rate:",r
Request "Enter time:",t
p·r·t/100→i
Disp "Interest is",i
Disp "Amount is",p+i
EndPrgm
```

```
Define compound()=
Prgm
Request "Enter Principal:",p
Request "Enter interest rate:",r
Request "Enter time:",t
p·(1+r/100)^t→a
Disp "Interest is",round(a-p,2)
Disp "Amount is",round(a,2)
EndPrgm
```

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Spreadsheet

Copy and complete the spreadsheets as shown, to compile a simple interest and compound interest sheet.

	A	B	C	D	E
1	Interest calculator				
2		Principal		Rate	
3		4000		=5.4/100/12	
4					
5		Simple interest		Compound interest	
6	Time (months)	Interest	Amount	Interest	Amount
7	0	0	=B$3+B7	0	=B$3*(1+D$3)^A7
8	=A7+1	=B$3*D$3	=C7+B8	=E8-E7	=B$3*(1+D$3)^A8
9	=A8+1	=B$3*D$3	=C8+B9	=E9-E8	=B$3*(1+D$3)^A9
10	=A9+1	=B$3*D$3	=C9+B10	=E10-E9	=B$3*(1+D$3)^A10
11	=A10+1	=B$3*D$3	=C10+B11	=E11-E10	=B$3*(1+D$3)^A11
12	=A11+1	=B$3*D$3	=C11+B12	=E12-E11	=B$3*(1+D$3)^A12
13	=A12+1	=B$3*D$3	=C12+B13	=E13-E12	=B$3*(1+D$3)^A13
14					
15					
16					
17					

Fill in the principal in B3 and the rate per period in D3. For example, for \$4000 invested at 5.4% p.a. paid monthly, B3 will be 4000 and D3 will be $\frac{0.054}{12}$.

- Recall the simple interest formula from previous years: $I = \frac{Prt}{100}$ where I is the total amount of interest, P is the initial amount or principal, r is percentage interest rate and t is the time.

BUILDING UNDERSTANDING

1. Which is better on an investment of \$100 for 2 years:
   A simple interest calculated at 20% p.a. or
   B compound interest calculated at 20% p.a. and paid annually?
2. State the values of P, r and n for an investment of \$750 at 7.5% p.a., compounded annually for 5 years.
3. State the values of I, P, r and t for an investment of \$300 at 3% p.a. simple interest over 300 months.
4. Use the simple interest formula $I = \frac{Prt}{100}$ to find the simple interest on an investment of \$2000 at 4% p.a. over 3 years.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 32 Comparing simple and compound interest using technology

Find the total amount of the following investments, using technology.

a \$5000 at 5% p.a., compounded annually for 3 years

b \$5000 at 5% p.a., simple interest for 3 years

SOLUTION	EXPLANATION
a \$5788.13	$A = P\left(1 + \frac{r}{100}\right)^n$ using $P = 5000$, $r = 5$ and $n = 3$. Alternatively, use a spreadsheet or computer program. Refer to the Key ideas.
b \$5750	Total $= P + \frac{Prt}{100}$ using $P = 5000$, $r = 5$ and $t = 3$.

Now you try

Find the total amount of the following investments, using technology.

a \$4000 at 6% p.a., compounded annually for 3 years

b \$4000 at 6% p.a., simple interest for 3 years

Exercise 3M

FLUENCY	1–3	2, 3	2, 3

1 Find the total amount of the following investments, using technology.

Example 32a **a** \$7000 at 4% p.a., compounded annually for 5 years

Example 32b **b** \$7000 at 4% p.a., simple interest for 5 years

Example 32

2 **a** Find the total amount of the following investments, using technology.

- **i** \$6000 at 6% p.a., compounded annually for 3 years
- **ii** \$6000 at 3% p.a., compounded annually for 5 years
- **iii** \$6000 at 3.4% p.a., compounded annually for 4 years
- **iv** \$6000 at 10% p.a., compounded annually for 2 years
- **v** \$6000 at 5.7% p.a., compounded annually for 5 years

b Which of the above yields the most interest?

3 **a** Find the total amount of the following investments, using technology where possible.

- **i** \$6000 at 6% p.a. simple interest for 3 years
- **ii** \$6000 at 3% p.a. simple interest for 6 years
- **iii** \$6000 at 3.4% p.a. simple interest for 7 years
- **iv** \$6000 at 10% p.a. simple interest for 2 years
- **v** \$6000 at 5.7% p.a. simple interest for 5 years

b Which of the above yields the most interest?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

PROBLEM-SOLVING	4	4	4

4 a Determine the total simple and compound interest accumulated in the following cases.
  i $4000 at 6% p.a. payable annually for:
    I 1 year II 2 years III 5 years IV 10 years.
  ii $4000 at 6% p.a. payable bi-annually for:
    I 1 year II 2 years III 5 years IV 10 years.
  iii $4000 at 6% p.a. payable monthly for:
    I 1 year II 2 years III 5 years IV 10 years.
b Would you prefer the same rate of compound interest or simple interest if you were investing money?
c Would you prefer the same rate of compound interest or simple interest if you were borrowing money and paying off the loan in instalments?

REASONING	5	5	6

5 a Copy and complete the following table when simple interest is applied.

Principal	Rate	Overall time	Interest	Amount
$7000		5 years		$8750
$7000		5 years		$10 500
	10%	3 years	$990	
	10%	3 years	$2400	
$9000	8%	2 years		
$18 000	8%	2 years		

b Explain the effect on the interest when we double the:
  i rate ii period iii overall time.

6 Copy and complete the following table when compound interest is applied. You may need to use a calculator and trial and error to find some of the missing numbers. Write your answers correct to two decimal places where necessary.

Principal	Rate	Period	Overall time	Interest	Amount
$7000		annually	5 years		$8750
$7000		annually	5 years		$10 500
$9000	8%	fortnightly	2 years		
$18 000	8%	fortnightly	2 years		

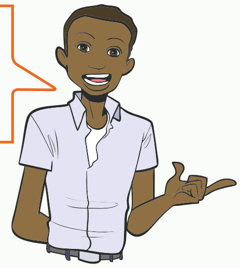
ENRICHMENT: Changing the parameters	–	–	7, 8

7 If you invest $5000, determine the interest rate per annum (to two decimal places) if the total amount is approximately $7500 after 5 years and if:
a interest is compounded annually
b interest is compounded quarterly
c interest is compounded weekly
Comment on the effect of changing the period for each payment on the rate needed to achieve the same total amount in a given time.

8 a Determine, to one decimal place, the equivalent simple interest rate for the following investments over 3 years.
  i $8000 at 4%, compounded annually ii $8000 at 8%, compounded annually
b If you double or triple the compound interest rate, how is the simple interest rate affected?

Problems and challenges

Up for a challenge? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.



1 Write $3^{n-1} + 3^{n-1} + 3^{n-1}$ as a single term with base 3.

2 Simplify.

a $\dfrac{25^6 \times 5^4}{125^5}$

b $\dfrac{8^x \times 3^x}{6^x \times 9^x}$

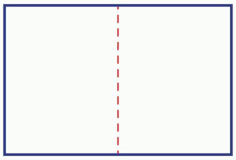
3 Solve $3^{2x} \times 27^{x+1} = 81$.

4 Simplify.

a $\dfrac{2^{n+1} - 2^{n+2}}{2^{n-1} - 2^{n-2}}$

b $\dfrac{2^{a+3} - 4 \times 2^a}{2^{2a+1} - 4^a}$

5 A rectangular piece of paper has an area of $100\sqrt{2}$ cm^2. The piece of paper is such that, when it is folded in half along the dashed line as shown, the new rectangle is similar (i.e. of the same shape) to the original rectangle. What are the dimensions of the piece of paper?



6 Simplify the following, leaving your answer with a rational denominator.

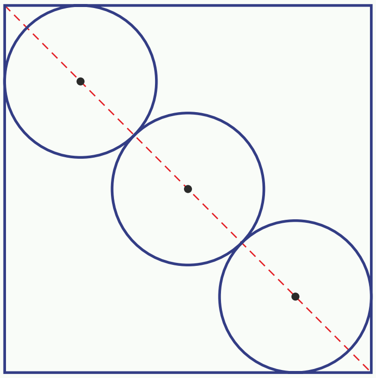
$$\frac{\sqrt{2}}{2\sqrt{2}+1} + \frac{2}{\sqrt{3}+1}$$

7 Simplify.

a $\dfrac{x^{\frac{1}{2}}y^{-\frac{1}{2}} - x^{-\frac{1}{2}}y^{\frac{1}{2}}}{\sqrt{xy}}$

b $\dfrac{x^{\frac{1}{2}}y^{-\frac{1}{2}} - x^{-\frac{1}{2}}y^{\frac{1}{2}}}{x^{-1}y^{-1}}$

8 Three circles, each of radius 1 unit, fit inside a square such that the two outer circles touch the middle circle and the sides of the square, as shown. Given the centres of the circle lie on the diagonal of the square, find the exact area of the square.



9 Given that $5^{x+1} - 5^{x-2} = 620\sqrt{5}$, find the value of x.

ISBN 978-1-108-77290-7

Chapter summary

Indices and surds

Negative indices

$a^{-m} = \frac{1}{a^m}$

$\frac{1}{a^{-m}} = a^m$

Scientific notation

Numbers (large and small) in scientific notation are of the form $a \times 10^m$, where $1 \leq a < 10$ or $-10 < a \leq -1$ and m is an integer.

e.g. $3.45 \times 10^5 = 345\,000$

$2.1 \times 10^{-3} = 0.0021$

e.g 1475 231 in scientific notation, using three significant figures is 1.48×10^6.

Index laws

Law 1: $a^m \times a^n = a^{m+n}$

Law 2: $a^m \div a^n = a^{m-n}$

Law 3: $(a^m)^n = a^{mn}$

Law 4: $(ab)^m = a^m b^m$

Law 5: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Zero power : $a^0 = 1,\ a \neq 0$

Rational Indices (10A)

$\sqrt{a} = a^{\frac{1}{2}}$

$\sqrt[n]{a} = a^{\frac{1}{n}}$

$\sqrt[n]{a^m} = a^{\frac{m}{n}}$

e.g. $27^{\frac{1}{3}} = \sqrt[3]{27}$

$= 3$

Simplifying surds (10A)

A surd uses symbol $\sqrt{\ }$ and as a decimal is infinite and non-recurring
Simplify surds: use the highest square factor.

e.g. **1** $\sqrt{20} = \sqrt{4 \times 5}$

$= \sqrt{4} \times \sqrt{5}$

$= 2\sqrt{5}$

2 $4\sqrt{27} = 4 \times \sqrt{9} \times \sqrt{3}$

$= 12\sqrt{3}$

Exponential equations (10A)

If $a^x = a^y$, then $x = y$.

e.g. **1** $2^x = 16$

$2^x = 2^4$

$\therefore\ x = 4$

2 $27^{x-4} = 9^x$

$(3^3)^{x-4} = (3^2)^x$

$3^{3x-12} = 3^{2x}$

$\therefore\ 3x - 12 = 2x$

$x = 12$

Multiply/Divide (10A)

$\sqrt{x} \times \sqrt{y} = \sqrt{xy}$

$a\sqrt{x} \times b\sqrt{y} = ab\sqrt{xy}$

$\sqrt{x} \div \sqrt{y} = \sqrt{\frac{x}{y}}$

$a\sqrt{x} \div (b\sqrt{y}) = \frac{a}{b}\sqrt{\frac{x}{y}}$

Add/Subtract (10A)

Like surds only

e.g. **1** $3\sqrt{2} + 4\sqrt{2} = 7\sqrt{2}$

$4\sqrt{2} + \sqrt{5} - \sqrt{2} = 3\sqrt{2} + \sqrt{5}$

Simplify first

e.g. **2** $3\sqrt{8} - 4\sqrt{2}$

$= 3 \times \sqrt{4} \times \sqrt{2} - 4\sqrt{2}$

$= 6\sqrt{2} - 4\sqrt{2}$

$= 2\sqrt{2}$

Exponential growth/decay

Rule: amount $A = A_0\left(1 \pm \frac{r}{100}\right)^n$

A_0 = initial amount

r = rate of growth/decay, as a %

n = time period

Use + for growth.

Use – for decay.

Compound interest

$A = P\left(1 + \frac{r}{100}\right)^n$

A (amount): total value

P (principal): initial amount

r (interest rate): per period

n (period): number of periods

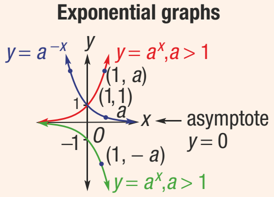
Rationalise the denominator (10A)

Express with a whole number in the denominato r

$\frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$

$= \frac{2\sqrt{5}}{5}$

Exponential graphs



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter checklist

Chapter checklist: Success criteria

		✔
3A	**1. I can decide if a number is rational or irrational.** e.g. Express $\sqrt{6}$ as a decimal and decide if it is rational or irrational. 10A	
3A	**2. I can simplify a surd using the highest square number factor.** e.g. Simplify $\sqrt{75}$. 10A	
3A	**3. I can express a surd as a square root of an integer.** e.g. Express $3\sqrt{7}$ in the form $\sqrt{a}$ where a is an integer. 10A	
3B	**4. I can add and subtract expressions involving like surds.** e.g. Simplify $\sqrt{3} + 5\sqrt{2} + 3\sqrt{3} - 2\sqrt{2}$. 10A	
3B	**5. I can simplify surds to add or subtract.** e.g. Simplify $3\sqrt{12} + 2\sqrt{27}$. 10A	
3C	**6. I can multiply surds.** e.g. Simplify $3\sqrt{5} \times 2\sqrt{10}$. 10A	
3C	**7. I can simplify surds using division.** e.g. Simplify $4\sqrt{15} \div \sqrt{3}$. 10A	
3C	**8. I can apply the distributive law to expressions involving surds.** e.g. Expand and simplify $2\sqrt{5}\left(3\sqrt{3} - \sqrt{5}\right)$. 10A	
3D	**9. I can rationalise a denominator.** e.g. Rationalise $\frac{4\sqrt{2}}{\sqrt{7}}$. 10A	
3E	**10. I can apply index laws to multiply common bases.** e.g. Simplify $3x^2y^3 \times 2xy^4$.	
3E	**11. I can apply index laws in division.** e.g. Simplify $3x^5y^3 \div (12x^2y)$.	
3E	**12. I can simplify using a number of index laws.** e.g. Simplify $3(x^2y)^3 \times \left(\frac{2}{x}\right)^2$ using index laws.	
3E	**13. I can use the zero power.** e.g. Evaluate $(3a)^0 - 5a^0$.	
3F	**14. I can rewrite an expression using positive indices.** e.g. Express $3x^2y^{-3}$ using positive indices.	
3F	**15. I can rewrite an expression with a negative power in the denominator using positive indices.** e.g. Express $\frac{3}{y^{-4}}$ using positive indices.	
3F	**16. I can simplify expressions and apply index laws to negative indices.** e.g. Simplify $\frac{(x^{-1}y)^{-3}}{4x^{-2}y^3}$ and express using positive indices.	

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter checklist

			✔
3G	**17. I can convert from scientific notation to a basic numeral.** e.g. Write 3.07×10^4 and 4.1×10^{-3} as basic numerals.		
3G	**18. I can convert to scientific notation using significant figures.** e.g. Write 0.0035892 in scientific notation using three significant figures.		
3G	**19. I can use technology to perform calculations in scientific notation.** e.g. Evaluate $\sqrt{3.02 \times 10^{24}}$, answering in scientific notation using three significant figures.		
3H	**20. I can write roots in index form.** e.g. Express $\sqrt[3]{x^6}$ in index form.	10A	
3H	**21. I can write rational indices in surd form.** e.g. Express $4^{\frac{3}{2}}$ in surd form.	10A	
3H	**22. I can evaluate numbers with rational indices.** e.g. Evaluate $25^{-\frac{1}{2}}$ without a calculator.	10A	
3I	**23. I can solve exponential equations using a common base.** e.g. Solve $3^x = 27$ for x.	10A	
3I	**24. I can solve exponential equations that require a common base to be found.** e.g. Solve $25^x = 125^{x-2}$.	10A	
3J	**25. I can sketch a graph of an exponential equation.** e.g. Sketch $y = 3^x$ labelling the y-intercept and one other point.		
3J	**26. I can sketch exponential graphs involving reflections.** e.g. Sketch $y = 2^x$, $y = -2^x$ and $y = 2^{-x}$ on the same axes.		
3J	**27. I can find the intersection of horizontal lines and exponential graphs.** e.g. Find the intersection of the graphs of $y = 3^x$ and $y = 27$.		
3K	**28. I can form an exponential rule for a situation.** e.g. Write an exponential rule for the value of Scott's car purchased for \$35 000 and decreasing in value by 15% per year.		
3K	**29. I can apply exponential rules.** e.g. The value of a house in n years' time is given by $V = 590\,000(1.06)^n$. What will be the value in 4 years' time and find when it will be valued at \$1 000 000.		
3L	**30. I can calculate compound interest using the formula.** e.g. Determine the amount after 3 years if \$6000 is compounded annually at 4%. Round to the nearest cent.		
3L	**31. I can find compounded amounts using different time periods.** e.g. An investment of \$2000 is compounded at 4.8% over 3 years. Determine the amount after 3 years if the interest is paid monthly.		
3M	**32. I can compare interest using technology.** e.g. Use technology to compare the total amount of the investment of \$6000 for 4 years at 5% p.a. using simple interest and compound interest.		

Chapter review

Short-answer questions

3A 10A

1 Simplify the following surds.

a $\sqrt{24}$
b $\sqrt{72}$
c $3\sqrt{200}$
d $4\sqrt{54}$
e $\sqrt{\dfrac{4}{49}}$
f $\sqrt{\dfrac{8}{9}}$
g $\dfrac{5\sqrt{28}}{2}$
h $\dfrac{2\sqrt{45}}{15}$

3B/C 10A

2 Simplify the following.

a $2\sqrt{3} + 4 + 5\sqrt{3}$
b $6\sqrt{5} - \sqrt{7} - 4\sqrt{5} + 3\sqrt{7}$
c $\sqrt{8} + 3\sqrt{2}$
d $4\sqrt{3} + 2\sqrt{18} - 4\sqrt{2}$
e $2\sqrt{5} \times \sqrt{6}$
f $-3\sqrt{2} \times 2\sqrt{10}$
g $\dfrac{2\sqrt{15}}{\sqrt{3}}$
h $\dfrac{5\sqrt{14}}{15\sqrt{2}}$
i $\dfrac{\sqrt{27}}{3} - \sqrt{3}$

3C 10A

3 Expand and simplify.

a $\sqrt{2}(2\sqrt{3} + 4)$
b $2\sqrt{3}(2\sqrt{15} - \sqrt{3})$
c $(\sqrt{11})^2$
d $(4\sqrt{3})^2$

3D 10A

4 Rationalise the denominator.

a $\dfrac{1}{\sqrt{6}}$
b $\dfrac{10}{\sqrt{2}}$
c $\dfrac{6\sqrt{3}}{\sqrt{2}}$
d $\dfrac{4\sqrt{7}}{\sqrt{2}}$
e $\dfrac{3\sqrt{3}}{2\sqrt{6}}$
f $\dfrac{5\sqrt{5}}{4\sqrt{10}}$
g $\dfrac{\sqrt{5} + 2}{\sqrt{2}}$
h $\dfrac{4\sqrt{2} - \sqrt{3}}{\sqrt{3}}$

3E/F

5 Simplify the following, expressing all answers with positive indices when required.

a $(5y^3)^2$
b $7m^0 - (5n)^0$
c $4x^2y^3 \times 5x^5y^7$
d $3x^2y^{-4}$
e $\left(\dfrac{3x}{y^{-3}}\right)^2 \times \dfrac{x^{-5}}{6y^2}$
f $\dfrac{3(a^2b^{-4})^2}{(2ab^2)^2} \div \dfrac{(ab)^{-2}}{(3a^{-2}b)^2}$

3H 10A

6 Express in index form.

a $\sqrt{21}$
b $\sqrt[3]{x}$
c $\sqrt[3]{m^5}$
d $\sqrt[5]{a^2}$
e $\sqrt{10x^3}$
f $\sqrt[3]{2a^9b}$
g $7\sqrt{7}$
h $4\sqrt[3]{4}$

3H 10A

7 Evaluate these without using a calculator.

a $25^{\frac{1}{2}}$
b $64^{\frac{1}{3}}$
c $\left(\dfrac{1}{8}\right)^{\frac{1}{3}}$
d $49^{-\frac{1}{2}}$
e $100^{-\frac{1}{2}}$
f $125^{-\frac{1}{3}}$

3G

8 a Write the following numbers as a basic numeral.

i 3.21×10^3
ii 4.024×10^6
iii 7.59×10^{-3}
iv 9.81×10^{-5}

b Write the following numbers in scientific notation, using three significant figures.

i 0.0003084
ii 0.0000071753
iii 5 678 200
iv 119 830 000

3K

9 Form exponential rules for the following situations.

a An antique bought for \$800 is expected to grow in value by 7% per year.
b A balloon with volume 3000 cm^3 is leaking air at a rate of 18% per minute.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

3L/M

10 Determine the final amount after 4 years if:
- **a** \$1000 is compounded annually at 5%
- **b** \$3000 is compounded monthly at 4%
- **c** \$5000 is compounded daily at 3%.

3I 10A

11 Solve the following exponential equations for x.
- **a** $3^x = 27$
- **b** $7^x = 49$
- **c** $4^{2x+1} = 64$
- **d** $2^{x-2} = 16$
- **e** $9^x = \frac{1}{81}$
- **f** $5^x = \frac{1}{125}$
- **g** $36^x = 216$
- **h** $8^{x+1} = 32$
- **i** $7^{3x-4} = 49^x$
- **j** $11^{x-5} = \frac{1}{121}$
- **k** $100^{x-2} = 1000^x$
- **l** $9^{3-2x} = 27^{x+2}$

3J

12 Sketch the following graphs, labelling the y-intercept and the point where $x = 1$.
- **a** $y = 4^x$
- **b** $y = -3^x$
- **c** $y = 5^{-x}$

Multiple-choice questions

3A 10A

1 Which of the following is a surd?

A $\sqrt{36}$ **B** π **C** $\sqrt{7}$ **D** $\sqrt[3]{8}$ **E** $1.\dot{6}$

3A 10A

2 A square has an area of 75 square units. Its side length, in simplified form, is:

A $3\sqrt{5}$ **B** 8.5 **C** $25\sqrt{3}$ **D** $5\sqrt{3}$ **E** $6\sqrt{15}$

3A 10A

3 $4\sqrt{5}$ is equivalent to:

A $\sqrt{100}$ **B** $\sqrt{80}$ **C** $2\sqrt{10}$ **D** $\sqrt{20}$ **E** $\sqrt{40}$

3B 10A

4 $3\sqrt{12} + 7 - 4\sqrt{3}$ simplifies to:

A $7 + \sqrt{3}$ **B** $8\sqrt{3} + 7$ **C** $6\sqrt{6} + 3\sqrt{3}$ **D** $3\sqrt{3}$ **E** $2\sqrt{3} + 7$

3C 10A

5 The expanded form of $2\sqrt{5}(5 - 3\sqrt{3})$ is:

A $10\sqrt{5} - 6\sqrt{15}$ **B** $7\sqrt{5} - 5\sqrt{15}$ **C** $10\sqrt{5} - 12\sqrt{2}$ **D** $10 - 5\sqrt{15}$ **E** $7\sqrt{5} - 5\sqrt{3}$

3D 10A

6 $\frac{2\sqrt{5}}{\sqrt{6}}$ is equivalent to:

A $\frac{2\sqrt{30}}{\sqrt{6}}$ **B** $\frac{5\sqrt{6}}{3}$ **C** $2\sqrt{5}$ **D** $\frac{\sqrt{30}}{3}$ **E** $\frac{\sqrt{30}}{10}$

3E

7 The simplified form of $\frac{(6xy^3)^2}{3x^3y^2 \times 4x^4y^0}$ is:

A $\frac{y^4}{2x^6}$ **B** $\frac{3y^3}{x^{10}}$ **C** $\frac{y^6}{x^5}$ **D** $\frac{3y^4}{x^5}$ **E** $\frac{y^6}{2x^6}$

3F

8 $\frac{8a^{-1}b^{-2}}{12a^3b^{-5}}$ expressed with positive indices is:

A $\frac{2a^2}{3b^3}$ **B** $\frac{a^2b^3}{96}$ **C** $\frac{2b^3}{3a^4}$ **D** $-\frac{2b^7}{3a^2}$ **E** $\frac{3}{2}a^4b^7$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter review

3G **9** The radius of the Earth is approximately 6 378 137 m. In scientific notation, using three significant figures, this is:

A 6.378×10^6 m **B** 6.38×10^6 m **C** 6.4×10^5 m
D 6.37×10^6 m **E** 6.36×10^6 m

3H 10A **10** $\sqrt{8x^6}$ in index form is:

A $8x^3$ **B** $8x^2$ **C** $4x^3$ **D** $8^{\frac{1}{2}}x^3$ **E** $8^{\frac{1}{2}}x^4$

3I 10A **11** The solution to $3^{2x-1} = 9^2$ is:

A $x = \frac{3}{2}$ **B** $x = 2$ **C** $x = \frac{5}{2}$ **D** $x = 6$ **E** $x = 3$

3K **12** A rule for the amount $\$A$ in an account after n years for an initial investment of \$5000 that is increasing at 7% per annum is:

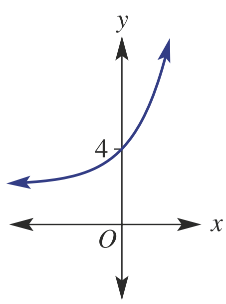
A $A = 5000(1.7)^n$ **B** $A = 5000(0.93)^n$ **C** $A = 5000(0.3)^n$
D $A = 5000(1.07)^n$ **E** $A = 5000(0.7)^n$

3J **13** The graph of $y = 3^x$ intersects the y-axis at:

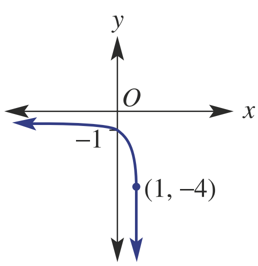
A $(0, 3)$ **B** $(3, 0)$ **C** $(0, 1)$ **D** $(1, 3)$ **E** $\left(0, \frac{1}{3}\right)$

3J **14** The graph of $y = 4^{-x}$ is:

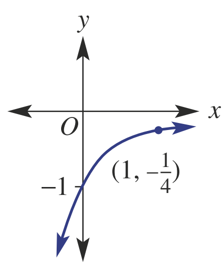
A



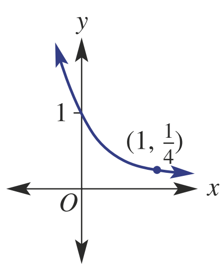
B



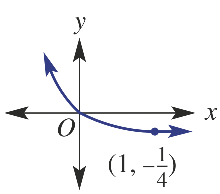
C



D



E



3J **15** The graph of $y = 3^x$ and $y = \frac{1}{3}$ intersect at the point:

A $(1, 3)$ **B** $\left(-1, \frac{1}{3}\right)$ **C** $(-1, 3)$ **D** $\left(\frac{1}{9}, \frac{1}{3}\right)$ **E** $\left(1, \frac{1}{3}\right)$

ISBN 978-1-108-77290-7

Extended-response questions

10A **1** A small rectangular jewellery box has a base with dimensions $3\sqrt{15}$ cm by $(12 + \sqrt{3})$ cm and a height of $(2\sqrt{5} + 4)$ cm.

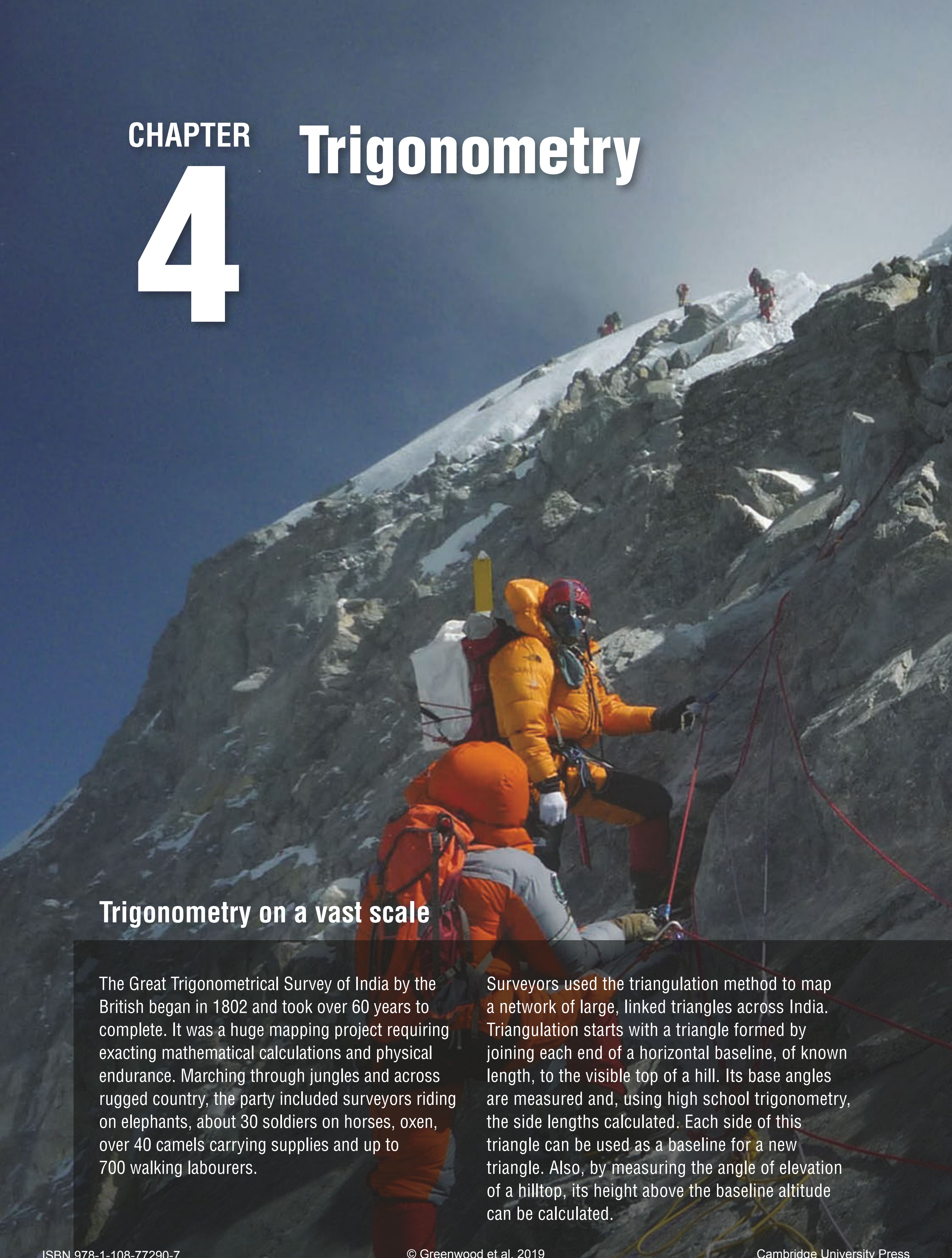
- **a** Determine the exact area of the base of the box, in expanded and simplified form.
- **b** What is the exact volume of the box?
- **c** Julie's earring boxes occupy an area of $9\sqrt{5}$ cm^2. What is the exact number that would fit across the base of the jewellery box? Give your answer with a rational denominator.
- **d** The surface of Julie's rectangular dressing table has dimensions $(\sqrt{2} - 1)$ m by $(\sqrt{2} + 1)$ m.
  - **i** Find the area of the dressing table, in square centimetres.
  - **ii** What percentage of the area of the dressing table does the jewellery box occupy? Give your answer to one decimal place.

2 Georgia invests \$10 000 in shares in a new company. She has been told that their value is expected to increase at 6.5% per year.

- **a** Write a rule for Georgia's expected value, V dollars, in shares after n years.
- **b** Use your rule to find the value she expects the shares to be after:
  - **i** 2 years
  - **ii** 5 years
- **c** When her shares are valued at \$20 000 Georgia plans to cash them in. According to this rule, how many years will it take to reach this amount? Give your answer to one decimal place.
- **d** After 6 years there is a downturn in the market and the shares start to drop, losing value at 3% per year.
  - **i** What is the value of Georgia's shares prior to the downturn in the market? Give your answer to the nearest dollar.
  - **ii** Using your answer from part **d i**, write a rule for the expected value, V dollars, of Georgia's shares t years after the market downturn.
  - **iii** Ten years after Georgia initially invested in the shares the market is still falling at this rate. She decides it's time to sell her shares. What is their value, to the nearest dollar? How does this compare with the original amount of \$10 000 she invested?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022



CHAPTER 4

Trigonometry

Trigonometry on a vast scale

The Great Trigonometrical Survey of India by the British began in 1802 and took over 60 years to complete. It was a huge mapping project requiring exacting mathematical calculations and physical endurance. Marching through jungles and across rugged country, the party included surveyors riding on elephants, about 30 soldiers on horses, oxen, over 40 camels carrying supplies and up to 700 walking labourers.

Surveyors used the triangulation method to map a network of large, linked triangles across India. Triangulation starts with a triangle formed by joining each end of a horizontal baseline, of known length, to the visible top of a hill. Its base angles are measured and, using high school trigonometry, the side lengths calculated. Each side of this triangle can be used as a baseline for a new triangle. Also, by measuring the angle of elevation of a hilltop, its height above the baseline altitude can be calculated.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Online resources

A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

In this chapter

Victorian Curriculum

MEASUREMENT AND GEOMETRY
Pythagoras and trigonometry

Solve right-angled triangle problems including those involving direction and angles of elevation and depression (VCMMG346)

(10A) Establish the sine, cosine and area rules for any triangle and solve related problems (VCMMG367)

(10A) Use the unit circle to define trigonometric functions as functions of a real variable, and graph them with and without the use of digital technologies (VCMMG368)

(10A) Solve simple trigonometric equations (VCMMG369)

(10A) Apply Pythagoras' theorem and trigonometry to solving three-dimensional problems in right-angled triangles (VCMMG370)

The surveying team gradually travelled from South India 2400 km north to the Himalayas in Nepal. After measuring Mt Everest from six different locations, Radhanath Sikdar declared it the highest mountain on Earth at 8840 m (29 002 feet). It was an accurate measurement: using GPS (Global Positioning System) Mt Everest has since been found to have a height of 8850 m (29 035 feet). GPS also involves a process of triangulation using radio waves sent between satellites and Earth.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

4A Trigonometric ratios

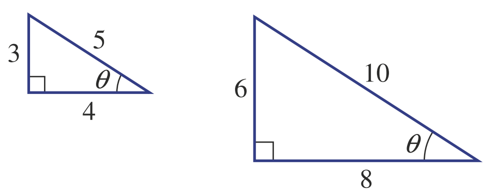
Learning intentions

- To understand how the trigonometric ratios relate the angles and side lengths of right-angled triangles
- To know the trigonometric ratios involving sine, cosine and tangent
- To be able to identify which trigonometric ratio to apply based on the information in a given right-angled triangle
- To be able to use trigonometry to find an unknown side length in a right-angled triangle

The study of trigonometry explores the relationship between the angles and side lengths of triangles. Trigonometry can be applied to simple problems, such as finding the angle of elevation of a kite, to solving complex problems in surveying and design.

Engineers use trigonometry to determine the horizontal and vertical components of the forces acting on bridge trusses and cables. Equations are formed by equating forces in opposite directions at joints; solving simultaneously calculates each load.

Trigonometry is built upon the three ratios sine, cosine and tangent. These ratios do not change for triangles that are similar in shape

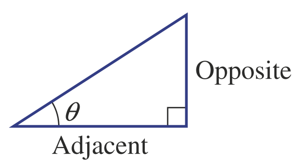


$\sin\theta = \frac{3}{5}$ $\quad\quad$ $\sin\theta = \frac{6}{10} = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$ $\quad\quad$ $\cos\theta = \frac{8}{10} = \frac{4}{5}$

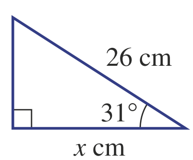
$\tan\theta = \frac{3}{4}$ $\quad\quad$ $\tan\theta = \frac{6}{8} = \frac{3}{4}$

LESSON STARTER Which ratio?

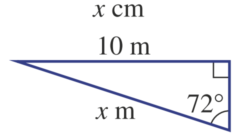


In a group or with a partner, see if you can recall some facts from Year 9 trigonometry to answer the following questions.

- What is the name given to the longest side of a right-angled triangle?
- $\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$ is one trigonometric ratio. What are the other two?
- Which ratio would be used to find the value of x in this triangle? Can you also find the answer?



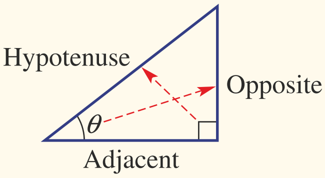
- Which ratio would be used to find the value of x in this triangle? Can you also find the answer?



ISBN 978-1-108-77290-7

KEY IDEAS

- The **hypotenuse** is the longest side of a right-angled triangle. It is opposite the right angle.
- Given a right-angled triangle and another angle θ, the three trigonometric ratios are:
  - sine of angle θ (**sin θ**) $= \dfrac{\text{length of the opposite side}}{\text{length of the hypotenuse}}$
  - cosine of angle θ (**cos θ**) $= \dfrac{\text{length of the adjacent side}}{\text{length of the hypotenuse}}$
  - tangent of angle θ (**tan θ**) $= \dfrac{\text{length of the opposite side}}{\text{length of the adjacent side}}$



- Many people like to use SOHCAHTOA to help remember the three ratios.
  - $\sin\theta = \dfrac{O}{H}$ $\cos\theta = \dfrac{A}{H}$ $\tan\theta = \dfrac{O}{A}$
- To find an unknown length on a right-angled triangle:
  - Choose a trigonometric ratio that links one known angle and a known side length with the unknown side length.
  - Solve for the unknown side length.

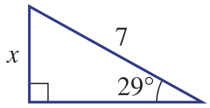
BUILDING UNDERSTANDING

1 Use a calculator to evaluate the following, correct to three decimal places.

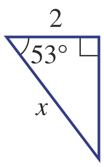
a cos 37° **b** sin 72° **c** tan 50° **d** cos 21.4°

2 Decide which ratio $\left(\text{i.e. } \sin\theta = \dfrac{O}{H}, \cos\theta = \dfrac{A}{H} \text{ or } \tan\theta = \dfrac{O}{A}\right)$ would be best to help find the value of x in these triangles. Do not find the value of x.

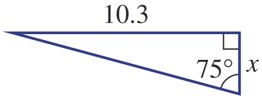
a



b



c



3 Solve for x in these equations, correct to two decimal places.

a $\tan 31° = \dfrac{x}{3}$ **b** $\cos 54° = \dfrac{x}{5}$

c $\sin 15.6° = \dfrac{x}{12.7}$ **d** $\sin 57° = \dfrac{2}{x}$

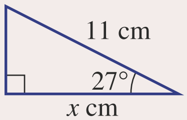
e $\cos 63.4° = \dfrac{10}{x}$ **f** $\tan 71.6° = \dfrac{37.5}{x}$

ISBN 978-1-108-77290-7

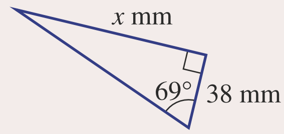
Example 1 Solving for an unknown in the numerator

Find the value of x in these right-angled triangles, correct to two decimal places.

a



b



SOLUTION

EXPLANATION

a

$$\cos \theta = \frac{A}{H}$$

Choose the ratio $\cos \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$.

$$\cos 27^\circ = \frac{x}{11}$$

Multiply both sides by 11, then use a calculator.

$$\therefore x = 11 \times \cos 27^\circ$$
$$= 9.80 \text{ (to 2 d.p.)}$$

Round your answer as required.

b

$$\tan \theta = \frac{O}{A}$$

The tangent ratio uses the opposite and the adjacent sides.

$$\tan 69^\circ = \frac{x}{38}$$

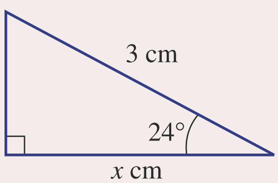
$$\therefore x = 38 \times \tan 69^\circ$$
$$= 98.99 \text{ (to 2 d.p.)}$$

Multiply both sides by 38.

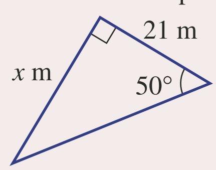
Now you try

Find the value of x in these right-angled triangles, correct to two decimal places.

a



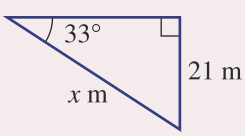
b



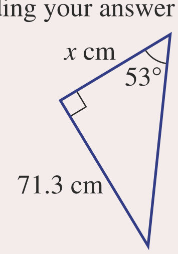
Example 2 Solving for an unknown in the denominator

Find the value of x in these right-angled triangles, rounding your answer to two decimal places.

a



b



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

SOLUTION

EXPLANATION

a

$$\sin\theta = \frac{O}{H}$$

$$\sin 33^\circ = \frac{21}{x}$$

Choose the sine ratio since the adjacent side is not marked.

$$x \times \sin 33^\circ = 21$$

$$x = \frac{21}{\sin 33^\circ}$$

Multiply both sides by x to remove the fraction, then divide both sides by sin 33°.

$$= 38.56 \text{ (to 2 d.p.)}$$

Evaluate using a calculator and round your answer as required.

b

$$\tan\theta = \frac{O}{A}$$

$$\tan 53^\circ = \frac{71.3}{x}$$

The hypotenuse is unmarked, so use the tangent ratio.

$$x \times \tan 53^\circ = 71.3$$

$$x = \frac{71.3}{\tan 53^\circ}$$

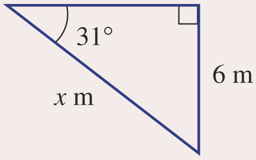
$$= 53.73 \text{ (to 2 d.p.)}$$

Multiply both sides by x, then solve by dividing both sides by tan 53°.

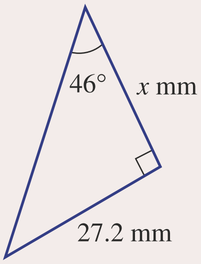
Now you try

Find the value of x in these right-angled triangles, rounding your answer to two decimal places.

a



b



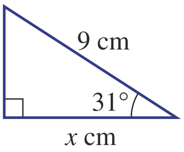
Exercise 4A

FLUENCY	1, 2–3(1/2)	2–3(1/2)	2–3(1/3), 4

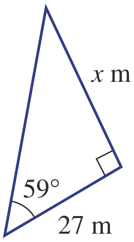
Example 1

1 Find the value of x in these right-angled triangles, correct to two decimal places.

a



b



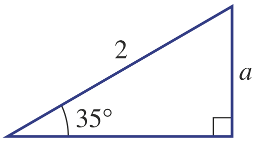
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

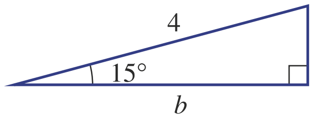
Example 1

2 Use trigonometric ratios to find the values of the pronumerals, to two decimal places.

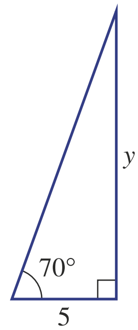
a



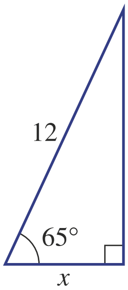
b



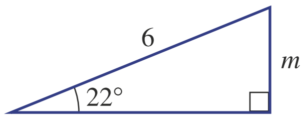
c



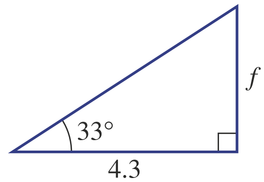
d



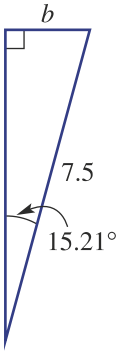
e



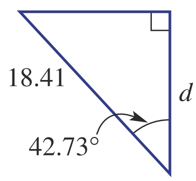
f



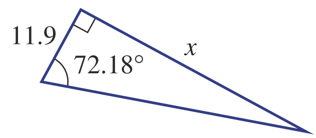
g



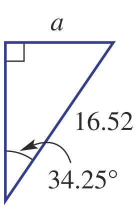
h



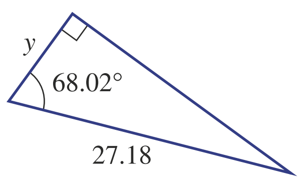
i



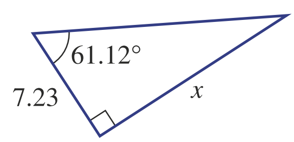
j



k



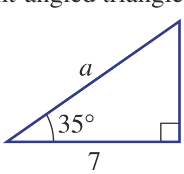
l



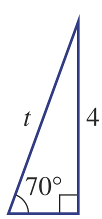
Example 2

3 Use trigonometric ratios to find the values of the pronumerals, to two decimal places, for these right-angled triangles.

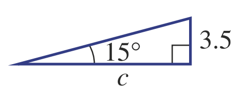
a



b



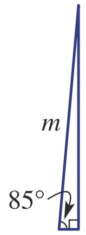
c



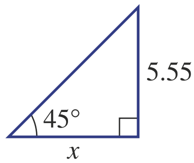
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

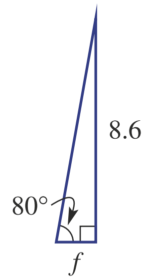
d



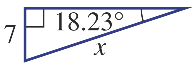
e



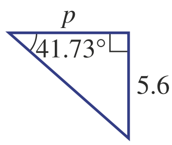
f



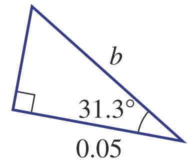
g



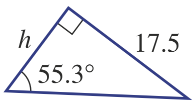
h



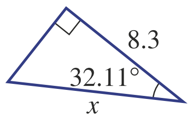
i



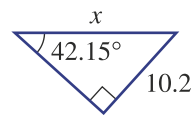
j



k

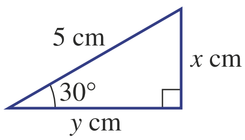


l

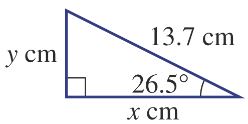


4 Find the unknown side lengths for these right-angled triangles, correct to two decimal places where necessary.

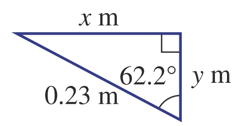
a



b



c

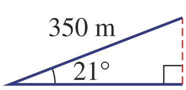


PROBLEM-SOLVING	5, 6	7–9	8–11

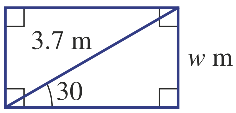
5 A 4WD climbs a 350 m straight slope at an angle of 21° to the horizontal.

a Find the vertical distance travelled, correct to the nearest metre.

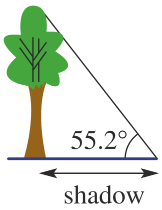
b Find the horizontal distance travelled, correct to the nearest metre.



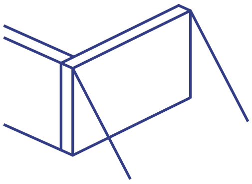
6 A diagonal wall brace of length 3.7 metres is at an angle of 30° to the horizontal. Find the width (w m) of the face of the wall, to the nearest centimetre.



7 The angle from the horizontal of the line of sight from the end of a tree's shadow to the top of the tree is 55.2°. The length of the shadow is 15.5 m. Find the height of the tree, correct to one decimal place.



8 On a construction site, large concrete slabs of height 5.6 metres are supported at the top by steel beams positioned at an angle of 42° from the vertical. Find the length of the steel beams, to two decimal places.



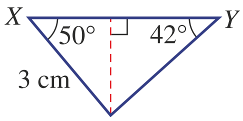
ISBN 978-1-108-77290-7

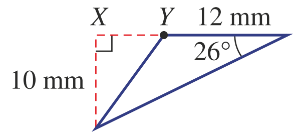
Cambridge University Press
Updated June 2022

9 By measuring the diagonals, a surveyor checks the dimensions of a rectangular revegetation area of length 25 metres. If the angle of the diagonal to the side length is 28.6°, find the length of the diagonals, correct to one decimal place.

10 A right-angled triangular flag is made for the premiers of a school competition. The second-longest edge of the flag is 25 cm and the largest non-right angle on the flag is 71°. Find the length of the longest edge of the flag, to the nearest millimetre.

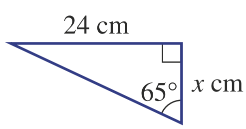
11 Find the length XY in these diagrams, correct to one decimal place.

a 

b 

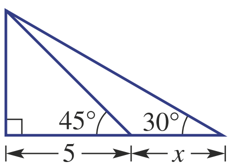
REASONING	12	12	12, 13

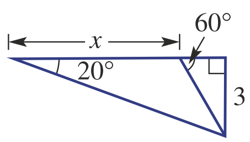
12 A student solves for x, to two decimal places, in the given triangle and gets 11.21, as shown. But the answer is 11.19. Explain the student's error.

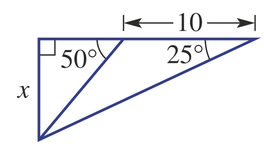


$$\tan 65° = \frac{24}{x}$$
$$x \tan 65° = 24$$
$$x = \frac{24}{\tan 65°}$$
$$= \frac{24}{2.14}$$
$$= 11.21$$

13 Find the value of x, correct to one decimal place, in these triangles.

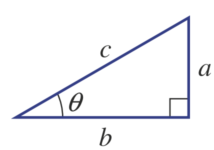
a 

b 

c 

ENRICHMENT: Exploring identities	–	–	14

14 For the following proofs, consider the right-angled triangle shown.



a Show that $\tan \theta = \frac{\sin \theta}{\cos \theta}$ by completing these steps.

i Write a in terms of c and θ.

ii Write b in terms of c and θ.

iii Write $\tan \theta$ in terms of a and b.

iv Substitute your expressions from parts i and ii into your expression for $\tan \theta$ in part iii.

Simplify to prove $\tan \theta = \frac{\sin \theta}{\cos \theta}$.

v Can you find a different way of proving the rule described above?

b Show that $(\sin \theta)^2 + (\cos \theta)^2 = 1$ by completing these steps.

i Write a in terms of c and θ.

ii Write b in terms of c and θ.

iii State Pythagoras' theorem using a, b and c.

iv Use your results from parts i, ii and iii to show that $(\sin \theta)^2 + (\cos \theta)^2 = 1$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

4B Finding unknown angles

Learning intentions

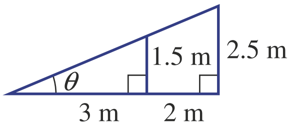
- To know that the inverse trigonometric functions are used to find angles in right-angled triangles
- To be able to use the inverse trigonometric functions to find an angle in a right-angled triangle given two side lengths

The three trigonometric ratios discussed earlier can also be used to find unknown angles in right-angled triangles if at least two side lengths are known. If, for example, $\cos\theta = \frac{1}{2}$ then we use the inverse trigonometric function for cosine, $\cos^{-1}\left(\frac{1}{2}\right)$, to find θ. Calculators are used to obtain these values.

The Eleanor Schonell Bridge in Brisbane is a cable-stayed bridge in which each cable forms a right-angled triangle with the pylons and the bridge deck. Trigonometry and geometry are essential tools for engineers.

LESSON STARTER The ramp

A ski ramp is 2.5 m high and 5 m long (horizontally) with a vertical strut of 1.5 m placed as shown.



- Discuss which triangle could be used to find the angle of incline, θ. Does it matter which triangle is used?
- Which trigonometric ratio is to be used and why?
- How does $\tan^{-1}$ on a calculator help to calculate the value of θ?
- Discuss how you can check if your calculator is in degree mode.

KEY IDEAS

- **Inverse trigonometric functions** are used to find angles in right-angled triangles.

  If $\sin\theta = k$ then $\theta = \sin^{-1}(k)$.

  If $\cos\theta = k$ then $\theta = \cos^{-1}(k)$.

  If $\tan\theta = k$ then $\theta = \tan^{-1}(k)$.

  where $-1 \leqslant k \leqslant 1$ for $\sin\theta$ and $\cos\theta$.

BUILDING UNDERSTANDING

1 State the missing part in each sentence.

a If $\cos 60° = 0.5$, then $\cos^{-1}(0.5) =$ ______.

b If $\sin 30° = \frac{1}{2}$, then $\sin^{-1}($______$) = 30°$.

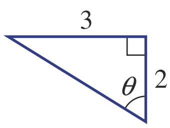
c If $\tan 37° \approx 0.75$, then $\tan^{-1}($______$) \approx 37°$.

2 Find θ in the following, rounding your answer to two decimal places where necessary.

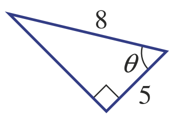
a $\sin \theta = 0.4$ **b** $\cos \theta = 0.5$ **c** $\tan \theta = 0.2$ **d** $\sin \theta = 0.1$

3 Decide which trigonometric ratio (i.e. sine, cosine or tangent) would be used to find θ in these triangles.

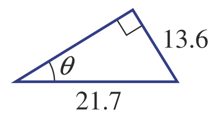
a



b



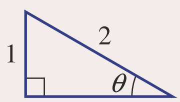
c



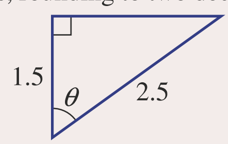
Example 3 Finding angles

Find the value of θ in the following right-angled triangles, rounding to two decimal places in part **b**.

a



b



SOLUTION

a $\sin \theta = \frac{1}{2}$

$\therefore \theta = \sin^{-1}\left(\frac{1}{2}\right)$

$= 30°$

b $\cos \theta = \frac{1.5}{2.5}$

$\therefore \theta = \cos^{-1}\left(\frac{1.5}{2.5}\right)$

$= 53.13°$ (to 2 d.p.)

EXPLANATION

Use $\sin \theta$, as the opposite side and the hypotenuse are given.

Use inverse sine on a calculator to find the angle.

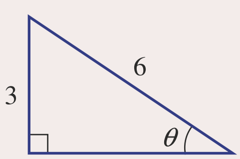
The adjacent side and the hypotenuse are given, so use $\cos \theta$.

Use inverse cosine on a calculator to find the angle and round your answer to two decimal places.

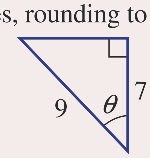
Now you try

Find the value of θ in the following right-angled triangles, rounding to two decimal places in part **b**.

a



b



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 4 Working with simple applications

A long, straight mine tunnel is sunk into the ground. Its final depth is 120 m and the end of the tunnel is 100 m horizontally from the ground entrance. Find the angle the tunnel makes with the horizontal (θ), correct to one decimal place.

SOLUTION

$$\tan\theta = \frac{120}{100}$$

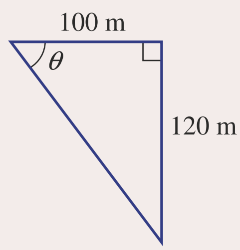
$$\theta = \tan^{-1}\left(\frac{120}{100}\right)$$

$$= 50.2^\circ \text{ (to 1 d.p.)}$$

$\therefore$ 50.2° is the angle the tunnel makes with the horizontal.

EXPLANATION

Start with a labelled diagram.



Now you try

A straight rabbit burrow is dug into the ground. Its final depth is 4 m and the end of the burrow is 5 m horizontally from the ground entrance. Find the angle the burrow makes with the horizontal (θ), correct to one decimal place.

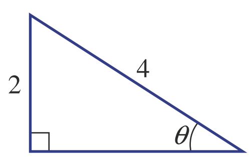
Exercise 4B

FLUENCY	1, 2–3(1/2)	2–4(1/2)	2–4(1/3)

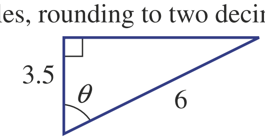
Example 3

1 Find the value of θ in the following right-angled triangles, rounding to two decimal places in part **b**.

a



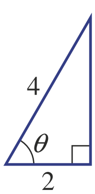
b



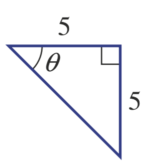
Example 3

2 Find the value of θ in the following right-angled triangles, rounding your answer to two decimal places where necessary.

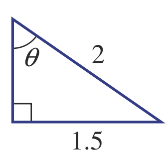
a



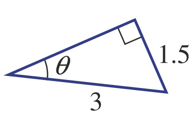
b



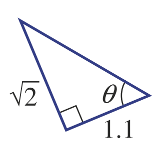
c



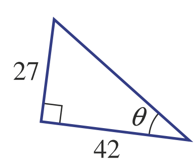
d



e

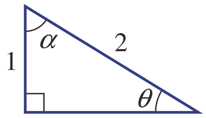


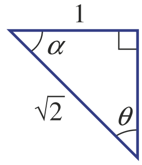
f

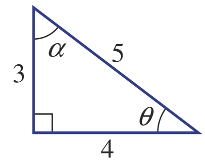


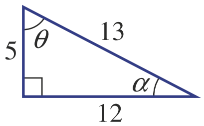
ISBN 978-1-108-77290-7

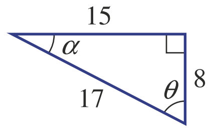
3 Find the value of α and θ, to one decimal place where necessary, for these special triangles.

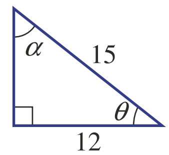
a


b


c


d


e


f


4 The lengths of two sides of a right-angled triangle are provided. Use this information to find the size of the two interior acute angles, and round each answer to one decimal place.

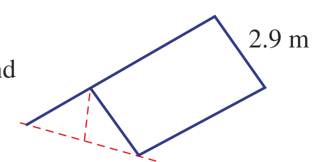
a hypotenuse 5 cm, opposite 3.5 cm
b hypotenuse 7.2 m, adjacent 1.9 m
c hypotenuse 0.4 mm, adjacent 0.21 mm
d opposite 2.3 km, adjacent 5.2 km
e opposite 0.32 cm, adjacent 0.04 cm
f opposite $\sqrt{5}$ cm, hypotenuse $\sqrt{11}$ cm

PROBLEM-SOLVING	5, 6	6, 7	7, 8

Example 4

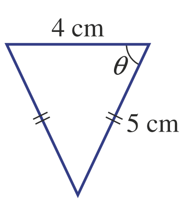
5 A ladder reaches 5.5 m up a wall and sits 2 m from the base of the wall. Find the angle the ladder makes with the horizontal, correct to two decimal places.

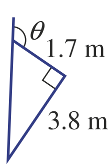
6 A tarpaulin with a simple A-frame design is set up as a shelter. The width of half of the tarpaulin is 2.9 metres, as shown. Find the angle to the ground that the sides of the tarpaulin make if the height at the middle of the shelter is 1.5 metres. Round your answer to the nearest 0.1 of a degree.

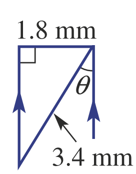


7 A diagonal cut of length 2.85 metres is to be made on a rectangular wooden slab from one corner to the other. The front of the slab measures 1.94 metres. Calculate the angle with the front edge at which the carpenter needs to begin the cut. Round your answer to one decimal place.

8 Find the value of θ in these diagrams, correct to one decimal place.

a


b


c


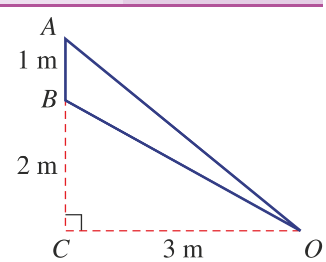
REASONING	9	9, 10	10, 11

9 Consider $\triangle OAC$ and $\triangle OBC$.

a Find, correct to one decimal place where necessary:

i $\angle AOC$ **ii** $\angle BOC$

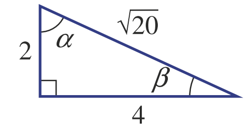
b Hence, find the angle $\angle AOB$.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

10 This triangle includes the unknown angles α and β.



a Explain why only one inverse trigonometric ratio needs to be used to find the values of both α and β.

b Find α and β, correct to one decimal place, using your method from part **a**.

11 **a** Draw a right-angled isosceles triangle and show all the internal angles.

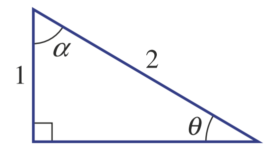
b If one of the shorter sides is of length x, show that $\tan 45° = 1$.

c Find the exact length of the hypotenuse in terms of x.

d Show that $\sin 45° = \cos 45°$.

ENRICHMENT: A special triangle	–	–	12

12 Consider this special triangle.



a Find the value of θ.

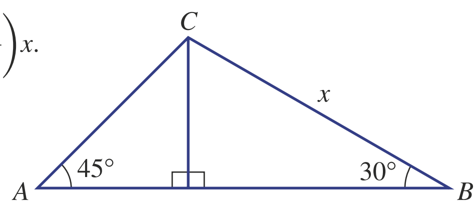
b Find the value of α.

c Use Pythagoras' theorem to find the exact length of the unknown side, in surd form.

d Hence, write down the exact value for the following, in surd form.

i $\sin 30°$ **ii** $\cos 60°$ **iii** $\sin 60°$ **iv** $\cos 30°$ **v** $\tan 30°$ **vi** $\tan 60°$

e For the diagram on the right, show that $AB = \left(\frac{\sqrt{3} + 1}{2}\right)x$.



A map showing the triangles and transects used in the Great Trigonometric Survey of India, produced in 1870

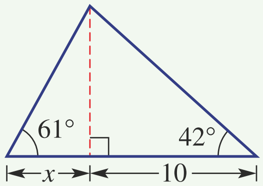
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

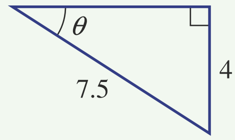
Using calculators in trigonometry

Find the value of the unknowns in these triangles, correct to two decimal places.

a

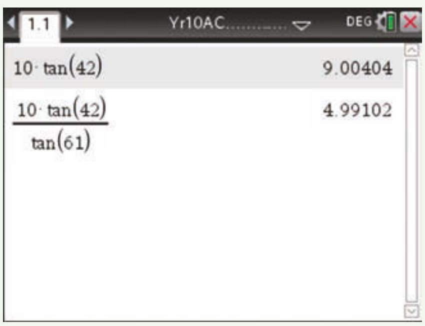


b



Using the TI-Nspire:

a First, find the height of the triangle using tan. Do not round this value before using it for the next step. Then use this result to find the value of x also using tan. Ensure your General Settings include Degree and Approximate (decimal) modes.



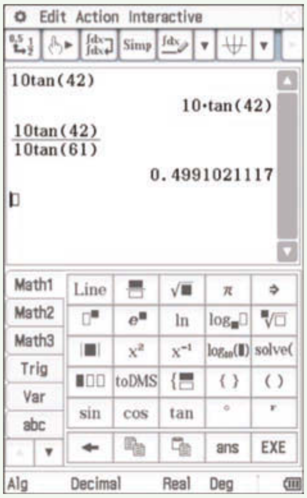
Hint: use the trig key to access tan.
Hint: you can also include a degree symbol (ctrl and select °) in your entries if desired.

b Use the inverse sine function in Degree mode.

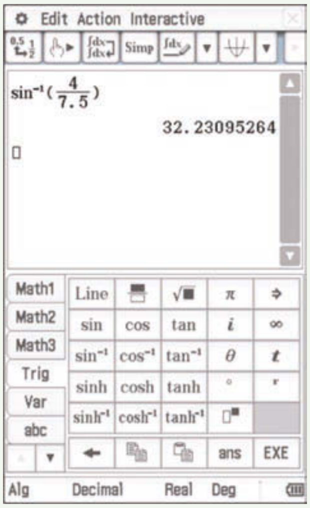
Hint: use the trig key to access $\sin^{-1}$.

Using the ClassPad:

a In **Standard Degree** mode, first find the height of the triangle using tan. Use this result to find the value of x also using tan. Do this calculation in **Decimal Degree** mode.



b Use the inverse sine function in **Decimal Degree** mode.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

4C Applications using angles of elevation and depression

Learning intentions

- To know how angles of elevation and depression are measured
- To be able to draw and label an appropriate diagram from a word problem description and identify a right-angled triangle
- To know how to apply the correct trigonometric relationship to solve a problem

There are many situations where a two-dimensional right-angled triangle can be drawn so that trigonometry can be used to solve a problem. An angle of elevation or depression is commonly used in such triangles.

Pilots are trained in trigonometry. Starting the final descent, a pilot will check that the plane's altitude and its horizontal distance from the runway allow for the required angle of descent (i.e. depression) of 3° below the horizontal.

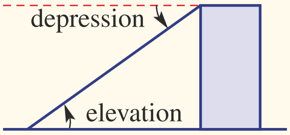
LESSON STARTER Mountain peaks

Two mountain peaks in Victoria are Mt Stirling (1749 m) and Mt Buller (1805 m). A map shows a horizontal distance between them of 6.8 km.

- Discuss if you think there is enough information to find the angle of elevation of Mt Buller from Mt Stirling.
- What diagram can be used to summarise the information?
- Show how trigonometry can be used to find this angle of elevation.
- Discuss what is meant by the words *elevation* and *depression* in this context.

KEY IDEAS

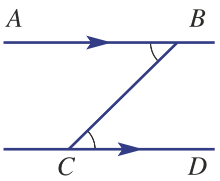
- The **angle of elevation** is measured *up* from the horizontal.
- The **angle of depression** is measured *down* from the horizontal.
  - On the same diagram, the angle of elevation and the angle of depression are equal. They are alternate angles in parallel lines.



- To solve more complex problems involving trigonometry:
  - Visualise and draw a right-angled triangle and add any given information.
  - Use a trigonometric ratio to find the unknown.
  - Answer the question in words.

BUILDING UNDERSTANDING

1 Name the two marked angles which are equal in this diagram.



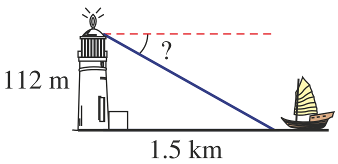
2 Choose the diagram (**A**, **B** or **C**) which matches the description (**a**, **b** or **c**). (Do not try to find the answer.)

a A cable of length 45 metres is anchored from the ground to the top of a communications mast. The angle of elevation of the cable to the top of the mast is 60°. Find the height of the communications mast.

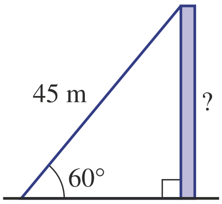
b The angle of depression from the top of a 25 metre tall viewing tower to a crocodile on the ground is 62°. Find the direct distance from the top of the tower to the crocodile.

c Find the angle of depression from a lighthouse beacon that is 112 metres above sea level to a boat that is at a horizontal distance of 1.5 kilometres from the lighthouse.

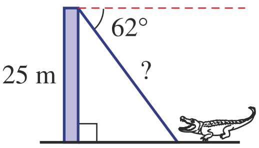
A



B

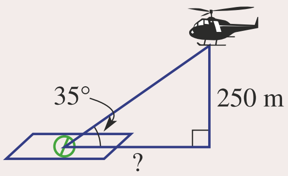


C



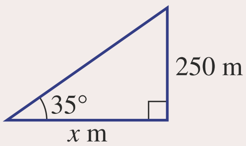
Example 5 Applying trigonometry in worded problems

A helicopter is hovering at an altitude of 250 metres. The angle of elevation from the helipad to the helicopter is 35°. Find the horizontal distance of the helicopter from the helipad, to the nearest centimetre.



SOLUTION

Let x metres be the horizontal distance from the helicopter to the helipad.



$$\tan 35° = \frac{250}{x}$$
$$\therefore x \times \tan 35° = 250$$
$$x = \frac{250}{\tan 35°}$$
$$= 357.04$$

The horizontal distance from the helicopter to the helipad is 357.04 m.

EXPLANATION

Use $\tan\theta = \frac{\text{O}}{\text{A}}$ since the opposite and adjacent sides are being used. Solve for x.

There are 100 cm in 1 m, so round to two decimal places for the nearest centimetre.

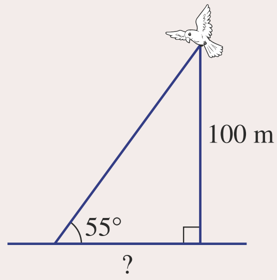
Answer the question in words.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Now you try

A bird is hovering at an altitude of 100 m. The angle of elevation from the observation point to the bird is 55°. Find the horizontal distance of the bird from the observation point, to the nearest centimetre.



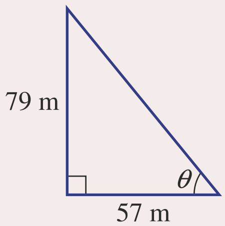
Example 6 Combining trigonometry with problem solving

Two vertical buildings 57 metres apart are 158 metres and 237 metres high. Find the angle of elevation from the top of the shorter building to the top of the taller building, correct to two decimal places.

SOLUTION

Let θ be the angle of elevation from the top of the shorter building to the top of the taller building.

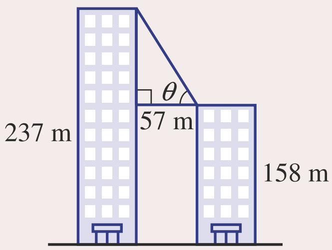
$$\begin{aligned}\text{Height difference} &= 237 - 158\\ &= 79 \text{ m}\end{aligned}$$



$$\begin{aligned}\tan\theta &= \frac{79}{57}\\ \theta &= \tan^{-1}\left(\frac{79}{57}\right)\\ &= 54.19^\circ \text{ (to 2 d.p.)}\end{aligned}$$

The angle of elevation from the top of the shorter building to the top of the taller building is 54.19°.

EXPLANATION



Draw the relevant right-angled triangle separately. We are given the opposite (O) and the adjacent (A) sides; hence, use tan.

Use the inverse tan function to find θ, correct to two decimal places.

Answer the question in words.

Now you try

Two vertical poles 32 metres apart are 62 metres and 79 metres high. Find the angle of elevation from the top of the shorter pole to the top of the taller pole, correct to two decimal places.

ISBN 978-1-108-77290-7

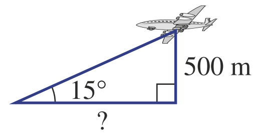
Cambridge University Press
Updated June 2022

Exercise 4C

FLUENCY	1–4	2–4	3–5

Example 5

1 The altitude of an aeroplane is 500 metres, and the angle of elevation from the runway to the aeroplane is 15°. Find the horizontal distance from the aeroplane to the runway, to the nearest centimetre.



Example 5

2 The distance between two buildings is 24.5 metres. Find the height of the taller building, to the nearest metre, if the angle of elevation from the top of the shorter building to the top of the taller building is 85° and the height of the shorter building is 40 m.

3 The angle of depression from one mountain summit to another is 15.9°. If the two mountains differ in height by 430 metres, find the horizontal distance between the two summits, to the nearest centimetre.

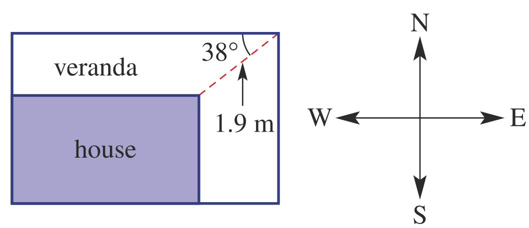
Example 6

4 Two vertical buildings positioned 91 metres apart are 136 metres and 192 metres tall, respectively. Find the angle of elevation from the top of the shorter building to the top of the taller building, to the nearest degree.

5 An L-shaped veranda has dimensions as shown. Find the width, to the nearest centimetre, of the veranda for the following sides of the house:

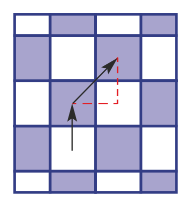
a north side

b east side.



PROBLEM-SOLVING	6	6, 7	7, 8

6 A knight on a chessboard is moved forward 3.6 cm from the centre of one square to another, then diagonally across at 45° to the centre of the destination square. How far did the knight move in total? Give your answer to two decimal places.



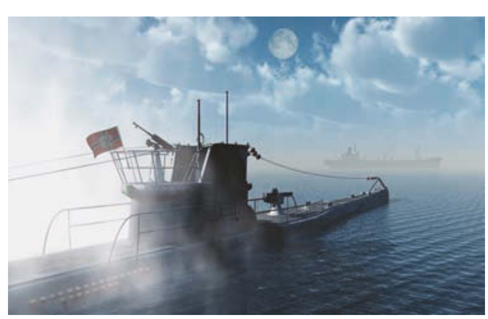
7 Two unidentified flying discs are detected by a receiver. The angle of elevation from the receiver to each disc is 39.48°. The discs are hovering at a direct distance of 826 m and 1.296 km from the receiver. Find the difference in height between the two unidentified flying discs, to the nearest metre.

8 Initially a ship and a submarine are stationary at sea level, positioned 1.78 kilometres apart. The submarine then manoeuvres to position A, 45 metres directly below its starting point. In a second manoeuvre, the submarine dives a further 62 metres to position B. Give all answers to two decimal places.

a Find the angle of elevation of the ship from the submarine when the submarine is at position A.

b Find the angle of elevation of the ship from the submarine when the submarine is at position B.

c Find the difference in the angles of elevation from the submarine to the ship when the submarine is at positions A and B.



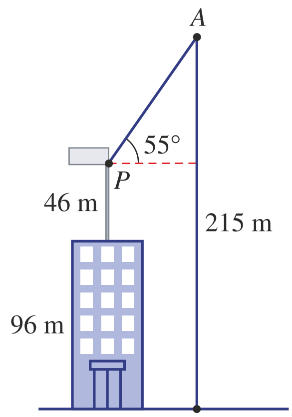
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

REASONING	9	9, 10	10–12

9 A communications technician claims that when the horizontal distance between two television antennas is less than 12 metres, then an interference problem will occur. The heights of two antennas above ground level are 7.5 metres and 13.9 metres, respectively, and the angle of elevation from the top of the shorter antenna to the top of the taller antenna is 29.5°. According to the technician's claim, will there be an interference problem for these two antennas?

10 The pivot point (P) of the main supporting arm (AP) of a construction crane is 46 metres above the top of a 96 metre tall office building. When the supporting arm is at an angle of 55° to the horizontal, the length of cable dropping from the point A to the ground is 215 metres. Find the length of the main supporting arm (AP), to the nearest centimetre.



11 Consider a regular hexagon with internal angles of 120° and side lengths of 10 cm.

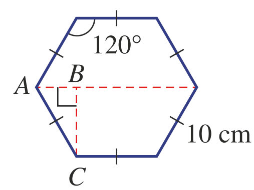
a For the given diagram find, to the nearest millimetre, the lengths:

i BC **ii** AB

b Find the distance, to the nearest millimetre, between:

i two parallel sides **ii** two opposite vertices.

c Explore and describe how changing the side lengths of the hexagon changes the answers to part **b**.



12 An aeroplane is flying horizontally, directly towards the city of Melbourne at an altitude of 400 metres. At a given time the pilot views the city lights of Melbourne at an angle of depression of 1.5°. Two minutes later the angle of depression of the city lights is 5°. Find the speed of the aeroplane in km/h, correct to one decimal place.

ENRICHMENT: Vegetable garden design	–	–	13

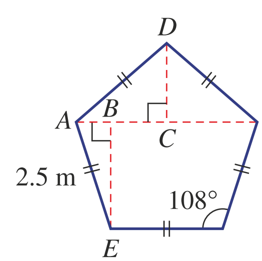
13 A vegetable garden is to be built in the shape of a regular pentagon using redgum sleepers of length 2.5 metres, as shown. It is known that the internal angles of a regular pentagon are 108°.

a Find the size of the following angles.

i $\angle AEB$ **ii** $\angle EAB$ **iii** $\angle CAD$ **iv** $\angle ADC$

b Find these lengths, to two decimal places.

i AB **ii** BE **iii** AC **iv** CD



c Find the distance between a vertex on the border of the vegetable garden and the centre of its opposite side, to two decimal places.

d Find the distance between any two non-adjacent vertices on the border of the vegetable garden, to two decimal places.

e Show that when the length of the redgum sleepers is x metres, the distance between a vertex and the centre of its opposite side of the vegetable garden will be $1.54x$ metres, using two decimal places.

ISBN 978-1-108-77290-7

4D Bearings

Learning intentions

- To understand how true bearings are measured and written
- To be able to state a true bearing and its opposite direction from a diagram
- To be able to apply bearings in word problems using a diagram and trigonometry

True bearings are used to communicate direction and therefore are important in navigation. Ship and aeroplane pilots, bushwalkers and military personnel all use bearings to navigate and communicate direction.

Accurate navigation is vital to military personnel, ship and plane pilots, geologists and bushwalkers, who all use bearings and maps to navigate and communicate direction. GPS signals are weak, unreliable and not accurate enough for precise navigation.

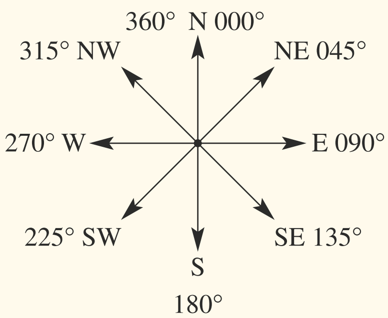
LESSON STARTER Navigating a square

A mining surveyor starts walking from base camp to map out an area for soil testing. She starts by walking 2 km on a true bearing of 020° and wants to map out an area that is approximately square.

- Draw a diagram showing the first leg of the walk and the direction of north.
- If the surveyor turns right for the next leg, what will be the true bearing for this section?
- List the direction (as a true bearing) and the distance for all four legs of the walk. Remember that the mapped area must be a square.

KEY IDEAS

- **True bearings** (°T) are measured clockwise from due north. Some angles and directions are shown in this diagram; for example, NE means north-east.
  - True bearings are usually written using three digits.
  - Opposite directions differ by 180°.



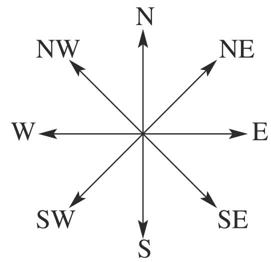
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

BUILDING UNDERSTANDING

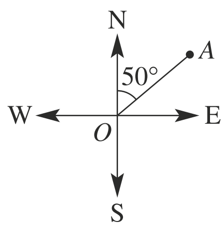
1 Give the true bearing for each of these directions.

a N **b** NE
c E **d** SE
e S **f** SW
g W **h** NW

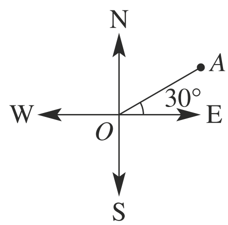


2 For each diagram, give the true bearing from O to A.

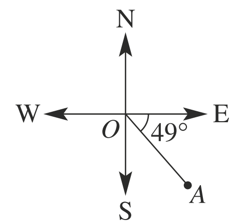
a



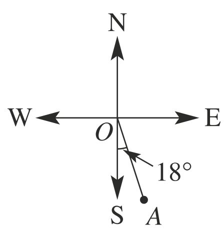
b



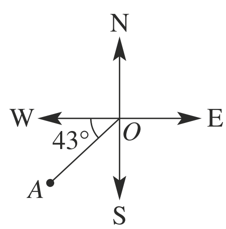
c



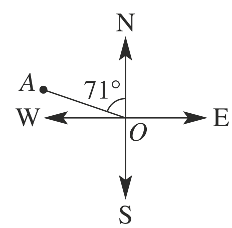
d



e



f



3 State the bearing that is the opposite direction to the following.

a 020°T **b** 262°T **c** 155°T **d** 344°T

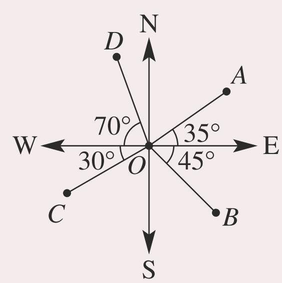
Example 7 Stating a direction

A, B, C and D are four points, as shown.

a Give the true bearing of each point from the origin, O, in this diagram.

b Give the true bearing of:

i O from A **ii** O from D.



SOLUTION

a The bearing of A is $90° - 35° = 055°$T.

The bearing of B is $90° + 45° = 135°$T.

The bearing of C is $270° - 30° = 240°$T.

The bearing of D is $270° + 70° = 340°$T.

EXPLANATION

East is 090° so subtract 35° from 90°.

B is 90° plus the additional 45° in a clockwise direction.

West is 270° so subtract 30° from 270°.

Alternatively for D, subtract 20° from 360°.

Continued on next page

ISBN 978-1-108-77290-7

Cambridge University Press

Updated June 2022

b **i** The bearing of O from A is $180^\circ + 55^\circ = 235^\circ$T

The bearing of A from O is 055°T and an opposite direction differs by 180°.

ii The bearing of O from D is $340^\circ - 180^\circ = 160^\circ$T

Subtract 180° from the opposite direction (340°T).

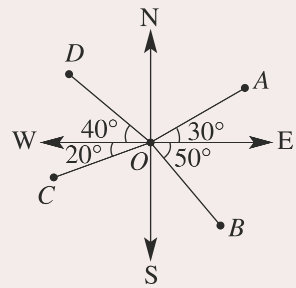
Now you try

A, B, C and D are four points, as shown.

a Give the true bearing of each point from the origin, O, in this diagram.

b Give the true bearing of:

i O from A **ii** O from D



Example 8 Using bearings with trigonometry

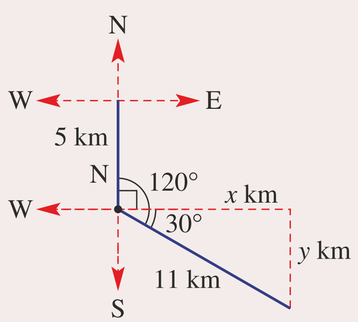
A ship travels due south for 5 km, then on a true bearing of 120° for 11 km.

a Find how far east the ship is from its starting point, correct to two decimal places.

b Find how far south the ship is from its starting point.

SOLUTION

a



EXPLANATION

Draw a clear diagram, labelling all relevant angles and lengths. Draw a compass at each change of direction. Clearly show a right-angled triangle, which will help to solve the problem.

$$\cos 30^\circ = \frac{x}{11}$$

$$x = 11 \times \cos 30^\circ$$

$$= 9.53 \text{ (to 2 d.p.)}$$

As x is adjacent to 30° and the hypotenuse has length 11 km, use cos.

The ship is 9.53 km east of its initial position.

Answer in words.

b $\sin 30° = \dfrac{y}{11}$

$y = 11 \times \sin 30°$

$= 5.5$

Distance south $= 5.5 + 5 = 10.5$ km

The ship is 10.5 km south of its initial position.

Use sine for opposite and hypotenuse. Use the value provided rather than your answer from part **a**.

Multiply both sides by 11.

Find total distance south by adding the initial 5 km. Answer in words.

Now you try

A ship travels due south for 8 km, then on a bearing of 160° for 12 km.

a Find how far east the ship is from its starting point, correct to two decimal places.

b Find how far south the ship is from its starting point, correct to two decimal places.

Exercise 4D

FLUENCY	1–5	2–6	2, 4, 6, 7

Example 7a

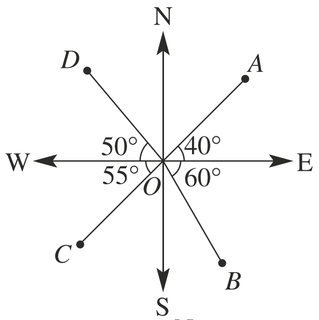
Example 7b

1 A, B, C and D are four points, as shown.

a Give the true bearing of each point from the origin, O, in this diagram.

b Give the true bearing of:

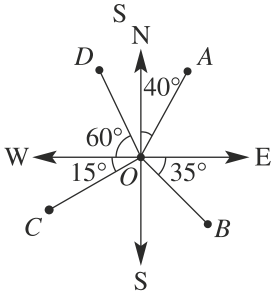
i O from A **ii** O from D.



Example 7

2 Find the true bearing of O from each of the following points, shown in this simple map. (*Hint*: First, find the bearing of each point from O.)

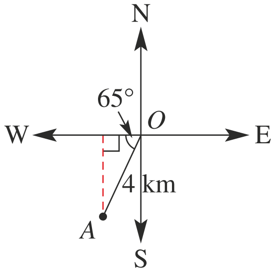
a A **b** B **c** C **d** D



3 For this simple map, find the following, correct to one decimal place.

a How far west is point A from O?

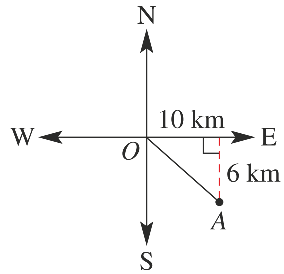
b How far south is point A from O?



4 Find the true bearing, correct to the nearest degree, of:

a point A from O

b point O from A.



Example 8

5 A ship travels due south for 3 km, then on a true bearing of 130° for 5 km.

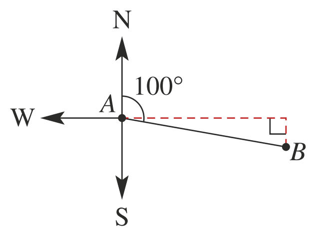
a Find how far east the ship is from its starting point, correct to two decimal places.

b Find how far south the ship is from its starting point, correct to two decimal places.

6 Two points, A and B, positioned 15 cm apart, are such that B is on a true bearing of 100° from A.

a Find how far east point B is from A, correct to two decimal places.

b Find how far south point B is from A, correct to the nearest millimetre.



7 An aeroplane flies 138 km in a southerly direction from a military air base to a drop-off point. The drop-off point is 83 km west of the air base. Find the true bearing, correct to the nearest degree, of:

a the drop-off point from the air base

b the air base from the drop-off point.

PROBLEM-SOLVING	8, 9	8–10	10, 11

8 A bushwalker hikes due north from a resting place for 1.5 km to a waterhole and then on a true bearing of 315° for 2 km to base camp.

a Find how far west the base camp is from the waterhole, to the nearest metre.

b Find how far north the base camp is from the waterhole, to the nearest metre.

c Find how far north the base camp is from the initial resting place, to the nearest metre.

9 On a map, point C is 4.3 km due east of point B, whereas point B is 2.7 km on a true bearing of 143° from point A. Give your answer to two decimal places for the following.

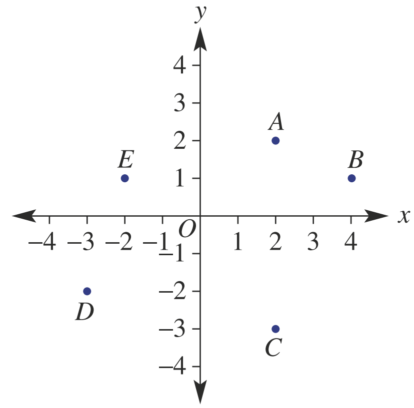
a Find how far east point B is from A.

b Find how far east point C is from A.

c Find how far south point C is from A.

10 A military desert tank manoeuvres 13.5 km from point A on a true bearing of 042° to point B. From point B, how far due south must the tank travel to be at a point due east of point A? Give the answer correct to the nearest metre.

11 Consider the points O, A, B, C, D and E on this Cartesian plane. Round the answers to one decimal place.



a Find the true bearing of:

i A from O **ii** D from O

iii B from C **iv** E from C

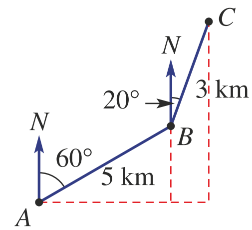
b Find the true bearing from:

i O to E **ii** A to B

iii D to C **iv** B to D

REASONING	12	12, 13	12–14

12 An overall direction and distance of a journey can be calculated by considering two (or more) smaller parts (or legs). Find the bearing of C from A and the length AC in this journey by answering these parts.



a Find, correct to two decimal places where necessary, how far north:

i point B is from A

ii point C is from B

iii point C is from A.

b Find, correct to two decimal places, how far east:

i point B is from A

ii point C is from B

iii point C is from A.

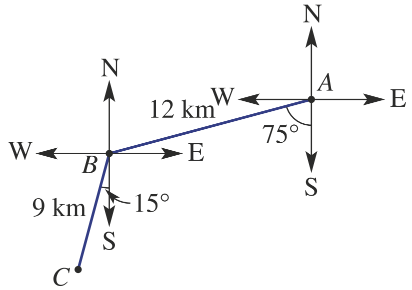
c Now use your answers above to find the following, correct to one decimal place.

i the true bearing of C from A

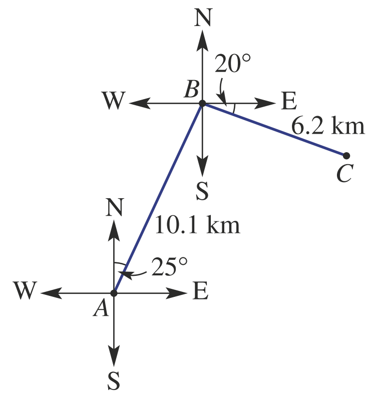
ii the distance from A to C. (*Hint*: Use Pythagoras' theorem.)

13 Use the technique outlined in Question 12 to find the distance AC and the bearing of C from A in these diagrams. Give your answers correct to one decimal place.

a



b



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

14 Tour groups A and B view a rock feature from different positions on a road heading east–west.

Group A views the rock at a distance of 235 m on a bearing of 155° and group B views the rock feature on a bearing of 162° at a different point on the road. Round all answers to two decimal places in the following.

a Find how far south the rock feature is from the road.
b Find how far east the rock feature is from:
  i group A
  ii group B.
c Find the distance between group A and group B.

ENRICHMENT: Navigation challenges	–	–	15, 16

15 A light aeroplane is flown from a farm airstrip to a city runway that is 135 km away. The city runway is due north from the farm airstrip. To avoid a storm, the pilot flies the aeroplane on a bearing of 310° for 50 km, and then due north for 45 km. The pilot then heads directly to the city runway. Round your answers to two decimal places in the following.

a Find how far west the aeroplane diverged from the direct line between the farm airstrip and the city runway.
b Find how far south the aeroplane was from the city runway before heading directly to the city runway on the final leg of the flight.
c Find the bearing the aeroplane was flying on when it flew on the final leg of the flight.

16 A racing yacht sails from the start position to a floating marker on a bearing of 205.2° for 2.82 km, then to a finish line on a bearing of 205.9° for 1.99 km. Round each of the following to two decimal places.

a Find how far south the finish line is from the start position.
b Find how far west the finish line is from the start position.
c Use Pythagoras' theorem to find the distance between the finish line and the start position.

ISBN 978-1-108-77290-7

4E Applications in three dimensions 10A

Learning intentions

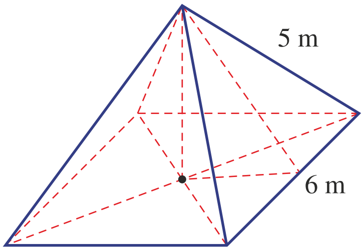
- To be able to visualise right-angled triangles in 3D objects
- To be able to draw and label right-angled triangles formed in 3D objects
- To know how to apply the trigonometric ratios to find an unknown and relate this to the original object

Although a right-angled triangle is a two-dimensional shape, it can also be used to solve problems in three dimensions. Being able to visualise right-angled triangles included in three-dimensional diagrams is an important part of the process of finding angles and lengths associated with three-dimensional objects.

Surveyors use trigonometry to calculate distances and angles between points in three dimensions. Surveyors accurately locate corners for new buildings, boundaries of property for legal ownership, and the placement of roads, bridges, dams, water pipes, power pylons, etc.

LESSON STARTER How many right-angled triangles?

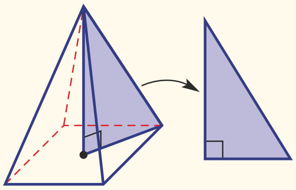
A right square-based pyramid has the apex above the centre of the base. In this example, the base length is 6 m and slant height is 5 m. Other important lines are dashed.



- Using the given dashed lines and the edges of the pyramid, how many different right-angled triangles can you draw?
- Is it possible to determine the exact side lengths of all your right-angled triangles?
- Is it possible to determine all the angles inside all your right-angled triangles?

KEY IDEAS

- Using trigonometry to solve problems in three dimensions involves:
  - visualising and drawing any relevant two-dimensional triangles
  - using trigonometric ratios to find unknowns
  - relating answers from two-dimensional diagrams to the original three-dimensional object.

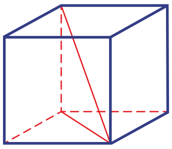


ISBN 978-1-108-77290-7

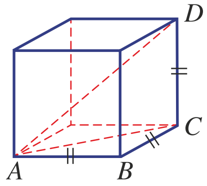
Cambridge University Press
Updated June 2022

BUILDING UNDERSTANDING

1 By considering only the lines drawn inside this rectangular prism, how many right-angled triangles are formed?



2 The cube shown here has side length 2 m.



a Draw the right-angled triangle ABC and find and label all the side lengths. Pythagoras' theorem can be used. Answer using exact values (e.g. $\sqrt{5}$).

b Draw the right-angled triangle ACD and find and label all the side lengths. Pythagoras' theorem can be used. Answer using exact values.

c Use trigonometry to find $\angle DAC$, correct to one decimal place.

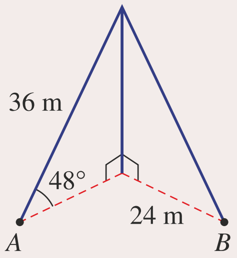
d Find the size of $\angle CAB$.

Example 9 Applying trigonometry in 3D

A vertical mast is supported at the top by two cables reaching from two points, A and B. The cable reaching from point A is 36 metres long and is at an angle of 48° to the horizontal. Point B is 24 metres from the base of the mast.

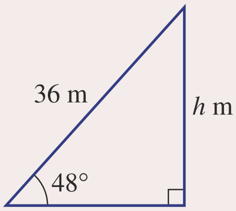
a Find the height of the mast, correct to three decimal places.

b Find the angle to the horizontal of the cable reaching from point B, to two decimal places.



SOLUTION

a Let h be the height of the mast, in metres.



$$\sin 48^\circ = \frac{h}{36}$$

$$h = 36 \times \sin 48^\circ$$
$$= 26.753 \text{ (to 3 d.p.)}$$

The height of the mast is 26.753 m.

EXPLANATION

First, draw the right-angled triangle, showing the information given.

The opposite (O) and hypotenuse (H) are given, so use sine.

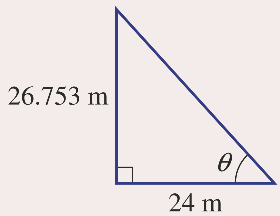
Multiply both sides by 36 and round to three decimal places.

Answer the question in words.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b



Draw the second triangle, including the answer from part **a**.

$$\tan\theta = \frac{26.753\ldots}{24}$$

$$\theta = \tan^{-1}\left(\frac{26.753\ldots}{24}\right)$$

$$= 48.11° \text{ (to 2 d.p.)}$$

More precisely, use $\tan\theta = \dfrac{36 \times \sin 48°}{24}$.

$$\theta = \tan^{-1}\left(\frac{36 \times \sin 48°}{24}\right)$$

The cable reaching from point B is at an angle of 48.11° to the horizontal.

Answer the question in words, rounding your answer appropriately.

Now you try

A vertical mast is supported at the top by two cables reaching from two points, A and B. The cable reaching from point A is 53 metres long and is at an angle of 37° to the horizontal. Point B is 29 metres from the base of the mast.

a Find the height of the mast, correct to three decimal places.

b Find the angle to the horizontal of the cable reaching from point B, to two decimal places.

Exercise 4E

FLUENCY	1–4	2–4	3–5

Example 9

1 A vertical mast is supported at the top by two cables reaching from two points, A and B. The cable reaching from point A is 43 metres long and is at an angle of 61° to the horizontal. Point B is 37 metres from the base of the mast.

a Find the height of the mast, correct to three decimal places.

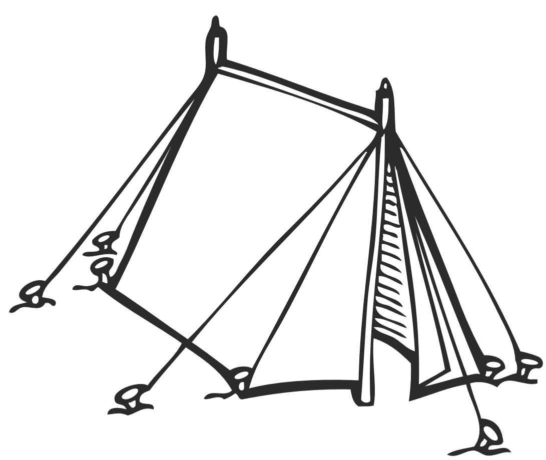
b Find the angle to the horizontal of the cable reaching from point B, to two decimal places.

Example 9

2 A vertical tent pole is supported at the top by two ropes reaching from two pegs, A and B. The rope reaching from peg A is 3 m long and is at an angle of 39° to the horizontal. Peg B is 2 m from the base of the pole.

a Find the height of the pole correct to three decimal places.

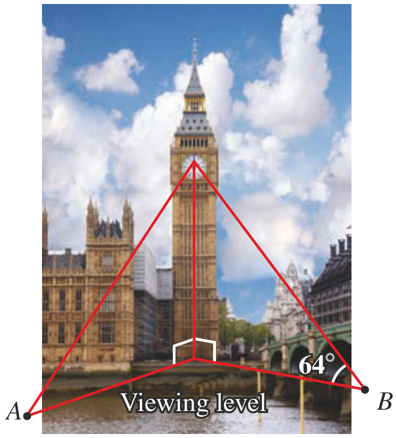
b Find the angle to the horizontal of the cable reaching from peg B, to two decimal places.



3 Viewing points A and B are at a horizontal distance from a clock tower of 36 metres and 28 metres, respectively. The viewing angle to the clockface at point B is 64°.

a Find the height of the clockface above the viewing level, to three decimal places.

b Find the viewing angle to the clockface at point A, to two decimal places.

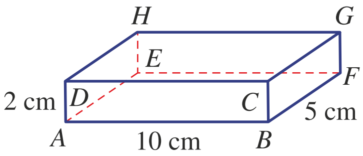


4 A rectangular prism, $ABCDEFGH$, is 5 cm wide, 10 cm long and 2 cm high.

a By drawing the triangle ABF find, to two decimal places:

i $\angle BAF$ ii AF

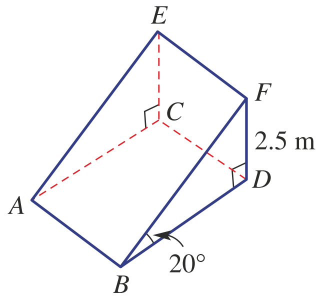
b By drawing the triangle AGF, find $\angle GAF$, to two decimal places.



5 A ramp, $ABCDEF$, rests at an angle of 20° to the horizontal and the highest point on the ramp is 2.5 metres above the ground, as shown. Give your answers to two decimal places in the following questions.

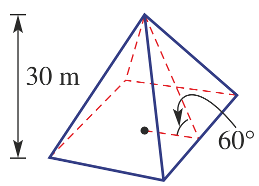
a Find the length of the ramp BF.

b Find the length of the horizontal BD.



PROBLEM-SOLVING	6, 7	6–9	7–9

6 The triangular faces of a right square-based pyramid are at an angle of 60° to the base. The height of the pyramid is 30 m. Find the perimeter of the base of the pyramid, correct to one decimal place.



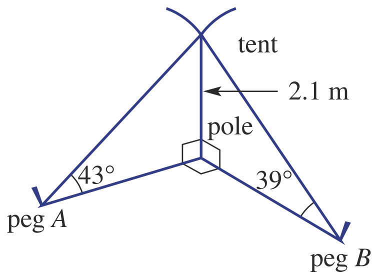
7 A tent pole 2.1 metres tall is secured by ropes in two directions. The ropes are held by pegs A and B at angles of 43° and 39°, respectively, from the horizontal. The line from the base of the pole to peg A is at right angles to the line from the base of the pole to peg B. Round your answers to two decimal places in these questions.

a Find the distance from the base of the tent pole to:

i peg A ii peg B

b Find the angle at peg A formed by peg A, peg B and the base of the pole.

c Find the distance between peg A and peg B.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

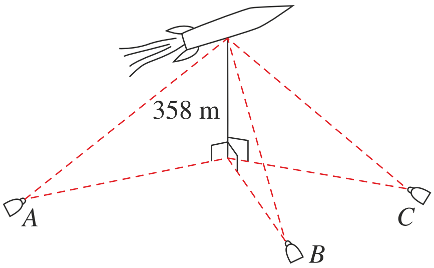
8 The communities of Wood Town and Green Village live in a valley. Communication between the two communities is enhanced by a repeater station on the summit of a nearby mountain. It is known that the angles of depression from the repeater station to Wood Town and Green Village are 44.6° and 58.2°, respectively. Also, the horizontal distances from the repeater to Wood Town and Green Village are 1.35 km and 1.04 km, respectively.

a Find the vertical height, to the nearest metre, between the repeater station and:

i Wood Town **ii** Green Village.

b Find the difference in height between the two communities, to the nearest metre.

9 Three cameras operated at ground level view a rocket being launched into space.



At 5 seconds immediately after launch, the rocket is 358 m above ground level and the three cameras, A, B and C, are positioned at an angle of 28°, 32° and 36°, respectively, to the horizontal.

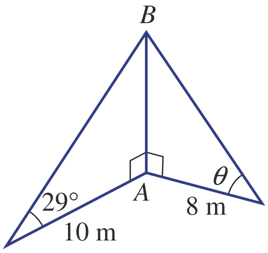
At the 5 second mark, find:

a which camera is closest to the rocket

b the distance between the rocket and the closest camera, to the nearest centimetre.

REASONING	10	10	10, 11

10 It is important to use a high degree of accuracy for calculations that involve multiple parts.



For this 3D diagram complete these steps.

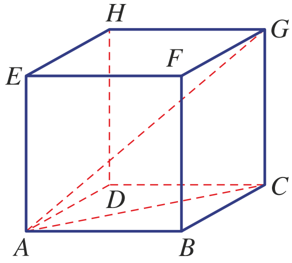
a Find AB, correct to one decimal place.

b Use your answer from part **a** to find θ, correct to one decimal place.

c Now recalculate θ using a more accurate value for AB. Round θ to one decimal place.

d What is the difference between the answers for parts **b** and **c**?

11 For a cube, $ABCDEFGH$, of side length 1 unit, as shown, use trigonometry to find the following, correct to two decimal places where necessary. Be careful that errors do not accumulate.



a $\angle BAC$ **b** AC

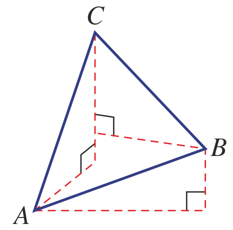
c $\angle CAG$ **d** AG

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

ENRICHMENT: Three points in 3D	–	–	12, 13

12 Three points, A, B and C, in three-dimensional space are such that $AB = 6$, $BC = 3$ and $AC = 5$.



The angles of elevation from A to B and from B to C are $15°$ and $25°$, respectively. Round your answer to two decimal places in the following.

a Find the vertical difference in height between:
- **i** A and B
- **ii** B and C
- **iii** A and C.

b Find the angle of elevation from A to C.

13 The points A, B and C in 3D space are such that:
- $AB = 10$ mm, $AC = 17$ mm and $BC = 28$ mm.
- The angle of elevation from A to B is $20°$.
- The angle of elevation from A to C is $55°$.

Find the angle of elevation from B to C, to the nearest degree.

Triangulation points or 'trig stations' such as this are used in geodetic surveying to mark points at which measurements are made to calculate local altitude. The calculations involved are similar to those in the Enrichment questions above.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

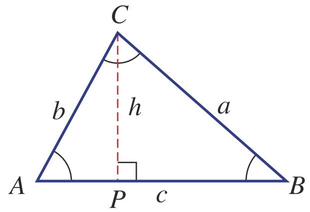
4F The sine rule 10A

Learning intentions

- To know how the sine rule relates the ratio of sides and angles in non right-angled triangles
- To know the criteria of a triangle that must be known to apply the sine rule
- To know how to apply the sine rule to find an angle or a side length in non right-angled triangles
- To understand that some information of triangles can lead to two possible triangles involving either an acute angle or an obtuse angle
- To be able to use the sine rule to find acute or obtuse angles

The use of sine, cosine and tangent functions can be extended to non right-angled triangles.

First consider the triangle below with sides a, b and c and with opposite angles $\angle A$, $\angle B$ and $\angle C$. Height h is also shown.



Pilots need to compensate for cross-winds. In a triangle *ABC*, if *AB* shows a plane's speed and direction and *BC* the wind's speed and direction, then side *AC* gives the plane's resultant speed and direction, calculated using the sine and cosine rules.

$$\begin{aligned} &\text{From } \triangle CPB, && \sin B = \frac{h}{a} \\ &\text{so} && h = a\sin B \\ &\text{From } \triangle CPA, && \sin A = \frac{h}{b} \\ &\text{so} && h = b\sin A \end{aligned}$$

$$\therefore\ a\sin B = b\sin A \text{ or } \frac{a}{\sin A} = \frac{b}{\sin B}$$

Similarly, it can be shown that $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$ and $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

In this section we will consider the sine of angles larger than 90°. This will be discussed in more detail in **Section 4I** but for the moment we will accept such angles with our trigonometric functions.

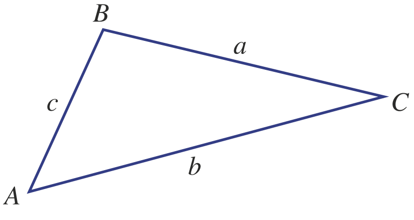
LESSON STARTER Explore the sine rule

Use a ruler and a protractor to measure the side lengths (a, b and c) in centimetres, correct to one decimal place, and the angles (A, B and C), correct to the nearest degree, for this triangle.

- Calculate the following.

  a $\dfrac{a}{\sin A}$ **b** $\dfrac{b}{\sin B}$ **c** $\dfrac{c}{\sin C}$

- What do you notice about the three answers above?
- Draw your own triangle and check to see if your observations are consistent for any triangle.

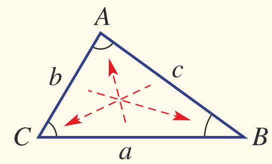


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

KEY IDEAS

- When using the sine rule, label triangles with capital letters for vertices and the corresponding lower-case letter for the side opposite the angle.

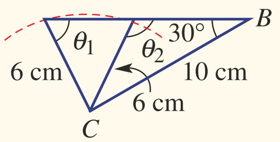


- The **sine rule** states that the ratios of each side of a triangle to the sine of the opposite angle are equal.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

  - The sine rule holds true for both acute- and obtuse-angled triangles.
  - Use the sine rule when you know:
    - one side length and
    - the angle opposite that side length and
    - another side length or angle.

- The **ambiguous case** can arise when we are given two sides and an angle that is not the included angle.
  - This example shows a diagram with two given side lengths and one angle. Two triangles are possible.

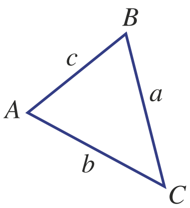


  - Using $\frac{6}{\sin 30°} = \frac{10}{\sin \theta}$ could give two results for θ (i.e. θ_1 or θ_2). You will need to choose the correct angle (i.e. acute or obtuse) to suit your triangle (if known).
  - θ_1 and θ_2 are supplementary so to find the obtuse angle θ_2 use $\theta_2 = 180° - \theta_1$.

BUILDING UNDERSTANDING

1 State the missing parts of the sine rule for this triangle.

$$\frac{a}{} = \frac{}{\sin B} = \frac{c}{}$$



2 Solve each equation for a or b, correct to one decimal place.

a $\frac{a}{\sin 47°} = \frac{2}{\sin 51°}$ **b** $\frac{5}{\sin 63°} = \frac{b}{\sin 27°}$

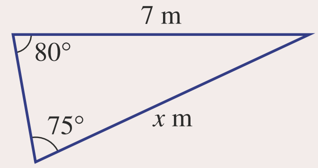
3 Find θ, correct to one decimal place, if θ is acute.

a $\frac{4}{\sin 38°} = \frac{5}{\sin \theta}$ **b** $\frac{1.2}{\sin \theta} = \frac{1.8}{\sin 47°}$

ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

Example 10 Finding a side length using the sine rule

Find the value of x in this triangle, correct to one decimal place.



SOLUTION

$$\frac{x}{\sin 80^\circ} = \frac{7}{\sin 75^\circ}$$

$$x = \frac{7}{\sin 75^\circ} \times \sin 80^\circ$$

$$= 7.1 \text{ (to 1 d.p.)}$$

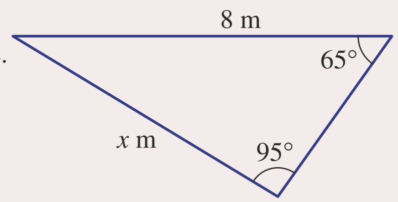
EXPLANATION

Use the sine rule $\frac{a}{\sin A} = \frac{b}{\sin B}$.

Multiply both sides by sin 80°.

Now you try

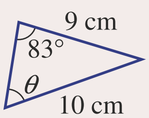
Find the value of x in this triangle, correct to one decimal place.



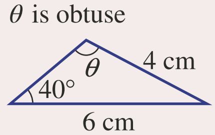
Example 11 Finding an angle using the sine rule

Find the value of θ in these triangles, correct to one decimal place.

a θ is acute



b θ is obtuse



SOLUTION

a $$\frac{10}{\sin 83^\circ} = \frac{9}{\sin \theta}$$

$$10 \times \sin \theta = 9 \times \sin 83^\circ$$

$$\sin \theta = \frac{9 \times \sin 83^\circ}{10}$$

$$\theta = \sin^{-1}\left(\frac{9 \times \sin 83^\circ}{10}\right)$$

$$= 63.3^\circ \text{ (to 1 d.p.)}$$

EXPLANATION

Alternatively, use $\frac{\sin A}{a} = \frac{\sin B}{b}$.

So $\frac{\sin \theta}{9} = \frac{\sin 83^\circ}{10}$.

$$\sin \theta = \frac{9 \times \sin 83^\circ}{10}$$

Use $\sin^{-1}$ on your calculator to find the value of θ.

Continued on next page

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b $\dfrac{4}{\sin 40^\circ} = \dfrac{6}{\sin \theta}$

$4 \times \sin \theta = 6 \times \sin 40^\circ$

$\sin \theta = \dfrac{6 \times \sin 40^\circ}{4}$

$\theta = \sin^{-1}\left(\dfrac{6 \times \sin 40^\circ}{4}\right)$

$\theta = 74.6^\circ$ or $180^\circ - 74.6^\circ = 105.4^\circ$

θ is obtuse, so $\theta = 105.4^\circ$ (to 1 d.p.).

Alternatively, use $\dfrac{\sin \theta}{6} = \dfrac{\sin 40^\circ}{4}$.

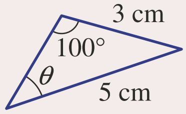
$\sin \theta = \dfrac{6 \times \sin 40^\circ}{4}$

This is an example of the ambiguous case of the sine rule but as θ is obtuse, you will need to choose the supplement of 74.6°.

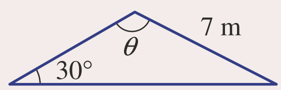
Now you try

Find the value of θ in this triangle, correct to one decimal place.

a



b θ is obtuse.

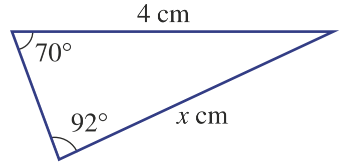


Exercise 4F

FLUENCY	1, 2–3(1/2)	2–3(1/2)	2–3(1/3)

Example 10

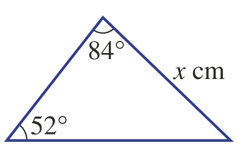
1 Find the value of x in this triangle, correct to one decimal place.



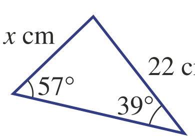
Example 10

2 Find the value of x in these triangles, correct to one decimal place.

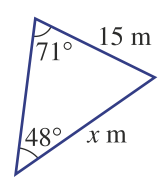
a



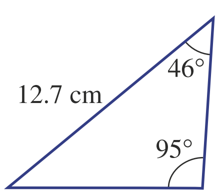
b



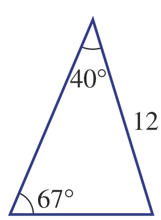
c



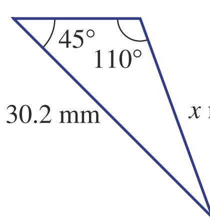
d



e

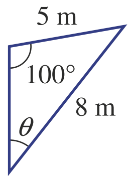


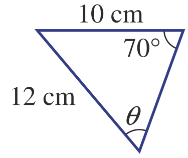
f

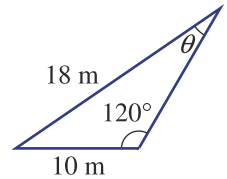


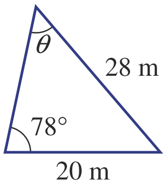
Example 11a

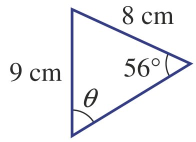
3 Find the value of θ, correct to one decimal place, if θ is acute.

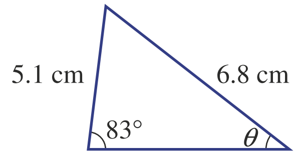
a


b


c


d


e


f


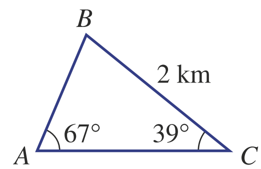
PROBLEM-SOLVING	4, 5	5–7	6–8

4 Three markers, A, B and C, map out the course for a cross-country race. The angles at A and C are 67° and 39°, respectively, and BC is 2 km.

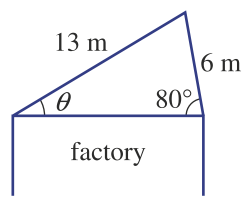
a Find the length AB, correct to three decimal places.

b Find the angle at B.

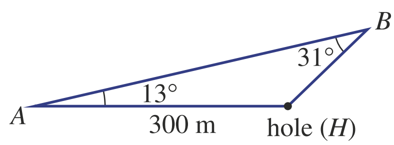
c Find the length AC, correct to three decimal places.



5 A factory roof has a steep 6 m section at 80° to the horizontal and another 13 m section. What is the angle of elevation of the 13 m section of roof? Give your answer to one decimal place.



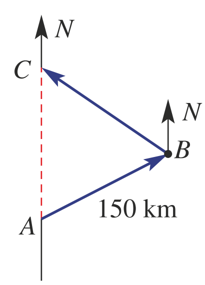
6 A golf ball is hit off-course by 13° to point B. The shortest distance to the hole is 300 m and the angle formed by the new ball position is 31°, as shown. Find the new distance to the hole (BH), correct to one decimal place.



7 An aeroplane is flying due north but, to avoid a storm, it flies 150 km on a bearing of 060°T and then on a bearing of 320°T until it reaches its original course.

a Find the angles $\angle ABC$ and $\angle ACB$.

b As a result of the diversion, how much farther did the aeroplane have to fly? Round your answer to the nearest kilometre.



ISBN 978-1-108-77290-7

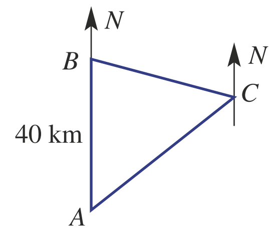
Cambridge University Press
Updated June 2022

8 A ship heads due north from point A for 40 km to point B, and then heads on a true bearing of 100° to point C. The bearing from C to A is 240°.

a Find $\angle ABC$.

b Find the distance from A to C, correct to one decimal place.

c Find the distance from B to C, correct to one decimal place.

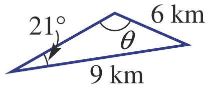


REASONING	9(1/2)	9(1/2), 10	9(1/3), 10, 11

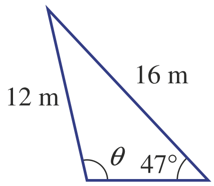
Example 11b

9 Find the value of θ, correct to one decimal place, if θ is obtuse.

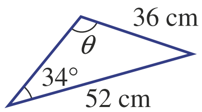
a



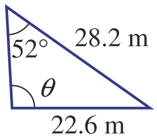
b



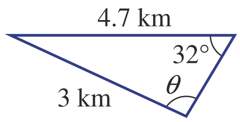
c



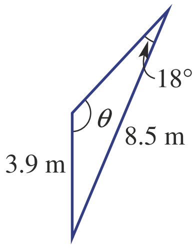
d



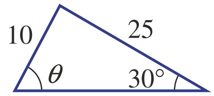
e



f



10 Try to find the angle θ in this triangle. What do you notice? Can you explain this result?



11 A triangle ABC has $\angle C = 25°$, $AC = 13$ cm and $AB = 9$ cm. Find all possible values of $\angle B$, correct to one decimal place.

ENRICHMENT: More on the ambiguous case	–	–	12

12 When finding a missing angle θ in a triangle, the number of possible solutions for θ can be one or two, depending on the given information.

Two solutions: A triangle ABC has $AB = 3$ cm, $AC = 2$ cm and $\angle B = 35°$.

a Find the possible values of $\angle C$, correct to one decimal place.

b Draw a triangle for each angle for $\angle C$ in part **a**.

One solution: A triangle ABC has $AB = 6$ m, $AC = 10$ m and $\angle B = 120°$.

c Find the possible values of $\angle C$, correct to one decimal place.

d Explain why there is only one solution for $\angle C$ and not the extra supplementary angle, as in parts **a** and **b** above.

e Draw a triangle for your solution to part **c**. See the poster on the inside back cover of your book for more information about solving unfamiliar problems.

ISBN 978-1-108-77290-7

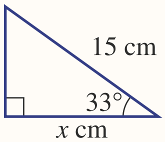
Cambridge University Press
Updated June 2022

Progress quiz

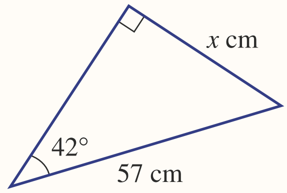
4A

1 Find the value of x in these right-angled triangles, rounding your answer to two decimal places.

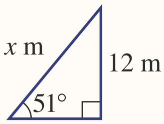
a



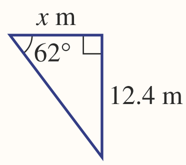
b



c



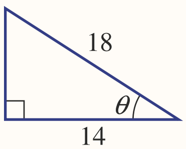
d



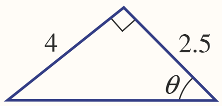
4B

2 Find the value of θ in the following right-angled triangles, correct to the nearest degree.

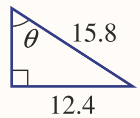
a



b

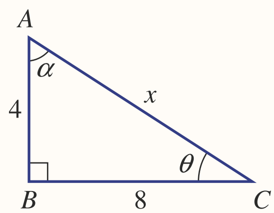


c



4A/B

3 For triangle ABC, find:



a the exact value of:

i x **ii** $\sin \alpha$ **iii** $\sin \theta$

b θ, correct to one decimal place.

4B

4 At what angle to the horizontal must a 4.5 m ladder be placed against a wall if it must reach up to just below a window that is 4 m above the level ground? Round your answer to the nearest degree.

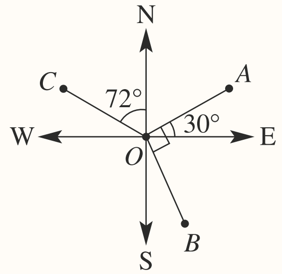
4C

5 The angle of depression from the top of a 20 m building to a worker standing on the ground below is 40°. Find the distance of the worker from the base of the building, correct to two decimal places.

Progress quiz

4D

6 Give the true bearing of A, B and C from the origin, O, in the given diagram.



4D

7 A man leaves camp C at 11 a.m. and walks 12 km on a true bearing of 200°. He then stops. A woman also leaves camp C at 11 a.m. However, she walks on a true bearing of 110° for 6.5 km before stopping.

a How far apart are the man and the woman once they stop? Give your answer correct to two decimal places.

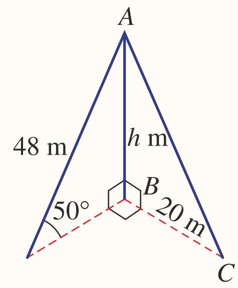
b If the man changes direction and walks to where the woman is waiting, on what bearing should he walk? Round your answer to one decimal place.

4E 10A

8 Consider the given 3D diagram on the right.

a Find the value of h, correct to two decimal places.

b Find $\angle ACB$, to the nearest degree.

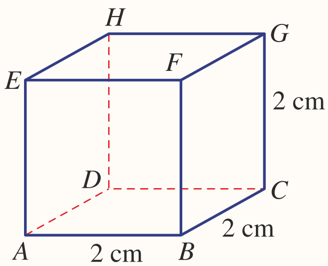


4E 10A

9 A cube has vertices A, B, C, D, E, F, G and H and has side length 2 cm.

a Use Pythagoras' theorem to find the length AC as an exact value.

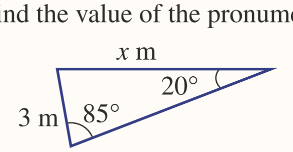
b Find the angle of elevation of the diagonal AG, i.e. find $\angle CAG$. Round to the nearest degree.



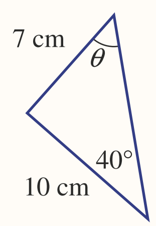
4F 10A

10 Find the value of the pronumeral, correct to one decimal place, using the sine rule.

a



b θ is acute.

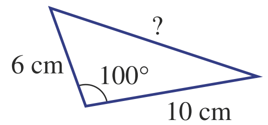


ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

4G The cosine rule 10A

Learning intentions
- To know that the cosine rule relates one angle and three sides of any triangle
- To be able to use the cosine rule to find any angle (given all three sides) or a third side of a triangle (given two sides and the included angle)

When a triangle is defined by two sides and the included angle, the sine rule is unhelpful in finding the length of the third side because at least one of the other two angles is needed.



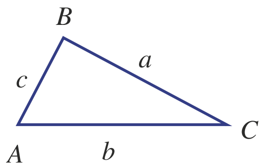
In such situations a new rule called the cosine rule can be used. It relates all three side lengths and the cosine of one angle. This means that the cosine rule can also be used to find an angle inside a triangle when given all three sides.

The proof of the cosine rule will be considered in the Enrichment question of this section.

LESSON STARTER Cosine rule in three ways

One way to write the cosine rule is like this:

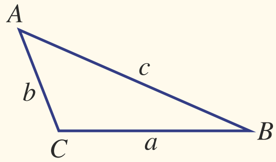
$c^2 = a^2 + b^2 - 2ab \cos C$, where c^2 is the subject of the formula.



- Rewrite the cosine rule by replacing c with a, a with c and C with A.
- Rewrite the cosine rule by replacing c with b, b with c and C with B.

KEY IDEAS

- The **cosine rule** relates one angle and three sides of any triangle.
- The cosine rule is used to find:
  - the third side of a triangle when given two sides and the included angle
  - an angle when given three sides.



$$c^2 = a^2 + b^2 - 2ab \cos C \quad \text{or} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

- If θ is obtuse, then note that $\cos \theta$ is negative. This will be discussed in more detail in **Section 4I**.

ISBN 978-1-108-77290-7

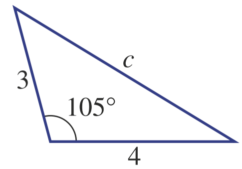
Cambridge University Press
Updated June 2022

BUILDING UNDERSTANDING

1 State the missing parts to the cosine rule for each triangle.

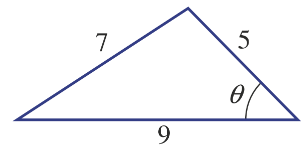
a $c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = 3^2 + \_\_\_ - 2 \times \_\_\_\_ \times \_\_\_\_ \times \cos\_\_\_\_$



b $c^2 = a^2 + b^2 - 2ab\cos C$

$\_\_\_\_\_ = 5^2 + 9^2 - 2 \times \_\_\_\_ \times \_\_\_\_ \times \cos\theta$



2 Simplify and solve for the unknown (i.e. c or θ) in these equations, correct to one decimal place.

a $c^2 = 4^2 + 7^2 - 2 \times 4 \times 7 \times \cos 120°$

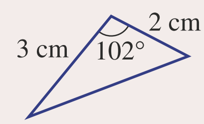
b $c^2 = 1.5^2 + 1.1^2 - 2 \times 1.5 \times 1.1 \times \cos 70°$

c $10^2 = 7^2 + 6^2 - 2 \times 7 \times 6 \times \cos\theta$

d $18^2 = 21^2 + 30^2 - 2 \times 21 \times 30 \times \cos\theta$

Example 12 Finding a side length using the cosine rule

Find the length of the third side in this triangle, correct to two decimal places.

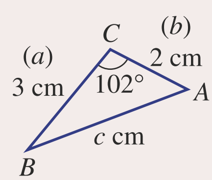


SOLUTION

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$= 3^2 + 2^2 - 2(3)(2)\cos 102°$$
$$= 13 - 12\cos 102°$$
$$= 15.49494\ldots$$
$$\therefore c = 3.94 \text{ (to 2 d.p.)}$$

The length of the third side is 3.94 cm.

EXPLANATION

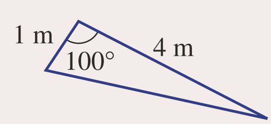


Let c be the length of the unknown side, so $a = 3$ and $b = 2$. Alternatively, let $b = 3$ and $a = 2$.

$c = \sqrt{15.494904\ldots}$

Now you try

Find the length of the third side in this triangle, correct to two decimal places.

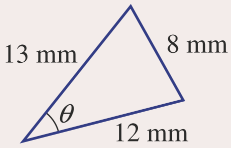


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 13 Finding an angle using the cosine rule

Find the angle θ in this triangle, correct to two decimal places.



SOLUTION

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ 8^2 &= 13^2 + 12^2 - 2(13)(12)\cos\theta \\ 64 &= 313 - 312\cos\theta \\ 312\cos\theta &= 249 \\ \cos\theta &= \frac{249}{312} \\ \theta &= \cos^{-1}\left(\frac{249}{312}\right) \\ &= 37.05^\circ \text{ (to 2 d.p.)} \end{aligned}$$

EXPLANATION

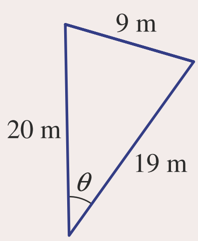
Choose θ to represent $\angle C$, so this makes $c = 8$.

Alternatively, use $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$ to give the same result.

$313 - 64 = 249$

Now you try

Find the angle θ in this triangle, correct to two decimal places.

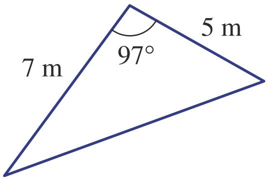


Exercise 4G

FLUENCY	1, 2–3(1/2)	2–3(1/2)	2–3(1/3)

Example 12

1 Find the length of the third side in this triangle, correct to two decimal places.



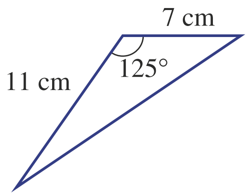
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

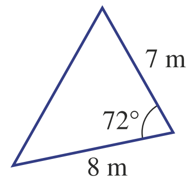
Example 12

2 Find the length of the third side, correct to two decimal places.

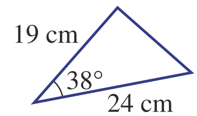
a



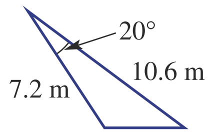
b



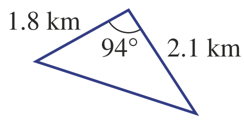
c



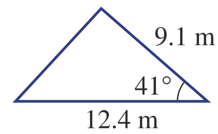
d



e



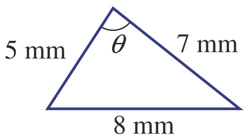
f



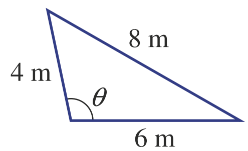
Example 13

3 Find the angle θ, correct to two decimal places.

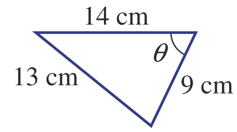
a



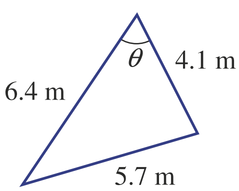
b



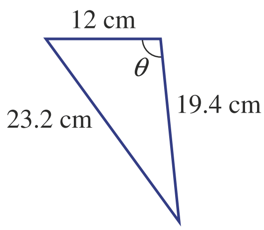
c



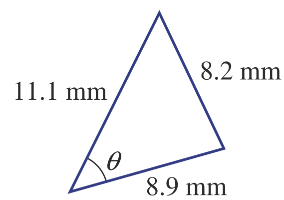
d



e



f



PROBLEM-SOLVING	4–6	5–7	6–8

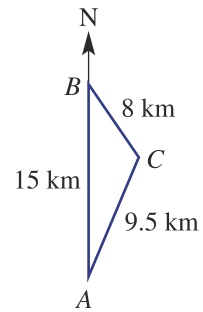
4 A triangular goat paddock has two sides of lengths 320 m and 170 m, and a 71° angle between them. Find the length of the third side, correct to the nearest metre.

5 Find the size of all three angles in a triangle that has side lengths 10 m, 7 m and 13 m. Round each angle to one decimal place.

6 Three camp sites, A, B and C, are planned for a hike and the distances between the camp sites are 8 km, 15 km and 9.5 km, as shown. If camp site B is due north of camp site A, find the following, correct to one decimal place.

a The bearing from camp site B to camp site C.

b The bearing from camp site C to camp site A.

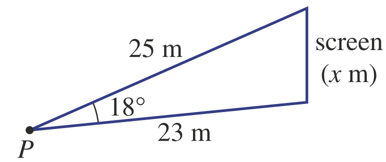


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

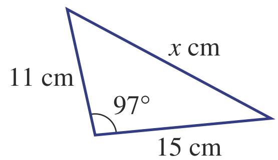
7 A helicopter on a joy flight over Kakadu National Park travels due east for 125 km, then on a bearing of 215°T for 137 km before returning to its starting point. Find the total length of the journey, correct to the nearest kilometre.

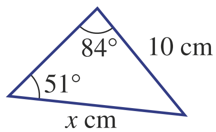
8 The viewing angle to a vertical screen is 18° and the distances between the viewing point, P, and the top and bottom of the screen are 25 m and 23 m, respectively. Find the height of the screen (x m), correct to the nearest centimetre.

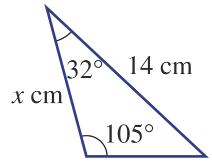


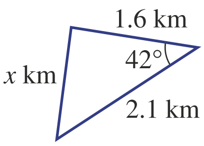
REASONING	9(1/2)	9(1/2), 10	9(1/3), 10, 11

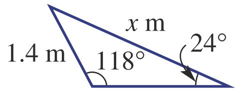
9 Decide whether the cosine rule or sine rule would be used to calculate the value of x in these triangles.

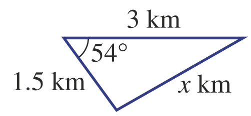
a


b


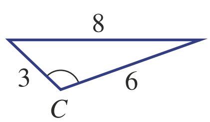
c


d


e


f


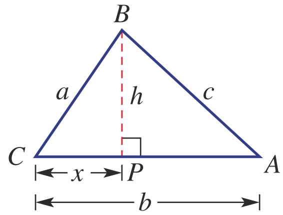
10 a Rearrange $c^2 = a^2 + b^2 - 2ab\cos C$ to make $\cos C$ the subject.
b Use your rule to find angle C in this triangle, correct to one decimal place.



11 A student uses the cosine rule to find an angle in a triangle and simplifies the equation to $\cos\theta = -0.17$. Is the triangle acute or obtuse? Give a reason.

ENRICHMENT: Proof of the cosine rule	–	–	12

12 Triangle ABC shown here includes point P such that $PB \perp CA$, $BP = h$ and $CP = x$.



a Write an expression for length AP.
b Use Pythagoras' theorem and $\triangle CBP$ to write an equation in a, x and h.
c Use Pythagoras' theorem and $\triangle APB$ to write an equation in b, c, x and h.
d Combine your equations from parts **b** and **c** to eliminate h. Simplify your result.
e Use $\cos\theta = \dfrac{\text{Adjacent}}{\text{Hypotenuse}}$ to write an expression for $\cos C$.
f Combine your equations from parts **d** and **e** to prove $c^2 = a^2 + b^2 - 2ab\cos C$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Applications and problem-solving

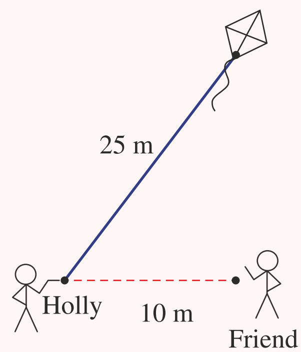
The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

Flying a kite

1 Holly and her friend are flying a kite on a 25 m long string (pulled tight).

Holly and her friend are interested in the relationship between the angle of elevation and the height of the kite.

a Holly's friend is standing 10 m away from her and notices the kite is directly overhead. Determine the angle of elevation of the kite from Holly's hand. Round to one decimal place.



b The kite rises with a gush of wind and the angle of elevation becomes 75°. How high is the kite (vertically) above Holly's hand level correct to one decimal place?

c The kite can range from being 0 m to 25 m above Holly's hand level. At what height is the kite when the angle of elevation is 45°? Round to one decimal place.

d Determine the angle of elevation when the kite is 12.5 m above Holly's hand level.

e If the string is x m long, find in terms of x:

i the height of the kite at an angle of elevation of 30°

ii the angle of elevation when the kite's vertical height is $\frac{3}{4}$ the length of the string. Round to one decimal place.

Off the beaten track

2 A walker travels off a straight-line track that runs east–west and then later returns to a point further west along the track.

The walker is interested in the relationship between the chosen bearing heading off the track and the return bearing as well as the distances that need to be walked in order to complete the journey.

a Initially the walker takes a true bearing off the track at 200° for 5 km.

i To return to the track they travel on a true bearing of 340°. What distance do they need to walk to meet up with the track? Give reasons.

ii Another walker walks for x km from the track on a true bearing of 230°. They head back to the track on a true bearing of 310°. What distance do they need to walk to meet up with the track?

iii If a walker was to walk in a south-westerly direction from the track for x km on a true bearing of q°, where $180 < q < 270$, on what true bearing do they need to travel for x km to arrive back at the track?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b A walker heads from the track on a true bearing of 210° for 4 km.

i Determine three different bearings and distances that would get them back on the track. Include diagrams to represent these walks.

3 km west along the track from where they start begins a 500 m section of the track that is not accessible. To avoid this section the walker must rejoin the track either before or after this section.

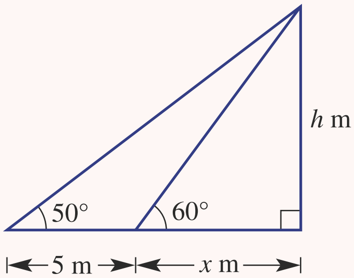
ii On what possible bearings does the walker need to travel to get back to the track but avoid the inaccessible part? Assume that the walker does not want to backtrack, and round bearings to the nearest whole number.

How high is the building?

3 To determine the height of a building two angles of elevation are recorded on level ground a set distance apart.

By applying basic right-angled trigonometry we can investigate the height of the building using angles of elevation and the distance between the points at ground level.

a For the case shown, with two angles of elevation 50° and 60°, 5 m apart:



i Write down two equations involving x and h (one from each right-angled triangle).

ii Use the expressions from part **i** to solve for h correct to the nearest centimetre.

b **i** Repeat part **a**, with the two angles of elevation 50° and 60°, now taken d m apart, to find h in terms of d.

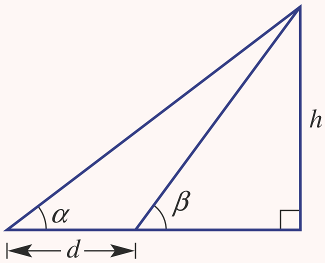
ii Use your answer from part **i**, to confirm your answer to part **a** and to find the height of the building if the measurements were taken 8 m apart. Round to the nearest centimetre.

c Using the diagram from part **a**:

i make use of the sine rule to find the value of h correct to the nearest centimetre.

ii compare your working from part **i** with part **a**. Is either method preferable?

d For the general case shown below, use the method from part **a** to find an expression for h in terms of α, β and d. Check by using your values from part **a**.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

4H Area of a triangle 10A

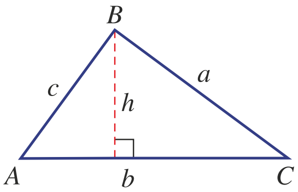
Learning intentions

- To understand how the area of a triangle can be found when two sides and the included angle are known
- To be able to use the area of a triangle formula, $A = \frac{1}{2}ab\sin C$

We can use trigonometry to establish a rule for the area of a triangle using two sides and the included angle.

We can see in this triangle that $\sin C = \frac{h}{a}$, so $h = a\sin C$.

$\therefore A = \frac{1}{2}bh$ becomes $A = \frac{1}{2}ba\sin C$.



A polygon's area can be found by dividing it into oblique triangles and measuring relevant angles. This method is useful for finding polygon-shaped areas such as irregular farm paddocks, blocks of land or a space that is to be landscaped or paved.

LESSON STARTER Calculating area in two ways

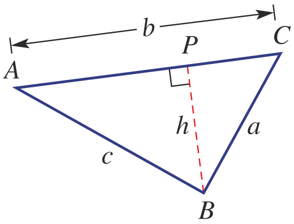
Draw any triangle ABC and construct the height PB. Measure the following as accurately as possible.

a AC **b** BC **c** BP **d** $\angle C$

Now calculate the area using:

- Area $= \frac{1}{2}bh$
- Area $= \frac{1}{2}ab\sin C$

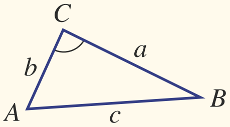
How close are your answers? They should be equal!



KEY IDEAS

- The area of a triangle is equal to half the product of two sides and the sine of the included angle.

  Area $= \frac{1}{2}ab\sin C$



ISBN 978-1-108-77290-7

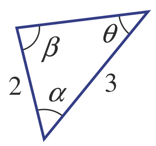
Cambridge University Press
Updated June 2022

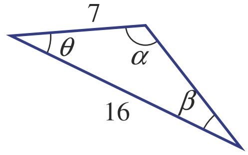
BUILDING UNDERSTANDING

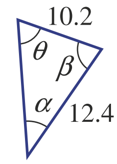
1 Evaluate $\frac{1}{2}ab \sin C$, correct to one decimal place, for the given values of a, b and C.

a $a = 3, b = 4, C = 38°$ **b** $a = 15, b = 7, C = 114°$

2 Which angle pronumeral (i.e. α, β or θ) represents the included angle between the two given sides in these triangles?

a


b


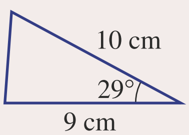
c


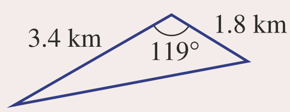
3 Solve these equations for C. Round your answer to two decimal places.

a $10 = \frac{1}{2} \times 4 \times 6 \times \sin C$ **b** $25 = \frac{1}{2} \times 7 \times 10 \times \sin C$

Example 14 Finding the area of a triangle

Find the area of these triangles, correct to one decimal place.

a


b


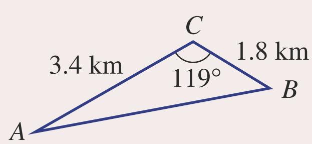
SOLUTION

a
$$\begin{aligned} \text{Area} &= \frac{1}{2}ab \sin C \\ &= \frac{1}{2} \times 9 \times 10 \times \sin 29° \\ &= 21.8 \text{ cm}^2 \text{ (to 1 d.p.)} \end{aligned}$$

b
$$\begin{aligned} \text{Area} &= \frac{1}{2}ab \sin C \\ &= \frac{1}{2} \times 1.8 \times 3.4 \times \sin 119° \\ &= 2.7 \text{ km}^2 \text{ (to 1 d.p.)} \end{aligned}$$

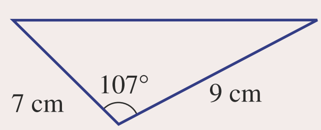
EXPLANATION

Substitute the two sides (a and b) and the included angle (C) into the rule.



Now you try

Find the area of this triangle, correct to one decimal place.

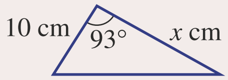


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 15 Finding a side length given the area

Find the value of x, correct to two decimal places, given that the area of this triangle is 70 cm^2.



SOLUTION

$$\text{Area} = \frac{1}{2}ab\sin C$$

$$70 = \frac{1}{2} \times 10 \times x \times \sin 93^\circ$$

$$14 = x \sin 93^\circ$$

$$x = \frac{14}{\sin 93^\circ}$$

$$= 14.02 \text{ (to 2 d.p.)}$$

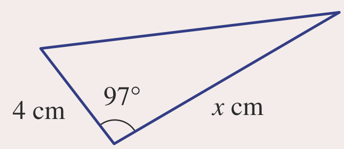
EXPLANATION

Substitute all the given information into the rule, letting $a = 10$ and $b = x$. Use $\angle C = 93^\circ$ as the included angle.

$\frac{1}{2} \times 10 = 5$, so divide both sides by 5 $(70 \div 5 = 14)$ and then solve for x.

Now you try

Find the value of x, correct to two decimal places, given that the area of this triangle is 37 cm^2.

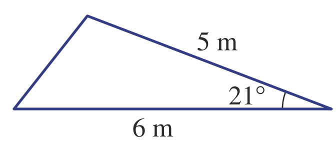


Exercise 4H

FLUENCY	1, 2(1/2), 4(1/2)	2(1/2), 3, 4(1/2)	2–4(1/3)

Example 14

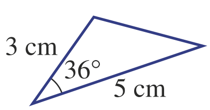
1 Find the area of this triangle, correct to one decimal place.



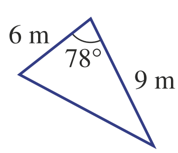
Example 14

2 Find the area of these triangles, correct to one decimal place.

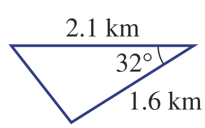
a



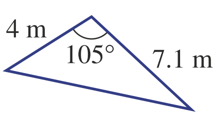
b



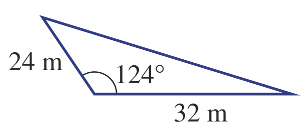
c



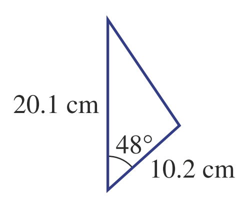
d



e



f



ISBN 978-1-108-77290-7

3 Find the area of these triangles, correct to one decimal place.

a $\triangle XYZ$ if $XY = 5$ cm, $XZ = 7$ cm and $\angle X = 43°$

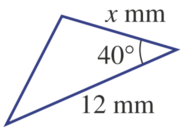
b $\triangle STU$ if $ST = 12$ m, $SU = 18$ m and $\angle S = 78°$

c $\triangle EFG$ if $EF = 1.6$ km, $FG = 2.1$ km and $\angle F = 112°$

Example 15

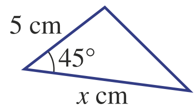
4 Find the value of x, correct to one decimal place, for these triangles with given areas.

a



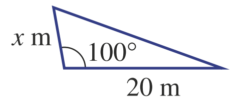
Area = 22 mm^2

b



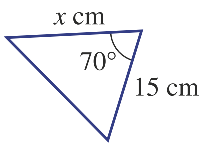
Area = 14 cm^2

c



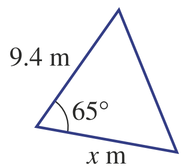
Area = 90 m^2

d



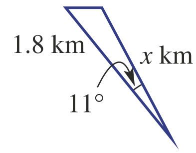
Area = 128 cm^2

e



Area = 45 m^2

f

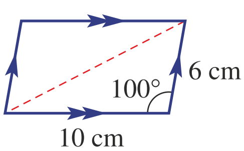


Area = 0.23 km^2

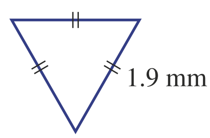
PROBLEM-SOLVING	5, 6	5–7	5–8(1/3)

5 Find the area of these shapes, correct to two decimal places.

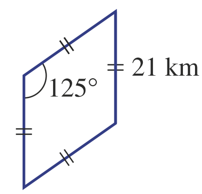
a



b

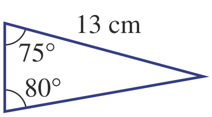


c

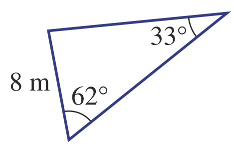


6 First use the sine rule to find another side length, and then find the area of these triangles, correct to two decimal places.

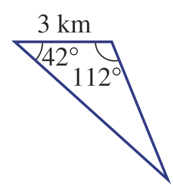
a



b

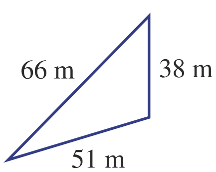


c

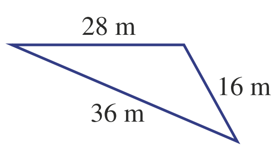


7 First use the cosine rule to find an angle, and then calculate the area of these triangles, correct to two decimal places.

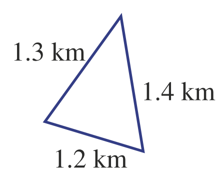
a



b

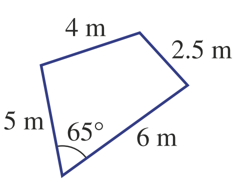


c

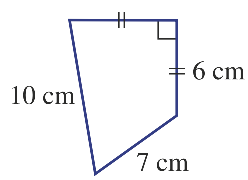


8 Find the area of these quadrilaterals, correct to one decimal place.

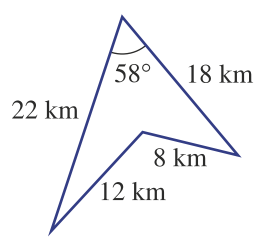
a



b



c



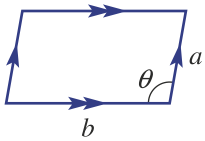
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

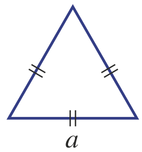
REASONING	9	9, 10	9–11

9 Write a rule for the area of these shapes, using the given pronumerals.

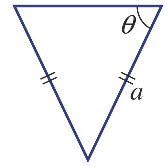
a



b

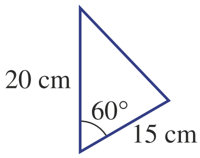


c

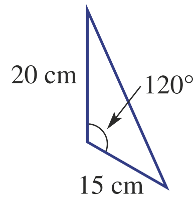


10 **a** Find the area of these two triangles, correct to one decimal place.

i

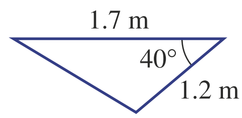


ii



b What do you notice about your answers in part **a**? How can you explain this?

c Draw another triangle that has the same two given lengths and area as the triangle on the right.



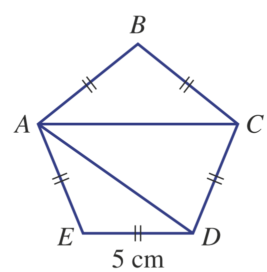
11 **a** Use the rule Area $= \frac{1}{2}ab\sin C$ to find the two possible values of θ (one acute and one obtuse) in the triangle detailed below. Round your answer to one decimal place.

$$\triangle ABC \text{ with } AB = 11\text{ m},\ AC = 8\text{ m, included angle } \theta \text{ and Area} = 40\text{ m}^2.$$

b Draw the two triangles for the two sets of results found in part **a**.

ENRICHMENT: Polygon areas	–	–	12

12 You will recall that the sum (S) of the interior angles of a polygon with n sides is given by $S = 180(n - 2)$.



a This regular pentagon has each side measuring 5 cm.

i Calculate the angle sum of a pentagon.

ii Calculate the size of one interior angle of a regular pentagon.

iii Find the area of $\triangle AED$, correct to two decimal places.

iv Find the length AD, correct to two decimal places.

v Find $\angle ADC$ and $\angle DAC$.

vi Find the area of $\triangle ADC$, correct to two decimal places.

vii Find the total area of the pentagon, correct to one decimal place.

b Use a similar approach to find the area of a regular hexagon of side length 5 cm, correct to one decimal place.

c Can this method be used for other regular polygons? Explore and give examples.

ISBN 978-1-108-77290-7

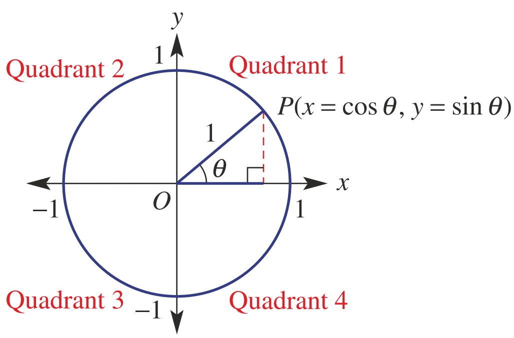
Cambridge University Press
Updated June 2022

4I The unit circle 10A

Learning intentions

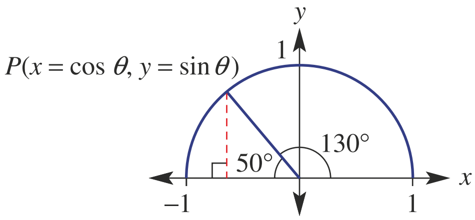
- To know what the unit circle represents
- To understand how a point on a unit circle can be defined by coordinates related to the cosine of the angle in its triangle and the sine of the angle
- To know the four quadrants of the unit circle and the sign and symmetry properties in these quadrants for the trigonometric ratios
- To be able to identify in which quadrant an angle lies and determine whether its different trigonometric ratios will be positive or negative
- To be able to write an angle in terms of its reference angle in the first quadrant
- To know how tan can be expressed in terms of sin and cos

From early trigonometry we calculated $\sin\theta$, $\cos\theta$ and $\tan\theta$ using acute angles. We will now extend this to include the four quadrants of the unit circle, using $0° \leqslant \theta \leqslant 360°$.

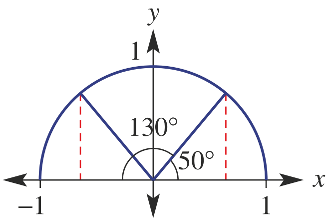


Note the following.

- The unit circle has radius one unit and has centre (0, 0) on a number plane.
- θ is defined anticlockwise from the positive x-axis.
- There are four quadrants, as shown.
- Using a point $P(x, y)$ on the unit circle we define the three trigonometric ratios
  - $\sin\theta = \dfrac{\text{Opposite}}{\text{Hypotenuse}} = \dfrac{y}{1} = y$
  - $\cos\theta = \dfrac{\text{Adjacent}}{\text{Hypotenuse}} = \dfrac{x}{1} = x$
  - $\tan\theta = \dfrac{\text{Opposite}}{\text{Adjacent}} = \dfrac{y}{x} = \dfrac{\sin\theta}{\cos\theta}$
- The coordinates of P, a point on the unit circle, are $(x, y) = (\cos\theta, \sin\theta)$.
- In the second quadrant $\sin\theta$ is positive, $\cos\theta$ is negative and $\tan\theta$ is negative. In this diagram we can see $P(\cos 130°, \sin 130°)$, where $\cos 130°$ is negative, $\sin 130°$ is positive and so $\tan 130°$ will be negative.



In the diagram at right showing 130°, a 50° angle (180° − 130°) drawn in the first quadrant can help relate trigonometric values from the second quadrant to the first quadrant. By symmetry we can see that $\sin 130° = \sin 50°$ and $\cos 130° = -\cos 50°$. This 50° angle is called the **reference angle** (or related angle).



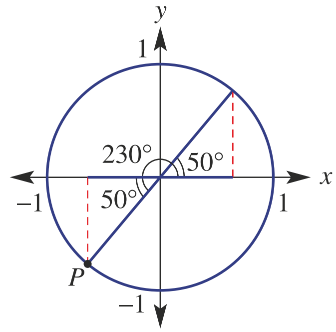
In this section we explore these symmetries and reference angles in the second, third and fourth quadrants.

LESSON STARTER Positive or negative

For the angle 230°, the reference angle is 50° and $P = (\cos 230°, \sin 230°)$.

Since P is in the third quadrant, we can see that:

- $\cos 230° = -\cos 50°$, which is negative.
- $\sin 230° = -\sin 50°$, which is negative.



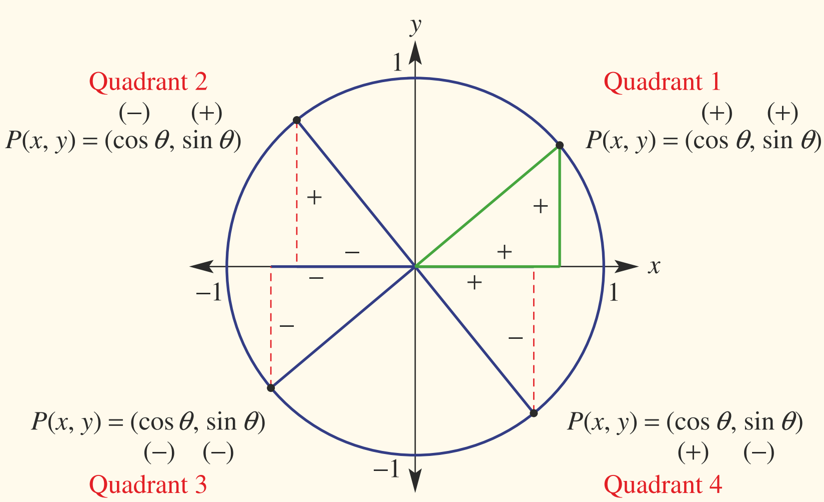
Now determine the following for each value of θ given below.
You should draw a unit circle for each.

- What is the reference angle?
- Is $\cos\theta$ positive or negative?
- Is $\sin\theta$ positive or negative?
- Is $\tan\theta$ positive or negative?

a $\theta = 240°$ **b** $\theta = 210°$ **c** $\theta = 335°$ **d** $\theta = 290°$ **e** $\theta = 162°$

KEY IDEAS

- Every point $P(x, y)$ on the unit circle can be described in terms of the angle θ such that: $x = \cos\theta$ and $y = \sin\theta$, where $-1 \leqslant \sin\theta \leqslant 1$ and $-1 \leqslant \cos\theta \leqslant 1$.



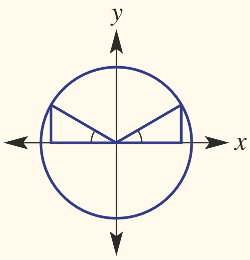
- For different quadrants, $\cos\theta$ and $\sin\theta$ can be positive or negative.
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- ASTC means:
  - Quadrant 1: All $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive.
  - Quadrant 2: Only Sin θ is positive.
  - Quadrant 3: Only Tan θ is positive.
  - Quadrant 4: Only Cos θ is positive.

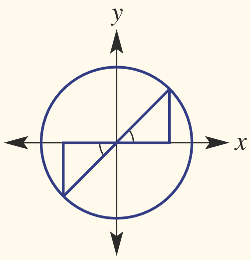
ISBN 978-1-108-77290-7

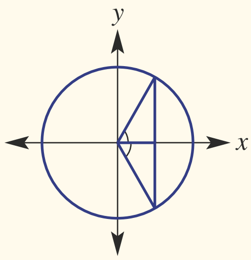
Cambridge University Press
Updated June 2022

- A **reference angle** (sometimes called a related angle) is an acute angle that helps to relate cos θ and sin θ to the first quadrant.

Angle θ	90° to 180°	180° to 270°	270° to 360°
Reference angle	$180° - \theta$	$\theta - 180°$	$360° - \theta$







- Multiples of 90°.

θ	0°	90°	180°	270°	360°
sin θ	0	1	0	–1	0
cos θ	1	0	–1	0	1
tan θ	0	undefined	0	undefined	0

BUILDING UNDERSTANDING

1. Which quadrant in the unit circle corresponds to these values of θ?
   - a $0° < \theta < 90°$
   - b $180° < \theta < 270°$
   - c $270° < \theta < 360°$
   - d $90° < \theta < 180°$
2. Decide which quadrants make the following true.
   - a sin θ is positive
   - b tan θ is negative
   - c cos θ is negative
   - d cos θ is positive
   - e tan θ is positive
   - f sin θ is negative
3. State the missing values in this table.

θ	0°	90°	180°	270°	360°
sin θ					0
cos θ			–1		
tan θ		undefined			

4. Use a calculator to evaluate the following, correct to three decimal places.
   - a sin 172°
   - b sin 212°
   - c cos 143°
   - d cos 255°
   - e tan 222°
   - f tan 134°

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 16 Choosing supplementary angles

Choose an obtuse angle to complete each statement.

a $\sin 30° = \sin$ ____ **b** $\cos 57° = -\cos$ ____ **c** $\tan 81° = -\tan$ ____

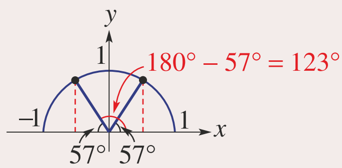
SOLUTION

EXPLANATION

a $\sin 30° = \sin 150°$

Choose the supplement of 30°, which is $180° - 30° = 150°$.

b $\cos 57° = -\cos 123°$



c $\tan 81° = -\tan 99°$

The supplement of 81° is 99°.

Now you try

Choose an obtuse angle to complete each statement.

a $\sin 40° = \sin$ ____ **b** $\cos 74° = -\cos$ ____ **c** $\tan 47° = -\tan$ ____

Example 17 Positioning a point on the unit circle

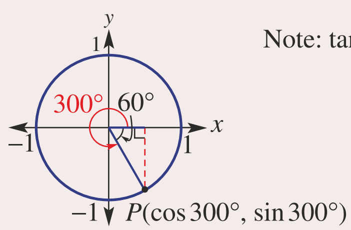
Decide in which quadrant θ lies and state whether $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive or negative.

a $\theta = 300°$ **b** $\theta = 237°$

SOLUTION

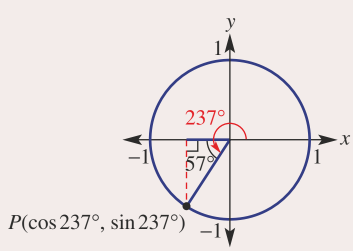
EXPLANATION

a $\theta = 300°$ is in quadrant 4.
$\sin\theta$ is negative
$\cos\theta$ is positive
$\tan\theta$ is negative



Note: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

b $\theta = 237°$ is in quadrant 3.
$\sin\theta$ is negative
$\cos\theta$ is negative
$\tan\theta$ is positive



Now you try

Decide in which quadrant θ lies and state whether $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive or negative.

a $\theta = 320°$ **b** $\theta = 215°$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 18 Using a reference angle

Write the following using their reference angle.

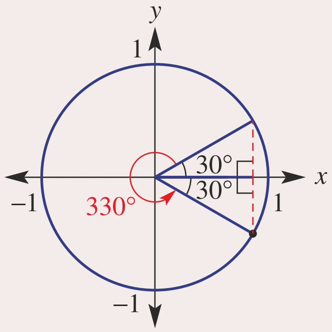
a $\sin 330°$ **b** $\cos 162°$ **c** $\tan 230°$

SOLUTION

EXPLANATION

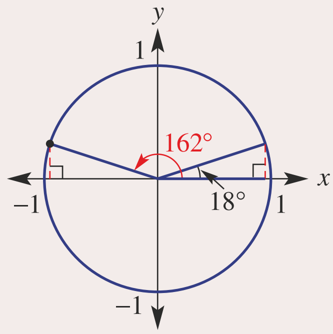
a $\sin 330° = -\sin 30°$

$\sin 330°$ is negative and the reference angle is $360° - 330° = 30°$.



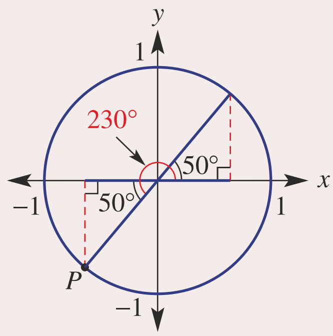
b $\cos 162° = -\cos 18°$

$\cos 162°$ is negative and the reference angle is $180° - 162° = 18°$.



c $\tan 230° = \tan 50°$

$\tan 230°$ is positive (negative ÷ negative) and the reference angle is $230° - 180° = 50°$.



Now you try

Write the following using their reference angle.

a $\sin 310°$ **b** $\cos 126°$ **c** $\tan 260°$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 4I

FLUENCY	1, 2–5(1/2)	2–6(1/2)	2–6(1/3)

1 Choose an obtuse angle to complete each statement.

Example 16a **a** **i** $\sin 20° = \sin$ ____ **ii** $\sin 75° = \sin$ ____

Example 16b **b** **i** $\cos 70° = -\cos$ ____ **ii** $\cos 42° = -\cos$ ____

Example 16c **c** **i** $\tan 35° = -\tan$ ____ **ii** $\tan 79° = -\tan$ ____

Example 16 **2** Choose an obtuse angle to complete each statement.

a $\sin 40° = \sin$ ____ **b** $\sin 65° = \sin$ ____
c $\cos 25° = -\cos$ ____ **d** $\cos 81° = -\cos$ ____
e $\tan 37° = -\tan$ ____ **f** $\tan 8° = -\tan$ ____

3 Choose an acute angle to complete each statement.

a $\sin 150° = \sin$ ____ **b** $\sin 94° = \sin$ ____
c $-\cos 110° = \cos$ ____ **d** $-\cos 171° = \cos$ ____
e $-\tan 159° = \tan$ ____ **f** $-\tan 143° = \tan$ ____

Example 17 **4** Decide in which quadrant θ lies and state whether $\sin \theta$, $\cos \theta$ and $\tan \theta$ are positive or negative.

a $\theta = 172°$ **b** $\theta = 295°$ **c** $\theta = 252°$ **d** $\theta = 73°$
e $\theta = 318°$ **f** $\theta = 154°$ **g** $\theta = 197°$ **h** $\theta = 221°$
i $\theta = 210°$ **j** $\theta = 53°$ **k** $\theta = 346°$ **l** $\theta = 147°$

Example 18 **5** Write each of the following using its reference angle.

a $\sin 280°$ **b** $\cos 300°$ **c** $\tan 220°$ **d** $\sin 140°$
e $\cos 125°$ **f** $\tan 315°$ **g** $\sin 345°$ **h** $\cos 238°$
i $\tan 227°$ **j** $\sin 112°$ **k** $\cos 294°$ **l** $\tan 123°$

6 If θ is acute, find the value of θ.

a $\sin 150° = \sin \theta$ **b** $\sin 240° = -\sin \theta$ **c** $\sin 336° = -\sin \theta$
d $\cos 220° = -\cos \theta$ **e** $\cos 109° = -\cos \theta$ **f** $\cos 284° = \cos \theta$
g $\tan 310° = -\tan \theta$ **h** $\tan 155° = -\tan \theta$ **i** $\tan 278° = -\tan \theta$

PROBLEM-SOLVING	7, 8	7(1/2), 8	8, 9

7 Write the reference angle (i.e. related angle) in the first quadrant for these angles.

a 138° **b** 227° **c** 326° **d** 189°
e 213° **f** 298° **g** 194° **h** 302°

8 For what values of θ, in degrees, are the following true?

a All of $\sin \theta$, $\cos \theta$ and $\tan \theta$ are positive.
b Only $\sin \theta$ is positive.
c Only $\cos \theta$ is positive.
d Only $\tan \theta$ is positive.

ISBN 978-1-108-77290-7

9 Complete the table by finding a second angle, θ_2, that gives the same value for the trigonometric function as θ_1. Use the unit circle to help in each case and assume $0 \leqslant \theta_2 \leqslant 360°$.

Trigonometric function	$\sin\theta$	$\cos\theta$	$\sin\theta$	$\tan\theta$	$\cos\theta$	$\tan\theta$	$\cos\theta$	$\sin\theta$	$\tan\theta$
θ_1	30°	45°	190°	15°	125°	320°	260°	145°	235°
θ_2									

REASONING

10	10, 11, 13(½)	10(½), 11, 12, 13(½)

10 Decide which quadrant suits the given information.

a $\sin\theta < 0$ and $\cos\theta > 0$
b $\tan\theta > 0$ and $\cos\theta > 0$
c $\tan\theta < 0$ and $\cos\theta < 0$
d $\sin\theta > 0$ and $\tan\theta < 0$
e $\sin\theta > 0$ and $\tan\theta > 0$
f $\sin\theta < 0$ and $\cos\theta < 0$

11 Explain why $\tan\theta > 0$ when $180° < \theta < 270°$.

12 Explain why $\tan 90°$ and $\tan 270°$ are undefined.

13 By considering a unit circle, state whether the following are true or false.

a $\sin 10° < \cos 10°$
b $\sin 50° < \tan 50°$
c $\cos 80° > \sin 80°$
d $\cos 90° = \sin 0°$
e $\tan 180° = \sin 180°$
f $\cos 170° > \sin 170°$
g $\sin 120° > \tan 120°$
h $\sin 90° = \cos 180°$
i $\tan 230° < \cos 230°$
j $\cos 350° < \sin 85°$
k $\sin 260° < \cos 110°$
l $\tan 270° = \cos 180°$

ENRICHMENT: Trigonometric identities

–	–	14, 15

14 You will recall that complementary angles sum to 90°. Answer these questions to explore the relationship between sine and cosine ratios of complementary angles.

a Evaluate the following, correct to two decimal places.

i $\sin 10°$ **ii** $\cos 80°$ **iii** $\sin 36°$ **iv** $\cos 54°$
v $\cos 7°$ **vi** $\sin 83°$ **vii** $\cos 68°$ **viii** $\sin 22°$

b Describe what you notice from part **a**.

c Complete the following.

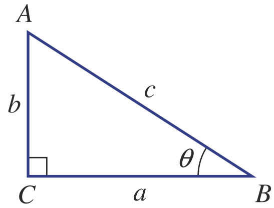
i $\cos\theta = \sin(\_\_\_\_)$ **ii** $\sin\theta = \cos(\_\_\_\_)$

d State the value of θ if θ is acute.

i $\sin 20° = \cos\theta$ **ii** $\sin 85° = \cos\theta$
iii $\cos 71° = \sin\theta$ **iv** $\cos 52° = \sin\theta$

e For this triangle $\angle B = \theta$.

i Write $\angle A$ in terms of θ.
ii Write a ratio for $\sin\theta$ in terms of b and c.
iii Write a ratio for $\cos(90° - \theta)$ in terms of b and c.

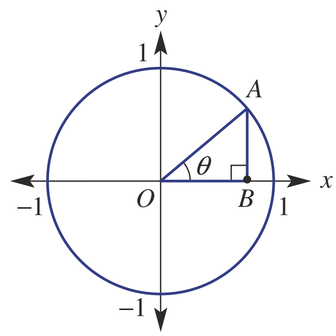


f If $\cos(90° - \theta) = \dfrac{2}{3}$, find $\tan\theta$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

15 Trigonometric identities are mathematical statements that may involve $\sin\theta$ and/or $\cos\theta$ and/or $\tan\theta$ and hold true for all values of θ. In previous exercises you will have already considered the trigonometric identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.



a Consider the triangle OAB in the unit circle shown.

i Given that $OA = 1$, $OB = \cos\theta$ and $AB = \sin\theta$, use Pythagoras' theorem to prove the trigonometric identity: $\sin^2\theta + \cos^2\theta = 1$. (Note: $\sin^2\theta = (\sin\theta)^2$.)

ii Check your identity using a calculator to see if it holds true for $\theta = 30°, 145°, 262°$ and $313°$.

b **i** Evaluate the given pairs of numbers using a calculator.
$(\sin 60°, \cos 30°)$, $(\sin 80°, \cos 10°)$, $(\sin 110°, \cos -20°)$, $(\sin 195°, \cos -105°)$

ii What do you notice about the value of each number in the pairs above? Drawing a unit circle illustrating each pair of values may help.

iii What is the relationship between θ in $\sin\theta$ and θ in $\cos\theta$ that is true for all pairs in part **b i**?

iv In terms of θ, complete this trigonometric identity: $\sin\theta = \cos(\_\_\_\_)$.

v Check this identity for $\theta = 40°, 155°, 210°$ and $236°$.

c Explore other trigonometric identities by drawing diagrams and checking different angles, such as:

i $\sin\theta = \sin(180° - \theta)$

ii $\cos\theta = \cos(360° - \theta)$

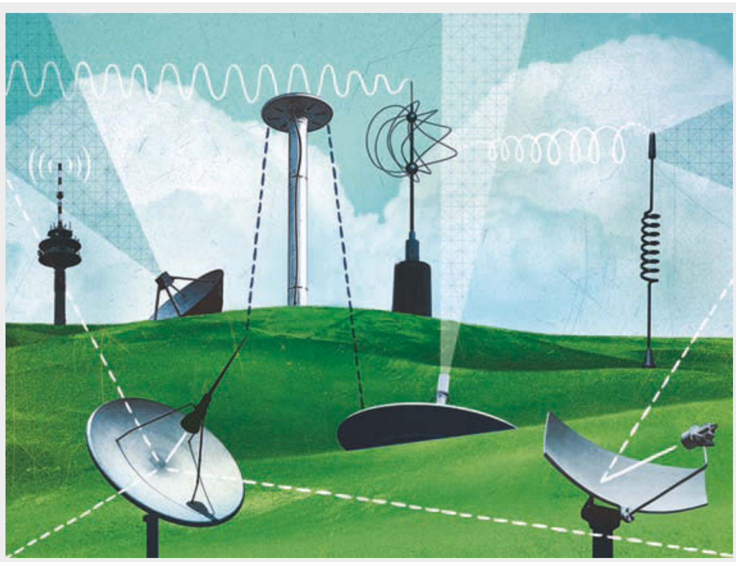
iii $\tan\theta = \tan(180° + \theta)$

iv $\sin 2\theta = 2\sin\theta\cos\theta$

v $\cos 2\theta = \cos^2\theta - \sin^2\theta$

vi $\cos 2\theta = 2\cos^2\theta - 1$

vii $\cos 2\theta = 1 - 2\sin^2\theta$



Extending trigonometry to any sized angle led to the discovery that their values regularly repeat, like the periodic change in the height of a wave. A world-changing application of trigonometry is the modelling of electromagnetic waves.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

4J Exact values and radians 10A

Learning intentions

- To understand the radian unit of measure for an angle
- To be able to convert between radians and degrees for angles
- To know the exact values of 30°, 45° and 60° for the three trigonometric ratios
- To be able to write certain angles in terms of a reference angle of 30°, 45° or 60° and determine their exact value

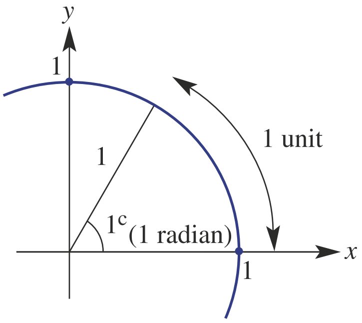
We know that the common unit for angles is the degree and that one full revolution is 360°. This unit of measure dates back thousands of years and relates to ancient number systems which used the number 360 and its factors as its base.

A more mathematically 'pure' unit of measure for an angle is called the radian. One radian (1^c) is the angle subtended by an arc of length 1 on the unit circle. This along with exact values for selected trigonometric ratios will be studied in this section.

Radian measure has numerous applications in engineering design, as radians relate arc length to the circle's central angle and its radius. Curves in roads, railway tracks and racing car circuits are mostly arc shaped and radians are used to calculate their lengths.

LESSON STARTER What is a radian?

This 1 st quadrant diagram shows 1 radian (1^c) subtended by an arc of length 1 on the unit circle.



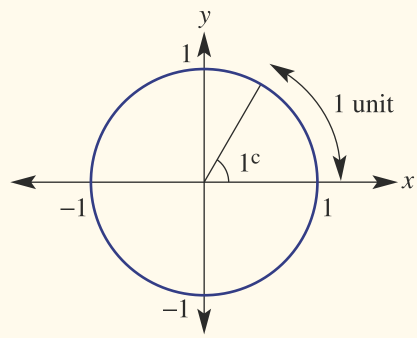
- First estimate the 1 radian angle using degrees.
- Find the circumference of the unit circle as an exact value.
  Hence state the number of radians in one full revolution (360°).
- State the number of radians equivalent to:
  a 180° **b** 90° **c** 270° **d** 45° **e** 30°
- Discuss how you might:
  a convert an angle from degrees to radians
  b convert an angle from radians to degrees.

ISBN 978-1-108-77290-7

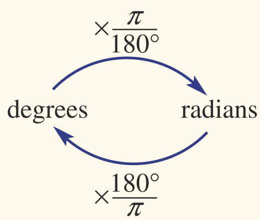
Cambridge University Press
Updated June 2022

KEY IDEAS

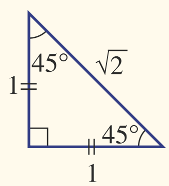
- An angle of one **radian** (1^c) is subtended by an arc length of one unit on the unit circle.
  - $360° = 2\pi^c$
  - $180° = \pi^c$
  - $90° = \dfrac{\pi^c}{2}$
  - $60° = \dfrac{\pi^c}{3}$
  - $45° = \dfrac{\pi^c}{4}$
  - $30° = \dfrac{\pi^c}{6}$

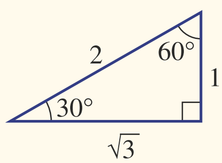


- Conversion
  - $1° = \dfrac{\pi^c}{180}$
  - $1^c = \dfrac{180°}{\pi}$



- Exact values for $\sin\theta$, $\cos\theta$ and $\tan\theta$ can be obtained using two special triangles. Pythagoras' theorem can be used to confirm the length of each side.





$\sin 45° = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ $\quad\sin 30° = \dfrac{1}{2}$ $\quad\sin 60° = \dfrac{\sqrt{3}}{2}$

$\cos 45° = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ $\quad\cos 30° = \dfrac{\sqrt{3}}{2}$ $\quad\cos 60° = \dfrac{1}{2}$

$\tan 45° = \dfrac{1}{1} = 1$ $\quad\tan 30° = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$ $\quad\tan 60° = \dfrac{\sqrt{3}}{1} = \sqrt{3}$

- Exact values for $\sin\theta$, $\cos\theta$ and $\tan\theta$ for angles of 0°, 30°, 45°, 60° and 90° are given in this table.

$\theta°$	θ^c	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0^c	0	1	0
30°	$\dfrac{\pi^c}{6}$	$\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{\sqrt{3}}{3}$
45°	$\dfrac{\pi^c}{4}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{2}}{2}$	1
60°	$\dfrac{\pi^c}{3}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{2}$	$\sqrt{3}$
90°	$\dfrac{\pi^c}{2}$	1	0	undefined

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

BUILDING UNDERSTANDING

1 State the missing values to complete this table.

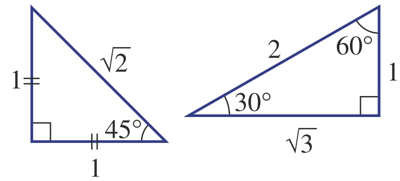
Degrees	360°		90°		45°		15°
Radians		π^c		$\frac{\pi^c}{3}$		$\frac{\pi^c}{6}$	

2 State the missing fraction.

a To convert from degrees to radians we multiply by ________.

b To convert from radians to degrees we multiply by ________.

3 Use trigonometric ratios (SOHCAHTOA) with these triangles to state the exact value for:



a $\sin 45°$ **b** $\cos 45°$ **c** $\tan 45°$

d $\cos 30°$ **e** $\sin 30°$ **f** $\tan 30°$

g $\tan 60°$ **h** $\cos 60°$ **i** $\sin 60°$

4 State the missing values to complete this table.

$\theta°$	0	30	45	60	90	180	270	360
θ^c								
$\sin\theta$								
$\cos\theta$								
$\tan\theta$								

Example 19 Converting degrees and radians

Convert the following to the units given in the brackets.

a 120° (radians) **b** $\frac{5\pi^c}{4}$ (degrees)

SOLUTION

a $120° = 120 \times \frac{\pi}{180}$

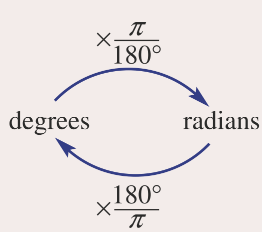
$= \frac{2\pi^c}{3}$

b $\frac{5\pi^c}{4} = \frac{5\pi}{4} \times \frac{180}{\pi}$

$= 225°$

EXPLANATION

$1° = \frac{\pi^c}{180}$ so $120° = 120 \times \frac{\pi}{180}$



Now you try

Convert the following to the units given in the brackets.

a 135° (radians) **b** $\frac{5\pi^c}{6}$ (degrees)

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 20 Using exact values

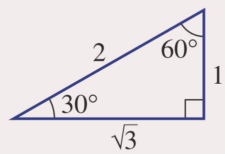
Find the exact value of each of the following.

a $\cos 60^\circ$ **b** $\sin 150^\circ$ **c** $\tan 135^\circ$

SOLUTION

a $\cos 60^\circ = \frac{1}{2}$

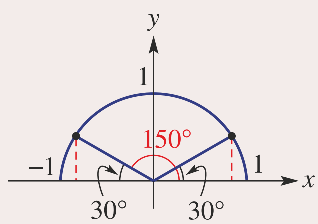
EXPLANATION



$\cos 60^\circ = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{1}{2}$

b $\sin 150^\circ = \sin 30^\circ$

$= \frac{1}{2}$



The sine of supplementary angles are equal and the exact value of $\sin 30^\circ$ is $\frac{1}{2}$.

c $\tan 135^\circ = -\tan 45^\circ$

$= -1$

45° and 135° are supplementary angles.
Also $\tan (180^\circ - \theta) = -\tan \theta$ and $\tan 45^\circ = 1$.

Now you try

Find the exact value of each of the following.

a $\cos 30^\circ$ **b** $\sin 135^\circ$ **c** $\tan 150^\circ$

Exercise 4J

FLUENCY	1, 2–3(1/2), 4	2–3(1/2), 4, 5(1/2)	2–3(1/3), 4, 5(1/3)

1 Convert the following to the units given in the brackets.

Example 19a **a** **i** 90° (radians) **ii** 225° (radians)

Example 19b **b** **i** $\frac{3\pi^c}{4}$ (degrees) **ii** $\frac{7\pi^c}{6}$ (degrees)

Example 19 **2** Convert the following to the units given in the brackets.

a 60° (radians) **b** 150° (radians) **c** 225° (radians) **d** 330° (radians)

e $\frac{3\pi^c}{4}$ (degrees) **f** $\frac{7\pi^c}{6}$ (degrees) **g** $\frac{5\pi^c}{3}$ (degrees) **h** $\frac{11\pi^c}{12}$ (degrees)

ISBN 978-1-108-77290-7

Example 20

3 Find an exact value for each of the following.

a $\cos 30°$ **b** $\sin 45°$ **c** $\tan 60°$ **d** $\cos 45°$
e $\cos 150°$ **f** $\tan 120°$ **g** $\sin 135°$ **h** $\cos 135°$
i $\sin 120°$ **j** $\tan 150°$ **k** $\cos 120°$ **l** $\sin 150°$
m $\tan 135°$ **n** $\sin 90°$ **o** $\cos 90°$ **p** $\tan 90°$

4 Recall the exact sine, cosine and tangent values for 30°, 45° and 60°.

θ	30°	45°	60°
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

a State the reference angle for 225°.
b Hence, give the exact value of the following.
  i $\sin 225°$ **ii** $\cos 225°$ **iii** $\tan 225°$
c State the reference angle for 330°.
d Hence, give the exact value of the following.
  i $\sin 330°$ **ii** $\cos 330°$ **iii** $\tan 330°$
e State the reference angle for 120°.
f Hence, give the exact value of the following.
  i $\sin 120°$ **ii** $\cos 120°$ **iii** $\tan 120°$

5 Give the exact value of the following.

a $\sin 135°$ **b** $\tan 180°$ **c** $\cos 150°$ **d** $\sin 240°$
e $\tan 315°$ **f** $\cos 210°$ **g** $\sin 330°$ **h** $\tan 120°$
i $\cos 225°$ **j** $\sin 270°$ **k** $\tan 330°$ **l** $\cos 300°$
m $\sin 180°$ **n** $\tan 270°$ **o** $\tan 225°$ **p** $\cos 180°$

PROBLEM-SOLVING 6–7(1/2) 6–8(1/2) 6–8(1/3), 9

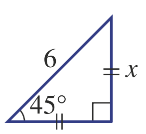
6 Write down the reference angle for these radian measures.

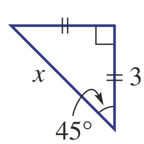
a $\frac{2\pi^c}{3}$ **b** $\frac{3\pi^c}{4}$ **c** $\frac{7\pi^c}{6}$ **d** $\frac{4\pi^c}{3}$
e $\frac{5\pi^c}{3}$ **f** $\frac{11\pi^c}{6}$ **g** $\frac{11\pi^c}{12}$ **h** $\frac{17\pi^c}{12}$

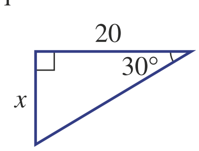
7 Use reference angles and exact values to evaluate the following.

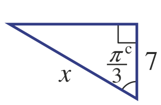
a $\sin\frac{2\pi^c}{3}$ **b** $\sin\frac{5\pi^c}{4}$ **c** $\sin\frac{11\pi^c}{6}$ **d** $\cos\frac{3\pi^c}{4}$
e $\cos\frac{4\pi^c}{3}$ **f** $\cos\frac{5\pi^c}{3}$ **g** $\tan\frac{5\pi^c}{6}$ **h** $\tan\frac{4\pi^c}{3}$

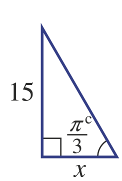
8 Use trigonometric ratios to find the exact value of x. Calculators are not required.

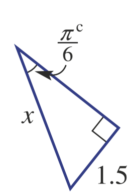
a


b


c


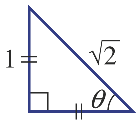
d


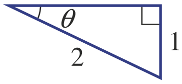
e


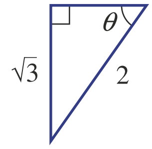
f


ISBN 978-1-108-77290-7

9 Find the exact value of θ without the use of a calculator. Give a radian answer.

a 

b 

c 

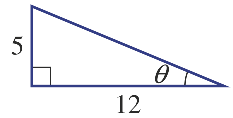
REASONING	10(1/2)	10(1/2), 11	10(1/2), 11

10 Explain why each of the following is true.

a $\sin\dfrac{5\pi^c}{6} = \sin\dfrac{\pi^c}{6}$ **b** $\cos\dfrac{2\pi^c}{3} = -\cos\dfrac{\pi^c}{3}$ **c** $\cos\dfrac{11\pi^c}{6} = \cos\dfrac{\pi^c}{6}$

d $\tan\dfrac{3\pi^c}{4} = -\tan\dfrac{\pi^c}{4}$ **e** $\tan\dfrac{5\pi^c}{4} = \tan\dfrac{\pi^c}{4}$ **f** $\sin\dfrac{4\pi^c}{3} = -\sin\dfrac{\pi^c}{3}$

11 This right-angled triangle has its two shorter sides of length 5 and 12.



a Use Pythagoras' theorem to find the length of the hypotenuse.

b Find:

i $\sin\theta$ **ii** $\cos\theta$ **iii** $\tan\theta$

c Use $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ to verify your result from part **b iii**.

ENRICHMENT: Symmetry and equations	–	–	12

12 We know that $\sin 150° = \sin 30° = \dfrac{1}{2}$ so to solve $\sin\theta = \dfrac{1}{2}$ for $0° \leqslant \theta \leqslant 360°$ the solutions are 30° and 150°. Alternatively, to solve $\sin\theta = \dfrac{1}{2}$ for $0^c \leqslant \theta \leqslant 2\pi^c$ the solutions are $\dfrac{\pi^c}{6}$ and $\dfrac{5\pi^c}{6}$.

a Write the solutions for these simple equations for $0° \leqslant \theta \leqslant 360°$.

i $\sin\theta = \dfrac{\sqrt{3}}{2}$ **ii** $\sin\theta = \dfrac{\sqrt{2}}{2}$ **iii** $\cos\theta = \dfrac{1}{2}$

iv $\cos\theta = -\dfrac{\sqrt{3}}{2}$ **v** $\tan\theta = 1$ **vi** $\tan\theta = -\sqrt{3}$

b Write the solutions for these simple equations for $0^c \leqslant \theta \leqslant 2\pi^c$.

i $\sin\theta = \dfrac{\sqrt{3}}{2}$ **ii** $\sin\theta = -\dfrac{\sqrt{2}}{2}$ **iii** $\cos\theta = -\dfrac{\sqrt{2}}{2}$

iv $\cos\theta = \dfrac{\sqrt{3}}{2}$ **v** $\tan\theta = \dfrac{1}{\sqrt{3}}$ **vi** $\tan\theta = -1$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

4K Graphs of trigonometric functions 10A

Learning intentions
- To know the meaning of the terms amplitude and period and be able to relate them to sine and cosine graphs
- To understand the shape of the sine and cosine graphs and their periodic nature
- To be able to use a sine or cosine graph to find the approximate solution of an equation
- To be able to use symmetry of the unit circle and graphs to compare the trigonometric ratios of angles

As the angle θ increases from 0° to 360°, the values of sin θ, cos θ and tan θ increase or decrease depending on the value of θ. Graphing the values of sin θ, cos θ and tan θ against θ gives a clear picture of this.

These wave-like graphs based on trigonometric functions are used to model many variables from the height of the tide on a beach to the width of a soundwave giving a high or low pitch sound.

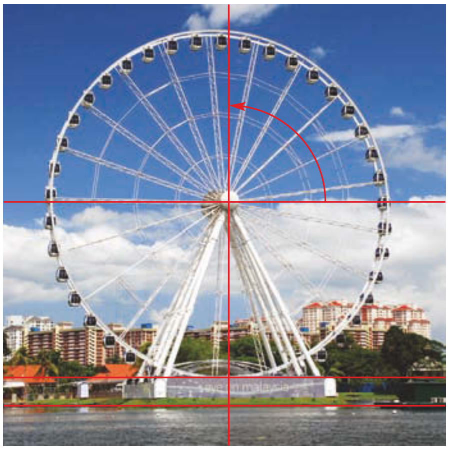
Incoming solar energy is essential for agriculture and solar power production. Local light intensity is determined by the sun's angle of elevation, which has a periodic variation. Trigonometric graphs can model light intensity versus day of the year, time of the day and latitude.

LESSON STARTER Ferris wheel ride

Have you ever had a ride on a Ferris wheel? Imagine yourself riding a Ferris wheel again. The wheel rotates at a constant rate, but on which part of the ride will your vertical upwards movement be fastest? On which part of the ride will your vertical movement be slowest?

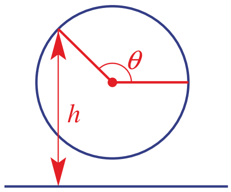
Work in groups to discuss these questions and help each other to complete the table and graph.

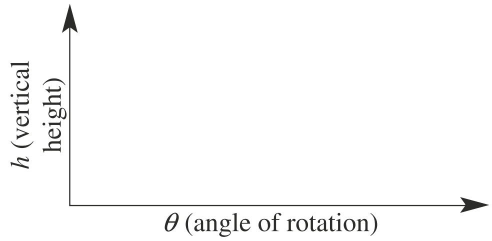
For this example, assume that the bottom of the Ferris wheel is 2 m above the ground and the diameter of the wheel is 18 m. Count the start of a rotation from halfway up on the right, as shown. The wheel rotates in an anticlockwise direction.



Position	Angle of rotation, θ, from halfway up	Vertical height, h, above ground level (m)
Halfway up	0°	
Top	90°	
Halfway down		
Bottom		2
Halfway up		
Top		
Halfway down		

Now draw a graph of the **vertical height** (h) above the ground (vertical axis) versus the **angle** (θ) of anticlockwise rotation for two complete turns of the Ferris wheel.

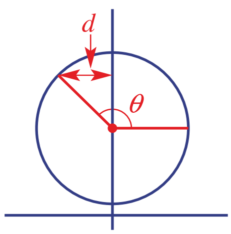


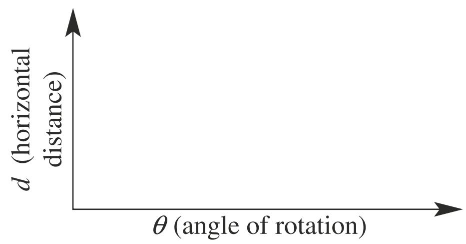


As a group, discuss some of the key features of the graph.

- What are the maximum and minimum values for the height?
- Discuss any symmetry you see in your graph. How many values of θ (rotation angle) have the same value for height? Give some examples.

Discuss how the shape would change for a graph of the **horizontal distance** (d) from the circumference (where you sit) to the central **vertical** axis of the Ferris wheel versus the angle (θ) of rotation. Sketch this graph.

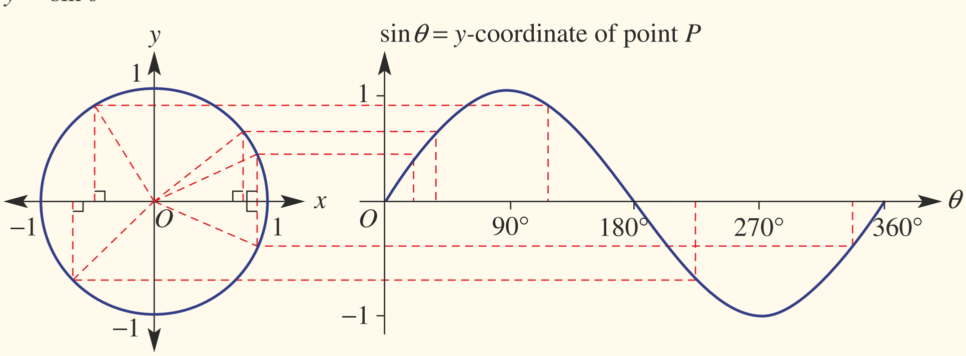




The shapes of the Ferris wheel graphs you have drawn are examples of **periodic functions** because the graph shape continuously repeats one cycle (for each period of 360°) as the wheel rotates. The graph of height above the ground illustrates a sine function (sin θ). The graph of the distance from a point on the circumference to the central vertical axis of the Ferris wheel illustrates a cosine function (cos θ).

KEY IDEAS

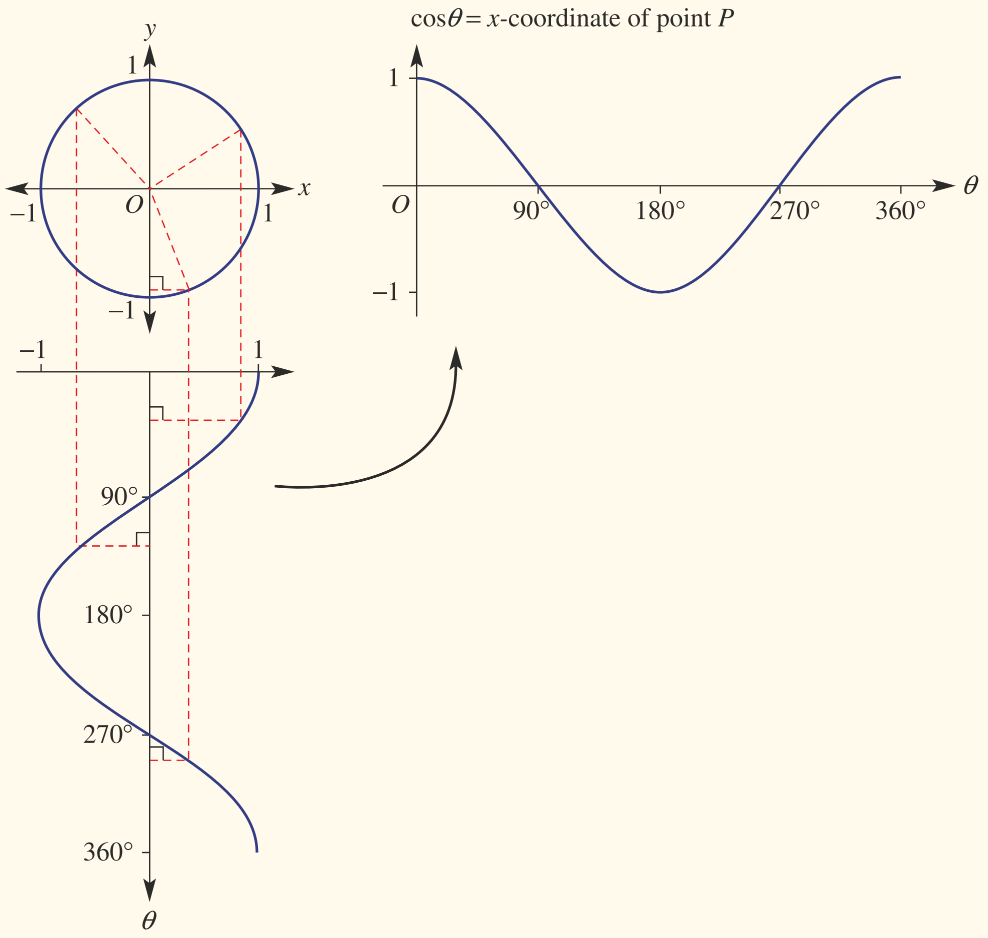
- By plotting θ on the x-axis and sin θ on the y-axis we form the graph of sin θ.
  sin θ = y-coordinate of point P on the unit circle.
  - $y = \sin\theta$



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

- By plotting θ on the x-axis and $\cos\theta$ on the y-axis we form the graph of $\cos\theta$.
  $\cos\theta = x$-coordinate of point P on the unit circle.
  - When we write $y = \cos\theta$, the y variable is not to be confused with the y-coordinate of the point P on the unit circle.
  - $y = \cos\theta$

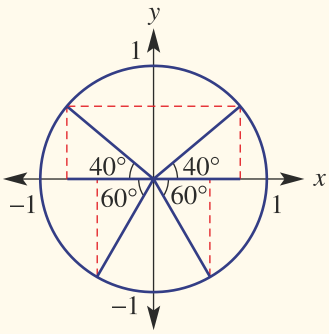


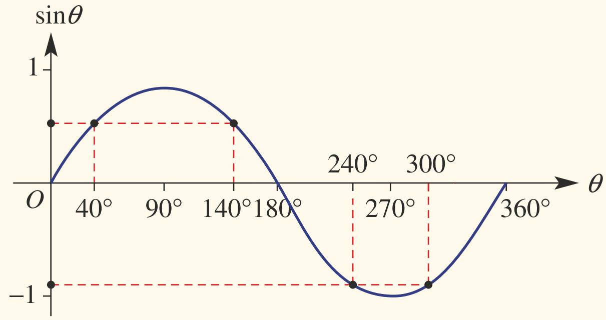
- **Amplitude** is the maximum displacement of the graph from a reference level (here it is the x-axis).
- The **period** of a graph is the time taken (or number of degrees) to make one complete cycle.
- Both $y = \sin\theta$ and $y = \cos\theta$ have Amplitude = 1 and Period = 360°.
- **Symmetry** within the unit circle using reference angles can be illustrated using graphs of trigonometric functions.

ISBN 978-1-108-77290-7

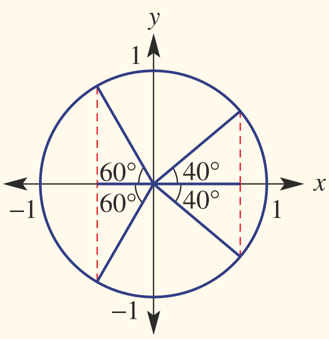
Cambridge University Press
Updated June 2022

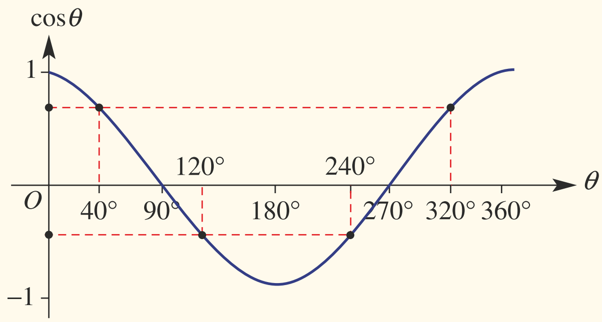
- This shows $\sin 40° = \sin 140°$ (reference angle 40°) and $\sin 240° = \sin 300°$ (reference angle 60°).

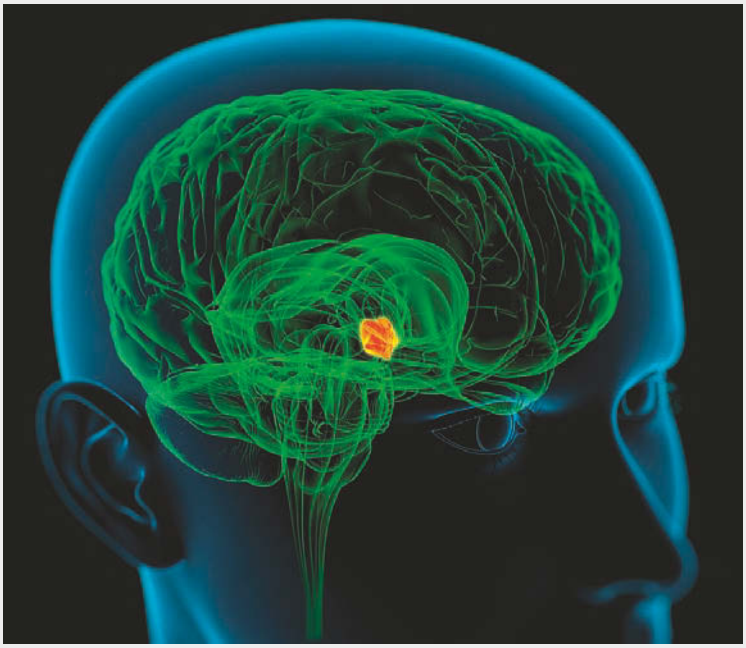




- This shows $\cos 40° = \cos 320°$ (reference angle 40°) and $\cos 120° = \cos 240°$ (reference angle 60°).







Circadian rhythms, such as the brain's 24-hour sleep-wake cycle, can be modelled with trigonometric graphs. Research shows that digital devices' blue light causes the hypothalamus to suppress the sleep hormone, delaying the brain's sleep waves.

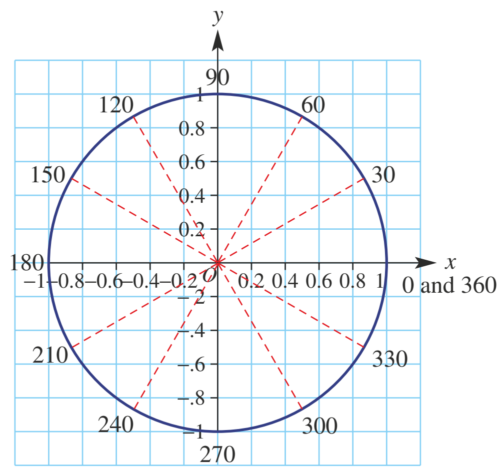
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

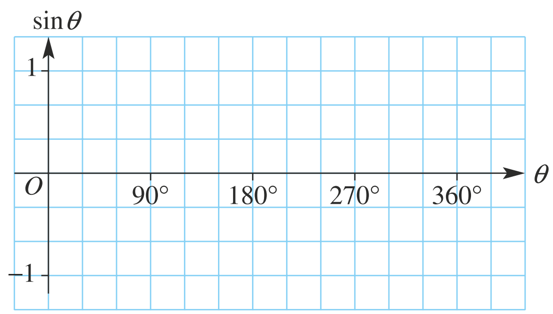
BUILDING UNDERSTANDING

1 **a** Find the missing values in the table below for $\sin\theta$, stating the y-coordinate of each point at which the angle intersects the unit circle (shown below).

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin\theta$	0	0.5			0.87			−0.5					



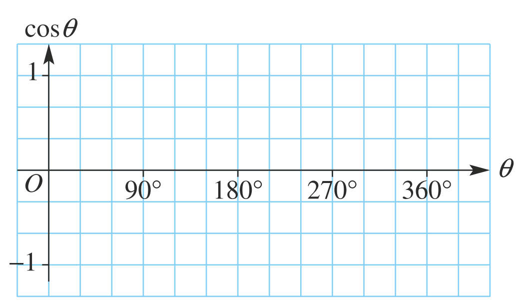
b Graph the points above and join them to make a smooth curve for $\sin\theta$.



2 **a** Using the unit circle diagram in Question **1**, find the missing values in the table below for $\cos\theta$, stating the x-coordinate of each point at which the angle intersects the unit circle.

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\cos\theta$	1	0.87			−0.5			−0.87					

b Graph the points above and join them to make a smooth curve for $\cos\theta$.



3 **a** For the graph of $\sin\theta$ and using $0° \leqslant \theta \leqslant 360°$, state:

i the maximum and minimum values of $\sin\theta$

ii the values of θ for which $\sin\theta = 0$.

b For the graph of $\cos\theta$ and using $0° \leqslant \theta \leqslant 360°$, state:

i the maximum and minimum values of $\cos\theta$

ii the values of θ for which $\cos\theta = 0$.

c State the values of θ for which:

i $\cos\theta < 0$

ii $\sin\theta < 0$

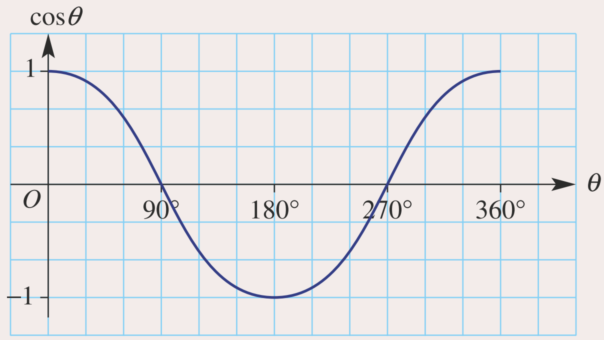
ISBN 978-1-108-77290-7

Example 21 Reading off a trigonometric graph

Use this graph of cos θ to estimate:

a the value of cos θ for $\theta = 200°$

b the two values of θ for which cos $\theta = 0.7$



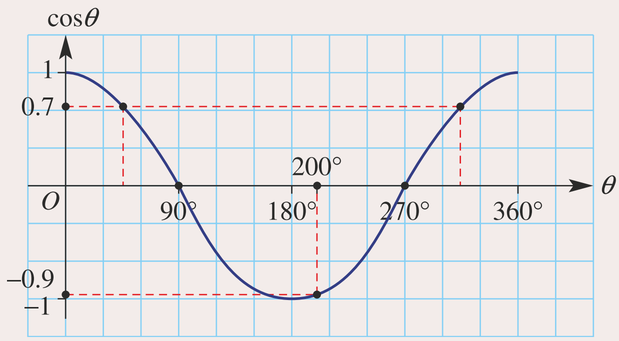
SOLUTION

a $\cos 200° \approx -0.9$

b $\cos \theta = 0.7$

$\theta \approx 46°$ or $314°$

EXPLANATION



Now you try

Use the above graph of cos θ to estimate:

a the value of cos θ for $\theta = 100°$

b the two values of θ for which cos $\theta = -0.6$

Example 22 Comparing the size of the sine of angles

Use the graph of $y = \sin \theta$ to state whether or not the following are true or false.

a $\sin 70° < \sin 130°$

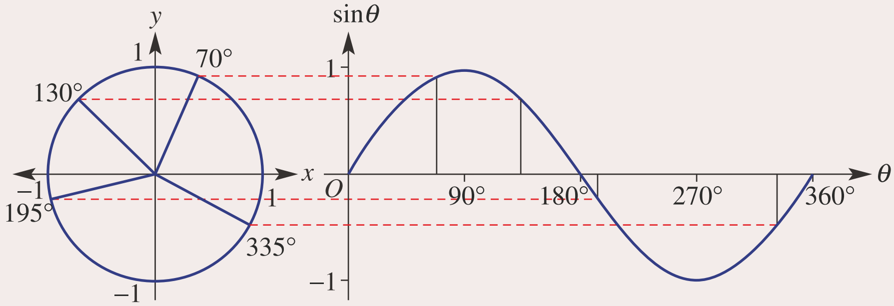
b $\sin 195° > \sin 335°$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

SOLUTION EXPLANATION

a false
b true



Now you try

Use the graph of $y = \sin\theta$ to state whether or not the following are true or false.

a $\sin 30° > \sin 140°$

b $\sin 240° > \sin 290°$

Exercise 4K

FLUENCY	1–3	2, 3, 4(½)	2, 3, 4(⅓)

Example 21

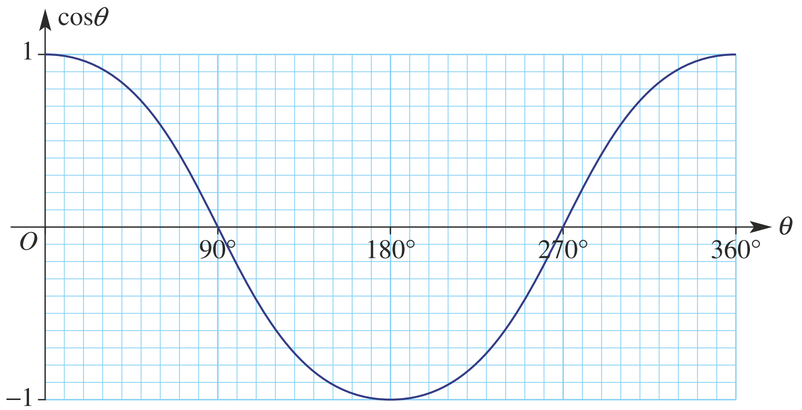
1 Use the graph of $\cos\theta$ shown in Question **2** below to estimate:

a the value of $\cos\theta$ for $\theta = 140°$

b the two values of θ for which $\cos\theta = 0.4$

Example 21

2 This graph shows $\cos\theta$ for $0° \leqslant \theta \leqslant 360°$.



a Use this graph to estimate the value of $\cos\theta$ for the following.

i $\theta = 35°$ **ii** $\theta = 190°$ **iii** $\theta = 330°$ **iv** $\theta = 140°$

v $\theta = 260°$ **vi** $\theta = 75°$ **vii** $\theta = 115°$ **viii** $\theta = 305°$

b Use the same graph to estimate the two values of θ for each of the following.

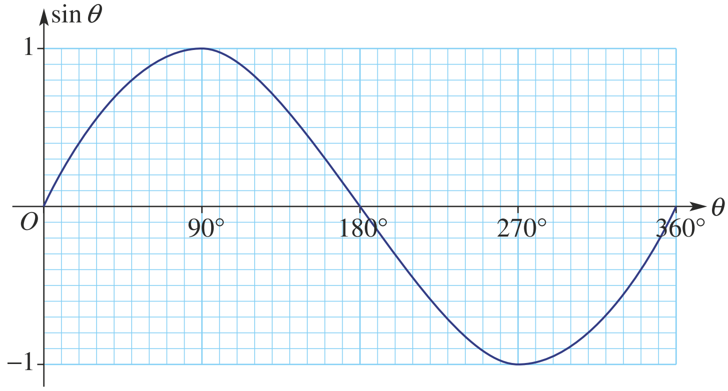
i $\cos\theta = 0.8$ **ii** $\cos\theta = 0.6$ **iii** $\cos\theta = 0.3$ **iv** $\cos\theta = 0.1$

v $\cos\theta = -0.4$ **vi** $\cos\theta = -0.2$ **vii** $\cos\theta = -0.8$ **viii** $\cos\theta = -0.6$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

3 This graph shows $\sin\theta$ for $0° \leqslant \theta \leqslant 360°$.



a Use this graph to estimate the value of $\sin\theta$ for the following.

i $\theta = 25°$ **ii** $\theta = 115°$ **iii** $\theta = 220°$ **iv** $\theta = 310°$
v $\theta = 160°$ **vi** $\theta = 235°$ **vii** $\theta = 320°$ **viii** $\theta = 70°$

b Use the same graph to estimate the two values of θ for each of the following.

i $\sin\theta = 0.6$ **ii** $\sin\theta = 0.2$ **iii** $\sin\theta = 0.3$ **iv** $\sin\theta = 0.9$
v $\sin\theta = -0.4$ **vi** $\sin\theta = -0.8$ **vii** $\sin\theta = -0.7$ **viii** $\sin\theta = -0.1$

Example 22

4 By considering the graphs of $y = \sin\theta$ and $y = \cos\theta$, state whether the following are true or false.

a $\sin 60° > \sin 200°$ **b** $\sin 100° < \sin 300°$ **c** $\sin 135° < \sin 10°$
d $\sin 200° = \sin 340°$ **e** $\cos 70° < \cos 125°$ **f** $\cos 315° > \cos 135°$
g $\cos 310° = \cos 50°$ **h** $\cos 95° > \cos 260°$ **i** $\sin 90° = \cos 360°$
j $\cos 180° = \sin 180°$ **k** $\sin 210° > \sin 285°$ **l** $\cos 15° > \cos 115°$

PROBLEM-SOLVING

5–6(1/2)	5–7(1/2)	5–8(1/4)

5 For each of the following angles, state the second angle between 0° and 360° that gives the same value for $\sin\theta$.

a 70° **b** 120° **c** 190° **d** 280°
e 153° **f** 214° **g** 307° **h** 183°

6 For each of the following angles, state the second angle between 0° and 360° that gives the same value for $\cos\theta$.

a 80° **b** 10° **c** 165° **d** 285°
e 224° **f** 147° **g** 336° **h** 199°

7 Give the reference angle in the first quadrant that matches these angles.

a 150° **b** 120° **c** 195° **d** 290°
e 235° **f** 260° **g** 125° **h** 205°
i 324° **j** 252° **k** 117° **l** 346°

8 Use a calculator to find the two values of θ for $0° \leqslant \theta \leqslant 360°$, correct to one decimal place, for these simple equations.

a $\sin\theta = 0.3$ **b** $\sin\theta = 0.7$ **c** $\cos\theta = 0.6$
d $\cos\theta = 0.8$ **e** $\sin\theta = -0.2$ **f** $\sin\theta = -0.8$
g $\cos\theta = -0.4$ **h** $\cos\theta = 0.65$ **i** $\sin\theta = 0.48$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

REASONING	9	9, 10	10, 11

9 **a** How many values of θ satisfy $\sin\theta = 2$? Give a reason.
b How many values of θ satisfy $\cos\theta = -4$? Give a reason.

10 Recall the exact values for $\sin\theta$ and $\cos\theta$ for $30°$, $45°$ and $60°$ in the first quadrant.

a Complete this table.

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$		$\frac{1}{2}$			
$\cos\theta$					0

b Using the reference angles in the table above, state the exact value of the following.

i $\sin 150°$	**ii** $\cos 120°$	**iii** $\cos 225°$	**iv** $\sin 180°$
v $\cos 300°$	**vi** $\sin 240°$	**vii** $\cos 270°$	**viii** $\sin 135°$
ix $\cos 210°$	**x** $\sin 330°$	**xi** $\sin 315°$	**xii** $\cos 240°$
xiii $\sin 225°$	**xiv** $\sin 120°$	**xv** $\cos 150°$	**xvi** $\cos 330°$

11 For θ between $0°$ and $360°$, find the two values of θ that satisfy the following.

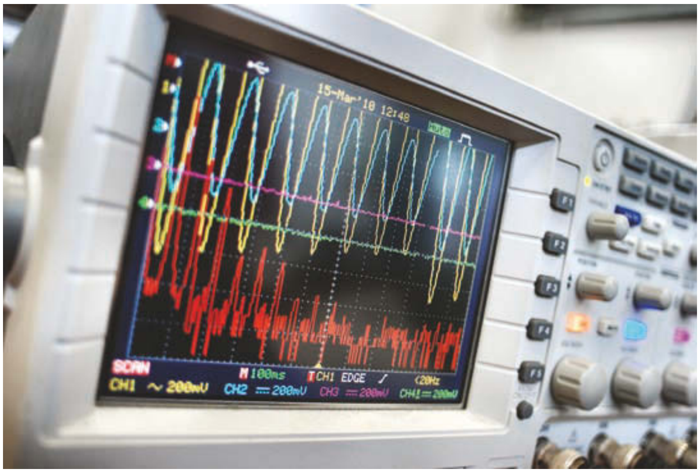
a $\cos\theta = \frac{\sqrt{2}}{2}$ **b** $\sin\theta = \frac{\sqrt{3}}{2}$ **c** $\sin\theta = \frac{1}{2}$

d $\sin\theta = -\frac{1}{2}$ **e** $\cos\theta = -\frac{1}{2}$ **f** $\cos\theta = -\frac{\sqrt{3}}{2}$

ENRICHMENT: Trigonometric functions with technology	–	–	12

12 Use technology to sketch the graph of the following families of curves on the same axes, and then write a sentence describing the effect of the changing constant.

a **i** $y = \sin x$ **ii** $y = -\sin x$
b **i** $y = \cos x$ **ii** $y = -\cos x$
c **i** $y = \sin x$ **ii** $y = 3\sin x$ **iii** $y = \frac{1}{2}\sin x$
d **i** $y = \cos x$ **ii** $y = \cos(2x)$ **iii** $y = \cos\left(\frac{x}{3}\right)$
e **i** $y = \sin x$ **ii** $y = \sin(x) + 2$ **iii** $y = \sin(x) - 1$
f **i** $y = \cos x$ **ii** $y = \cos(x - 45°)$ **iii** $y = \cos(x + 60°)$



ISBN 978-1-108-77290-7

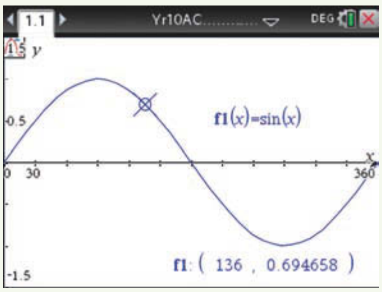
Cambridge University Press
Updated June 2022

Using calculators to graph trigonometric functions

1. Sketch the graph of $y = \sin(x)$ for $0° \leqslant x \leqslant 360°$ and trace to explore the behaviour of y.
2. Sketch the graph of $y = \cos(x)$ and $y = \cos(2x)$ for $0° \leqslant x \leqslant 360°$ on the same set of axes.

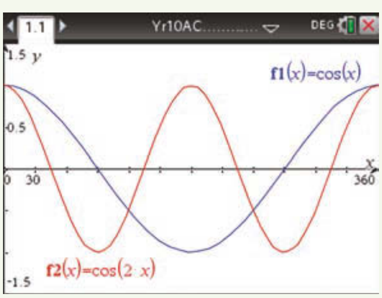
Using the TI-Nspire:

1. In a **Graphs** page, define $f1(x) = \sin(x)$ and press enter. Use menu **>Window/Zoom> Window Settings** and set x from 0 to 360 and y from –1.5 to 1.5. Use menu **>Trace>Graph Trace** and scroll along the graph.



Note: Ensure you are in Degree mode. This setting can be accessed using menu **>Settings** whilst in the **Graphs** page.
Hint: you can double click on the end axes values and edit if preferred.

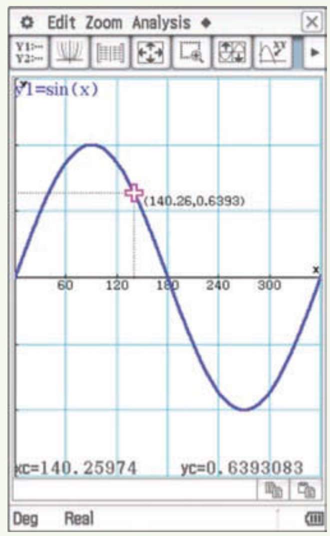
2. In a **Graphs** page, define $f1(x) = \cos(x)$ and $f2(x) = \cos(2x)$. Use the same settings as before.



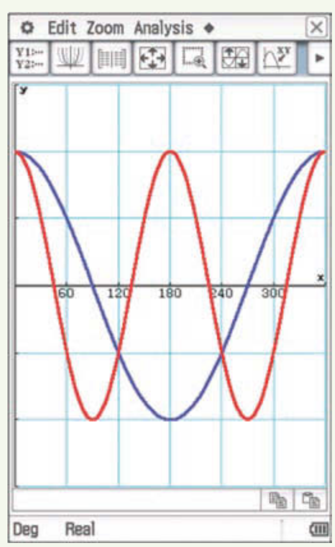
Hint: if the graph entry line is not showing, press tab or double click in an open area.

Using the ClassPad:

1. With the calculator in **Degree** mode, go to the **Graph&Table** application. Enter the rule $y1 = \sin(x)$ followed by **EXE**. Tap the graph icon to see the graph. Tap the window icon and set x from 0 to 360 with a scale of 60 and y from about –1.5 to 1.5 with a scale of 0.5. Tap **Analysis**, **Trace** and then scroll along the graph.



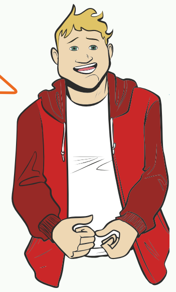
2. In the **Graph&Table** application, enter the rules $y1 = \cos(x)$ and $y2 = \cos(2x)$ followed by **EXE**. Tap the graph icon. Use settings as before.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

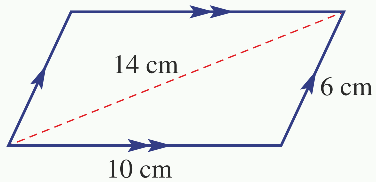
Up for a challenge? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.



1 Two adjacent sides of a parallelogram have lengths of 6 cm and 10 cm. If the length of the longer diagonal is 14 cm, find:

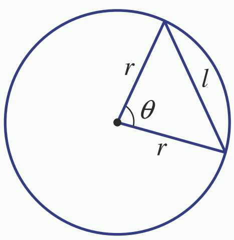
a the size of the internal angles of the parallelogram

b the length of the other diagonal, to one decimal place.



2 Two cyclists, Stuart and Cadel, start their ride from the same starting point. Stuart travels 30 km on a bearing of 025°T, while Cadel travels the same distance but in a direction of 245°T. What is Cadel's bearing from Stuart after they have travelled the 30 km?

3 Show that for a circle of radius r the length of a chord l that subtends an angle θ at the centre of the circle is given by $l = \sqrt{2r^2(1 - \cos\theta)}$.



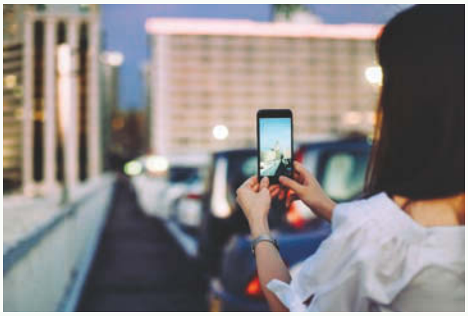
4 Akira measures the angle of elevation to the top of a mountain to be 20°. He walks 800 m horizontally towards the mountain and finds the angle of elevation has doubled. What is the height of the mountain above Akira's position, to the nearest metre?

5 A walking group sets out due east from the town hall at 8 km/h. At the same time, another walking group leaves from the town hall along a different road in a direction of 030°T at 5 km/h.

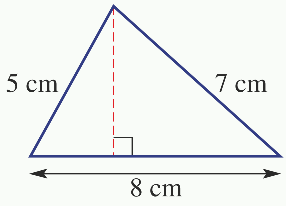
a How long will it be before the groups are 15 km apart? Give your answer to the nearest minute.

b What is the true bearing of the second group from the first group, to the nearest degree, at any time?

6 Edwina stands due south of a building 40 m tall to take a photograph of it. The angle of elevation to the top of the building is 23°. What is the angle of elevation, correct to two decimal places, after she walks 80 m due east to take another photo?



7 Calculate the height of the given triangle, correct to two decimal places.



ISBN 978-1-108-77290-7

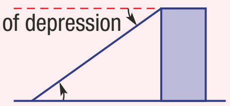
Cambridge University Press
Updated June 2022

Chapter summary

Trigonometry

Elevation and depression

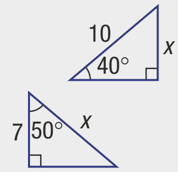
- Draw right-angled triangle with key information.
- Use trigonometry to find unknown.



Finding angles

If $\sin\theta = k$
$\theta = \sin^{-1}(k)$
If $\cos\theta = k$
$\theta = \cos^{-1}(k)$
If $\tan\theta = k$
$\theta = \tan^{-1}(k)$
$-1 \le k \le 1$ for $\sin\theta$ and $\cos\theta$.

Finding lengths

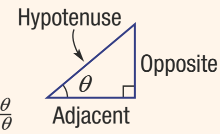


$\sin 40° = \frac{x}{10}$
$\therefore\ x = 10 \times \sin 40°$
$\cos 50° = \frac{7}{x}$
$x \times \cos 50° = 7$
$x = \frac{7}{\cos 50°}$

Trigonometric ratios

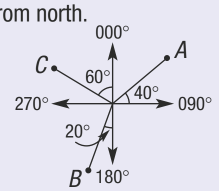
SOHCAHTOA

$\sin\theta = \frac{O}{H}$
$\cos\theta = \frac{A}{H}$
$\tan\theta = \frac{O}{A} = \frac{\sin\theta}{\cos\theta}$



Bearings

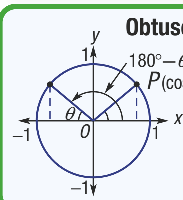
True bearings are measured clockwise from north.



A: 050°T
B: 200°T
C: 300°T

Opposite directions differ by 180°.

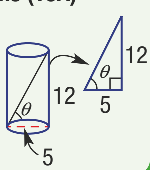
Obtuse angles (10A)



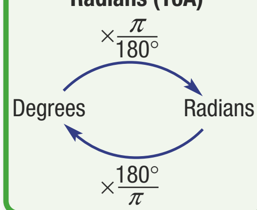
$\sin(180° - \theta) = \sin\theta$
$\cos(180° - \theta) = -\cos\theta$
$\tan(180° - \theta) = -\tan\theta$

3D applications (10A)

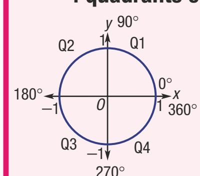
- Identify and redraw right-angled triangle.
- Use trigonometric ratios.
- Answer in words.



Radians (10A)



4 quadrants of unit circle (10A)

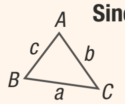


Q1: All are positive
Q2: only sin positive
Q3: only tan positive
Q4: only cos positive

The reference angle relates an angle to its acute angle in quadrant 1.
Symmetry can then be used to find $\sin\theta$ or $\cos\theta$ or $\tan\theta$.
e.g. sin 210° in quadrant 3 is negative
$210° - 180° = 30°$
$\sin 210° = -\sin 30°$
$= -\frac{1}{2}$

To find reference angle of θ in Q1:
Q2: $180° - \theta$
Q3: $\theta - 180°$
Q4: $360° - \theta$

Sine rule (10A)

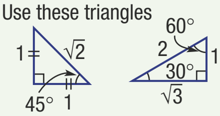


$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

When finding angles, determine whether you require the acute or the obtuse angle.

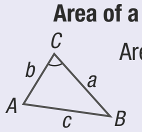
Exact values (10A)

Use these triangles



θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Area of a triangle (10A)



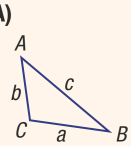
$\text{Area} = \frac{1}{2}ab\sin C$

Cosine rule (10A)

$c^2 = a^2 + b^2 - 2ab\cos C$

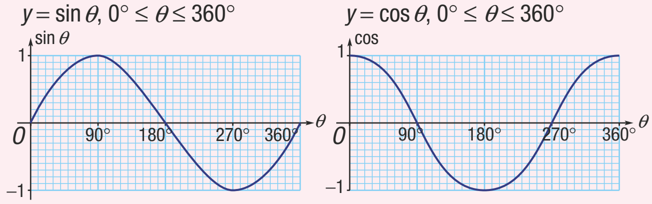
Rule is used to find:

- third side, given two sides and included angle
  or
- an angle, given all three sides.



Graphs of $\sin\theta$ and $\cos\theta$ (10A)

$y = \sin\theta,\ 0° \le \theta \le 360°$ $\quad$ $y = \cos\theta,\ 0° \le \theta \le 360°$



Symmetry of the unit circle can be observed in graphs.

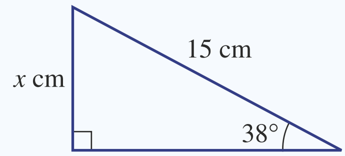
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

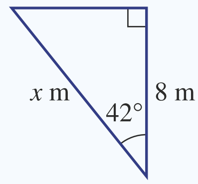
Chapter checklist: Success criteria

✔

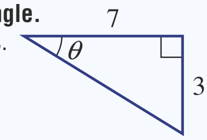
4A **1. I can use trigonometry to solve for a side length in the numerator.**
e.g. Find the value of x in the triangle shown, correct to two decimal places.



4A **2. I can use trigonometry to solve for a side length in the denominator.**
e.g. Find the value of x in the triangle shown, correct to two decimal places.



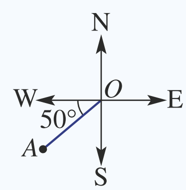
4B **3. I can use trigonometry to find the value of an angle in a right-angled triangle.**
e.g. For the triangle shown, find the value of θ correct to two decimal places.



4C **4. I can apply trigonometry in word problems involving angle of elevation and depression.**
e.g. From a boat at sea level to the top of a lighthouse, the angle of elevation is 25°. If the boat is a horizontal distance of 45 m from the base of the lighthouse, determine the height of the top of the lighthouse above sea level, correct to the nearest metre.

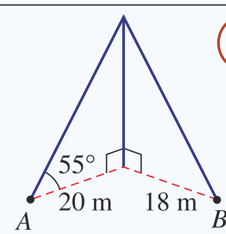
4C **5. I can apply trigonometry to problem solve in word problems.**
e.g. Two buildings 33 metres apart are 82 metres and 56 metres high. Find the angle of elevation from the top of the shorter building to the top of the taller building, correct to two decimal places.

4D **6. I can state a direction and an opposite direction.**
e.g. In the diagram, give the true bearing of A from O and O from A.

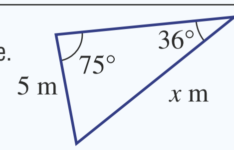


4D **7. I can use bearings with trigonometry.**
e.g. A hiker walks due south for 3 km, then on a true bearing of 220° for 2 km. Find how far west the hiker is from his starting point, correct to two decimal places.

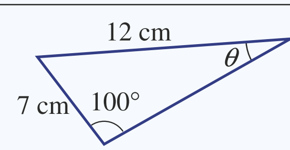
4E **8. I can apply trigonometry in 3D.** 10A
e.g. A vertical flagpole is supported by two wires reaching from points A and B on level ground as shown. The wire from point A is attached 20 m from the base of the flag pole and the wire from point B is attached 18 m from the base of the flag pole. If the angle of elevation from point A to the top of the flag pole is 55°, find the angle of elevation from B to the top of the flag pole correct to one decimal place.



4F **9. I can find a side length using the sine rule.** 10A
e.g. Find the value of x in this triangle, correct to one decimal place.

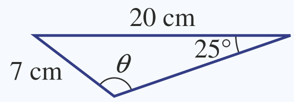
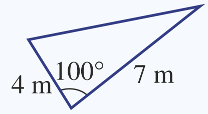
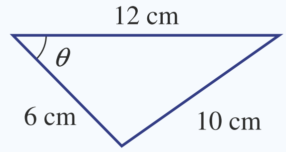
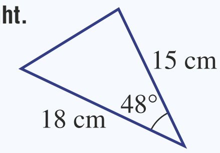
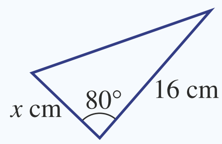
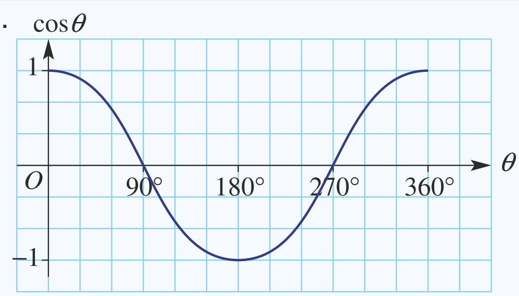


4F **10. I can find an acute angle using the sine rule.** 10A
e.g. Find the value of θ in this triangle.



ISBN 978-1-108-77290-7

Chapter checklist

			✔
4F	**11. I can find an obtuse angle using the sine rule.** e.g. Find the value of θ in this triangle if θ is obtuse. 	10A	
4G	**12. I can use the cosine rule to find a side length.** e.g. Find the length of the third side in this triangle, correct to two decimal places. 	10A	
4G	**13. I can use the cosine rule to find an angle.** e.g. Find the value of θ in this triangle, correct to two decimal places. 	10A	
4H	**14. I can find the area of a triangle without knowing the vertical height.** e.g. Find the area of this triangle, correct to one decimal place. 	10A	
4H	**15. I can use the area of a triangle to find an unknown side length.** e.g. Find the value of x in this triangle, correct to two decimal places, given that the area of the triangle is 49 cm^2. 	10A	
4I	**16. I can choose an obtuse angle to relate a trigonometric ratio to an acute angle.** e.g. Choose an obtuse angle to complete the statement: $\cos 33° = -\cos$ ____ .	10A	
4I	**17. I can position a point on the unit circle.** e.g. Decide in which quadrant $\theta = 250°$ lies and state whether $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive or negative.	10A	
4I	**18. I can rewrite a trigonometric value using its reference angle.** e.g. Write cos 300° and sin 195° using their reference angles.	10A	
4J	**19. I can convert between degrees and radians.** e.g. Convert 150° to radians and $\frac{3\pi^c}{4}$ to degrees.	10A	
4J	**20. I can use exact values.** e.g. Find the exact value of sin 120° and cos 315°.	10A	
4K	**21. I can use the graph of sine or cosine.** e.g. Use this graph of $y = \cos\theta$ to estimate the value of cos 130° and the two values of θ for which $\cos\theta = 0.4$. 	10A	
4K	**22. I can use a graph to estimate the size of sine or cosine of an angle.** e.g. Use the graph of $y = \cos\theta$ to state whether or not $\cos 50° > \cos 310°$.	10A	

ISBN 978-1-108-77290-7

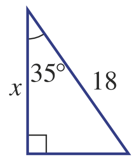
Cambridge University Press
Updated June 2022

Short-answer questions

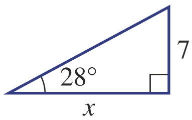
4A

1 Find the value of each pronumeral, rounding your answer to two decimal places.

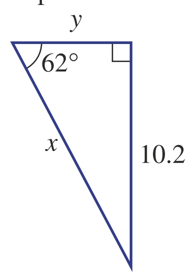
a



b



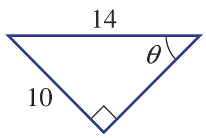
c



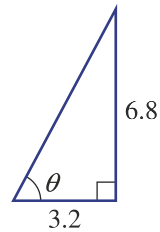
4B

2 Find the value of θ, correct to one decimal place.

a



b



4C

3 An escalator in a shopping centre from level 1 to level 2 is 22 m in length and has an angle of elevation of 16°. Determine how high level 2 is above level 1, to one decimal place.

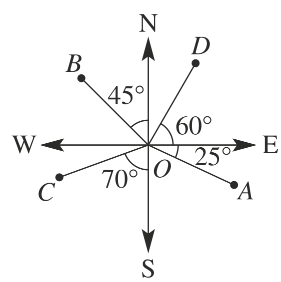
4D

4 **a** Write each bearing A–D as a true bearing.

b Give the true bearing of:

i O from A

ii O from C.



4D

5 A helicopter flies due south for 160 km and then on a bearing of 125°T for 120 km. Answer the following, to one decimal place.

a How far east is the helicopter from its start location?

b How far south is the helicopter from its start location?

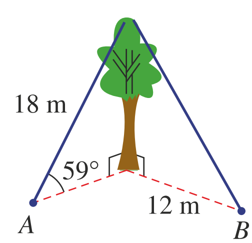
c What bearing must it fly on to return directly to the start location?

4E 10A

6 A tree is being supported by two ropes, as shown. The rope to point A is 18 m long and makes an angle of 59° with the ground. Point B is 12 m from the base of the tree.

a Find the height of the tree, to two decimal places.

b Find the angle the rope to point B makes with the ground, to the nearest degree.



ISBN 978-1-108-77290-7

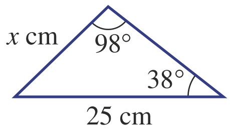
Chapter review

4F/H 10A

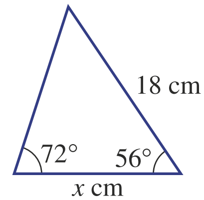
7 For these triangles, find the following, correct to one decimal place.

i the value of x **ii** the area of the triangle

a

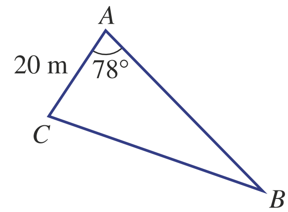


b



4H 10A

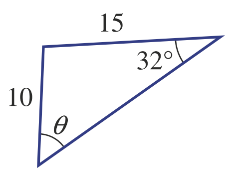
8 Three fences are used to form a triangular pig pen with known dimensions, as shown in the diagram. If the area of the pig pen is 275 m^2, what is the length AB? Round your answer to one decimal place.



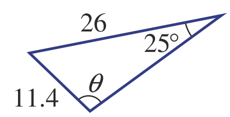
4F 10A

9 Use the sine rule to find the value of θ, correct to one decimal place.

a θ is acute



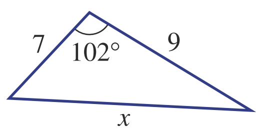
b θ is obtuse



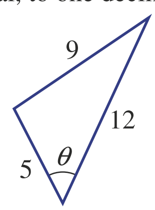
4G 10A

10 Use the cosine rule to find the value of the pronumeral, to one decimal place.

a



b



4I/J 10A

11 **a** Rewrite the following using their reference angle.

i sin 120° **ii** cos 210° **iii** tan 315° **iv** sin 225°

b Hence, give the exact value of each part in **a**.

c State whether the following are positive or negative.

i cos 158° **ii** tan 231° **iii** sin 333° **iv** cos 295°

4J 10A

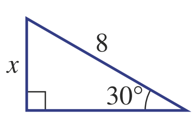
12 Convert the following to the units shown in the brackets.

a 60° (radians) **b** 225° (radians) **c** $\frac{5\pi^c}{6}$ (degrees) **d** $\frac{5\pi^c}{3}$ (degrees)

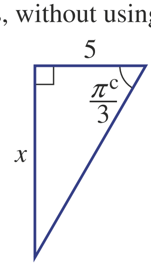
4J 10A

13 Use exact values to find the value of the pronumerals, without using a calculator.

a



b

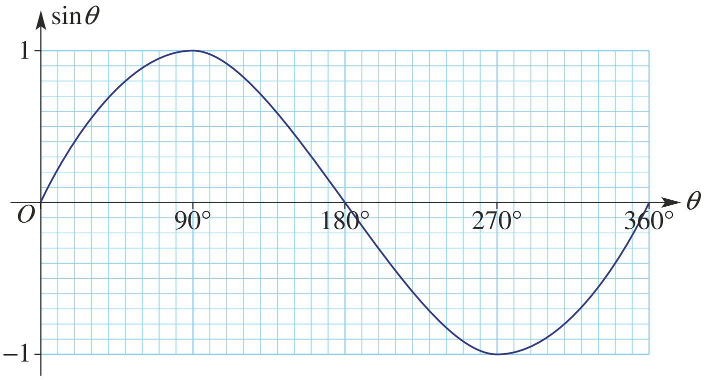


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

4K 10A

14 Use the graph of $\sin\theta$ shown to complete the following.



a Estimate the value of:
  i $\sin 130°$ **ii** $\sin 255°$

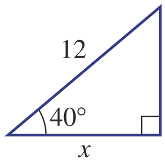
b Find the values of θ between $0°$ and $360°$ such that:
  i $\sin\theta = 0.8$ **ii** $\sin\theta = -0.3$ **iii** $\sin\theta = 1.5$

c State if the following are true or false.
  i $\sin 90° = 1$ **ii** $\sin 75° > \sin 140°$ **iii** $\sin 220° < \sin 250°$

Multiple-choice questions

4A

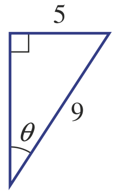
1 The value of x in the diagram shown is equal to:



A $\dfrac{12}{\cos 40°}$ **B** $12\sin 40°$ **C** $\dfrac{\sin 40°}{12}$
D $12\cos 40°$ **E** $\dfrac{12}{\tan 40°}$

4B

2 The angle θ, correct to one decimal place, is:



A $56.3°$ **B** $33.7°$ **C** $29.1°$
D $60.9°$ **E** $42.4°$

4C

3 The angle of depression from the top of a communications tower measuring 44 m tall to the top of a communications tower measuring 31 m tall is $18°$. The horizontal distance between the two towers is closest to:

A 12 m **B** 4 m **C** 14 m **D** 42 m **E** 40 m

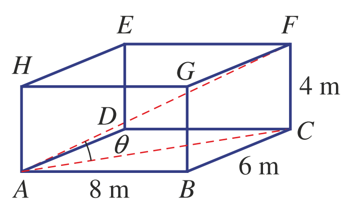
4D

4 A yacht is sailed from A to B on a bearing of $196°$T. To sail from B directly back to A the true bearing would be:

A $074°$T **B** $096°$T **C** $164°$T **D** $016°$T **E** $286°$T

4E 10A

5 The angle θ that AF makes with the base of the rectangular prism is closest to:

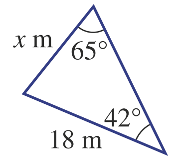


A $22°$ **B** $68°$ **C** $16°$
D $24°$ **E** $27°$

ISBN 978-1-108-77290-7

4F 10A

6 The side length x in the triangle shown, correct to one decimal place, is:



A 10.9 B 29.7 C 13.3
D 12.6 E 17.1

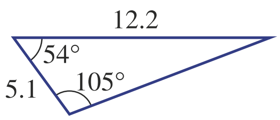
4G 10A

7 The smallest angle in the triangle with side lengths 8 cm, 13 cm and 19 cm, to the nearest degree is:

A 19° B 33° C 52° D 24° E 29°

4H 10A

8 The area of the triangle shown can be determined by calculating:



A $\frac{1}{2} \times 5.1 \times 12.2 \times \cos 54°$ B $\frac{1}{2} \times 5.1 \times 12.2 \times \sin 105°$
C $\frac{1}{2} \times 12.2 \times 5.1$ D $\frac{1}{2} \times 12.2 \times 5.1 \times \sin 54°$
E $\frac{1}{2} \times 6.1 \times 5.1 \times \sin 21°$

4I 10A

9 The incorrect statement below is:

A $\cos 110° = -\cos 70°$ B $\cos 246°$ is negative C $\tan 130°$ is positive
D $\sin 150° = \sin 30°$ E $\sin 300°$ is negative and $\cos 300°$ is positive

4J 10A

10 The exact value of $\sin \frac{2\pi^c}{3}$ is

A $\frac{1}{2}$ B $-\frac{\sqrt{3}}{2}$ C $\frac{\sqrt{3}}{2}$ D $\frac{\sqrt{2}}{2}$ E $-\frac{1}{2}$

Extended-response questions

1 A group of friends set out on a hike to a waterfall in a national park. They are given the following directions to walk from the park's entrance to the waterfall to avoid having to cross a river.

Walk 5 km on a bearing of 325°T and then 3 km due north.

Round each answer to one decimal place.

a Draw and label a diagram to represent this hike.
b Determine how far east or west the waterfall is from the entrance.
c Find the direct distance from the park's entrance to the waterfall.

The friends set up tents on level ground at the base of the waterfall at points A (35 m from the base of the waterfall) and B (28 m from the base of the waterfall). The angle of elevation from A to the top of the waterfall is 32°.

10A d Determine:
i the height of the waterfall
ii the angle of elevation from B to the top of the waterfall.

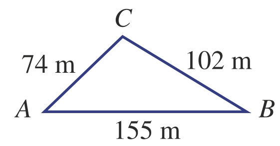
ISBN 978-1-108-77290-7

2 A paddock, ABC, is fenced off, as shown in the figure.

10A

a Find $\angle A$, to three decimal places.

b Hence, find the area enclosed by the fences. Round your answer to two decimal places.



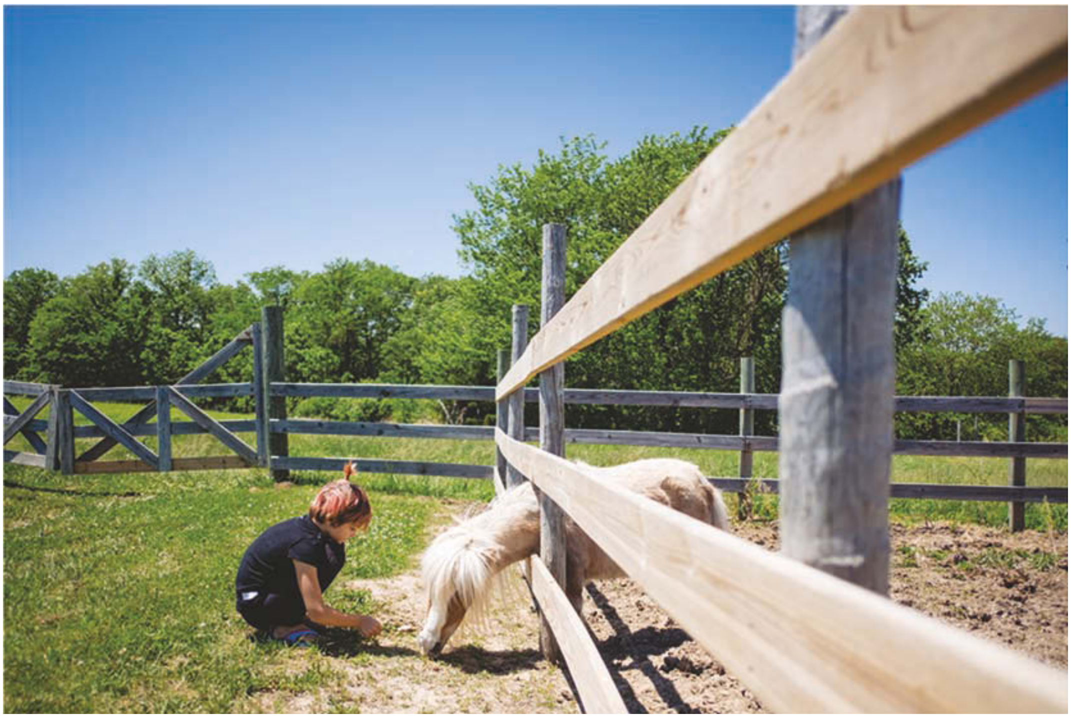
It is planned to divide the paddock into two triangular paddocks by constructing a fence from point C to meet AB at right angles at a point D.

c Determine how many metres of fencing will be required along CD, to the nearest centimetre.

d How far is point D from point A, to the nearest centimetre?

e The person who constructs the fence CD misinterprets the information and builds a fence that does not meet AB at right angles. The fence is 45 metres long.

- **i** Determine, to two decimal places, the two possible angles (i.e. acute and obtuse), this fence line makes with AB.
- **ii** Hence, find the two possible distances of fence post D from A. Round your answer to one decimal place.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

CHAPTER

5 Quadratic expressions and equations

Mathematics of flight

If you blow air across the top of a small piece of paper it will lift rather than be forced down. Daniel Bernoulli, an 18th-century Swiss mathematician and scientist, discovered the relationship between fluid pressure and fluid speed, simply represented by this quadratic equation:

$P = -\frac{1}{2}v^2 + c$, where P is internal pressure, v is speed and c is a constant.

As air is a fluid, Bernoulli's law shows us that with increased air speed there is decreased internal air pressure. This explains why, when a cyclonic wind blows across a house roof, the stronger air pressure from inside the house can push the roof off. The aerofoil shape of a bird or plane wing (i.e. concave-down on the top) causes air to flow at a higher speed over the wing's upper surface and hence air pressure is decreased. The air of higher pressure under the wing helps to lift the plane or bird. Bernoulli's quadratic equation shows us that wing lift is proportional to the square of the airspeed.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Wing loading in kg/m^2 is calculated by dividing the fully loaded weight of a plane by the wing area. Planes with a large wing loading need large wings or a high airspeed or both to achieve lift. The gigantic Antonov cargo plane has a wing loading of around 660 kg/m^2 and can lift enormous weights due to its huge wing area of 905 m^2 and airspeed of 800 km/h. A hang glider has large wings and a low mass giving a wing loading of only 6 kg/m^2. Hence it can still obtain lift at very slow speeds.

Online resources

A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

In this chapter

Victorian Curriculum

NUMBER AND ALGEBRA
Patterns and algebra

Factorise algebraic expressions by taking out a common algebraic factor (VCMNA329)

Expand binomial products and factorise monic quadratic expressions using a variety of strategies (VCMNA332)

Substitute values into formulas to determine an unknown and re-arrange formulas to solve for a particular term (VCMNA333)

Linear and non-linear relationships

Solve simple quadratic equations using a range of strategies (VCMNA341)

(10A) Factorise monic and non-monic quadratic expressions and solve a wide range of quadratic equations derived from a variety of contexts (VCMNA362)

5A Expanding expressions CONSOLIDATING

Learning intentions
- To review how to apply the distributive law to expand brackets
- To be able to expand binomial products including perfect squares
- To be able to form a difference of perfect squares by expansion

You will recall that expressions that include numerals and pronumerals are central to the topic of algebra. Sound skills in algebra are essential for solving most mathematical problems and this includes the ability to expand expressions involving brackets. This includes binomial products, perfect squares and the difference of perfect squares. Exploring how projectiles fly subject to the Earth's gravity, for example, can be modelled with expressions with and without brackets.

Business analysts develop profit equations, which are quadratics, when sales and profit/item are linear relations of the selling price, e.g. \$$p$/ice-cream:

$$\begin{aligned}\text{Profit/week} &= \text{weekly sales} \times \text{profit/item}\\ &= 150(10 - p) \times (p - 2)\\ &= -150(p^2 - 12p + 20)\end{aligned}$$

LESSON STARTER Five key errors

Here are five expansion problems with incorrect answers. Discuss what error has been made and then give the correct expansion.

- $-2(x - 3) = -2x - 6$
- $(x + 3)^2 = x^2 + 9$
- $(x - 2)(x + 2) = x^2 + 4x - 4$
- $5 - 3(x - 1) = 2 - 3x$
- $(x + 3)(x + 5) = x^2 + 8x + 8$

KEY IDEAS

- **Like terms** have the same pronumeral part.
  - They can be collected (i.e. added and subtracted) to form a single term.
    For example: $7x - 11x = -4x$ and $4a^2b - 7ba^2 = -3a^2b$
- The **distributive law** is used to expand brackets.
  - $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$
  - $(a + b)(c + d) = ac + ad + bc + bd$
  - $(a + b)(c + d)$ is called a binomial product because each expression in the brackets has two terms.

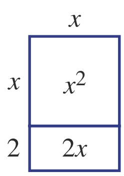
ISBN 978-1-108-77290-7

- **Perfect squares**
  - $(a + b)^2 = (a + b)(a + b)$
    $= a^2 + 2ab + b^2$
  - $(a - b)^2 = (a - b)(a - b)$
    $= a^2 - 2ab + b^2$
- **Difference of perfect squares (DOPS)**
  - $(a + b)(a - b) = a^2 - b^2$
- By definition, a perfect square is an integer that is the square of an integer; however, the rules above also apply for a wide range of values for a and b, including all real numbers.

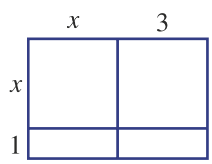
BUILDING UNDERSTANDING

1 Use each diagram to help expand the expressions.

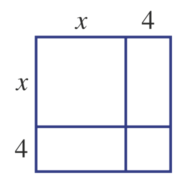
a $x(x + 2)$



b $(x + 3)(x + 1)$



c $(x + 4)^2$



2 Simplify these expressions.

a $2 \times 3x$ **b** $-4 \times 5x$ **c** $-x \times 4x$ **d** $5x \div 10$

e $-6x^2 \div (2x)$ **f** $3x - 21x$ **g** $-3x + 8x$ **h** $-5x - 8x$

Example 1 Expanding simple expressions

Expand and simplify where possible.

a $-3(x - 5)$ **b** $2x(1 - x)$ **c** $\frac{2}{7}(14x + 3)$ **d** $x(2x - 1) - x(3 - x)$

SOLUTION	EXPLANATION
a $-3(x - 5) = -3x + 15$	Use the distributive law: $a(b - c) = ab - ac$. $-3 \times x = -3x$ and $-3 \times (-5) = 15$
b $2x(1 - x) = 2x - 2x^2$	Recall that $2x \times (-x) = -2x^2$.
c $\frac{2}{7}(14x + 3) = \frac{2}{7} \times 14x + \frac{2}{7} \times 3$	When multiplying fractions cancel before multiplying numerators and denominators.
$= 4x + \frac{6}{7}$	Recall that $3 = \frac{3}{1}$.

Continued on next page

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

d $x(2x-1)-x(3-x) = 2x^2 - x - 3x + x^2$ $= 3x^2 - 4x$	Apply the distributive law to each set of brackets first, then simplify by collecting like terms. Recall that $-x \times (-x) = x^2$.

Now you try

Expand and simplify where possible.

a $-2(x-4)$ **b** $5x(4-x)$ **c** $\frac{3}{5}(10x+1)$ **d** $x(5x-1)-x(2-3x)$

Example 2 Expanding binomial products, perfect squares and difference of perfect squares

Expand the following.

a $(x+5)(x+4)$ **b** $(x-4)^2$ **c** $(2x+1)(2x-1)$

SOLUTION	EXPLANATION
a $(x+5)(x+4) = x^2 + 4x + 5x + 20$ $= x^2 + 9x + 20$	For binomial products use $(a+b)(c+d) = ac + ad + bc + bd$. Simplify by collecting like terms.
b $(x-4)^2 = (x-4)(x-4)$ $= x^2 - 4x - 4x + 16$ $= x^2 - 8x + 16$	Rewrite and expand using the distributive law.
Alternatively: $(x-4)^2 = x^2 - 2(x)(4) + 4^2$ $= x^2 - 8x + 16$	Alternatively for perfect squares $(a-b)^2 = a^2 - 2ab + b^2$. Here $a = x$ and $b = 4$.
c $(2x+1)(2x-1) = 4x^2 - 2x + 2x - 1$ $= 4x^2 - 1$	Expand, recalling that $2x \times 2x = 4x^2$. Cancel the $-2x$ and $+2x$ terms.
Alternatively: $(2x+1)(2x-1) = (2x)^2 - (1)^2$ $= 4x^2 - 1$	Alternatively for difference of perfect squares $(a-b)(a+b) = a^2 - b^2$. Here $a = 2x$ and $b = 1$.

Now you try

Expand the following.

a $(x+2)(x+5)$ **b** $(x-2)^2$ **c** $(3x+2)(3x-2)$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 3 Expanding more binomial products

Expand and simplify.

a $(2x - 1)(3x + 5)$ **b** $2(x - 3)(x - 2)$ **c** $(x + 2)(x + 4) - (x - 2)(x - 5)$

SOLUTION

a $(2x - 1)(3x + 5) = 6x^2 + 10x - 3x - 5$
$= 6x^2 + 7x - 5$

EXPLANATION

Expand using the distributive law and simplify.
Note: $2x \times 3x = 2 \times 3 \times x \times x = 6x^2$.

b $2(x - 3)(x - 2) = 2(x^2 - 2x - 3x + 6)$
$= 2(x^2 - 5x + 6)$
$= 2x^2 - 10x + 12$

First expand the brackets using the distributive law, simplify and then multiply each term by 2.

c $(x + 2)(x + 4) - (x - 2)(x - 5)$
$= (x^2 + 4x + 2x + 8) - (x^2 - 5x - 2x + 10)$
$= (x^2 + 6x + 8) - (x^2 - 7x + 10)$
$= x^2 + 6x + 8 - x^2 + 7x - 10$
$= 13x - 2$

Expand each binomial product.
Remove brackets in the last step before simplifying.
$-(x^2 - 7x + 10) = -1 \times x^2 + (-1) \times (-7x) + (-1) \times 10$
$= -x^2 + 7x - 10$

Now you try

Expand and simplify.

a $(3x - 1)(2x + 7)$ **b** $3(x - 1)(x - 4)$ **c** $(x + 3)(x + 1) - (x - 3)(x - 4)$

Exercise 5A

FLUENCY	1, 2–6(1/3)	2–6(1/3)	2–6(1/4)

1 Expand and simplify where possible.

Example 1a
a **i** $-4(x - 1)$ **ii** $-2(x - 6)$

Example 1b
b **i** $3x(2 - x)$ **ii** $7x(5 - x)$

Example 1c
c **i** $\frac{4}{5}(15x + 2)$ **ii** $\frac{7}{9}(18x - 1)$

Example 1a–c
2 Expand and simplify where possible.

a $2(x + 5)$ **b** $3(x - 4)$ **c** $-5(x + 3)$ **d** $-4(x - 2)$
e $3(2x - 1)$ **f** $4(3x + 1)$ **g** $-2(5x - 3)$ **h** $-5(4x + 3)$
i $x(2x + 5)$ **j** $x(3x - 1)$ **k** $2x(1 - x)$ **l** $3x(2 - x)$
m $-2x(3x + 2)$ **n** $-3x(6x - 2)$ **o** $-5x(2 - 2x)$ **p** $-4x(1 - 4x)$
q $\frac{2}{5}(10x + 4)$ **r** $\frac{3}{4}(8x - 5)$ **s** $-\frac{1}{3}(6x + 1)$ **t** $-\frac{1}{2}(4x - 3)$
u $-\frac{3}{8}(24x - 1)$ **v** $-\frac{2}{9}(9x + 7)$ **w** $\frac{3x}{4}(3x + 8)$ **x** $\frac{2x}{5}(7 - 3x)$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 1d

3 Expand and simplify.

a $x(3x-1)+x(4-x)$ **b** $x(5x+2)+x(x-5)$ **c** $x(4x-3)-2x(x-5)$
d $3x(2x+4)-x(5-2x)$ **e** $4x(2x-1)+2x(1-3x)$ **f** $2x(2-3x)-3x(2x-7)$

Example 2a

4 Expand the following.

a $(x+2)(x+8)$ **b** $(x+3)(x+4)$ **c** $(x+7)(x+5)$
d $(x+8)(x-3)$ **e** $(x+6)(x-5)$ **f** $(x-2)(x+3)$
g $(x-7)(x+3)$ **h** $(x-4)(x-6)$ **i** $(x-8)(x-5)$

Example 2b,c

5 Expand the following.

a $(x+5)^2$ **b** $(x+7)^2$ **c** $(x+6)^2$
d $(x-3)^2$ **e** $(x-8)^2$ **f** $(x-10)^2$
g $(x+4)(x-4)$ **h** $(x+9)(x-9)$ **i** $(2x-3)(2x+3)$
j $(3x+4)(3x-4)$ **k** $(4x-5)(4x+5)$ **l** $(8x-7)(8x+7)$

Example 3a

6 Expand the following using the distributive law.

a $(2x+1)(3x+5)$ **b** $(4x+5)(3x+2)$ **c** $(5x+3)(2x+7)$
d $(3x+2)(3x-5)$ **e** $(5x+3)(4x-2)$ **f** $(2x+5)(3x-5)$
g $(4x-5)(4x+5)$ **h** $(2x-9)(2x+9)$ **i** $(5x-7)(5x+7)$
j $(7x-3)(2x-4)$ **k** $(5x-3)(5x-6)$ **l** $(7x-2)(8x-2)$
m $(2x+5)^2$ **n** $(5x+6)^2$ **o** $(7x-1)^2$

PROBLEM-SOLVING	7–8(1/2)	7–9(1/3)	7–9(1/4), 10

7 Write the missing number.

a $(x+?)(x+2)=x^2+5x+6$ **b** $(x+?)(x+5)=x^2+8x+15$
c $(x+7)(x-?)=x^2+4x-21$ **d** $(x+4)(x-?)=x^2-4x-32$
e $(x-6)(x-?)=x^2-7x+6$ **f** $(x-?)(x-8)=x^2-10x+16$

Example 3b

8 Expand the following.

a $2(x+3)(x+4)$ **b** $3(x+2)(x+7)$ **c** $-2(x+8)(x+2)$
d $-4(x+9)(x+2)$ **e** $5(x-3)(x+4)$ **f** $3(x+5)(x-3)$
g $-3(a+2)(a-7)$ **h** $-5(a+2)(a-8)$ **i** $4(a-3)(a-6)$
j $3(y-4)(y-5)$ **k** $-2(y-3)(y-8)$ **l** $-6(y-4)(y-3)$
m $3(2x+3)(2x+5)$ **n** $6(3x-4)(x+2)$ **o** $-2(x+4)(3x-7)$
p $2(x+3)^2$ **q** $4(m+5)^2$ **r** $2(a-7)^2$
s $-3(y-5)^2$ **t** $3(2b-1)^2$ **u** $-3(2y-6)^2$

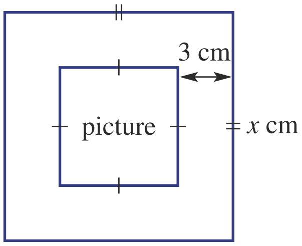
Example 3c

9 Expand and simplify the following.

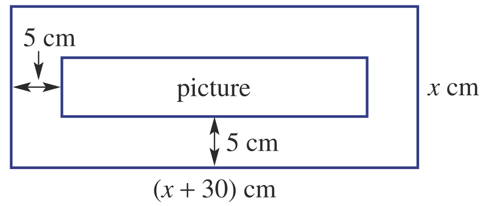
a $(x+1)(x+3)+(x+2)(x+4)$ **b** $(x+8)(x+3)+(x+4)(x+5)$
c $(y+3)(y-1)+(y-2)(y-4)$ **d** $(y-7)(y+4)+(y+5)(y-3)$
e $(2a+3)(a-5)-(a+6)(2a+5)$ **f** $(4b+8)(b+5)-(3b-5)(b-7)$
g $(x+5)^2-7$ **h** $(x-7)^2-9$
i $3-(2x-9)^2$ **j** $14-(5x+3)^2$

10 Find an expanded expression for the area of the pictures centred in these rectangular frames.

a



b



REASONING	12–13(½)	11–13(½)	11–14(½)

11 Prove the following by expanding the left-hand side.

a $(a+b)(a-b)=a^2-b^2$ **b** $(a+b)^2=a^2+2ab+b^2$

c $(a-b)^2=a^2-2ab+b^2$ **d** $(a+b)^2-(a-b)^2=4ab$

12 Use the distributive law to evaluate the following without the use of a calculator.
For example: $4 \times 102 = 4 \times 100 + 4 \times 2 = 408$.

a 6×103 **b** 4×55 **c** 9×63 **d** 8×208

e 7×198 **f** 3×297 **g** 8×495 **h** 5×696

13 Each problem below has an incorrect answer. Find the error and give the correct answer.

a $-x(x-7)=-x^2-7x$ **b** $3a-7(4-a)=-4a-28$

c $(2x+3)^2=4x^2+9$ **d** $(x+2)^2-(x+2)(x-2)=0$

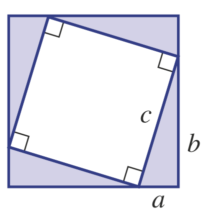
14 Expand these cubic expressions.

a $(x+2)(x+3)(x+1)$ **b** $(x+4)(x+2)(x+5)$ **c** $(x+3)(x-4)(x+3)$

d $(x-4)(2x+1)(x-3)$ **e** $(x+6)(2x-3)(x-5)$ **f** $(2x-3)(x-4)(3x-1)$

ENRICHMENT: Expanding to prove	–	–	15

15 One of the ways to prove Pythagoras' theorem is to arrange four congruent right-angled triangles around a square to form a larger square, as shown.



a Find an expression for the total area of the four shaded triangles by multiplying the area of one triangle by 4.

b Find an expression for the area of the four shaded triangles by subtracting the area of the inner square from the area of the outer square.

c By combining your results from parts **a** and **b**, expand and simplify to prove Pythagoras' theorem: $a^2+b^2=c^2$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

5B Factorising expressions

Learning intentions

- To understand what it means to write an expression in factorised form
- To know to always look for a common factor before trying other factorising techniques
- To be able to recognise a difference of perfect squares including ones involving surds
- To be able to factorise using a common factor or a difference of perfect squares
- To be able to use the grouping technique to factorise

A common and key step in the simplification and solution of equations involves factorisation. Factorisation is the process of writing a number or expression as a product of its factors.

In this section we look at expressions in which all terms have a common factor, expressions that are a difference of perfect squares and four-term expressions, which can be factorised by grouping.

After a car accident, crash investigators use the length of tyre skid marks to determine a vehicle's speed before braking. The quadratic equation $u^2 + 2as = 0$ relates to speed, u, to a known braking distance, s, and deceleration $a = -10$ m/s^2 on a dry, flat bitumen road.

LESSON STARTER But there are no common factors!

An expression such as $xy + 4x + 3y + 12$ has no common factors across all four terms, but it can still be factorised. The method of grouping can be used.

- Complete this working to show how to factorise the expression.

$$\begin{aligned} xy + 4x + 3y + 12 &= x(\_\_\_\_\_\_) + 3(\_\_\_\_\_\_) \\ &= (\_\_\_\_\_\_)(x + 3) \end{aligned}$$

- Now repeat with the expression rearranged.

$$\begin{aligned} xy + 3y + 4x + 12 &= y(\_\_\_\_\_\_) + 4(\_\_\_\_\_\_) \\ &= (\_\_\_\_\_\_)(\_\_\_\_\_\_) \end{aligned}$$

- Are the two results equivalent?

KEY IDEAS

- **Factorise** expressions with **common factors** by 'taking out' the common factors.
  For example: $-5x - 20 = -5(x + 4)$ and $4x^2 - 8x = 4x(x - 2)$.
- Factorise a **difference of perfect squares** (DOPS) using $a^2 - b^2 = (a + b)(a - b)$.
  - We use surds when a^2 or b^2 is not a perfect square, such as 1, 4, 9, …
    For example: $x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5})$ using $(\sqrt{5})^2 = 5$.
- Factorise four-term expressions if possible by **grouping** terms and factorising each pair.
  For example: $$\begin{aligned} x^2 + 5x - 2x - 10 &= x(x + 5) - 2(x + 5) \\ &= (x + 5)(x - 2) \end{aligned}$$

ISBN 978-1-108-77290-7

BUILDING UNDERSTANDING

1. Determine the highest common factor of these pairs of terms.
   a $7x$ and 14 b $-5y$ and -25 c $12a^2$ and $9a$ d $-3x^2y$ and $-6xy$
2. State the missing parts.
   a If $x(x - 1) = x^2 - x$, then $x^2 - x = x($______$)$
   b If $2(1 - x) = 2 - 2x$, then $2 - 2x =$ ______
   c If $(x + 2)(x - 2) =$ ______, then $x^2 - 4 =$ ______
   d If $(3x - 7)($______$) = 9x^2 - 49$, then $9x^2 - 49 =$ ______

Example 4 Taking out common factors

Factorise by taking out common factors.

a $-3x - 12$ b $20a^2 + 30a$ c $2(x + 1) - a(x + 1)$

SOLUTION

a $-3x - 12 = -3(x + 4)$

EXPLANATION

-3 is common to both $-3x$ and -12. Divide each term by -3 to determine the terms in the brackets. Expand to check.

b $20a^2 + 30a = 10a(2a + 3)$

The HCF of $20a^2$ and $30a$ is $10a$.

c $2(x + 1) - a(x + 1) = (x + 1)(2 - a)$

$(x + 1)$ is a common factor to both parts of the expression.

Now you try

Factorise by taking out common factors.

a $-2x - 8$ b $15a^2 + 20a$ c $3(x + 2) - a(x + 2)$

Example 5 Factorising difference of perfect squares

Factorise the following difference of perfect squares. You may need to look for a common factor first.

a $x^2 - 16$ b $9a^2 - 4b^2$ c $12y^2 - 1200$ d $(x + 3)^2 - 4$

SOLUTION

a $x^2 - 16 = (x)^2 - (4)^2$
$= (x + 4)(x - 4)$

EXPLANATION

Use $a^2 - b^2 = (a + b)(a - b)$, where $a = x$ and $b = 4$.

Continued on next page

b $9a^2 - 4b^2 = (3a)^2 - (2b)^2$
$= (3a + 2b)(3a - 2b)$

$9a^2 = (3a)^2$ and $4b^2 = (2b)^2$.

c $12y^2 - 1200 = 12(y^2 - 100)$
$= 12(y + 10)(y - 10)$

First, take out the common factor of 12.
$100 = 10^2$, use $a^2 - b^2 = (a + b)(a - b)$.

d $(x + 3)^2 - 4 = (x + 3)^2 - (2)^2$
$= (x + 3 + 2)(x + 3 - 2)$
$= (x + 5)(x + 1)$

Use $a^2 - b^2 = (a + b)(a - b)$, where $a = x + 3$ and $b = 2$. Simplify each bracket.

Now you try

Factorise the following difference of perfect squares. You may need to look for a common factor first.

a $x^2 - 25$ **b** $16a^2 - 9b^2$ **c** $2y^2 - 98$ **d** $(x + 2)^2 - 36$

Example 6 Factorising DOPS using surds

Factorise these DOPS using surds.

a $x^2 - 10$ **b** $x^2 - 24$ **c** $(x - 1)^2 - 5$

SOLUTION

EXPLANATION

a $x^2 - 10 = x^2 - (\sqrt{10})^2$
$= (x + \sqrt{10})(x - \sqrt{10})$

Recall that $(\sqrt{10})^2 = 10$.

b $x^2 - 24 = x^2 - (\sqrt{24})^2$
$= (x + \sqrt{24})(x - \sqrt{24})$
$= (x + 2\sqrt{6})(x - 2\sqrt{6})$

Use $(\sqrt{24})^2 = 24$. Simplify:
$\sqrt{24} = \sqrt{4 \times 6} = \sqrt{4} \times \sqrt{6} = 2\sqrt{6}$

c $(x - 1)^2 - 5 = (x - 1)^2 - (\sqrt{5})^2$
$= (x - 1 + \sqrt{5})(x - 1 - \sqrt{5})$

Use $a^2 - b^2 = (a + b)(a - b)$, where $a = x - 1$ and $b = \sqrt{5}$.

Now you try

Factorise these DOPS using surds.

a $x^2 - 7$ **b** $x^2 - 32$ **c** $(x - 5)^2 - 2$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 7 Factorisation by grouping

Factorise by grouping $x^2 - x + ax - a$.

SOLUTION

$$\begin{aligned} x^2 - x + ax - a &= x(x-1) + a(x-1) \\ &= (x-1)(x+a) \end{aligned}$$

EXPLANATION

Factorise two pairs of terms, then take out the common binomial factor $(x - 1)$.

Now you try

Factorise by grouping $x^2 - 2x + ax - 2a$.

Exercise 5B

FLUENCY	1, 2–6(1/2)	2–7(1/3)	2–7(1/4)

1 Factorise by taking out common factors.

Example 4a **a** i $-4x - 12$ ii $-9x - 36$

Example 4b **b** i $10a^2 + 40a$ ii $17a^2 + 34a$

Example 4c **c** i $4(x + 2) - a(x + 2)$ ii $11(x + 5) - a(x + 5)$

Example 4a,b

2 Factorise by taking out the common factors.

a $3x - 18$ **b** $4x + 20$ **c** $7a + 7b$ **d** $9a - 15$
e $-5x - 30$ **f** $-4y - 2$ **g** $-12a - 3$ **h** $-2ab - bc$
i $4x^2 + x$ **j** $5x^2 - 2x$ **k** $6b^2 - 18b$ **l** $14a^2 - 21a$
m $10a - 5a^2$ **n** $12x - 30x^2$ **o** $-2x - x^2$ **p** $-4y - 8y^2$
q $ab^2 - a^2b$ **r** $2x^2yz - 4xy$ **s** $-12m^2n - 12mn^2$ **t** $6xyz^2 - 3z^2$

Example 4c

3 Factorise, noting the common binomial factor. (*Hint*: For parts **g**-**i**, insert a 1 where appropriate.)

a $5(x - 1) - a(x - 1)$ **b** $b(x + 2) + 3(x + 2)$ **c** $a(x + 5) - 4(x + 5)$
d $x(x + 2) + 5(x + 2)$ **e** $x(x - 4) - 2(x - 4)$ **f** $3(x + 1) - x(x + 1)$
g $a(x + 3) + (x + 3)$ **h** $x(x - 2) - (x - 2)$ **i** $(x - 6) - x(x - 6)$

Example 5a,b

4 Factorise the following difference of perfect squares.

a $x^2 - 9$ **b** $x^2 - 25$ **c** $y^2 - 49$ **d** $y^2 - 1$
e $4x^2 - 9$ **f** $36a^2 - 25$ **g** $1 - 81y^2$ **h** $100 - 9x^2$
i $25x^2 - 4y^2$ **j** $64x^2 - 25y^2$ **k** $9a^2 - 49b^2$ **l** $144a^2 - 49b^2$

Example 5c,d

5 Factorise the following.

a $2x^2 - 32$ **b** $5x^2 - 45$ **c** $6y^2 - 24$ **d** $3y^2 - 48$
e $3x^2 - 75y^2$ **f** $3a^2 - 300b^2$ **g** $12x^2 - 27y^2$ **h** $63a^2 - 112b^2$
i $(x + 5)^2 - 16$ **j** $(x - 4)^2 - 9$ **k** $(a - 3)^2 - 64$ **l** $(a - 7)^2 - 1$
m $(3x + 5)^2 - x^2$ **n** $(2y + 7)^2 - y^2$ **o** $(5x + 11)^2 - 4x^2$ **p** $(3x - 5y)^2 - 25y^2$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 6

6 Factorise using surds and remember to simplify surds where possible.

a $x^2 - 7$ b $x^2 - 5$ c $x^2 - 19$
d $x^2 - 21$ e $x^2 - 14$ f $x^2 - 30$
g $x^2 - 15$ h $x^2 - 11$ i $x^2 - 8$
j $x^2 - 18$ k $x^2 - 45$ l $x^2 - 20$
m $x^2 - 32$ n $x^2 - 48$ o $x^2 - 50$
p $x^2 - 200$ q $(x + 2)^2 - 6$ r $(x + 5)^2 - 10$
s $(x - 3)^2 - 11$ t $(x - 1)^2 - 7$ u $(x - 6)^2 - 15$
v $(x + 4)^2 - 21$ w $(x + 1)^2 - 19$ x $(x - 7)^2 - 26$

Example 7

7 Factorise by grouping.

a $x^2 + 4x + ax + 4a$ b $x^2 + 7x + bx + 7b$ c $x^2 - 3x + ax - 3a$
d $x^2 + 2x - ax - 2a$ e $x^2 + 5x - bx - 5b$ f $x^2 + 3x - 4ax - 12a$
g $x^2 - ax - 4x + 4a$ h $x^2 - 2bx - 5x + 10b$ i $3x^2 - 6ax - 7x + 14a$

PROBLEM-SOLVING	8(1/2)	8–9(1/2)	8–10(1/3)

8 Factorise fully and simplify surds.

a $x^2 - \frac{2}{9}$ b $x^2 - \frac{3}{4}$ c $x^2 - \frac{7}{16}$ d $x^2 - \frac{5}{36}$
e $(x - 2)^2 - 20$ f $(x + 4)^2 - 27$ g $(x + 1)^2 - 75$ h $(x - 7)^2 - 40$
i $3x^2 - 4$ j $5x^2 - 9$ k $7x^2 - 5$ l $6x^2 - 11$
m $-9 + 2x^2$ n $-16 + 5x^2$ o $-10 + 3x^2$ p $-7 + 13x^2$

9 Factorise by first rearranging.

a $xy - 6 - 3x + 2y$ b $ax - 12 + 3a - 4x$ c $ax - 10 + 5x - 2a$
d $xy + 12 - 3y - 4x$ e $2ax + 3 - a - 6x$ f $2ax - 20 + 8a - 5x$

10 Factorise fully.

a $5x^2 - 120$ b $3x^2 - 162$ c $7x^2 - 126$ d $2x^2 - 96$
e $2(x + 3)^2 - 10$ f $3(x - 1)^2 - 21$ g $4(x - 4)^2 - 48$ h $5(x + 6)^2 - 90$

REASONING	11(1/2)	11(1/2), 12	11(1/2), 13, 14

11 Evaluate the following, without the use of a calculator, by first factorising.

a $16^2 - 14^2$ b $18^2 - 17^2$ c $13^2 - 10^2$ d $15^2 - 11^2$
e $17^2 - 15^2$ f $11^2 - 9^2$ g $27^2 - 24^2$ h $52^2 - 38^2$

12 a Show that $4 - (x + 2)^2 = -x(x + 4)$ by factorising the left-hand side.

b Now factorise the following.

i $9 - (x + 3)^2$ ii $16 - (x + 4)^2$ iii $25 - (x - 5)^2$
iv $25 - (x + 2)^2$ v $49 - (x - 1)^2$ vi $100 - (x + 4)^2$

13 a Prove that, in general, $(x + a)^2 \neq x^2 + a^2$.

b Are there any values of x for which $(x + a)^2 = x^2 + a^2$? If so, what are they?

14 Show that $x^2 - \frac{4}{9} = \frac{1}{9}(3x + 2)(3x - 2)$ using two different methods.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

ENRICHMENT: Hidden DOPS	–	–	15(1/2), 16

15 Factorise and simplify the following without initially expanding the brackets.

a $(x + 2)^2 - (x + 3)^2$
b $(y - 7)^2 - (y + 4)^2$
c $(a + 3)^2 - (a - 5)^2$
d $(b + 5)^2 - (b - 5)^2$
e $(s - 3)^2 - (s + 3)^2$
f $(y - 7)^2 - (y + 7)^2$
g $(2w + 3x)^2 - (3w + 4x)^2$
h $(d + 5e)^2 - (3d - 2e)^2$
i $(4f + 3j)^2 - (2f - 3j)^2$
j $(3r - 2p)^2 - (2p - 3r)^2$

16 **a** Is it possible to factorise $x^2 + 5y - y^2 + 5x$? Can you show how?

b Also try factorising:

i $x^2 + 7x + 7y - y^2$
ii $x^2 - 2x - 2y - y^2$
iii $4x^2 + 4x + 6y - 9y^2$
iv $25y^2 + 15y - 4x^2 + 6x$

Factorising is a key component of the proof of Fermat's last theorem, which states that there are no solutions to $x^n + y^n = z^n$ for $n \geqslant 3$. Although it looks simple, it took the best mathematicians on Earth 358 years to find a proof of this theorem. It was finally proved in 1994 by Andrew Wiles, and his proof is almost 130 pages long!

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Using calculators to expand and factorise

1 Expand and simplify $3(2x - 3)(5x + 1) - (4x - 1)^2$.

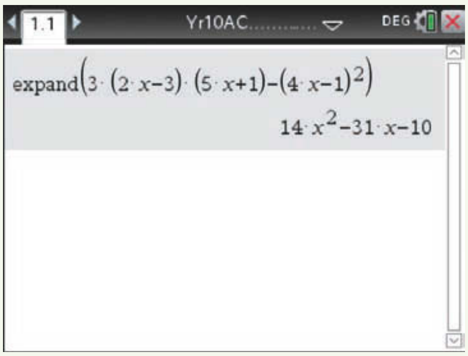
2 Factorise.

a $ax^2 + x - ax - 1$

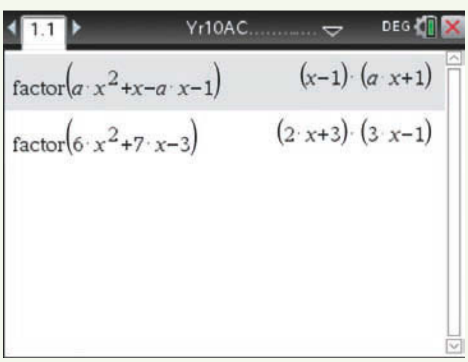
b $6x^2 + 7x - 3$

Using the TI-Nspire:

1 In a **calculator** page use menu **>Algebra>Expand** and type as shown.



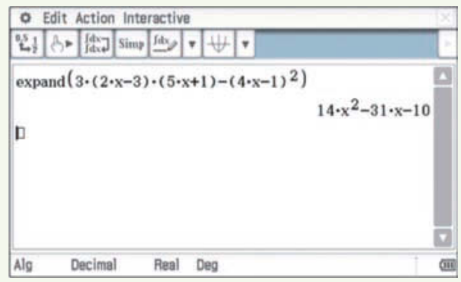
2 In a **Calculator** page use menu **>Algebra>Factor** and type as shown.



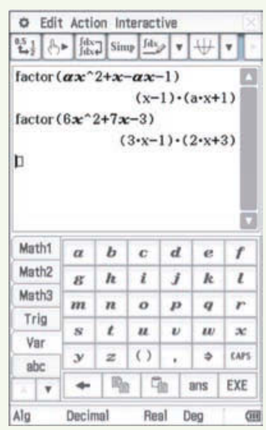
Note: Use a multiplication sign between the *a* and *x*.

Using the ClassPad:

1 In the **Main** application, type and highlight the expression, then tap **Interactive**, **Transformation**, **expand** and type in as shown below.



2 Use the **VAR** keyboard to type the expression as shown. Highlight the expression and tap **Interactive**, **Transformation**, **factor**.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

5C Factorising monic quadratic trinomials

Learning intentions

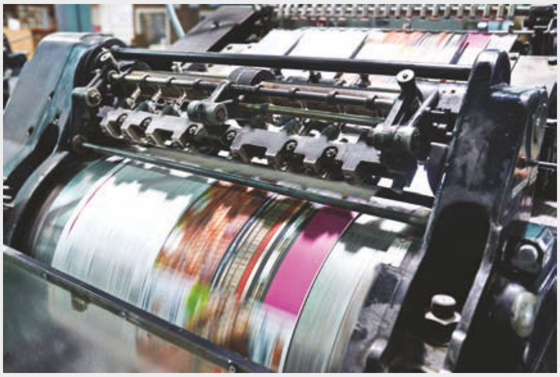
- To be able to identify a monic quadratic trinomial
- To understand the relationship between expanding brackets to form a trinomial and factorising a monic trinomial
- To know how to factorise a monic quadratic trinomial
- To be able to simplify algebraic fractions by first factorising and cancelling common factors

A quadratic trinomial of the form $x^2 + bx + c$ is called a monic quadratic because the coefficient of x^2 is 1.

Now consider:

$$\begin{aligned}(x+m)(x+n) &= x^2 + xn + mx + mn \\ &= x^2 + (m+n)x + mn\end{aligned}$$

We can see from this expansion that mn gives the constant term (c) and $m + n$ is the coefficient of x. This tells us that to factorise a monic quadratic trinomial we should look for factors of the constant term (c) that add to give the coefficient of the middle term (b).



Trinomial quadratics can model the revenue and profits from book publishing. Market research and past sales are used to develop unique quadratic models which find the book's selling price that predicts maximum revenue.

LESSON STARTER Factorising $x^2 - 6x - 72$

Discuss what is wrong with each of these statements when trying to factorise $x^2 - 6x - 72$.

- Find factors of 72 that add to 6.
- Find factors of 72 that add to -6.
- Find factors of -72 that add to 6.
- $-18 \times 4 = -72$ so $x^2 - 6x - 72 = (x - 18)(x + 4)$
- $-9 \times 8 = -72$ so $x^2 - 6x - 72 = (x - 9)(x + 8)$

Can you write a correct statement that correctly factorises $x^2 - 6x - 72$?

KEY IDEAS

- **Monic quadratics** have a coefficient of x^2 equal to 1.
- Monic quadratics of the form $x^2 + bx + c$ can be factorised by finding the two numbers that multiply to give the constant term (c) and add to give the coefficient of x (i.e. b).
  $x^2 + \underbrace{(m + n)}_{b}x + \underbrace{mn}_{c} = (x + m)(x + n)$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

BUILDING UNDERSTANDING

1 Find two integers that multiply to give the first number and add to give the second number.

a 18, 11 **b** 20, 12 **c** −15, 2 **d** −12, 1
e −24, −5 **f** −30, −7 **g** 10, −7 **h** 36, −15

2 A number (except zero) divided by itself always equals 1.

For example: $\frac{\cancel{a}^1}{\cancel{a}^1} = 1$, $\frac{2\cancel{(x-3)}^1}{\cancel{(x-3)}^1} = 2$, $\frac{\cancel{(a+5)}^1}{2\cancel{(a+5)}^1} = \frac{1}{2}$

Invent some algebraic fractions that are equal to:

a 1 **b** 3 **c** −5 **d** $\frac{1}{3}$

3 Simplify by cancelling common factors. For parts **f** to **g**, first factorise the numerator.

a $\frac{2x}{4}$ **b** $\frac{6a}{2a}$ **c** $\frac{3(x+1)}{9(x+1)}$ **d** $\frac{2(x-2)}{8(x-2)}$

e $\frac{8(x+4)}{12(x+4)}$ **f** $\frac{x^2+x}{x}$ **g** $\frac{x^2-2x}{x}$ **h** $\frac{x^2-3x}{2x}$

Example 8 Factorising trinomials of the form $x^2 + bx + c$

Factorise.

a $x^2 + 8x + 15$ **b** $x^2 - 5x + 6$ **c** $2x^2 - 10x - 28$ **d** $x^2 - 8x + 16$

SOLUTION

a $x^2 + 8x + 15 = (x + 3)(x + 5)$

EXPLANATION

$3 \times 5 = 15$ and $3 + 5 = 8$

Check: $(x + 3)(x + 5) = x^2 + 5x + 3x + 15$
$= x^2 + 8x + 15$

b $x^2 - 5x + 6 = (x - 3)(x - 2)$

$-3 \times (-2) = 6$ and $-3 + (-2) = -5$

Check: $(x - 3)(x - 2) = x^2 - 2x - 3x + 6$
$= x^2 - 5x + 6$

c $2x^2 - 10x - 28 = 2(x^2 - 5x - 14)$
$= 2(x - 7)(x + 2)$

First, take out the common factor of 2.

$-7 \times 2 = -14$ and $-7 + 2 = -5$

d $x^2 - 8x + 16 = (x - 4)(x - 4)$
$= (x - 4)^2$

$-4 \times (-4) = 16$ and $-4 + (-4) = -8$

$(x - 4)(x - 4) = (x - 4)^2$ is a perfect square.

Now you try

Factorise:

a $x^2 + 7x + 12$ **b** $x^2 - 10x + 24$ **c** $2x^2 - 2x - 12$ **d** $x^2 - 6x + 9$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 9 Simplifying algebraic fractions

Use factorisation to simplify these algebraic fractions.

a $\dfrac{x^2 - x - 6}{x + 2}$

b $\dfrac{x^2 - 9}{x^2 - 2x - 15} \times \dfrac{x^2 - 4x - 5}{2x - 6}$

SOLUTION

a $\dfrac{x^2 - x - 6}{x + 2} = \dfrac{(x - 3)\cancel{(x + 2)}^1}{\cancel{(x + 2)}^1}$

$= x - 3$

First, factorise $x^2 - x - 6$ and then cancel $(x + 2)$.

b $\dfrac{x^2 - 9}{x^2 - 2x - 15} \times \dfrac{x^2 - 4x - 5}{2x - 6}$

$= \dfrac{\cancel{(x + 3)}^1\cancel{(x - 3)}^1}{\cancel{(x - 5)}^1\cancel{(x + 3)}^1} \times \dfrac{\cancel{(x - 5)}^1(x + 1)}{2\cancel{(x - 3)}^1}$

$= \dfrac{x + 1}{2}$

EXPLANATION

First, factorise all expressions in the numerators and denominators. Cancel to simplify where possible.

Now you try

Use factorisation to simplify these algebraic fractions.

a $\dfrac{x^2 - 2x - 8}{x + 2}$

b $\dfrac{x^2 - 4}{x^2 + x - 2} \times \dfrac{x^2 + 3x - 4}{2x - 4}$

Exercise 5C

FLUENCY	1, 2–4(1/4)	2–5(1/3)	2–5(1/4)

1 Factorise.

Example 8a — **a** i $x^2 + 3x + 2$ ii $x^2 + 6x + 5$

Example 8b — **b** i $x^2 - 4x + 3$ ii $x^2 - 11x + 30$

Example 8c — **c** i $2x^2 - 8x - 10$ ii $3x^2 - 9x - 30$

Example 8d — **d** i $x^2 - 4x + 4$ ii $x^2 - 10x + 25$

Example 8a,b

2 Factorise these quadratic trinomials.

a $x^2 + 7x + 6$ **b** $x^2 + 5x + 6$ **c** $x^2 + 6x + 9$ **d** $x^2 + 7x + 10$

e $x^2 + 7x + 12$ **f** $x^2 + 11x + 18$ **g** $x^2 + 5x - 6$ **h** $x^2 + x - 6$

i $x^2 + 2x - 8$ **j** $x^2 + 3x - 4$ **k** $x^2 + 7x - 30$ **l** $x^2 + 9x - 22$

m $x^2 - 7x + 10$ **n** $x^2 - 6x + 8$ **o** $x^2 - 7x + 12$ **p** $x^2 - 2x + 1$

q $x^2 - 9x + 18$ **r** $x^2 - 11x + 18$ **s** $x^2 - 4x - 12$ **t** $x^2 - x - 20$

u $x^2 - 5x - 14$ **v** $x^2 - x - 12$ **w** $x^2 + 4x - 32$ **x** $x^2 - 3x - 10$

Example 8c

3 Factorise by first taking out the common factor.

a $2x^2 + 14x + 20$ **b** $3x^2 + 21x + 36$ **c** $2x^2 + 22x + 36$

d $5x^2 - 5x - 10$ **e** $4x^2 - 16x - 20$ **f** $3x^2 - 9x - 30$

g $-2x^2 - 14x - 24$ **h** $-3x^2 + 9x - 6$ **i** $-2x^2 + 10x + 28$

j $-4x^2 + 4x + 8$ **k** $-5x^2 - 20x - 15$ **l** $-7x^2 + 49x - 42$

Example 8d

4 Factorise these perfect squares.

a $x^2 - 4x + 4$ b $x^2 + 6x + 9$ c $x^2 + 12x + 36$ d $x^2 - 14x + 49$
e $x^2 - 18x + 81$ f $x^2 - 20x + 100$ g $2x^2 + 44x + 242$ h $3x^2 - 24x + 48$
i $5x^2 - 50x + 125$ j $-3x^2 + 36x - 108$ k $-2x^2 + 28x - 98$ l $-4x^2 - 72x - 324$

Example 9a

5 Use factorisation to simplify these algebraic fractions. In some cases, you may need to remove a common factor first.

a $\dfrac{x^2 - 3x - 54}{x - 9}$ b $\dfrac{x^2 + x - 12}{x + 4}$ c $\dfrac{x^2 - 6x + 9}{x - 3}$

d $\dfrac{x + 2}{x^2 + 9x + 14}$ e $\dfrac{x - 3}{x^2 - 8x + 15}$ f $\dfrac{x + 1}{x^2 - 5x - 6}$

g $\dfrac{2(x + 12)}{x^2 + 4x - 96}$ h $\dfrac{x^2 - 5x - 36}{3(x - 9)}$ i $\dfrac{x^2 - 15x + 56}{5(x - 8)}$

PROBLEM-SOLVING	6(1/2)	6–7(1/3)	6–8(1/3)

Example 9b

6 Simplify by first factorising.

a $\dfrac{x^2 - 4}{x^2 - x - 6} \times \dfrac{5x - 15}{x^2 + 4x - 12}$ b $\dfrac{x^2 + 3x + 2}{x^2 + 4x + 3} \times \dfrac{x^2 - 9}{3x + 6}$

c $\dfrac{x^2 + 2x - 3}{x^2 - 25} \times \dfrac{2x - 10}{x + 3}$ d $\dfrac{x^2 - 9}{x^2 - 5x + 6} \times \dfrac{4x - 8}{x^2 + 8x + 15}$

e $\dfrac{x^2 - 4x + 3}{x^2 + 4x - 21} \times \dfrac{4x + 4}{x^2 - 1}$ f $\dfrac{x^2 + 6x + 8}{x^2 - 4} \times \dfrac{6x - 24}{x^2 - 16}$

g $\dfrac{x^2 - x - 6}{x^2 + x - 12} \times \dfrac{x^2 + 5x + 4}{x^2 - 1}$ h $\dfrac{x^2 - 4x - 12}{x^2 - 4} \times \dfrac{x^2 - 6x + 8}{x^2 - 36}$

7 Simplify these expressions that involve surds.

a $\dfrac{x^2 - 7}{x + \sqrt{7}}$ b $\dfrac{x^2 - 10}{x - \sqrt{10}}$ c $\dfrac{x^2 - 12}{x + 2\sqrt{3}}$

d $\dfrac{\sqrt{5}x + 3}{5x^2 - 9}$ e $\dfrac{\sqrt{3}x - 4}{3x^2 - 16}$ f $\dfrac{7x^2 - 5}{\sqrt{7}x + \sqrt{5}}$

g $\dfrac{(x + 1)^2 - 2}{x + 1 + \sqrt{2}}$ h $\dfrac{(x - 3)^2 - 5}{x - 3 - \sqrt{5}}$ i $\dfrac{(x - 6)^2 - 6}{x - 6 + \sqrt{6}}$

8 Simplify using factorisation.

a $\dfrac{x^2 + 2x - 3}{x^2 - 25} \div \dfrac{3x - 3}{2x + 10}$ b $\dfrac{x^2 + 3x + 2}{x^2 + 4x + 3} \div \dfrac{4x + 8}{x^2 - 9}$ c $\dfrac{x^2 - x - 12}{x^2 - 9} \div \dfrac{x^2 - 16}{3x + 12}$

d $\dfrac{x^2 - 49}{x^2 - 3x - 28} \div \dfrac{4x + 28}{6x + 24}$ e $\dfrac{x^2 + 5x - 14}{x^2 + 2x - 3} \div \dfrac{x^2 + 9x + 14}{x^2 + x - 2}$ f $\dfrac{x^2 + 8x + 15}{x^2 + 5x - 6} \div \dfrac{x^2 + 6x + 5}{x^2 + 7x + 6}$

REASONING	9	9, 10(1/2)	10(1/2), 11, 12

9 A businessman is showing off his new formula to determine the company's profit, in millions of dollars, after t years.

$$\text{Profit} = \frac{t^2 - 49}{5t - 40} \times \frac{t^2 - 5t - 24}{2t^2 - 8t - 42}$$

Show that this is really the same as

$$\text{Profit} = \frac{t + 7}{10}.$$

ISBN 978-1-108-77290-7

10 Note that an expression with a perfect square can be simplified as shown.

$$\frac{(x+3)^2}{x+3} = \frac{(x+3)\cancel{(x+3)}^1}{\cancel{x+3}^1}$$
$$= x + 3$$

Use this idea to simplify the following.

a $\dfrac{x^2 - 6x + 9}{x - 3}$ **b** $\dfrac{x^2 + 2x + 1}{x + 1}$ **c** $\dfrac{x^2 - 16x + 64}{x - 8}$

d $\dfrac{6x - 12}{x^2 - 4x + 4}$ **e** $\dfrac{4x + 20}{x^2 + 10x + 25}$ **f** $\dfrac{x^2 - 14x + 49}{5x - 35}$

11 **a** Prove that $\dfrac{a^2 + 2ab + b^2}{a^2 + ab} \div \dfrac{a^2 - b^2}{a^2 - ab} = 1$.

b Make up your own expressions, like the one in part **a**, which equal 1. Ask a classmate to check them.

12 Simplify.

a $\dfrac{a^2 + 2ab + b^2}{a(a + b)} \div \dfrac{a^2 - b^2}{a^2 - 2ab + b^2}$ **b** $\dfrac{a^2 - 2ab + b^2}{a^2 - b^2} \div \dfrac{a^2 - b^2}{a^2 + 2ab + b^2}$

c $\dfrac{a^2 - b^2}{a^2 - 2ab + b^2} \div \dfrac{a^2 - b^2}{a^2 + 2ab + b^2}$ **d** $\dfrac{a^2 + 2ab + b^2}{a(a + b)} \div \dfrac{a(a - b)}{a^2 - 2ab + b^2}$

ENRICHMENT: Addition and subtraction with factorisation	–	–	13(1/2)

13 Factorisation can be used to help add and subtract algebraic fractions. Here is an example.

$$\begin{aligned}\frac{3}{x-2} + \frac{x}{x^2 - 6x + 8} &= \frac{3}{x-2} + \frac{x}{(x-2)(x-4)} \\ &= \frac{3(x-4)}{(x-2)(x-4)} + \frac{x}{(x-2)(x-4)} \\ &= \frac{3x - 12 + x}{(x-2)(x-4)} \\ &= \frac{4x - 12}{(x-2)(x-4)} \\ &= \frac{4(x-3)}{(x-2)(x-4)}\end{aligned}$$

Now simplify the following.

a $\dfrac{2}{x + 3} + \dfrac{x}{x^2 - x - 12}$ **b** $\dfrac{4}{x + 2} + \dfrac{3x}{x^2 - 7x - 18}$

c $\dfrac{3}{x + 4} - \dfrac{2x}{x^2 - 16}$ **d** $\dfrac{4}{x^2 - 9} - \dfrac{1}{x^2 - 8x + 15}$

e $\dfrac{x + 4}{x^2 - x - 6} - \dfrac{x - 5}{x^2 - 9x + 18}$ **f** $\dfrac{x + 3}{x^2 - 4x - 32} - \dfrac{x}{x^2 + 7x + 12}$

g $\dfrac{x + 1}{x^2 - 25} - \dfrac{x - 2}{x^2 - 6x + 5}$ **h** $\dfrac{x + 2}{x^2 - 2x + 1} - \dfrac{x + 3}{x^2 + 3x - 4}$

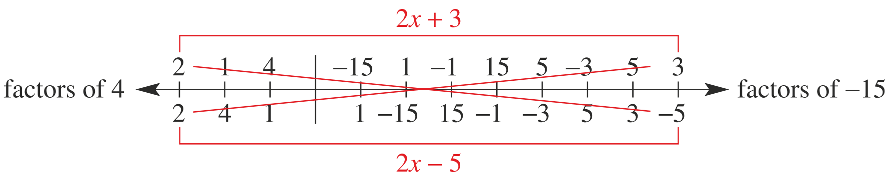
ISBN 978-1-108-77290-7

5D Factorising non-monic quadratic trinomials 10A

Learning intentions

- To understand the relationship between expansion and factorisation for binomial products
- To know and be able to apply the process for factorising non-monic quadratic trinomials

There are a number of ways of factorising non-monic quadratic trinomials of the form $ax^2 + bx + c$, where $a \neq 1$. The cross method, for example, uses lists of factors of a and c so that a correct combination can be found. For example, to factorise $4x^2 - 4x - 15$:



$2 \times (-5) + 2 \times 3 = -4$, so choose $(2x + 3)$ and $(2x - 5)$.
$\therefore\ 4x^2 - 4x - 15 = (2x + 3)(2x - 5)$

The method outlined in this section, however, uses grouping.

LESSON STARTER Does the order matter?

To factorise the non-monic quadratic $4x^2 - 4x - 15$ using grouping, we multiply a by c, which is $4 \times (-15) = -60$. Then we look for numbers that multiply to give -60 and add to give -4 (the coefficient of x).

- What are the two numbers that multiply to give -60 and add to give -4?
- Complete the following using grouping.

$$\begin{aligned} 4x^2 - 4x - 15 &= 4x^2 - 10x + 6x - 15 \qquad -10 \times 6 = -60, \qquad -10 + 6 = -4 \\ &= 2x(\_\_\_\_\_\_) + 3(\_\_\_\_\_\_) \\ &= (2x - 5)(\_\_\_\_\_\_) \end{aligned}$$

- If we changed the order of the $-10x$ and $+6x$ do you think the result would change? Copy and complete to find out.

$$\begin{aligned} 4x^2 - 4x - 15 &= 4x^2 + 6x - 10x - 15 \qquad 6 \times (-10) = -60, \qquad 6 + (-10) = -4 \\ &= 2x(\_\_\_\_\_\_) - 5(\_\_\_\_\_\_) \\ &= (\_\_\_\_\_\_)(\_\_\_\_\_\_) \end{aligned}$$

KEY IDEAS

- To factorise a **non-monic** trinomial of the form $ax^2 + bx + c$, follow these steps:
  - Find two numbers that multiply to give $a \times c$ and add to give b.
    For $15x^2 - x - 6$, $a \times c = 15 \times (-6) = -90$.
    The factors of -90 that add to -1 (b) are -10 and 9.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

- Use the two numbers shown in the example above to split bx, then factorise by grouping.

$$\begin{aligned}15x^2 - x - 6 &= 15x^2 - 10x + 9x - 6\\ &= 5x(3x - 2) + 3(3x - 2) = (3x - 2)(5x + 3)\end{aligned}$$

- There are other valid methods that can be used to factorise non-monic trinomials. The cross method is illustrated in the introduction.

BUILDING UNDERSTANDING

1 State the missing numbers in this table.

$ax^2 + bx + c$	$a \times c$	Two numbers that multiply to give $a \times c$ and add to give b
$6x^2 + 13x + 6$	36	9 and ______
$8x^2 + 18x + 4$	32	
$12x^2 + x - 6$		-8 and ______
$10x^2 - 11x - 6$		
$21x^2 - 20x + 4$		-6 and ______
$15x^2 - 13x + 2$		

2 Factorise by grouping pairs.

a $x^2 + 2x + 5x + 10$
b $x^2 - 7x - 2x + 14$
c $6x^2 - 8x + 3x - 4$
d $8x^2 - 4x + 6x - 3$
e $5x^2 + 20x - 2x - 8$
f $12x^2 - 6x - 10x + 5$

Example 10 Factorising non-monic quadratics

Factorise.

a $6x^2 + 19x + 10$
b $9x^2 + 6x - 8$

SOLUTION

a
$$\begin{aligned}6x^2 + 19x + 10 &= 6x^2 + 15x + 4x + 10\\ &= 3x(2x + 5) + 2(2x + 5)\\ &= (2x + 5)(3x + 2)\end{aligned}$$

b
$$\begin{aligned}9x^2 + 6x - 8 &= 9x^2 + 12x - 6x - 8\\ &= 3x(3x + 4) - 2(3x + 4)\\ &= (3x + 4)(3x - 2)\end{aligned}$$

EXPLANATION

a $a \times c = 6 \times 10 = 60$; choose 15 and 4 since $15 \times 4 = 60$ and $15 + 4 = 19$ (b).
Factorise by grouping.

b $a \times c = 9 \times (-8) = -72$; choose 12 and -6 since $12 \times (-6) = -72$ and $12 + (-6) = 6$ (b).

Now you try

Factorise.

a $6x^2 + 11x + 3$
b $8x^2 + 10x - 3$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 11 Simplifying algebraic fractions involving quadratic expressions

Simplify $\dfrac{4x^2 - 9}{10x^2 + 13x - 3} \times \dfrac{25x^2 - 10x + 1}{10x^2 - 17x + 3}$.

SOLUTION	EXPLANATION
$\dfrac{4x^2 - 9}{10x^2 + 13x - 3} \times \dfrac{25x^2 - 10x + 1}{10x^2 - 17x + 3}$	First, use the range of factorising techniques to factorise all quadratics.
$= \dfrac{\cancel{(2x+3)}^1\cancel{(2x-3)}^1}{\cancel{(2x+3)}^1\cancel{(5x-1)}^1} \times \dfrac{\cancel{(5x-1)}^1\cancel{(5x-1)}^1}{\cancel{(2x-3)}^1\cancel{(5x-1)}^1}$	Cancel to simplify.
$= 1$	

Now you try

Simplify $\dfrac{9x^2 - 4}{12x^2 - 17x + 6} \times \dfrac{16x^2 - 24x + 9}{12x^2 - x - 6}$.

Exercise 5D

FLUENCY	1, 2(1/2)	2–3(1/3)	2–3(1/4)

1 Factorise.

Example 10a
a **i** $8x^2 + 14x + 3$ **ii** $10x^2 + 19x + 6$

Example 10b
b **i** $6x^2 + 13x - 5$ **ii** $8x^2 + 2x - 3$

Example 10
2 Factorise the following.

a $3x^2 + 10x + 3$ **b** $2x^2 + 3x + 1$ **c** $3x^2 + 8x + 4$ **d** $3x^2 - 5x + 2$
e $2x^2 - 11x + 5$ **f** $5x^2 + 2x - 3$ **g** $3x^2 - 11x - 4$ **h** $3x^2 - 2x - 1$
i $7x^2 + 2x - 5$ **j** $2x^2 - 9x + 7$ **k** $3x^2 + 2x - 8$ **l** $2x^2 + 5x - 12$
m $2x^2 - 9x - 5$ **n** $13x^2 - 7x - 6$ **o** $5x^2 - 22x + 8$ **p** $8x^2 - 14x + 5$
q $6x^2 + x - 12$ **r** $10x^2 + 11x - 6$ **s** $6x^2 + 13x + 6$ **t** $4x^2 - 5x + 1$
u $8x^2 - 14x + 5$ **v** $8x^2 - 26x + 15$ **w** $6x^2 - 13x + 6$ **x** $9x^2 + 9x - 10$

3 Factorise the following.

a $18x^2 + 27x + 10$ **b** $20x^2 + 39x + 18$ **c** $21x^2 + 22x - 8$
d $30x^2 + 13x - 10$ **e** $40x^2 - x - 6$ **f** $28x^2 - 13x - 6$
g $24x^2 - 38x + 15$ **h** $45x^2 - 46x + 8$ **i** $25x^2 - 50x + 16$

PROBLEM-SOLVING	4(1/2), 6	4–5(1/3), 6	4–5(1/3), 6

4 Factorise by first taking out the common factor.

a $6x^2 + 38x + 40$ **b** $6x^2 - 15x - 36$ **c** $48x^2 - 18x - 3$
d $32x^2 - 88x + 60$ **e** $16x^2 - 24x + 8$ **f** $90x^2 + 90x - 100$
g $-50x^2 - 115x - 60$ **h** $12x^2 - 36x + 27$ **i** $20x^2 - 25x + 5$

5 Simplify by first factorising.

a $\frac{6x^2 - x - 35}{3x + 7}$ b $\frac{8x^2 + 10x - 3}{2x + 3}$ c $\frac{9x^2 - 21x + 10}{3x - 5}$ d $\frac{10x - 2}{15x^2 + 7x - 2}$

e $\frac{4x + 6}{14x^2 + 17x - 6}$ f $\frac{20x - 12}{10x^2 - 21x + 9}$ g $\frac{2x^2 + 11x + 12}{6x^2 + 11x + 3}$ h $\frac{12x^2 - x - 1}{8x^2 + 14x + 3}$

i $\frac{10x^2 + 3x - 4}{14x^2 - 11x + 2}$ j $\frac{9x^2 - 4}{15x^2 + 4x - 4}$ k $\frac{14x^2 + 19x - 3}{49x^2 - 1}$ l $\frac{8x^2 - 2x - 15}{16x^2 - 25}$

6 A cable is suspended across a farm channel. The height (h), in metres, of the cable above the water surface is modelled by the equation $h = 3x^2 - 19x + 20$, where x metres is the distance from one side of the channel.

a Factorise the right-hand side of the equation.
b Determine the height of the cable when $x = 3$. Interpret this result.
c Determine where the cable is at the level of the water surface.

REASONING	7(1/2)	7–8(1/2)	7–8(1/3), 9

Example 11

7 Combine all your knowledge of factorising to simplify the following.

a $\frac{9x^2 - 16}{x^2 - 6x + 9} \times \frac{x^2 + x - 12}{3x^2 + 8x - 16}$ b $\frac{4x^2 - 1}{6x^2 - x - 2} \times \frac{9x^2 - 4}{8x - 4}$

c $\frac{1 - x^2}{15x + 9} \times \frac{25x^2 + 30x + 9}{5x^2 + 8x + 3}$ d $\frac{20x^2 + 21x - 5}{16x^2 + 8x - 15} \times \frac{16x^2 - 24x + 9}{25x^2 - 1}$

e $\frac{100x^2 - 25}{2x^2 - 9x - 5} \div \frac{2x^2 - 7x + 3}{5x^2 - 40x + 75}$ f $\frac{3x^2 - 12}{30x + 15} \div \frac{2x^2 - 3x - 2}{4x^2 + 4x + 1}$

g $\frac{9x^2 - 6x + 1}{6x^2 - 11x + 3} \div \frac{9x^2 - 1}{6x^2 - 7x - 3}$ h $\frac{16x^2 - 25}{4x^2 - 7x - 15} \div \frac{4x^2 - 17x + 15}{16x^2 - 40x + 25}$

8 Find a method to show how $-12x^2 - 5x + 3$ factorises to $(1 - 3x)(4x + 3)$. Then factorise the following.

a $-8x^2 + 2x + 15$ b $-6x^2 + 11x + 10$ c $-12x^2 + 13x + 4$
d $-8x^2 + 18x - 9$ e $-14x^2 + 39x - 10$ f $-15x^2 - x + 6$

9 Make up your own complex expression like those in Question 7, which simplifies to 1. Check your expression with your teacher or a classmate.

ENRICHMENT: Non-monics with addition and subtraction	–	–	10(1/2)

10 Factorise the quadratics in the expressions and then simplify using a common denominator.

a $\frac{2}{2x - 3} + \frac{x}{8x^2 - 10x - 3}$ b $\frac{3}{3x - 1} - \frac{x}{6x^2 + 13x - 5}$

c $\frac{4x}{2x - 5} + \frac{x}{8x^2 - 18x - 5}$ d $\frac{4x}{12x^2 - 11x + 2} - \frac{3x}{3x - 2}$

e $\frac{2}{4x^2 - 1} + \frac{1}{6x^2 - x - 2}$ f $\frac{2}{9x^2 - 25} - \frac{3}{9x^2 + 9x - 10}$

g $\frac{4}{8x^2 - 18x - 5} - \frac{2}{12x^2 - 5x - 2}$ h $\frac{1}{10x^2 - 19x + 6} + \frac{2}{4x^2 + 8x - 21}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

5E Factorising by completing the square

Learning intentions

- To know the expanded form of a perfect square
- To be able to carry out the process of completing the square
- To know how to factorise by first completing the square
- To understand that not all quadratic expressions can be factorised and to be able to identify those that can't

Consider the quadratic expression $x^2 + 6x + 1$. We cannot factorise this using the methods we have established in the previous exercises because there are no factors of 1 that add to 6.

We can, however, use our knowledge of perfect squares and the difference of perfect squares to help find factors using surds.

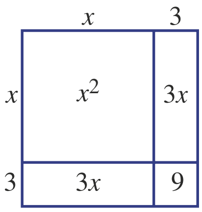
The statistical analysis of agricultural research data has found that quadratic equations model harvest yields (kg/ha) versus the quantity of nitrogen fertiliser (kg/ha) used. The CSIRO provides Australian farmers with numerous mathematical models.

LESSON STARTER Make a perfect square

This diagram is a square. Its sides are $x + 3$ and its area is given by $x^2 + 6x + 9 = (x + 3)^2$.

Use a similar diagram to help make a perfect square for the following and determine the missing number for each.

- $x^2 + 8x + ?$
- $x^2 + 12x + ?$



Can you describe a method for finding the missing number without drawing a diagram?

KEY IDEAS

- Recall for a perfect square $(x + a)^2 = x^2 + 2ax + a^2$ and $(x - a)^2 = x^2 - 2ax + a^2$.
- To **complete the square** for $x^2 + bx$, add $\left(\frac{b}{2}\right)^2$.
  - $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$
- To factorise by completing the square:
  - Add $\left(\frac{b}{2}\right)^2$ and balance by subtracting $\left(\frac{b}{2}\right)^2$.
  - Factorise the perfect square and simplify.
  - Factorise using DOPS: $a^2 - b^2 = (a + b)(a - b)$; surds can be used.

$$\begin{aligned} x^2 + 6x + 1 &= x^2 + 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 1 \\ &= \left(x + \frac{6}{2}\right)^2 - 8 \\ &= (x + 3)^2 - (\sqrt{8})^2 \\ &= (x + 3 + \sqrt{8})(x + 3 - \sqrt{8}) \\ &= (x + 3 + 2\sqrt{2})(x + 3 - 2\sqrt{2}) \end{aligned}$$

- Not all quadratic expressions factorise. This will be seen when you end up with expressions such as $(x + 3)^2 + 6$, which is *not* a difference of two perfect squares.

ISBN 978-1-108-77290-7 Cambridge University Press
 Updated June 2022

BUILDING UNDERSTANDING

1. These expressions are of the form $x^2 + bx$. Evaluate $\left(\frac{b}{2}\right)^2$ for each one.
   - a $x^2 + 6x$
   - b $x^2 + 2x$
   - c $x^2 - 4x$
   - d $x^2 - 8x$
   - e $x^2 + 5x$
   - f $x^2 - 9x$
2. Factorise these perfect squares.
   - a $x^2 + 4x + 4$
   - b $x^2 + 8x + 16$
   - c $x^2 + 10x + 25$
   - d $x^2 - 12x + 36$
   - e $x^2 - 6x + 9$
   - f $x^2 - 18x + 81$
3. Factorise using surds. Recall that $a^2 - b^2 = (a + b)(a - b)$.
   - a $(x + 1)^2 - 5$
   - b $(x + 4)^2 - 10$
   - c $(x - 3)^2 - 11$

Example 12 Completing the square

Decide what number must be added to these expressions to complete the square. Then factorise the resulting perfect square.

a $x^2 + 10x$ b $x^2 - 7x$

SOLUTION	EXPLANATION
a $\left(\frac{10}{2}\right)^2 = 5^2 = 25$	For $x^2 + bx$, add $\left(\frac{b}{2}\right)^2$.
	Here $b = 10$, and evaluate $\left(\frac{b}{2}\right)^2$.
$x^2 + 10x + 25 = (x + 5)^2$	$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$
b $\left(\frac{-7}{2}\right)^2 = \frac{49}{4}$	In $x^2 - 7x$, $b = -7$ and evaluate $\left(\frac{b}{2}\right)^2$.
$x^2 - 7x + \frac{49}{4} = \left(x - \frac{7}{2}\right)^2$	Factorise the perfect square.

Now you try

Decide what number must be added to these expressions to complete the square. Then factorise the resulting perfect square.

a $x^2 + 12x$ b $x^2 - 9x$

Example 13 Factorising by completing the square

Factorise the following by completing the square if possible.

a $x^2 + 8x - 3$ b $x^2 - 2x + 8$

Continued on next page

ISBN 978-1-108-77290-7

SOLUTION

a $x^2 + 8x - 3 = \left(x^2 + 8x + \left(\frac{8}{2}\right)^2\right) - \left(\frac{8}{2}\right)^2 - 3$

$= \left(x + \frac{8}{2}\right)^2 - 16 - 3$

$= (x + 4)^2 - 19$

$= (x + 4)^2 - (\sqrt{19})^2$

$= (x + 4 - \sqrt{19})(x + 4 + \sqrt{19})$

EXPLANATION

Add $\left(\frac{b}{2}\right)^2$ to complete the square and balance by subtracting $\left(\frac{b}{2}\right)^2$ also.

Factorise the resulting perfect square and simplify.

Express 19 as $(\sqrt{19})^2$ to set up a DOPS. Apply $a^2 - b^2 = (a + b)(a - b)$ using surds.

b $x^2 - 2x + 8 = \left(x^2 - 2x + \left(\frac{2}{2}\right)^2\right) - \left(\frac{2}{2}\right)^2 + 8$

$= \left(x - \frac{2}{2}\right)^2 + 7$

$= (x - 1)^2 + 7$

$\therefore x^2 - 2x + 8$ cannot be factorised.

Add $\left(\frac{2}{2}\right)^2 = (1)^2$ to complete the square and balance by subtracting $(1)^2$ also.

Factorise the perfect square and simplify.

$(x - 1)^2 + 7$ is not a *difference* of perfect squares.

Now you try

Factorise the following by completing the square if possible.

a $x^2 + 6x - 1$

b $x^2 - 4x + 7$

Example 14 Factorising with fractions

Factorise $x^2 + 3x + \frac{1}{2}$.

SOLUTION

$x^2 + 3x + \frac{1}{2} = \left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) - \left(\frac{3}{2}\right)^2 + \frac{1}{2}$

$= \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{1}{2}$

$= \left(x + \frac{3}{2}\right)^2 - \frac{7}{4}$

$= \left(x + \frac{3}{2}\right)^2 - \left(\sqrt{\frac{7}{4}}\right)^2$

$= \left(x + \frac{3}{2} - \frac{\sqrt{7}}{2}\right)\left(x + \frac{3}{2} + \frac{\sqrt{7}}{2}\right)$

$= \left(x + \frac{3 - \sqrt{7}}{2}\right)\left(x + \frac{3 + \sqrt{7}}{2}\right)$

EXPLANATION

Add $\left(\frac{3}{2}\right)^2$ to complete the square and balance by subtracting $\left(\frac{3}{2}\right)^2$. Leave in fraction form.

Factorise the perfect square and simplify.

$-\frac{9}{4} + \frac{1}{2} = -\frac{9}{4} + \frac{2}{4} = -\frac{7}{4}$

Recall that $\sqrt{\frac{7}{4}} = \frac{\sqrt{7}}{\sqrt{4}} = \frac{\sqrt{7}}{2}$ and use DOPS.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Now you try

Factorise $x^2 + 5x + \frac{1}{2}$.

Exercise 5E

FLUENCY	1, 2–3(1/2)	2–4(1/2)	2–4(1/3)

1 Decide what number must be added to these expressions to complete the square. Then factorise the resulting perfect square.

Example 12a **a** i $x^2 + 8x$ ii $x^2 + 14x$

Example 12b **b** i $x^2 - 5x$ ii $x^2 - 11x$

Example 12 **2** Decide what number must be added to these expressions to complete the square. Then factorise the resulting perfect square.

a $x^2 + 6x$ **b** $x^2 + 12x$ **c** $x^2 + 4x$ **d** $x^2 + 8x$
e $x^2 - 10x$ **f** $x^2 - 2x$ **g** $x^2 - 8x$ **h** $x^2 - 12x$
i $x^2 + 5x$ **j** $x^2 + 9x$ **k** $x^2 + 7x$ **l** $x^2 + 11x$
m $x^2 - 3x$ **n** $x^2 - 7x$ **o** $x^2 - x$ **p** $x^2 - 9x$

Example 13a **3** Factorise by completing the square.

a $x^2 + 4x + 1$ **b** $x^2 + 6x + 2$ **c** $x^2 + 2x - 4$
d $x^2 + 10x - 4$ **e** $x^2 - 8x + 13$ **f** $x^2 - 12x + 10$
g $x^2 - 4x - 3$ **h** $x^2 - 8x - 5$ **i** $x^2 + 14x + 6$

Example 13b **4** Factorise, if possible.

a $x^2 + 6x + 11$ **b** $x^2 + 4x + 7$ **c** $x^2 + 8x + 1$
d $x^2 + 4x + 2$ **e** $x^2 + 10x + 3$ **f** $x^2 + 4x - 6$
g $x^2 - 10x + 30$ **h** $x^2 - 6x + 6$ **i** $x^2 - 12x + 2$
j $x^2 - 2x + 2$ **k** $x^2 - 8x - 1$ **l** $x^2 - 4x + 6$

PROBLEM-SOLVING	5(1/2)	5–6(1/2)	5–7(1/3)

Example 14 **5** Factorise the following.

a $x^2 + 3x + 1$ **b** $x^2 + 7x + 2$ **c** $x^2 + 5x - 2$ **d** $x^2 + 9x - 3$
e $x^2 - 3x + \frac{1}{2}$ **f** $x^2 - 5x + \frac{1}{2}$ **g** $x^2 - 5x - \frac{3}{2}$ **h** $x^2 - 9x - \frac{5}{2}$

6 Factorise by first taking out the common factor.

a $2x^2 + 12x + 8$ **b** $3x^2 + 12x - 3$ **c** $4x^2 - 8x - 16$
d $3x^2 - 24x + 6$ **e** $-2x^2 - 4x + 10$ **f** $-3x^2 - 30x - 3$
g $-4x^2 - 16x + 12$ **h** $-2x^2 + 16x + 4$ **i** $-3x^2 + 24x - 15$

7 Factorise the following.

a $3x^2 + 9x + 3$ **b** $5x^2 + 15x - 35$ **c** $2x^2 - 10x + 4$
d $4x^2 - 28x + 12$ **e** $-3x^2 - 21x + 6$ **f** $-2x^2 - 14x + 8$
g $-4x^2 + 12x + 20$ **h** $-3x^2 + 9x + 6$ **i** $-2x^2 + 10x + 8$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

REASONING	8	8	8, 9

8 A student factorises $x^2 - 2x - 24$ by completing the square.

a Show the student's working to obtain the factorised form of $x^2 - 2x - 24$.

b Now that you have seen the answer from part a, what would you suggest is a better way to factorise $x^2 - 2x - 24$?

9 a Explain why $x^2 + 9$ cannot be factorised using real numbers.

b Decide whether the following can or cannot be factorised.

i $x^2 - 25$
ii $x^2 - 10$
iii $x^2 + 6$
iv $x^2 + 11$
v $(x + 1)^2 + 4$
vi $(x - 2)^2 - 8$
vii $(x + 3)^2 - 15$
viii $(2x - 1)^2 + 1$

c For what values of m can the following be factorised, using real numbers?

i $x^2 + 4x + m$
ii $x^2 - 6x + m$
iii $x^2 - 10x + m$

ENRICHMENT: Non-monic quadratics and completing the square	–	–	10(½)

10 A non-monic quadratic such as $2x^2 - 5x + 1$ can be factorised in the following way.

$$\begin{aligned}2x^2 - 5x + 1 &= 2\left(x^2 - \frac{5}{2}x + \frac{1}{2}\right)\\ &= 2\left(x^2 - \frac{5}{2}x + \left(\frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2 + \frac{1}{2}\right) \qquad \text{Note: } \frac{5}{2} \div 2 = \frac{5}{4}\\ &= 2\left(\left(x - \frac{5}{4}\right)^2 - \frac{25}{16} + \frac{8}{16}\right)\\ &= 2\left(\left(x - \frac{5}{4}\right)^2 - \frac{17}{16}\right)\\ &= 2\left(x - \frac{5}{4} + \frac{\sqrt{17}}{4}\right)\left(x - \frac{5}{4} - \frac{\sqrt{17}}{4}\right)\end{aligned}$$

Factorise these using a similar technique.

a $2x^2 + 5x - 12$
b $3x^2 + 4x - 3$
c $4x^2 - 7x - 16$
d $3x^2 - 2x + 6$
e $-2x^2 - 3x + 4$
f $-3x^2 - 7x - 3$
g $-4x^2 + 11x - 24$
h $-2x^2 + 3x + 4$
i $2x^2 + 5x - 7$
j $3x^2 + 4x - 5$
k $-2x^2 - 3x + 5$
l $-3x^2 - 7x - 4$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Progress quiz

5A

1 Expand brackets and simplify where possible.

a $-\frac{2x}{3}(12x - 5)$
b $a(3a - 2) - a(5 - a)$
c $(m + 2)(m + 5)$
d $(k - 3)^2$
e $(3m - 2)(3m + 2)$
f $(4h + 7)(2h - 5)$
g $5(x - 4)(x - 3)$
h $(p + 5)(p + 4) - (p - 2)(p - 8)$

5B

2 Factorise the following.

a $4a - 20$
b $-12m^2 + 18m$
c $4(x + 5) - x(x + 5)$
d $a^2 - 81$
e $16a^2 - 121b^2$
f $5m^2 - 125$
g $(k + 2)^2 - 49$
h $(x - 1)^2 - 4$
i $x^2 - 20$ (use surds)
j $(h + 3)^2 - 7$ (use surds)
k $x^2 + 5x + ax + 5a$
l $4x^2 - 8mx - 5x + 10m$

5C

3 Factorise.

a $x^2 + x - 20$
b $a^2 - 10a + 21$
c $3k^2 - 21k - 54$
d $m^2 - 12m + 36$

5C

4 Use factorisation to simplify these algebraic fractions.

a $\frac{x^2 + 2x - 15}{x + 5}$

b $\frac{x^2 - 25}{x^2 - 9x + 20} \times \frac{x^2 + 3x - 28}{2x + 14}$

5E

5 Complete the square and factorise, if possible.

a $x^2 + 8x + 3$
b $x^2 - 12x + 26$
c $x^2 + 14x + 50$
d $x^2 + 5x - \frac{1}{2}$

5D 10A

6 Factorise.

a $6a^2 + 19a + 10$
b $8m^2 - 6m - 9$
c $15x^2 - 22x + 8$
d $6k^2 - 11k - 35$

5D 10A

7 Simplify $\frac{9x^2 - 49}{3x^2 - 4x - 7} \times \frac{2x^2 + 7x + 5}{6x^2 + 5x - 21}$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

5F Solving quadratic equations using factorisation

Learning intentions

- To be able to recognise a quadratic equation
- To understand that for the product of two or more numbers to be zero, then one or both of the numbers must be zero
- To know how to rearrange a quadratic equation equal to zero
- To be able to apply the steps required for solving a quadratic equation using the Null Factor Law
- To understand that a quadratic equation can have 0, 1 or 2 solutions

The result of multiplying a number by zero is zero. Consequently, if an expression equals zero then at least one of its factors must be zero. This is called the Null Factor Law and it provides us with an important method that can be utilised to solve a range of mathematical problems involving quadratic equations.

Galileo (17th century) discovered that the path of a thrown or launched object under the influence of gravity follows a precise mathematical rule, the quadratic equation. The flight time, maximum height and range of projectiles could now be calculated.

LESSON STARTER Does factorisation beat trial and error?

Set up two teams.

Team A: Trial and error

Team B: Factorisation

Instructions:

- Team A must try to find the two solutions of $3x^2 - x - 2 = 0$ by guessing and checking values for x that make the equation true.
- Team B must solve the same equation $3x^2 - x - 2 = 0$ by first factorising the left-hand side.

Which team was the first to find the two solutions for x? Discuss the methods used.

KEY IDEAS

- The **Null Factor Law** states that if the product of two numbers is zero, then either or both of the two numbers is zero.
  - If $a \times b = 0$, then either $a = 0$ or $b = 0$.
    For example, if $x(x - 3) = 0$, then either $x = 0$ or $x - 3 = 0$ (i.e. $x = 0$ or $x = 3$).
- To solve a quadratic equation, write it in standard form (i.e. $ax^2 + bx + c = 0$) and factorise. Then use the Null Factor Law.
  - If the coefficients of all the terms have a common factor, then first divide by that common factor.

BUILDING UNDERSTANDING

1 State the solutions to these equations, which are already in factorised form.

a $x(x+1)=0$ **b** $2x(x-4)=0$ **c** $(x-3)(x+2)=0$

d $(x+\sqrt{3})(x-\sqrt{3})=0$ **e** $(2x-1)(3x+7)=0$ **f** $(8x+3)(4x+3)=0$

2 Rearrange and state in standard form $ax^2+bx+c=0$ with $a>0$. Do not solve.

a $x^2+2x=3$ **b** $x^2-5x=-6$ **c** $4x^2=3-4x$

d $2x(x-3)=5$ **e** $x^2=4(x-3)$ **f** $-4=x(3x+2)$

3 How many different solutions for x will these equations have?

a $(x-2)(x-1)=0$ **b** $(x+1)(x+1)=0$ **c** $(x+\sqrt{2})(x-\sqrt{2})=0$

d $(x+8)(x-\sqrt{5})=0$ **e** $(x+2)^2=0$ **f** $3(2x+1)^2=0$

Example 15 Solving quadratic equations using the Null Factor Law

Solve the following quadratic equations.

a $x^2-2x=0$ **b** $x^2-15=0$ **c** $2x^2=50$

SOLUTION

a
$$x^2-2x=0$$
$$x(x-2)=0$$
$$\therefore x=0 \text{ or } x-2=0$$
$$\therefore x=0 \text{ or } x=2$$

EXPLANATION

Factorise by taking out the common factor x. Apply the Null Factor Law: if $a\times b=0$, then $a=0$ or $b=0$.

Solve for x.

Check your solutions by substituting back into the equation.

b
$$x^2-15=0$$
$$(x+\sqrt{15})(x-\sqrt{15})=0$$
$$\therefore x+\sqrt{15}=0 \text{ or } x-\sqrt{15}=0$$
$$\therefore x=-\sqrt{15} \text{ or } x=\sqrt{15}$$

Factorise $a^2-b^2=(a-b)(a+b)$ using surds.

Alternatively, add 15 to both sides to give $x^2=15$, then take the positive and negative square root.

So $x=\pm\sqrt{15}$.

c
$$2x^2=50$$
$$2x^2-50=0$$
$$2(x^2-25)=0$$
$$2(x+5)(x-5)=0$$
$$\therefore x+5=0 \text{ or } x-5=0$$
$$\therefore x=-5 \text{ or } x=5$$

First, write in standard form (i.e. $ax^2+bx+c=0$).

Take out the common factor of 2 and then factorise using $a^2-b^2=(a+b)(a-b)$.

Alternatively, divide first by 2 to give $x^2=25$ and $x=\pm5$.

Now you try

Solve the following quadratic equations.

a $x^2-3x=0$ **b** $x^2-11=0$ **c** $3x^2=27$

ISBN 978-1-108-77290-7

Example 16 Solving $ax^2 + bx + c = 0$

Solve the following quadratic equations.

a $x^2 - 5x + 6 = 0$ **b** $x^2 + 2x + 1 = 0$ 10A **c** $10x^2 - 13x - 3 = 0$

SOLUTION

a
$$x^2 - 5x + 6 = 0$$
$$(x-3)(x-2) = 0$$
$$\therefore x - 3 = 0 \text{ or } x - 2 = 0$$
$$\therefore x = 3 \text{ or } x = 2$$

EXPLANATION

Factorise by finding two numbers that multiply to 6 and add to -5: $-3 \times (-2) = 6$ and $-3 + (-2) = -5$. Apply the Null Factor Law and solve for x.

b
$$x^2 + 2x + 1 = 0$$
$$(x+1)(x+1) = 0$$
$$(x+1)^2 = 0$$
$$\therefore x + 1 = 0$$
$$\therefore x = -1$$

$1 \times 1 = 1$ and $1 + 1 = 2$
$(x+1)(x+1) = (x+1)^2$ is a perfect square.
This gives one solution for x.

c
$$10x^2 - 13x - 3 = 0$$
$$10x^2 - 15x + 2x - 3 = 0$$
$$5x(2x-3) + (2x-3) = 0$$
$$(2x-3)(5x+1) = 0$$
$$\therefore 2x - 3 = 0 \text{ or } 5x + 1 = 0$$
$$\therefore 2x = 3 \text{ or } 5x = -1$$
$$\therefore x = \frac{3}{2} \text{ or } x = -\frac{1}{5}$$

First, factorise using grouping or another method.
$10 \times (-3) = -30$, $-15 \times 2 = -30$ and $-15 + 2 = -13$.

Solve using the Null Factor Law.

Check your solutions by substitution.

Now you try

Solve the following quadratic equations.

a $x^2 - x - 12 = 0$ **b** $x^2 + 6x + 9 = 0$ **c** $6x^2 + x - 2 = 0$

Example 17 Solving disguised quadratics

Solve the following by first writing in the form $ax^2 + bx + c = 0$.

a $x^2 = 4(x + 15)$ **b** $\frac{x+6}{x} = x$

SOLUTION

a
$$x^2 = 4(x+15)$$
$$x^2 = 4x + 60$$
$$x^2 - 4x - 60 = 0$$
$$(x-10)(x+6) = 0$$
$$\therefore x - 10 = 0 \text{ or } x + 6 = 0$$
$$\therefore x = 10 \text{ or } x = -6$$

EXPLANATION

First expand and then write in standard form by subtracting $4x$ and 60 from both sides.
Factorise and apply the Null Factor Law:
$-10 \times 6 = -60$ and $-10 + 6 = -4$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b $\dfrac{x + 6}{x} = x$

First multiply both sides by x and then write in standard form.

$x + 6 = x^2$

$0 = x^2 - x - 6$

$0 = (x - 3)(x + 2)$

Factorise and solve using the Null Factor Law.

$\therefore x - 3 = 0$ or $x + 2 = 0$

$\therefore x = 3$ or $x = -2$

Check your solutions.

Now you try

Solve the following by first writing in the form $ax^2 + bx + c = 0$.

a $x^2 = 2(x + 24)$ **b** $\dfrac{x + 20}{x} = x$

Exercise 5F

FLUENCY	1, 2–3(1/2)	2–4(1/2)	2–4(1/3)

1 Solve the following quadratic equations.

Example 15a **a** **i** $x^2 - 5x = 0$ **ii** $x^2 - 12x = 0$

Example 15b **b** **i** $x^2 - 13 = 0$ **ii** $x^2 - 19 = 0$

Example 15c **c** **i** $2x^2 = 18$ **ii** $4x^2 = 64$

Example 15

2 Solve the following quadratic equations.

a $x^2 - 4x = 0$ **b** $x^2 - 3x = 0$ **c** $x^2 + 2x = 0$

d $3x^2 - 12x = 0$ **e** $2x^2 - 10x = 0$ **f** $4x^2 + 8x = 0$

g $x^2 - 7 = 0$ **h** $x^2 - 11 = 0$ **i** $3x^2 - 15 = 0$

j $x^2 = 2x$ **k** $x^2 = -5x$ **l** $7x^2 = -x$

m $5x^2 = 20$ **n** $3x^2 = 27$ **o** $2x^2 = 72$

Example 16a,b

3 Solve the following quadratic equations.

a $x^2 + 3x + 2 = 0$ **b** $x^2 + 5x + 6 = 0$ **c** $x^2 - 6x + 8 = 0$

d $x^2 - 7x + 10 = 0$ **e** $x^2 + 4x - 12 = 0$ **f** $x^2 + 2x - 15 = 0$

g $x^2 - x - 20 = 0$ **h** $x^2 - 5x - 24 = 0$ **i** $x^2 - 12x + 32 = 0$

j $x^2 + 4x + 4 = 0$ **k** $x^2 + 10x + 25 = 0$ **l** $x^2 - 8x + 16 = 0$

m $x^2 - 14x + 49 = 0$ **n** $x^2 - 24x + 144 = 0$ **o** $x^2 + 18x + 81 = 0$

Example 16c

10A

4 Solve the following quadratic equations.

a $2x^2 + 11x + 12 = 0$ **b** $4x^2 + 16x + 7 = 0$ **c** $2x^2 - 17x + 35 = 0$

d $2x^2 - 23x + 11 = 0$ **e** $3x^2 - 4x - 15 = 0$ **f** $5x^2 - 7x - 6 = 0$

g $6x^2 + 7x - 20 = 0$ **h** $7x^2 + 25x - 12 = 0$ **i** $20x^2 - 33x + 10 = 0$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

PROBLEM-SOLVING	6(1/2)	5–6(1/2)	5–7(1/3)

5 Solve by first taking out a common factor.

a $2x^2 + 16x + 24 = 0$
b $2x^2 - 20x - 22 = 0$
c $3x^2 - 18x + 27 = 0$
d $5x^2 - 20x + 20 = 0$
10A e $-8x^2 - 4x + 24 = 0$
10A f $18x^2 - 57x + 30 = 0$

Example 17a

6 Solve the following by first writing in the form $ax^2 + bx + c = 0$.

a $x^2 = 2(x + 12)$
b $x^2 = 4(x + 8)$
c $x^2 = 3(2x - 3)$
d $x^2 + 7x = -10$
e $x^2 - 8x = -15$
f $x(x + 4) = 4x + 9$
g $2x - 16 = x(2 - x)$
h $x^2 + 12x + 10 = 2x + 1$
i $x^2 + x - 9 = 5x - 4$
j $x^2 - 5x = -15x - 25$
k $x^2 - 14x = 2x - 64$
l $x(x + 4) = 4(x + 16)$
m $2x(x - 2) = 6$
10A n $3x(x + 6) = 4(x - 2)$
10A o $4x(x + 5) = 6x - 4x^2 - 3$

Example 17b

7 Solve the following by first writing in the form $ax^2 + bx + c = 0$.

a $\dfrac{5x + 84}{x} = x$
b $\dfrac{9x + 70}{x} = x$
c $\dfrac{18 - 7x}{x} = x$
10A d $\dfrac{20 - 3x}{x} = 2x$
10A e $\dfrac{6x + 8}{5x} = x$
10A f $\dfrac{7x + 10}{2x} = 3x$
g $\dfrac{3}{x} = x + 2$
10A h $\dfrac{1}{x} = 3 - 2x$
i $\dfrac{4}{x - 2} = x + 1$

REASONING	8	8, 9	9, 10

8 a Write down the solutions to the following equations.
i $2(x - 1)(x + 2) = 0$
ii $(x - 1)(x + 2) = 0$

b What difference has the common factor of 2 made to the solutions in the first equation?

c Explain why $x^2 - 5x - 6 = 0$ and $3x^2 - 15x - 18 = 0$ have the same solutions.

9 Explain why $x^2 + 16x + 64 = 0$ has only one solution.

10 When solving $x^2 - 2x - 8 = 7$ a student writes the following.

$x^2 - 2x - 8 = 7$
$(x - 4)(x + 2) = 7$
$x - 4 = 7$ or $x + 2 = 7$
$x = 11$ or $x = 5$

Discuss the problem with this solution and then write a correct solution.

ENRICHMENT: More algebraic fractions with quadratics	–	–	11(1/2)

11 Solve these equations by first multiplying by an appropriate expression.

a $x + 3 = -\dfrac{2}{x}$
b $-\dfrac{1}{x} = x - 2$
c $-\dfrac{5}{x} = 2x - 11$
d $\dfrac{x^2 - 48}{x} = 2$
e $\dfrac{x^2 + 12}{x} = -8$
f $\dfrac{2x^2 - 12}{x} = -5$
g $\dfrac{x - 5}{4} = \dfrac{6}{x}$
h $\dfrac{x - 2}{3} = \dfrac{5}{x}$
i $\dfrac{x - 4}{2} = -\dfrac{2}{x}$
j $\dfrac{x + 4}{2} - \dfrac{3}{x - 3} = 1$
k $\dfrac{x}{x - 2} - \dfrac{x + 1}{x + 4} = 1$
l $\dfrac{1}{x - 1} - \dfrac{1}{x + 3} = \dfrac{1}{3}$

ISBN 978-1-108-77290-7

Using calculators to solve quadratic equations

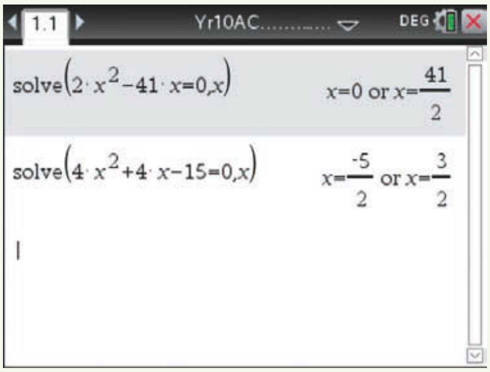
1 Solve:

a $2x^2 - 41x = 0$ **b** $4x^2 + 4x - 15 = 0$

2 Solve $ax^2 + bx + c = 0$.

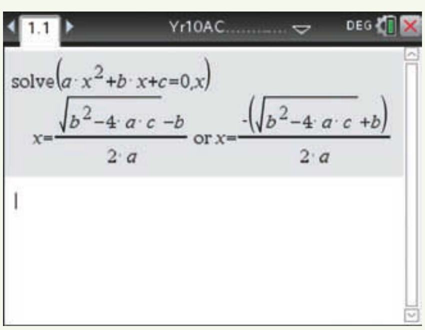
Using the TI-Nspire:

1 In a **Calculator** page use menu **>Algebra>Solve** and type as shown ending with:, x.



Note: if your answers are decimal then change the **Calculation Mode** to **Auto** in **Settings** on the Home screen.

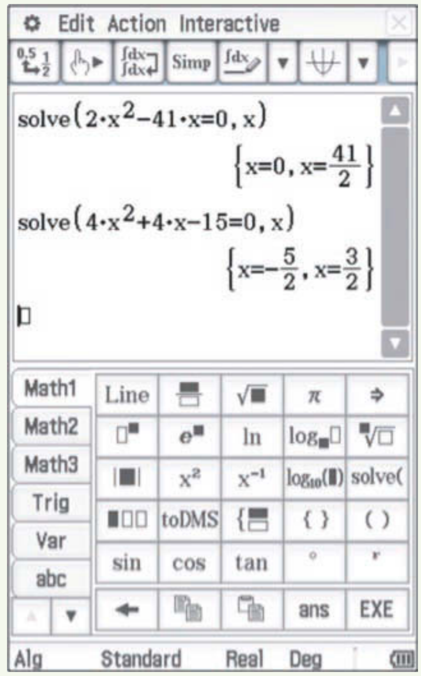
2 Use menu >**Algebra>Solve** and type as shown.



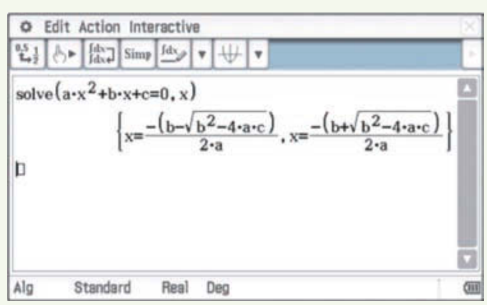
Note: use a multiplication sign between a and x^2 in ax^2 and b and x in bx.
This gives the general quadratic formula studied in **Section 5I**.

Using the ClassPad:

1 In the **Main** application, type and highlight the equation then tap **Interactive**, **Advanced**, **Solve**, **OK**.



2 Use the **VAR** keyboard to type the equation as shown. Highlight the equation and tap **Interactive**, **Advanced**, **Solve**, **OK**. This gives the general quadratic formula studied in **Section 5I**.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

5G Applications of quadratics

Learning intentions

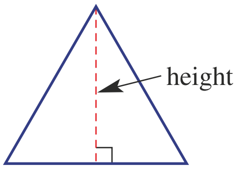
- To be able to set up a quadratic equation from a word problem
- To know how to apply the steps for solving a quadratic equation
- To understand and check the validity of solutions in the context of the given problem

Defining variables, setting up equations, solving equations and interpreting solutions are all important elements of applying quadratic equations to problem solving. The area of a rectangular paddock, for example, that is fenced off using a fixed length of fencing can be found by setting up a quadratic equation, solving it and then interpreting the solutions.

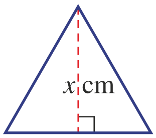
Aerospace engineers model the trajectory of a rocket under the influence of gravity using a quadratic equation of height, h, versus time. The solutions to $h = 0$ are the times when the rocket is at ground level and give its flight time.

LESSON STARTER The 10 cm^2 triangle

There are many base and height measurements for a triangle that give an area of 10 cm^2.



- Draw three different triangles that have a 10 cm^2 area. Include the measurements for the base and the height.
- Do any of your triangles have a base length that is 1 cm more than the height? Find the special triangle with area 10 cm^2 that has a base 1 cm more than its height by following these steps.
  - Let x cm be the height of the triangle.
  - Write an expression for the base length.
  - Write an equation if the area is 10 cm^2.
  - Solve the equation to find two solutions for x.
  - Which solution is to be used to describe the special triangle? Why?



KEY IDEAS

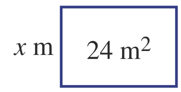
- When applying quadratic equations, follow these steps.
  - Define a variable; i.e. ‘Let x be …’.
  - Write an equation.
  - Solve the equation.
  - Choose the solution(s) that solves the equation and answers the question in the context in which it was given. Check that the solutions seem reasonable.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

BUILDING UNDERSTANDING

1 A rectangle has an area of 24 m^2. Its length is 5 m longer than its width.



- **a** Complete this sentence: 'Let x m be the ________________.
- **b** State an expression for the rectangle's length.
- **c** State an equation using the rectangle's area.
- **d** Rearrange your equation from part **c** in standard form (i.e. $ax^2 + bx + c = 0$) and solve for x.
- **e** Find the dimensions of the rectangle.

2 Repeat all the steps in Question **1** to find the dimensions of a rectangle with the following properties.

- **a** Its area is 60 m^2 and its length is 4 m more than its width.
- **b** Its area is 63 m^2 and its length is 2 m less than its width.

Example 18 Finding dimensions using quadratics

The area of a rectangle is fixed at 28 m^2 and its length is 3 metres more than its width. Find the dimensions of the rectangle.

SOLUTION

Let x m be the width of the rectangle.

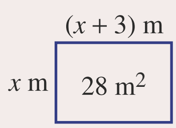
$$\text{Length} = (x + 3)\text{ m}$$
$$x(x + 3) = 28$$
$$x^2 + 3x - 28 = 0$$
$$(x + 7)(x - 4) = 0$$
$$x + 7 = 0 \text{ or } x - 4 = 0$$
$$\therefore x = -7 \text{ or } x = 4$$

$x > 0$ so, choose $x = 4$.

Rectangle has width 4 m and length 7 m.

EXPLANATION

Draw a diagram to help.



Write an equation using the given information.

Then write in standard form and solve for x.

Disregard $x = -7$ because $x > 0$.

Answer the question in full. Note: Length is $4 + 3 = 7$.

Now you try

The area of a rectangle is fixed at 48 m^2 and its length is 2 metres more than its width. Find the dimensions of the rectangle.

Exercise 5G

FLUENCY	1–5	2–5	3–6

Example 18 **1** The area of a rectangle is fixed at 12 m^2 and its length is 1 metre more than its width. Find the dimensions of the rectangle.

Example 18 **2** The area of a rectangle is fixed at 54 m^2 and its length is 3 metres more than its width. Find the dimensions of the rectangle.

ISBN 978-1-108-77290-7

Cambridge University Press Updated June 2022

3 Find the height and base lengths of a triangle that has an area of 24 cm^2 and height 2 cm more than its base.

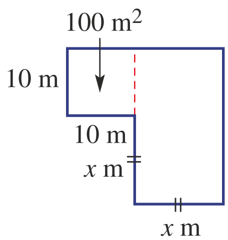
4 Find the height and base lengths of a triangle that has an area of 7 m^2 and height 5 m less than its base.

5 The product of two consecutive numbers is 72. Use a quadratic equation to find the two sets of numbers.

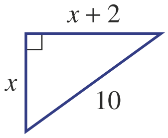
6 The product of two consecutive, even positive numbers is 168. Find the two numbers.

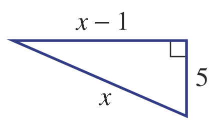
PROBLEM-SOLVING	7, 8	7–10	8–11

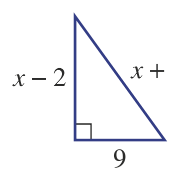
7 A 100 m^2 hay shed is to be extended to give 475 m^2 of floor space in total, as shown. All angles are right angles. Find the value of x.



8 Solve for x in these right-angled triangles, using Pythagoras' theorem.

a


b


c


9 A square hut of side length 5 m is to be surrounded by a veranda of width x metres. Find the width of the veranda if its area is to be 24 m^2.

10 A father's age is the square of his son's age (x). In 20 years' time the father will be three times as old as his son. What are the ages of the father and son?

11 A rectangular painting is to have a total area (including the frame) of 1200 cm^2. If the painting is 30 cm long and 20 cm wide, find the width of the frame.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

REASONING	12	12, 13	13, 14

12 The sum of the first n positive integers is given by $\frac{1}{2}n(n+1)$.

- a Find the sum of the first 10 positive integers (i.e. use $n = 10$).
- b Find the value of n if the sum of the first n positive integers is:
  - i 28
  - ii 91
  - iii 276

13 A ball is thrust vertically upwards from a machine on the ground. The height (h metres) after t seconds is given by $h = t(4 - t)$.

- a Find the height after 1.5 seconds.
- b Find when the ball is at a height of 3 metres.
- c Why are there two solutions to part b?
- d Find when the ball is at ground level. Explain.
- e Find when the ball is at a height of 4 metres.
- f Why is there only one solution for part e?
- g Is there a time when the ball is at a height of 5 metres? Explain.

14 The height (h metres) of a golf ball is given by $h = -0.01x(x - 100)$, where x metres is the horizontal distance from where the ball was hit.

- a Find the values of x when $h = 0$.
- b Interpret your answer from part a.
- c Find how far the ball has travelled horizontally when the height is 1.96 metres.

ENRICHMENT: Fixed perimeter and area	–	–	15, 16

15 A small rectangular block of land has a perimeter of 100 m and an area of 225 m^2. Find the dimensions of the block of land.

16 A rectangular farm has perimeter 700 m and area 30000 m^2. Find its dimensions.

ISBN 978-1-108-77290-7

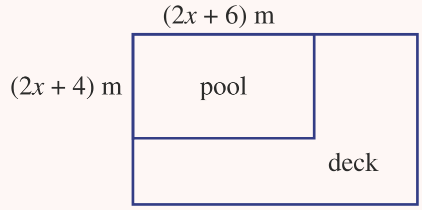
Cambridge University Press
Updated June 2022

Applications and problem-solving

The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

On the pool deck

1 Designs for a rectangular pool are being considered with the dimensions shown such that the length is 2 m more than the width, as shown. The pool will also have a deck built around it as shown. The length and width of the combined rectangular area will be an increase of 50% of the length and width of the pool.



The pool designer wants to explore the areas of possible decks in comparison to the area of the pool.

- **a** Give the length and width of the combined pool and deck area in terms of x.
- **b** Find the area of the deck in m^2 in terms of x.
- **c** If the area of the deck is 100 m^2, determine the dimensions of the pool by first finding the value of x.
- **d** Use your answer to **b** to determine what fraction the pool area is of the deck area.
- **e** Repeat parts **a** and **b** to determine what fraction the pool area is of the deck area, if the deck increases the length and width of the rectangular area by 25%.

Round-robin tournament

2 A round-robin tournament with n teams, where every team plays each other once, requires $\frac{n^2 - n}{2}$ games.

Using this rule, the tournament organisers wish to explore the number of games that need to be scheduled and the number of teams required for a given number of games.

- **a** How many games are played in a round-robin tournament with 6 teams?
- **b** A round-robin tournament has 28 games, solve an appropriate equation to find the number of teams in the competition.

ISBN 978-1-108-77290-7

c Investigate if doubling the number of teams, doubles the number of matches required. Prove algebraically.

d Give a simplified expression in terms of n for the number of games required for $n + 1$ teams.

e A tournament has n teams. How many more games are required in the tournament if the number of teams increases by

i 1 team? **ii** 2 teams? **iii** x teams?

Kayaking along the river

3 A kayaker is paddling up a river which is flowing at a certain speed. He travels 15 km up the river and then back down the river to where he started, kayaking at the same still-water speed, x km/h. The trip takes 4 hours to go up and down the river.

You wish to investigate the effect of the varying river flow speed on the speed of the kayaker who needs to complete the trip of fixed distance in the given time frame.

a If the river is flowing at a rate of 2 km/h and the man is kayaking at a rate of x km/h, find:

i expressions, in terms of x, for the rate the kayaker is moving upstream and the rate the kayaker is moving downstream

ii the value of x for this 4 hour journey.

Consider the same journey, taking the same time, with the river flowing at y km/h.

b Find a rule for the speed of the kayaker in still-water, x km/h, in terms of y.

c Use your rule from part **b** to confirm your answer to part **a** and to find the kayaker's speed if there was no current.

5H Solving quadratic equations by completing the square

Learning intentions

- To understand that completing the square can be used to help factorise a quadratic equation when integers cannot be found
- To be able to solve an equation by using the completing the square method to factorise
- To recognise a form of a quadratic equation that gives no solutions

In **Section 5E** we saw that some quadratics cannot be factorised using integers but instead could be factorised by completing the square. Surds were also used to complete the factorisation. We can use this method to solve many quadratic equations.

In the 9th century, the great Persian mathematician Al-Khwarizmi first solved quadratic equations by completing the square. His *Al-jabr* book was the principal maths textbook in European universities for 500 years, introducing algebra, algorithms and surds.

LESSON STARTER Where does $\sqrt{6}$ come in?

Consider the equation $x^2 - 2x - 5 = 0$ and try to solve it by discussing these points.

- Are there any common factors that can be taken out?
- Are there any integers that multiply to give -5 and add to give -2?
- Try completing the square on the left-hand side. Does this help and how?
- Show that the two solutions contain the surd $\sqrt{6}$.

KEY IDEAS

- To solve quadratic equations of the form $ax^2 + bx + c = 0$ for which you cannot factorise using integers:
  - Complete the square for the quadratic expression and factorise if possible.
  - Solve the quadratic equation using the Null Factor Law or an alternate method.
- Expressions such as $x^2 + 5$ and $(x - 1)^2 + 7$ cannot be factorised further and therefore give no solutions as they cannot be expressed as a difference of two squares.

BUILDING UNDERSTANDING

1 What number must be added to the following expressions to form a perfect square?

a $x^2 + 2x$ **b** $x^2 + 20x$ **c** $x^2 - 4x$ **d** $x^2 + 5x$

2 Factorise using surds.

a $x^2 - 3 = 0$ **b** $x^2 - 10 = 0$ **c** $(x + 1)^2 - 5 = 0$

3 Solve these equations.

a $(x - \sqrt{2})(x + \sqrt{2}) = 0$ **b** $(x - \sqrt{7})(x + \sqrt{7}) = 0$

c $(x - 3 + \sqrt{5})(x - 3 - \sqrt{5}) = 0$ **d** $(x + 5 + \sqrt{14})(x + 5 - \sqrt{14}) = 0$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 19 Solving quadratic equations by completing the square

Solve these quadratic equations by first completing the square.

a $x^2 + 4x + 2 = 0$ **b** $x^2 + 6x - 11 = 0$ **c** $x^2 - 3x + 1 = 0$

SOLUTION / EXPLANATION

a

$$x^2 - 4x + 2 = 0$$
$$x^2 - 4x + 4 - 4 + 2 = 0$$
$$(x-2)^2 - 2 = 0$$
$$(x - 2 + \sqrt{2})(x - 2 - \sqrt{2}) = 0$$
$$\therefore x - 2 + \sqrt{2} = 0 \quad \text{or} \quad x - 2 - \sqrt{2} = 0$$
$$\therefore x = 2 - \sqrt{2} \quad \text{or} \quad x = 2 + \sqrt{2}$$

Alternate method, from

$$(x-2)^2 - 2 = 0$$
$$(x-2)^2 = 2$$
$$x - 2 = \pm\sqrt{2}$$
$$x = 2 \pm \sqrt{2}$$

Complete the square: $\left(\frac{-4}{2}\right)^2 = 4$.

$x^2 - 4x + 4 = (x-2)(x-2) = (x-2)^2$

Use $a^2 - b^2 = (a+b)(a-b)$.

Apply the Null Factor Law and solve for x.

The solutions can also be written as $2 \pm \sqrt{2}$.

An alternate approach after completing the square is to add 2 to both sides and then take the square root of both sides

$\pm\sqrt{2}$ since $(+\sqrt{2})^2 = 2$ and $(-\sqrt{2})^2 = 2$.

b

$$x^2 + 6x - 11 = 0$$
$$x^2 + 6x + 9 - 9 - 11 = 0$$
$$(x+3)^2 - 20 = 0$$
$$(x + 3 - \sqrt{20})(x + 3 + \sqrt{20}) = 0$$
$$(x + 3 - 2\sqrt{5})(x + 3 + 2\sqrt{5}) = 0$$
$$\therefore x + 3 - 2\sqrt{5} = 0 \quad \text{or} \quad x + 3 + 2\sqrt{5} = 0$$
$$\therefore x = -3 + 2\sqrt{5} \quad \text{or} \quad x = -3 - 2\sqrt{5}$$

Alternatively, $x = -3 \pm 2\sqrt{5}$.

Complete the square: $\left(\frac{6}{2}\right)^2 = 9$.

Use difference of perfect squares with surds.

Recall that $\sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}$.

Apply the Null Factor Law and solve for x.

$(x+3)^2 = 20$ can also be solved by taking the square root of both sides.

Alternatively, write solutions using the $\pm$ symbol.

c

$$x^2 - 3x + 1 = 0$$
$$x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 1 = 0$$
$$\left(x - \frac{3}{2}\right)^2 - \frac{5}{4} = 0$$
$$\left(x + \frac{3}{2} + \sqrt{\frac{5}{4}}\right)\left(x - \frac{3}{2} - \sqrt{\frac{5}{4}}\right) = 0$$
$$x - \frac{3}{2} + \sqrt{\frac{5}{4}} = 0 \quad \text{or} \quad x - \frac{3}{2} - \sqrt{\frac{5}{4}} = 0$$
$$\therefore x = \frac{3}{2} - \frac{\sqrt{5}}{2} \quad \text{or} \quad x = \frac{3}{2} + \frac{\sqrt{5}}{2}$$
$$x = \frac{3 - \sqrt{5}}{2} \quad \text{or} \quad x = \frac{3 + \sqrt{5}}{2}$$

So $x = \dfrac{3 \pm \sqrt{5}}{2}$

$\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$

$a^2 - b^2 = (a+b)(a-b)$

Use the Null Factor Law.

Recall that $\sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{\sqrt{4}} = \frac{\sqrt{5}}{2}$

Combine using the $\pm$ symbol.

Continued on next page

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Now you try

Solve these quadratic equations by first completing the square.

a $x^2 - 6x + 2 = 0$ b $x^2 + 4x - 14 = 0$ c $x^2 - 5x + 2 = 0$

Exercise 5H

FLUENCY	1, 2–4(1/2)	2–4(1/3)	2–4(1/3)

1 Solve these quadratic equations by first completing the square.

Example 19a
a i $x^2 - 8x + 3 = 0$ ii $x^2 - 12x + 7 = 0$

Example 19b
b i $x^2 + 4x - 4 = 0$ ii $x^2 + 10x - 7 = 0$

Example 19a
2 Solve by first completing the square.

a $x^2 + 6x + 3 = 0$ b $x^2 + 4x + 2 = 0$ c $x^2 + 10x + 15 = 0$
d $x^2 + 4x - 2 = 0$ e $x^2 + 8x - 3 = 0$ f $x^2 + 6x - 5 = 0$
g $x^2 - 8x - 1 = 0$ h $x^2 - 12x - 3 = 0$ i $x^2 - 2x - 16 = 0$
j $x^2 - 10x + 18 = 0$ k $x^2 - 6x + 4 = 0$ l $x^2 - 8x + 9 = 0$
m $x^2 + 6x - 4 = 0$ n $x^2 + 20x + 13 = 0$ o $x^2 - 14x - 6 = 0$

Example 19b
3 Solve by first completing the square.

a $x^2 + 8x + 4 = 0$ b $x^2 + 6x + 1 = 0$ c $x^2 - 10x + 5 = 0$
d $x^2 - 4x - 14 = 0$ e $x^2 - 10x - 3 = 0$ f $x^2 + 8x - 8 = 0$
g $x^2 - 2x - 31 = 0$ h $x^2 + 12x - 18 = 0$ i $x^2 + 6x - 41 = 0$

Example 19c
4 Solve by first completing the square.

a $x^2 + 5x + 2 = 0$ b $x^2 + 3x + 1 = 0$ c $x^2 + 7x + 5 = 0$
d $x^2 - 3x - 2 = 0$ e $x^2 - x - 3 = 0$ f $x^2 + 5x - 2 = 0$
g $x^2 - 7x + 2 = 0$ h $x^2 - 9x + 5 = 0$ i $x^2 + x - 4 = 0$
j $x^2 + 9x + 9 = 0$ k $x^2 - 3x - \frac{3}{4} = 0$ l $x^2 + 5x + \frac{5}{4} = 0$

PROBLEM-SOLVING	5(1/2), 8	5–7(1/3), 8	5–7(1/3), 9

5 Decide how many solutions there are to these equations. Try factorising the equations if you are unsure.

a $x^2 - 2 = 0$ b $x^2 - 10 = 0$ c $x^2 + 3 = 0$
d $x^2 + 7 = 0$ e $(x - 1)^2 + 4 = 0$ f $(x + 2)^2 - 7 = 0$
g $(x - 7)^2 - 6 = 0$ h $x^2 - 2x + 6 = 0$ i $x^2 - 3x + 10 = 0$
j $x^2 + 2x - 4 = 0$ k $x^2 + 7x + 1 = 0$ l $x^2 - 2x + 17 = 0$

6 Solve the following, if possible, by first factoring out the coefficient of x^2 and then completing the square.

a $2x^2 - 4x + 4 = 0$ b $4x^2 + 20x + 8 = 0$ c $2x^2 - 10x + 4 = 0$
d $3x^2 + 27x + 9 = 0$ e $3x^2 + 15x + 3 = 0$ f $2x^2 - 12x + 8 = 0$

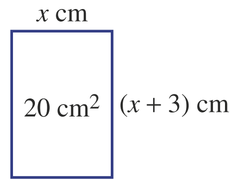
7 Solve the following quadratic equations, if possible.

a $x^2 + 3x = 5$ b $x^2 + 5x = 9$ c $x^2 + 7x = -15$
d $x^2 - 8x = -11$ e $x^2 + 12x + 10 = 2x + 5$ f $x^2 + x + 9 = 5x - 3$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

8 A rectangle's length is 3 cm more than its width. Find the dimensions of the rectangle if its area is 20 cm^2.



9 The height, h km, of a ballistic missile launched from a submarine at sea level is given by $h = \dfrac{x(400 - x)}{20000}$, where x km is the horizontal distance travelled.

a Find the height of a missile that has travelled the following horizontal distances.
   i 100 km   **ii** 300 km

b Find how far the missile has travelled horizontally when the height is:
   i 0 km   **ii** 2 km

c Find the horizontal distance the missile has travelled when its height is 1 km. (*Hint*: Complete the square.)

REASONING	10	10, 11	11, 12

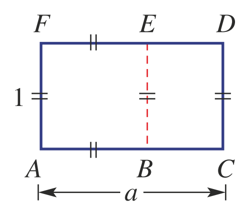
10 Complete the square to show that the following have no (real) solutions.

a $x^2 + 4x + 5 = 0$ **b** $x^2 - 3x = -3$

11 A friend starts to solve $x^2 + x - 30 = 0$ by completing the square but you notice there is a much quicker way. What method do you describe to your friend?

12 This rectangle is a golden rectangle.

- $ABEF$ is a square.
- Rectangle $BCDE$ is similar to rectangle $ACDF$.



a Show that $\dfrac{a}{1} = \dfrac{1}{a - 1}$.

b Find the exact value of a (which will give you the golden ratio) by completing the square.

ENRICHMENT: Completing the square with non-monics	–	–	13($\frac{1}{2}$)

13 In the Enrichment section of **Exercise 5E** we looked at a method to factorise non-monic quadratics by completing the square. It involved taking out the coefficient of x^2. Dividing both sides by that number is possible in these equations and this makes the task easier. Use this technique to solve the following equations.

a $2x^2 + 4x - 1 = 0$ **b** $3x^2 + 6x - 12 = 0$

c $-2x^2 + 16x - 10 = 0$ **d** $3x^2 - 9x + 3 = 0$

e $4x^2 + 20x + 8 = 0$ **f** $5x^2 + 5x - 15 = 0$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

5I Solving quadratic equations using the quadratic formula

Learning intentions

- To know the quadratic formula and when to apply it
- To be able to use the quadratic formula to solve a quadratic equation
- To know what the discriminant is and what it can be used for
- To be able to use the discriminant to determine the number of solutions of a quadratic equation

A general formula for solving quadratic equations can be found by completing the square for the general case.

Consider $ax^2 + bx + c = 0$, where a, b, c are constants and $a \neq 0$. Start by dividing both sides by a.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right) = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Surveyors regularly subdivide land into house blocks. When dimensions are linear expressions of the same variable, an area formula forms a quadratic equation. For a given area, surveyors can solve this equation using the quadratic formula.

This formula now gives us a mechanism to solve quadratic equations and to determine how many solutions the equation has.

The expression under the root sign, $b^2 - 4ac$, is called the discriminant ($\triangle$) and helps us to identify the number of solutions. A quadratic equation can have 0, 1 or 2 solutions.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

LESSON STARTER How many solutions?

Complete this table to find the number of solutions for each equation.

$ax^2 + bx + c = 0$	a	b	c	$b^2 - 4ac$	$\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$	$\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$
$2x^2 + 7x + 1 = 0$						
$9x^2 - 6x + 1 = 0$						
$x^2 - 3x + 4 = 0$						

Discuss under what circumstances a quadratic equation has:
- 2 solutions
- 1 solution
- 0 solutions.

KEY IDEAS

- If $ax^2 + bx + c = 0$ (where a, b, c are constants and $a \neq 0$), then

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

  - This is called the **quadratic formula**.
  - The quadratic formula is useful when a quadratic cannot be factorised easily.
- The **discriminant** is $\triangle = b^2 - 4ac$.
  - When $\triangle < 0$, the quadratic equation has 0 real solutions (since $\sqrt{\triangle}$ is undefined when $\triangle$ is negative).
  - When $\triangle = 0$, the quadratic equation has 1 real solution $\left(x = -\dfrac{b}{2a}\right)$.
  - When $\triangle > 0$, the quadratic equation has 2 real solutions $\left(x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$.

BUILDING UNDERSTANDING

1 For these quadratic equations in the form $ax^2 + bx + c = 0$, state the values of a, b and c.
  a $3x^2 + 2x + 1 = 0$
  b $5x^2 + 3x - 2 = 0$
  c $2x^2 - x - 5 = 0$
  d $-3x^2 + 4x - 5 = 0$

2 Find the value of the discriminant ($b^2 - 4ac$) for each part in Question 1 above.

3 State the number of solutions of a quadratic that has:
  a $b^2 - 4ac = 0$
  b $b^2 - 4ac < 0$
  c $b^2 - 4ac > 0$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 20 Using the discriminant

Determine the number of solutions to the following quadratic equations using the discriminant.

a $x^2 + 5x - 3 = 0$

b $2x^2 - 3x + 4 = 0$

c $x^2 + 6x + 9 = 0$

SOLUTION	EXPLANATION
a $a = 1, b = 5, c = -3$	State the values of a, b and c in $ax^2 + bx + c = 0$.
$\triangle = b^2 - 4ac$ $= (5)^2 - 4(1)(-3)$ $= 25 + 12$ $= 37$	Calculate the value of the discriminant by substituting values.
$\triangle > 0$, so there are 2 solutions.	Interpret the result with regard to the number of solutions.
b $a = 2, b = -3, c = 4$ $\triangle = b^2 - 4ac$ $= (-3)^2 - 4(2)(4)$ $= 9 - 32$ $= -23$	State the values of a, b and c and substitute to evaluate the discriminant. Recall that $(-3)^2 = -3 \times (-3) = 9$.
$\triangle < 0$, so there are no solutions.	Interpret the result.
c $a = 1, b = 6, c = 9$ $\triangle = b^2 - 4ac$ $= (6)^2 - 4(1)(9)$ $= 36 - 36$ $= 0$ $\triangle = 0$, so there is 1 solution.	Substitute the values of a, b and c to evaluate the discriminant and interpret the result. Note: $x^2 + 6x + 9 = (x + 3)^2$ is a perfect square.

Now you try

Determine the number of solutions to the following quadratic equations using the discriminant.

a $x^2 + 7x - 1 = 0$

b $3x^2 - x + 2 = 0$

c $x^2 + 8x + 16 = 0$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 21 Solving quadratic equations using the quadratic formula

Find the exact solutions to the following using the quadratic formula.

a $x^2 + 5x + 3 = 0$

b $2x^2 - 2x - 1 = 0$

SOLUTION

a $a = 1, b = 5, c = 3$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-5 \pm \sqrt{(5)^2 - 4(1)(3)}}{2(1)} \\ &= \frac{-5 \pm \sqrt{25 - 12}}{2} \\ &= \frac{-5 \pm \sqrt{13}}{2} \end{aligned}$$

b $a = 2, b = -2, c = -1$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)} \\ &= \frac{2 \pm \sqrt{4 + 8}}{4} \\ &= \frac{2 \pm \sqrt{12}}{4} \\ &= \frac{2 \pm 2\sqrt{3}}{4} \\ &= \frac{1 \pm \sqrt{3}}{2} \end{aligned}$$

EXPLANATION

a Determine the values of a, b and c in $ax^2 + bx + c = 0$. Write out the quadratic formula and substitute the values.

Simplify.

Two solutions: $x = \frac{-5 - \sqrt{13}}{2}, \frac{-5 + \sqrt{13}}{2}$.

b Determine the values of a, b and c.

Simplify: $\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$.

Cancel using the common factor:

$$\begin{aligned} \frac{2 \pm 2\sqrt{3}}{4} &= \frac{2(1 \pm \sqrt{3})}{4} \\ &= \frac{1 \pm \sqrt{3}}{2} \end{aligned}$$

Now you try

Find the exact solutions to the following using the quadratic formula.

a $x^2 + 3x + 1 = 0$

b $4x^2 - 2x - 3 = 0$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 5I

FLUENCY	1, 2–3(1/2)	2–4(1/3)	2–4(1/3)

Example 20

1 Determine the number of solutions to the following quadratic equations using the discriminant.

a $x^2 + 3x - 1 = 0$ **b** $4x^2 - 2x + 5 = 0$ **c** $x^2 + 4x + 4 = 0$

Example 20

2 Using the discriminant, determine the number of solutions for these quadratic equations.

a $x^2 + 5x + 3 = 0$ **b** $x^2 + 3x + 4 = 0$ **c** $x^2 + 6x + 9 = 0$
d $x^2 + 7x - 3 = 0$ **e** $x^2 + 5x - 4 = 0$ **f** $x^2 + 4x - 4 = 0$
g $4x^2 + 5x + 3 = 0$ **h** $4x^2 + 3x + 1 = 0$ **i** $2x^2 + 12x + 9 = 0$
j $-x^2 - 6x - 9 = 0$ **k** $-2x^2 + 3x - 4 = 0$ **l** $-4x^2 - 6x + 3 = 0$

Example 21a

3 Find the exact solutions to the following quadratic equations, using the quadratic formula.

a $x^2 + 3x - 2 = 0$ **b** $x^2 + 7x - 4 = 0$ **c** $x^2 - 7x + 5 = 0$
d $x^2 - 8x + 16 = 0$ **e** $-x^2 - 5x - 4 = 0$ **f** $-x^2 - 8x - 7 = 0$
g $4x^2 + 7x - 1 = 0$ **h** $3x^2 + 5x - 1 = 0$ **i** $3x^2 - 4x - 6 = 0$
j $-2x^2 + 5x + 5 = 0$ **k** $-3x^2 - x + 4 = 0$ **l** $5x^2 + 6x - 2 = 0$

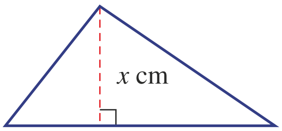
Example 21b

4 Find the exact solutions to the following quadratic equations, using the quadratic formula.

a $x^2 + 4x + 1 = 0$ **b** $x^2 - 6x + 4 = 0$ **c** $x^2 + 6x - 2 = 0$
d $-x^2 - 3x + 9 = 0$ **e** $-x^2 + 4x + 4 = 0$ **f** $-3x^2 + 8x - 2 = 0$
g $2x^2 - 2x - 3 = 0$ **h** $3x^2 - 6x - 1 = 0$ **i** $-5x^2 + 8x + 3 = 0$

PROBLEM-SOLVING	5, 6(1/2)	5, 6(1/2), 8	6(1/2), 7, 9

5 A triangle's base is 5 cm more than its height of x cm. Find its height if the triangle's area is 10 cm^2.

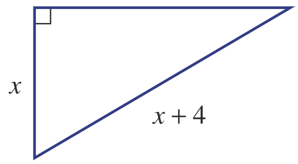


6 Solve the following using the quadratic formula.

a $3x^2 = 1 + 6x$ **b** $2x^2 = 3 - 4x$ **c** $5x = 2 - 4x^2$
d $2x - 5 = -\frac{1}{x}$ **e** $\frac{3}{x} = 3x + 4$ **f** $-\frac{5}{x} = 2 - x$
g $5x = \frac{2x + 2}{x}$ **h** $x = \frac{3x + 4}{2x}$ **i** $3x = \frac{10x - 1}{2x}$

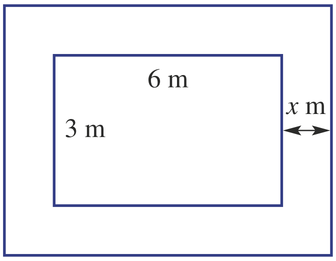
7 Two positive numbers differ by 3 and their product is 11. Find the numbers.

8 Find the exact perimeter of this right-angled triangle.



ISBN 978-1-108-77290-7

9 A rectangular pool measuring 6 m by 3 m is to have a path surrounding it. If the total area of the pool and path is to be 31 m^2, find the width (x m) of the path, correct to the nearest centimetre.



REASONING	10	10, 11	11, 12

10 Explain why the rule $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ gives only one solution when $b^2 - 4ac = 0$.

11 Make up three quadratic equations that have:

a no solutions
b 1 solution
c 2 solutions.

12 For what two values of k does $x^2 + kx + 9 = 0$ have only one solution?

ENRICHMENT: *k* determines the number of solutions	–	–	13(1/2)

13 The discriminant for $x^2 + 2x + k = 0$ is $4 - 4k$, so there:

- are no solutions for $4 - 4k < 0, \therefore k > 1$
- is 1 solution for $4 - 4k = 0, \therefore k = 1$
- are 2 solutions for $4 - 4k > 0, \therefore k < 1$

a For what values of k does $x^2 + 4x + k = 0$ have:
i no solutions?
ii 1 solution?
iii 2 solutions?

b For what values of k does $kx^2 + 3x + 2 = 0$ have:
i no solutions?
ii 1 solution?
iii 2 solutions?

c For what values of k does $x^2 + kx + 1 = 0$ have:
i no solutions?
ii 1 solution?
iii 2 solutions?

d For what values of k does $3x^2 + kx - 1 = 0$ have:
i no solutions?
ii 1 solution?
iii 2 solutions?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Problems and challenges

1 Find the monic quadratic in the form $x^2 + bx + c = 0$ with solutions $x = 2 - \sqrt{3}$ and $x = 2 + \sqrt{3}$.

Up for a challenge? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.

2 If $x + \dfrac{1}{x} = 7$, what is $x^2 + \dfrac{1}{x^2}$?

3 Find all the solutions to each equation. (*Hint*: Consider letting $a = x^2$ in each equation.)

a $x^4 - 5x^2 + 4 = 0$

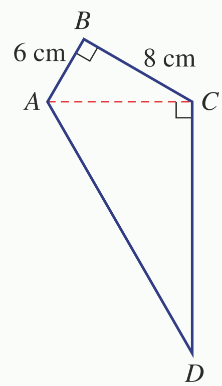
b $x^4 - 7x^2 - 18 = 0$

4 Make a substitution as you did in Question **3** to obtain a quadratic equation to help you solve the following.

a $3^{2x} - 4 \times 3^x + 3 = 0$

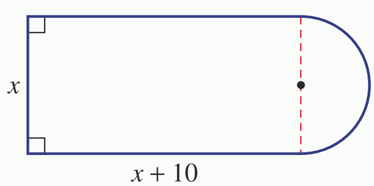
b $4 \times 2^{2x} - 9 \times 2^x + 2 = 0$

5 Quadrilateral *ABCD* has a perimeter of 64 cm with measurements as shown. What is the area of the quadrilateral?



6 A cyclist in a charity ride rides 300 km at a constant average speed. If the average speed had been 5 km/h faster, the ride would have taken 2 hours less. What was the average speed of the cyclist?

7 Find the value of x, correct to one decimal place, in this diagram if the area is to be 20 square units.



8 Prove that $x^2 - 2x + 2 > 0$ for all values of x.

9 A square has the same perimeter as a rectangle of length x cm and width y cm. Determine a simplified expression for the difference in their areas and, hence, show that when the perimeters are equal the square has the greatest area.

10 The equation $x^2 + wx + t = 0$ has solutions α and β, where the equation $x^2 + px + q = 0$ has solutions 3α and 3β. Determine the ratios $w : p$ and $t : q$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Quadratic equations

Factorising non-monic quadratics $ax^2 + bx + c$ (10A)

Use grouping, split up bx using two numbers that multiply to $a \times c$ and add to give b:

e.g. $6x^2 - 5x - 4$

$a \times c = 6 \times (-4) = -24$

$-8 \times 3 = -24$

$-8 + 3 = -5$

$\therefore \quad 6x^2 + 3x - 8x - 4$

$= 3x(2x + 1) - 4(2x + 1)$

$= (2x + 1)(3x - 4)$

Factorising $x^2 + bx + c$

Two numbers $\times$ to give c.

Two numbers $+$ to give b.

e.g. $x^2 - 7x - 18 = (x - 9)(x + 2)$

$-9 \times 2 = -18$

$-9 + 2 = -7$

Factorising and DOPS

Always take out common factors first.

Difference of perfect squares

$a^2 - b^2 = (a - b)(a + b)$

e.g. $4x^2 - 9 = (2x)^2 - (3)^2$

$= (2x - 3)(2x + 3)$

$x^2 - 7 = (x - \sqrt{7})(x + \sqrt{7})$

Factorising by completing the square

e.g. $x^2 + 4x - 3$

$= (x^2 + 4x + (\frac{4}{2})^2) - (\frac{4}{2})^2 - 3$

$= (x + \frac{4}{2})^2 - 4 - 3$

$= (x + 2)^2 - 7$

$= (x + 2 - \sqrt{7})(x + 2 + \sqrt{7})$

Note, for example, $(x + 2)^2 + 5$ cannot be factorised.

Expanding brackets

$a(b + c) = ab + ac$

$(a + b)(c + d) = ac + ad + bc + bd$

$(a + b)(a - b) = a^2 - b^2$

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

Solving quadratic equations

Null Factor Law:

If $ab = 0$ then $a = 0$ or $b = 0$.

Write each quadratic in standard form $ax^2 + bx + c = 0$, factorise then apply the Null Factor Law to solve.

e.g. **1** $x^2 - 4x = 0$

$x(x - 4) = 0$

$x = 0$ or $x - 4 = 0$

$x = 0$ or $x = 4$

2 $x^2 = 3x - 10$

$x^2 - 3x + 10 = 0$

$(x - 5)(x + 2) = 0$

$x - 5 = 0$ or $x + 2 = 0$

$x = 5$ or $x = -2$

Quadratic formula

If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant $\Delta = b^2 - 4ac$ tells us how many solutions:

$\Delta > 0$	2 solutions
$\Delta = 0$	1 solution
$\Delta < 0$	0 solutions

Applications

1. Define the variable.
2. Set up the equation.
3. Solve by factorising and using the Null Factor Law or quadratic formula.
4. Determine the suitable answer(s).

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter checklist

Chapter checklist: Success criteria

		✔
5A	**1. I can apply the distributive law to expand and simplify.** e.g. Expand and simplify $2x(3x - 5) - 3(3x - 5)$.	
5A	**2. I can expand a binomial product.** e.g. Expand and simplify $(2x - 3)(x + 4)$.	
5A	**3. I can expand to form a difference of perfect squares.** e.g. Expand $(3x + 2)(3x - 2)$.	
5A	**4. I can expand a perfect square.** e.g. Expand $(x + 5)^2$.	
5B	**5. I can factorise by taking out a common factor.** e.g. Factorise $12x^2 - 18x$.	
5B	**6. I can factorise a difference of perfect squares.** e.g. Factorise $9x^2 - 16$.	
5B	**7. I can factorise a difference of perfect squares involving surds.** e.g. Factorise $x^2 - 7$ using surds.	
5B	**8. I can factorise using grouping.** e.g. Factorise $x^2 - ax + 2x - 2a$ by grouping.	
5C	**9. I can factorise a monic trinomial.** e.g. Factorise $x^2 - 8x - 20$.	
5C	**10. I can factorise a trinomial with a common factor.** e.g. Factorise $3x^2 - 24x + 45$.	
5C	**11. I can multiply and divide algebraic fractions by first factorising.** e.g. Simplify by first factorising $\frac{x^2 - 4}{x + 2} \times \frac{3x + 12}{x^2 + 2x - 8}$.	
5D	**12. I can factorise a non-monic quadratic.** 10A e.g. Factorise $5x^2 + 13x - 6$.	
5E	**13. I can factorise by completing the square.** e.g. Factorise $x^2 + 6x + 2$ by completing the square.	
5E	**14. I can recognise when a quadratic cannot be factorised.** e.g. Factorise $x^2 - 3x + 4$ by completing the square if possible.	
5F	**15. I can solve a quadratic equation by factorising and applying the Null Factor Law.** e.g. Solve $3x^2 - 9x = 0$.	
5F	**16. I can solve a quadratic equation by first rearranging into standard form.** e.g. Solve $x^2 = 2x + 3$.	
5G	**17. I can solve a word problem using a quadratic model.** e.g. The area of a rectangle is 60 m^2 and its length is 4 metres more than its width. Find the dimensions of the rectangle.	
5H	**18. I can solve a quadratic equation using completing the square.** e.g. Solve $x^2 + 4x + 22 = 0$ by first completing the square.	
5I	**19. I can determine the number of solutions of a quadratic equation.** e.g. Use the discriminant to determine the number of solutions of the equation $2x^2 - 3x - 5 = 0$.	
5I	**20. I can use the quadratic formula to solve a quadratic equation.** e.g. Find the exact solutions to $2x^2 + 3x - 4 = 0$ using the quadratic formula.	

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

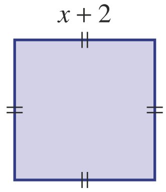
Short-answer questions

5A **1** Expand the following and simplify where possible.

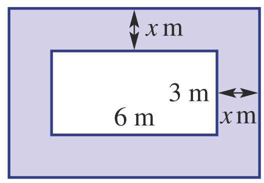
a $2(x + 3) - 4(x - 5)$ **b** $(x + 5)(3x - 4)$ **c** $(5x - 2)(5x + 2)$
d $(x - 6)^2$ **e** $(x + 4)^2 - (x + 3)(x - 2)$ **f** $(3x - 2)(4x - 5)$

5A **2** Write, in expanded form, an expression for the shaded areas. All angles are right angles.

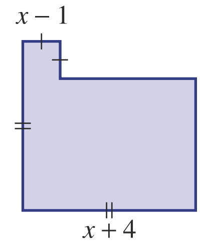
a



b



c



5B **3** Factorise the following difference of perfect squares. Remember to look for a common factor first.

a $x^2 - 49$ **b** $9x^2 - 16$ **c** $4x^2 - 1$
d $3x^2 - 75$ **e** $2x^2 - 18$ **f** $x^2 - 11$
g $-2x^2 + 40$ **h** $(x + 1)^2 - 16$ **i** $(x - 3)^2 - 10$

5C **4** Factorise these quadratic trinomials.

a $x^2 - 8x + 12$ **b** $x^2 + 10x - 24$ **c** $-3x^2 + 21x - 18$

5D 10A **5** Factorise these non-monic quadratic trinomials.

a $3x^2 + 17x + 10$ **b** $4x^2 + 4x - 15$ **c** $12x^2 - 16x - 3$ **d** $12x^2 - 23x + 10$

5C/D **6** Simplify.

a $\dfrac{12x}{x^2 + 2x - 3} \times \dfrac{x^2 - 1}{6x + 6}$ 10A **b** $\dfrac{4x^2 - 9}{2x^2 + x - 6} \div \dfrac{8x + 12}{x^2 - 2x - 8}$

5E **7** Factorise the following by completing the square.

a $x^2 + 8x + 10$ **b** $x^2 + 10x - 4$ **c** $x^2 - 6x - 3$
d $x^2 + 3x - 2$ **e** $x^2 + 5x + 3$ **f** $x^2 + 7x + \dfrac{9}{2}$

5F **8** Solve these quadratic equations by factorising and applying the Null Factor Law.

a $x^2 + 4x = 0$ **b** $3x^2 - 9x = 0$ **c** $x^2 - 25 = 0$
d $x^2 - 10x + 21 = 0$ **e** $x^2 - 8x + 16 = 0$ **f** $x^2 + 5x - 36 = 0$
10A **g** $2x^2 + 3x - 2 = 0$ 10A **h** $6x^2 + 11x - 10 = 0$ 10A **i** $18x^2 + 25x - 3 = 0$

5F **9** Solve the following quadratic equations by first writing them in standard form.

a $3x^2 = 27$ **b** $x^2 = 4x + 5$
c $2x^2 - 28 = x(x - 3)$ **d** $\dfrac{3x + 18}{x} = x$

5G **10** A rectangular sandpit is 2 m longer than it is wide. If it occupies an area of 48 m^2, determine the dimensions of the sandpit by solving a suitable equation.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter review

5H **11** Solve these quadratic equations by first completing the square.

a $x^2 + 4x - 3 = 0$ **b** $x^2 - 6x + 1 = 0$
c $x^2 - 3x - 2 = 0$ **d** $x^2 + 5x - 5 = 0$

5I **12** For each quadratic equation, determine the number of solutions by finding the value of the discriminant.

a $x^2 + 2x + 1 = 0$ **b** $x^2 - 3x - 3 = 0$
c $2x^2 - 4x + 3 = 0$ **d** $-3x^2 + x + 5 = 0$

5I **13** Use the quadratic formula to give exact solutions to these quadratic equations.

a $x^2 + 3x - 6 = 0$ **b** $x^2 - 2x - 4 = 0$
c $2x^2 - 4x - 5 = 0$ **d** $-3x^2 + x + 3 = 0$

Multiple-choice questions

5A **1** $(x + 5)^2$ is equivalent to:

A $x^2 + 25$ **B** $x^2 + 5x$ **C** $x^2 + 5x + 25$
D $x^2 + 10x + 25$ **E** $x^2 + 50$

5A **2** $2(2x - 1)(x + 4)$ is equivalent to:

A $4x^2 + 15x - 4$ **B** $4x^2 + 14x - 8$ **C** $8x^2 + 28x - 16$
D $8x^2 + 18x - 4$ **E** $4x^2 + 10x + 8$

5B **3** $4x^2 - 25$ in factorised form is:

A $4(x - 5)(x + 5)$ **B** $(2x - 5)^2$ **C** $(2x - 5)(2x + 5)$
D $(4x + 5)(x - 5)$ **E** $2(2x + 1)(x - 25)$

5C **4** The fully factorised form of $2x^2 - 10x - 28$ is:

A $2(x + 2)(x - 7)$ **B** $(2x + 7)(x + 4)$ **C** $2(x - 4)(x - 1)$
D $(2x - 2)(x + 14)$ **E** $(x - 2)(x + 7)$

5C **5** $\dfrac{x^2 + x - 20}{8x} \times \dfrac{2x + 8}{x^2 - 16}$ simplifies to:

A $\dfrac{x - 20}{8}$ **B** $\dfrac{x + 5}{4x}$ **C** $\dfrac{x + 5}{x - 4}$ **D** $x - 5$ **E** $\dfrac{x^2 - 20}{16}$

5E **6** The term that needs to be added to make $x^2 - 6x$ a perfect square is:

A 18 **B** -9 **C** -3 **D** 9 **E** 3

5F **7** The solution(s) to $2x^2 - 8x = 0$ are:

A $x = 0, x = -4$ **B** $x = 2$ **C** $x = 0, x = 4$
D $x = 4$ **E** $x = 0, x = 2$

5F 10A **8** For $8x^2 - 14x + 3 = 0$, the solutions for x are:

A $\dfrac{1}{8}, -\dfrac{1}{3}$ **B** $\dfrac{3}{4}, -\dfrac{1}{2}$ **C** $\dfrac{1}{4}, \dfrac{3}{2}$
D $\dfrac{3}{4}, -\dfrac{1}{2}$ **E** $-\dfrac{1}{2}, -\dfrac{3}{8}$

ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

5F **9** When written in the standard form $ax^2 + bx + c = 0$, $\dfrac{x-3}{x} = 2x$ is:

A $x^2 + 2x + 3 = 0$ **B** $x^2 + 3 = 0$ **C** $2x^2 + x - 3 = 0$
D $2x^2 - x - 3 = 0$ **E** $2x^2 - x + 3 = 0$

5G **10** The product of two consecutive numbers is 72. If x is the smaller number, an equation to represent this would be:

A $x^2 + x + 72 = 0$ **B** $2x - 71 = 0$ **C** $x^2 + x - 72 = 0$
D $x^2 + 1 = 72$ **E** $x^2 = x + 72$

5H **11** For $(x - 7)^2 - 3 = 0$, the solutions for x are:

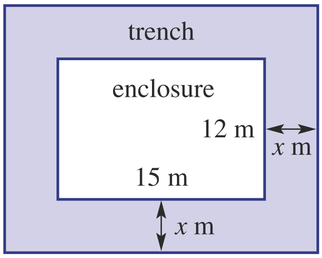
A $7 - \sqrt{3}, 7 + \sqrt{3}$ **B** $-7 - \sqrt{3}, -7 + \sqrt{3}$ **C** $7, -3$
D $-7 - \sqrt{3}, 7 + \sqrt{3}$ **E** $4, 10$

5I **12** If $ax^2 + bx + c = 0$ has exactly two solutions, then:

A $b^2 - 4ac = 0$ **B** $b^2 - 4ac > 0$ **C** $b^2 - 4ac \leqslant 0$
D $b^2 - 4ac \neq 0$ **E** $b^2 - 4ac < 0$

Extended-response questions

1 A zoo enclosure for a rare tiger is rectangular in shape and has a trench of width x m all the way around it to ensure the tiger doesn't get far if it tries to escape. The dimensions are as shown.



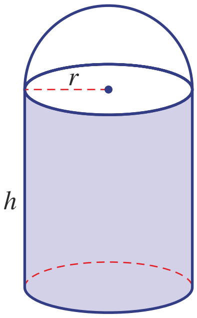
a Write an expression in terms of x for:
  i the length of the enclosure and trench combined
  ii the width of the enclosure and trench combined.

b Use your answers from part **a** to find the area of the overall enclosure and trench, in expanded form.

c Hence, find an expression for the area of the trench alone.

d Zoo restrictions state that the trench must have an area of at least 58 m^2. By solving a suitable equation, find the minimum width of the trench.

2 The surface area S of a cylindrical tank with a hemispherical top is given by the equation $S = 3\pi r^2 + 2\pi rh$, where r is the radius and h is the height of the cylinder.



a If the radius of a tank with height 6 m is 3 m, determine its exact surface area.

b If the surface area of a tank with radius 5 m is 250 m^2, determine its height, to two decimal places.

c The surface area of a tank of height 6 m is found to be 420 m^2.
  i Substitute the values and rewrite the equation in terms of r only.
  ii Rearrange the equation and write it in the form $ar^2 + br + c = 0$.
  iii Solve for r using the quadratic formula and round your answer to two decimal places.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Semester review 1

Linear relations

Short-answer questions

1 Simplify:

a $\dfrac{12 - 8x}{4}$

b $\dfrac{5x - 10}{3} \times \dfrac{12}{x - 2}$

c $\dfrac{3}{4} - \dfrac{2}{a}$

d $\dfrac{4}{x + 2} + \dfrac{5}{x - 3}$

2 a Solve these equations for x.

i $2 - 3x = 14$ ii $2(2x + 3) = 7x$ iii $\dfrac{x - 3}{2} = 5$ iv $\dfrac{3x - 2}{4} = \dfrac{2x + 1}{5}$

b Solve these inequalities for x and graph their solutions on a number line.

i $3x + 2 \leqslant 20$ ii $2 - \dfrac{x}{3} > 1$

3 a Find the gradient and y-intercept for these linear relations and sketch each graph.

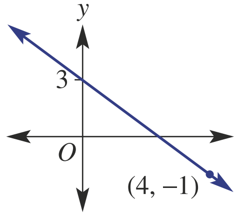
i $y = 3x - 2$ ii $4x + 3y = 6$

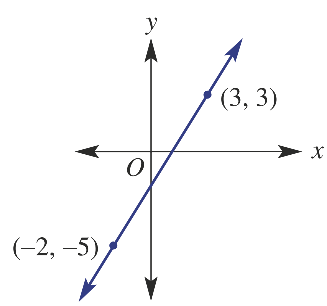
b Sketch by finding the x- and y-intercepts where applicable.

i $y = 2x - 6$ ii $3x + 5y = 15$

iii $x = 3$ iv $y = -2x$

4 Find the equation of the straight lines shown.

a


b


5 Find the value(s) of a in each of the following when:

a The lines $y = ax - 3$ and $y = -3x + 2$ are parallel.

b The gradient of the line joining the points $(3, 2)$ and $(5, a)$ is -3.

c The distance between $(3, a)$ and $(5, 4)$ is $\sqrt{13}$.

d The lines $y = ax + 4$ and $y = \dfrac{1}{4}x - 3$ are perpendicular.

6 Solve these pairs of simultaneous equations.

a $y = 2x - 1$, $y = 5x + 8$

b $2x - 3y = 8$, $y = x - 2$

c $2x + y = 2$, $5x + 3y = 7$

d $3x - 2y = 19$, $4x + 3y = -3$

7 At a fundraising event, two hot dogs and three cans of soft drink cost \$13, and four hot dogs and two cans of soft drink cost \$18. What are the individual costs of a hot dog and a can of soft drink?

Ext

8 Sketch the half planes for these linear inequalities.

a $y \geqslant 3 - 2x$ b $3x - 2y < 9$ c $y > -3$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Multiple-choice questions

1 The simplified form of $2x(4 - 3y) - 3(3x - 4xy)$ is:

A $6xy - x$ **B** $2xy$ **C** $x - 18xy$
D $12xy - 7x$ **E** $2x - 3y - 4xy$

2 The point that is not on the line $y = 3x - 2$ is:

A $(-1, -5)$ **B** $(1, 1)$ **C** $(-2, -4)$
D $(4, 10)$ **E** $(0, -2)$

3 The length, d, and midpoint, M, of the line segment joining the points $(-2, 4)$ and $(3, -2)$ are:

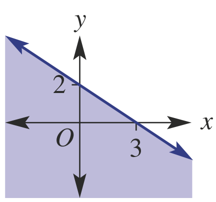
A $d = \sqrt{5}, M = (0.5, 1)$ **B** $d = \sqrt{61}, M = (2.5, 3)$ **C** $d = \sqrt{29}, M = (1, 1)$
D $d = \sqrt{61}, M = (0.5, 1)$ **E** $d = \sqrt{11}, M = (1, 2)$

4 The equation of the line that is perpendicular to the line with equation $y = -2x - 1$ and passes through the point $(1, -2)$ is:

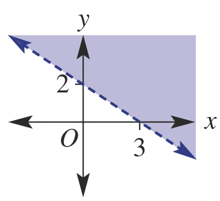
A $y = -\frac{1}{2}x + \frac{3}{2}$ **B** $y = 2x - 2$ **C** $y = -2x - 4$
D $y = x - 2$ **E** $y = \frac{1}{2}x - \frac{5}{2}$

Ext **5** The graph of $3x + 2y < 6$ is:

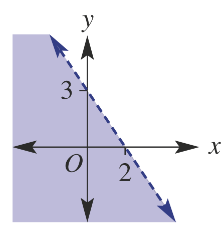
A



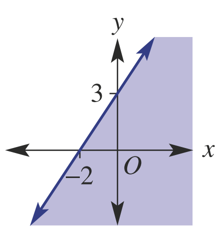
B



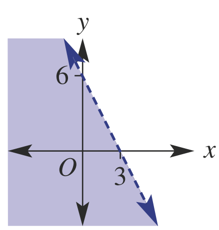
C



D



E



Extended-response question

A block of land is marked on a map with coordinate axes and with boundaries given by the equations $y = 4x - 8$ and $3x + 2y = 17$.

a Solve the two equations simultaneously to find their point of intersection.
b Sketch each equation on the same set of axes, labelling axis intercepts and the point of intersection.

The block of land is determined by the intersecting region $x \geqslant 0$, $y \geqslant 0$, $y \geqslant 4x - 8$ and $3x + 2y \leqslant 17$.

Ext **c** Shade the area of the block of land (i.e. the intersecting region on the graph in part **b**).
d Find the area of the block of land if 1 unit represents 100 metres.

ISBN 978-1-108-77290-7

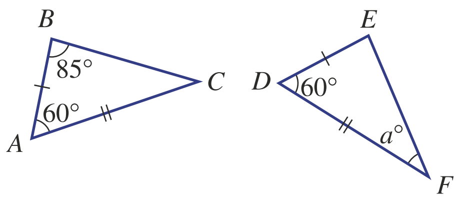
Cambridge University Press
Updated June 2022

Geometry

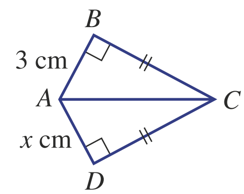
Short-answer questions

1 Prove the following congruence statements, giving reasons, and use this to find the value of the pronumerals.

a $\triangle ABC \equiv \triangle DEF$



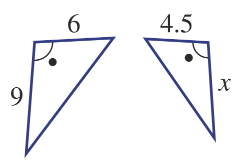
b $\triangle ABC \equiv \triangle ADC$



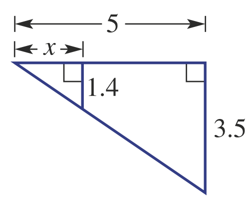
2 Use congruence to prove that a parallelogram (with opposite parallel sides) has equal opposite sides.

3 Find the value of the pronumeral, given these pairs of triangles are similar.

a

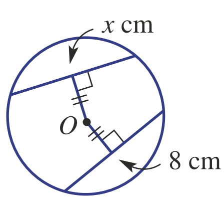


b

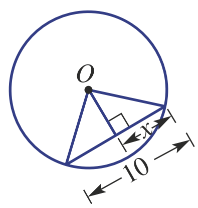


10A **4** Use the chord and circle theorems to find the value of each pronumeral.

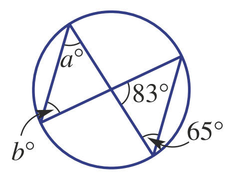
a



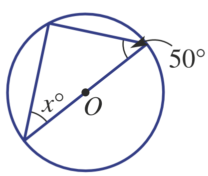
b



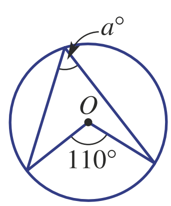
c



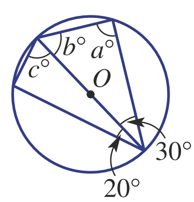
d



e

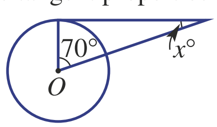


f

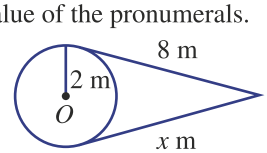


Ext **5** Use tangent properties to find the value of the pronumerals.

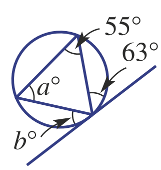
a



b

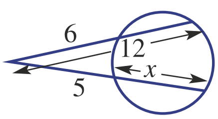


c

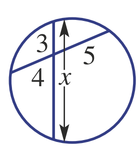


Ext **6** Find the value of x in each figure.

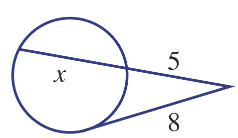
a



b



c

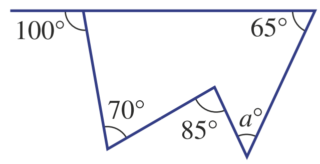


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

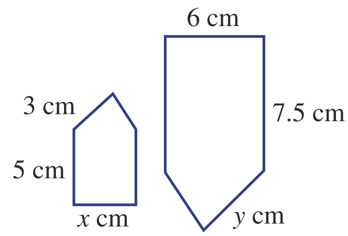
Multiple-choice questions

1 The value of a in the diagram shown is:



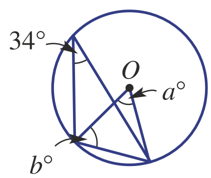
A 40 **B** 25 **C** 30
D 50 **E** 45

2 The values of x and y in these similar figures are:



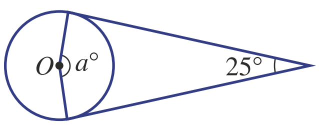
A $x = 2.6, y = 5$
B $x = 4, y = 4.5$
C $x = 4, y = 7.5$
D $x = 3, y = 6$
E $x = 3.5, y = 4.5$

10A **3** The values of the pronumerals in this diagram are:



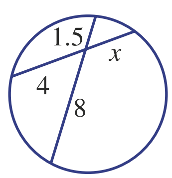
A $a = 17, b = 56$ **B** $a = 34, b = 73$
C $a = 68, b = 56$ **D** $a = 34, b = 34$
E $a = 68, b = 34$

Ext **4** The value of angle a in this diagram is:



A 115 **B** 165 **C** 140
D 130 **E** 155

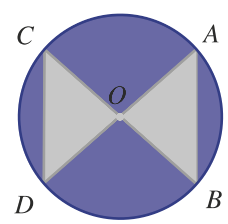
Ext **5** The value of x in this diagram is:



A 5.5 **B** 0.75 **C** 3
D 4 **E** 8

Extended-response question

10A A logo for a car manufacturing company is silver and purple and shaped as shown, with O indicating the centre of the circle.



The radius of the logo is 5 cm and chord AB is 6 cm. Given the two chords are equidistant from the centre of the circle, complete the following.

a What is the length of CD? Give a reason.
b Hence, prove that $\triangle OAB \equiv \triangle OCD$.
c By first finding the length of OM, where M is the point such that $OM \perp AB$, find the area of $\triangle OAB$.
d Hence, determine what percentage of the logo is occupied by the silver portion, given the area of a circle is πr^2. Answer correct to one decimal place.
e Given that $\angle OCD = 53.1°$, what is the angle between the two triangles (i.e. $\angle BOD$)?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Indices and surds

Short-answer questions

10A **1** Simplify:

a $\sqrt{54}$ b $4\sqrt{75}$ c $\frac{3\sqrt{24}}{2}$

d $\sqrt{5} \times \sqrt{2}$ e $3\sqrt{7} \times \sqrt{7}$ f $3\sqrt{6} \times 4\sqrt{8}$

g $\sqrt{15} \div \sqrt{5}$ h $\frac{3\sqrt{30}}{9\sqrt{6}}$ i $\sqrt{\frac{200}{49}}$

10A **2** Simplify fully.

a $2\sqrt{5} + 3\sqrt{7} + 5\sqrt{5} - 4\sqrt{7}$ b $\sqrt{20} - 2\sqrt{5}$ c $\sqrt{18} - 4 + 6\sqrt{2} - 2\sqrt{50}$

10A **3** Expand and simplify these expressions.

a $2\sqrt{3}(\sqrt{5} - 2)$ b $(3\sqrt{5})^2$

10A **4** Rationalise the denominator.

a $\frac{3}{\sqrt{2}}$ b $\frac{2\sqrt{3}}{5\sqrt{6}}$ c $\frac{2 - \sqrt{5}}{\sqrt{5}}$

5 Use the index laws to simplify the following. Express all answers with positive indices.

a $(2x^2)^3 \times 3x^4y^2$ b $\left(\frac{3a}{b^4}\right)^2 \times \frac{2b^{10}}{6(2a^5)^0}$ c $3a^{-5}b^2$ d $\frac{4x^{-2}y^3}{10x^{-4}y^6}$

6 Convert:

a to a basic numeral

i 3.72×10^4 ii 4.9×10^{-6}

b to scientific notation, using three significant figures

i 0.000072973 ii 4 725 400 000

10A **7** a Express in index form.

i $\sqrt{10}$ ii $\sqrt{7x^6}$ iii $4\sqrt[5]{x^3}$ iv $15\sqrt{15}$

b Express in surd form.

i $6^{\frac{1}{2}}$ ii $20^{\frac{1}{5}}$ iii $7^{\frac{3}{4}}$

10A **8** Evaluate without using a calculator.

a 5^{-1} b 2^{-4} c $81^{\frac{1}{4}}$ d $8^{-\frac{1}{3}}$

10A **9** Solve these exponential equations for x.

a $4^x = 64$ b $7^{-x} = \frac{1}{49}$ c $9^x = 27$ d $5^{5x+1} = 125^x$

10 Sketch the graphs.

a $y = 2^x$ b $y = 2^{-x}$ c $y = -2^x$

11 Determine the final amount after 3 years if:

a \$2000 is compounded annually at 6%

b \$7000 is compounded monthly at 3%.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Multiple-choice questions

10A **1** The simplified form of $2\sqrt{45}$ is:

A $\sqrt{90}$ **B** $6\sqrt{5}$ **C** $10\sqrt{3}$
D $18\sqrt{5}$ **E** $6\sqrt{15}$

10A **2** $7\sqrt{3} - 4\sqrt{2} + \sqrt{12} + \sqrt{2}$ simplifies to:

A $7\sqrt{3} - 3\sqrt{2}$ **B** $5\sqrt{3} + 3\sqrt{2}$ **C** $\sqrt{3} - 3\sqrt{2}$
D $9\sqrt{3} - 3\sqrt{2}$ **E** $3\sqrt{3} + 5\sqrt{2}$

10A **3** The expanded form of $2\sqrt{3}(\sqrt{3} + 1)$:

A 7 **B** 12 **C** $9 + \sqrt{3}$
D $18 + 2\sqrt{3}$ **E** $6 + 2\sqrt{3}$

4 The simplified form of $\frac{12(a^3)^{-2}}{(2ab)^2 \times a^2b^{-1}}$, when written using positive indices, is:

A $\frac{6}{a^2b}$ **B** $3a^2$ **C** $\frac{6a}{b}$
D $\frac{3}{a^2b^3}$ **E** $\frac{3}{a^{10}b}$

5 0.00032379 in scientific notation, using three significant figures, is:

A 3.23×10^{-4} **B** 3.24×10^{4} **C** 3.24×10^{-4}
D 32.4×10^{3} **E** 0.324×10^{-5}

Extended-response question

Lachlan's share portfolio is rising at 8% per year and is currently valued at \$80 000.

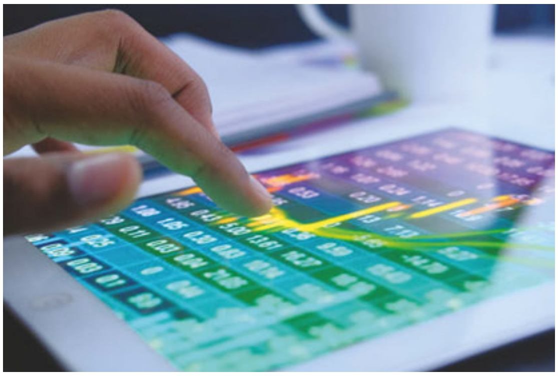
a Determine a rule for the value of Lachlan's share portfolio (V dollars) in n years' time.

b What will be the value of the portfolio, to the nearest dollar:

i next year? **ii** in 4 years' time?

c Use trial and error to find when, to two decimal places, the share portfolio will be worth \$200 000.

d After 4 years, however, the market takes a downwards turn and the share portfolio begins losing value. Two years after the downturn, Lachlan sells his shares for \$96 170. If the market was declining in value at a constant percentage per year, what was this rate of decline, to the nearest percentage?



ISBN 978-1-108-77290-7

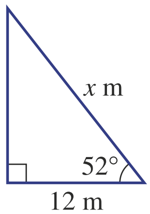
Cambridge University Press
Updated June 2022

Trigonometry

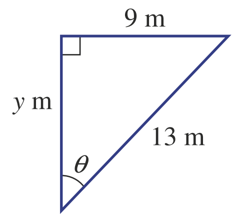
Short-answer questions

1 Find the value of the pronumeral in each right-angled triangle, correct to one decimal place.

a



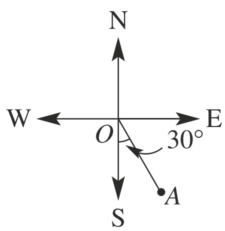
b



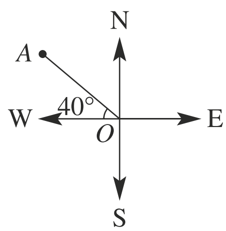
2 For the following bearings, give the true bearing of:

i A from O **ii** O from A

a



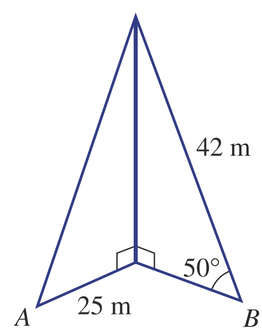
b



10A **3** Two wires reach from the top of an antenna to points A and B on the ground, as shown. Point A is 25 m from the base of the antenna, and the wire from point B is 42 m long and makes an angle of 50° with the ground.

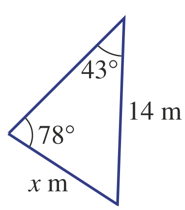
a Find the height of the antenna, to three decimal places.

b Find the angle the wire at point A makes with the ground, to one decimal place.

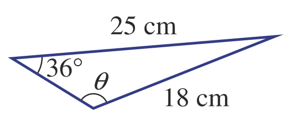


10A **4** Find the value of the pronumeral, correct to one decimal place.

a



b θ is obtuse



10A **5** Find the largest angle, correct to one decimal place, in a triangle with side lengths 8 m, 12 m and 15 m.

10A **6** Convert:

a 50° to radians **b** $\dfrac{3\pi^c}{4}$ to degrees

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

10A **7** **a** If $\theta = 223°$, state which of $\sin \theta$, $\cos \theta$ and $\tan \theta$ are positive?

b Choose the angle θ to complete each statement.

i $\sin 25° = \sin \theta$, where θ is obtuse.

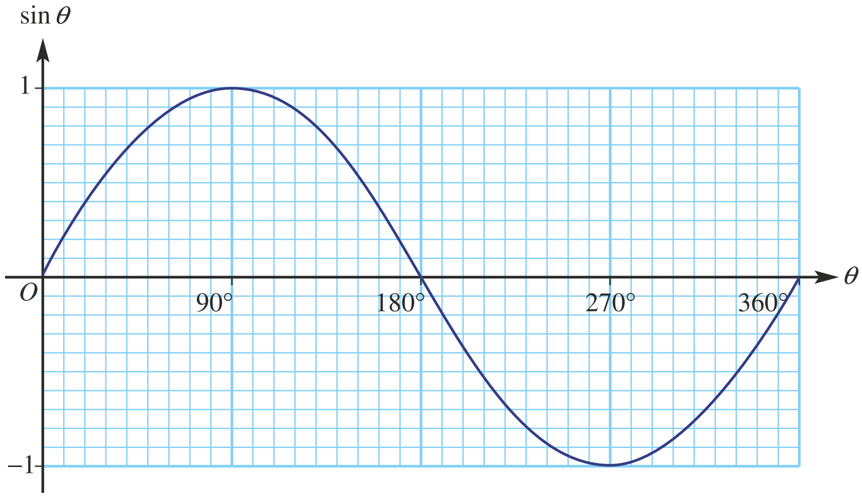
ii $\tan 145° = -\tan \theta$, where θ is acute.

iii $\cos 318° = \cos \theta$, where θ is the reference angle.

c State the exact value of:

i $\cos \frac{\pi^c}{3}$ **ii** $\sin 135°$ **iii** $\tan 330°$

10A **8** Use the graph of $\sin \theta$ shown to answer the following.



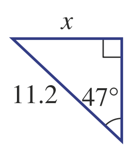
a Estimate the value of $\sin \theta$ for $\theta = 160°$.

b Estimate the two values of θ for which $\sin \theta = -0.8$.

c Is $\sin 40° < \sin 120°$?

Multiple-choice questions

1 The value of x in the triangle shown is approximately:



A 7.6 **B** 12.0 **C** 10.4
D 6.5 **E** 8.2

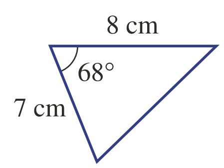
2 A bird 18 m up in a tree spots a worm on the ground 12 m from the base of the tree. The angle of depression from the bird to the worm is closest to:

A 41.8° **B** 56.3° **C** 61.4°
D 33.7° **E** 48.2°

3 A walker travels due south for 10 km and then on a true bearing of 110° for 3 km. The total distance south from the starting point to the nearest kilometre is:

A 11 km **B** 1 km **C** 9 km
D 13 km **E** 15 km

10A **4** The area of the triangle shown is closest to:



A 69 cm^2 **B** 52 cm^2 **C** 28 cm^2
D 26 cm^2 **E** 10 cm^2

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

10A **5** Choose the *incorrect* statement.

A $\theta = 290°$ is in quadrant 4

B $\sin 120° = \frac{\sqrt{3}}{2}$

C $\cos 110° = -\cos 20°$

D $\tan \theta$ is positive for $200° < \theta < 250°$.

E $\sin 230° = -\sin 50°$

Extended-response question

A group of walkers set out on a trek to get to the base of a mountain range. The mountains have two peaks, which are 112 m and 86 m above ground level from the base. The angle of elevation from the peak of the smaller mountain to the peak of the taller mountain is 14°.

a Find the horizontal distance between the two mountain peaks, correct to one decimal place.

10A To get to the base of the mountain range, the walkers set out from the national park entrance on a bearing of 52°T for a distance of 13 km and then turn on a bearing of 340°T for the last 8 km of the trek.

b Draw a diagram representing the trek. Label all known measurements.

c If the walkers are able to trek directly from their start location to their endpoint, what distance would they cover? Round your answer to three decimal places.

d After they have explored the mountains, the group will be taken by bus back along the direct path from their end location to the park entrance. Determine the true bearing on which they will travel. Round your answer to the nearest degree.

Quadratic expressions and equations

Short-answer questions

1 Expand and simplify.

a $(3x + 1)(3x - 1)$ **b** $(2x - 5)^2$ **c** $(2x + 3)(x + 5) - (3x - 5)(x - 4)$

2 Factorise fully these quadratics. Remember to take out any common factors first.

a $4x^2 - y^2$ **b** $(x + 2)^2 - 7$ **c** $3x^2 - 48$

d $x^2 + 5x - 14$ **e** $x^2 - 10x + 25$ **f** $2x^2 - 16x + 24$

10A **3** Factorise these non-monic quadratics.

a $3x^2 - 2x - 8$ **b** $6x^2 + 7x - 3$ **c** $10x^2 - 23x + 12$

4 Solve these quadratic equations using the Null Factor Law.

a $2x(x - 3) = 0$ **b** $(x + 4)(2x - 1) = 0$

c $x^2 + 5x = 0$ **d** $x^2 - 16 = 0$

e $x^2 - 7 = 0$ **f** $x^2 - 4x + 4 = 0$

g $x^2 - 5x - 24 = 0$ 10A **h** $3x^2 + 5x - 2 = 0$

5 Solve these quadratic equations by first writing them in standard form.

a $x^2 = 40 - 3x$ **b** $x(x - 6) = 4x - 21$ **c** $\frac{x + 20}{x} = x$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

6 **a** Factorise by completing the square.

  i $x^2 - 6x + 4$ **ii** $x^2 + 4x + 7$ **iii** $x^2 + 3x + 1$

b Use your answers to part **a** to solve these equations, if possible.

  i $x^2 - 6x + 4 = 0$ **ii** $x^2 + 4x + 7 = 0$ **iii** $x^2 + 3x + 1 = 0$

7 Solve these quadratic equations using the quadratic formula. Leave your answers in exact surd form.

  a $2x^2 + 3x - 6 = 0$ **b** $x^2 - 4x - 6 = 0$

Multiple-choice questions

1 The expanded form of $2(2x - 3)(3x + 2)$ is:

A $12x^2 - 5x - 6$ **B** $12x^2 - 12$ **C** $12x^2 - 10x - 12$

D $24x^2 - 20x - 24$ **E** $12x^2 - x - 6$

2 The factorised form of $25y^2 - 9$ is:

A $(5y - 3)^2$ **B** $(5y - 3)(5y + 3)$ **C** $(25y - 3)(y + 3)$

D $(5y - 9)(5y + 1)$ **E** $5(y + 1)(y - 9)$

3 $\dfrac{x^2 - 4}{x^2 - x - 6} \times \dfrac{x^2 - 4x + 3}{4x - 8}$ simplifies to:

A $x - 2$ **B** $\dfrac{x + 3}{12}$ **C** $\dfrac{x^2 + 1}{2x}$

D $\dfrac{x - 1}{4}$ **E** $\dfrac{x^2}{x - 2}$

4 The solution(s) to the quadratic equation $x^2 - 4x + 4 = 0$ is/are:

A $x = 0, 4$ **B** $x = 2$ **C** $x = 1, 4$

D $x = 2, -2$ **E** $x = -1, 4$

10A **5** A quadratic equation $ax^2 + bx + c = 0$ has a discriminant equal to 17. This tells us that:

A The equation has a solution $x = 17$.

B The equation has no solutions.

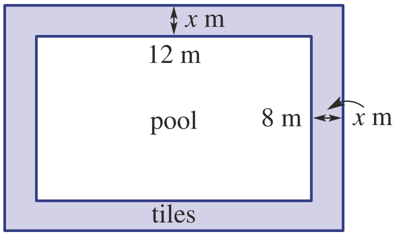
C $a + b + c = 17$

D The equation has two solutions.

E The equation has one solution.

Extended-response question

A rectangular backyard swimming pool, measuring 12 metres by 8 metres, is surrounded by a tiled path of width x metres, as shown.



a Find a simplified expression for the area of the tiled path.

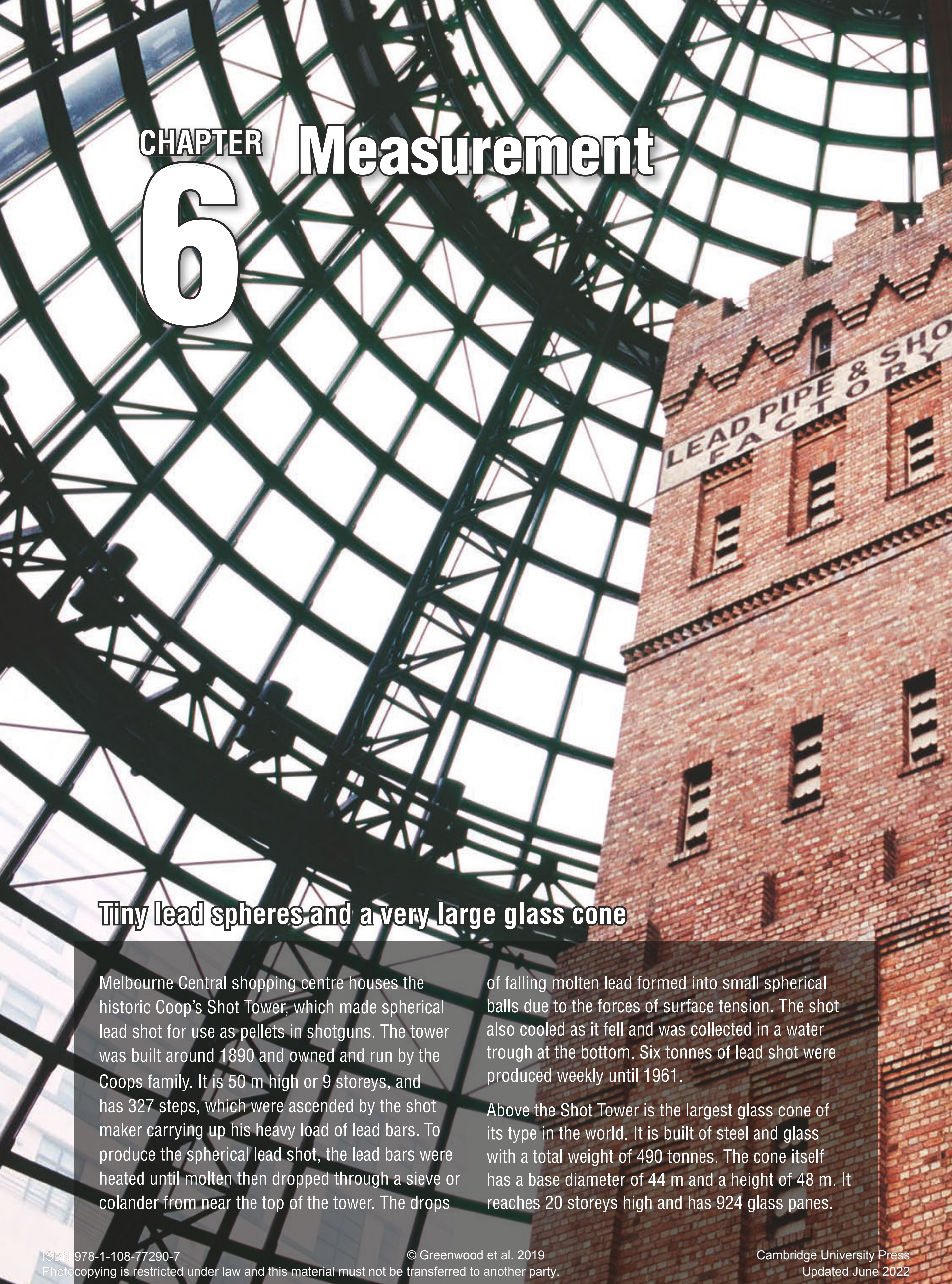
b If $x = 1$, what is the tiled area?

c Solve an appropriate equation to determine the width, x metres, if the tiled area is $156\ \text{m}^2$.

d Find the width, x metres, if the tiled area is $107.36\ \text{m}^2$. Use the quadratic formula.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022



CHAPTER 6 Measurement

Tiny lead spheres and a very large glass cone

Melbourne Central shopping centre houses the historic Coop's Shot Tower, which made spherical lead shot for use as pellets in shotguns. The tower was built around 1890 and owned and run by the Coops family. It is 50 m high or 9 storeys, and has 327 steps, which were ascended by the shot maker carrying up his heavy load of lead bars. To produce the spherical lead shot, the lead bars were heated until molten then dropped through a sieve or colander from near the top of the tower. The drops of falling molten lead formed into small spherical balls due to the forces of surface tension. The shot also cooled as it fell and was collected in a water trough at the bottom. Six tonnes of lead shot were produced weekly until 1961.

Above the Shot Tower is the largest glass cone of its type in the world. It is built of steel and glass with a total weight of 490 tonnes. The cone itself has a base diameter of 44 m and a height of 48 m. It reaches 20 storeys high and has 924 glass panes.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Online resources

A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

In this chapter

Victorian Curriculum

MEASUREMENT AND GEOMETRY
Using units of measurement

Solve problems involving surface area and volume for a range of prisms, cylinders and composite solids (VCMMG343)

(10A) Solve problems involving surface area and volume of right pyramids, right cones, spheres and related composite solids (VCMMG365)

Pythagoras and trigonometry

Solve right-angled triangle problems including those involving direction and angles of elevation and depression (VCMMG346)

Apply Pythagoras' theorem and trigonometry to solve three-dimensional problems in right-angled triangles (VCMMG370)

6A Review of length CONSOLIDATING

Learning intentions
- To know how to convert between metric units of length
- To know the meaning of the terms perimeter, circumference and sector
- To review how to find the perimeter of a closed shape
- To be able to find the circumference of a circle and the perimeter of a sector
- To be able to find both exact and rounded answers to problems involving perimeters

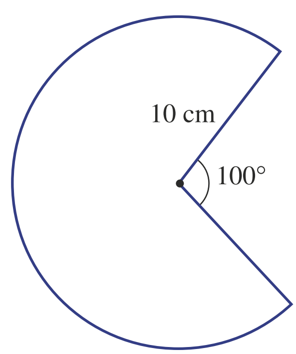
Length measurements are common in many areas of mathematics, science and engineering, and are clearly associated with the basic measures of perimeter and circumference, which will be studied here.

Auto engineers apply arc geometry in vehicle steering design. When turning, the outside and inside wheels follow arcs of differing radii and length; hence these wheels rotate at different rates and are steered at slightly different angles.

LESSON STARTER The simple sector

This sector looks simple enough but can you describe how to find its perimeter? Discuss these points to help.

- Recall the rule for the circumference of a circle.
- What is a definition of perimeter?
- What fraction of a circle is this sector?
- Find the perimeter using both exact and rounded numbers.



ISBN 978-1-108-77290-7 et a 19
Photocopying is restricted under law and this m st n o another pa
Cambridge University Press
Updated June 2022

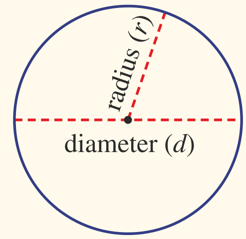
KEY IDEAS

- Converting between metric units of length

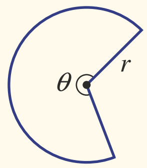
$$\text{km} \xrightarrow{\times 1000} \text{m} \xrightarrow{\times 100} \text{cm} \xrightarrow{\times 10} \text{mm}$$

$$\text{km} \xleftarrow{\div 1000} \text{m} \xleftarrow{\div 100} \text{cm} \xleftarrow{\div 10} \text{mm}$$

- **Perimeter** is the distance around the outside of a closed shape.
- The **circumference** of a circle is the distance around the circle.
  - $C = 2\pi r = \pi d$, where $d = 2r$.



- A **sector** of a circle is a portion of a circle enclosed by two radii and the arc between them.
- Perimeter of a sector
  - $P = 2r + \dfrac{\theta}{360} \times 2\pi r$

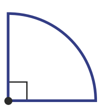


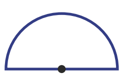
BUILDING UNDERSTANDING

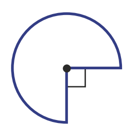
1 Convert the following length measurements to the units given in brackets.

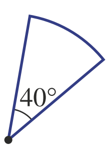
a 4 cm (mm) **b** 0.096 m (cm) **c** 0.001 km (m)
d 800 cm (m) **e** 297 m (km) **f** 5102 mm (cm)

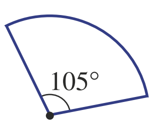
2 What fraction of a circle (in simplest form) is shown in these sectors?

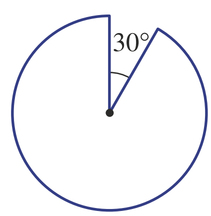
a


b


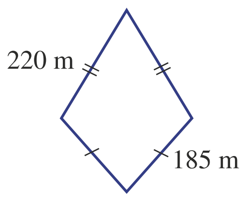
c


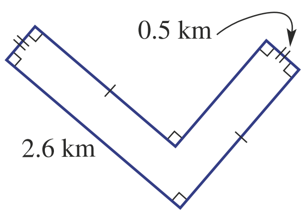
d


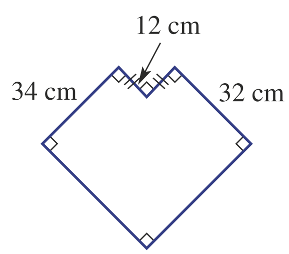
e


f


3 Find the perimeter of these shapes.

a


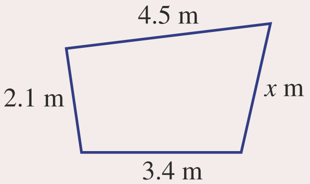
b


c


Example 1 Finding the perimeter of polygons

Consider the given two-dimensional shape.

a Find the perimeter of the shape when $x = 2.6$.
b Find x when the perimeter is 11.9 m.
c Write an expression for x in terms of the perimeter, P.



SOLUTION

a Perimeter $= 4.5 + 2.1 + 3.4 + 2.6$
$= 12.6$ m

b $11.9 = 4.5 + 2.1 + 3.4 + x$
$= 10 + x$
$\therefore x = 1.9$

c $P = 4.5 + 2.1 + 3.4 + x$
$= 10 + x$
$\therefore x = P - 10$

EXPLANATION

Simply add the lengths of all four sides using $x = 2.6$.

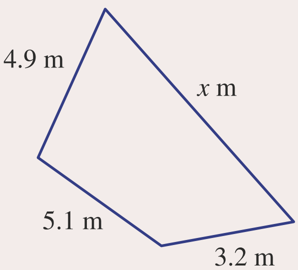
Add all four sides and set equal to the perimeter 11.9. Simplify and solve for x by subtracting 10 from both sides.

Use P for the perimeter and add all four sides. Simplify and rearrange to make x the subject.

Now you try

Consider the given two-dimensional shape.

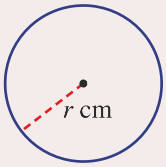
a Find the perimeter of the shape when $x = 7$.
b Find x when the perimeter is 21.3 m.
c Write an expression for x in terms of the perimeter, P.



Example 2 Using the formula for the circumference of a circle

If a circle has radius r cm, find the following, rounding the answer to two decimal places where necessary.

a the circumference of a circle when $r = 2.5$
b a rule for r in terms of the circumference, C
c the radius of a circle with a circumference of 10 cm



SOLUTION

a Circumference $= 2\pi r$
$= 2\pi(2.5)$
$= 15.71$ cm (to 2 d.p.)

EXPLANATION

Write the rule for circumference and substitute $r = 2.5$, then evaluate and round as required.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b $C = 2\pi r$

$\therefore r = \dfrac{C}{2\pi}$

Write the rule for circumference, then divide both sides by 2π to make r the subject.

c $r = \dfrac{C}{2\pi}$

$= \dfrac{10}{2\pi}$

$= 1.59$ cm (to 2 d.p.)

Substitute $C = 10$ into the rule from part **b** and evaluate.

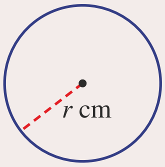
Now you try

If a circle has radius r cm, find the following, rounding the answer to two decimal places where necessary.

a the circumference of a circle when $r = 3.5$

b a rule for r in terms of the circumference, C

c the radius of a circle with a circumference of 12 cm

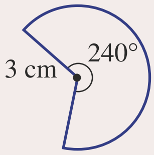


Example 3 Finding perimeters of sectors

This sector has a radius of 3 cm.

a Find the sector's exact perimeter.

b Find the perimeter, correct to one decimal place.



SOLUTION

a $P = 2r + \dfrac{\theta}{360} \times 2\pi r$

$= 2 \times 3 + \dfrac{240}{360} \times 2 \times \pi \times 3$

$= 6 + \dfrac{2}{3} \times 2 \times \pi \times 3$

$= 6 + 4\pi$ cm

b $P = 6 + 4\pi$

$= 18.6$ cm (to 1 d.p.)

EXPLANATION

The perimeter of a sector consists of two radii and a fraction $\left(\dfrac{240}{360} = \dfrac{2}{3}\right)$ of the circumference of a circle.

$6 + 4\pi$ is the exact value.

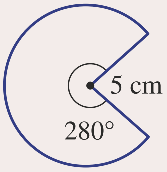
Round to the required one decimal place, using a calculator.

Now you try

This sector has a radius of 5 cm.

a Find the sector's exact perimeter.

b Find the perimeter, correct to one decimal place.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 6A

FLUENCY	1, 2, 3(½), 4	2, 3(½), 4, 5(½)	2, 3(½), 4, 5(½)

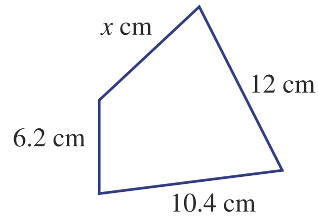
Example 1

1 Consider the given two-dimensional shape.

a Find the perimeter of the shape when $x = 8$.

b Find x when the perimeter is 33.7 cm.

c Write an expression for x in terms of the perimeter, P.



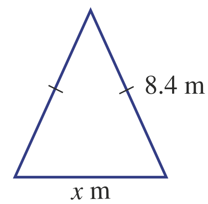
Example 1

2 Consider the given two-dimensional shape.

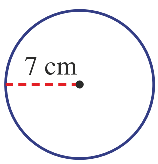
a Find the perimeter of the shape when $x = 5$.

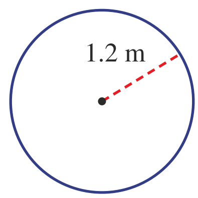
b Find x when the perimeter is 20 m.

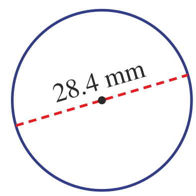
c Write an expression for x in terms of the perimeter, P.

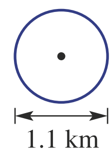


3 Find the circumference of these circles, correct to two decimal places.

a


b


c


d


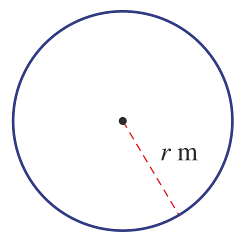
Example 2

4 If a circle has radius r m, find the following, rounding to two decimal places, where necessary.

a The circumference of a circle when $r = 12$.

b A rule for r in terms of the circumference, C.

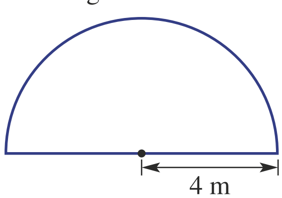
c The radius of a circle with a circumference of 35 m.

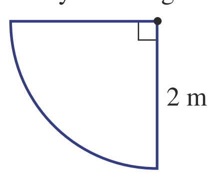


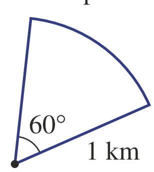
Example 3

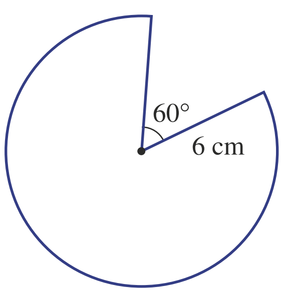
5 Find the perimeter of these sectors:

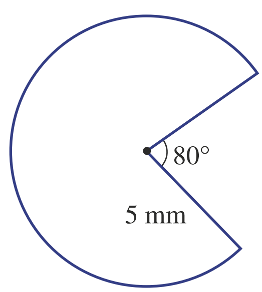
i using exact values **ii** by rounding the answer to one decimal place.

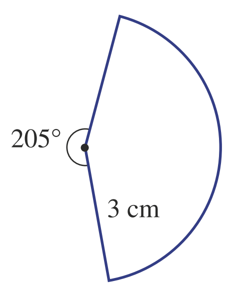
a


b


c


d


e


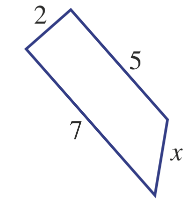
f


ISBN 978-1-108-77290-7

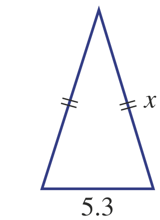
Cambridge University Press
Updated June 2022

PROBLEM-SOLVING			
	6(1/2), 7	6(1/2), 7, 8(1/2)	6(1/3), 8, 9

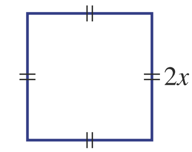
6 Find the value of x for these shapes with the given perimeters.

a 

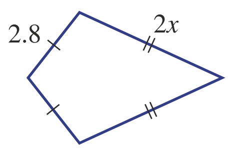
Perimeter = 17

b 

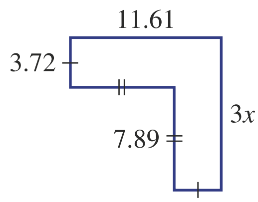
Perimeter = 22.9

c 

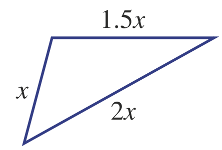
Perimeter = 0.072

d 

Perimeter = 16.2

e 

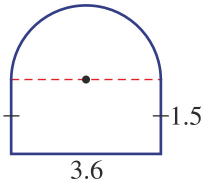
Perimeter = 46.44

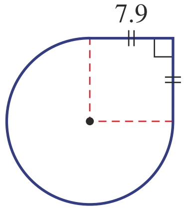
f 

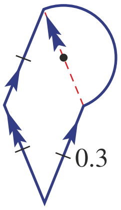
Perimeter = 10.8

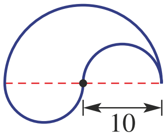
7 A rectangular rose garden of length 15 m and width 9 m is surrounded by a path of width 1.2 m. Find the distance around the outside of the path.

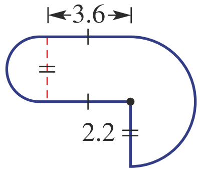
8 Find the perimeter of these composite shapes, correct to two decimal places.

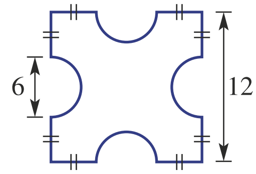
a 

b 

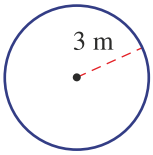
c 

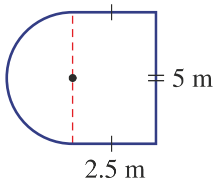
d 

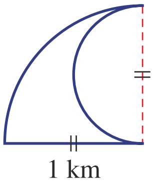
e 

f 

9 Find the perimeter of these shapes, giving your answers as exact values.

a 

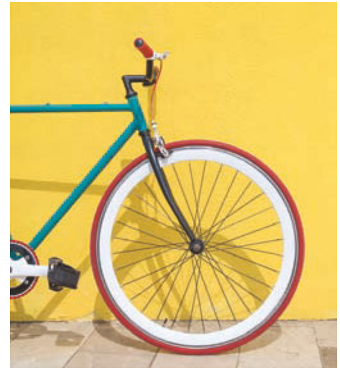
b 

c 

ISBN 978-1-108-77290-7

REASONING	10	10, 11	11–13

10 A bicycle has wheels with diameter 64 cm.



a Find how far, correct to the nearest centimetre, the bicycle moves when the wheels turn:

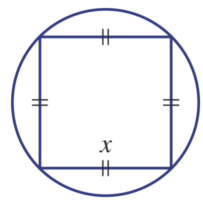
- **i** one rotation
- **ii** five rotations

b How many rotations are required for the bike to travel 10 km? Round your answer to the nearest whole number.

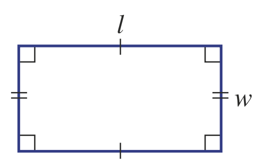
c Find an expression for the number of rotations required to cover 10 km if the wheel has a diameter of d cm.

11 A square of side length n has the same perimeter as a circle. What is the radius of the circle? Give an expression in terms of n.

12 A square of side length x just fits inside a circle. Find the exact circumference of the circle in terms of x.



13 Consider a rectangle with perimeter P, length l and width w.



a Express l in terms of w and P.

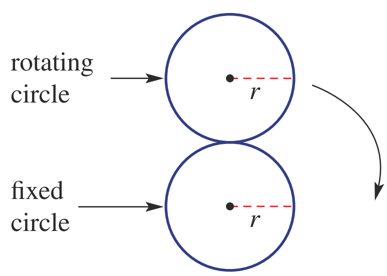
b Express l in terms of w when $P = 10$.

c If $P = 10$, state the range of all possible values of w.

d If $P = 10$, state the range of all possible values of l.

ENRICHMENT: Rotating circles	–	–	14

14 When a circle rolls around the outside of another circle it will rotate by a certain angle.



For these problems the fixed circle will have radius r. Given the following conditions, by how many degrees will the moving circle rotate if it rolls around the fixed circle once?

a Assume the rotating circle has radius r (shown).

b Assume the rotating circle has radius $\frac{1}{2}r$.

c Assume the rotating circle has radius $2r$.

d Assume the rotating circle has radius $\frac{1}{3}r$.

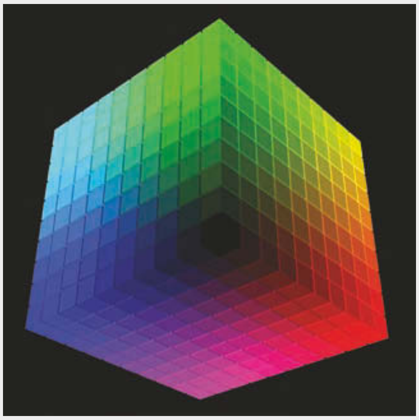
ISBN 978-1-108-77290-7

6B Pythagoras' theorem

Learning intentions

- To know the relationship between the square of the hypotenuse of a right-angled triangle and the sum of the squares of the other two side lengths
- To be able to apply Pythagoras' theorem to find a missing side length of a right-angled triangle
- To be able to identify right-angled triangles in 3D objects and apply Pythagoras' theorem

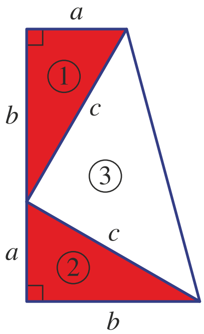
You will recall that for any right-angled triangle we can connect thelength of the three sides using Pythagoras' theorem. When given two of the sides, we can work out the length of the remaining side. This has applications in all sorts of two- and three-dimensional problems.



In a colour cube, each colour has coordinates (x, y, z). Colour specialists use Pythagoras' theorem in 3D to find the shortest distance between any two colours. Applications include print and digital advertising, web page design and image editing.

LESSON STARTER President Garfield's proof

Five years before he became president of the United States of America in 1881, James Garfield discovered a proof of Pythagoras' theorem. It involves arranging two identical right-angled triangles (① and ②) to form a trapezium, as shown.



- Use the formula for the area of a trapezium $\left(\frac{1}{2}(a+b)h\right)$ to find an expression for the area of the entire shape.
- Explain why the third triangle ③ is right-angled.
- Find an expression for the sum of the areas of the three triangles.
- Hence, prove that $c^2 = a^2 + b^2$.

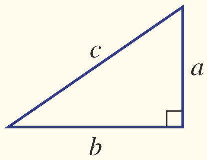
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

KEY IDEAS

- **Pythagoras' theorem** states that:
  The sum of the squares of the two shorter sides of a right-angled triangle equals the square of the hypotenuse.

  $a^2 + b^2 = c^2$

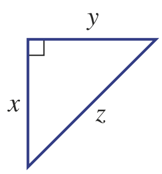


- To write an answer using an exact value, use a square root sign where possible (e.g. $\sqrt{3}$).

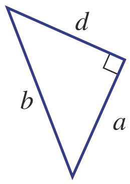
BUILDING UNDERSTANDING

1. Find the value of a in these equations. Express your answer in exact form using a square root sign. Assume $a > 0$.
   - **a** $a^2 + 3^2 = 8^2$
   - **b** $2^2 + a^2 = 9^2$
   - **c** $a^2 + a^2 = 2^2$
   - **d** $a^2 + a^2 = 10^2$
2. State an equation connecting the pronumerals in these right-angled triangles.

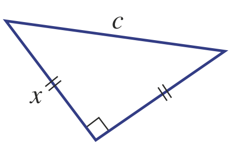
a



b



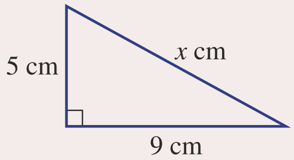
c



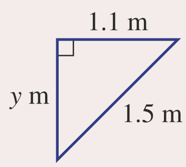
Example 4 Finding side lengths using Pythagoras' theorem

Find the length of the unknown side in these right-angled triangles, correct to two decimal places.

a



b



SOLUTION	EXPLANATION
a $c^2 = a^2 + b^2$	
$\therefore x^2 = 5^2 + 9^2$	x cm is the length of the hypotenuse. Substitute the two shorter sides $a = 5$ and $b = 9$ (or $a = 9$ and $b = 5$).
$= 106$	
$\therefore x = \sqrt{106}$	Find the square root of both sides and round your answer as required.
$= 10.30$ (to 2 d.p.)	
The length of the unknown side is 10.30 cm.	

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b $a^2 + b^2 = c^2$

$y^2 + 1.1^2 = 1.5^2$

$y^2 = 1.5^2 - 1.1^2$

$= 2.25 - 1.21$

$= 1.04$

$\therefore y = \sqrt{1.04}$

$= 1.02$ (to 2 d.p.)

The length of the unknown side is 1.02 m.

Substitute the shorter side $b = 1.1$ and the hypotenuse $c = 1.5$.

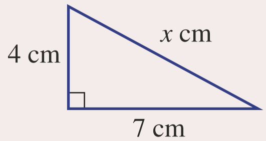
Subtract 1.1^2 from both sides.

Find the square root of both sides and evaluate.

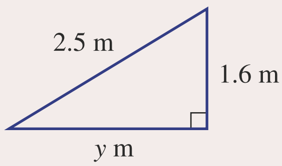
Now you try

Find the length of the unknown side in these right-angled triangles, correct to two decimal places.

a



b



For centuries builders, carpenters and landscapers have used Pythagoras' theorem to construct right angles for their foundations and plots. The ancient Egyptians used three stakes joined by a rope to make a triangular shape with side lengths of 3, 4 and 5 units, which form a right angle when the rope is taut.

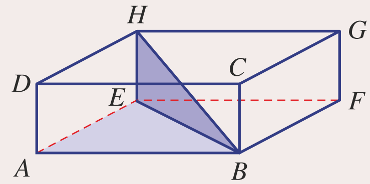
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 5 Using Pythagoras' theorem in 3D

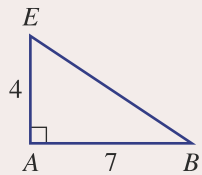
Consider a rectangular prism $ABCDEFGH$ with the side lengths $AB = 7$, $AE = 4$ and $EH = 2$. Find:

a BE, leaving your answer in exact form
b BH, correct to two decimal places.



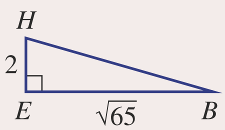
SOLUTION

a



$$c^2 = a^2 + b^2$$
$$\therefore BE^2 = 4^2 + 7^2$$
$$= 65$$
$$\therefore BE = \sqrt{65}$$

b



$$BH^2 = HE^2 + EB^2$$
$$= 2^2 + (\sqrt{65})^2$$
$$= 4 + 65$$
$$= 69$$
$$\therefore BE = \sqrt{69}$$
$$= 8.31 \text{ (to 2 d.p.)}$$

EXPLANATION

Draw the appropriate right-angled triangle with two known sides.

Substitute $a = 4$ and $b = 7$.
Solve for BE exactly.

Leave intermediate answers in surd form to reduce the chance of accumulating errors in further calculations.

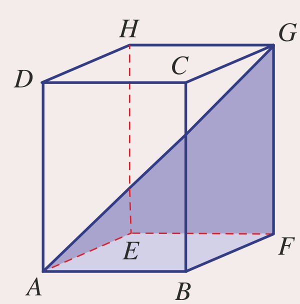
Draw the appropriate triangle.

Substitute $HE = 2$ and $EB = \sqrt{65}$.
Note: $(\sqrt{65})^2 = \sqrt{65} \times \sqrt{65} = 65$.

Now you try

Consider a rectangular prism $ABCDEFGH$ with the side lengths $AB = 5$, $BF = 6$ and $FG = 7$. Find:

a AF, leaving your answer in exact form
b AG, correct to two decimal places.



ISBN 978-1-108-77290-7

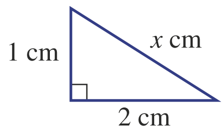
Exercise 6B

FLUENCY	1, 2–4(1/2)	2–5(1/2)	2–3(1/3), 4(1/2), 5(1/3)

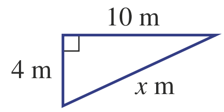
1 Find the length of the unknown side in these right-angled triangles, correct to two decimal places.

Example 4a

a **i**

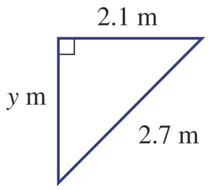


ii

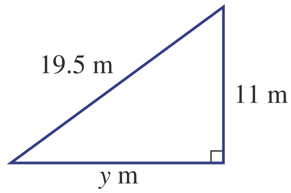


Example 4b

b **i**



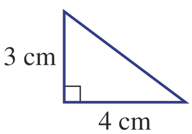
ii



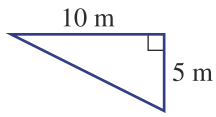
Example 4a

2 Use Pythagoras' theorem to find the length of the hypotenuse for these right-angled triangles. Round your answers to two decimal places where necessary.

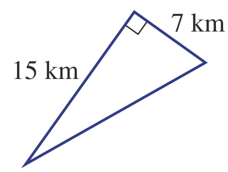
a



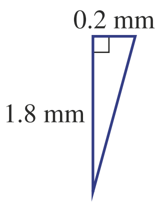
b



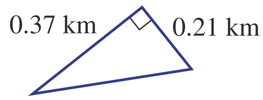
c



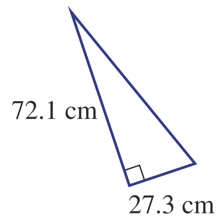
d



e



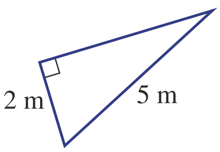
f



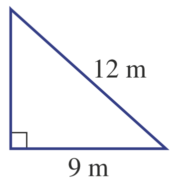
Example 4b

3 Find the length of the unknown side in these right-angled triangles, correct to two decimal places.

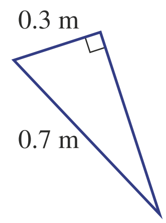
a



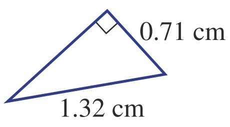
b



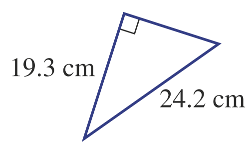
c



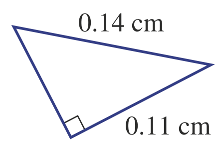
d



e



f



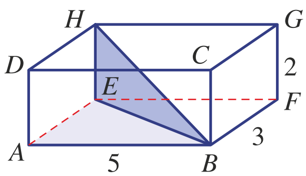
Example 5

4 For each of these rectangular prisms, find:

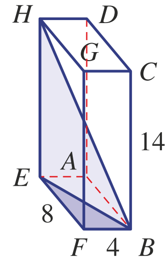
i BE, leaving your answer in exact form

ii BH, correct to two decimal places.

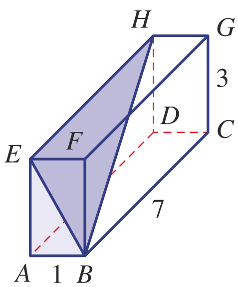
a



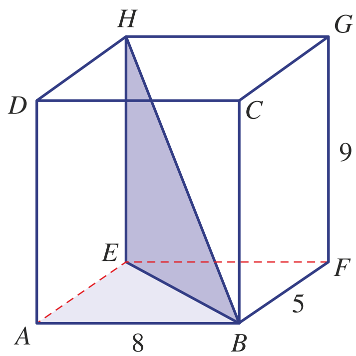
b



c

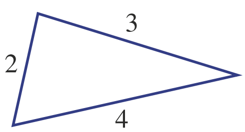


d

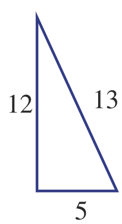


5 Use Pythagoras' theorem to help decide whether these triangles are right-angled. They may not be drawn to scale.

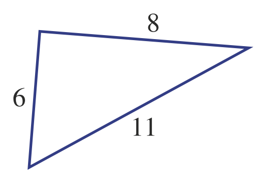
a



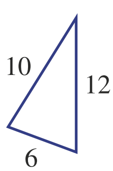
b



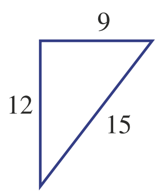
c



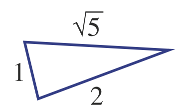
d



e



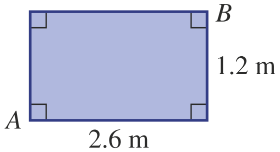
f



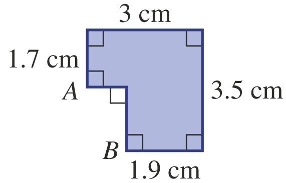
PROBLEM-SOLVING	6(½), 7	6(½), 7, 8(½)	8–10

6 Use Pythagoras' theorem to find the distance between points A and B in these diagrams, correct to two decimal places.

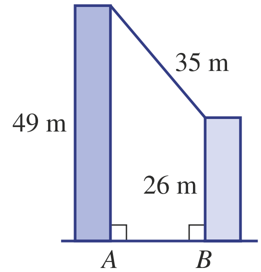
a



b

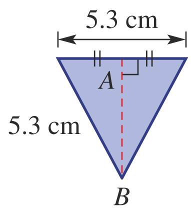


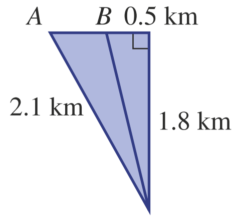
c

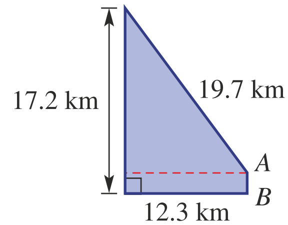


ISBN 978-1-108-77290-7

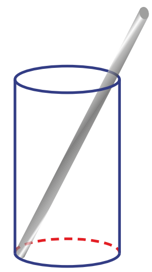
Cambridge University Press
Updated June 2022

d 

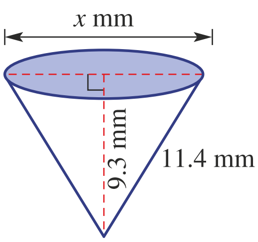
e 

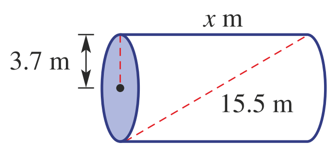
f 

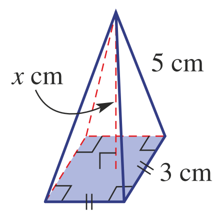
7 A 20 cm drinking straw sits diagonally in a glass of radius 3 cm and height 10 cm. What length of straw protrudes from the glass? Round your answer to one decimal place.

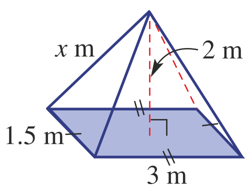


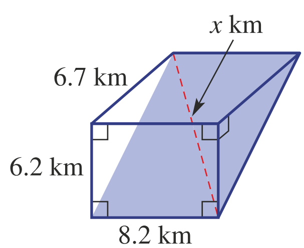
8 Find the value of x, correct to two decimal places, in these three-dimensional solids.

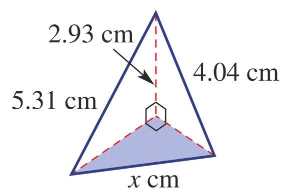
a 

b 

c 

d 

e 

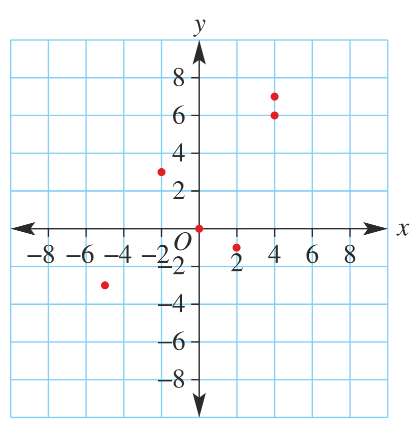
f 

9 Find the exact distance between these pairs of points on a number plane.

a (0, 0) and (4, 6)

b (−2, 3) and (2, −1)

c (−5, −3) and (4, 7)



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

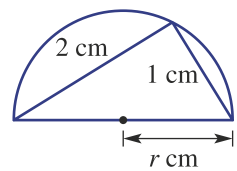
10 **a** Find the length of the longest rod that will fit inside these objects. Give your answer correct to one decimal place.

- **i** a cylinder with diameter 10 cm and height 20 cm
- **ii** a rectangular prism with side lengths 10 cm, 20 cm and 10 cm

b Investigate the length of the longest rod that will fit in other solids, such as triangular prisms, pentagonal prisms, hexagonal prisms and truncated rectangular pyramids. Include some three-dimensional diagrams.

REASONING	11	11, 12	12, 13

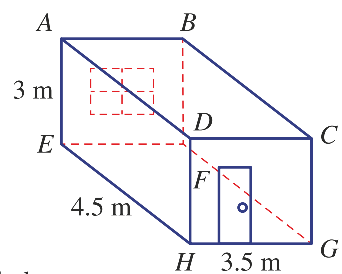
11 Two joining chords in a semicircle have lengths 1 cm and 2 cm, as shown. Find the exact radius, r cm, of the semicircle. Give reasons.



12 The diagonals of a rectangle are 10 cm long. Find the exact dimensions of the rectangle if:

- **a** the length is twice the width
- **b** the length is three times the width
- **c** the length is ten times the width.

13 Streamers are used to decorate the interior of a rectangular room that is 4.5 m long, 3.5 m wide and 3 m high, as shown.



a Find the length of streamer, correct to two decimal places, required to connect from:

- **i** A to H
- **ii** E to B
- **iii** A to C
- **iv** A to G via C
- **v** E to C via D
- **vi** E to C directly.

b Find the shortest length of streamer required, correct to two decimal places, to reach from A to G if the streamer is not allowed to reach across open space. (*Hint*: Consider a net of the prism.)

ENRICHMENT: How many proofs?	–	–	14

14 There are hundreds of proofs of Pythagoras' theorem.

- **a** Research some of these proofs using the internet and pick one you understand clearly.
- **b** Write up the proof, giving full reasons.
- **c** Present your proof to a friend or the class. Show all diagrams, algebra and reasons.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

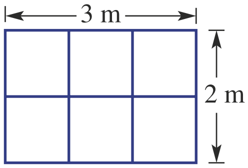
6C Review of area CONSOLIDATING

Learning intentions

- To understand the meaning of square units and the definition of area
- To know how to convert between metric units of area
- To know how to find the area of a square, rectangle, triangle, rhombus, parallelogram, trapezium, kite, circle and sector
- To be able to use the formulas of regular shapes to find areas of composite shapes using addition or subtraction

Area is a measure of surface and is expressed as a number of square units.

By the inspection of a simple diagram like the one shown, a rectangle with side lengths 2 m and 3 m has an area of 6 square metres or 6 m^2.



Area = 6 m^2

For rectangles and other basic shapes, we can use area formulas to help us calculate the number of square units.

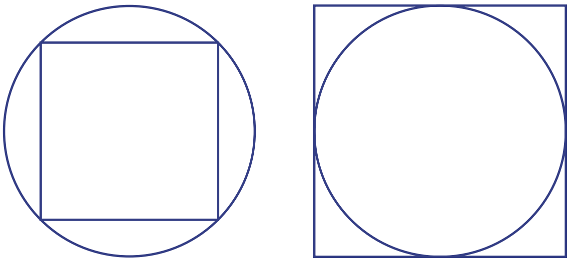
Some common metric units for area include square kilometres (km^2), square metres (m^2), square centimetres (cm^2) and square millimetres (mm^2).

Architects apply circle sector geometry to design spiral staircases. A circle sector with the stairwell's diameter and arc length equal to the spiral's outer length is used. This sector is divided into equal smaller sectors for the steps.

LESSON STARTER Pegs in holes

Discuss, with reasons relating to the area of the shapes, which is the better fit:

- a square peg in a round hole?
- a round peg in a square hole?



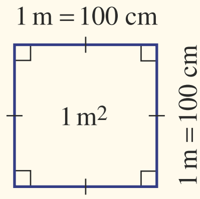
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

KEY IDEAS

- Conversion of units of area

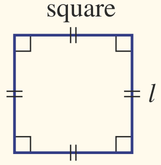
$$km^2 \underset{\div 1000^2}{\overset{\times 1000^2}{\rightleftarrows}} m^2 \underset{\div 100^2}{\overset{\times 100^2}{\rightleftarrows}} cm^2 \underset{\div 10^2}{\overset{\times 10^2}{\rightleftarrows}} mm^2$$



$1\text{ m}^2 = 100\text{ cm} \times 100\text{ cm}$
$= 100^2\text{ cm}^2$

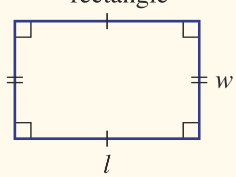
- 1 hectare (1 ha) $= 10\,000\text{ m}^2$
- The area of a two-dimensional shape can be defined as the number of square units contained within its boundaries. Some common area formulas are given.

square



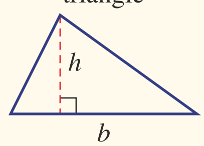
Area $= l^2$

rectangle



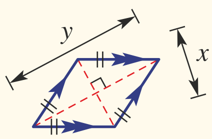
Area $= lw$

triangle



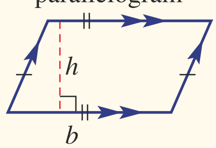
Area $= \frac{1}{2}bh$

rhombus



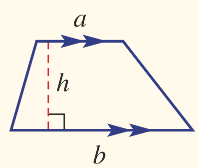
Area $= \frac{1}{2}xy$

parallelogram



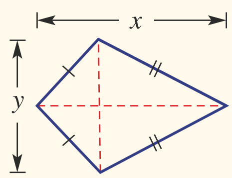
Area $= bh$

trapezium



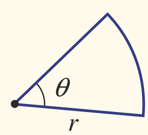
Area $= \frac{1}{2}(a + b)h$

kite



Area $= \frac{1}{2}xy$

- The rule for the area of a circle is:
  Area $= \pi r^2$, where r is the radius.
- The rule for the area of a sector is $A = \frac{\theta}{360}\pi r^2$.



BUILDING UNDERSTANDING

1 State the formula for the area of these shapes.

a circle **b** sector **c** square
d rectangle **e** kite **f** trapezium
g triangle **h** rhombus **i** parallelogram
j semicircle **k** quadrant (quarter circle)

2 Decide how many:

a **i** mm in 1 cm **ii** mm^2 in 1 cm^2
b **i** cm in 1 m **ii** cm^2 in 1 m^2
c **i** m in 1 km **ii** m^2 in 1 km^2
d m^2 in 1 ha

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 6 Converting between units of area

Convert these areas to the units shown in the brackets.

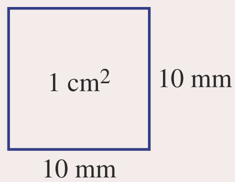
a $2.5\ \text{cm}^2$ (mm^2) **b** $2\,000\,000\ \text{cm}^2$ (km^2)

SOLUTION

a $2.5\ \text{cm}^2 = 2.5 \times 10^2\ \text{mm}^2$
$= 2.5 \times 100\ \text{mm}^2$
$= 250\ \text{mm}^2$

b $2\,000\,000\ \text{cm}^2 = 2\,000\,000 \div 100^2\ \text{m}^2$
$= 200\ \text{m}^2$
$= 200 \div 1000^2\ \text{km}^2$
$= 0.0002\ \text{km}^2$

EXPLANATION



$1\ \text{cm}^2 = 10 \times 10\ \text{mm}^2 = 100\ \text{mm}^2$

$\text{km}^2 \xleftarrow{\div 1000^2} \text{m}^2 \xleftarrow{\div 100^2} \text{cm}^2$

$100^2 = 10\,000$ and $1000^2 = 1\,000\,000$

Now you try

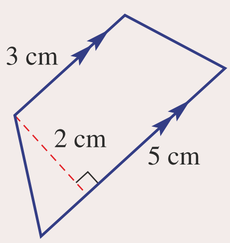
Convert these areas to the units shown in the brackets.

a $3.5\ \text{m}^2$ (cm^2) **b** $50\,000\ \text{mm}^2$ (m^2)

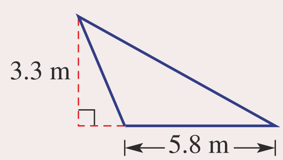
Example 7 Finding the area of basic shapes

Find the area of these basic shapes, correct to two decimal places where necessary.

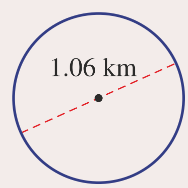
a



b



c



SOLUTION

a $A = \frac{1}{2}(a + b)h$

$= \frac{1}{2}(3 + 5)2$

$= 8\ \text{cm}^2$

b $A = \frac{1}{2}bh$

$= \frac{1}{2}(5.8)(3.3)$

$= 9.57\ \text{m}^2$

EXPLANATION

a The shape is a trapezium, so use this formula.

Substitute $a = 3$, $b = 5$ and $h = 2$.

Simplify and include the correct units.

b The shape is a triangle.

Substitute $b = 5.8$ and $h = 3.3$.

Simplify and include the correct units.

Continued on next page

ISBN 978-1-108-77290-7

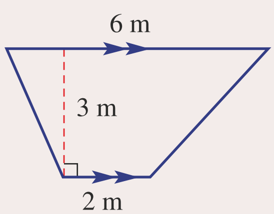
c $A = \pi r^2$
$= \pi(0.53)^2$
$= 0.88 \text{ km}^2$ (to 2 d.p.)

The shape is a circle.
The radius, r, is half the diameter;
i.e $1.06 \div 2 = 0.53$
Evaluate using a calculator and round your answer to the required number of decimal places.

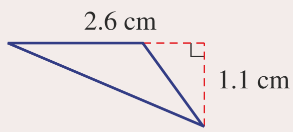
Now you try

Find the area of these basic shapes, correct to two decimal places where necessary.

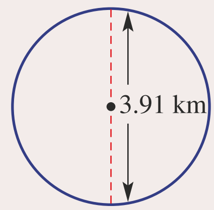
a



b



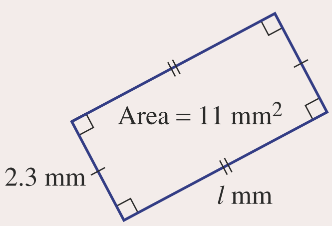
c



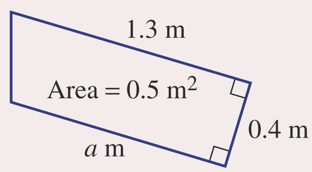
Example 8 Using area to find unknown lengths

Find the value of the pronumeral for these basic shapes, rounding to two decimal places where necessary.

a



b



SOLUTION

a $A = lw$
$11 = l \times 2.3$
$\therefore l = \dfrac{11}{2.3}$
$= 4.78$ (to 2 d.p.)

b $A = \dfrac{1}{2}(a + b)h$
$0.5 = \dfrac{1}{2}(a + 1.3) \times 0.4$
$0.5 = 0.2(a + 1.3)$
$2.5 = a + 1.3$
$\therefore a = 1.2$

EXPLANATION

a Use the rectangle area formula.
Substitute $A = 11$ and $w = 2.3$.
Divide both sides by 2.3 to solve for l.

b Use the trapezium area formula.
Substitute $A = 0.5$, $b = 1.3$ and $h = 0.4$.
Simplify $\left(\dfrac{1}{2} \times 0.4 = 0.2\right)$, then divide both sides by 0.2 and solve for a.

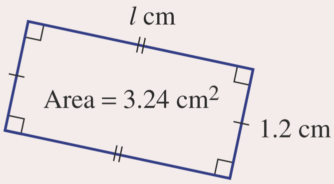
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

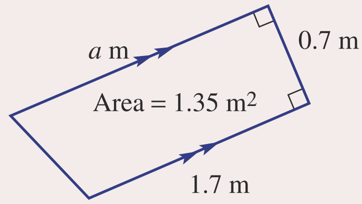
Now you try

Find the value of the pronumeral for these basic shapes, rounding to two decimal places where necessary.

a



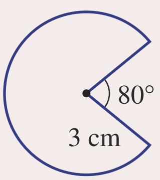
b



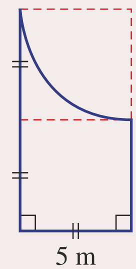
Example 9 Finding areas of sectors and composite shapes

Find the area of this sector and composite shape. Write your answer as an exact value and as a decimal, correct to two decimal places.

a



b



SOLUTION

a $A = \dfrac{\theta}{360} \times \pi r^2$

$= \dfrac{280}{360} \times \pi \times 3^2$

$= 7\pi$

$= 21.99\ \text{cm}^2$ (to 2 d.p.)

b $A = 2 \times 5^2 - \dfrac{1}{4} \times \pi \times 5^2$

$= 50 - \dfrac{25\pi}{4}$

$= 30.37\ \text{m}^2$ (to 2 d.p.)

EXPLANATION

Write the formula for the area of a sector.

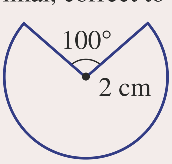
Sector angle $= 360° - 80° = 280°$.
Simplify to express as an exact value (7π), then round as required.

The area consists of two squares minus a quarter circle with radius 5 m.
$50 - \dfrac{25\pi}{4}$ is the exact value.

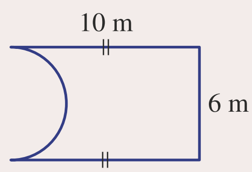
Now you try

Find the area of this sector and composite shape. Write your answer as an exact value and as a decimal, correct to two decimal places.

a



b



ISBN 978-1-108-77290-7

Exercise 6C

FLUENCY	1, 2–4(½)	2–5(½)	2–5(⅓)

Example 6a

Example 6b

1 Convert these areas to the units shown in the brackets.

a i $1.5\ \text{cm}^2$ (mm^2) ii $5\ \text{m}^2$ (cm^2) iii $0.2\ \text{km}^2$ (m^2)

b i $7\,000\,000\ \text{cm}^2$ (km^2) ii $450\,000\ \text{mm}^2$ (m^2) iii $6\,000\,000\,000\ \text{mm}^2$ (km^2)

Example 6

2 Convert the following area measurements to the units given in brackets.

a $3000\ \text{mm}^2$ (cm^2) b $29\,800\ \text{cm}^2$ (m^2) c $205\,000\ \text{m}^2$ (km^2)

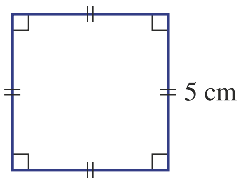
d $0.5\ \text{m}^2$ (cm^2) e $5\ \text{km}^2$ (m^2) f $0.0001\ \text{km}^2$ (m^2)

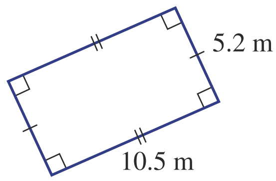
g $0.023\ \text{m}^2$ (cm^2) h $537\ \text{cm}^2$ (mm^2) i $0.0027\ \text{km}^2$ (m^2)

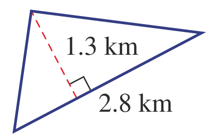
j $10\ \text{m}^2$ (mm^2) k $0.00022\ \text{km}^2$ (cm^2) l $145\,000\,000\ \text{mm}^2$ (km^2)

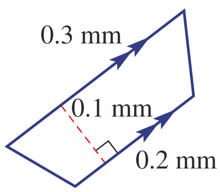
Example 7

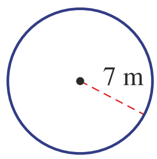
3 Find the area of these basic shapes, rounding to two decimal places where necessary.

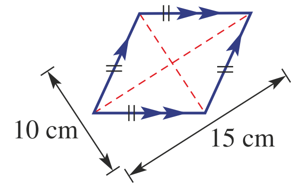
a


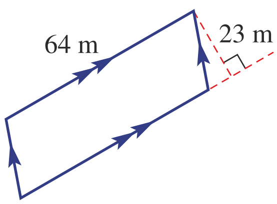
b


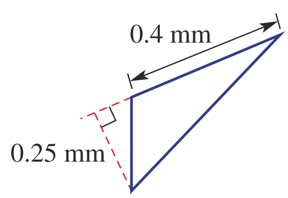
c


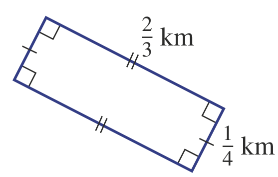
d


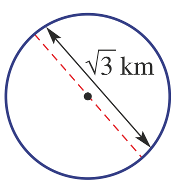
e


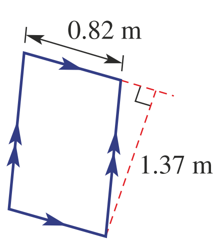
f


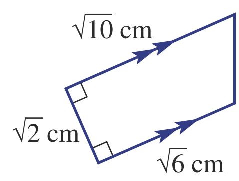
g


h


i


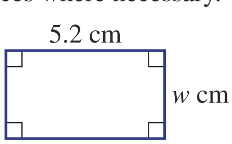
j


k


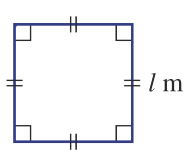
l


Example 8

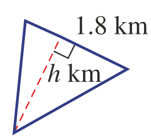
4 Find the value of the pronumeral for these basic shapes with given areas, rounding to two decimal places where necessary.

a


Area = $15\ \text{cm}^2$

b


Area = $206\ \text{m}^2$

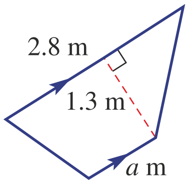
c


Area = $1.3\ \text{km}^2$

ISBN 978-1-108-77290-7

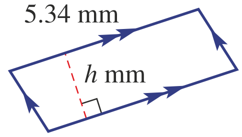
Cambridge University Press
Updated June 2022

d



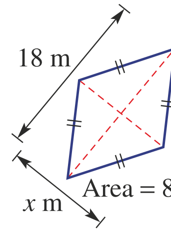
Area = 2.5 m^2

e



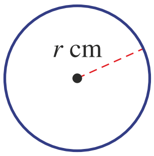
Area = 10.21 mm^2

f



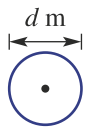
Area = 80 m^2

g



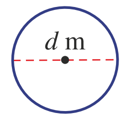
Area = 5 cm^2

h



Area = 0.21 m^2

i

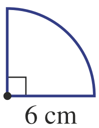


Area = 26 km^2

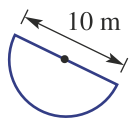
Example 9a

5 Find the area of each sector. Write your answer as an exact value and as a decimal rounded to two decimal places.

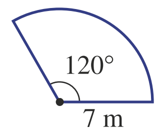
a



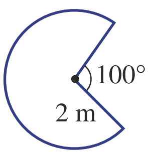
b



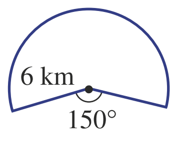
c



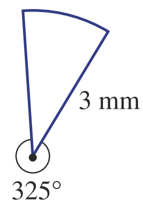
d



e

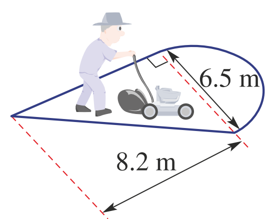


f



PROBLEM-SOLVING	6, 7	6, 7(½)	7(½), 8

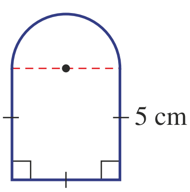
6 A lawn area is made up of a semicircular region with diameter 6.5 metres and a triangular region of length 8.2 metres, as shown. Find the total area of lawn, to one decimal place.



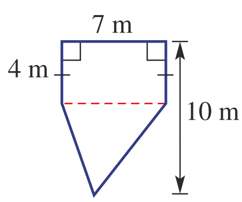
Example 9b

7 Find the area of these composite shapes. Write your answers as exact values and as decimals, correct to two decimal places.

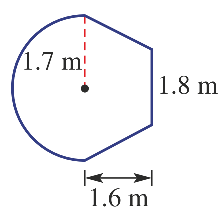
a



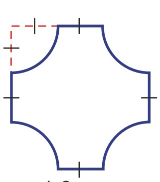
b



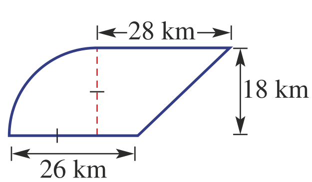
c



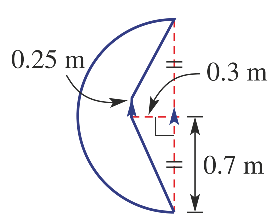
d



e



f



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

8 An L-shaped concrete slab being prepared for the foundation of a new house is made up of two rectangles with dimensions 3 m by 2 m and 10 m by 6 m.

a Find the total area of the concrete slab.

b If two bags of cement are required for every 5 m^2 of concrete, how many whole bags of cement will need to be purchased for the job?

REASONING	9	9, 10	10, 11

9 1 hectare (1 ha) is 10000 m^2 and an acre is $\frac{1}{640}$ square miles (1 mile $\approx$ 1.61 km).
Find how many:

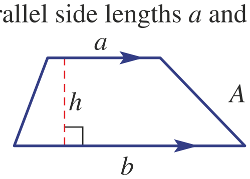
a hectares in 1 km^2

b square metres in 20 hectares

c hectares in 1 acre (round to one decimal place)

d acres in 1 hectare (round to one decimal place).

10 Consider a trapezium with area A, parallel side lengths a and b and height h.



$$A = \frac{1}{2}(a + b)h$$

a Rearrange the area formula to express a in terms of A, b and h.

b Hence, find the value of a for these given values of A, b and h.

i $A = 10, b = 10, h = 1.5$ ii $A = 0.6, b = 1.3, h = 0.2$ iii $A = 10, b = 5, h = 4$

c Sketch the trapezium with the dimensions found in part b iii above. What shape have you drawn?

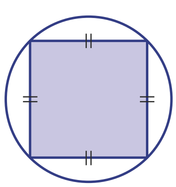
11 Provide a proof of the following area formulas, using only the area formulas for rectangles and triangles.

a parallelogram b kite c trapezium

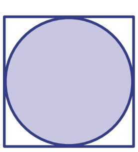
ENRICHMENT: Percentage areas	–	–	12

12 Find, correct to one decimal place, the percentage areas for these situations.

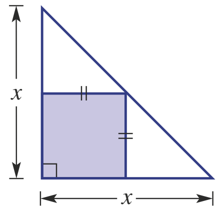
a The largest square inside a circle.



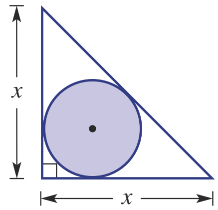
b The largest circle inside a square.



c The largest square inside a right isosceles triangle.



d The largest circle inside a right isosceles triangle.



ISBN 978-1-108-77290-7

6D Surface area of prisms and cylinders

Learning intentions

- To know what defines a prism and a cylinder
- To know the meaning of the term surface area
- To know how to use a net to identify the surfaces of prisms and cylinders
- To be able to find the surface area of prisms
- To know how the formula for the surface area of a cylinder is developed and be able to apply it
- To be able to identify visible surfaces of a composite solid to include in surface area calculations

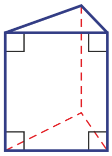
Knowing how to find the area of simple shapes combined with some knowledge about three-dimensional objects helps us to find the total surface area of a range of solids.

A cylindrical can, for example, has two circular ends and a curved surface that could be rolled out to form a rectangle. Finding the sum of the two circles and the rectangle will give the total surface area of the cylinder.

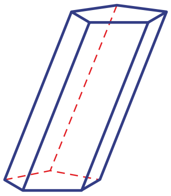
Steel cans used for food are coated with tin-plate (2% tin), as tin doesn't corrode. Cans are manufactured by cutting a rectangle, forming a tube, attaching the base, sterilising, filling with food and then joining the circular top.

You will recall the following information about prisms and cylinders.

- A **prism** is a polyhedron with a uniform cross-section and two congruent ends.
  - A prism is named by the shape of the cross-section.
  - The remaining sides are parallelograms.
- A **cylinder** has a circular cross-section.
  - A cylinder is similar to a prism in that it has a uniform cross-section and two congruent ends.



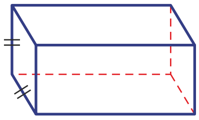
right triangular prism



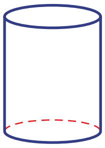
oblique pentagonal prism

LESSON STARTER Drawing nets

Drawing or visualising a net can help when finding the total surface area of a solid. Try drawing a net for these solids.



square prism



cylinder

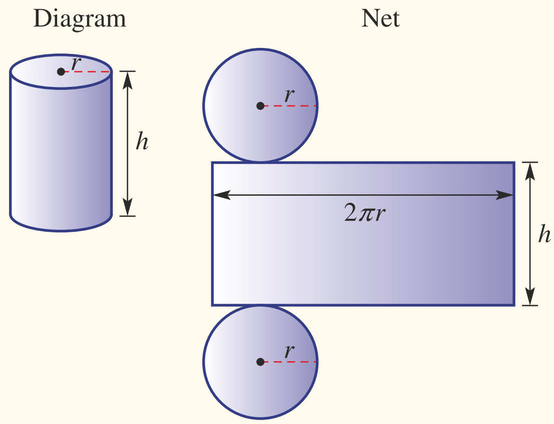
By labelling the dimensions, can you come up with a formula for the total surface area of these solids?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

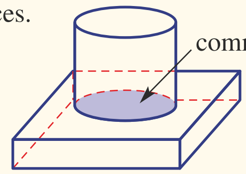
KEY IDEAS

- The **total surface area** (TSA) of a three-dimensional object can be found by finding the sum of the areas of each of the shapes that make up the surface of the object.
- A **net** is a two-dimensional illustration of all the surfaces of a solid object.
- Given below are the net and surface area of a **cylinder**.



$$\begin{aligned}\text{TSA} &= 2\text{ circles} + 1\text{ rectangle}\\ &= 2\pi r^2 + 2\pi rh\\ &= 2\pi r(r+h)\end{aligned}$$

- **Composite solids** are solids made up of two or more basic solids.
  - To find a total surface area do not include any common faces.
    - In this example, the top circular face area of the cylinder is equal to the common face area, so the TSA = surface area of prism + curved surface area of cylinder.



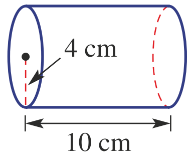
BUILDING UNDERSTANDING

1 Draw an example of these solids.

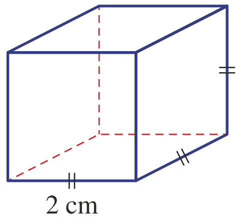
a cylinder **b** rectangular prism **c** triangular prism

2 Draw a net for each of these solids.

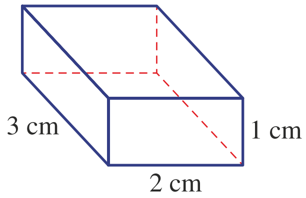
a



b



c



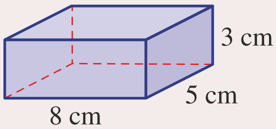
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

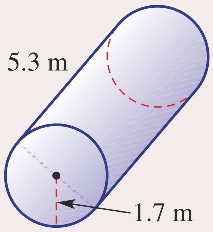
Example 10 Finding the surface area of prisms and cylinders

Find the total surface area of this rectangular prism and cylinder. Round your answer to two decimal places where necessary.

a



b

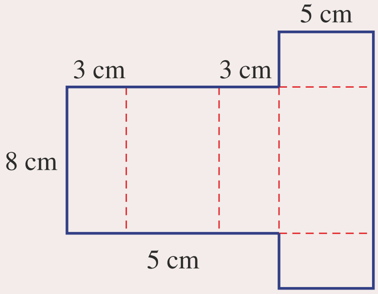


SOLUTION

a $\text{TSA} = 2 \times (8 \times 3) + 2 \times (5 \times 3) + 2 \times (8 \times 5)$
$= 48 + 30 + 80$
$= 158 \text{ cm}^2$

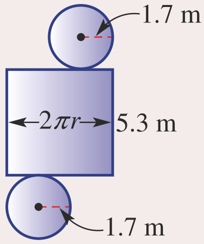
EXPLANATION

Draw the net of the solid if needed to help you. Sum the areas of the rectangular surfaces.



b $\text{TSA} = 2\pi r^2 + 2\pi rh$
$= 2\pi(1.7)^2 + 2\pi(1.7) \times 5.3$
$= 74.77 \text{ m}^2$ (to 2 d.p.)

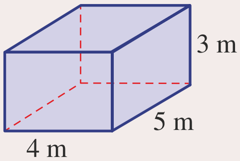
Write the formula and substitute the radius and height.



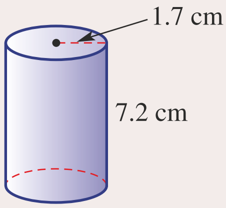
Now you try

Find the total surface area of this rectangular prism and cylinder. Round your answer to two decimal places where necessary.

a



b

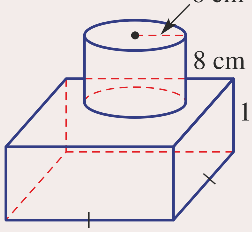


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 11 Finding the surface area of composite solids

A composite object consists of a square-based prism and a cylinder, as shown. Find the total surface area, correct to one decimal place.



SOLUTION

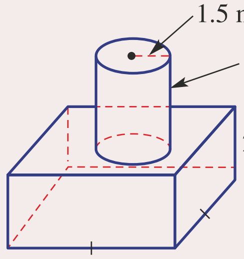
$$\begin{aligned}\text{TSA} &= 4 \times (20 \times 10) + 2 \times (20 \times 20) + 2 \times \pi \times 6 \times 8 \\ &\quad + \pi(6)^2 - \pi(6)^2 \\ &= 1600 + 96\pi \\ &= 1901.6\ \text{cm}^2 \text{ (to 1 d.p.)}\end{aligned}$$

EXPLANATION

The common circular area, which should not be included, is added back on with the top of the cylinder. So the total surface area of the prism is added to only the curved area of the cylinder.

Now you try

A composite object consists of a square-based prism and a cylinder, as shown. Find the total surface area, correct to one decimal place.

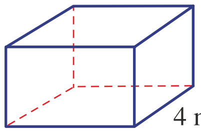


Exercise 6D

FLUENCY	1, 2–3(1/2)	2–3(1/2), 4	2–3(1/3), 4, 5

Example 10 **1** Find the total surface area of this rectangular prism and cylinder. Round your answers to two decimal places where necessary.

a



b

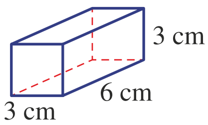
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

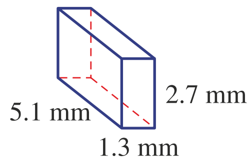
Example 10

2 Find the total surface area of these solids. Round your answers to two decimal places where necessary.

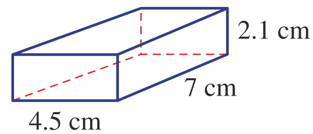
a



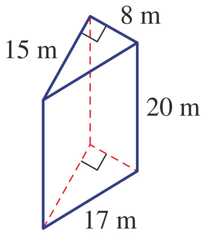
b



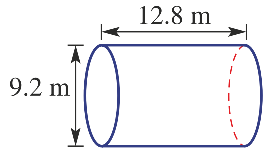
c



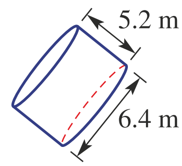
d



e

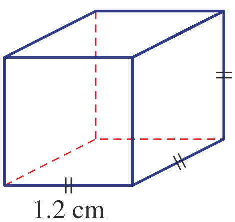


f

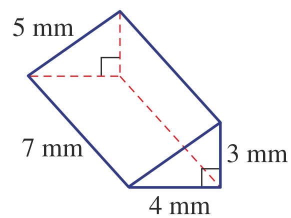


3 Find the total surface area of these solids.

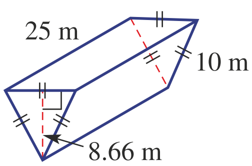
a



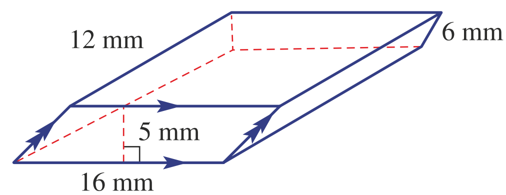
b



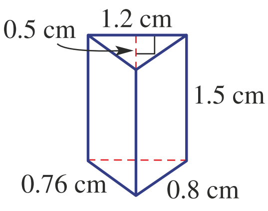
c



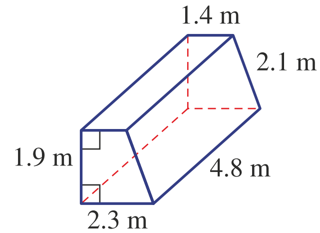
d



e



f



4 Find the total surface area, in square metres, of the outer surface of an open pipe with radius 85 cm and length 4.5 m, correct to two decimal places.

5 What is the minimum area of paper required to wrap a box with dimensions 25 cm wide, 32 cm long and 20 cm high?

ISBN 978-1-108-77290-7

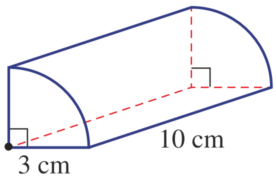
Cambridge University Press
Updated June 2022

PROBLEM-SOLVING

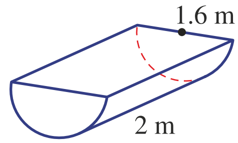
PROBLEM-SOLVING	6–7(1/2)	6–8(1/2)	6–8(1/2), 9

6 The cross-sections of these solids are sectors. Find the total surface area, rounding to one decimal place.

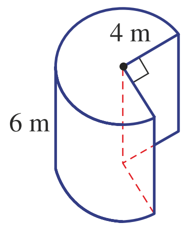
a



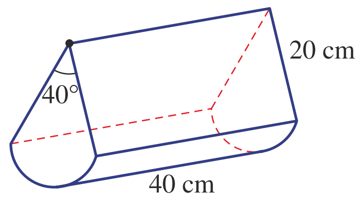
b



c



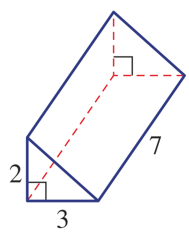
d



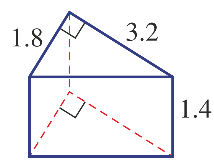
10A

7 Use Pythagoras' theorem to determine any unknown side lengths and find the total surface area of these solids, correct to one decimal place.

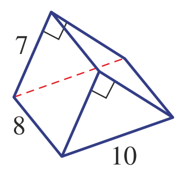
a



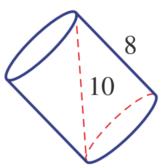
b



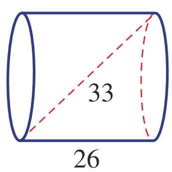
c



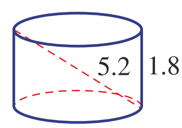
d



e



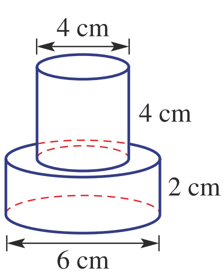
f



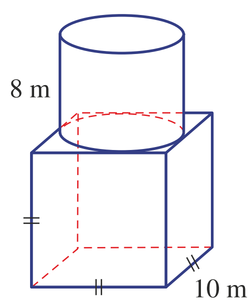
Example 11

8 Find the total surface area of these composite solids. Answer correct to one decimal place.

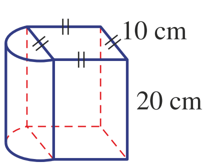
a



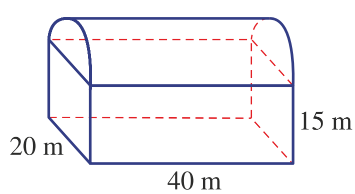
b



c



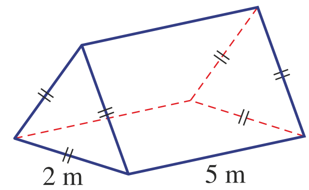
d



ISBN 978-1-108-77290-7

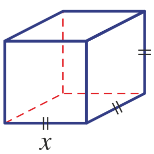
Cambridge University Press
Updated June 2022

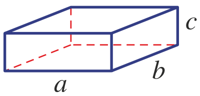
9 Find the total surface area of this triangular prism, correct to one decimal place.

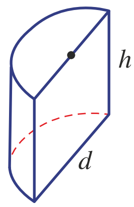


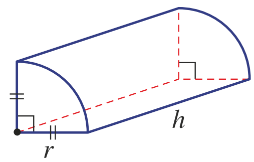
REASONING	10	10, 11	11–13

10 Find a formula for the total surface area of these solids, using the given pronumerals.

a


b


c


d


11 Find the exact total surface area for a cylinder with the given dimensions. Your exact answer will be in terms of π.

a $r = 1$ and $h = 2$ **b** $r = \frac{1}{2}$ and $h = 5$

12 The total surface area of a cylinder is given by the rule:

$$\text{Total surface area} = 2\pi r(r + h)$$

Find the height, to two decimal places, of a cylinder that has a radius of 2 m and a total surface area of:

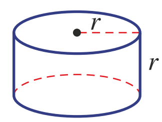
a 35 m^2 **b** 122 m^2

13 Can you find the exact radius of the base of a cylinder if its total surface area is 8π cm^2 and its height is 3 cm?

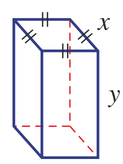
ENRICHMENT: Deriving formulas for special solids	–	–	14

14 Derive the formulas for the total surface area of the following solids.

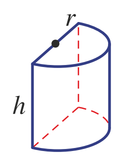
a a cylinder with its height equal to its radius r



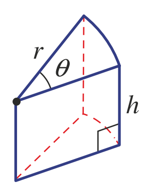
b a square-based prism with square side length x and height y



c a half cylinder with radius r and height h



d a solid with a sector cross-section, radius r, sector angle θ and height h



ISBN 978-1-108-77290-7

6E Surface area of pyramids and cones 10A

Learning intentions

- To know the shape of pyramids and cones and their associated nets
- To know the formula for the surface area of a cone
- To be able to find the surface area of a pyramid and a cone
- To be able to use Pythagoras' theorem to find the vertical height or slant height of a cone

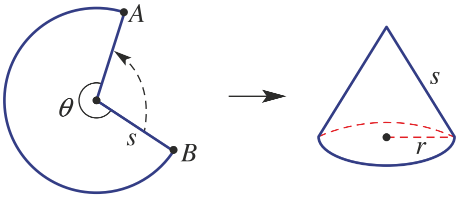
Pyramids and cones are solids for which we can also calculate the total surface area by finding the sum of the areas of all the outside surfaces.

The total surface area of a pyramid involves finding the sum of the areas of the base and its triangular faces. The rule for the surface area of a cone can be developed after drawing a net that includes a circle (base) and sector (curved surface).

Mechanical engineers and sheet metal workers apply surface area and volume formulas when designing and constructing stainless steel equipment for the food and beverage industries. Cylinder and cone formulas are used when designing brewery vats.

LESSON STARTER The cone formula

Use a pair of compasses to construct a large sector. Use any sector angle θ that you like. Cut out the sector and join the points A and B to form a cone of radius r.

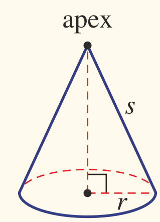


- Give the rule for the area of the base of the cone.
- Give the rule for the circumference of the base of the cone.
- Give the rule for the circumference of a circle with radius s.
- Use the above to find an expression for the area of the base of the cone as a fraction of the area πs^2.
- Hence, explain why the rule for the surface area of a cone is given by:
  Total surface area $= \pi r^2 + \pi rs$.

ISBN 978-1-108-77290-7

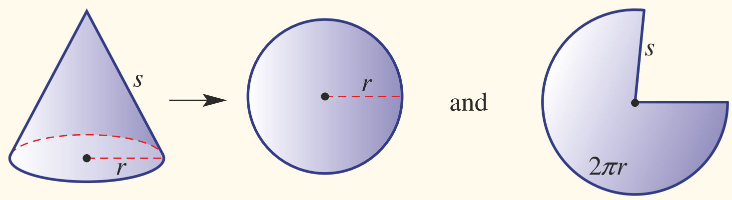
Cambridge University Press
Updated June 2022

KEY IDEAS



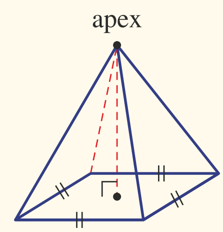
right cone

- A **cone** is a solid with a circular base and a curved surface that reaches from the base to a point called the **apex**.
  - A right cone has its apex directly above the centre of the base.
  - The pronumeral s is used for the slant height and r is the radius of the base.
  - Cone total surface area is given by:



Area(base) $= \pi r^2$ Area $= \dfrac{2\pi r}{2\pi s} \times \pi s^2 = \pi rs$

$\therefore$ TSA (cone) $= \pi r^2 + \pi rs = \pi r(r + s)$



right square-based pyramid

- A **pyramid** has a base that is a polygon and its remaining faces are triangles that meet at the apex.
  - A pyramid is named by the shape of its base.
  - A right pyramid has its apex directly above the centre of the base.

BUILDING UNDERSTANDING

1 State the rule for the following.

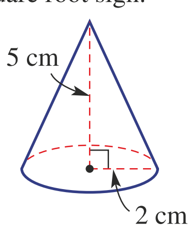
a area of a triangle

b surface area of the base of a cone with radius r

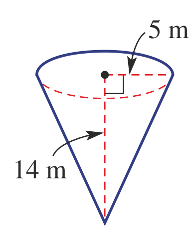
c surface area of the curved part of a cone with slant height s and radius r

2 Find the exact slant height for these cones, using Pythagoras' theorem. Express exactly, using a square root sign.

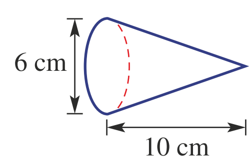
a



b

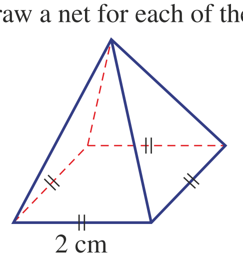


c

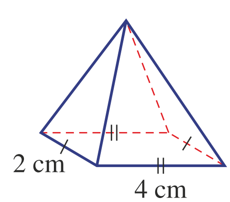


3 Draw a net for each of these solids.

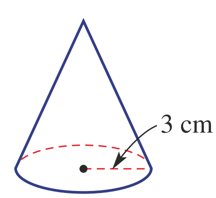
a



b



c



ISBN 978-1-108-77290-7

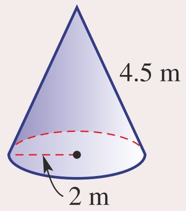
Cambridge University Press
Updated June 2022

Example 12 Finding the total surface area of a cone and pyramid

Find the total surface area of these solids, using two decimal places for part **a**.

a a cone with radius 2 m and slant height 4.5 m

b a square-based pyramid with square-base length 25 mm and triangular face height 22 mm



SOLUTION

a TSA $= \pi r^2 + \pi rs$

$= \pi(2)^2 + \pi(2) \times (4.5)$

$= 40.84 \text{ m}^2$ (to 2 d.p.)

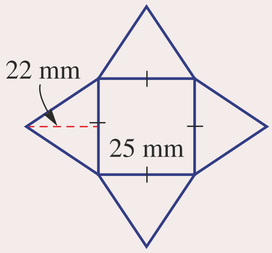
b TSA $= l^2 + 4 \times \frac{1}{2}bh$

$= 25^2 + 4 \times \frac{1}{2} \times 25 \times 22$

$= 1725 \text{ mm}^2$

EXPLANATION

The cone includes the circular base plus the curved part. Substitute $r = 2$ and $s = 4.5$.

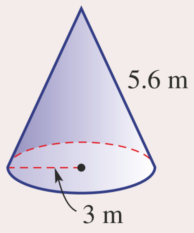


Now you try

Find the total surface area of these solids, using two decimal places for part **a**.

a a cone with radius 3 m and slant height 5.6 m

b a square-based pyramid with square-base length 20 mm and triangular face height 19 mm



Example 13 Finding the slant height and vertical height of a cone

A cone with radius 3 cm has a curved surface area of 100 cm^2.

a Find the slant height of the cone, correct to one decimal place.

b Find the height of the cone, correct to one decimal place.

SOLUTION

a Surface area $= \pi rs$

$100 = \pi \times 3 \times s$

$s = \frac{100}{3\pi}$

$= 10.6$ cm (to 1 d.p.)

EXPLANATION

Substitute the given information into the rule for the curved surface area of a cone and solve for s.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

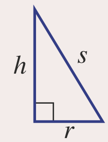
b $h^2 + r^2 = s^2$

$$h^2 + 3^2 = \left(\frac{100}{3\pi}\right)^2$$

$$h^2 = \left(\frac{100}{3\pi}\right)^2 - 9$$

$$h = \sqrt{\left(\frac{100}{3\pi}\right)^2 - 9}$$

$= 10.2$ cm (to 1 d.p.)



Identify the right-angled triangle within the cone and use Pythagoras' theorem to find the height h. Use the exact value of s from part **a** to avoid accumulating errors.

Now you try

A cone with radius 2 cm has a curved surface area of 80 cm^2.

a Find the slant height of the cone, correct to one decimal place.

b Find the height of the cone, correct to one decimal place.

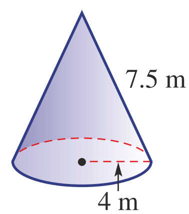
Exercise 6E

FLUENCY	1–4	2–5	2–5

1 Find the total surface area of these solids, using two decimal places for part **a**.

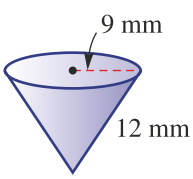
Example 12a **a** A cone with radius 4 m and slant height 7.5 m.

Example 12b **b** A square-based pyramid with square-base length 30 mm and triangular face height 20 mm.

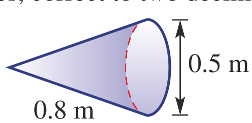


Example 12a **2** Find the total surface area of these cones, correct to two decimal places.

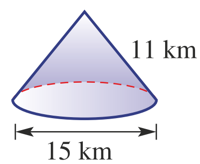
a



b

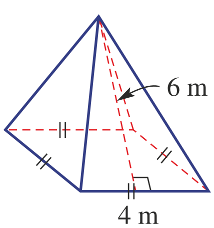


c

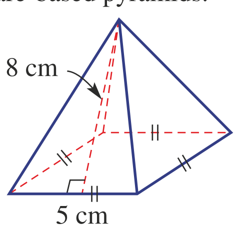


Example 12b **3** Find the total surface area of these square-based pyramids.

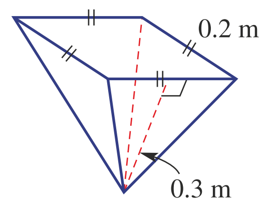
a



b

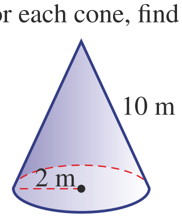


c

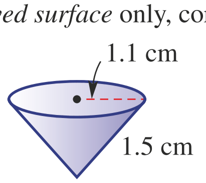


4 For each cone, find the area of the *curved surface* only, correct to two decimal places.

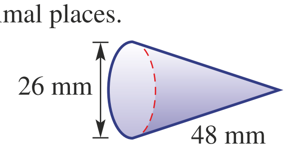
a



b



c



ISBN 978-1-108-77290-7

5 A cone has height 10 cm and radius 3 cm.

a Use Pythagoras' theorem to find the slant height of the cone, rounding your answer to two decimal places.

b Find the total surface area of the cone, correct to one decimal place.

PROBLEM-SOLVING	6, 7	7–9	8, 9, 10($^1/_2$)

Example 13

6 A cone with radius 5 cm has a curved surface area of 400 cm^2.

a Find the slant height of the cone, correct to one decimal place.

b Find the height of the cone, correct to one decimal place.

7 A cone with radius 6.4 cm has a curved surface area of 380 cm^2.

a Find the slant height of the cone, correct to one decimal place.

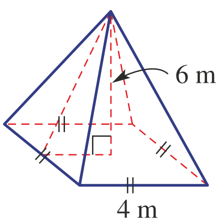
b Find the height of the cone, correct to one decimal place.

8 Party hats A and B are in the shape of open cones with no base. Hat A has radius 7 cm and slant height 25 cm, and hat B has radius 9 cm and slant height 22 cm. Which hat has the greater surface area?

9 This right square-based pyramid has base side length 4 m and vertical height 6 m

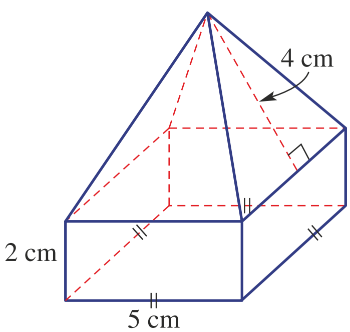
a Find the height of the triangular faces, correct to one decimal place.

b Find the total surface area, correct to one decimal place.

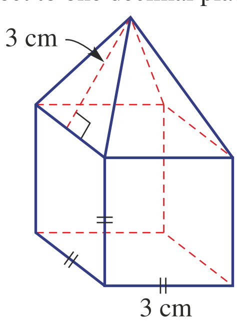


10 Find the total surface area of these composite solids, correct to one decimal place as necessary.

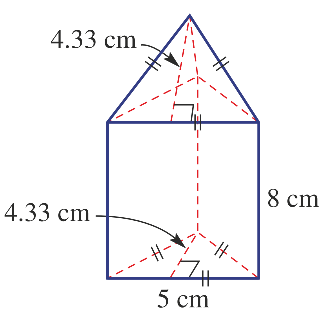
a



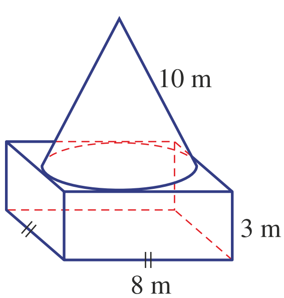
b



c

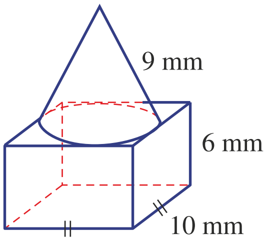


d

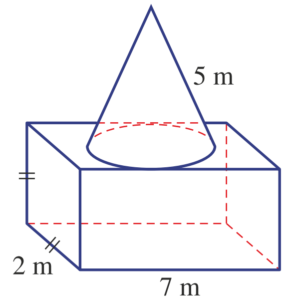


ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

e



f

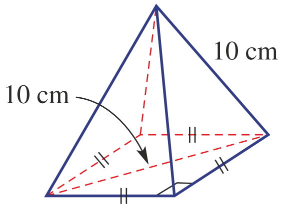


REASONING	11	11, 12	12, 13

11 Explain why the surface area of a cone with radius r and height h is given by the expression $\pi r(r + \sqrt{r^2 + h^2})$.

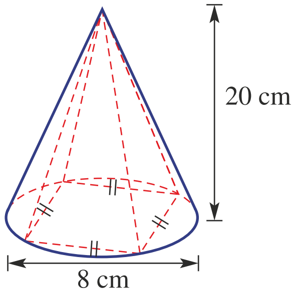
12 A cone has a height equal to its radius (*i.e.* $h = r$). Show that its total surface area is given by the expression $\pi r^2(1 + \sqrt{2})$.

13 There is enough information in this diagram to find the total surface area, although the side length of the base and the height of the triangular faces are not given. Find the total surface area, correct to one decimal place.



ENRICHMENT: Carving pyramids from cones	–	–	14

14 A woodworker uses a rotating lathe to produce a cone with radius 4 cm and height 20 cm. From that cone the woodworker then cut slices off the sides of the cone to produce a square-based pyramid of the same height.



a Find the exact slant height of the cone.
b Find the total surface area of the cone, correct to two decimal places.
c Find the exact side length of the base of the square-based pyramid.
d Find the height of the triangular faces of the pyramid, correct to three decimal places.
e Find the total surface area of the pyramid, correct to two decimal places.
f Express the total surface area of the pyramid as a percentage of the total surface area of the cone. Give the answer correct to the nearest whole percentage.

ISBN 978-1-108-77290-7

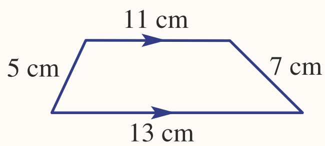
Progress quiz

In the following questions round all answers to two decimal places, where necessary.

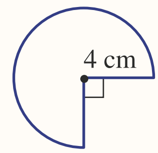
6A

1 Find the perimeter of these shapes. (Note: In part **c**, all angles are right angles.)

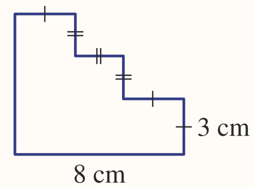
a



b



c

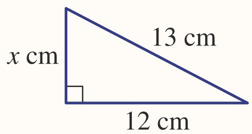


6B

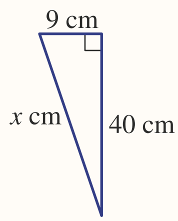
2 For each of these shapes, find (correct to two decimal places where necessary):

i the value of x **ii** the perimeter **iii** the area.

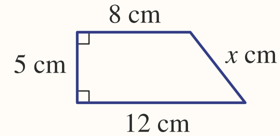
a



b



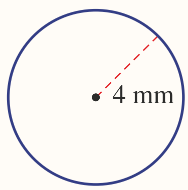
c



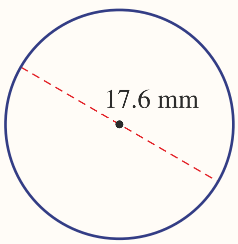
6A/C

3 Calculate the circumference and area of these circles.

a



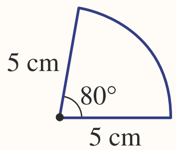
b



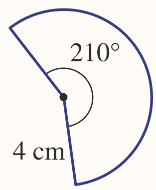
6C

4 Find the area of each of these sectors.

a



b



6C

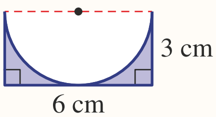
5 Convert 4.5 cm^2 to the following units.

a mm^2 **b** m^2

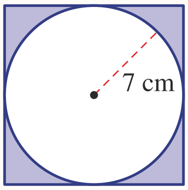
6C

6 Find the shaded (purple) area of these shapes (correct to two decimal places).

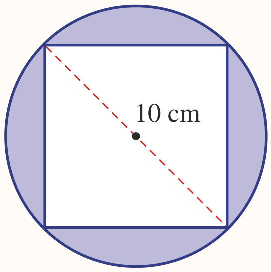
a



b



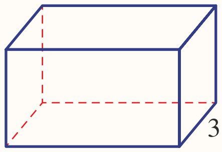
c

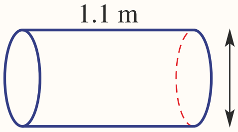


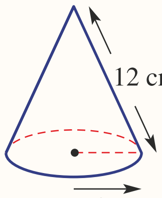
ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

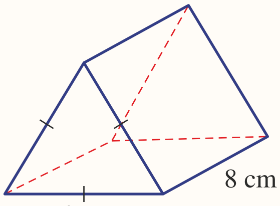
6D/E

7 Calculate the surface area of these solids (correct to two decimal places where necessary).

a


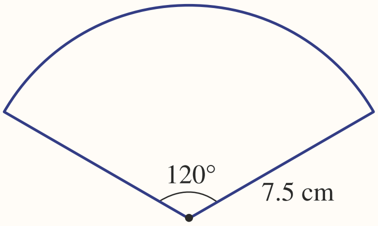
b


10A c


d


6E

8 A cone is made from a sector of a circle with radius 7.5 cm and central angle 120°.



a Calculate the curved surface area of the cone, correct to two decimal places.

b Calculate the radius of the circular base of the cone.

c Calculate the perpendicular height of the cone, correct to two decimal places.

6B

9 Find the diagonal length of a rectangular prism with dimensions 10 cm by 8 cm by 5 cm, correct to two decimal places.

ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

6F Volume of prisms and cylinders

Learning intentions

- To understand the concept of volume and capacity of an object
- To know how to use the cross-section of a prism or cylinder to find its volume
- To know the meaning of the term oblique prism
- To be able to find the volume of right and oblique prisms and cylinders
- To be able to identify the regular 3D shapes that comprise a composite solid and find its volume

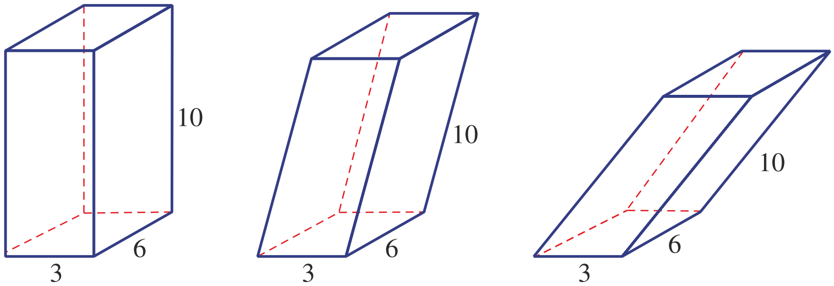
Volume is the amount of space contained within the outer surfaces of a three-dimensional object and is measured in cubic units.

The common groups of objects considered in this section are the prisms and the cylinders.

The volume of grain or cereal that a silo can store is calculated using the volume formulas for cylinders and cones.

LESSON STARTER Right and oblique prisms

Recall that to find the volume of a right prism, you would first find the area of the base and multiply by the height. Here is a right rectangular prism and two oblique rectangular prisms with the same side lengths.



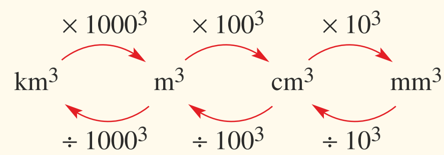
- What is the volume of the right rectangular prism? (The one on the left.)
- Do you think the volume of the two oblique prisms would be calculated in the same way as that for the right rectangular prism and using exactly the same lengths?
- Would the volume of the oblique prisms be equal to or less than that of the rectangular prism?
- Discuss what extra information is required to find the volume of the oblique prisms.
- How does finding the volume of oblique prisms (instead of a right prism) compare with finding the area of a parallelogram (instead of a rectangle)?

ISBN 978-1-108-77290-7

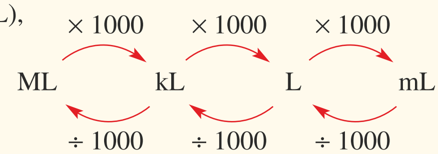
Cambridge University Press
Updated June 2022

KEY IDEAS

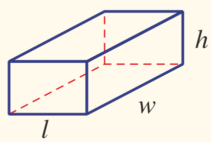
- Metric units for **volume** include cubic kilometres (km^3), cubic metres (m^3), cubic centimetres (cm^3) and cubic millimetres (mm^3).



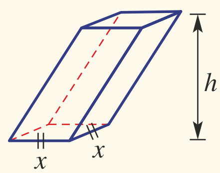
- Units for **capacity** include megalitres (ML), kilolitres (kL), litres (L) and millilitres (mL).
  - 1 cm^3 = 1 mL



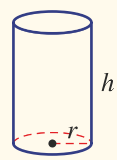
- For right and oblique prisms and cylinders, the volume is given by $V = Ah$, where:
  - A is the area of the base
  - h is the perpendicular height.



right rectangular prism
$V = Ah$
$\quad = lwh$



oblique square prism
$V = Ah$
$\quad = x^2h$

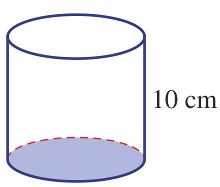


right cylinder
$V = Ah$
$\quad = \pi r^2 h$

BUILDING UNDERSTANDING

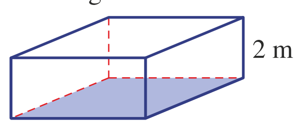
1 Find the volume of these solids with the given base areas.

a



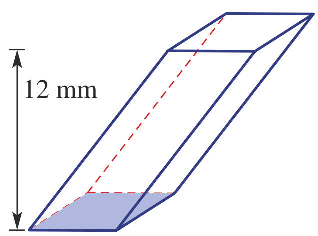
$A = 8\text{ cm}^2$

b



$A = 16\text{ m}^2$

c



$A = 9\text{ mm}^2$

2 State the rule for the volume of the following.

a right rectangular prism with length a, width b and height c

b oblique square prism with base dimensions y by y and perpendicular height h

c cylinder with radius r and height h

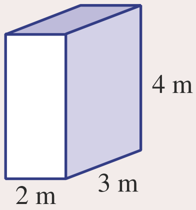
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

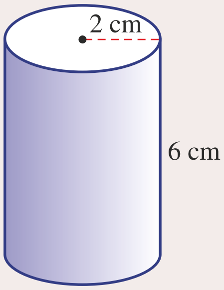
Example 14 Finding the volume of right prisms and cylinders

Find the volume of these solids, rounding to two decimal places for part **b**.

a



b



SOLUTION

a $V = lwh$
$= 2 \times 3 \times 4$
$= 24 \text{ m}^3$

b $V = \pi r^2 h$
$= \pi(2)^2 \times 6$
$= 75.40 \text{ cm}^3$ (to 2 d.p.)

EXPLANATION

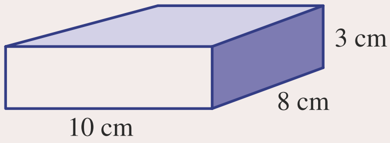
Write the volume formula for a rectangular prism.
Substitute $l = 2$, $w = 3$ and $h = 4$

The prism is a cylinder with base area πr^2.
Substitute $r = 2$ and $h = 6$.
Evaluate and round your answer as required.

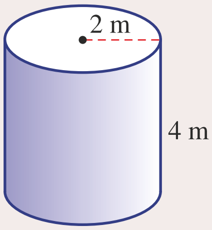
Now you try

Find the volume of these solids, rounding to two decimal places for part **b**.

a

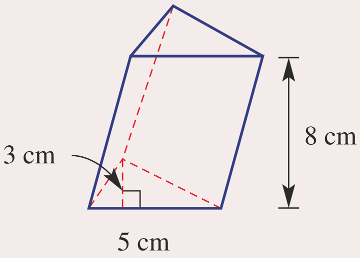


b



Example 15 Finding the volume of an oblique prism

Find the volume of this oblique prism.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

SOLUTION

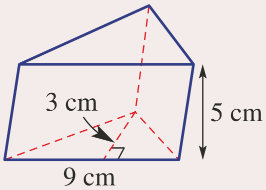
$$V = \frac{1}{2}bh \times 8$$
$$= \frac{1}{2} \times 5 \times 3 \times 8$$
$$= 60 \text{ cm}^3$$

EXPLANATION

The base is a triangle, so multiply the area of the base triangle by the perpendicular height.

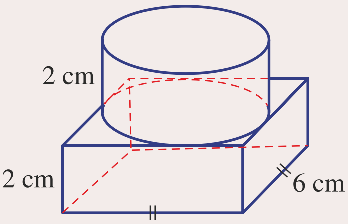
Now you try

Find the volume of this oblique prism.



Example 16 Finding the volume of a composite solid

Find the volume of this composite solid, correct to one decimal place.



SOLUTION

Radius of cylinder $= \frac{6}{2} = 3$ cm

$$V = lwh + \pi r^2 h$$
$$= 6 \times 6 \times 2 + \pi \times 3^2 \times 2$$
$$= 72 + 18\pi$$
$$= 128.5 \text{ cm}^3 \text{ (to 1 d.p.)}$$

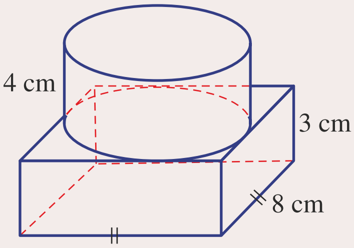
EXPLANATION

First, find the radius length, which is half the side length of the square base.

Add the volume of the square-based prism and the volume of the cylinder.

Now you try

Find the volume of this composite solid, correct to one decimal place.



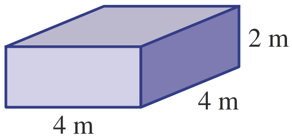
Exercise 6F

FLUENCY	1, 2($\frac{1}{2}$), 3–5	2($\frac{1}{2}$), 3–5, 6($\frac{1}{2}$)	2($\frac{1}{2}$), 3–5, 6($\frac{1}{2}$)

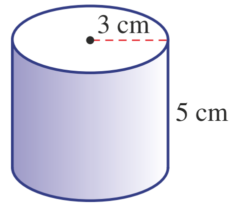
Example 14

1 Find the volume of these solids, rounding to two decimal places for part **b**.

a



b



2 Convert these volume measurements to the units given in brackets. (Refer to the Key ideas for help.)

a 2 cm^3 (mm^3) **b** 0.2 m^3 (cm^3) **c** 0.015 km^3 (m^3)

d 5700 mm^3 (cm^3) **e** 28 300 000 m^3 (km^3) **f** 762 000 cm^3 (m^3)

g 0.13 m^3 (cm^3) **h** 0.000001 km^3 (m^3) **i** 2.094 cm^3 (mm^3)

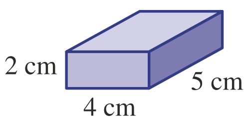
j 2.7 L (mL) **k** 342 kL (ML) **l** 35 L (kL)

m 5.72 ML (kL) **n** 74 250 mL (L) **o** 18.44 kL (L)

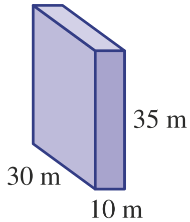
Example 14a

3 Find the volume of each rectangular prism.

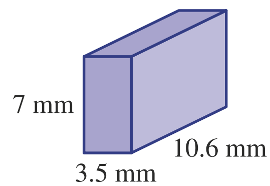
a



b



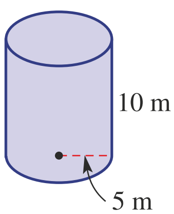
c



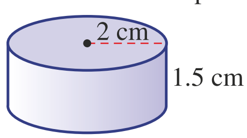
Example 14b

4 Find the volume of each cylinder, correct to two decimal places.

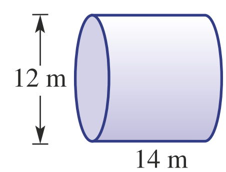
a



b



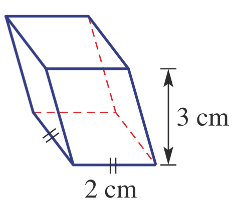
c



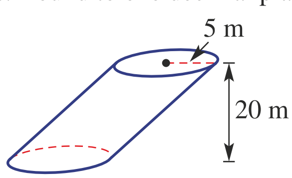
Example 15

5 Find the volume of these oblique solids. Round to one decimal place for part **b**.

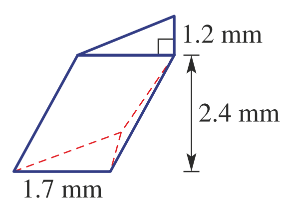
a



b



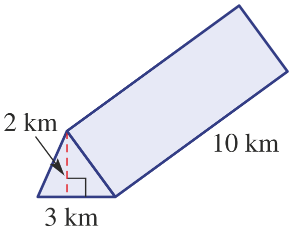
c



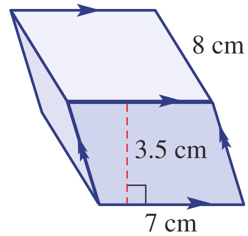
ISBN 978-1-108-77290-7

6 Find the volume of these solids, rounding your answers to three decimal places where necessary.

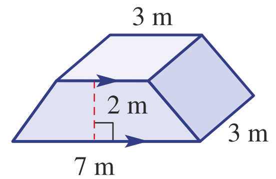
a



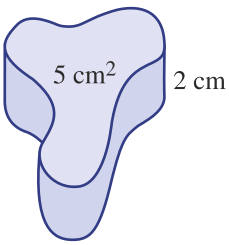
b



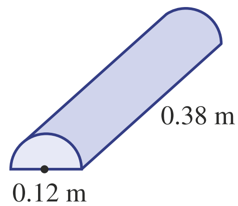
c



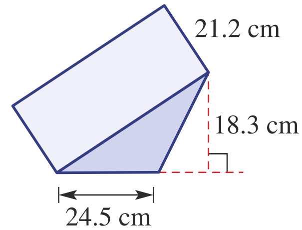
d



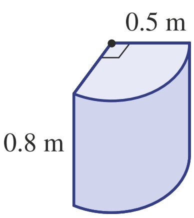
e



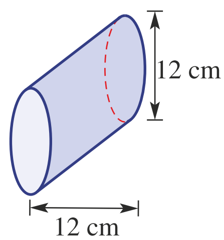
f



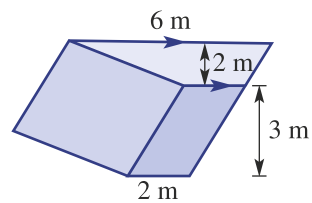
g



h



i



PROBLEM-SOLVING	7, 8	8, 9(1/2)	9(1/2), 10

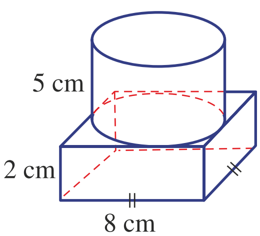
7 How many containers holding 1000 cm^3 (1 L) of water are needed to fill 1 m^3?

8 How many litres of water are required to fill a rectangular fish tank that is 1.2 m long, 80 cm wide and 50 cm high?

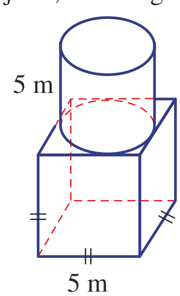
Example 16

9 Find the volume of these composite objects, rounding to two decimal places where necessary.

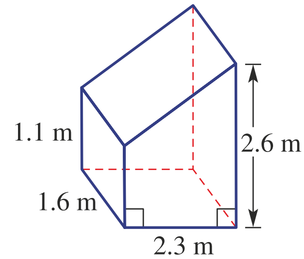
a



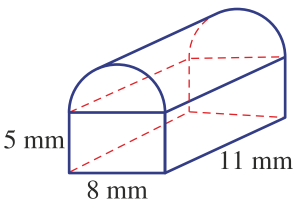
b



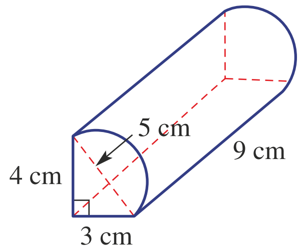
c



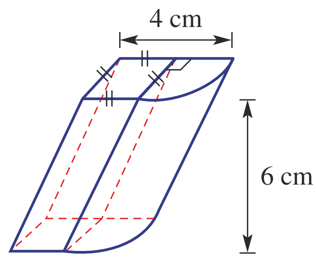
d



e



f



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

10 Find the exact volume of a cube if its surface area is:

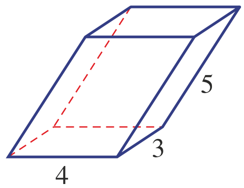
a 54 cm^2

b 18 m^2

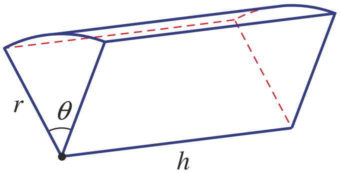
REASONING	11	11, 12	12–14

11 Use the rule $V = \pi r^2 h$ to find the height of a cylinder, to one decimal place, with radius 6 cm and volume 62 cm^3.

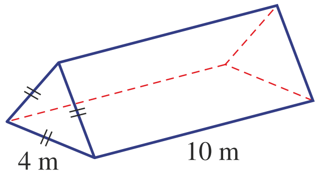
12 A fellow student says the volume of this prism is given by $V = 4 \times 3 \times 5$. Explain his error.



13 Find a formula for the volume of a cylindrical portion with angle θ, radius r and height h, as shown.



14 Decide whether there is enough information in this diagram of a triangular prism to find its volume. If so, find the volume, correct to one decimal place.

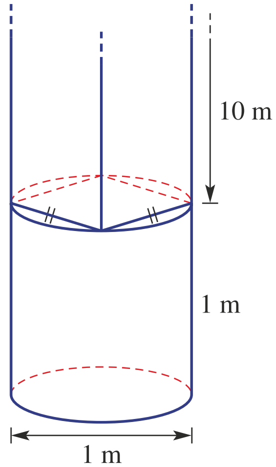


ENRICHMENT: Concrete poles	–	–	15

15 A concrete support structure for a building is made up of a cylindrical base and a square-based prism as the main column. The cylindrical base is 1 m in diameter and 1 m high, and the square prism is 10 m long and sits on the cylindrical base as shown.

a Find the exact side length of the square base of the prism.

b Find the volume of the entire support structure, correct to one decimal place.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

6G Volume of pyramids and cones 10A

Learning intentions

- To understand that the volume of a pyramid or cone is a fraction of the volume of the prism or cylinder with the same base area
- To know the formulas for the volume of pyramids and cones
- To be able to find the volume of pyramids and cones

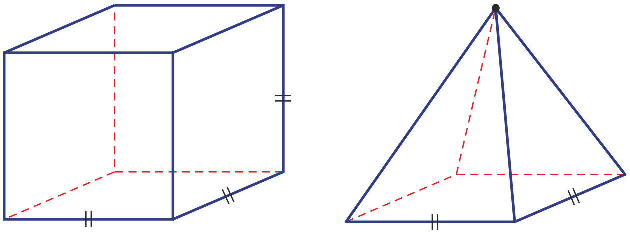
The volume of a cone or pyramid is a certain fraction of the volume of a prism with the same base area.

This particular fraction is the same for both cones and pyramids and will be explored in this section.

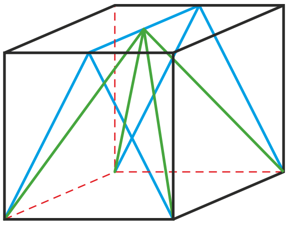
When applying fertiliser, farmers use a container called a spreader, made in a pyramid or cone shape. Agricultural equipment engineers calculate a spreader's volume using pyramid or cone formulas.

LESSON STARTER Is a pyramid half the volume of a prism?

Here is a cube and a square pyramid with equal base side lengths and equal heights.



- Discuss whether or not you think the pyramid is half the volume of the cube.
- Now consider this diagram of the cube with the pyramid inside.



The cube is black.
The pyramid is green.
The triangular prism is blue.

- Compared to the cube, what is the volume of the triangular prism (blue)? Give reasons.
- Is the volume of the pyramid (green) more or less than the volume of the triangular prism (blue)?
- Do you know what the volume of the pyramid is as a fraction of the volume of the cube?

ISBN 978-1-108-77290-7

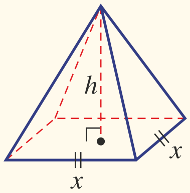
Cambridge University Press Updated June 2022

KEY IDEAS

- For pyramids and cones the volume is given by $V = \frac{1}{3}Ah$,

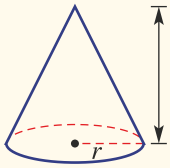
  where A is the area of the base and h is the perpendicular height.

right square pyramid



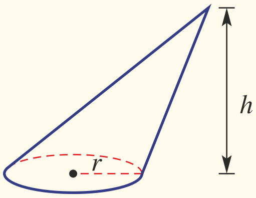
$$V = \frac{1}{3}Ah$$
$$= \frac{1}{3}x^2h$$

right cone



$$V = \frac{1}{3}Ah$$
$$= \frac{1}{3}\pi r^2h$$

oblique cone

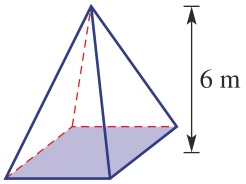


$$V = \frac{1}{3}Ah$$
$$= \frac{1}{3}\pi r^2h$$

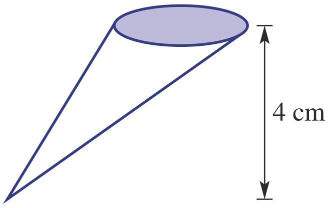
BUILDING UNDERSTANDING

1. A cylinder has volume 12 cm^3. What will be the volume of a cone with the same base area and perpendicular height?
2. A pyramid has volume 5 m^3. What will be the volume of a prism with the same base area and perpendicular height?
3. State the volume of these solids with the given base areas.

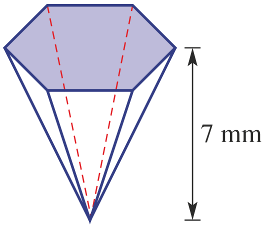
a



b



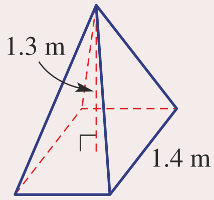
c



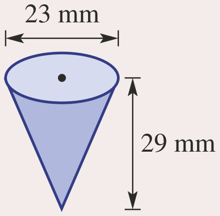
Example 17 Finding the volume of pyramids and cones

Find the volume of this rectangular-based pyramid and cone. Give the answer for part **b**, correct to two decimal places.

a



b



Continued on next page

SOLUTION

a $V = \frac{1}{3}Ah$

$= \frac{1}{3}(l \times w) \times h$

$= \frac{1}{3}(1.4 \times 1.2) \times 1.3$

$= 0.728\ \text{m}^3$

EXPLANATION

The pyramid has a rectangular base with area $l \times w$.

Substitute $l = 1.4$, $w = 1.2$ and $h = 1.3$.

b $V = \frac{1}{3}Ah$

$= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(11.5)^2 \times 29$

$= 4016.26\ \text{mm}^3$ (to 2 d.p.)

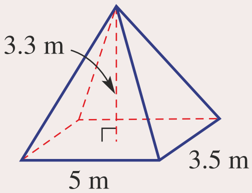
The cone has a circular base of area πr^2.

Substitute $r = \frac{23}{2} = 11.5$ and $h = 29$.

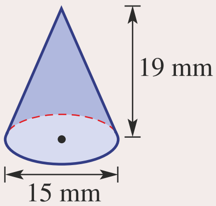
Now you try

Find the volume of this rectangular-based pyramid and cone. Give the answer for part **b**, correct to two decimal places.

a



b



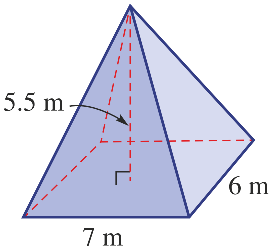
Exercise 6G

FLUENCY	1, 2–3(1/2)	2–3(1/2)	2–3(1/3)

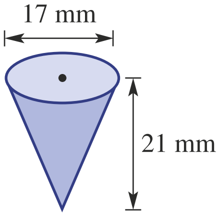
Example 17

1 Find the volume of this rectangular-based pyramid and cone. Give the answer for part **b**, correct to two decimal places.

a



b



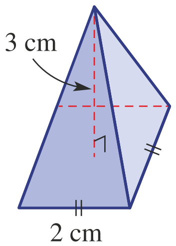
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

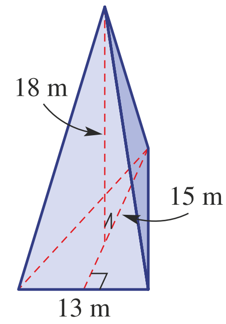
Example 17a

2 Find the volume of the following pyramids. For the oblique pyramids (i.e. parts **d**, **e**, **f**) use the given perpendicular height.

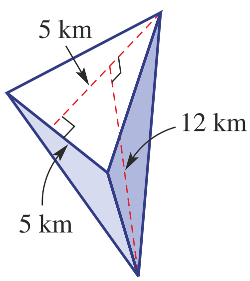
a



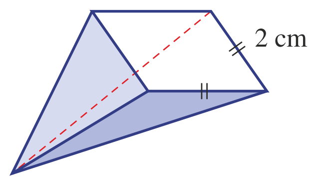
b



c

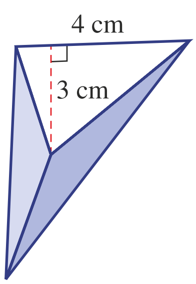


d



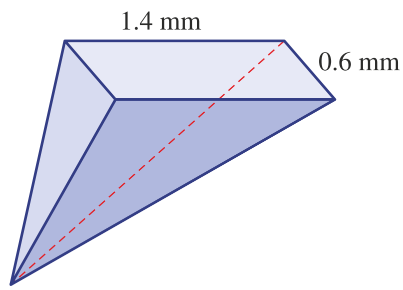
Perpendicular height = 2 cm

e



Perpendicular height = 4 cm

f

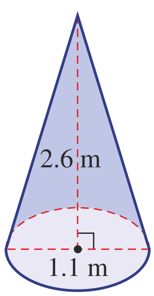


Perpendicular height = 1.2 mm

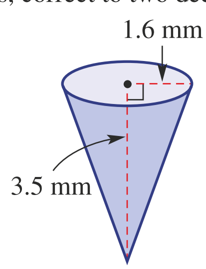
Example 17b

3 Find the volume of the following cones, correct to two decimal places.

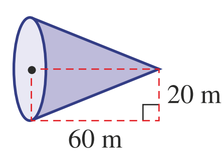
a



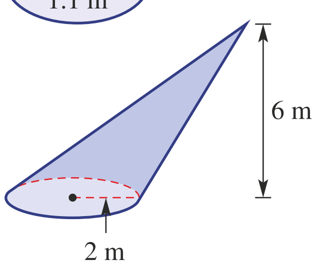
b



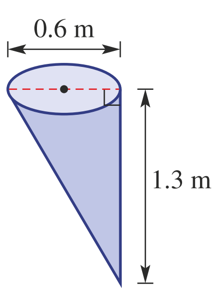
c



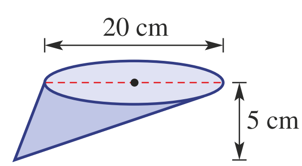
d



e



f



ISBN 978-1-108-77290-7

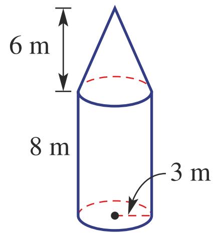
Cambridge University Press
Updated June 2022

PROBLEM-SOLVING	4, 5(1/2)	4, 5(1/2)	5(1/2), 6

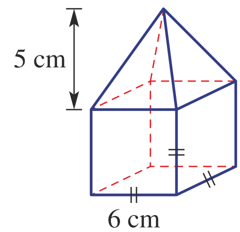
4 A medicine cup is in the shape of a cone with base radius 3 cm and height 5 cm. Find its capacity in millilitres, correct to the nearest millilitre.

5 Find the volume of these composite objects, rounding to two decimal places where necessary.

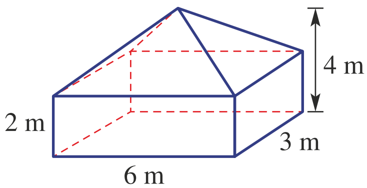
a



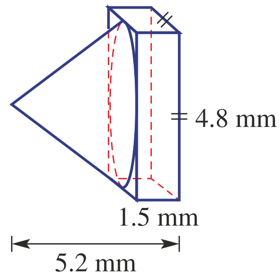
b



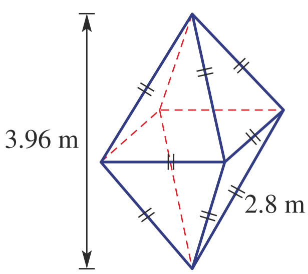
c



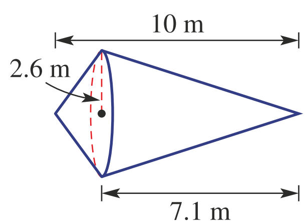
d



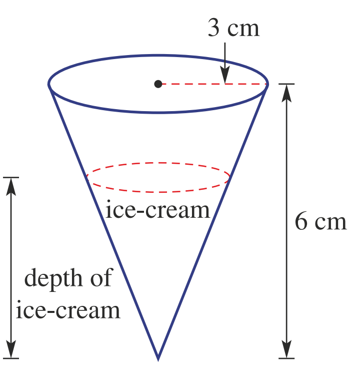
e



f



6 The volume of ice-cream in the cone is half the volume of the cone. The cone has a 3 cm radius and 6 cm height. What is the depth of the ice-cream, correct to two decimal places?



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

REASONING	7	7, 8	8, 9

7 A wooden cylinder is carved to form a cone that has the same base area and the same height as the original cylinder. What fraction of the wooden cylinder is wasted? Give a reason.

8 A square-based pyramid and a cone are such that the diameter of the cone is equal to the length of the side of the square base of the pyramid. They also have the same height.

a Using x as the side length of the pyramid and h as its height, write a rule for:

i the volume of the pyramid in terms of x and h

ii the volume of the cone in terms of x and h.

b Express the volume of the cone as a fraction of the volume of the pyramid. Give an exact answer.

9 a Use the rule $V = \frac{1}{3}\pi r^2 h$ to find the base radius of a cone, to one decimal place, with height 23 cm and volume 336 cm^3.

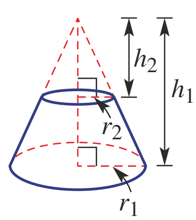
b Rearrange the rule $V = \frac{1}{3}\pi r^2 h$ to write:

i h in terms of V and r

ii r in terms of V and h.

ENRICHMENT: Truncated cones	–	–	10

10 A truncated cone is a cone that has its apex cut off by an intersecting plane. In this example, the top has radius r_2, the base has radius r_1 and the two circular ends are parallel.



a Give reasons why $\frac{r_1}{r_2} = \frac{h_1}{h_2}$.

b Find a rule for the volume of a truncated cone.

c Find the volume, to one decimal place, of a truncated cone when $r_1 = 2$ cm, $h_1 = 5$ cm and h_2 equals:

i $\frac{1}{2}h_1$

ii $\frac{2}{3}h_1$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

6H Surface area and volume of spheres 10A

Learning intentions

- To know the shape of a sphere and a hemisphere
- To know the formulas for the surface area and volume of a sphere and be able to use them
- To be able to use the formulas to find the volume and surface area of composite solids and spherical portions

Planets are spherical in shape due to the effects of gravity. This means that we can describe a planet's size using only one measurement – its diameter or radius. Mars, for example, has a diameter of about half that of the Earth, which is about 12 756 km. The Earth's volume is about 9 times that of Mars and this is because the volume of a sphere varies with the cube of the radius. The surface area of the Earth is about 3.5 times that of Mars because the surface area of a sphere varies with the square of the radius.

Spherical tanks store compressed or liquid gases for the petroleum and chemical industries. The spherical shape uses the smallest land area for the storage volume and distributes pressure evenly over the sphere's surface area.

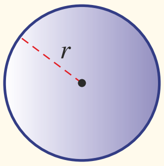
LESSON STARTER What percentage of a cube is a sphere?

A sphere of radius 1 unit just fits inside a cube.

- First, guess the percentage of space occupied by the sphere.
- Draw a diagram showing the sphere inside the cube.
- Calculate the volume of the cube and the sphere. For the sphere, use the formula $V = \frac{4}{3}\pi r^3$.
- Now calculate the percentage of space occupied by the sphere. How close was your guess?

KEY IDEAS

- The surface area of a **sphere** depends on its radius, r, and is given by:
  Surface area $= 4\pi r^2$
- The volume of a sphere depends on its radius, r, and is given by:
  Volume $= \frac{4}{3}\pi r^3$

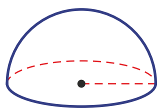
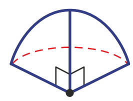
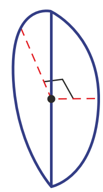


BUILDING UNDERSTANDING

1. Evaluate and round your answer to two decimal places.

   a $4 \times \pi \times 5^2$ **b** $4 \times \pi \times \left(\frac{1}{2}\right)^2$ **c** $\frac{4}{3} \times \pi \times 2.8^3$ **d** $\frac{4\pi(7)^3}{3}$

2. Rearrange $12 = 4\pi r^2$ to write r as the subject.
3. Rearrange $8 = \frac{4}{3}\pi r^3$ to write r as the subject.
4. What fraction of a sphere is shown in these diagrams?

   a
   
   b
   
   c
   

Example 18 Finding the surface area and volume of a sphere

Find the surface area and volume of a sphere of radius 7 cm, correct to two decimal places.

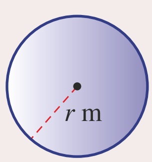
SOLUTION	EXPLANATION
$\text{TSA} = 4\pi r^2$	Write the rule for the surface area of a sphere and
$= 4\pi(7)^2$	substitute $r = 7$.
$= 615.75\ \text{cm}^2$ (to 2 d.p.)	Evaluate and round the answer.
$V = \frac{4}{3}\pi r^3$	Write the rule for the volume of a sphere and substitute $r = 7$.
$= \frac{4}{3}\pi(7)^3$	
$= 1436.76\ \text{cm}^3$ (to 2 d.p.)	Evaluate and round the answer.

Now you try

Find the surface area and volume of a sphere of radius 5 cm, correct to two decimal places.

Example 19 Finding the radius of a sphere

Find the radius of a sphere with volume 10 m^3, correct to two decimal places.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

SOLUTION

$$V = \frac{4}{3}\pi r^3$$

$$10 = \frac{4}{3}\pi r^3$$

$$30 = 4\pi r^3$$

$$\frac{15}{2\pi} = r^3$$

$$\therefore\ r = \sqrt[3]{\frac{15}{2\pi}}$$

$$= 1.34 \text{ (to 2 d.p.)}$$

$\therefore$ The radius is 1.34 m.

EXPLANATION

Substitute $V = 10$ into the formula for the volume of a sphere.

Solve for r^3 by multiplying both sides by 3 and then dividing both sides by 4π. Simplify $\frac{30}{4\pi} = \frac{15}{2\pi}$.

Take the cube root of both sides to make r the subject and evaluate.

Now you try

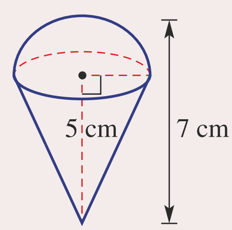
Find the radius of a sphere with volume 6 m^3, correct to two decimal places.

Example 20 Finding the surface area and volume of composite solids with spherical portions

This composite object includes a hemisphere and cone, as shown.

a Find the surface area, rounding to two decimal places.

b Find the volume, rounding to two decimal places.



SOLUTION

a Radius $r = 7 - 5 = 2$

$$s^2 = 5^2 + 2^2$$

$$s^2 = 29$$

$$s = \sqrt{29}$$

$$\text{TSA} = \frac{1}{2} \times 4\pi r^2 + \pi rs$$

$$= \frac{1}{2} \times 4\pi(2)^2 + \pi(2)(\sqrt{29})$$

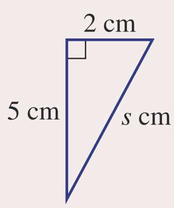
$$= 8\pi + 2\sqrt{29}\pi$$

$$= 58.97 \text{ cm}^2 \text{ (to 2 d.p.)}$$

EXPLANATION

First find the radius, r cm, of the hemisphere.

Calculate the slant height, s, of the cone using Pythagoras' theorem.



Write the rules for the surface area of each component and note that the top shape is a hemisphere (i.e. half sphere).

Only the curved surface of the cone is required.

Substitute $r = 2$ and $h = 5$.

Simplify and then evaluate, rounding as required.

Continued on next page

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b $V = \frac{1}{2} \times \frac{4}{3}\pi r^3 + \frac{1}{3}\pi r^2 h$ — Volume (object) $= \frac{1}{2}$Volume (sphere) + Volume (cone)

$= \frac{1}{2} \times \frac{4}{3}\pi(2)^3 + \frac{1}{3}\pi(2)^2(5)$ — Substitute $r = 2$ and $h = 5$.

$= \frac{16\pi}{3} + \frac{20\pi}{3}$ — Simplify and then evaluate, rounding as required.

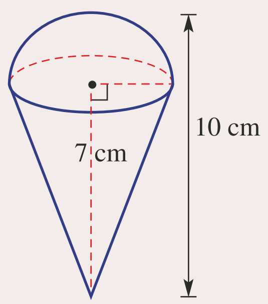
$= \frac{36\pi}{3}$

$= 12\pi$

$= 37.70\text{ cm}^3$ (to 2 d.p.)

Now you try

This composite object includes a hemisphere and cone, as shown.



a Find the surface area, rounding to two decimal places.
b Find the volume, rounding to two decimal places.

Exercise 6H

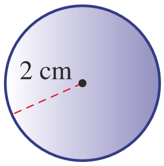
FLUENCY	1, 2–3(1/2)	2–4(1/2)	2–4(1/3)

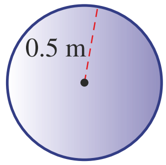
Example 18

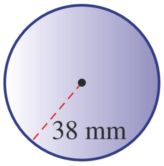
1 Find the surface area and volume of a sphere of radius 4 cm, correct to two decimal places.

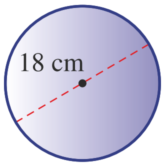
Example 18

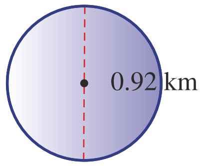
2 Find the surface area and volume of the following spheres, correct to two decimal places.

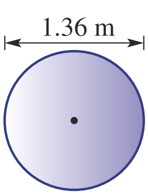
a


b


c


d


e


f


ISBN 978-1-108-77290-7

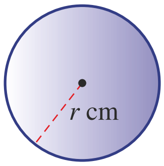
Cambridge University Press
Updated June 2022

3 Find the total surface area and volume of a sphere with the given dimensions. Give the answer correct to two decimal places.

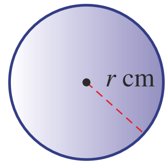
a radius 3 cm
b radius 4 m
c radius 7.4 m
d diameter $\sqrt{5}$ mm
e diameter $\sqrt{7}$ m
f diameter 2.2 km

Example 19

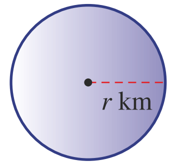
4 a Find the radius of these spheres with the given volumes, correct to two decimal places.

i


$V = 15\text{ cm}^3$

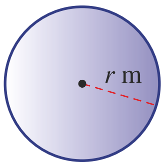
ii


$V = 180\text{ cm}^3$

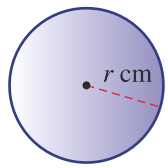
iii


$V = 0.52\text{ km}^3$

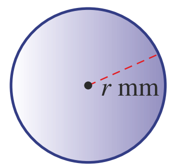
b Find the radius of these spheres with the given surface area, correct to two decimal places.

i


$S = 10\text{ m}^2$

ii


$S = 120\text{ cm}^2$

iii


$S = 0.43\text{ mm}^2$

PROBLEM-SOLVING	5–7	8–10, 11(1/2)	11–12(1/2), 13, 14

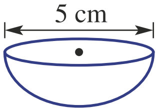
5 A box with dimensions 30 cm long, 30 cm wide and 30 cm high holds 50 tennis balls of radius 3 cm. Find:

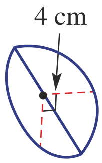
a the volume of one tennis ball, correct to two decimal places
b the volume of 50 tennis balls, correct to one decimal place
c the volume of the box not taken up by the tennis balls, correct to one decimal place.

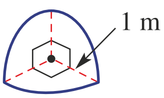
6 An expanding spherical storage bag has 800 cm^3 of water pumped into it. Find the diameter of the bag, correct to one decimal place, after all the water has been pumped in.

7 A sphere just fits inside a cube. What is the surface area of the sphere as a percentage of the surface area of the cube? Round your answer to the nearest whole percentage.

8 Find the volume of these portions of a sphere, correct to two decimal places.

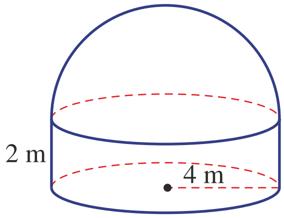
a


b


c


9 Two sports balls have radii 10 cm and 15 cm. Find the difference in their total surface areas, correct to one decimal place.

10 A monolithic structure has a cylindrical base of radius 4 m and height 2 m and a hemispherical top.



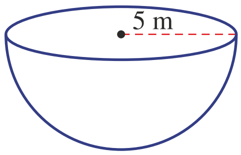
a What is the radius of the hemispherical top?
b Find the total volume of the entire monolithic structure, correct to one decimal place.

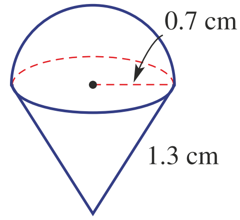
ISBN 978-1-108-77290-7

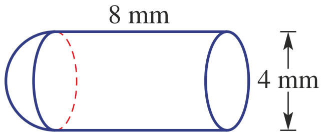
Cambridge University Press
Updated June 2022

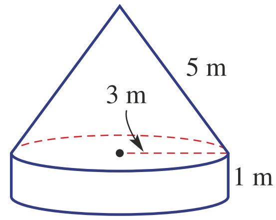
Example 20a

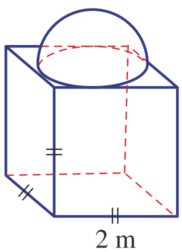
11 Find the total surface area for these solids, correct to two decimal places.

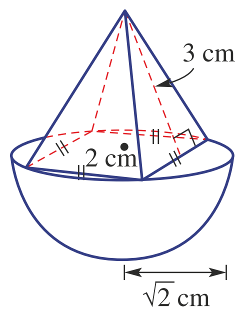
a


b


c


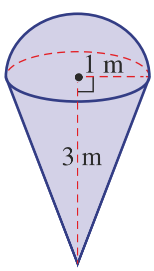
d


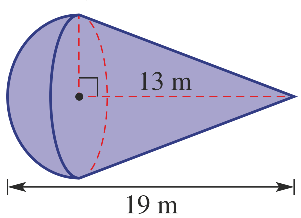
e


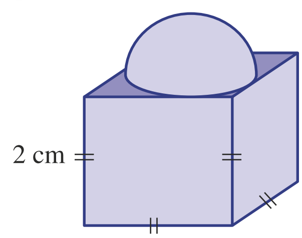
f


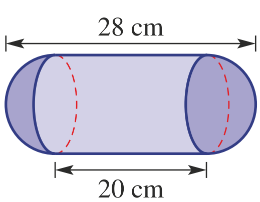
Example 20b

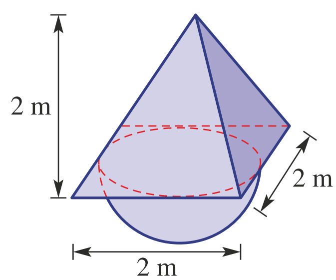
12 Find the volume of the following composite objects, correct to two decimal places.

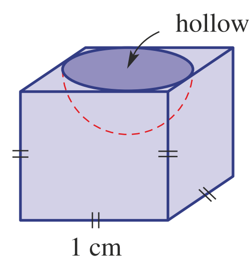
a


b


c


d


e


f


13 A spherical party balloon is blown up to help decorate a room.

a Find the volume of air, correct to two decimal places, needed for the balloon to be:

i 10 cm wide **ii** 20 cm wide **iii** 30 cm wide

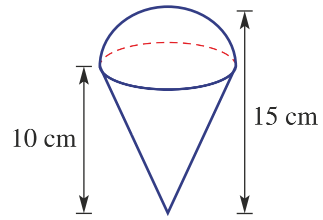
b If the balloon pops when the volume of air reaches 120 000 cm^3, find the diameter of the balloon at that point, correct to one decimal place.

14 A hemisphere sits on a cone and two height measurements are given as shown. Find:

a the radius of the hemisphere

b the exact slant height of the cone in surd form

c the total surface area of the solid, correct to one decimal place.



REASONING	15	15, 16	16–18

15 **a** Find a rule for the radius of a sphere with surface area S.

b Find a rule for the radius of a sphere with volume V.

16 A ball's radius is doubled.

a By how much does its surface area change?

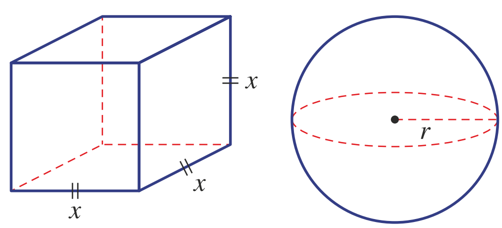
b By how much does its volume change?

17 Show that the volume of a sphere is given by $V = \frac{1}{6}\pi d^3$, where d is the diameter.

18 A cylinder and a sphere have the same radius, r, and volume, V. Find a rule for the height of the cylinder in terms of r.

ENRICHMENT: Comparing surface areas	–	–	19

19 Imagine a cube and a sphere that have the same volume.



a If the sphere has volume 1 unit^3, find:

i the exact radius of the sphere

ii the exact surface area of the sphere

iii the value of x (i.e. the side length of the cube)

iv the surface area of the cube

v the surface area of the sphere as a percentage of the surface area of the cube, correct to one decimal place.

b Now take the radius of the sphere to be r units. Write:

i the rule for the surface area of the sphere

ii the rule for x in terms of r, given the volumes are equal

iii the surface area of the cube in terms of r.

c Now write the surface area of the sphere as a fraction of the surface area of the cube, using your results from part **b**.

Simplify to show that the result is $\sqrt[3]{\frac{\pi}{6}}$.

d Compare your answers from part **a v** with that of part **c** (i.e. as a percentage).

ISBN 978-1-108-77290-7

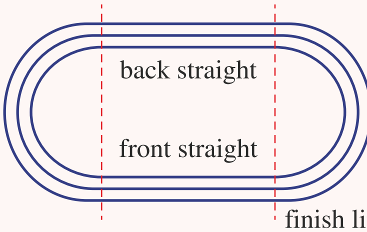
Cambridge University Press
Updated June 2022

Applications and problem-solving

The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

Athletics stagger

1 An athletics 400 m track is made up of two straight sections of equal length and two semicircular bends. The first two lanes of the eight lanes of the track are shown.



International regulations state that the radius of the semicircle to the inside edge of the track is 36.500 m.

A coach is interested in the length of each lane of the running track and how a staggered start is used to ensure that each runner has the same distance to cover.

a The 400 m distance for lane 1 is measured for the *running line* of that lane. The *running line* is taken at 300 mm in from the lane's inside edge. By first finding the radius for the lane 1 *running line*, calculate the length of the straight sections, correct to three decimal places.

Each lane is 1.22 m wide, with the *running line* for each lane after lane 1 considered to be 200 mm in from the lane's inside edge.

b If the competitor from lane 2 started from the lane 1 start line, how far would they be required to run, based on the *running lines*, to complete one lap? Round to two decimal places.

To ensure everyone runs 400 m, competitors need a staggered start.

c From your answer to part **b**, what should be the stagger for the lane 2 competitor on their *running line*, correct to two decimal places?

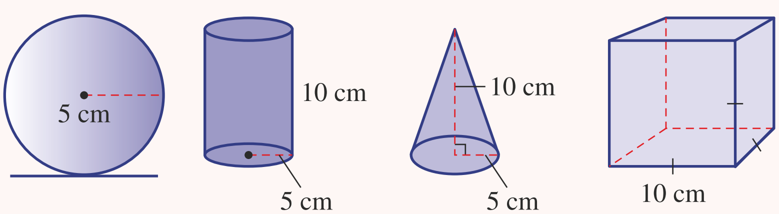
d Calculate the stagger for each of the competitors in lanes 3 to 8, correct to two decimal places.

e Determine a rule for the stagger, s m, of lane number l on the *running line*.

Perfume volume

2 The different containers shown below are used as perfume bottles for a brand of perfume.

The cylinder, sphere and cone each have a radius of 5 cm. The cube has a side length of 10 cm. The heights of the cylinder and the cone are both 10 cm.



A perfume company wishes to compare the volume of each of the container types and the amount of glass used to make each one. They will also consider adjusting the dimensions of the containers so that they have the same capacity.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

a Determine the volume of each container, rounding to two decimal places where necessary.

b If the same amount of money is being charged for each bottle, rank them in order from best to worst value.

c The company is looking to improve the value of the cone-shaped container. Determine what height the cone would need to have so that it has the same volume as the sphere if they kept the radius of 5 cm.

d Consider a cylinder and cone such that their height is equal to twice their radius, r cm.

  i Give the ratios of the volume of the cylinder to the volume of a sphere and the volume of the cone to the volume of a sphere, if they all have the same radius, r cm.

  ii Determine what factor the height of the cylinder and the height of the cone need to be multiplied by so that they have the same volume as the sphere.

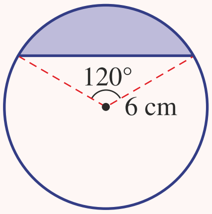
e The bottles are made of glass. Give the ratio of their surface areas, sphere : cone : cylinder, if they have the same volume and same radius r. Which container uses the most glass?

f What height would a square-based pyramid, with side length equal to a sphere's diameter, need to be if it has the same volume as the sphere? Answer in terms of r.

Drifting log

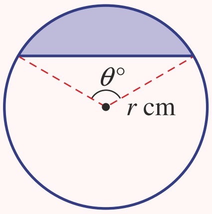
3 A cylindrical Huon pine log of length 3 m and radius 20 cm is floating horizontally in the water.

A timber engineer wants to investigate the amount of pine wood that sits above water level. They consider a cross-section of the log, including an angle at the centre and the vertical distance between the top of the log and the water level.



a A cross-section of a smaller cylindrical log is as shown, with radius 6 cm. Find the area of the shaded region of the cross-section correct to two decimal places. Recall the formula $\frac{1}{2}ab\sin C$ for the area of a triangle.

b Hence, determine a rule for the shaded region shown below right in terms of r and θ.



c The Huon pine log floating in the water (length 3 m, radius 20 cm) is such that the top of the log is 5 cm above the water.

  i Draw a diagram to represent this, labelling all known lengths.

  ii Use trigonometry to find the required angle θ for the rule in part **b**. Answer correct to three decimal places.

  iii Find the area of the end of the log that is above water, correct to three decimal places.

  iv Hence, find the volume of the log that is not under water, correct to the nearest cubic centimetre.

d Find the volume of the log in part **c** not under water if the top of the log is 25 cm above the water. Round to the nearest cubic centimetre.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

6I Limits of accuracy EXTENDING

Learning intentions
- To understand the difficulty in obtaining exact measurements
- To know how to find the upper and lower boundaries (limits of accuracy) for the true measurement

Humans and machines measure many different things, such as the time taken to swim a race, the length of timber needed for a building and the volume of cement needed to lay a concrete path around a swimming pool. The degree or level of accuracy required usually depends on the intended purpose of the measurement.

All measurements are approximate. Errors can happen as a result of the equipment being used or the person using the measuring device.

Accuracy is a measure of how close a recorded measurement is to the exact measurement. Precision is the ability to obtain the same result over and over again.

Major track events are electronically timed to the millisecond and rounded to hundredths. An electronic beep has replaced the pistol sound that took 0.15 s to reach the farthest athlete. A camera scans the finish line 2000 times/s and signals the timer as athletes finish.

LESSON STARTER Rounding a decimal

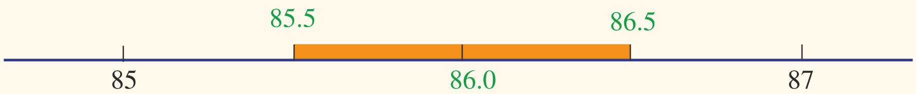
- A piece of timber is measured to be 86 cm, correct to the nearest centimetre.
  - **a** What is the smallest measurement possible that rounds to 86 cm when rounded to the nearest cm?
  - **b** What is the largest measurement possible that rounds to 86 when rounded to the nearest cm?
- If a measurement is recorded as 6.0 cm, correct to the nearest millimetre, then:
  - **a** What units were used when measuring?
  - **b** What is the smallest decimal that could be rounded to this value?
  - **c** What is the largest decimal that would have resulted in 6.0 cm?
- Consider a square with side length 7.8941 cm.
  - **a** What is the perimeter of the square if the side length is:
    - **i** used with the four decimal places?
    - **ii** rounded to one decimal place?
    - **iii** truncated at one decimal place (i.e. 7.8)?
  - **b** What is the difference between the perimeters if the decimal is rounded to two decimal places or truncated at two decimal places or written with two significant figures?

ISBN 978-1-108-77290-7

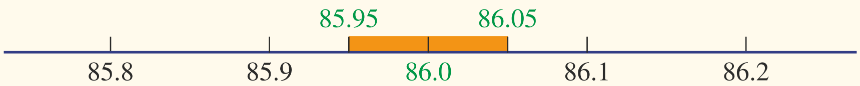
Cambridge University Press
Updated June 2022

KEY IDEAS

- The limits of accuracy tell you what the upper and lower boundaries are for the true measurement.
  - Usually, it is $\pm 0.5 \times$ the smallest unit of measurement.
    For example, when measuring to the nearest centimetre, 86 cm has limits from 85.5 cm up to (but not including) 86.5 cm.



  - When measuring to the nearest millimetre, the limits of accuracy for 86.0 cm are 85.95 cm to 86.05 cm.



BUILDING UNDERSTANDING

1. State a decimal that gives 3.4 when rounded from two decimal places.
2. State a measurement of 3467 mm, correct to the nearest:
   a centimetre **b** metre
3. What is the smallest decimal that could result in an answer of 6.7 when rounded to one decimal place?

Example 21 Finding limits of accuracy

Give the limits of accuracy for these measurements.

a 72 cm **b** 86.6 mm

SOLUTION

a $72 \pm 0.5 \times 1$ cm
$= 72 - 0.5$ cm to $72 + 0.5$ cm
$= 71.5$ cm to 72.5 cm

EXPLANATION

Smallest unit of measurement is one whole cm.
Error $= 0.5 \times 1$ cm
This error is subtracted and added to the given measurement to find the limits of accuracy.

SOLUTION

b $86.6 \pm 0.5 \times 0.1$ mm
$= 86.6 \pm 0.05$ mm
$= 86.6 - 0.05$ mm to $86.6 + 0.05$ mm
$= 86.55$ mm to 86.65 mm

EXPLANATION

Smallest unit of measurement is 0.1 mm.
Error $= 0.5 \times 0.1$ mm $= 0.05$ mm
This error is subtracted and added to the given measurement to find the limits of accuracy.

Now you try

Give the limits of accuracy for these measurements.

a 45 cm **b** 15.7 mm

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 22 Considering the limits of accuracy

Janis measures each side of a square as 6 cm. Find:

a the upper and lower limits for the sides of the square
b the upper and lower limits for the perimeter of the square
c the upper and lower limits for the square's area.

SOLUTION	EXPLANATION
a $6 \pm 0.5 \times 1$ cm $= 6 - 0.5$ cm to $6 + 0.5$ cm $= 5.5$ cm to 6.5 cm	Smallest unit of measurement is one whole cm. Error $= 0.5 \times 1$ cm
b Lower limit $P = 4 \times 5.5$ $= 22$ cm Upper limit $P = 4 \times 6.5$ $= 26$ cm	The lower limit for the perimeter uses the lower limit for the measurement taken and the upper limit for the perimeter uses the upper limit of 6.5 cm.
c Lower limit $= 5.5^2$ $= 30.25\text{ cm}^2$ Upper limit $= 6.5^2$ $= 42.25\text{ cm}^2$	The lower limit for the area is 5.5^2, whereas the upper limit will be 6.5^2.

Now you try

Janis measures each side of a square as 9 cm. Find:

a the upper and lower limits for the sides of the square
b the upper and lower limits for the perimeter of the square
c the upper and lower limits for the square's area.

Exercise 6I

FLUENCY	1, 2–3(1/2), 4	2–3(1/2), 4, 5(1/2), 6	2–3(1/3), 4, 5(1/2), 6

1 Give the limits of accuracy for these measurements.

Example 21a
a **i** 55 cm **ii** 32 cm

Example 21b
b **i** 32.3 mm **ii** 108.4 mm

Example 21
2 For each of the following:
i Give the smallest unit of measurement (e.g. 0.1 cm is the smallest unit in 43.4 cm).
ii Give the limits of accuracy.

a 45 cm **b** 6.8 mm **c** 12 m **d** 15.6 kg
e 56.8 g **f** 10 m **g** 673 h **h** 9.84 m
i 12.34 km **j** 0.987 km **k** 1.65 L **l** 9.03 mL

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

3 Give the limits of accuracy for the following measurements.

a 5 m **b** 8 cm **c** 78 mm
d 5 mL **e** 2 km **f** 34.2 cm
g 3.9 kg **h** 19.4 kg **i** 457.9 L
j 18.65 m **k** 7.88 km **l** 5.05 s

4 What are the limits of accuracy for the amount $4500 when it is written:

a to two significant figures?
b to three significant figures?
c to four significant figures?

5 Write the following as a measurement, given that the lower and upper limits of these measurements are as follows.

a 29.5 m to 30.5 m **b** 14.5 g to 15.5 g
c 4.55 km to 4.65 km **d** 8.95 km to 9.05 km
e 985 g to 995 g **f** 989.5 g to 990.5 g

6 Martha writes down the length of her fabric as 150 cm. As Martha does not give her level of accuracy, give the limits of accuracy of her fabric if it was measured correct to the nearest:

a centimetre **b** 10 centimetres **c** millimetre

PROBLEM-SOLVING	7, 8	7, 8	8, 9

7 A length of copper pipe is given as 25 cm, correct to the nearest centimetre.

a What are the limits of accuracy for this measurement?
b If 10 pieces of copper, each with a given length of 25 cm, are joined end to end, what is the minimum length that it could be?
c What is the maximum length for the 10 pieces of pipe in part **b**?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 22

8 The side of a square is recorded as 9.2 cm, correct to two significant figures.

a What is the minimum length that the side of this square could be?

b What is the maximum length that the side of this square could be?

c Find the upper and lower boundaries for this square's perimeter.

d Find the upper and lower limits for the area of this square.

9 The side of a square is recorded as 9.20 cm, correct to three significant figures.

a What is the minimum length that the side of this square could be?

b What is the maximum length that the side of this square could be?

c Find the upper and lower boundaries for this square's perimeter.

d Find the upper and lower limits for the area of this square.

e How has changing the level of accuracy from 9.2 cm (see Question 8) to 9.20 cm affected the calculation of the square's perimeter and area?

REASONING	10	10	10, 11

10 Cody measures the mass of an object to be 6 kg. Jacinta says the same object is 5.8 kg and Luke gives his answer as 5.85 kg.

a Explain how all three people could have different answers for the same measurement.

b Write down the level of accuracy being used by each person.

c Are all their answers correct? Discuss.

11 Write down a sentence explaining the need to accurately measure items in our everyday lives and the accuracy required for each of your examples. Give three examples of items that need to be measured correct to the nearest:

a kilometre　b millimetre　c millilitre　d litre.

ENRICHMENT: Percentage error	–	–	12(½)

12 To calculate the percentage error of any measurement, the error (i.e. ± the smallest unit of measurement) is compared to the given or recorded measurement and then converted to a percentage.

For example: 5.6 cm

Error $= \pm 0.5 \times 0.1 = \pm 0.05$

$$\begin{aligned}\text{Percentage error} &= \frac{\pm 0.05}{5.6} \times 100\% \\ &= \pm 0.89\% \text{ (to two significant figures)}\end{aligned}$$

Find the percentage error for each of the following. Round to two significant figures.

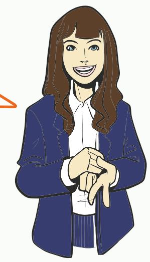
a 28 m　b 9 km　c 8.9 km　d 8.90 km

e 178 mm　f $8.96　g 4.25 m　h 701 mL

ISBN 978-1-108-77290-7

Problems and challenges

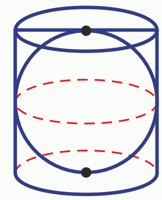
Up for a challenge? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.



1 A cube has a surface area that has the same value as its volume. What is the side length of this cube?

2 The wheels of a truck travelling at 60 km/h make 4 revolutions per second. What is the diameter of each wheel in metres, correct to one decimal place?

3 A sphere fits exactly inside a cylinder, and just touches the top, bottom and curved surface.

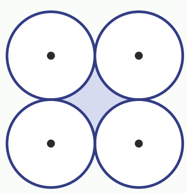


a Show that the surface area of the sphere equals the curved surface area of the cylinder.

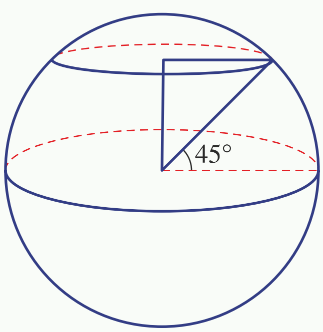
b What percentage of the volume of the cylinder is taken up by the sphere? Round your answer to the nearest whole percentage.

4 A sphere and cone with the same radius, r, have the same volume. Find the height of the cone in terms of r.

5 Four of the same circular coins of radius r are placed such that they are just touching, as shown. What is the area of the shaded region enclosed by the coins in terms of r?



6 Find the exact ratio of the equator to the distance around the Earth at latitude 45° north. (Assume that the Earth is a perfect sphere.)



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter summary

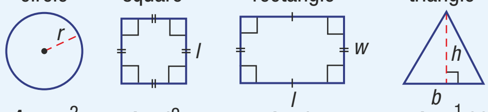
Measurement

Area

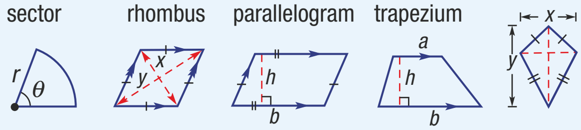
Units of area

$$\text{km}^2 \underset{\div 1000^2}{\overset{\times 1000^2}{\rightleftarrows}} \text{m}^2 \underset{\div 100^2}{\overset{\times 100^2}{\rightleftarrows}} \text{cm}^2 \underset{\div 10^2}{\overset{\times 10^2}{\rightleftarrows}} \text{mm}^2$$

Formulas

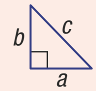


circle $A = \pi r^2$ | square $A = l^2$ | rectangle $A = lw$ | triangle $A = \frac{1}{2}bh$



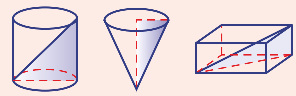
sector $A = \frac{\theta}{360} \times \pi r^2$ | rhombus $A = \frac{1}{2}xy$ | parallelogram $A = bh$ | trapezium $A = \frac{1}{2}(a + b)h$ | kite $A = \frac{1}{2}xy$

Pythagoras' theorem



$c^2 = a^2 + b^2$

Can occur in 3D shapes



Length

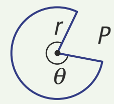
Units of length

$$\text{km} \underset{\div 1000}{\overset{\times 1000}{\rightleftarrows}} \text{m} \underset{\div 100}{\overset{\times 100}{\rightleftarrows}} \text{cm} \underset{\div 10}{\overset{\times 10}{\rightleftarrows}} \text{mm}$$

Perimeter is the distance around the outside of a closed shape.

Circumference of a circle
$C = 2\pi r = \pi d$

Perimeter of a sector

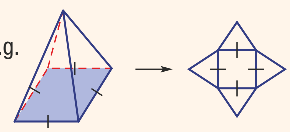


$P = 2r + \frac{\theta}{360} \times 2\pi r$

Surface area

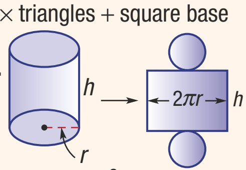
For prisms and pyramids draw the net and add the areas of all the faces.

e.g.



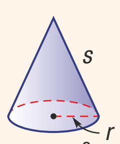
TSA = 4 × triangles + square base

Cylinder



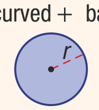
$\text{TSA} = \underset{\text{curved surface}}{2\pi rh} + \underset{\text{base ends}}{2\pi r^2}$

Cone (10A)



$\text{TSA} = \underset{\text{curved}}{\pi rs} + \underset{\text{base}}{\pi r^2}$

Sphere (10A)



$\text{TSA} = 4\pi r^2$

For composite solids consider which surfaces are exposed.

Limits of accuracy (Ext)

Usually ±0.5 × the smallest unit of measurement
e.g. 72 cm is 71.5 cm to 72.5 cm.

Volume

Units of volume

$$\text{km}^3 \underset{\div 1000^3}{\overset{\times 1000^3}{\rightleftarrows}} \text{m}^3 \underset{\div 100^3}{\overset{\times 100^3}{\rightleftarrows}} \text{cm}^3 \underset{\div 10^3}{\overset{\times 10^3}{\rightleftarrows}} \text{mm}^3$$

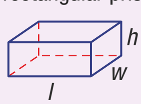
Units of capacity

$$\underset{\text{(ML)}}{\text{megalitres}} \underset{\div 1000}{\overset{\times 1000}{\rightleftarrows}} \underset{\text{(kL)}}{\text{kilolitres}} \underset{\div 1000}{\overset{\times 1000}{\rightleftarrows}} \underset{\text{(L)}}{\text{litres}} \underset{\div 1000}{\overset{\times 1000}{\rightleftarrows}} \underset{\text{(mL)}}{\text{millilitres}}$$

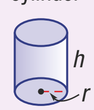
1 cm^3 = 1 mL

For right and oblique prisms and cylinders
$V = Ah$, where A is the area of the base and h is the perpendicular height.

rectangular prism | cylinder



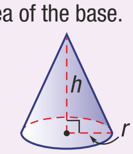
$V = lwh$



$V = \pi r^2 h$

For pyramids and cones: (10A)
$V = \frac{1}{3}Ah$, where A is the area of the base.

Cone: $V = \frac{1}{3}\pi r^2 h$



Sphere: $V = \frac{4}{3}\pi r^3$ (10A)

ISBN 978-1-108-77290-7

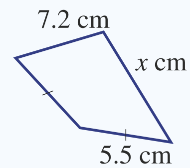
Cambridge University Press
Updated June 2022

Chapter checklist

Chapter checklist: Success criteria

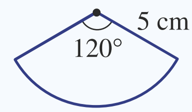
✔

6A **1. I can find the perimeter of a polygon.**
e.g. Find the value of x when the perimeter is 26.6 cm.

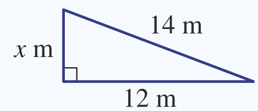


6A **2. I can use the formula for the circumference of a circle.**
e.g. A circle has a radius of r cm, if its circumference is 12 cm, find r correct to two decimal places.

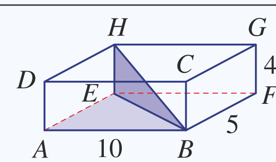
6A **3. I can find the perimeter of a sector.**
e.g. Find the exact perimeter of this sector.



6B **4. I can find a side length using Pythagoras' theorem.**
e.g. Find the length of the unknown side correct to two decimal places.

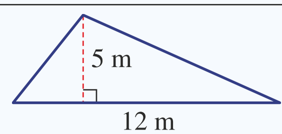


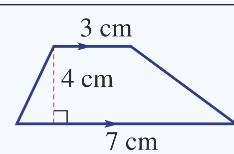
6B **5. I can use Pythagoras' theorem in 3D.**
e.g. For the rectangular prism shown find BE in exact form and find BH correct to two decimal places.



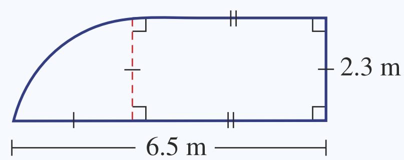
6C **6. I can convert between units of area.**
e.g. Convert 1200 cm^2 to m^2.

6C **7. I can work with area formulas.**
e.g. Find the area of these shapes.

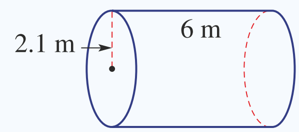




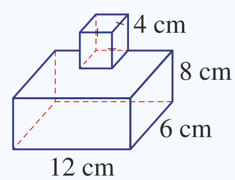
6C **8. I can find the area of composite shapes involving sectors.**
e.g. Find the area of this composite shape. Give your answer as an exact value and as a decimal correct to two decimal places.



6D **9. I can find the surface area of a cylinder.**
e.g. Find total surface area of the cylinder shown correct to two decimal places.



6D **10. I can find the surface area of composite solids.**
e.g. A composite solid consists of a rectangular prism and a cube as shown. Find the total surface area of the object.

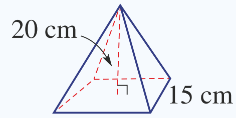


6E **11. I can find the surface area of a cone.** 10A
e.g. Find the total surface area of a cone with radius 4 cm and slant height 10 cm. Round to two decimal places.

ISBN 978-1-108-77290-7

Chapter checklist

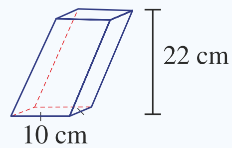
6E **12. I can find the surface area of a pyramid.** 10A
e.g. Find the total surface area of the square based pyramid shown.



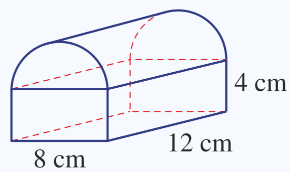
6E **13. I can find the slant height or vertical height of a cone.** 10A
e.g. A cone has radius 5 cm and a curved surface area of 120 cm^2. Find the vertical height of the cone correct to one decimal place.

6F **14. I can find the volume of a cylinder or prism.**
e.g. Find the volume of a cylinder with height 10 cm and radius 2 cm. Round to two decimal places.

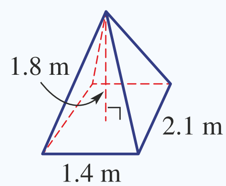
6F **15. I can find the volume of an oblique prism.**
e.g. Find the volume of this oblique prism.



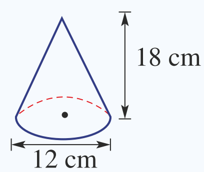
6F **16. I can find the volume of a composite solid.**
e.g. Find the volume of the composite solid correct to one decimal place.



6G **17. I can find the volume of a pyramid.** 10A
e.g. Find the volume of this rectangular based pyramid.



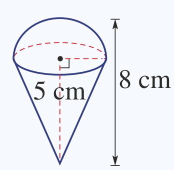
6G **18. I can find the volume of a cone.** 10A
e.g. Find the volume of this cone correct to two decimal places.



6H **19. I can find the surface area and volume of a sphere.** 10A
e.g. Find the volume and surface area of a sphere of radius 3 cm, correct to two decimal places.

6H **20. I can find the radius of a sphere.** 10A
e.g. Find the radius of a sphere with volume 12 m^3, correct to two decimal places.

6H **21. I can work with composite solids with spherical portions.**
e.g. Find the surface area and volume of this cone and hemisphere correct to two decimal places.



6I **22. I can find limits of accuracy.** Ext
e.g. Give the limits of accuracy for 85 cm.

6I **23. I can work with limits of accuracy.** Ext
e.g. Each side of a square is measured as 10 cm. Find the upper and lower limits for the perimeter of the square.

ISBN 978-1-108-77290-7

Short-answer questions

6A/C/F

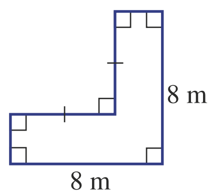
1 Convert the given measurements to the units in the brackets.

a 0.23 m (cm)
b 270 mm^2 (cm^2)
c 2.6 m^3 (cm^3)
d 8.372 litres (mL)
e 638 250 mm^2 (m^2)
f 0.0003 km^2 (cm^2)

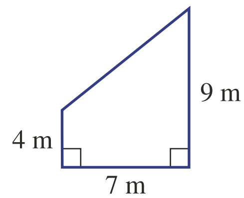
6B

2 Find the perimeter of these shapes, correct to one decimal place where necessary. (Note: Pythagoras' theorem may be required in part **b**.)

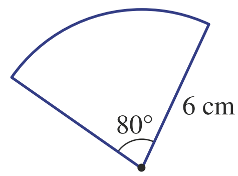
a



b



c



6A/C

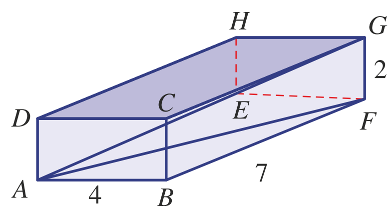
3 A floral clock at the Botanic Gardens is in the shape of a circle and has a circumference of 14 m.

a Find the radius of the clock, in exact form.
b Hence, find the area occupied by the clock. Answer to two decimal places.

6B

4 For the rectangular prism with dimensions as shown, use Pythagoras' theorem to find:

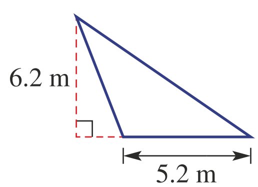
a AF, leaving your answer in exact form
b AG, to two decimal places.



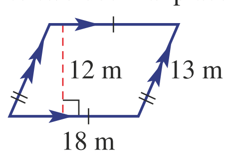
6C

5 Find the area of these shapes. Round to two decimal places where necessary.

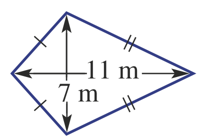
a



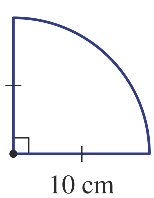
b



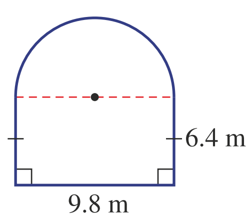
c



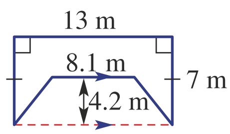
d



e



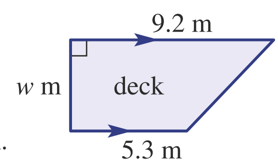
f



6C

6 A backyard deck, as shown, has an area of 34.8 m^2.

a Find the width, w metres, of the deck.
b Calculate the perimeter of the deck, correct to two decimal places. Pythagoras' theorem will be required to calculate the missing length.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

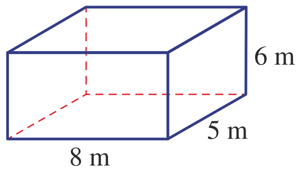
Chapter review

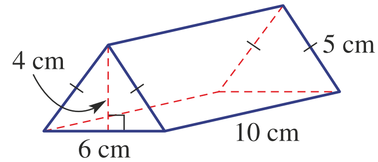
6D–H

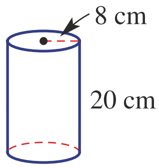
7 For each of these solids find, correct to two decimal places where necessary:

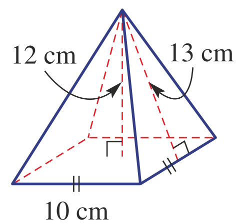
i the total surface area

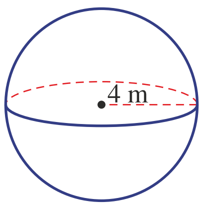
ii the volume

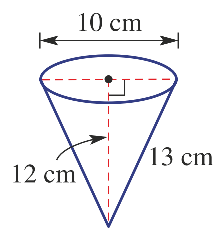
a


b


c


10A d


10A e


10A f


6E 10A

8 A cone has a radius of 6 cm and a curved surface area of 350 cm^2.

a Find the slant height of the cone, in exact form.

b Find the height of the cone, correct to one decimal place.

6E/G 10A

9 A papier mâché model of a square-based pyramid with base length 30 cm has volume 5400 cm^3.

a What is the height of the pyramid?

b Use Pythagoras' theorem to find the exact height of the triangular faces.

c Hence, find the total surface area of the model, correct to one decimal place.

6F/G 10A

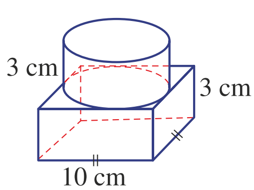
10 A cylinder and a cone each have a base radius of 1 m. The cylinder has a height of 4 m. Determine the height of the cone if the cone and cylinder have the same volume.

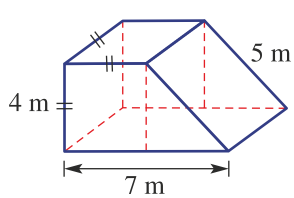
6D/F

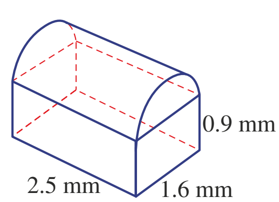
11 For each of the following composite solids find, correct to two decimal places where necessary:

i the total surface area

ii the volume.

a


b


c


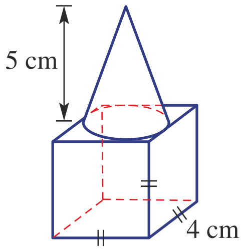
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

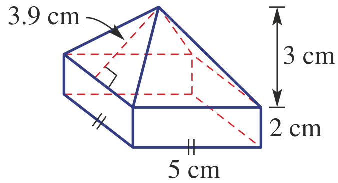
6E-H 10A

12 For each of the following composite solids find, correct to two decimal places where necessary:
  i the total surface area **ii** the volume

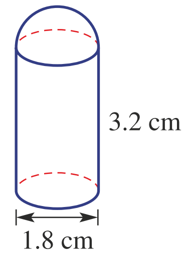
a



b

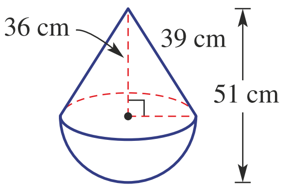


c



6G/H 10A

13 A water buoy is in the shape shown. Find:



a the volume of air inside the buoy, in exact form
b the surface area of the buoy, in exact form.

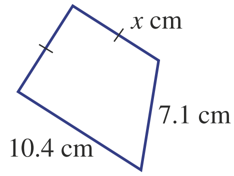
6I Ext

14 Give the limits of accuracy for these measurements.
a 8 m **b** 10.3 kg **c** 4.75 L

Multiple-choice questions

6A

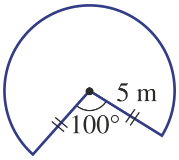
1 If the perimeter of this shape is 30.3 cm, then the value of x is:



A 12.8
B 6.5
C 5.7
D 6.4
E 3.6

6A

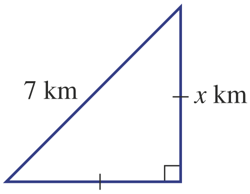
2 The perimeter of the sector shown, rounded to one decimal place, is:



A 8.7 m
B 22.7 m
C 18.7 m
D 56.7 m
E 32.7 m

6B

3 The value of x in this triangle is closest to:



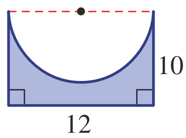
A 4.9
B 3.5
C 5.0
D 4.2
E 3.9

ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

Chapter review

6C

4 The exact shaded (purple) area in square units is:



A $32 - 72\pi$
B $120 - 36\pi$
C $32 + 6\pi$
D $120 - 18\pi$
E $48 + 18\pi$

6C

5 $0.128\ m^2$ is equivalent to:

A $12.8\ cm^2$ **B** $128\ mm^2$ **C** $1280\ cm^2$ **D** $0.00128\ cm^2$ **E** $1280\ mm^2$

6D

6 A cube has a total surface area of $1350\ cm^2$. The side length of the cube is:

A 15 cm **B** 11 cm **C** 18 cm **D** 12 cm **E** 21 cm

6D

7 A cylindrical tin of canned food has a paper label glued around its curved surface. If the can is 14 cm high and has a radius of 4 cm, the area of the label is closest to:

A $452\ cm^2$ **B** $352\ cm^2$ **C** $126\ cm^2$ **D** $704\ cm^2$ **E** $235\ cm^2$

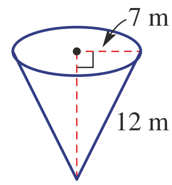
6E 10A

8 The exact surface area of a cone of diameter 24 cm and slant height 16 cm is:

A $216\pi\ cm^2$ **B** $960\pi\ cm^2$ **C** $528\pi\ cm^2$ **D** $336\pi\ cm^2$ **E** $384\pi\ cm^2$

6B

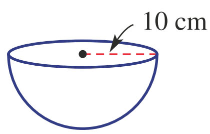
9 A cone has a radius of 7 m and a slant height of 12 m. The cone's exact height, in metres, is:



A $\sqrt{52}$
B $\sqrt{193}$
C $\sqrt{85}$
D $\sqrt{137}$
E $\sqrt{95}$

6H 10A

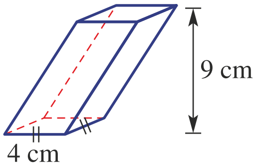
10 The volume of the hemisphere shown, correct to the nearest cubic centimetre, is:



A $1257\ cm^3$
B $628\ cm^3$
C $2094\ cm^3$
D $1571\ cm^3$
E $4189\ cm^3$

6F

11 The volume of this oblique square-based prism is:



A $72\ cm^3$
B $48\ cm^3$
C $176\ cm^3$
D $144\ cm^3$
E $120\ cm^3$

ISBN 978-1-108-77290-7

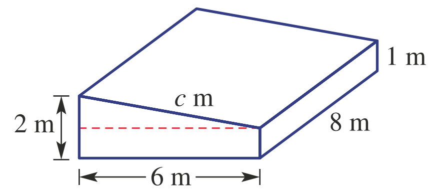
Cambridge University Press
Updated June 2022

6H 10A

12 The volume of air in a sphere is 100 cm^3. The radius of the sphere, correct to two decimal places, is:
A 1.67 cm **B** 10.00 cm **C** 2.82 cm **D** 23.87 cm **E** 2.88 cm

Extended-response questions

1 A waterski ramp consists of a rectangular flotation container and a triangular angled section, as shown.

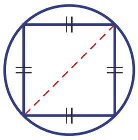


a What volume of air is contained within the entire ramp structure?
b Find the length of the angled ramp (c metres), in exact surd form.

The entire structure is to be painted with a waterproof paint costing $20 per litre. One litre of paint covers 25 square metres.

c Find the total surface area of the ramp, correct to one decimal place.
d Find the number of litres and the cost of paint required for the job. Assume you can purchase only 1 litre tins of paint.

2 A circular school oval of radius 50 metres is marked with spray paint to form a square pitch, as shown.

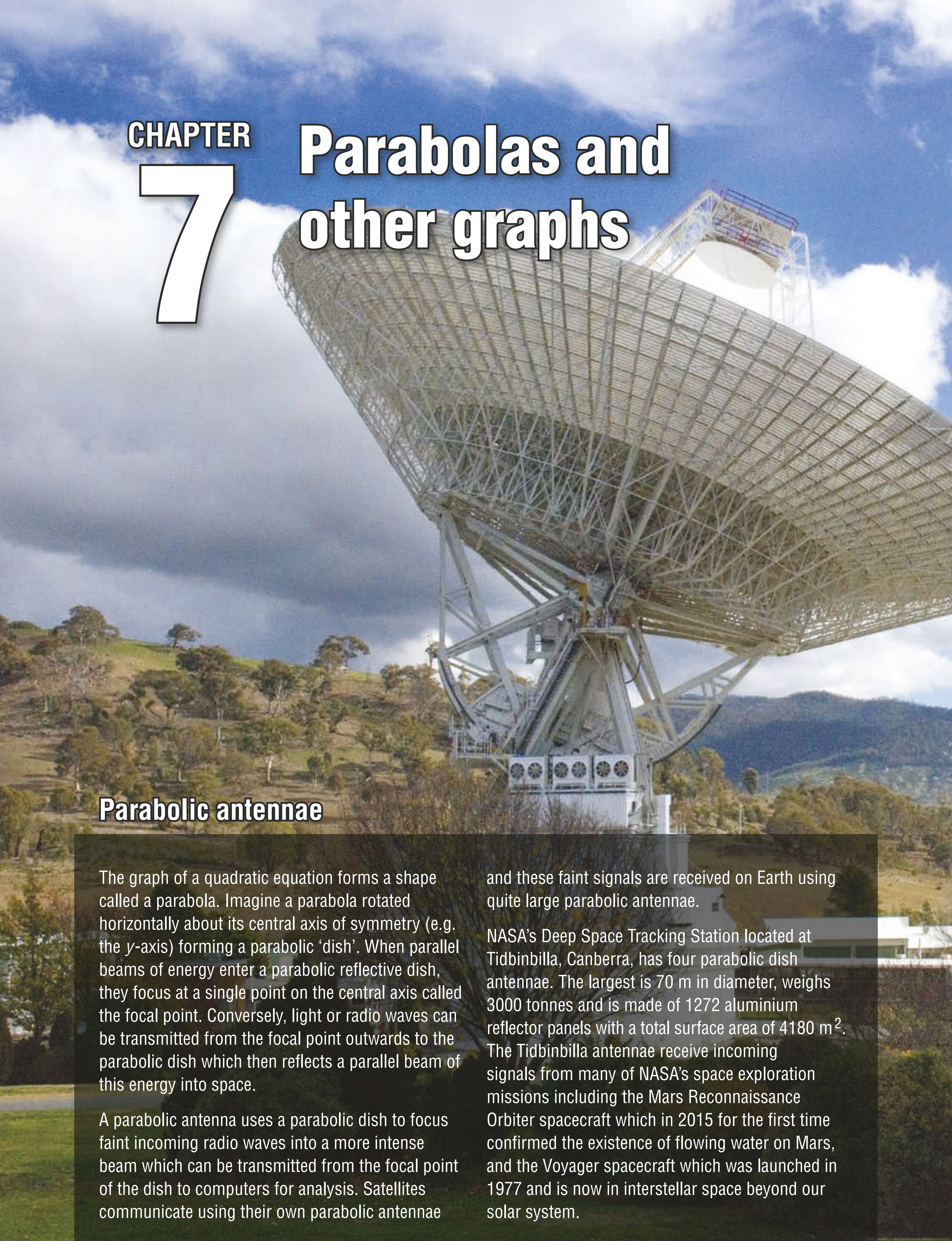


a State the diagonal length of the square.
b Use Pythagoras' theorem to find the side length of the square, in exact surd form.
c Find the area of the square pitch.
d Find the percentage area of the oval that is not part of the square pitch. Round your answer to the nearest whole percentage.

Two athletes challenge each other to a one-lap race around the oval. Athlete A runs around the outside of the oval at an average rate of 10 metres per second. Athlete B runs around the outside of the square at an average rate of 9 metres per second. Athlete B's average running speed is less because of the need to slow down at each corner.

e Find who comes first and the difference in times, correct to the nearest hundredth of a second.

ISBN 978-1-108-77290-7



CHAPTER 7 Parabolas and other graphs

Parabolic antennae

The graph of a quadratic equation forms a shape called a parabola. Imagine a parabola rotated horizontally about its central axis of symmetry (e.g. the y-axis) forming a parabolic 'dish'. When parallel beams of energy enter a parabolic reflective dish, they focus at a single point on the central axis called the focal point. Conversely, light or radio waves can be transmitted from the focal point outwards to the parabolic dish which then reflects a parallel beam of this energy into space.

A parabolic antenna uses a parabolic dish to focus faint incoming radio waves into a more intense beam which can be transmitted from the focal point of the dish to computers for analysis. Satellites communicate using their own parabolic antennae and these faint signals are received on Earth using quite large parabolic antennae.

NASA's Deep Space Tracking Station located at Tidbinbilla, Canberra, has four parabolic dish antennae. The largest is 70 m in diameter, weighs 3000 tonnes and is made of 1272 aluminium reflector panels with a total surface area of 4180 m^2. The Tidbinbilla antennae receive incoming signals from many of NASA's space exploration missions including the Mars Reconnaissance Orbiter spacecraft which in 2015 for the first time confirmed the existence of flowing water on Mars, and the Voyager spacecraft which was launched in 1977 and is now in interstellar space beyond our solar system.

ISBN 978-1-108-77290-7

Updated June 2022

Online resources

A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

In this chapter

Victorian Curriculum

NUMBER AND ALGEBRA
Patterns and algebra

Factorise algebraic expressions by taking out a common factor (VCMNA329)

Expand binomial products and factorise monic quadratic expressions using a variety of strategies (VCMNA332)

Substitute values into formulas to determine an unknown and re-arrange formulas to solve for a particular term (VCMNA333)

Linear and non-linear relationships

Explore the connection between algebraic and graphical representations of relations such as simple quadratic, reciprocal, circle and exponential, using digital technology as appropriate (VCMNA339)

Solve simple quadratic equations using a range of strategies (VCMNA341)

(10A) Describe, interpret and sketch parabolas, hyperbolas, circles and exponential functions and their transformations (VCMNA359)

(10A) Factorise monic and non-monic quadratic expressions and solve a wide range of quadratic equations derived from a variety of contexts (VCMNA362)

(10A) Use function notation to describe the relationship between dependent and independent variables in modelling contexts (VCMNA363)

Real numbers

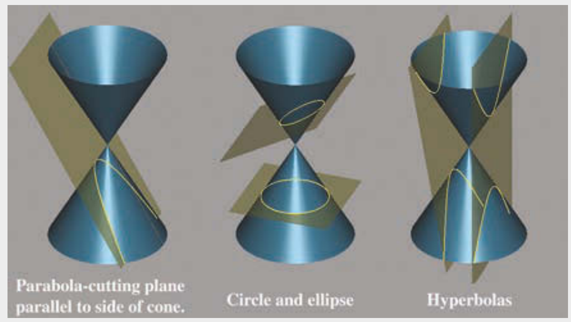
Solve simple problems involving inverse proportion (VCMNA327)

7A Exploring parabolas

Learning intentions

- To know the shape and symmetry of the basic parabola $y = x^2$
- To be able to identify the key features of a parabola from a graph
- To be able to observe the impact of transformations of $y = x^2$

One of the simplest and most important non-linear graphs is the parabola. When a ball is thrown or water streams up and out from a garden hose, the path followed has a parabolic shape. The parabola is the graph of a quadratic relation with the basic rule $y = x^2$. Quadratic rules, such as $y = (x - 1)^2$, $y = 2x^2 - x - 3$ and $y = (x + 4)^2 - 7$, also give graphs that are parabolas and are transformations of the graph of $y = x^2$.

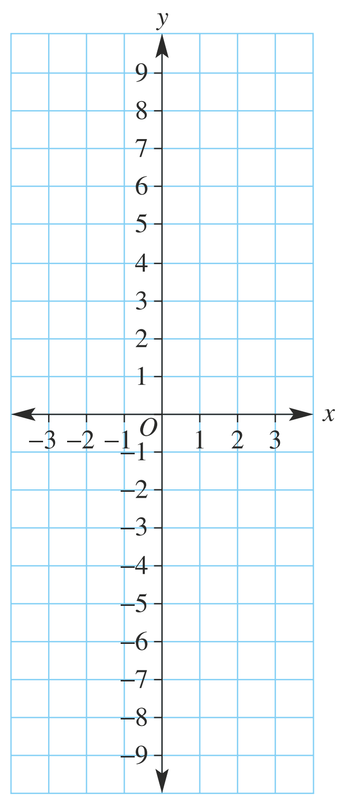


Intersecting a cone with a plane forms curves called the conic sections. Greek scholars analysed the conics using geometry. From the 17th century, Descartes' newly discovered Cartesian geometry enabled a more powerful algebraic analysis.

LESSON STARTER To what effect?

To see how different quadratic rules compare to the graph of $y = x^2$, complete this table and plot the graph of each equation on the same set of axes.

x	−3	−2	−1	0	1	2	3
$y_1 = x^2$	9	4					
$y_2 = -x^2$	−9						
$y_3 = (x - 2)^2$							
$y_4 = x^2 - 3$							



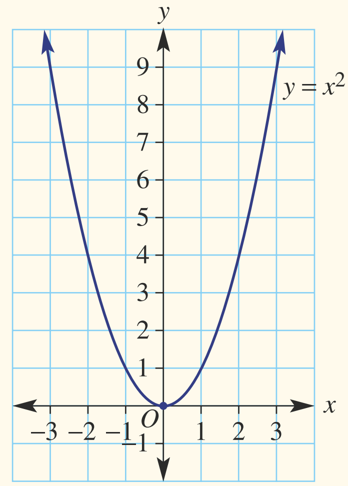
- For all the graphs, find such features as the:
  - turning point
  - axis of symmetry
  - y-intercept
  - x-intercepts.
- Discuss how each of the graphs of y_2, y_3 and y_4 compare to the graph of $y = x^2$. Compare the rule with the position of the graph.

ISBN 978-1-108-77290-7

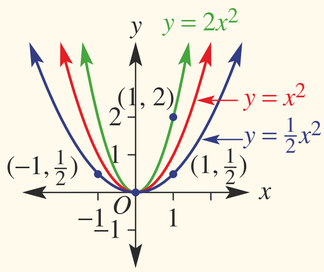
Cambridge University Press
Updated June 2022

KEY IDEAS

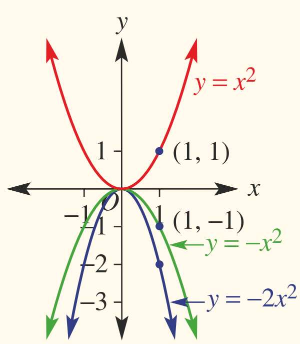
- A **parabola** is the graph of a quadratic relation. The basic parabola has the rule $y = x^2$.
  - The vertex (or turning point) is (0, 0).
  - It is a minimum turning point.
  - Axis of symmetry is $x = 0$.
  - y-intercept is 0.
  - x-intercept is 0.



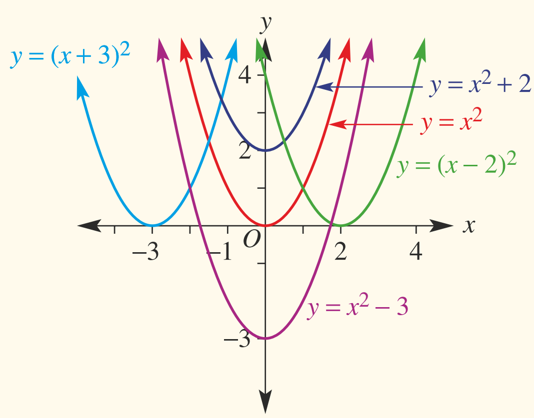
- Simple transformations of the graph of $y = x^2$ include:
  - dilation



  - reflection



  - translation



BUILDING UNDERSTANDING

1 Complete the features of this graph.

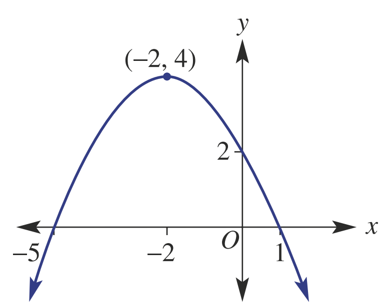
a The parabola has a __________ (maximum or minimum).

b The coordinates of the turning point are __________.

c The y-intercept is __________.

d The x-intercepts are __________ and __________.

e The axis of symmetry is __________.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2 Complete the features of this graph.

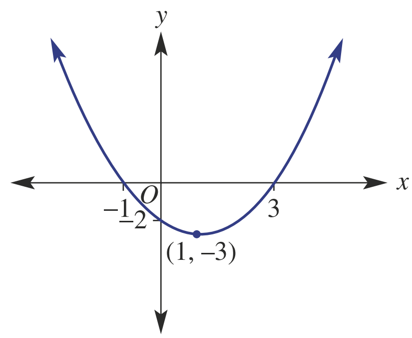
a The parabola has a __________ (maximum or minimum).

b The coordinates of the turning point are __________.

c The y-intercept is __________.

d The x-intercepts are __________ and __________.

e The axis of symmetry is __________



Example 1 Identifying key features of parabolas

Determine the following key features of each of the given graphs.

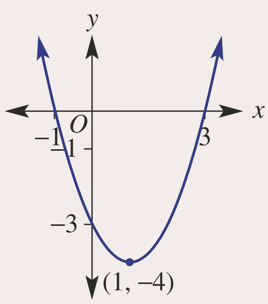
i turning point and whether it is a maximum or minimum

ii axis of symmetry

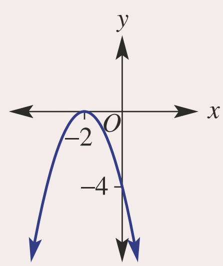
iii x-intercepts

iv y-intercept

a



b



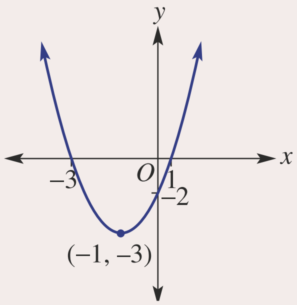
SOLUTION	EXPLANATION
a i Turning point is a minimum at $(1, -4)$.	Lowest point of graph is at $(1, -4)$.
ii Axis of symmetry is $x = 1$.	Line of symmetry is through the x-coordinate of the turning point.
iii x-intercepts are -1 and 3.	x-intercepts lie on the x-axis $(y = 0)$ and the
iv y-intercept is -3.	y-intercept on the y-axis $(x = 0)$.
b i Turning point is a maximum at $(-2, 0)$.	Graph has a highest point at $(-2, 0)$.
ii Axis of symmetry is $x = -2$.	Line of symmetry is through the x-coordinate of the turning point.
iii x-intercept is -2.	Turning point is also the one x-intercept.
iv y-intercept is -4.	

Now you try

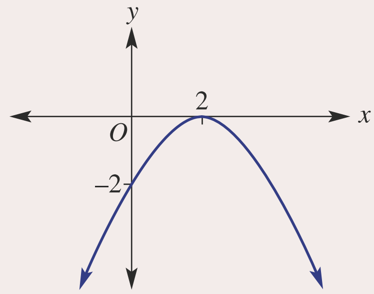
Determine the following key features of each of the given graphs.

i turning point and whether it is a maximum or minimum
ii axis of symmetry
iii x-intercepts
iv y-intercepts

a



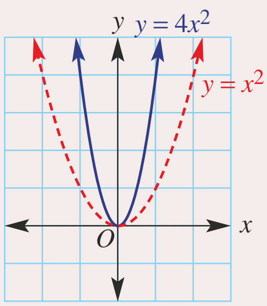
b



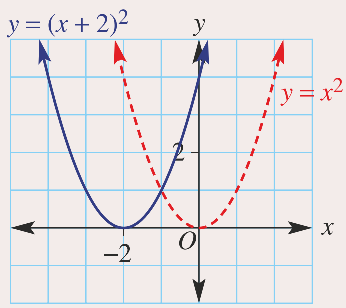
Example 2 Transforming parabolas

Copy and complete the table for the following graphs.

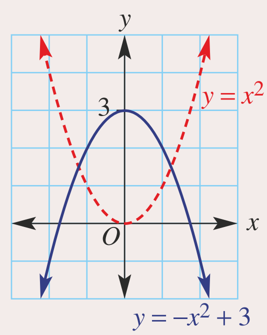
a



b



c



	Formula	Maximum or minimum	Reflected in the x-axis (yes/no)	Turning point	y-value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = 4x^2$					
b	$y = (x + 2)^2$					
c	$y = -x^2 + 3$					

SOLUTION

	Formula	Maximum or minimum	Reflected in the x-axis (yes/no)	Turning point	y-value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = 4x^2$	minimum	no	(0, 0)	4	narrower
b	$y = (x + 2)^2$	minimum	no	(–2, 0)	9	same
c	$y = -x^2 + 3$	maximum	yes	(0, 3)	2	same

EXPLANATION

Read features from graphs and consider the effect of each change in equation on the graph.

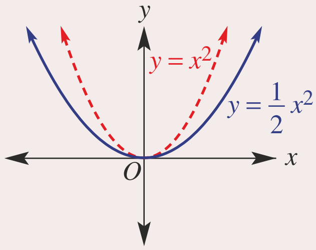
Continued on next page

ISBN 978-1-108-77290-7

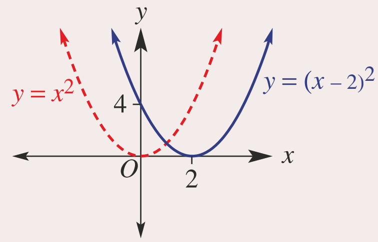
Now you try

Copy and complete the table for the following graphs.

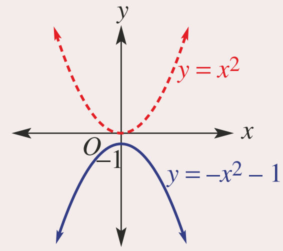
a



b



c



	Formula	Maximum or minimum	Reflected in the x-axis (yes/no)	Turning point	y-value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = \frac{1}{2}x^2$					
b	$y = (x - 2)^2$					
c	$y = -x^2 - 1$					

Exercise 7A

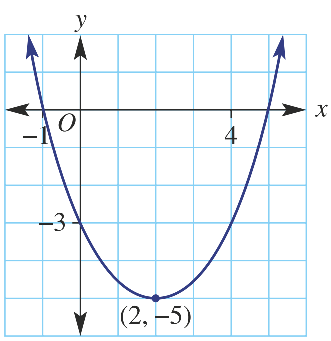
FLUENCY	1–4	1–4(½)	2–4(½)

Example 1

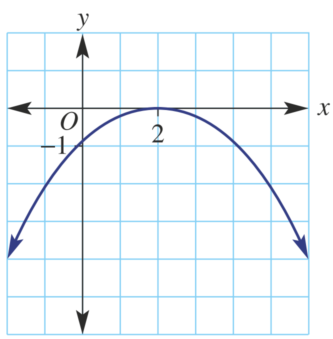
1 Determine these key features of the following graphs.

  i turning point and whether it is a maximum or minimum
  ii axis of symmetry **iii** x-intercepts **iv** y-intercept

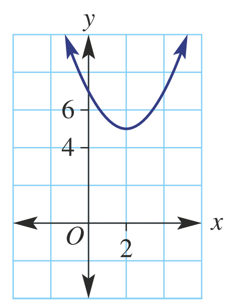
a



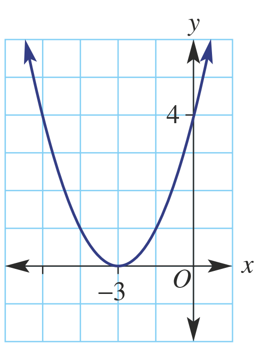
b



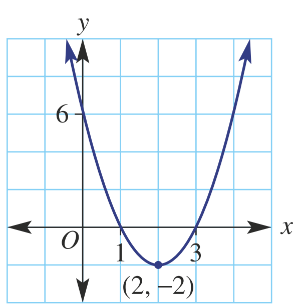
c



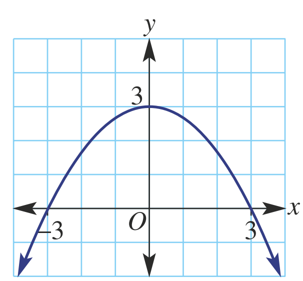
d



e



f



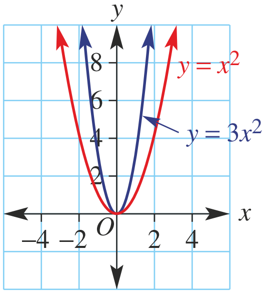
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

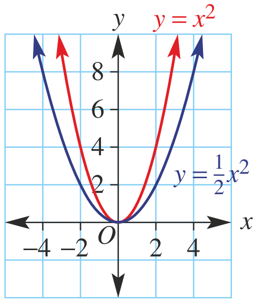
Example 2a

2 Copy and complete the table below for the following graphs.

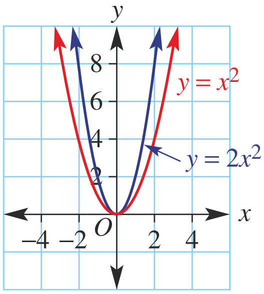
a



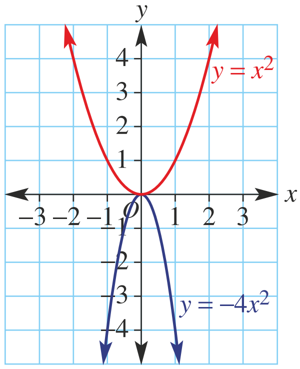
b



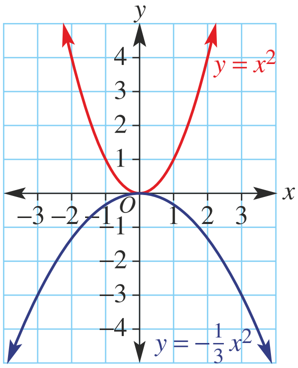
c



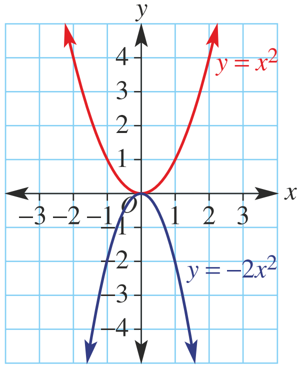
d



e



f



	Formula	Maximum or minimum	Reflected in the x-axis (yes/no)	Turning point	y-value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = 3x^2$					
b	$y = \frac{1}{2}x^2$					
c	$y = 2x^2$					
d	$y = -4x^2$					
e	$y = -\frac{1}{3}x^2$					
f	$y = -2x^2$					

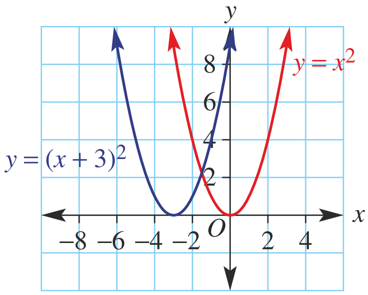
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

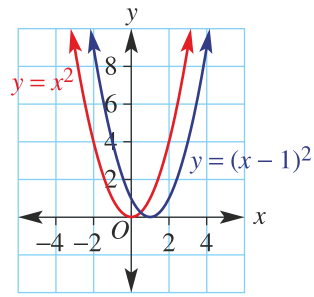
Example 2b

3 Copy and complete the table below for the following graphs.

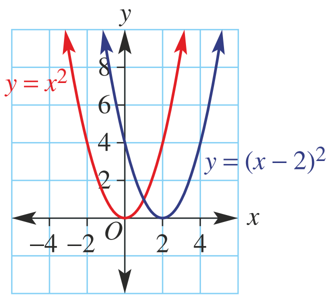
a



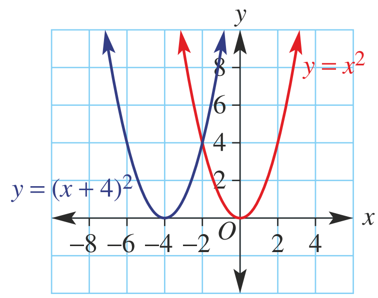
b



c



d

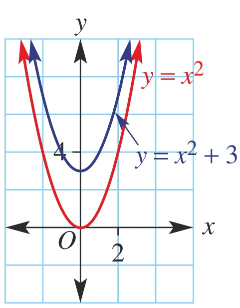


	Formula	Turning point	Axis of symmetry	y-intercept ($x = 0$)	x-intercept
a	$y = (x + 3)^2$				
b	$y = (x - 1)^2$				
c	$y = (x - 2)^2$				
d	$y = (x + 4)^2$				

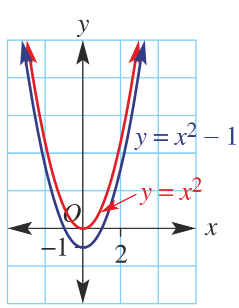
Example 2c

4 Copy and complete the table for the following graphs.

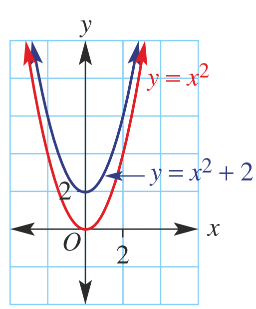
a



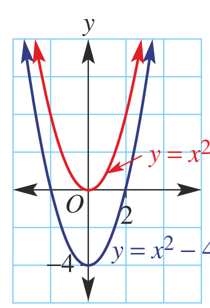
b



c



d



	Formula	Turning point	y-intercept ($x = 0$)	y-value when $x = 1$
a	$y = x^2 + 3$			
b	$y = x^2 - 1$			
c	$y = x^2 + 2$			
d	$y = x^2 - 4$			

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

PROBLEM-SOLVING	5–7(1/2)	5–7(1/2), 8	5–7(1/3), 8

5 Write down the equation of the axis of symmetry for the graphs of these rules.

a $y = x^2$ **b** $y = x^2 + 7$ **c** $y = -2x^2$

d $y = -3x^2$ **e** $y = x^2 - 4$ **f** $y = (x - 2)^2$

g $y = (x + 1)^2$ **h** $y = -(x + 3)^2$ **i** $y = -x^2 - 3$

j $y = \frac{1}{2}x^2 + 2$ **k** $y = x^2 - 16$ **l** $y = -(x + 4)^2$

6 Write down the coordinates of the turning point for the graphs of the equations in Question **5**.

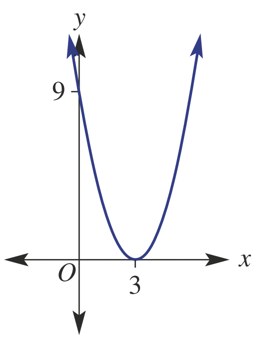
7 Find the y-intercept (i.e. when $x = 0$) for the graphs of the equations in Question **5**.

8 Match each of the following equations to one of the graphs below.

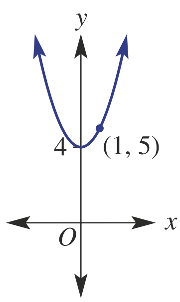
a $y = 2x^2$ **b** $y = x^2 - 6$ **c** $y = (x + 2)^2$ **d** $y = -5x^2$

e $y = (x - 3)^2$ **f** $y = \frac{1}{4}x^2$ **g** $y = x^2 + 4$ **h** $y = (x - 2)^2 + 3$

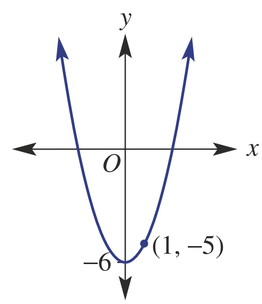
A



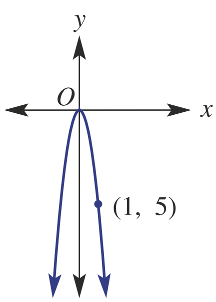
B



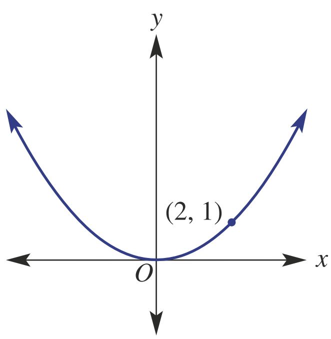
C



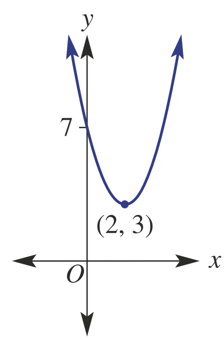
D



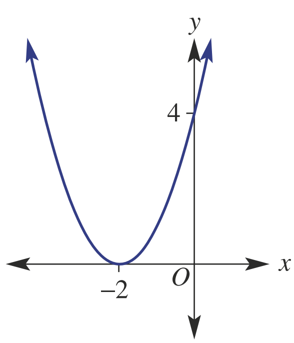
E



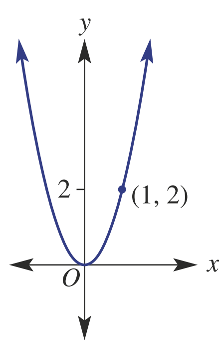
F



G



H



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

REASONING	9–11	9–11	9–11, 12

9 a Using technology, plot the following pairs of graphs on the same set of axes for $-5 \leqslant x \leqslant 5$ and compare their tables of values.

i $y = x^2$ and $y = 4x^2$
ii $y = x^2$ and $y = \frac{1}{3}x^2$
iii $y = x^2$ and $y = 6x^2$
iv $y = x^2$ and $y = \frac{1}{4}x^2$
v $y = x^2$ and $y = 7x^2$
vi $y = x^2$ and $y = \frac{2}{5}x^2$

b Suggest how the constant a in $y = ax^2$ transforms the graph of $y = x^2$.

10 a Using technology, plot the following sets of graphs on the same set of axes for $-5 \leqslant x \leqslant 5$ and compare the turning point of each.

i $y = x^2$, $y = (x + 1)^2$, $y = (x + 2)^2$, $y = (x + 3)^2$
ii $y = x^2$, $y = (x - 1)^2$, $y = (x - 2)^2$, $y = (x - 3)^2$

b Explain how the constant h in $y = (x + h)^2$ transforms the graph of $y = x^2$.

11 a Using technology, plot the following sets of graphs on the same set of axes for $-5 \leqslant x \leqslant 5$ and compare the turning point of each.

i $y = x^2$, $y = x^2 + 1$, $y = x^2 + 2$, $y = x^2 + 3$
ii $y = x^2$, $y = x^2 - 1$, $y = x^2 - 3$, $y = x^2 - 5$

b Explain how the constant k in $y = x^2 + k$ transforms the graph of $y = x^2$.

12 Write down an example of a quadratic equation whose graph has:

a two x-intercepts
b one x-intercept
c no x-intercepts

ENRICHMENT: Finding the rule	–	–	13(½), 14

13 Find a quadratic rule that satisfies the following information.

a turning point (0, 2) and another point (1, 3)
b turning point (0, 2) and another point (1, 1)
c turning point (−1, 0) and y-intercept 1
d turning point (2, 0) and y-intercept 4
e turning point (0, 0) and another point (2, 8)
f turning point (0, 0) and another point (−1, −3)
g turning point (−1, 2) and y-intercept 3
h turning point (4, −2) and y-intercept 0

14 Plot a graph of the parabola $x = y^2$ for $-3 \leqslant y \leqslant 3$ and describe its features.

This parabolic solar power collecting array is such that a fluid is heated by the sun and its heat is converted into electricity.

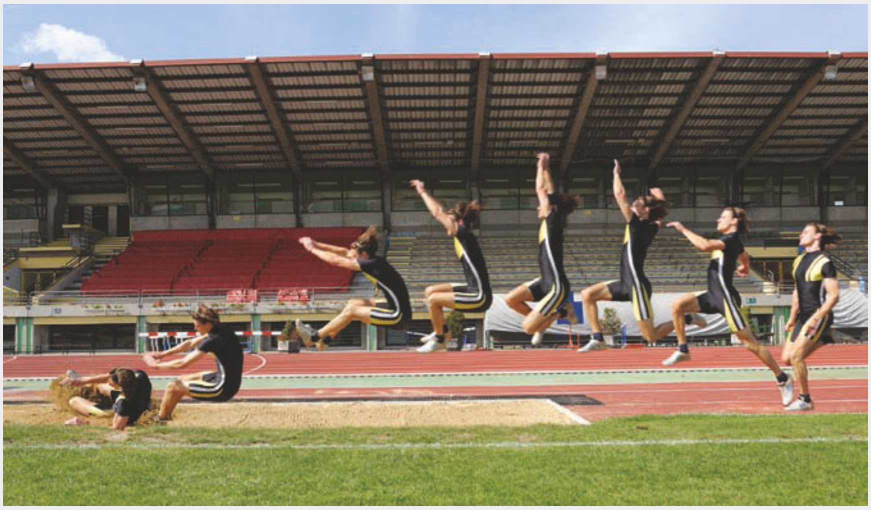
ISBN 978-1-108-77290-7

7B Sketching parabolas using transformations

Learning intentions

- To know the types of transformations: dilation, reflection and translation
- To understand the effect of these transformations on the graph of $y = x^2$
- To know how to determine the turning point of a quadratic rule from turning point form
- To be able to sketch a quadratic from turning point form
- To be able to find the rule of a quadratic graph given the turning point and another point

Previously we have explored simple transformations of the graph of $y = x^2$ and plotted these on a number plane. We will now formalise these transformations and sketch graphs showing key features without the need to plot every point.



Parabolic flight paths occur in athletic jumping and throwing events. Using photography and parabola transformations, sports scientists can find quadratic equations for specific trajectories. Comparing actual and ideal parabolas may reveal areas for technique improvement.

LESSON STARTER So where is the turning point?

Consider the quadratic rule $y = -(x - 3)^2 + 7$.

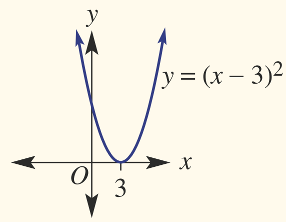
- Discuss the effect of the negative sign in $y = -x^2$ compared with $y = x^2$.
- Discuss the effect of -3 in $y = (x - 3)^2$ compared with $y = x^2$.
- Discuss the effect of $+7$ in $y = x^2 + 7$ compared with $y = x^2$.
- Now for $y = -(x - 3)^2 + 7$, find:
  - the coordinates of the turning point
  - the axis of symmetry
  - the y-intercept.
- What would be the coordinates of the turning point in these quadratics?
  - $y = (x - h)^2 + k$
  - $y = -(x - h)^2 + k$

ISBN 978-1-108-77290-7

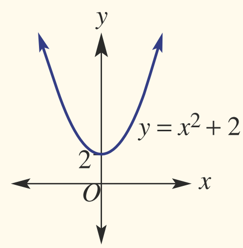
Cambridge University Press
Updated June 2022

KEY IDEAS

- To sketch a parabola, draw a parabolic curve and label key features including:
  - turning point
  - axis of symmetry
  - y-intercept (substitute $x = 0$)
- For $y = ax^2$, a **dilates** the graph of $y = x^2$.
  - Turning point is $(0, 0)$.
  - y-intercept and x-intercept are both 0.
  - Axis of symmetry is $x = 0$.
  - When $a > 0$, the parabola is **upright**.
  - When $a < 0$, the parabola is **inverted**.
- For $y = (x - h)^2$, h **translates** the graph of $y = x^2$ horizontally.
  - When $h > 0$, the graph is translated h units to the right.
  - When $h < 0$, the graph is translated h units to the left.



- For $y = x^2 + k$, k translates the graph of $y = x^2$ vertically.
  - When $k > 0$, the graph is translated k units up.
  - When $k < 0$, the graph is translated k units down.
- The **turning point form** of a quadratic is $y = a(x - h)^2 + k$.
  - The turning point is (h, k).
  - The axis of symmetry is $x = h$.



BUILDING UNDERSTANDING

1 Give the coordinates of the turning point for the graphs of these rules.

a $y = x^2$ **b** $y = x^2 + 3$ **c** $y = -x^2 - 4$

d $y = (x - 2)^2$ **e** $y = (x + 5)^2$ **f** $y = -\frac{1}{3}x^2$

2 Substitute $x = 0$ to find the y-intercept of the graphs with these equations.

a $y = x^2 + 5$ **b** $y = -x^2 - 3$ **c** $y = (x + 2)^2$ **d** $y = (x + 1)^2 + 1$

3 Choose the word: *left*, *right*, *up* or *down* to suit.

a Compared with the graph of $y = x^2$, the graph of $y = x^2 + 3$ is translated _____________.

b Compared with the graph of $y = x^2$, the graph of $y = (x - 3)^2$ is translated _____________.

c Compared with the graph of $y = x^2$, the graph of $y = (x + 1)^2$ is translated _____________.

d Compared with the graph of $y = x^2$, the graph of $y = x^2 - 6$ is translated _____________.

e Compared with the graph of $y = -x^2$, the graph of $y = -x^2 - 2$ is translated _____________.

f Compared with the graph of $y = -x^2$, the graph of $y = -(x + 3)^2$ is translated _____________.

g Compared with the graph of $y = -x^2$, the graph of $y = -(x - 2)^2$ is translated _____________.

h Compared with the graph of $y = -x^2$, the graph of $y = -x^2 + 4$ is translated _____________.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

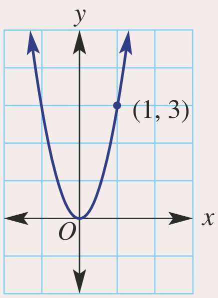
Example 3 Sketching with transformations

Sketch graphs of the following quadratic relations, labelling the turning point and the y-intercept.

a $y = 3x^2$ **b** $y = -x^2 + 4$ **c** $y = (x - 2)^2$

SOLUTION

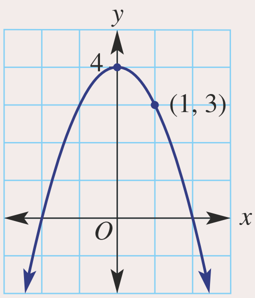
a



EXPLANATION

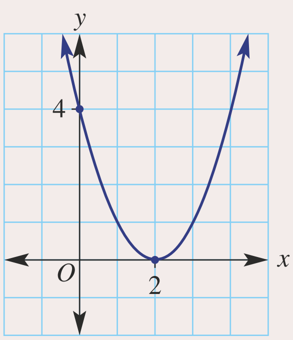
$y = 3x^2$ is upright and narrower than $y = x^2$. The turning point and y-intercept are at the origin $(0, 0)$.
Substitute $x = 1$ to label a second point.

b



$y = -x^2 + 4$ is inverted (i.e. has a maximum) and is translated 4 units up compared with $y = -x^2$. The turning point is at $(0, 4)$ and the y-intercept (i.e. when $x = 0$) is 4.

c



$y = (x - 2)^2$ is upright (i.e. has a minimum) and is translated 2 units right compared with $y = x^2$. Thus, the turning point is at $(2, 0)$.
Substitute $x = 0$ for the y-intercept: $y = (0 - 2)^2$
$= (-2)^2$
$= 4$

Now you try

Sketch graphs of the following quadratic relations, labelling the turning point and the y-intercept.

a $y = 2x^2$ **b** $y = -x^2 + 3$ **c** $y = (x + 1)^2$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

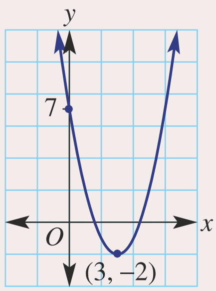
Example 4 Using turning point form

Sketch the graphs of the following, labelling the turning point and the y-intercept.

a $y = (x - 3)^2 - 2$ **b** $y = -(x + 1)^2 + 4$

SOLUTION

a



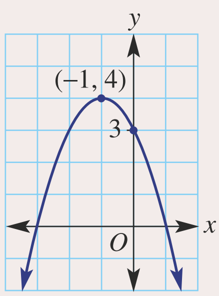
EXPLANATION

In $y = a(x - h)^2 + k$, $h = 3$ and $k = -2$, so the vertex is $(3, -2)$.

Substitute $x = 0$ to find the y-intercept:

$$\begin{aligned} y &= (0 - 3)^2 - 2 \\ &= 9 - 2 \\ &= 7 \end{aligned}$$

b



The graph is inverted since $a = -1$.

$h = -1$ and $k = 4$, so the vertex is $(-1, 4)$.

When $x = 0$: $\begin{aligned} y &= -(0 + 1)^2 + 4 \\ &= -1 + 4 \\ &= 3 \end{aligned}$

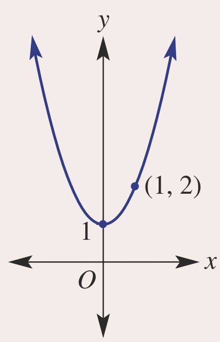
Now you try

Sketch the graphs of the following, labelling the turning point and the y-intercept.

a $y = (x + 1)^2 - 2$ **b** $y = -(x - 2)^2 + 3$

Example 5 Finding a rule from a simple graph

Find a rule for this parabola with turning point $(0, 1)$ and another point $(1, 2)$.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

SOLUTION

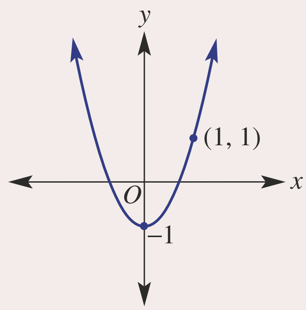
$y = ax^2 + 1$
When $x = 1$, $y = 2$ so:
$2 = a(1)^2 + 1$
$\therefore a = 1$
So $y = x^2 + 1$.

EXPLANATION

In $y = a(x - h)^2 + k$, $h = 0$ and $k = 1$ so the rule is $y = ax^2 + 1$.
We need $y = 2$ when $x = 1$, so $a = 1$.

Now you try

Find a rule for this parabola with turning point $(0, -1)$ and another point $(1, 1)$.



Exercise 7B

FLUENCY	1, 2–4(1/2)	2–4(1/2)	2–4(1/3)

Example 3 **1** Sketch graphs of the following quadratic relations, labelling the turning point and the y-intercept.
- **a** $y = 4x^2$
- **b** $y = -x^2 + 2$
- **c** $y = (x - 4)^2$

Example 3 **2** Sketch graphs of the following quadratics, labelling the turning point and the y-intercept. If the turning point is on the y-axis, also label the point where $x = 1$
- **a** $y = 2x^2$
- **b** $y = -3x^2$
- **c** $y = \frac{1}{2}x^2$
- **d** $y = -\frac{1}{3}x^2$
- **e** $y = x^2 + 2$
- **f** $y = x^2 - 4$
- **g** $y = -x^2 + 1$
- **h** $y = -x^2 - 3$
- **i** $y = (x + 3)^2$
- **j** $y = (x - 1)^2$
- **k** $y = -(x + 2)^2$
- **l** $y = -(x - 3)^2$

3 State the coordinates of the turning point for the graphs of these rules.
- **a** $y = (x + 3)^2 + 1$
- **b** $y = (x + 2)^2 - 4$
- **c** $y = (x - 1)^2 + 3$
- **d** $y = (x - 4)^2 - 2$
- **e** $y = (x - 3)^2 - 5$
- **f** $y = (x - 2)^2 + 2$
- **g** $y = -(x - 3)^2 + 3$
- **h** $y = -(x - 2)^2 + 6$
- **i** $y = -(x + 1)^2 + 4$
- **j** $y = -(x - 2)^2 - 5$
- **k** $y = -(x + 1)^2 - 1$
- **l** $y = -(x - 4)^2 - 10$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 4

4 Sketch graphs of the following quadratics, labelling the turning point and the y-intercept.

a $y = (x + 1)^2 + 1$	**b** $y = (x + 2)^2 - 1$	**c** $y = (x + 3)^2 + 2$
d $y = (x - 1)^2 + 2$	**e** $y = (x - 4)^2 + 1$	**f** $y = (x - 1)^2 - 4$
g $y = -(x - 1)^2 + 3$	**h** $y = -(x - 2)^2 + 1$	**i** $y = -(x + 3)^2 - 2$
j $y = -(x - 2)^2 + 1$	**k** $y = -(x - 4)^2 - 2$	**l** $y = -(x + 2)^2 + 2$

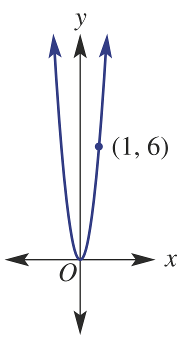
PROBLEM-SOLVING	6(½)	5–6(½)	6(½), 7

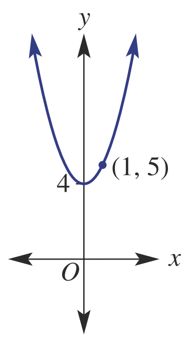
5 Write the rule for each graph when $y = x^2$ is transformed by the following.

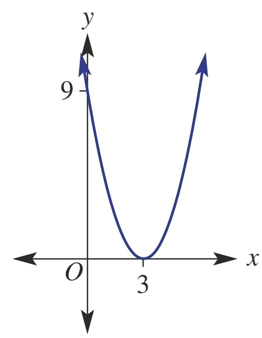
a reflected in the x-axis
b translated 2 units to the left
c translated 5 units down
d translated 4 units up
e translated 1 unit to the right
f reflected in the x-axis and translated 2 units up
g reflected in the x-axis and translated 3 units left
h translated 5 units left and 3 units down
i translated 6 units right and 1 unit up

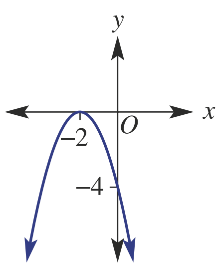
Example 5

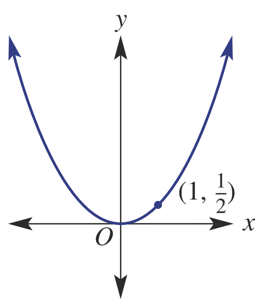
6 Determine the rule for the following parabolas.

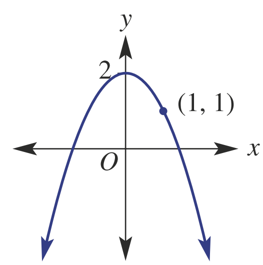
a


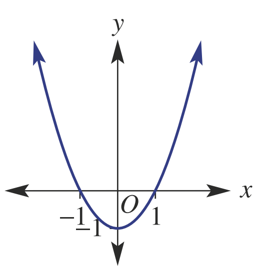
b


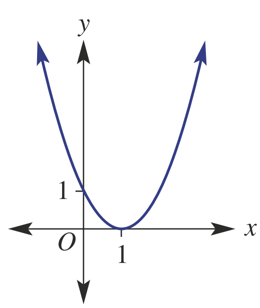
c


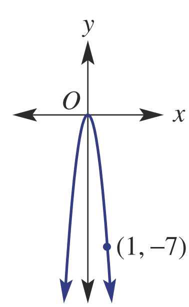
d


e


f


g


h


i


ISBN 978-1-108-77290-7

7 The path of a basketball is given by $y = -(x - 5)^2 + 25$, where y metres is the height and x metres is the horizontal distance.

a Is the turning point a maximum or a minimum?
b What are the coordinates of the turning point?
c What is the y-intercept?
d What is the maximum height of the ball?
e What is the height of the ball at these horizontal distances?
i $x = 3$
ii $x = 7$
iii $x = 10$

REASONING	8(1/2)	8–9(1/2)	9(1/2), 10

8 Recall that $y = (x - h)^2 + k$ and $y = a(x - h)^2 + k$ both have the same turning point coordinates. State the coordinates of the turning point for the graphs of these rules.
a $y = 2(x - 1)^2$
b $y = 3(x + 2)^2$
c $y = -4(x + 3)^2$
d $y = 3x^2 - 4$
e $y = 5x^2 - 2$
f $y = -2x^2 + 5$
g $y = 6(x + 4)^2 - 1$
h $y = 2(x + 2)^2 + 3$
i $y = 3(x - 5)^2 + 4$
j $y = -4(x + 2)^2 + 3$
k $y = -2(x + 3)^2 - 5$
l $y = -5(x - 3)^2 - 3$

9 Describe the transformations that take $y = x^2$ to:
a $y = (x - 3)^2$
b $y = (x + 2)^2$
c $y = x^2 - 3$
d $y = x^2 + 7$
e $y = -x^2$
f $y = (x + 2)^2 - 4$
g $y = (x - 5)^2 + 8$
h $y = -(x + 3)^2$
i $y = -x^2 + 6$

10 For $y = a(x - h)^2 + k$, write:
a the coordinates of the turning point
b the y-intercept.

ENRICHMENT: Sketching with many transformations	–	–	11(1/2)

11 Sketch the graph of the following, showing the turning point and y-intercept.
a $y = 2(x - 3)^2 + 4$
b $y = 3(x + 2)^2 + 5$
c $y = -2(x - 3)^2 + 4$
d $y = -2(x + 3)^2 - 4$
e $y = \frac{1}{2}(x - 3)^2 + 4$
f $y = -\frac{1}{2}(x - 3)^2 + 4$
g $y = 4 - x^2$
h $y = -3 - x^2$
i $y = 5 - 2x^2$
j $y = 2 + \frac{1}{2}(x - 1)^2$
k $y = 1 - 2(x + 2)^2$
l $y = 3 - 4(x - 2)^2$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

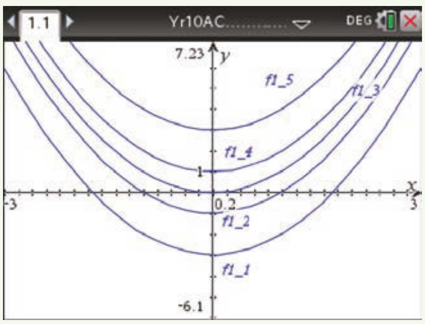
Using calculators to sketch parabolas

1 Sketch the graph of the family $y = x^2 + k$, using $k = \{-3, -1, 0, 1, 3\}$.

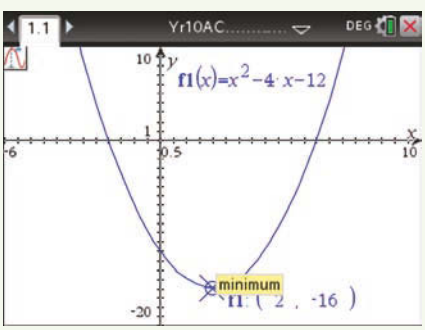
2 Sketch a graph of $y = x^2 - 4x - 12$ and show the x-intercepts and the turning point.

Using the TI-Nspire:

1 In a **Graphs** page, type the rule in $f1(x)$ using the **given** symbol (|) which is accessed using ctrl =.
$f1(x) = x^2 + k \mid k = \{-3, -1, 0, 1, 3\}$

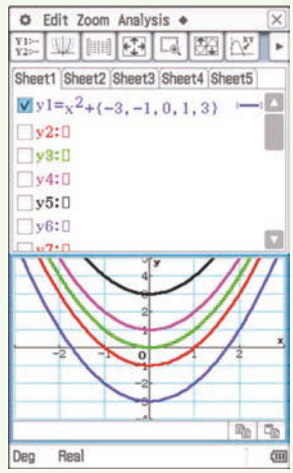


2 Enter the rule $f1(x) = x^2 - 4x - 12$. Change the scale using the window settings. Use menu **>Trace>Graph Trace** and scroll along the graph to show significant points.

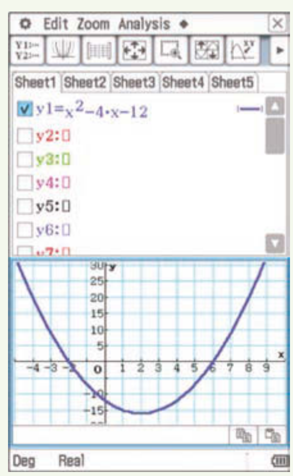


Using the ClassPad:

1 In the **Graph&Table** application enter the rule $y1 = x^2 + \{-3, -1, 0, 1, 3\}$ followed by **EXE**. Tap to see the graph.



2 Enter the rule $y1 = x^2 - 4x - 12$. Tap and set an appropriate scale. Tap **Analysis**, **G-Solve**, **root** to locate the x-intercepts. Tap **Analysis**, **G-Solve**, **Min** to locate the turning point.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

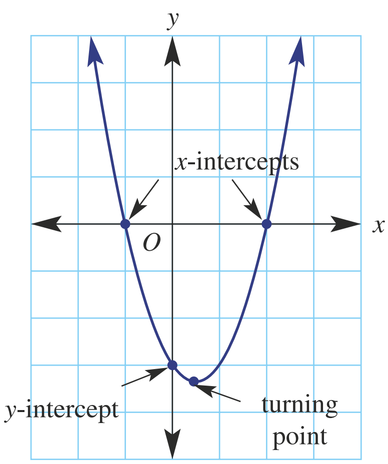
7C Sketching parabolas using factorisation

Learning intentions

- To know the steps for sketching a quadratic graph in the form $y = x^2 + bx + c$
- To understand that quadratic graphs can have 0, 1 or 2 x-intercepts
- To know how to use factorisation to determine the x-intercepts of a quadratic graph
- To know how to use symmetry to locate the turning point once the x-intercepts are known

A quadratic relation written in the form $y = x^2 + bx + c$ differs from that of turning point form, $y = a(x - h)^2 + k$, and so the transformations of the graph of $y = x^2$ to give $y = x^2 + bx + c$ are less obvious. To address this, we have a number of options. We can first try to factorise to find the x-intercepts and then use symmetry to find the turning point or, alternatively, we can complete the square and express the quadratic relation in turning point form. The second of these methods will be studied in the next section.

Engineers can develop equations of parabolic shapes, such as the St Louis archway, by reversing the factorisation procedure. A sketch is labelled with measurements, the x-intercepts form the factors and expanding the brackets gives the basic quadratic equation.



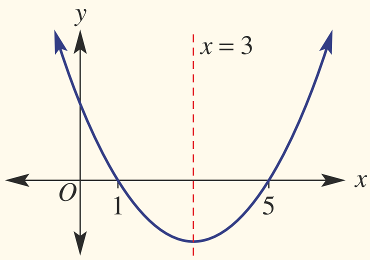
LESSON STARTER Why does the turning point of $y = x^2 - 2x - 3$ have coordinates $(1, -4)$?

- Factorise $y = x^2 - 2x - 3$.
- Hence, find the x-intercepts.
- Discuss how symmetry can be used to locate the turning point.
- Hence, confirm the coordinates of the turning point.

ISBN 978-1-108-77290-7

KEY IDEAS

- To sketch a graph of $y = x^2 + bx + c$:
  - Find the y-intercept by substituting $x = 0$.
  - Find the x-intercept(s) by substituting $y = 0$. Factorise where possible and use the Null Factor Law (if $a \times b = 0$, then $a = 0$ or $b = 0$).
- Once the x-intercepts are known, the turning point can be found using symmetry.
  - The axis of symmetry (also the x-coordinate of the turning point) lies halfway between the x-intercepts: $x = \frac{1+5}{2} = 3$.



  - Substitute this x-coordinate into the rule to find the y-coordinate of the turning point.

BUILDING UNDERSTANDING

1 Use the Null Factor Law to find the x-intercepts ($y = 0$) for these factorised quadratics.
a $y = (x + 1)(x - 2)$ **b** $y = x(x - 3)$ **c** $y = -3x(x + 2)$

2 Factorise these quadratics.
a $y = x^2 - 4x$ **b** $y = x^2 + 2x - 8$ **c** $y = x^2 - 8x + 16$ **d** $y = x^2 - 25$

3 Find the y-intercept for the quadratics in Question 2.

4 Use the given rule and x-intercepts to find the coordinates of the turning point.
a $y = x^2 - 8x + 12$, x-intercepts: 2 and 6
b $y = -x^2 - 2x + 8$, x-intercepts: -4 and 2

Example 6 Using the x-intercepts to find the turning point

Sketch the graph of the following quadratics by using the x-intercepts to help determine the coordinates of the turning point.

a $y = x^2 - 2x$ **b** $y = x^2 - 6x + 5$

SOLUTION	EXPLANATION
a y-intercept at $x = 0$: $y = 0$ x-intercepts at $y = 0$: $0 = x^2 - 2x$ $0 = x(x - 2)$ $x = 0$ or $x - 2 = 0$ $\therefore x = 0, x = 2$ x-intercepts are 0 and 2.	Identify key features of the graph: y-intercept (when $x = 0$), x-intercepts (when $y = 0$), then factorise by noting the common factor and solve by applying the Null Factor Law. Recall that if $a \times b = 0$, then $a = 0$ or $b = 0$.

ISBN 978-1-108-77290-7

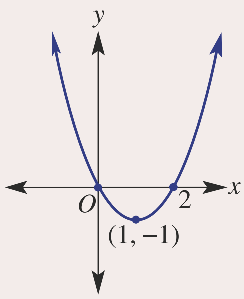
Cambridge University Press
Updated June 2022

Turning point at $x = \frac{0 + 2}{2} = 1$.

$y = 1^2 - 2(1)$
$= -1$

Turning point is a minimum at $(1, -1)$.

Using symmetry the x-coordinate of the turning point is halfway between the x-intercepts.
Substitute $x = 1$ into $y = x^2 - 2x$ to find the y-coordinate of the turning point.
It is a minimum turning point since the coefficient of x^2 is positive.



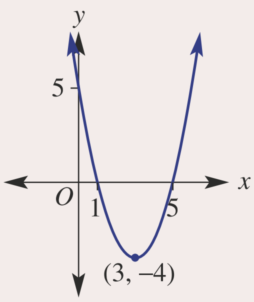
Label key features on the graph and join points in the shape of a parabola.

b y-intercept at $x = 0$: $y = 0$
x-intercepts at $y = 0$: $0 = x^2 - 6x + 5$
$0 = (x - 5)(x - 1)$
$x - 5 = 0$ or $x - 1 = 0$
$\therefore x = 5, x = 1$
x-intercepts are 1 and 5.

Identify key features of the graph: y-intercept and x-intercepts by factorising and applying the Null Factor Law.

Turning point at $x = \frac{1 + 5}{2} = 3$.
$y = (3)^2 - 6 \times (3) + 5$
$= 9 - 18 + 5$
$= -4$
Turning point is a minimum at $(3, -4)$.

Using symmetry the x-coordinate of the turning point is halfway between the x-intercepts.
Substitute $x = 3$ into $y = x^2 - 6x + 5$ to find the y-coordinate of the turning point.
It is a minimum turning point since the coefficient of x^2 is positive.



Label key features on the graph and join points in the shape of a parabola.

Now you try

Sketch the graph of the following quadratics by using the x-intercepts to help determine the coordinates of the turning point.

a $y = x^2 - 6x$

b $y = x^2 - 8x + 7$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 7 Sketching a perfect square

Sketch the graph of the quadratic $y = x^2 + 6x + 9$.

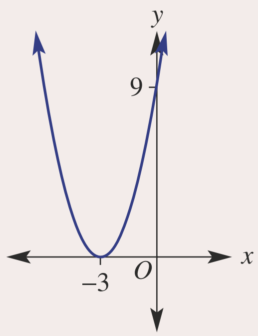
SOLUTION

y-intercept at $x = 0$: $y = 9$

x-intercepts at $y = 0$:
$$\begin{aligned} 0 &= x^2 + 6x + 9 \\ 0 &= (x+3)^2 \\ x + 3 &= 0 \\ \therefore x &= -3 \end{aligned}$$

x-intercept is -3.

Turning point is at $(-3, 0)$.



EXPLANATION

For y-intercept substitute $x = 0$.

For x-intercepts substitute $y = 0$ and factorise: $(x + 3)(x + 3) = (x + 3)^2$. Apply the Null Factor Law to solve for x.

As there is only one x-intercept, it is also the turning point.

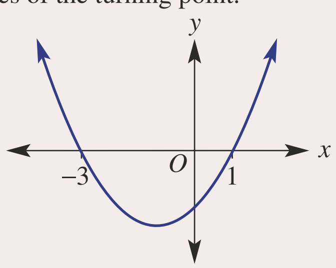
Label key features on the graph.

Now you try

Sketch the graph of $y = x^2 + 8x + 16$.

Example 8 Finding a turning point from a graph

The equation of this graph is of the form $y = (x + a)(x + b)$. Use the x-intercepts to find the values of a and b, then find the coordinates of the turning point.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

SOLUTION

$a = 3$ and $b = -1$.

$y = (x + 3)(x - 1)$

x-coordinate of the turning point is $\dfrac{-3 + 1}{2} = -1$.

y-coordinate is $(-1 + 3)(-1 - 1) = 2 \times (-2) = -4$.

Turning point is $(-1, -4)$.

EXPLANATION

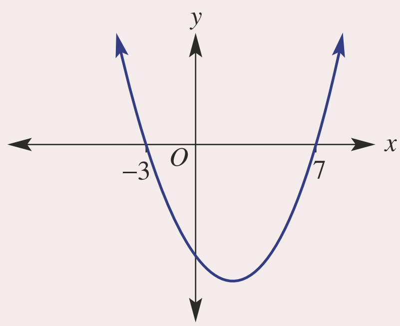
Using the Null Factor Law, $(x + 3)(x - 1) = 0$ gives $x = -3$ and $x = 1$, so $a = 3$ and $b = -1$.

Find the average of the two x-intercepts to find the x-coordinate of the turning point.

Substitute $x = -1$ into the rule to find the y-value of the turning point.

Now you try

The equation of this graph is of the form $y = (x + a)(x + b)$. Use the x-intercepts to find the values of a and b, then find the coordinates of the turning point.



Exercise 7C

FLUENCY	1, 2–5(1/2)	2–6(1/2)	2–6(1/3)

Example 6

1 Sketch the graph of the following quadratics by using the x-intercepts to help determine the coordinates of the turning point.

a $y = x^2 - 8x$ **b** $y = x^2 - 4x + 3$

Example 6a

2 Sketch by first finding x-intercepts.

a $y = x^2 + 2x$ **b** $y = x^2 + 6x$ **c** $y = x^2 - 4x$
d $y = x^2 - 5x$ **e** $y = x^2 + 3x$ **f** $y = x^2 + 7x$

Example 6b

3 Sketch the graphs of the following quadratics.

a $y = x^2 - 6x + 8$ **b** $y = x^2 - 8x + 12$
c $y = x^2 + 8x + 15$ **d** $y = x^2 - 6x - 16$
e $y = x^2 - 2x - 8$ **f** $y = x^2 - 4x - 21$
g $y = x^2 + 8x + 7$ **h** $y = x^2 - 12x + 20$

4 Sketch graphs of the following quadratics.

a $y = x^2 - 9x + 20$ **b** $y = x^2 - 5x + 6$ **c** $y = x^2 - 13x + 12$ **d** $y = x^2 + 11x + 30$
e $y = x^2 + 5x + 4$ **f** $y = x^2 + 13x + 12$ **g** $y = x^2 - 4x - 12$ **h** $y = x^2 - x - 2$
i $y = x^2 - 5x - 14$ **j** $y = x^2 + 3x - 4$ **k** $y = x^2 + 7x - 30$ **l** $y = x^2 + 9x - 22$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 7

5 Sketch graphs of the following perfect squares.

a $y = x^2 + 4x + 4$
b $y = x^2 + 8x + 16$
c $y = x^2 - 10x + 25$
d $y = x^2 + 20x + 100$

6 Sketch graphs of the following quadratics that include a difference of perfect squares.

a $y = x^2 - 9$
b $y = x^2 - 16$
c $y = x^2 - 4$

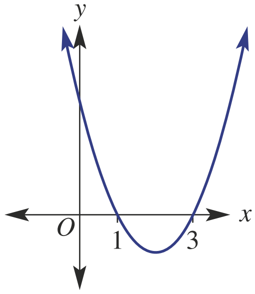
PROBLEM-SOLVING	7(½), 8	7(½), 8, 9	8, 10, 11

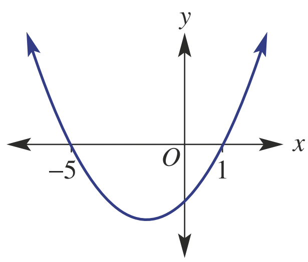
7 Determine the turning points of the following quadratics.

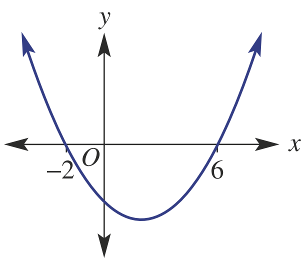
a $y = 2(x^2 - 7x + 10)$
b $y = 3(x^2 - 7x + 10)$
c $y = 3x^2 + 18x + 24$
d $y = 4x^2 + 24x + 32$
e $y = 4(x^2 - 49)$
f $y = -4(x^2 - 49)$
g $y = 3x^2 - 6x + 3$
h $y = 5x^2 - 10x + 5$
i $y = 2x^2 - 4x + 10$

Example 8

8 The equations of these graphs are of the form $y = (x + a)(x + b)$. Use the x-intercepts to find the values of a and b, and then find the coordinates of the turning point.

a


b


c


9 State the x-intercepts and turning point for these quadratics.

a $y = x^2 - 2$
b $y = x^2 - 11$
c $y = x^2 - 50$

10 Sketch a graph of these quadratics.

a $y = 9 - x^2$
b $y = 1 - x^2$
c $y = 4x - x^2$
d $y = 3x - x^2$
e $y = -x^2 + 2x + 8$
f $y = -x^2 + 8x + 9$

11 If the graph of $y = a(x + 2)(x - 4)$ passes through the point $(2, 16)$, determine the value of a and the coordinates of the turning point for this parabola.

REASONING	12	12, 13	13–15

12 Explain why $y = (x - 3)(x - 5)$ and $y = 2(x - 3)(x - 5)$ both have the same x-intercepts.

13 **a** Explain why $y = x^2 - 2x + 1$ has only one x-intercept.
b Explain why $y = x^2 + 2$ has zero x-intercepts.

14 Consider the quadratics $y = x^2 - 2x - 8$ and $y = -x^2 + 2x + 8$.

a Show that both quadratics have the same x-intercepts.
b Find the coordinates of the turning points for both quadratics.
c Compare the positions of the turning points.

15 A quadratic has the rule $y = x^2 + bx$. Give:

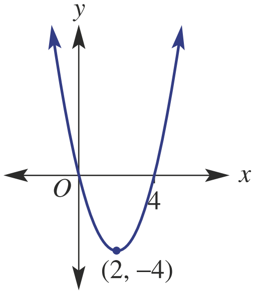
a the y-intercept
b the x-intercepts
c the coordinates of the turning point.

ISBN 978-1-108-77290-7

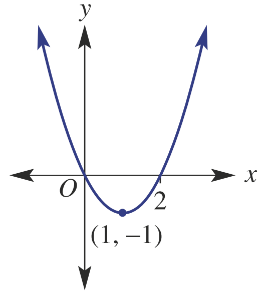
ENRICHMENT: More rules from graphs	–	–	16(1/2)

16 Determine the equation of each of these graphs in factorised form; for example, $y = 2(x - 3)(x + 2)$.

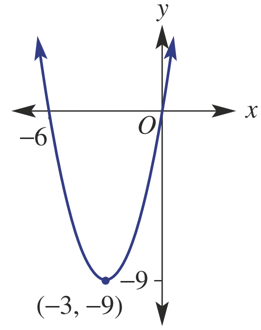
a



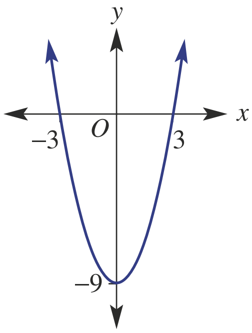
b



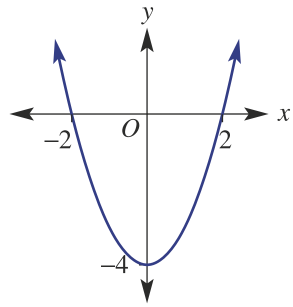
c



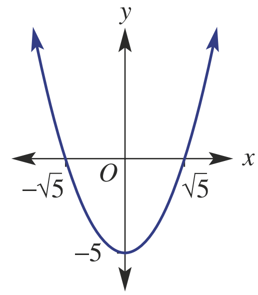
d



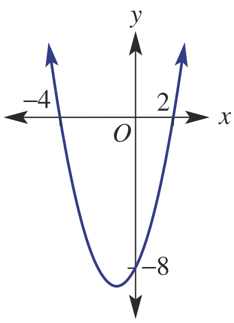
e



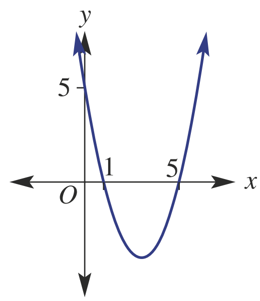
f



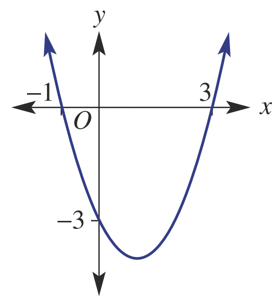
g



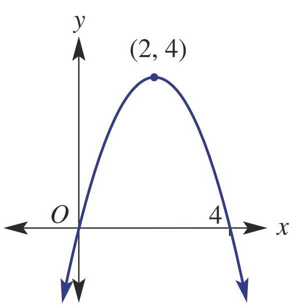
h



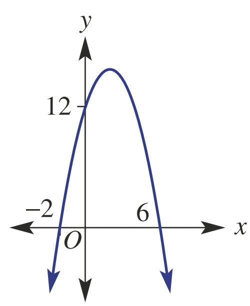
i



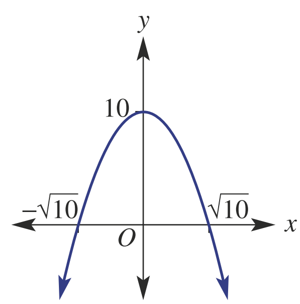
j



k



l



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

7D Sketching parabolas by completing the square

Learning intentions
- To know that completing the square can be used to express any quadratic in turning point form
- To be able to find any x-intercepts from the turning point form of a quadratic
- To be able to sketch a quadratic equation in turning point form, labelling key features

We have learnt previously that the turning point of a parabola can be read directly from a rule in the form $y = a(x - h)^2 + k$. This form of quadratic rule can be obtained by completing the square.

Businesses use mathematical modelling to analyse profits. Quadratic equations model revenue vs selling price and its graph is an inverted parabola. With rising prices, revenue grows until the turning point, then it decreases due to declining sales.

LESSON STARTER I forgot how to complete the square!

To make $x^2 + 6x$ a perfect square we need to add 9 (from $\left(\frac{6}{2}\right)^2$) since $x^2 + 6x + 9 = (x + 3)^2$. So to complete the square for $x^2 + 6x + 2$ we write

$$x^2 + 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 2 = (x + 3)^2 - 7.$$

- Discuss the rules for completing the square and explain how $x^2 + 6x + 2$ becomes $(x + 3)^2 - 7$.
- What does the turning point form of $x^2 + 6x + 2$ tell us about its graph?
- How can you use the turning point form of $x^2 + 6x + 2$ to help find the x-intercepts of $y = x^2 + 6x + 2$?

KEY IDEAS

- By **completing the square**, all quadratics in the form $y = ax^2 + bx + c$ can be expressed in turning point form; i.e. $y = a(x - h)^2 + k$.
- To sketch a quadratic in the form $y = a(x - h)^2 + k$, follow these steps.
  - Determine the coordinates of the turning point (h, k).
    - When a is positive, the parabola has a minimum turning point.
    - When a is negative, the parabola has a maximum turning point.
  - Determine the y-intercept by substituting $x = 0$.
  - Determine the x-intercepts, if any, by substituting $y = 0$ and solving the equation.
- To solve $x^2 = a$, $a > 0$, take the square root of both sides: $x = \pm\sqrt{a}$.
  For any perfect square, say $(x + 1)^2 = 16$, take the square root of both sides:

$$\begin{aligned} x + 1 &= \pm 4 \\ x &= -1 \pm 4 \\ x &= 3 \text{ or } x = -5 \end{aligned}$$

BUILDING UNDERSTANDING

1 By completing the square, state the coordinates of the turning point (TP).

a
$$\begin{aligned} y &= x^2 + 2x - 5 \\ &= x^2 + 2x + \_\_\_\_ - \_\_\_\_ - \_\_\_\_ \\ &= (\_\_\_\_)^2 - \_\_\_\_ \end{aligned}$$
TP = (____, ____)

b
$$\begin{aligned} y &= x^2 - 6x + 10 \\ &= x^2 - 6x + \_\_\_\_ - \_\_\_\_ + \_\_\_\_ \\ &= \_\_\_\_\_\_ \end{aligned}$$
TP = (____, ____)

2 Solve these equations for x, giving exact answers.

a $x^2 = 9$ **b** $x^2 = 3$ **c** $(x - 1)^2 = 16$ **d** $(x + 4)^2 = 2$

Example 9 Finding key features of quadratics in turning point form

For $y = -4(x - 1)^2 + 16$:

a Determine the coordinates of its turning point and state whether it is a maximum or minimum.

b Determine the y-intercept.

c Determine the x-intercepts (if any).

SOLUTION

a Turning point is a maximum at (1, 16).

EXPLANATION

For $y = a(x - h)^2 + k$ the turning point is at (h, k). As $a = -4$ is negative, the parabola has a maximum turning point.

SOLUTION

b y-intercept at $x = 0$:
$$\begin{aligned} y &= -4(0 - 1)^2 + 16 \\ &= -4 + 16 \\ &= 12 \end{aligned}$$
$\therefore$ y-intercept is 12.

EXPLANATION

Substitute $x = 0$ to find the y-intercept. Recall that $(0 - 1)^2 = (-1)^2 = 1$.

SOLUTION

c x-intercepts at $y = 0$:
$$\begin{aligned} 0 &= -4(x - 1)^2 + 16 \\ 0 &= (x - 1)^2 - 4 \\ (x - 1)^2 &= 4 \\ x - 1 &= \pm 2 \\ x &= 1 \pm 2 \\ x &= -1, 3 \end{aligned}$$
x-intercepts are -1 and 3.

EXPLANATION

Substitute $y = 0$ for x-intercepts.
Divide both sides by -4.
Add 4 to both sides, and take the square root of both sides.
Answers are ± 2, since $2^2 = 4$ and $(-2)^2 = 4$.
$1 - 2 = -1$ and $1 + 2 = 3$ are the x-intercepts.
Note: Check that the x-intercepts are evenly spaced either side of the turning point.
Alternatively, use DOPS to write in factorised form:
$(x - 1)^2 - 2^2 = 0$
$(x - 1 - 2)(x - 1 + 2) = 0$
and apply the Null Factor Law to solve for x.

Continued on next page

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Now you try

For $y = -2(x + 1)^2 + 18$:

a Determine the coordinates of its turning point and state whether it is a maximum or minimum.

b Determine the y-intercept.

c Determine the x-intercepts (if any).

Example 10 Sketching by completing the square

Sketch these graphs by completing the square, giving the x-intercepts in exact form.

a $y = x^2 + 6x + 15$ **b** $y = x^2 - 3x - 1$

SOLUTION

a Turning point form:

$$\begin{aligned} y &= x^2 + 6x + 15 \\ &= x^2 + 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 15 \\ &= (x + 3)^2 + 6 \end{aligned}$$

Turning point is a minimum at $(-3, 6)$.

y-intercept at $x = 0$:

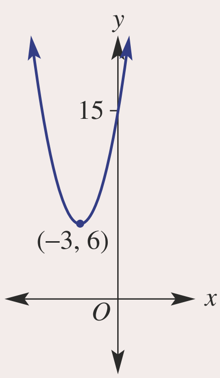
$$\begin{aligned} y &= (0)^2 + 6(0) + 15 \\ &= 15 \end{aligned}$$

$\therefore$ y-intercept is 15.

x-intercepts at $y = 0$:

$0 = (x + 3)^2 + 6$

There is no solution and there are no x-intercepts.



EXPLANATION

To change the equation into turning point form, complete the square by adding and subtracting $\left(\frac{6}{2}\right)^2 = 9$.

Read off the turning point, which is a minimum, as $a = 1$ is positive.

For the y-intercept, substitute $x = 0$ into the original equation.

For the x-intercepts, substitute $y = 0$ into the turning point form.

This cannot be solved as $(x + 3)^2$ cannot equal -6, hence there are no x-intercepts. Note also that the turning point is a minimum with a lowest y-coordinate of 6, telling us there are no x-intercepts.

Sketch the graph, showing the key points.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b Turning point form:

$y = x^2 - 3x - 1$

$= x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 1$

$= \left(x - \frac{3}{2}\right)^2 - \frac{13}{4}$

Complete the square to write in turning point form: $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$ and $-\frac{9}{4} - 1 = -\frac{9}{4} - \frac{4}{4} = -\frac{13}{4}$.

Turning point is a minimum at $\left(\frac{3}{2}, -\frac{13}{4}\right)$.

y-intercept at $x = 0$:

$y = (0)^2 - 3(0) - 1$

$= -1$

$\therefore$ y-intercept is -1.

Substitute $x = 0$ to find the y-intercept.

x-intercepts at $y = 0$:

$0 = \left(x - \frac{3}{2}\right)^2 - \frac{13}{4}$

$\left(x - \frac{3}{2}\right)^2 = \frac{13}{4}$

$x - \frac{3}{2} = \pm\frac{\sqrt{13}}{2}$

$x = \frac{3 + \sqrt{13}}{2}, x = \frac{3 - \sqrt{13}}{2}$

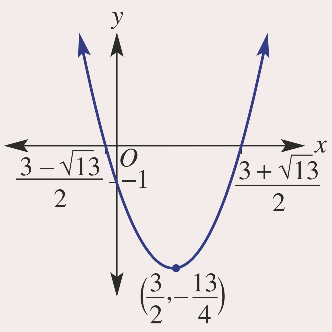
Substitute $y = 0$ to find the x-intercepts.

Add $\frac{13}{4}$ to both sides and take the square root.

$\sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{\sqrt{4}} = \frac{\sqrt{13}}{2}$

$x = \frac{3}{2} \pm \frac{\sqrt{13}}{2}$ can also be expressed as

$x = \frac{3 \pm \sqrt{13}}{2}$.



Label key features on the graph, using exact values.

Now you try

Sketch these graphs by completing the square, giving the x-intercepts in exact form.

a $y = x^2 - 2x + 2$

b $y = x^2 - 5x + 1$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 7D

FLUENCY	1, 2–6(1/2)	2–7(1/2)	2–7(1/3)

Example 9

1 For $y = -3(x-2)^2 + 3$:

- **a** Determine the coordinates of its turning point and state whether it is a maximum or minimum.
- **b** Determine the y-intercept.
- **c** Determine the x-intercepts (if any).

Example 9a

2 State whether the turning points of the following are a maximum or a minimum and give the coordinates.

- **a** $y = 2(x-3)^2 + 5$
- **b** $y = -2(x-1)^2 + 3$
- **c** $y = -4(x+1)^2 - 2$
- **d** $y = 6(x+2)^2 - 5$
- **e** $y = 3(x+5)^2 + 10$
- **f** $y = -4(x-7)^2 + 2$
- **g** $y = -5(x-3)^2 + 8$
- **h** $y = 2(x-3)^2 - 7$

Example 9b

3 Determine the y-intercept of each of the following.

- **a** $y = (x+1)^2 + 5$
- **b** $y = (x+2)^2 - 6$
- **c** $y = (x-3)^2 - 2$
- **d** $y = (x-4)^2 - 7$
- **e** $y = -(x+5)^2 + 9$
- **f** $y = -(x-7)^2 - 6$
- **g** $y = x^2 + 6x + 3$
- **h** $y = x^2 + 5x + 1$
- **i** $y = x^2 + 7x - 5$
- **j** $y = x^2 + x - 8$
- **k** $y = x^2 - 5x + 13$
- **l** $y = x^2 - 12x - 5$

Example 9c

4 Determine the x-intercepts (if any) of the following.

- **a** $y = (x-3)^2 - 4$
- **b** $y = (x+4)^2 - 9$
- **c** $y = (x-3)^2 - 36$
- **d** $y = 2(x+2)^2 - 10$
- **e** $y = -3(x-1)^2 + 30$
- **f** $y = (x-5)^2 - 3$
- **g** $y = (x-4)^2$
- **h** $y = (x+6)^2$
- **i** $y = 2(x-7)^2 + 18$
- **j** $y = -2(x-3)^2 - 4$
- **k** $y = -(x-2)^2 + 5$
- **l** $y = -(x-3)^2 + 10$

5 Determine the x-intercepts (if any) by first completing the square and rewriting the equation in turning point form. Give exact answers.

- **a** $y = x^2 + 6x + 5$
- **b** $y = x^2 + 6x + 2$
- **c** $y = x^2 + 8x - 5$
- **d** $y = x^2 + 2x - 6$
- **e** $y = x^2 - 4x + 14$
- **f** $y = x^2 - 12x - 5$

6 Sketch the graphs of the following. Label the turning point and intercepts.

- **a** $y = (x-2)^2 - 4$
- **b** $y = (x+4)^2 - 9$
- **c** $y = (x+4)^2 - 1$
- **d** $y = (x-3)^2 - 4$
- **e** $y = (x+8)^2 + 16$
- **f** $y = (x+7)^2 + 2$
- **g** $y = (x-2)^2 + 1$
- **h** $y = (x-3)^2 + 6$
- **i** $y = -(x-5)^2 - 4$
- **j** $y = -(x+4)^2 - 9$
- **k** $y = -(x+9)^2 + 25$
- **l** $y = -(x-2)^2 + 4$

Example 10a

7 Sketch these graphs by completing the square. Label the turning point and intercepts.

- **a** $y = x^2 + 4x + 3$
- **b** $y = x^2 - 2x - 3$
- **c** $y = x^2 + 6x + 9$
- **d** $y = x^2 - 8x + 16$
- **e** $y = x^2 - 2x - 8$
- **f** $y = x^2 - 2x - 15$
- **g** $y = x^2 + 8x + 7$
- **h** $y = x^2 + 6x + 5$
- **i** $y = x^2 + 12x$

PROBLEM-SOLVING	8(1/2)	8–9(1/2)	8–10(1/3)

Example 10b

8 Sketch these graphs by completing the square. Label the turning point and intercepts with exact values.

- **a** $y = x^2 + 4x + 1$
- **b** $y = x^2 + 6x - 5$
- **c** $y = x^2 - 2x + 6$
- **d** $y = x^2 - 8x + 20$
- **e** $y = x^2 + 4x - 4$
- **f** $y = x^2 - 3x + 1$
- **g** $y = x^2 + 5x + 2$
- **h** $y = x^2 - x - 2$
- **i** $y = x^2 + 3x + 3$

ISBN 978-1-108-77290-7

9 Complete the square and decide if the graphs of the following quadratics will have zero, one or two x-intercepts.

a $y = x^2 - 4x + 2$ **b** $y = x^2 - 4x + 4$ **c** $y = x^2 + 6x + 9$

d $y = x^2 + 2x + 6$ **e** $y = x^2 - 3x + 4$ **f** $y = x^2 - 5x + 5$

10 Take out a common factor and complete the square to find the x-intercepts for these quadratics.

a $y = 2x^2 + 4x - 10$ **b** $y = 3x^2 - 12x + 9$ **c** $y = 2x^2 - 12x - 14$

d $y = 4x^2 + 16x - 24$ **e** $y = 5x^2 + 20x - 35$ **f** $y = 2x^2 - 6x + 2$

REASONING	11(1/2)	11(1/2), 12	11(1/3), 12, 13

11 To sketch a graph of the form $y = -x^2 + bx + c$ we can complete the square by taking out a factor of -1. Here is an example.

$$\begin{aligned} y &= -x^2 - 2x + 5 \\ &= -(x^2 + 2x - 5) \\ &= -\left(x^2 + 2x + \left(\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2 - 5\right) \\ &= -((x + 1)^2 - 6) \\ &= -(x + 1)^2 + 6 \end{aligned}$$

So the turning point is a maximum at $(-1, 6)$.

Sketch the graph of these quadratics using the technique above.

a $y = -x^2 - 4x + 3$ **b** $y = -x^2 + 2x + 2$ **c** $y = -x^2 + 6x - 4$

d $y = -x^2 + 8x - 8$ **e** $y = -x^2 - 3x - 5$ **f** $y = -x^2 - 5x + 2$

12 For what values of k will the graph of $y = (x - h)^2 + k$ have:

a zero x-intercepts? **b** one x-intercept? **c** two x-intercepts?

13 Show that $x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \frac{b^2 - 4c}{4}$.

ENRICHMENT: Completing the square with non-monics	–	–	14(1/2)

14 This example shows how to complete the square with non-monic quadratics of the form $y = ax^2 + bx + c$.

$$\begin{aligned} y &= 3x^2 + 6x + 1 \\ &= 3\left(x^2 + 2x + \frac{1}{3}\right) \\ &= 3\left(x^2 + 2x + \left(\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2 + \frac{1}{3}\right) \\ &= 3\left((x + 1)^2 - \frac{2}{3}\right) \\ &= 3(x + 1)^2 - 2 \end{aligned}$$

Key features:

The turning point is $(-1, -2)$.

y-intercept is 1.

x-intercepts:

$$\begin{aligned} 0 &= 3(x + 1)^2 - 2 \\ (x + 1)^2 &= \frac{2}{3} \\ x &= \pm\sqrt{\frac{2}{3}} - 1 \end{aligned}$$

Use this technique to sketch the graphs of these non-monic quadratics.

a $y = 4x^2 + 8x + 3$ **b** $y = 3x^2 - 12x + 10$ **c** $y = 2x^2 + 12x + 1$

d $y = 2x^2 + x - 3$ **e** $y = 2x^2 - 7x + 3$ **f** $y = 4x^2 - 8x + 20$

g $y = 6x^2 + 5x + 9$ **h** $y = 5x^2 - 3x + 7$ **i** $y = 5x^2 + 12x$

j $y = 7x^2 + 10x$ **k** $y = -3x^2 - 9x + 2$ **l** $y = -4x^2 + 10x - 1$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

7E Sketching parabolas using the quadratic formula

Learning intentions

- To know the quadratic formula and how it can be used to find the solutions of a quadratic equation
- To be able to use the quadratic formula to determine the x-intercepts of a quadratic graph
- To understand how the discriminant can be used to determine the number of x-intercepts of a quadratic graph
- To be able to use the axis of symmetry rule to locate the turning point of a quadratic graph

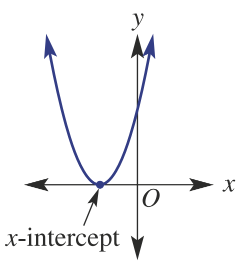
So far we have found x-intercepts for parabolas by factorising (and using the Null Factor Law) and by completing the square. An alternative method is to use the quadratic formula, which states that if $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The discriminant $\Delta = b^2 - 4ac$ determines the number of solutions to the equation.

If $\Delta = 0$, i e. $b^2 - 4ac = 0$.

The solution to the equation becomes $x = -\dfrac{b}{2a}$.

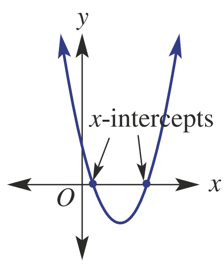
There is one solution and one x-intercept.



If $\Delta > 0$, i.e. $b^2 - 4ac > 0$.

The solution to the equation becomes $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

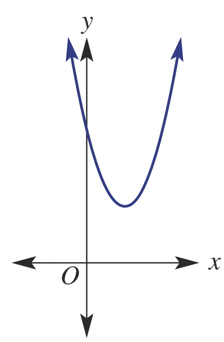
There are two solutions and two x-intercepts.



If $\Delta < 0$, i.e. $b^2 - 4ac < 0$.

Square roots exist for positive numbers only.

There are no solutions or x-intercepts.



LESSON STARTER No working required

Three students set to work to find the x-intercepts for $y = x^2 - 2x + 3$:

Student A finds the intercepts by factorising.

Student B finds the intercepts by completing the square.

Student C uses the discriminant in the quadratic formula.

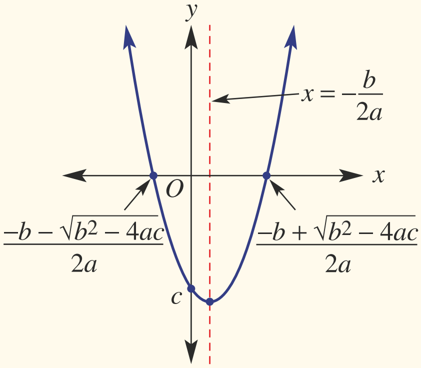
- Try the method for student A. What do you notice?
- Try the method for student B. What do you notice?
- What is the value of the discriminant for student C? What does this tell them about the number of x-intercepts for the quadratic?
- What advice would student C give students A and B?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

KEY IDEAS

- To sketch the graph of $y = ax^2 + bx + c$, find the following points.
  - y-intercept at $x = 0$: $y = a(0)^2 + b(0) + c = c$
  - x-intercepts when $y = 0$:
    For $0 = ax^2 + bx + c$, use the **quadratic formula**:
    $$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \text{ or } \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$
    Alternatively, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
  - Turning point: The x-coordinate lies halfway between the x-intercepts, so $x = -\dfrac{b}{2a}$.
    The y-coordinate is found by substituting the x-coordinate into the original equation.



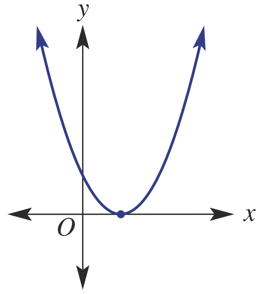
- $x = -\dfrac{b}{2a}$ is the **axis of symmetry**.
- To determine if there are zero, one or two solutions x-intercepts, use the **discriminant** $\Delta = b^2 - 4ac$.
  If $\Delta < 0 \rightarrow$ no x-intercepts.
  If $\Delta = 0 \rightarrow$ one x-intercept.
  If $\Delta > 0 \rightarrow$ two x-intercepts.

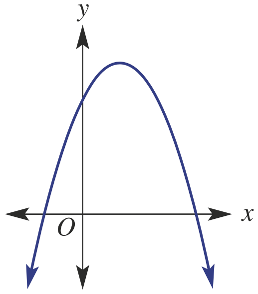
BUILDING UNDERSTANDING

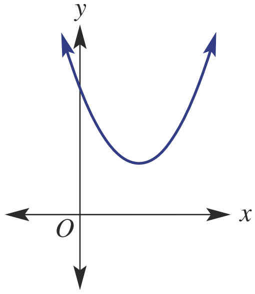
1 A graph has the rule $y = ax^2 + bx + c$. Determine the number of x-intercepts it will have if:
a $b^2 - 4ac > 0$ **b** $b^2 - 4ac < 0$ **c** $b^2 - 4ac = 0$

2 Give the exact value of $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ when:
a $a = 1, b = 2, c = -1$ **b** $a = -2, b = 3, c = 5$ **c** $a = -2, b = -1, c = 2$

3 For the following graphs, state whether the discriminant of these quadratics would be zero, positive or negative.

a


b


c


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 11 Using the discriminant and using $x = -\frac{b}{2a}$ to find the turning point

Consider the parabola given by the quadratic equation $y = 3x^2 - 6x + 5$.

a Determine the number of x-intercepts.

b Determine the y-intercept.

c Use $x = -\frac{b}{2a}$ to determine the turning point.

SOLUTION

a $\Delta = b^2 - 4ac$
$= (-6)^2 - 4(3)(5)$
$= -24$
$\Delta < 0$, so there are no x-intercepts.

b y-intercept is 5.

c $x = -\frac{b}{2a}$
$= -\frac{(-6)}{2(3)} = 1$
$y = 3(1)^2 - 6(1) + 5$
$= 2$
$\therefore$ Turning point is at (1, 2).

EXPLANATION

Use the discriminant $\Delta = b^2 - 4ac$ to find the number of x-intercepts. In $3x^2 - 6x + 5$, $a = 3$, $b = -6$ and $c = 5$.
Interpret the result.

Substitute $x = 0$ for the y-intercept.

For the x-coordinate of the turning point use $x = -\frac{b}{2a}$ with $a = 3$ and $b = -6$, as above.
Substitute the x-coordinate into $y = 3x^2 - 6x + 5$ to find the corresponding y-coordinate of the turning point.

Now you try

Consider the parabola given by the quadratic equation $y = 2x^2 - 4x + 1$.

a Determine the number of x-intercepts.

b Determine the y-intercept.

c Use $x = -\frac{b}{2a}$ to determine the turning point.

Example 12 Sketching graphs using the quadratic formula

Sketch the graph of the quadratic $y = 2x^2 + 4x - 3$, labelling all significant points. Round the x-intercepts to two decimal places.

SOLUTION

y-intercept is -3.

EXPLANATION

Identify key features; i.e. x- and y-intercepts and the turning point. Substitute $x = 0$ for the y-intercept.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

x-intercepts ($y = 0$):

$2x^2 + 4x - 3 = 0$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-4 \pm \sqrt{4^2 - 4(2)(-3)}}{2(2)} \\ &= \frac{-4 \pm \sqrt{40}}{4} \\ &= \frac{\cancel{-4}^{-2} \pm \cancel{2}^{1}\sqrt{10}}{\cancel{4}^{2}} \\ &= \frac{-2 \pm \sqrt{10}}{2} \end{aligned}$$

$x = 0.58, -2.58$ (to 2 d.p.)

$\therefore$ x-intercepts are $0.58, -2.58$.

Use the quadratic formula to find the x-intercepts. For $y = 2x^2 + 4x - 3$, $a = 2$, $b = 4$ and $c = -3$.

Simplify $\sqrt{40} = \sqrt{4 \times 10} = \sqrt{4} \times \sqrt{10} = 2\sqrt{10}$, then cancel the common factor of 2.

Use a calculator to round to two decimal places.

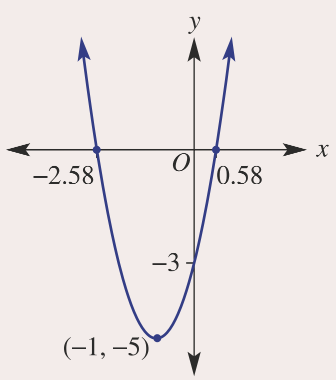
$$\begin{aligned} \text{Turning point is at } x &= -\frac{b}{2a} \\ &= -\frac{(4)}{2(2)} \\ &= -1 \end{aligned}$$

and $\therefore y = 2(-1)^2 + 4(-1) - 3 = -5$.

$\therefore$ Turning point is at $(-1, -5)$.

Substitute $x = -1$ into $y = 2x^2 + 4x - 3$ to find the y-coordinate of the turning point.

Label the key features on the graph and sketch



Now you try

Sketch the graph of the quadratic $y = 3x^2 - 6x + 1$, labelling all significant points. Round the x-intercepts to two decimal places.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 7E

FLUENCY	1, 2–4(1/2)	2–5(1/2)	2–5(1/3)

Example 11

1 Consider the parabola given by the quadratic equation $y = 2x^2 - 8x + 9$.
- **a** Determine the number of x-intercepts.
- **b** Determine the y-intercept.
- **c** Use $x = -\frac{b}{2a}$ to determine the turning point.

Example 11a

2 Use the discriminant to determine the number of x-intercepts for the parabolas given by the following quadratics.
- **a** $y = x^2 + 4x + 4$
- **b** $y = x^2 - 3x + 5$
- **c** $y = -x^2 + 4x + 2$
- **d** $y = 3x^2 - 4x - 2$
- **e** $y = 2x^2 - x + 2$
- **f** $y = 2x^2 - 12x + 18$
- **g** $y = 3x^2 - 2x$
- **h** $y = 3x^2 + 5x$
- **i** $y = -3x^2 - 2x$
- **j** $y = 3x^2 + 5$
- **k** $y = 4x^2 - 2$
- **l** $y = -5x^2 + x$

Example 11b

3 Determine the y-intercept for the parabolas given by the following quadratics.
- **a** $y = x^2 + 2x + 3$
- **b** $y = x^2 - 4x + 5$
- **c** $y = 4x^2 + 3x - 2$
- **d** $y = 5x^2 - 2x - 4$
- **e** $y = -2x^2 - 5x + 8$
- **f** $y = -2x^2 + 7x - 10$
- **g** $y = 3x^2 + 8x$
- **h** $y = -4x^2 - 3x$
- **i** $y = 5x^2 - 7$

Example 11c

4 Use $x = -\frac{b}{2a}$ to determine the coordinates of the turning points for the parabolas defined by the following quadratics.
- **a** $y = x^2 + 2x + 4$
- **b** $y = x^2 + 4x - 1$
- **c** $y = x^2 - 4x + 3$
- **d** $y = -x^2 + 2x - 6$
- **e** $y = -x^2 - 3x + 4$
- **f** $y = -x^2 + 7x - 7$
- **g** $y = 2x^2 + 3x - 4$
- **h** $y = 4x^2 - 3x$
- **i** $y = -4x^2 - 9$
- **j** $y = -4x^2 + 2x - 3$
- **k** $y = -3x^2 - 2x$
- **l** $y = -5x^2 + 2$

Example 12

5 Sketch the graph of these quadratics, labelling all significant points. Round the x-intercepts to two decimal places.
- **a** $y = 2x^2 + 8x - 5$
- **b** $y = 3x^2 + 6x - 2$
- **c** $y = 4x^2 - 2x - 3$
- **d** $y = 2x^2 - 4x - 9$
- **e** $y = 2x^2 - 8x - 11$
- **f** $y = 3x^2 + 9x - 10$
- **g** $y = -3x^2 + 6x + 8$
- **h** $y = -2x^2 - 4x + 7$
- **i** $y = -4x^2 + 8x + 3$
- **j** $y = -2x^2 - x + 12$
- **k** $y = -3x^2 - 2x$
- **l** $y = -5x^2 - 10x - 4$

PROBLEM-SOLVING	7(1/2)	6–7(1/2)	6–7(1/3), 8

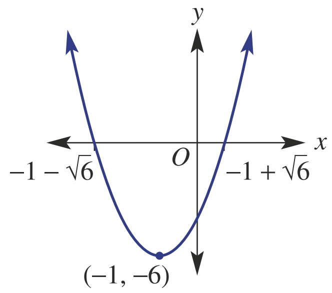
6 Sketch the graphs of these quadratics.
- **a** $y = 4x^2 + 12x + 9$
- **b** $y = 9x^2 - 6x + 1$
- **c** $y = -4x^2 - 20x - 25$
- **d** $y = -9x^2 + 30x - 25$
- **e** $y = -2x^2 + 8x - 11$
- **f** $y = -3x^2 + 12x - 16$
- **g** $y = 4x^2 + 4x + 3$
- **h** $y = 3x^2 + 4x + 2$

7 Give the exact x-intercepts of the graphs of these parabolas. Simplify any surds.
- **a** $y = 3x^2 - 6x - 1$
- **b** $y = -2x^2 - 4x + 3$
- **c** $y = -4x^2 + 8x + 6$
- **d** $y = 2x^2 + 6x - 3$
- **e** $y = 2x^2 - 8x + 5$
- **f** $y = 5x^2 - 10x - 1$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

8 Find a rule in the form $y = ax^2 + bx + c$ that matches this graph.



REASONING	9	9, 10	10, 11

9 Write down two rules in the form $y = ax^2 + bx + c$ that have:
a two x-intercepts **b** one x-intercept **c** no x-intercepts

10 Explain why the quadratic formula gives only one solution when the discriminant $b^2 - 4ac = 0$.

11 Write down the quadratic formula for monic quadratic equations (i.e. where $a = 1$).

ENRICHMENT: Some proof	–	–	12, 13

12 Substitute $x = -\frac{b}{2a}$ into $y = ax^2 + bx + c$ to find the general rule for the y-coordinate of the turning point in terms of a, b and c.

13 Prove the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ by solving $ax^2 + bx + c = 0$.
(*Hint*: Divide both sides by a and complete the square.)

One of the most important applications of quadratic equations is in modelling acceleration, first formulated by Galileo in the early 1600s, and shown in action here in the present day.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

7F Applications of parabolas

Learning intentions

- To be able to set up a quadratic model to solve a word problem
- To know how to apply the processes of quadratics to identify key features of a graph and relate them to real-life contexts
- To be able to identify the possible values of a variable in a given context

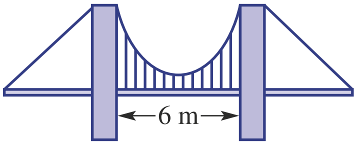
Quadratic equations and their graphs can be used to solve a range of practical problems. These could involve, for example, the path of a projectile or the shape of a bridge's arch. We can relate quantities with quadratic rules and use their graphs to illustrate key features. For example, x-intercepts show where one quantity (y) is equal to zero, and the turning point is where a quantity is a maximum or minimum.

Engineers could design a suspension bridge by placing the road on the x-axis and a pylon on the y-axis. The support cable forms a parabola and its quadratic equation is used to find the heights of the evenly spaced, vertical supports.

LESSON STARTER The civil engineer

Michael, a civil engineer, designs a model for the curved cable of a 6 m suspension bridge using the equation $h = (d - 3)^2 + 2$, where h metres is the height of the hanging cables above the road for a distance d metres from the left pillar.

- What are the possible values for d?
- Sketch the graph of $h = (d - 3)^2 + 2$ for appropriate values of d.
- What is the height of the pillars above the road?
- What is the minimum height of the cable above the road?
- Discuss how key features of the graph have helped to answer the questions above.



KEY IDEAS

- Applying quadratics to solve problems may involve:
  - defining variables
  - forming equations
  - solving equations
  - deciding on a suitable range of values for the variables
  - sketching graphs showing key features
  - finding the maximum or minimum turning point.

ISBN 978-1-108-77290-7

BUILDING UNDERSTANDING

1 A ball is thrown upwards from ground level and reaches a height of h metres after t seconds, given by the formula $h = 20t - 5t^2$.

- **a** Sketch a graph of the rule for $0 \leqslant t \leqslant 4$ by finding the t-intercepts (x-intercepts) and the coordinates of the turning point.
- **b** What maximum height does the ball reach (refer to the turning point)?
- **c** How long does it take the ball to return to ground level ($h = 0$)?

2 The path of a javelin thrown by Jo is given by the formula $h = -\frac{1}{16}(d - 10)^2 + 9$, where h metres is the height of the javelin above the ground and d metres is the horizontal distance travelled.

- **a** Sketch the graph of the rule for $0 \leqslant d \leqslant 22$ by finding the intercepts and the coordinates of the turning point.
- **b** What is the maximum height the javelin reaches (refer to the turning point)?
- **c** What horizontal distance does the javelin travel (i.e. when is $h = 0$)?

Example 13 Applying quadratics in problems

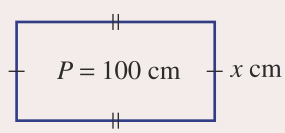
A piece of wire measuring 100 cm in length is bent into the shape of a rectangle. Let x cm be the width of the rectangle.

- **a** Use the perimeter to write an expression for the length of the rectangle in terms of x.
- **b** Write an equation for the area of the rectangle (A cm^2) in terms of x.
- **c** Decide on the suitable values of x.
- **d** Sketch the graph of A versus x for suitable values of x.
- **e** Use the graph to determine the maximum area that can be formed.
- **f** What will be the dimensions of the rectangle to achieve its maximum area?

SOLUTION

a
$$\begin{aligned} 2 \times \text{length} + 2x &= 100 \\ 2 \times \text{length} &= 100 - 2x \\ \therefore\ \text{Length} &= 50 - x \end{aligned}$$

EXPLANATION



100 cm of wire will form the perimeter.
Length is half of $(100 - 2 \times \text{width})$.

b $A = x(50 - x)$

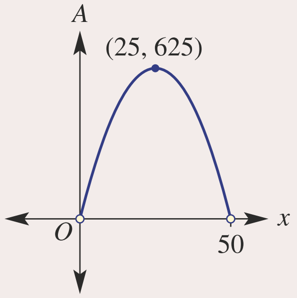
Area of a rectangle = length × width.

c Length and width must be positive, so we require:
$x > 0$ and $50 - x > 0$
i.e. $x > 0$ and $50 > x$
i.e. $0 < x < 50$

Require each dimension to be positive, solve for x.

Continued on next page

d



Sketch the graph, labelling the intercepts and turning point, which has x-coordinate halfway between the x-intercepts; i.e. $x = 25$. Substitute $x = 25$ into the area formula to find the maximum area: $A = 25(50 - 25) = 625$. Note open circles at $x = 0$ and $x = 50$ as these points are not included in the range of x-values.

e The maximum area that can be formed is 625 cm^2.

Read from the graph. The maximum area is the y-coordinate of the turning point.

f Maximum occurs when width $x = 25$ cm, so

$$\text{Length} = 50 - 25 = 25 \text{ cm}$$

Dimensions that give maximum area are 25 cm by 25 cm, which is, in fact, a square.

From turning point, $x = 25$ gives the maximum area. Substitute to find the corresponding length. Length $= 50 - x$.

Now you try

A piece of wire measuring 80 cm in length is bent into the shape of a rectangle. Let x cm be the width of the rectangle.

- **a** Use the perimeter to write an expression for the length of the rectangle in terms of x.
- **b** Write an equation for the area of the rectangle (A cm^2) in terms of x.
- **c** Decide on the suitable values of x.
- **d** Sketch the graph of A versus x for suitable values of x.
- **e** Use the graph to determine the maximum area that can be formed.
- **f** What will be the dimensions of the rectangle to achieve its maximum area?

Exercise 7F

FLUENCY	1–3	1–4	1, 3–5

Example 13

1 A 20 cm piece of wire is bent to form a rectangle. Let x cm be the width of the rectangle.

- **a** Use the perimeter to write an expression for the length of the rectangle in terms of x.
- **b** Write an equation for the area of the rectangle (A cm^2) in terms of x.
- **c** Decide on suitable values of x.
- **d** Sketch the graph of A versus x for suitable values of x.
- **e** Use the graph to determine the maximum area that can be formed.
- **f** What will be the dimensions of the rectangle to achieve its maximum area?

2 A wood turner carves out a bowl according to the formula $d = \frac{1}{3}x^2 - 27$, where d cm is the depth of the bowl and x cm is the distance from the centre of the bowl.

- **a** Sketch a graph for $-9 \leqslant x \leqslant 9$, showing x-intercepts and the turning point.
- **b** What is the width of the bowl?
- **c** What is the maximum depth of the bowl?

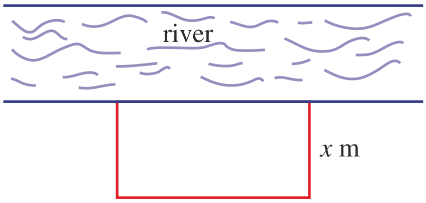
ISBN 978-1-108-77290-7

3 The equation for the arch of a particular bridge is given by $h = -\frac{1}{500}(x - 100)^2 + 20$, where h m is the height above the base of the bridge and x m is the distance from the left side.

a Determine the coordinates of the turning point of the graph.
b Determine the x-intercepts of the graph.
c Sketch the graph of the arch for appropriate values of x.
d What is the span of the arch?
e What is the maximum height of the arch?

Richmond Bridge, Tasmania, is the oldest bridge in Australia that is still in use.

4 A farmer has 100 m of fencing to form a rectangular paddock with a river on one side (that does not require fencing), as shown.



a Use the perimeter to write an expression for the length of the paddock in terms of the width, x metres.
b Write an equation for the area of the paddock (A m^2) in terms of x.
c Decide on suitable values of x.
d Sketch the graph of A versus x for suitable values of x.
e Use the graph to determine the maximum paddock area that can be formed.
f What will be the dimensions of the paddock to achieve its maximum area?

5 The sum of two positive numbers is 20 and x is the smaller number.

a Write the second number in terms of x.
b Write a rule for the product, P, of the two numbers in terms of x.
c Sketch a graph of P vs x.
d Find the values of x when:
  i $P = 0$ **ii** P is a maximum.
e What is the maximum value of P?

PROBLEM-SOLVING	6	6, 7	7, 8

6 The equation for a support span is given by $h = -\frac{1}{40}(x - 20)^2$, where h m is the distance below the base of a bridge and x m is the distance from the left side.

a Determine the coordinates of the turning point of the graph.
b Sketch a graph of the equation using $0 \leqslant x \leqslant 40$.
c What is the width of the support span?
d What is the maximum height of the support span?

7 Jordie throws a rock from the top of a 30 metre high cliff and its height (h metres) above the sea is given by $h = 30 - 5t^2$, where t is in seconds.

a Find the exact time it takes for the rock to hit the water.
b Sketch a graph of h vs t for appropriate values of t.
c What is the exact time it takes for the rock to fall to a height of 20 metres?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

8 A bird dives into the water to catch a fish. It follows a path given by $h = t^2 - 8t + 7$, where h is the height in metres above sea level and t is the time in seconds.

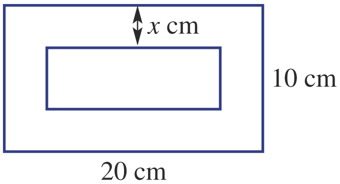
- **a** Sketch a graph of h vs t, showing intercepts and the turning point.
- **b** Find the time when the bird:
  - **i** enters the water
  - **ii** exits the water
  - **iii** reaches a maximum depth.
- **c** What is the maximum depth to which the bird dives?
- **d** At what times is the bird at a depth of 8 metres?

REASONING	9	9–11	11, 12

9 The height, h metres, of a flying kite is given by the rule $h = t^2 - 6t + 10$ for t seconds.
- **a** Find the minimum height of the kite during this time.
- **b** Does the kite ever hit the ground during this time? Give reasons.

10 The sum of two numbers is 64. Show that their product has a maximum of 1024.

11 A rectangular framed picture has a total length and width of 20 cm and 10 cm, respectively. The frame has width x cm.



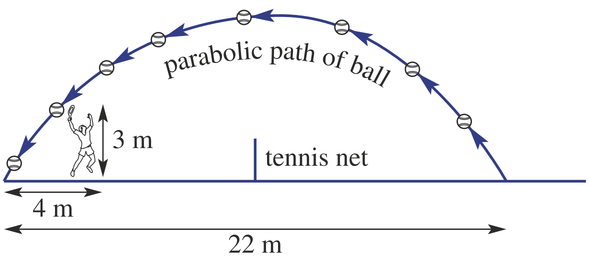
- **a** Find the rule for the area (A cm^2) of the picture inside.
- **b** What are the minimum and maximum values of x?
- **c** Sketch a graph of A vs x using suitable values of x.
- **d** Explain why there is no turning point for your graph, using suitable values of x.
- **e** Find the width of the frame if the area of the picture is 144 cm^2.

12 A dolphin jumping out of the water follows a path described by $h = -\frac{1}{2}(x^2 - 10x + 16)$, where h is the vertical height, in metres, and x metres is the horizontal distance travelled.
- **a** How far horizontally does the dolphin travel out of the water?
- **b** Does the dolphin ever reach a height of 5 metres above water level? Give reasons.

ENRICHMENT: The highway and the river and the lobbed ball	–	–	13, 14

13 The path of a river is given by the rule $y = \frac{1}{10}x(x - 100)$ and all units are given in metres. A highway is to be built near or over the river on the line $y = c$.
- **a** Sketch a graph of the path of the river, showing key features.
- **b** For the highway with equation $y = c$, decide how many bridges will need to be built if:
  - **i** $c = 0$
  - **ii** $c = -300$
- **c** Locate the coordinates of the bridge, correct to one decimal place, if:
  - **i** $c = -200$
  - **ii** $c = -10$
- **d** Describe the situation when $c = -250$.

14 A tennis ball is lobbed from ground level and must cover a horizontal distance of 22 m if it is to land just inside the opposite end of the court. If the opponent is standing 4 m from the baseline and he can hit any ball less than 3 m high, what is the lowest maximum height the lob must reach to win the point?



ISBN 978-1-108-77290-7

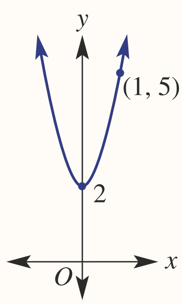
7B

1 Sketch graphs of the following quadratic relations, labelling the turning point and the y-intercept. (The x-intercepts are *not* required.)

a $y = 2x^2$ **b** $y = -x^2 + 3$ **c** $y = (x - 3)^2$ **d** $y = -(x + 2)^2 - 1$

7B

2 Find a rule for this parabola with turning point $(0, 2)$ and another point $(1, 5)$.



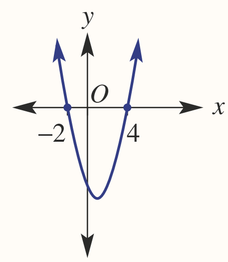
7C

3 Sketch graphs of the following quadratics, label the x- and y-intercepts and determine the coordinates of the turning point, using symmetry.

a $y = x^2 - 2x - 3$ **b** $y = x^2 - 4x + 4$

7C

4 The equation of this graph is of the form $y = (x + a)(x + b)$. Use the x-intercepts to find the values of a and b, then find the coordinates of the turning point.



7D

5 For $y = -2(x - 3)^2 + 8$ determine the:

a coordinates of the turning point and state whether it is a maximum or minimum

b y-intercept **c** x-intercepts (if any).

7D

6 Sketch these graphs by completing the square to write the equation in turning point form. Label the exact x- and y-intercepts and turning point on the graph.

a $y = x^2 - 4x + 3$ **b** $y = x^2 - 2x - 6$

7E

7 For the parabolas given by the following quadratic equations:

i Use the discriminant to determine the number of x-intercepts.

ii Determine the y-intercept.

iii Use $x = -\frac{b}{2a}$ to determine the turning point.

a $y = x^2 - 4x + 5$ **b** $y = x^2 + 6x - 7$ **c** $y = -x^2 - 8x - 16$

7E

8 Sketch the graph of the quadratic $y = 2x^2 - 8x + 5$, labelling all significant points. Give the x-intercepts, rounded to two decimal places.

7F

9 A farmer has 44 metres of fencing to build three sides of a rectangular animal pen with the wall of the farm shed forming the other side. The width of the pen is x metres.

a Write an equation for the area of the pen (A m^2) in terms of x.

b Sketch the graph of A versus x for suitable values of x.

c Determine the maximum area that can be formed and state the dimensions.

Progress quiz

ISBN 978-1-108-77290-7

7G Intersection of lines and parabolas 10A

Learning intentions

- To understand how a line can intersect a parabola at 0, 1 or 2 points
- To be able to find the points of intersection of a line and a parabola using substitution
- To know that the discriminant can be used to determine the number of points of intersection of a line and a parabola

We have seen previously when simultaneously solving a pair of linear equations that there is one solution provided that the graphs of these linear equations are not parallel. Graphically, this represents the point of intersection for the two straight lines.

For the intersection of a parabola and a line we can have either zero, one or two points of intersection. As we have done for linear simultaneous equations, we can use the method of substitution to solve a linear equation and a non-linear equation simultaneously.

This image shows why the Infinity Bridge, England, is named after the symbol ∞. Architects develop and solve equations in three dimensions to determine the intersection points of lines (support cables) and parabolic curves (the arches).

LESSON STARTER How many times does a line cut a parabola?

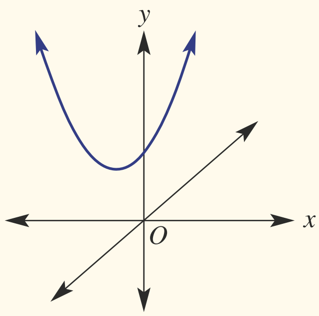
- Use computer graphing software to plot a graph of $y = x^2$.
- By plotting lines of the form $x = h$, determine how many points of intersection a vertical line will have with the parabola.
- By plotting lines of the form $y = c$, determine how many points of intersection a horizontal line could have with the parabola.
- By plotting straight lines of the form $y = 2x + k$ for various values of k, determine the number of possible intersections between a line and a parabola.
- State some values of k for which the line above intersects the parabola:
  - twice
  - never
- Can you find the value of k for which the line intersects the parabola exactly once?

KEY IDEAS

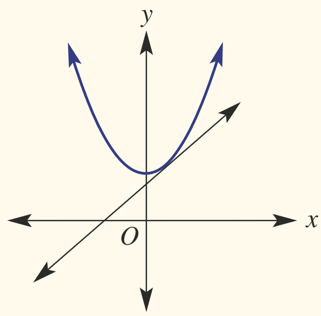
- When solving a pair of simultaneous equations involving a parabola and a line, we may obtain zero, one or two solutions. Graphically, this represents zero, one or two points of intersection between the parabola and the line.
  - A line that intersects a curve twice is called a **secant**.
  - A line that intersects a curve in exactly one place is called a **tangent**.

ISBN 978-1-108-77290-7

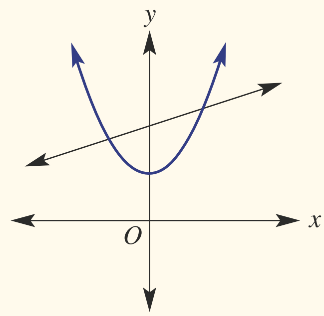
Cambridge University Press
Updated June 2022



zero points of intersection

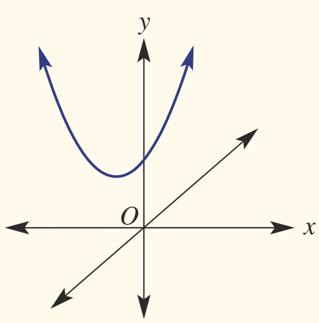


one point of intersection

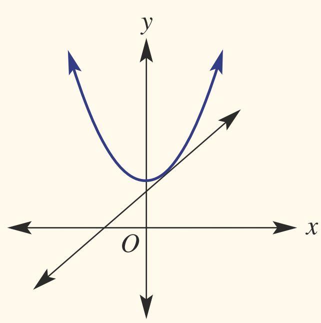


two points of intersection

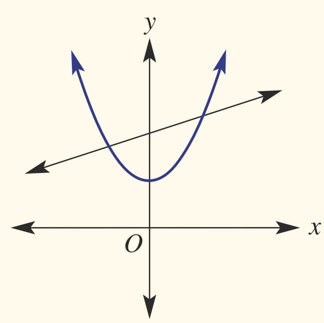
- The method of substitution is used to solve the equations simultaneously.
  - Substitute one equation into the other.
  - Rearrange the resulting equation into the form $ax^2 + bx + c = 0$.
  - Solve for x by factorising and applying the Null Factor Law or use the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
  - Substitute the x-values into one of the original equations to find the corresponding y-value.
- After substituting the equations and rearranging, we arrive at an equation of the form $ax^2 + bx + c = 0$. Hence, the discriminant, $b^2 - 4ac$, can be used to determine the number of solutions (i.e points of intersection) of the two equations.



zero solutions
$b^2 - 4ac < 0$



one solution
$b^2 - 4ac = 0$



two solutions
$b^2 - 4ac > 0$

BUILDING UNDERSTANDING

1 **a** Find the coordinates of the point where the vertical line $x = 2$ intersects the parabola $y = 2x^2 + 5x - 6$.

b Find the coordinates of the point where the vertical line $x = -1$ intersects the parabola $y = x^2 + 3x - 1$.

2 Rearrange the following into the form $ax^2 + bx + c = 0$, where $a > 0$.

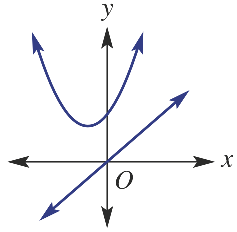
a $x^2 + 5x = 2x - 6$ **b** $x^2 - 3x + 4 = 2x + 1$ **c** $x^2 + x - 7 = -2x + 5$

ISBN 978-1-108-77290-7

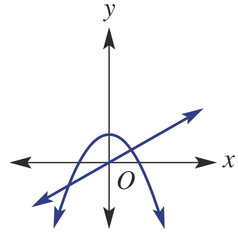
Cambridge University Press
Updated June 2022

3 What do we know about the discriminant, $b^2 - 4ac$, of the resulting equation from solving the equations which correspond to these graphs simultaneously?

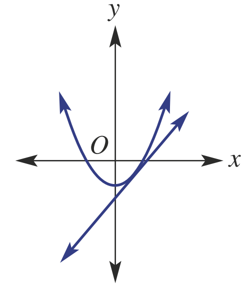
a



b



c



Example 14 Finding points of intersection of a parabola and a horizontal line

Find any points of intersection of these parabolas and lines.

a $y = x^2 - 3x$
$y = 4$

b $y = x^2 + 2x + 4$
$y = -2$

SOLUTION

a By substitution

$$x^2 - 3x = 4$$
$$x^2 - 3x - 4 = 0$$
$$(x - 4)(x + 1) = 0$$
$$x - 4 = 0 \text{ or } x + 1 = 0$$
$$x = 4 \text{ or } x = -1$$

$\therefore$ The points of intersection are at $(4, 4)$ and $(-1, 4)$.

EXPLANATION

Substitute $y = 4$ from the second equation into the first equation.

Write in the form $ax^2 + bx + c = 0$ by subtracting 4 from both sides.

Factorise and apply the Null Factor Law to solve for x.

As the points are on the line $y = 4$, the y-coordinate of the points of intersection is 4.

SOLUTION

b By substitution:

$$x^2 + 2x + 4 = -2$$
$$x^2 + 2x + 6 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(6)}}{2(1)}$$
$$= \frac{-2 \pm \sqrt{-20}}{2}$$

$\therefore$ There are no points of intersection.

EXPLANATION

Substitute $y = -2$ into the first equation.

Apply the quadratic formula to solve $x^2 + 2x + 6 = 0$, where $a = 1$, $b = 2$ and $c = 6$.

$\sqrt{-20}$ has no real solutions.

The parabola $y = x^2 + 2x + 4$ and the line $y = -2$ do not intersect.

Now you try

Find any points of intersection of these parabolas and lines.

a $y = x^2 - x$
$y = 2$

b $y = x^2 + 4x + 1$
$y = -4$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 15 Solving simultaneous equations involving a line and a parabola

Solve the following equations simultaneously.

a $y = x^2$
$y = 2x$

b $y = -4x^2 - x + 6$
$y = 3x + 7$

c $y = x^2 + 1$
$2x - 3y = -4$

SOLUTION	EXPLANATION
a By substitution:	Substitute $y = 2x$ into $y = x^2$.
$x^2 = 2x$ $x^2 - 2x = 0$	Rearrange the equation so that it is equal to 0.
$x(x-2) = 0$	Factorise by removing the common factor x.
$x = 0$ or $x - 2 = 0$ $x = 0$ or $x = 2$	Apply the Null Factor Law to solve for x.
When $x = 0$, $y = 2 \times (0) = 0$. When $x = 2$, $y = 2 \times (2) = 4$.	Substitute the x-values into $y = 2x$ to obtain the corresponding y-value. Alternatively, the equation $y = x^2$ can be used to find the y-values or it can be used to check the y-values.
$\therefore$ The solutions are $x = 0, y = 0$ and $x = 2, y = 4$.	The points $(0, 0)$ and $(2, 4)$ lie on both the line $y = 2x$ and the parabola $y = x^2$.
b By substitution: $-4x^2 - x + 6 = 3x + 7$ $-x + 6 = 4x^2 + 3x + 7$ $6 = 4x^2 + 4x + 7$ $0 = 4x^2 + 4x + 1$ $\therefore (2x+1)(2x+1) = 0$ $2x + 1 = 0$ $x = -\frac{1}{2}$	Substitute $y = 3x + 7$ into the first equation. When rearranging the equation equal to 0, gather the terms on the side that makes the coefficient of x^2 positive, as this will make the factorising easier. Hence, add $4x^2$ to both sides, then add x to both sides and subtract 6 from both sides. Factorise and solve for x.
When $x = -\frac{1}{2}$, $y = 3 \times \left(-\frac{1}{2}\right) + 7$ $= \frac{11}{2}$ or $5\frac{1}{2}$	Substitute the x-value into $y = 3x + 7$ (or $y = -4x^2 - x + 6$ but $y = 3x + 7$ is a simpler equation).
$\therefore$ The only solution is $x = -\frac{1}{2}$, $y = \frac{11}{2}$.	Finding only one solution indicates that this line is a tangent to the parabola.

Continued on next page

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

c By substitution:

$$\begin{aligned} 2x - 3(x^2 + 1) &= -4 \\ 2x - 3x^2 - 3 &= -4 \\ 2x - 3 &= 3x^2 - 4 \\ 2x &= 3x^2 - 1 \end{aligned}$$

Replace y in $2x - 3y = -4$ with $x^2 + 1$, making sure you include brackets.

Expand the brackets and then rearrange into the form $ax^2 + bx + c = 0$.

$$\therefore 3x^2 - 2x - 1 = 0$$
$$(3x + 1)(x - 1) = 0$$

Factorise and solve for x.

$$3x + 1 = 0 \text{ or } x - 1 = 0$$
$$x = -\frac{1}{3} \text{ or } x = 1$$

When $x = -\frac{1}{3}$, $y = \left(-\frac{1}{3}\right)^2 + 1$
$= \frac{1}{9} + 1$
$= \frac{10}{9}$

Substitute the x-values into one of the two original equations to solve for y.

When $x = 1$, $y = (1)^2 + 1$
$= 2$

$\therefore$ The solutions are $x = -\frac{1}{3}$, $y = \frac{10}{9}$ and $x = 1$, $y = 2$.

The line and parabola intersect in two places.

Now you try

Solve the following equations simultaneously.

a $y = x^2$
$y = 4x$

b $y = -x^2 + 6x + 7$
$y = 4x + 8$

c $y = x^2 - 1$
$3x + 2y = 0$

Example 16 Solving simultaneous equations with the quadratic formula

Solve the equations $y = x^2 + 5x - 5$ and $y = 2x$ simultaneously. Round your values to two decimal places.

SOLUTION

By substitution:
$x^2 + 5x - 5 = 2x$
$x^2 + 3x - 5 = 0$

EXPLANATION

Rearrange into standard form.
$x^2 + 3x - 5$ does not factorise with whole numbers.

ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

Using the quadratic formula:

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-5)}}{2(1)}$$
$$= \frac{-3 \pm \sqrt{9 + 20}}{2}$$
$$= \frac{-3 \pm \sqrt{29}}{2}$$
$$= 1.19258\ldots \text{ or } -4.19258\ldots$$

Quadratic formula: If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Here, $a = 1$, $b = 3$ and $c = -5$.

Use a calculator to evaluate $\dfrac{-3 - \sqrt{29}}{2}$ and $\dfrac{-3 + \sqrt{29}}{2}$. Recall that if the number under the square root is negative, then there will be no real solutions.

In exact form, $y = 2x$

$$= 2 \times \left(\frac{-3 \pm \sqrt{29}}{2}\right)$$
$$= -3 \pm \sqrt{29}$$

Substitute exact x-values into $y = 2x$.

$\therefore$ The solutions are $x = 1.19$, $y = 2.39$ and $x = -4.19$, $y = -8.39$ (to 2 d.p.).

Round your values to two decimal places, as required.

Now you try

Solve the equations $y = x^2 + 2x - 1$ and $y = 3x$ simultaneously. Round your values to two decimal places.

Example 17 Determining the number of solutions of simultaneous equations

Determine the number of solutions (points of intersection) of the following pairs of equations.

a $y = x^2 + 3x - 1$
$y = x - 2$

b $y = 2x^2 - 3x + 8$
$y = 5 - 2x$

SOLUTION

a By substitution:

$x^2 + 3x - 1 = x - 2$

$x^2 + 2x + 1 = 0$

Using the discriminant:

$$b^2 - 4ac = (2)^2 - 4(1)(1)$$
$$= 4 - 4$$
$$= 0$$

$\therefore$ There is one solution to the pair of equations.

EXPLANATION

Start as if solving the equations simultaneously. Once the equation is in the form $ax^2 + bx + c = 0$, the discriminant $b^2 - 4ac$ can be used to determine the *number* of solutions. Here, $a = 1$, $b = 2$ and $c = 1$.

Recall: $b^2 - 4ac > 0$ means two solutions.
$b^2 - 4ac = 0$ means one solution.
$b^2 - 4ac < 0$ means no solutions.

Continued on next page

b By substitution:

$2x^2 - 3x + 8 = 5 - 2x$

$2x^2 - x + 3 = 0$

Using the discriminant:

$$\begin{aligned} b^2 - 4ac &= (-1)^2 - 4(2)(3) \\ &= 1 - 24 \\ &= -23 < 0 \end{aligned}$$

$\therefore$ There is no solution to the pair of equations.

Substitute and rearrange into the form $ax^2 + bx + c = 0$.

Calculate the discriminant. Here, $a = 2, b = -1$ and $c = 3$.

$b^2 - 4ac < 0$ means no solutions.

Now you try

Determine the number of solutions (points of intersection) of the following pairs of equations.

a $y = x^2 + 7x - 4$
$y = x - 1$

b $y = 3x^2 - x + 6$
$y = 5 - 2x$

Exercise 7G

FLUENCY	1, 2–4(1/2), 5	2–4(1/2), 5, 6(1/2)	2–4(1/3), 5, 6(1/3)

Example 14

1 Find the points of intersection of these parabolas and horizontal lines.

a $y = x^2 - 2x$
$y = 8$

b $y = x^2 + 5x + 1$
$y = -7$

Example 14

2 Find the points of intersection of these parabolas and horizontal lines.

a $y = x^2 + x$
$y = 6$

b $y = x^2 - 4x$
$y = 12$

c $y = x^2 + 3x + 6$
$y = 1$

d $y = 2x^2 + 7x + 1$
$y = -2$

e $y = 4x^2 - 12x + 9$
$y = 0$

f $y = 3x^2 + 2x + 9$
$y = 5$

Example 15a,b

3 Solve these simultaneous equations using substitution.

a $y = x^2$
$y = 3x$

b $y = x^2$
$y = -2x$

c $y = x^2$
$y = 3x + 18$

d $y = x^2 - 2x + 5$
$y = x + 5$

e $y = -x^2 - 11x + 4$
$y = -3x + 16$

f $y = x^2 + 3x - 1$
$y = 4x + 5$

g $y = x^2 - 2x - 4$
$y = -2x - 5$

h $y = -x^2 + 3x - 5$
$y = 3x - 1$

i $y = 2x^2 + 4x + 10$
$y = 1 - 7x$

j $y = 3x^2 - 2x - 20$
$y = 2x - 5$

k $y = -x^2 - 4x + 3$
$y = 2x + 12$

l $y = x^2 + x + 2$
$y = 1 - x$

Example 15c

4 Solve these simultaneous equations by first substituting.

a $y = x^2$
$2x + y = 8$

b $y = x^2$
$x - y = -2$

c $y = x^2$
$2x + 3y = 1$

d $y = x^2 + 3$
$5x + 2y = 4$

e $y = x^2 + 2x$
$2x - 3y = -4$

f $y = -x^2 + 9$
$6x - y = 7$

ISBN 978-1-108-77290-7

Example 16

5 Solve the following simultaneous equations, making use of the quadratic formula.

a Give your answers to one decimal place where necessary.

i $y = 2x^2 + 3x + 6$
$y = x + 4$

ii $y = x^2 + 1$
$y = 2x + 3$

iii $y = -2x^2 + x + 3$
$y = 3x + 2$

iv $y = 2x^2 + 4x + 5$
$y = 3 - 2x$

b Give your answers in exact surd form.

i $y = x^2 + 2x - 5$
$y = x$

ii $y = x^2 - x - 1$
$y = 2x$

iii $y = -x^2 - 3x + 3$
$y = -2x$

iv $y = x^2 + 3x - 3$
$y = 2x + 1$

Example 17

6 Determine the number of solutions to the following simultaneous equations.

a $y = x^2 + 2x - 3$
$y = x + 4$

b $y = 2x^2 + x$
$y = 3x - 1$

c $y = 3x^2 - 7x + 3$
$y = 1 - 2x$

d $y = x^2 + 5x + 1$
$y = 2x - 3$

e $y = -x^2$
$y = 2x + 1$

f $y = -x^2 + 2x$
$y = 3x - 1$

PROBLEM-SOLVING	7, 8(1/2)	7, 8(1/2)	8, 9

7 Ben, a member of an indoor cricket team, playing a match in a gymnasium, hits a ball that follows a path given by $y = -0.1x^2 + 2x + 1$, where y is the height above ground, in metres, and x is the horizontal distance travelled by the ball, in metres.

The ceiling of the gymnasium is 10.6 metres high. Will this ball hit the roof? Explain.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

8 Solve the following equations simultaneously.

a $y = x^2 + 2x - 1$

$y = \dfrac{x-3}{2}$

b $y = x(x - 4)$

$y = \dfrac{1}{2}x - 5$

c $y = (x - 2)^2 + 7$

$y = 9 - x$

d $y = \dfrac{8 - x^2}{2}$

$y = 2(x - 1)$

9 A train track is to be constructed over a section of a lake. On a map, the edge of the lake that the train track will pass over is modelled by the equation $y = 6 - 2x^2$. The segment of train track is modelled by the equation $y = x + 5$.

The section of track to be constructed will start and end at the points at which this track meets the lake.

a Determine the location (i.e. coordinates) of the points on the map where the framework for the track will start and end.

b If 1 unit represents 100 metres, determine the length of track that must be built over the lake, correct to the nearest metre.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

REASONING	10	10, 11	11, 12

10 Consider the parabola with equation $y = x^2 - 6x + 5$.

a Use any suitable method to determine the coordinates of the turning point of this parabola.

b Hence, state for which values of c the line $y = c$ will intersect the parabola:

i twice

ii once

iii not at all

11 Consider the parabola with equation $y = x^2$ and the family of lines $y = x + k$.

a Determine the discriminant, in terms of k, obtained when solving these equations simultaneously.

b Hence, determine for which values of k the line will intersect the parabola:

i twice

ii once

iii not at all

12 a Use the discriminant to show that the line $y = 2x + 1$ does not intersect the parabola $y = x^2 + 3$.

b Determine for which values of k the line $y = 2x + k$ does intersect the parabola $y = x^2 + 3$.

ENRICHMENT: Multiple tangents?	–	–	13

13 The line $y = mx$ is a tangent to the parabola $y = x^2 - 2x + 4$ (i.e the line touches the parabola at just one point).

a Find the possible values of m.

b Can you explain why there are two possible values of m? (*Hint*: A diagram may help.)

c If the value of m is changed so that the line now intersects the parabola in two places, what is the set of possible values for m?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Applications and problem-solving

The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

Designer jeans

1 A popular pair of designer jeans sells for \$300. A franchise sells 1200 pairs a month. The company carried out some research and discovered that for every \$10 decrease in price it can sell 100 more pairs a month.

The franchise is interested in maximising profit based on possible changes to the price of the jeans.

a If the jeans are sold for \$290 one month:

i how many sales are expected?

ii how much is made in sales (revenue)? How does this compare to the revenue in a month when the jeans are sold for \$300?

b Complete a table like the one shown below, to look at the revenue as the price of jeans is decreased.

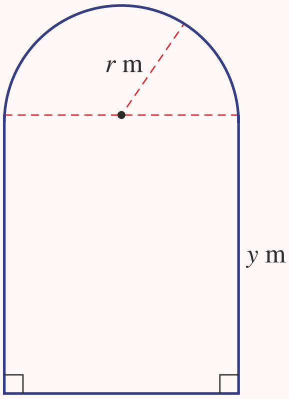
Number of \$10 price decreases	Price of jeans (\$)	Number of sales	Revenue (\$)
0	300	1200	$300 \times 1200 = 360\,000$
1	290		
2			
3			

c Use your table to help establish a rule for the revenue, R dollars, based on the number of \$10 price decreases, x.

d Use your rule to determine the price to sell the jeans for to maximise the revenue and state this maximum revenue.

Stained-glass windows

2 Many objects, such as windows, are composite shapes. The stained-glass window shown, for example, is made up of a rectangle and semicircle.



ISBN 978-1-108-77290-7

A window company is interested in exploring the relationship between the perimeter of the window and its area. It also wants to look at maximising the area for a fixed perimeter.

a If the perimeter of the window is fixed at 6 m, find:

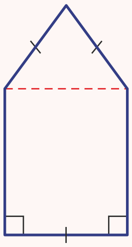
i an expression for the height of the rectangular section, y m, in terms of r

ii a rule for the area, A m^2, of the window in the form $A = ar^2 + br$ where a and b are constants

iii the dimensions of the window that maximise its area and the maximum area.

b Repeat part **a** for a perimeter of P m. Confirm your result by checking your answer to part **a iii** with $P = 6$.

c Investigate a second stained-glass window, as shown, with an equilateral triangle top section. Compare its maximum area with part **a**, for the same perimeter of 6 m.



On the lake

3 The parabolic curve can be used to model many shapes in the natural world. A certain lake is represented on a Cartesian plane by the region bound by the curves $y = \frac{1}{2}x^2 - 4x$ and $y = 3x - \frac{1}{2}x^2$.

You will investigate the use of the parabola to model the boundary of the lake and a path of a boat. You will calculate distances on the lake and find minimum distances between objects on or beside the lake.

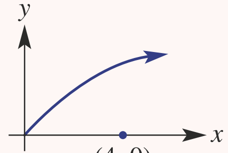
a **i** Sketch the region represented by the lake including the points of intersection.

ii At $x = 2$, determine the vertical distance across the lake.

iii Find a rule in terms of x for the vertical distance (y direction) across the lake.

iv Hence, find the maximum vertical distance across the lake.

b A speed boat on the lake is following a path as shown on the right. A spectator stands on the sidelines at (4, 0) with 1 unit representing 10 m. The rule for the speedboat's path is given by $y = \sqrt{x}$.



i Find the direct distance, in metres, from the spectator to the speedboat when the speedboat is at (1, 1).

ii Find a rule in terms of x for the distance between (4, 0) and a point (x, y) on the speed boat path. (*Hint*: $y = \sqrt{x}$.)

iii As x increases, explain what happens to the value of $\sqrt{x}$.

iv Complete the table below by using the minimums of the quadratics in **I** and **III** to infer the minimum of their square root graph.

Rule	x-value of minimum	Minimum value
I $y = x^2 + 2$	0	2
II $y = \sqrt{x^2 + 2}$		$\sqrt{2}$
III $y = x^2 - 4x + 7$		
IV $y = \sqrt{x^2 - 4x + 7}$		

v Hence, if the minimum value of a quadratic rule y is n at $x = m$, give the minimum value of $\sqrt{y}$ and for which x-value it occurs.

vi Use the ideas above and your rule from **ii** to determine the coordinates on the speedboat's path, $y = \sqrt{x}$, where it will be closest to the spectator. What is this minimum distance?

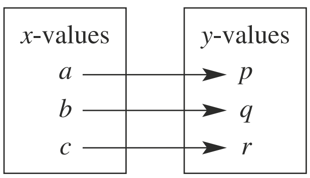
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

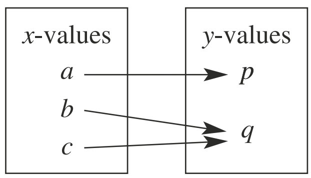
7H Functions 10A

Learning intentions
- To understand what defines a mathematical relation and a function
- To know how to recognise or test for a function
- To be able to use the standard notation for functions
- To know the meaning of the domain and range of a function
- To be able to find the set of allowable x-values (domain) and resulting y-values (range) of a function

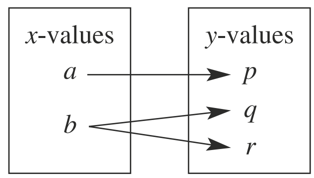
On the Cartesian plane a relationship between the variables x and y can be shown. When this relation has a unique (i.e. only one) y-value for each of its x-values, it is called a function.



Function



Function

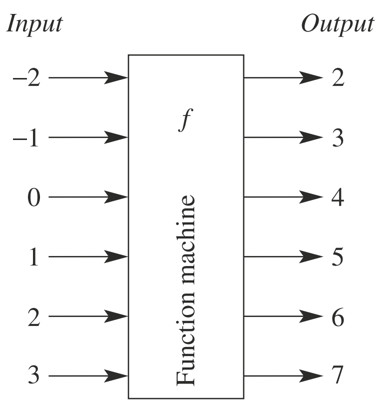


Not a function since $x = b$ gives two y-values.

All functions, including parabolas, use a special notation where the y is replaced by $f(x)$. $y = x^2$ becomes $f(x) = x^2$ (i.e. y is a function of x).

LESSON STARTER A function machine

Consider the input and output of the following machine.



The name f is given to the function and it is written
$f(\text{input}) = \text{output}$

This beautiful, old walking bridge in Kromlau, Germany, forms a circle with its reflection. The equation of the full circle is a relation; however, the two semicircle equations, one modelling the bridge and the other its reflection, are functions.

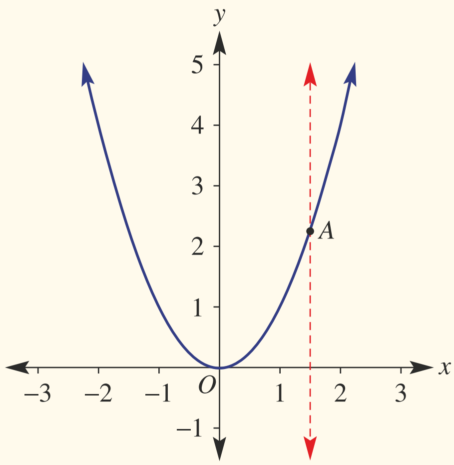
- Using the above idea, complete the following.
  $f(-2) =$ _____, $f(-1) =$ _____, $f(0) =$ _____, $f(1) =$ _____, $f(2) =$ _____,
  $f(3) =$ _____ and, hence $f(x) =$ ______.

ISBN 978-1-108-77290-7

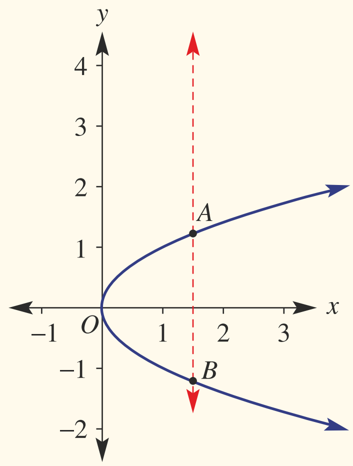
Cambridge University Press
Updated June 2022

KEY IDEAS

- Any set of ordered pairs is called a **relation**.
- A relation in which each x-value produces only one y-value is called a **function** and can use function notation, $f(x)$.
- A relation that is a function has only one y-value for each x-value. Graphically, any vertical line drawn through the graph of a function will cut it only once.
  - For example:



This relation is a function as any vertical line shows that one x-value links to only one y-value.



This relation is *not* a function as a vertical line shows that one x-value links to more than one y-value.

- $f(x)$ is the notation used to replace y if the relation is a function.
  - $f(x) = 3x - 1$ can be written instead of $y = 3x - 1$.
  - The parabola $y = x^2$ is a function, so can be written as $f(x) = x^2$. Also, $f(-2) = 4$ can be written to describe the point $(-2, 4)$.
- The set of allowable x-coordinates (i.e. the input) in a relation is also called the **domain**.
- The **range** is the term given to the set of resulting y-coordinates (output) in the relation.

BUILDING UNDERSTANDING

1 Give the following functions using function notation.

a $y = 8x$ **b** $y = 9 - x^2$ **c** $y = 2^x$

2 For each of the following, state whether they are true or false.

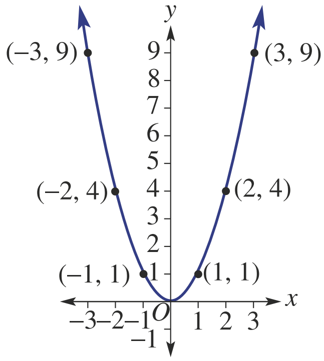
a All parabolas are functions.
b Any vertical line will cut $y = 2x - 1$ only once.
c Only positive x-values can be used as the input in $f(x) = x^2$.
d All straight lines are functions.
e A circle is not a function.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

3 Use this sketch to decide on the permissible y-values for $y = x^2$, given that the following x-values are allowed.

a $x \geqslant 0$
b $x > 0$
c $x > 3$
d $-1 \leqslant x \leqslant 1$
e all real x-values



Example 18 Using function notation

For $f(x) = x^2 - 3x + 1$, find:

a $f(0)$ **b** $f(-3)$ **c** $f(c)$

SOLUTION

a $f(0) = (0)^2 - 3(0) + 1$
$= 1$

b $f(-3) = (-3)^2 - 3(-3) + 1$
$= 9 + 9 + 1$
$= 19$

c $f(c) = c^2 - 3c + 1$

EXPLANATION

The input of 0 is substituted into the rule for each x-value.

$x = -3$ is substituted into the rule on the right-hand side.
Note that $(-3)^2 \neq -3^2$

The x has been replaced by c. Therefore, replace each x with c in the rule.

Now you try

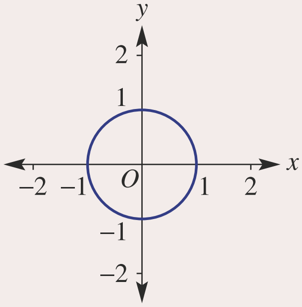
For $f(x) = x^2 - 5x + 2$, find:

a $f(0)$ **b** $f(-2)$ **c** $f(k)$

Example 19 Recognising a function

From the following, identify which are functions.

a $\{(1, 4), (2, 8), (4, 16), (5, 20)\}$ **b** $y = x^2 + 3$ **c**



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

SOLUTION

a A function since each x-value has only one y-value.

EXPLANATION

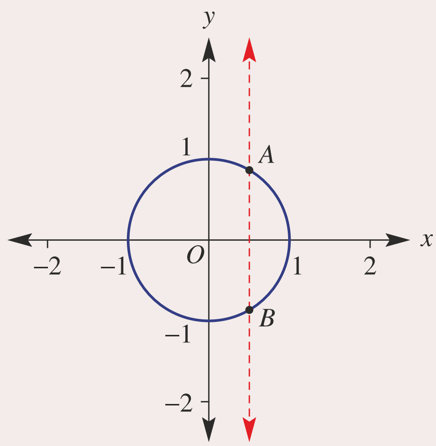
Each of the x-values; i.e. $x = 1, 2, 4$, and 5, occurs only once. So the coordinates represent a function.

b $y = x^2 + 3$ is a function since each x-value will produce only one y-value.

As the rule represents a parabola, each x-value will produce only one y-value and so it is a function.

c Not a function because a vertical line shows that one x-value links to more than one y-value.

A vertical line drawn anywhere through the graph will cross in more than one place, therefore it is not a function.



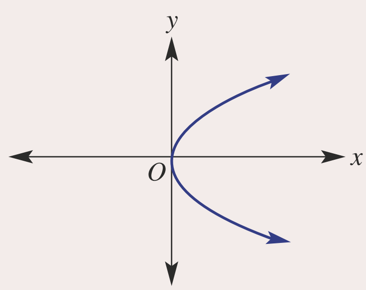
Now you try

From the following, identify which are functions.

a $\{(-1, 4), (0, 2), (1, 0), (2, -2)\}$

b $y = -x^2 + 1$

c



Example 20 Determining domain and range

Write down the allowable x-values (domain) and the resulting y-values (range) for each of these functions.

a $y = 4x - 1$

b $y = x^2 - 4$

SOLUTION

a Domain is the set of all real x-values.
Range is the set of all real y-values.

EXPLANATION

The function is a straight line. The input (i.e. x-values) can be any number and will produce any number as an output value.

Continued on next page

b Domain is the set of all real x-values.
Range is the set of y-values, where $y \geqslant -4$.

It is possible to square any value of x.
As squaring a negative number makes it positive, the smallest y-value possible as an output is -4.

Now you try

Write down the permissible x- and y-values for each of these functions.

a $y = -x + 3$ **b** $y = -x^2 + 2$

Exercise 7H

FLUENCY	1, 2–5(1/2), 7–8(1/2)	2–8(1/2)	2–8(1/3)

Example 18

1 For $f(x) = x^2 - 2x + 3$, find:

a $f(0)$ **b** $f(-2)$ **c** $f(c)$

Example 18

2 Given $f(x) = 2x^2 - x + 4$, find:

a $f(0)$ **b** $f(2)$ **c** $f(8)$

d $f\left(\frac{1}{2}\right)$ **e** $f(-2)$ **f** $f(a)$

Example 19a,b

3 From the following, identify which are functions.

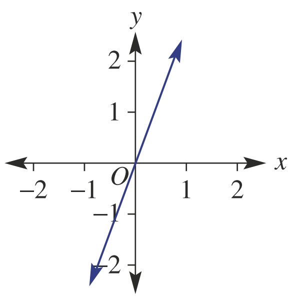
a $\{(1, 2), (2, 4), (3, 6)\}$ **b** $\{(1, 0), (-1, 4), (2, 0)\}$ **c** $\{(0, 4), (1, 3), (2, -1), (1, 5)\}$

d $y = 2x$ **e** $y = -x^2 - 4$ **f** $y = 3^x$

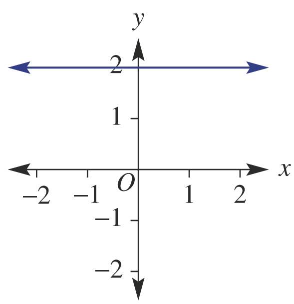
Example 19c

4 Use the vertical line test to decide which of the following graphs represent a function.

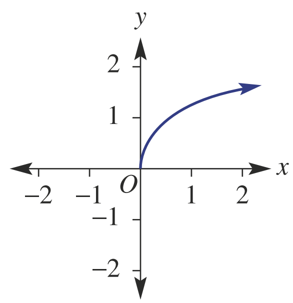
a



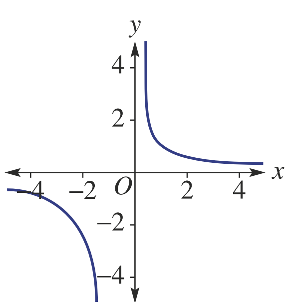
b



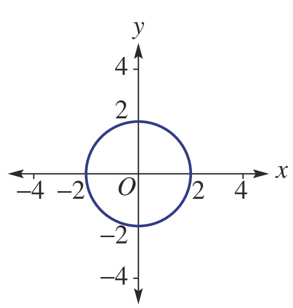
c



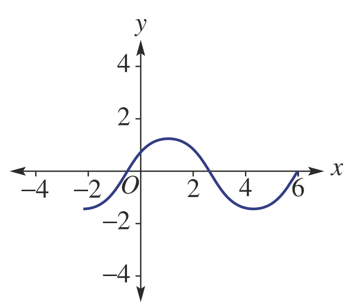
d



e



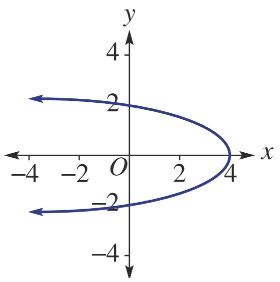
f



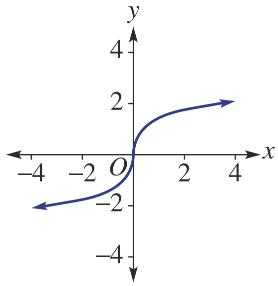
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

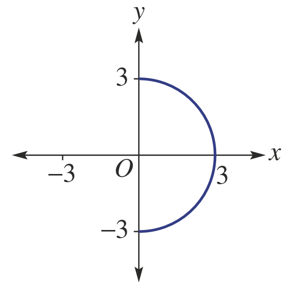
g



h



i



5 Given $f(x) = 2x^3 - x^2 + x$, find:

a $f(0)$ **b** $f(1)$ **c** $f(-1)$ **d** $f(5)$ **e** $f(0.2)$ **f** $f(k)$

6 Find $f(0)$, $f(2)$, $f(-4)$, $f(a)$ and $f(a + 1)$ for each of the following functions.

a $f(x) = 4x$ **b** $f(x) = 1 - x^2$ **c** $f(x) = (x - 2)(x + 6)$ **d** $f(x) = 4x^2 + 9$

Example 20a

7 Find the set of allowable x-values for each function. (Note: This is the domain of each function.)

a $f(x) = 2 - x$ **b** $f(x) = x^2$ **c** $f(x) = 3x^2$ **d** $f(x) = 2 - x^2$

Example 20b

8 For each function in Question **7**, write down the set of y-values (i.e. the range) that it has as its resulting output.

PROBLEM-SOLVING	9	9, 10	9–11

9 Given $f(x) = 7x - 9$ and $g(x) = 6 - 2x$:

a Find:

i $f(2)$ **ii** $g(4)$ **iii** $f(2) + g(4)$

iv $f(-2) + 2g(1)$ **v** $f(g(2))$ **vi** $g(f(2))$

b The value of a is such that $f(a) = g(a)$. Explain the significance of $x = a$ in terms of the two graphs $y = f(x)$ and $y = g(x)$.

10 Answer the following as true or false for each of these functions.

i $f(x) = 2x - 2$ **ii** $f(x) = x^2 + 4$

a $f(3a) = 3f(a)$ **b** $f(a) = f(-a)$ **c** $f(a) + f(b) = f(a + b)$

11 Given the function $f(x) = 2x^2 - 3x - 1$, simplify $\dfrac{f(x + h) - f(x)}{h}$.

REASONING	12	12, 13	12–15

12 **a** Explain why all parabolas of the form $y = ax^2 + bx + c$ are functions.

b What type of straight line is not a function and why is it not a function?

c Why is finding the coordinates of the vertex of a parabola used when finding the range of the function?

d Considering your response to part **c**, find the range of the following quadratic functions.

i $y = x^2 + 4x$ **ii** $y = x^2 - 5x - 6$ **iii** $y = 1 - x - 2x^2$ **iv** $y = x^2 + 6x + 10$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

13 Given that $\frac{a}{0}$ is undefined, which value of x is not permissible for each of the following functions?

a $f(x) = \dfrac{3}{x-1}$ **b** $f(x) = \dfrac{3}{2x+1}$ **c** $f(x) = \dfrac{-2}{1-x}$

14 Given that you cannot take the square root of a negative number, write down the domain of:

a $y = \sqrt{x}$ **b** $y = \sqrt{x-2}$ **c** $y = \sqrt{x+2}$ **d** $y = \sqrt{2-x}$

15 For $f(x) = x^2 + \dfrac{1}{x^2}$:

a Find $f(a)$ and $f(-a)$.

b Find the values of $f(-3)$, $f(-2)$, $f(-1)$ and $f(0)$. Hence, sketch the graph of $y = f(x)$, in the domain $-3 \leqslant x \leqslant 3$.

c Comment on any symmetry you notice.

ENRICHMENT: Sketching hybrid functions	–	–	16

16 a Sketch the following functions.

i $f(x) = \begin{cases} 2x & \text{for } x \geqslant 0 \\ -2x & \text{for } x < 0 \end{cases}$

ii $f(x) = \begin{cases} 4 & \text{for } x \geqslant 2 \\ x^2 & \text{for } -2 < x < 2 \\ 4 & \text{for } x \leqslant -2 \end{cases}$

iii $f(x) = \begin{cases} 2x+4 & \text{for } x > 0 \\ -(x+4) & \text{for } x \leqslant 0 \end{cases}$

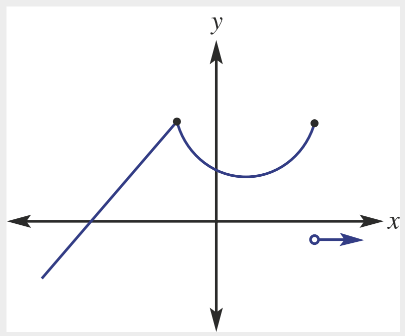
b A function is said to be continuous if its entire graph can be drawn without lifting the pen from the page. Which of the functions in part **a** are discontinuous?

c For the functions sketched in part **a**, write down the range.

d For each of the functions in part **a**, find the value of:

i $f(2) + f(0) + f(-2)$

ii $f(3) - 2f(1) + 4f(-4)$



A hybrid function includes two or more rules for various parts of the domain.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

7I Graphs of circles

Learning intentions
- To know the form of the Cartesian equation of a circle centred at the origin with radius r
- To be able to sketch a graph of a circle centred at the origin, using the radius to label intercepts
- To know how to find the points of intersection of a line and a circle

We know of the circle as a common shape in geometry, but we can also describe a circle using an equation and as a graph on the Cartesian plane.

Programs that manipulate a robot's arm include circle equations. Working in 2D: with the lower arm secured at its base, the joint can move in a semicircle; each point on this semicircle can be the centre of a circular path that the hand can follow.

LESSON STARTER Plotting the circle

A graph has the rule $x^2 + y^2 = 9$.

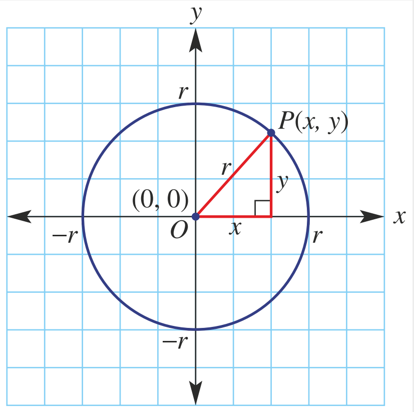
- When $x = 0$, what are the two values of y?
- When $x = 1$, what are the two values of y?
- When $x = 4$, are there any values of y? Discuss.
- Complete this table of values.

x	-3	-2	-1	0	1	2	3
y		$\pm\sqrt{5}$					

- Now plot all your points on a number plane and join them to form a smooth curve.
- What shape have you drawn and what are its features?
- How does the radius of your circle relate to the equation?

KEY IDEAS

- The Cartesian equation of a circle with centre $(0, 0)$ and radius r is given by $x^2 + y^2 = r^2$.
- Making x or y the subject:
  - $y = \pm\sqrt{r^2 - x^2}$
  - $x = \pm\sqrt{r^2 - y^2}$
- To find the intersection points of a circle and a line, use the method of substitution.



Using Pythagoras' theorem, $a^2 + b^2 = c^2$ gives $x^2 + y^2 = r^2$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

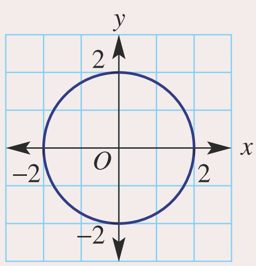
BUILDING UNDERSTANDING

1. Draw a circle on the Cartesian plane with centre (0, 0) and radius 2.
2. Solve these equations for the unknown variable. There are two solutions for each.
   a $x^2 + 2^2 = 9$
   b $x^2 + 3^2 = 25$
   c $5^2 + y^2 = 36$
3. A circle has equation $x^2 + y^2 = r^2$. Complete these sentences.
   a The centre of the circle is _____.
   b The radius of the circle is _____.

Example 21 Sketching a circle

For the equation $x^2 + y^2 = 4$, complete the following.

a State the coordinates of the centre.
b State the radius.
c Find the values of y when $x = 1$.
d Find the values of x when $y = \frac{1}{2}$.
e Sketch a graph showing intercepts.

SOLUTION	EXPLANATION
a (0, 0)	(0, 0) is the centre for all circles $x^2 + y^2 = r^2$.
b $r = 2$	$x^2 + y^2 = r^2$, so $r^2 = 4$.
c $x^2 + y^2 = 4$	
$1^2 + y^2 = 4$	Substitute $x = 1$ and solve for y.
$y^2 = 3$	
$y = \pm\sqrt{3}$	Recall that $(\sqrt{3})^2$ and $(-\sqrt{3})^2$ both equal 3.
d $x^2 + \left(\frac{1}{2}\right)^2 = 4$	Substitute $y = \frac{1}{2}$.
$x^2 + \frac{1}{4} = 4$	$4 - \frac{1}{4} = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$
$x^2 = \frac{15}{4}$	
$x = \pm\sqrt{\frac{15}{4}}$	$\sqrt{\frac{15}{4}} = \frac{\sqrt{15}}{\sqrt{4}} = \frac{\sqrt{15}}{2}$
$x = \pm\frac{\sqrt{15}}{2}$	
e 	Draw a circle with centre (0, 0) and radius 2. Label intercepts.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Now you try

For the equation $x^2 + y^2 = 16$, complete the following.

a State the coordinates of the centre.

b State the radius.

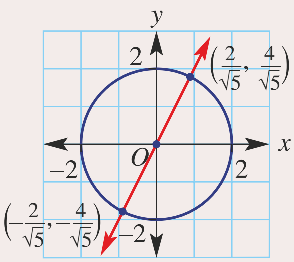
c Find the values of y when $x = 1$.

d Find the values of x when $y = \frac{1}{2}$.

e Sketch a graph showing intercepts.

Example 22 Intersecting circles and lines

Find the coordinates of the points where $x^2 + y^2 = 4$ intersects $y = 2x$. Sketch a graph showing the exact intersection points.

SOLUTION	EXPLANATION
$x^2 + y^2 = 4$ and $y = 2x$	Substitute $y = 2x$ into $x^2 + y^2 = 4$ and solve for x.
$x^2 + (2x)^2 = 4$	Recall that $(2x)^2 = 2x \times 2x = 4x^2$.
$x^2 + 4x^2 = 4$	
$5x^2 = 4$	
$x^2 = \frac{4}{5}$	
$x = \pm\sqrt{\frac{4}{5}}$	$\sqrt{\frac{4}{5}} = \frac{\sqrt{4}}{\sqrt{5}} = \frac{2}{\sqrt{5}}$
$x = \pm\frac{2}{\sqrt{5}}$	
If $y = 2x$:	Substitute both values of x into $y = 2x$ to find the y-coordinate.
$x = \frac{2}{\sqrt{5}}$ gives $y = 2 \times \frac{2}{\sqrt{5}} = \frac{4}{\sqrt{5}}$	
$x = -\frac{2}{\sqrt{5}}$ gives $y = 2 \times \left(-\frac{2}{\sqrt{5}}\right) = -\frac{4}{\sqrt{5}}$	
	For $x^2 + y^2 = 4$, $r = 2$. Mark the intersection points and sketch $y = 2x$.

Now you try

Find the coordinates of the points where $x^2 + y^2 = 25$ intersects $y = x$. Sketch a graph showing the exact intersection points.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 7I

FLUENCY	1, 3–5(1/2)	2, 3–6(1/2)	2, 3–6(1/3)

Example 21

1 A circle has equation $x^2 + y^2 = 9$. Complete the following.

a State the coordinates of the centre.

b State the radius.

c Find the values of y when $x = 2$.

d Find the values of x when $y = \frac{3}{2}$.

e Sketch a graph showing intercepts.

Example 21

2 For the equation $x^2 + y^2 = 25$ complete the following.

a State the coordinates of the centre.

b State the radius.

c Find the values of y when $x = \frac{9}{2}$.

d Find the values of x when $y = 4$.

e Sketch a graph showing intercepts.

3 Give the radius of the circles with these equations.

a $x^2 + y^2 = 36$ **b** $x^2 + y^2 = 81$ **c** $x^2 + y^2 = 144$

d $x^2 + y^2 = 5$ **e** $x^2 + y^2 = 14$ **f** $x^2 + y^2 = 20$

4 Write the equation of a circle with centre (0, 0) and the given radius.

a 2 **b** 7 **c** 100 **d** 51

e $\sqrt{6}$ **f** $\sqrt{10}$ **g** 1.1 **h** 0.5

5 For the circle with equation $x^2 + y^2 = 4$ find the exact coordinates where:

a $x = 1$ **b** $x = -1$ **c** $x = \frac{1}{2}$

d $y = -\frac{1}{2}$ **e** $y = -2$ **f** $y = 0$

6 Without showing any working steps, write down the x- and y-intercepts of these circles.

a $x^2 + y^2 = 1$ **b** $x^2 + y^2 = 16$ **c** $x^2 + y^2 = 3$ **d** $x^2 + y^2 = 11$

PROBLEM-SOLVING	7(1/2), 8	7(1/2), 8–10	7(1/2), 9–12

7 Write down the radius of these circles.

a $x^2 + y^2 - 8 = 0$ **b** $x^2 - 4 = -y^2$ **c** $y^2 = 9 - x^2$

d $10 - y^2 - x^2 = 0$ **e** $3 + x^2 + y^2 = 15$ **f** $17 - y^2 = x^2 - 3$

Example 22

8 Find the coordinates of the points where $x^2 + y^2 = 9$ intersects $y = x$. Sketch a graph showing the intersection points.

9 Find the coordinates of the points where $x^2 + y^2 = 10$ intersects $y = 3x$. Sketch a graph showing the intersection points.

10 Find the coordinates of the points where $x^2 + y^2 = 6$ intersects $y = -\frac{1}{2}x$. Sketch a graph showing the intersection points.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

11 Determine the exact length of the chord formed by the intersection of $y = x - 1$ and $x^2 + y^2 = 5$. Sketch a graph showing the intersection points and the chord.

10A **12** For the circle $x^2 + y^2 = 4$ and the line $y = mx + 4$, determine the exact values of the gradient, m, so that the line:

- **a** is a tangent to the circle
- **b** intersects the circle in two places
- **c** does not intersect the circle.

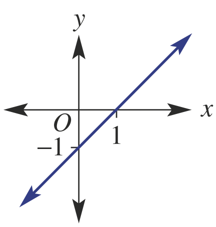
REASONING	13	13, 14	14, 15

13 Match equations **a–f** with graphs **A–F**.

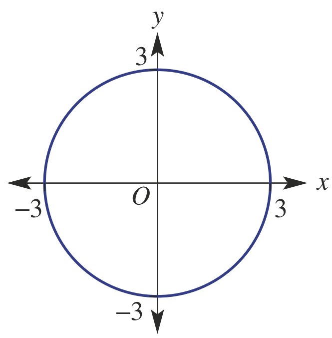
a $x^2 + y^2 = 7$ **b** $y = x - 1$ **c** $y = -2x + 2$

d $y = x^2 - 1$ **e** $y = -x(x + 2)$ **f** $x^2 + y^2 = 9$

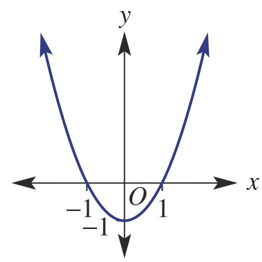
A



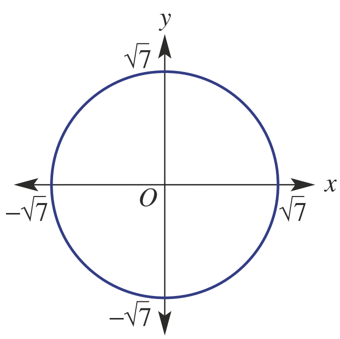
B



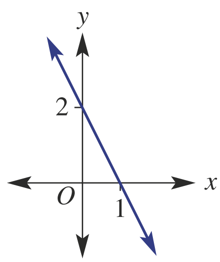
C



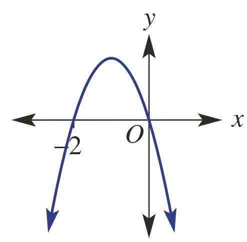
D



E



F



14 **a** Write $x^2 + y^2 = 16$ in the form $y = \pm\sqrt{r^2 - x^2}$.

b Write $x^2 + y^2 = 3$ in the form $x = \pm\sqrt{r^2 - y^2}$.

15 **a** Explain why the graphs of $y = 3$ and $x^2 + y^2 = 4$ do not intersect.

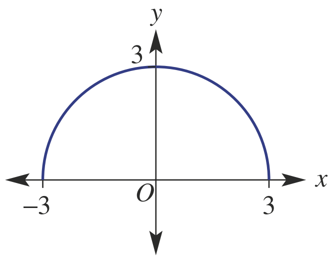
b Explain why the graphs of $x = -2$ and $x^2 + y^2 = 1$ do not intersect.

ISBN 978-1-108-77290-7

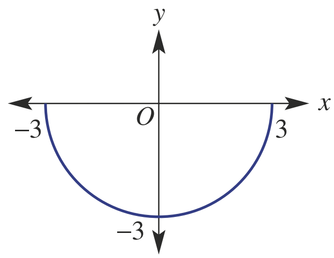
Cambridge University Press
Updated June 2022

ENRICHMENT: Half circles	–	–	16–17($^1/_2$)

16 When we write $x^2 + y^2 = 9$ in the form $y = \pm\sqrt{9 - x^2}$, we define two circle halves.



$y = \sqrt{9 - x^2}$



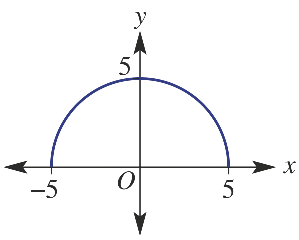
$y = -\sqrt{9 - x^2}$

Sketch the graphs of these half circles.

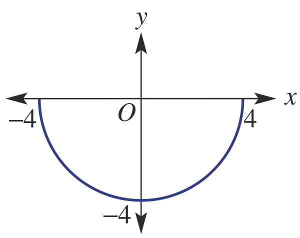
a $y = \sqrt{4 - x^2}$
b $y = \sqrt{25 - x^2}$
c $y = -\sqrt{1 - x^2}$
d $y = -\sqrt{10 - x^2}$
e $y = \sqrt{16 - x^2}$
f $x = -\sqrt{36 - y^2}$
g $x = -\sqrt{7 - y^2}$
h $x = \sqrt{5 - y^2}$
i $x = \sqrt{12 - y^2}$

17 Write the rules for these half circles.

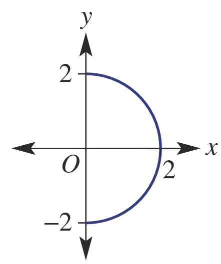
a



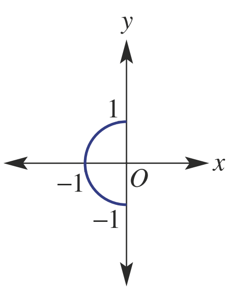
b



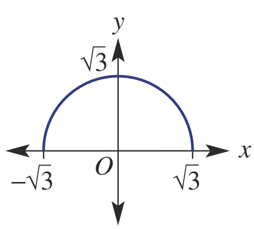
c



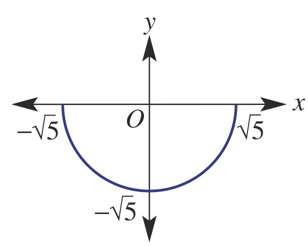
d



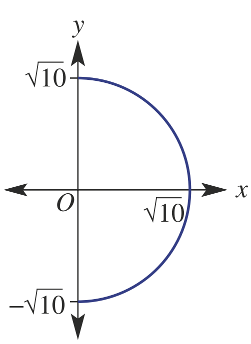
e



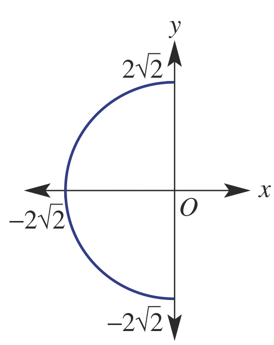
f



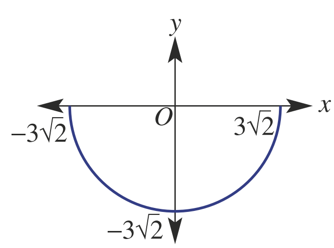
g



h



i



ISBN 978-1-108-77290-7
Cambridge University Press
Updated June 2022

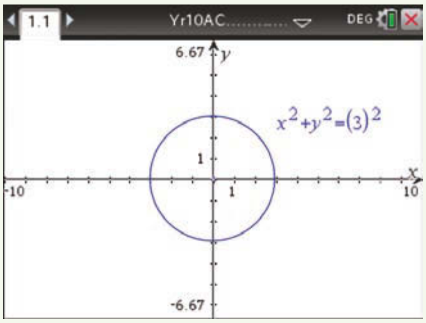
Using calculators to graph circles and other graphs

1 Sketch a graph of $x^2 + y^2 = r^2$ using $r = \{1, 3\}$.

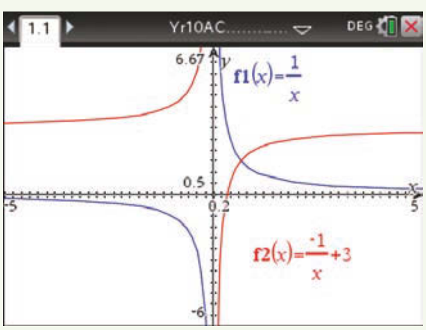
2 Sketch a graph of $y = \frac{1}{x}$ and $y = -\frac{1}{x} + 3$ on the same set of axes.

Using the TI-Nspire:

1 In a **Graphs** page, use menu **>Graph Entry/Edit>Equation>Circle>centre form** and enter $(x - 0)^2 + (y - 0)^2 = 3^2$ for $r = 3$.

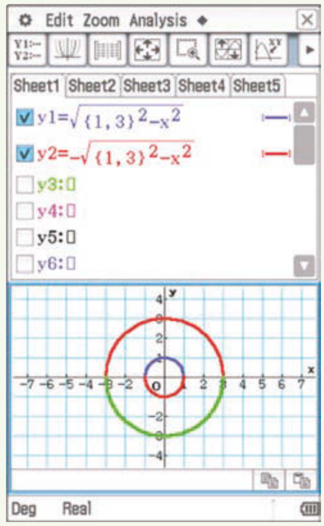


2 Enter the rules as $f1(x)$ and $f2(x)$. Change the scale using the window setting.
Arrow up or down to toggle between graphs.

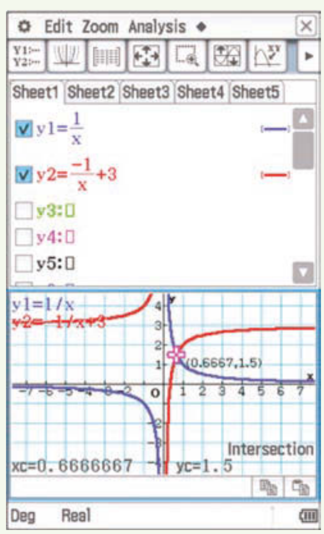


Using the ClassPad:

1 In the **graph&Table** application enter the rule $y1 = p(\{1, 3\}^2 - x^2)$ followed by **EXE**. Enter the rule $y2 = -p(\{1, 3\}^2 - x^2)$ followed by **EXE**. Tap [graph icon] to see the graph. Select **Zoom**, **Square**.



2 Enter the rules as $y1$ and $y2$. Change the scale by tapping [icon]. Use the **Analysis**, **G-Solve** menu to show significant points. Arrow up or down to toggle between graphs.



7J Graphs of hyperbolas

Learning intentions

- To know the general equation form of a rectangular hyperbola: $y = \frac{a}{x}$
- To know how to sketch and label the key features of a rectangular hyperbola, including asymptotes
- To be able to sketch reflections of hyperbolas
- To be able to find point(s) of intersection of a line and a hyperbola

A simple rectangular hyperbola is the graph of the equation $y = \frac{1}{x}$. These types of equations are common in many mathematical and practical situations.

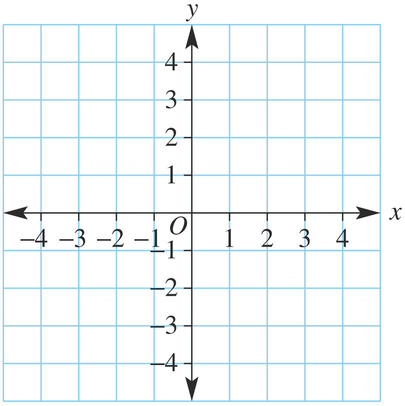
When two stones are thrown into a pond, the resulting concentric ripples intersect at a set of points that together form the graph of a hyperbola. In a similar way, when signals are received from two different satellites, a ship's navigator can map the hyperbolic shape of the intersecting signals and help to determine the ship's position.

Two signal pulses emitted simultaneously will spread in overlapping concentric circles. All circle intersection points with the same difference in distance (or time) from the centres form a hyperbola shape. Using signal time differences and hyperbolic charts, navigators can locate a ship's position more reliably than using GPS.

LESSON STARTER How many asymptotes?

Consider the rule for the simple hyperbola $y = \frac{1}{x}$. First, complete the table and graph, and then discuss the points below.

x	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y										



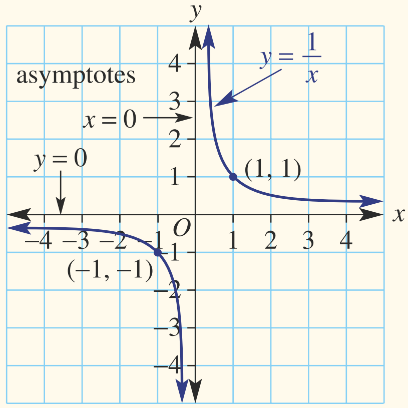
- Discuss the shape of the graph of the hyperbola.
- What would the values of y approach as x increases to infinity or negative infinity?
- What would the values of y approach as x decreases to zero from the left or from the right?
- What are the equations of the asymptotes for $y = \frac{1}{x}$?

ISBN 978-1-108-77290-7

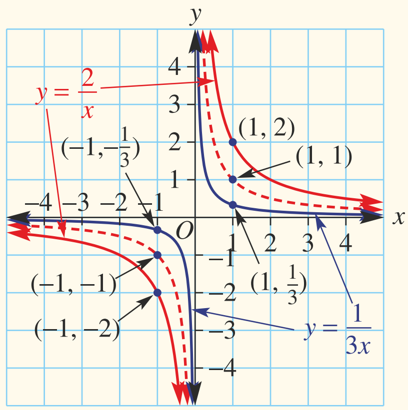
Cambridge University Press
Updated June 2022

KEY IDEAS

- A hyperbola has two asymptotes. Recall that an **asymptote** is a straight line that a curve approaches more and more closely but never quite reaches.



- A **rectangular hyperbola** is the graph of the rule $y = \frac{a}{x}$, $a \neq 0$.
  - $y = \frac{1}{x}$ is the basic rectangular hyperbola.
  - $x = 0$ (y-axis) and $y = 0$ (x-axis) are its asymptotes.
  - For $a > 1$ the hyperbola will be further out from the asymptotes.
  - For $0 < a < 1$ the hyperbola will be closer in to the asymptotes.



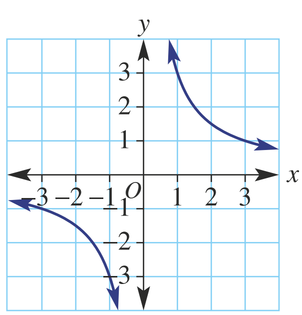
- The graph of $y = -\frac{a}{x}$ is a reflection of the graph of $y = \frac{a}{x}$ in the x- (or y-) axis.
- To find the intersection points of a hyperbola and a line, use the method of substitution.

BUILDING UNDERSTANDING

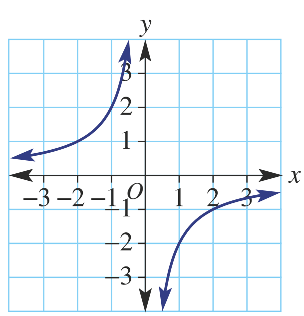
1 Match the rules **a**, **b** and **c** with the graphs **A**, **B** and **C**.

a $y = \frac{1}{x}$ **b** $y = \frac{3}{x}$ **c** $y = -\frac{2}{x}$

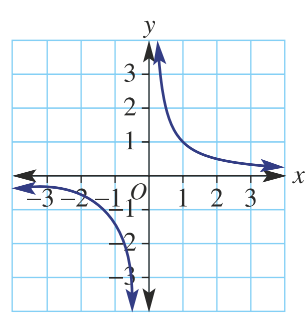
A



B



C



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2 **a** Give in ascending order: $1 \div 0.1$, $1 \div 0.001$, $1 \div 0.01$, $1 \div 0.00001$.

b For $y = \frac{1}{x}$, which of the following x-values will give the largest value of y:

$\frac{1}{5}, \frac{1}{10}, \frac{1}{2}$ or $\frac{1}{100}$?

c For $y = \frac{1}{x}$, calculate the difference in the y-values for $x = 10$ and $x = 1000$.

d For $y = \frac{1}{x}$, calculate the difference in the y-values for $x = -\frac{1}{2}$ and $x = -\frac{1}{1000}$.

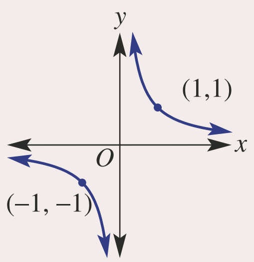
Example 23 Sketching a hyperbola

Sketch the graphs of each hyperbola, labelling the points where $x = 1$ and $x = -1$.

a $y = \frac{1}{x}$ **b** $y = \frac{2}{x}$ **c** $y = -\frac{3}{x}$

SOLUTION

a $y = \frac{1}{x}$

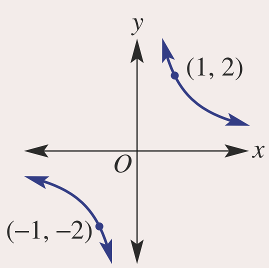


EXPLANATION

Draw the basic shape of a rectangular hyperbola $y = \frac{1}{x}$.

Substitute $x = 1$ and $x = -1$ to find the two points.

b $y = \frac{2}{x}$



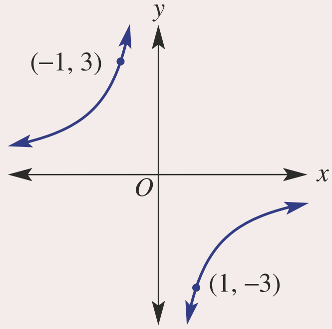
For $x = 1$, $y = \frac{2}{1} = 2$.

For $x = -1$, $y = \frac{2}{-1} = -2$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

c $y = -\frac{3}{x}$



$y = -\frac{3}{x}$ is a reflection of $y = \frac{3}{x}$ in either the x- or y-axis.

When $x = 1$, $y = -\frac{3}{1} = -3$.

When $x = -1$, $y = -\frac{3}{-1} = 3$.

Now you try

Sketch the graphs of each hyperbola, labelling the points where $x = 1$ and $x = -1$.

a $y = \frac{1}{x}$ **b** $y = \frac{3}{x}$ **c** $y = -\frac{2}{x}$

Example 24 Intersecting with hyperbolas

Find the coordinates of the points where $y = \frac{1}{x}$ intersects these lines.

a $y = 3$ **b** $y = 4x$

SOLUTION

a $y = \frac{1}{x}$ and $y = 3$

$3 = \frac{1}{x}$

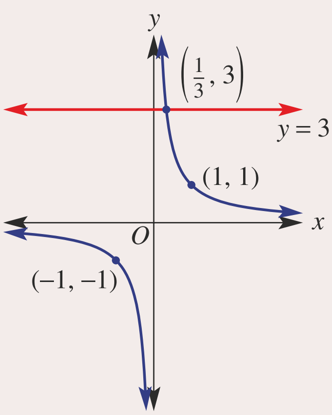
$3x = 1$

$x = \frac{1}{3}$

$\therefore$ Intersection point is $\left(\frac{1}{3}, 3\right)$.

EXPLANATION

Substitute $y = 3$ into $y = \frac{1}{x}$ and solve.



Continued on next page

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b $y = \frac{1}{x}$ and $y = 4x$

$4x = \frac{1}{x}$

$4x^2 = 1$

$x^2 = \frac{1}{4}$

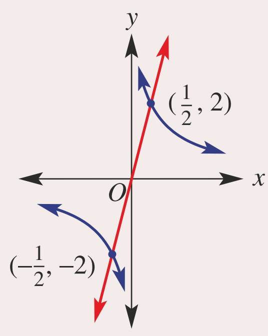
$x = \pm\frac{1}{2}$

$y = 4 \times \left(\frac{1}{2}\right) = 2$ and $y = 4 \times \left(-\frac{1}{2}\right) = -2$

$\therefore$ Intersection points are $\left(\frac{1}{2}, 2\right)$ and $\left(-\frac{1}{2}, -2\right)$.

Substitute and solve by multiplying both sides by x. Note that $\sqrt{\frac{1}{4}} = \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2}$.

Find the corresponding y-values.



Now you try

Find the coordinates of the points where $y = \frac{1}{x}$ intersects these lines.

a $y = 4$ **b** $y = 3x$

Exercise 7J

FLUENCY	1, 2–4(1/2)	2–5(1/2)	2–5(1/2)

Example 23

1 Sketch the graphs of each hyperbola, labelling the points where $x = 1$ and $x = -1$.

a $y = \frac{1}{x}$ **b** $y = \frac{4}{x}$ **c** $y = -\frac{4}{x}$

Example 23

2 Sketch the graphs of these hyperbolas, labelling the points where $x = 1$ and $x = -1$.

a $y = \frac{1}{x}$ **b** $y = \frac{2}{x}$ **c** $y = \frac{3}{x}$

d $y = -\frac{1}{x}$ **e** $y = -\frac{2}{x}$ **f** $y = -\frac{3}{x}$

3 Find the coordinates on the graph of $y = \frac{2}{x}$, where:

a $x = 2$ **b** $x = 4$ **c** $x = -1$ **d** $x = -6$

4 Find the coordinates on the graph of $y = -\frac{5}{x}$, where:

a $x = 10$ **b** $x = -4$ **c** $x = -7$ **d** $x = 9$

5 Find the coordinates on the graph of $y = \frac{3}{x}$, where:

a $y = 3$ **b** $y = 1$ **c** $y = -2$ **d** $y = -6$

ISBN 978-1-108-77290-7

PROBLEM-SOLVING	6–7(½)	6–7(½)	7–8(½)

6 **a** Decide whether the point $(1, 3)$ lies on the hyperbola $y = \frac{3}{x}$.

b Decide whether the point $(1, -5)$ lies on the hyperbola $y = -\frac{5}{x}$.

c Decide whether the point $(2, 1)$ lies on the hyperbola $y = -\frac{2}{x}$.

d Decide whether the point $(-3, 6)$ lies on the hyperbola $y = -\frac{6}{x}$.

Example 24

7 Find the coordinates of the points where $y = \frac{1}{x}$ intersects these lines.

a $y = 2$ **b** $y = 6$ **c** $y = -1$ **d** $y = -10$
e $y = x$ **f** $y = 4x$ **g** $y = 2x$ **h** $y = 5x$

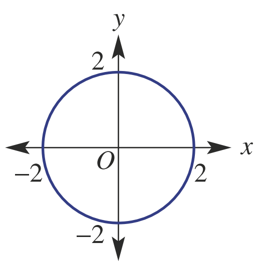
8 Find the coordinates of the points where $y = -\frac{2}{x}$ intersects these lines.

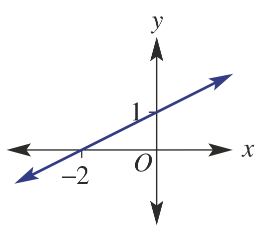
a $y = -3$ **b** $y = 4$ **c** $y = -\frac{1}{2}$ **d** $y = \frac{1}{3}$
e $y = -2x$ **f** $y = -8x$ **g** $y = -\frac{1}{2}x$ **h** $y = -x$

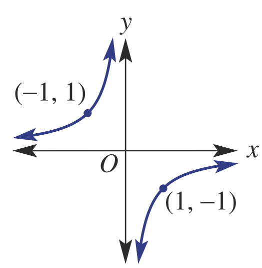
REASONING	9	9, 10	9–12

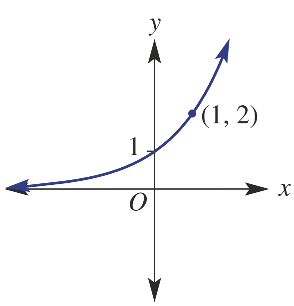
9 Match equations **a**–**f** with graphs **A**–**F**.

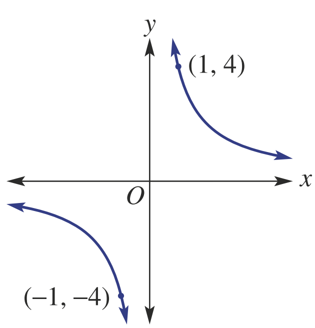
a $y = \frac{4}{x}$ **b** $y = -\frac{1}{x}$ **c** $y = 2^x$
d $y = \frac{1}{2}x + 1$ **e** $x^2 + y^2 = 4$ **f** $y = 3^{-x}$

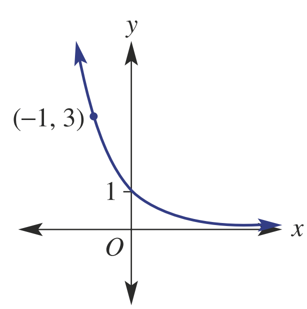
A


B


C


D


E


F


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

10 Is it possible for a line on a number plane to not intersect the graph of $y = \frac{1}{x}$? If so, give an example.

11 Write the missing word (*zero* or *infinity*) for these sentences.

a For $y = \frac{1}{x}$, when x approaches infinity, y approaches ________.

b For $y = \frac{1}{x}$, when x approaches negative infinity, y approaches ________.

c For $y = \frac{1}{x}$, when x approaches zero from the right, y approaches ________.

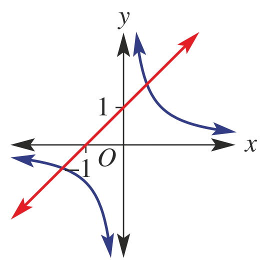
d For $y = \frac{1}{x}$, when x approaches zero from the left, y approaches ________.

12 Compare the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{2x}$. Describe the effect of the coefficient of x in $y = \frac{1}{2x}$.

ENRICHMENT: To intersect or not!	–	–	13

10A

13 The graphs of $y = \frac{1}{x}$ and $y = x + 1$ intersect at two points. To find the points we set:



$\frac{1}{x} = x + 1$

$1 = x(x + 1)$

$0 = x^2 + x - 1$

Using the quadratic formula, $x = \frac{-1 \pm \sqrt{5}}{2}$ and $y = \frac{1 \pm \sqrt{5}}{2}$.

a Find the exact coordinates of the intersection of $y = \frac{1}{x}$ and these lines.

i $y = x - 1$ **ii** $y = x - 2$ **iii** $y = x + 2$

b Try to find the coordinates of the intersection of $y = \frac{1}{x}$ and $y = -x + 1$. What do you notice?

What part of the quadratic formula confirms this result?

c Write down the equations of the two straight lines (which have gradient -1) that intersect $y = \frac{1}{x}$ only once.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

7K Direct and inverse proportion and other rates of change

Learning intentions

- To understand the relationship between two variables that are directly proportional or inversely proportional
- To understand that the shape of a graph shows how a variable and its rate of change varies
- To be able to find and use rules involving direct and inverse proportion
- To know how to model and interpret a situation using a distance–time graph

Two variables are said to be directly related if they are in a constant (i.e. unchanged) ratio. If two variables are in direct proportion, as one variable increases so does the other. For example, consider the relationship between speed and distance travelled in a given time. In 1 hour, a car can travel 50 km at 50 km/h, 100 km at 100 km/h, etc.

For two variables in inverse or indirect variation, as one variable increases the other decreases. For example, consider a beach house that costs $2000 per week to rent. As the number of people renting the house increases, then the cost per person decreases.

The shape of a graph shows how y varies and also how the rate of change of y (i.e. the gradient) varies. For example, when a car brakes to stop at traffic lights, it decelerates. The distance travelled per second is increasingly smaller until the car stops.

Renting a holiday house can be expensive. But, when sharing, the rent per person is inversely proportional to the number of people. However, the total rent increases in direct proportion to the length of stay.

LESSON STARTER Movement graphs

This is a whole class activity. Two volunteers are needed: the 'walker' who completes a journey between the front and back of the classroom and the 'grapher' who graphs the journey on the whiteboard.

1 The 'walker' will complete a variety of journeys but these are not stated to the class. For example:
   - Walk slowly from the front of the room, stop halfway for a few seconds and then walk steadily to the back of the room. Stop and then return to the front at a fast steady pace.
   - Start quickly from the back of the room and gradually slow down until stopping at the front.
   - Start slowly from the front and gradually increase walking speed all the way to the back. Stop for a few seconds and then return at a steady speed to the front of the room.

2 The 'grapher' draws a graph of distance versus time on the whiteboard at the same time as the 'walker' moves. The distance is measured from the front of the room. No numbers are needed.

3 The class members also each draw their own distance–time graph as the 'walker' moves.

4 After each walk, discuss how well the 'grapher' has modelled the 'walker's' movement.

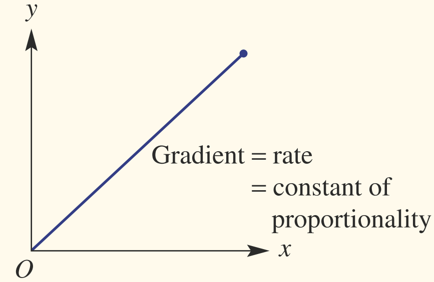
5 This activity can also be done in reverse. The 'grapher' draws a distance–time graph on the board and the 'walker' moves to match the graph. The class checks that the 'walker' is following the graph correctly.

ISBN 978-1-108-77290-7

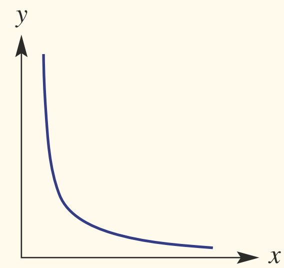
KEY IDEAS

- y varies directly with x if their relationship is

  $y = kx$ or $\frac{y}{x} = k$.
  - k is a constant, and is called the constant of proportionality.
  - The graph of y versus x gives a straight line that passes through the origin, O or $(0, 0)$, where k is the gradient.
  - We write: $y \propto x$ which means that $y = kx$.
  - We say: y varies directly as x or y is **directly proportional** to x.



- y varies inversely with x if their relationship is $y = \frac{k}{x}$ or $xy = k$.
  - The graph of y versus x gives a hyperbola.
  - We write: $y \propto \frac{1}{x}$, which means that $y = \frac{k}{x}$ or $xy = k$.
  - We say: y varies inversely as x or y is **inversely proportional** to x.

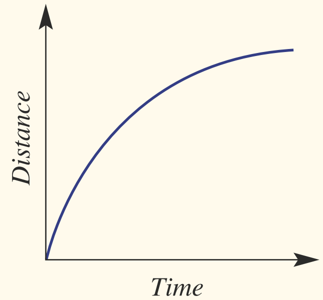


- The shape of a graph shows how both y and the rate of change of y (i.e. the gradient) varies.
  - If y increases as x increases, then the rate of change (the gradient) is positive.
  - If y decreases as x increases, then the rate of change (the gradient) is negative.
  - If y does not change as x increases, then the rate of change (the gradient) is zero.
  - Straight lines have a constant rate of change. There is a fixed change in y for each unit increase in x.
  - Curves have a varying rate of change. The change in y varies for each unit increase in x.
  - Analysing a graph and describing how both y and the rate of change of y varies allows us to check whether a given graph models a situation accurately.

  For example, these distance–time graphs show various journeys from 'home' (distance = 0 at home).

Journey A

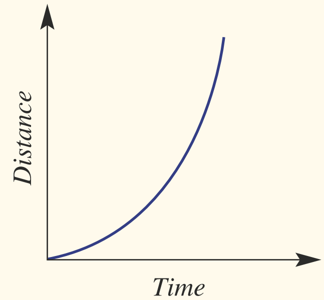
Decelerating away from home.



Distance from home is increasing at a decreasing rate.

Journey B

Accelerating away from home.



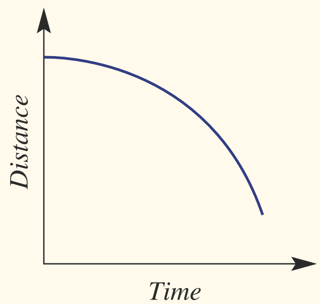
Distance from home is increasing at an increasing rate.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Journey C

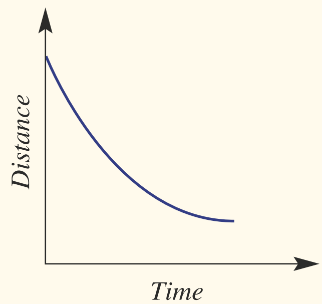
Accelerating towards home.



Distance from home is decreasing at an increasing rate.

Journey D

Decelerating towards home.



Distance from home is decreasing at a decreasing rate.

BUILDING UNDERSTANDING

1 For each pair of variables, state whether they are in direct or inverse proportion or neither.

a The number of *hours* worked and *wages* earned at a fixed rate per hour.

b The *volume* of remaining fuel in a car and the *cost* of filling the fuel tank.

c The *speed* and *time* taken to drive a certain distance.

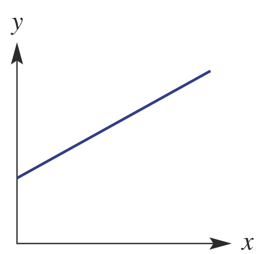
d The *size* of a movie file and the *time* for downloading it to a computer at a constant rate of kB/s.

e The *cost* of a taxi ride and the *distance* travelled. The cost includes flag fall (i.e. a starting charge) and a fixed $/km.

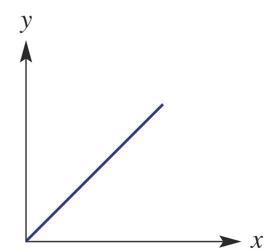
f The *rate* of typing in words per minute and the *time* needed to type a particular assignment.

2 State the main features of each graph and whether it shows direct proportion or inverse (i.e. indirect) proportion or neither

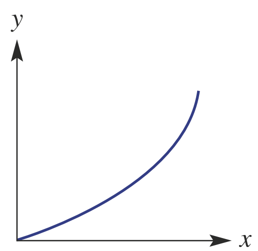
a



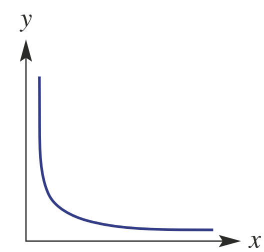
b



c



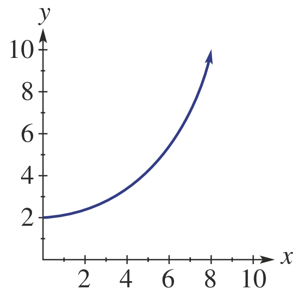
d



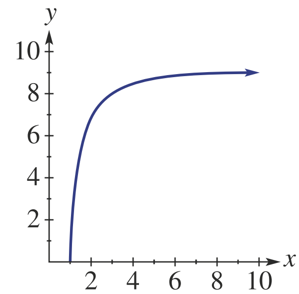
3 For each graph below, copy and complete this sentence by supplying the words *increasing* or *decreasing*.

y is ______________ at a ______________ rate.

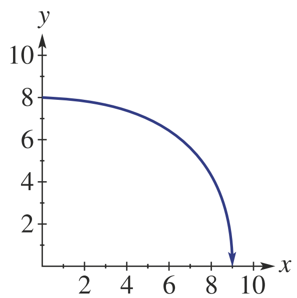
a



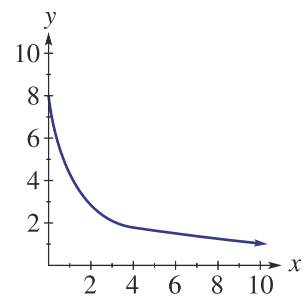
b



c



d



Example 25 Finding and using a direct variation rule

If m is directly proportional to h and $m = 90$ when $h = 20$, determine:

a the relationship between m and h

b m when $h = 16$

c h when $m = 10$

SOLUTION	EXPLANATION
a $m \propto h$	First, write the variation statement.
$m = kh$	Write the equation, including k.
$90 = k \times 20$	Substitute $m = 90$ and $h = 20$.
$k = \frac{90}{20}$	Divide both sides by 20 to find k.
$k = \frac{9}{2}$	Simplify.
$\therefore m = \frac{9}{2}h$	Write the rule using the value of k found.
b $m = \frac{9}{2} \times 16$	Substitute $h = 16$
$\therefore m = 72$	Simplify to find m.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

c $10 = \frac{9}{2} \times h$ Substitute $m = 10$.

$10 \times \frac{2}{9} = h$ Solve for h.

$\therefore h = \frac{20}{9}$

Now you try

If a is directly proportional to b and $a = 120$ when $b = 80$, determine:

a the relationship between a and b

b a when $b = 40$

c b when $a = 12$

Example 26 Finding and using an inverse proportion rule

If x and y are inversely proportional and $y = 6$ when $x = 10$, determine:

a the constant of proportionality, k, and write the rule

b y when $x = 15$

c x when $y = 12$

SOLUTION	EXPLANATION
a $y \propto \frac{1}{x}$	Write the variation statement.
$y = \frac{k}{x}$	Write the equation, including k.
$k = xy$	The constant of proportionality, $k = xy$:
$k = 10 \times 6$	Use $x = 10$, $y = 6$.
$k = 60$	
$y = \frac{60}{x}$	Substitute $k = 60$ into the rule $y = \frac{k}{x}$.
b $y = \frac{60}{15}$	Substitute $x = 15$ into the rule $y = \frac{60}{x}$.
$y = 4$	
c $12 = \frac{60}{x}$	Substitute $y = 12$ into the rule $y = \frac{60}{x}$.
$12x = 60$	Multiply both sides by x.
$x = 5$	Divide both sides by 12.

Now you try

If m and n are inversely proportional and $n = 10$ when $m = 6$, determine:

a the constant of proportionality, k, and write the rule

b n when $m = 12$

c m when $n = 30$

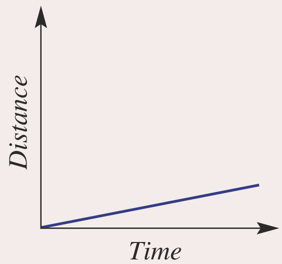
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

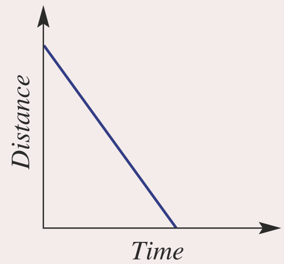
Example 27 Matching descriptions to distance–time graphs

These distance–time graphs show various journeys, each with distance measured from 'home' (i.e distance = 0). For each graph, select and copy one correct description from each category of how the distance from home, the gradient and the speed are varying.

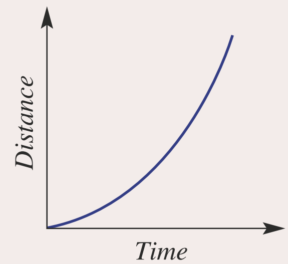
A



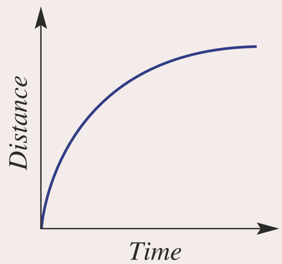
B



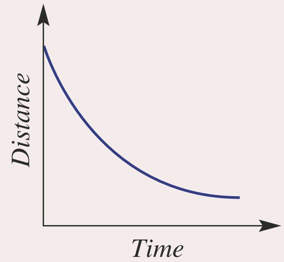
C



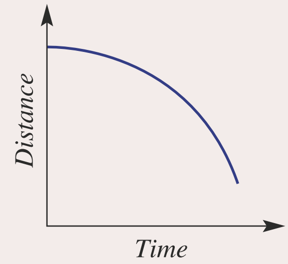
D



E



F



Distance from home
Increasing distance from home
Decreasing distance from home
Fixed distance from home

Gradient of graph
Positive constant gradient
Positive varying gradient
Negative constant gradient
Negative varying gradient
Zero gradient

Speed
Stationary
Lower constant speed
Higher constant speed
Decreasing speed, decelerating
Increasing speed, accelerating

SOLUTION	EXPLANATION
Graph A	
Increasing distance from home. Positive constant gradient. Lower constant speed.	Distance from home increases at a constant rate (speed), so an upward sloping straight line has a positive constant gradient. Lines that are less steep mean a lower constant speed.
Graph B	
Decreasing distance from home. Constant negative gradient. Higher constant speed.	Distance from home decreases at a constant rate (speed), so a downward sloping straight line has a constant negative gradient. Steeper lines mean a higher constant speed.
Graph C	
Increasing distance from home. Positive varying gradient. Increasing speed, accelerating.	Distance from home is increasing at an increasing rate, so the curve has a positive varying gradient. As the curve becomes steeper the rate of change (speed) increases (i.e. more distance in a given time). Increasing speed indicates accelerating.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Graph D

Increasing distance from home.
Positive varying gradient.
Decreasing speed, decelerating.

Distance from home is increasing at a decreasing rate, so the curve has a positive varying gradient. As the curve becomes flatter the rate of change (speed) decreases (i.e. less distance in a given time). Decreasing speed indicates decelerating.

Graph E

Decreasing distance from home.
Negative varying gradient.
Decreasing speed, decelerating.

Distance from home is decreasing at a decreasing rate, so the curve has a negative varying gradient. As the curve becomes flatter the rate of change (speed) decreases (i.e. less distance in a given time). Decreasing speed indicates decelerating.

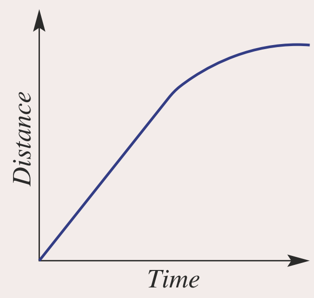
Graph F

Decreasing distance from home.
Negative varying gradient.
Increasing speed, accelerating.

Distance from home is decreasing at an increasing rate, so the curve has a negative varying gradient. As the curve becomes steeper the rate of change (speed) increases (i.e. more distance in a given time). Increasing speed indicates accelerating.

Now you try

Describe how the distance from home, the gradient and the speed are varying for this graph.



Exercise 7K

FLUENCY	1–4	2–5	2, 3, 5, 6

Example 25

1 a If m is directly proportional to h and $m = 50$ when $h = 10$, determine:
- **i** the relationship between m and h
- **ii** m when $h = 6$
- **iii** h when $m = 8$

Example 26

b If x and y are inversly proportional and $y = 4$ when $x = 6$, determine:
- **i** the constant of proportionality, k, and write the rule
- **ii** y when $x = 8$
- **iii** x when $y = 12$

Example 25

2 a If p is directly proportional to q and $p = 40$ when $q = 10$, determine:
- **i** the relationship between p and q
- **ii** p when $q = 15$
- **iii** q when $p = 100$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b If p is directly proportional to q and $p = 100$ when $q = 2$, determine:
- **i** the relationship between p and q
- **ii** p when $q = 15$
- **iii** q when $p = 200$

Example 26

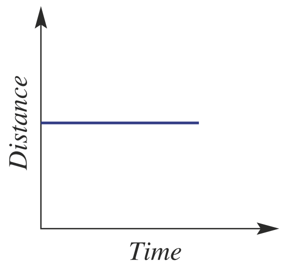
3 **a** If x and y are inversely proportional and $y = 12$ when $x = 6$, determine:
- **i** the constant of proportionality, k, and write the rule
- **ii** y when $x = 36$
- **iii** x when $y = 3$

b If y varies inversely with x and $y = 10$ when $x = 5$, determine:
- **i** the constant of proportionality, k, and write the rule
- **ii** y when $x = 100$
- **iii** x when $y = 100$

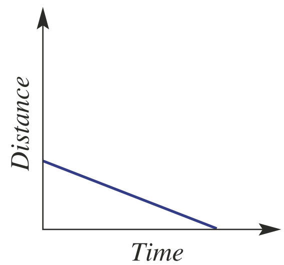
Example 27

4 The distance–time graphs below show various journeys, each with distance measured from 'home' (i.e distance $= 0$). For each graph, select and copy one correct description from each category below of how the distance from home, the gradient and the speed are varying.

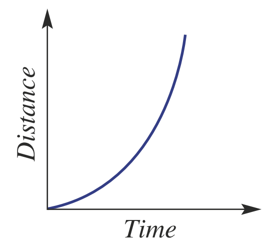
a



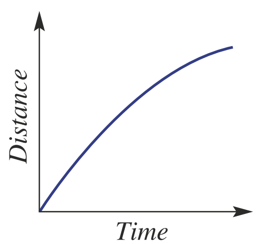
b



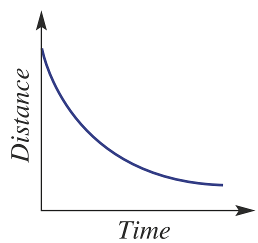
c



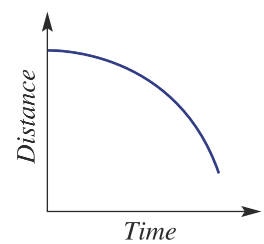
d



e



f



Distance from home
Increasing distance from home
Decreasing distance from home
Fixed distance from home

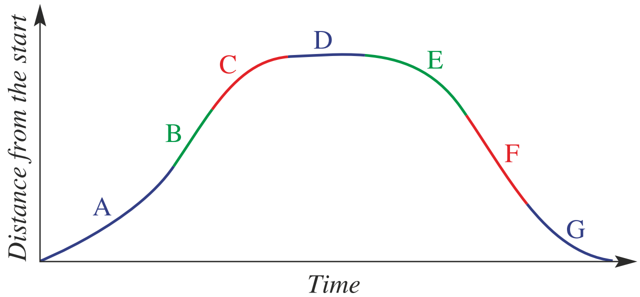
Gradient of graph
Positive constant gradient
Positive varying gradient
Negative constant gradient
Negative varying gradient
Zero gradient

Speed
Stationary
Lower constant speed
Higher constant speed
Decreasing speed, decelerating
Increasing speed, accelerating

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

5 From the lists below, select and copy the correct description for the rate and speed of each segment of this distance–time graph. The rate of change of distance with respect to time is the gradient.



Rate of change of distance with respect to time	**Speed**
Positive constant rate of change	Stationary
Positive varying rate of change	Constant speed
Negative constant rate of change	Decreasing speed, decelerating
Negative varying rate of change	Increasing speed, accelerating
Zero rate of change	

6 The amount that a farmer earns from selling wheat is in direct proportion to the number of tonnes harvested.

a Find the constant of proportionality, k, given that a farmer receives \$8296 for 34 tonnes of wheat.
b Write the direct proportion equation relating selling price, P, and number of tonnes, n.
c Calculate the selling price of 136 tonnes of wheat.
d Calculate the number of tonnes of harvested wheat that is sold for \$286 700.

PROBLEM-SOLVING	7, 8	7–9	8, 9

7 A 30-seater school bus costs 20 students \$3.70 each to hire for a day. The overall cost of the bus remains the same regardless of the number of students.

a Write a relationship between the cost per student (c) and the number of students (s)
b If only 15 students use the bus, what would be their individual cost, to the nearest cent?
c What is the minimum a student would be charged, to the nearest cent?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

8 For each relationship described below:

i write a suitable equation.

ii sketch the graph, choosing appropriate values for the initial and final points on the graph.

a The distance that a car travels in 1 hour is directly proportional to the speed of the car. The roads have a 100 km/h speed limit.

b The cost per person of hiring a yacht is inversely proportional to the number of people sharing the total cost. A yacht in the Whitsunday Islands can be hired for $320 per day for a maximum of eight people on board.

c There is a direct proportional relationship between a measurement given in metric units and in imperial units. A weight measured in pounds is 2.2 times the value of the weight in kilograms.

d The time taken to type 800 words is inversely proportional to the typing speed in words per minute.

9 Sketch a population–time graph from each of these descriptions

a A population of bilbies is decreasing at a decreasing rate.

b A population of Tasmanian devils is decreasing at an increasing rate.

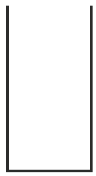
c A population of camels is increasing at a decreasing rate.

d A population of rabbits is increasing at an increasing rate.

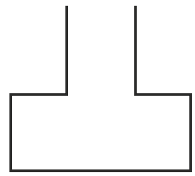
REASONING	10	10, 11	10–12

10 Water is poured at a constant rate into each of the containers shown below. For each container, draw a graph of the water depth versus time. Numbers are not required

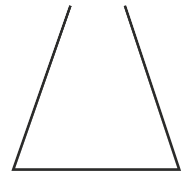
a



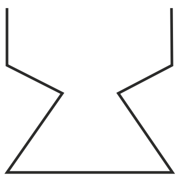
b



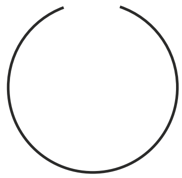
c



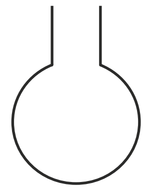
d



e

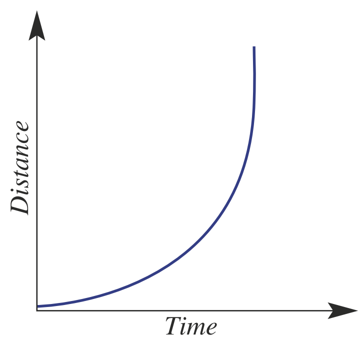


f

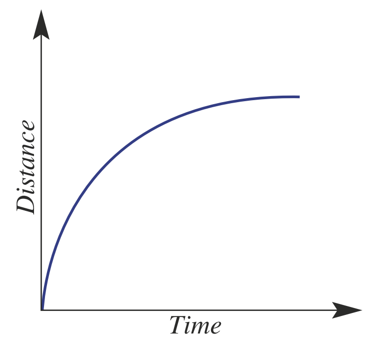


11 Which of the following graphs don’t match the journey description correctly or are not physically possible? For each graph, explain the feature that is incorrect and redraw it correctly. Distances are to be interpreted as distance from the original starting point (displacement)

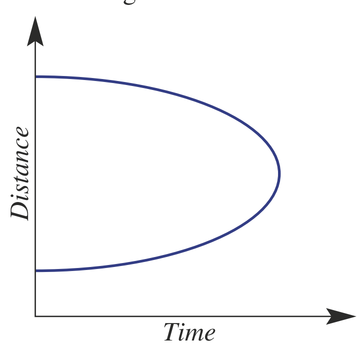
a Stopped, then accelerating to a very high speed.



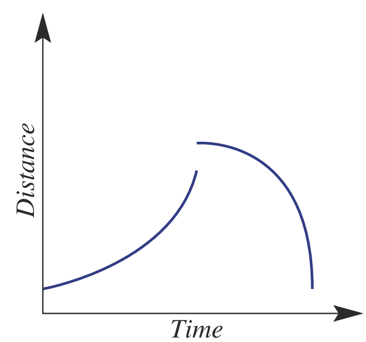
b Moving at very high speed and decelerating to a stop.



c Accelerating, changing direction, then decelerating.



d Accelerating, changing direction, then decelerating.

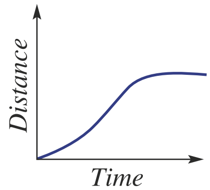


ISBN 978-1-108-77290-7

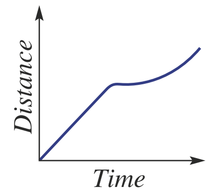
Cambridge University Press
Updated June 2022

12 Match each distance–time graph (**A–D**) with the correct journey described (**a–d**). Give reasons for each choice, describing how the changing distance and varying rate relates to the movement of the object.

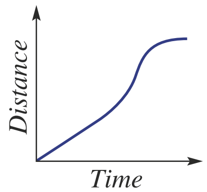
A



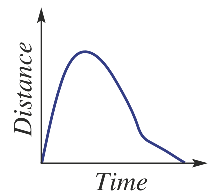
B



C



D



a A soccer player runs at a steady speed across the field, stops briefly to avoid a tackle and then accelerates farther away.

b A rocket stage 1 booster accelerates to huge speed, then detaches and quickly decelerates, then accelerates as it falls towards Earth, finally a parachute opens and it slows, falling to Earth at a steady speed.

c A motorbike moves at a steady speed, then accelerates to pass a car, then brakes and decelerates, coming to a stop at traffic lights.

d A school bus accelerates away from the bus stop, then moves at a steady speed and then decelerates and stops as it arrives at the next bus stop.

ENRICHMENT: Creating distance–time graphs	–	–	13

13 Work in small groups to develop distance–time graphs from recorded data. Equipment: 100 m tape measure, stopwatch, recording materials, video camera.

a Determine a suitable method for recording the distance a student has moved after every 5 seconds over a 30 second period.

b Select a variety of activities for the moving student to do in each 30 second period. For example, walking slowly, running fast, starting slowly then speeding up, etc.

c Record and graph distance versus time for each 30 second period.

d For each graph, using sentences with appropriate vocabulary, describe how the distance and rate of change of distance is varying.

e Analyse how accurately each graph has modelled that student's movement.

ISBN 978-1-108-77290-7

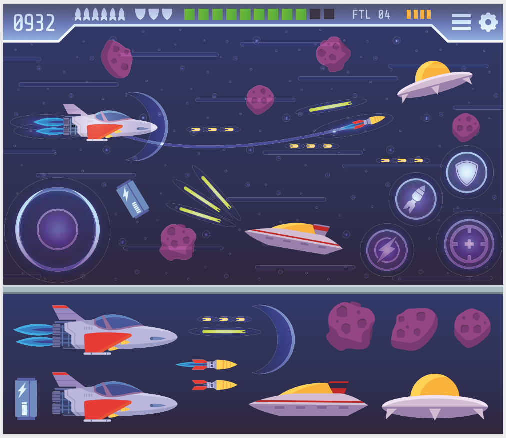
Cambridge University Press
Updated June 2022

7L Further transformations of graphs 10A

Learning intentions

- To understand the effect of translations on graphs of circles, exponentials and rectangular hyperbolas
- To be able to determine the centre of a circle and asymptotes of exponentials and hyperbolas from the equation of the transformed graph
- To be able to sketch graphs using transformations

Earlier in this chapter we considered a wide range of transformations of parabolas but a limited number of transformations of circles, exponentials and hyperbolas. We will now look more closely at translations of these relations and the key features of their graphs.



Transformation maths is the basis for coding animations in digital games. Translations, reflections and rotations will move a figure, object or graph without changing its shape. However, dilations change a shape by shrinking or enlargement.

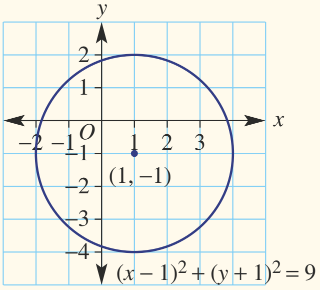
LESSON STARTER Translations, translations, translations

Use technology to assist in the discussion of these questions.

- How does the graph of $(x-1)^2+(y+2)^2=9$ compare with that of $x^2+y^2=9$?
- What is the effect of h, k and r in $(x-h)^2+(y-k)^2=r^2$?
- How does the graph of $y=2^{x-2}+1$ compare with that of $y=2^x$?
- What is the effect of h and k in $y=2^{x-h}+k$?
- How does the graph of $y=\dfrac{1}{x+2}-1$ compare with that of $y=\dfrac{1}{x}$?
- What is the effect of h and k in $y=\dfrac{1}{x-h}+k$?

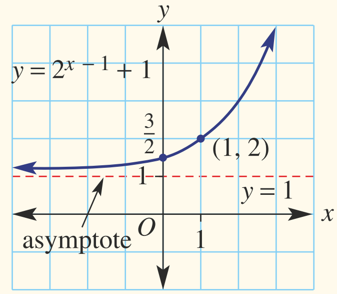
KEY IDEAS

- The equation of a **circle** in standard form is $(x-h)^2+(y-k)^2=r^2$.
  - (h, k) is the centre.
  - r is the radius.



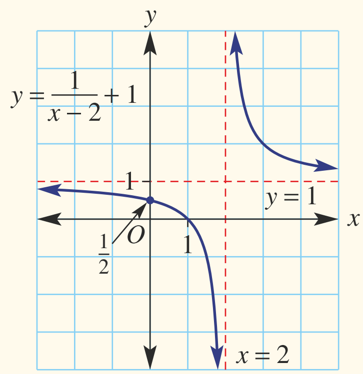
- For the graph of the **exponential** equation $y = a^{x-h} + k$ the graph of $y = a^x$ is:
  - translated h units to the right
  - translated k units up.

  The equation of the asymptote is $y = k$.



- For the graph of the **hyperbola** $y = \frac{1}{x-h} + k$ the graph of $y = \frac{1}{x}$ is:
  - translated h units to the right
  - translated k units up.

  The asymptotes are $x = h$ and $y = k$.



BUILDING UNDERSTANDING

1 Choose the word *left, right, up* or *down* to complete each sentence.

a The graph of $y = 2^x - 3$ is the translation of the graph of $y = 2^x$ _____ by 3 units.
b The graph of $y = 2^{x-4}$ is the translation of the graph of $y = 2^x$ _____ by 4 units.
c The graph of $y = \frac{1}{x+2}$ is the translation of the graph of $y = \frac{1}{x}$ _____ by 2 units.
d The graph of $y = \frac{1}{x} - 6$ is the translation of the graph of $y = \frac{1}{x}$ _____ by 6 units.
e The graph of $(x+3)^2 + y^2 = 1$ is the translation of the graph of $x^2 + y^2 = 1$ _____ by 3 units.
f The graph of $x^2 + (y-2)^2 = 1$ is the translation of the graph of $x^2 + y^2 = 1$ _____ by 2 units.

2 What is the value of k in the equation of the asymptote $y = k$ for the following?

a $y = 2^{x-1} + 3$ **b** $y = 3^{x+2} - 1$ **c** $y = -5^{x+2} + 4$

3 What are the values of h and k in the asymptotes $x = h$ and $y = k$ for the following hyperbolas?

a $y = \frac{1}{x} + 2$ **b** $y = \frac{1}{x-3}$ **c** $y = \frac{1}{x+2} - 1$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 28 Sketching with transformations

Sketch the graphs of the following relations. Label important features.

a $(x-2)^2+(y+3)^2=9$ **b** $y=2^{x+2}-3$ **c** $y=\dfrac{1}{x+1}+2$

SOLUTION

a $(x-2)^2+(y+3)^2=9$

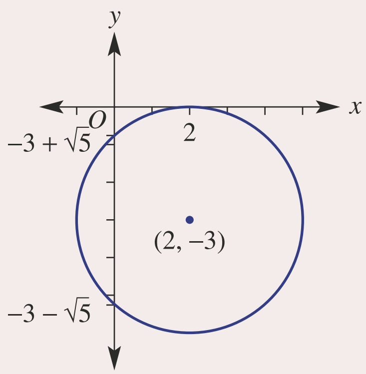
Centre $(2, -3)$

Radius $= 3$

x-intercept is 2.

y-intercepts at $x=0$:

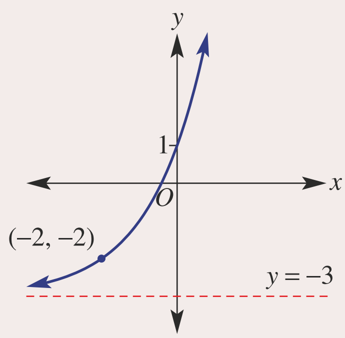
$$
\begin{aligned}
(0-2)^2+(y+3)^2&=9\\
4+(y+3)^2&=9\\
(y+3)^2&=5\\
y+3&=\pm\sqrt{5}\\
y&=-3\pm\sqrt{5}
\end{aligned}
$$



b $y=2^{x+2}-3$

y-intercept is 1.

Asymptote is $y=-3$.



EXPLANATION

For $(x-h)^2+(y-k)^2=r^2$, (h, k) is the centre and r is the radius.

Radius is 3 so, from centre $(2, -3)$, point on circle is $(2, 0)$.

Find the y-intercepts by substituting $x=0$ and solving for y.

Label centre and axes intercepts.

For $y=2^{x-h}+k$, $y=k$ is the equation of the asymptote.

Substitute $x=0$ to find the y-intercept. (We will leave the x-intercept until we study logarithms later.)

At $x=0$, $y=2^2-3=1$.

$(0, 1)$ in $y=2^x$ is translated 2 units to the left and 3 units down to $(-2, -2)$.

Alternatively, substitute $x=1$ to label a second point.

Continued on next page

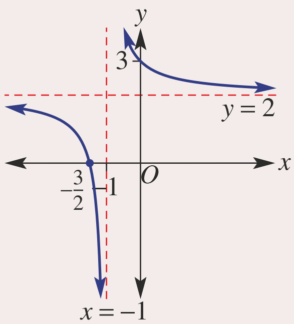
c $y = \dfrac{1}{x+1} + 2$

At $x = 0$, $y = \dfrac{1}{1} + 2 = 3$.

y-intercept is 3.

x-intercept is $-\dfrac{3}{2}$.

Asymptotes: $x = -1$, $y = 2$



For $y = \dfrac{1}{x-h} + k$, $h = -1$ and $k = 2$, so the asymptotes are $x = -1$ and $y = 2$.

Substitute to find the x- and y-intercepts.

x-intercepts ($y = 0$):

At $y = 0$, $0 = \dfrac{1}{x+1} + 2$

$$-2 = \frac{1}{x+1}$$

$$x + 1 = -\frac{1}{2}$$

$$x = -\frac{3}{2}$$

Now you try

Sketch the graphs of the following relations. Label important features.

a $(x + 1)^2 + (y + 2)^2 = 9$ **b** $y = 2^{x-1} - 1$ **c** $y = \dfrac{1}{x-1} - 2$

Exercise 7L

FLUENCY	1, 2–4(1/2)	2–4 (1/2)	2–4 (1/3)

Example 28

1 Sketch the graphs of the following relations. Label important features.

a $(x + 2)^2 + (y - 1)^2 = 4$ **b** $y = 2^{x+1} - 1$ **c** $y = \dfrac{1}{x-2} - 1$

Example 28a

2 Sketch the graph of the following circles. Label the coordinates of the centre and find the x- and y-intercepts, if any.

a $(x - 3)^2 + (y + 1)^2 = 1$ **b** $(x + 2)^2 + (y - 3)^2 = 4$ **c** $(x - 1)^2 + (y + 3)^2 = 25$

d $(x + 3)^2 + (y - 2)^2 = 25$ **e** $(x + 2)^2 + (y - 1)^2 = 9$ **f** $x^2 + (y - 4)^2 = 36$

g $(x + 1)^2 + y^2 = 9$ **h** $(x - 2)^2 + (y - 5)^2 = 64$ **i** $(x + 3)^2 + (y - 1)^2 = 5$

Example 28b

3 Sketch the graph of these exponentials. Label the asymptote and the y-intercept.

a $y = 2^x - 2$ **b** $y = 2^x + 1$ **c** $y = 2^x - 5$

d $y = 2^{x-1}$ **e** $y = 2^{x+3}$ **f** $y = 2^{x+1}$

g $y = 2^{x-1} + 1$ **h** $y = 2^{x+2} - 3$ **i** $y = 2^{x-3} - 4$

Example 28c

4 Sketch the graph of these hyperbolas. Label the asymptotes and find the x-and y-intercepts.

a $y = \dfrac{1}{x} + 2$ **b** $y = \dfrac{1}{x} - 1$ **c** $y = \dfrac{1}{x+3}$

d $y = \dfrac{1}{x-2}$ **e** $y = \dfrac{1}{x+1} + 1$ **f** $y = \dfrac{1}{x-1} - 3$

g $y = \dfrac{1}{x-3} + 2$ **h** $y = \dfrac{1}{x+4} - 1$ **i** $y = \dfrac{1}{x-5} + 6$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

PROBLEM-SOLVING	5(1/2), 6	5(1/2), 6, 7	5(1/3), 6, 7, 8(1/2)

5 The graphs of these exponentials involve a number of transformations. Sketch their graphs, labelling the y-intercept and the equation of the asymptote.

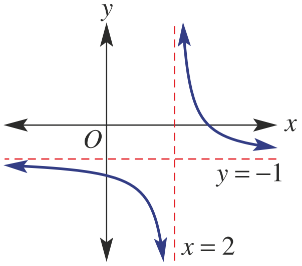
a $y = -2^x + 1$ **b** $y = -2^x - 3$ **c** $y = -2^{x+3}$
d $y = -2^{x-2}$ **e** $y = -2^{x+1} - 1$ **f** $y = -2^{x+2} + 5$

6 Sketch these hyperbolas, labelling asymptotes and intercepts.

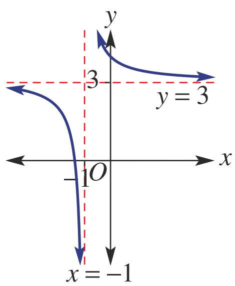
a $y = \dfrac{-1}{x+1} + 2$ **b** $y = \dfrac{-2}{x+2} - 1$ **c** $y = \dfrac{-2}{x-3} - 2$

7 The following hyperbolas are of the form $y = \dfrac{1}{x-h} + k$. Write the rule for each graph.

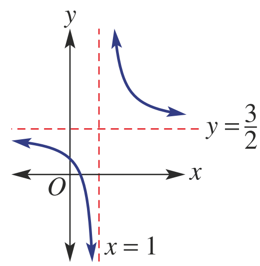
a



b



c



8 Find the coordinates of the intersection of the graphs of these equations.

a $y = \dfrac{1}{x+1}$ and $y = x + 2$ **b** $y = \dfrac{1}{x-2} + 1$ and $y = x + 3$
c $y = \dfrac{-1}{x+2} - 3$ and $y = -2x - 1$ **d** $(x-1)^2 + y^2 = 4$ and $y = 2x$
e $(x+2)^2 + (y-3)^2 = 16$ and $y = -x - 3$ **f** $x^2 + (y+1)^2 = 10$ and $y = \dfrac{1}{3}x - 1$

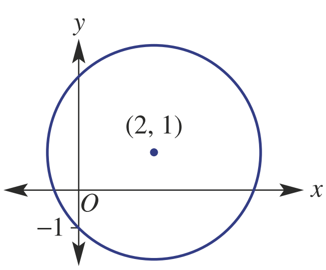
REASONING	9	9, 10	10, 11

9 A circle has equation $(x-3)^2 + (y+2)^2 = 4$. Without sketching a graph, state the minimum and maximum values for:

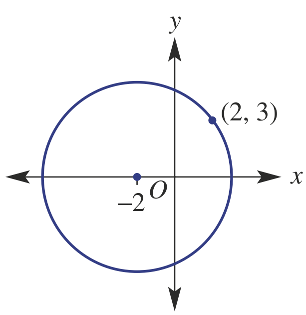
a x **b** y

10 Find a rule for each graph.

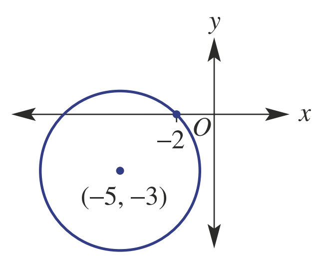
a



b



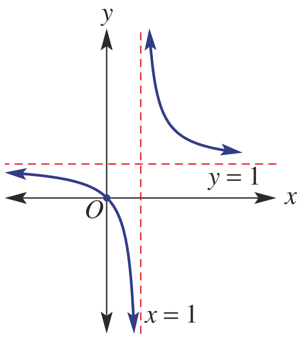
c



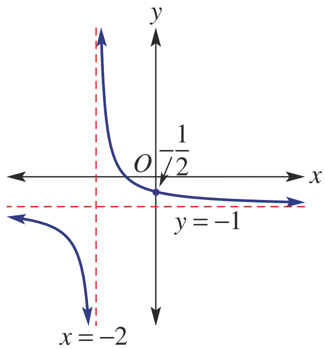
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

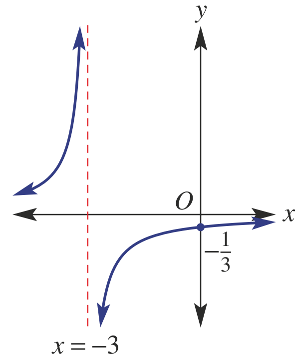
d



e



f



11 Explain why the graphs of the following pairs of relations do not intersect.

a $y = \frac{1}{x}$ and $y = -x$

b $(x-1)^2 + (y+2)^2 = 4$ and $y = 1$

c $y = 2^{x-1} + 3$ and $y = x - 3$

d $y = \frac{2}{x+3} - 1$ and $y = \frac{1}{3x}$

ENRICHMENT: Circles by completing the square	–	–	12(1/2), 13

12 By expanding brackets of an equation in the standard form of a circle, we can write:

$$(x-1)^2 + (y+2)^2 = 4 \qquad (1)$$

$$x^2 - 2x + 1 + y^2 + 4y + 4 = 4$$

$$x^2 - 2x + y^2 + 4y + 1 = 0 \qquad (2)$$

Note that in equation (2) it is not obvious that the centre is (1, −2) and that the radius is 2. It is therefore preferable to write the equation of a circle in standard form (i.e. as in equation (1)).

If given an equation such as (2), we can complete the square in both x and y to write the equation of the circle in standard form.

$$\begin{aligned} x^2 - 2x + y^2 + 4y + 1 &= 0 \\ (x-1)^2 - 1^2 + (y+2)^2 - 2^2 + 1 &= 0 \\ (x-1)^2 - 1 + (y+2)^2 - 4 + 1 &= 0 \\ (x-1)^2 + (y+2)^2 &= 4 \end{aligned}$$

The radius is 2 and centre (1, −2).

Write these equations of circles in standard form. Then state the coordinates of the centre and the radius.

a $x^2 + 4x + y^2 - 2y + 1 = 0$

b $x^2 + 8x + y^2 + 10y + 5 = 0$

c $x^2 - 6x + y^2 - 4y - 3 = 0$

d $x^2 - 2x + y^2 + 6y - 5 = 0$

e $x^2 + 10x + y^2 + 8y + 17 = 0$

f $x^2 + 6x + y^2 + 6y = 0$

g $x^2 + 3x + y^2 - 6y + 4 = 0$

h $x^2 + 5x + y^2 - 4y - 2 = 0$

i $x^2 - x + y^2 + 3y + 1 = 0$

j $x^2 - 3x + y^2 - 5y - 4 = 0$

13 Give reasons why $x^2 + 4x + y^2 - 6y + 15 = 0$ is not the equation of a circle.

ISBN 978-1-108-77290-7

Up for a challenge? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.

1 Solve these inequalities for x.
  a $6x^2 + x - 2 \leqslant 0$
  b $12x^2 + 5x - 3 > 0$
  c $x^2 - 7x + 2 < 0$

2 Sketch the region defined by $x^2 - 4x + y^2 - 6y - 3 \leqslant 0$.

3 Prove the following.
  a The graphs of $y = x - 4$ and $y = x^2 - x - 2$ do not intersect.
  b The graphs of $y = -3x + 2$ and $y = 4x^2 - 7x + 3$ touch at one point.
  c The graphs of $y = 3x + 3$ and $y = x^2 - 2x + 4$ intersect at two points.

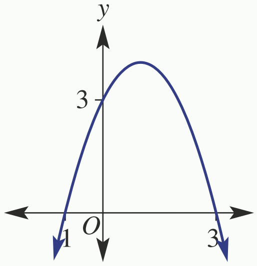
4 Prove that there are no points (x, y) that satisfy $x^2 - 4x + y^2 + 6y + 15 = 0$.

5 For what values of k does the graph of $y = kx^2 - 2x + 3$ have:
  a one x-intercept?
  b two x-intercepts?
  c no x-intercepts?

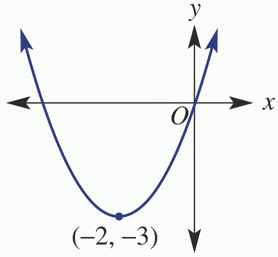
6 For what values of k does the graph of $y = 5x^2 + kx + 1$ have:
  a one x-intercept?
  b two x-intercepts?
  c no x-intercepts?

7 Find the rules for these parabolas.

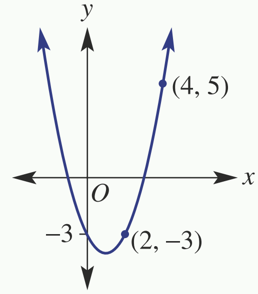
a



b



c



8 A graph of $y = ax^2 + bx + c$ passes through the points $A(0, -8)$, $B(-1, -3)$ and $C(1, -9)$. Use your knowledge of simultaneous equations to find the values of a, b and c and, hence, find the turning point for this parabola, stating the answer using fractions.

9 Determine the maximum vertical distance between these two parabolas at any given x-value between the points where they intersect:
$y = x^2 + 3x - 2$ and $y = -x^2 - 5x + 10$

10 Two points, P and Q, are on the graph of $y = x^2 + x - 6$. The origin $(0, 0)$ is the midpoint of the line segment PQ. Determine the exact length of PQ.

11 A parabola, $y_1 = (x - 1)^2 + 2$, is reflected in the x-axis to become y_2. Now y_1 and y_2 are each translated 2 units horizontally but in opposite directions, forming y_3 and y_4. If $y_5 = y_3 + y_4$, sketch the graphs of the possible equations for y_5.

ISBN 978-1-108-77290-7

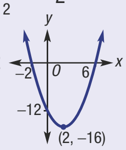
Cambridge University Press
Updated June 2022

Chapter summary

Parabolas

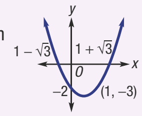
Sketching $y = x^2 + bx + c$

e.g. $y = x^2 - 4x - 12$
y-intercept: $x = 0$, $y = -12$
x-intercepts: $y = 0$, $x^2 - 4x - 12 = 0$
Factorise $(x - 6)(x + 2) = 0$
Null Factor Law
$x - 6 = 0$ or $x + 2 = 0$
$x = 6$ or $x = -2$
Turning point: halfway between x-intercepts
$$x = \frac{6 + (-2)}{2} = 2$$
Substitute $x = 2$:
$y = 2^2 - 4(2) - 12$
$= -16$
$(2, -16)$ minimum



Sketching $y = a(x - h)^2 + k$

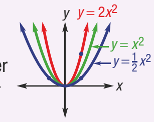
Any quadratic can be written in turning point form by completing the square.
e.g. sketch $y = (x - 1)^2 - 3$
Turning point $(1, -3)$ minimum
y-intercept: $x = 0$, $y = 1 - 3$
$= -2$



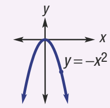
x-intercepts: $y = 0$
$0 = (x - 1)^2 - 3$
$3 = (x - 1)^2$
$\pm\sqrt{3} = x - 1$
$\therefore\ x = 1 + \sqrt{3},\ 1 - \sqrt{3}$
(alternatively use the Null Factor Law)

Transformations and turning point form

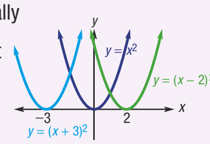
$y = ax^2$ dilates graph compared to $y = x^2$
$0 < a < 1$ graph is wider
$a > 1$ graph is narrower



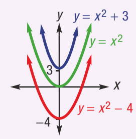
$a < 0$ graph is inverted (reflected in x-axis)



$y = (x - h)^2$ translates $y = x^2$ h units horizontally
$h > 0$ move right
$h < 0$ move left



$y = x^2 + k$ translates $y = x^2$ k units vertically
$k > 0$ move up
$k < 0$ move down

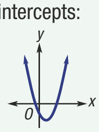


Turning point form
$y = a(x - h)^2 + k$ has turning point at (h, k)

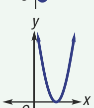
Quadratic formula

For $y = ax^2 + bx + c$, the quadratic formula can be used to find x-intercepts.
i.e. if $ax^2 + bx + c = 0$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
The discriminant $\Delta = b^2 - 4ac$ tells how many x-intercepts:

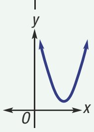
$\Delta > 0$
two x-intercepts



$\Delta = 0$
one x-intercept

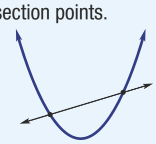


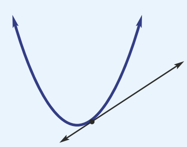
$\Delta < 0$
no x-intercepts

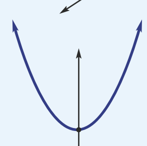


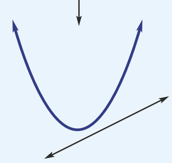
Lines and parabolas (10A)

Using substitution we can solve simultaneously to find 0, 1 or 2 intersection points.

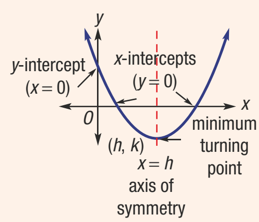


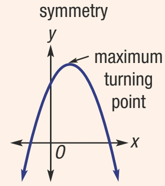






Key features





ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Relation *vs* function

Any collection of points is a relation. A function is a special relation. $y = x^2$ is an example of a function because every x-value gives only one y-value.

Relations and functions (10A)

Domain

The set of all allowable x-values

Range

The set of all resulting y-values

Function notation

$y = x^2$ can be written as $f(x) = x^2$.

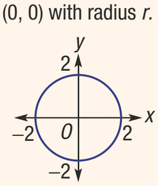
Vertical line test

A relation is a function when any vertical line drawn through the graph cuts it only once.

Other graphs

Circles

$x^2 + y^2 = r^2$ is a circle centred at (0, 0) with radius r.



$x^2 + y^2 = 4$
$r^2 = 4$
$\therefore r = 2$

The circle rule can be rearranged to
$y = \pm\sqrt{r^2 - x^2}$ or $x = \pm\sqrt{r^2 - y^2}$

Circle with equation **(10A)**
$(x - h)^2 + (y - k)^2 = r^2$
is centred at (h, k) with radius r.

Direct and inverse proportion

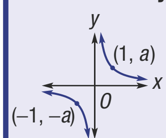
Direct variation: $y = kx$
- Both variables increase or decrease together.
- Graph is a straight line through (0, 0).

Inverse (indirect) variation: $y = \frac{k}{x}$ or $xy = k$
- If one variable increases, the other decreases.
- Graph is a hyperbola shape.

Description of graphs:
- y is increasing or decreasing or constant.
- Rate of change is positive or negative or zero.
- Rate of change is increasing or decreasing or constant.

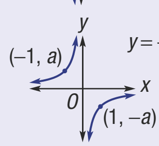
Hyperbolas

$y = \frac{a}{x}$

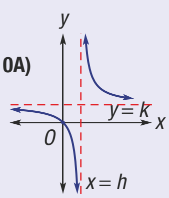


Asymptotes
$x = 0$ and $y = 0$

$y = -\frac{a}{x}$ (reflect in x- or y-axis)



$y = \frac{a}{x - h} + k$ **(10A)**
find asymptotes and intercepts to sketch

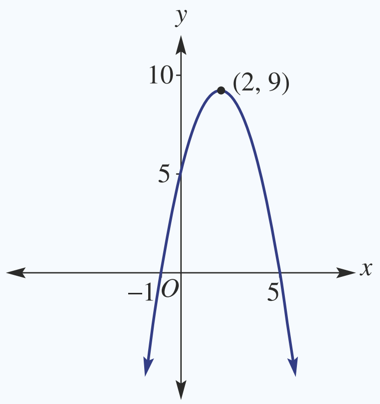


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter checklist

Chapter checklist: Success criteria

		✔
7A	**1. I can identify the key features of a parabola.** e.g. For the graph shown determine the turning point (and whether it's a maximum or a minimum), the axis of symmetry, the x-intercepts and the y-intercept. 	
7B	**2. I can sketch a quadratic relation involving a dilation or reflection.** e.g. Sketch the graph of $y = -4x^2$, labelling the turning point and one other point.	
7B	**3. I can sketch a quadratic relation involving translations.** e.g. Sketch the graph of $y = (x - 1)^2 + 3$, labelling the turning point and y-intercept.	
7C	**4. I can use the x-intercepts to find the turning point and sketch a quadratic.** e.g. Sketch the graph of $y = x^2 - 2x - 3$, labelling the intercepts and the turning point.	
7C	**5. I can sketch a quadratic graph that is a perfect square.** e.g. Sketch the graph of $y = x^2 - 4x + 4$, labelling the intercepts and turning point.	
7D	**6. I can determine the key features of a graph in turning point form.** e.g. For $y = 2(x - 1)^2 - 18$ determine the turning point, y-intercept and any x-intercepts.	
7D	**7. I can sketch a quadratic graph by first completing the square.** e.g. Sketch $y = x^2 + 8x + 20$ by first completing the square.	
7E	**8. I can use the discriminant to find the number of x-intercepts of a quadratic graph.** e.g. Determine the number of x-intercepts of the parabola given by $y = 2x^2 + 3x + 6$.	
7E	**9. I can find the turning point of a quadratic using $x = -\frac{b}{2a}$.** e.g. Determine the turning point coordinates of the parabola given by $y = 2x^2 + 8x - 5$.	
7E	**10. I can sketch a quadratic graph using the quadratic formula.** e.g. Sketch the graph of the quadratic $y = 3x^2 - 6x - 2$, labelling significant points. Round the x-intercepts to two decimal places.	
7F	**11. I can apply quadratic models in word problems.** e.g. A piece of wire measuring 80 cm in length is bent into the shape of a rectangle. Let x cm be the width of the rectangle. i Use the perimeter to find an expression for the length of the rectangle in terms of x. ii Hence, find a rule for the area of the rectangle, A cm^2, in terms of x and sketch its graph for suitable values of x. iii Use the graph to determine the maximum area that can be formed and the dimensions of the rectangle that give this area.	
7G	**12. I can find the points of intersection of a line and a parabola.** 10A e.g. Find the points of intersection of $y = 2x^2 - 5$ and $y = x + 1$.	

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter checklist

✔

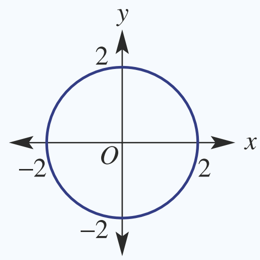
7G **13. I can determine the number of points of intersection of a line and a parabola.** 10A
e.g. Determine the number of solutions (points of intersection) of the equations $4x + y = -4$ and $y = x^2 + 2x + 5$.

7H **14. I can use function notation.** 10A
e.g. If $f(x) = x^2 - 2x + 4$, find $f(3)$.

7H **15. I can recognise if a relation is a function.** 10A
e.g. Which of the following are functions?

i $y = x^2 - 2x$,

ii



iii $\{(1, 3), (2, 5), (4, 3), (2, -1)\}$

7H **16. I can determine the domain and range of a function.** 10A
e.g. Write down the allowable x-values (domain) and the resulting y-values (range) of $y = x^2 + 6$.

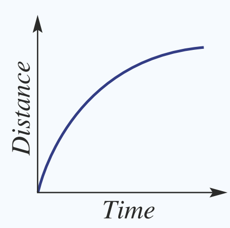
7I **17. I can sketch the graph of a circle by finding its centre and radius.**
e.g. For the equation $x^2 + y^2 = 16$, state the coordinates of its centre, state its radius and sketch its graph labelling intercepts.

7I **18. I can find the points of intersection of a circle and a line.**
e.g. Find the coordinates of the points where $x^2 + y^2 = 9$ intersects $y = 2x$.

7J **19. I can sketch the graph of a hyperbola.**
e.g. Sketch the graphs of $y = \frac{3}{x}$ and $y = -\frac{2}{x}$, labelling the points where $x = 1$ and $x = -1$.

7J **20. I can find the intersection of a line and a parabola.**
e.g. Find the coordinates of the points where $y = \frac{1}{x}$ meets the line $y = 9x$.

7K **21. I can interpret a distance–time graph.**
e.g. For the distance–time graph showing a journey from home, describe the journey in relation to distance from home, gradient of graph and speed.



7K **22. I can work with direct variation.**
e.g. If y is directly proportional to x and $y = 45$ when $x = 25$, determine the relationship between y and x and use this to find y when $x = 40$.

7K **23. I can work with inverse variation.**
e.g. If x and y are inversely proportional and $y = 5$ when $x = 4$, determine the constant of proportionality k and write the rule.

7L **24. I can sketch the graphs of circles, exponentials and hyperbolas with transformations.** 10A
e.g. Sketch the graphs of $(x - 1)^2 + (y + 2)^2 = 4$ and $y = \frac{1}{x - 2} + 1$, labelling key features.

ISBN 978-1-108-77290-7

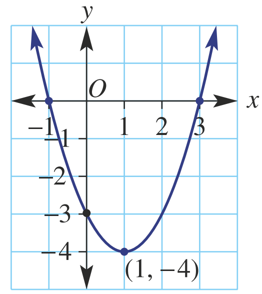
Cambridge University Press
Updated June 2022

Chapter review

Short-answer questions

7A

1 State the following features of the quadratic graph shown.



a turning point and whether it is a maximum or a minimum
b axis of symmetry
c x-intercepts
d y-intercept

7A/B

2 State whether the graphs of the following quadratics have a maximum or a minimum turning point and give its coordinates.

a $y = (x - 2)^2$
b $y = -x^2 + 5$
c $y = -(x + 1)^2 - 2$
d $y = 2(x - 3)^2 + 4$

7C

3 Sketch the quadratics below by first finding:

i the y-intercept
ii the x-intercepts, using factorisation
iii the turning point

a $y = x^2 - 4$
b $y = x^2 + 8x + 16$
c $y = x^2 - 2x - 8$

7D

4 Complete the following for each quadratic below.

i State the coordinates of the turning point and whether it is a maximum or a minimum.
ii Find the y-intercept.
iii Find the x-intercepts (if any).
iv Sketch the graph, labelling the features above.

a $y = -(x - 1)^2 - 3$
b $y = 2(x + 3)^2 - 8$

7D

5 Sketch the following quadratics by completing the square. Label all key features with exact coordinates.

a $y = x^2 - 4x + 1$
b $y = x^2 + 3x - 2$

7E

6 State the number of x-intercepts of the following quadratics either by using the discriminant or by inspection where applicable.

a $y = (x + 4)^2$
b $y = (x - 2)^2 + 5$
c $y = x^2 - 2x - 5$
d $y = 2x^2 + 3x + 4$

7E

7 For the following quadratics:

i Find the y-intercept.
ii Use $x = -\frac{b}{2a}$ to find the coordinates of the turning point.
iii Use the quadratic formula to find the x-intercepts, rounding to one decimal place.
iv Sketch the graph.

a $y = 2x^2 - 8x + 5$
b $y = -x^2 + 3x + 4$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

7G 10A

8 Solve these equations simultaneously.

a $y = x^2 + 4x - 2$
$y = 10$

b $y = 2x^2 + 5x + 9$
$y = -x + 4$

c $y = x^2 + 1$
$2x + 3y = 4$

7G 10A

9 Use the discriminant to show that the line $y = x + 4$ intersects the parabola $y = x^2 - x + 5$ in just one place.

7H 10A

10 If $f(x) = 2x^2 - x + 3$, find:

a $f(2)$ **b** $f(-3)$ **c** $f(0.5)$ **d** $f(k)$

7H 10A

11 For each of the following, state the domain (i.e. the set of allowable x-values) and range (i.e. resulting y-values).

a $y = 2x - 8$ **b** $y = 4$ **c** $x = 1$

d $y = \frac{1}{x}$ **e** $y = x^2 - 3$ **f** $y = x^2 - x$

7I

12 Sketch these circles. Label the centre and axes intercepts.

a $x^2 + y^2 = 25$ **b** $x^2 + y^2 = 7$

7I

13 Find the exact coordinates of the points of intersection of the circle $x^2 + y^2 = 9$ and the line $y = 2x$. Sketch the graphs, showing the points of intersection.

7J

14 Sketch these hyperbolas, labelling the points where $x = 1$ and $x = -1$.

a $y = \frac{2}{x}$ **b** $y = -\frac{3}{x}$

7J

15 Find the points of intersection of the hyperbola $y = \frac{4}{x}$ and these lines.

a $y = 3$ **b** $y = 2x$

7K

16 **a** If y varies directly with x and $y = 10$ when $x = 2$, find the rule linking y with x.

b If $y = \frac{k}{x}$ and $x = 6$ when $y = 12$, determine:

i the value of k **ii** y when $x = 4$ **iii** x when $y = 0.7$

7L 10A

17 Sketch the following graphs, labelling key features with exact coordinates.

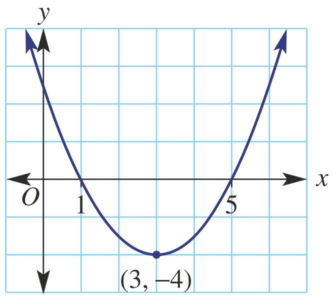
a $(x + 1)^2 + (y - 2)^2 = 4$ **b** $y = 2^{x-1} + 3$ **c** $y = \frac{1}{x + 2} - 3$

Multiple-choice questions

7A

1 The equation of the axis of symmetry of the graph shown is:

A $y = -4$
B $x = 3$
C $x = -4$
D $y = 3$
E $y = 3x$



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

7B **2** Compared to the graph of $y = x^2$, the graph of $y = (x - 3)^2$ is:
A translated 3 units down
B translated 3 units left
C dilated by a factor of 3
D translated 3 units right
E translated 3 units up

7B **3** The coordinates and type of turning point of $y = -(x + 2)^2 + 1$ is:
A a minimum at $(-2, -1)$
B a maximum at $(2, 1)$
C a minimum at $(2, 1)$
D a maximum at $(2, -1)$
E a maximum at $(-2, 1)$

7B **4** The y-intercept of $y = 3(x - 1)^2 + 4$ is:
A 1 **B** 4 **C** $\frac{1}{3}$ **D** 7 **E** 3

7C **5** The x-intercept(s) of the graph of $y = x^2 + 3x - 10$ are:
A $2, -5$ **B** -10 **C** $5, -2$ **D** $-5, -2$ **E** $5, 2$

7C **6** A quadratic graph has x-intercepts at $x = -7$ and $x = 2$. The x-coordinate of the turning point is:
A $x = -\frac{5}{2}$ **B** $x = -\frac{7}{2}$ **C** $x = \frac{9}{3}$ **D** $x = \frac{5}{2}$ **E** $x = -\frac{9}{2}$

7D **7** The quadratic rule $y = x^2 - 4x - 3$, when written in turning point form, is:
A $y = (x - 2)^2 - 3$
B $y = (x - 4)^2 + 1$
C $y = (x - 2)^2 - 7$
D $y = (x + 4)^2 - 19$
E $y = (x + 2)^2 - 1$

7E **8** A quadratic graph $y = ax^2 + bx + c$ has two x-intercepts. This tells us that:
A The graph has a maximum turning point.
B $-\frac{b}{2a} < 0$
C There is no y-intercept.
D $b^2 - 4ac > 0$
E $b^2 - 4ac = 0$

7F **9** A toy rocket follows the path given by $h = -t^2 + 4t + 6$, where h is the height above ground, in metres, t seconds after launch. The maximum height reached by the rocket is:
A 10 metres **B** 2 metres **C** 6 metres **D** 8 metres **E** 9 metres

7H 10A **10** The restrictions on x and y in the rule $y = \frac{1}{x + 1}$ are:
A $x \neq 1, y \neq 0$ **B** $x \neq 1, y \neq 1$ **C** $x \neq 0, y \neq 0$
D $x \neq -1, y \neq 0$ **E** $x \neq -1, y \neq -1$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

7I **11** The equation of a circle centred at the origin with radius 4 units is:

A $y = 4x^2$ **B** $x^2 + y^2 = 4$ **C** $x^2 + y^2 = 8$ **D** $y = 4^x$ **E** $x^2 + y^2 = 16$

7K **12** If y is inversely proportional to x, the equation is of the form:

A $y = kx$ **B** $y = kx + c$ **C** $y = \frac{x}{k}$ **D** $y = kx^2$ **E** $y = \frac{k}{x}$

7L 10A **13** The graph of $y = \frac{1}{x-1} + 2$ has asymptote(s) at:

A $x = 1, y = 2$ **B** $y = 2$ **C** $x = -1$
D $x = 2, y = 1$ **E** $x = -1, y = 2$

7L 10A **14** The circle with equation $(x + 1)^2 + (y - 3)^2 = 16$ has centre coordinates and radius r:

A $(1, -3), r = 4$ **B** $(1, -3), r = 16$ **C** $(-1, 3), r = 4$
D $(-1, 3), r = 16$ **E** $(-1, -3), r = 4$

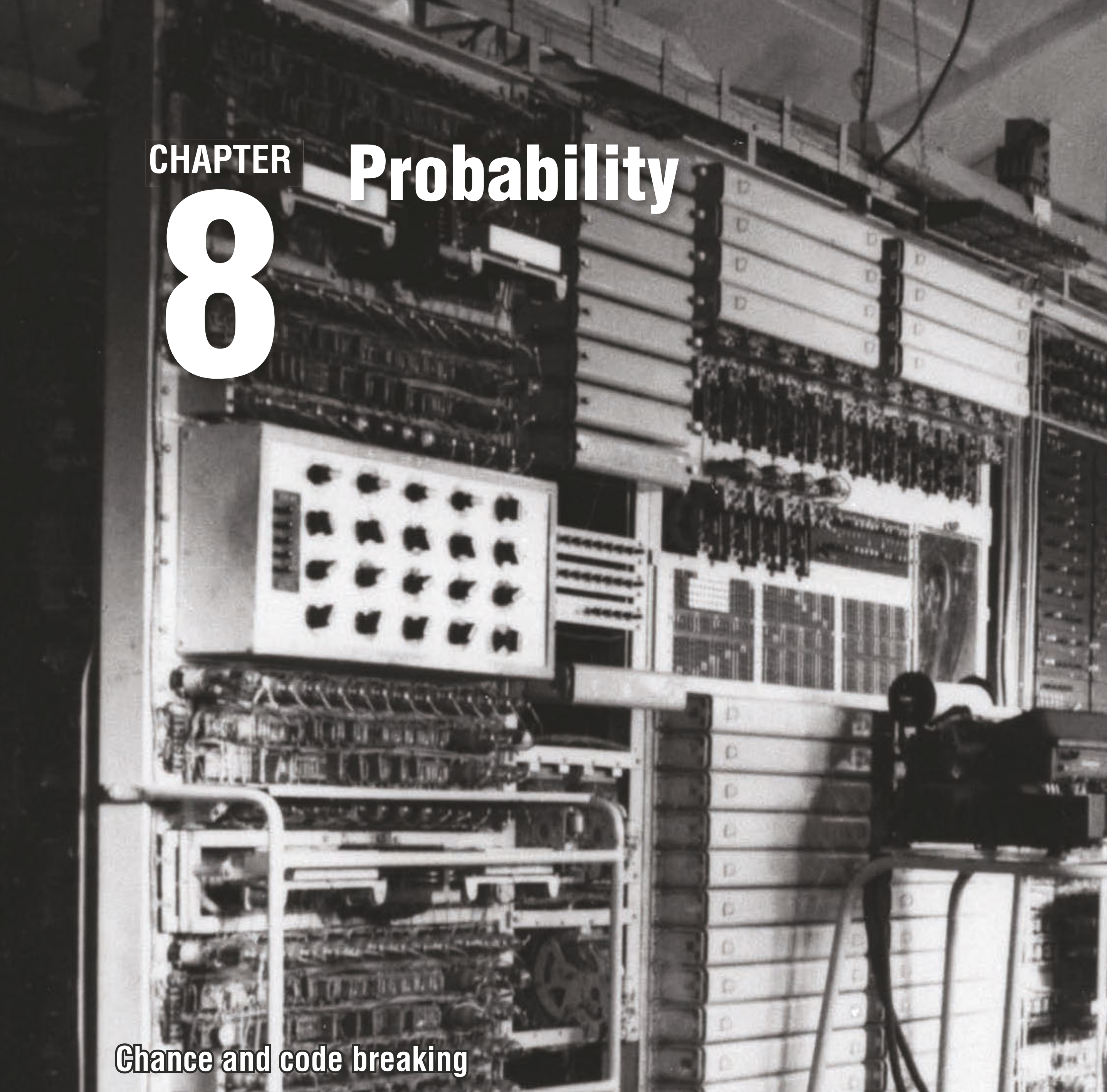
Extended-response questions

1 The cable for a suspension bridge is modelled by the equation $h = \frac{1}{800}(x - 200)^2 + 30$, where h metres is the distance above the base of the bridge and x metres is the distance from the left side of the bridge.

a Determine the turning point of the graph of the equation.
b If the bridge is symmetrical, determine the suitable values of x.
c Determine the range of values of h.
d Sketch a graph of the equation for the suitable values of x.
e What horizontal distance does the cable span?
f What is the closest distance of the cable from the base of the bridge?
g What is the greatest distance of the cable from the base of the bridge?

2 200 metres of fencing is to be used to form a rectangular paddock. Let x metres be the width of the paddock.
a Write an expression for the length of the paddock in terms of x.
b Write an equation for the area of the paddock (A m^2) in terms of x.
c Decide on the suitable values of x.
d Sketch the graph of A versus x for suitable values of x.
e Use the graph to determine the maximum paddock area that can be formed.
f What will be the dimensions of the paddock to achieve its maximum area?

ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022



CHAPTER 8 Probability

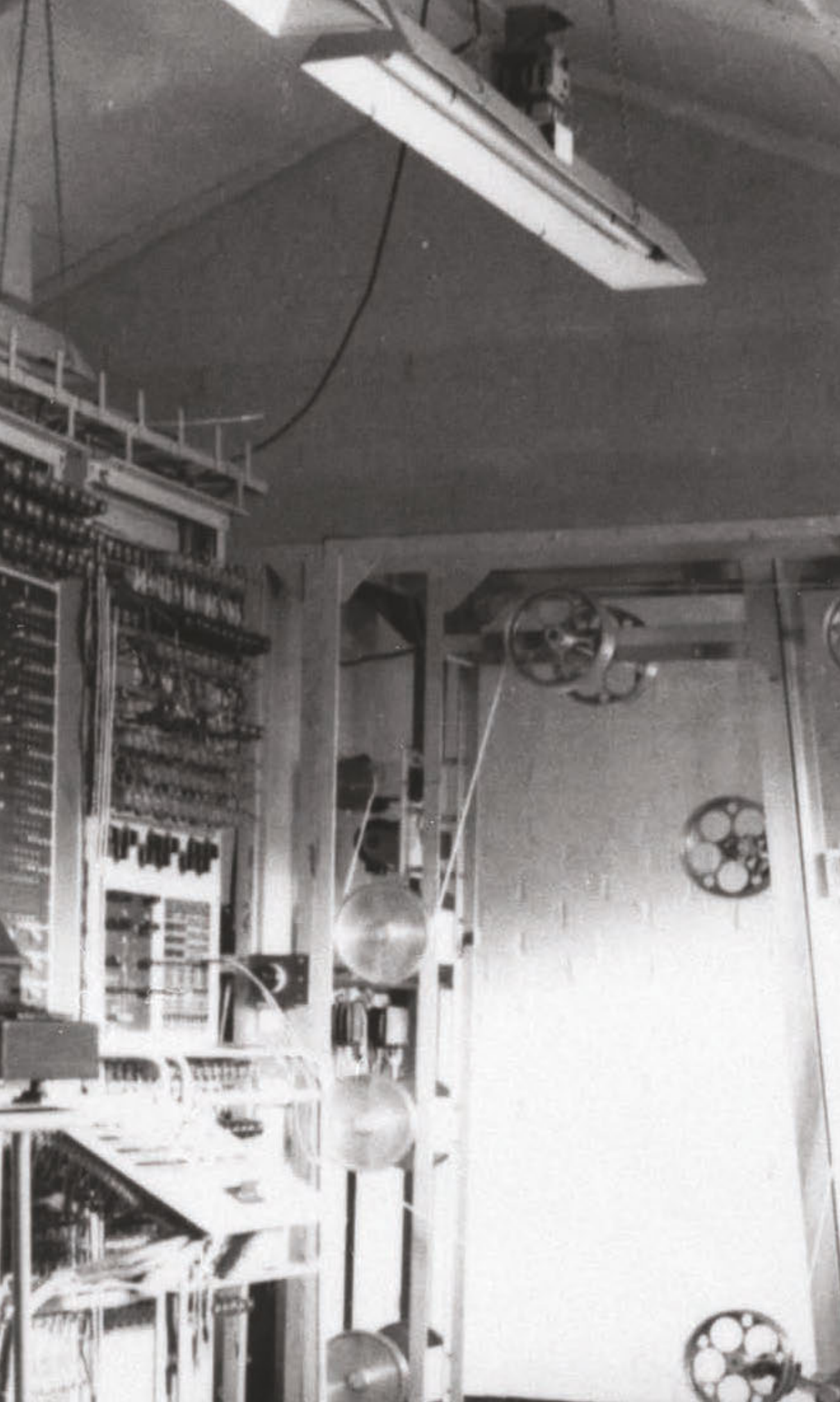
Chance and code breaking

A four-digit combination lock has 10 possibilities for each number, providing $10^4 = 10\,000$ combinations. So when it is locked, there is a one in ten thousand chance of guessing the correct code. To methodically test all the possibilities with, say, one second per tryout, it would take 10 000 seconds, which is over 2.5 hours.

The probability of guessing a computer password varies with its length and random nature. The probability of cracking a password of 8 lower case letters with one try is 1 in 26^8, or 1 in 208 827 064 576. Despite this very low chance for a single guess, a powerful computer can tryout billions of combinations per second and might crack it in minutes.

During the Second World War, English mathematicians and engineers designed various machines to break the codes of German encrypted messages. Colossus, built in 1943 to decode the settings of the German Lorenz machine, was

ISBN 978-1-108-77290-7 Cambridge University Press
 Updated June 2022



Online resources

A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

In this chapter

8A Review of probability (CONSOLIDATING)
8B Unions and intersections
8C The addition rule
8D Conditional probability
8E Two-step experiments using tables
8F Using tree diagrams
8G Independent events

Victorian Curriculum

STATISTICS AND PROBABILITY
Chance

Describe the results of two- and three-step chance experiments, both with and without replacements, assign probabilities to outcomes and determine probabilities of events.

Investigate the concept of independence (VCMSP347)

Use the language of 'if… then', 'given', 'of', 'knowing that' to investigate conditional statements and identify common mistakes in interpreting such language (VCMSP348)

(10A) Investigate reports of studies in digital media and elsewhere for information on their planning and implementation (VCMSP371)

the first electronic, digital, manually programmable computer ever built. Colossus weighed 5 tonnes and filled a room with its 8 racks of glass vaccum tubes and resistor all wired together. It processed 5000 entries per second and could decode encrypted messages in hours rather than the weeks it took for manual decoding. Ten Colossus computers were built and their success likely shortened the war by two years.

8A Review of probability CONSOLIDATING

Learning intentions
- To review the key terms of probability: trial, sample space, event and outcome
- To understand the possible values of a probability and how they describe the level of chance
- To know how to calculate theoretical probabilities for equally likely outcomes
- To be able to calculate and use an experimental probability

Probability is an area of mathematics concerned with the likelihood of particular random events. In some situations, such as rolling a die, we can determine theoretical probabilities because we know the total number of possible outcomes and the number of favourable outcomes. In other cases, we can use statistics and experimental results to describe the chance that an event will occur. The chance that a particular soccer team will win its next match, for example, could be estimated using various results from preceding games.

A soccer team could win, lose or draw the next match it plays, but these three outcomes do not necessarily have the same probability.

LESSON STARTER Name the event

For each number below, describe an event that has that exact or approximate probability. If you think it is exact, then give a reason.

- $\frac{1}{2}$
- 25%
- 0.2
- 0.00001
- $\frac{99}{100}$

KEY IDEAS

- Definitions
  - A **trial** is a single experiment, such as a single roll of a die.
  - The **sample space** is the list of all possible outcomes from an experiment. For example, when rolling a 6-sided die the sample space is $\{1, 2, 3, 4, 5, 6\}$.
  - An **outcome** is a possible result of an experiment.
  - An **event** is the collection of favourable outcomes.
  - **Equally likely outcomes** are outcomes that have the same chance of occurring.

- In the study of probability, a numerical value based on a scale from 0 to 1 is used to describe levels of **chance**.

0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
impossible	unlikely				even chance			likely		certain

ISBN 978-1-108-77290-7

- The probability of an event in which outcomes are **equally likely** is calculated as:

$$\text{Pr(event)} = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

- **Experimental probability** is calculated in the same way as theoretical probability but uses the results of an experiment:

$$\text{Pr(event)} = \frac{\text{number of favourable outcomes}}{\text{total number of trials}}$$

  - The **long run proportion** is the experimental probability for a sufficiently large number of trials.

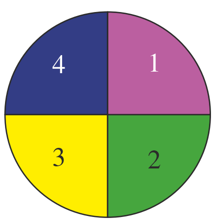
BUILDING UNDERSTANDING

1 A coin is flipped once.

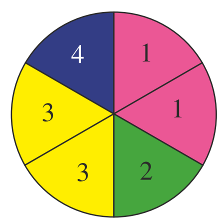
a How many different outcomes are possible from a single flip of the coin?
b What is the sample space from a single flip of the coin.
c Are the possible outcomes equally likely?
d What is the probability of obtaining a tail?
e What is the probability of not obtaining a tail?
f What is the probability of obtaining a tail or a head?

2 For the following spinners, find the probability that the outcome will be a 4.

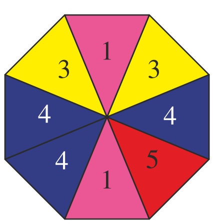
a



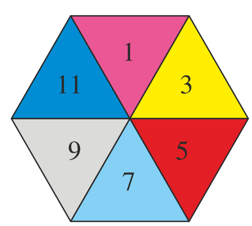
b



c



d



3 Given a spinner has a probability of $\frac{1}{3}$ of spinning a 4, estimate how many 4s you could expect in:

a 12 spins **b** 60 spins **c** 300 spins.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 1 Calculating simple theoretical probabilities

A letter is chosen from the word TELEVISION. Find the probability that the letter is:

a a V b an E c not an E d an E or a V.

SOLUTION

a $\Pr(V) = \dfrac{1}{10}$ (= 0.1)

EXPLANATION

$\Pr(V) = \dfrac{\text{number of Vs}}{\text{total number of letters}}$

b $\Pr(E) = \dfrac{2}{10}$

There are 2 Es in the word TELEVISION.

$= \dfrac{1}{5}$ (= 0.2)

Simplify the fraction.

c $\Pr(\text{not an E}) = \dfrac{8}{10}$

$= \dfrac{4}{5}$ (= 0.8)

If there are 2 Es in the word TELEVISION, which has 10 letters, then there must be 8 letters that are not E. This is the same as $1 - \Pr(E)$.

d $\Pr(\text{an E or a V}) = \dfrac{3}{10}$ (= 0.3)

The number of letters that are either E or V is 3.

Now you try

A letter is chosen from the word CALCULATION. Find the probability that the letter is:

a a T b an A c not an A d an A or a T.

Example 2 Calculating simple experimental probabilities

An experiment involves tossing three coins and counting the number of heads. Here are the results after running the experiment 100 times.

Number of heads	0	1	2	3
Frequency	11	40	36	13

Find the experimental probability of obtaining:

a zero heads
b two heads
c fewer than two heads
d at least one head.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

SOLUTION

EXPLANATION

a $\Pr(0 \text{ heads}) = \frac{11}{100}$
$= 0.11$

$\Pr(0 \text{ heads}) = \frac{\text{number of times 0 heads are observed}}{\text{total number of trials}}$

b $\Pr(2 \text{ heads}) = \frac{36}{100}$
$= 0.36$

$\Pr(2 \text{ heads}) = \frac{\text{number of times 2 heads are observed}}{\text{total number of trials}}$

c $\Pr(\text{fewer than 2 heads}) = \frac{11 + 40}{100}$
$= \frac{51}{100}$
$= 0.51$

Fewer than 2 heads means to observe 0 or 1 head.

d $\Pr(\text{at least one head}) = \frac{40 + 36 + 13}{100}$
$= \frac{89}{100}$
$= 0.89$

At least 1 head means that 1, 2 or 3 heads can be observed. This is the same as 1 − Pr(no heads).

Now you try

A experiment involves checking second-hand bicycles for faults. Here are the results after checking 100 bicycles.

Number of faults	0	1	2	3	4
Frequency	10	32	45	9	4

Find the experimental probability that a randomly selected bicycle will have:

a 0 faults **b** 3 faults **c** fewer than 3 faults **d** at least one fault.

Exercise 8A

FLUENCY	1–4	2–5	2–5(½)

Example 1

1 A letter is chosen from the word TEACHER. Find the probability that the letter is:

a an R **b** an E **c** not an E **d** an R or an E.

Example 1

2 A letter is chosen from the word EXPERIMENT. Find the probability that the letter is:

a an E **b** a vowel **c** not a vowel **d** an X or a vowel.

3 A 10-sided die numbered 1 to 10 is rolled once. Find these probabilities.

a Pr(8) **b** Pr(odd)
c Pr(even) **d** Pr(less than 6)
e Pr(prime) (Remember that 1 is not prime.) **f** Pr(3 or 8)
g Pr(8, 9 or 10) **h** Pr(greater than 9)

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 2

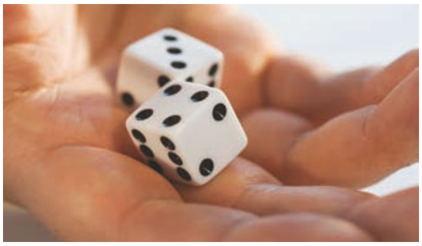
4 An experiment involves tossing three coins and counting the number of heads. Here are the results after running the experiment 100 times.

Number of heads	0	1	2	3
Frequency	9	38	43	10

Find the experimental probability of obtaining:

a zero heads
b two heads
c fewer than two heads
d at least one head.

5 An experiment involves rolling two dice and counting the number of sixes. Here are the results after running the experiment 100 times.



Number of sixes	0	1	2
Frequency	62	35	3

Find the experimental probability of obtaining:

a zero sixes
b two sixes
c fewer than two sixes
d at least one six.

PROBLEM-SOLVING

6, 7 | 7, 8 | 8, 9

6 Thomas is a prizewinner in a competition and will be randomly awarded a single prize chosen from a collection of 50 prizes. The type and number of prizes to be handed out are listed below.

Prize	car	holiday	iPad	DVD
Number	1	4	15	30

Find the probability that Thomas will be awarded the following.

a a car
b an iPad
c a prize that is not a car

7 Find the probability of choosing a red counter if a counter is chosen from a box that contains the following counters.

a 3 red and 3 yellow
b 3 red and 5 yellow
c 1 red, 1 yellow and 2 blue
d 5 red, 12 green and 7 orange
e 10 red only
f 6 blue and 4 green

8 Many of the 50 cars inspected at an assembly plant contained faults. The results of the inspection are as follows.

Number of faults	0	1	2	3	4
Number of cars	30	12	4	3	1

Find the experimental probability that a car selected from the assembly plant will have:

a one fault
b four faults
c fewer than two faults
d one or more faults
e three or four faults
f at least two faults.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

9 A quality control inspector examines clothing at a particular factory on a regular basis and records the number of faulty items identified each day. After 20 visits to the factory over the course of the year, the results are summarised in a table.

Number of faulty items	0	1	2	3	4
Frequency	14	4	1	0	1

a Estimate the probability that the inspector will identify the following numbers of faulty items on any particular day.

i 0 **ii** 1 **iii** 2 **iv** 3 **v** 4

b If the factory is fined when two or more faulty items are found, estimate the probability that the factory will be fined on the next inspection.

REASONING	10	10, 11	11, 12

10 A bag contains red and yellow counters. A counter is drawn from the bag and then replaced. This happens 100 times with 41 of the counters drawn being red.

a How many counters drawn were yellow?

b If there were 10 counters in the bag, how many do you expect were red? Give a reason.

c If there were 20 counters in the bag, how many do you expect were red? Give a reason.

11 A card is chosen from a standard deck of 52 playing cards that includes 4 aces, 4 kings, 4 queens and 4 jacks. Find the following probabilities.

a Pr(heart) **b** Pr(king) **c** Pr(king of hearts)

d Pr(heart or club) **e** Pr(king or jack) **f** Pr(heart or king)

g Pr(not a king) **h** Pr(neither a heart nor a king)

12 The probability of selecting a white chocolate from a box is $\frac{1}{5}$ and the probability of selecting a dark chocolate from the same box is $\frac{1}{3}$. The other chocolates are milk chocolates.

a Find the probability of selecting a milk chocolate.

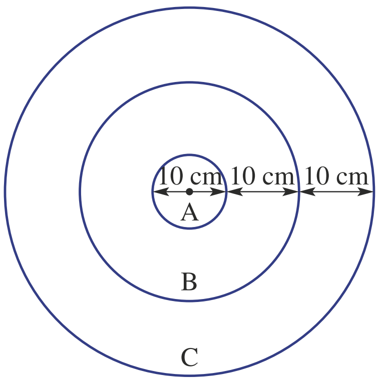
b How many chocolates in total could be in the box? Give reasons. Is there more than one answer?

ENRICHMENT: Target probability	–	–	13

13 A target board is made up of three rings (A, B and C) that are 10 cm apart, as shown.

An experienced archer shoots an arrow at the board and is guaranteed to hit it, with an equal chance of doing so at any point.

Recall that the area of a circle $= \pi r^2$.



a Calculate the total area of the target and express your answer as an exact value (e.g. 10π).

b Calculate, using exact values, the area of the regions labelled:

i A **ii** B **iii** C.

c Calculate the probability that the region in which the archer's arrow will hit will be:

i A **ii** B **iii** C **iv** A or B

v B or C **vi** A or C **vii** A, B or C **viii** not B.

d Investigate whether changing the width of each ring in the target by the same amount changes the answers to part **c**.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

8B Unions and intersections

Learning intentions

- To know the symbols for set notation for union, intersection and complement and what sets they represent
- To know how to use a Venn diagram or two-way table to display the outcomes of two or more events
- To be able to use Venn diagrams and two-way tables to find associated probabilities

When we consider two or more events it is possible that there are outcomes that are common to both events. A TV network, for example, might be collecting statistics regarding whether or not a person watches cricket and/or tennis or neither over the Christmas holidays. The estimated probability that a person will watch cricket *or* tennis will therefore depend on how many people responded yes to watching both cricket *and* tennis.

TV ratings come from the programs viewed in 3000 randomly selected homes. Statistical analysis gives the proportion of each age group who watch specific programs. This information impacts TV advertising, program development and scheduling.

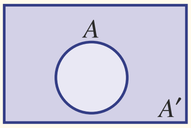
LESSON STARTER Duplication in cards

Imagine that you randomly draw one card from a standard deck of 52 playing cards.

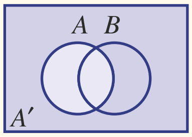
- Discuss what a standard deck includes.
- What is the probability of selecting a heart?
- What is the probability of selecting a king?
- Now find the probability that the card is a king and a heart. Is this possible?
- Find the probability that the card is a king or a heart. Discuss why the probability is not just equal to $\frac{4}{52} + \frac{13}{52} = \frac{17}{52}$.

KEY IDEAS

- Set notation
  - A **set** is a collection or group of elements that can include numbers, letters or other objects.
  - The **sample space**, denoted by S, Ω, ∪ or ξ, is the set of all possible elements or objects considered in a particular situation. This is also called the **universal set**.
  - A **Venn diagram** illustrates how all elements in the sample space are distributed among the events.



or



  - A **null** or **empty set** is a set with no elements and is symbolised by { } or ∅.
  - All elements that belong to both *A and B* make up the **intersection**: $A \cap B$.
  - All elements that belong to either events *A or B* make up the **union**: $A \cup B$.
  - Two sets A and B are **mutually exclusive** if they have no elements in common, meaning $A \cap B = \varnothing$.

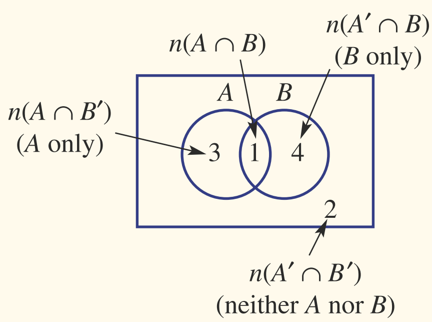
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

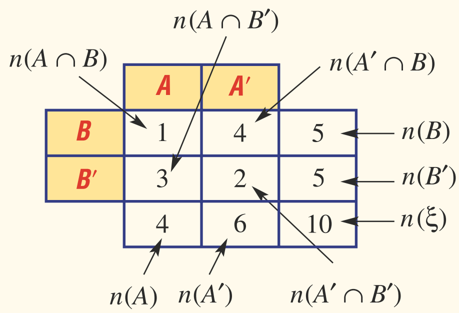
- For an event A, the **complement** of A is A' (or 'not A').
- $\Pr(A') = 1 - \Pr(A)$
- A **only** (or $A \cap B'$) is defined as all the elements in A but not in any other set.
- $n(A)$ is the number of elements in set A.

■ Venn diagrams and two-way tables are useful tools when considering two or more events.

Venn diagram

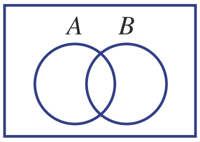


Two-way table



BUILDING UNDERSTANDING

1 On a Venn diagram like the one shown describe the region represented by each of the following.



- **a** A
- **b** B
- **c** $A \cap B$
- **d** $A \cup B$
- **e** A only
- **f** B only
- **g** A'
- **h** neither A nor B

2 Which symbols $\cup$, $\cap$, or $\varnothing$, would be used to rewrite the following.

- **a** null set
- **b** A and B
- **c** A or B
- **d** A or B or C

3 Decide if the events A and B are mutually exclusive.

- **a** $A = \{1, 3, 5, 7\}$
  $B = \{5, 8, 11, 14\}$
- **b** $A = \{-3, -2, \ldots, 4\}$
  $B = \{-11, -10, \ldots, -4\}$
- **c** $A = \{\text{prime numbers}\}$
  $B = \{\text{even numbers}\}$

ISBN 978-1-108-77290-7

Example 3 Listing sets

Consider the given events A and B that involve numbers taken from the first 10 positive integers.

$A = \{1, 2, 3, 4, 5, 6\}$ $B = \{1, 3, 7, 8\}$

a Represent the two events A and B in a Venn diagram, showing the number of elements belonging to each region.

b List the following sets.

i $A \cap B$ **ii** $A \cup B$

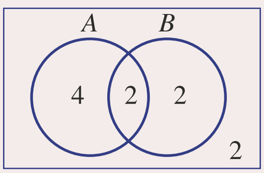
c If a number from the first 10 positive integers is randomly selected, find the probability that the following events occur.

i A **ii** $A \cap B$ **iii** $A \cup B$

d Are the events A and B mutually exclusive? Why or why not?

SOLUTION

a



The two elements 1 and 3 are common to both sets A and B, $A \cap B$.

The two elements 9 and 10 belong to neither set A nor set B, $(A' \cap B')$. A has 6 elements, with 2 in the intersection, so A only has 4 elements.

b i $A \cap B = \{1, 3\}$

$A \cap B$ is the intersection of sets A and B.

ii $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$A \cup B$ contains elements in either A or B.

c i $\Pr(A) = \frac{6}{10} = \frac{3}{5}$

There are 6 elements in A.

ii $\Pr(A \cap B) = \frac{2}{10} = \frac{1}{5}$

$A \cap B$ contains 2 elements.

iii $\Pr(A \cup B) = \frac{8}{10} = \frac{4}{5}$

$A \cup B$ contains 8 elements.

d The sets A and B are not mutually exclusive since $A \cap B \neq \varnothing$.

The set $A \cap B$ contains at least 1 element.

Now you try

Consider the given events A and B that involve numbers taken from the first 10 positive integers.

$A = \{2, 3, 4, 5, 6, 7, 8\}$ and $B = \{1, 2, 3, 5, 7\}$

a Represent the two events A and B in a Venn diagram, showing the number of elements belonging to each region.

b List the following sets.

i $A \cap B$ **ii** $A \cup B$

c If a number from the first 10 positive integers is randomly selected, find the probability that the following events occur.

i A **ii** $A \cap B$ **iii** $A \cup B$

d Are the events A and B mutually exclusive? Why or why not?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 4 Using Venn diagrams

From a class of 30 students, 12 enjoy cricket (C), 14 enjoy netball (N) and 6 enjoy both cricket and netball.

a Illustrate this information in a Venn diagram.

b State the number of students who enjoy:

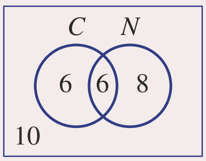
i netball only **ii** neither cricket nor netball.

c Find the probability that a student chosen randomly from the class will enjoy:

i netball **ii** netball only **iii** both cricket and netball.

SOLUTION / EXPLANATION

a



First, write 6 in the intersection (i.e. 6 enjoy cricket and netball), then determine the other values according to the given information. Cricket only is $12 - 6 = 6$.

The total must be 30.

b i $n(N \text{ only}) = 8$

Includes elements in N but not in C.

ii $n(\text{neither } C \text{ nor } N) = 10$

These are the elements outside both C and N.

c i $\Pr(N) = \frac{14}{30} = \frac{7}{15}$

14 of the 30 students enjoy netball.

ii $\Pr(N \text{ only}) = \frac{8}{30} = \frac{4}{15}$

8 of the 30 students enjoy netball but not cricket.

iii $\Pr(C \cap N) = \frac{6}{30} = \frac{1}{5}$

6 students enjoy both cricket and netball.

Now you try

From a pack of 20 dogs, 9 enjoy fresh meat (M), 12 enjoy dry food (D) and 7 enjoy both fresh meat and dry food.

a Illustrate this information in a Venn diagram.

b State the number of dogs who enjoy:

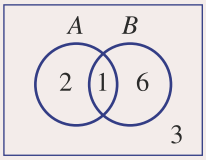
i fresh meat only **ii** neither fresh meat nor dry food.

c Find the probability that a dog chosen at random from the pack will enjoy:

i fresh meat **ii** dry food only **iii** both fresh meat and dry food.

Example 5 Using two-way tables

The Venn diagram shows the distribution of elements in two sets, A and B.



a Transfer the information in the Venn diagram to a two-way table.

b Find:

i $n(A \cap B)$ **ii** $n(A' \cap B)$ **iii** $n(A \cap B')$ **iv** $n(A' \cap B')$

v $n(A)$ **vi** $n(B')$ **vii** $n(A \cup B)$

c Find:

i $\Pr(A \cap B)$ **ii** $\Pr(A')$ **iii** $\Pr(A \cap B')$

SOLUTION

a

	A	A'	
B	1	6	7
B'	2	3	5
	3	9	12

b **i** $n(A \cap B) = 1$

ii $n(A' \cap B) = 6$

iii $n(A \cap B') = 2$

iv $n(A' \cap B') = 3$

v $n(A) = 3$

vi $n(B') = 5$

vii $n(A \cup B) = 9$

c **i** $\Pr(A \cap B) = \dfrac{1}{12}$

ii $\Pr(A') = \dfrac{9}{12} = \dfrac{3}{4}$

iii $\Pr(A \cap B') = \dfrac{2}{12} = \dfrac{1}{6}$

EXPLANATION

	A	A'	
B	$n(A \cap B)$	$n(A' \cap B)$	$n(B)$
B'	$n(A \cap B')$	$n(A' \cap B')$	$n(B')$
	$n(A)$	$n(A')$	$n(\xi)$

$n(A \cap B)$ is the intersection of A and B.

$n(A' \cap B)$ is B only.

$n(A \cap B')$ is A only.

$n(A' \cap B')$ is neither A nor B.

$n(\mathrm{A}) = n(A \cap B') + n(A \cap B)$

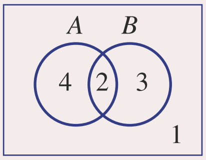
$n(B') = n(A \cap B') + n(A' \cap B')$

$n(A \cup B) = n(A \cap B) + n(A \cap B') + n(A' \cap B)$

When calculating probabilities, you will need to divide the number of elements in each set by the number of elements in the sample space, which is 12.

Now you try

The Venn diagram shows the distribution of elements in two sets, A and B.



a Transfer the information in the Venn diagram to a two-way table.

b Find:

i $n(A \cap B)$ **ii** $n(A' \cap B)$ **iii** $n(A \cap B')$ **iv** $n(A' \cap B')$

v $n(A)$ **vi** $n(B')$ **vii** $n(A \cup B)$

c Find:

i $\Pr(A \cap B)$ **ii** $\Pr(A')$ **iii** $\Pr(A \cap B')$

ISBN 978-1-108-77290-7

Exercise 8B

FLUENCY	1, 3, 4	2–5	2, 4–6

Example 3

1 Consider the given events A and B, which involve numbers taken from the first 10 positive integers.

$A = \{1, 2, 4, 5, 7, 8, 10\}$ $B = \{2, 3, 5, 6, 8, 9\}$

a Represent events A and B in a Venn diagram, showing the number of elements belonging to each region.

b List the following sets.

i $A \cap B$ ii $A \cup B$

c If a number from the first 10 positive integers is randomly selected, find the probability that the following events occur.

i A ii $A \cap B$ iii $A \cup B$

d Are the events A and B mutually exclusive? Why/why not?

Example 3

2 The elements of the events A and B described below are numbers taken from the first 10 prime numbers.

$A = \{2, 5, 7, 11, 13\}$ $B = \{2, 3, 13, 17, 19, 23, 29\}$

a Represent events A and B in a Venn diagram.

b List the elements belonging to the following.

i A and B ii A or B

c Find the probability that these events occur.

i A ii B iii $A \cap B$ iv $A \cup B$

Example 4

3 From a group of 50 adults, 35 enjoy reading fiction (F), 20 enjoy reading non-fiction (N) and 10 enjoy reading both fiction and non-fiction.

a Illustrate the information in a Venn diagram.

b State the number of people who enjoy reading:

i fiction only ii neither fiction nor non-fiction.

c Find the probability that a person chosen at random will enjoy reading:

i non-fiction ii non-fiction only

iii both fiction and non-fiction.

4 At a show, 45 children have the choice of riding on the Ferris wheel (F) and/or the Big Dipper (B). Thirty-five of the children wish to ride on the Ferris wheel, 15 children want to ride on the Big Dipper and 10 children want to ride on both.

a Illustrate the information in a Venn diagram.

b Find:

i $n(F \text{ only})$

ii $n(\text{neither } F \text{ nor } B)$

c For a child chosen at random from the group, find the following probabilities.

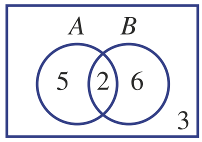
i $\Pr(F)$ ii $\Pr(F \cap B)$ iii $\Pr(F \cup B)$

iv $\Pr(F')$ v $\Pr(\text{neither } F \text{ nor } B)$

ISBN 978-1-108-77290-7

Example 5

5 The Venn diagram below shows the distribution of elements in two sets, A and B.



a Transfer the information in the Venn diagram to a two-way table.

b Find:

i $n(A \cap B)$ **ii** $n(A' \cap B)$ **iii** $n(A \cap B')$ **iv** $n(A' \cap B')$
v $n(A)$ **vi** $n(B')$ **vii** $n(A \cup B)$ **viii** $n(\xi)$

c Find:

i $\Pr(A \cap B)$ **ii** $\Pr(A')$ **iii** $\Pr(A \cap B')$

6 From a total of 10 people, 5 like apples (A), 6 like bananas (B) and 4 like both apples and bananas.

a Draw a Venn diagram for the 10 people.

b Draw a two-way table.

c Find:

i $n(A' \cap B)$ **ii** $n(A' \cap B')$ **iii** $\Pr(A \cap B)$ **iv** $\Pr(A \cup B)$

PROBLEM-SOLVING

7, 8	7, 9	8–10

7 Decide which of the elements would need to be removed from event A if the two events A and B described below are to become mutually exclusive.

a $A = \{1, 2, 3, 4\}$
$B = \{4, 5, 6, 7\}$

b $A = \{10, 12, 14, 16, 18\}$
$B = \{9, 10, 11, 12\}$

c $A = \{\text{a, b, c, d, e}\}$
$B = \{\text{a, c, e, g}\}$

d $A = \{1, 3, 5, 8, 10, 15, 20, 22, 23\}$
$B = \{7, 9, 14, 16, 19, 21, 26\}$

8 A letter is chosen at random from the word COMPLEMENTARY and two events, C and D, are as follows.

C: choosing a letter belonging to the word COMPLETE
D: choosing a letter belonging to the word CEMENT

a Represent the events C and D in a Venn diagram. Ensure that your Venn diagram includes *all* the letters that make up the word COMPLEMENTARY.

b Find the probability that the randomly chosen letter will:

i belong to C **ii** belong to C and D **iii** belong to C or D
iv not belong to C **v** belong to neither C nor D.

9 Complete the following two-way tables.

a

	A	A'	
B		3	6
B'			
		4	11

b

	A	A'	
B	2	7	
B'			3
	4		

10 In a group of 12 chefs, all enjoy baking cakes and/or tarts. In fact, 7 enjoy baking cakes and 8 enjoy baking tarts. Find out how many chefs enjoy baking both cakes and tarts.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

REASONING	11	11, 12	12, 13

11 If events A and B are mutually exclusive and $\Pr(A) = a$ and $\Pr(B) = b$, write expressions for:

a $\Pr(\text{not } A)$ **b** $\Pr(A \text{ or } B)$ **c** $\Pr(A \text{ and } B)$

12 Use diagrams to show that $(A \cup B)' = A' \cap B'$.

13 Mario and Erin are choosing a colour to paint the interior walls of their house. They have six colours to choose from: white (w), cream (c), navy (n), sky blue (s), maroon (m) and violet (v).

Mario would be happy with white or cream and Erin would be happy with cream, navy or sky blue. As they can't decide, a colour is chosen at random for them.

Let M be the event that Mario will be happy with the colour and let E be the event that Erin will be happy with the colour.

a Represent the events M and E in a Venn diagram.

b Find the probability that the following events occur.

i Mario will be happy with the colour choice; i.e. find $\Pr(M)$.

ii Mario will not be happy with the colour choice.

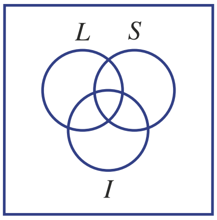
iii Both Mario and Erin will be happy with the colour choice.

iv Mario or Erin will be happy with the colour choice.

v Neither Mario nor Erin will be happy with the colour choice.

ENRICHMENT: Triple Venn diagrams	–	–	14, 15

14 Of 15 chosen courier companies, 9 offer a local service (L), 7 offer an interstate service (S) and 6 offer an international service (I). Two companies offer all three services, 3 offer both local and interstate services, 5 offer only local services and 1 offers only an international service.



a Draw a Venn diagram displaying the given information.

b Find the number of courier companies that offer neither a local, interstate nor international service.

c If a courier is chosen at random from the 15 examined initially, find the following probabilities.

i $\Pr(L)$

ii $\Pr(L \text{ only})$

iii $\Pr(L \text{ or } S)$

iv $\Pr(L \text{ and } S \text{ only})$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

15 Thirty-eight people were interviewed about their travelling experience in the past 12 months. Although the interviewer did not write down the details of the interviews, she remembers the following information.

In the past 12 months:

- Two people travelled overseas, interstate and within their own state.
- Two people travelled overseas and within their own state only.
- Seven people travelled interstate only.
- 22 people travelled within their own state.
- Three people did not travel at all.
- The number of people who travelled interstate and within their own state only was twice the number of people who travelled overseas and interstate only.
- The number of people who travelled overseas was equal to the number of people who travelled within their own state only.

a Use a Venn diagram to represent the information that the interviewer remembers.

b By writing down equations using the variables x (the number of people who travelled overseas and interstate only) and y (the number of people who travelled overseas only), solve simultaneously and find:

i the number of people who travelled interstate and overseas only

ii the number of people who travelled overseas.

c If one person from the 38 is chosen at random, find the probability that the person will have travelled to the following places:

i within their own state only

ii overseas only

iii interstate only

iv overseas or interstate or within their own state

v interstate or overseas.

Airlines employ mathematicians to use probability and statistics to predict passenger numbers so they can ensure seats on most flights are filled.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

8C The addition rule

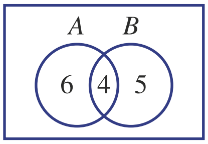
Learning intentions

- To understand and know the addition rule for finding the probability of the union of two events
- To be able to apply the addition rule to find unknown probabilities
- To know the meaning of the term mutually exclusive

When two events are mutually exclusive we know that the probability of the union of the events can be found by simply adding the probabilities of each of the individual events. If they are not mutually exclusive then we need to take the intersection into account.

Restaurants improve customer satisfaction and efficiency by analysing data from orders, loyalty programs, etc. Menus and marketing can use data such as the proportion of brunch customers who order the crab omelette or blueberry pancakes or both.

If we take 15 people who like apples (A) or bananas (B), for example, we could illustrate this with the following possible Venn diagram.



$$\Pr(A) = \frac{10}{15}$$
$$\Pr(B) = \frac{9}{15}$$

Clearly, the probability that a person likes apples or bananas is not $\frac{10}{15} + \frac{9}{15} = \frac{19}{15}$ as this is impossible. The intersection needs to be taken into account because, in the example above, this has been counted twice. This consideration leads to the addition rule, which will be explained in this section.

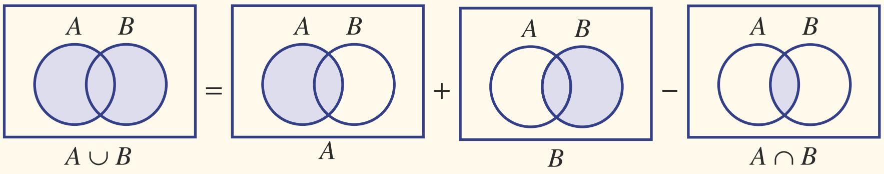
LESSON STARTER What's the intersection?

Two events, A and B, are such that $\Pr(A) = 0.5$, $\Pr(B) = 0.4$ and $\Pr(A \cup B) = 0.8$.

- Are the events mutually exclusive? Why?
- Is it possible to find $\Pr(A \cap B)$? If so, find $\Pr(A \cap B)$.
- Can you write a rule connecting $\Pr(A \cup B)$, $\Pr(A)$, $\Pr(B)$ and $\Pr(A \cap B)$?
- Does your rule hold true for mutually exclusive events?

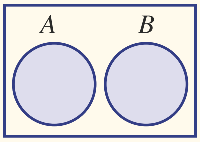
KEY IDEAS

- The **addition rule** for two events, A and B, is:
  $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$



ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

- If A and B are mutually exclusive then:
  - $\Pr(A \cap B) = 0$
  - $\Pr(A \cup B) = \Pr(A) + \Pr(B)$



BUILDING UNDERSTANDING

1 A fair 6-sided die is rolled.

- Event A is rolling a number greater than 3.
- Event B is rolling an even number.

a State the sets.

i A **ii** B

iii A or B (i.e. $A \cup B$) **iv** A and B (i.e. $A \cap B$)

b Are events A and B mutually exclusive? Give a reason.

c Find $\Pr(A \cup B)$.

2 Use the given information and the addition rule to find $\Pr(A \cup B)$.

a $\Pr(A) = 0.7, \Pr(B) = 0.5, \Pr(A \cap B) = 0.4$

b $\Pr(A) = 0.65, \Pr(B) = 0.4, \Pr(A \cap B) = 0.35$

3 Use the addition rule to find $\Pr(A \cap B)$ if $\Pr(A \cup B) = 0.9, \Pr(A) = 0.5$ and $\Pr(B) = 0.45$.

Example 6 Applying the addition rule

A card is selected from a standard deck of 52 playing cards (4 suits, no jokers). Let A be the event 'the card is a diamond' and B be the event 'the card is a jack'.

a Find:

i $n(A)$ **ii** $n(B)$ **iii** $n(A \cap B)$

b Find:

i $\Pr(A)$ **ii** $\Pr(A')$ **iii** $\Pr(A \cap B)$

c Use the addition rule to find $\Pr(A \cup B)$.

d Find the probability that the card is a jack or not a diamond.

SOLUTION	EXPLANATION
a i $n(A) = 13$	One-quarter of the cards is the diamond suit.
ii $n(B) = 4$	There is one jack in each suit.
iii $n(A \cap B) = 1$	Only one card is both a diamond and a jack.
b i $\Pr(A) = \frac{13}{52} = \frac{1}{4}$	13 out of the 52 cards are diamond.
ii $\Pr(A') = 1 - \frac{1}{4}$ $= \frac{3}{4}$	The complement of A is A'.
iii $\Pr(A \cap B) = \frac{1}{52}$	There is one jack of diamonds out of the 52 cards.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

c $\Pr(A \cup B)$
$= \Pr(A) + \Pr(B) - \Pr(A \cap B)$
$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52}$
$= \frac{16}{52}$
$= \frac{4}{13}$

Substitute $\Pr(A)$, $\Pr(B)$ and $\Pr(A \cap B)$ into the addition rule to find $\Pr(A \cup B)$.

d Pr(jack *or* not a diamond)
$= \Pr(B \cup A')$
$= \Pr(B) + \Pr(A') - \Pr(B \cap A')$
$= \frac{4}{52} + \frac{39}{52} - \frac{3}{52}$
$= \frac{40}{52}$
$= \frac{10}{13}$

Use the addition rule.

There are 4 jacks and 39 cards that are not diamonds.
There are 3 cards that are both jacks and not diamonds.

Now you try

A card is selected from a standard deck of 52 playing cards (4 suits, no jokers). Let A be the event 'the card is a club' and B be the event 'the card is a queen'.

a Find:
i $n(A)$ **ii** $n(B)$ **iii** $n(A \cap B)$

b Find:
i $\Pr(A)$ **ii** $\Pr(A')$ **iii** $\Pr(A \cap B)$

c Use the addition rule to find $\Pr(A \cup B)$.

d Find the probability that the card is a queen or not a club.

Example 7 Using the addition rule

Two events, A and B, are such that $\Pr(A) = 0.4$, $\Pr(B) = 0.8$ and $\Pr(A \cup B) = 0.85$.
Find:
a $\Pr(A \cap B)$ **b** $\Pr(A' \cap B')$

SOLUTION

a
$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$
$$0.85 = 0.4 + 0.8 - \Pr(A \cap B)$$
$$0.85 = 1.2 - \Pr(A \cap B)$$
$$\therefore \Pr(A \cap B) = 1.2 - 0.85$$
$$= 0.35$$

EXPLANATION

Write the addition rule and substitute the given information.
Simplify and solve for $\Pr(A \cap B)$.

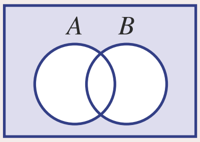
Continued on next page

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b $\Pr(A' \cap B') = 1 - 0.85$
$= 0.15$

$A' \cap B' = (A \cup B)'$



Now you try

Two events, A and B, are such that $\Pr(A) = 0.5$, $\Pr(B) = 0.6$ and $\Pr(A \cup B) = 0.75$.
Find:

a $\Pr(A \cap B)$ **b** $\Pr(A' \cap B')$

Exercise 8C

FLUENCY	1–4	2–5	2–5

Example 6

1 A card is selected from a standard deck of 52 playing cards. Let A be the event 'the card is a spade' and B be the event 'the card is an ace'.

a Find:
  i $n(A)$ **ii** $n(B)$ **iii** $n(A \cap B)$

b Find:
  i $\Pr(A)$ **ii** $\Pr(A')$ **iii** $\Pr(A \cap B)$

c Use the addition rule to find $\Pr(A \cup B)$.

d Find the probability that the card is an ace or not a spade.

Example 6

2 A number is chosen from the set $\{1, 2, 3, \ldots, 20\}$. Let A be the event 'choosing a multiple of 3' and let B be the event 'choosing a prime number'.

a List set:
  i A **ii** B

b Find:
  i $\Pr(A \cap B)$ **ii** $\Pr(A \cup B)$

c Find the probability that the number is a prime and not a multiple of 3.

3 In a class of 24 students, 13 like Mathematics, 9 like English and 3 like both.

a Find the probability that a randomly selected student in this class likes both Mathematics and English.

b Find the probability that a randomly selected student in this class likes neither Mathematics nor English.

Example 7

4 Two events, A and B, are such that $\Pr(A) = 0.3$, $\Pr(B) = 0.6$ and $\Pr(A \cup B) = 0.8$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(A' \cap B')$

5 Two events, A and B, are such that $\Pr(A) = 0.45$, $\Pr(B) = 0.75$ and $\Pr(A \cup B) = 0.9$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(A' \cap B')$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

PROBLEM-SOLVING	6, 7	6, 7	7, 8

6 Of 32 cars at a show, 18 cars have four-wheel drive, 21 are sports cars and 27 have four-wheel drive or are sports cars.

a Find the probability that a randomly selected car at the show is both four-wheel drive and a sports car.

b Find the probability that a randomly selected car at the show is neither four-wheel drive nor a sports car.

7 A card is selected from a standard deck of 52 playing cards. Find the probability that the card is:

a a heart or a king
b a club or a queen
c a black card or an ace
d a diamond or not a king
e a king or not a heart
f a 10 or not a spade.

8 **a** Find $\Pr(A \cap B')$ when $\Pr(A \cup B) = 0.8$, $\Pr(A) = 0.5$ and $\Pr(B) = 0.4$.

b Find $\Pr(A' \cap B)$ when $\Pr(A \cup B) = 0.76$, $\Pr(A) = 0.31$ and $\Pr(B) = 0.59$.

REASONING	9	9, 10	10, 11

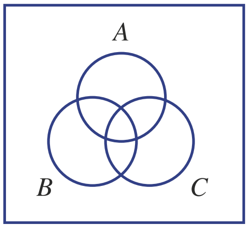
9 Why does the addition rule become $\Pr(A \cup B) = \Pr(A) + \Pr(B)$ for mutually exclusive events?

10 Explain why the following represent impossible events.

a $\Pr(A) = 0.3, \Pr(B) = 0.5, \Pr(A \cap B) = 0.4$

b $\Pr(A \cup B) = 0.75, \Pr(A) = 0.32, \Pr(B) = 0.39$

11 Write down an addition rule for $\Pr(A \cup B \cup C)$ using sets A, B and C.



ENRICHMENT: Divisibility and the addition rule	–	–	12, 13

12 A number is randomly selected from the first 20 positive integers. Find the probability that it is divisible by:

a 3
b 4
c 2 and 3
d 2 or 3
e 3 or 5
f 2 or 5

13 A number is randomly selected from the first 500 positive integers. Find the probability that it is divisible by:

a 4
b 7
c 3 and 5
d 2 and 7
e 3 and 8
f 3, 7 and 9

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

8D Conditional probability

Learning intentions

- To understand the notion of conditional probability and that extra information can alter a probability
- To know how to use a Venn diagram or two-way table to determine a conditional probability
- To be able to identify a conditional probability scenario in a word problem

The mathematics associated with the probability that an event occurs given that another event has already occurred is called conditional probability.

Consider, for example, a group of primary school students who have bicycles for a special cycling party. Some of the bicycles have gears, some have suspension and some have both gears and suspension. Consider these two questions.

- What is the probability that a randomly selected bicycle has gears?
- What is the probability that a randomly selected bicycle has gears given that it has suspension?

The second question is conditional in that we already know that the bicycle has suspension.

Loyalty programs that track customers' buying habits assist with targeted advertising. Analysing customer data, a clothing retailer could find the fraction of customers aged under 30 who spend over $100 on one piece of clothing.

LESSON STARTER Gears and suspension

Suppose that in a group of 18 bicycles, 9 have gears, 11 have suspension and 5 have both gears and suspension. Discuss the solution to the following question by considering the points below.

What is the probability that a randomly selected bicycle will have gears given that it has suspension?

- Illustrate the information on a Venn diagram.
- How many of the bicycles that have suspension have gears?
- Which areas in the Venn diagram are to be considered when answering the question? Give reasons.
- What would be the answer to the question in reverse; i.e. what is the probability that a bicycle will have suspension given that it has gears?

KEY IDEAS

- The probability of event A occurring given that event B has occurred is denoted by $\Pr(A|B)$, which reads 'the probability of A given B'. This is known as **conditional probability**.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \text{ and } \Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}$$

- For problems in this section these rules can be simplified to:

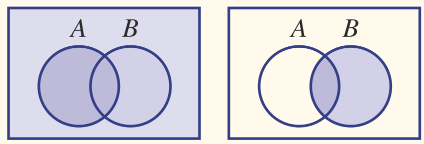
$$\Pr(A|B) = \frac{n(A \cap B)}{n(B)} \text{ and } \Pr(B|A) = \frac{n(A \cap B)}{n(A)}$$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

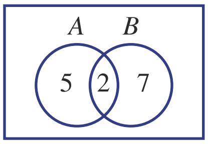
- $\Pr(A|B)$ differs from $\Pr(A)$ in that the sample space is reduced to the set B, as shown in these Venn diagrams.

$\Pr(A)$ $\Pr(A|B)$



BUILDING UNDERSTANDING

1. In a group of 20 people, 15 are wearing jackets and 10 are wearing hats; 5 are wearing both a jacket and a hat.
   a. What fraction of the people who are wearing jackets are wearing hats?
   b. What fraction of the people who are wearing hats are wearing jackets?
2. Use this Venn diagram to answer these questions.



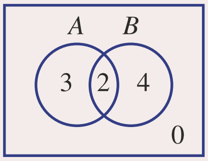
   a. i. Find $n(A \cap B)$. ii. Find $n(B)$.
   b. Find $\Pr(A|B)$ using $\Pr(A|B) = \frac{n(A \cap B)}{n(B)}$.
3. Use this two-way table to answer these questions.

	A	A'	
B	7	5	12
B'	3	1	4
	10	6	16

   a. i. Find $n(A \cap B)$. ii. Find $n(A)$.
   b. Find $\Pr(B|A)$ using $\Pr(B|A) = \frac{n(A \cap B)}{n(A)}$.
   c. Find $\Pr(A|B)$.

Example 8 Finding conditional probabilities using a Venn diagram

Consider this Venn diagram displaying the number of elements belonging to the events A and B.



Find the following probabilities.

a $\Pr(A)$ b $\Pr(A \cap B)$ c $\Pr(A|B)$ d $\Pr(B|A)$

SOLUTION

a $\Pr(A) = \frac{5}{9}$

b $\Pr(A \cap B) = \frac{2}{9}$

c $\Pr(A|B) = \frac{2}{6} = \frac{1}{3}$

d $\Pr(B|A) = \frac{2}{5}$

EXPLANATION

There are 5 elements in A and 9 in total.

There are 2 elements common to A and B.

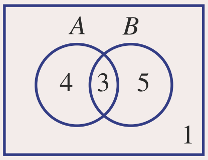
2 of the 6 elements in B are in A.

2 of the 5 elements in A are in B.

Continued on next page

Now you try

Consider this Venn diagram displaying the number of elements belonging to the events A and B.



Find the following probabilities.

a $\Pr(A)$ **b** $\Pr(A \cap B)$ **c** $\Pr(A|B)$ **d** $\Pr(B|A)$

Example 9 Finding conditional probabilities using a two-way table

From a group of 15 hockey players at a game of hockey, 13 played on the field, 7 sat on the bench and 5 both played and sat on the bench.

A hockey player is chosen at random from the team.

Let A be the event 'the person played on the field' and let B be the event 'the person sat on the bench'.

a Represent the information in a two-way table.
b Find the probability that the person only sat on the bench.
c Find the probability that the person sat on the bench given that they played on the field.
d Find the probability that the person played on the field given that they sat on the bench.

SOLUTION

a

	A	A'	
B	5	2	7
B'	8	0	8
	13	2	15

EXPLANATION

$n(A \cap B) = 5, n(A) = 13, n(B) = 7$. The total is 15. Insert these values and then fill in the other places to ensure the rows and columns give the required totals.

b $\Pr(B \cap A') = \dfrac{2}{15}$

Two people sat on the bench and did not play on the field.

c $\Pr(B|A) = \dfrac{5}{13}$

$n(B \cap A) = 5$ and $n(A) = 13$.

d $\Pr(A|B) = \dfrac{5}{7}$

$n(A \cap B) = 5$ and $n(B) = 7$.

Now you try

In a group of 23 movie goers, 13 bought popcorn, 15 bought a Coke and 9 bought popcorn and a Coke. One of the movie goers is selected at random.

Let A be the event 'the person bought popcorn' and B be the event 'the person bought a Coke'.

a Represent the information in a two-way table.
b Find the probability that the person only bought popcorn.
c Find the probability that the person bought popcorn given that they bought a Coke.
d Find the probability that the person bought a Coke given that they bought popcorn.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

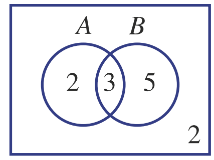
Exercise 8D

FLUENCY	1, 2–3(1/2)	2–3(1/2), 4	2–3(1/4), 4, 5

Example 8

1 Consider this Venn diagram displaying the number of elements belonging to the events A and B. Find the following probabilities.

a $\Pr(A)$
b $\Pr(A \cap B)$
c $\Pr(A|B)$
d $\Pr(B|A)$

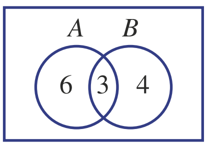


Example 8

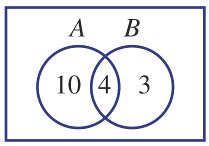
2 The following Venn diagrams display information about the number of elements associated with the events A and B. For each Venn diagram, find:

i $\Pr(A)$
ii $\Pr(A \cap B)$
iii $\Pr(A|B)$
iv $\Pr(B|A)$

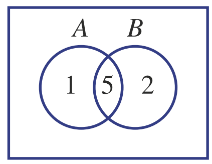
a



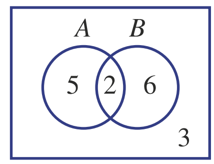
b



c



d



3 The following two-way tables show information about the number of elements in the events A and B. For each two-way table, find:

i $\Pr(A)$
ii $\Pr(A \cap B)$
iii $\Pr(A|B)$
iv $\Pr(B|A)$

a

	A	A'	
B	2	8	10
B'	5	3	8
	7	11	18

b

	A	A'	
B	1	4	5
B'	3	1	4
	4	5	9

c

	A	A'	
B	7	3	10
B'	1	6	7
	8	9	17

d

	A	A'	
B	4	2	6
B'	8	2	10
	12	4	16

Example 9

4 Of a group of 20 English cricket fans at a match, 13 purchased a pie, 15 drank beer and 9 both purchased a pie and drank beer.
Let A be the event 'the fan purchased a pie'.
Let B be the event 'the fan drank beer'.

a Represent the information in a two-way table.
b Find the probability that a fan in the group only purchased a pie (and did not drink beer).
c Find the probability that a fan in the group purchased a pie given that they drank beer.
d Find the probability that a fan in the group drank beer given that they purchased a pie.

ISBN 978-1-108-77290-7

5 Of 15 musicians surveyed to find out whether they play the violin or the piano, 5 play the violin, 8 play the piano and 2 play both instruments.

- **a** Represent the information in a Venn diagram.
- **b** How many of the musicians surveyed do not play either the violin or the piano?
- **c** Find the probability that one of the 15 musicians surveyed plays piano given that they play the violin.
- **d** Find the probability that one of the 15 musicians surveyed plays the violin given that they play the piano.

PROBLEM-SOLVING	6, 7	6, 7	8, 9

6 On a car production line, 30 cars are due to be completed by the end of the day. Fifteen of the cars have cruise control and 20 have airbags, and 6 have both cruise control and airbags.

- **a** Represent the information provided in a Venn diagram or two-way table.
- **b** Find the probability that a car chosen at random will contain the following.
  - **i** cruise control only
  - **ii** airbags only
- **c** Given that the car chosen has cruise control, find the probability that the car will have airbags.
- **d** Given that the car chosen has airbags, find the probability that the car will have cruise control.

7 For each of the following, complete the given two-way tables and find:

i $n(A' \cap B')$ **ii** $\Pr(B|A)$ **iii** $\Pr(A|B)$

a

	A	A'	
B	2		4
B'			
	5		8

b

	A	A'	
B	3		16
B'			
	8		27

8 A card is drawn from a standard deck of 52 playing cards. Find the probability that:

- **a** the card is a king given that it is a heart
- **b** the card is a jack given that it is a red card
- **c** the card is a diamond given that it is a queen
- **d** the card is a black card given that it is an ace.

9 A number is chosen from the first 24 positive integers. Find the probability that:

- **a** the number is divisible by 3 given that it is divisible by 4
- **b** the number is divisible by 6 given that is divisible by 3.

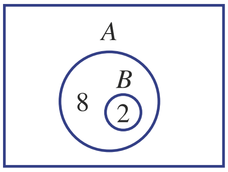
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

REASONING	10	10, 11	11, 12

10 Two events, A and B, are mutually exclusive. What can be said about the probability of A given B (i.e. $\Pr(A|B)$) or the probability of B given A (i.e. $\Pr(B|A)$)? Give a reason.

11 Two events, A and B, are such that B is a subset of A, as shown in this Venn diagram.



a Find $\Pr(A|B)$. **b** Find $\Pr(B|A)$.

12 a Rearrange the rule $\Pr(B|A) = \dfrac{\Pr(A \cap B)}{\Pr(A)}$ to make $\Pr(A \cap B)$ the subject.

b Hence, find $\Pr(A \cap B)$ when $\Pr(B|A) = 0.3$ and $\Pr(A) = 0.6$.

ENRICHMENT: Investment workshops	–	–	13

13 People aged between 20 and 50 years attended a workshop on shares, property or cash at an investment conference. The number of people attending each workshop is shown in this table.

Workshop	20–29 years	30–39 years	40–50 years
Shares	40	85	25
Property	18	57	6
Cash	5	32	61

a How many people attended the conference?

b Find the probability that a randomly selected person at the conference is aged between 30 and 39 years.

c Find the probability that a randomly selected person at the conference attends the property workshop.

d Find the probability that a randomly selected person at the conference attends the property workshop given they are not in the 30–39 age group.

e Find the probability that a randomly selected person at the conference is aged between 40 and 50 years given that they do not attend the cash workshop.

f Find the probability that a randomly selected person at the conference does not attend the shares workshop given they are not in the 30–39 age group.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

8A **1** A letter is selected at random from the word SUPERCALIFRAGILISTIC. Find the probability that the letter is:

a an S **b** an A **c** a P **d** a vowel **e** not a vowel.

8A **2** An experiment involves choosing a card from a standard deck of 52 playing cards and recording its suit. Here are the results of running the experiment 100 times.

Suit	heart	club	spade	diamond
Frequency	17	29	38	16

Find the experimental probability of obtaining:

a a heart **b** a club **c** a red card **d** a black card.

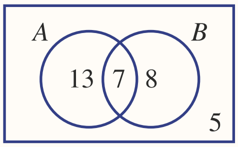
8A **3** A card is chosen from a standard deck of 52 playing cards. Find the theoretical probability of the card being:

a a king **b** a red king **c** the queen of hearts
d a black 10 or a red queen **e** an ace **f** not an ace.

8B **4** From a class of 50 students, 32 enjoy tennis and 20 enjoy squash. If 3 students enjoy both sports:

a Display this information in a Venn diagram and a two-way table.
b State the number of students who enjoy only tennis.
c Write down the probability of choosing one student from the group and that student enjoying neither tennis nor squash.

8B **5** The Venn diagram shows the distribution of elements in two sets, A and B. Find:

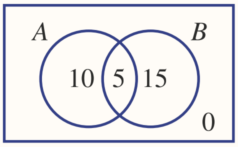


a $n(A \cap B)$ **b** $n(A)$ **c** $n(A' \cap B')$
d $\Pr(A \cap B)$ **e** $\Pr(A')$ **f** $\Pr(A \cup B)$

8C **6** Two events, A and B, are such that $\Pr(A) = 0.7$, $\Pr(B) = 0.45$ and $\Pr(A \cap B) = 0.32$. Find:

a $\Pr(A \cup B)$ **b** $\Pr(A' \cap B')$

8D **7** Consider the Venn diagram displaying the number of elements belonging to events A and B. Find:



a $\Pr(A)$ **b** $\Pr(A \cap B)$ **c** $\Pr(A|B)$ **d** $\Pr(B|A)$

8D **8** From a survey of 40 students in the playground, 35 like soft drink, 20 like water and 15 like both soft drink and water.

a Display this information in a two-way table.
b Find the probability that a student chosen from the group dislikes soft drink given that they like water.

ISBN 978-1-108-77290-7

8E Two-step experiments using tables

Learning intentions

- To be able to construct a table to systematically display the outcomes of a two-step experiment
- To understand the difference between 'with replacement' and 'without replacement' and their impact on the possible outcomes of a two-step experiment
- To be able to calculate probabilities from a sample space in a table

When an experiment involves two or more components, like flipping a coin twice or selecting 3 chocolates from a box, we are dealing with multi-stage experiments. The outcomes for such an event depend on whether or not they are conducted with or without replacement. For two-step experiments, tables are helpful when listing all the possible outcomes.

Geneticists use 'two-parent' tables to find proportions of inherited traits, such as the maize kernel colours purple (9/16), red (3/16) and white (4/16). Research of inheritance patterns in maize led to revolutionary findings of how chromosomes change during reproduction.

LESSON STARTER Does replacement matter?

From the digits { 1, 2, 3, 4, 5 } you select two of these to form a two-digit number.

- How many numbers can be formed if selections are made with replacement?
- How many numbers can be formed if selections are made without replacement?

32 14 25

22 53

31 34 42

not allowed
without replacement

- Find the probability that the number 35 is formed if selections are made with replacement.
- Find the probability that the number 35 is formed if selections are made without replacement.

KEY IDEAS

- Tables (or arrays) can be used to list the sample space for **two-step experiments**.
- If **replacement** is allowed then outcomes from each selection can be repeated.
- If selections are made **without replacement** then outcomes from each selection cannot be repeated.
  For example: Two selections are made from the digits { 1, 2, 3 }.

With replacement

		1st		
		1	2	3
	1	(1, 1)	(2, 1)	(3, 1)
2nd	2	(1, 2)	(2, 2)	(3, 2)
	3	(1, 3)	(2, 3)	(3, 3)

Without replacement

		1st		
		1	2	3
	1	×	(2, 1)	(3, 1)
2nd	2	(1, 2)	×	(3, 2)
	3	(1, 3)	(2, 3)	×

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

BUILDING UNDERSTANDING

1 Two letters are chosen from the word DOG.
These tables list the sample space if selections are made:

with replacement

		1st		
		D	O	G
	D	(D, D)	(O, D)	(G, D)
2nd	O	(D, O)	(O, O)	(G, O)
	G	(D, G)	(O, G)	(G, G)

without replacement

		1st		
		D	O	G
	D	×	(O, D)	(G, D)
2nd	O	(D, O)	×	(D, G)
	G	(D, G)	(O, G)	×

a State the total number of outcomes if selection is made:
  i with replacement **ii** without replacement.
b If selection is made with replacement, find the probability that:
  i the two letters are the same **ii** there is at least one D **iii** there is not an O
c If selection is made without replacement, find the probability that:
  i the two letters are the same **ii** there is at least one D **iii** there is not an O

2 Two digits are selected from the set $\{2, 3, 4\}$ to form a two-digit number. Find the number of two-digit numbers that can be formed if the digits are selected:
a with replacement **b** without replacement.

Example 10 Constructing a table with replacement

A fair 6-sided die is rolled twice.
a List the sample space, using a table.
b State the total number of outcomes.
c Find the probability of obtaining the outcome $(1, 5)$.
d Find:
  i Pr(double) **ii** Pr(sum of at least 10) **iii** Pr(sum not equal to 7)
e Find the probability of a sum of 12, given that the sum is at least 10.

SOLUTION

a

				Roll 1			
		1	2	3	4	5	6
	1	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
	2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
	3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
Roll 2	4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
	5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
	6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

EXPLANATION

Be sure to place the number from roll 1 in the first position for each outcome.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b 36 outcomes

There is a total of $6 \times 6 = 36$ outcomes.

c $\Pr(1, 5) = \frac{1}{36}$

Only one outcome is $(1, 5)$.

d **i** $\Pr(\text{double}) = \frac{6}{36} = \frac{1}{6}$

Six outcomes have the same number repeated.

ii $\Pr(\text{sum of at least } 10) = \frac{6}{36} = \frac{1}{6}$

Six outcomes have a sum of either 10, 11 or 12.

iii $\Pr(\text{sum not equal to } 7) = 1 - \frac{6}{36} = \frac{5}{6}$

This is the complement of having a sum of 7. Six outcomes have a sum of 7.

e $\Pr(\text{sum of } 12 \mid \text{sum of at least } 10) = \frac{1}{6}$

One of the 6 outcomes with a sum of at least 10 has a sum of 12.

Now you try

A fair 6-sided die is rolled twice.

a Find the probability of obtaining the outcome $(6, 4)$.

b Find:

i $\Pr((3, 2) \text{ or } (2, 3))$

ii $\Pr(\text{sum of at least } 9)$

iii $\Pr(\text{sum less than } 4)$

c Find the probability of a sum of 3 given that the sum is at most 4.

Example 11 Constructing a table without replacement

Two letters are chosen from the word KICK without replacement.

a Construct a table to list the sample space.

b Find the probability of:

i obtaining the outcome (K, C)

ii selecting two Ks

iii selecting a K and a C

iv selecting two Ks given that at least one K is selected.

Continued on next page

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

SOLUTION

a

		1st			
		K	I	C	K
2nd	K	×	(I, K)	(C, K)	(K, K)
	I	(K, I)	×	(C, I)	(K, I)
	C	(K, C)	(I, C)	×	(K, C)
	K	(K, K)	(I, K)	(C, K)	×

b **i** $\Pr(\text{K, C}) = \frac{2}{1} = \frac{1}{6}$

ii $\Pr(\text{K, K}) = \frac{2}{12} = \frac{1}{6}$

iii $\Pr(\text{K} \cap \text{C}) = \frac{4}{12} = \frac{1}{3}$

iv $\Pr(2 \text{ Ks} \mid \text{at least } 1 \text{ K}) = \frac{2}{10} = \frac{1}{5}$

EXPLANATION

Selection is without replacement, so the same letter (from the same position) cannot be chosen twice.

Two of the 12 outcomes are (K, C).

Two of the outcomes are K and K, which use different Ks from the word KICK.

Four outcomes contain a K and a C.

There are 10 outcomes with at least one K, two of which have two Ks.

Now you try

Two cars are chosen from a group of four without replacement. Of the four cars, two are red (R), one is blue (B) and one is white (W).

a Construct a table to list the sample space.

b Find the probability of:

i obtaining a red car first then a white car

ii selecting two red cars

iii selecting one red car and one blue car

iv selecting two red cars given that at least one of them is a red car.

Exercise 8E

FLUENCY	1–3	1, 3, 4	2–4

Example 10a–d

1 A fair 4-sided die is rolled twice.

a List the sample space, using a table.

b State the total number of possible outcomes.

c Find the probability of obtaining the outcome (2, 4).

d Find the probability of:

i a double **ii** a sum of at least 5 **iii** a sum not equal to 4.

Example 10a–d

2 Two coins are tossed, each landing with a head (H) or tail (T).

a List the sample space, using a table.

b State the total number of possible outcomes.

c Find the probability of obtaining the outcome (H, T).

d Find the probability of obtaining:

i one tail **ii** at least one tail.

e If the two coins are tossed 1000 times, how many times would you expect to get two tails?

ISBN 978-1-108-77290-7
Cambridge University Press
Updated June 2022

Example 11

3 Two letters are chosen from the word SET without replacement.

- **a** Show the sample space, using a table.
- **b** Find the probability of:
  - **i** obtaining the outcome (E, T)
  - **ii** selecting one T
  - **iii** selecting at least one T
  - **iv** selecting an S and a T
  - **v** selecting an S or a T.

4 A letter is chosen from the word LEVEL without replacement and then a second letter is chosen from the same word.

- **a** Draw a table displaying the sample space for the pair of letters chosen.
- **b** State the total number of outcomes possible.
- **c** State the number of outcomes that contain exactly one of the following letters.
  - **i** V
  - **ii** L
  - **iii** E
- **d** Find the probability that the outcome will contain exactly one of the following letters.
  - **i** V
  - **ii** L
  - **iii** E
- **e** Find the probability that the two letters chosen will be the same.

PROBLEM-SOLVING

5, 6	5, 6	6, 7

5 In a quiz, Min guesses that the probability of rolling a sum of 10 or more from two fair 6-sided dice is 10%. Complete the following to decide whether or not this guess is correct.

		Die 1					
		1	2	3	4	5	6
Die 2	1	2	3	…			
	2	3	…				
	3	4					
	4	⋮					
	5	⋮					
	6						

- **a** Copy and complete the table representing all the outcomes for possible totals that can be obtained.
- **b** State the total number of outcomes.
- **c** Find the number of the outcomes that represent a sum of:
  - **i** 3
  - **ii** 7
  - **iii** less than 7.
- **d** Find the probability that the following sums are obtained.
  - **i** 7
  - **ii** less than 5
  - **iii** greater than 2
  - **iv** at least 11
- **e** Find the probability that the sum is at least 10, and decide whether or not Min's guess is correct.

6 A letter is randomly chosen from the word OLD and then a second letter is chosen from the word COLLEGE.

- **a** Draw a table illustrating all possible pairs of letters that can be chosen.
- **b** State the total number of outcomes.
- **c** If a double represents selecting the same letter, find the probability of selecting a double.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

7 The 10 students who completed a special flying course are waiting to see if they will be awarded the one Distinction or the one Merit award for their efforts.

a In how many ways can the two awards be given if:
  i the same student can receive both awards?
  ii the same student cannot receive both awards?

b Assuming that a student cannot receive both awards, find the probability that a particular student receives:
  i the Distinction award
  ii the Merit award
  iii neither award.

c Assuming that a student can receive both awards, find the probability that they receive at least one award.

REASONING	8	8, 9	8, 10

Example 10e

8 Two fair 4-sided dice are rolled and the sum is noted.

a Find the probability of:
  i a sum of 5
  ii a sum of less than 6.

b i Find the probability of a sum of 5 given that the sum is less than 6.
  ii Find the probability of a sum of 2 given that the sum is less than 6.
  iii Find the probability of a sum of 7 given that the sum is at least 7.

9 Decide whether the following situations would naturally involve selections with replacement or without replacement.

a selecting two people to play in a team
b tossing a coin twice
c rolling two dice
d choosing two chocolates to eat

10 In a game of chance, six cards numbered 1 to 6 are lying face down on a table. Two cards are selected without replacement and the sum of both numbers is noted.

a State the total number of outcomes.

b Find the probability that the total sum is:
  i equal to 3
  ii equal to 4
  iii at least 10
  iv no more than 5.

c What would have been the answer to part b i if the experiment had been conducted with replacement?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

ENRICHMENT: Random weights	–	–	11

11 In a gym, Justin considers choosing two weights to fit onto a rowing machine to make the load heavier There are four different weights to choose from: 2.5 kg, 5 kg, 10 kg and 20 kg, and there are plenty of each weight available. After getting a friend to randomly choose both weights, Justin attempts to operate the machine.

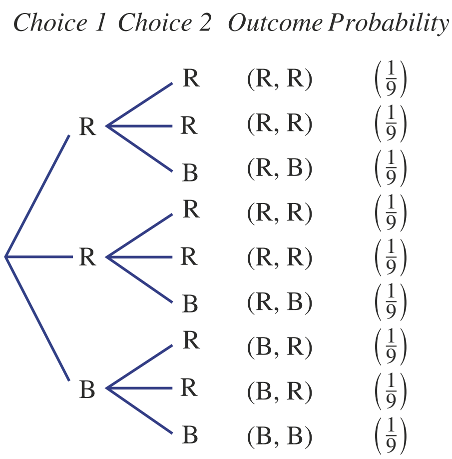
- **a** Complete a table that displays all possible total weights that could be placed on the machine.
- **b** State the total number of outcomes.
- **c** How many of the outcomes deliver a total weight described by the following?
  - **i** equal to 10 kg
  - **ii** less than 20 kg
  - **iii** at least 20 kg
- **d** Find the probability that Justin will be attempting to lift the following total weight.
  - **i** 20 kg
  - **ii** 30 kg
  - **iii** no more than 10 kg
  - **iv** less than 10 kg
- **e** If Justin is unable to lift more than 22 kg, what is the probability that he will not be able to operate the rowing machine?

ISBN 978-1-108-77290-7

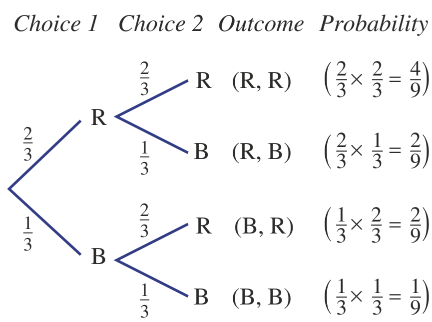
8F Using tree diagrams

Learning intentions
- To be able to draw a tree diagram to show the sample space of two or more stage experiments
- To understand when it is appropriate to use a tree diagram to display outcomes
- To know how to determine probabilities on tree diagram branches using with or without replacement
- To be able to find probabilities of event outcomes using a tree diagram

Suppose a bag contains two red counters and one blue counter and that two counters are selected at random with replacement. One way to display the outcomes is with a tree diagram in which all equally likely outcomes are listed in columns, as shown below left. A more efficient way, however, is to group similar outcomes and write their corresponding probabilities on the branches, as shown below right.



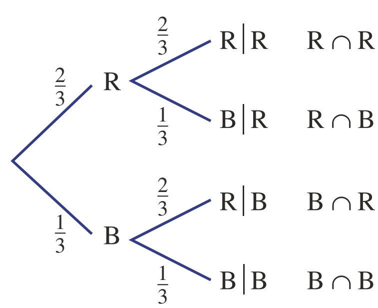
$\Pr(\text{R, B}) = \frac{2}{9}$



$\Pr(\text{R, B}) = \frac{2}{9}$

You will note that in the tree diagram on the right the probability of each outcome is obtained by multiplying the branch probabilities. The reason for this relates to conditional probabilities.

Using conditional probabilities, the tree diagram on the right above can be redrawn like this (right).



We know from conditional probability that:

- $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$

Using B and R we could write:

- $\Pr(R|B) = \dfrac{\Pr(B \cap R)}{\Pr(B)}$

By rearranging we have:

$$\begin{aligned}\Pr(B \cap R) &= \Pr(B) \times \Pr(R|B)\\ &= \frac{1}{3} \times \frac{2}{3}\\ &= \frac{2}{9}\end{aligned}$$

This explains why we multiply branches on tree diagrams. This also applies when selection is made without replacement.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

LESSON STARTER Prize probability

Two lucky door prizes are randomly awarded to a group of 7 male and 3 female partygoers.

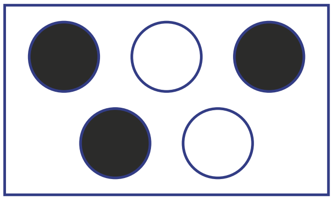
- Use a tree diagram with branch probabilities to show how selection with replacement can be displayed.
- Use a tree diagram with branch probabilities to show how selection without replacement can be displayed.
- Which of these situations has a higher probability?
  - **a** A male and a female receive one prize each if selection is made with replacement.
  - **b** A male and a female receive one prize each if selection is made without replacement.

KEY IDEAS

- **Tree diagrams** can be used to list the sample space for experiments involving two or more stages.
  - Branch probabilities are used to describe the chance of each outcome at each step.
  - Each outcome for the experiment is obtained by multiplying the branch probabilities.
  - Branch probabilities will depend on whether selection is made with or without replacement.

BUILDING UNDERSTANDING

1 A box contains 2 white (W) and 3 black (B) counters.



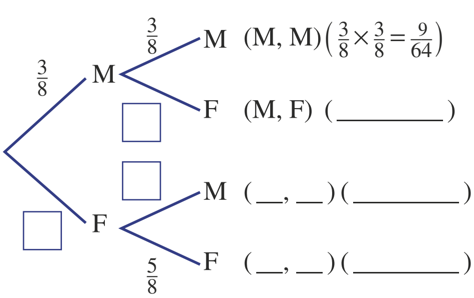
a A single counter is drawn at random. Find the probability that it is:
i white **ii** black

b Two counters are now drawn at random. The first one is replaced before the second one is drawn. Find the probability that the second counter is:
i white **ii** black

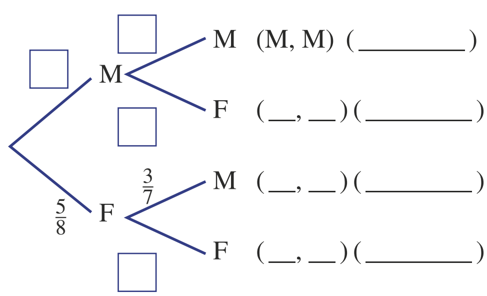
c Two counters are drawn and the first counter is not replaced before the second one is drawn. If the first counter is white, find the probability that the second counter is:
i white **ii** black

2 Two prizes are awarded to a group of 3 male (M) and 5 female (F) candidates. State the missing components in these tree diagrams.

a with replacement



b without replacement



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 12 Constructing a tree diagram for multi-stage experiments

Boxes A and B contain 4 counters each. Box A contains 2 red and 2 green counters and box B contains 1 red and 3 green counters. A box is chosen at random and then a single counter is selected.

a If box A is chosen, what is the probability that a red counter is chosen from it?

b If box B is chosen, what is the probability that a red counter is chosen from it?

c Represent the options available as a tree diagram that shows all possible outcomes and related probabilities.

d What is the probability of selecting box B and a red counter?

e What is the probability of selecting a red counter?

SOLUTION

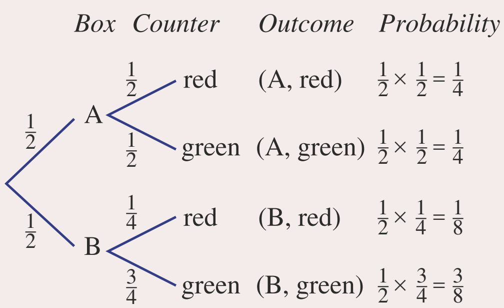
a $\text{Pr(red from box A)} = \dfrac{2}{4} = \dfrac{1}{2}$

Two of the 4 counters in box A are red.

b $\text{Pr(red from box B)} = \dfrac{1}{4}$

One of the 4 counters in box B is red.

c



First selection is a box followed by a counter. Multiply each of the probabilities along the branch pathways to find the probability of each outcome.

d $\text{Pr(B, red)} = \dfrac{1}{2} \times \dfrac{1}{4}$

$= \dfrac{1}{8}$

The probability of choosing box B is $\dfrac{1}{2}$ and a red counter from box B is $\dfrac{1}{4}$, so multiply these probabilities for the outcome (B, red).

e $\text{Pr(1 red)} = \dfrac{1}{2} \times \dfrac{1}{2} + \dfrac{1}{2} \times \dfrac{1}{4}$

$= \dfrac{1}{4} + \dfrac{1}{8}$

$= \dfrac{3}{8}$

The outcomes (A, red) and (B, red) both contain 1 red counter, so add together the probabilities for these two outcomes.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Now you try

Boxes A and B contain 5 counters each. Box A contains 3 red and 2 green counters and box B contains 1 red and 4 green counters. A box is chosen at random and then a single counter is selected.

a If box A is chosen, what is the probability that a red counter is chosen from it?
b If box B is chosen, what is the probability that a red counter is chosen from it?
c Represent the options available as a tree diagram that shows all possible outcomes and related probabilities.
d What is the probability of selecting box B and a red counter?
e What is the probability of selecting a red counter?

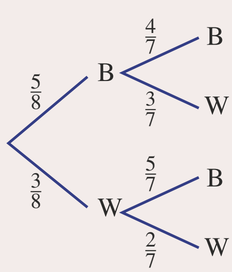
Example 13 Using a tree diagram for experiments without replacement

A bag contains 5 blue (B) and 3 white (W) marbles and 2 marbles are selected without replacement.

a Draw a tree diagram showing all outcomes and probabilities.
b Find the probability of selecting:
  i a blue marble followed by a white marble (B, W)
  ii 2 blue marbles
  iii exactly 1 blue marble.
c If the experiment is repeated with replacement, find the answers to each question in part **b**.

SOLUTION

a *Selection 1* *Selection 2* *Outcome* *Probability*



Outcome	Probability
(B, B)	$\frac{5}{8} \times \frac{4}{7} = \frac{5}{14}$
(B, W)	$\frac{5}{8} \times \frac{3}{7} = \frac{15}{56}$
(W, B)	$\frac{3}{8} \times \frac{5}{7} = \frac{15}{56}$
(W, W)	$\frac{3}{8} \times \frac{2}{7} = \frac{3}{28}$

EXPLANATION

After 1 blue marble is selected there are 7 marbles remaining: 4 blue and 3 white.

After 1 white marble is selected there are 7 marbles remaining: 5 blue and 2 white.

b i $\Pr(\text{B, W}) = \frac{5}{8} \times \frac{3}{7}$

$= \frac{15}{56}$

Multiply the probabilities on the (B, W) pathway.

ii $\Pr(\text{B, B}) = \frac{5}{8} \times \frac{4}{7}$

$= \frac{5}{14}$

Only 4 blue marbles remain after the first selection. Multiply the probabilities on the (B, B) pathway.

Continued on next page

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

iii $\Pr(1\text{ blue}) = \frac{5}{8} \times \frac{3}{7} + \frac{3}{8} \times \frac{5}{7}$

$= \frac{30}{56}$

$= \frac{15}{28}$

The outcomes (B, W) and (W, B) both have one blue marble. Multiply probabilities to find individual probabilities, then sum for the final result.

c **i** $\Pr(B, W) = \frac{5}{8} \times \frac{3}{8}$

$= \frac{15}{64}$

When selecting objects with replacement, remember that the number of marbles in the bag remains the same for each selection.

That is, $\Pr(B) = \frac{5}{8}$ and $\Pr(W) = \frac{3}{8}$ throughout.

ii $\Pr(B, B) = \frac{5}{8} \times \frac{5}{8}$

$= \frac{25}{64}$

iii $\Pr(1\text{ blue}) = \frac{5}{8} \times \frac{3}{8} + \frac{3}{8} \times \frac{5}{8}$

$= \frac{30}{64}$

$= \frac{15}{32}$

One blue marble corresponds to the (B, W) or (W, B) outcomes.

Now you try

A bag contains 4 blue (B) and 5 white (W) marbles and 2 marbles are selected without replacement.

a Draw a tree diagram showing all outcomes and probabilities.

b Find the probability of selecting:

i a blue marble followed by a white marble (B, W)

ii 2 blue marbles

iii exactly 1 blue marble.

c If the experiment is repeated with replacement, find the answers to each question in part **b**.

The spread of an infectious disease can be simply modelled with a tree diagram. If each ill person infects 4 other people, after 10 infection 'waves' over 1 million people are infected. Fortunately, vaccination strengthens the immune system, reducing infection rates.

ISBN 978-1-108-77290-7

Exercise 8F

FLUENCY	1–3	2–4	3, 4

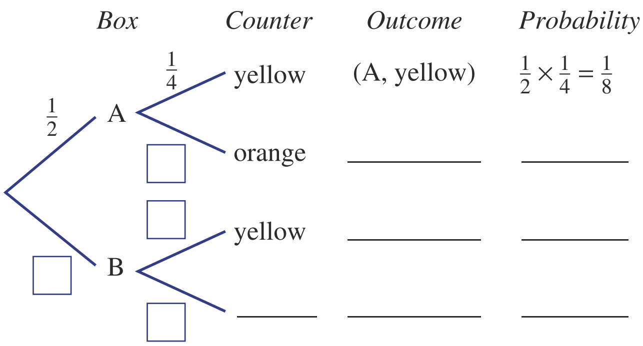
Example 12

1 Boxes A and B contain 4 counters each. Box A contains 1 yellow and 3 orange counters and box B contains 3 yellow and 1 orange counter. A box is chosen at random and then a singlecounter is selected.

a If box A is chosen, what is the probability of selecting a yellow counter?

b If box B is chosen, what is the probability of selecting a yellow counter?

c Represent the options available by completing this tree diagram.



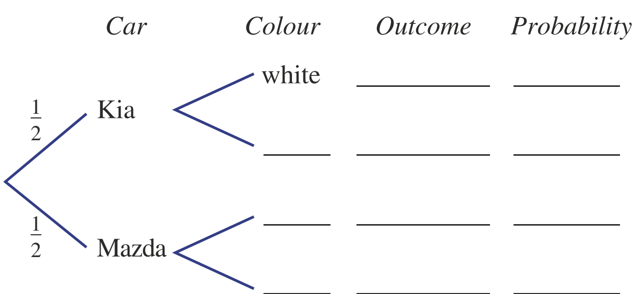
d What is the probability of selecting box B and a yellow counter?

e What is the probability of selecting a yellow counter?

Example 12

2 As part of a salary package an employee randomly selects a Kia or a Mazda. There are 3 white Kias and 1 silver Kia and 2 white Mazdas and 1 red Mazda to choose from.

a Complete a tree diagram showing a random choice of a car make and then a colour.



b Find the probability that the employee chooses:

i a white Kia

ii a red Mazda

iii a white car

iv a car that is not white

v a silver car or a white car

vi a car that is neither a Kia nor red.

Example 13

3 A bag contains 4 red (R) and 2 white (W) marbles, and 2 marbles are selected without replacement.

a Draw a tree diagram showing all outcomes and probabilities.

b Find the probability of selecting:

i a red marble and then a white marble (R, W)

ii 2 red marbles

iii exactly 1 red marble.

c The experiment is repeated with replacement. Find the answers to each question in part **b**.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 13

4 Two students are selected from a group of 3 males (M) and 4 females (F) without replacement.

a Draw a tree diagram to find the probability of selecting:

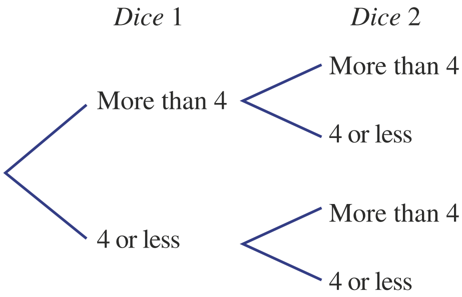
i 2 males
ii 2 females
iii 1 male and 1 female.
iv 2 people either both male or both female.

b The experiment is repeated with replacement. Find the answers to each question in part a.

PROBLEM-SOLVING	5, 6	5, 7	7, 8

5 Two dice are rolled and it is noted when the dice show a number more than 4 or not more than 4.

a Complete a tree diagram, showing the outcomes of the two dice.



b Find the probability that the two dice are:

i both more than 4
ii both 4 or less
iii not both 4 or less
iv one 4 or less and one more than 4.

6 Two bottles of wine are randomly selected for tasting from a box containing 2 red and 2 white wines. Use a tree diagram to help answer the following.

a If the first bottle is replaced before the second is selected, find:

i Pr(2 red)
ii Pr(1 red)
iii Pr(not 2 white)
iv Pr(at least 1 white)

b If the first bottle is not replaced before the second is selected, find:

i Pr(2 red)
ii Pr(1 red)
iii Pr(not 2 white)
iv Pr(at least 1 white)

7 Cans of sliced peaches produced by 'Just peaches' are sometimes underweight. A box of 10 cans is selected from the factory and then 2 cans from the 10 are tested without replacement. This particular box of 10 cans is known to have 2 cans that are underweight.

a State the probability that the first can chosen will be:

i underweight
ii not underweight.

b Use a tree diagram to find the probability that:

i both cans are underweight
ii one can is underweight
iii at most 1 can is underweight.

c The factory passes the inspection if no cans are found to be underweight. Find the chance that this will occur and express your answer as a percentage, rounded to one decimal place.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

8 The probability of rain on any particular day is 0.2. However, the probability of rain on a day after a rainy day is 0.85, whereas the probability of rain on a day after a non-rainy day is 0.1.

- **a** On two consecutive days, find the probability of having:
  - **i** two rainy days
  - **ii** exactly one rainy day
  - **iii** at least one dry day.
- **b** On three consecutive days, find the probability of having:
  - **i** three rainy days
  - **ii** exactly one dry day
  - **iii** at most two rainy days.

REASONING	9	9, 10	10, 11

9 Two socks are selected at random from a drawer containing 4 red and 4 yellow socks.

- **a** Find the probability that the two socks will be of the same colour if the socks are drawn without replacement.
- **b** Find the probability that the two socks will not be of the same colour if the socks are drawn without replacement.

10 A box contains 2 red (R) and 3 blue (B) counters and three counters are selected without replacement.

- **a** Use a tree diagram to find:
  - **i** Pr(R, R, B)
  - **ii** Pr(2 red)
  - **iii** Pr(3 red)
  - **iv** Pr(at least 1 red)
  - **v** Pr(at most 2 blue)
- **b** If a fourth selection is made without replacement, find the probability that:
  - **i** at least 1 red is selected
  - **ii** 3 blue are selected.

11 Containers A, B and C hold 4 marbles each, all of which are the same size. The following table illustrates the marble colours in each container.

	Container A	Container B	Container C
Purple	1	2	3
Green	3	2	1

A container is chosen at random and then a marble is selected from the container.

- **a** Draw a tree diagram to help determine all the possible outcomes and the associated probabilities. *Suggestion*: You will need three branches to start (representing the three different containers that can be chosen), followed by two branches for each of A, B and C (to represent the choice of either a purple or a green marble).
- **b** State the total number of outcomes.
- **c** Find the following probabilities.
  - **i** Pr(A, purple)
  - **ii** Pr(B, green)
  - **iii** Pr(C, purple)
- **d** Find the probability of obtaining a green marble.

ISBN 978-1-108-77290-7

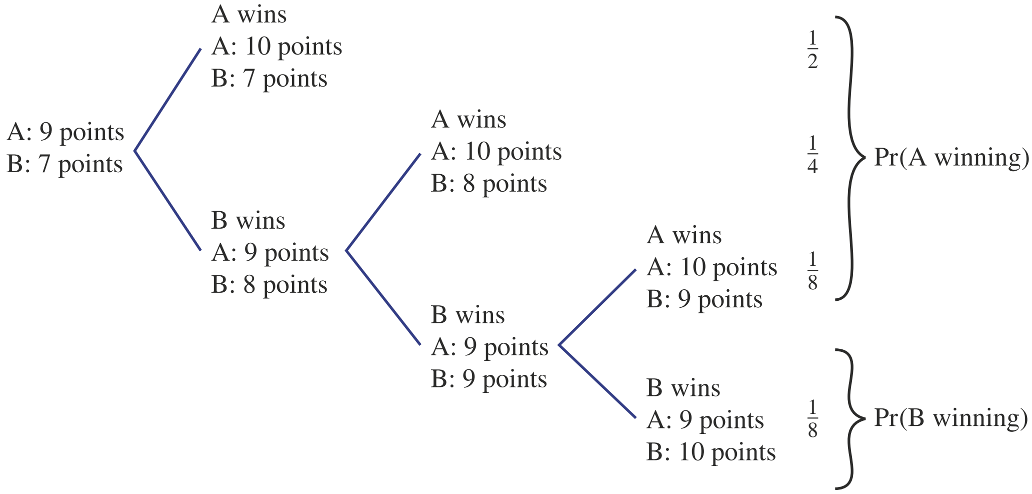
Cambridge University Press
Updated June 2022

ENRICHMENT: Fermat and Pascal	–	–	12

12 The French mathematicians Pierre de Fermat and Blaise Pascal inspired the development of mathematical probability through their consideration of simple games. Here's one of their first problems.

> Two equally skilled people play a game in which the first to earn 10 points wins \$100 and each player has an equal chance of winning a point. At some point in the game, however, one of the players has to leave and the game must be stopped. If the game score is 9 points to 7, how should the \$100 be divided between the two players?

This diagram shows the number of ways the game could have been completed.



a Use this diagram to help calculate the probability that:

- **i** player A wins the game
- **ii** player B wins the game.

b Based on your answers from part **a**, describe how the \$100 should be divided between players A and B.

c Investigate how the \$100 should be divided between players A and B if the game is stopped with the following number of points. You will need to draw a new tree diagram each time.

- **i** player A: 8 points, player B: 7 points
- **ii** player A: 7 points, player B: 7 points
- **iii** player A: 8 points, player B: 6 points
- **iv** player A: 6 points, player B: 7 points
- **v** Choose your own pair of game points and investigate.

ISBN 978-1-108-77290-7

Applications and problem-solving

The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

TV news popularity

1 In a country town, viewers have a choice between a 6 p.m. news bulletin on Channel A and another on Channel B. Two independent research companies take a sample of 40 households in the town regarding their news viewing habits over the course of a month.

In both surveys, 3 households didn't watch a 6 p.m. news at all, while 5 households said they watched both channels over the course of the month.

You wish to analyse the given sample data to investigate the news viewing habits of a number of people.

a In the first survey, the number of households that watch only Channel B is 3 times the number of households that watch only Channel A. Complete a diagram or table to determine how many households watched only Channel A.

b In the second survey, twice as many households watched Channel B from time to time compared to Channel A. How many households claimed to watch Channel B?

In the city there are three channels with a 6 p.m. news bulletin, Channels A, B and C. A survey of 400 households showed that over the course of a month, 30 households watched channels A and B but not C, 21 watched channels A and C but not B, 33 watched channels B and C only. 158 households in total watched Channel C, 190 watched Channel B and 55 watched Channel A only. 43 households did not watch a 6 p.m. news bulletin.

c From the survey results, calculate the probability that a randomly selected household watched Channel B only.

Spin to win

2 In a game at a fair, two spinners are spun and their numbers multiplied together to produce a total.

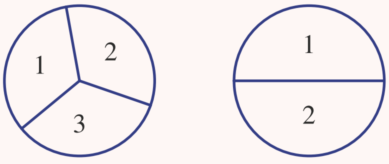
The game costs $5 to play. An odd total sees you win $10, while an even total means that you lose your money.

The manager of the game wants to explore its profitability by calculating various probabilities. He also wishes to make adjustments to the game so that the chances of winning can be increased in the hope that more people might play.

a Consider the game which has the two spinners shown.

i Complete the table below using a tree diagram and use it to determine the probability of winning $10.

Total	1	2			6
Probability					



Spinner 1 Spinner 2

ISBN 978-1-108-77290-7

ii Hence, fill in the table for the profit from one game. What outcome does the game favour?

Profit	\$5	−\$5
Probability		

iii In one particular game the player wins. What is the probability they spun an even number on the first spinner?

Word spreads around the fair that not many people are winning at the game. To get customers back, the manager decides to make the game a 50% chance of winning. He leaves the first spinner as is and the second spinner is adjusted to have the odd and even number in unequal portions.

b Let p be the proportion of the second spinner occupied by the odd number.

i Determine the probability of achieving an odd total and an even total in terms of p.

ii For the game to be fair (equal chance of win or lose), what should be the value of p?

c Design a pair of spinners such that:

i the probability of an odd total and an even total is equal

ii the probability of an odd total is twice the probability of an even total.

d If p is the proportion of the first spinner that has odd numbers and q is the proportion of the second spinner that has odd numbers, what is the requirement for pq for parts **c i** and **ii**?

Rolling 6s

3 A simple dice game involves three rolls of a regular 6-sided die. Points are awarded as follows:

- 10 points for a 6 on the first roll
- 6 points for a 6 on the second roll
- 2 points for a 6 on the third roll
- 5 point bonus if all three rolls are 6s.

You are to investigate the probability of obtaining a certain number of points using a fair die and then reconsider the game if a biased die is used.

a Consider the following game probabilities.

i What is the probability of obtaining each of 10 points in a round, 6 points in a round and 2 points in a round?

ii What is probability of obtaining 16 points in a round?

iii What is the probability of obtaining the maximum 23 points?

iv Two 6s occur in the three rolls of a dice. What is the probability of this occurring and what are the possible points obtained?

v A competitor needs to score at least one point in the last round of the game to win. What is the probability that they win?

b A brother challenges his sister to another game but with a biased die. This die has a probability of p of obtaining a 6. Answer in terms of p in parts **i–iii**.

i What is the probability of rolling no 6s in a round with this die?

ii Hence, what is the probability of scoring points in a round?

iii If the probability of obtaining maximum points in a round is $\frac{1}{27}$, what is the value of p?

iv If the probability of obtaining exactly 16 points in a round is 0.032 while the probability of obtaining exactly 10 points is 0.128, determine the value of p.

ISBN 978-1-108-77290-7

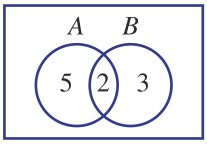
Cambridge University Press
Updated June 2022

8G Independent events

Learning intentions

- To understand what it means for two events to be independent
- To be able to determine mathematically if two events are independent
- To know that selections made with replacement will be independent

In previous sections we have looked at problems involving conditional probability. This Venn diagram, for example, gives the following results.

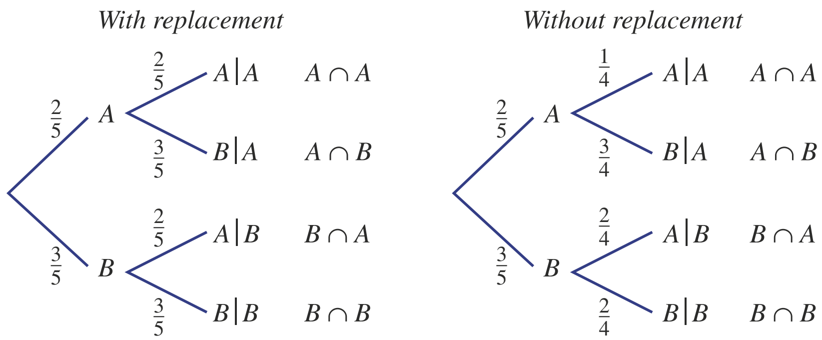


$$\Pr(A) = \frac{7}{10} \quad \text{and} \quad \Pr(A|B) = \frac{2}{5}.$$

Clearly the condition B in $\Pr(A|B)$ has changed the probability of A. The events A and B are therefore not independent.

For multiple events we can consider events either with or without replacement.

These tree diagrams, for example, show two selections of marbles from a bag of 2 aqua (A) and 3 blue (B) marbles.



In the first tree diagram we can see that $\Pr(A|B) = \Pr(A)$, so the events are independent. In the second tree diagram we can see that $\Pr(A|B) \neq \Pr(A)$, so the events are not independent.

So, for independent events we have:

$\Pr(A|B) = \Pr(A) \qquad (*)$

This implies that $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{Pr(B)}$ becomes $\Pr(A) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$.

Using (*) and rearranging gives:

$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$.

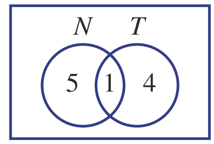
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

LESSON STARTER Are they independent?

Recall that two events are independent if the outcome of one event does not affect the probability of the other event. Discuss whether or not you think the following pairs of events are independent. Give reasons.

- Tossing two coins with the events:
  - getting a tail on the first coin
  - getting a tail on the second coin.
- Selecting two mugs without replacement from a drawer in which there are 3 red and 2 blue mugs and obtaining the events:
  - first is a blue mug
  - second is a red mug.
- Selecting a person from a group of 10 who enjoys playing netball (N) and/or tennis (T), as in the Venn diagram shown.
  - selecting a person from the group who enjoys netball
  - selecting a person from the group who enjoys tennis.



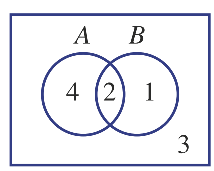
KEY IDEAS

- Two events are **independent** if the outcome of one event does not change the probability of obtaining the other event.
  - $\Pr(A|B) = \Pr(A)$ or $\Pr(B|A) = \Pr(B)$
  - $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$
- For multi-stage experiments with selection made **with replacement**, successive events are independent.
- For multi-stage experiments with selection made **without replacement**, successive events are not independent.

BUILDING UNDERSTANDING

1 A fair coin is tossed twice. Let A be the event 'the first toss gives a tail' and let B be the event 'the second toss gives a tail'.
 a Find:
  i $\Pr(A)$
  ii $\Pr(B)$
 b Would you say that events A and B are independent?
 c What is $\Pr(B|A)$?

2 This Venn diagram shows the number of elements in events A and B.
 a Find:
  i $\Pr(B)$
  ii $\Pr(B|A)$
 b Is $\Pr(B|A) = \Pr(B)$?
 c Are the events A and B independent?



3 Complete each sentence.
 a For multistage experiments, successive events are independent if selections are made _________ replacement.
 b For multistage experiments, successive events are not independent if selections are made _________ replacement.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

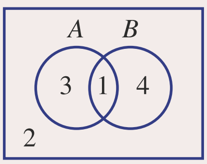
Example 14 Checking for independent events

A selection of 10 mobile phone offers includes four with free connection and five with a free second battery, whereas one offer has both free connection and a free second battery.
Let A be the event 'choosing a mobile phone with free connection'.
Let B be the event 'choosing a mobile phone with a free second battery'.

a Summarise the information about the 10 mobile phone offers in a Venn diagram.
b Find:
  i $\Pr(A)$ **ii** $\Pr(A|B)$
c State whether or not the events A and B are independent.

SOLUTION

a



EXPLANATION

Start with the 1 element that belongs to both sets A and B and complete the diagram according to the given information.

b **i** $\Pr(A) = \frac{4}{10}$

$= \frac{2}{5}$

4 of the 10 elements belong to set A.

ii $\Pr(A|B) = \frac{1}{5}$

1 of the 5 elements in set B belongs to set A.

c The events A and B are not independent.

$\Pr(A|B) \neq \Pr(A)$

Now you try

A selection of 14 hotel offers includes 8 with free Wifi and 9 with a free breakfast, whereas 3 offer both free Wifi and a free breakfast.
Let A be the event 'choosing a hotel with free Wifi'.
Let B be the event 'choosing a hotel with a free breakfast'.

a Summarise the information about the 14 hotel offers in a Venn diagram.
b Find:
  i $\Pr(A)$ **ii** $\Pr(A|B)$
c State whether or not the events A and B are independent.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 8G

FLUENCY	1, 2, 3–4(½)	2, 3–4(½)	2, 3–4(½)

Example 14

1 A selection of 8 offers for computer printers includes 3 with a free printer cartridge and 4 with a free box of paper, whereas 2 have both a free printer cartridge and a free box of paper.
Let A be the event ‘choosing a printer with a free printer cartridge’.
Let B be the event ‘choosing a printer with a free box of paper’.

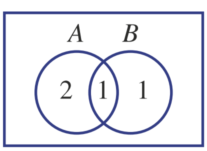
a Summarise the given information about the 8 computer printer offers in a Venn diagram.
b Find:
  i $\Pr(A)$
  ii $\Pr(A|B)$
c State whether or not the events A and B are independent.

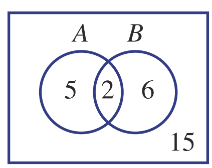
Example 14

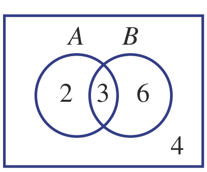
2 A selection of 6 different baby strollers includes 3 with a free rain cover, 4 with a free sun shade, and 2 offer both a free rain cover and a free sun shade.
Let A be the event ‘choosing a stroller with a free sun shade’.
Let B be the event ‘choosing a stroller with a free rain cover’.

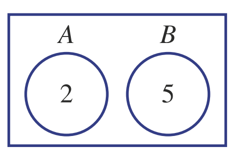
a Summarise the given information about the 6 baby strollers in a Venn diagram.
b Find:
  i $\Pr(A)$
  ii $\Pr(A|B)$
c State whether or not the events A and B are independent.

3 From events A and B in the given Venn diagrams:
  i Find $\Pr(A)$ and $\Pr(A|B)$.
  ii Hence, decide whether or not events A and B are independent.

a


b


c


d


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

4 For the events A and B, with details provided in the given two-way tables, find $\Pr(A)$ and $\Pr(A|B)$ and decide whether or not the events A and B are independent.

a

	A	A'	
B	1	1	2
B'	3	3	6
	4	4	8

b

	A	A'	
B	1	3	4
B'	2	4	6
	3	7	10

c

	A	A'	
B	3	17	20
B'	12	4	16
	15	21	36

d

	A	A'	
B	1		9
B'			
	5		45

PROBLEM-SOLVING	5	5, 6	6, 7

5 Of 17 leading accountants, 15 offer advice on tax (T), whereas 10 offer advice on business growth (G). Eight of the accountants offer advice on both tax and business growth. One of the 17 accountants is chosen at random.

a Use a Venn diagram or two-way table to help find:
i $\Pr(T)$ **ii** $\Pr(T \text{ only})$ **iii** $\Pr(T|G)$

b Are the events T and G independent?

6 A fair coin is tossed 5 times. Find the probability of obtaining:

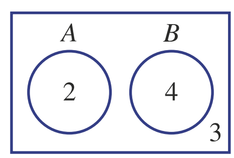
a 5 heads **b** at least 1 tail **c** at least 1 head.

7 A fair 6-sided die is rolled three times. Find the probability that the sum of the three dice is:

a 18 **b** 3 **c** 4 **d** 5

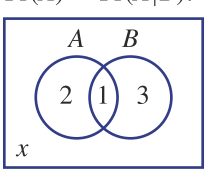
REASONING	8	8, 9	8, 9

8 Use this diagram to help decide if this statement is true or false. If two events, A and B, are mutually exclusive then they are also independent.

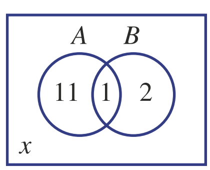


9 Consider the events A and B with the number of elements contained in each event given in the Venn diagrams below. In each case, find the value of x so that the events A and B are independent; i.e. $\Pr(A) = \Pr(A|B)$.

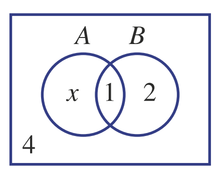
a



b



c



ENRICHMENT: Independence and the addition rule	–	–	10, 11

10 For two independent events A and B, recall that $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$. Two independent events A and B are such that $\Pr(A) = 0.6$ and $\Pr(B) = 0.4$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(A \cup B)$

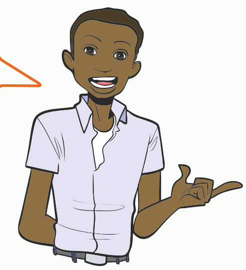
11 For two independent events A and B, we are given $\Pr(A \cup B) = 0.9$ and $\Pr(A) = 0.4$. Find $\Pr(B)$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Problems and challenges

Up for a challenge? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.



1 A women's tennis match is won by the first player to win two sets. Andrea has a 0.4 chance of winning in each set against. Elisa. Find the following probabilities.

- a Andrea wins in two sets.
- b Andrea wins in three sets.
- c Elisa wins after losing the first set.

2 Find $\Pr(A)$ if $\Pr(A \cup B) = 0.74$ and $\Pr(B) = 0.36$, assuming that A and B are independent events.

3 A fair coin is tossed 3 times. Find the probability that:

- a at least 1 head is obtained
- b at least 1 head is obtained given that the first toss is a head
- c at least 2 heads are obtained given that there is at least 1 head.

4 Two digits are chosen without replacement from the set $\{1, 2, 3, 4\}$ to form a two-digit number. Find the probability that the two-digit number is:

a 32 b even c less than 40 d at least 22.

5 A fair coin is tossed 6 times. What is the probability that at least one tail is obtained?

6 What is the chance of choosing the correct six numbers in a 49-ball lottery game?

7 Two leadership positions are to be filled from a group of two girls and three boys. What is the probability that the positions will be filled by one girl and one boy?

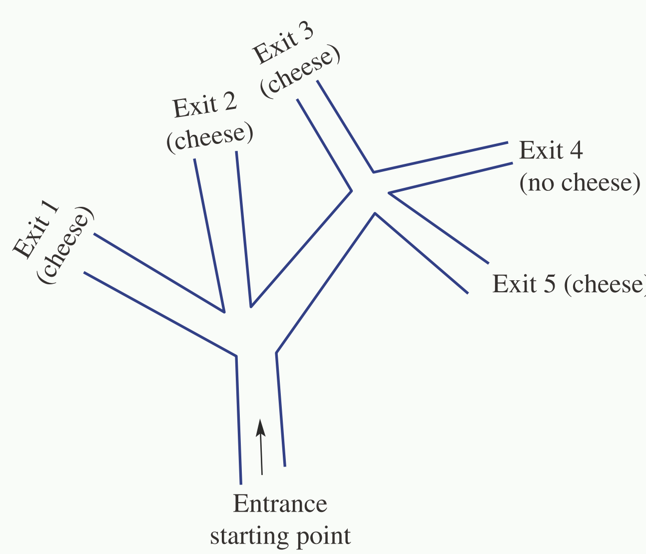
8 The letters of the word DOOR are jumbled randomly. What is the probability that the final arrangement will spell DOOR?

9 True or false? In a group of 23 people, the probability that at least two people have the same birthday is more than 0.5.

10 In an experiment, a mouse runs into a maze and randomly chooses one of the three paths at each fork.

Cheese is located at four of the five exit points.

What is the probability that the mouse finds its way to an exit containing cheese?



ISBN 978-1-108-77290-7

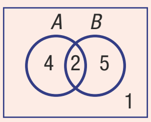
Chapter summary

Probability

Review

- Sample space is the list of all possible outcomes.
- $\text{Pr(event)} = \dfrac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$

Venn diagram

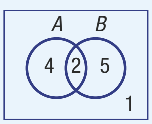


Two-way table

	A	A'	
B	2	5	7
B'	4	1	5
	6	6	12

Conditional probability

$\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$ or $\dfrac{n(A \cap B)}{n(B)}$



	A	A'	
B	2	5	7
B'	4	1	5
	6	6	12

$\Pr(A \mid B) = \dfrac{2}{7}$

Independent events

- $\Pr(A \mid B) = \Pr(A)$
- $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

Tables

With replacement

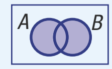
	A	B	C
A	(A, A)	(B, A)	(C, A)
B	(A, B)	(B, B)	(C, B)
C	(A, C)	(B, C)	(C, C)

Without replacement

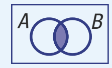
	A	B	C
A	×	(B, A)	(C, A)
B	(A, B)	×	(C, B)
C	(A, C)	(B, C)	×

Unions and intersections

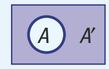
- Union $A \cup B$ (A or B)



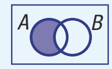
- Intersection $A \cap B$ (A and B)



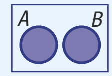
- Complement of A is A' (not A)



- A only is $A \cap B'$



- Mutually exclusive events $A \cap B = \varnothing$



Addition rule

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

If A and B are mutually exclusive:
$\Pr(A \cap B) = 0$ and
$\Pr(A \cup B) = \Pr(A) + \Pr(B)$

Tree diagrams

3 white
4 black

With replacement

Choice 1	*Choice 2*	*Outcome*	*Probability*
$\frac{3}{7}$ W	$\frac{3}{7}$ W	(W, W)	$\frac{9}{49}$
	$\frac{4}{7}$ B	(W, B)	$\frac{12}{49}$
$\frac{4}{7}$ B	$\frac{3}{7}$ W	(B, W)	$\frac{12}{49}$
	$\frac{4}{7}$ B	(B, B)	$\frac{16}{49}$

Without replacement

Choice 1	*Choice 2*	*Outcome*	*Probability*
$\frac{3}{7}$ W	$\frac{2}{6}$ W	(W, W)	$\frac{1}{7}$
	$\frac{4}{6}$ B	(W, B)	$\frac{2}{7}$
$\frac{4}{7}$ B	$\frac{3}{6}$ W	(B, W)	$\frac{2}{7}$
	$\frac{3}{6}$ B	(B, B)	$\frac{2}{7}$

$$\Pr(1 \text{ white, } 1 \text{ black}) = \frac{2}{7} + \frac{2}{7} = \frac{4}{7}$$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter checklist

Chapter checklist: Success criteria

✔

8A **1. I can calculate simple theoretical probabilities.**
e.g. A letter is chosen from the word CHRISTMAS. Find the probability that the letter is an S.

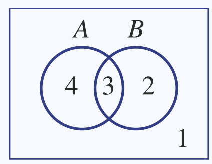
8A **2. I can calculate simple experimental probabilities.**
e.g. An experiment involves rolling a regular 6-sided die 3 times and counting the number of 6s. The results after running the experiment 100 times are:

Number of 6s	0	1	2	3
Frequency	54	32	12	2

Find the experimental probability of obtaining more than 1 six.

8B **3. I can construct and use a Venn diagram.**
e.g. From a class of 20 music students, 14 can play the piano (P), 8 can play the guitar (G) and 5 can play both. Illustrate this information in a Venn diagram and use this to find the probability that a student randomly chosen from the class plays neither the piano nor the guitar.

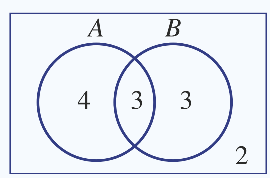
8B **4. I can use a two-way table.**
e.g. The Venn diagram shows the distribution of elements in two sets, A and B.
Transfer the information in the Venn diagram to a two-way table and find $n(A' \cap B)$ and $\Pr(A \cup B)$.



8C **5. I can apply the addition rule.**
e.g. A card is selected from a standard deck of 52 playing cards (4 suits, no jokers). Let A be the event 'the card is red' and B be the event 'the card is an Ace'. Use the addition rule to the find the probability that the card is an Ace or red ($\Pr(A \cup B)$).

8C **6. I can use the addition rule.**
e.g. Two events, A and B, are such that $\Pr(A) = 0.3$, $\Pr(B) = 0.6$ and $\Pr(A \cup B) = 0.8$. Find $\Pr(A \cap B)$.

8D **7. I can find conditional probability from a Venn diagram.**
e.g. Consider the Venn diagram displaying the number of elements belonging to the events A and B.
Find $\Pr(A|B)$ and $\Pr(B|A)$.



8D **8. I can find conditional probabilities using a two-way table.**
e.g. In a team of 11 cricketers in a match, 6 players bowled (A), 5 players batted (B) and 2 both batted and bowled. Represent the information in a two-way table and find the probability that a randomly selected person bowled given that they batted.

8E **9. I can construct a table for a two-step experiment with replacement.**
e.g. A fair 4-sided die is rolled twice. List the sample space of the sum of the faces of the two die using a table and find Pr(sum of at least 5).

8E **10. I can construct a table for a two-step experiment without replacement.**
e.g. A bag contains 2 blue marbles, 1 red and 1 green. Two marbles are chosen from the bag without replacement. Construct a table to list the sample space and find the probability of selecting a blue marble and a green marble.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter checklist

		✔
8F	**11. I can construct a tree diagram for multi-stage experiments.** e.g. Boxes A and B contain 5 counters each. Box A contains 3 blue and 2 white counters and box B contains 4 blue and 1 white counter. A box is chosen at random followed by a single counter. Draw a tree diagram that shows all possible outcomes and related probabilities. Use this to find the probability of selecting a blue counter.	
8F	**12. I can use a tree diagram for experiments without replacement.** e.g. A box contains 5 hard chocolates (H) and 3 soft chocolates (S). Two chocolates are selected at random without replacement. Draw a tree diagram showing all outcomes and find the probability of selecting one of each.	
8G	**13. I can check for independent events.** e.g. A selection of 10 gym memberships includes 3 with 24-hour gym access and 5 with personal training sessions, while 2 of the memberships have both of these. Let A be the event 'choosing a membership with 24-hour gym access' and B be the event 'choosing a membership with personal training sessions'. Display the information in a Venn diagram or two-way table and determine whether or not events A and B are independent.	

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter review

Short-answer questions

8A **1** A letter is chosen from the word INTEREST. Find the probability that the letter will be:

a an I **b** an E **c** a vowel
d not a vowel **e** E or T.

8A **2** A letter is chosen from the word POSITIVE. Find the probability that the letter also belongs to these words.

a NEGATIVE **b** ADDITION **c** DIVISION

8A **3** Belinda, an engineer, inspects 20 houses in a street for cracks. The results are summarised in this table.

Number of cracks	0	1	2	3	4
Frequency	8	5	4	2	1

a From these results estimate the probability that the next house inspected in the street will have the following number of cracks.

i 0 **ii** 1 **iii** 2 **iv** 3 **v** 4

b Estimate the probability that the next house will have:

i at least 1 crack **ii** no more than 2 cracks.

8B **4** Of 36 people, 18 have an interest in cars, 11 have an interest in homewares and 6 have an interest in both cars and homewares.

a Represent this information using a Venn diagram.
b Represent this information using a two-way table.
c State the number of people surveyed who do not have an interest in either cars or homewares.
d If a person is chosen at random from the group, find the probability that the person will:

i have an interest in cars and homewares
ii have an interest in homewares only
iii not have any interest in cars.

8C **5** All 26 birds in an aviary have clipped wings and/or a tag. In total, 18 birds have tags and 14 have clipped wings.

a Find the number of birds that have both a tag and clipped wings.
b Find the probability that a bird chosen at random will have a tag only.

8B/C **6** A card is selected from a standard deck of 52 playing cards. Let A be the event 'the card is a heart' and let B be the event 'the card is a king'.

a Find:

i $n(A)$ **ii** $n(B)$ **iii** $n(A \cap B)$

b Find:

i $\Pr(A')$ **ii** $\Pr(A \cap B)$

c Use the addition rule to find $\Pr(A \cup B)$.
d Find the probability that the card is a king or not a diamond.

ISBN 978-1-108-77290-7

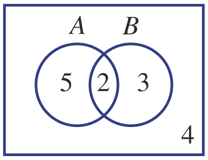
Cambridge University Press
Updated June 2022

8C **7** Two events, A and B, are such that $\Pr(A) = 0.25$, $\Pr(B) = 0.35$ and $\Pr(A \cup B) = 0.5$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(A' \cap B')$

8D **8** For these probability diagrams, find $\Pr(A|B)$.

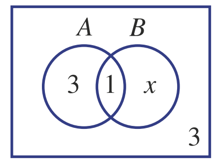
a



b

	A	A'	
B	1		
B'	2	2	
			9

8D/G **9** Two events, A and B, are represented on the following Venn diagram. Also, $n(B \text{ only}) = x$, where x is a positive integer.



a If $x = 4$, find:

i $\Pr(A)$ **ii** $\Pr(B)$ **iii** $\Pr(A|B)$

b If $x = 4$, decide whether the events A and B are independent.

c If $x = 1$, find:

i $\Pr(A)$ **ii** $\Pr(B)$ **iii** $\Pr(A|B)$

d If $x = 1$, decide if the events A and B are independent.

8E **10** A letter is chosen at random from the word HAPPY and a second letter is chosen from the word HEY.

a List the sample space, using a table.

b State the total number of outcomes.

c Find the probability that the two letters chosen will be:

i H then E **ii** the same **iii** not the same.

8F **11** A fair 4-sided die is rolled twice and the total is noted.

a Use a tree diagram to list the sample space, including all possible totals.

b Find these probabilities.

i $\Pr(2)$ **ii** $\Pr(5)$ **iii** $\Pr(1)$ **iv** $\Pr(\text{not } 1)$

8F **12** Two people are selected from a group of two females and three males, without replacement. Use a tree diagram to find the probability of selecting:

a a female on the first selection

b a male on the second selection given that a female is chosen on the first selection

c two males **d** one male **e** at least one female.

8G **13** Two independent events, A and B, are such that $\Pr(A) = 0.4$ and $\Pr(B) = 0.3$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(A \cup B)$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter review

Multiple-choice questions

8A **1** A letter is chosen from the word SUCCESS. The probability that the letter is neither C nor S is:

A $\frac{2}{7}$ **B** $\frac{3}{5}$ **C** $\frac{5}{7}$ **D** $\frac{4}{7}$ **E** $\frac{3}{7}$

8A **2** The number of manufacturing errors spotted in a car plant on 20 randomly selected days is given by this table.

Number of errors	0	1	2	3
Frequency	11	6	2	1

An estimate of the probability that on the next day at least one error will be observed is:

A $\frac{3}{10}$ **B** $\frac{9}{20}$ **C** $\frac{11}{20}$ **D** $\frac{17}{20}$ **E** $\frac{3}{20}$

8B **3** From the list of the first 10 positive integers, $A = \{1, 3, 5, 7, 9\}$ and B is the set of primes less than 10. Therefore, $\Pr(A')$ and $\Pr(A \text{ only})$ are, respectively:

A $\frac{1}{3}, \frac{1}{5}$ **B** $\frac{1}{2}, \frac{1}{2}$ **C** $\frac{1}{2}, \frac{3}{10}$

D $\frac{1}{2}, \frac{1}{5}$ **E** $\frac{1}{3}, \frac{2}{5}$

8B **4** For this two-way table, $\Pr(A \cap B')$ is:

A $\frac{2}{3}$ **B** $\frac{1}{4}$ **C** $\frac{1}{7}$

D $\frac{1}{3}$ **E** $\frac{3}{4}$

	A	A'	
B	2		3
B'			4
		4	

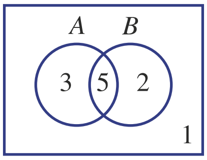
8C **5** The sets A and B are known to be mutually exclusive. Which of the following is therefore true?

A $n(A) = n(B)$ **B** $n(A \cap B) = 0$ **C** $A = \varnothing$

D $\Pr(A \cap B) = 1$ **E** $\Pr(A \cup B) = 0$

8D **6** For this Venn diagram, $\Pr(A|B)$ is:

A $\frac{5}{7}$ **B** $\frac{5}{2}$ **C** $\frac{5}{8}$ **D** $\frac{5}{3}$ **E** $\frac{3}{11}$



8E **7** A letter is chosen from each of the words CAN and TOO. The probability that the pair of letters will not have an O is:

A $\frac{2}{3}$ **B** $\frac{1}{2}$ **C** $\frac{1}{3}$ **D** $\frac{1}{9}$ **E** $\frac{5}{9}$

8E **8** A box has 3 red and 2 blue counters. If a red counter is selected and not replaced, then the probability that a blue counter will be observed on the second selection is:

A $\frac{1}{2}$ **B** $\frac{2}{5}$ **C** $\frac{2}{3}$ **D** $\frac{1}{4}$ **E** **F** $\frac{3}{4}$

8F **9** The number of times a coin must be tossed to give 16 possible outcomes is:

A 8 **B** 2 **C** 16 **D** 3 **E** 4

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

8G **10** Two events are independent when:

A $\Pr(A) = \Pr(B)$ **B** $\Pr(A') = \varnothing$ **C** $\Pr(A \cup B) = 0$
D $\Pr(A|B) = \Pr(B)$ **E** $\Pr(A) = \Pr(A|B)$

Extended-response questions

1 Of 15 people surveyed to find out whether they run or swim for exercise, 6 said they run, 4 said they swim and 8 said they neither run nor swim.

a How many people surveyed run and swim?
b One of the 15 people is selected at random. Find the probability that they:
- **i** run or swim
- **ii** only swim.

c Represent the information in a two-way table.
d Find the probability that:
- **i** a person swims given that they run
- **ii** a person runs given that they swim.

2 A bakery sells three types of bread: raisin (R) at \$2 each, sourdough (S) at \$3 each, and white (W) at \$1.50 each. Lillian is in a hurry. She randomly selects two loaves and takes them quickly to the counter. Each type of loaf has an equal chance of being selected.

a Draw a table showing the possible combination of loaves that Lillian could have selected.
b Find the probability that Lillian selects:
- **i** two raisin loaves
- **ii** two loaves that are the same
- **iii** at least one white loaf
- **iv** not a sourdough loaf.

Lillian has only \$4 in her purse.

c How many different combinations of bread will Lillian be able to afford?
d Find the probability that Lillian will not be able to afford her two chosen loaves.

On the next day, there are only two raisin, two sourdough and three white loaves available. Lillian chooses two loaves without replacement from the limited number of loaves.

e Use a tree diagram showing branch probabilities to find:
- **i** Pr(2 raisin loaves)
- **ii** Pr(1 sourdough loaf)
- **iii** Pr(not more than 1 white loaf)
- **iv** Pr(2 loaves that are not the same)

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

CHAPTER 9 Statistics

Statistics helping the environment

To classify and map the mosaic of plant or animal life in any ecological community requires the collection and analysis of large quantities of data. To organise raw data, a spreadsheet matrix can be developed using the columns for sample sites, rows for species and an 'abundance score' (i.e. frequency count) in each cell. Statistical cluster analysis bunches similar samples to create a tree-like graph. Multi-dimensional scaling creates 2D or 3D drawings of points showing samples grouped according to similarities or differences.

An important use of statistics is to determine the significance or importance of a set of results. For example, a pollution incident in a stream might result in some species, such as fish, shrimps or crayfish, slowly dying out. A scientist can measure the diversity of life in the stream with regular netting and counting the various species caught. Diversity versus time can be graphed and a trend line drawn through the scattered points. A downward sloping trend line suggests a decline in diversity. Statistical analysis is used to determine if the deviation of data

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

points from the trend line is small enough for the trend to be significant. Using the procedure, a scientist has discovered that a species of fairy shrimp, *Branchinella latzi*, is now extinct in the pools on Uluru (Ayers Rock) due to human waste pollution.

Online resources

A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

In this chapter

Victorian Curriculum

STATISTICS AND PROBABILITY
Data representation and interpretation

Determine quartiles and interquartile range and investigate the effect of individual data values, including outliers on the interquartile range (VCMSP349)

Construct and interpret box plots and use them to compare data sets (VCMSP350)

Compare shapes of box plots to corresponding histograms and dot plots and discuss the distribution of data (VCMSP351)

Use scatter plots to investigate and comment on relationships between two numerical variables (VCMSP352)

Investigate and desribe bivariate numerical data, including where the independent variable is time (VCMSP353)

Evaluate statistical reports in the media and other places by linking claims to displays, statistics and representative data (VCMSP354)

(10A) Calculate and interpret the mean and standard deviation of data and use these to compare data sets. Investigate the effect of individual data values including outliers, on the standard deviation (VCMSP372)

(10A) Use digital technology to investigate bivariate numerical data sets. Where appropriate use a straight line to describe the relationship allowing for variation, make predictions based on this straight line and discuss limitations (VCMSP373)

9A Collecting and using data

Learning intentions
- To understand how surveys work and the necessary considerations for their construction
- To understand the difference between a population and a sample
- To know how to describe types of data using the key words: categorical (nominal or ordinal) or numerical (discrete or continuous)
- To be able to decide if a survey sample is representative

There are many reports on television and radio that begin with the words 'A recent study has found that …'. These are usually the result of a survey or investigation that a researcher has conducted to collect information about an important issue, such as unemployment, crime or obesity.

Sometimes the results of these surveys are used to persuade people to change their behaviour. Sometimes they are used to pressure the government into changing the laws or to change the way the government spends public money.

Results of surveys and other statistics can sometimes be misused or displayed in a way to present a certain point of view.

Niche marketing is when a product or service is advertised to a specific group, such as people who train for obstacle competitions or dog-owners who use luxury dog groomers. Surveys provide valuable data for niche marketing and sales.

LESSON STARTER Improving survey questions

Here is a short survey. It is not very well constructed.

Question 1: How old are you?

Question 2: How much time did you spend sitting in front of the television or a computer yesterday?

Question 3: Some people say that teenagers like you are lazy and spend way too much time sitting around when you should be outside exercising. What do you think of that comment?

Have a class discussion about the following.

- What will the answers to Question 1 look like? How could they be displayed?
- What will the answers to Question 2 look like? How could they be displayed?
- Is Question 2 going to give a realistic picture of your normal daily activity?
- Do you think Question 2 could be improved somehow?
- What will the answers to Question 3 look like? How could they be displayed?
- Do you think Question 3 could be improved somehow?

ISBN 978-1-108-77290-7

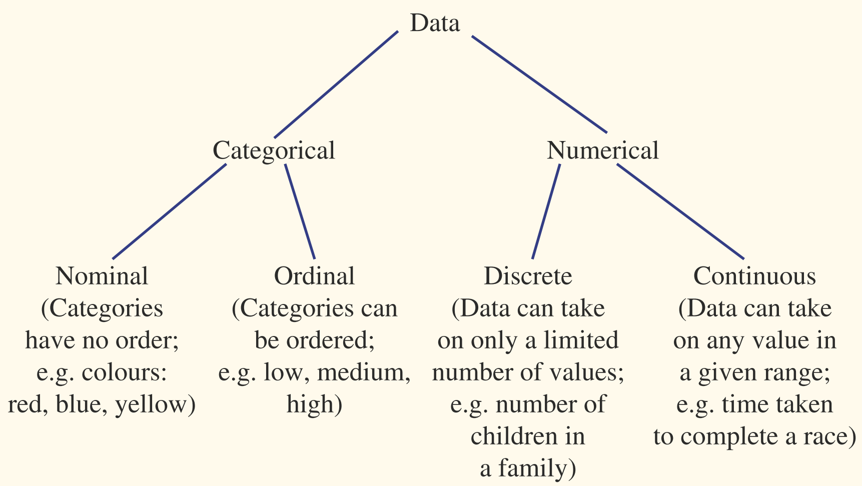
Cambridge University Press
Updated June 2022

KEY IDEAS

- **Surveys** are used to collect statistical data.
  - Survey questions need to be constructed carefully so that the person knows exactly what sort of answer to give. Survey questions should use simple language and should not be ambiguous.
  - Survey questions should not be worded so that they deliberately try to provoke a certain kind of response.
  - If the question contains an option to be chosen from a list, the number of options should be an odd number, so that there is a ‘neutral’ choice. For example, the options could be:

strongly agree	agree	unsure	disagree	strongly disagree

- A **population** is a group of people, animals or objects with something in common. Some examples of populations are:
  - all the people in Australia on Census Night
  - all the students in your school
  - all the boys in your maths class
  - all the tigers in the wild in Sumatra
  - all the cars in Brisbane
  - all the wheat farms in NSW.
- A **sample** is a group that has been chosen from a population. Sometimes information from a sample is used to describe the whole population, so it is important to choose the sample carefully.
- **Statistical data** can be divided into subgroups.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

BUILDING UNDERSTANDING

1 Match each word (a–e) with its definition (A–E).

a	population	A	a group chosen from a population
b	census	B	a tool used to collect statistical data
c	sample	C	all the people or objects in question
d	survey	D	statistics collected from every member of the population
e	data	E	the factual information collected from a survey or other source

2 Match each word (a–f) with its definition (A–F).

a	numerical	A	categorical data that has no order
b	continuous	B	data that are numbers
c	discrete	C	numerical data that take on a limited number of values
d	categorical	D	data that can be divided into categories
e	ordinal	E	numerical data that take any value in a given range
f	nominal	F	categorical data that can be ordered

3 Classify each set of data as categorical or numerical.
- a 4.7, 3.8, 1.6, 9.2, 4.8
- b red, blue, yellow, green, blue, red
- c low, medium, high, low, low, medium

4 Which one of the following survey questions would generate categorical data?
- A How many times do you eat at your favourite fast-food place in a typical week?
- B How much do you usually spend buying your favourite fast food?
- C How many items did you buy last time you went to your favourite fast-food place?
- D Which is your favourite fast-food?

Example 1 Describing types of data

What type of data would the following survey questions generate?
- a How many televisions do you have in your home?
- b To what type of music do you most like to listen?

SOLUTION	EXPLANATION
a numerical and discrete	The answer to the question is a number with a limited number of values; in this case, a whole number.
b categorical and nominal	The answer is a type of music and these categories have no order.

Now you try

What type of data would the following survey questions generate?
- a How tall are the students in Year 10?
- b What is your level of satisfaction (low, medium and high) with a meal at a restaurant?

ISBN 978-1-108-77290-7

Example 2 Choosing a survey sample

A survey is carried out on the internet to determine Australia's favourite musical performer. Why will this sample not necessarily be representative of Australia's views?

SOLUTION

An internet survey is restricted to people with a computer and internet access, ruling out some sections of the community from participating in the survey.

EXPLANATION

The sample may not include some of the older members of the community or those in areas without access to the internet. Also, the survey would need to be set up so that people can do it only once so that 'fake' surveys are not completed.

Now you try

A survey is carried out in a library to determine typical study habits of Year 12 students. Why will this sample not necessarily be representative of all Year 12 students?

Exercise 9A

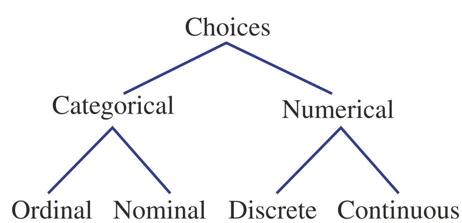
FLUENCY	1–3	2, 3	2, 3

Example 1 **1** What type of data would the following survey questions generate?

- **a** How many people are there in each office room?
- **b** What was the time taken to complete the task?
- **c** What colour are the jackets on a rack?
- **d** How would you rate the movie: good, don't care, bad?

Example 1 **2** Year 10 students were asked the following questions in a survey. Describe what type of data each question generates.

- **a** How many people under the age of 18 years are there in your immediate family?
- **b** How many letters are there in your first name?
- **c** Which company is the carrier of your mobile telephone calls? Optus/Telstra/Vodafone/Virgin/Other (Please specify.)
- **d** What is your height?
- **e** How would you describe your level of application in Maths? (Choose from very high, high, medium or low.)



Example 2 **3** Decide if the following surveys would be representative of the entire Australian population:

- **a** a survey via social media to find out people's favourite news program
- **b** a survey to find out the average number of pets in a household from people entering a pet store
- **c** using census data to determine the average household income
- **d** making 10 000 random phone calls to find out who is likely to win the next federal election

ISBN 978-1-108-77290-7

PROBLEM-SOLVING	4, 5	4–6	5–7

4 The principal decides to survey Year 10 students to determine their opinion of Mathematics.
  a In order to increase the chance of choosing a representative sample, the principal should:
    A Give a survey form to the first 30 Year 10 students who arrive at school.
    B Give a survey form to all the students studying the most advanced Maths subject.
    C Give a survey form to five students in every Maths class.
    D Give a survey form to 20% of the students in every class.
  b Explain your choice of answer in part a. Describe what is wrong with the other three options.

5 Discuss some of the problems with the selection of a survey sample for each given topic.
  a A survey at the train station of how Australians get to work.
  b An email survey on people's use of computers.
  c Phoning people on the electoral roll to determine Australia's favourite sport.

Is a train station survey of how people get to work representative?

6 Choose a topic in which you are especially interested, such as football, cricket, movies, music, cooking, food, computer games or social media.

Make up a survey about your topic that you could give to the people in your class.

It must have *four* questions.
Question 1 must produce data that are categorical and ordinal.
Question 2 must produce data that are categorical and nominal.
Question 3 must produce data that are numerical and discrete.
Question 4 must produce data that are numerical and continuous.

7 A television news reporter surveyed four companies and found that the profits of three of these companies had reduced over the past year. They report that this means the country is facing an economic downturn and that only one in four companies is making a profit.
  a What are some of the problems in this media report?
  b How could the news reporter improve their sampling methods?
  c Is it correct to say that only one in four companies is making a profit? Explain.

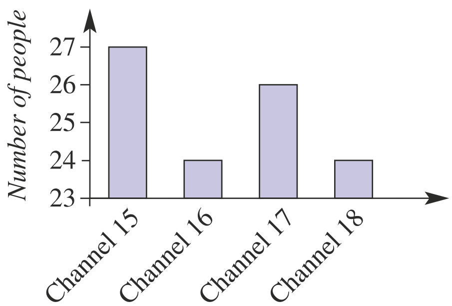
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

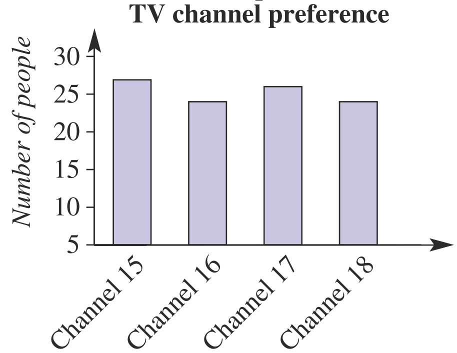
REASONING	8	8, 9	9, 10

8 Here are two column graphs, each showing the same results of a survey that asked people which TV channel they preferred.

Graph A
TV channel preference



Graph B
TV channel preference



- **a** Which graph could be titled 'Channel 15 is clearly most popular'?
- **b** Which graph could be titled 'All TV channels have similar popularity'?
- **c** What is the difference between the two graphs?
- **d** Which graph is misleading and why?

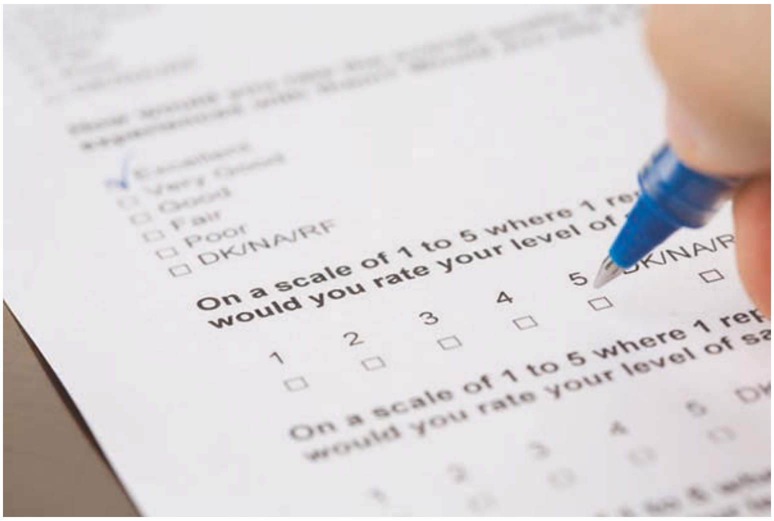
9 Describe three ways that graphs or statistics could be used to mislead people and give a false impression about the data.

10 Search the internet or newspaper for 'misleading graphs' and 'how to lie with statistics'. Explain why they are misleading.

ENRICHMENT: The 2016 Australian Census	–	–	11, 12

11 Research the 2016 Australian Census on the website of the Australian Bureau of Statistics. Find out something interesting from the results of the 2016 Australian Census and write a short news report.

12 It is often said that Australia has an ageing population. What does this mean?
Search the internet for evidence showing that the 'average' Australian is getting older every year.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

9B Review of statistical graphs CONSOLIDATING

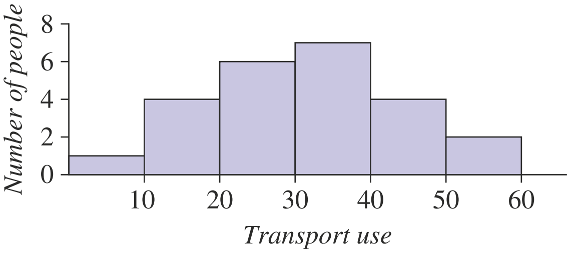
Learning intentions
- To review the types of graphs that can be used to display categorical data or numerical data
- To know how to construct a frequency table and histogram from numerical data using class intervals
- To know how to find the measures of centre, mean and median, of a set of data

Statistical graphs are an essential element in the analysis and representation of data. Graphs can help to show the most frequent category, the range of values, the shape of the distribution and the centre of the data. By looking at statistical graphs the reader can quickly draw conclusions about the numbers or categories in the data set and interpret this within the context of the data.

People who specialise in medical biostatistics apply statistical techniques to analyse results from health-related research, such as in genetics, medicine and pharmacy. Data presentation includes using bar charts, line charts, histograms and scatter plots.

LESSON STARTER Public transport

A survey was carried out to find out how many times people in the group had used public transport in the past month. The results are shown in this histogram.



Discuss what the histogram tells you about this group of people and their use of public transport. You may wish to include these points:

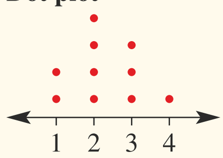
- How many people were surveyed?
- Is the data symmetrical or skewed?
- Is it possible to work out the exact mean? Why/why not?
- Do you think these people were selected from a group in your own community? Give reasons.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

KEY IDEAS

- The types of **statistical data** that we saw in the previous section; i.e. categorical (nominal or ordinal) and numerical (discrete or continuous), can be displayed using different types of graphs to represent the different data.
- Graphs for a single set of categorical or discrete data
  - **Dot plot**

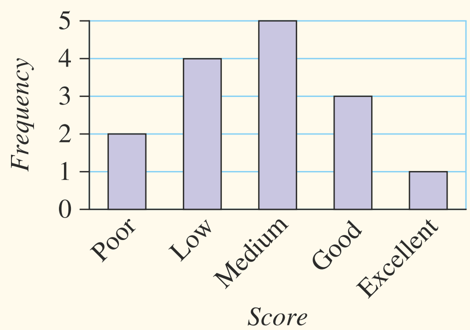


  - **Stem-and-leaf plot**

Stem	Leaf
0	1 3
1	2 5 9
2	1 4 6 7
3	0 4

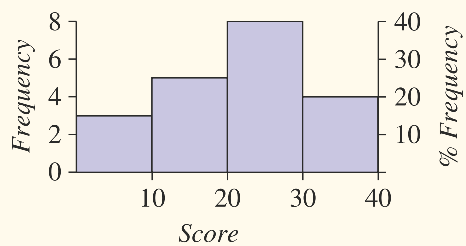
2|4 means 24

  - **Column graph**



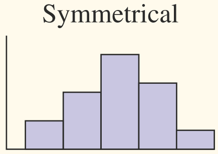
- **Histograms** can be used for grouped discrete or continuous numerical data.
  The interval 10– includes all numbers from 10 (including 10) to fewer than 20.

Class interval	Frequency	Percentage frequency
0–	3	15
10–	5	25
20–	8	40
30–40	4	20

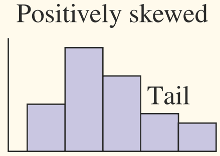


- Measures of centre include:
  - **mean** ($\bar{x}$) $\bar{x} = \dfrac{\text{sum of all data values}}{\text{number of data values}}$
  - **median** the middle value when data are placed in order
- The **mode** of a data set is the most common value.
- Data can be **symmetrical** or **skewed**.

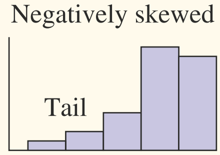
Symmetrical



Positively skewed



Negatively skewed

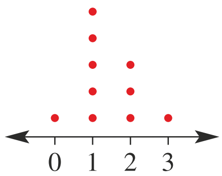


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

BUILDING UNDERSTANDING

1 A number of families were surveyed to find the number of children in each. The results are shown in this dot plot.



a How many families were surveyed?
b Find the mean number of children in the families surveyed.
c State the median number of children in the families surveyed.
d State the mode for the number of children in the families surveyed.
e What percentage of the families have, at most, two children?

2 State the missing values in this frequency table.

Class interval	Frequency	Percentage frequency
0–	2	
10–	1	
20–	5	
30–40	2	
Total		

Example 3 Presenting and analysing data

Twenty people were surveyed to find out how many times they use the internet in a week. The raw data are listed.

21, 19, 5, 10, 15, 18, 31, 40, 32, 25
11, 28, 31, 29, 16, 2, 13, 33, 14, 24

a Organise the data into a frequency table using class intervals of 10. Include a percentage frequency column.
b Construct a histogram for the data, showing both the frequency and percentage frequency on the one graph.
c Construct a stem-and-leaf plot for the data.
d Use your stem-and-leaf plot to find the median.

SOLUTION

a

Class interval	Frequency	Percentage frequency
0–	2	10
10–	8	40
20–	5	25
30–	4	20
40–50	1	5
Total	20	100

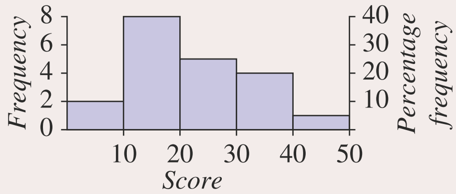
EXPLANATION

Calculate each percentage frequency by dividing the frequency by the total (i.e. 20) and multiplying by 100.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b Number of times the internet is accessed



Transfer the data from the frequency table to the histogram. Axis scales are evenly spaced and the histogram bar is placed across the boundaries of the class interval. There is no space between the bars.

c

Stem	Leaf
0	2 5
1	0 1 3 4 5 6 8 9
2	1 4 5 8 9
3	1 1 2 3
4	0

3|1 means 31

Order the data in each leaf and also show a key (e.g. 3|1 means 31).

d $\text{Median} = \dfrac{19 + 21}{2}$

$= 20$

After counting the scores in order from the lowest value (i.e. 2), the two middle values are 19 and 21, so the median is the mean of these two numbers.

Now you try

Sixteen people were surveyed to find out how many phone texts they send in one day. The raw data are as follows.

10, 7, 2, 5, 22, 14, 7, 9, 11, 29, 32, 18, 5, 24, 12, 14

a Organise the data into a frequency table using class intervals of 10. Include a percentage frequency column.
b Construct a histogram for the data, showing both the frequency and percentage frequency on the one graph.
c Construct a stem-and-leaf plot for the data.
d Use your stem-and-leaf plot to find the median.

Exercise 9B

FLUENCY	1–4	1, 3, 4	1, 3, 4

Example 3

1 The number of wins scored this season is given for 20 hockey teams. Here are the raw data.
4, 8, 5, 12, 15, 9, 9, 7, 3, 7,
10, 11, 1, 9, 13, 0, 6, 4, 12, 5
a Organise the data into a frequency table using class intervals of 5 and include a percentage frequency column.
b Construct a histogram for the data, showing both the frequency and percentage frequency on the one graph.
c Construct a stem-and-leaf plot for the data.
d Use your stem-and-leaf plot to find the median.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2 This frequency table displays the way in which 40 people travel to and from work.

Type of transport	Frequency	Percentage frequency
Car	16	
Train	6	
Tram	8	
Walking	5	
Bicycle	2	
Bus	3	
Total	**40**	

a Copy and complete the table.

b Use the table to find:

i the frequency of people who travel by train

ii the most popular form of transport

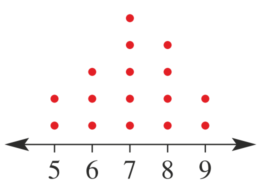
iii the percentage of people who travel by car

iv the percentage of people who walk or cycle to work

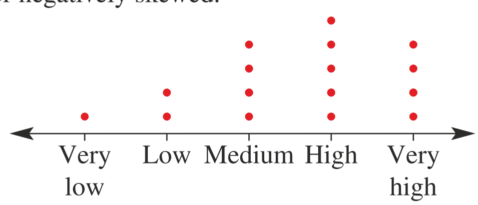
v the percentage of people who travel by public transport, including trains, buses and trams.

3 Describe each graph as symmetrical, positively skewed or negatively skewed.

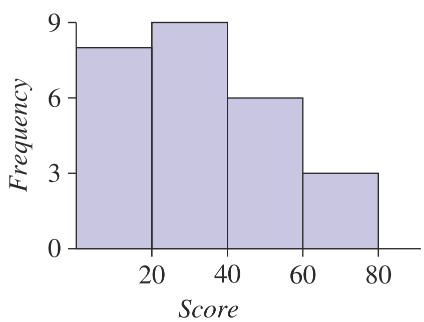
a



b



c



d

Stem	Leaf
4	1 6
5	0 5 4 8
6	1 8 9 9 9
7	2 7 8
8	3 8

4 | 6 means 46

4 For the data in these stem-and-leaf plots, find:

i the mean (rounded to one decimal place)

ii the median

iii the mode

a

Stem	Leaf
2	1 3 7
3	2 8 9 9
4	4 6

3 | 2 means 32

b

Stem	Leaf
0	4
1	0 4 9
2	1 7 8
3	2

2 | 7 means 27

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

PROBLEM-SOLVING	5, 6	6, 7	7, 8

5 Two football players, Nick and Jack, compare their personal tallies of the number of goals scored for their team over a 12-match season. Their tallies are as follows.

Game	1	2	3	4	5	6	7	8	9	10	11	12
Nick	0	2	2	0	3	1	2	1	2	3	0	1
Jack	0	0	4	1	0	5	0	3	1	0	4	0

a Draw a dot plot to display Nick's goal-scoring achievement.
b Draw a dot plot to display Jack's goal-scoring achievement.
c How would you describe Nick's scoring habits?
d How would you describe Jack's scoring habits?

6 Three different electric sensors, A, B and C, are used to detect movement in Harvey's backyard over a period of 3 weeks. An in-built device counts the number of times the sensor detects movement each night. The results are as follows.

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Sensor A	0	0	1	0	0	1	1	0	0	2	0	0	0	0	0	1	1	0	0	1	0
Sensor B	0	15	1	2	18	20	2	1	3	25	0	0	1	15	8	9	0	0	2	23	2
Sensor C	4	6	8	3	5	5	5	4	8	2	3	3	1	2	2	1	5	4	0	4	9

a Using class intervals of 3 and starting at 0, draw up a frequency table for each sensor.
b Draw histograms for each sensor.
c Given that it is known that stray cats consistently wander into Harvey's backyard, how would you describe the performance of:
  i sensor A?
  ii sensor B?
  iii sensor C?

Possums could set off the sensors.

7 This tally records the number of mice that were weighed and categorised into particular mass intervals for a scientific experiment.

a Construct a table using these column headings: Mass, Frequency and Percentage frequency.
b Find the total number of mice weighed in the experiment.
c State the percentage of mice that were in the 20– gram interval.
d Which was the most common weight interval?
e What percentage of mice were in the most common mass interval?
f What percentage of mice had a mass of 15 grams or more?

Mass (grams)	Tally
10–	𝍷𝍷𝍷
15–	𝍸 𝍷
20–	𝍸 𝍸 𝍸 𝍷
25–	𝍸 𝍸 𝍸 𝍸 𝍷
30–35	𝍷𝍷𝍷𝍷

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

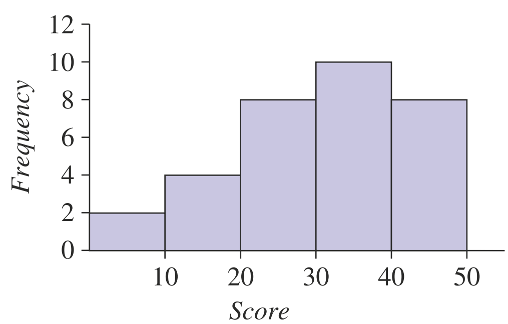
8 A school symphony orchestra contains four musical sections: strings, woodwind, brass and percussion. The number of students playing in each section is summarised in this tally.

Section	Tally
String	𝍸 𝍸 𝍸 𝍸 𝍷
Woodwind	𝍸 𝍷𝍷𝍷
Brass	𝍸 𝍷𝍷
Percussion	𝍷𝍷𝍷𝍷

a Construct and complete a percentage frequency table for the data.
b What is the total number of students in the school orchestra?
c What percentage of students play in the string section?
d What percentage of students do not play in the string section?
e If the number of students in the string section increases by 3, what will be the percentage of students who play in the percussion section? Round your answer to one decimal place.
f What will be the percentage of students in the string section of the orchestra if the entire woodwind section is absent? Round your answer to one decimal place.

REASONING	9	9, 10	10, 11

9 This histogram shows the distribution of test scores for a class. Explain why the percentage of scores in the 20–30 range is 25%.



10 Explain why the exact value of the mean, median and mode cannot be determined directly from a histogram.

11 State the possible values of a, b and c in this ordered stem-and-leaf plot.

Stem	Leaf
3	2 3 a 7
4	b 4 8 9 9
5	0 1 4 9 c
6	2 6

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

ENRICHMENT: Cumulative frequency curves and percentiles	–	–	12

12 Cumulative frequency is obtained by adding a frequency to the total of its predecessors. It is sometimes referred to as a 'running total'.

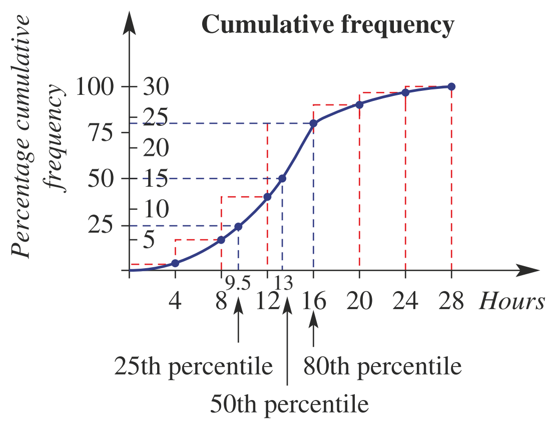
$$\text{Percentage cumulative frequency} = \frac{\text{cumulative frequency}}{\text{total number of data elements}} \times 100$$

A cumulative frequency graph is one in which the heights of the columns are proportional to the corresponding cumulative frequencies.

The points in the upper right-hand corners of these rectangles join to form a smooth curve called the cumulative frequency curve.

If a percentage scale is added to the vertical axis, the same graph can be used as a percentage cumulative frequency curve, which is convenient for the reading of percentiles.

Number of hours	Frequency	Cumulative frequency	Percentage cumulative frequency
0–	1	1	3.3
4–	4	5	16.7
8–	7	12	40.0
12–	12	24	80.0
16–	3	27	90.0
20–	2	29	96.7
24–28	1	30	100.0



The following information relates to the amount, in dollars, of winter gas bills for houses in a suburban street.

Amount ($)	Frequency	Cumulative frequency	Percentage cumulative frequency
0–	2		
40–	1		
80–	12		
120–	18		
160–	3		
200–240	1		

a Copy and complete the table. Round the percentage cumulative frequency to one decimal place.

b Find the number of houses that have gas bills of less than $120.

c Construct a cumulative frequency curve for the gas bills.

d Estimate the following percentiles.

i 50th ii 20th iii 80th

e In this street, 95% of households pay less than what amount?

f What percentage of households pay less than $100?

ISBN 978-1-108-77290-7

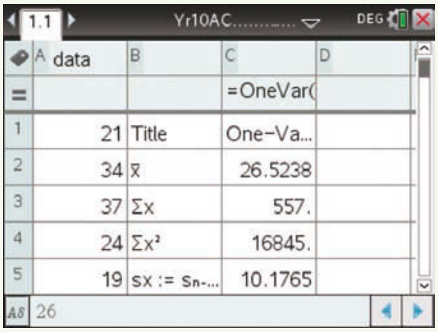
Using calculators to graph grouped data

1 Enter the following data in a list called *data* and find the mean and median.
21, 34, 37, 24, 19, 11, 15, 26, 43, 38, 25, 16, 9, 41, 36, 31, 24, 21, 30, 39, 17

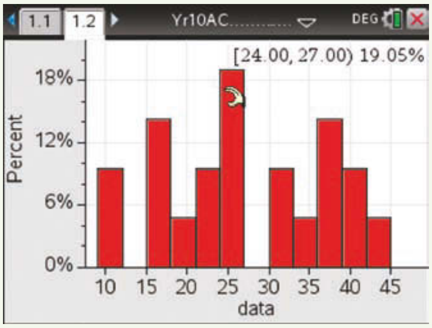
2 Construct a histogram using intervals of 3 and percentage frequency for the data above.

Using the TI-Nspire:

1 In a **Lists and spreadsheets** page type in the list name *data* and enter the values as shown. Use menu **>Statistics>Stat Calculations>One-Variable Statistics** and press enter. Scroll to view the statistics.

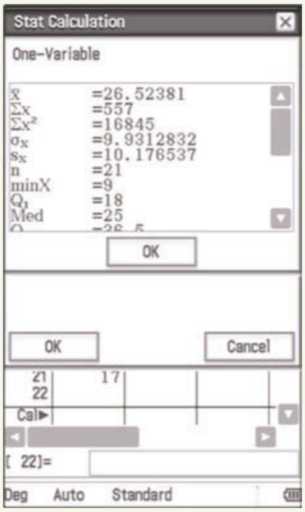


2 Insert a **Data and Statistics** page and select the *data* variable for the horizontal axis. Use menu**> Plot Type>Histogram**. Then use menu**>Plot Properties> Histogram Properties>Bin Settings>Equal Bin Width**. Choose the **Width** to be 3 and **Alignment** to be 0. Use menu**>Window/ Zoom>Zoom-Data** to auto rescale.
Use menu **>Plot Properties>Histogram Properties>Histogram Scale>Percent** to show the percentage frequency.

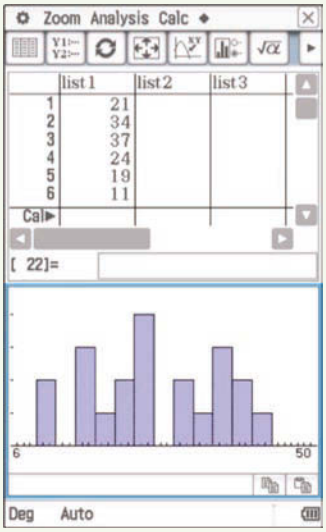


Using the ClassPad:

1 In the **Statistics** application enter the data into list1. Tap **Calc, One-Variable** and then **OK**. Scroll to view the statistics.



2 Tap **SetGraph**, ensure StatGraph1 is ticked and then tap **Setting**. Change the **Type** to **Histogram**, set **XList** to **list1**, **Freq** to **1** and then tap on **Set**. Tap the graph icon and set **HStart** to 9 and **HStep** to 3.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

9C Summary statistics

Learning intentions
- To understand the concept of quartiles for a set of data
- To be able to find the five-figure summary for a set of data
- To understand how the range and interquartile range describe the spread of a data set
- To know how to determine the outliers of a set of data

In addition to the median of a single set of data, there are two related statistics called the upper and lower quartiles. When data are placed in order, then the lower quartile is central to the lower half of the data and the upper quartile is central to the upper half of the data. These quartiles are used to calculate the interquartile range, which helps to describe the spread of the data, and determine whether or not any data points are outliers.

Australians who rent have a wide spread of ages: roughly 27% are 15–25 years; 31% are 25–35 years; 20% are 35–45 years; 15% are 45–55 years; and 7% are older than 55. A five-figure summary and a box plot would more effectively show this age spread.

LESSON STARTER House prices

A real estate agent tells you that the median house price for a suburb in 2019 was \$753 000 and the mean was \$948 000.

- Is it possible for the median and the mean to differ by so much?
- Under what circumstances could this occur? Discuss.

KEY IDEAS

- **Five-figure summary**
  - **Minimum value** (min): the minimum value
  - **Lower quartile** (Q_1): the number above 25% of the ordered data
  - **Median** (Q_2): the middle value above 50% of the ordered data
  - **Upper quartile** (Q_3): the number above 75% of the ordered data
  - **Maximum value** (max): the maximum value
- **Measures of spread**
  - **Range** = max value − min value
  - **Interquartile range** (IQR)

    $$\begin{aligned}\text{IQR} &= \text{upper quartile} - \text{lower quartile}\\ &= Q_3 - Q_1\end{aligned}$$
  - Odd number

    1 2 2 3) 5 (6 6 7 9

    ↓ ↓ ↓

    Q_1 (2) Q_2 (5) Q_3 (6.5)

    IQR (4.5)

    Even number

    2 3 3 4 7 | 8 8 9 9 9

    ↓ ↓ ↓

    Q_1 (3) Q_2 (7.5) Q_3 (9)

    IQR (6)
  - The **standard deviation** is discussed in **Section 9E**.

- **Outliers** are data elements outside the vicinity of the rest of the data. More formally, a data point is an outlier when it is below the **lower fence** (i.e. lower limit) or above the **upper fence** (i.e. upper limit).
  - $\text{Lower fence} = Q_1 - 1.5 \times \text{IQR}$
  - $\text{Upper fence} = Q_3 + 1.5 \times \text{IQR}$
  - An outlier does not significantly affect the median of a data set.
  - An outlier does significantly affect the mean of a data set.

BUILDING UNDERSTANDING

1 a State the types of values that must be calculated for a five-figure summary.
b Explain the difference between the range and the interquartile range.
c What is an *outlier*?
d How do you determine if a score in a single data set is an outlier?

2 This data set shows the number of cars in 13 families surveyed.
0, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 4, 8
a Find the median (i.e. the middle value).
b By first removing the middle value, determine:
i the lower quartile Q_1 (middle of lower half)
ii the upper quartile Q_3 (middle of upper half).
c Determine the interquartile range (IQR).
d Calculate $Q_1 - 1.5 \times \text{IQR}$ and $Q_3 + 1.5 \times \text{IQR}$.
e Are there any values that are outliers (numbers below $Q_1 - 1.5 \times \text{IQR}$ or above $Q_3 + 1.5 \times \text{IQR}$)?

3 The number of ducks spotted in eight different flocks are given in this data set.
2, 7, 8, 10, 11, 11, 13, 15
a i Find the median (i.e. average of the middle two numbers).
ii Find the lower quartile (i.e. middle of the smallest four numbers).
iii Find the upper quartile (i.e. middle of the largest four numbers).
b Determine the IQR.
c Calculate $Q_1 - 1.5 \times \text{IQR}$ and $Q_3 + 1.5 \times \text{IQR}$.
d Are there any outliers (i.e. numbers below $Q_1 - 1.5 \times \text{IQR}$ or above $Q_3 + 1.5 \times \text{IQR}$)?

Example 4 Finding the range and IQR

Determine the range and IQR for these data sets by finding the five-figure summary.

a 2, 2, 4, 5, 6, 8, 10, 13, 16, 20

b 1.6, 1.7, 1.9, 2.0, 2.1, 2.4, 2.4, 2.7, 2.9

SOLUTION	EXPLANATION
a Range = 20 − 2 = 18	Range = max − min
2 2 4 5 6 | 8 10 13 16 20 ↑ at 4: Q_1; ↑ at the split: Q_2 (7); ↑ at 13: Q_3	First, split the ordered data in half to locate the median, which is $\frac{6+8}{2} = 7$.
$Q_2 = 7$, so $Q_1 = 4$ and $Q_3 = 13$. IQR = 13 − 4 = 9	Q_1 is the median of the lower half and Q_3 is the median of the upper half. IQR = $Q_3 - Q_1$
b Range = 2.9 − 1.6 = 1.3	Max = 2.9, min = 1.6
1.6 1.7 | 1.9 2.0 2.1 2.4 2.4 | 2.7 2.9 Q_1 Q_2 Q_3	Leave the median out of the upper and lower halves when locating Q_1 and Q_3.
$Q_1 = \frac{1.7+1.9}{2} = 1.8$ $Q_3 = \frac{2.4+2.7}{2} = 2.55$ IQR = 2.55 − 1.8 = 0.75	Average the two middle values of the lower and upper halves to find Q_1 and Q_3.

Now you try

Determine the range and IQR for these data sets by finding the five-figure summary.

a 3, 5, 5, 6, 7, 9, 10, 12

b 3.8, 3.9, 4.0, 4.2, 4.5, 4.5, 4.7

Example 5 Finding the five-figure summary and outliers

The following data set represents the number of flying geese spotted on each day of a 13-day tour of England.

5, 1, 2, 6, 3, 3, 18, 4, 4, 1, 7, 2, 4

a For the data, find:

- **i** the minimum and maximum number of geese spotted
- **ii** the median
- **iii** the upper and lower quartiles
- **iv** the IQR
- **v** any outliers by determining the lower and upper fences.

b Can you give a possible reason for why the outlier occurred?

Continued on next page

SOLUTION

a **i** Min = 1, max = 18

ii 1, 1, 2, 2, 3, 3, 4, 4, 4, 5, 6, 7, 18

$\therefore$ Median = 4

iii Lower quartile $= \dfrac{2+2}{2}$

$= 2$

Upper quartile $= \dfrac{5+6}{2}$

$= 5.5$

iv $\text{IQR} = 5.5 - 2$

$= 3.5$

v Lower fence $= Q_1 - 1.5 \times \text{IQR}$

$= 2 - 1.5 \times 3.5$

$= -3.25$

Upper fence $= Q_3 + 1.5 \times \text{IQR}$

$= 5.5 + 1.5 \times 3.5$

$= 10.75$

$\therefore$ The outlier is 18.

b Perhaps a flock of geese was spotted that day.

EXPLANATION

Look for the largest and smallest numbers and order the data:

1 1 2 | 2 3 3) 4 (4 4 5 | 6 7 18

Q_1 Q_2 Q_3

Since Q_2 falls on a data value, it is not included in the lower or upper halves when Q_1 and Q_3 are calculated.

$\text{IQR} = Q_3 - Q_1$

A data point is an outlier when it is less than $Q_1 - 1.5 \times \text{IQR}$ or greater than $Q_3 + 1.5 \times \text{IQR}$.

There are no numbers less than -3.25 but 18 is greater than 10.75.

Now you try

The following data set represents the number of people on 11 buses in a local area.

36, 24, 15, 23, 26, 0, 19, 24, 26, 33, 19

a For the data, find:

i the minimum and maximum number of people on the buses.

ii the median

iii the upper and lower quartiles

iv the IQR

v any outliers by determining the lower and upper fences.

b Can you give a possible reason for why the outlier occurred?

Exercise 9C

FLUENCY	1–3	1(½), 2, 3	2–4

Example 4

1 Determine the range and IQR for these data sets by finding the five-figure summary.

a 3, 4, 6, 8, 8, 10, 13

b 10, 10, 11, 14, 14, 15, 16, 18

c 1.2, 1.8, 1.9, 2.3, 2.4, 2.5, 2.9, 3.2, 3.4

d 41, 49, 53, 58, 59, 62, 62, 65, 66, 68

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 5

2 The following numbers of cars, travelling on a quiet suburban street, were counted on each day for 15 days.

10, 9, 15, 14, 10, 17, 15, 0, 12, 14, 8, 15, 15, 11, 13

For the given data, find:

a the minimum and maximum number of cars counted
b the median
c the lower and upper quartiles
d the IQR
e any outliers by determining the lower and upper fences
f a possible reason for the outlier.

3 Summarise the data sets below by finding:

i the minimum and maximum values
ii the median (Q_2)
iii the lower and upper quartiles (Q_1 and Q_3)
iv the IQR
v any outliers.

a 4, 5, 10, 7, 5, 14, 8, 5, 9, 9
b 24, 21, 23, 18, 25, 29, 31, 16, 26, 25, 27

4 The number 20 is an outlier in this data set 1, 2, 2, 3, 4, 20.

a Calculate the mean if the outlier is:
i included
ii excluded
b Calculate the median if the outlier is:
i included
ii excluded
c By how much does including the outlier increase the:
i mean?
ii median?

PROBLEM-SOLVING	5, 6(½)	6(½), 7	7, 8

5 Twelve different calculators had the following numbers of buttons.

36, 48, 52, 43, 46, 53, 25, 60, 128, 32, 52, 40

a For the given data, find:
i the minimum and maximum number of buttons on the calculators
ii the median
iii the lower and upper quartiles
iv the IQR
v any outliers
vi the mean.
b Which is a better measure of the centre of the data, the mean or the median? Explain.
c Can you give a possible reason why the outlier has occurred?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

6 Using the definition of an outlier, decide whether or not any outliers exist in the following sets of data. If so, list them.

a 3, 6, 1, 4, 2, 5, 9, 8, 6, 3, 6, 2, 1
b 8, 13, 12, 16, 17, 14, 12, 2, 13, 19, 18, 12, 13
c 123, 146, 132, 136, 139, 141, 103, 143, 182, 139, 127, 140
d 2, 5, 5, 6, 5, 4, 5, 6, 7, 5, 8, 5, 5, 4

7 For the data in this stem-and-leaf plot, find:

a the IQR
b any outliers
c the median if the number 37 is added to the list
d the median if the number 22 is added to the list instead of 37.

Stem	Leaf
0	1
1	6 8
2	0 4 6
3	2 3

2 | 4 means 24

8 Three different numbers have median 2 and range 2. Find the three numbers.

REASONING	9	9, 10	10–12

9 Explain what happens to the mean of a data set if all the values are:

a increased by 5
b multiplied by 2
c divided by 10.

10 Explain what happens to the IQR of a data set if all values are:

a increased by 5
b multiplied by 2
c divided by 10.

11 Give an example of a small data set that satisfies the following.

a median = mean
b median = upper quartile
c range = IQR

12 Explain why, in many situations, the median is preferred to the mean as a measure of centre.

ENRICHMENT: Some research	–	–	13

13 Use the internet to search for data about a topic that interests you. Try to choose a single set of data that includes between 15 and 50 values.

a Organise the data using:
  i a stem-and-leaf plot
  ii a frequency table and histogram.
b Find the mean and the median.
c Find the range and the interquartile range.
d Write a brief report describing the centre and spread of the data, referring to parts **a** to **c** above.
e Present your findings to your class or a partner.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

9D Box plots

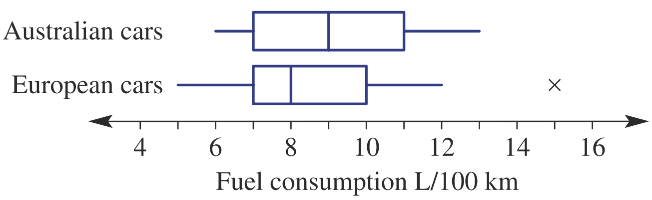
Learning intentions
- To understand the features of a box plot in describing the spread of a set of data
- To know how to construct a box plot with outliers
- To be able to compare data sets using parallel box plots

The five-figure summary (min, Q_1, Q_2, Q_3, max) can be represented in graphical form as a box plot. Box plots are graphs that summarise single data sets. They clearly display the minimum and maximum values, the median, the quartiles and any outliers. Box plots also give a clear indication of how data are spread, as the IQR is shown by the width of the central box.

Medical researchers analyse data about the health of babies and mothers. Parallel box plots comparing birth weights of full-term babies born to smoking and non-smoking mothers show significantly lower weights for babies whose mothers smoke.

LESSON STARTER Fuel consumption

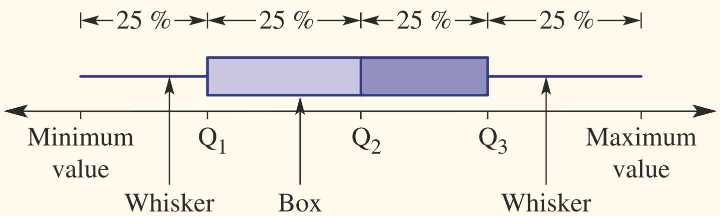
This parallel box plot summarises the average fuel consumption (litres per 100 km) for a group of Australian-made and European-made cars.



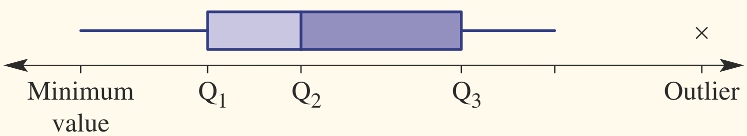
- What do the box plots say about how the fuel consumption compares between Australian-made and European-made cars?
- What does each part of the box plot represent?
- What do you think the cross (×) represents on the European cars box plot?

KEY IDEAS

- A **box plot** (also called a box-and-whisker plot) can be used to summarise a data set.
  - The number of data values in each quarter (25%) are approximately equal.



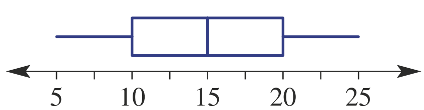
- An **outlier** is marked with a cross (×).
  - An outlier is greater than $Q_3 + 1.5 \times \text{IQR}$ or less than $Q_1 - 1.5 \times \text{IQR}$.
  - The whiskers stretch to the lowest and highest data values that are not outliers.



- **Parallel box plots** are two or more box plots drawn on the same scale. They are used to compare data sets within the same context.

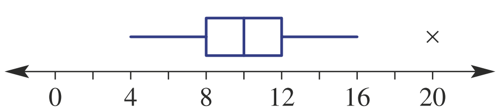
BUILDING UNDERSTANDING

1 For this simple box plot, state:



a the median (Q_2) **b** the minimum
c the maximum **d** the range
e the lower quartile (Q_1) **f** the upper quartile (Q_3)
g the interquartile range (IQR).

2 Complete the following for this box plot.



a Find the IQR.
b Calculate $Q_1 - 1.5 \times \text{IQR}$.
c Calculate $Q_3 + 1.5 \times \text{IQR}$.
d State the value of the outlier.
e Check that the outlier is greater than $Q_3 + 1.5 \times \text{IQR}$.

ISBN 978-1-108-77290-7

Example 6 Constructing box plots

Consider the given data set:

5, 9, 4, 3, 5, 6, 6, 5, 7, 12, 2, 3, 5

a Determine whether any outliers exist by first finding Q_1 and Q_3.

b Draw a box plot to summarise the data, marking outliers if they exist.

SOLUTION

a $$2\ \ 3\ \ \underbrace{3\ \ 4}_{\substack{\uparrow \\ Q_1}}\ \ 5\ \ 5)\underset{\substack{\uparrow \\ Q_2}}{5}(5\ \ 6\ \ \underbrace{6\ \ 7}_{\substack{\uparrow \\ Q_3}}\ \ 9\ \ 12$$

$$Q_1 = \frac{3+4}{2} = 3.5 \qquad Q_3 = \frac{6+7}{2} = 6.5$$

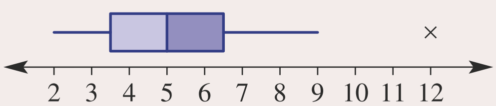
$$\therefore \text{IQR} = 6.5 - 3.5 = 3$$

$$Q_1 - 1.5 \times \text{IQR} = 3.5 - 1.5 \times 3 = -1$$

$$Q_3 + 1.5 \times \text{IQR} = 6.5 + 1.5 \times 3 = 11$$

$\therefore$ 12 is an outlier.

b



EXPLANATION

Order the data to help find the quartiles.

Locate the median Q_2 then split the data in half above and below this value.

Q_1 is the middle value of the lower half and Q_3 the middle value of the upper half.
Determine IQR = $Q_3 - Q_1$.

Check for any outliers; i.e. values below $Q_1 - 1.5 \times \text{IQR}$ or above $Q_3 + 1.5 \times \text{IQR}$.

There are no data values below -1 but $12 > 11$.
Draw a line and mark in a uniform scale reaching from 2 to 12. Sketch the box plot by marking the minimum 2 and the outlier 12 and Q_1, Q_2 and Q_3. The end of the five-point summary is the nearest value below 11; i.e. 9.

Now you try

Consider the given data set:

12, 8, 19, 13, 22, 15, 1, 17, 24, 19

a Determine whether any outliers exist by first finding Q_1 and Q_3.

b Draw a box plot to summarise the data, marking outliers if they exist.

Exercise 9D

FLUENCY	1	1–2(1/2)	1–2(1/2)

Example 6

1 Consider the data sets below.

i Determine whether any outliers exist by first finding Q_1 and Q_3.

ii Draw a box plot to summarise the data, marking outliers if they exist.

a 4, 6, 5, 2, 3, 4, 4, 13, 8, 7, 6

b 1.8, 1.7, 1.8, 1.9, 1.6, 1.8, 2.0, 1.1, 1.4, 1.9, 2.2

c 21, 23, 18, 11, 16, 19, 24, 21, 23, 22, 20, 31, 26, 22

d 0.04, 0.04, 0.03, 0.03, 0.05, 0.06, 0.07, 0.03, 0.05, 0.02

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2 First, find Q_1, Q_2 and Q_3 and then draw box plots for the given data sets. Remember to find outliers and mark them on your box plot if they exist.

a 11, 15, 18, 17, 1, 2, 8, 12, 19, 15

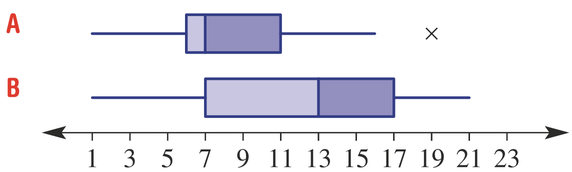
b 37, 48, 52, 51, 51, 42, 48, 47, 39, 41, 65

c 0, 1, 5, 4, 4, 4, 2, 3, 3, 1, 4, 3

d 124, 118, 73, 119, 117, 120, 120, 121, 118, 122

PROBLEM-SOLVING	3, 4	3, 4	4, 5

3 Consider these parallel box plots, A and B.



a What statistical measure do these box plots have in common?

b Which data set (A or B) has a wider range of values?

c Find the IQR for:

i data set A　　ii data set B.

d How would you describe the main difference between the two sets of data from which the parallel box plots have been drawn?

4 The following masses, in kilograms, of 15 Madagascan lemurs are recorded as part of a conservation project.

14.4, 15.5, 17.3, 14.6, 14.7
15.0, 15.8, 16.2, 19.7, 15.3
13.8, 14.6, 15.4, 15.7, 14.9

a Find Q_1, Q_2 and Q_3.

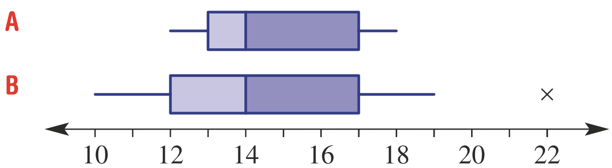
b Which masses, if any, would be considered outliers?

c Draw a box plot to summarise the lemurs' masses.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

5 Two data sets can be compared using parallel box plots on the same scale, as shown below.



a What statistical measures do these box plots have in common?

b Which data set (A or B) has a wider range of values?

c Find the IQR for:

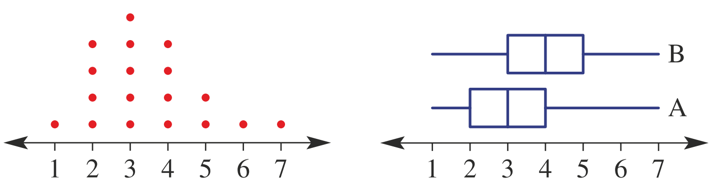
  i data set A **ii** data set B.

d How would you describe the main difference between the two sets of data from which the parallel box plots have been drawn?

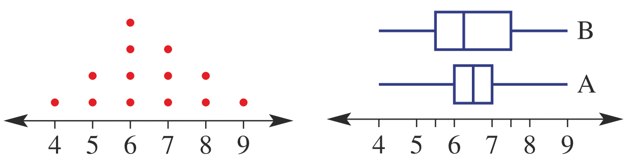
REASONING	6	6, 7	7, 8

6 Select the box plot (A or B) that best matches the given dot plot or histogram.

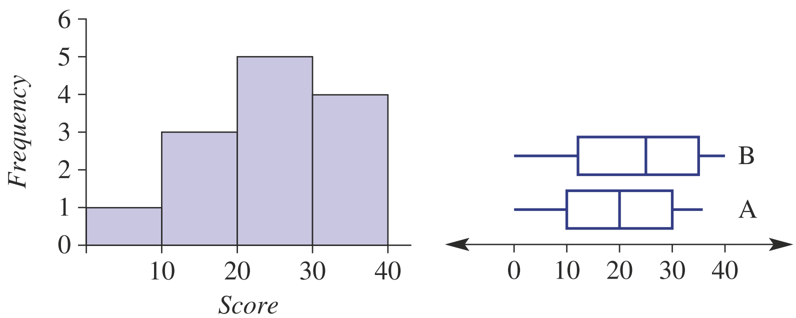
a



b



c



7 Fifteen essays are marked for spelling errors by a particular examiner and the following numbers of spelling errors are counted.

3, 2, 4, 6, 8, 4, 6, 7, 6, 1, 7, 12, 7, 3, 8

The same 15 essays are marked for spelling errors by a second examiner and the following numbers of spelling errors are counted.

12, 7, 9, 11, 15, 5, 14, 16, 9, 11, 8, 13, 14, 15, 13

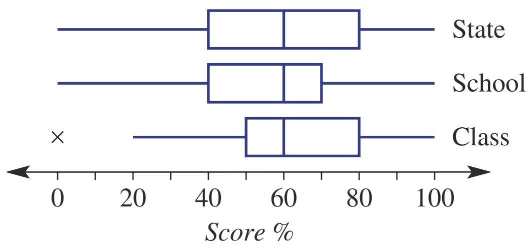
a Draw parallel box plots for the data.

b Do you believe there is a major difference in the way the essays were marked by the two examiners? If yes, describe this difference.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

8 The results for a Year 12 class are to be compared with the Year 12 results of the school and the State, using the parallel box plots shown.



a Describe the main differences between the performance of:
   i the class against the school
   ii the class against the State
   iii the school against the State.

b Why is an outlier shown on the class box plot but not shown on the school box plot?

ENRICHMENT: Creating your own parallel box plots	–	–	9

9 **a** Choose an area of study for which you can collect data easily, for example:
- heights or weights of students
- maximum temperatures over a weekly period
- amount of pocket money received each week for a group of students.

b Collect at least two sets of data for your chosen area of study – perhaps from two or three different sources, including the internet.
Examples:
- Measure student heights in your class and from a second class in the same year level.
- Record maximum temperatures for 1 week and repeat for a second week to obtain a second data set.
- Use the internet to obtain the football scores of two teams for each match in the previous season.

c Draw parallel box plots for your data.

d Write a report on the characteristics of each data set and the similarities and differences between the data sets collected.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Using calculators to draw box plots

1 Type these data into lists and define them as Test A and Test B.

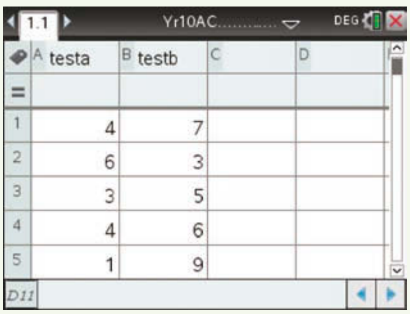
Test A: 4, 6, 3, 4, 1, 3, 6, 4, 5, 3, 4, 3

Test B: 7, 3, 5, 6, 9, 3, 6, 7, 4, 1, 4, 6

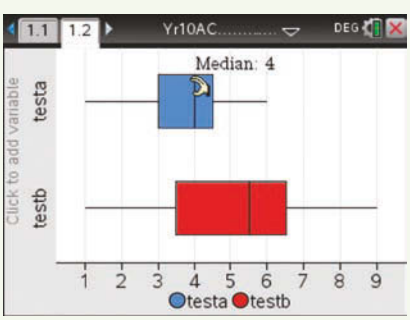
2 Draw parallel box plots for the data.

Using the TI-Nspire:

1 In a **Lists and spreadsheets** page type in the list names ***testa*** and ***testb*** and enter the values as shown.

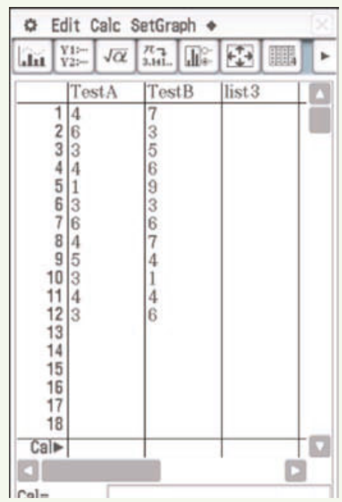


2 Insert a **Data and Statistics** page and select the ***testa*** variable for the horizontal axis. Change to a box plot using menu **>Plot Type>Box Plot**. Trace (or hover over) to reveal the statistical measures. To show the box plot for ***testb***, use menu **>Plot Properties>Add X Variable** and select ***testb***.

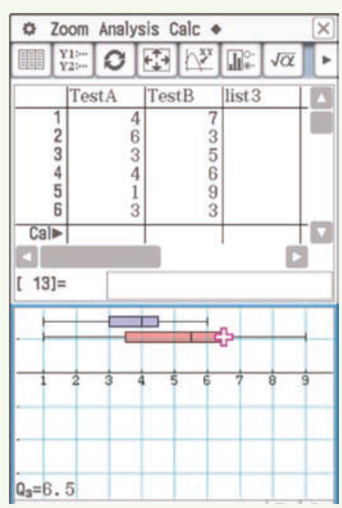


Using the ClassPad:

1 In the **Statistics** application enter the data into the lists. Give each column a title.



2 Tap . For graph **1**, set **Draw** to **On**, **Type** to **MedBox**, **XList** to **mainTestA** and **Freq** to **1**.
For graph **2**, set **Draw** to **On**, **Type** to **MedBox**, **XList** to **mainTestB** and **Freq** to **1**. Tap **Set**. Tap .



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

9E Standard deviation 10A

Learning intentions

- To understand that standard deviation is a number that describes the spread of the data about the mean
- To know that a small standard deviation means data are concentrated about the mean
- To know how to calculate the standard deviation for a small set of data
- To be able to compare two sets of data referring to the mean and standard deviation

For a single data set we have already discussed the range and interquartile range to describe the spread of the data. Another statistic commonly used to describe spread is standard deviation. The standard deviation is a number that describes how far data values are from the mean. A data set with a relatively small standard deviation will have data values concentrated about the mean, and if a data set has a relatively large standard deviation then the data values will be more spread out from the mean.

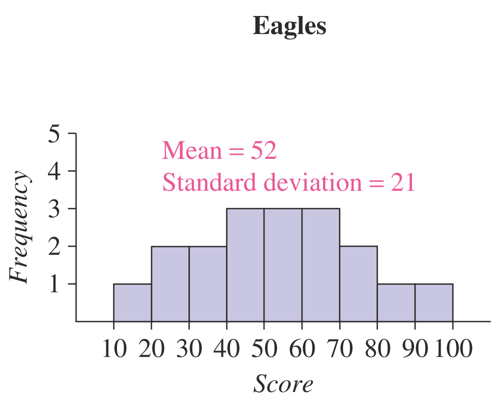
The standard deviation can be calculated by hand but, given the tedious nature of the calculation, technology can be used for more complex data sets. In this section technology is not required but you will be able to find a function on your calculator (often denoted s or σ) that can be used to find the standard deviation.

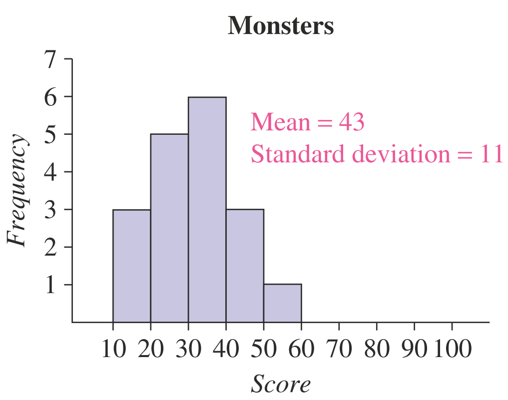
When selecting a sportsperson for a competition, the average and standard deviation of past results are useful. Two cricketers may have equal average runs per game, but the player with the smaller standard deviation is the more consistent batter.

LESSON STARTER Which is the better team?

These histograms show the number of points scored by the Eagles and the Monsters basketball teams in an 18-round competition. The mean and standard deviation are given for each team.

- Which team has the higher mean? What does this say about the team's performance?
- Which team has the smaller standard deviation? What does this say about the team's performance? Discuss.





ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

KEY IDEAS

- The **standard deviation** is a number that describes the spread of data about the mean.
  - The sample standard deviation is for a sample data set drawn from the population.
  - If every data value from a population is used, then we calculate the population standard deviation.
- To calculate the **sample standard deviation** (s), follow these steps.
  1. Find the mean ($\bar{x}$).
  2. Find the difference between each value and the mean (called the deviation).
  3. Square each deviation.
  4. Sum the squares of each deviation.
  5. Divide by the number of data values less 1 (i.e. $n - 1$).
  6. Take the square root.

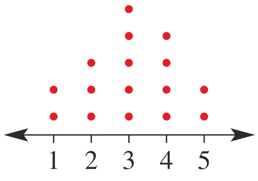
$$s = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n - 1}}$$

- If the data represent the complete population, then divide by n instead of $(n - 1)$. This would give the **population standard deviation** (σ). Dividing by $(n - 1)$ for the sample standard deviation gives a better estimate of the population standard deviation.
- If data are concentrated about the mean, then the standard deviation is relatively small.
- If data are spread out from the mean, then the standard deviation is relatively large.
- In many common situations we can expect 95% of the data to be within two standard deviations of the mean.

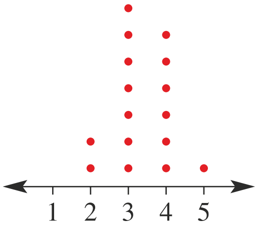
BUILDING UNDERSTANDING

1. Use the word *smaller* or *larger* to complete each sentence.
   a. If data are more spread out from the mean, then the standard deviation is ___________.
   b. If data are more concentrated about the mean, then the standard deviation is ____________.
2. Here are two dot plots, A and B.

A



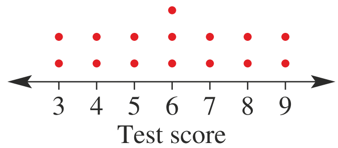
B



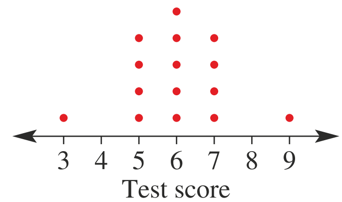
   a. Which data set (A or B) would have the higher mean?
   b. Which data set (A or B) would have the higher standard deviation?

3 These dot plots show the results for a class of 15 students who sat tests A and B. Both sets of results have the same mean and range.

A



B



Which data set (A or B) would have the higher standard deviation? Give a reason.

4 This back-to-back stem-and-leaf plot compares the number of trees or shrubs in the backyards of homes in the suburbs of Gum Heights and Oak Valley.

a Which suburb has the smaller mean number of trees or shrubs? Do not calculate the actual means.

b Without calculating the actual standard deviations, which suburb has the smaller standard deviation?

Gum Heights Leaf	Stem	Oak Valley Leaf
7 3 1	0	
8 6 4 0	1	0
9 8 7 2	2	0 2 3 6 8 8 9
9 6 4	3	4 6 8 9
	4	3 6

2 | 8 means 28

Example 7 Calculating the standard deviation

Calculate the mean and sample standard deviation for this small data set, correct to one decimal place.

2, 4, 5, 8, 9

SOLUTION

$$\bar{x} = \frac{2 + 4 + 5 + 8 + 9}{5}$$

$$= 5.6$$

$$s = \sqrt{\frac{(2 - 5.6)^2 + (4 - 5.6)^2 + (5 - 5.6)^2 + (8 - 5.6)^2 + (9 - 5.6)^2}{5 - 1}}$$

$$= \sqrt{\frac{(-3.6)^2 + (-1.6)^2 + (-0.6)^2 + (2.4)^2 + (3.4)^2}{4}}$$

$$= 2.9 \text{ (to 1 d.p.)}$$

EXPLANATION

Sum all the data values and divide by the number of data values (i.e. 5) to find the mean.

Deviation 1 is $2 - 5.6$ (the difference between the data value and the mean).

Sum the square of all the deviations, divide by $(n - 1)$ (i.e. 4) and then take the square root.

Now you try

Calculate the mean and sample standard deviation for this small data set, correct to one decimal place.

1, 2, 2, 4, 5

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 8 Interpreting the standard deviation

This back-to-back stem-and-leaf plot shows the distribution of distances that 17 people in Darwin and Sydney travel to work. The means and standard deviations are given.

Darwin Leaf	Stem	Sydney Leaf	
8 7 4 2	0	1 5	Sydney
9 9 5 5 3	1	2 3 7	$\bar{x} = 27.9$
8 7 4 3 0	2	0 5 5 6	$s = 15.1$
5 2 2	3	2 5 9 9	Darwin
	4	4 4 6	$\bar{x} = 19.0$
	5	2	$s = 10.1$

3 | 5 means 35 km

Consider the position and spread of the data and then answer the following.

a By looking at the stem-and-leaf plot, suggest why Darwin's mean is less than that of Sydney.
b Why is Sydney's standard deviation larger than that of Darwin?
c Give a practical reason for the difference in centre and spread for the data for Darwin and Sydney.

SOLUTION

a The maximum score for Darwin is 35. Sydney's mean is affected by several values larger than 35.

b The data for Sydney are more spread out from the mean. Darwin's scores are more closely clustered near its mean.

c Sydney is a larger city and more spread out, so people have to travel farther to get to work.

EXPLANATION

The mean depends on every value in the data set.

Sydney has more scores with a large distance from its mean. Darwin's scores are closer to the Darwin mean.

Higher populations often lead to larger cities and longer travel distances.

Now you try

This stem-and-leaf plot shows the distribution of hours of television watched by 20 students from each of Year 7 and Year 12 over a 1-month period. The means and standard deviations are given.

Year 7	Stem	Year 12	Year 7
9	0	4 7	$\bar{x} = 30.1$
9 5 2	1	0 1 3 4 4 6 7 9	$s = 10.7$
9 8 8 4 1	2	1 2 2 4 5 7 8 9	
9 8 7 5 3 2 0	3	3 5	Year 12
6 3 2 2	4		$\bar{x} = 19.6$
			$s = 8.5$

2 | 4 means 24 hours

Consider the position and spread of the data and then answer the following.

a Why is the mean for Year 12 less than that for Year 7?
b Why is Year 7's standard deviation larger than that for Year 12?
c Give a practical reason for the difference in centre and spread for the Year 7 and Year 12 data.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 9E

FLUENCY	1, 2(1/2)	2(1/2), 3	2(1/2), 3

Example 7

1 Calculate the mean and sample standard deviation for this small data set, correct to one decimal place.

1, 2, 4, 5, 7

Example 7

2 Calculate the mean and sample standard deviation for these small data sets. Use the formula for the sample standard deviation. Round the standard deviation to one decimal place where necessary.

a 3, 5, 6, 7, 9

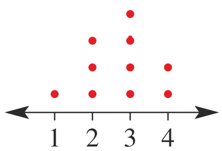
b 1, 1, 4, 5, 7

c 2, 5, 6, 9, 10, 11, 13

d 28, 29, 32, 33, 36, 37

3 Calculate the mean and sample standard deviation for the data in these graphs, correct to one decimal place.

a



b

Stem	Leaf
0	4
1	1 3 7
2	0 2

1 | 7 means 17

PROBLEM-SOLVING	4, 5	4, 5	5, 6

Example 8

4 This back-to-back stem-and-leaf plot shows the distribution of distances travelled by students at an inner-city and an outer-suburb school. The means and standard deviations are given.

Inner-city Leaf	Stem	Outer-suburb Leaf
9 6 4 3 1 1	0	3 4 9
9 4 2 0	1	2 8 8 9
7 1	2	1 3 4
	3	4
	4	1

2 | 4 means 24 km

Inner-city
$\bar{x} = 10.6$
$s = 8.0$

Outer-suburb
$\bar{x} = 18.8$
$s = 10.7$

Consider the position and spread of the data and then answer the following.

a Why is the mean for the outer-suburb school larger than that for the inner-city school?

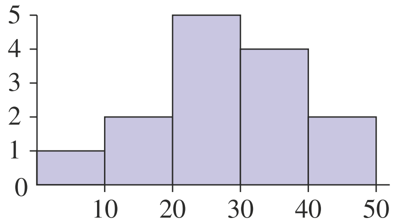
b Why is the standard deviation for the inner-city school smaller than that for the outer-suburb school?

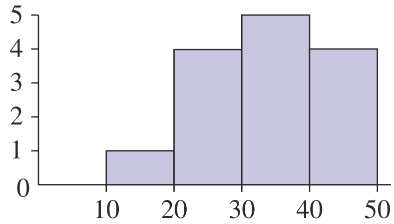
c Give a practical reason for the difference in centre and spread for the two schools.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

5 Consider these two histograms, and then state whether the following are true or false.

A


B


a The mean for set A is greater than the mean for set B.

b The range for set A is greater than the range for set B.

c The standard deviation for set A is greater than the standard deviation for set B.

6 Find the mean and sample standard deviation for the scores in these frequency tables. Round the standard deviations to one decimal place.

a

Score	Frequency
1	3
2	1
3	3

b

Score	Frequency
4	1
5	4
6	3

REASONING	7	7, 8	8, 9

7 Two simple data sets, A and B, are identical except for the maximum value, which is an outlier for set B.

A: 4, 5, 7, 9, 10

B: 4, 5, 7, 9, 20

a Is the range for set A equal to the range for set B?

b Is the mean for each data set the same?

c Is the median for each data set the same?

d Would the standard deviation be affected by the outlier? Explain.

8 Data sets 1 and 2 have means $\bar{x}_1$ and $\bar{x}_2$, and standard deviations s_1 and s_2.

a If $\bar{x}_1 > \bar{x}_2$, does this necessarily mean that $s_1 > s_2$? Give a reason.

b If $s_1 < s_2$ does this necessarily mean that $\bar{x}_1 < \bar{x}_2$?

9 Data sets A and B each have 20 data values and are very similar except for an outlier in set A. Explain why the interquartile range might be a better measure of spread than the range or the standard deviation.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

ENRICHMENT: Study scores	–	–	10

10 The Mathematics study scores (out of 100) for 50 students in a school are as listed.

71, 85, 62, 54, 37, 49, 92, 85, 67, 89
96, 44, 67, 62, 75, 84, 71, 63, 69, 81
57, 43, 64, 61, 52, 59, 83, 46, 90, 32
94, 84, 66, 70, 78, 45, 50, 64, 68, 73
79, 89, 80, 62, 57, 83, 86, 94, 81, 65

The mean ($\bar{x}$) is 69.16 and the sample standard deviation (s) is 16.0.

a Calculate:
- **i** $\bar{x} + s$
- **ii** $\bar{x} - s$
- **iii** $\bar{x} + 2s$
- **iv** $\bar{x} - 2s$
- **v** $\bar{x} + 3s$
- **vi** $\bar{x} - 3s$

b Use your answers from part **a** to find the percentage of students with a score within:
- **i** one standard deviation from the mean
- **ii** two standard deviations from the mean
- **iii** three standard deviations from the mean.

c **i** Research what it means when we say that the data are ‘normally distributed’. Give a brief explanation.
- **ii** For data that are normally distributed, find out what percentage of data are within one, two and three standard deviations from the mean. Compare this with your results for part **b** above.

ISBN 978-1-108-77290-7

Progress quiz

9A

1 What type of data would these survey questions generate?

a How many pets do you have?

b What is your favourite ice-cream flavour?

9B

2 A Year 10 class records the length of time (in minutes) each student takes to travel from home to school. The results are listed here.

15	32	6	14	44	28	15	9	25	18
8	16	13	20	19	27	23	12	38	15

a Organise the data into a frequency table, using class intervals of 10. Include a percentage frequency column.

b Construct a histogram for the data, showing both the frequency and percentage frequency on the one graph.

c Construct a stem-and-leaf plot for the data.

d Use your stem-and-leaf plot to find the median.

9C

3 Determine the range and IQR for these data sets by finding the five-figure summary.

a 4, 9, 12, 15, 16, 18, 20, 23, 28, 32

b 4.2, 4.3, 4.7, 5.1, 5.2, 5.6, 5.8, 6.4, 6.6

9C

4 The following numbers of parked cars were counted in the school car park and adjacent street each day at morning recess for 14 school days.

36, 38, 46, 30, 69, 31, 40, 37, 55, 34, 44, 33, 47, 42

a For the data, find:

i the minimum and maximum number of cars

ii the median

iii the upper and lower quartiles

iv the IQR

v any outliers.

b Can you give a possible reason for why the outlier occurred?

9D

5 The ages of a team of female gymnasts are given in this data set:

18, 23, 14, 28, 21, 19, 15, 32, 17, 18, 20, 13, 21

a Determine whether any outliers exist by first finding Q_1 and Q_3.

b Draw a box plot to summarise the data, marking outliers if they exist.

9E

10A

6 Find the sample standard deviation for this small data set, correct to one decimal place.

2, 3, 5, 6, 9

Use the sample standard deviation formula.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

9F Time-series data

Learning intentions
- To understand that time-series data are data recorded at regular time intervals
- To know how to plot a time-series graph with time on the horizontal axis
- To be able to use a time-series plot to describe any trend in the data

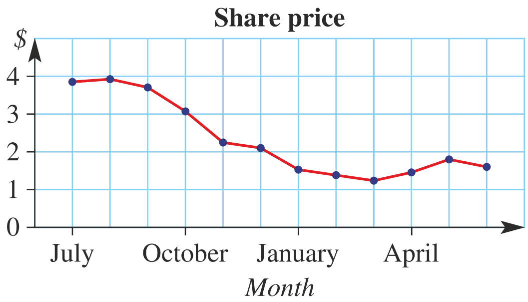
A time series is a sequence of data values that are recorded at regular time intervals. Examples include temperature recorded on the hour, speed recorded every second, population recorded every year and profit recorded every month. A line graph can be used to represent time-series data and these can help to analyse the data, describe trends and make predictions about the future.

The BOM (Bureau of Meteorology) publishes time-series graphs of Australian annual and monthly mean temperature anomalies, i.e. deviations from the overall average. Over recent decades, these graphs show an upward trend of positive and increasing anomalies.

LESSON STARTER Share price trends

A company's share price is recorded at the end of each month of the financial year, as shown in this time-series graph.

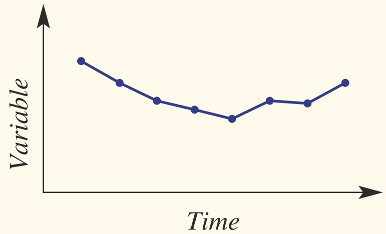
- Describe the trend in the data at different times of the year.
- At what time of year do you think the company starts reporting bad profit results?
- Does it look like the company's share price will return to around $4 in the next year? Why?



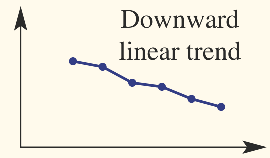
ISBN 978-1-108-77290-7

KEY IDEAS

- **Time-series data** are recorded at regular time intervals.
- The graph or plot of a time series uses:
  - time on the horizontal axis as the **independent** variable
  - line segments connecting points on the graph.
  - the variable being considered on the vertical axis as the **dependent** variable



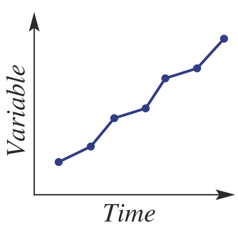
- If the time-series plot results in points being on or near a straight line, then we say that the trend is **linear**.



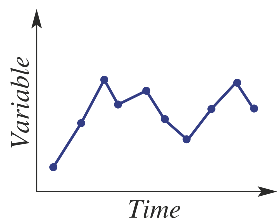
BUILDING UNDERSTANDING

1 Describe the following time-series plots as having a linear (i.e. straight-line trend), non-linear trend (i.e. a curve) or no trend.

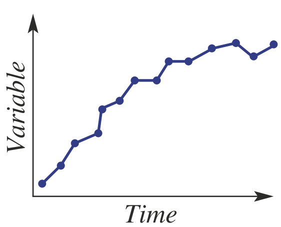
a



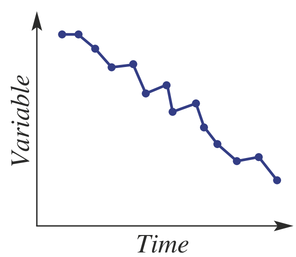
b



c



d



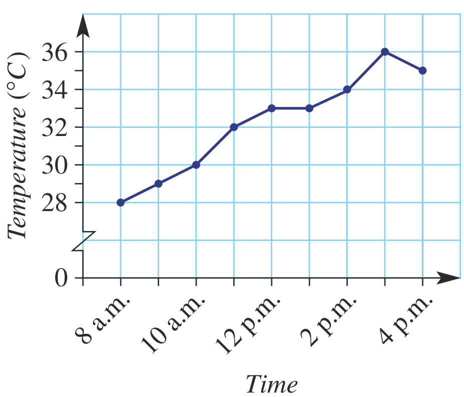
2 This time-series graph shows the temperature over the course of an 8-hour school day.

a State the temperature at:
 i 8 a.m. **ii** 12 p.m.
 iii 1 p.m. **iv** 4 p.m.

b What was the maximum temperature?

c During what times did the temperature:
 i stay the same? **ii** decrease?

d Describe the general trend in the temperature for the 8-hour school day.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 9 Plotting and interpreting a time-series plot

The approximate population of an outback town is recorded from 1990 to 2005.

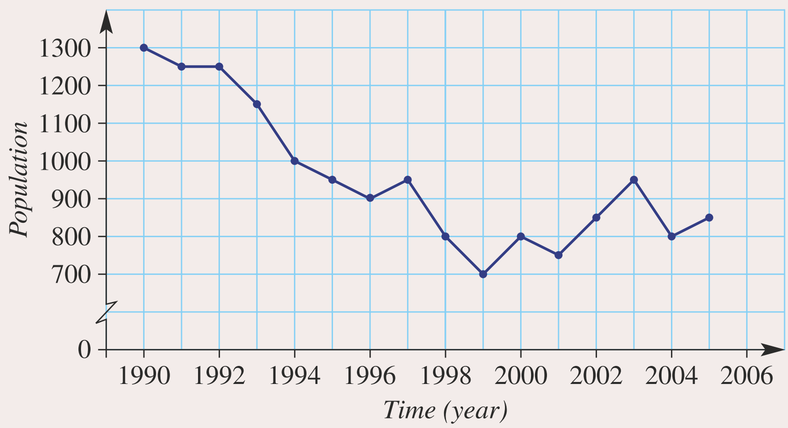
Year	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005
Population	1300	1250	1250	1150	1000	950	900	950	800	700	800	750	850	950	800	850

a Plot the time series.

b Describe the trend in the data over the 16 years.

SOLUTION

a



EXPLANATION

Use time on the horizontal axis. Break the y-axis so as to not include 0–700. Join points with line segments.

b The population declines steadily for the first 10 years. The population rises and falls in the last 6 years, resulting in a slight upwards trend.

Interpret the overall rise and fall of the lines on the graph.

Now you try

The average price of lambs at a market over 14 weeks is given in this table.

Week	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Price ($)	82	80	85	89	91	87	93	104	100	111	108	105	112	119

a Plot the time series.

b Describe the trend in the data over the 14 weeks.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 9F

FLUENCY	1, 2	2, 3	2, 3

Example 9 **1** The approximate population of a small village is recorded from 2005 to 2015.

Year	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015
Population	550	500	550	600	700	650	750	750	850	950	900

a Plot the time-series graph.
b Describe the general trend in the data over the 11 years.
c For the 11 years, what was the:
i minimum population? **ii** maximum population?

Example 9 **2** A company's share price over 12 months is recorded in this table.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Price ($)	1.30	1.32	1.35	1.34	1.40	1.43	1.40	1.38	1.30	1.25	1.22	1.23

a Plot the time-series graph. Break the y-axis to exclude values from \$0 to \$1.20.
b Describe the way in which the share price has changed over the 12 months.
c What is the difference between the maximum and minimum share price in the 12 months?

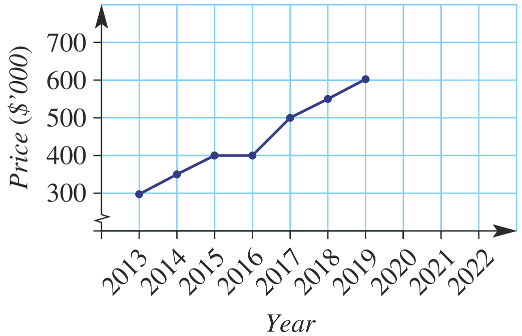
3 The pass rate (%) for a particular examination is given in a table over 10 years.

Year	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004
Pass rate (%)	74	71	73	79	85	84	87	81	84	83

a Plot the time-series graph for the 10 years.
b Describe the way in which the pass rate for the examination has changed in the given time period.
c In what year was the pass rate a maximum?
d By how much had the pass rate improved from 1995 to 1999?

PROBLEM-SOLVING	4, 5	4, 5	5, 6

4 This time-series plot shows the upwards trend of house prices in an Adelaide suburb over 7 years from 2013 to 2019.



a Would you say that the general trend in house prices is linear or non-linear?
b Assuming the trend in house prices continues for this suburb, what would you expect the house price to be in:
i 2020? **ii** 2022?

ISBN 978-1-108-77290-7

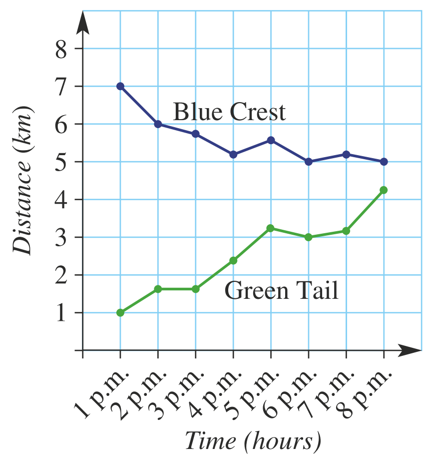
Cambridge University Press
Updated June 2022

5 The two top-selling book stores for a company list their sales figures for the first 6 months of the financial year. Sales amounts are in thousands of dollars.

	July	August	September	October	November	December
City Central ($'000)	12	13	12	10	11	13
Southbank ($'000)	17	19	16	12	13	9

a What was the difference in the sales volume for:
 i August? ii December?
b In how many months did the City Central store sell more books than the Southbank store?
c Construct a time-series plot for both stores on the same set of axes.
d Describe the trend of sales for the 6 months for:
 i City Central ii Southbank
e Based on the trend for the sales for the Southbank store, what would you expect the approximate sales volume to be in January?

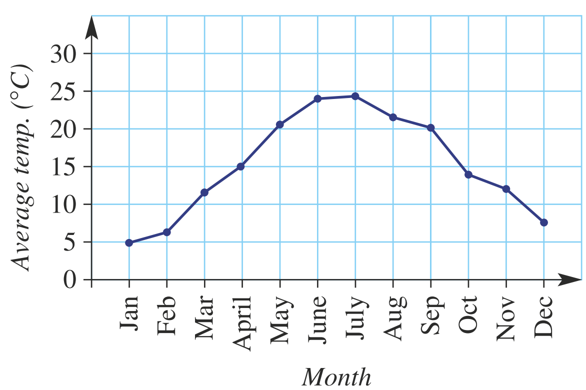
6 Two pigeons (Green Tail and Blue Crest) each have a beacon that communicates with a recording machine. The distance of each pigeon from the machine is recorded every hour for 8 hours.



a State the distance from the machine at 3 p.m. for:
 i Blue Crest ii Green Tail
b Describe the trend in the distance from the recording machine for:
 i Blue Crest ii Green Tail
c Assuming that the given trends continue, predict the time when the pigeons will be the same distance from the recording machine.

REASONING | 7 | 7, 8 | 8, 9

7 The average monthly maximum temperature for a city is illustrated in this graph.



a Explain why the average maximum temperature for December is close to the average maximum temperature for January.
b Do you think this graph is for an Australian city?
c Do you think the data are for a city in the Northern Hemisphere or the Southern Hemisphere? Give a reason.

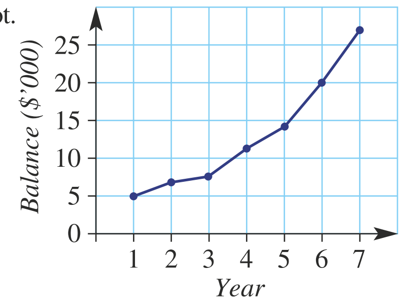
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

8 The balance of an investment account is shown in this time-series plot.

a Describe the trend in the account balance over the 7 years.

b Give a practical reason for the shape of the curve that models the trend in the graph.



9 A drink at room temperature is placed in a fridge that is at 4°C.

a Sketch a time-series plot that might show the temperature of the drink after it has been placed in the fridge.

b Would the temperature of the drink ever get to 3°C? Why?

ENRICHMENT: Moving run average

	–	–	10

10 In this particular question, a moving average is determined by calculating the average of all data values up to a particular time or place in the data set.

Consider a batsman in cricket with the following runs scored from 10 completed innings.

Innings	1	2	3	4	5	6	7	8	9	10
Score	26	38	5	10	52	103	75	21	33	0
Moving average	26	32	23							

a Complete the table by calculating the moving average for innings 4–10. Round to the nearest whole number where required.

b Plot the score and moving averages for the batter on the same set of axes.

c Describe the behaviour of the:

i score graph

ii moving average graph.

d Describe the main difference in the behaviour of the two graphs. Give reasons.

ISBN 978-1-108-77290-7

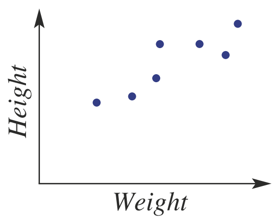
Cambridge University Press
Updated June 2022

9G Bivariate data and scatter plots

Learning intentions

- To understand that bivariate data involve data about two variables in a given context
- To know how to draw a scatter plot to compare data from two variables
- To be able to use a scatter plot to describe the correlation between the two variables using key terms

When we collect information about two variables in a given context, we are collecting bivariate data. As there are two variables involved in bivariate data, we use a number plane to graph the data. These graphs are called scatter plots and are used to illustrate a relationship that may exist between the variables. Scatter plots make it very easy to see the strength of the association between the two variables.



Market research analysts find a positive correlation in scatter plots of advertising spending versus product sales. AI (artificial intelligence) algorithms use automated marketing to create highly effective digital advertising, specifically targeted to each person's online presence.

LESSON STARTER A relationship or not?

Consider the two variables in each part below.

- Would you expect there to be some relationship between the two variables in each of these cases?
- If you think a relationship exists, would you expect the second listed variable to increase or to decrease as the first variable increases?

a Height of person and Weight of person
b Temperature and Life of milk
c Length of hair and IQ
d Depth of topsoil and Brand of motorcycle
e Years of education and Income
f Spring rainfall and Crop yield
g Size of ship and Cargo capacity
h Fuel economy and CD track number
i Amount of traffic and Travel time
j Cost of 2 litres of milk and Ability to swim
k Background noise and Amount of work completed

How might the size of a ship and its cargo capacity be related?

ISBN 978-1-108-77290-7

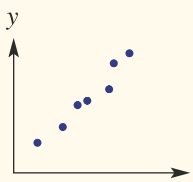
Cambridge University Press
Updated June 2022

KEY IDEAS

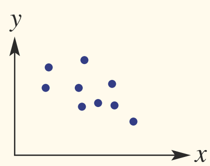
- **Bivariate data** include data for two variables.
  - The two variables are usually related; for example, height and weight.
  - The variable that is changed or controlled is the independent variable and is on the x-axis.
  - The variable being tested or measured is the dependent variable and is on the y-axis.
- A **scatter plot** is a graph on a number plane in which the axes variables correspond to the two variables from the bivariate data.
- The words *relationship*, *correlation* and *association* are used to describe the way in which variables are related.
- Types of correlation:

  Examples

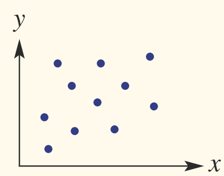
  Strong positive correlation



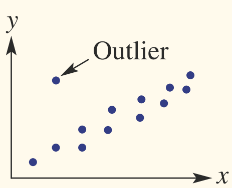
  Weak negative correlation



  No correlation



- An **outlier** can clearly be identified as a data point that is isolated from the rest of the data.



BUILDING UNDERSTANDING

1 Decide if it is likely for there to be a strong correlation between these pairs of variables.

- **a** Height of door and Thickness of door handle
- **b** Weight of car and Fuel consumption
- **c** Temperature and Length of phone calls
- **d** Size of textbook and Number of textbook
- **e** Diameter of flower and Number of bees
- **f** Amount of rain and Size of vegetables in the vegetable garden

2 For each of the following sets of bivariate data with variables x and y, decide whether y generally increases or decreases as x increases.

a

x	1	2	3	4	5	6	7	8	9	10
y	3	2	4	4	5	8	7	9	11	12

b

x	0.1	0.3	0.5	0.9	1.0	1.1	1.2	1.6	1.8	2.0	2.5
y	10	8	8	6	7	7	7	6	4	3	1

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 10 Constructing and interpreting scatter plots

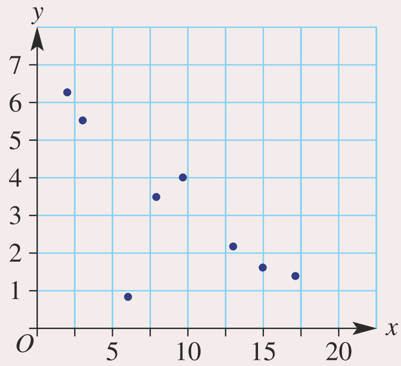
Consider this simple bivariate data set.

x	13	9	2	17	3	6	8	15
y	2.1	4.0	6.2	1.3	5.5	0.9	3.5	1.6

a Draw a scatter plot for the data.
b Describe the correlation between x and y as positive or negative.
c Describe the correlation between x and y as strong or weak.
d Identify any outliers.

SOLUTION

a



b Negative correlation

c Strong correlation

d The outlier is (6, 0.9).

EXPLANATION

Plot each point using a • on graph paper.

As x increases, y decreases.

The downwards trend in the data is clearly defined.

This point defies the trend.

Now you try

Consider this simple bivariate data set.

x	12	2	15	10	4	5	8	13	7
y	4.0	1.3	4.5	3.6	1.8	2.0	2.5	2.0	2.9

a Draw a scatter plot for the data.
b Describe the correlation between x and y as positive or negative.
c Describe the correlation between x and y as strong or weak.
d Identify any outliers.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 9G

FLUENCY	1–4	2–4	2–4

Example 10

1 Consider this simple bivariate data set. (Use technology to assist if desired. See page 709.)

x	1	2	3	4	5	6	7	8
y	1.0	1.1	1.3	1.3	1.4	1.6	1.8	1.0

a Draw a scatter plot for the data.
b Describe the correlation between x and y as positive or negative.
c Describe the correlation between x and y as strong or weak.
d Identify any outliers.

Example 10

2 Consider this simple bivariate data set. (Use technology to assist if desired. See page 709.)

x	14	8	7	10	11	15	6	9	10
y	4	2.5	2.5	1.5	1.5	0.5	3	2	2

a Draw a scatter plot for the data.
b Describe the correlation between x and y as positive or negative.
c Describe the correlation between x and y as strong or weak.
d Identify any outliers.

3 By completing scatter plots (by hand or using technology) for each of the following data sets, describe the correlation between x and y as positive, negative or none.

a

x	1.1	1.8	1.2	1.3	1.7	1.9	1.6	1.6	1.4	1.0	1.5
y	22	12	19	15	10	9	14	13	16	23	16

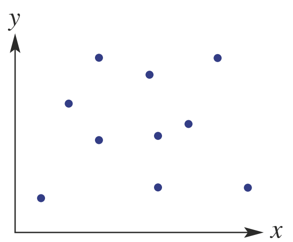
b

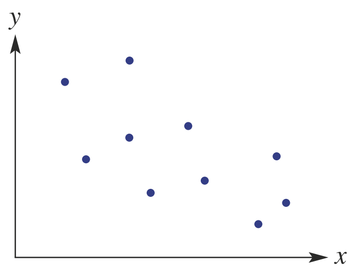
x	4	3	1	7	8	10	6	9	5	5
y	115	105	105	135	145	145	125	140	120	130

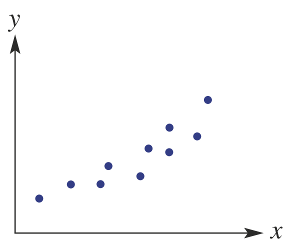
c

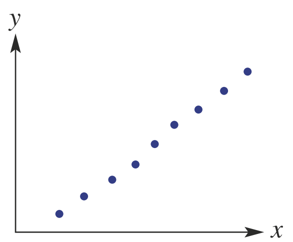
x	28	32	16	19	21	24	27	25	30	18
y	13	25	22	21	16	9	19	25	15	12

4 For the following scatter plots, describe the correlation between x and y.

a


b


c


d


PROBLEM-SOLVING	5, 6	6, 7	6, 8

5 For common motor vehicles, consider the two variables *Engine size* (cylinder volume) and *Fuel economy* (number of kilometres travelled for every litre of petrol).

- **a** Do you expect there to be some relationship between these two variables?
- **b** As the engine size increases, would you expect the fuel economy to increase or decrease?
- **c** The following data were collected for 10 vehicles.

Car	A	B	C	D	E	F	G	H	I	J
Engine size	1.1	1.2	1.2	1.5	1.5	1.8	2.4	3.3	4.2	5.0
Fuel economy	21	18	19	18	17	16	15	20	14	11

  - **i** Do the data generally support your answers to parts **a** and **b**?
  - **ii** Which car gives a fuel economy reading that does not support the general trend?

6 A tomato grower experiments with a new organic fertiliser and sets up five separate garden beds: A, B, C, D and E. The grower applies different amounts of fertiliser to each bed and records the diameter of each tomato picked.

The average diameter of a tomato from each garden bed and the corresponding amount of fertiliser are recorded below.

Bed	A	B	C	D	E
Fertiliser (grams per week)	20	25	30	35	40
Average diameter (cm)	6.8	7.4	7.6	6.2	8.5

- **a** Draw a scatter plot for the data with 'Diameter' on the vertical axis and 'Fertiliser' on the horizontal axis. Label the points *A*, *B*, *C*, *D* and *E*.
- **b** Which garden bed appears to go against the trend?
- **c** According to the given results, would you be confident in saying that the amount of fertiliser fed to tomato plants does affect the size of the tomato produced?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

7 In a newspaper, the number of photos and number of words were counted for 15 different pages. Here are the results.

Page	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of photos	3	2	1	2	6	4	5	7	4	5	2	3	1	0	1
Number of words	852	1432	1897	1621	912	1023	817	436	1132	1201	1936	1628	1403	2174	1829

a Sketch a scatter plot using 'Number of photos' on the horizontal axis and 'Number of words' on the vertical axis.

b From your scatter plot, describe the general relationship between the number of photos and the number of words per page. Use the words positive, negative, strong correlation or weak correlation.

8 On 14 consecutive days, a local council measures the volume of sound heard from a freeway at various points in a local suburb. The volume of sound, in decibels, is recorded against the distance (in metres) between the freeway and the point in the suburb.

Distance (m)	200	350	500	150	1000	850	200	450	750	250	300	1500	700	1250
Volume (dB)	4.3	3.7	2.9	4.5	2.1	2.3	4.4	3.3	2.8	4.1	3.6	1.7	3.0	2.2

a Draw a scatter plot of *Volume* against *Distance*, plotting *Volume* on the vertical axis and *Distance* on the horizontal axis.

b Describe the correlation between *Distance* and *Volume* as positive, negative or none.

c Generally, as *Distance* increases does *Volume* increase or decrease?

REASONING	9	9, 10	10, 11

9 A government department is interested in convincing the electorate that a larger number of police on patrol leads to a lower crime rate. Two separate surveys are completed over a one-week period and the results are listed in this table.

	Area	A	B	C	D	E	F	G
Survey 1	Number of police	15	21	8	14	19	31	17
	Incidence of crime	28	16	36	24	24	19	21
Survey 2	Number of police	12	18	9	12	14	26	21
	Incidence of crime	26	25	20	24	22	23	19

a By using scatter plots, determine whether or not there is a relationship between the number of police on patrol and the incidence of crime, using the data in:

i survey 1 **ii** survey 2.

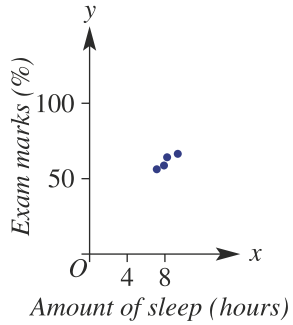
b Which survey results do you think the government will use to make its point? Why?

10 A student collects some data and finds that there is a positive correlation between height and the ability to play tennis. Does that mean that if you are tall you will be better at tennis? Explain.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

11 A person presents you with this scatter plot and suggests a strong correlation between the amount of sleep and exam marks. What do you suggest is the problem with the person's graph and conclusions?



ENRICHMENT: Does television provide a good general knowledge?

–	–	12

12 A university graduate is conducting a test to see whether a student's general knowledge is in some way linked to the number of hours of television watched.

Twenty Year 10 students sit a written general knowledge test marked out of 50. Each student also provides the graduate with details about the number of hours of television watched per week. The results are given in the table below.

Student	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
Hours of TV	11	15	8	9	9	12	20	6	0	15	9	13	15	17	8	11	10	15	21	3
Test score	30	4	13	35	26	31	48	11	50	33	31	28	27	6	39	40	36	21	45	48

a Which two students performed best on the general knowledge test, having watched TV for the following numbers of hours?

i fewer than 10 **ii** more than 4

b Which two students performed worst on the general knowledge test, having watched TV for the following numbers of hours?

i fewer than 10 **ii** more than 4

c Which four students best support the argument that the more hours of TV watched, the better your general knowledge will be?

d Which four students best support the argument that the more hours of TV watched, the worse your general knowledge will be?

e From the given data, would you say that the graduate should conclude that a student's general knowledge is definitely linked to the number of hours of TV watched per week?

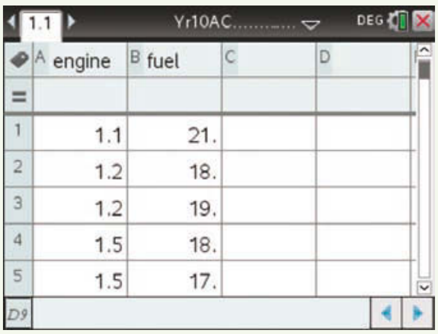
Using calculators to draw scatter plots

Type the following data about car fuel economy into two lists and draw a scatter plot.

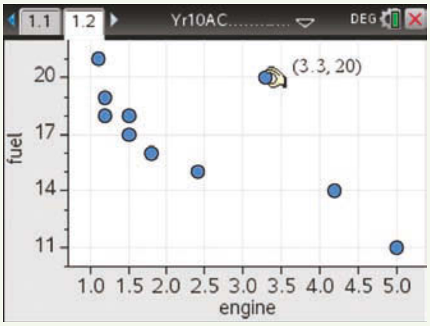
Car	A	B	C	D	E	F	G	H	I	J
Engine size	1.1	1.2	1.2	1.5	1.5	1.8	2.4	3.3	4.2	5.0
Fuel economy	21	18	19	18	17	16	15	20	14	11

Using the TI-Nspire:

1 In a **Lists and spreadsheets** page type in the list names ***engine*** and ***fuel*** and enter the values as shown.

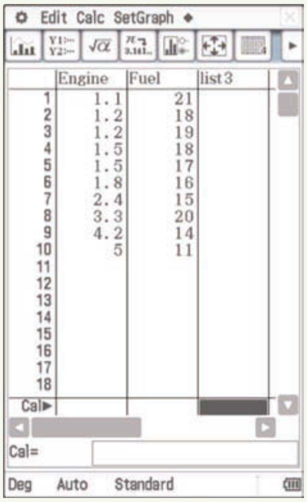


2 Insert a **Data and Statistics** page and select the ***engine*** variable for the horizontal axis and ***fuel*** for the vertical axis. Hover over points to reveal coordinates.

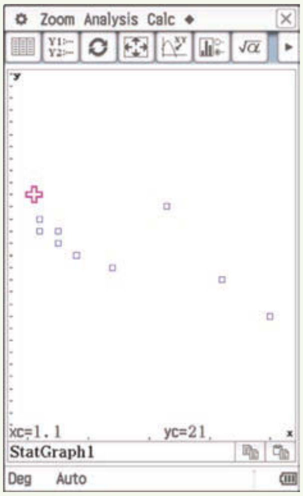


Using the ClassPad:

1 In the **Statistics** application, assign a title to each column then enter the data into the lists.



2 Tap . For graph 1 set **Draw** to **On**, **Type** to **Scatter**, **XList** to **mainEngine**, **YList** to **mainFuel**, **Freq** to **1** and **Mark** to **square**. Tap **Set**. Tap . Tap **Analysis**, **Trace** to reveal coordinates.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Applications and problem-solving

The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

Twenty20

1 Two teams, the Auckland Aces and the Sunrisers Hyderabad, are part of an international 20/20 cricket tournament. They each play 10 round-robin matches and their batting totals are shown below.

Aces	148	172	186	179	194	132	112	154	142	177
Sunrisers	147	160	166	182	171	163	170	155	152	166

You are to compare the statistics of the two cricket teams using box plots and discuss each team's performance in terms of the number of runs and the consistency of the run scoring across the season.

a Draw parallel box plots for these two data sets.

b Compare the box plots of the two teams, commenting on which team appears capable of getting higher scores and which team appears more consistent.

c The Auckland Aces' lowest two scores were the result of rain delays and the restricted number of overs that they faced. If these two innings were increased by 40 runs each, what changes occur on the box plot?

d In their first final, the Sunrisers Hyderabad's batting total would be an outlier if included in their above set of scores. What possible scores did they get in this innings?

Salaries and payrise

2 A small business has 20 employees with the following monthly salaries.

Salary ($)	Number of employees
4500	5
5400	8
5800	5
6400	2

The small business wishes to calculate measures of centre and spread for its salary data and then investigate the impact on these summary statistics given changes in some specific salaries.

a **i** Calculate the mean, median, range and standard deviation (to the nearest dollar) of these salaries.

ii The top two earning employees are given an increase of $\$x$ per month. Describe the impact on the mean, median and range in terms of x.

iii Describe the impact on the standard deviation from part **ii**.

ISBN 978-1-108-77290-7

b Employees at another small business think they are paid less given their mean monthly salary is \$4800 with standard deviation \$800.

- **i** In this company 95% of salaries lie within two standard deviations of the mean. What would employees who are in the top or bottom 2.5% of earners be earning?
- **ii** If each employee in this business is given a pay rise of $\$x$, give the new mean and standard deviation of employee salaries in terms of x where appropriate.
- **iii** The employees instead decide to give each person a percentage increase in their salary. If each person's salary is increased by a factor of k, give the new mean and standard deviation of the salaries in terms of k.

Winter getaway

3 A family is planning to escape the winter cold and spend July in Noosa. They wish to be prepared for varying temperatures throughout the day and compare the daily maximum and minimum temperatures of a recent July as shown in the table below.

Min. temp (°C)	19	17	17	16	18	19	18	11	12	14	14	15	15	12	15	15
Max. temp (°C)	23	21	20	20	23	23	24	23	19	17	17	19	20	21	21	23

Min. temp (°C)	14	16	16	16	13	14	15	16	16	16	17	17	17	17	15
Max. temp (°C)	23	23	24	24	22	19	19	23	24	24	24	25	25	26	23

The family is interested in the relationship between the maximum and minimum temperatures for the month of July and use this to make predictions for their upcoming holiday.

- **a** Prepare a scatter plot of these data with the minimum temperature on the horizontal axis.
- **b** To make predictions the family use a straight line to model the data. If the line passes through the points shown in red, find the equation of this line by completing the following:
  max. temp = ______ × min. temp + ______.
- **c** Use your equation in part **b** to find the likely:
  - **i** max. temp on a day with a min. temp of 13°C, rounding to the nearest degree
  - **ii** min. temp on a day with a max temp of 28°C, rounding to the nearest degree.
- **d** Which of your results in part **c** seem the most accurate? Why?
- **e** Select two other points on the graph that a straight line modelling the data could reasonably pass through. Find the equation of this line and repeat part **c**. Comment on the similarities or differences in your results for part **c** using the two different equations.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

9H Line of best fit by eye

Learning intentions

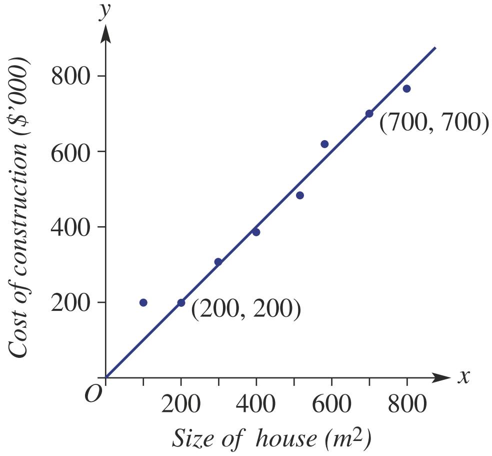
- To understand that a line of best fit can be used as a model for the data when there is a strong linear association
- To know how to fit a line of best fit by eye
- To know how to find the equation of a line of best fit
- To be able to use the line of best fit and its equation to estimate data values within and outside the data range

When bivariate data have a strong linear correlation, we can model the data with a straight line. This line is called a trend line or line of best fit. When we fit the line 'by eye', we try to balance the number of data points above the line with the number of points below the line. This trend line and its equation can then be used to construct other data points within and outside the existing data points.

A scatter plot of product price (y) versus demand (x) shows a negative correlation, with a downward sloping trend line. Businesses use demand equations to forecast sales and make informed decisions about future stock and staffing levels.

LESSON STARTER Size versus cost

This scatter plot shows the estimated cost of building a house of a given size, as quoted by a building company. The given trend line passes through the points (200, 200) and (700, 700).



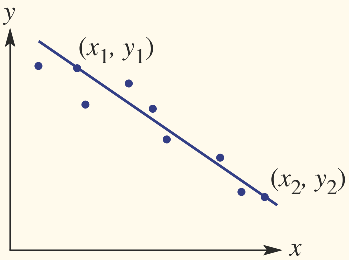
- Do you think the trend line is a good fit to the points on the scatter plot? Why?
- How can you find the equation of the trend line?
- How can you predict the cost of a house of 1000 m^2 with this building company?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

KEY IDEAS

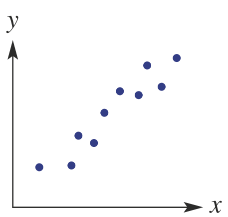
- A **line of best fit** or **trend line** is positioned by eye by balancing the number of points above the line with the number of points below the line.
  - The distance of each point from the trend line also must be taken into account.
- The equation of the line of best fit can be found using two points that are on the line of best fit.
- For $y = mx + c$:
  $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and substitute a point to find the value of c.
  - Alternatively, use $y - y_1 = m(x - x_1)$.
- The line of best fit and its equation can be used for:
  - **interpolation**: constructing points within the given data range
  - **extrapolation**: constructing points outside the given data range.



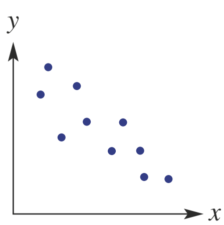
BUILDING UNDERSTANDING

1 Practise fitting a line of best fit on these scatter plots by trying to balance the number of points above the line with the numbers of points below the line. (Use the side of a ruler if you don't want to draw a line.)

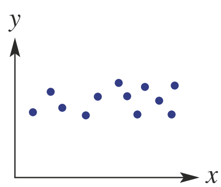
a



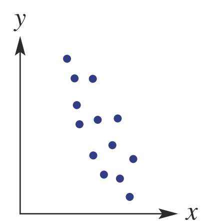
b



c

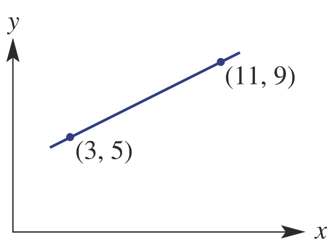


d

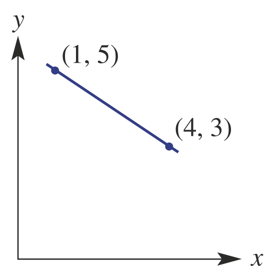


2 For each graph find the equation of the line in the form $y = mx + c$. First, find the gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and then subsititute a point.

a



b



3 Using $y = \dfrac{5}{4}x - 3$, find:

a y when:
  i $x = 16$
  ii $x = 7$

b x when:
  i $y = 4$
  ii $y = \dfrac{1}{2}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 11 Fitting a line of best fit

Consider the variables x and y and the corresponding bivariate data.

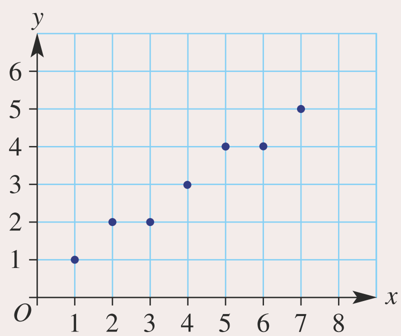
x	1	2	3	4	5	6	7
y	1	2	2	3	4	4	5

a Draw a scatter plot for the data.
b Is there positive, negative or no correlation between x and y?
c Fit a line of best fit by eye to the data on the scatter plot.
d Use your line of best fit to estimate:
  i y when $x = 3.5$ **ii** y when $x = 0$
  iii x when $y = 1.5$ **iv** x when $y = 5.5$

SOLUTION

EXPLANATION

a

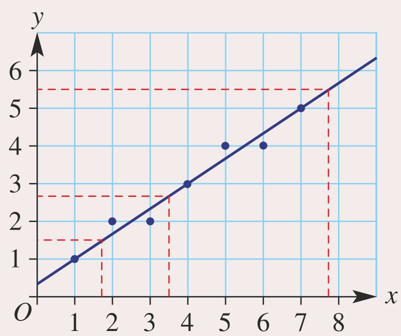


Plot the points on graph paper.

b Positive correlation

As x increases, y increases.

c



Since a relationship exists, draw a line on the plot, keeping as many points above as below the line. (There are no outliers in this case.)

d **i** $y \approx 2.7$
ii $y \approx 0.4$
iii $x \approx 1.7$
iv $x \approx 7.8$

Extend vertical and horizontal lines from the values given and read off your solution. As they are approximations, we use the $\approx$ sign and not the $=$ sign.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Now you try

Consider the variables x and y and the corresponding bivariate data.

x	1	2	3	4	5	6
y	10	8	8	6	5	3

a Draw a scatter plot for the data.

b Is there positive, negative or no correlation between x and y?

c Fit a line of best fit by eye to the data on the scatter plot.

d Use your line of best fit to estimate:

i y when $x = 3.5$ **ii** y when $x = 0$

iii x when $y = 1.5$ **iv** x when $y = 5.5$

Example 12 Finding the equation of a line of best fit

This scatter plot shows a linear relationship between English marks and Literature marks in a small class of students. A trend line passes through (40, 30) and (60, 60).

a Find the equation of the trend line.

b Use your equation to estimate a Literature score if the English score is:

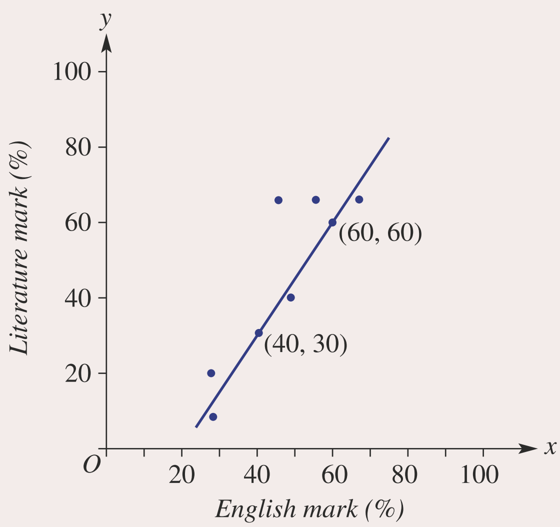
i 50

ii 86

c Use your equation to estimate the English score if the Literature score is:

i 42

ii 87



SOLUTION

a
$$y = mx + c$$
$$m = \frac{60 - 30}{60 - 40} = \frac{30}{20} = \frac{3}{2}$$
$$\therefore\ y = \frac{3}{2}x + c$$
$$(40, 30): 30 = \frac{3}{2}(40) + c$$
$$30 = 60 + c$$
$$c = -30$$
$$\therefore\ y = \frac{3}{2}x - 30$$

EXPLANATION

Use $m = \frac{y_2 - y_1}{x_2 - x_1}$ for the two given points.

Substitute either (40, 30) or (60, 60) to find c.

Continued on next page

ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

b **i** $y = \frac{3}{2}(50) - 30 = 45$ — Substitute $x = 50$ and find the value of y.

$\therefore$ Literature score is 45.

ii $y = \frac{3}{2}(86) - 30 = 99$ — Repeat for $x = 86$.

$\therefore$ Literature score is 99.

c **i** $42 = \frac{3}{2}x - 30$ — Substitute $y = 42$ and solve for x.

$72 = \frac{3}{2}x$

$x = 48$

$\therefore$ English score is 48.

ii $87 = \frac{3}{2}x - 30$ — Repeat for $y = 87$.

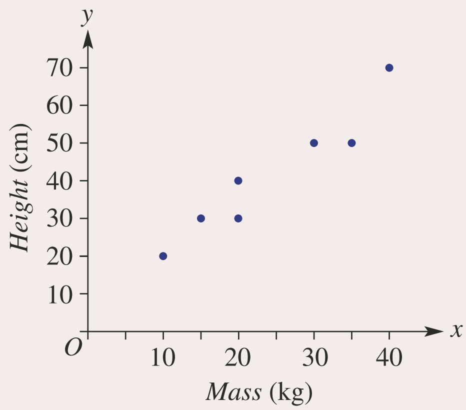
$117 = \frac{3}{2}x$

$x = 78$

$\therefore$ English score is 78.

Now you try

This scatter plot shows a linear relationship between the mass and height of a small number of dogs. A trend line passes through (10, 20) and (40, 70).



a Find the equation of the trend line.

b Use your equation to estimate a dog height if its mass is:

i 25 kg **ii** 52 kg

c Use your equation to estimate a dog mass if its height is:

i 60 cm **ii** 80 cm

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 9H

FLUENCY	1–3	1, 2(½), 3	1, 3

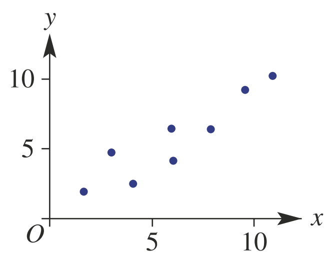
Example 11

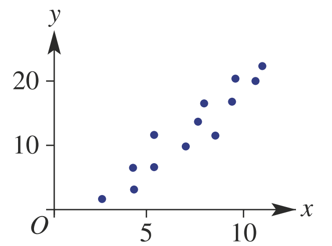
1 Consider the variables x and y and the corresponding bivariate data.

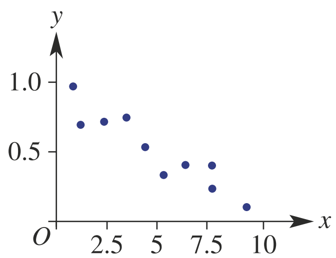
x	1	2	3	4	5	6	7
y	2	2	3	4	4	5	5

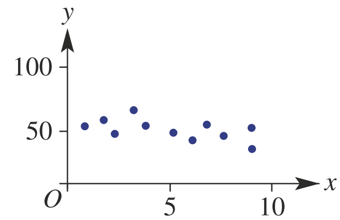
a Draw a scatter plot for the data.
b Is there positive, negative or no correlation between x and y?
c Fit a line of best fit by eye to the data on the scatter plot.
d Use your line of best fit to estimate:
  i y when $x = 3.5$ **ii** y when $x = 0$
  iii x when $y = 2$ **iv** x when $y = 5.5$

2 For the following scatter plots, pencil in a line of best fit by eye, and then use your line to estimate the value of y when $x = 5$.

a


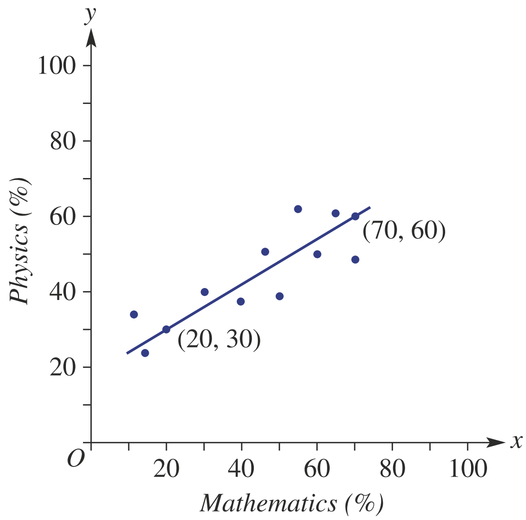
b


c


d


Example 12

3 This scatter plot shows a linear relationship between Mathematics marks and Physics marks in a small class of students. A trend line passes through (20, 30) and (70, 60).
a Find the equation of the trend line.
b Use your equation to find the Physics score if the Mathematics score is:
  i 40 **ii** 90
c Use your equation to find the Mathematics score if the Physics score is:
  i 36 **ii** 78



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

PROBLEM-SOLVING	4	4, 5	4, 5

4 Over eight consecutive years, a city nursery has measured the growth of an outdoor bamboo species for that year. The annual rainfall in the area where the bamboo is growing was also recorded. The data are listed in the table.

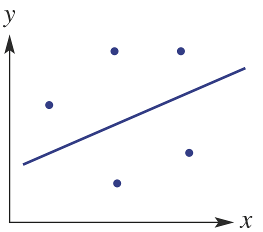
Rainfall (mm)	450	620	560	830	680	650	720	540
Growth (cm)	25	45	25	85	50	55	50	20

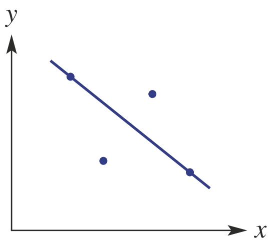
a Draw a scatter plot for the data, showing growth on the vertical axis.
b Fit a line of best fit by eye.
c Use your line of best fit to estimate the growth expected for the following rainfall readings.
You do not need to find the equation of the line.
i 500 mm **ii** 900 mm
d Use your line of best fit to estimate the rainfall for a given year if the growth of the bamboo was:
i 30 cm **ii** 60 cm

5 A line of best fit for a scatter plot, relating the weight (kg) and length (cm) of a group of dogs, passes through the points (15, 70) and (25, 120). Assume weight is on the x-axis.
a Find the equation of the trend line.
b Use your equation to estimate the length of an 18 kg dog.
c Use your equation to estimate the weight of a dog that has a length of 100 cm.

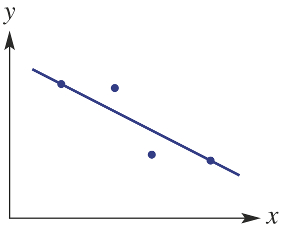
REASONING	6	6, 7	7, 8

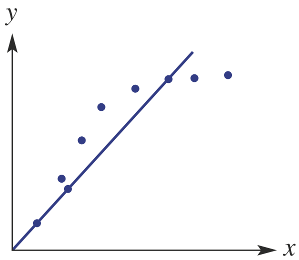
6 Describe the problem when using each trend line below for interpolation.

a


b


7 Describe the problem when using each trend line below for extrapolation.

a


b


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

8 A trend line relating the percentage scores for Music performance (y) and Music theory (x) is given by $y = \frac{4}{5}x + 10$.

a Find the value of x when:

i $y = 50$ **ii** $y = 98$

b What problem occurs in predicting Music theory scores when using high Music performance scores?

ENRICHMENT: Heart rate and age – – 9

9 Two independent scientific experiments confirmed a correlation between *Maximum heart rate* (in beats per minute or b.p.m.) and *Age* (in years). The data for the two experiments are as follows.

Experiment 1													
Age (years)	15	18	22	25	30	34	35	40	40	52	60	65	71
Max. heart rate (b.p.m.)	190	200	195	195	180	185	170	165	165	150	125	128	105

Experiment 2													
Age (years)	20	20	21	26	27	32	35	41	43	49	50	58	82
Max. heart rate (b.p.m.)	205	195	180	185	175	160	160	145	150	150	135	140	90

a Sketch separate scatter plots for experiment 1 and experiment 2.

b By fitting a line of best fit by eye to your scatter plots, estimate the maximum heart rate for a person aged 55 years, using the results from:

i experiment 1 **ii** experiment 2

c Estimate the age of a person who has a maximum heart rate of 190, using the results from:

i experiment 1 **ii** experiment 2

d For a person aged 25 years, which experiment estimates a lower maximum heart rate?

e Research the average maximum heart rate of people according to age and compare with the results given above.

Different watches are used by people to record heart rate.

ISBN 978-1-108-77290-7

9I Linear regression using technology 10A

Learning intentions
- To understand that there are different methods for fitting a straight line to bivariate data
- To know how to use technology to find the least squares regression line
- To be able to use the regression line equation as a model to make predictions

In **Section 9H** we used a line of best fit by eye to describe a general linear (i.e. straight line) trend for bivariate data. In this section we look at the more formal methods for fitting straight lines to bivariate data. This is called linear regression. There are many different methods used by statisticians to model bivariate data. One of the most common methods is called least squares regression. This is best handled with the use of technology.

Data scientists use machine learning algorithms, such as multiple linear regression, to extract relationships from big data. Predictive modelling applications include insurance premiums, financial services, healthcare, stock market trading and effects of climate change.

LESSON STARTER What can my calculator or software do?

Explore the menus of your chosen technology to see what kind of regression tools are available. For CAS calculator users, refer to page 722.

- Can you find the least squares regression tools?
- Use your technology to try **Example 13**.

KEY IDEAS

- **Linear regression** involves using a method to fit a straight line to bivariate data.
  - The result is a straight line equation that can be used for interpolation and extrapolation.
- The **least squares** regression line minimises the sum of the square of the deviations of each point from the line.
  - Outliers have an effect on the least squares regression line because all deviations are included in the calculation of the equation of the line.

ISBN 978-1-108-77290-7

BUILDING UNDERSTANDING

1 A regression line for a bivariate data set is given by $y = 2.3x - 4.1$. Use this equation to find:

a the value of y when x is:

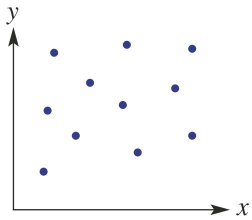
i 7 **ii** 3.2

b the value of x when y is:

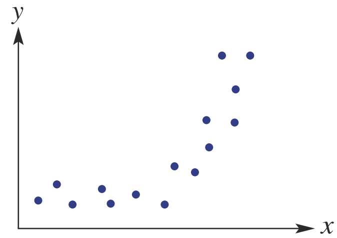
i 12 **ii** 0.5

2 Give a brief reason why a linear regression line is not very useful in the following scatter plots.

a



b



Example 13 Finding and using regression lines

Consider the following data and use a graphics or CAS calculator or software to help answer the questions below. Round answers to two decimal places where necessary.

x	1	2	2	4	5	5	6	7	9	11
y	1.8	2	1.5	1.6	1.7	1.3	0.8	1.1	0.8	0.7

a Construct a scatter plot for the data.

b Find the equation of the least squares regression line.

c Sketch the graph of the regression line onto the scatter plot.

d Use the least squares regression line to estimate the value of y when x is:

i 4.5 **ii** 15

Now you try

Consider the following data and use a graphics or CAS calculator or software to help answer the questions below. Round answers to two decimal places where necessary.

x	1	3	3	4	6	7	9	10
y	0.5	2	1.6	3	5	6.5	5.6	8

a Construct a scatter plot for the data.

b Find the equation of the least squares regression line.

c Sketch the graph of the regression line onto the scatter plot.

d Use the least squares regression line to estimate the value of y when x is:

i 4.5 **ii** 15

ISBN 978-1-108-77290-7

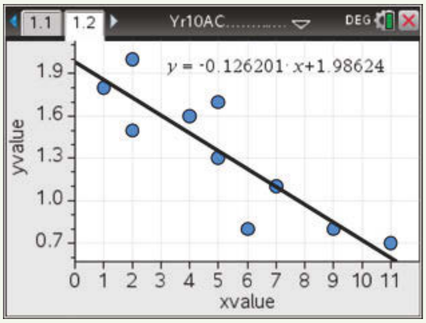
Cambridge University Press
Updated June 2022

Using calculators to find equations of regression

Using the TI-Nspire:

a, b, c In a **Lists & Spreadsheet** page enter the data in the lists named *xvalue* and *yvalue*. Insert a **Data & Statistics** page and select *xvalue* as the variable on the horizontal axis and *yvalue* as the variable on the vertical axis.

To show the linear regression line and equation use menu **>Analyze>Regression>Show Linear(mx + b)**

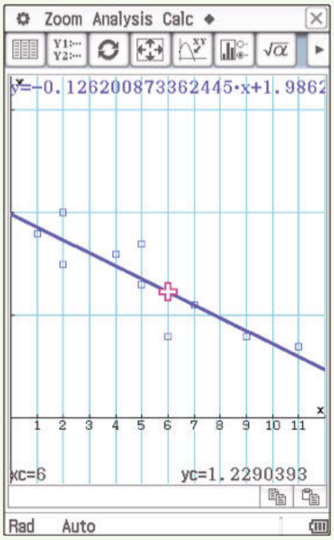


Least squares: $y = -0.126201x + 1.986245$

Using the ClassPad:

a, b, c In the **Statistics** application enter the data into the lists. Tap **Calc**, **Regression**, **Linear Reg** and set **XList** to **list1**, **YList** to **list2**, **Freq** to **1, Copy Formula** to **y1** and **Copy Residual** to **Off**. Tap **OK** to view the regression equation. Tap on OK again to view the regression line.

Tap **Analysis**, **Trace** and then scroll along the regression line.



d **i** $y \approx 1.42$
ii $y \approx 0.09$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 9I

FLUENCY	1, 2	1, 2	1, 2

Example 13

1 Consider the data in tables **A–C** and use a graphics or CAS calculator or software to help answer the following questions. Round answers to two decimal places where necessary.

A

x	1	2	3	4	5	6	7	8
y	3.2	5	5.6	5.4	6.8	6.9	7.1	7.6

B

x	3	6	7	10	14	17	21	26
y	3.8	3.7	3.9	3.6	3.1	2.5	2.9	2.1

C

x	0.1	0.2	0.5	0.8	0.9	1.2	1.6	1.7
y	8.2	5.9	6.1	4.3	4.2	1.9	2.5	2.1

a Construct a scatter plot for the data.
b Find the equation of the least squares regression line.
c Sketch the graphs of the regression lines onto the scatter plot.
d Use the least squares regression line to estimate the value of y when x is:
i 7 **ii** 12

2 The values and ages of 14 cars are summarised in these tables.

Age (years)	5	2	4	9	10	8	7
Price ($'000)	20	35	28	14	11	12	15

Age (years)	11	2	1	4	7	6	9
Price ($'000)	5	39	46	26	19	17	14

a Using Age for the x-axis and rounding your coefficients to two decimal places, find the least squares regression line.
b Use your least squares regression line to estimate the value of a 3-year-old car, correct to the nearest dollar.
c Use your least squares regression line to estimate the age of a $15 000 car, correct to the nearest year.

ISBN 978-1-108-77290-7

PROBLEM-SOLVING	3, 4	3, 4	4, 5

3 A factory that produces denim jackets does not have air-conditioning. It was suggested that high temperatures inside the factory were having an effect on the number of jackets able to be produced, so a study was completed and data collected on 14 consecutive days.

Max. daily temp. inside factory (°C)	28	32	36	27	24	25	29	31	34	38	41	40	38	31
Number of jackets produced	155	136	120	135	142	148	147	141	136	118	112	127	136	132

Use a graphics or CAS calculator to complete the following.

a Draw a scatter plot for the data.

b Find the equation of the least squares regression line, rounding coefficients to two decimal places.

c Graph the line onto the scatter plot.

d Use the regression line to estimate how many jackets, correct to the nearest whole number, would be able to be produced if the maximum daily temperature in the factory was:

i 30°C **ii** 35°C **iii** 45°C

4 A particular brand of electronic photocopier is considered for scrap once it has broken down more than 50 times or if it has produced more than 200 000 copies. A study of one particular copier gave the following results.

Number of copies (×1000)	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150
Total number of breakdowns	0	0	1	2	2	5	7	9	12	14	16	21	26	28	33

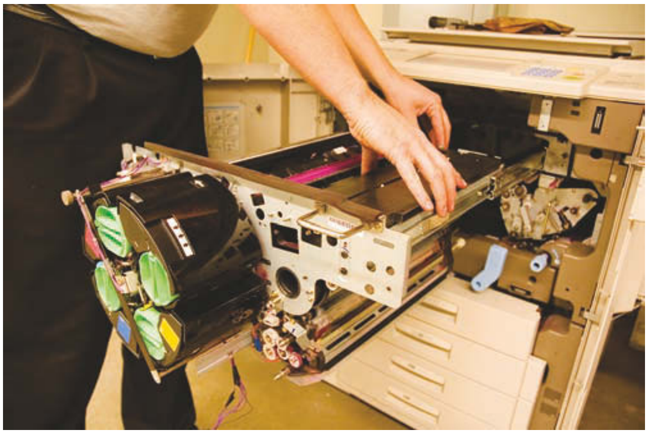
a Sketch a scatter plot for the data.

b Find the equation of the least squares regression line.

c Graph the least squares regression line onto the scatter plot.

d Using your regression line, estimate the number of copies the photocopier will have produced at the point when you would expect 50 breakdowns.

e Would you expect this photocopier to be considered for scrap because of the number of breakdowns or the number of copies made?



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

5 At a suburban sports club, the distance record for the hammer throw has increased over time. The first recorded value was 72.3 m in 1967 and the most recent record was 118.2 m in 1996.

Further details are as follows.

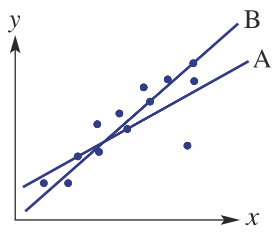
Year	1967	1968	1969	1976	1978	1983	1987	1996
New record (m)	72.3	73.4	82.7	94.2	99.1	101.2	111.6	118.2

a Draw a scatter plot for the data.
b Find the equation of the least squares regression line.
c Use your regression equation to estimate the distance record for the hammer throw for:
i 2000 ii 2020
d Would you say that it is realistic to use your regression equation to estimate distance records beyond 2020? Why?

REASONING	6	6, 7	6, 7

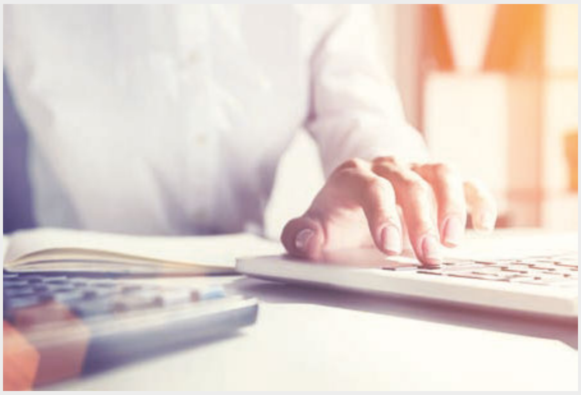
6 Briefly explain why the least squares regression line is affected by outliers.

7 This scatter plot shows both the least squares regression line and another type of regression line. Which line (i.e. A or B) do you think is the least squares line? Give a reason.



ENRICHMENT: Correlation coefficient	–	–	8

8 Use the internet to find out about the Pearson correlation coefficient and then answer these questions.
a What is the coefficient used for?
b Do most calculators include the coefficient as part of their statistical functions?
c What does a relatively large or small correlation coefficient mean?



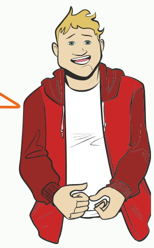
Statisticians work in many fields of industry, business, finance, research, government and social services. As computers are used to process the data, they can spend more time on higher-level skills, such as designing statistical investigations and data analysis.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Problems and challenges

Up for a challenges? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.



1 The mean mass of six boys is 71 kg, and the mean mass of five girls is 60 kg. Find the average mass of all 11 people put together.

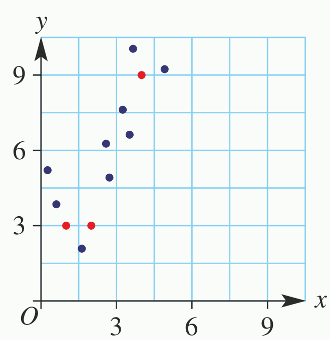
2 Sean has a current four-topic average of 78% for Mathematics. What score does he need in the fifth topic to have an overall average of 80%?

3 A single-ordered data set includes the following data.
   2, 4, 5, 6, 8, 10, x
   What is the largest possible value of x if it is not an outlier?

4 Find the interquartile range for a set of data if 75% of the data are above 2.6 and 25% of the data are above 3.7.

5 A single data set has 3 added to every value. Describe the change in:
   a the mean
   b the median
   c the range
   d the interquartile range
   e the standard deviation.

6 Three key points on a scatter plot have coordinates (1, 3), (2, 3) and (4, 9). Find a quadratic equation that fits these three points exactly.



7 Six numbers are written in ascending order: 1.4, 3, 4.7, 5.8, a, 11.
   Find all possible values of a if the number 11 is considered to be an outlier.

8 The class mean, $\bar{x}$, and standard deviation, s, for some Year 10 term tests are:
   Maths ($\bar{x} = 70\%$, $s = 9\%$); Physics ($\bar{x} = 70\%$, $s = 6\%$); Biology ($\bar{x} = 80\%$, $s = 6.5\%$).
   If Emily gained 80% in each of these subjects, which was her best and worst result? Give reasons for your answer.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter summary

Statistics

Collecting data
A survey can be used to collect data from a population. The sample of the population chosen to be surveyed should be selected without bias and should be representative of the population.

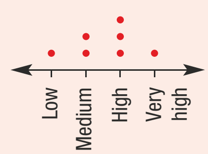
Data

Categorical
- Nominal (red, blue, ...)
- Ordinal (low, medium, ...)

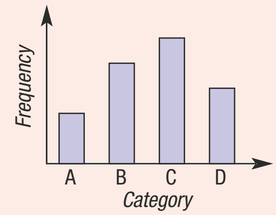
Numerical
- Discrete (1, 2, 3, ...)
- Continuous (0.31, 0.481, ...)

Graphs for single set of categorical or discrete data

Dot plot



Column graphs



Stem-and-leaf plot

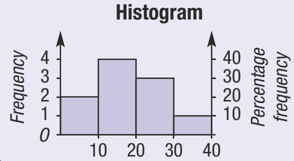
Stem	Leaf
0	1 6
1	2 7 9
2	3 8
3	4

2 | 3 means 23

Grouped data

Class interval	Frequency	Percentage frequency
0–	2	20
10–	4	40
20–	3	30
30–40	1	10
Total	10	100

Histogram



Measures of centre
- Mean $(\bar{x}) = \dfrac{\text{sum of all values}}{\text{number of scores}}$
- Median (Q_2) = middle value

(Mode = most common value)

Measures of spread
- Range = max – min
- Interquartile range (IQR) $= Q_3 - Q_1$
- Sample standard deviation (s) for n data values (10A)

$$= \sqrt{\frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+\ldots+(x_n-\bar{x})^2}{n-1}}$$

- If s is relatively small the data are concentrated about the mean.
- If s is relatively large the data are spread out from the mean.

Quartiles
Q_1: above 25% of the data

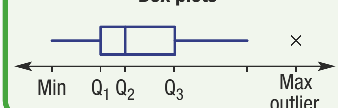
Q_3: above 75% of the data

$$2\ \ 3\ \ \underset{\uparrow Q_1}{5}\ \ 7\ \ \underbrace{8\,|\,9}_{\uparrow\, Q_2 = 8.5}\ \ 11\ \ \underset{\uparrow Q_3}{12}\ \ 14\ \ 15$$

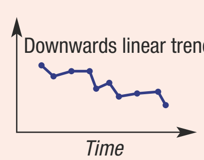
$$1\ \ \underset{\uparrow Q_1}{4}\ \ 7)\ \ \underset{\uparrow Q_2}{8}\ \ (12\ \ \underset{\uparrow Q_3}{16}\ \ 21$$

Outliers
- Single data set
  - less than $Q_1 - 1.5 \times \text{IQR}$ or more than $Q_3 + 1.5 \times \text{IQR}$
- Bivariate
  - not in the vicinity of the rest of the data

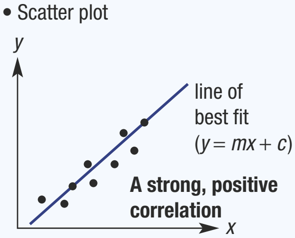
Box plots



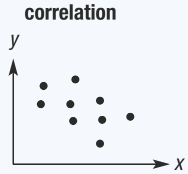
Time-series data



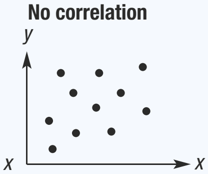
Bivariate data
- Two related variables
- Scatter plot



Weak, negative correlation



No correlation



Linear regression using technology (10A)
- Least squares line

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter checklist

Chapter checklist: Success criteria

		✔
9A	**1. I can describe types of data.** e.g. What type of data would the survey question 'How many pairs of shoes do you own?' generate?	
9A	**2. I can choose a survey sample.** e.g. A survey is carried out by calling people listed in the phone book, to determine their voting preferences for a state election. Why will this sample not necessarily be representative of the state's views?	
9B	**3. I can present data in a histogram.** e.g. 15 people were surveyed to find out how many hours they spend on the internet in a week. The data are: 7, 12, 14, 20, 2, 26, 8, 11, 17, 12, 21, 5, 6, 18, 14 Construct a histogram for the data using class intervals of 5, showing both the frequency and percentage frequency on the one graph.	
9B	**4. I can analyse data in a statistical graph.** e.g. For the stem and leaf plot shown below, find the mean correct to one decimal place, the median and the mode. **Stem** | **Leaf** 0 | 2 5 7 1 | 1 1 4 6 2 | 0 3 9 3 | 2 5 2|3 means 23	
9C	**5. I can find the five-figure summary and interquartile range.** e.g. For the data set below find the minimum, maximum, median, upper and lower quartiles and the range and IQR. 7, 10, 12, 12, 14, 18, 22, 25, 26, 30	
9C	**6. I can find any outliers in a data set.** e.g. The following data represent the number of aces by a tennis player in 11 grand slam matches for the year: 15, 12, 22, 2, 10, 18, 16, 14, 15, 20, 16 For the data find the upper and lower quartiles and use these to help determine if there are any outliers.	
9D	**7. I can construct a box plot.** e.g. For the data set: 5, 8, 2, 1, 6, 3, 3, 1, 4, 18, 2, 8, 5, draw a box plot to summarise the data, marking outliers if they exist.	
9E	**8. I can calculate the standard deviation.** 10A e.g. For the data set 10, 5, 4, 7, 2, calculate the mean and standard deviation correct to one decimal place.	

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter checklist

✔

9E

9. I can interpret a standard deviation value.

10A

e.g. This back-to-back stem-and-leaf plot shows the average monthly maximum temperatures for a year in New York and Melbourne. The mean and standard deviation are given.

New York Leaf	Stem	Melbourne Leaf
7 5 4	0	
8 7 2 0	1	3 4 5 7 7
9 9 7 5 2	2	0 0 2 4 4 6 6

1 | 7 means 17°C

Melbourne: $\bar{x} = 19.8$, $s = 4.6$
New York: $\bar{x} = 17.1$, $s = 9.4$

By looking at the stem-and-leaf plot, suggest why New York's mean is less than that of Melbourne and why New York's standard deviation is larger than that of Melbourne.

9F

10. I can plot and interpret a time-series plot.

e.g. The approximate number of DVD rental stores in a city over a 10-year period is shown below.

Year	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019
Number of DVD stores	72	65	56	56	31	22	14	14	8	6

Plot the time series and describe the trend in the data over the 10 years.

9G

11. I can construct and interpret a scatter plot.

e.g. For the bivariate data set below, draw a scatter plot and describe the correlation between x and y as positive or negative and strong or weak.

x	5	8	12	4	6	15	11	3
y	4.4	6	11	4.7	5.3	11.6	10.3	2.4

9H

12. I can fit a line of best fit by eye and use the line to make predictions.

e.g. For the scatter plot from the data set above, fit a line of best fit by eye on the scatter plot and use it to estimate y when $x = 10$ and x when $y = 8$.

9H

13. I can find the equation of a line of best fit.

e.g. A scatter plot shows a linear relationship between two variables x and y. If the trend line passes through (20, 15) and (40, 25), find the equation of the trend line and use it to estimate x when $y = 50$.

9I

14. I can find and use a regression line using technology.

10A

e.g. For the data set below, use technology to construct a scatter plot for the data and find the equation of the least squares regression line. Use the equation to estimate the value of y when x is 12.

x	2	3	5	6	7	8	8	9
y	10.8	10.6	9.2	4.7	7.3	5.6	6.2	4.1

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Short-answer questions

9B

1 A group of 16 people was surveyed to find the number of hours of television they watch in a week. The raw data are listed:

6, 5, 11, 13, 24, 8, 1, 12
7, 6, 14, 10, 9, 16, 8, 3

a Organise the data into a table with class intervals of 5 and include a percentage frequency column.
b Construct a histogram for the data, showing both the frequency and percentage frequency on the graph.
c Would you describe the data as symmetrical, positively skewed or negatively skewed?
d Construct a stem-and-leaf plot for the data, using 10s as the stem.
e Use your stem-and-leaf plot to find the median.

9D

2 For each set of data below, complete the following tasks.

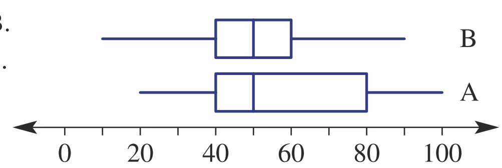
i Find the range.
ii Find the lower quartile (Q_1) and the upper quartile (Q_3).
iii Find the interquartile range.
iv Locate any outliers.
v Draw a box plot.

a 2, 2, 3, 3, 3, 4, 5, 6, 12
b 11, 12, 15, 15, 17, 18, 20, 21, 24, 27, 28
c 2.4, 0.7, 2.1, 2.8, 2.3, 2.6, 2.6, 1.9, 3.1, 2.2

9D

3 Compare these parallel box plots, A and B, and answer the following as true or false.

a The range for A is greater than the range for B.
b The median for A is equal to the median for B.
c The interquartile range is smaller for B.
d 75% of the data for A sit below 80.



9G

4 Consider the simple bivariate data set.

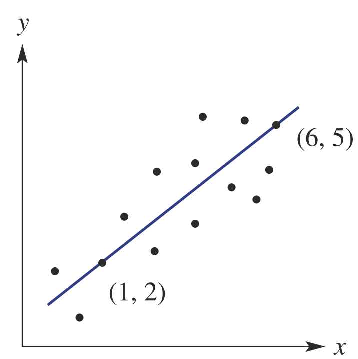
x	1	4	3	2	1	4	3	2	5	5
y	24	15	16	20	22	11	5	17	6	8

a Draw a scatter plot for the data.
b Describe the correlation between x and y as positive or negative.
c Describe the correlation between x and y as strong or weak.
d Identify any outliers.

9H

5 The line of best fit passes through the two points labelled on this graph.

a Find the equation of the line of best fit.
b Use your equation to estimate the value of y when:
i $x = 4$ **ii** $x = 10$
c Use your equation to estimate the value of x when:
i $y = 3$ **ii** $y = 12$



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

9E 10A

6 Calculate the mean and sample standard deviation for these small data sets. Round the standard deviation to one decimal place.

a 4, 5, 7, 9, 10 **b** 1, 1, 3, 5, 5, 9

9E

7 The Cats and The Vipers basketball teams compare their number of points per match for a season.

The data are presented in this back-to-back stem-and-leaf plot.

The Cats Leaf	Stem	The Vipers Leaf
	0	9
2	1	9
8 3	2	0 4 8 9
7 4	3	2 4 7 8 9
9 7 4 1 0	4	2 8
7 6 2	5	0

2|4 means 24

State which team has:

a the higher range

b the higher mean

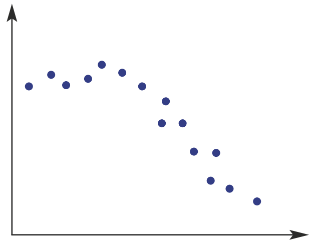
c the higher median

10A **d** the higher standard deviation.

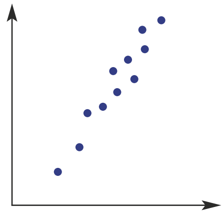
9F

8 Describe the trend in these time-series plots as linear, non-linear or no trend.

a



b



9I 10A

9 For the simple bivariate data set in Question **4**, rounding coefficients to two decimal places, use technology to find the equation of the least squares regression line.

Multiple-choice questions

9A

1 The type of data generated by the survey question *What is your favourite food?* is:

A numerical and discrete

B numerical and continuous

C a sample

D categorical and nominal

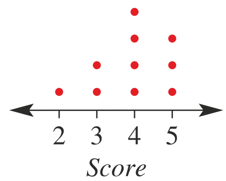
E categorical and ordinal

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter review

Questions **2**–**4** refer to the dot plot shown at right.

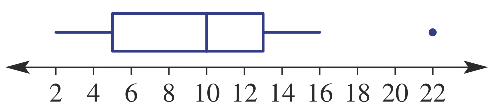


9B **2** The mean of the scores in the data is:
- **A** 3.5
- **B** 3.9
- **C** 3
- **D** 4
- **E** 5

9B **3** The mode for the data is:
A 3.5 **B** 2 **C** 3 **D** 4 **E** 5

9B **4** The dot plot is:
- **A** symmetrical
- **B** positively skewed
- **C** negatively skewed
- **D** bimodal
- **E** correlated

Questions **5** and **6** refer to this box plot.

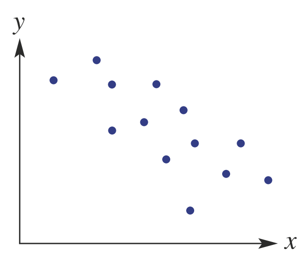


9D **5** The interquartile range is:
A 8 **B** 5 **C** 3 **D** 20 **E** 14

9D **6** The range is:
A 5 **B** 3 **C** 20 **D** 14 **E** 8

9E 10A **7** The sample standard deviation for the small data set 1, 1, 2, 3, 3 is:
A 0.8 **B** 2 **C** 1 **D** 0.9 **E** 2.5

9G **8** The variables x and y in this scatter plot could be described as having:
- **A** no correlation
- **B** a strong, positive correlation
- **C** a strong, negative correlation
- **D** a weak, negative correlation
- **E** a weak, positive correlation



9H **9** The equation of the line of best fit for a set of bivariate data is given by $y = 2.5x - 3$. An estimate for the value of x when $y = 7$ is:
A -1.4 **B** 1.2 **C** 1.6 **D** 7 **E** 4

9H **10** The equation of the line of best fit connecting the points (1, 1) and (4, 6) is:
- **A** $y = 5x + 3$
- **B** $y = \frac{5}{3}x - \frac{2}{3}$
- **C** $y = -\frac{5}{3}x + \frac{8}{3}$
- **D** $y = \frac{5}{3}x - \frac{8}{3}$
- **E** $y = \frac{3}{5}x - \frac{2}{3}$

ISBN 978-1-108-77290-7

Extended-response questions

1 The number of flying foxes taking refuge in two different fig trees was recorded over a period of 14 days. The data collected are given here.

Tree 1	56	38	47	59	63	43	49	51	60	77	71	48	50	62
Tree 2	73	50	36	82	15	24	73	57	65	86	51	32	21	39

a Find the IQR for:
 i tree 1
 ii tree 2

b Identify any outliers for:
 i tree 1
 ii tree 2

c Draw parallel box plots for the data.

d By comparing your box plots, describe the difference in the ways the flying foxes use the two fig trees for taking refuge.

2 The approximate number of shoppers in an air-conditioned shopping plaza was recorded for 14 days, along with the corresponding maximum daily outside temperatures for those days.

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Max. daily temp. (T) (°C)	27	26	28	33	38	36	28	30	32	25	25	27	29	33
No. of shoppers (N)	1050	950	1200	1550	1750	1800	1200	1450	1350	900	850	700	950	1250

a Draw a scatter plot for the number of shoppers versus the maximum daily temperatures, with the number of shoppers, correct to the nearest whole number, on the vertical axis, and describe the correlation between the variables as either positive, negative or none.

b Use technology to determine the least squares regression line for the data, rounding coefficients to two decimal places.

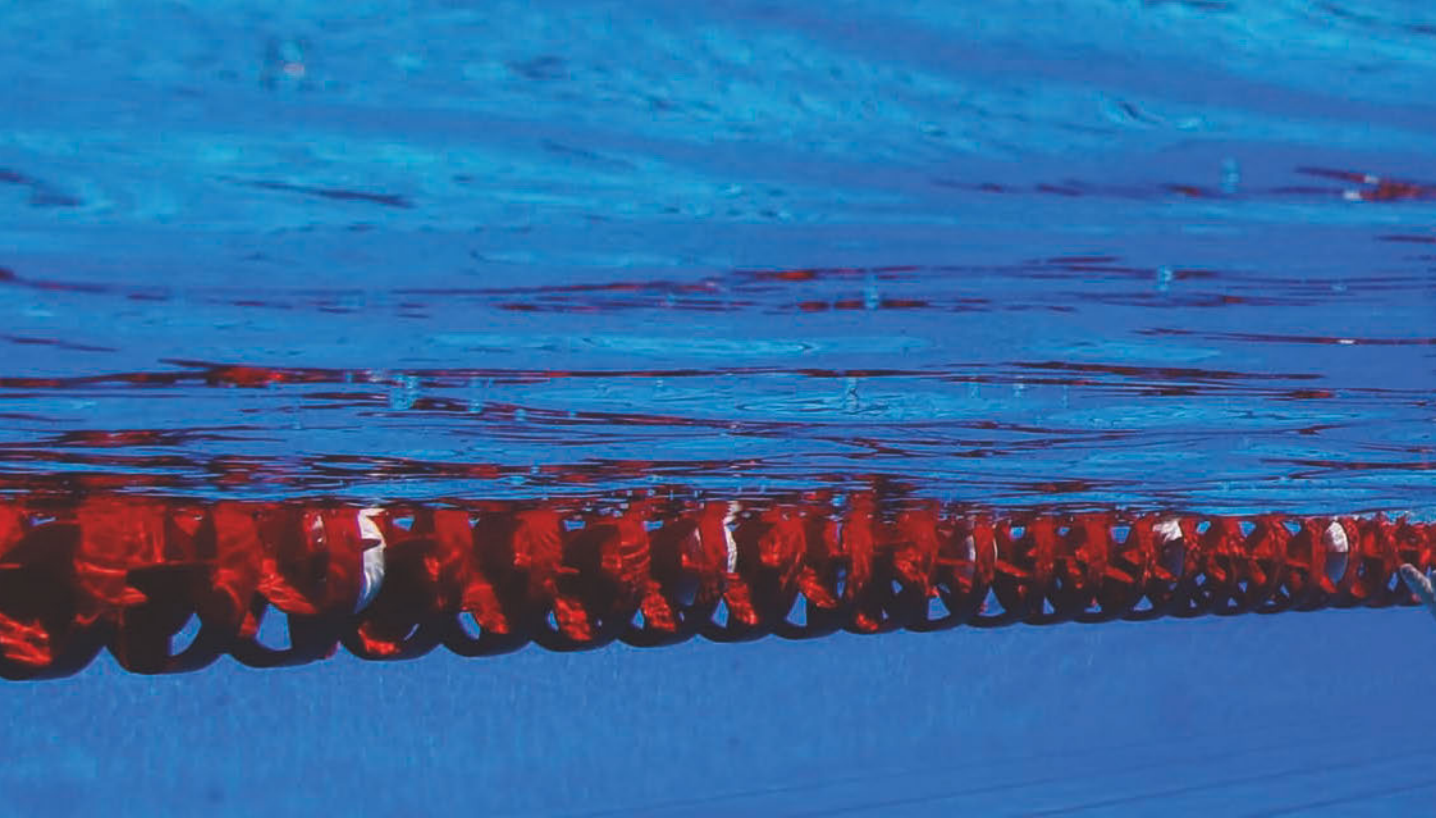
c Use your least squares regression equation to estimate:
 i the number of shoppers on a day, correct to the nearest whole number, with a maximum daily temperature of 24°C
 ii the maximum daily temperature, correct to one decimal place, if the number of shoppers at the plaza is 1500.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

CHAPTER

10 Logarithms and polynomials



The chemistry of Olympic pools

The inverse of the exponential function, called the logarithm function, is used in many fields. The common logarithm (know as the base 10 logarithm) is helpful in representing very big and very small numbers, and in chemistry, logarithms are used when describing and measuring pH levels of a solution: $\text{pH} = -\log_{10}$ (hydrogen ion in solution).

The term pH comes from the Latin *potentia hydrogenii*, or 'capacity of hydrogen'. The pH scale has values from 0 (very acidic) to 14 (very alkaline). As the pH scale is a logarithmic scale, a pH value of 5 is ten times more acidic than a pH value of 6.

Pure water has a pH value of 7 and is classified as neutral. However, adding chemicals such as chlorine can make the water either more acidic or more basic. In Olympic swimming pools, a pH level of 7.4 is ideal, as it matches the pH level of the human eye. Pool water with pH within a range of 7.2 to 7.8 allows there to be enough chlorine to act on any germs, while still making the water comfortable for the swimmers (avoiding skin irritation or the dreaded 'eye burn') and preventing damage to the lining of the pool or its mechanical parts.

Online resources

A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

In this chapter

Victorian Curriculum

NUMBER AND ALGEBRA
Real numbers

(10A) Use the definition of a logarithm to establish and apply the laws of logarithms and investigate logarithmic scales in measurement (VCMNA356)

Patterns and algebra

(10A) Investigate the concepts of a polynomial and apply the factor and remainder theorems to solve problems (VCMN357)

Linear and non-linear relationships

(10A) Apply understanding of polynomials to sketch a range of curves and describe the features of these curves from their equation (VCMNA361)

10A Introducing logarithms 10A

Learning intentions
- To understand the form of a logarithm and its relationship with index form
- To be able to convert between equivalent index and logarithmic forms
- To be able to evaluate simple logarithms both with and without technology
- To be able to solve simple logarithmic equations

Logarithms (‘logical arithmetic’) are an important idea in mathematics and were invented by John Napier in the 17th century to simplify arithmetic calculations. Logarithms are linked directly to exponentials and can be used to solve a range of exponential equations.

Recall that $2^3 = 8$ (2 to the power 3 equals 8). We can also say that the logarithm of 8 to the base 2 equals 3 and we write $\log_2 8 = 3$. So for exponential equations such as $y = 2^x$, a logarithm finds x for a given value of y.

A logarithm can often be evaluated by hand but calculators can also be used.

Logarithms can also be used to create logarithmic scales, which are commonly used in science, economics and engineering. For example, the Richter scale, and the moment magnitude scale that replaced it, are logarithmic scales that illustrate the strength of an earthquake.

Seismologists calculate the magnitude of an earthquake using the logarithm of its intensity. The 2004 Sumatra earthquake of Richter magnitude 9.3 had 1000 times more intense shaking than the Richter magnitude 6.3 earthquake in Christchurch in 2011.

LESSON STARTER Can you work out logarithms?

We know that $3^2 = 9$, so $\log_3 9 = 2$. This means that $\log_3 9$ is equal to the index that makes 3 to the power of that index equal 9. Similarly, $10^3 = 1000$ so $\log_{10} 1000 = 3$.
Now find the value of the following.

- $\log_{10} 100$
- $\log_2 64$
- $\log_{10} 10\,000$
- $\log_3 27$
- $\log_2 16$
- $\log_4 64$

KEY IDEAS

- A **logarithm** of a number to a given base is the power (or index) to which the base is raised to give the number.
  - For example: $\log_2 16 = 4$ since $2^4 = 16$.
  - The base a is written as a subscript to the operator word ‘log’; i.e. $\log_a$.
- In general, if $a^x = y$ then $\log_a y = x$ with $a > 0$ and $y > 0$.
  - We say ‘the logarithm of y to the base a is x’.

ISBN 978-1-108-77290-7

BUILDING UNDERSTANDING

1 State the missing values in this table.

x	0	1	2	3	4	5
2^x						
3^x						243
4^x					256	
5^x		5				
10^x			100			

2 State the value of the unknown number for each statement.

a 2 to the power of what number gives 16?
b 3 to the power of what number gives 81?
c 7 to the power of what number gives 343?
d 10 to the power of what number gives 10 000?

3 Give these numbers as fractions.

a 0.0001 **b** 0.5 **c** 2^{-2} **d** 3^{-3}

Example 1 Writing equivalent statements involving logarithms

Write an equivalent statement to the following.

a $\log_{10} 1000 = 3$ **b** $2^5 = 32$

SOLUTION	EXPLANATION
a $10^3 = 1000$	$\log_a y = x$ is equivalent to $a^x = y$.
b $\log_2 32 = 5$	$a^x = y$ is equivalent to $\log_a y = x$.

Now you try

Write an equivalent statement to the following.

a $\log_{10} 100 = 2$ **b** $3^4 = 81$

Example 2 Evaluating logarithms

a Evaluate the following logarithms.

i $\log_2 8$ **ii** $\log_5 625$

b Evaluate the following.

i $\log_3 \frac{1}{9}$ **ii** $\log_{10} 0.001$

c Evaluate, correct to three decimal places, using a calculator.

i $\log_{10} 7$ **ii** $\log_{10} 0.5$

Continued on next page

SOLUTION

a i $\log_2 8 = 3$

ii $\log_5 625 = 4$

b i $\log_3 \frac{1}{9} = -2$

ii $\log_{10} 0.001 = -3$

c i $\log_{10} 7 = 0.845$ (to 3 d.p.)

ii $\log_{10} 0.5 = -0.301$ (to 3 d.p.)

EXPLANATION

Ask the question '2 to what power gives 8?'
Note: $2^3 = 8$

$5^4 = 5 \times 5 \times 5 \times 5 = 625$

$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

$10^{-3} = \frac{1}{10^3} = \frac{1}{1000} = 0.001$

Use the log button on a calculator and use base 10. (Some calculators will give log base 10 by pressing the log button.)

Use the log button on a calculator.

Now you try

a Evaluate the following logarithms.

i $\log_2 16$ ii $\log_3 243$

b Evaluate the following.

i $\log_2 \frac{1}{8}$ ii $\log_{10} 0.01$

c Evaluate, correct to three decimal places, using a calculator.

i $\log_{10} 5$ ii $\log_{10} 0.45$

Example 3 Solving simple logarithmic equations

Find the value of x in these equations.

a $\log_4 64 = x$ b $\log_2 x = 6$

SOLUTION

a $\log_4 64 = x$
$4^x = 64$
$x = 3$

b $\log_2 x = 6$
$2^6 = x$
$x = 64$

EXPLANATION

$\log_a y = x$ then $a^x = y$.

$4^3 = 64$

Write in index form:
$2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$

Now you try

Find the value of x in these equations.

a $\log_3 81 = x$ b $\log_5 x = 3$

ISBN 978-1-108-77290-7

Exercise 10A

FLUENCY	1–3, 4–5(1/2)	2–6(1/2)	2–6(1/3)

Example 1a
Example 1b

1 Write an equivalent statement to the following.

a **i** $\log_{10} 10\,000 = 4$ **ii** $\log_2 32 = 5$

b **i** $2^3 = 8$ **ii** $3^4 = 81$

Example 1a

2 Write the following in index form.

a $\log_2 16 = 4$ **b** $\log_{10} 100 = 2$ **c** $\log_3 27 = 3$

d $\log_2 \frac{1}{4} = -2$ **e** $\log_{10} 0.1 = -1$ **f** $\log_3 \frac{1}{9} = -2$

Example 1b

3 Write the following in logarithmic form.

a $2^3 = 8$ **b** $3^4 = 81$ **c** $2^5 = 32$ **d** $4^2 = 16$ **e** $10^{-1} = \frac{1}{10}$ **f** $5^{-3} = \frac{1}{125}$

Example 2a

4 Evaluate the following logarithms.

a $\log_2 16$ **b** $\log_2 4$ **c** $\log_2 64$ **d** $\log_3 27$

e $\log_3 3$ **f** $\log_4 16$ **g** $\log_5 125$ **h** $\log_{10} 1000$

i $\log_7 49$ **j** $\log_{11} 121$ **k** $\log_{10} 100\,000$ **l** $\log_9 729$

m $\log_2 1$ **n** $\log_5 1$ **o** $\log_{37} 1$ **p** $\log_1 1$

Example 2b

5 Evaluate the following.

a $\log_2 \frac{1}{8}$ **b** $\log_2 \frac{1}{4}$ **c** $\log_3 \frac{1}{9}$ **d** $\log_{10} \frac{1}{1000}$

e $\log_7 \frac{1}{49}$ **f** $\log_3 \frac{1}{81}$ **g** $\log_5 \frac{1}{625}$ **h** $\log_8 \frac{1}{8}$

i $\log_{10} 0.1$ **j** $\log_{10} 0.001$ **k** $\log_{10} 0.00001$ **l** $\log_2 0.5$

m $\log_2 0.125$ **n** $\log_5 0.2$ **o** $\log_5 0.04$ **p** $\log_3 0.\dot{1}$

Example 2c

6 Evaluate, correct to three decimal places, using a calculator.

a $\log_{10} 5$ **b** $\log_{10} 47$ **c** $\log_{10} 162$ **d** $\log_{10} 0.8$ **e** $\log_{10} 0.17$ **f** $\log_{10} \frac{1}{27}$

PROBLEM-SOLVING	7(1/2), 8	7(1/2), 8	7(1/4), 9

Example 3

7 Find the value of x in these equations.

a $\log_3 27 = x$ **b** $\log_2 32 = x$ **c** $\log_2 64 = x$ **d** $\log_5 625 = x$

e $\log_{10} 1000 = x$ **f** $\log_6 36 = x$ **g** $\log_2 x = 4$ **h** $\log_3 x = 4$

i $\log_{10} x = 3$ **j** $\log_3 x = -2$ **k** $\log_4 x = -1$ **l** $\log_7 x = -3$

m $\log_x 27 = 3$ **n** $\log_x 32 = 5$ **o** $\log_x 64 = 3$ **p** $\log_x 64 = 2$

q $\log_x 81 = 4$ **r** $\log_x 10\,000 = 4$ **s** $\log_x 0.5 = -1$ **t** $\log_4 0.25 = x$

8 A single bacterium cell divides into two every minute.

a Complete this cell population table.

Time (minutes)	0	1	2	3	4	5
Population	1	2				

b Write a rule for the population, P, after t minutes.

c Use your rule to find the population after 8 minutes.

d Use trial and error to find the time (correct to the nearest minute) for the population to rise to 10 000.

e Write the exact answer to part **d** as a logarithm.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

9 Evaluate:

a $\log_2 4 \times \log_3 9 \times \log_4 16 \times \log_5 25$

b $2 \times \log_3 27 - 5 \times \log_8 64 + 10 \times \log_{10} 1000$

c $\dfrac{4 \times \log_5 125}{\log_2 64} + \dfrac{2 \times \log_3 9}{\log_{10} 10}$

REASONING	10	10, 11	11, 12

10 Consider a bacteria population growing such that the total increases 10-fold every hour.

a Complete this table for the population (P) and $\log_{10} P$ for 5 hours (h).

h	0	1	2	3	4	5
P	1	10	100			
$\log_{10}P$						

b Plot a graph of $\log_{10} P$ (y-axis) against hours (x-axis). What do you notice?

c Find a rule linking $\log_{10} P$ with h.

11 The Richter magnitude of an earthquake is determined from a logarithm of the amplitude of waves recorded by a seismograph. It uses log base 10. So for example, an earthquake of magnitude 3 is 10 times more powerful than one with magnitude 2 and an earthquake of magnitude 7 is 100 times more powerful than one with magnitude 5.

a Write the missing number. An earthquake of magnitude 6 is:

i $\square$ times more powerful than one of magnitude 5.

ii $\square$ times more powerful than one of magnitude 4.

iii $\square$ times more powerful than one of magnitude 2.

b Write the missing number. An earthquake of magnitude 9 is:

i $\square$ times more powerful than one of magnitude 8.

ii 1000 times more powerful than one of magnitude $\square$.

iii 10^6 times more powerful than one of magnitude $\square$.

12 Is it possible for a logarithm (of the form $\log_a b$) to give a negative result? If so, give an example and reasons.

ENRICHMENT: Fractional logarithms	–	–	13(½)

13 We know that we can write $\sqrt{2} = 2^{\frac{1}{2}}$, so $\log_2\sqrt{2} = \frac{1}{2}$ and $\log_2 \sqrt[3]{2} = \frac{1}{3}$. Now evaluate the following without the use of a calculator.

a $\log_2 \sqrt[4]{2}$

b $\log_2 \sqrt[5]{2}$

c $\log_3 \sqrt{3}$

d $\log_3 \sqrt[3]{3}$

e $\log_7 \sqrt{7}$

f $\log_{10} \sqrt[3]{10}$

g $\log_{10} \sqrt[3]{100}$

h $\log_2 \sqrt[3]{16}$

i $\log_3 \sqrt[4]{9}$

j $\log_5 \sqrt[4]{25}$

k $\log_2 \sqrt[5]{64}$

l $\log_3 \sqrt[7]{81}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

10B Laws of logarithms 10A

Learning intentions

- To know how to combine logarithms with the same base using the logarithm laws for addition and subtraction
- To know properties of logarithms involving powers and the logarithm of 1
- To be able to use logarithm properties to simplify expressions

From the study of indices you will recall a number of index laws that can be used to manipulate expressions involving powers. Similarly, we have laws for logarithms and these can be derived using the index laws.

Recall index law 1: $a^m \times a^n = a^{m+n}$

Now let let $x = a^m$ and $y = a^n$ (1)

So $m = \log_a x$ and $n = \log_a y$ (2)

From equation (1) $xy = a^m \times a^n$
$= a^{m+n}$ (using index law 1)

So: $m + n = \log_a (xy)$
From (2) $m + n = \log_a x + \log_a y$
So: $\log_a (xy) = \log_a x + \log_a y$

Audiologists measure the loudness of sound in decibels (dB), a logarithmic scale. Permanent hearing loss occurs after listening to 88 dB music 4 hours/day. Each 3 dB increase halves the safe time; at 100 dB hearing loss occurs in 15 minutes/day.

This is a proof for one of the logarithm laws and we will develop the others later in this section.

LESSON STARTER Proving a logarithm law

In the introduction above there is a proof of the first logarithm law, which is considered in this section. It uses the first index law.

- Now complete a similar proof for the second logarithm law, $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$, using the second index law.

KEY IDEAS

- Law 1: $\log_a x + \log_a y = \log_a (xy)$
  - This relates to index law 1: $a^m \times a^n = a^{m+n}$.
- Law 2: $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$
  - This relates to index law 2: $a^m \div a^n = a^{m-n}$.
- Law 3: $\log_a (x^n) = n \log_a x$
  - This relates to index law 3: $(a^m)^n = a^{m \times n}$.
- Other properties of logarithms.
  - $\log_a 1 = 0$, $(a \neq 1)$ using $a^0 = 1$
  - $\log_a a = 1$, using $a^1 = a$
  - $\log_a \frac{1}{x} = \log_a x^{-1} = -\log_a x$ from law 3.

ISBN 978-1-108-77290-7

BUILDING UNDERSTANDING

1 Complete the rules for logarithms using the given pronumerals.

a $\log_b (xy) = \log_b x + \_\_\_\_\_\_$

b $\log_b \left(\frac{x}{y}\right) = \_\_\_\_\_\_ - \_\_\_\_\_\_$

c $\log_a b^m = m \times \_\_\_\_\_\_$

d $\log_a a = \_\_\_\_\_\_$

e $\log_c 1 = \_\_\_\_\_\_$

f $\log_a \frac{1}{b} = \_\_\_\_\_\_$

2 State the missing numbers.

a $\log_2 \boxed{} = 1$

b $\log_3 \boxed{} = 0$

c $\log_a 2 + \log_a \boxed{} = \log_a 8$

d $\log_a 36 - \log_a \boxed{} = \log_a 3$

e $\log_a 3^4 = \boxed{} = \log_a 3$

f $\boxed{} \log_a 3 = \log_a \frac{1}{3}$

3 Evaluate:

a $\log_{10} 100$

b $\log_2 32$

c $\log_3 27$

d $-2 \log_5 25$

e $4 \log_{10} 1000$

f $-6 \log_5 1$

Example 4 Simplifying logarithmic expressions

Simplify the following.

a $\log_a 4 + \log_a 5$

b $\log_a 22 - \log_a 11$

c $3 \log_a 2$

SOLUTION

a $\log_a 4 + \log_a 5 = \log_a 20$

EXPLANATION

This is logarithm law 1:

$\log_a x + \log_a y = \log_a (xy)$

SOLUTION

b $\log_a 22 - \log_a 11 = \log_a 2$

EXPLANATION

This uses logarithm law 2:

$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$

Note: $\log_a \frac{22}{11} = \log_a 2$

SOLUTION

c $3 \log_a 2 = \log_a 2^3$

$= \log_a 8$

EXPLANATION

Using logarithm law 3: $n \log_a x = \log_a x^n$

Now you try

Simplify the following

a $\log_a 3 + \log_a 8$

b $\log_a 32 - \log_a 16$

c $2 \log_a 4$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 5 Evaluating logarithmic expressions

Simplify and evaluate the following expressions.

a $\log_2 1$ **b** $\log_5 5$ **c** $\log_6 \frac{1}{36}$ **d** $\log_2 6 - \log_2 3$

SOLUTION

a $\log_2 1 = 0$

b $\log_5 5 = 1$

c $\log_6 \frac{1}{36} = \log_6 6^{-2}$
$= -2 \times \log_6 6$
$= -2 \times 1$
$= -2$

d $\log_2 6 - \log_2 3 = \log_2 2$
$= 1$

EXPLANATION

$2^0 = 1$

$5^1 = 5$

Alternatively, use the rule $\log_a \frac{1}{x} = -\log_a x$.

So $\log_6 \frac{1}{36} = -\log_6 36$
$= -2$

$\log_2 \left(\frac{6}{3}\right) = \log_2 2$ and $2^1 = 2$

Now you try

Simplify and evaluate the following expressions.

a $\log_4 1$ **b** $\log_7 7$ **c** $\log_3 \frac{1}{9}$ **d** $\log_3 15 - \log_3 5$

Exercise 10B

FLUENCY	1, 2–5(1/2)	2–5(1/2)	2–5(1/3)

1 Simplify the following.

Example 4a
a **i** $\log_a 2 + \log_a 7$ **ii** $\log_a 10 + \log_a 0.5$

Example 4b
b **i** $\log_a 12 - \log_a 6$ **ii** $\log_a 77 - \log_a 11$

Example 4c
c **i** $2 \log_a 3$ **ii** $3 \log_a 4$

Example 4a
2 Simplify using the first logarithm law.

a $\log_a 3 + \log_a 2$ **b** $\log_a 5 + \log_a 3$ **c** $\log_a 7 + \log_a 4$
d $\log_b 6 + \log_b 3$ **e** $\log_b 15 + \log_b 1$ **f** $\log_b 1 + \log_b 17$

Example 4b
3 Simplify using the second logarithm law.

a $\log_a 10 - \log_a 5$ **b** $\log_a 36 - \log_a 12$ **c** $\log_a 100 - \log_a 10$
d $\log_b 28 - \log_b 14$ **e** $\log_b 3 - \log_b 2$ **f** $\log_b 7 - \log_b 5$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 4c

4 Simplify using the third logarithm law.

a $2\log_a 3$ **b** $2\log_a 5$ **c** $3\log_a 3$
d $4\log_a 2$ **e** $5\log_a 2$ **f** $3\log_a 10$

Example 5a,b

5 Evaluate:

a $\log_3 1$ **b** $\log_7 1$ **c** $\log_x 1$
d $\log_4 4$ **e** $\log_{18} 18$ **f** $\log_a a$
g $5\log_2 1$ **h** $3\log_4 4$ **i** $\frac{1}{3}\log_7 7$
j $\frac{2}{3}\log_{10} 10$ **k** $\frac{\log_{15} 225}{2}$ **l** $\frac{\log_3 243}{10}$

PROBLEM-SOLVING	6–7(1/2)	6–7(1/2)	6–8(1/3)

Example 5c

6 Simplify and evaluate.

a $\log_2 \frac{1}{4}$ **b** $\log_3 \frac{1}{27}$ **c** $\log_4 \frac{1}{64}$
d $\log_5 \frac{1}{5}$ **e** $\log_{10} \frac{1}{100}$ **f** $\log_{10} \frac{1}{100\,000}$

Example 5d

7 Simplify and evaluate.

a $\log_2 10 - \log_2 5$ **b** $\log_3 30 - \log_3 10$ **c** $\log_4 128 - \log_4 2$
d $\log_4 8 + \log_4 2$ **e** $\log_8 16 + \log_8 4$ **f** $\log_{10} 50 + \log_{10} 2$

8 Simplify using a combination of log law number 3 with law 1 and 2.

a $2\log_3 2 + \log_3 5$ **b** $4\log_{10} 2 + \log_{10} 3$
c $3\log_{10} 2 - \log_{10} 4$ **d** $5\log_7 2 - \log_7 16$
e $\frac{1}{2}\log_3 4 + 2\log_3 2$ **f** $\log_5 3 - \frac{1}{2}\log_5 9$
g $\frac{1}{3}\log_2 27 - \frac{1}{3}\log_2 64$ **h** $\frac{1}{4}\log_5 16 + \frac{1}{5}\log_5 243$

REASONING	9	9, 10	9(1/2), 10, 11

9 Recall that $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[3]{x} = x^{\frac{1}{3}}$ and in general $\sqrt[n]{x} = x^{\frac{1}{n}}$. Use this to simplify the following.

a $\log_2 \sqrt{8}$ **b** $\log_2 \sqrt{32}$
c $\log_2 \sqrt[3]{16}$ **d** $\log_{10} \sqrt{1000}$
e $\log_7 \sqrt[3]{7}$ **f** $\log_5 \sqrt[5]{625}$

10 Prove that:

a $\log_a \frac{1}{x} = -\log_a x$ using logarithm law 2

b $\log_a \frac{1}{x} = -\log_a x$ using logarithm law 3

11 Prove that $\log_a \sqrt[n]{x} = \frac{\log_a x}{n}$ using logarithm law 3.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

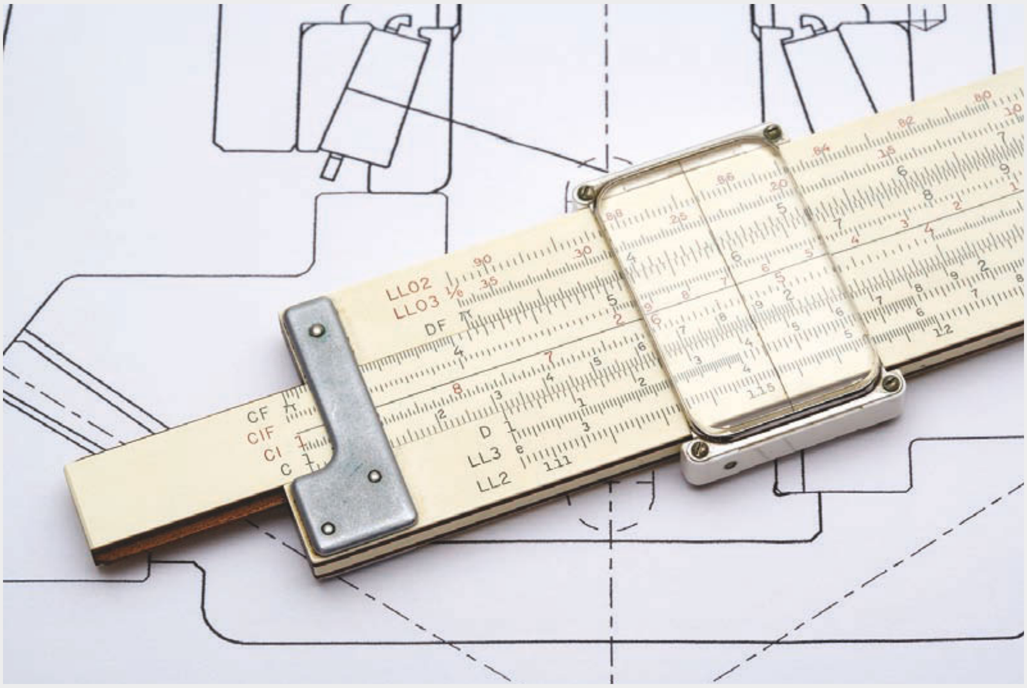
ENRICHMENT: Proving the laws for logarithms	–	–	12

12 Read the proof for logarithm law 1 in the introduction and then complete the following tasks.

a Complete a proof giving all reasons for logarithm law 1: $\log_a (xy) = \log_a x + \log_a y$.

b Complete a proof for logarithm law 2: $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$.

c Complete a proof for logarithm law 3: $\log_a x^n = n \log_a x$.



Before the invention of the electronic calculator, multiplication and division of numbers with many digits was done with tables of logarithms or slide rules with logarithmic scales.

10C Solving exponential equations using logarithms 10A

Learning Intentions

- To know how to solve exponential equations by rewriting in logarithmic form using the given base
- To be able to solve an exponential equation using base 10
- To be able to use technology to evaluate logarithms

When solving a simple exponential equation like $2^x = 16$ we know that the solution is $x = 4$ because $2^4 = 16$. Solving $2^x = 10$, however, is trickier and requires the use of logarithms. Depending on what calculator functions you have, one of two different methods can be chosen. These methods can be used to solve many types of problems in science and finance.

The many applications of solving exponential equations include medical scientists calculating when a radioactive tracer has decayed; financiers determining when an investment doubles; and food scientists calculating the time for a bacteria population to reach food-poisoning levels.

LESSON STARTER Trial and error versus logarithms

Consider the equation $10^x = 20$.

- First, use a calculator and trial and error to find a value of x (correct to three decimal places) that satisfies the equation.
- Now write $10^x = 20$ in logarithmic form and use the log function on your calculator to find the value of x.
- Check the accuracy of your value of x obtained by trial and error.

KEY IDEAS

- Solving for x if $a^x = y$
  - Using the given base: $x = \log_a y$
  - Using base 10:

$$\begin{aligned} a^x &= y \\ \log_{10} a^x &= \log_{10} y && \text{(taking } \log_{10} \text{ of both sides)} \\ x \log_{10} a &= \log_{10} y && \text{(using law 3)} \\ x &= \frac{\log_{10} y}{\log_{10} a} && \text{(dividing by } \log_{10} a\text{)} \end{aligned}$$

- Most calculators can evaluate using log base 10, but CAS calculators can work with any base.

ISBN 978-1-108-77290-7

BUILDING UNDERSTANDING

1 Give the logarithmic form of these equations.

a $2^3 = 8$ **b** $4^{\frac{1}{2}} = 2$ **c** $3^x = 10$

2 State the missing number.

a $5^{\square} = 125$ **b** $10^{\square} = 10000$

c $\log_2 \boxed{} = 3$ **d** $\log_4 \boxed{} = \frac{1}{2}$

3 Use a calculator to evaluate the following, correct to three decimal places.

a $\log_{10} 7$ **b** $\log_{10} 0.6$ **c** $\log_{10} \frac{3}{4}$ **d** $\frac{\log_{10} 12}{\log_{10} 7}$

Example 6 Solving using the given base

Solve the following using the given base. Round your answer to three decimal places.

a $2^x = 7$ **b** $50 \times 1.1^x = 100$

SOLUTION

a
$$2^x = 7$$
$$x = \log_2 7$$
$$= 2.807 \text{ (to 3 d.p.)}$$

EXPLANATION: If $a^x = y$ then $x = \log_a y$.
This method can be used on calculators that have a log function $\log_a y$, where both a and y can be entered.

b
$$50 \times 1.1^x = 100$$
$$1.1^x = 2$$ (Divide both sides by 50.)
$$x = \log_{1.1} 2$$
$$= 7.273 \text{ (to 3 d.p.)}$$

EXPLANATION: Divide both sides by 50.
Write in logarithmic form, then use a calculator for the approximation.

Now you try

Solve the following using the given base. Round your answer to three decimal places.

a $3^x = 10$ **b** $20 \times 1.2^x = 60$

Example 7 Solving using base 10

Solve using base 10 and evaluate, correct to three decimal places.

a $3^x = 5$ **b** $1000 \times 0.93^x = 100$

SOLUTION

a
$$3^x = 5$$
$$\log_{10} 3^x = \log_{10} 5$$
$$x \log_{10} 3 = \log_{10} 5$$
$$x = \frac{\log_{10} 5}{\log_{10} 3}$$
$$= 1.465 \text{ (to 3 d.p.)}$$

EXPLANATION: Take $\log_{10}$ of both sides.
Use law 3: $\log_a x^n = n\log_a x$.
Divide by $\log_{10} 3$.
Use the log function on a calculator.

Continued on next page

ISBN 978-1-108-77290-7

b $1000 \times 0.93^x = 100$

$0.93^x = 0.1$ — Divide both sides by 1000.

$\log_{10} 0.93^x = \log_{10} 0.1$ — Take $\log_{10}$ of both sides.

$x \log_{10} 0.93 = \log_{10} 0.1$ — Use law 3 and solve for x by dividing both sides by $\log_{10} 0.93$.

$x = \dfrac{\log_{10} 0.1}{\log_{10} 0.93}$ — Use the log function on a calculator.

$= 31.729$ (to 3 d.p.)

Now you try

Solve using base 10 and evaluate, correct to three decimal places.

a $2^x = 11$ **b** $200 \times 0.85^x = 50$

Exercise 10C

FLUENCY	1, 2–3(1/2)	2–4(1/2)	2–4(1/3)

Example 6a, Example 6b

1 Solve the following using the given base. Round your answer to three decimal places.

a **i** $4^x = 5$ **ii** $3^x = 13$

b **i** $20 \times 1.3^x = 80$ **ii** $10 \times 1.6^x = 70$

Example 6a

2 Solve the following using the given base and round to three decimal places where necessary.

a $3^x = 5$ **b** $2^x = 11$ **c** $5^x = 13$

d $1.2^x = 3.5$ **e** $2.9^x = 3.5$ **f** $0.2^x = 0.04$

Example 6b

3 Solve the following using the given base and round to three decimal places where necessary.

a $10 \times 2^x = 20$ **b** $25 \times 3^x = 75$ **c** $4 \times 1.5^x = 20$

d $3.8 \times 1.7^x = 9.5$ **e** $300 \times 0.9^x = 150$ **f** $7.3 \times 0.4^x = 1.8$

Example 7

4 Solve using base 10 and evaluate, correct to three decimal places.

a $2^x = 6$ **b** $3^x = 8$ **c** $5^x = 7$ **d** $11^x = 15$

e $1.8^x = 2.5$ **f** $0.9^x = 0.5$ **g** $10 \times 2^x = 100$ **h** $7 \times 3^x = 28$

i $130 \times 7^x = 260$ **j** $4 \times 1.5^x = 20$ **k** $100 \times 0.8^x = 50$ **l** $30 \times 0.7^x = 20$

PROBLEM-SOLVING	5	5, 6	6, 7

5 The rule modelling a population (P) of mosquitoes is given by $P = 8^t$, where t is measured in days. Find the number of days, correct to three decimal places where necessary, required for the population to reach:

a 64

b 200

c 1000

ISBN 978-1-108-77290-7

6 An investment of $10000 is expected to grow by 5% p.a. so the balance $$A$ is given by the rule $A = 10000 \times 1.05^n$, where n is the number of years. Find the time (to two decimal places) for the investment to grow to:

a $20000 b $32000 c $100000

7 50 kg of a radioactive isotope in a set of spent nuclear fuel rods is decaying at a rate of 1% per year. The mass of the isotope (m kg) is therefore given by $m = 50 \times 0.99^n$, where n is the number of years. Find the time (to two decimal places) when the mass of the isotope reduces to:

a 45 kg b 40 kg c 20 kg

REASONING	8	8, 9	9, 10

8 The value of a bank balance increases by 10% per year. The initial amount is $2000.

a Write a rule connecting the balance $$A$ with the time (n years).

b Find the time, correct to the nearest year, when the balance is double the original amount.

9 The value of a Ferrari is expected to reduce by 8% per year. The original cost is $300000.

a Find a rule linking the value of the Ferrari ($$F$) and the time ($n$ years).

b Find the time it takes for the value of the Ferrari to reduce to $150000. Round your answer to one decimal place.

10 The half-life of a substance is the time it takes for the substance to reduce to half its original mass. Round answers to the nearest year.

a Find the half-life of a 10 kg rock if its mass reduces by 1% per year.

b Find the half-life of a 20 g crystal if its mass reduces by 0.05% per year.

ENRICHMENT: Change of base formula	–	–	11

11 If $a^x = y$ then we can write $x = \log_a y$. Alternatively, if $a^x = y$ we can find the logarithm of both sides, as shown here.

$$\begin{aligned} a^x &= y \\ \log_b a^x &= \log_b y \\ x \log_b a &= \log_b y \\ x &= \frac{\log_b y}{\log_b a} \\ \therefore\ \log_a y &= \frac{\log_b y}{\log_b a} \end{aligned}$$

This is the change of base formula.

a Use the change of base formula to write the following with base 10.

i $\log_2 7$ ii $\log_3 16$ iii $\log_5 1.3$

b Change to log base 10 and simplify.

i $\log_5 10$ ii $\log_2 1000$ iii $\log_3 0.1$

c Make x the subject and then change to base 10. Round your answer to three decimal places.

i $3^x = 6$ ii $9^x = 13$ iii $2 \times 1.3^x = 1.9$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

10D Introducing polynomials 10A

Learning intentions

- To know the general form of a polynomial
- To know the meaning of the degree of a polynomial and the names of common polynomials
- To be able to use function notation for a polynomial

We are familiar with linear expressions such as $3x - 1$ and $4 + \frac{x}{2}$ and with quadratic expressions such as $x^2 - 3$ and $-4x^2 + 2x - 4$. These expressions are in fact part of a larger group called polynomials, which are sums of powers of a variable using whole number powers $\{0, 1, 2, \ldots\}$. For example, $2x^3 - 3x^2 + 4$ is a cubic polynomial and $1 - 4x^3 + 3x^7$ is a polynomial of degree 7. The study of polynomials opens up many ideas in the analysis of functions and graphing that are studied in many senior mathematics courses.

All calculators perform calculations like log 43, sin 65, etc. by substituting numbers into polynomials. A calculator can't possibly store all potential results, so a specific polynomial from the Taylor series is coded for each calculator function button. (You might encounter Taylor series if you study mathematics at university; it refers to the fact that many different types of functions can be represented by an infinite sum of special terms.)

LESSON STARTER Is it a polynomial?

A polynomial is an expression that includes sums of powers of x with whole number powers $\{0, 1, 2, \ldots\}$. Decide, with reasons, whether the following are polynomials.

- $5 + 2x + x^2$
- $\sqrt{x} + x^2$
- $\frac{2}{x} + 3$
- $4x^4 - x^2 - 6$
- $4x^{\frac{1}{3}} + 2x^2 + 1$
- 5

KEY IDEAS

- A **polynomial** is an expression of the form $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where:
  - n is a positive integer or zero $\{0, 1, 2, \ldots\}$
  - $a_n, a_{n-1}, \ldots a_0$ are **coefficients** which can be any real number.
  - a_0 is the **constant term**
  - $a_n x^n$ is the **leading term**

- Naming polynomials
  Polynomials are named by the highest power of x. This is called the **degree** of the polynomial.
  - **constant** For example: 2
  - **linear** For example: $3x - 7$
  - **quadratic** For example: $2x^2 - 4x + 11$
  - **cubic** For example: $-4x^3 + 6x^2 - x + 3$
  - **quartic** For example: $\frac{1}{2}x^4 - x^2 - 2$
  - **of degree** 8 For example: $3x^8 - 4x^5 + x - 3$
- **Function notation**
  - A polynomial in x can be called $P(x)$
    For example: $P(x) = 2x^3 - x$ is a cubic polynomial.
  - $P(k)$ is the value of the polynomial at $x = k$.
    For example: If $P(x) = 2x^3 - x$, then:

    $P(3) = 2(3)^3 - (3)$ and $P(-1) = 2(-1)^3 - (-1)$

    $ = 51$ $\qquad\qquad\qquad\quad = -2 + 1$

    $\qquad\qquad\qquad\qquad\qquad\quad = -1$

BUILDING UNDERSTANDING

1 A polynomial expression is given by $3x^4 - 2x^3 + x^2 - x + 2$.
  a How many terms does the polynomial have?
  b State the coefficient of:
    i x^4 ii x^3 iii x^2 iv x
  c What is the value of the constant term?

2 Decide if these polynomials are constant, linear, quadratic, cubic or quartic.
  a $2x - 5$ b $x^2 - 3$ c $x^4 + 2x^3 + 1$
  d $1 + x + 3x^2$ e 6 f $4x - x^3 + x^2$

3 State the degree of each of these polynomials.
  a $2x^3 + 4x^2 - 2x + 1$ b $x^4 - 2x^2 - 2$ c $-3x^6 + 2x^4 - 9x^2 + 1$

Example 8 Evaluating polynomials

If $P(x) = x^3 - 3x^2 - x + 2$, find:
a $P(2)$ b $P(-3)$

SOLUTION	EXPLANATION
a $P(2) = (2)^3 - 3(2)^2 - (2) + 2$ $= 8 - 12 - 2 + 2$ $= -4$	Substitute $x = 2$ and evaluate.
b $P(-3) = (-3)^3 - 3(-3)^2 - (-3) + 2$ $= -27 - 27 + 3 + 2$ $= -49$	Substitute $x = -3$ and note $(-3)^3 = -27$ and $(-3)^2 = 9$.

Continued on next page

ISBN 978-1-108-77290-7

Now you try

If $P(x) = x^3 + 2x^2 - 3x + 4$, find:

a $P(2)$

b $P(-1)$

Example 9 Deciding if an expression is a polynomial

Decide if the following expressions are polynomials.

a $4x^2 - 1 + 7x^4$

b $2x^2 - \sqrt{x} + \frac{2}{x}$

SOLUTION

a yes

b no

EXPLANATION

a Powers of x are whole numbers $\{0, 1, 2, \ldots\}$.

b $2x^2 - \sqrt{x} + \frac{2}{x} = 2x^2 - x^{\frac{1}{2}} + 2x^{-1}$

Powers include $\frac{1}{2}$ and -1, which are not allowed in the polynomial family.

Now you try

Decide if the following expressions are polynomials.

a $5x^2 - x + 4x^3$

b $\frac{1}{x} + \sqrt{x} + 1$

Exercise 10D

FLUENCY	1–3, 4(½)	2–4	2, 4

Example 8

1 If $P(x) = x^3 - 3x^2 - 2x + 3$, find:

a $P(2)$ b $P(4)$ c $P(-1)$ d $P(-3)$

Example 8

2 If $P(x) = 2x^4 - 3x^3 + 5x - 4$, find:

a $P(1)$ b $P(3)$ c $P(-1)$ d $P(-2)$

3 Evaluate the quadratic polynomial $x^2 - x + 2$, using:

a $x = 4$ b $x = 10$ c $x = -2$ d $x = -1$

Example 9

4 Decide if the following are polynomials.

a $3x^3 + x^2 - x + 3$

b $2x^4 - x^2 - 4$

c $\frac{2}{x} - \frac{3}{x} + 2$

d $\frac{7}{x^2} - \frac{1}{x} + 2$

e $x^4 - x^3 + \frac{2}{x^3}$

f $4 - 7x^8$

g $\sqrt{x} + 2 - x^2$

h $\sqrt[4]{x} + \sqrt[3]{x} + \sqrt{x}$

i $x^3 + \frac{1}{\sqrt{x}}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

PROBLEM-SOLVING	$5(1/2)$, 8	$5(1/2)$, 7, 8	$5(1/2)$, 6–8

5 If $P(x) = x^3 - x^2$ and $Q(x) = 4 - 3x$, find:

a $P(1) + Q(2)$ **b** $P(3) + Q(-1)$ **c** $P(-2) - Q(-2)$

d $Q(1) - P(3)$ **e** $(P(2))^2 + (Q(1))^2$ **f** $(P(-1))^3 - (Q(-1))^3$

6 Find the coefficient of x^2 in these polynomials.

a $P(x) = \dfrac{4 - 2x^2}{4}$ **b** $P(x) = \dfrac{x^3 + 7x^2 + x - 3}{-7}$ **c** $P(x) = \dfrac{x^3 - 4x^2}{-8}$

7 Evaluate $P(-2)$ for these polynomials.

a $P(x) = (x + 2)^2$

b $P(x) = (x - 2)(x + 3)(x + 1)$

c $P(x) = x^2(x + 5)(x - 7)$

8 The height (P metres) of a roller coaster track above a platform is given by the equation $P(x) = x^3 - 12x^2 + 35x$, where x metres is the horizontal distance from the beginning of the platform.

a Find the height of the track using:

i $x = 2$ **ii** $x = 3$ **iii** $x = 7$

b Does the track height ever fall below the level of the platform? If so, find a value of x for which this occurs.

Polynomials are used in the design of roller coasters.

REASONING	9	9, $10(1/2)$	$10–11(1/2)$

9 **a** What is the maximum number of terms in a polynomial of degree 7?

b What is the maximum number of terms in a polynomial of degree n?

c What is the minimum number of terms in a polynomial of degree 5?

d What is the minimum number of terms in a polynomial of degree n?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

10 If $P(x) = x^3 - x^2 - 2x$, evaluate and simplify these without the use of a calculator.

a $P\left(\frac{1}{2}\right)$
b $P\left(\frac{1}{3}\right)$
c $P\left(-\frac{1}{2}\right)$
d $P\left(-\frac{1}{4}\right)$
e $P\left(-\frac{2}{3}\right)$
f $P\left(\frac{4}{5}\right)$
g $P\left(-\frac{1}{2}\right) + P\left(\frac{1}{2}\right)$
h $P\left(-\frac{3}{4}\right) + P\left(\frac{3}{4}\right)$

11 If $P(x) = 2x^3 - x^2 - 5x - 1$, find the following and simplify where possible.

a $P(k)$
b $P(b)$
c $P(2a)$
d $P(-a)$
e $P(-2a)$
f $P(-3k)$
g $P(ab)$
h $P(-ab)$

ENRICHMENT: Finding unknown coefficients	–	–	12

12 If $P(x) = x^3 - 2x^2 + bx + 3$ and $P(1) = 4$, we can find the value of b as follows.

$$\begin{aligned} P(1) &= 4 \\ (1)^3 - 2(1)^2 + b(1) + 3 &= 4 \\ 2 + b &= 4 \\ b &= 2 \end{aligned}$$

a Use this method to find the value of b if $P(x) = x^3 - 4x^2 + bx - 2$ and if:

i $P(1) = 5$
ii $P(2) = -6$
iii $P(-1) = -8$
iv $P(-2) = 0$
v $P(-1) = 2$
vi $P(-3) = -11$

b If $P(x) = x^4 - 3x^3 + kx^2 - x + 2$, find k if:

i $P(1) = 2$
ii $P(-2) = 0$
iii $P(-1) = -15$

c If $P(x) = x^3 + ax^2 + bx - 3$ and $P(1) = -1$ and $P(-2) = -1$, find the values of a and b.

ISBN 978-1-108-77290-7

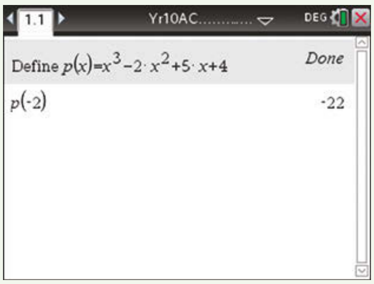
Cambridge University Press
Updated June 2022

Using calculators to work with polynomials

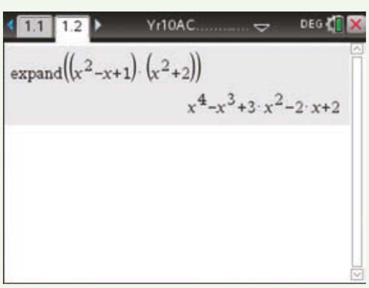
1 Define the polynomial $P(x) = x^3 - 2x^2 + 5x + 4$ and evaluate at $x = -2$.
2 Expand and simplify $(x^2 - x + 1)(x^2 + 2)$.

Using the TI-Nspire:

1 In a **Calculator** page define the polynomial using menu **>Actions>Define** as shown. Evaluate for $x = 2$, $p(-2)$.

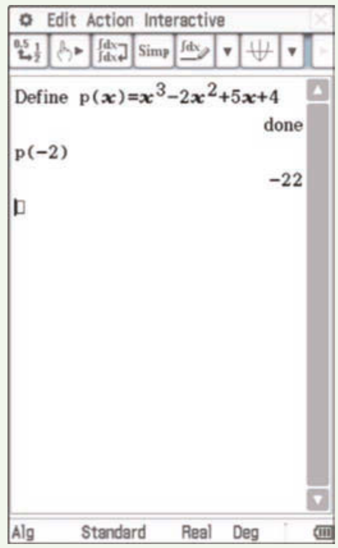


2 Use menu **>Algebra>Expand**, then type in the expression and press enter.

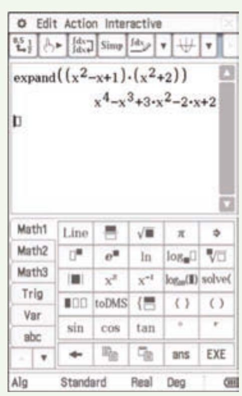


Using the ClassPad:

1 In the **Main** application, type and highlight the polynomial expression. Tap **Interactive, Define**, **OK**. Evaluate by typing $p(-2)$.



2 In the **Main** application, type and highlight expression. Tap **Interactive**, **Transformation**, **expand**. **OK**. **EXE**.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

10E Expanding and simplifying polynomials 10A

Learning intentions
- To be able to apply the rules of expanding brackets to multiply polynomials
- To understand that multiplying polynomials results in a polynomial of higher degree

From your work on quadratics, you will remember how to use the distributive law to expand brackets. For example, $(2x-1)(x+5)$ expands to $2x^2+10x-x-5$, and after collecting like terms this simplifies to $2x^2+9x-5$. In a similar way we can expand the product of two or more polynomials of any degree. To do this we also multiply every term in one polynomial with every term in the next polynomial.

Polynomial 'secret key' technology enables secure communication between and within groups and is more efficient than standard encryption/decryption systems. Applications include video conferencing, military communications, and between the components of the IoT (Internet of Things).

LESSON STARTER The product of two quadratics

The equation $(x^2-x+3)(2x^2+x-1)=2x^4-x^3+4x^2+4x-3$ is written on the board.

- Is the equation true for $x=1$?
- Is the equation true for $x=-2$?
- How can you prove the equation to be true for all values of x?

KEY IDEAS

- Expand products of polynomials by multiplying each term in one polynomial by each term in the next polynomial.
- Simplify by collecting like terms.

BUILDING UNDERSTANDING

1 Expand and simplify these quadratics.
a $x(x+2)$ **b** $(x-5)(x+11)$ **c** $(4x-3)(2x-5)$

2 Collect like terms to simplify.
a $2x^4-3x^3+x^2-1-x^4-2x^3+3x^2-2$
b $5x^6+2x^4-x^2+5-5x^4+x^3+8-6x^6$

3 Use substitution to confirm that this equation is true for the given x-values.
$(x^3-x+3)(x^2+2x-1)=x^5+2x^4-2x^3+x^2+7x-3$
a $x=1$ **b** $x=0$ **c** $x=-2$

ISBN 978-1-108-77290-7

Example 10 Expanding polynomials

Expand and simplify.

a $x^3(x - 4x^2)$

b $(x^2 + 1)(x^3 - x + 1)$

SOLUTION

a $x^3(x - 4x^2) = x^4 - 4x^5$

b $(x^2 + 1)(x^3 - x + 1)$
$= x^2(x^3 - x + 1) + 1(x^3 - x + 1)$
$= x^5 - x^3 + x^2 + x^3 - x + 1$
$= x^5 + x^2 - x + 1$

EXPLANATION

$x^3 \times x^1 = x^4$ and $x^3 \times (-4x^2) = -4x^5$ using index law 1.

$(x^2 + 1)(x^3 - x + 1)$

$-x^3$ cancels with x^3.

Now you try

Expand and simplify.

a $x^2(3x^2 - x)$

b $(x - 2)(x^3 + 4x - 3)$

Example 11 Expanding $P(x) \times Q(x)$

If $P(x) = x^2 + x - 1$ and $Q(x) = x^3 + 2x + 3$, expand and simplify the following.

a $P(x) \times Q(x)$

b $(Q(x))^2$

SOLUTION

a $P(x) \times Q(x)$
$= (x^2 + x - 1)(x^3 + 2x + 3)$
$= x^2(x^3 + 2x + 3) + x(x^3 + 2x + 3) - 1(x^3 + 2x + 3)$
$= x^5 + 2x^3 + 3x^2 + x^4 + 2x^2 + 3x - x^3 - 2x - 3$
$= x^5 + x^4 + x^3 + 5x^2 + x - 3$

b $(Q(x))^2$
$= (x^3 + 2x + 3)^2$
$= (x^3 + 2x + 3)(x^3 + 2x + 3)$
$= x^3(x^3 + 2x + 3) + 2x(x^3 + 2x + 3) + 3(x^3 + 2x + 3)$
$= x^6 + 2x^4 + 3x^3 + 2x^4 + 4x^2 + 6x + 3x^3 + 6x + 9$
$= x^6 + 4x^4 + 6x^3 + 4x^2 + 12x + 9$

EXPLANATION

Each term in the first polynomial is multiplied by each term in the second polynomial.

$(Q(x))^2 = Q(x) \times Q(x)$
Expand to gain 9 terms then collect and simplify.

Now you try

If $P(x) = x^3 - x + 3$ and $Q(x) = x^2 + x - 4$, expand and simplify the following.

a $P(x) \times Q(x)$

b $(P(x))^2$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 10E

FLUENCY	1, 2–3(½), 4	2–3(½), 4	2–3(⅓), 5

1 Expand and simplify.

Example 10a

a i $x^3(x - 2x^2)$

ii $x^2(x^2 + 4x)$

Example 10b

b i $(x^2 + 1)(x^3 + 2x - 3)$

ii $(x - 1)(x^4 + x^2 - 1)$

Example 10a

2 Expand and simplify.

a $x^2(x - 3)$
b $x^2(x^2 - 1)$
c $2x^2(1 + 3x)$
d $x^3(1 - x)$
e $x^3(x^2 + 3x)$
f $-3x^2(x^4 - x)$
g $-2x^3(x^2 + x)$
h $-x^2(x^5 - x^2)$
i $-4x^3(x^4 - 2x^7)$

Example 10b

3 Expand and simplify.

a $(x^2 + 1)(x^3 + 2)$
b $(x^2 - 1)(x^3 + x)$
c $(x^2 - x)(x^3 - 3x)$
d $(x^2 - 2)(x^3 + x - 2)$
e $(x^3 - x)(x^2 + 2x + 3)$
f $(x^3 - x^2)(x^2 - x + 4)$
g $(x^3 - x^2 - 1)(x^3 + x - 2)$
h $(x^3 - 5x^2 + 2)(x^3 - x + 1)$
i $(x^4 - x^2 + 1)(x^4 + x - 3)$

Example 11

4 If $P(x) = x^2 - 2x + 1$ and $Q(x) = x^3 + x - 1$, expand and simplify.

a $P(x) \times Q(x)$
b $(Q(x))^2$
c $(P(x))^2$

5 If $P(x) = x^3 + 2x^2 - x - 4$ and $Q(x) = x^2 + x - 2$, expand and simplify.

a $P(x) \times Q(x)$
b $(Q(x))^2$
c $(P(x))^2$

PROBLEM-SOLVING	6(½)	6(½), 7	6(½), 7, 8

6 If $P(x) = x^2 - 5x + 1$ and $Q(x) = x^3 + x$, simplify.

a $P(x) + Q(x)$
b $Q(x) - P(x)$
c $5P(x) + 2Q(x)$
d $1 - P(x)Q(x)$
e $4 - (Q(x))^2$
f $(P(x))^2 - (Q(x))^2$

7 Find the square of $P(x)$ in expanded form when $P(x) = (x^2 + x - 1)^2$.

8 Show that $(x^2 - x - 1)^2 - (x^2 - x + 1)^2 = 4x - 4x^2$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

REASONING	9	9, 10	10, 11

9 If $P(x)$ and $Q(x)$ are polynomials, does $P(x)Q(x) = Q(x)P(x)$ for all values of x?

10 Give the degree of the polynomial $P(x) \times Q(x)$ when:
- **a** $P(x)$ is quadratic and $Q(x)$ is linear
- **b** $P(x)$ is quadratic and $Q(x)$ is cubic
- **c** $P(x)$ is cubic and $Q(x)$ is quartic
- **d** $P(x)$ is of degree 7 and $Q(x)$ is of degree 5.

11 If $P(x)$ is of degree m and $Q(x)$ is of degree n and $m > n$, what is the highest possible degree of the following polynomials?
- **a** $P(x) + Q(x)$
- **b** $P(x) - Q(x)$
- **c** $P(x) \times Q(x)$
- **d** $(P(x))^2$
- **e** $(P(x))^2 - Q(x)$
- **f** $(Q(x))^3$

ENRICHMENT: Triple expansions	–	–	12

12 Expand and simplify.
- **a** $x(x^2 + 1)(x - 1)$
- **b** $x^3(x + 3)(x - 1)$
- **c** $(x + 2)(x - 1)(x + 3)$
- **d** $(x + 4)(2x - 1)(3x + 1)$
- **e** $(5x - 2)(x - 2)(3x + 5)$
- **f** $(x^2 + 1)(x^2 - 2)(x + 3)$

ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

Progress quiz

10A

10A **1** Express as a logarithm.
a $2^5 = 32$ **b** $10^3 = 1000$ **c** $a^1 = a$

10A **2** Rewrite using exponential notation.
a $\log_{10} 100 = 2$ **b** $\log_2 8 = 3$ **c** $\log_7 1 = 0$

10A **3** Find the value of x.
a $\log_2 16 = x$ **b** $\log_3 81 = x$
c $\log_{12} 1 = x$ **d** $\log_5 x = 4$
e $\log_{10} 100000 = x$ **f** $\log_{10} 0.000001 = x$

10B **4** Simplify the following, using logarithm laws.
a $\log_{10} 20 + \log_{10} 50$ **b** $\log_2 \frac{1}{3} + \log_2 12$
c $\log_3 18 - \log_3 2$ **d** $\log_3 \frac{1}{9}$
e $\log_{10} \sqrt{10}$ **f** $\log_4 24 - (\log_4 2 + \log_4 3)$

10C **5** Solve and round each answer to three decimal places. Use either the given base or base 10.
a $3^x = 7$
b $1.2^x = 200$
c $500(1.09)^x = 1000$

10D **6** Explain why $4x^3 - \sqrt{x} + 6x - 1$ is not a polynomial.

10D **7** Consider the polynomial $P(x) = 3x^4 - 2x^3 + x^2 + 7x + 8$. Find:
a $P(0)$
b $P(-1)$
c $P(k)$

10D **8** For the polynomial $P(x) = 4x^3 + 3x^2 + 2x + 1$, state:
a the degree of the polynomial
b the constant term
c the coefficent of x^2
d the leading term.

10E **9** Expand and simplify the following.
a $x^4(x^3 - 2x + 1)$
b $(x^2 - 1)(x^2 + 2x + 6)$

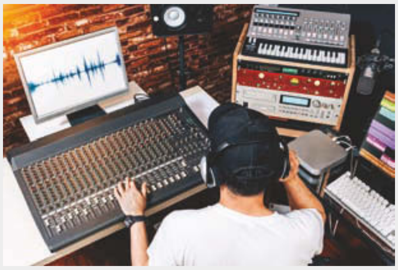
10E **10** If $P(x) = x^3 + x + 2$ and $Q(x) = x^4 + 2x$, find the following in their simplest forms.
a $P(x) + Q(x)$
b $P(x) - Q(x)$
c $(Q(x))^2$
d $P(x) \times Q(x)$

ISBN 978-1-108-77290-7

10F Division of polynomials 10A

Learning Intentions
- To know the long division algorithm
- To be able to carry out the long division algorithm to divide polynomials
- To know how to express a polynomial using the quotient, divisor and remainder

Division of polynomials requires the use of the long division algorithm. You may have used this algorithm for the division of whole numbers in primary school.



Acoustic engineers use complex mathematical procedures, including polynomial division, to analyse and electronically reproduce the vibrations that make sound; for designing headphones and synthesisers; and to analyse the architecture required for a superb concert sound.

Recall that 7 divided into 405 can be calculated in the following way.

	57	
	$7\overline{)405}$	7 into 4 does not go.
5×7	35	7 into 40 gives 5 and $5 \times 7 = 35$.
	55	Then subtract $405 - 350$.
7×7	49	7 into 55 gives 7 and $7 \times 7 = 49$.
	6	Subtract to give remainder 6.

So $405 \div 7 = 57$ and 6 remainder. The 57 is called the quotient.

Another way to write this is $405 = 7 \times 57 + 6$.

We use this technique to divide polynomials.

LESSON STARTER Recall long division

Use long division to find the quotient and remainder for the following.

- $832 \div 3$
- $2178 \div 7$

KEY IDEAS

- We use the long division algorithm to divide polynomials.
- The result is not necessarily a polynomial.

  Example:

  $$\text{dividend} \rightarrow \frac{x^3 - x^2 + x - 1}{\underbrace{x + 2}_{\text{divisor}}} = \underbrace{x^2 - 3x + 7}_{\text{quotient}} - \frac{15}{x + 2} \leftarrow \text{remainder}$$

  We can write this as:

  $$\underbrace{x^3 - x^2 + x - 1}_{\text{dividend}} = \underbrace{(x + 2)}_{\text{divisor}}\underbrace{(x^2 - 3x + 7)}_{\text{quotient}} - \underbrace{15}_{\text{remainder}}$$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

BUILDING UNDERSTANDING

1 Use long division to find the remainder.

a $208 \div 9$ **b** $143 \div 7$ **c** $2184 \div 3$

2 Complete the equation with the missing numbers.

a If $182 \div 3 = 60$ remainder 2, then $182 = \_\_\_\_ \times 60 + \_\_\_\_$.

b If $2184 \div 5 = 436$ remainder 4, then $2184 = \_\_\_\_ \times 436 + \_\_\_\_$.

c If $617 \div 7 = 88$ remainder 1, then $617 = 7 \times \_\_\_\_ + \_\_\_\_$.

Example 12 Dividing polynomials

a Divide $P(x) = x^3 + 2x^2 - x + 3$ by $(x - 2)$ and write in the form $P(x) = (x - 2)Q(x) + R$, where R is the remainder.

b Divide $P(x) = 2x^3 - x^2 + 3x - 1$ by $(x + 3)$ and write in the form $P(x) = (x + 3)Q(x) + R$.

SOLUTION

a

$$\begin{array}{rr}
 & x^2 + 4x + 7 \\
x - 2\,\big) & \overline{x^3 + 2x^2 - x + 3} \\
x^2(x-2) & \underline{x^3 - 2x^2 \qquad\qquad} \\
 & 4x^2 - x + 3 \\
4x(x-2) & \underline{4x^2 - 8x \qquad} \\
 & 7x + 3 \\
7(x-2) & \underline{7x - 14} \\
 & 17
\end{array}$$

$\therefore x^3 + 2x^2 - x + 3 = (x - 2)(x^2 + 4x + 7) + 17$

EXPLANATION

First, divide x from $(x - 2)$ into the leading term (i.e. x^3). So divide x into x^3 to give x^2.

$x^2(x - 2)$ gives $x^3 - 2x^2$ and subtract from $x^3 + 2x^2 - x + 3$.

After subtraction, divide x into $4x^2$ to give $4x$ and repeat the process above.

After subtraction, divide x into $7x$ to give 7.

Subtract to give the remainder 17.

SOLUTION

b

$$\begin{array}{rr}
 & 2x^2 - 7x + 24 \\
x + 3\,\big) & \overline{2x^3 - x^2 + 3x - 1} \\
2x^2(x+3) & \underline{2x^3 + 6x^2 \qquad\qquad} \\
 & -7x^2 + 3x - 1 \\
-7x(x+3) & \underline{-7x^2 - 21x \qquad} \\
 & 24x - 1 \\
24(x+3) & \underline{24x + 72} \\
 & -73
\end{array}$$

$\therefore 2x^3 - x^2 + 3x - 1 = (x + 3)(2x^2 - 7x + 24) - 73$

EXPLANATION

First, divide x from $(x + 3)$ into the leading term. So divide x into $2x^3$ to give $2x^2$.

After subtraction, divide x into $-7x^2$ to give $-7x$.

After subtraction, divide x into $24x$ to give 24.

Subtract to give the remainder -73.

Now you try

a Divide $P(x) = x^3 + x^2 - 4x + 3$ by $(x - 1)$ and write in the form $P(x) = (x - 1)Q(x) + R$, where R is the remainder.

b Divide $P(x) = 3x^3 - 2x^2 + 5x - 2$ by $(x + 2)$ and write in the form $P(x) = (x + 2)Q(x) + R$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 10F

FLUENCY	1–3	2, 3, 4(1/2)	2, 3, 4(1/3)

Example 12a **1** **a** Divide $P(x) = x^3 + x^2 - 3x + 2$ by $(x - 1)$ and write in the form $P(x) = (x - 1)Q(x) + R$, where R is the remainder.

Example 12b **b** Divide $P(x) = 2x^3 - x^2 + 4x - 2$ by $(x + 2)$ and write in the form $P(x) = (x + 2)(Q(x)) + R$.

Example 12a **2** Divide $P(x) = x^3 + x^2 - 2x + 3$ by $(x - 1)$ and write in the form $P(x) = (x - 1)Q(x) + R$, where R is the remainder.

Example 12b **3** Divide $P(x) = 3x^3 - x^2 + x + 2$ by $(x + 1)$ and write in the form $P(x) = (x + 1)Q(x) + R$, where R is the remainder.

4 For each of the following, express in this form:
Dividend = divisor × quotient + remainder (as in the examples)

- **a** $(2x^3 - x^2 + 3x - 2) \div (x - 2)$
- **b** $(2x^3 + 2x^2 - x - 3) \div (x + 2)$
- **c** $(5x^3 - 2x^2 + 7x - 1) \div (x + 3)$
- **d** $(-x^3 + x^2 - 10x + 4) \div (x - 4)$
- **e** $(-2x^3 - 2x^2 - 5x + 7) \div (x + 4)$
- **f** $(-5x^3 + 11x^2 - 2x - 20) \div (x - 3)$

PROBLEM-SOLVING	5	5, 6(1/2)	5, 6(1/2)

5 Divide and write in this form:
Dividend = divisor × quotient + remainder

- **a** $(6x^4 - x^3 + 2x^2 - x + 2) \div (x - 3)$
- **b** $(8x^5 - 2x^4 + 3x^3 - x^2 - 4x - 6) \div (x + 1)$

6 Divide the following and express in the usual form.

- **a** $(x^3 - x + 1) \div (x + 2)$
- **b** $(x^3 + x^2 - 3) \div (x - 1)$
- **c** $(x^4 - 2) \div (x + 3)$
- **d** $(x^4 - x^2) \div (x - 4)$

REASONING	7	7, 8	8–10

7 There are three values of k for which $P(x) = x^3 - 2x^2 - x + 2$ divided by $(x - k)$ gives a remainder of zero. Find the three values of k.

8 Prove that $(6x^3 - 37x^2 + 32x + 15) \div (x - 5)$ leaves remainder 0.

9 Find the remainder when $P(x)$ is divided by $(2x - 1)$ given that:

- **a** $P(x) = 2x^3 - x^2 + 4x + 2$
- **b** $P(x) = -3x^3 + 2x^2 - 7x + 5$

10 Find the remainder when $P(x) = -3x^4 - x^3 - 2x^2 - x - 1$ is divided by these expressions.

- **a** $x - 1$
- **b** $2x + 3$
- **c** $-3x - 2$

ENRICHMENT: When the remainder is not a constant	–	–	11

11 Divide the following and express in the form $P(x) =$ divisor $\times Q(x) + R$, where R is a function of x.

- **a** $(x^3 - x^2 + 3x + 2) \div (x^2 - 1)$
- **b** $(2x^3 + x^2 - 5x - 1) \div (x^2 + 3)$
- **c** $(5x^4 - x^2 + 2) \div (x^3 - 2)$

Applications and problem-solving

The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

Accumulating ants

1 When worker ants look for food they leave a scent along their path so that other ants can find the food source. This can lead to ants accumulating quickly in an area away from their nest, like around small crumbs they find in a household kitchen.

Scientists interested in the growth of the population of ants use exponential relations to describe this behaviour. They will use rules to predict ant numbers and model the population of ants by constructing suitable equations.

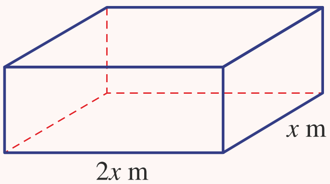
- a A rule for a population, P, of ants which has found some food in a kitchen pantry is given by $P = 10 \times 2^{2t}$ where t is in hours after the food is first found.
  - i What was the initial number of ants in the pantry when the food is first found?
  - ii How many ants were in the pantry 2 hours after the food was found?
  - iii After how many hours did the ant population reach 1000? Use logarithms and round to one decimal place.
  - iv By what factor does the population increase each hour according to this rule?
- b Another group of ants has found the cat food in the laundry. The rule for the growth of this population of ants is given by $P = P_0 \times 3^{2t}$ where t is in hours and P_0 is the initial number of ants that found the cat food.
  - i Give a rule for t in terms of P and P_0 using logarithms with the given base.
  - ii Use your rule to find the number of hours it takes for the initial ant population to triple.
- c A general ant population model around a food source is given by $P = P_0 \times a^{bt}$, where a and b are constants.
  - i $\text{Log}_a P$ can be expressed in the form $\log_a P = mt + c$. Find m and c in terms of P_0, a and b and write the rule for $\log_a P$.
  - ii What does this tell you about the graph of $\log_a P$ against t?

Rabbit hutch design

2 Parents are designing a rectangular rabbit hutch for their daughter's rabbit.

The parents wish to use a fixed amount of material to construct a special type of hutch and try to maximise its volume. They also want to consider changing the design and the amount of material to maintain a fixed volume and improve the overall conditions for the rabbit.

- a They initially have 12 m of wood to make the edges of the frame, the sides of which they will then cover with chicken wire. The base area will be twice as long as it is wide as shown.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

- **i** Find an expression for the allowable height of the hutch in terms of x.
- **ii** Hence, give a rule for the volume, $V(x)$ m^3, of the hutch in expanded form and state the possible values of x.
- **iii** Use technology to find the maximum possible volume of this hutch and the dimensions that give this volume. Round values to one decimal place.
- **iv** Repeat parts **i** and **ii** for p metres of wood to find a rule for the volume in terms of x and p.

b The parents have decided they may need to purchase extra wood to give the rabbit the space it needs. They redraw their original design so that the base dimensions are $(x + 10)$ cm and $(x + 80)$ cm. The volume of their hutch, in cm^3, is given by $V(x) = x^3 + 60x^2 - 1900x - 24000$.

- **i** Determine the height, in cm, of the design in terms of x, using division.
- **ii** If the parents settle on dimensions that use an x-value of 90, how many extra metres of wood will they need?

Increasing sound

3 Sound is measured in decibels, dB, with rule given by $d = 10\log_{10}\dfrac{P}{P_0}$ where P is the power or intensity of the sound measured in watts/cm^2 and P_0 is the weakest sound that the human ear can hear in watts/cm^2.

You will investigate the relationship between decibels and sound intensity for a human ear under certain conditions. You will consider properties of sound in common situations including rock concerts, street noise and home speakers.

a Use the rule with $P_0 = 10^{-16}$ to find:

- **i** the sound in decibels when P is 10^{-4}, the maximum intensity the human ear can tolerate before experiencing pain
- **ii** the intensity of sound at a rock concert when the sound is recorded as 100 decibels
- **iii** what 0 decibels represents in terms of the power P.

b A sound is being recorded at different time intervals in a suburban street. Over the course of the day the sound ranges from 50 dB to 70 dB. Use $P_0 = 10^{-16}$.

- **i** Find the range of the intensity P throughout the day.
- **ii** Describe the change in the intensity range compared to the change in decibel range.

c Two speakers emit sound intensity power of P_1 and P_2 where $P_2 > P_1$.

- **i** Give a simplified rule, c, for the difference in decibels between the two speakers.
- **ii** If speaker 2 emits twice the power of speaker 1, what is the difference in decibels to the nearest decibel?
- **iii** Use your rule for c to complete the table on the right.

Speaker 2 power compared to speaker 1	$\times 10$	$\times 100$	$\times 1000$
Change in decibels, c			

- **iv** Hence, if the increase in intensity is by a factor of 10^n, give a rule for the change in decibels c in terms of n.
- **v** Using your rule what is the difference in decibels for a sound that is 100 000 times more intense?

10G The remainder and factor theorems 10A

Learning intentions
- To know how to use the remainder theorem to find the remainder when a polynomial is divided by a linear expression
- To understand that a remainder of zero implies that a divisor is a factor
- To be able to use the factor theorem to decide if a linear expression is a factor of a polynomial

Using long division we can show, after dividing $(x-2)$ into $P(x) = x^3 - x^2 + 2x - 3$, that $P(x) = x^3 - x^2 + 2x - 3 = (x-2)(x^2 + x + 4) + 5$, where 5 is the remainder.

The polynomial remainder theorem is used in algorithms for detecting and correcting errors that can occur in digital data transmissions, such as from GPS satellites to a phone. Polynomials are exchanged using codes formed from coefficients.

Using the right-hand side to evaluate $P(2)$, we have:

$$\begin{aligned} P(2) &= (2-2)(2^2 + 2 + 4) + 5 \\ &= 0 \times (2^2 + 2 + 4) + 5 \\ &= 0 + 5 \\ &= 5 \end{aligned}$$

This shows that the remainder when $P(x)$ is divided by $(x-2)$ is $P(2)$.

More generally, when $P(x)$ is divided by $(x-a)$ we obtain:

$$P(x) = (x-a)Q(x) + R$$

So $$P(a) = 0 \times Q(a) + R = R$$

So the remainder is $P(a)$ and this result is called the remainder theorem. This means that we can find the remainder when dividing $P(x)$ by $(x-a)$ simply by evaluating $P(a)$.

We also know that when factors are divided into a number there is zero remainder. So if $P(x)$ is divided by $(x-a)$ and the remainder $P(a)$ is zero, then $(x-a)$ is a factor of $P(x)$. This result is called the factor theorem.

LESSON STARTER Which way is quicker?

A polynomial $P(x) = x^3 - 3x^2 + 6x - 4$ is divided by $(x-2)$.

- Show, using long division, that the remainder is 4.
- Find $P(2)$. What do you notice?
- Explain how you can find the remainder when $P(x)$ is divided by:
  a $x - 3$ **b** $x - 5$
- Show that when $P(x)$ is divided by $(x+1)$ the remainder is -14.
- What would be the remainder when $P(x)$ is divided by $(x-1)$? What do you notice and what does this say about $(x-1)$ in relation to $P(x)$?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

KEY IDEAS

- **Remainder theorem**: When a polynomial $P(x)$ is divided by $(x - a)$ the remainder is $P(a)$.
  - When dividing by $(x - 3)$ the remainder is $P(3)$.
  - When dividing by $(x + 2)$ the remainder is $P(-2)$.
- **Factor theorem**: When $P(x)$ is divided by $(x - a)$ and the remainder is zero (i.e. $P(a) = 0$), then $(x - a)$ is a factor of $P(x)$.

$$\begin{aligned} P(x) &= x^3 - 3x^2 - 3x + 10 \\ &= \underbrace{(x-2)}_{\text{factor}}\underbrace{(x^2 - x - 5)}_{\text{quotient}} \end{aligned}$$

$P(2) = 0$

$(x - 2)$ is a factor with zero remainder.

BUILDING UNDERSTANDING

1. If $P(x) = 2x^3 - x^2 - x - 1$, find the value of the following.
   a $P(1)$ b $P(3)$ c $P(-2)$ d $P(-4)$
2. What value of x do you substitute into $P(x)$ to find the remainder when a polynomial $P(x)$ is divided by:
   a $x - 3$? b $x + 2$?
3. What is the remainder when an expression is divided by one of its factors?

Example 13 Using the remainder theorem

Find the remainder when $P(x) = x^3 - 5x^2 - x + 4$ is divided by:

a $x - 2$ b $x + 1$

SOLUTION

a
$$\begin{aligned} P(x) &= x^3 - 5x^2 - x + 4 \\ P(2) &= (2)^3 - 5(2)^2 - 2 + 4 \\ &= 8 - 20 - 2 + 4 \\ &= -10 \end{aligned}$$
The remainder is -10.

EXPLANATION

For $(x - 2)$ substitute $x = 2$.

Using the remainder theorem, $P(2)$ gives the remainder.

SOLUTION

b
$$\begin{aligned} P(x) &= x^3 - 5x^2 - x + 4 \\ P(-1) &= -1 - 5 + 1 + 4 \\ &= -1 \end{aligned}$$
The remainder is -1.

EXPLANATION

For $(x + 1)$ substitute $x = -1$.

Note: $(-1)^3 = -1$, $(-1)^2 = 1$ and $-(-1) = 1$.

Now you try

Find the remainder when $P(x) = x^3 - 4x^2 + 6x - 1$ is divided by:

a $x - 1$ b $x + 2$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 14 Finding a linear factor

Decide whether each of the following is a factor of $P(x) = x^3 + x^2 - 3x - 6$.

a $x + 1$ **b** $x - 2$

SOLUTION

a $P(x) = x^3 + x^2 - 3x - 6$

$P(-1) = -1 + 1 + 3 - 6$

$= -3$

$\therefore (x + 1)$ is not a factor.

b $P(x) = x^3 + x^2 - 3x - 6$

$P(2) = 8 + 4 - 6 - 6$

$= 0$

$\therefore (x - 2)$ is a factor.

EXPLANATION

If $(x + 1)$ is a factor of $P(x)$, then $P(-1) = 0$.
This is not true as the remainder is -3.

Substitute $x = 2$ to evaluate $P(2)$.
Since $P(2) = 0$, $(x - 2)$ is a factor of $P(x)$.

Now you try

Decide whether each of the following is a factor of $P(x) = 2x^3 - 3x^2 - 11x + 6$.

a $x + 1$ **b** $x - 3$

Example 15 Applying the remainder theorem

Find the value of k such that $(x^3 - x^2 + 2x + k) \div (x - 1)$ has a remainder of 5.

SOLUTION

Let $P(x) = x^3 - x^2 + 2x + k$.

$P(1) = 5$

$(1)^3 - (1)^2 + 2(1) + k = 5$

$2 + k = 5$

$k = 3$

EXPLANATION

The remainder is $P(1)$, which is 5.
Substitute $x = 1$ and solve for k.

Now you try

Find the value of k such that $(x^3 + 2x^2 - x + k) \div (x - 2)$ has a remainder of 12.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 10G

FLUENCY	1, 2–4(1/2)	2–5(1/2)	2–5(1/2)

Example 13

1 Find the remainder when $P(x) = x^3 - 2x^2 + 3x - 1$ is divided by:
a $x - 2$ **b** $x + 1$ **c** $x + 3$

Example 13

2 Find the remainder when $P(x) = x^3 - 2x^2 + 7x - 3$ is divided by:
a $x - 1$ **b** $x - 2$ **c** $x - 3$ **d** $x - 4$
e $x + 4$ **f** $x + 2$ **g** $x + 1$ **h** $x + 3$

3 Find the remainder when $P(x) = x^4 - x^3 + 3x^2$ is divided by:
a $x - 1$ **b** $x - 2$ **c** $x + 2$ **d** $x + 1$

Example 14

4 Decide which of the following are factors of $P(x) = x^3 - 4x^2 + x + 6$.
a $x - 1$ **b** $x + 1$ **c** $x - 2$ **d** $x + 2$
e $x - 3$ **f** $x + 3$ **g** $x - 4$ **h** $x + 4$

5 Decide which of the following are factors of $P(x) = x^4 - 2x^3 - 25x^2 + 26x + 120$.
a $x - 2$ **b** $x + 2$ **c** $x + 3$ **d** $x - 3$
e $x - 4$ **f** $x + 4$ **g** $x - 5$ **h** $x + 5$

PROBLEM-SOLVING	6(1/2)	6–7(1/2)	6–7(1/2)

6 Use the factor theorem and trial and error to find a linear factor of these polynomials.
a $P(x) = x^3 + 2x^2 + 7x + 6$ **b** $P(x) = x^3 + 2x^2 - x - 2$
c $P(x) = x^3 + x^2 + x + 6$ **d** $P(x) = x^3 - 2x - 4$

7 Use the factor theorem to find all three linear factors of these polynomials.
a $P(x) = x^3 - 2x^2 - x + 2$ **b** $P(x) = x^3 - 2x^2 - 5x + 6$
c $P(x) = x^3 - 4x^2 + x + 6$ **d** $P(x) = x^3 - 2x^2 - 19x + 20$

REASONING	8	8, 9, 10(1/2)	9–11

Example 15

8 For what value of k will $(x^3 - 2x^2 + 5x + k) \div (x - 1)$ have the following remainders?
a 0 **b** 2 **c** -10 **d** 100

9 For what value of k will $(x^4 - 2x^3 + x^2 - x + k) \div (x + 2)$ have zero remainder?

10 Find the value of k in these polynomials.
a $P(x) = x^3 + 2x^2 + kx - 4$ and when divided by $(x - 1)$ the remainder is 4.
b $P(x) = x^3 - x^2 + kx - 3$ and when divided by $(x + 1)$ the remainder is -6.
c $P(x) = 2x^3 + kx^2 + 3x - 4$ and when divided by $(x + 2)$ the remainder is -6.
d $P(x) = kx^3 + 7x^2 - x - 4$ and when divided by $(x - 2)$ the remainder is -2.

11 Find the value of k when:
a $(x + 2)$ is a factor of $x^3 - kx^2 - 2x - 4$ **b** $(x - 3)$ is a factor of $2x^3 + 2x^2 - kx - 3$

ENRICHMENT: Simultaneous coefficients	–	–	12

12 Use simultaneous equations and the given information to find the value of a and b in these cubics.
a $P(x) = x^3 + ax^2 + bx - 3$ and $P(1) = -1$ and $P(2) = 5$
b $P(x) = 2x^3 - ax^2 - bx - 1$ and $P(-1) = -10$ and $P(-2) = -37$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

10H Solving polynomial equations 10A

Learning intentions
- To know how to find a factor of a polynomial using the factor theorem
- To be able to factorise a polynomial using division by a known factor
- To be able to apply the Null Factor Law to solve a polynomial equation in factorised form

We know from our work with quadratics that the Null Factor Law can be used to solve a quadratic equation in factorised form.

For example:
$$x^2 - 3x - 40 = 0$$
$$(x-8)(x+5) = 0$$

Using the Null Factor Law:

$$x - 8 = 0 \quad \text{or} \quad x + 5 = 0$$
$$x = 8 \quad \text{or} \quad x = -5$$

We can also apply this method to solve higher degree polynomials.

Solving complex, realistic polynomial equations occurs in civil, aerospace, electrical, industrial and mechanical engineering. Architects apply polynomial modelling to solve 3D structural problems, such as the curved supports in the Disney Concert Hall, Los Angeles.

If a polynomial is not in a factorised form, we use the remainder and factor theorems to help find its factors. Long division can also be used in this process.

LESSON STARTER Solving a cubic

Consider the cubic equation $P(x) = 0$, where $P(x) = x^3 + 6x^2 + 5x - 12$.

- Explain why $(x - 1)$ is a factor of $P(x)$.
- Use long division to find $P(x) \div (x - 1)$.
- Write $P(x)$ in the form $(x - 1)Q(x)$.
- Now complete the factorisation of $P(x)$.
- Show how the Null Factor Law can be used to solve $P(x) = 0$. Why are there three solutions?

KEY IDEAS

- A **polynomial equation** of the form $P(x) = 0$ can be solved by:
  - factorising $P(x)$
  - using the Null Factor law: If $a \times b \times c = 0$ then $a = 0$, $b = 0$ or $c = 0$.
- To factorise a polynomial follow these steps.
  - Find one factor using the remainder and factor theorems. Start with $(x - 1)$ using $P(1)$ or $(x + 1)$ using $P(-1)$. If required, move to $(x - 2)$ or $(x + 2)$ etc.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

- A good idea is to first consider factors of the constant term of the polynomial to reduce the number of trials.
- Use long division to find the quotient after dividing by the factor.
- Factorise the quotient (if possible).
- Continue until $P(x)$ is fully factorised.

BUILDING UNDERSTANDING

1. Give a reason why $(x + 1)$ is a factor of $P(x) = x^3 - 7x - 6$. (*Hint*: Find $P(-1)$.)
2. Use the Null Factor Law to solve these quadratic equations.
   a $(x - 1)(x + 3) = 0$
   b $x^2 - x - 12 = 0$

Example 16 Using the Null Factor Law

Solve for x.

a $(x - 1)(x + 2)(x + 5) = 0$

b $(2x - 3)(x + 7)(3x + 1) = 0$

SOLUTION

a $(x - 1)(x + 2)(x + 5) = 0$

$x - 1 = 0$ or $x + 2 = 0$ or $x + 5 = 0$

$x = 1$ $\quad x = -2$ $\quad x = -5$

EXPLANATION

Using the Null Factor Law, if $a \times b \times c = 0$ then $a = 0$ or $b = 0$ or $c = 0$.

SOLUTION

b $(2x - 3)(x + 7)(3x + 1) = 0$

$2x - 3 = 0$ or $x + 7 = 0$ or $3x + 1 = 0$

$2x = 3$ $\quad x = -7$ $\quad 3x = -1$

$x = \frac{3}{2}$ $\quad x = -7$ $\quad x = -\frac{1}{3}$

EXPLANATION

Equate each factor to 0 and solve for the three values of x.

Now you try

Solve for x.

a $(x - 2)(x + 1)(x + 6) = 0$

b $(2x - 1)(x + 3)(5x + 2) = 0$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 17 Factorising and solving

Solve $x^3 + 2x^2 - 5x - 6 = 0$.

SOLUTION

Let $P(x) = x^3 + 2x^2 - 5x - 6$.

$P(1) = 1 + 2 - 5 - 6 \neq 0$

$P(-1) = -1 + 2 + 5 - 6 = 0$

$\therefore x + 1$ is a factor.

$$\begin{array}{rr} & x^2 + x - 6 \\ x+1 \, \big) & \overline{x^3 + 2x^2 - 5x - 6} \\ x^2(x+1) & \underline{x^3 + x^2 \qquad\qquad} \\ & x^2 - 5x - 6 \\ x(x+1) & \underline{x^2 + x \qquad} \\ & -6x - 6 \\ -6(x+1) & \underline{-6x - 6} \\ & 0 \end{array}$$

$\therefore P(x) = (x + 1)(x^2 + x - 6)$

$= (x + 1)(x + 3)(x - 2)$

Solve $P(x) = 0$:

$(x + 1)(x + 3)(x - 2) = 0$

$x + 1 = 0$ or $x + 3 = 0$ or $x - 2 = 0$

$x = -1$ $\quad x = -3$ $\quad x = 2$

EXPLANATION

Try to find a factor using the remainder and factor theorems. Start with $(x - 1)$ using $P(1)$ or $(x + 1)$ using $P(-1)$. If required, move to $(x - 2)$ or $(x + 2)$ or others using factors of 6.

$P(-1) = 0$ so $(x + 1)$ is a factor.

Divide $(x + 1)$ into $P(x)$ to find the quotient using long division.

Note: The remainder is 0, as expected $(P(-1) = 0)$.

$P(x) = (x + 1)Q(x) + R$ but $R = 0$.

$x^2 + x - 6$ factorises to $(x + 3)(x - 2)$.

Use the Null Factor Law to now solve for x.

Now you try

Solve $x^3 + 2x^2 - 11x - 12 = 0$.

Exercise 10H

FLUENCY	1, 2–3(1/2)	2–3(1/2)	2–3(1/3)

1 Solve for x.

Example 16a
a **i** $(x + 1)(x - 4)(x - 2) = 0$ **ii** $(x + 5)(x - 7)(x + 3) = 0$

Example 16b
b **i** $(2x - 5)(x + 4)(3x + 2)$ **ii** $(5x + 1)(3x - 8)(x - 2) = 0$

Example 16
2 Solve for x using the Null Factor Law.

a $(x + 3)(x - 2)(x - 1) = 0$

b $(x + 2)(x + 7)(x - 1) = 0$

c $(x - 4)(x + 4)(x - 3) = 0$

d $\left(x + \frac{1}{2}\right)(x - 3)\left(x + \frac{1}{3}\right) = 0$

e $(2x + 1)(x - 3)(3x + 2) = 0$

f $(4x - 1)(5x - 2)(7x + 2) = 0$

g $\left(x + \frac{1}{2}\right)(3x + 11)(11x + 12) = 0$

h $(5x + 3)(19x + 2)\left(x - \frac{1}{2}\right) = 0$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 17

3 For each of the following cubic equations, follow these steps as in **Example 17**.

- Use the factor theorem to find a factor.
- Use long division to find the quotient.
- Factorise the quotient.
- Write the polynomial in a fully factorised form.
- Use the Null Factor Law to solve for x.

a $x^3 - 4x^2 + x + 6 = 0$ **b** $x^3 + 6x^2 + 11x + 6 = 0$
c $x^3 - 6x^2 + 11x - 6 = 0$ **d** $x^3 - 8x^2 + 19x - 12 = 0$
e $x^3 - 3x^2 - 16x - 12 = 0$ **f** $x^3 + 6x^2 - x - 30 = 0$

PROBLEM-SOLVING	4	4, 5	5, 6

4 Use the quadratic formula to solve for x, expressing your answers in exact form.
a $(x - 1)(x^2 - 2x - 4) = 0$ **b** $(x + 2)(x^2 + 6x + 10) = 0$

5 Solve by first taking out a common factor.
a $2x^3 - 14x^2 + 14x + 30 = 0$ **b** $3x^3 + 12x^2 + 3x - 18 = 0$

6 Solve for x.
a $x^3 - 13x + 12 = 0$ **b** $x^3 - 7x - 6 = 0$

REASONING	7	7, 8	7, 8(½), 9, 10

7 State the maximum number of solutions to $P(x) = 0$ when $P(x)$ is of degree:
a 3 **b** 4 **c** n

8 Show that the following equations can be factorised easily without the use of long division, and then give the solutions.
a $x^3 - x^2 = 0$ **b** $x^3 + x^2 = 0$
c $x^3 - x^2 - 12x = 0$ **d** $2x^5 + 4x^4 + 2x^3 = 0$

9 Explain why $x^4 + x^2 = 0$ has only one solution.

10 Explain why $(x - 2)(x^2 - 3x + 3) = 0$ has only one solution.

ENRICHMENT: Quartics with four factors	–	–	11

11 Factorising a quartic may require two applications of the factor theorem and long division. Solve these quartics by factorising the left-hand side first.
a $x^4 + 8x^3 + 17x^2 - 2x - 24 = 0$ **b** $x^4 - 2x^3 - 11x^2 + 12x + 36 = 0$
c $x^4 + x^3 - 11x^2 - 9x + 18 = 0$ **d** $2x^4 - 3x^3 - 7x^2 + 12x - 4 = 0$

ISBN 978-1-108-77290-7

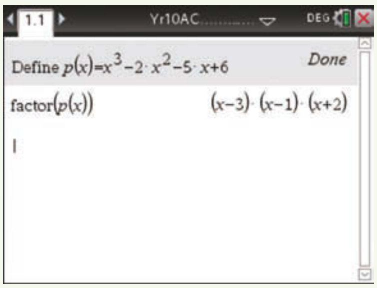
Cambridge University Press
Updated June 2022

Using calculators to factorise and solve polynomials

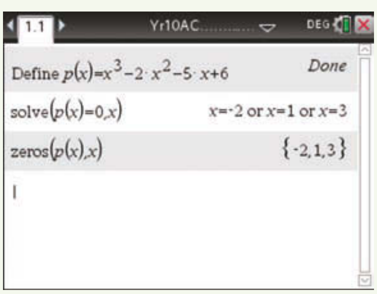
1. Define the polynomial $P(x) = x^3 - 2x^2 - 5x + 6$ and factorise.
2. Solve $x^3 - 2x^2 - 5x + 6 = 0$.

Using the TI-Nspire:

1. In a **Calculator** page define the polynomial using menu **>Actions>Define**. Factor the polynomial using menu **>Algebra>Factor** as shown.

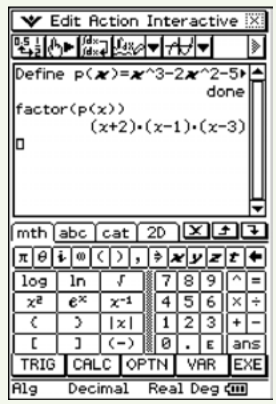


2. Solve using menu **>Algebra>Solve**. Then type $p(x) = 0$, x as shown.
Alternatively, use Solve using menu **>Algebra>Zeros** for solving equations equalling zero.



Using the ClassPad:

1. In the **Main** application, type and highlight the polynomial. Tap **Interactive**, **Define**. Type p for function name and tap *OK*. Type $p(x)$ in the next entry line. Highlight and tap **Interactive**, **Transformation**, **factor**, **factor**.



2. Type and highlight $p(x) = 0$, Tap **Interactive**, **Advanced**, **Solve**.

ISBN 978-1-108-77290-7

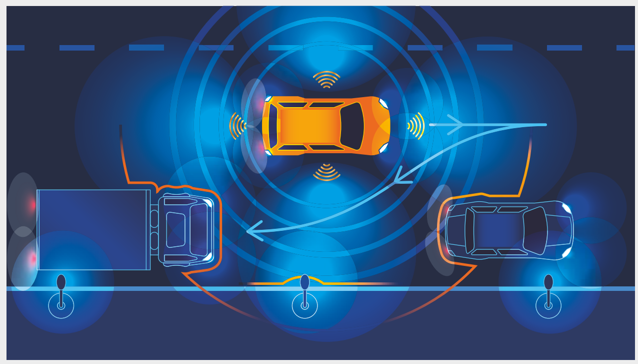
Cambridge University Press
Updated June 2022

10I Graphs of polynomials 10A

Learning intentions

- To know the shape of the graphs of the basic polynomials $y = x^3$ and $y = x^4$
- To know the shape of a graph of a cubic polynomial with three different factors
- To be able to find the axis intercepts of a cubic graph
- To know how to use the shape and intercepts to sketch a cubic graph

So far in Year 10 we have studied graphs of linear equations (straight lines) and graphs of quadratic equations (parabolas). We have also looked at graphs of circles, exponentials and hyperbolas. In this section we introduce the graphs of polynomials by focusing on those of degree 3 and 4. We start by considering the basic cubic $y = x^3$ and quartic $y = x^4$, and then explore other cubics and quartics in factorised form.

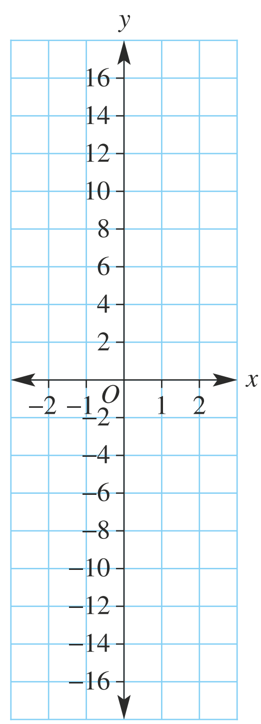


A spline is a curve formed by joining sections of various polynomial graphs. Splines are used in motion planning algorithms including for robots avoiding obstacles, self-driving cars parallel parking, and industrial robots' joint and hand trajectories.

LESSON STARTER Plotting $y = x^3$ and $y = x^4$

Complete the table before plotting points and considering the discussion points below.

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
$y = x^2$							
$y = x^3$							
$y = x^4$							



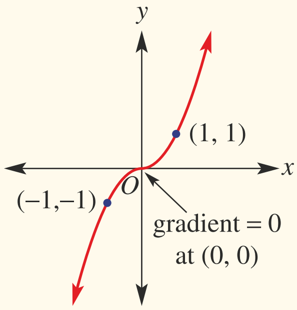
- Describe the features and shape of each graph.
- Describe the differences between the graphs of $y = x^2$ and $y = x^4$. Where do they intersect?

ISBN 978-1-108-77290-7

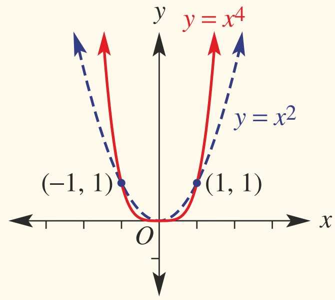
Cambridge University Press
Updated June 2022

KEY IDEAS

- Graphs of basic polynomials
  - $y = x^3$

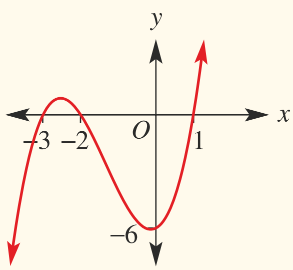


  - $y = x^4$

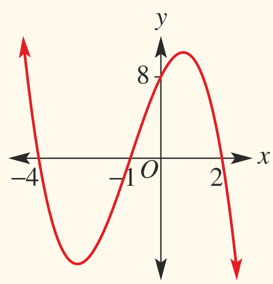


- To sketch cubic graphs in factorised form with three different factors:
  - Find the three x-intercepts using the Null Factor Law.
  - Find the y-intercept.
  - Connect points to sketch a positive or negative cubic graph.

Positive cubic
(The coefficient of x^3 is positive.)
$y = (x - 1)(x + 2)(x + 3)$



Negative cubic
(The coefficient of x^3 is negative.)
$y = -(x + 4)(x - 2)(x + 1)$

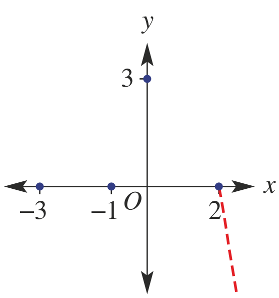


- Further consideration is needed to find turning points of cubics, as they are not located symmetrically between x-intercepts. This will be studied at more senior levels of mathematics.

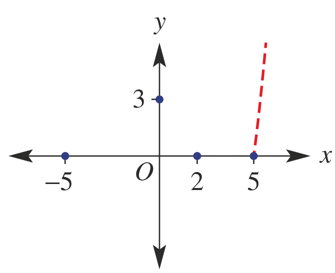
BUILDING UNDERSTANDING

1 Join the given x- and y-intercepts to form a smooth cubic curve. Each graph has been started for you on the right-hand side.

a



b



2 Find the x- and y-intercepts of the graphs of these cubics.

a $y = (x + 1)(x - 3)(x - 4)$

b $y = -2x(x + 7)(x - 5)$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 18 Sketching cubic graphs

Sketch the graphs of the following by finding the x- and y-intercepts.

a $y = (x + 2)(x - 1)(x - 3)$

b $y = -x(x + 3)(x - 2)$

SOLUTION

a $y = (x + 2)(x - 1)(x - 3)$

y-intercept at $x = 0$:

$y = (2)(-1)(-3)$

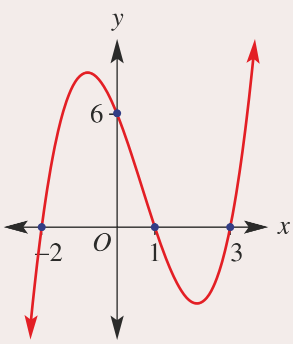
$= 6$

x-intercepts at $y = 0$:

$0 = (x + 2)(x - 1)(x - 3)$

$\therefore x + 2 = 0$ or $x - 1 = 0$ or $x - 3 = 0$

$x = -2$ $\quad x = 1$ $\quad x = 3$



b $y = -x(x + 3)(x - 2)$

y-intercept at $x = 0$:

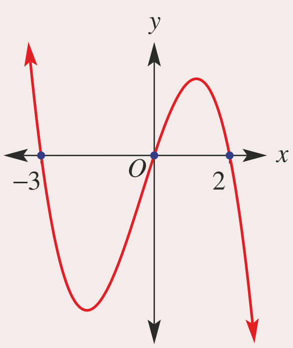
$y = -0(3)(-2) = 0$

x-intercepts at $y = 0$:

$0 = -x(x + 3)(x - 2)$

$\therefore -x = 0$ or $x + 3 = 0$ or $x - 2 = 0$

$x = 0$ $\quad x = -3$ $\quad x = 2$



EXPLANATION

Substitute $x = 0$ to find the y-intercept.

Substitute $y = 0$ to find the x-intercepts. Use the Null Factor Law.

Mark the four intercepts and connect to form a positive cubic graph. The coefficient of x^3 in the expansion of y is positive, so the graph points upwards to the right.

Find the y-intercept using $x = 0$.

The three factors are $-x$, $x + 3$ and $x - 2$.

The coefficient of x^3 in the expansion of y is negative, so the graph points downwards at the right.

Continued on next page

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Now you try

Sketch the graphs of the following by finding the x- and y-intercepts.

a $y = (x + 3)(x + 1)(x - 2)$ **b** $y = -x(x + 4)(x - 1)$

Exercise 10I

FLUENCY	1, 2(½), 3	2(½), 3	2(⅓), 3

Example 18

1 Sketch the graphs of the following by finding the x- and y-intercepts.

a $y = (x + 3)(x - 2)(x - 5)$ **b** $y = -x(x + 1)(x - 3)$

Example 18

2 Sketch the graphs of the following by finding x- and y-intercepts.

a $y = (x + 2)(x - 1)(x - 3)$ **b** $y = (x - 3)(x - 4)(x + 1)$

c $y = (x - 5)(x - 1)(x + 2)$ **d** $y = \frac{1}{2}(x + 3)(x - 2)(x - 1)$

e $y = x(x - 2)(x + 3)$ **f** $y = x(x - 5)(x + 1)$

g $y = -2x(x - 1)(x + 3)$ **h** $y = -\frac{1}{3}x(x + 1)(x - 3)$

i $y = -(x + 2)(x + 4)(x - 1)$ **j** $y = -(x + 3)\left(x - \frac{1}{2}\right)(x + 1)$

3 Sketch $y = x^2$, $y = x^3$ and $y = x^4$ on the same set of axes.

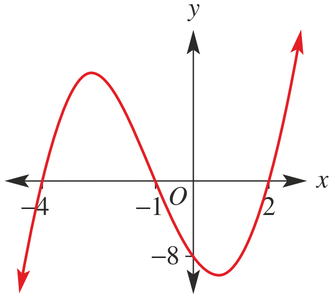
PROBLEM-SOLVING	4, 5	4, 5	4–6

4 Sketch the graph of:

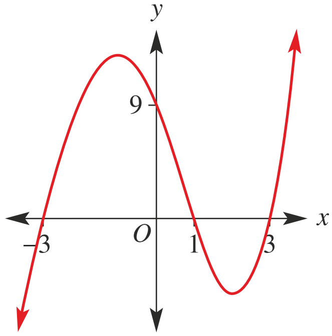
a $y = -x^3$ **b** $y = -x^4$

5 Find a cubic rule for these graphs.

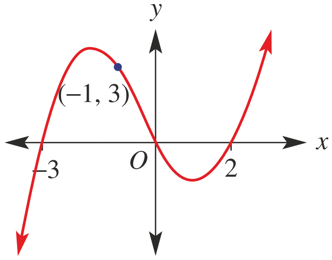
a



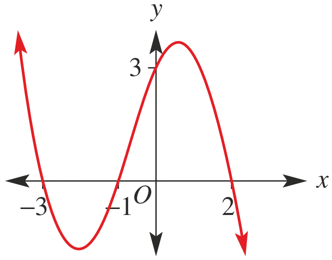
b



c



d



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

6 Sketch these quartics, making use of the Null Factor Law for x-intercepts.

a $y = (x - 5)(x - 3)(x + 1)(x + 2)$

b $y = -x(x + 4)(x + 1)(x - 4)$

REASONING	7	7–8(½)	7–8(½)

7 We know that the graph of $y = (x - 2)^2 - 1$ is the graph $y = x^2$ translated 2 units to the right and 1 unit down. Use this idea to sketch graphs of the following.

a $y = (x - 2)^3 - 1$

b $y = (x + 2)^3$

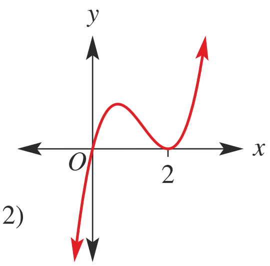
c $y = x^3 - 2$

d $y = x^4 - 1$

e $y = (x + 3)^4$

f $y = (x - 2)^4 - 3$

8 If a polynomial has a repeated factor $(x - a)$, then the point at $x = a$ is an x-intercept and also a turning point; e.g. $y = x(x - 2)^2$ as shown.



Now sketch these polynomials.

a $y = x(x - 3)^2$

b $y = -2x(x + 1)^2$

c $y = -(x + 2)^2(x - 3)$

d $y = (x + 4)(x + 1)^2$

e $y = (2 - x)(x + 1)^2$

f $y = -x^2(x + 2)(x - 2)$

ENRICHMENT: Polynomial with the lot	–	–	9

9 To sketch a graph of a polynomial that is not in factorised form you must factorise the polynomial to help find the x-intercepts.

Complete the following for each polynomial.

i Find the y-intercept.

ii Factorise the polynomial using the factor theorem and long division.

iii Find the x-intercepts.

iv Sketch the graph.

a $y = x^3 + 4x^2 + x - 6$

b $y = x^3 - 7x^2 + 7x + 15$

c $y = x^4 + 2x^3 - 9x^2 - 2x + 8$

d $y = x^4 - 34x^2 + 225$

ISBN 978-1-108-77290-7

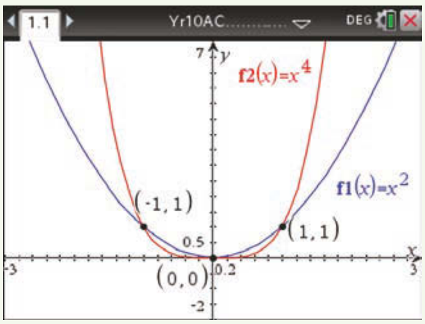
Cambridge University Press
Updated June 2022

Using calculators to sketch polynomials

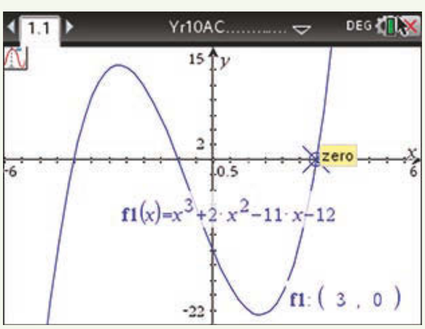
1. Sketch the graphs of $y = x^2$ and $y = x^4$ on the same set of axes and find the intersection points.
2. Sketch $P(x) = x^3 + 2x^2 - 11x - 12$ and find the x-intercepts.

Using the TI-Nspire:

1. In a **Graphs** page, enter the rules $y = x^2$ and $y = x^4$. Adjust the scale using **Window Settings** and find their intersection points using menu **>Analyze >Intersection**. Alternatively use menu **>Geometry>Points & Lines >Intersection Point(s)** to display all three intersections simultaneously as shown.

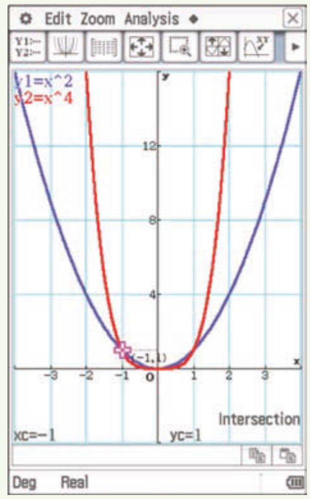


2. Enter the rule $P(x) = x^3 + 2x^2 - 11x - 12$. Adjust the scale using **Window Settings**. Find the x-intercepts using **Trace>Graph Trace** or using **Analyze Graph>Zero** and set the lower and upper bounds by scrolling left and right.

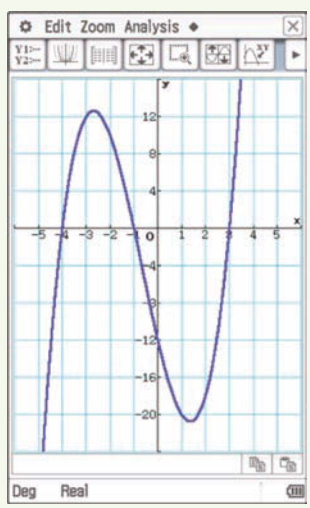


Using the ClassPad:

1. In the **Graph&Table** application, enter the rules $y1 = x^2$ and $y2 = x^4$. Tap to adjust the scale. Tap to see the graph. Tap **Analysis**, **G-Solve**, **Intersect**.



2. Enter the rule $y1 = x^3 + 2x^2 - 11x - 12$. Tap to see the graph. Adjust the scale by tapping on . Tap **Analysis**, **G-Solve**, **root** to find x-intercepts.



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Up for a challenge? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.

1 Simplify the following without the use of a calculator.

a $2\log_3 4 - \log_3 \frac{16}{9}$

b $-\log_2 \frac{1}{4} + 3\log_2 4$

c $\log_5 \sqrt{125} + \log_3 \frac{1}{3}$

d $2\log_2 27 \div \log_2 9$

2 Solve these equations using log base 10. Round your answers to two decimal places.

a $5^{x-1} = 2$ b $0.2^x = 10$ c $2^x = 3^{x+1}$

3 Solve for x: $2\log_{10} x = \log_{10}(5x + 6)$

4 Given that $\log_a 3 = p$ and $\log_a 2 = q$, find an expression for $\log_a(4.5a^2)$.

5 Solve these inequalities using log base 10. Round your answers to two decimal places.

a $3^x > 10$ b $0.5^x \leqslant 7$

6 If $y = a \times 2^{bx}$ and the graph of y passes through $(-1, 2)$ and $(3, 6)$, find the exact values of a and b.

7 An amount of money is invested at 10% p.a., compound interest. How long will it take for the money to double? Give an exact value.

8 Find the remainder when $x^4 - 3x^3 + 6x^2 - 6x + 6$ is divided by $(x^2 + 2)$.

9 $x^3 + ax^2 + bx - 24$ is divisible by $(x + 3)$ and $(x - 2)$. Find the values of a and b.

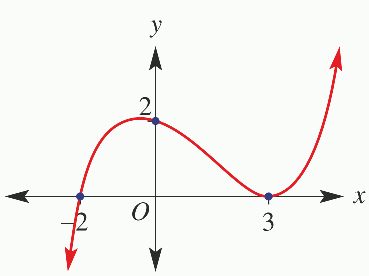
10 Prove the following, using division.

a $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$ b $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

11 Solve for x.

a $(x + 1)(x - 2)(x - 5) \leqslant 0$ b $x^3 - x^2 - 16x + 16 > 0$

12 A cubic graph has a y-intercept at 2, a turning point at $(3, 0)$ and another x-intercept at -2. Find the rule for the graph.



13 Given that $x^2 - 5x + 1 = 0$, find the value of $x^4 - 2x^3 - 16x^2 + 13x + 14$ without solving the first equation.

14 A quartic graph has a turning point at $(0, 0)$ and two x-intercepts at 3 and -3. Find the rule for the graph if it also passes through $(2, 2)$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter summary

Logarithms (10A)

Definition

$\log_a y = x$ is equivalent to $a^x = y$, $a > 0$

- $\log_2 8 = 3$ since $2^3 = 8$
- $\log_{10} 100 = 2$ since $10^2 = 100$
- $\log_3 \frac{1}{27} = -3$ since $3^{-3} = \frac{1}{27}$
- $\log_{10} 0.001 = -3$ since $10^{-3} = 0.001$

Other properties

- $\log_a 1 = 0 \quad (a^0 = 1)$
- $\log_a a = 1 \quad (a^1 = a)$
- $\log_a \frac{1}{x} = -\log_a x \quad (\frac{1}{x} = x^{-1})$

Log law 1

$\log_a x + \log_a y = \log_a(xy)$

e.g. $\log_a 5 + \log_a 6 = \log_a 30$

Log law 2

$\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$

e.g. $\log_3 18 - \log_3 2 = \log_3 9$
$= 2$

Log law 3

$\log_a x^n = n \log_a x$

e.g. $\log_2 8^{\frac{1}{2}} = \frac{1}{2}\log_2 8$
$= \frac{1}{2} \times 3$
$= \frac{3}{2}$

Solving exponential equations

$2^x = 5$
$x = \log_2 5$

or

$2^x = 5$
$\log_{10} 2^x = \log_{10} 5$
$x \log_{10} 2 = \log_{10} 5$
$x = \dfrac{\log_{10} 5}{\log_{10} 2}$

Polynomials (10A)

Polynomial of degree n

$a_n x^n + a_{n-1}x^{n-1} + ... + a_0 x^0$

e.g. cubic $x^3 - 2x^2 + 3x - 4$

Powers of x are whole numbers.

Expanding polynomials

Example:

$(x^2 - 1)(x^2 + 2x - 3)$
$= x^2(x^2 + 2x - 3) - 1(x^2 + 2x - 3)$
$= x^4 + 2x^3 - 3x^2 - x^2 - 2x + 3$
$= x^4 + 2x^3 - 4x^2 - 2x + 3$

Remainder theorem

If $P(x)$ is divided by $(x - a)$ then the remainder is $P(a)$.

Factor theorem

If $P(x)$ is divided by $(x - a)$ and $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$.

Division

$(x^3 - 2x^2 + 5x + 1) \div (x - 1)$

$$
\begin{array}{rl}
 & \quad x^2 - x + 4 \\
x-1 & \overline{)\, x^3 - 2x^2 + 5x + 1} \\
x^2(x-1) & \; x^3 - x^2 \\
 & \quad\; -x^2 + 5x + 1 \\
-x(x-1) & \quad\; -x^2 + x \\
 & \qquad\quad 4x + 1 \\
4(x-1) & \qquad\quad 4x - 1 \\
 & \qquad\qquad\; 5
\end{array}
$$

$\therefore \underbrace{x^3 - 2x^2 + 5x + 1}_{\text{dividend}} = \underbrace{(x - 1)}_{\text{divisor}}\underbrace{(x^2 - x + 4)}_{\text{quotient}} + \underbrace{5}_{\text{remainder}}$

Null Factor Law

$(x + 2)(x - 3)(2x + 1) = 0$

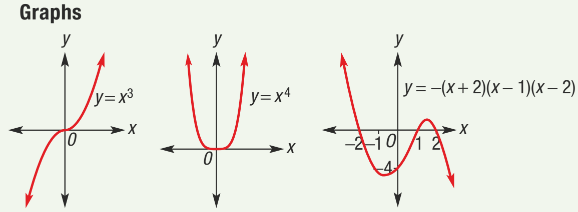
$x + 2 = 0$ or $x - 3 = 0$ or $2x + 1 = 0$

$x = -2 \qquad x = 3 \qquad x = -\frac{1}{2}$

Factorising

- Find one factor using the factor theorem.
- Use long division to find the quotient.
- Factorise the quotient.

Graphs



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter checklist: Success criteria

Chapter checklist

			✔
10A	**1. I can convert between index form and logarithmic form.** e.g. Write an equivalent statement in index form for $\log_2 8 = 3$ and in logarithm form for $3^4 = 81$.	10A	
10A	**2. I can evaluate a logarithm.** e.g. Evaluate $\log_4 64$.	10A	
10A	**3. I can evaluate a logarithm using a calculator.** e.g. Evaluate $\log_{10} 8$ correct to three decimal places.	10A	
10A	**4. I can solve a simple logarithmic equation.** e.g. Find the value of x in the equation $\log_3 x = 5$.	10A	
10B	**5. I can apply logarithm laws.** e.g. Simplify $\log_2 5 + \log_2 3$ and $2 \log_2 3 + \log_2 7$.	10A	
10B	**6. I can evaluate logarithmic expressions by simplifying.** e.g. Simplify and evaluate $\log_5 10 - \log_5 2 + \log_7 1$.	10A	
10C	**7. I can solve an exponential equation using logarithms with the given base.** e.g. Solve $3^x = 20$ correct to three decimal places using the given base.	10A	
10C	**8. I can solve exponential equations using base 10.** e.g. Solve $2 \times 1.1^x = 6$ using base 10 and correct to three decimal places.	10A	
10D	**9. I can identify a polynomial.** e.g. Which of the following expressions are polynomials? $2x^2 - \frac{5}{\sqrt{x}}$ and $3x^3 - 2x + 7$.	10A	
10D	**10. I can evaluate a polynomial for a given value.** e.g. If $P(x) = 2x^3 - x^2 + 3$, find $P(-2)$.	10A	
10E	**11. I can expand and simplify polynomials.** e.g. Expand and simplify $(x^3 + 3x - 2)(2x^2 - x + 4)$.	10A	
10F	**12. I can divide polynomials.** e.g. Divide $P(x) = x^3 - 2x^2 + 3x + 6$ by $(x + 1)$ and write in the form $P(x) = (x + 1)Q(x) + R$, where R is the remainder.	10A	
10G	**13. I can find the remainder using the remainder theorem.** e.g. Find the remainder when $P(x) = x^3 - 2x^2 + 3x - 4$ is divided by $(x - 3)$.	10A	
10G	**14. I can decide whether a linear expression is a factor of a polynomial using the factor theorem.** e.g. Decide if $(x + 2)$ is a factor of $P(x) = x^3 + 3x^2 - x - 6$.	10A	
10G	**15. I can apply the remainder theorem to find a missing value.** e.g. Find the value of k such that $(x^3 - 2x^2 + kx - 2) \div (x - 3)$ has a remainder of 4.	10A	
10H	**16. I can apply the Null Factor Law to solve polynomial equations.** e.g. Solve $(2x + 5)(x - 3)(x + 2) = 0$ for x.	10A	
10H	**17. I can factorise and solve a cubic equation.** e.g. Solve $x^3 - 5x^2 + 2x + 8 = 0$.	10A	
10I	**18. I can sketch a cubic graph labelling intercepts.** e.g. Sketch $y = (x - 5)(x + 1)(x - 2)$ labelling x- and y-intercepts.	10A	

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter review

10A Short-answer questions

10A **1** Write the following in logarithmic form.
a $2^4 = 16$ **b** $10^3 = 1000$ **c** $3^{-2} = \frac{1}{9}$

10A **2** Write the following in index form.
a $\log_3 81 = 4$ **b** $\log_4 \frac{1}{16} = -2$ **c** $\log_{10} 0.1 = -1$

10A **3** Evaluate the following.
a $\log_{10} 1000$ **b** $\log_3 81$ **c** $\log_2 16$
d $\log_7 1$ **e** $\log_3 \frac{1}{27}$ **f** $\log_5 \frac{1}{125}$
g $\log_4 0.25$ **h** $\log_{10} 0.0001$ **i** $\log_3 0.\dot{1}$

10B **4** Simplify using the laws for logarithms.
a $\log_a 4 + \log_a 2$ **b** $\log_b 7 + \log_b 3$ **c** $\log_b 24 + \log_b 6$
d $\log_a 1000 - \log_a 100$ **e** $2\log_a 2$ **f** $3\log_a 10$
g $\log_{10} 25 + \log_{10} 4$ **h** $\log_3 60 - \log_3 20$ **i** $\log_2 \sqrt{8}$

10C **5** Solve these equations using logarithms with the given base.
a $3^x = 6$ **b** $20 \times 1.2^x = 40$

10C **6** Solve for x, in exact form, using base 10.
a $2^x = 13$ **b** $100 \times 0.8^x = 200$

10D **7** If $P(x) = x^3 - x^2 - x - 1$, find:
a $P(0)$ **b** $P(2)$ **c** $P(-1)$ **d** $P(-3)$

10E **8** Expand and simplify.
a $(x^2 + 2)(x^2 + 1)$ **b** $x^3(x^2 - x - 3)$
c $(x^2 + x - 3)(x^3 - 1)$ **d** $(x^3 + x - 3)(x^3 + x - 1)$

10F **9** Use long division to express each of the following in this form:

Dividend = divisor × quotient + remainder

a $(x^3 + x^2 + 2x + 3) \div (x - 1)$ **b** $(x^3 - 3x^2 - x + 1) \div (x + 1)$
c $(2x^3 - x^2 + 4x - 7) \div (x + 2)$ **d** $(-2x^3 - x^2 - 3x - 4) \div (x - 3)$

10G **10** Use the remainder theorem to find the remainder when $P(x) = 2x^3 - 2x^2 + 4x - 7$ is divided by:
a $x - 1$ **b** $x + 2$ **c** $x + 3$ **d** $x - 3$

10G **11** Using the factor theorem, decide if the following are factors of $P(x) = x^3 - 2x^2 - 11x + 12$.
a $x + 1$ **b** $x - 1$ **c** $x - 4$ **d** $x + 3$

10H **12** Solve these cubic equations.
a $(x - 3)(x - 1)(x + 2) = 0$ **b** $(x - 5)(2x - 3)(3x + 1) = 0$

10H **13** Factorise and solve these cubic equations.
a $x^3 + 4x^2 + x - 6 = 0$ **b** $x^3 - 9x^2 + 8x + 60 = 0$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

10I

14 Sketch the graphs of these polynomials.

a $y = x^3$

b $y = (x + 1)(x - 1)(x - 4)$

c $y = -x(x - 3)(x + 2)$

d $y = x^4$

10A

Multiple-choice questions

10A

1 Which of the following is equivalent to $5^3 = 125$?

A $\log_3 125 = 5$ B $\log_3 5 = 125$ C $\log_5 125 = 3$ D $125^3 = 5$ E $\log_{125} 3 = 5$

10A

2 If $\log_2 64 = x$, then x is equal to:

A 5 B 6 C 32 D 128 E 64^2

10A

3 If $5^x = 7$, then x is equal to:

A $\log_{10} \frac{7}{5}$ B $\log_x 5$ C $\log_{10} 7$ D $\log_7 5$ E $\log_5 7$

10B

4 $\log_6 \frac{1}{6}$ simplifies to:

A -1 B 1 C 36 D 6 E 0

10D

5 $x^6 - 2x^2 + 1$ is a polynomial of degree:

A 1 B -2 C 2 D 6 E 0

10D

6 Which of these is a polynomial?

A $x^{\frac{1}{2}} + x^2$ B $\sqrt[3]{x} + x^2$ C $x^3 + x^2 + \frac{1}{x^2}$ D $4x^5 - 2x^3 - 1$ E $\frac{1}{x} - x$

10D

7 If $P(x) = x^3 - x$ and $Q(x) = -2x^2 + 1$, then $P(-1) - Q(1)$ is equal to:

A 1 B -1 C 2 D 3 E -3

10G

8 The remainder when $P(x) = 2x^3 + 4x^2 - x - 5$ is divided by $(x + 1)$ is:

A -4 B -2 C -10 D 0 E -1

10H

9 The three solutions to $(x - 3)(x + 5)(2x - 1) = 0$ are x equals:

A $\frac{1}{3}, -\frac{1}{5}$ and 2 B $5, -3$ and $-\frac{1}{2}$ C $-5, 3$ and $-\frac{1}{2}$

D $-5, 3$ and 1 E $-5, 3$ and $\frac{1}{2}$

10I

10 The equation of this graph could be:

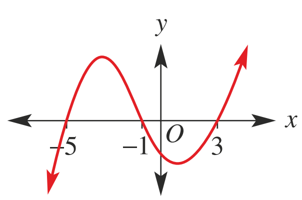
A $y = x(x + 5)(x - 1)$

B $y = (x + 5)(x + 1)(x + 3)$

C $y = (x - 5)(x - 1)(x + 3)$

D $y = (x - 5)(x + 1)(x - 3)$

E $y = (x + 5)(x + 1)(x - 3)$



ISBN 978-1-108-77290-7
Cambridge University Press
Updated June 2022

Chapter review

10A

Extended-response questions

1 A share portfolio initially valued at \$100000 is invested, compounding continuously at a rate equivalent to 10% per annum.

a If \$$A$ is the value of the investment and n is the number of years, which of the following is the correct rule linking A and n?

A $A = 100000 \times 0.1^n$ **B** $A = 100000 \times 1.1^n$ **C** $A = \dfrac{1.1^n}{100000}$

b Find the value of the investment, correct to the nearest dollar, after:

i 2 years **ii** 18 months **iii** 10.5 years

c Find the time, correct to two decimal places, when the investment is expected to have increased in value to:

i \$200000 **ii** \$180000 **iii** \$0.5 million

2 A cubic polynomial has the rule $P(x) = x^3 - 5x^2 - 17x + 21$.

a Find:

i $P(-1)$ **ii** $P(1)$

b Explain why $(x - 1)$ is a factor of $P(x)$.

c Divide $P(x)$ by $(x - 1)$ to find the quotient.

d Factorise $P(x)$ completely.

e Solve $P(x) = 0$.

f Find $P(0)$.

g Sketch a graph of $P(x)$, labelling x- and y-intercepts.

ISBN 978-1-108-77290-7

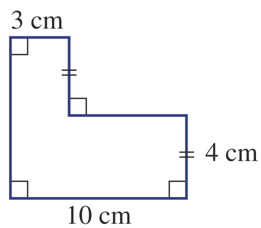
Cambridge University Press
Updated June 2022

Measurement

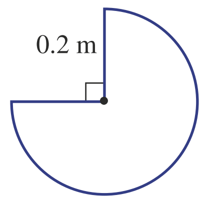
Short-answer questions

1 Find the perimeter and area of these shapes. Give answers correct to one decimal place where necessary. You will need to use Pythagoras' theorem for part **c**.

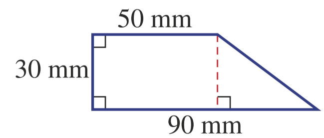
a



b

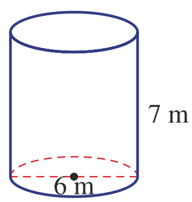


c

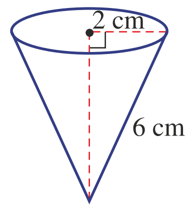


2 Find the surface area and volume for these solids. Give your answers to one decimal place.

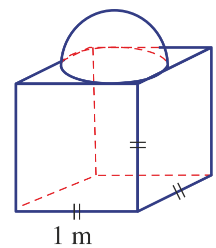
a



10A **b**



10A **c**



3 A rectangular prism has length 5 cm, width 3 cm and volume 27 cm^3.

a Find the height of the prism.

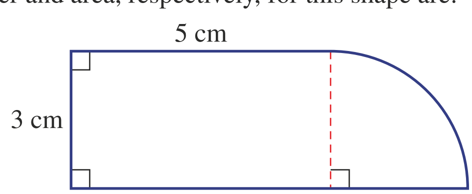
b Find the total surface area of the prism.

10A **4** A cone has volume 90 cm^3 and height 10 cm. Find the exact radius of the cone.

Multiple-choice questions

1 Correct to two decimal places, the perimeter and area, respectively, for this shape are:

A 12.71 cm, 16.77 cm^2
B 20.71 cm, 22.07 cm^2
C 25.42 cm, 29.14 cm^2
D 18.36 cm, 43.27 cm^2
E 17.71 cm, 17.25 cm^2



2 0.04 m^2 is equivalent to:

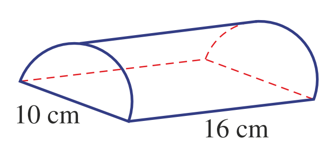
A 4 cm^2 **B** 40 mm^2 **C** 0.0004 cm^2 **D** 400 cm^2 **E** 0.00004 km^2

3 A square-based pyramid has base area 30 m^2 and height 7 m. Its volume is:

A 105 m^3 **B** 70 m^3 **C** 210π m^3 **D** 210 m^3 **E** 140 m^3

4 The curved surface area of this half cylinder, in exact form, is:

A 80π cm^2 **B** 105π cm^2 **C** 92.5π cm^2
D 120π cm^2 **E** 160π cm^2



10A **5** The volume of a sphere of diameter 30 cm is closest to:

A 113 097 cm^3 **B** 2827 cm^3 **C** 11 310 cm^3 **D** 14 137 cm^3 **E** 7069 cm^3

ISBN 978-1-108-77290-7

Semester review 2

Extended-response question

A cylindrical glass vase is packaged inside a box that is a rectangular prism, so that the vase touches the box on all four sides and is the same height as the box. The vase has a diameter of 8 cm and height 15 cm. Round your answers to two decimal places where necessary.

a Find the volume of the vase.
b Find the volume of space inside the box but outside the vase.
c A glass stirring rod is included in the vase. Find the length of the longest rod that can be packaged inside the vase.
d Find the difference in the length of rod in part c and the longest rod that can fit inside the empty box. Round your answer to two decimal places.

Parabolas and other graphs

Short-answer questions

1 Sketch the following parabolas and state the transformations from $y = x^2$.
  a $y = 3x^2$ b $y = -(x + 2)^2$ c $y = x^2 + 5$

2 Consider the quadratic $y = x^2 + 4x - 5$.
  a Find the y-intercept.
  b Find the x-intercepts by factorising.
  c Use symmetry to find the turning point.
  d Sketch the graph.

3 Consider the quadratic $y = -2(x - 3)^2 + 8$.
  a State the coordinates of the turning point and whether it is a maximum or minimum.
  b Find the y-intercept.
  c Find the x-intercepts by factorising.
  d Sketch the graph.

4 Sketch the following quadratics by first completing the square.
  a $y = x^2 + 6x + 2$ b $y = x^2 - 5x + 8$

5 Consider the quadratic $y = 2x^2 - 4x - 7$.
  a Use the discriminant to determine the number of x-intercepts of the graph.
  b Sketch its graph using the quadratic formula. Round x-intercepts to one decimal place.
  10A c Find the points of intersection of the quadratic and the line $y = -6x - 3$ by solving simultaneously.

6 Consider the function $f(x) = (x - 2)^2 + 3$
  a Explain why $f(x)$ is a function.
  b Evaluate
    i $f(2)$ ii $f(-1)$ iii $f(a)$
  c Give the allowable x-values (domain) and resulting y-values (range) of this function.

7 Sketch the following graphs, labelling key features.
  a $x^2 + y^2 = 4$ b $x^2 + y^2 = 10$ c $y = \frac{2}{x}$ d $y = -\frac{6}{x}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

8 Find the coordinates of the points of intersection of the graphs of the following.

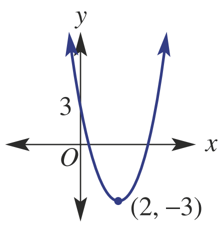
a $x^2 + y^2 = 15$ and $y = 2x$ b $y = \frac{2}{x}$ and $y = 8x$

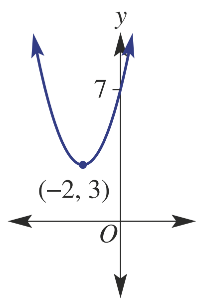
10A 9 Sketch the graphs of the following relations. Label important features.

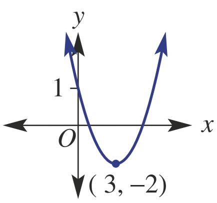
a $(x - 2)^2 + (y + 1)^2 = 16$ b $y = 2^{x+3} + 1$ c $y = \frac{1}{x+2} - 3$

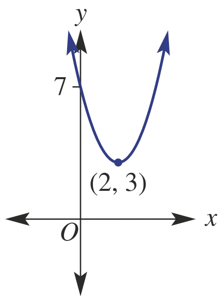
Multiple-choice questions

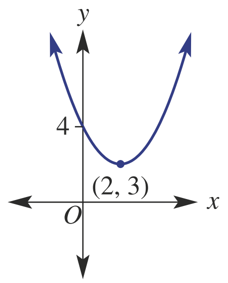
1 The graph of $y = (x - 2)^2 + 3$ could be:

A


B


C


D


E


2 The graph of $y = x^2 - 4x$ has a turning point with coordinates:

A $(2, -4)$ B $(0, -4)$ C $(4, 0)$ D $(-2, 12)$ E $(1, -3)$

3 For the quadratic $y = ax^2 + bx + c$, $b^2 - 4ac < 0$, we know that the graph has:

A a maximum turning point
B two x-intercepts
C no y-intercept
D no x-intercepts
E a minimum turning point

4 The graph with equation $x^2 + y^2 = 9$ is:

A a circle with radius 9
B a parabola with turning point $(0, 9)$
C a circle with radius 3
D a hyperbola with asymptote at $x = 3$
E an exponential curve with y-intercept $(0, 3)$

10A 5 The equations of the asymptotes of $y = \frac{1}{x} + 3$ are:

A $x = 0, y = 0$ B $x = 0, y = 3$ C $x = 3, y = 0$
D $x = 0, y = -3$ E $x = -3, y = 3$

ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

Extended-response question

A rollercoaster has a section modelled by the equation $h = \frac{1}{40}(x^2 - 120x + 1100)$, where h is the height above the ground and x is the horizontal distance from the start of the section. All distances are measured in metres and x can take all values between 0 and 200 metres.

a Sketch the graph of h vs x for $0 \leqslant x \leqslant 200$, labelling the endpoints.

b What is the height above ground at the start of the section?

c The rollercoaster travels through an underground tunnel. At what positions from the start will it enter and leave the tunnel?

d What is the maximum height the rollercoaster reaches?

e What is the maximum depth the rollercoaster reaches?

Probability

Short-answer questions

1 Consider events A and B. Event A is the set of letters in the word 'grape' and event B is the set of letters in the word 'apricot':

$A = \{g, r, a, p, e\}$ $\qquad$ $B = \{a, p, r, i, c, o, t\}$

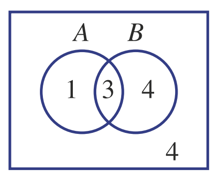
a Represent the two events A and B in a Venn diagram.

b If a letter is randomly selected from the alphabet, find:

i $\Pr(A)$ **ii** $\Pr(A \cap B)$ **iii** $\Pr(A \cup B)$ **iv** $\Pr(B')$

c Are the events A and B mutually exclusive? Why or why not?

2 The Venn diagram shows the distribution of elements in two sets, A and B.



a Transfer the information in the Venn diagram to a two-way table.

b Find:

i $n(A \cap B)$ **ii** $n(A' \cap B)$ **iii** $n(B')$ **iv** $n(A \cup B)$

c Find:

i $\Pr(A \cap B)$ **ii** $\Pr(A \cap B')$ **iii** $\Pr(B)$ **iv** $\Pr(B|A)$

3 Two events, A and B, are such that $\Pr(A) = 0.24$, $\Pr(B) = 0.57$ and $\Pr(A \cup B) = 0.63$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(A' \cap B')$

4 Two fair 4-sided dice numbered 1 to 4 are rolled and the total is noted.

a List the sample space as a table. **b** State the total number of outcomes.

c Find the probability of obtaining:

i a sum of 4 **ii** a sum of at least 5 **iii** a sum of 7, given the sum is at least 5.

5 In a group of 12 friends, 8 study German, 4 study German only and 2 study neither German nor Mandarin. Let A be the event 'studies German' and let B be the event 'studies Mandarin'.

a Summarise the information in a Venn diagram.

b Find:

i $\Pr(A)$ **ii** $\Pr(A|B)$

c State whether or not the events A and B are independent.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Multiple-choice questions

1 The number of tails obtained from 100 tosses of two fair coins is shown in the table.

Number of tails	0	1	2
Frequency	23	57	20

From this table, the experimental probability of obtaining two tails is:

A 0.23 **B** 0.25 **C** 0.2 **D** 0.5 **E** 0.77

2 From the given two-way table $\Pr(A \cap B')$ is:

A $\frac{1}{2}$ **B** $\frac{2}{3}$ **C** $\frac{1}{4}$

D $\frac{4}{5}$ **E** $\frac{1}{3}$

	A	A'	
B	2		
B'			5
	6		12

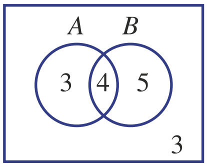
3 Two events, A and B, are such that $\Pr(A) = 0.7$, $\Pr(B) = 0.4$ and $\Pr(A \cap B) = 0.3$. $\Pr(A \cup B)$ is equal to:

A 1.4 **B** 0.8 **C** 0.6 **D** 0 **E** 0.58

4 From the information in the Venn diagram, $\Pr(A|B)$ is:

A $\frac{5}{12}$ **B** $\frac{4}{5}$ **C** $\frac{4}{7}$

D $\frac{4}{9}$ **E** $\frac{1}{3}$



5 A bag of 5 marbles contains 2 green ones. Two marbles are randomly selected without replacement. The probability of selecting the 2 green marbles is:

A $\frac{9}{20}$ **B** $\frac{2}{25}$ **C** $\frac{1}{10}$ **D** $\frac{2}{5}$ **E** $\frac{4}{25}$

Extended-response question

Lindiana Jones selects two weights from her pocket to sit on a weight-sensitive trigger device after removing the goblet of fire. Her pocket contains three weights, each weighing 200 g, and five weights, each weighing 250 g. The two weights are selected randomly without replacement. Use a tree diagram to help answer the following.

a Find the probability that Lindiana selects two weights totalling:

i 400 g **ii** 450 g **iii** 500 g

b If the total weight selected is less than 480 g, a poison dart will shoot from the wall. Find the probability that Lindiana is at risk from the poison dart.

c By feeling the weight of her selection, Lindiana knows that the total weight is more than 420 g. Given this information, what is the probability that the poison dart will be fired from the wall?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Statistics

Short-answer questions

1 Twenty people are surveyed to find out how many days in the past completed month they used public transport. The results are as follows.

7, 16, 22, 23, 28, 12, 18, 4, 0, 5
8, 19, 20, 22, 14, 9, 21, 24, 11, 10

- **a** Organise the data into a frequency table with class intervals of 5 and include a percentage frequency column.
- **b** Construct a histogram for the data, showing both the frequency and the percentage frequency on the one graph.
- **c** **i** State the frequency of people who used public transport on 10 or more days.
  - **ii** State the percentage of people who used public transport on fewer than 15 days.
  - **iii** State the most common interval of days for which public transport was used. Can you think of a reason for this?

2 By first finding quartiles and checking for outliers, draw box plots for the following data sets.

- **a** 8, 10, 2, 17, 6, 25, 12, 7, 12, 15, 4
- **b** 5.7, 4.8, 5.3, 5.6, 6.2, 5.7, 5.8, 5.1, 2.6, 4.8, 5.7, 8.3, 7.1, 6.8

3 Farsan's bank balance over 12 months is recorded below.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Balance ($)	1500	2100	2300	2500	2200	1500	1200	1600	2000	2200	1700	2000

- **a** Plot the time-series for the 12 months.
- **b** Describe the way in which the bank balance has changed over the 12 months.
- **c** Between which consecutive months did the biggest change in the bank balance occur?
- **d** What is the overall change in the bank balance over the year?

4 Consider the variables x and y and the corresponding bivariate data below.

x	4	5	6	7	8	9	10
y	2.1	2.5	3.1	2.8	4	3.6	4.9

- **a** Draw a scatter plot for the data.
- **b** Describe the correlation between x and y as either positive, negative or none.
- **c** Fit a line of good fit by eye to the data on the scatter plot.
- **d** Use your line of good fit to estimate:
  - **i** y when $x = 7.5$
  - **ii** x when $y = 5.5$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

10A 5 The back-to-back stem-and-leaf plot below shows the number of DVDs owned by people in two different age groups.

Over 40 Leaf	Stem	Under 40 Leaf
9 7 6 6 4 3 2	0	7 8
6 4 3 2 2 0	1	2 5 5 7 8
8 3	2	4 4 6
	3	2 6 9
	4	1 8

2|4 means 24

a By considering the centre and spread of the data, state with reasons:
  i Which data set will have the higher mean?
  ii Which data set will have the smaller standard deviation?

b Calculate the mean and sample standard deviation for each data set. Round your answers to one decimal place where necessary.

Multiple-choice questions

1 For the given stem-and-leaf plot, the range and median, respectively, of the data are:

Stem	Leaf
0	2 2 6 7
1	0 1 2 3 5 8
2	3 3 5 7 9

1|5 means 15

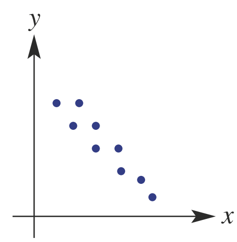
A 20, 12.5 **B** 7, 12 **C** 27, 12.5 **D** 29, 3 **E** 27, 13

2 The interquartile range (IQR) for the data set 2, 3, 3, 7, 8, 8, 10, 13, 15 is:

A 5 **B** 8.5 **C** 7 **D** 13 **E** 8

3 The best description of the correlation between the variables for the scatter plot shown is:

A weak, negative
B strong, positive
C strong, negative
D weak, positive
E no correlation



4 A line of best fit passes through the points (10, 8) and (15, 18). The equation of the line is:

A $y = \frac{2}{3}x + 8$ **B** $y = 2x - 12$ **C** $y = -\frac{1}{2}x + 13$

D $y = 2x + 6$ **E** $y = \frac{1}{2}x + 3$

10A **5** The mean and sample standard deviation of the small data set 2, 6, 7, 10 and 12, correct to one decimal place, are:

A $\bar{x} = 7.4$ and $s = 3.8$ **B** $\bar{x} = 7$ and $s = 3.7$ **C** $\bar{x} = 7.4$ and $s = 3.4$

D $\bar{x} = 7$ and $s = 7.7$ **E** $\bar{x} = 27.1$ and $s = 9.9$

10A

Extended-response question

After a month of watering with a set number of millilitres of water per day, the height of a group of the same species of plant is recorded below.

Water (mL)	8	5	10	14	12	15	18
Height (cm)	25	27	34	40	35	38	45

a Using 'Water' for the x-axis and rounding coefficients to two decimal places, find the equation of the least squares regression line.

b Use your least squares regression line to estimate the height (to the nearest cm) of a plant watered with 16 mL of water per day.

Logarithms and polynomials

10A

Short-answer questions

1 Simplify where necessary and evaluate.

a $\log_4 64$ **b** $\log_5 \frac{1}{25}$ **c** $\log_{10} 1000$ **d** $\log_7 1$

e $\log_4 2 + \log_4 8$ **f** $\log_3 54 - \log_3 6$ **g** $\log_8 8$ **h** $\log_a a^3$

2 Solve for x.

a $\log_6 216 = x$ **b** $\log_x 27 = 3$ **c** $\log_3 x = 4$

3 a Solve for x using the given base.

i $3^x = 30$ **ii** $15 \times 2.4^x = 60$

b Solve for x using base 10 and evaluate, correct to three decimal places.

i $7^x = 120$ **ii** $2000 \times 0.87^x = 500$

4 Consider the polynomials $P(x) = x^3 + 3x^2 - 4x - 6$ and $Q(x) = 2x^3 - 3x - 4$.

a Find:

i $P(2)$ **ii** $P(-1)$ **iii** $Q(-3)$ **iv** $Q(1)$

b Expand and simplify.

i $P(x) \times Q(x)$ **ii** $(Q(x))^2$

5 Divide $P(x) = x^3 - 4x^2 + 2x + 7$ by $(x - 3)$ and write in the form $P(x) = (x - 3)Q(x) + R$, where $Q(x)$ is the quotient and R is the remainder.

6 Find the remainder when $P(x) = x^3 - 2x^2 - 13x - 10$ is divided by each of the following and, hence, state if it is a factor.

a $x - 1$ **b** $x + 2$ **c** $x - 3$

7 Solve for x.

a $(x + 1)(x - 3)(x + 6) = 0$ **b** $-x(2x - 5)(3x + 2) = 0$

c $x^3 + 5x^2 + 2x - 8 = 0$ **d** $2x^3 - 3x^2 - 3x + 2 = 0$

ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

10A Multiple-choice questions

1 The value of $\log_2 16$ is:

A 8 **B** 256 **C** 2^{16} **D** $\frac{1}{4}$ **E** 4

2 An equivalent statement to $3^x = 20$ is:

A $x = \log_3 20$ **B** $20 = \log_3 x$ **C** $x = \log_{20} 3$

D $3 = \log_x 20$ **E** $x = \log_{10}\left(\frac{20}{3}\right)$

3 The expression that is *not* a polynomial is:

A $3x^2 + 1$ **B** $2 - 5x^5 + x$ **C** $7x - 5$

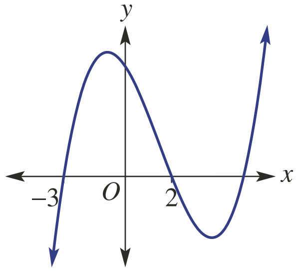
D $4x^3 + 2x - \frac{1}{x}$ **E** $5x^6 + 2x^4 - 3x^2 - x$

4 When $P(x) = x^4 - 3x^3 + 2x + 1$ is divided by $(x - 2)$, the remainder is:

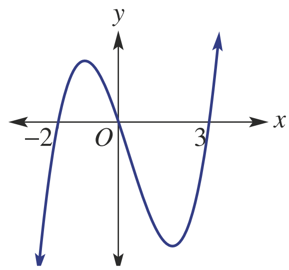
A 37 **B** 2 **C** -3 **D** 13 **E** 5

5 A possible graph of $y = -x(x + 3)(x - 2)$ is:

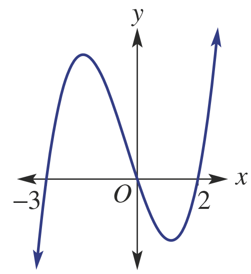
A



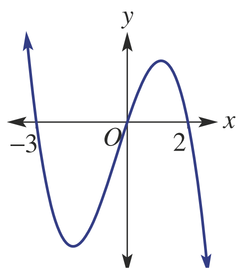
B



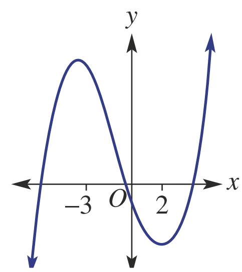
C



D



E



10A Extended-response question

A section of a train track that heads through a valley and then over a mountain is modelled by the equation $P(x) = -2x^3 + 3x^2 + 23x - 12$ for $-5 \leqslant x \leqslant 6$.

a Show that $(x + 3)$ is a factor of $P(x)$.

b Hence, factorise $P(x)$ using division.

c Sketch a graph of this section of the track, labelling axes intercepts and endpoints.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

CHAPTER 11

Algorithmic thinking

Driverless cars

The vast majority of accidents in motor vehicles are due to driver error, which is one of the reasons why companies such as Google are experimenting with driverless cars.

Such vehicles are currently being tested in a number of countries. The vehicles map their current position and use a range of sensors to determine the moving and stationary objects in the immediate area. This information is the input for computer-based algorithms which make predictions and test scenarios at a rapid rate. The efficiency of the computer code is critical in the running of the systems, which is why skilled programmers and mathematicians are responsible for their design. If the system detects a pedestrian, for example, the code needs to take into account the probability that this person could cross the road in front of the car. The software must choose the safest possible route at the safest speed.

ISBN 978-1-108-77290-7

Driverless cars have the potential to reduce road fatalities and increase the level of efficiency on our road networks. Parking would become less problematic, with a driverless car dropping you off at work and travelling to a remote parking spot somewhere else, before returning later in the day to pick you up.

Online resources

A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

In this chapter

Victorian Curriculum

NUMBER AND ALGEBRA
Patterns and algebra

Implement algorithms using data structures in a general-purpose programming language (VCMNA334)

(10A) Devise and use algorithms and simultaneous equations to solve mathematical problems (VCMNA358)

Linear and non-linear relationships

Solve equations using systematic guess-check-and-refine with digital technology (VCMNA342)

(10A) Solve simultaneous equations using systematic guess-check-and-refine with digital technology (VCMNA364)

Introduction

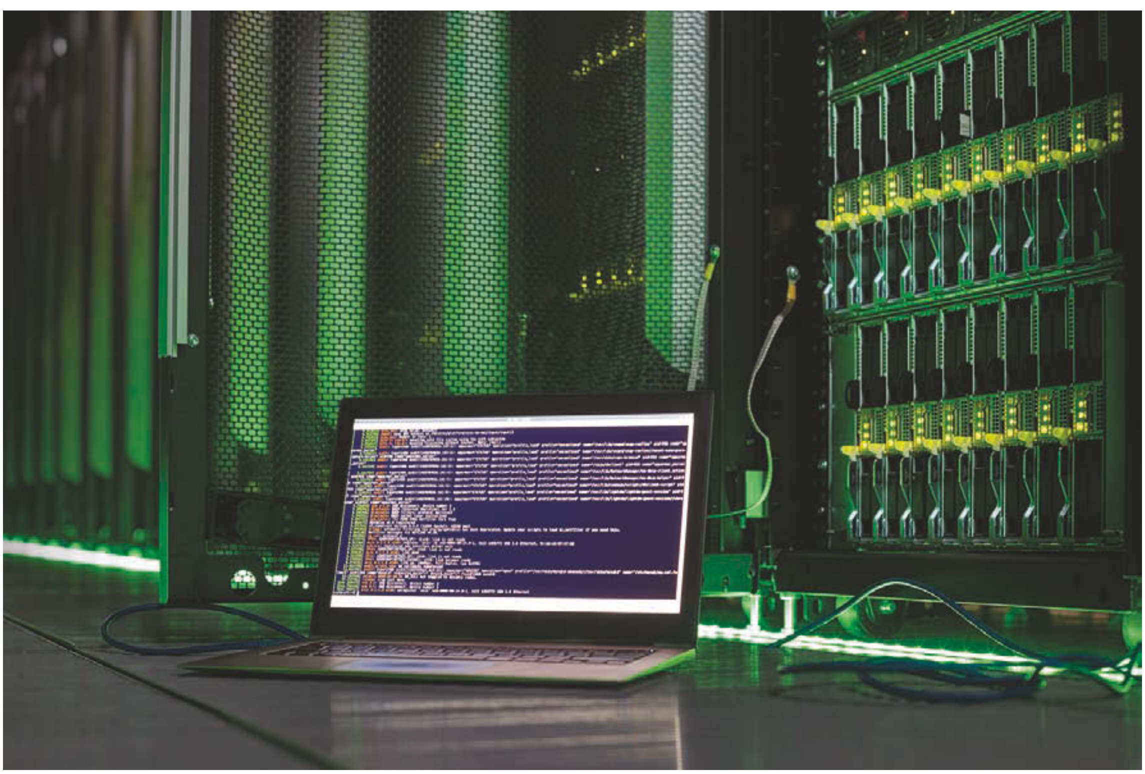
An **algorithm** is a sequence of steps that when followed, lead to the solution of a problem. It has a defined set of inputs and delivers an output. Each step in the algorithm leads to another step or completes the algorithm.

Algorithms occur in mathematics and computing, as well as in simple areas of daily life such as following a recipe. Algorithmic thinking is a type of thinking that involves designing algorithms to solve problems. The algorithms we design can then be written in a way that a computer program will understand, so that the computer does the hard computational work.

In the following activities you will carry out some algorithms as well as think about the design, analysis and implementation of your own algorithms.

The algorithms in these activities will be described through the use of spreadsheets, flow charts (a way of writing an algorithm in the form of a diagram), programming language and simulations. The following symbols will be used in the flow charts with arrows to connect each stage:

for input/output stages | for process stages | for decision stages



Activity 1: Using numerical methods to solve equations

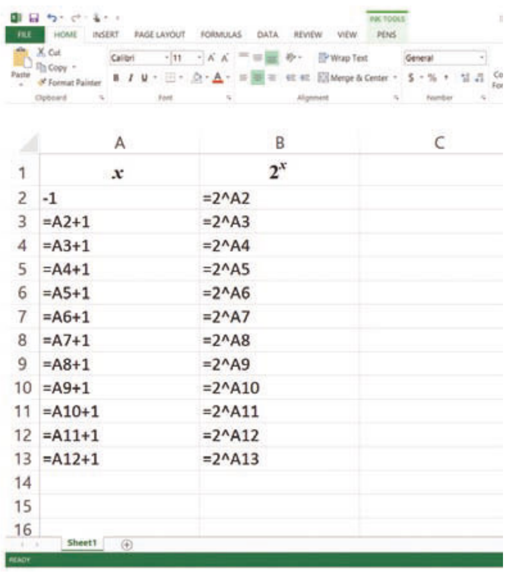
NUMBER AND ALGEBRA

Numerical methods using tables, spreadsheets, graphs and other algorithms can be used to solve both linear and non-linear equations, to find roots of equations and to compute other values. The following activities look at some common approaches.

1.1 Solving equations using tables and spreadsheets

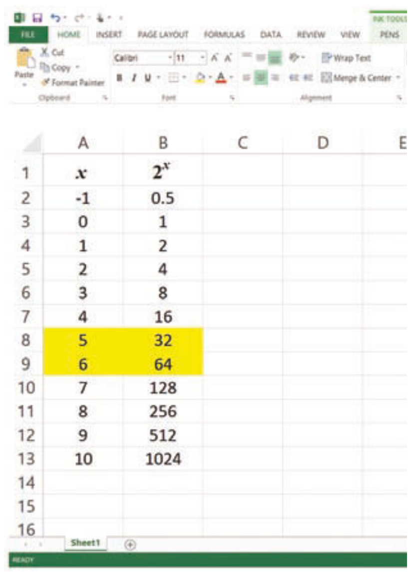
a To solve an equation such as $2^x = 40$, complete the following.

i Set up a spreadsheet like the one shown below, (*Figure 1*), entering the formulas and then filling down. Use an increment of 1 in column A (depending on the equation, you may also need to consider negative values of x) to find which integer values of x make the value of 2^x change from less than 40 to more than 40. In this case (*Figure 2*) you can see this is $x = 5$ and $x = 6$.



	A	B	C
1	x	2^x	
2	-1	=2^A2	
3	=A2+1	=2^A3	
4	=A3+1	=2^A4	
5	=A4+1	=2^A5	
6	=A5+1	=2^A6	
7	=A6+1	=2^A7	
8	=A7+1	=2^A8	
9	=A8+1	=2^A9	
10	=A9+1	=2^A10	
11	=A10+1	=2^A11	
12	=A11+1	=2^A12	
13	=A12+1	=2^A13	
14			
15			
16			

Figure 1



	A	B	C	D	E
1	x	2^x			
2	-1	0.5			
3	0	1			
4	1	2			
5	2	4			
6	3	8			
7	4	16			
8	5	32			
9	6	64			
10	7	128			
11	8	256			
12	9	512			
13	10	1024			
14					
15					
16					

Figure 2

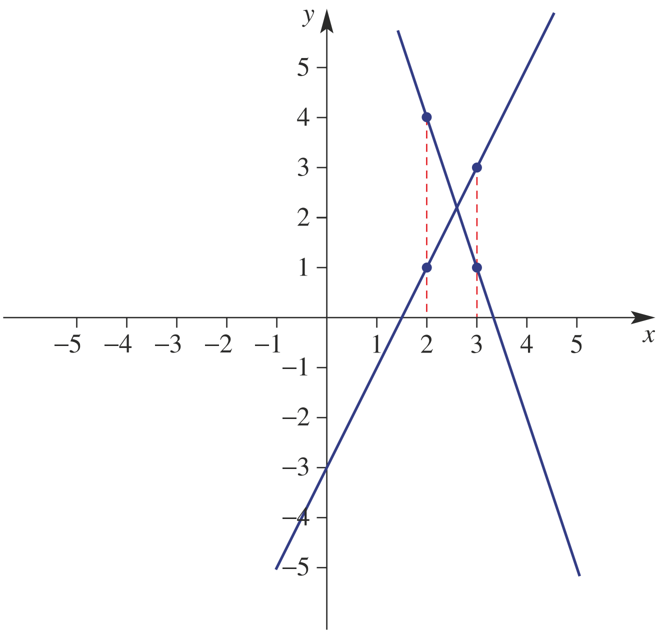
ii Now alter the spreadsheet to only consider x-values between $x = 5$ and $x = 6$, incrementing by 0.1. Fill down and find a solution for x, correct to one decimal place.

iii Now alter the spreadsheet to only consider x-values between $x = 5$ and $x = 6$, incrementing by 0.01. Fill down and find a solution for x, correct to two decimal places.

b A similar process can be used to solve a pair of simultaneous equations. To solve the equations $y = 2x - 3$ and $y = -3x + 10$ simultaneously, complete the following.

i Retain the x column like in part **a** and enter the two equations (= 2*A2 − 3 and = −3*A2 + 10) into columns B and C.

Fill down to look for the x-value that makes the two equations equal. If you first increment the x-values by 1, you are looking for when the column B values overtake the column C values or vice versa, to find the interval containing the point of intersection. This can be seen on the example graph shown.



ii Increase your accuracy by using smaller increments within a chosen interval. See if you can find the exact solution.

c To find the roots (or x-intercepts) of a quadratic equation, you can also use a spreadsheet or table. This time you are looking for the interval(s) (there may be two solutions) where the function value changes from positive to negative or vice versa.

Find the solutions to the quadratic equation $3x^2 - 4x - 2 = 0$, correct to two decimal places, using a spreadsheet or table. You may also need to consider negative values of x.

d The equations in parts **a** and **b** could also be treated as equations equal to 0 by rearranging to avoid having two columns to compare. For example, the solution to $2^x = 40$ is the same as the solution to $2^x - 40 = 0$. What is the equivalent equation equal to 0 that can be solved to find the x-value of the solution to $y = 2x - 3$ and $y = -3x + 10$?

This method will be used further in the next section.

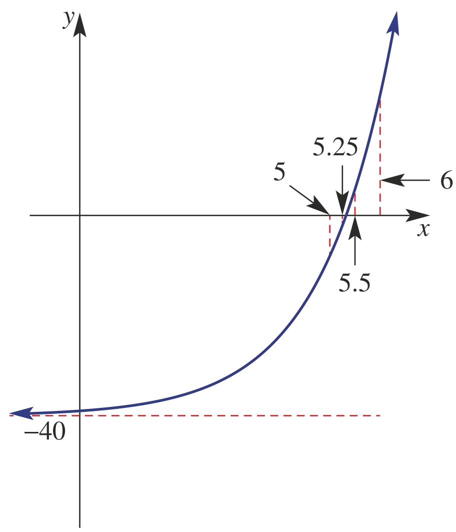
1.2 Solving equations using the bisection method

In the following we will look at the bisection method algorithm for solving the problems that were considered in **Section 1.1**. The process looks at solving an equation $f(x) = 0$ by finding smaller and smaller intervals of x-values within which the solution lies. Since the equation is being solved equal to 0, we will be looking for intervals where the function values are of different sign.

The process involves finding an average value of x between a pair of updated lower and upper bounds. The steps are:

- Find a lower and upper bound which are two x-values that contain the solution (i.e. one function value is positive and one negative).
- Now find the mean of your lower and upper bound.
- Find the value of the function for your mean.
- If the mean function value is zero the solution has been obtained. If it is the same sign as the lower bound, it becomes the new lower bound of the interval, otherwise it becomes the new upper bound.
- Repeating the above steps will deliver smaller and smaller intervals containing the solution.

a Work through the above algorithm to solve $2^x - 40 = 0$ until you are confident you have the answer correct to two decimal places.

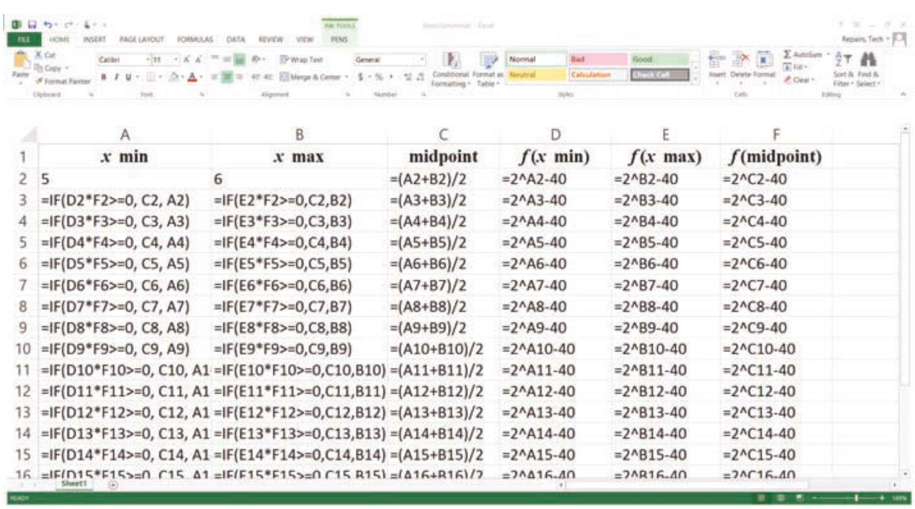


The process can take a while by hand, particularly for greater levels of accuracy. We will now consider a spreadsheet approach to this algorithm.

b Set up the following spreadsheet where $f(x) = 2^x - 40$ and the aim is to solve $f(x) = 0$, correct to four decimal places.

i Start the process by using the interval from $x = 5$ to $x = 6$ obtained in **Section 1.1**.

ii Enter in the formula for finding the midpoint and evaluate the function at the minimum, maximum and midpoint values (row 2).



	A	B	C	D	E	F
1	x min	x max	midpoint	$f(x$ min$)$	$f(x$ max$)$	f(midpoint)
2	5	6	=(A2+B2)/2	=2^A2-40	=2^B2-40	=2^C2-40
3	=IF(D2*F2>=0, C2, A2)	=IF(E2*F2>=0,C2,B2)	=(A3+B3)/2	=2^A3-40	=2^B3-40	=2^C3-40
4	=IF(D3*F3>=0, C3, A3)	=IF(E3*F3>=0,C3,B3)	=(A4+B4)/2	=2^A4-40	=2^B4-40	=2^C4-40
5	=IF(D4*F4>=0, C4, A4)	=IF(E4*F4>=0,C4,B4)	=(A5+B5)/2	=2^A5-40	=2^B5-40	=2^C5-40
6	=IF(D5*F5>=0, C5, A5)	=IF(E5*F5>=0,C5,B5)	=(A6+B6)/2	=2^A6-40	=2^B6-40	=2^C6-40
7	=IF(D6*F6>=0, C6, A6)	=IF(E6*F6>=0,C6,B6)	=(A7+B7)/2	=2^A7-40	=2^B7-40	=2^C7-40
8	=IF(D7*F7>=0, C7, A7)	=IF(E7*F7>=0,C7,B7)	=(A8+B8)/2	=2^A8-40	=2^B8-40	=2^C8-40
9	=IF(D8*F8>=0, C8, A8)	=IF(E8*F8>=0,C8,B8)	=(A9+B9)/2	=2^A9-40	=2^B9-40	=2^C9-40
10	=IF(D9*F9>=0, C9, A9)	=IF(E9*F9>=0,C9,B9)	=(A10+B10)/2	=2^A10-40	=2^B10-40	=2^C10-40
11	=IF(D10*F10>=0, C10, A1	=IF(E10*F10>=0,C10,B10)	=(A11+B11)/2	=2^A11-40	=2^B11-40	=2^C11-40
12	=IF(D11*F11>=0, C11, A1	=IF(E11*F11>=0,C11,B11)	=(A12+B12)/2	=2^A12-40	=2^B12-40	=2^C12-40
13	=IF(D12*F12>=0, C12, A1	=IF(E12*F12>=0,C12,B12)	=(A13+B13)/2	=2^A13-40	=2^B13-40	=2^C13-40
14	=IF(D13*F13>=0, C13, A1	=IF(E13*F13>=0,C13,B13)	=(A14+B14)/2	=2^A14-40	=2^B14-40	=2^C14-40
15	=IF(D14*F14>=0, C14, A1	=IF(E14*F14>=0,C14,B14)	=(A15+B15)/2	=2^A15-40	=2^B15-40	=2^C15-40
16	=IF(D15*F15>=0, C15, A1	=IF(E15*F15>=0,C15,B15)	=(A16+B16)/2	=2^A16-40	=2^B16-40	=2^C16-40

iii Use an IF statement (row 3) to determine which value the midpoint replaces in the interval. Here the same sign is determined by seeing if multiplying the function values results in a positive value (indicating both values are negative or both are positive). If the midpoint and minimum value produce function values of the same sign, the midpoint becomes the new minimum, otherwise the current minimum is maintained. The new maximum value is found in the same way.

iv Continue the process by filling down each of the columns until the desired accuracy is obtained. The result should be the spreadsheet values below where the solution of $2^x - 40 = 0$, correct to four decimal places, can be seen as $x = 5.3219$.

	A	B	C	D	E	F
1	*x* min	*x* max	midpoint	*f*(*x* min)	*f*(*x* max)	*f*(midpoint)
2	5	6	5.5	-8	24	5.254833996
3	5	5.5	5.25	-8	5.254833996	-1.94537232
4	5.25	5.5	5.375	-1.9453723	5.254833996	1.498865749
5	5.25	5.375	5.3125	-1.9453723	1.498865749	-0.26055001
6	5.3125	5.375	5.34375	-0.26055	1.498865749	0.60963063
7	5.3125	5.34375	5.328125	-0.26055	0.60963063	0.172184225
8	5.3125	5.328125	5.3203125	-0.26055	0.172184225	-0.04476873
9	5.3203125	5.328125	5.32421875	-0.0447687	0.172184225	0.063560892
10	5.3203125	5.32421875	5.32226563	-0.0447687	0.063560892	0.009359417
11	5.3203125	5.32226563	5.32128906	-0.0447687	0.009359417	-0.01771382
12	5.3212891	5.32226563	5.32177734	-0.0177138	0.009359417	-0.00417949
13	5.3217773	5.32226563	5.32202148	-0.0041795	0.009359417	0.00258939
14	5.3217773	5.32202148	5.32189941	-0.0041795	0.00258939	-0.00079519
15	5.3218994	5.32202148	5.32196045	-0.0007952	0.00258939	0.000897063
16	5.3218994	5.32196045	5.32192993	-0.0007952	0.000897063	5.09256E-05
17	5.3218994	5.32192993	5.32191467	-0.0007952	5.09256E-05	-0.00037214
18	5.3219147	5.32192993	5.3219223	-0.0003721	5.09256E-05	-0.00016061
19	5.3219223	5.32192993	5.32192612	-0.0001606	5.09256E-05	-5.484E-05

c Repeat the process from part **b** to solve the following equations, correct to four decimal places. Recall that to solve $f(x) = g(x)$ you can solve $f(x) - g(x) = 0$.

i $7x - 20 = 0$ **ii** $2x + 15 = 5x - 2$ **iii** $x^2 = x + 1$ (2 solutions)

d Investigate the Excel function *Goal seek* in the *Data* menu under *What-if analysis*. How could this be used to complete the above work?

1.3 Algorithms for finding square roots

The bisection method outlined in **Section 1.2** can also be used to find the square root of a number to a desired level of accuracy.

To find $\sqrt{5}$, for example, is the same as finding the positive solution $(x > 0)$ of the equation $x^2 = 5$ or $x^2 - 5 = 0$.

a Use the bisection method and a spreadsheet to find $\sqrt{5}$, correct to four decimal places.

While the bisection method is quite an efficient method for finding a square root, other numerical methods also exist.

The Babylonian algorithm, approaches the actual value of the square root very quickly and can achieve a high degree of accuracy within a few run-throughs.

The algorithm is an iterative process whereby each new approximation makes use of the previous approximation.

The basic algorithm for solving $x^2 = S$ is:

- Make an initial guess (x)
- Divide the number S by the guess and average the guess and the quotient $\frac{S}{x}$
- Make this average value the new guess and repeat the above step.

This algorithm can also be expressed as:

$$x_0 \approx \sqrt{S}$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{S}{x_n}\right)$$

where x_0 represents the initial guess and n is the number of iterations. As n increases the solution becomes more accurate.

b Try the method outlined above to evaluate the following square roots, correct to six decimal places.

i $\sqrt{5}$ **ii** $\sqrt{2}$ **iii** $\sqrt{45}$

c For $\sqrt{5}$, comment on the efficiency of the two algorithms (parts **a** and **b**) for obtaining an accurate approximation. Efficiency in this context relates to the number of algorithmic steps in a process and how quickly a solution is obtained.

d *Extension*:

i Show algebraically that $x^2 = S$ can be expressed as $x = \frac{1}{2}\left(x + \frac{S}{x}\right)$, which leads to the iterative formula seen in part **b**.

ii Use a similar process to part **i** to come up with an iterative formula to solve $x^3 = S$ and hence evaluate $\sqrt[3]{20}$, correct to six decimal places.

Activity 2: Pythagorean triples

MEASUREMENT AND GEOMETRY

In these activities you will work through some algorithms leading to writing a program to generate the Pythagorean triples. A Pythagorean triple is a set of integers x, y and z such that $x^2 + y^2 = z^2$.

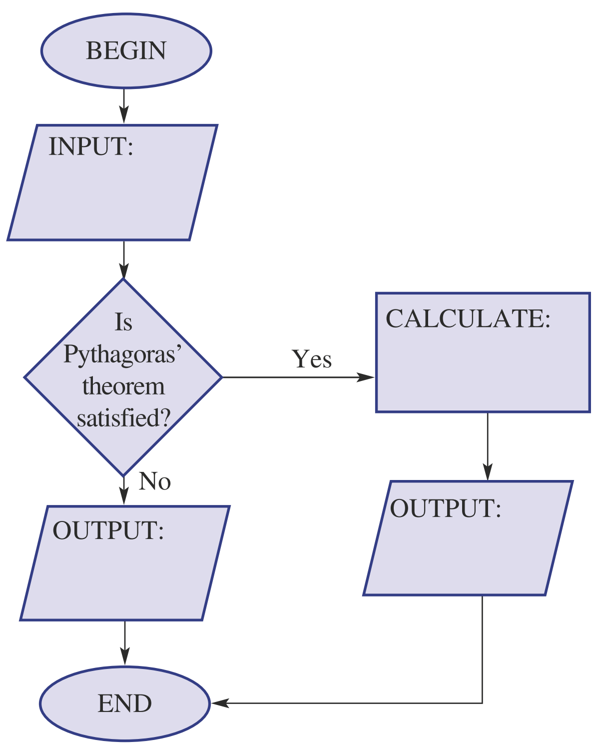
2.1 Is the triangle right-angled?

In this first activity you will design an algorithm to determine if a triangle is right-angled given the three side lengths of the triangle. If the triangle is right-angled, the program should say so and also give the two acute angles of the triangle.

a The flow chart below is set up for the algorithm for this program. Note the use of the following symbols:

for input/output stages   for process stages   for decision stages

Complete the flow chart by filling in the input, output and calculate boxes.

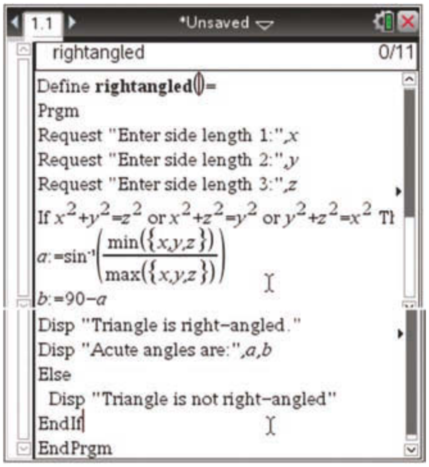


ISBN 978-1-108-77290-7

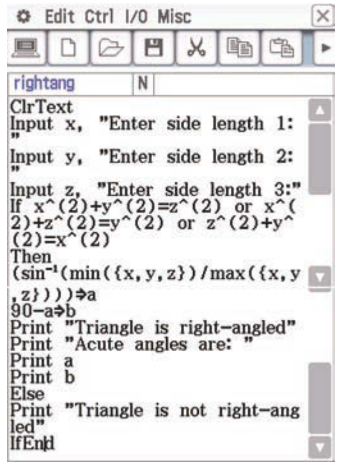
Cambridge University Press
Updated June 2022

b The screenshot below shows a program for implementing the algorithm in part **a**.

Using the TI-Nspire:



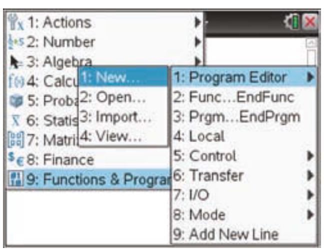
Using the ClassPad:



Read over the program and then enter it into your calculator by following these steps:

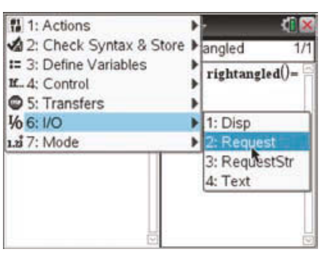
Using the TI-Nspire:

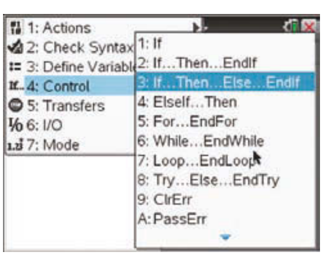
i Open a new document screen and select the **Functions & Programs** menu followed by **Program Editor** and **New**.



ii Give your program a name.

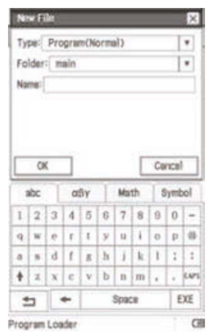
iii Type in the program. The input and output stages and the 'If Then' statement can be found in the **I/O** menu and in the **Control** menu or they can be typed directly.





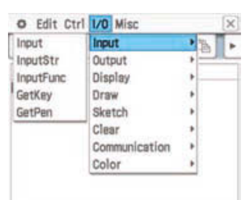
Using the ClassPad:

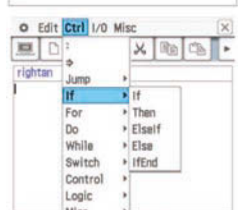
i From **Menu** choose **Program** followed by **New File**.



ii Give your program a name.

iii Type in the program. The input and output stages and the "If … Then" statement can be found in the **I/O** menu and in the **Ctrl** (Control) menu or they can be typed directly.

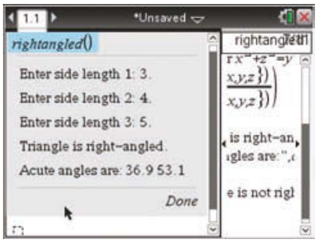




ISBN 978-1-108-77290-7 Cambridge University Press
 Updated June 2022

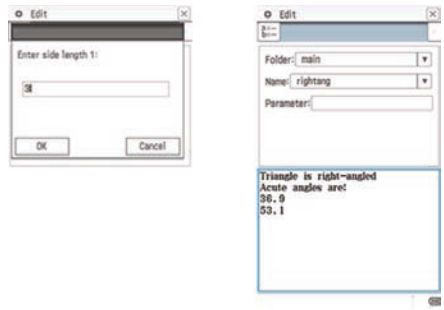
iv To save, from the menu select: **Check syntax and store.**

v To run the program, type the program name on the left-hand panel. You should be prompted for the inputs and the result will be displayed.



iv To save, from the **Edit** menu select Save or otherwise click on the disk icon.

v To run the program click on the computer icon and then the 'play' button: ▶. You will be prompted for the inputs and the result will be displayed as below for side lengths 3, 4 and 5.



vi *Extension.* Add to the above program so that it also displays the angles in a non-right-angled triangle.

The above program has a brute force approach to testing if the triangle is right-angled by testing if any of the possible combinations of side lengths satisfies Pythagoras' theorem. A more efficient algorithm would recognise that the hypotenuse has to be the longest side and that only one possible combination can work.

c The program can be adjusted so that it only tests the one case of Pythagoras' theorem by determining the longest of the three sides first and using this as the only possible hypotenuse. Complete these steps to achieve this:

i Assign the maximum of the input values as the hypotenuse c using $c:=max(\{x,y,z\})$ using the TI-Nspire or $max(\{x,y,z\})\Rightarrow c$ using the ClassPad.

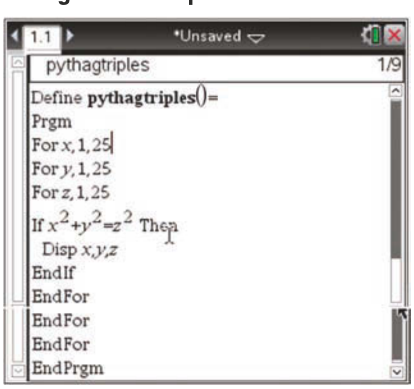
ii Use the *min* and *median* (since there are three lengths) functions to assign the two shorter side lengths a and b.

iii Test one case of Pythagoras' theorem, if $a^2+b^2=c^2$.

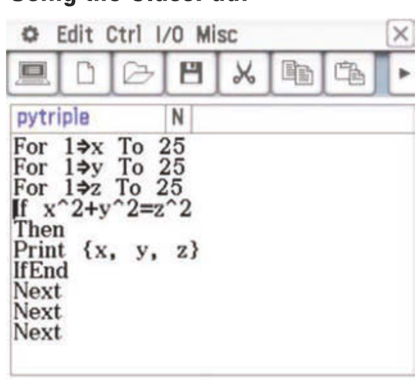
2.2 Pythagorean triples

a The program below demonstrates a brute force way of generating a list of Pythagorean triples. The first *For* loop, for instance, runs the part inside the *For* loop for $x=1$, then $x=2$ up until $x=25$.

Using the TI-Nspire:



Using the ClassPad:



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

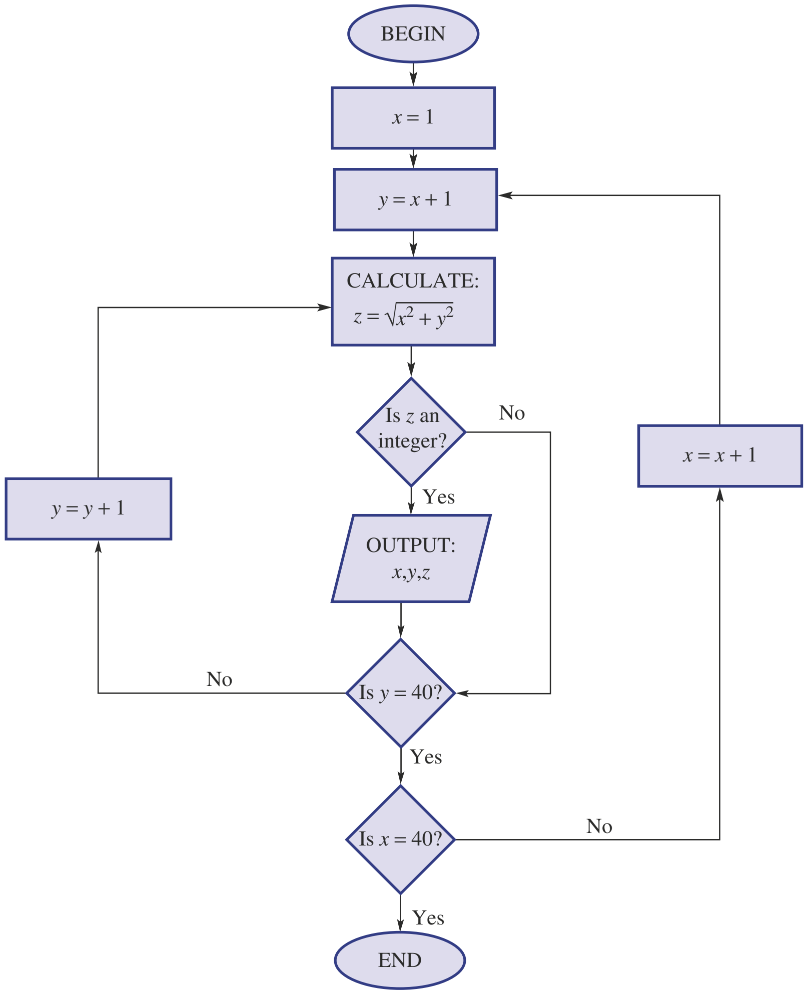
Type in and run the program and comment on any problems you see with the program both from an efficiency and an output point of view.

b Consider possible restrictions on the values of x, y and z to make the algorithm more efficient. We will keep the hypotenuse as having a longest side of 25. If x is to be designated as the shortest side it can be a *For* loop working through the values from 1 to 24.

- **i** If y is now the bigger of the two shorter sides, rewrite the *For* loop for y from its minimum possible value (this will be in terms of x) up to 24.
- **ii** Given z is the longest side, rewrite the *For* loop for z from its minimum possible value up to 25.
- **iii** Rerun your program to generate the Pythagorean triples with hypotenuse length 25 or less. What improvements do you notice from the program you ran in part **a**?

c The algorithm can have its efficiency increased further by taking out the *For* loop for z (the hypotenuse) and instead calculating the value for z based on the x- and y-values and checking if z is an integer. The flow chart for this is shown below. *For* loops can be used to run through the values of x and y and in this flow chart, we are using x- and y-values up to 40.

Implement this flow chart in a new program. (The function *fPart* may be useful as it returns the fraction part of a number. For an integer you would get *fPart* of the number equals 0.)



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2.3 Euclid's algorithm for Pythagorean triples

Euclid's formula is well known and generates the Pythagorean triples. The formula produces all the primitive triples. A primitive triple (or base triad) is one such that a, b and c have no common divisor. All multiples of these triples will also be Pythagorean triples. For example, $(3, 4, 5)$ is a primitive triple while $(6, 8, 10)$ is a Pythagorean triple that is not primitive.

The formula says that for positive integers m, n with $m > n$:

$a = m^2 - n^2$ $\qquad$ $b = 2mn$ $\qquad$ $c = m^2 + n^2$

a Show that $a^2 + b^2 = c^2$ using the formulas for a, b and c above.

To generate the primitive Pythagorean triples, it is required that $m - n$ is odd and that the greatest common divisor of m and n is 1.

b Design a flow chart to implement this algorithm and turn this into a program on your calculator. For $c \leqslant 100$ your program should generate 16 primitive Pythagorean triples. The calculator has a *gcd* function which returns the greatest common divisor of two numbers.

c *Extension.* Extend your program from part **b** to generate all the Pythagorean triples for $c \leqslant 100$.

Statue of Pythagoras on Samos Island.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Activity 3: Using simulations to find probabilities

STATISTICS AND PROBABILITY

In the past you may have seen some simple simulations carried out to determine probabilities. You would have seen that the more trials you run, the closer the experimental probability from the simulations will be to the theoretical probability. This is called the long run proportion.

Simulations can also be used to get an idea of the probability when the theoretical probability is not known or is difficult to find.

The cases below demonstrate some situations where the probability is not immediately obvious and where simulations can be used to see the event in action and the resulting probabilities.

3.1 Walk the plank

In the game of Walk the Plank, a numbered board represents the plank.

- Select a starting position somewhere along the plank.
- Toss a coin to determine whether you take one step forward (heads) or one step backwards (tails).
- Continue tossing the coin until you are either 'Safe' and back on board the ship or 'Overboard' and into the water.

a Play the game 20 times using a coin. Start from position 3 facing the deck. Count how many times you end up overboard and calculate the proportion of times you end up overboard.

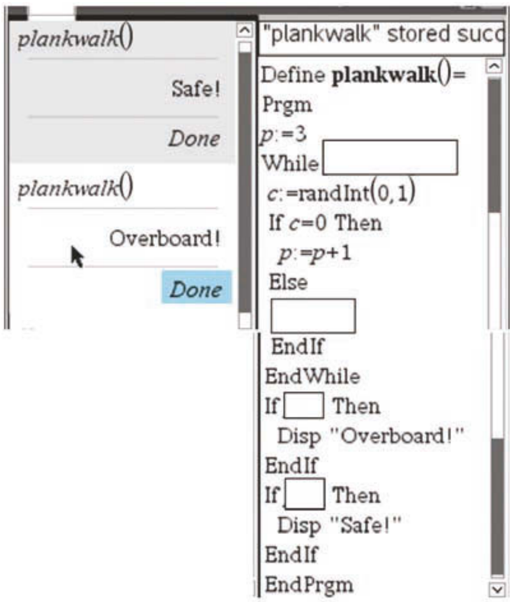
b Play the above game once more. Think about the processes involved if you were going to write this game as an algorithm. Discuss your thoughts with a partner.

ISBN 978-1-108-77290-7

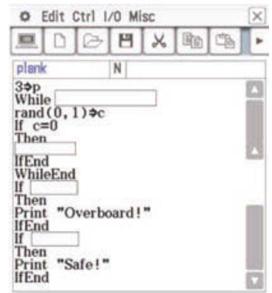
c Complete an algorithm flow chart to describe one runthrough of the game. Use the symbols used in **Activity 2**.

d By considering your flow chart above, fill in the boxes in the program below to complete it. The coin toss is simulated by generating a random number 0 or 1 and assigning 0 as heads and 1 as tails. p is the variable which stores the position on the plank.

Using the TI-Nspire:



Using the ClassPad:



Output screen

e Enter your program on your CAS calculator. The *While* loop can be found in the **Control** menu. It ensures the program keeps running while a condition is still being met.
Run your program 50 times and count the number of times overboard and hence calculate the proportion of times you end up overboard.

f Modify the above program to use a *For* loop to play the game multiple times and have a counter to count the number of times you end up overboard.

g Modify your program to cater for different start positions. For each start position, have 100 simulations of the game and record the number of times overboard in a table like the one below.

Start position, p	Overboard tally	Overboard frequency
1		
2		
3		
4		
5		

Construct a frequency histogram from your table. Comment on your results.

h *Extension*: Consider ways you could add extra elements to this game. Once you have some ideas, design the algorithm and implement the program to simulate your game.

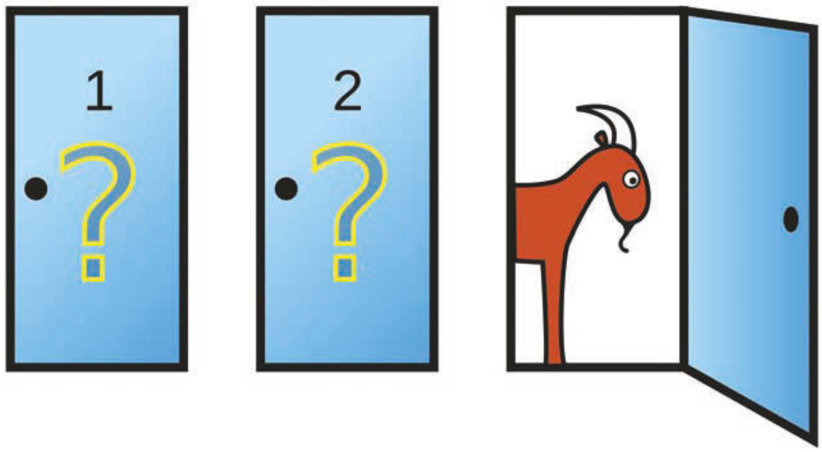
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

3.2 The Monty Hall Problem

The Monty Hall Problem is a probability puzzle based on a game show and is named after its original host.

In the game show you have three doors with a car prize behind one door and a goat behind each of the other two doors.



The contestant selects a door. The host (who knows what is behind each door) then opens one of the other two doors, always opening a door with a goat behind it. The host then asks the contestant if they want to stay with the door they have chosen or switch to the other unopened door.

The question is: Is it to your advantage to switch from your original choice or stay? Or, does it matter at all?

a With a partner, run the game above 10 times each where one of you is the host and one the contestant.

  i Record in a frequency table (like the one below) the number of times staying with the same door would have won and the number of times switching would have won out of the 20 games.

	Stay with choice		Switch choice	
	Tally	Frequency	Tally	Frequency
Winning move				

  ii From your results, come up with a hypothesis to test whether you are better off staying with your choice or switching.

  iii Combine your results with two or three other of pairs. Does your hypothesis hold up?

b To carry out the simulation 100 or more times, we can write a program to do this. Think about the processes involved in the game above. Use this to design an algorithm flow chart that runs the simulation a number of times and counts and displays how many times staying wins and how many times switching wins. Use the symbols from **Activity 2**.

ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

c A program for the Monty Hall simulation is shown below. The three doors are stored in a list. A 0 for a door represents a goat standing behind the door while a 1 indicates the prize is behind that door. Each door is initialised with a 0 in line 3 for the TI-Nspire and in the first line for the ClassPad.

Using the TI-Nspire:

```
* montyhall                                  0/20
Define montyhall()=
Prgm
doors:={0,0,0}
stay:=0
switch:=0
For i,1,100
 prize:=randInt(1,3)
 doors[prize]:=1
 choice:=randInt(1,3)
 shown:=randInt(1,3)
 While doors[shown]=1 or shown=choice
  shown:=randInt(1,3)
 EndWhile
 If doors[choice]=1 Then
  stay:=stay+1
 Else
  switch:=switch+1
 EndIf
 doors[prize]:=0
EndFor
Disp "Stay wins",stay
Disp "Switch wins",switch
EndPrgm
```

Using the ClassPad:

```
Edit Ctrl I/O Misc
monty  N
{0,0,0}⇒doors
0⇒stay
0⇒swit
For 1⇒j To 100
rand(1,3)⇒prize
1⇒doors[prize]
rand(1,3)⇒choice
rand(1,3)⇒shown
While doors[shown] =1 or sho
wn=choice
rand(1,3)⇒shown
WhileEnd
If doors[choice] =1
Then
stay+1⇒stay
Else
swit+1⇒swit
IfEnd
0⇒doors[prize]
Next
Print "Stay wins"
Print stay
stay+1⇒stay
Else
swit+1⇒swit
IfEnd
0⇒doors[prize]
Next
Print "Stay wins"
Print stay
Print "Switch wins"
Print swit
```

i How does it compare to the algorithm flow chart you prepared in part **b**?

ii Analyse the program and comment on the following:

- How many times does the simulation run?
- What is happening in line 8 for the TI-Nspire and in line 6 for the ClassPad?
- What is the purpose of the While loop (lines 11 and 12 for the TI-Nspire and lines 9–11 for the ClassPad) in this program?
- How are the stay and switch (swit) counters controlled?

iii Enter and run the program on a CAS calculator. You can alter the number of simulations to see how the results vary.

iv How do the results compare with part **a**? Does your hypothesis still hold? What would you now say in answer to the questions: Are you better off staying or switching? What is your chance of winning if you switch?

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Index

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

ISBN 978-1-108-77290-7

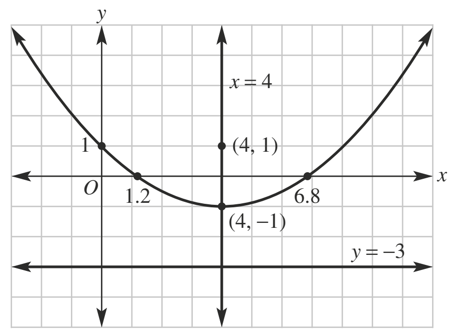
Cambridge University Press
Updated June 2022

Working with unfamiliar problems: Part I

1 $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$
2 99
3 a $x = 15$ b $x = 450$ c $x = 6$
4 $\frac{1}{8}$, \$56
5 24 cm
6 a i 11.8 seconds ii 6.5 seconds
b $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$
7 $XY = 5.6$ cm
8 35
9 72
10 Charlie 23 years, Bob 68 years
11 $\frac{D}{2}$
12 $b = 1\frac{1}{3}$
13 $k = 11$
14 $V = 27\text{ cm}^3$, TSA $= 54\text{ cm}^2$
15 4 cm < third side < 20 cm. Its length is between the addition and subtraction of the other two sides.
16 785
17 $n + 1$
18 10
19 3 : 5

Working with unfamiliar problems: Part 2

1 a i $P = 3 \times 4^5 \times \frac{x}{3^5}$ or $P = 3x\left(\frac{4}{3}\right)^5$;
$P = 3 \times 4^n \times \frac{x}{3^n}$
ii $A = \frac{\sqrt{3}}{4}x^2 + 3 \times \frac{\sqrt{3}}{4}\left(\frac{x}{3}\right)^2 +$
$3 \times 4 \times \frac{\sqrt{3}}{4}\left(\frac{x}{3^2}\right)^2 + 3 \times 4^2 \times \frac{\sqrt{3}}{4}\left(\frac{x}{3^3}\right)^2$
Area change $= 3 \times 4^{n-1} \times \frac{\sqrt{3}}{4}\left(\frac{x}{3^n}\right)^2$
b The perimeter increases indefinitely as $3x\left(\frac{4}{3}\right)^n \to \infty$ as $n \to \infty$. The area approaches a finite value as area change $\frac{\sqrt{3}}{4}x^2 \times \frac{3}{4}\left(\frac{4}{9}\right)^n \to 0$ as $n \to \infty$.
2 77 cm, 181 cm
3 $2k(2\sqrt{3} - 3)$
4 22°
5 $y = 3\frac{1}{4}$
6 $x = 0.9$, $y = 3.3$
7 $16\sqrt{10}$ cm
8 20 students; 6 with 100%, 7 with 75%, 7 with 76%, mean = 82.85%.
9 $9\frac{1}{8}$ units
10 a 11 b 28 c 20 d $\frac{3}{2}$
11 640
12 P to R: 145°, 1606 m; R to S: 295°, 789 m; S to Q: 51°, 1542 m; Q to P: 270°, 1400 m
13 $y = \frac{1}{8}x^2 - x + 1$ or $y = \frac{1}{8}(x - 4)^2 - 1$



14 a

Hours	1st and 6th	2nd and 5th	3rd and 4th
% change from equation	6.7%	18.3%	25%
% change from 'rule of thumb'	8.3%	16.7%	25%

The percentage change per hour for the 'rule of thumb' is 1.6 points higher for the 1st and 6th hours, 1.6 points lower in the 2nd and 5th hours and the same in the middle two hours. Overall, this is quite an accurate 'rule of thumb'.
b The proportion of tide height change
$= \frac{1}{2} \rightleftharpoons [\cos(30t_1) - \cos(30t_2)]$
15 $\frac{1 + \sqrt{5}}{2}$, 1.618034

Chapter 1

1A

Building understanding

1 C
2 D
3 a 1 b −4 c $-\frac{1}{5}$ d $-\frac{7}{3}$
4 a yes b yes c no
5 a 9 b −8 c −8 d −9

Now you try

Example 1
a $17a$ b $3ab^2$ c $2xy + 6x^2y$
Example 2
a $18ab$ b $-10x^3y^2$ c $-\frac{b}{2}$
Example 3
a $3x + 6$ b $-2x^2 + 2xy$ c $-4x + 9$

ISBN 978-1-108-77290-7

Example 4

a $2(x - 5)$ b $3x(x + 3)$

Example 5

13

Exercise 1A

1 a i $14a$ ii $5a$
 b i $2a^2b$ ii x^2y
 c i $xy + 5xy^2$ ii $4ab + 6ab^2$

2 a $10a$ b $15d$ c 0
 d $5xy$ e $4ab$ f $9t$
 g $9b$ h $-st^2$ i $-3m^2n$
 j $-0.7a^2b$ k $2gh + 5$ l $12xy - 3y$
 m $3a + 7b$ n $8jk - 7j$ o $ab^2 + 10a^2b$
 p $2mn - m^2n$ q $5st - s^2t$ r $3x^3y^4 + 2xy^2$

3 a $12ab$ b $25ab$ c $-6ad$
 d $-10hm$ e $30ht$ f $30bl$
 g $12s^2t$ h $-21b^2d^5$ i $8a^2b^4$
 j $24p^3q$ k $-18h^5i^5$ l $63m^2pr$
 m x n $3ab$ o $-\frac{a}{3}$
 p $-\frac{ab}{4}$ q $2b$ r $-3x$ s $-\frac{y}{2}$ t $-\frac{a}{2}$

4 a $5x + 5$ b $2x + 8$ c $3x - 15$
 d $-20 - 5b$ e $-2y + 6$ f $-7a - 7c$
 g $6m + 18$ h $4m - 12n + 20$ i $-2p + 6q + 4$
 j $2x^2 + 10x$ k $6a^2 - 24a$
 l $-12x^2 + 16xy$ m $15y^2 + 3yz - 24y$
 n $36g - 18g^2 - 45gh$ o $-8ab + 14a^2 - 20a$
 p $14y^2 - 14y^3 - 28y$ q $-6a^3 + 3a^2 + 3a$
 r $-5t^4 - 6t^3 - 2t$ s $6m^4 - 2m^3 + 10m^2$
 t $x^4 - x$ u $3s^4 - 6st$

5 a $5x + 23$ b $10a + 26$
 c $21y + 3$ d $15m + 6$
 e 10 f $11t - 1$
 g $3x^2 + 15x$ h $15z - 7$
 i $-11d^3$ j $9q^4 - 9q^3$

6 a $3(x - 3)$ b $4(x - 2)$ c $10(y + 2)$
 d $6(y + 5)$ e $x(x + 7)$ f $2a(a + 4)$
 g $5x(x - 1)$ h $9y(y - 7)$ i $xy(1 - y)$
 j $x^2y(1 - 4y)$ k $8a^2(b + 5)$ l $ab(7a + 1)$
 m $-5t(t + 1)$ n $-6mn(1 + 3n)$ o $-y(y + 8z)$

7 a -32 b 7 c 61 d 12
 e $-\frac{1}{2}$ f $\frac{13}{5}$ g $-\frac{7}{5}$ h 1

8 a $2x^2 + 6x$ b $x^2 - 5x$

9 a $P = 4x - 4, A = x^2 - 2x - 4$
 b $P = 4x + 2, A = 3x - 1$
 c $P = 4x + 14, A = 7x + 12$

10 a $(-2)(-2) = 4$, negative signs cancel
 b $a^2 > 0 \therefore -a^2 < 0$
 c $(-2)^3 = (-2)(-2)(-2) = -8$

11 a True b False, $1 - 2 \neq 2 - 1$
 c True d False, $\frac{1}{2} \neq \frac{2}{1}$
 e True f False, $3 - (2 - 1) \neq (3 - 2) - 1$
 g True h False, $8 \div (4 \div 2) \neq (8 \div 4) \div 2$

12 a $\frac{x + y}{2}$ or $x + \frac{y}{2}$
 b It could refer to either of the above, depending on interpretation.
 c 'Half of the sum of a and b' or 'a plus b all divided by 2'.

13 a $P = \left(4 + \frac{\pi}{2}\right)x + 2, A = \left(1 + \frac{\pi}{4}\right)x^2 + x$
 b $P = \left(6 + \frac{\pi}{2}\right)x - 6, A = \left(3 - \frac{\pi}{4}\right)x^2 - 3x$
 c $P = 2\pi x, A = \left(1 + \frac{\pi}{2}\right)x^2$

1B

Building understanding

1 a 1 b $\frac{5}{6}$ c 2 d $\frac{3}{4}$

2 a $\frac{2}{3}$ b $\frac{3}{7a}$ c $-\frac{7t}{4xy}$ d $-\frac{b^2c}{8x^2a}$

3 a $5x$ b $4x$ c $\frac{a}{4}$ d $\frac{1}{3a}$

Now you try

Example 6

a $3ab$ b $1 - 2x$

Example 7

a $\frac{a + 1}{2a}$ b 6

Exercise 1B

1 a i $3ab$ ii $2xy$
 b i $1 - 2x$ ii $1 - x$

2 a $5x$ b $-2x$ c $-9b$ d $-2y$
 e $-\frac{1}{2p}$ f $-\frac{4}{9st}$ g $-\frac{3x}{y}$ h $\frac{6b}{7}$

3 a $x + 2$ b $a - 5$ c $3x - 9$ d $1 - 3y$
 e $1 + 6b$ f $1 - 3x$ g $3 - t$ h $x - 4$
 i $x + 2$ j $3 - 2x$ k $a - 1$ l $\frac{1 + 2a}{3}$

4 a $\frac{x - 1}{2x}$ b $\frac{x + 4}{5x}$ c -4
 d $\frac{4}{9}$ e 5 f $\frac{5a}{2}$
 g 2 h 15 i $-\frac{1}{2}$

5 a 3 b 3 c $\frac{18}{5}$
 d $\frac{3}{4}$ e $\frac{4}{3}$ f $\frac{1}{25}$
 g $-\frac{5}{3}$ h $\frac{2}{5}$ i $-\frac{1}{3}$

6 a $x + 1$ b 2 c 4

7 a $3x$ b $\frac{4}{3a}$ c $\frac{x + 3}{5}$
 d $\frac{4}{x}$ e $\frac{4}{7x}$ f $\frac{2b^2}{b - 1}$

ISBN 978-1-108-77290-7

Cambridge University Press

Updated June 2022

8 a $\frac{10}{x+3}$ b $\frac{3x}{1-x}$ c $\frac{3(x+2)}{2}$
d $\frac{10x}{3}$ e $\frac{x-1}{2x}$ f $\frac{35x^2}{(2-x)(x-1)}$

9 a $x-1$ b $3(x+2)$ c $2(x-3)$
d $\frac{4}{x+2}$ e $\frac{-5}{1-x}$ f $4(x-1)$

10 a–c Factorise and cancel to 1.

11 a $1-x=-(x-1)$
b i $\frac{-7}{3}$ ii $\frac{-12}{x}$ iii $\frac{7}{2}$

12 a $\frac{x+2}{2}$ b x c $\frac{1}{3}$
d $\frac{3}{2(x+2)}$ e $4(a+1)$ f $\frac{1}{(a+1)(a-3)}$
g $\frac{x-y}{xy}$ h $\frac{(y+2)}{x}$

1C

Building understanding

1 a $2x-4$ b $-x-6$ c $-6x+12$

2 a $\frac{5}{6}$ b $\frac{17}{15}$ c $\frac{5}{14}$ d $\frac{17}{6}$

3 a 12 b 6 c 14 d $2x$

Now you try

Example 8

a $\frac{5-2a}{6}$ b $\frac{3a+8}{4a}$

Example 9

a $\frac{5x-4}{6}$ b $\frac{13x-6}{10}$

Example 10

$\frac{x+19}{(x-5)(x+1)}$

Exercise 1C

1 a i $\frac{1-2a}{4}$ ii $\frac{3-2a}{10}$
b i $\frac{a+6}{3a}$ ii $\frac{3a+35}{7a}$

2 a $\frac{3a+14}{21}$ b $\frac{4a+3}{8}$
c $\frac{3-15b}{10}$ d $\frac{4x+6}{15}$
e $\frac{1-6a}{9}$ f $\frac{2a}{15}$
g $\frac{3x}{20}$ h $\frac{11b}{14}$

3 a $\frac{2a+15}{3a}$ b $\frac{3a+8}{4a}$
c $\frac{7a-27}{9a}$ d $\frac{16-3b}{4b}$
e $\frac{4b-21}{14b}$ f $\frac{27-14y}{18y}$
g $\frac{-12-2x}{3x}$ h $\frac{-27-2x}{6x}$

4 a $\frac{9x+23}{20}$ b $\frac{7x+11}{12}$ c $\frac{3x+1}{4}$
d $\frac{4x+9}{9}$ e $\frac{8x-1}{6}$ f $\frac{8x+3}{10}$
g $\frac{7x+2}{24}$ h $\frac{5x-1}{5}$ i $\frac{x+1}{14}$

5 a $\frac{x+5}{6}$ b $\frac{6x+5}{12}$ c $\frac{-2x+38}{15}$
d $\frac{3x-23}{14}$ e $\frac{14x-8}{21}$
f $\frac{18x-9}{6}=\frac{6x-3}{2}$ g $\frac{x+14}{30}$
h $\frac{-14x-7}{15}$ i $\frac{-3x+10}{4}$

6 a $\frac{7x+22}{(x+1)(x+4)}$ b $\frac{7x-13}{(x-7)(x+2)}$
c $\frac{3x-1}{(x-3)(x+5)}$ d $\frac{x-18}{(x+3)(x-4)}$
e $\frac{-21}{(2x-1)(x-4)}$ f $\frac{14x-26}{(x-5)(3x-4)}$
g $\frac{41-7x}{(2x-1)(x+7)}$ h $\frac{3x+17}{(x-3)(3x+4)}$
i $\frac{14-17x}{(3x-2)(1-x)}$

7 a i a^2 ii x^2
b i $\frac{2a-3}{a^2}$ ii $\frac{a^2+a-4}{a^2}$ iii $\frac{3x+14}{4x^2}$

8 The 2 in the second numerator needs to be subtracted, $\frac{x-2}{6}$.

9 a $-(3-2x)=-3+2x\,(-1\times(-2x)=2x)$
b i $\frac{2}{x-1}$ ii $\frac{2x}{3-x}$ iii $\frac{x+3}{7-x}$

10 a -1 b $\frac{5a+2}{a^2}$ c $\frac{3x+5}{(x+1)^2}$
d $\frac{3x-x^2}{(x-2)^2}$ e $\frac{21x-9x^2}{14(x-3)^2}$ f $\frac{yz-xz-xy}{xyz}$

11 a 2 b 1

1D

Building understanding

1 a no b no c yes d yes

2 a true b false c false

3 a false b true c true

4 a 5 b 8 c -3 d 4

Now you try

Example 11

a $x=3$ b $x=-\frac{2}{3}$

Example 12

a $x=11$ b $x=-1$ c $x=-7$

Answers

1B

Exercise 1D

1 a i 3 ii 5
b i −3 ii −5

2 a $\frac{5}{2}$ b $\frac{11}{4}$ c $-\frac{1}{3}$
d $-\frac{11}{6}$ e −4 f $\frac{3}{2}$
g $-\frac{9}{2}$ h $-\frac{4}{3}$ i −2
j 7 k −2 l $\frac{11}{9}$

3 a 1 b 9 c $\frac{23}{2}$
d $-\frac{5}{6}$ e $-\frac{9}{11}$ f $\frac{2}{3}$
g 1 h 2 i 4
j 7 k −9 l 5
m 19 n 23 o 1

4 a 10 b 13 c −22 d 4
e −5 f 6 g 16 h 4
i −9 j 8 k 6 l −7
m 20 n 15 o −9 p 5

5 a $x + 3 = 7, x = 4$ b $x + 8 = 5, x = -3$
c $x - 4 = 5, x = 9$ d $15 - x = 22, x = -7$
e $2x + 5 = 13, x = 4$ f $2(x - 5) = -15, x = -\frac{5}{2}$
g $3x + 8 = 23, x = 5$ h $2x - 5 = x - 3, x = 2$

6 a 1 b 0 c −17 d $\frac{7}{2}$ e $\frac{27}{23}$
f $\frac{28}{5}$ g $\frac{13}{14}$ h $\frac{2}{5}$ i $\frac{86}{19}$

7 a 1 b 6 c 2 d 25

8 17 cm

9 17 and 18

10 24 km

11 a \$214
b \$582
c i 1 ii 10.5 iii 21

12 a 41 L
b 90 s = 1 min 30 s
c 250 s = 4 min 10 s

13 a 6 b 4 c −15 d 20 e 3
f 6 g 1 h −26 i −10

14 $x = 9$. Method 2 is better, expanding the brackets is unnecessary, given 2 is a factor of 8.

15 a $5 - a$ b $\frac{a}{6}$ c $\frac{5}{a}$
d $\frac{2a + 1}{a}$ e $\frac{3a + 1}{a}$ f $\frac{c - b}{a}$

16 a $a = \frac{c}{b + 1}$ b $a = \frac{b}{b + 1}$ c $a = \frac{1}{c - b}$
d $a = \frac{b}{b - 1}$ e $a = -b$ f $a = \frac{bc}{b - c}$

17 a $6a$ b $\frac{ab}{a + b}$ c $\frac{abc}{b - a}$

1E

Building understanding

1 a 3, 6, 10 (Answers may vary.)
b −4, −3, −2 (Answers may vary.)
c 5, 6, 7 (Answers may vary.)
d −8.5, −8.4, −8.3 (Answers may vary.)

2 a B b C c A

3 11, 12 or 13 rabbits

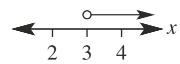
Now you try

Example 13

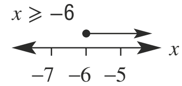
a $x < 2$ b $-4 < x \leqslant -1$

Example 14

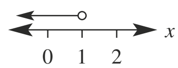
a $x > 3$



b $x \geqslant -6$



c $x < 1$

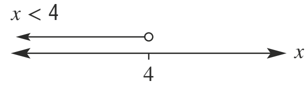


Exercise 1E

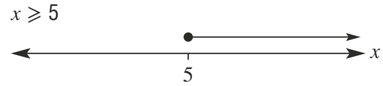
1 a i $x > 3$ ii $x \leqslant 5$
b i $-2 \leqslant x < 1$ ii $0 \leqslant x < 3$

2 a $x \geqslant 1$ b $x < 7$
c $x \leqslant 4$ d $x > -9$
e $-2 < x \leqslant 1$ f $8 < x \leqslant 11$
g $-9 < x < -7$ h $1.5 \leqslant x \leqslant 2.5$
i $-1 \leqslant x < 1$

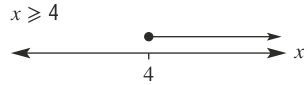
3 a $x < 4$



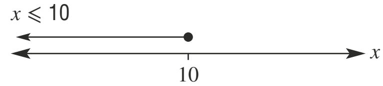
b $x \geqslant 5$



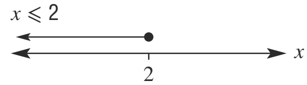
c $x \geqslant 4$



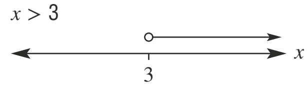
d $x \leqslant 10$



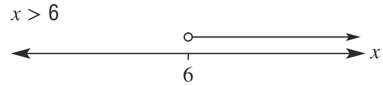
e $x \leqslant 2$



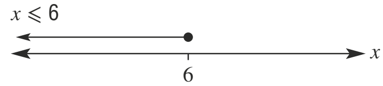
f $x > 3$



g $x > 6$

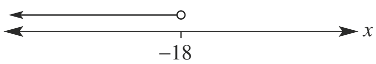


h $x \leqslant 6$

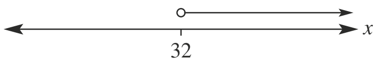


ISBN 978-1-108-77290-7 Cambridge University Press
 Updated June 2022

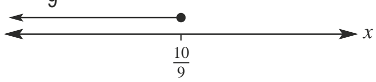
i $x < -18$



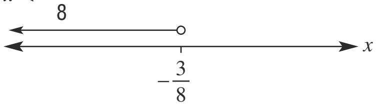
j $x > 32$



k $x \leqslant \frac{10}{9}$



l $x < -\frac{3}{8}$



4 a $x \geqslant -\frac{2}{5}$ b $x < 2$ c $x \leqslant -5$
d $x \leqslant -7$ e $x < -8$ f $x \geqslant 4$
g $x \geqslant -10$ h $x < -21$

5 a $x > 6$ b $x \leqslant 2$ c $x < \frac{5}{2}$
d $x \geqslant 10$ e $x \leqslant \frac{1}{16}$ f $x < \frac{11}{4}$

6 a $2x + 7 < 12, x < \frac{5}{2}$
b $4 - \frac{x}{2} \geqslant -2, x \leqslant 12$
c $3(x + 1) \geqslant 2, x \geqslant -\frac{1}{3}$
d $x + (x + 2) \leqslant 24, x \leqslant 10$ since x must be even
e $(x - 6) + (x - 4) + (x - 2) + x \leqslant 148, x \leqslant 40$

7 a i $C < \$1.30$ ii $C > \$2.30$
b i less than 9 min ii 16 min or more

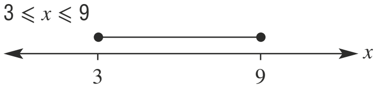
8 a $x < -5$ b $x \geqslant \frac{11}{4}$ c $x \geqslant \frac{11}{29}$
d $x \leqslant \frac{14}{5}$ e $x \geqslant \frac{27}{29}$ f $x < \frac{1}{2}$

9 a An infinite number of whole numbers (all the ones greater than 8).
b 1, 3 is the only whole number.

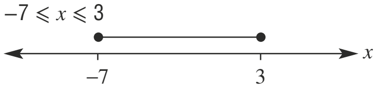
10 a $x \geqslant \frac{a + 3}{10}$
b $x < 2 - 4a$ if $a > 0$ and $x > 2 - 4a$ if $a < 0$
c $x < 1 - \frac{7}{a}$ or $x < \frac{a - 7}{a}$ if $a > 0$ and $x > 1 - \frac{7}{a}$ or $x > \frac{a - 7}{a}$ if $a < 0$.

11 a $-4 \leqslant x < 5$ b $-9.5 < x \leqslant -7$ c $x = 10$

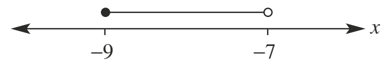
12 a $3 \leqslant x \leqslant 9$



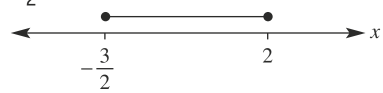
b $-7 \leqslant x \leqslant 3$



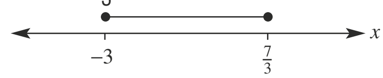
c $-9 \leqslant x < -7$



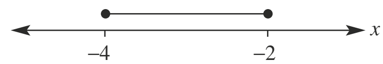
d $-\frac{3}{2} \leqslant x \leqslant 2$



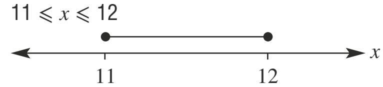
e $-3 \leqslant x \leqslant \frac{7}{3}$



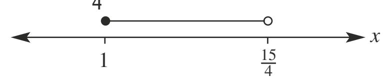
f $-4 \leqslant x \leqslant -2$



g $11 \leqslant x \leqslant 12$



h $1 \leqslant x < \frac{15}{4}$



13 a $x \geqslant 23$ b $x < \frac{19}{5}$ c $x \leqslant 1$

1F

Building understanding

1 a $y = -2x + 5, m = -2, c = 5$
b $y = 2x - 3, m = 2, c = -3$
c $y = x - 7, m = 1, c = -7$
d $y = -\frac{2x}{5} - \frac{3}{5}, m = -\frac{2}{5}, c = -\frac{3}{5}$

2 a i 3 ii 6 iii $\frac{21}{2}$
b i 2 ii 6 iii $\frac{8}{3}$

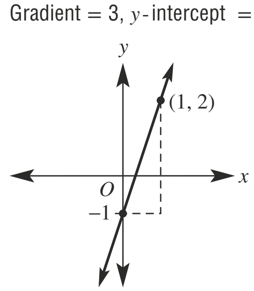
3 a A b D c B d C e E f F

4 a $x = 2$ b $y = 2$

Now you try

Example 15
a yes b no

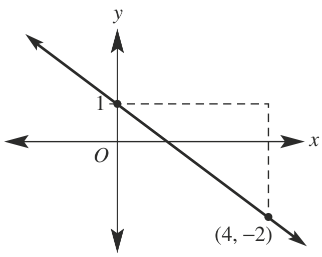
Example 16
a Gradient $= 3$, y-intercept $= -1$



ISBN 978-1-108-77290-7

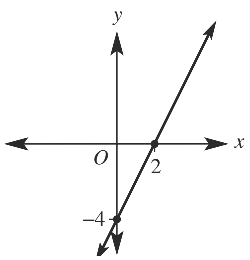
Cambridge University Press
Updated June 2022

b Gradient $= -\frac{3}{4}$, y-intercept $= 1$

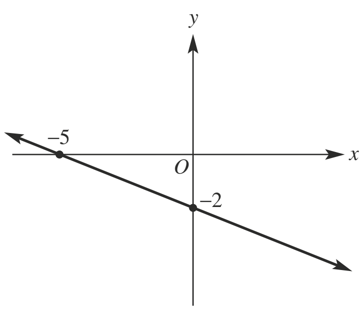


Example 17

a

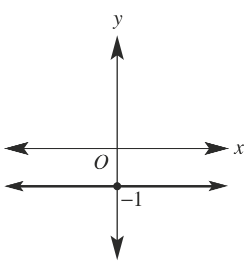


b

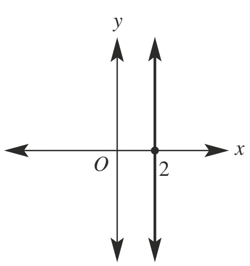


Example 18

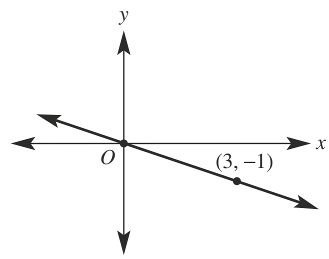
a



b



c

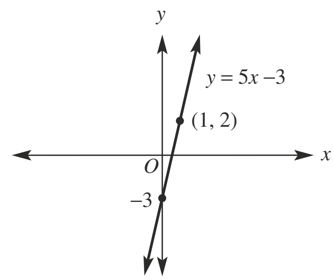


Exercise 1F

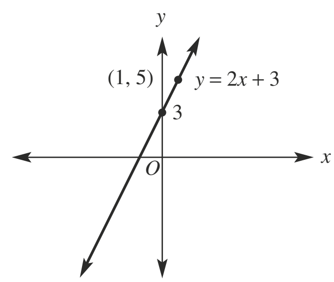
1 **a** **i** yes **ii** no
  b **i** no **ii** yes

2 **a** yes **b** yes **c** no
  d no **e** yes **f** no

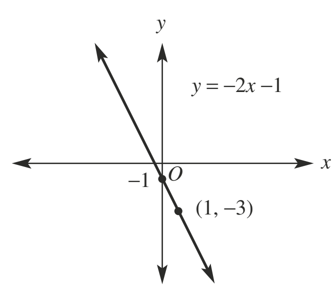
3 **a** $m = 5$, $c = -3$



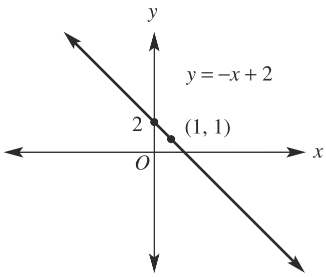
b $m = 2$, $c = 3$



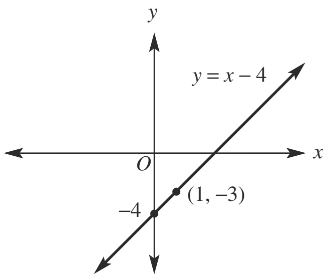
c $m = -2$, $c = -1$



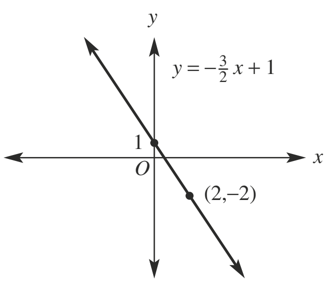
d $m = -1, c = 2$



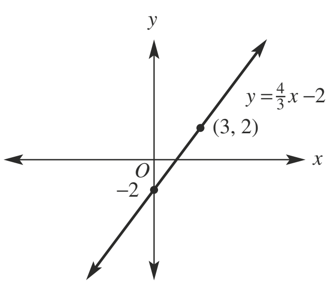
e $m = 1, c = -4$



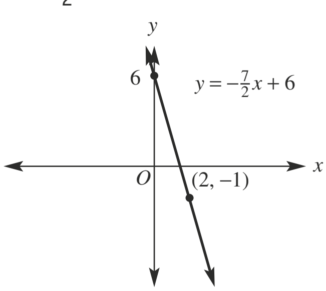
f $m = -\frac{3}{2}, c = 1$



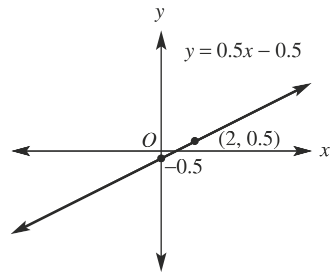
g $m = \frac{4}{3}, c = -2$



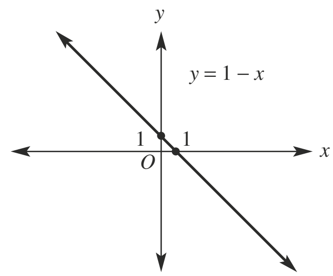
h $m = -\frac{7}{2}, c = 6$



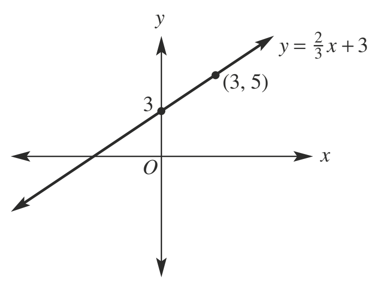
i $m = 0.5, c = -0.5$



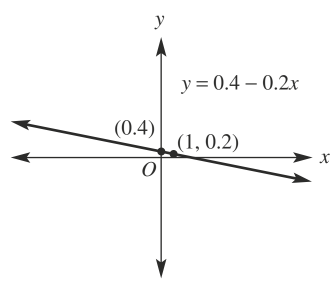
j $m = -1, c = 1$



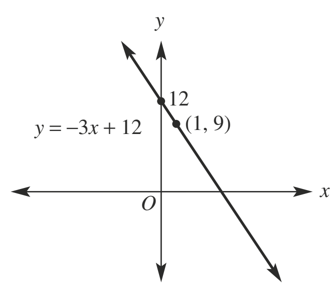
k $m = \frac{2}{3}, c = 3$



l $m = -0.2, c = 0.4$



4 a $m = -3, c = 12$



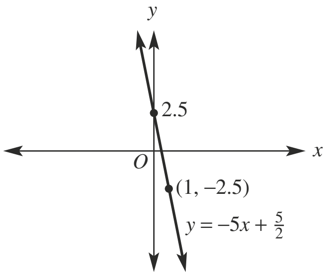
Answers

1F

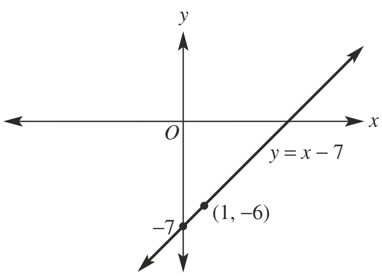
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

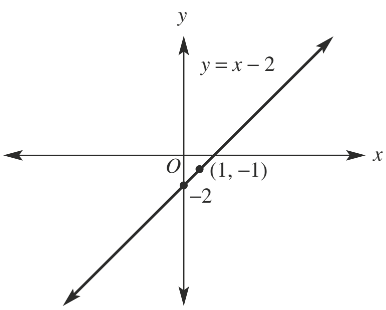
b $m = -5, c = \frac{5}{2}$



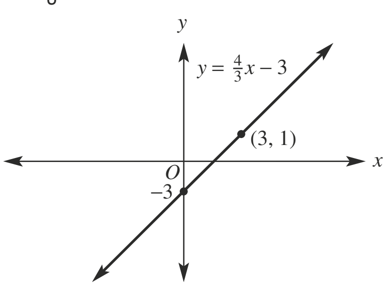
c $m = 1, c = -7$



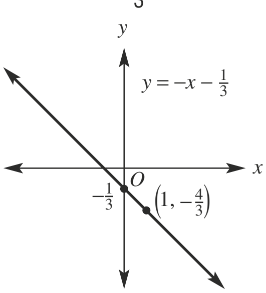
d $m = 1, c = -2$



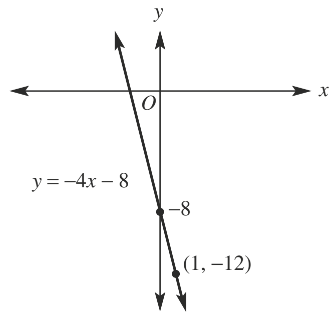
e $m = \frac{4}{3}, c = -3$



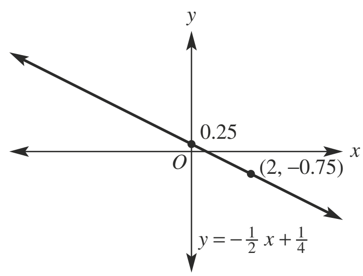
f $m = -1, c = -\frac{1}{3}$



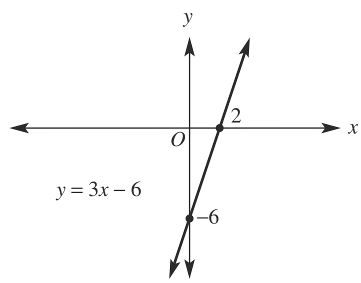
g $m = -4, c = -8$



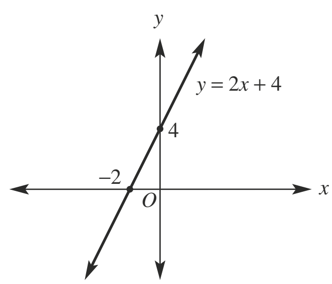
h $m = -\frac{1}{2}, c = \frac{1}{4}$



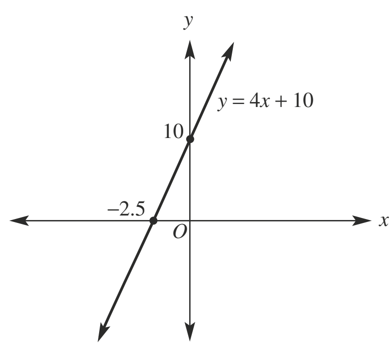
5 **a** $x = 2, y = -6$



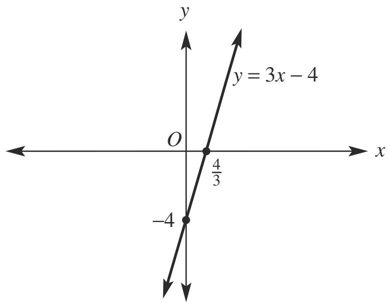
b $x = -2, y = 4$



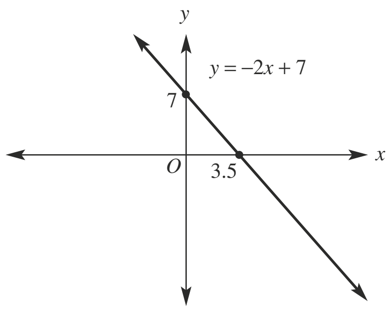
c $x = -2.5, y = 10$



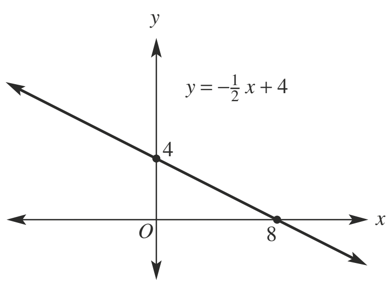
d $x = \frac{4}{3}, y = -4$



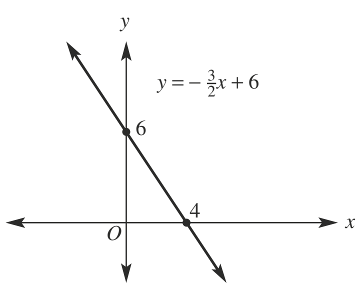
e $x = 3.5, y = 7$



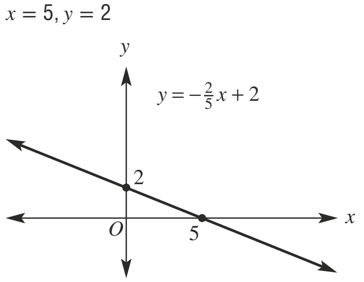
f $x = 8, y = 4$



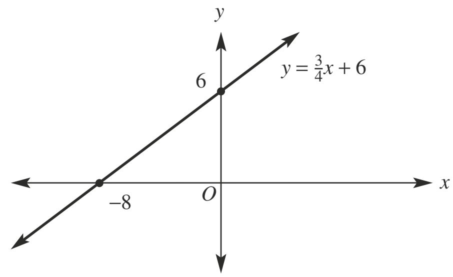
g $x = 4, y = 6$



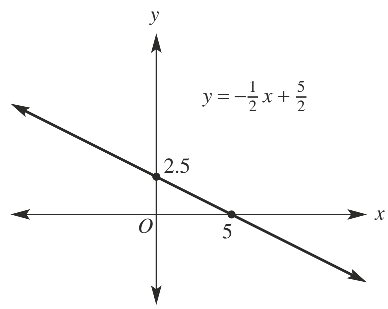
h $x = 5, y = 2$



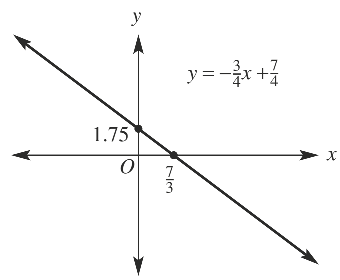
i $x = -8, y = 6$



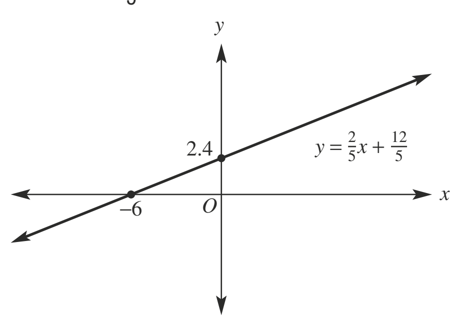
j $x = 5, y = 2.5$



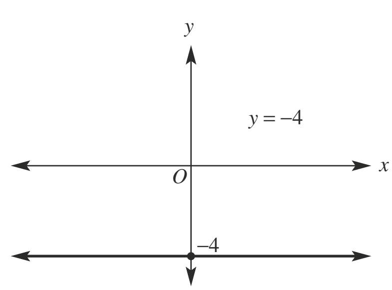
k $x = \frac{7}{3}, y = \frac{7}{4}$



l $x = -6, y = \frac{12}{5}$



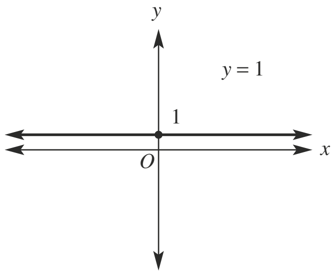
6 a



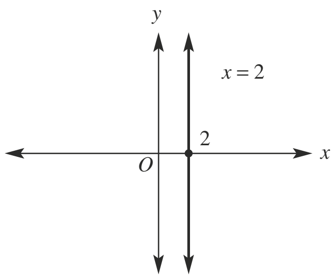
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

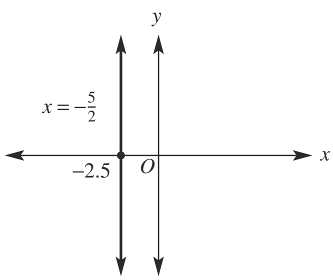
b



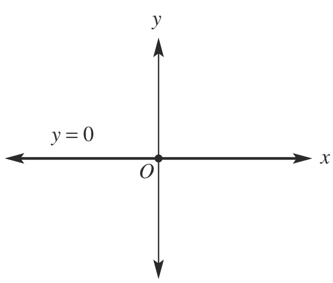
c



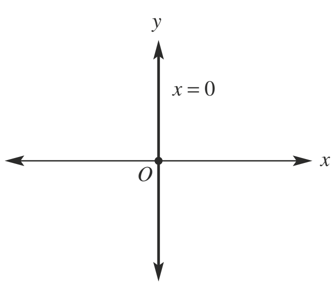
d



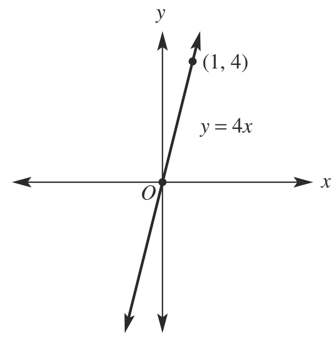
e



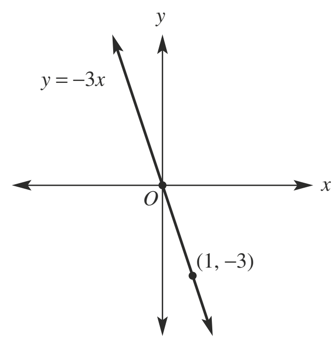
f



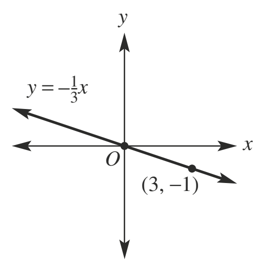
g



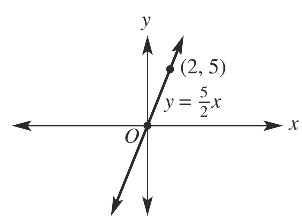
h



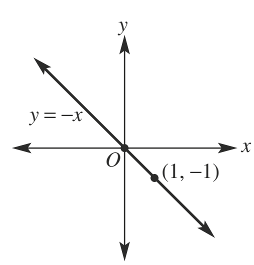
i



j



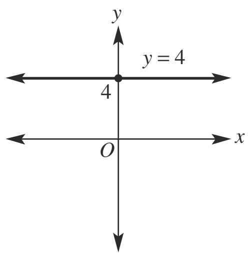
k



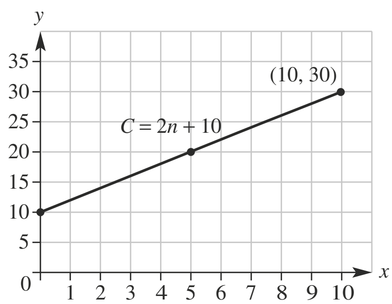
ISBN 978-1-108-77290-7

Cambridge University Press

Updated June 2022

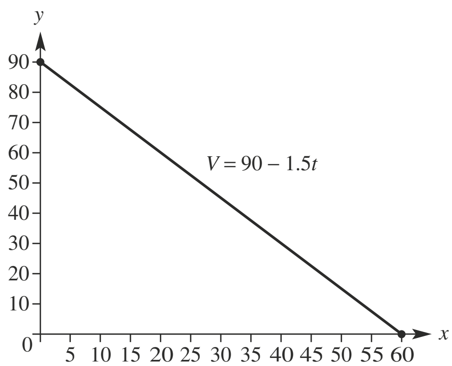
l 

7 a $C = 2n + 10$

b 

c i $28 ii 23.5 kg

8 a $V = 90 - 1.5t$

b 

c i 82.5 L ii 60 hours

9 a $7 per hour b $P = 7t$

10 a $0.05/km b $C = 0.05k$

c $C = 1200 + 0.05k$

11 a $m = 25$, 25 km per hour i.e. speed

b The cyclist started 30 km from home.

c (0, 30)

12 a $y = x + \frac{1}{2}$, gradient = 1

b $y = 0.5x + 1.5$, y-intercept = 1.5

c $y = -3x + 7$, gradient = –3

d $y = \frac{1}{2}x - 2$, gradient = $\frac{1}{2}$

13 a gradient = $\frac{3}{a}$, y-intercept = $\frac{7}{a}$

b gradient = a, y-intercept = $-b$

c gradient = $-\frac{a}{b}$, y-intercept = $\frac{3}{b}$

14 a $\frac{d}{a}$ b $\frac{d}{b}$ c $-\frac{a}{b}$

15 a 12 sq. units b 9 sq. units c $\frac{121}{4}$ sq. units

d $\frac{121}{5}$ sq. units e $\frac{32}{3}$ sq. units

1G

Building understanding

1 a 2 b 3 c 0

d –4 e –3 f undefined

2 a $c = 8$ b $c = -6$ c $c = 12$

Now you try

Example 19

$-\frac{7}{5}$

Example 20

$y = 2x + 1$

Example 21

$y = -\frac{5}{4}x + 12$

Exercise 1G

1 a –2 b $\frac{-6}{5}$

2 a $\frac{1}{4}$ b 2 c $\frac{5}{2}$ d 3

e 0 f 0 g –1 h $\frac{5}{2}$

i $-\frac{5}{7}$ j undefined k $\frac{3}{2}$ l $-\frac{3}{2}$

3 a $y = x + 3$ b $y = x - 2$ c $y = 3x + 6$

d $y = -3x + 4$ e $y = 4$ f $y = -7x - 10$

4 a $y = 2x + 4$ b $y = 4x - 5$ c $y = x - 4$

d $y = -2x + 12$ e $y = -3x - 4$ f $y = -3x - 2$

5 a $y = 3x + 5$ b $y = -2x + 4$

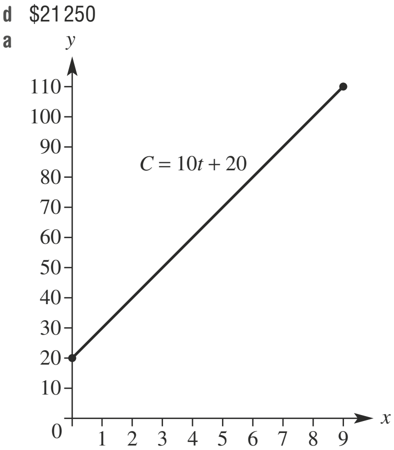
c $y = \frac{1}{2}x - \frac{3}{2}$ d $y = -2x - 2$

6 a $A = 500t + 15\,000$

b 15 000

c 4 years more, i.e 10 years from investment

d $21 250

7 a 

b $C = 10t + 20$

c i $10 per hour ii $20 up-front fee

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

8 a i $V = 4t$ ii $V = 3t$
iii $V = t + 1$ iv $V = 1.5t + 2$
b 1 L, 2 L
c Initially the flask contains b litres and it is losing 1 litre per minute.

9 a $m = \frac{-5}{5} = -1$
b $m = \frac{5}{-5} = -1$
c It doesn't matter in which pair of points is (x_1, y_1) and which is (x_2, y_2).

10 a $-\frac{4}{3}$ b $y = -\frac{4x}{3} + \frac{13}{3}$
c $y = -\frac{4x}{3} + \frac{13}{3}$
d The results from parts **b** and **c** are the same (when simplified). So it doesn't matter which point on the line is used in the formula $y - y_1 = m(x - x_1)$.

11 a i $\frac{1}{50} = 0.02$ ii $\frac{2}{50} = 0.04$
b i $y = 0.02x + 1.5$ ii $y = 0.04x + 1.5$
c The archer needs m to be between 0.02 and 0.04 to hit the target.

Progress quiz

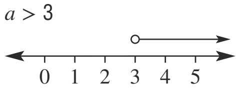
1 a $9a^2b + 2ab + 8b$ b $-12x^3y$ c $13m + 14$

2 a $4k$ b $a - 4$ c $\frac{7 - x}{2}$
d $\frac{m + 3}{3m}$ e $\frac{9a}{2}$ f $\frac{5}{2}$

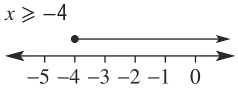
3 a $\frac{6 + m}{8}$ b $\frac{4x - 15}{6x}$
c $\frac{14 - 3a}{24}$ d $\frac{3m - 13}{(m - 1)(m - 3)}$

4 a $x = 5$ b $k = -\frac{3}{2}$ c $m = 30$ d $a = -\frac{9}{2}$

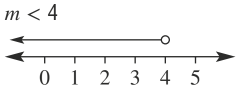
5 a $a > 3$



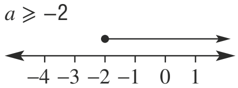
b $x \geqslant -4$



c $m < 4$

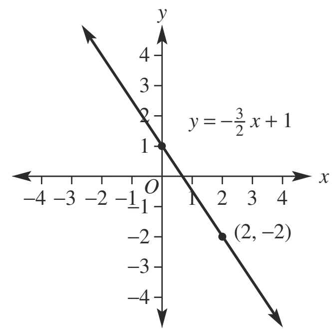


d $a \geqslant -2$

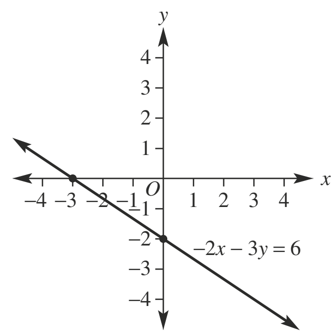


6 a (−3, 2) is not on the line.
b (−3, 2) is on the line.

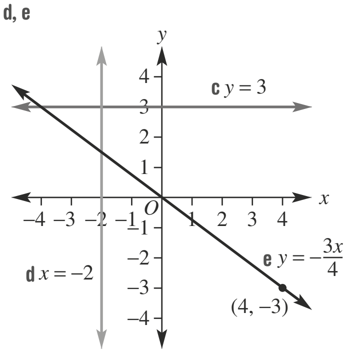
7 a gradient $= -\frac{3}{2}$, y-intercept $= 1$



b x-intercept $= -3$, y-intercept $= -2$



c, d, e



8 a $m = 3$, y-intercept $= -2$
b $m = -\frac{3}{5}$, y-intercept $= 3$

9 a $y = 2x + 3$ b $y = -\frac{3}{2}x + 8$ c $x = 5$

1H

Building understanding

1 a 4 b 5 c $\sqrt{41}$ d $\left(3, \frac{9}{2}\right)$

2 a 4 b 4
c $\sqrt{32} = 4\sqrt{2}$ d (0, −3)

3 a 2 b $\frac{11}{2}$ or 5.5
c 3 d −4

ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

Now you try

Example 22

a $\sqrt{5}$ **b** $\sqrt{89}$

Example 23

$\left(\frac{1}{2}, -4\right)$

Example 24

$a = 4$ or $a = 10$

Exercise 1H

1 **a** **i** $\sqrt{5}$ **ii** $\sqrt{20}$
b **i** $\sqrt{130}$ **ii** $\sqrt{90}$

2 **a** $\sqrt{29}$ **b** $\sqrt{58}$ **c** $\sqrt{37}$
d $\sqrt{65}$ **e** $\sqrt{37}$ **f** 15
g $\sqrt{101}$ **h** $\sqrt{193}$ **i** $\sqrt{37}$

3 **a** (1, 6.5) **b** (1.5, 2.5) **c** (–0.5, 1)
d (–1, 4.5) **e** (1, –1.5) **f** (–3.5, 3)
g (–3, –0.5) **h** (2, 2.5) **i** (–7, 10.5)

4 B and C are both 5 units away from (2, 3).

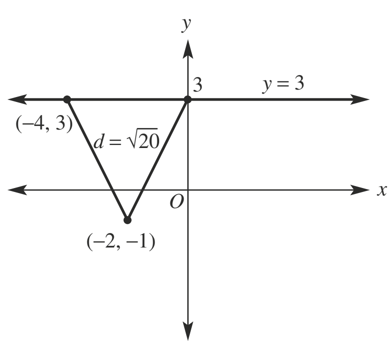
5 **a** $a = 3, b = 5$ **b** $a = -4, b = 5$
c $a = -2, b = 2$ **d** $a = 11, b = 2$

6 **a** 3, 7 **b** –1, 3 **c** –1, 9 **d** –6, 0

7 **a** 1478 m **b** 739 m

8 **a** (–0.5, 1) **b** (–0.5, 1)
c These are the same. The order of the points doesn't matter since addition is commutative $(x_1 + x_2 \ldots)$ $(x_1 + x_2 = x_2 + x_1)$.
d 5 **e** 5
f The order of the points doesn't matter $(x - y)^2 = (y - x)^2$, as $(-3)^2 = (3)^2$.

9 $a = -4, 0$



10 **a** $\left(\frac{1}{2}, 2\right)$ **b** $\left(-\frac{1}{3}, \frac{4}{3}\right)$ **c** $\left(\frac{4}{3}, \frac{8}{3}\right)$
d $\left(2, \frac{16}{5}\right)$ **e** $\left(-\frac{3}{4}, 1\right)$ **f** $\left(0, \frac{8}{5}\right)$

11 **a** $\sqrt{(x - 7)^2 + y^2}$
b $\sqrt{(x - 7)^2 + (x + 3)^2}$
c **i** 721 m **ii** 707 m
iii 721 m **iv** 762 m
d $x = 2$
e The distance will be a minimum when the dotted line joining Sarah to the fence is perpendicular to the fence (when it has gradient –1). The closest point is (2, 5).

1I

Building understanding

1 **a** 4 **b** –7 **c** $-\frac{3}{4}$ **d** $\frac{8}{7}$

2 **a** $-\frac{1}{3}$ **b** $\frac{1}{2}$ **c** $-\frac{8}{7}$ **d** $\frac{9}{4}$

3 **a** 5 **b** 4 **c** $y = 5x + 4$

4 **a** true **b** false

Now you try

Example 25

a perpendicular **b** neither **c** parallel

Example 26

a $y = -3x + 8$ **b** $y = -\frac{3}{2}x + 2$

Exercise 1I

1 **a** perpendicular **b** parallel **c** neither

2 **a** parallel **b** parallel **c** neither **d** neither
e perpendicular **f** perpendicular **g** parallel
h parallel **i** perpendicular **j** perpendicular

3 **a** $y = x + 4$ **b** $y = -x - 6$ **c** $y = -4x - 1$
d $y = \frac{2}{3}x - 6$ **e** $y = -\frac{4}{5}x + 7$ **f** $y = -\frac{1}{2}x + 6$
g $y = \frac{1}{4}x - 2$ **h** $y = -\frac{3}{2}x + 5$ **i** $y = -\frac{3}{4}x - 5$
j $y = \frac{7}{2}x + 31$

4 **a** $x = 6$ **b** $x = 0$ **c** $y = 11$ **d** $y = 8.4$
e $y = 3$ **f** $y = -3$ **g** $x = \frac{2}{3}$ **h** $x = -\frac{4}{11}$

5 **a** $y = \frac{2}{3}x + 5$ **b** $y = -\frac{5}{7}x + \frac{54}{7}$
c $y = \frac{2}{3}x + \frac{16}{3}$ **d** $y = 7x + 20$

6 **a** $y = -\frac{3}{2}x + 5$ **b** $y = \frac{7}{5}x + \frac{28}{5}$
c $y = -\frac{3}{2}x + 1$ **d** $y = -\frac{1}{7}x - \frac{10}{7}$

7 The second line has equation $y = -\frac{2}{3}x - \frac{5}{3}$. It cuts the x-axis at $x = -\frac{5}{2}$.

8 **a** m **b** $-\frac{a}{b}$ **c** $-\frac{1}{m}$ **d** $\frac{b}{a}$

9 **a** 14 **b** –2 **c** 5 **d** $\frac{9}{7}$

10 **a** $y = 2x + b - 2a$ **b** $y = mx + b - ma$
c $y = x + b - a$ **d** $y = -\frac{1}{m}x + b + \frac{a}{m}$

11 **a** **i** 1 **ii** –1 **iii** 1 **iv** –1
b AB is parallel to CD, BC is parallel to DA, AB and CD are perpendicular to BC and DA; i.e. opposite sides are parallel and adjacent sides are perpendicular.
c rectangle.

ISBN 978-1-108-77290-7

Cambridge University Press

Updated June 2022

12 a i $\frac{4}{3}$ ii $-\frac{3}{4}$ iii 0
b Right-angled triangle (AB is perpendicular to BC).
c 20

13 $y = -\frac{1}{2}x + 4$, x-intercept $= 8$

1J

Building understanding

1 a yes b yes c no d no e yes
2 a i Joe's: \$60, Paul's: \$150
ii Joe's: \$0.20 per km, Paul's: \$0.10 per km
iii Joe's: $C = 0.2k + 60$, Paul's: $C = 0.1k + 150$
iv 900 km
b Joe's Car Rental
c Paul's Motor Mart

Now you try

Example 27
a $(2, -2)$ b $(-1, 3)$

Example 28
$(2, -1)$

Exercise 1J

1 a $x = 2, y = -3$ b $x = -1, y = 3$
2 a $x = 2, y = 7$ b $x = 2, y = 5$
c $x = 3, y = 1$ d $x = 2, y = 1$
e $x = 1, y = 1$ f $x = 1, y = 1$
g $x = 5, y = 1$ h $x = 10, y = 4$
i $x = 1, y = 2$ j $x = 9, y = 2$
3 a $x = 2, y = 10$ b $x = 1, y = -5$
c $x = -3, y = 3$ d $x = 13, y = -2$
e $x = 3, y = 1$ f $x = 2, y = 1$
g $x = 1, y = 4$ h $x = 1, y = 3$
4 a i $E = 20t$ ii $E = 15t + 45$
b $t = 9, E = 180$
c i 9 hours ii \$180
5 a i $V = 62\,000 - 5000t$ ii $V = 40\,000 - 3000t$
b $t = 11, V = 7000$
c i 11 years ii \$7000
6 18 years
7 $197\,600\text{ m}^2$
8 a no b no c yes d yes
e no f yes g yes h no
9 a -4 b $\frac{3}{2}$ c 12
10 a $\left(\frac{k}{3}, \frac{2k}{3}\right)$ b $\left(\frac{k}{2}, -\frac{k}{2}\right)$
c $(-1 - k, -2 - k)$ d $\left(\frac{-2k-1}{3}, \frac{-2k-4}{3}\right)$
11 a $x = \frac{b}{a-b}, y = \frac{b^2}{a-b}$ b $x = \frac{-b}{a+b}, y = \frac{a}{a+b}$
c $x = \frac{a}{1+b}, y = \frac{-a}{1+b}$ d $x = \frac{b}{b-a}, y = \frac{b^2}{b-a}$
e $x = \frac{1}{a-2b}, y = \frac{(a-b)}{a-2b}$ f $x = \frac{c(1-b)}{a(b+1)}, y = \frac{2c}{b+1}$
g $x = \frac{ab}{a^2+b}, y = \frac{a^2b}{a^2+b}$ h $x = \frac{ab}{a^2+1}, y = \frac{b}{a^2+1}$
12 Answers will vary.

1K

Building understanding

1 a 0 b 0 c 0 d 0
2 a subtract b add c add d subtract
3 a $4x - 6y = 8$ b $6x - 9y = 12$
c $8x - 12y = 16$ d $20x - 30y = 40$

Now you try

Example 29
$x = 3, y = 1$

Example 30
a $x = 1, y = 3$ b $x = 3, y = -2$

Exercise 1K

1 a $x = 3, y = 2$ b $x = 4, y = 3$
2 a $x = 2, y = 5$ b $x = 2, y = 3$
c $x = 4, y = 2$ d $x = 2, y = 2$
e $x = 1, y = 1$ f $x = 2, y = 1$
g $x = 2, y = -1$ h $x = 2, y = 2$
i $x = 1, y = 2$ j $x = 2, y = 1$
k $x = 2, y = 1$ l $x = -1, y = 2$
3 a $x = 1, y = 1$ b $x = 4, y = 2$
c $x = 2, y = 1$ d $x = 4, y = -3$
e $x = \frac{1}{2}, y = 1$ f $x = -\frac{1}{2}, y = -\frac{1}{2}$
4 a $x = 4, y = -3$ b $x = 1, y = 1$
c $x = 3, y = 4$ d $x = 2, y = 2$
e $x = \frac{1}{2}, y = -1$ f $x = -3, y = \frac{1}{3}$
5 799 and 834
6 \$0.60
7 $A = \$15, C = \$11$
8 Should have been (1)–(2), to eliminate y: $-2y - (-2y) = 0$.
The correct solution is $(1, -1)$.
9 a $x = \frac{1}{a}, y = -1$ b $x = \frac{13}{3}, y = \frac{1}{3b}$
c $x = \frac{-2}{a}, y = \frac{2}{b}$ d $x = \frac{a+b}{2a}, y = \frac{a-b}{2b}$
e $x = \frac{c}{a+b}, y = \frac{c}{a+b}$
10 The two lines are parallel, they have the same gradient.
11 a $\frac{2}{x-1} - \frac{2}{x+1}$ b $\frac{2}{2x-3} - \frac{1}{x+2}$
c $\frac{3}{3x+1} - \frac{2}{2x-1}$ d $\frac{3}{3x-1} + \frac{2}{x+2}$
e $\frac{1}{x+3} + \frac{1}{x-4}$ f $\frac{1}{7(2x-1)} - \frac{3}{7(4-x)}$

ISBN 978-1-108-77290-7 © Greenwood et al. 2019 Cambridge University Press Updated June 2022

1L

Building understanding

1 a $x + y = 16, x - y = 2$
  b $x + y = 30, x - y = 10$
  c $x + y = 7, 2x + y = 12$
  d $2x + 3y = 11, 4x - 3y = 13$
2 $l = 3w$, $2l + 2w = 56$ or $l + w = 28$
3 a $5x$ dollars
  b $15y$ dollars
  c $3d + 4p$ dollars

Now you try

Example 31
Tim is 14, Tina is 6.

Example 32
A coffee is \$4 and a muffin is \$3.

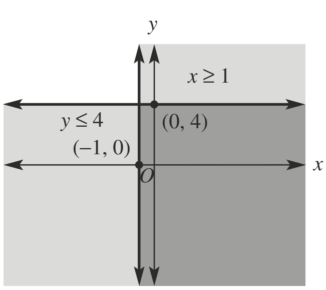
Exercise 1L

1 Nikki is 16, Travis is 8.
2 Cam is 33, Lara is 30.
3 Bolts cost \$0.10, washers cost \$0.30.
4 There were 2500 adults and 2500 children.
5 Thickshakes cost \$5, juices cost \$3.
6 There are 36 ducks and 6 sheep.
7 43
8 \$6.15 (mangoes cost \$1.10, apples cost \$0.65)
9 70
10 1 hour and 40 minutes
11 $\frac{1}{7}$ of an hour
12 200 m
13 $\frac{4}{17}$ L
14 $\frac{210}{19}$ L

1M

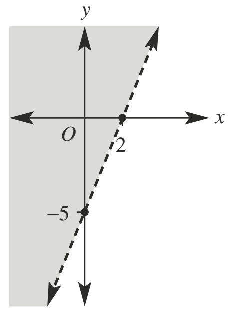
Building understanding

1 a no b yes c no
  d no e no f yes
2 a B b C c A
3 a–d $x \geqslant -1, y \leqslant 4$

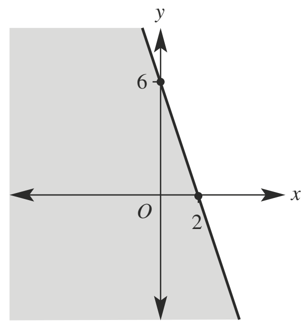


Now you try

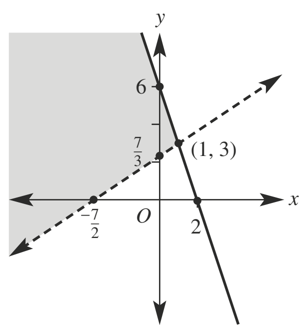
Example 33
a



b

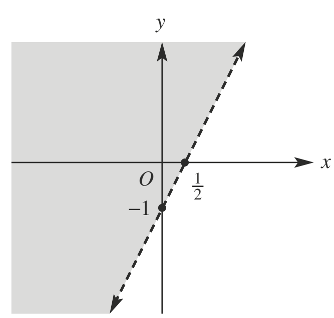


Example 34



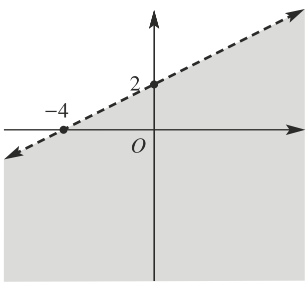
Exercise 1M

1 a i

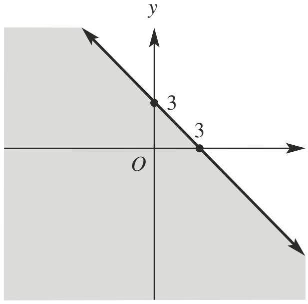


ISBN 978-1-108-77290-7

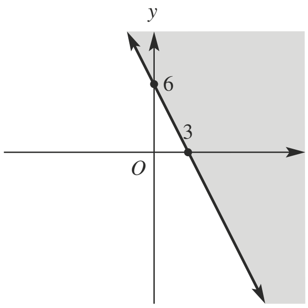
ii



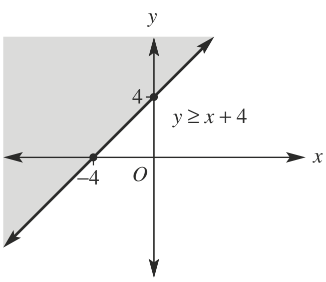
b **i**



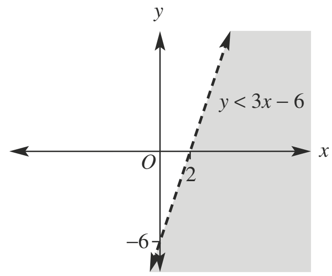
ii



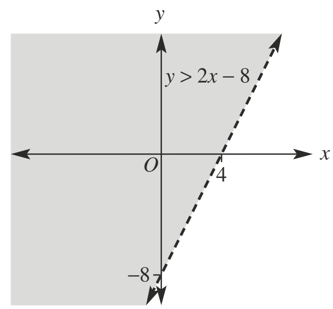
2 **a** $y \geqslant x + 4$



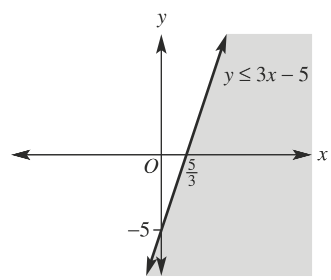
b $y < 3x - 6$



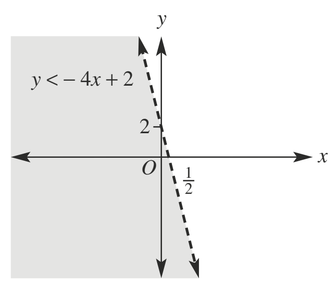
c $y > 2x - 8$



d $y \leqslant 3x - 5$



e $y < -4x + 2$

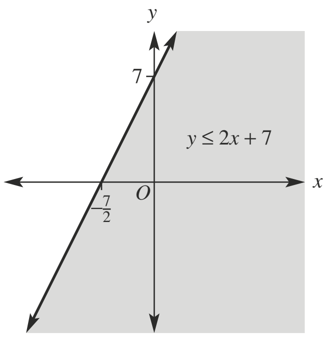


ISBN 978-1-108-77290-7

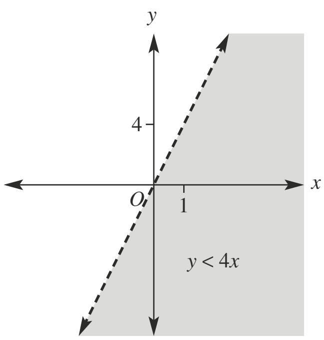
Cambridge University Press

Updated June 2022

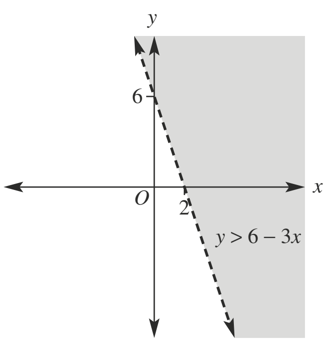
f $y \leqslant 2x + 7$



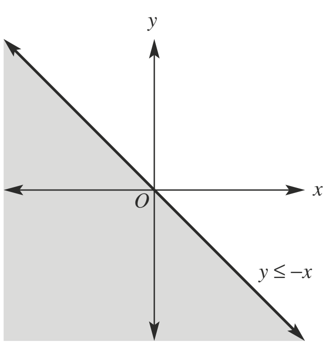
g $y < 4x$



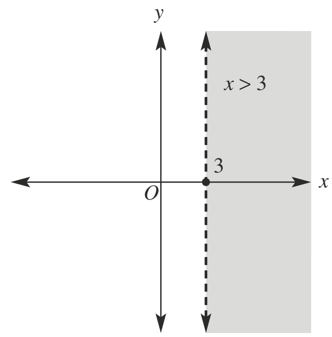
h $y > -3x + 6$



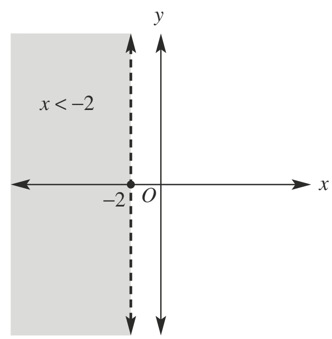
i $y \leqslant -x$



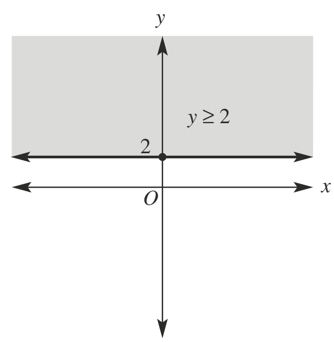
j $x > 3$



k $x < -2$



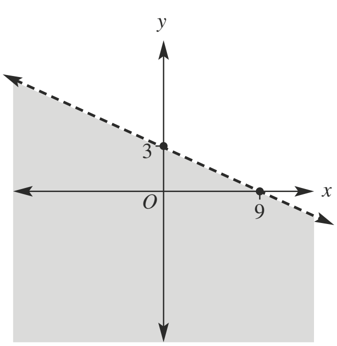
l $y \geqslant 2$



3 **a** yes **b** no **c** no **d** yes

4 **a** no **b** yes **c** no **d** no

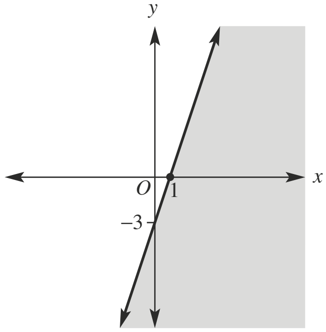
5 **a**



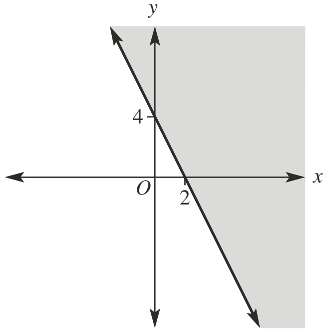
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

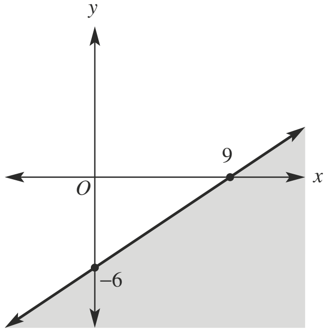
b



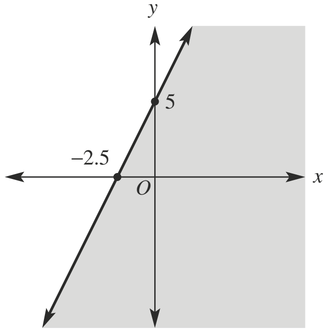
c



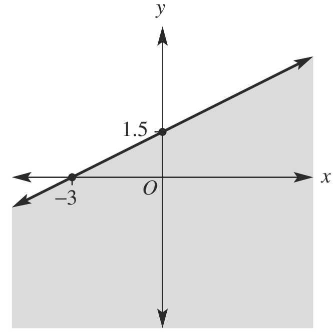
d



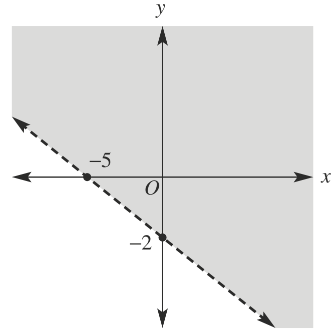
e



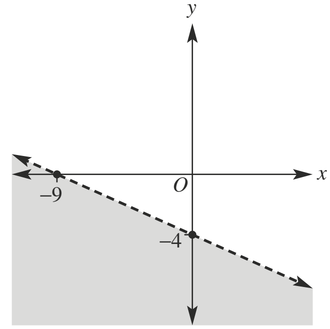
f



g



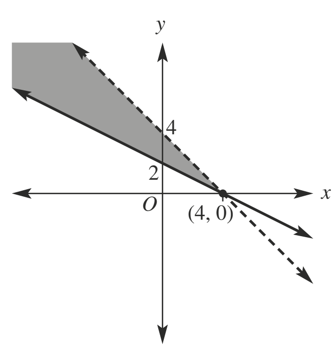
h



6 a $y \leqslant x + 3$ b $y \geqslant -2x + 2$

c $y < -\frac{3}{2}x - 3$ d $y > \frac{2}{5}x - 2$

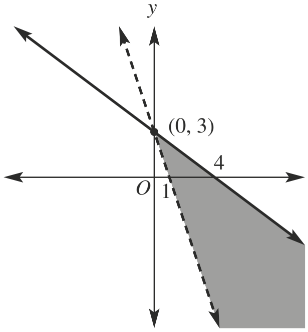
7 a



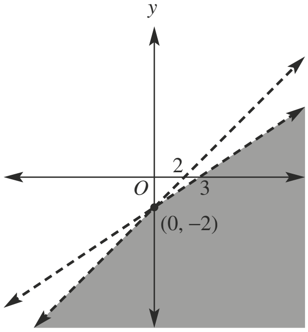
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

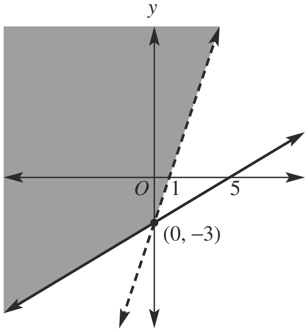
b



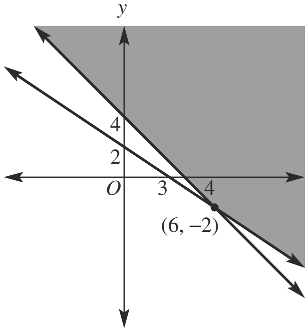
c



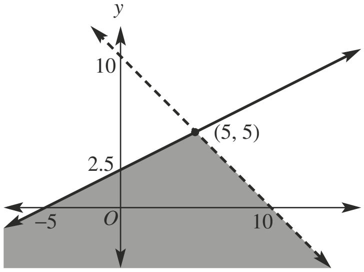
d



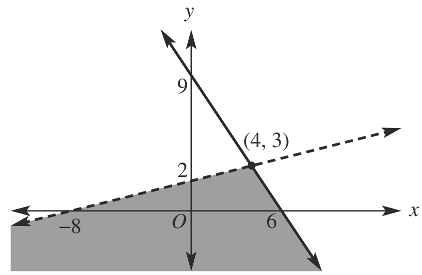
e



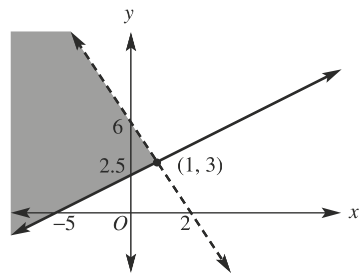
f



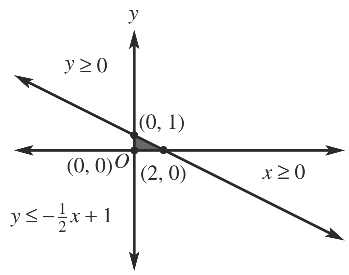
g



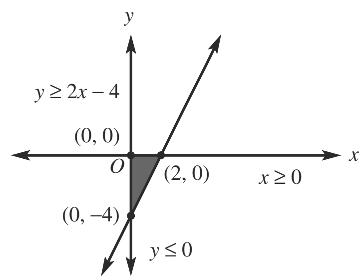
h



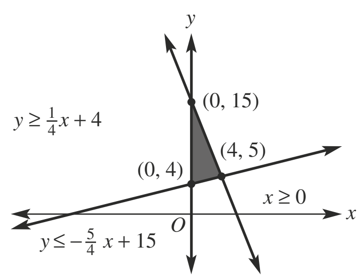
8 a



b

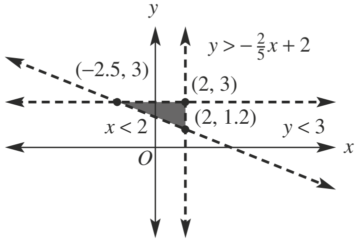


c

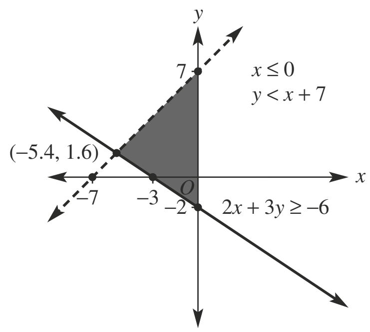


ISBN 978-1-108-77290-7

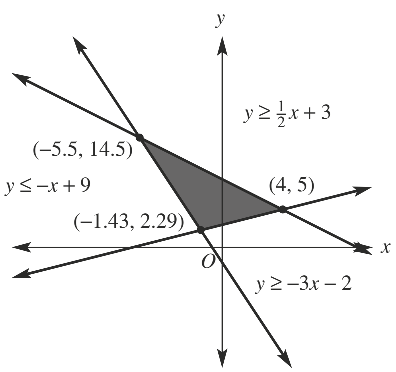
d



e



f



9 **a** $y \geqslant 0, y < 2x + 4, y \leqslant -x + 7$

b $y > -\frac{1}{2}x + 6, y \leqslant x + 3, x < 8$

10 **a** 1 **b** 4 **c** 22 **d** $\frac{81}{20}$

11 **a** **i** $\frac{115}{6}$ **ii** $\frac{578}{15}$

b Answers may vary; e.g. $x > 0, x < 3, y > 0, y < 2$

Problems and challenges

1 0.75 km

2 $\frac{6}{8}$

3 **a** The gradient from (2, 12) to (−2, 0) = the gradient from (−2, 0) to (−5, −9) = −3.

b The gradient from $(a, 2b)$ to $(2a, b)$ = the gradient from $(2a, b)$ to $(-a, 4b) = -\frac{b}{a}$.

4 The gradient of AC is $\frac{3}{5}$ and the gradient of AB is $-\frac{5}{3}$. So $\triangle ABC$ is a right-angled triangle, as AC is perpendicular to AB. Can also show that side lengths satisfy Pythagoras' theorem.

5 The missiles are travelling at $\frac{4840}{9}$ km/h and $\frac{9680}{9}$ km/h.

6 The distance between the two points and (2, 5) is 5 units.

7 The diagonals have equations $x = 0$ and $y = 3$. These lines are perpendicular and intersect at the midpoint (0, 3) of the diagonals. It is not a square since the angles at the corners are not 90°. In particular, AB is not perpendicular to $BC\left(m_{AB} \neq \frac{-1}{m_{BC}}\right)$.

8 $x = 2, y = -3, z = -1$

9 24 units^2

10 24, 15 years

Short-answer questions

1 **a** $5xy + 6x$ **b** $12a^2b$ **c** $\frac{3}{2}x$

d $3b + 21$ **e** $-2m^2 + 12m$ **f** $x + 2$

2 **a** $3x - 1$ **b** $\frac{2}{x + 2}$ **c** $\frac{3}{4}$

3 **a** $\frac{6 - 7a}{14}$ **b** $\frac{5a + 18}{6a}$

c $\frac{7x + 26}{30}$ **d** $\frac{11 - x}{(x + 1)(x - 3)}$

4 **a** $x = -3$ **b** $x = -\frac{3}{4}$

c $x = \frac{1}{5}$ **d** $x = 2$

5 **a** $x < 1$ **b** $x \geqslant -4$ **c** $-1 < x \leqslant 3$

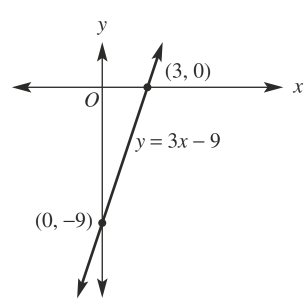
6 **a** $x > 5$ **b** $x \geqslant 10$

c $x > -3$ **d** $x \leqslant \frac{2}{7}$

7 **a** $V = 2 - 0.4t$ **b** 1.4 L

c 5 minutes **d** $\leqslant$ 3.5 minutes

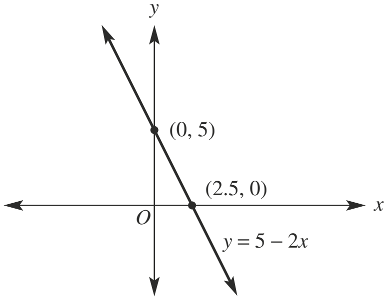
8 **a**

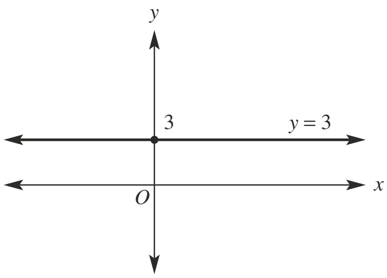


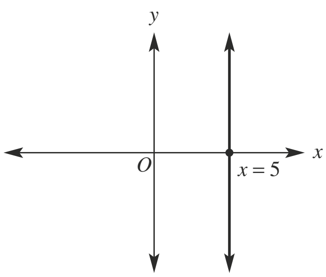
ISBN 978-1-108-77290-7

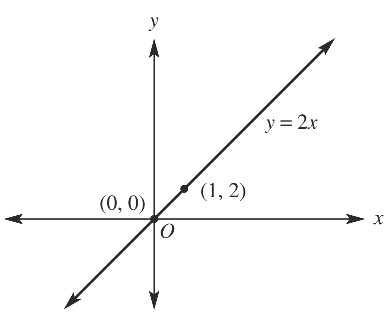
Cambridge University Press

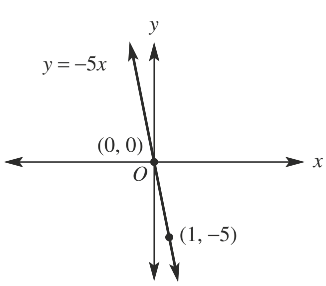
Updated June 2022

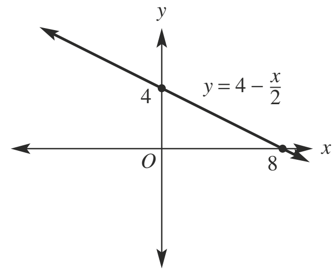
b


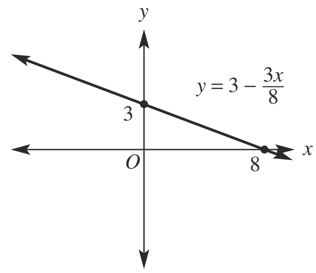
c


d


e


f


g


h


9 **a** $y = \frac{1}{2}x + 3$ **b** $y = \frac{5}{2}x + 5$

c $y = -\frac{3}{2}x + \frac{15}{2}$ **d** $y = 2x - 3$

10 **a** $m = -\frac{3}{5}$ **b** $y = -\frac{3}{5}x + \frac{34}{5}$

11 **a** $M = (4, 8), d = \sqrt{52} = 2\sqrt{13}$

b $M = \left(\frac{11}{2}, 1\right), d = \sqrt{61}$

c $M = \left(\frac{1}{2}, -\frac{5}{2}\right), d = \sqrt{18} = 3\sqrt{2}$

12 **a** $y = 3x - 2$ **b** $y = -1$

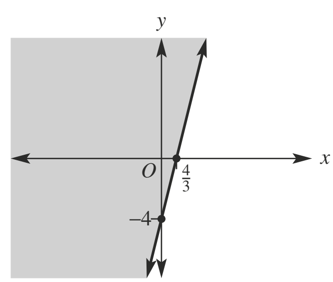
c $y = -\frac{1}{2}x + 5$ **d** $y = 3x - 1$

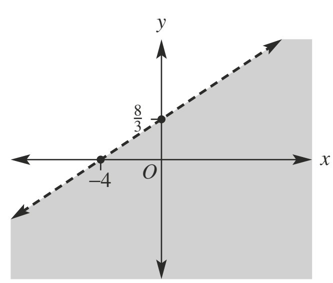
13 **a** $a = 7$ **b** $b = -8$ **c** $c = 0$ or 4

14 **a** $(-3, -1)$ **b** $(-8, -21)$

15 **a** $(-3, -1)$ **b** $(0, 2)$

16 A regular popcorn costs \$4 and a small drink costs \$2.50.

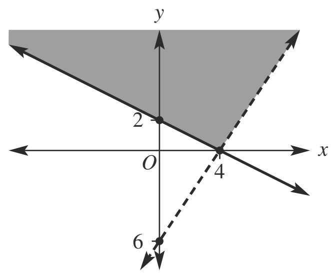
17 **a**


b


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

18 The point of intersection is (4, 0).

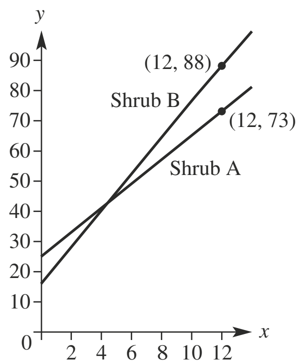


Multiple-choice questions

1 E	2 D	3 B	4 C
5 D	6 B	7 C	8 A
9 C	10 D	11 E	12 B
13 A	14 A	15 D	16 C

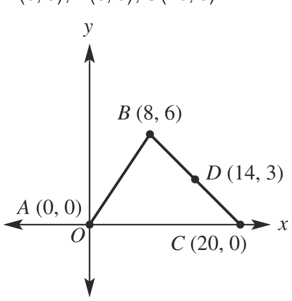
Extended-response questions

1 a i $h = 4t + 25$ ii $h = 6t + 16$
  b 16 cm
  c Shrub B because its gradient is greater.
  d



  e After 4.5 months
  f i 1.24 m   ii 26.25 months
    iii Between 8.75 and 11.25 months

2 a $A(0, 0)$, $B(8, 6)$, $C(20, 0)$



  b 43.4 km
  c The drink station is at (14, 3).
  d i $y = \frac{3}{4}x$ ii $y = -\frac{1}{2}x + 10$ iii $y = 0$
  e $y \geqslant 0$, $y \leqslant \frac{3}{4}x$, $y \leqslant -\frac{1}{2}x + 10$
  f $y = -\frac{4}{3}x + \frac{80}{3}$

Chapter 2

2A

Building understanding

1 triangle, quadrilateral, pentagon, hexagon, heptagon, octagon, nonagon, decagon
2 a false b true c true
  d true e false f false
  g true h false i true
3 a b b c c d d a
4 a $a = 110$ (angles on a line), $b = 70$ (vertically opposite)
  b $a = 140$ (angles in a revolution)
  c $a = 19$ (complementary)
  d $a = 113$ (cointerior angles in $\parallel$ lines), $b = 67$ (alternate angles in $\parallel$ lines), $c = 67$ (vertically opposite to b)
  e $a = 81$ (isosceles triangle), $b = 18$ (angles in a triangle)
  f $a = 17$ (angles in a triangle), $b = 102$ (angles at a line)

Now you try

Example 1
a $x = 70$ b $x = 100$

Example 2
a $a = 108$ b $a = 80$

Exercise 2A

1 a 45 b 110
2 a 72 b 60 c 56
3 a 60 (equilateral triangle)
  b 60 (exterior angle theorem)
  c 110 (isosceles, angles in a triangle)
  d 80 (angles in a triangle)
  e 10 (exterior angle theorem)
  f 20 (isosceles, angles in a triangle)
  g 109 (angles on a line)
  h 28 (diagonals meet at a right angle in a rhombus)
  i 23 (angles in a triangle)
  j 121 (vertically opposite to cointerior angle in $\parallel$ lines)
  k 71 (isosceles, cointerior angles in $\parallel$ lines)
  l 60 (isosceles, cointerior angles in $\parallel$ lines)
4 a 50 (angle sum in a quadrilateral)
  b 95 (angle sum in a quadrilateral)
  c 125 (angle sum in a pentagon)
  d 30 (angle sum in a pentagon)
  e 45 (angle sum in a hexagon)
  f 15 (angle sum in a quadrilateral)
5 a 108° b 135° c 144°
6 a 95 (alternate + cointerior)
  b 113 (2 × alternate)
  c 85 (alternate + cointerior)
  d 106 (cointerior)
  e 147, (cointerior, angles in a revolution)
  f 292, (angles in a revolution, alternate + cointerior)
7 a 176.4° b 3.6°

ISBN 978-1-108-77290-7

Cambridge University Press

Updated June 2022

8 a 12 b 20 c 48
9 $x = 36, y = 144$
10 115, equilateral and isocles triangle 60 + 55
11 a Expand the brackets. b $n = \frac{S + 360}{180}$
c $I = \frac{S}{n} = \frac{180(n-2)}{n}$ d $E = 180 - I = \frac{360}{n}$
12 a $\angle BCA = 180° - a° - b°$ (angles in a triangle)
b $c° = 180° - \angle BCA = a° + b°$ (angles at a line)
13 a alternate angles ($BA \parallel CD$)
b $\angle ABC + \angle BCD = 180°$ (cointerior), so $a + b + c = 180$.
c Angle sum of a triangle is 180°.
14 $\angle ACB = \angle DCE$ (vertically opposite), so $\angle CAB = \angle CBA = \angle CDE = \angle CED$ (isosceles) since $\angle CAB = \angle CED$ (alternate) $AB \parallel DE$.
15 Answers may vary.
16 a 15 (alternate angles in parallel lines)
b 315 (angle sum in an octagon)
17 Let M be the midpoint of AC. Then $\angle AMB = 60°$ ($\triangle ABM$ is equilateral). $\angle BMC = 120°$ (supplementary). Therefore, $\angle MBC = 30°$ ($\triangle MBC$ is isosceles). So $\angle ABC = \angle ABM + \angle MBC = 60° + 30° = 90°$.
18 Let $\angle AOB = x$ and $\angle COD = y$. $2x + 2y = 180°$ (angles at a line). So $\angle BOD = x + y = 90°$.

2B

Building understanding

1 a SAS b SSS c AAS
d SAS e RHS f RHS
2 a 5 b 4 c 3 d 5

Now you try

Example 3
a $AB = DE$ (given) S
$\angle ABC = \angle DEF$ (given) A
$BC = EF$ (given) S
$\therefore \triangle ABC \equiv \triangle DEF$ (SAS)
b $\angle ABC = \angle DEF$ (given) A
$\angle BAC = \angle EDF$ (given) A
$AC = DF$ (given) S
$\therefore \triangle ABC \equiv \triangle DEF$ (AAS)

Example 4
a $\angle A = \angle C = 90°$ (given) R
BD is common H
$AB = CB$ (given) S
$\therefore \triangle ABD \equiv \triangle CBD$ (RHS)
b $\triangle ABD \equiv \triangle CBD$ so $AD = CD$
c $CD = 4$ m

Exercise 2B

1 a $AB = DE$ (given) S
$\angle BAC = \angle EDF$ (given) A
$AC = DF$ (given) S
$\therefore \triangle ABC \equiv \triangle DEF$ (SAS)
b $\angle ABC = \angle DEF$ (given) A
$\angle BAC = \angle EDF$ (given) A
$BC = EF$ (given) S
$\therefore \triangle ABC \equiv \triangle DEF$ (AAS)
2 a $AB = DE$ (given) S
$\angle ABC = \angle DEF$ (given) A
$BC = EF$ (given) S
$\therefore \triangle ABC \equiv \triangle DEF$ (SAS)
b $\angle FED = \angle CBA = 90°$ (given) R
$FD = CA$ (given) H
$FE = CB$ (given) S
$\therefore \triangle FED \equiv \triangle CBA$ (RHS)
c $AC = DF$ (given) S
$BC = EF$ (given) S
$AB = DE$ (given) S
$\therefore \triangle ACB \equiv \triangle DFE$ (SSS)
d $\angle EDF = \angle BAC$ (given) A
$\angle DFE = \angle ACB$ (given) A
$EF = BC$ (given) S
$\therefore \triangle EDF \equiv \triangle BAC$ (AAS)
3 a $x = 7.3, y = 5.2$
b $x = 12, y = 11$
c $a = 2.6, b = 2.4$
d $x = 16, y = 9$
4 a $AD = CB$ (given) S
$DC = BA$ (given) S
AC is common; S
$\therefore \triangle ADC \equiv \triangle CBA$ (SSS)
b $\angle ADB = \angle CBD$ (given) A
$\angle ABD = \angle CDB$ (given) A
BD is common; S
$\therefore \triangle ADB \equiv \triangle CBD$ (AAS)
c $\angle BAC = \angle DEC$ (alternate, $AB \parallel DE$) A
$\angle CBA = \angle CDE$ (alternate, $AB \parallel DE$) A
$BC = DC$ (given) S
$\therefore \triangle BAC \equiv \triangle DEC$ (AAS)
d $DA = DC$ (given) S
$\angle ADB = \angle CDB$ (given) A
DB is common; S
$\therefore \triangle ADB \equiv \triangle CDB$ (SAS)
e $OA = OC$ (radii) S
$OB = OD$ (radii) S
$AB = CD$ (given) S
$\therefore \triangle OAB \equiv \triangle OCD$ (SSS)
f $\angle ADC = \angle ABC = 90°$ (given) R
AC is common; H
$DC = BC$ (given) S
$\therefore \triangle ADC \equiv \triangle ABC$ (RHS)
5 a $OA = OC$ (radii) S
$\angle AOB = \angle COB$ (given) A
OB is common; S
$\therefore \triangle AOB \equiv \triangle COB$ (SAS)
b $AB = BC$ (corresponding sides in congruent triangles)
c 10 mm

Answers

2A

6 a $BC = DC$ (given) S
$\angle BCA = \angle DCE$ (vertically opposite) A
$AC = EC$ (given) S
$\therefore \triangle ABC \equiv \triangle EDC$ (SAS)
b $AB = DE$ (corresponding sides in congruent triangles)
c $\angle ABC = \angle CDE$ (corresponding sides in congruent triangles).
$\angle ABC$ and $\angle CDE$ are alternate angles. $\therefore AB \parallel DE$.
d 5 cm

7 a $AB = CD$ (given) S
$AD = CB$ (given) S
BD is common; S
$\therefore \triangle ABD \equiv \triangle CDB$ (SSS)
b $\angle DBC = \angle BDA$ (corresponding angles in congruent triangles)
c $\angle DBC$ and $\angle BDA$ are alternate angles (and equal).
$\therefore AD \parallel BC$.

8 a $CB = CD$ (given) S
$\angle BCA = \angle DCE$ (vertically opposite) A
$CA = CE$ (given) S
$\therefore \triangle BCA \equiv \triangle DCE$ (SAS)
$\angle BAC = \angle DEC$ (corresponding angles in congruent triangles)
$\therefore$ Alternate angles are equal, so $AB \parallel DE$.
b $\angle OBC = \angle OBA = 90°$ (given) R
$OA = OC$ (radii) H
OB is common; S
$\therefore \triangle OAB \equiv \triangle OCB$ (RHS)
$AB = BC$ (corresponding sides in congruent triangles)
$\therefore OB$ bisects AC.
c $AB = CD$ (given) S
AC is common; S
$AD = CB$ (given) S
$\therefore \triangle ACD \equiv \triangle CAB$ (SSS)
$\angle DAC = \angle BCA$ (corresponding angles in congruent triangles)
$\therefore$ Alternate angles are equal, so $AD \parallel BC$.
d $AB = AE$ (given) S
$\angle ABC = \angle AED$ ($\triangle ABE$ is isosceles) A
$ED = BC$ (given) S
$\therefore \triangle ABC \equiv \triangle AED$ (SAS)
$AD = AC$ (corresponding sides in congruent triangles)
e $OD = OC$ (given) S
$\angle AOD = \angle BOC$ (vertically opposite) A
$OA = OB$ (given) S
$\therefore \triangle AOD \equiv \triangle BOC$ (SAS)
$\angle OAD = \angle OBC$ (corresponding angles in congruent triangles)
f $AD = AB$ (given) S
$\angle DAC = \angle BAC$ (given) A
AC is common; S
$\therefore \triangle ADC \equiv \triangle ABC$ (SAS)
$\angle ACD = \angle ACB$ (corresponding angles in congruent triangles)
$\angle ACD = \angle ACB$ are supplementary.
$\therefore \angle ACD = \angle ACB = 90°$
$\therefore AC \perp BD$

9 a $OA = OB$ (radii) S
OM is common; S
$AM = BM$ (M is midpoint) S
$\therefore \triangle OAM \equiv \triangle OBM$ (SSS)
$\angle OMA = \angle OMB$ (corresponding angles in congruent triangles)
$\angle OMA$ and $\angle OMB$ are supplementary.
$\therefore \angle OMA = \angle OMB = 90°$
$\therefore OM \perp AB$
b $OA = OB$ (radii of same circle) S
$CA = CB$ (radii of same circle) S
OC is common; S
$\therefore \triangle OAC \equiv OBC$ (SSS)
$\angle AOC = \angle BOC$ (corresponding angles in congruent triangles)
c $\angle CAB = \angle CBA = x$ ($\triangle ABC$ is isosceles)
$\angle EAB = \angle DBA = \frac{x}{2}$
$\therefore \triangle AFB$ is isosceles, so $AF = BF$.

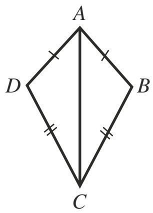
2C

Building understanding

1 a rectangle b parallelogram
c square d rhombus
2 a rectangle, square b rectangle, square
c parallelogram, rhombus, rectangle, square
d rhombus, square e rhombus, square
3 a A trapezium does not have both pairs of opposite sides parallel.
b A kite does not have two pairs of opposite sides parallel.

Now you try

Example 5



$AB = AD$ (given) S
$BC = DC$ (given) S
AC is common S
$\therefore \triangle ABC \equiv \triangle ADC$ (SSS)
$\therefore \angle ABC = \angle ADC$ (corresponding angles in congruent triangles)

Example 6
$AB = CD$ (given) S
$BC = DA$ (given) S
AC is common S
$\therefore \triangle ABC \equiv \triangle CDA$ (SSS)
$\therefore \angle BAC = \angle DCA$ so $AB \parallel CD$
and $\angle ACB = \angle CAD$ so $BC \parallel DA$
$\therefore ABCD$ is a parallelogram

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 2C

1 **a** $\angle BAC = \angle DCA$ (alternate angles)
$\angle BCA = \angle DAC$ (alternate angles)
AC is common.
$\therefore \triangle ABC \equiv \triangle CDA$ (AAS)

b As $\triangle ABC \equiv \triangle CDA$, $AD = CB$, $AB = CD$ (corresponding sides).

2 **a** $\angle ABE = \angle CDE$ (alternate angles)
$\angle BAE = \angle DCE$ (alternate angles)
$AB = CD$ (opposite sides of parallelogram)
$\therefore \triangle ABE \equiv \triangle CDE$ (AAS)

b $AE = CE$ (corresponding sides), $BE = DE$ (corresponding sides).

3 **a** $AB = CB$ (given)
$AD = CD$ (given)
BD is common.
$\therefore \triangle ABD \equiv \triangle CDB$ (SSS)

b $\angle ABD = \angle ADB = \angle CBD = \angle CDB$ (equal angles in congruent isosceles triangles). Therefore, BD bisects $\angle ABC$ and $\angle CDA$.

4 **a** $AE = CE$ (given)
$BE = DE$ (given)
$\angle AEB = \angle CED$ (vertically opposite angles)
$\therefore \triangle ABE \equiv \triangle CDE$ (SAS)

b $\angle ABE = \angle CDE$ (corresponding angles), $\angle BAE = \angle DCE$ (corresponding angles). Therefore, $AB \parallel DC$ (alternate angles are equal). $\angle ADE = \angle CBE$ (corresponding angles), $\angle DAE = \angle BCE$ (corresponding angles). Therefore, $AD \parallel BC$ (alternate angles are equal).

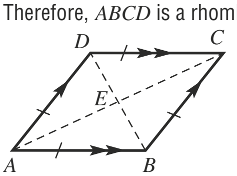
5 **a** $AD = CB$ (given)
$\angle DAC = \angle BCA$ (alternate angles)
AC is common.
$\therefore \triangle ABC \equiv \triangle CDA$ (SAS)

b $\angle BAC = \angle DCA$ (corresponding angles), therefore $AB \parallel DC$ (alternate angles are equal).

6 **a** $\triangle ABE \equiv \triangle CBE \equiv \triangle ADE \equiv \triangle CDE$ (SAS)

b $\angle ABE = \angle CDE$ (corresponding angles), $\angle BAE = \angle DCE$ (corresponding angles), therefore $AB \parallel CD$. $\angle ADE = \angle CBE$ (corresponding angles), $\angle DAE = \angle BCE$ (corresponding angles), therefore $AD \parallel CB$. Also, $AB = AD = CB = CD$ (corresponding sides). Therefore, $ABCD$ is a rhombus.

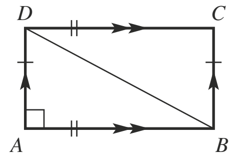
7 **a**



$\angle CAB = \angle ACD$ and $\angle CAD = \angle ACB$ (alternate angles).
So $\angle ECB = \angle ECD$ since $\triangle ABC$ and $\triangle ADC$ are isosceles.
$DC = BC$ (given)
EC is common.
$\therefore \triangle CDE \equiv \triangle CBE$ (SAS)
So $\angle CED = \angle CEB = 90°$.

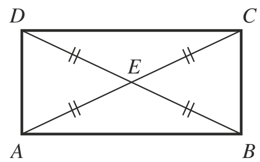
b From part **a**, $\angle ECD = \angle ECB$.

8



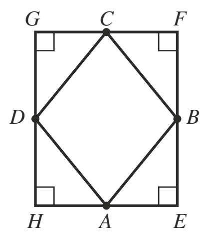
As $ABCD$ is a parallelogram, $\angle BDC = \angle DBA$ (alternate angles) and $\angle DBC = \angle BDA$ (alternate angles).
BD is common.
So $\triangle CBD \equiv \triangle ADB$ (AAS).
So $\angle BAD = \angle DCB = 90°$.
Similarly, $\angle ADC = \angle 180° - \angle BAD$ (cointerior angles) $= 90°$ and similarly for $\angle ABC$.

9



First, prove $\triangle AED \equiv \triangle BEC$ (SAS).
Hence, corresponding angles in the isosceles triangles are equal and $\triangle CED \equiv \triangle BEA$ (SAS).
Hence, corresponding angles in the isosceles triangles are equal.
So $\angle ADC = \angle DCB = \angle CBA = \angle BAC$, which sum to 360°.
Therefore, all angles are 90° and $ABCD$ is a rectangle.

10



First, prove all four corner triangles are congruent (SAS).
So $EF = FG = GH = HE$, so $EFGH$ is a rhombus.

2D

Building understanding

1 **a** Yes, both squares have all angles 90° and all sides of equal length.
b 3 **c** 15 cm

2 **a** 2 **b** $\frac{8}{5}$ **c** $\frac{4}{3}$ **d** $\frac{3}{2}$

3 **a** A **b** $\angle C$ **c** FD **d** $\triangle ABC \parallel\!\mid \triangle EFD$

Now you try

Example 7

a $ABCD \parallel\!\mid EFGH$ **b** $\frac{EH}{AD} = \frac{FG}{BC}$ **c** 2
d 6 **e** 4

Exercise 2D

1 **a** $ABCD \parallel\!\mid EFGH$ **b** $\frac{EH}{AD} = \frac{FG}{BC}$
c 2 **d** 8 **e** 4

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2 a $ABCDE \mid\mid\mid FGHIJ$ b $\frac{AB}{FG} = \frac{DE}{IJ}$
c $\frac{3}{2}$ d $\frac{3}{2}$ cm e $\frac{4}{3}$ cm
3 a $ABCD \mid\mid\mid EFGH$ b $\frac{EF}{AB} = \frac{GH}{CD}$
c $\frac{4}{3}$ d 12 m e 10.5 m
4 a 1.2 b 12.5 c 4.8
d 3.75 e 11.5 f 14.5
5 1.7 m
6 a 1.6 b 62.5 cm
7 a 2 b 1 c 1.875 d 4.3
8 a BC b $\triangle ABC \mid\mid\mid \triangle EDC$
c 1 d 4.5
9 a true b true c false d false
e false f false g false h false
i true j true
10 Yes, the missing angle in the first triangle is 20° and the missing angle in the second triangle is 75°, so all three angles are equal.
11 a $\frac{3}{2}$
b i 4 ii 9
c i 8 ii 27
d

Cube	Length	Area	Volume
Small	2	4	8
Large	3	9	27
Scale factor (fraction)	$\frac{3}{2}$	$\frac{9}{4}$	$\frac{27}{8}$

e Scale factor for area = (scale factor for length)^2;
Scale factor for volume = (scale factor for length)^3.
f i $\frac{b^2}{a^2}$ ii $\frac{b^3}{a^3}$
12 Answers will vary.

2E

Building understanding

1 a E b $\angle C$ c AB
d $\triangle ABC \mid\mid\mid \triangle DEF$
2 a $\angle D$ (alternate angles) b $\angle A$ (alternate angles)
c $\angle ECD$ d CA e $\triangle ABC \mid\mid\mid \triangle EDC$
3 a SAS b AAA c SAS d SSS

Now you try

Example 8
a $\frac{DE}{AB} = 1.5$ (ratio of corresponding sides) S
$\frac{DF}{AC} = 1.5$ (ratio of corresponding sides) S
$\angle BAC = \angle EDF$ (given) A
$\therefore \triangle ABC \mid\mid\mid \triangle DEF$ (SAS)
b $\angle ABC = \angle DEF$ (given) A
$\angle ACB = \angle DFE$ (given) A
$\therefore \triangle ABC \mid\mid\mid \triangle DEF$ (AAA)

Example 9
a $\angle BAC$ is common A
$\angle ABC = \angle ADE$ (corresponding angles in parallel lines) A
$\therefore \triangle ADE \mid\mid\mid \triangle ABC$ (AAA)
b $DE = 1$ m

Exercise 2E

1 a $\frac{DE}{AB} = \frac{4}{2} = 2$ (ratio of corresponding sides) S
$\frac{EF}{BC} = \frac{2}{1} = 2$ (ratio of corresponding sides) S
$\angle ABC = \angle DEF$ (given corresponding angles) A
$\therefore \triangle ABC \mid\mid\mid \triangle DEF$ (SAS).
b $\angle ABC = \angle DEF$ (given corresponding angles) A
$\angle ACB = \angle DFE$ (given corresponding angles) A
$\therefore \triangle ABC \mid\mid\mid \triangle DEF$ (AAA).
2 a $\angle ABC = \angle DEF = 65°$
$\angle BAC = \angle EDF = 70°$
$\therefore \triangle ABC \mid\mid\mid \triangle DEF$ (AAA).
b $\frac{DE}{AB} = \frac{2}{1} = 2$ (ratio of corresponding sides)
$\frac{EF}{BC} = \frac{6}{3} = 2$ (ratio of corresponding sides)
$\angle ABC = \angle DEF = 120°$
$\therefore \triangle ABC \mid\mid\mid \triangle DEF$ (SAS).
c $\frac{DF}{CA} = \frac{10}{5} = 2$ (ratio of corresponding sides)
$\frac{DE}{CB} = \frac{8}{4} = 2$ (ratio of corresponding sides)
$\angle ABC = \angle FED = 90°$
$\therefore \triangle ABC \mid\mid\mid \triangle FED$ (RHS).
d $\frac{AB}{DE} = \frac{28}{7} = 4$ (ratio of corresponding sides)
$\frac{BC}{EF} = \frac{16}{4} = 4$ (ratio of corresponding sides)
$\frac{AC}{DF} = \frac{32}{8} = 4$ (ratio of corresponding sides)
$\therefore \triangle ABC \mid\mid\mid \triangle DEF$ (SSS).
3 a 1.5
b 19.5
c 2.2
d $a = 4, b = 15$
e $x = 0.16, y = 0.325$
f $a = 43.2, b = 18$
4 a $\angle ABC = \angle EDC$ (alternate angles)
$\angle BAC = \angle DEC$ (alternate angles)
$\angle ACB = \angle ECD$ (vertically opposite angles)
$\therefore \triangle ABC \mid\mid\mid \triangle EDC$ (AAA).
b $\angle ABE = \angle ACD$ (corresponding angles)
$\angle AEB = \angle ADC$ (corresponding angles)
$\angle BAE = \angle CAD$ (common)
$\therefore \triangle ABE \mid\mid\mid \triangle ACD$ (AAA).
c $\angle DBC = \angle AEC$ (given)
$\angle BCD = \angle ECA$ (common)
$\therefore \triangle BCD \mid\mid\mid \triangle ECA$ (AAA).

ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

d $\frac{AB}{CB} = \frac{3}{7.5} = 0.4$

$\frac{EB}{DB} = \frac{2}{5} = 0.4$ (ratio of corresponding sides)

$\angle ABE = \angle CBD$ (vertically opposite angles)

$\therefore \triangle AEB \,|||\, \triangle CDB$ (SAS).

5 **a** $\angle EDC = \angle ADB$ (common)

$\angle CED = \angle BAD = 90°$

$\therefore \triangle EDC \,|||\, \triangle ADB$ (AAA).

b $\frac{4}{3}$ cm

6 **a** $\angle ACB = \angle DCE$ (common)

$\angle BAC = \angle EDC = 90°$

$\therefore \triangle BAC \,|||\, \triangle EDC$ (AAA).

b 1.25 m

7 1.90 m

8 4.5 m

9 **a** Yes, AAA for both.

b 20 m

c 20 m

d Less working required for May's triangles.

10 The missing angle in the smaller triangle is 47°, and the missing angle in the larger triangle is 91°. Therefore the two triangles are similar (AAA).

11 **a** $\angle AOD = \angle BOC$ (common)

$\angle OAD = \angle OBC$ (corresponding angles)

$\angle ODA = \angle OCB$ (corresponding angles)

So $\triangle OAD \,|||\, \triangle OBC$ (AAA).

$\frac{OC}{OD} = \frac{3}{1} = 3$ (ratio of corresponding sides), therefore

$\frac{OB}{OA} = 3$

$OB = 3OA$

b $\angle ABC = \angle EDC$ (alternate angles)

$\angle BAC = \angle DEC$ (alternate angles)

$\angle ACB = \angle ECD$ (vertically opposite)

So $\triangle ABC \,|||\, \triangle EDC$ (AAA).

$\frac{CE}{AC} = \frac{CD}{BC} = \frac{2}{5}$, therefore $\frac{AC + CE}{AC} = \frac{1 + 2}{5} = \frac{7}{5}$.

But $AC + CE = AE$, so $\frac{AE}{AC} = \frac{7}{5}$ and $AE = \frac{7}{5}AC$.

12 **a** $\angle BAD = \angle BCA = 90°$

$\angle ABD = \angle CBA$ (common)

So $\triangle ABD \,|||\, \triangle CBA$ (AAA).

Therefore, $\frac{AB}{CB} = \frac{BD}{AB}$.

$AB^2 = CB \times BD$

b $\angle BAD = \angle ACD = 90°$

$\angle ADB = \angle CDA$ (common)

So $\triangle ABD \,|||\, \triangle CAD$ (AAA).

Therefore, $\frac{AD}{CD} = \frac{BD}{AD}$.

$AD^2 = CD \times BD$

c Adding the two equations:

$$\begin{aligned} AB^2 + AD^2 &= CB \times BD + CD \times BD \\ &= BD(CB + CD) \\ &= BD \times BD \\ &= BD^2 \end{aligned}$$

Progress quiz

1 **a** $x = 78$ (exterior angle of a triangle)

b $w = 89$ (angle sum of a quadrilateral)

c $x = 120$ (interior angle of a regular hexagon)

d $x = 35$ (alternate angles in parallel lines)

e $x = 97$ (cointerior angles in parallel lines, vertically opposite angles equal)

f $w = 94$ (angle sum of an isosceles triangle)

2 **a** $AB = QB$ (given)

$\angle ABC = \angle QBP$ (vertically opposite)

$\angle CAB = \angle PQB$ (alternate angles $AC \parallel PQ$)

$\therefore \triangle ABC \equiv \triangle QBP$ (AAS)

b $CB = PB$ corresponding sides of congruent triangles and B is the midpoint of CP.

3 Let $ABCD$ be any rhombus with diagonals intersecting at P.

$AB = BC$ (sides of a rhombus equal)

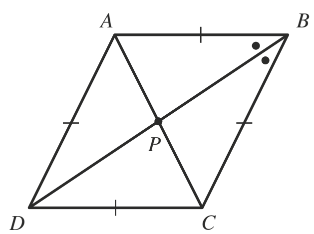
$\angle ABP = \angle CBP$ (diagonals of a rhombus bisect the interior angles through which they cross)

$\therefore \triangle ABP \equiv \triangle CBP$ (SAS)

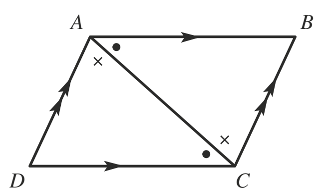
and $\angle APB = \angle BPC$ (corresponding angles of congruent triangles).

And $\angle APB + \angle BPC = 180°$ (straight line)

$\therefore$ diagonal $AC \perp$ diagonal DB.



4



Let $ABCD$ be any parallelogram with opposite sides parallel.

AC is common.

$\angle BAC = \angle ACD$ (alternate angles $AB \parallel CD$)

$\angle BCA = \angle DAC$ (alternate angles $AD \parallel CB$)

$\therefore \triangle ABC \equiv \triangle CDA$ (AAS)

and $AB = DC$ as well as $AD = BC$ (corresponding sides in congruent triangles).

5 **a** $\triangle ABE \,|||\, \triangle ACD$ (all angles equal)

b 2.5

c $x = 7.5$

6 **a** $\angle CAB = \angle FDE$ (given)

$\frac{AC}{DF} = \frac{AB}{DE} = \frac{1}{3}$ (ratio of corresponding sides)

$\therefore \triangle CAB \,|||\, \triangle FDE$ (SAS)

b $\angle BAO = \angle CDO$ (alternate angles $AB \parallel DC$)

$\angle AOB = \angle DOC$ (vertically opposite)

$\therefore \triangle ABO \,|||\, \triangle DCO$ (AAA)

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

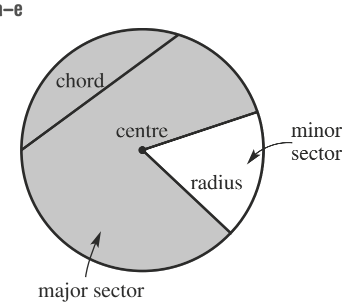
7 a $\angle D$ is common
$\angle ABD = \angle ECD$ (corresponding angles equal since $AB \parallel EC$)
$\therefore \triangle ABD \,|||\, \triangle ECD$ (AAA)
b 3 cm

8 $\angle A$ is common,
as Q and P are both midpoints.
$\frac{AP}{AB} = \frac{1}{2}$ and $\frac{AQ}{AC} = \frac{1}{2}$
$\therefore \triangle AQP \,|||\, \triangle ACB$ (SAS)
and $\frac{QP}{CB} = \frac{1}{2}$ (corresponding sides in the same ratio).
$\therefore QP = \frac{1}{2} CB$

2F

Building understanding

1 a–e



2 a 55° b 90°
c 75° d 140°

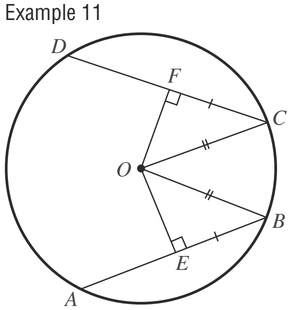
3 a 85° each
b $\angle AOB = \angle COD$ (chord theorem 1)
c 0.9 cm each
d $OE = OF$ (chord theorem 2)

4 a 1 cm each
b 52° each
c $AM = BM$ and $\angle AOM = \angle BOM$ (chord theorem 3)

Now you try

Example 10
a 2 m (chord theorem 2)
b 3 m, 60° (chord theorem 3)

Example 11



Since $CD = AB$ and E and F are midpoints (from theorem 3) then $CF = BE$ (S).
Also $OC = OB$ (radii) H
Also $\angle OFC = \angle OEB = 90°$ (from theorem 3) R
$\therefore \triangle OCF \equiv \triangle OBE$ (RHS).
$\therefore OF = OE$

Exercise 2F

1 a $OF = 4$ cm (using chord theorem 2)
b $AM = 3$ m
$\angle AOM = 50°$ (using chord theorem 3)

2 a $\angle DOC = 70°$ (chord theorem 1)
b $OE = 7.2$ cm (chord theorem 2)
c $XZ = 4$ cm and $\angle XOZ = 51°$ (chord theorem 3)

3 The perpendicular bisectors of two different chords of a circle intersect at the centre of the circle.

4 a 3.5 m b 9 m c 90° d 90°

5 a 140 b 40 c 19
d 72 e 30 f 54

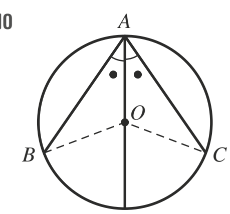
6 6 m

7 $3 + \sqrt{128}$ mm $= 3 + 8\sqrt{2}$ mm

8 a Triangles are congruent (SSS), so angles at the centre of the circle are corresponding, and therefore equal.
b Triangles are congruent (SAS), so chords are corresponding sides, and therefore equal.

9 a Triangles are congruent (SSS), so the angles formed by the chord and radius are corresponding, and therefore equal. Since these angles are also supplementary, they must be 90°.
b Triangles are congruent (SAS), so the angles formed by the chord and radius are corresponding, and therefore equal. Since these angles are also supplementary, they must be 90°.

10



First, prove $\triangle OAB \equiv \triangle OAC$ (AAS), which are isosceles.
So $AB = AC$, corresponding sides in congruent triangles.

11 a $AD = BD$ (radii of same circle)
$AC = BC$ (radii of same circle)
CD is common.
$\therefore \triangle ACD \equiv \triangle BCD$ (SSS).
b $AC = BC$ (radii of same circle)
$\angle ACE = \angle BCE$ (corresponding angles in congruent triangles)
CE is common.
$\therefore \triangle ACE \equiv \triangle BCE$ (SAS).
c Using the converse of chord theorem 3 since $\angle ACE = \angle BCE$, $CD \perp AB$.

ISBN 978-1-108-77290-7

2G

Building understanding

1 a $\angle ADC$ b $\angle ADC$ c $\angle ADC$
d $\angle AFC$ e $\angle AEC$ f $\angle AEC$

2 a $\angle AOB$ b $\angle ACB$ c 80° d 61°

3 a 180° b 90° c 60° d 7°

Now you try

Example 12
a 50 b 25

Example 13
60°

Exercise 2G

1 a 70 b 30

2 a 50 b 40 c 80
d 60 e 250 f 112.5
g 38 h 120 i 18

3 a 70 b 25 c 10

4 a $\angle ABC = 72°$, $\angle ABD = 22°$
b $\angle ABC = 70°$, $\angle ABD = 45°$
c $\angle ABC = 72°$, $\angle ABD = 35°$

5 a $\angle ADC = 75°$, $\angle ABC = 75°$
b $\angle ABC = 57.5°$, $\angle ADC = 57.5°$
c $\angle AOD = 170°$, $\angle ABD = 85°$

6 a 100° b 94.5° c 100°
d 119° e 70° f 66°

7 a 58° b 53° c 51°
d 45° e 19° f 21°

8 a 70° b 90°
c The angle in a semicircle is 90°.
d Theorem 2 is the specific case of theorem 1 when the angle at the centre is 180°.

9 a i false ii true iii true iv false
b i false ii true iii true iv false

10 a $2x°$ b $360 - 2x$

11 a $\angle AOC = 180° - 2x°$ ($\triangle AOC$ is isosceles)
b $\angle BOC = 180° - 2y°$ ($\triangle BOC$ is isosceles)
c $\angle AOB = 360° - \angle AOC - \angle BOC = 2x° + 2y°$
d $\angle AOB = 2(x° + y°) = 2\angle ACB$

12 a $\angle BOC = 180° - 2x°$ ($\triangle BOC$ is isosceles).
$\angle AOB = 180° - \angle BOC = 180° - (180° - 2x°) = 2x°$
b $\angle AOC = 180° - 2x°$ ($\triangle AOC$ is isosceles)
$\angle BOC = 180° - 2y°$ ($\triangle BOC$ is isosceles)
Reflex $\angle AOB = 360° - \angle AOC - \angle BOC$
$= 360° - (180° - 2x°) - (180° - 2y°)$
$= 2x° + 2y° = 2(x + y)° = 2\angle ACB.$
c $\angle OBC = x° + y°$ ($\triangle OCB$ is isosceles)
$\angle COB = 180° - 2(x + y)°$
$\angle AOB = 180° - 2x° - (180° - 2(x + y)°)$
$= 2y°$

13 $\angle AOB = 180° - 2x°$ ($\triangle AOB$ is isosceles)
$\angle BOC = 180° - 2y°$ ($\triangle BOC$ is isosceles)
$\angle AOB + \angle BOC = 180°$ (supplementary angles),
therefore $(180 - 2x) + (180 - 2y) = 180$
$$360 - 2x - 2y = 180$$
$$2x + 2y = 180$$
$$2(x + y) = 180$$
$$x + y = 90$$

2H

Building understanding

1 a $\angle ACD$ b $\angle ACD$ c $\angle ACD$

2 a $\angle ABD$ and $\angle ACD$ b 85°
c $\angle BAC$ and $\angle BDC$ d 17°

3 a Supplementary angles sum to 180°.
b 117°
c 109°
d Yes, 117° + 109° + 63° + 71° = 360°

Now you try

Example 14
a 37 b $a = 80, b = 95$

Exercise 2H

1 a 60 b $a = 84,\ b = 40$

2 a $x = 37$ b $x = 20$ c $x = 110$
d $x = 40$ e $x = 22.5$ f $x = 55$

3 a $x = 60$ b $x = 90$ c $x = 30$
d $x = 88$ e $x = 72, y = 108$ f $x = 123$

4 a 72 b 43 c 69
d 57 e 52 f 48
g 30 h 47 i 108

5 a $a = 30, b = 100$ b $a = 54, b = 90$
c $a = 105, b = 105, c = 75$ d $a = 55, b = 70$
e $a = 118, b = 21$ f $a = 45, b = 35$

6 a 80° b 71°
c $\angle CBE + \angle ABE = 180°$ (supplementary angles)
$\angle CBE + \angle CDE = 180°$ (circle theorem 4)
$\therefore \angle CBE + \angle ABE = \angle CBE + \angle CDE.$
$\therefore \angle ABE = \angle CDE$

7 a $\angle ACD = \angle ABD = x°$ and $\angle DAC = \angle DBC = y°$ (circle theorem 3)
b Using angle sum of $\triangle ACD$, $\angle ADC = 180° - (x° + y°)$.
c $\angle ABC$ and $\angle ADC$ are supplementary.

8 a i 80° ii 100° iii 80°
b $\angle BAF + \angle DCB = 180°$, therefore $AF \parallel CD$ (cointerior angles are supplementary).

9 a $\angle PCB = 90°$ (circle theorem 2)
b $\angle A = \angle P$ (circle theorem 3)
c $\sin P = \dfrac{a}{2r}$
d As $\angle A = \angle P$, $\sin A = \dfrac{a}{2r}$, therefore $2r = \dfrac{a}{\sin A}$.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2I

Building understanding

1 a Once b $90°$ c 5 cm
2 a $\angle BAP$ b $\angle BPX$ c $\angle ABP$ d $\angle APY$
3 a $180°$ b $360°$

Now you try

Example 15
a 10 b 30

Example 16
a $50°$ b $70°$

Exercise 2I

1 a 20 b 20
2 a $a = 19$ b $a = 62$ c $a = 70$
3 a $a = 50$ b $a = 28$ c $a = 25$
4 a $50°$ b $59°$
5 a $a = 73, b = 42, c = 65$
b $a = 26, b = 83, c = 71$
c $a = 69, b = 65, c = 46$
6 a 5 cm b 11.2 cm
7 a $a = 115$ b $a = 163$ c $a = 33$
d $a = 28$ e $a = 26$ f $a = 26$
g $a = 36$ h $a = 26$ i $a = 30$
8 a $a = 70$ b $a = 50$ c $a = 73$
d $a = 40$ e $a = 19$ f $a = 54$
9 4 cm
10 a OA and OB are radii of the circle.
b $\angle OAP = \angle OBP = 90°$
c $\angle OAP = \angle OBP = 90°$
OP is common
$OA = OB$
$\therefore \triangle OAP \equiv \triangle OBP$ (RHS)
d AP and BP are corresponding sides in congruent triangles.
11 a $\angle OPB = 90° - x°$, tangent meets radii at right angles
b $\angle BOP = 2x°$, using angle sum in an isocles triangle
c $\angle BAP = x°$, circle theorem 1
12 $\angle BAP = \angle BPY$ (alternate segment theorem)
$\angle BPY = \angle DPX$ (vertically opposite angles)
$\angle DPX = \angle DCP$ (alternate segment theorem)
$\therefore \angle BAP = \angle DCP$, so $AB \parallel DC$ (alternate angles are equal).
13 $AP = TP$ and $TP = BP$, hence $AP = BP$.
14 a Let $\angle ACB = x°$, therefore $\angle ABC = 90° - x°$.
Construct OP. $OP \perp PM$ (tangent). $\angle OPC = x°$ ($\triangle OPC$ is isosceles). Construct OM.
$\triangle OAM \equiv \triangle OPM$ (RHS), therefore $AM = PM$.
$\angle BPM = 180° - 90° - x° = 90° - x°$.
Therefore, $\triangle BPM$ is isosceles with $PM = BM$.
Therefore, $AM = BM$.
b Answers may vary.

2J

Building understanding

1 a 3 b 6 c 7 d 8
2 a $\frac{21}{2}$ b $\frac{5}{2}$ c $\frac{33}{7}$ d $\frac{27}{7}$
3 a $AP \times CP = BP \times DP$ b $AP \times BP = DP \times CP$
c $AP \times BP = CP^2$

Now you try

Example 17
a $\frac{4}{5}$ b $\frac{39}{5}$ c $\frac{5}{2}$

Exercise 2J

1 a $\frac{3}{2}$ b $\frac{83}{6}$ c $\frac{9}{4}$
2 a 5 b 10 c $\frac{112}{15}$
3 a $\frac{143}{8}$ b $\frac{178}{9}$ c $\frac{161}{9}$
4 a $\frac{32}{3}$ b $\frac{16}{3}$ c $\frac{35}{2}$
5 a $\sqrt{65}$ b $\sqrt{77}$
6 a $\frac{64}{7}$ b $\frac{209}{10}$ c $\frac{81}{7}$
d $\frac{74}{7}$ e $\frac{153}{20}$ f $\sqrt{65} - 1$
7 a $x(x + 5) = 7 \times 8, x^2 + 5x = 56, x^2 + 5x - 56 = 0$
b $x(x + 11) = 10 \times 22, x^2 + 11x = 220,$
$x^2 + 11x - 220 = 0$
c $x(x + 23) = 31^2, x^2 + 23x = 961, x^2 + 23x - 961 = 0$
8 For this diagram, the third secant rule states:
$AP^2 = DP \times CP$ and $BP^2 = DP \times CP$, so $BP = AP$.
9 $AP \times BP = DP \times CP$
$AP \times BP = AP \times CP$ since $AP = DP$.
$BP = CP$
10 a $\angle A = \angle D$ and $\angle B = \angle C$ (circle theorem 3)
b $\angle P$ is the same for both triangles (vertically opposite), so $\triangle ABP \parallel\!\mid \triangle DCP$ (AAA).
c $\frac{AP}{DP} = \frac{BP}{CP}$
d $\frac{AP}{DP} = \frac{BP}{CP}$, cross-multiplying gives $AP \times CP = BP \times DP$.
11 a $\angle B = \angle C$ (circle theorem 3)
b $\triangle PBD \parallel\!\mid \triangle PCA$ (AAA)
c $\frac{AP}{DP} = \frac{CP}{BP}$, so $AP \times BP = DP \times CP$.
12 a yes
b alternate segment theorem
c $\triangle BPC \parallel\!\mid \triangle CPA$ (AAA)
d $\frac{BP}{CP} = \frac{CP}{AP}$, so $CP^2 = AP \times BP$.
13 $d = \sqrt{4r_1r_2} = 2\sqrt{r_1r_2}$

ISBN 978-1-108-77290-7

Cambridge University Press

Updated June 2022

Problems and challenges

1 21 units^2
2 $BD = 5$ cm, $CE = 19$ cm
3 $\angle ADE = \angle ABE, \angle EFD = \angle BFA, \angle DEB = \angle DAB,$
$\angle DFB = \angle EFA, \angle CDB = \angle CAE, \angle DAE = \angle DBE,$
$\angle ADB = \angle AEB, \angle ABD = \angle AED = \angle CBD = \angle CEA$
4 42.5%
5 Check with your teacher.
6 a $\angle FDE = \angle DFC = \angle ABC$ (alternate and corresponding angles in parallel lines)
$\angle FED = \angle EFB = \angle ACB$ (alternate and corresponding angles in parallel lines)
$\angle DFE = \angle BAC$ (angle sum of a triangle)
$\triangle ABC \,|||\, \triangle FDE$ (AAA)
b i 4 : 1 ii 16 : 1
c 4^{n-1} : 1

Short-answer questions

1 a 65 b 120
c $x = 62, y = 118$ d 46
2 a 148° b 112°
3 a $AB = DE$ (given)
$\angle ABC = \angle DEF$ (given)
$\angle BAC = \angle EDF$ (given)
$\therefore \triangle ABC \equiv \triangle DEF$ (AAS).
b $AB = AD$ (given)
$\angle BAC = \angle DAC$ (given)
AC is common.
$\therefore \triangle ABC \equiv \triangle ADC$ (SAS).
c $AB = CD$ (given)
$AD = CB$ (given)
BD is common.
$\therefore \triangle ABD \equiv \triangle CDB$ (SSS).
4 a $AB = CD$ (given)
$\angle BAC = \angle DCA$ (alternate angles)
AC is common.
$\therefore \triangle ABC \equiv \triangle CDA$ (SAS).
b $\angle BCA = \angle DAC$ (alternate angles), therefore $AD \parallel BC$ (alternate angles are equal).
5 a $\frac{DE}{AB} = \frac{10.5}{7} = 1.5$
$\frac{EF}{BC} = \frac{14.7}{9.8} = 1.5$ (ratio of corresponding sides)
$\angle ABC = \angle DEF$ (given)
$\therefore \triangle ABC \,|||\, \triangle DEF$ (SAS)
$x = 19.5$
b $\angle EAB = \angle DAC$ (common)
$\angle EBA = \angle DCA$ (corresponding with $EB \parallel DC$)
$\therefore \triangle ABE \,|||\, \triangle ACD$ (AAA)
$x = 6$ 25
c $\angle BAC = \angle EDC$ (given)
$\angle ACB = \angle DCE$ (vertically opposite)
$\therefore \triangle ABC \,|||\, \triangle DEC$ (AAA)
$x = 8.82$
d $\angle ABD = \angle DBC$ (given)
$\angle DAB = \angle CDB = 35°$ (angle sum of triangle)
$\therefore \triangle ABD \,|||\, \triangle DBC$ (AAA)
$x = \frac{100}{7}$
6 a 65 (chord theorem 1)
b 7 (chord theorem 2)
c 6 (chord theorem 3)
7 a $a = 25$
b $a = 50, b = 40$
c $a = 70$
d $b = 54$
e $a = 115$
f $a = 30, b = 30$
8 a $x = 26, y = 58, z = 64$
b $a = 65, b = 130, c = 50, d = 8$
c $t = 63$
9 a 5 b 6 c $\frac{40}{3}$

Multiple-choice questions

1 C 2 B 3 B 4 C 5 B
6 A 7 E 8 C 9 D 10 B

Extended-response questions

1 a $\angle BAC = \angle BDE = 90°$
$\angle B$ is common.
$\therefore \triangle ABC \,|||\, \triangle DBE$ (AAA).
b 1.2 km
c i $\frac{AC}{DE} = \frac{3}{2}$
$\therefore \frac{AB}{DB} = \frac{3}{2}$ (ratio of corresponding sides in similar triangles)
$\frac{x+1}{x} = \frac{3}{2}$
$\therefore 2(x + 1) = 3x$
ii 2
d 44.4%
2 a i 106.26° ii 73.74°
b 12 cm c 25 cm d 70 cm

Chapter 3

3A

Building understanding

1 a irrational b root
c non-recurring d rational number
2 a 4 b 25
c 16 d 36

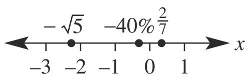
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Now you try

Example 1

a irrational
b rational
c rational



$-\sqrt{5} \approx -2.2$, $-40\% = -0.4$, $\frac{2}{7} \approx 0.29$

Example 2

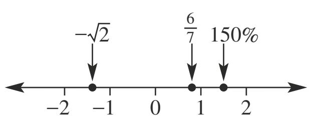
a $2\sqrt{5}$ b $20\sqrt{3}$ c $2\sqrt{3}$ d $\frac{5\sqrt{5}}{4}$

Example 3

a $\sqrt{18}$ b $\sqrt{75}$

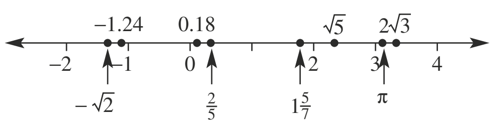
Exercise 3A

1 a irrational b rational c rational



$-\sqrt{2} \approx -1.4$, $150\% = 1.5$, $\frac{6}{7} \approx 0.86$

2 a irrational b rational
c rational d rational
e rational f irrational
g irrational h irrational



3 a yes b yes c no d no
e no f yes g yes h no

4 a $2\sqrt{3}$ b $3\sqrt{5}$ c $2\sqrt{6}$ d $4\sqrt{3}$
e $5\sqrt{3}$ f $10\sqrt{5}$ g $7\sqrt{2}$ h $3\sqrt{10}$
i $8\sqrt{2}$ j $6\sqrt{10}$ k $9\sqrt{2}$ l $4\sqrt{5}$

5 a $6\sqrt{2}$ b $6\sqrt{5}$ c $16\sqrt{3}$ d $6\sqrt{7}$
e $21\sqrt{2}$ f $20\sqrt{5}$ g $\sqrt{5}$ h $\sqrt{7}$
i $\frac{\sqrt{6}}{2}$ j $\frac{\sqrt{6}}{4}$ k $\frac{\sqrt{5}}{5}$ l $\frac{\sqrt{11}}{6}$
m $3\sqrt{11}$ n $2\sqrt{2}$ o $2\sqrt{2}$ p $\frac{2\sqrt{17}}{7}$
q $\frac{3\sqrt{3}}{2}$ r $4\sqrt{6}$ s $\frac{2\sqrt{3}}{3}$ t $\frac{3\sqrt{3}}{2}$

6 a $\frac{2\sqrt{2}}{3}$ b $\frac{2\sqrt{3}}{7}$ c $\frac{3\sqrt{2}}{5}$ d $\frac{\sqrt{11}}{5}$
e $\frac{\sqrt{10}}{3}$ f $\frac{\sqrt{3}}{6}$ g $\frac{\sqrt{13}}{4}$ h $\frac{1}{\sqrt{2}}$
i $\frac{\sqrt{5}}{3}$ j $\frac{3\sqrt{3}}{2}$ k $\frac{\sqrt{5}}{2\sqrt{2}}$ l $\frac{\sqrt{14}}{\sqrt{19}}$

7 a $\sqrt{12}$ b $\sqrt{32}$ c $\sqrt{50}$ d $\sqrt{27}$
e $\sqrt{45}$ f $\sqrt{108}$ g $\sqrt{128}$ h $\sqrt{700}$
i $\sqrt{810}$ j $\sqrt{125}$ k $\sqrt{245}$ l $\sqrt{363}$

8 a $15\sqrt{3}$ b $13\sqrt{7}$ c $19\sqrt{5}$ d $31\sqrt{3}$

9 a $4\sqrt{2}$ m b $2\sqrt{30}$ cm c $4\sqrt{15}$ mm

10 a radius $= 2\sqrt{6}$ cm, diameter $= 4\sqrt{6}$ cm
b radius $= 3\sqrt{6}$ m, diameter $= 6\sqrt{6}$ m
c radius $= 8\sqrt{2}$ m, diameter $= 16\sqrt{2}$ m

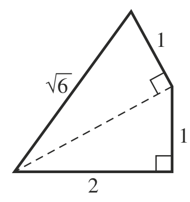
11 a $2\sqrt{5}$ cm b $3\sqrt{5}$ m c $\sqrt{145}$ mm
d $\sqrt{11}$ m e $\sqrt{11}$ mm f $2\sqrt{21}$ cm

12 $\sqrt{72} = \sqrt{36} \times 2$ (i.e. 36 is highest square factor of 72)
$= 6\sqrt{2}$

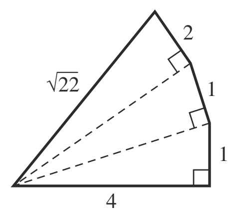
13 a 9, 25, 225 b $15\sqrt{2}$

14 a Draw triangle with shorter sides length 1 cm and 3 cm.
b Draw triangle with shorter sides length 2 cm and 5 cm.

c



d



15 Check with your teacher.

3B

Building understanding

1 a yes b no c no d yes
e yes f no g yes h yes

2 a $6x$ b $-5x$ c $17a$ d t

3 a $4\sqrt{3}$
b i $5\sqrt{3}$ ii $-3\sqrt{3}$ iii $17\sqrt{3}$

Now you try

Example 4

a $5\sqrt{5}$ b $\sqrt{7} + 7\sqrt{3}$

Example 5

a $5\sqrt{2}$ b $2\sqrt{3}$

Exercise 3B

1 a i $5\sqrt{7}$ ii $9\sqrt{3}$
b i $2\sqrt{5} + 5\sqrt{3}$ ii $2\sqrt{5} + 8\sqrt{11}$

2 a $6\sqrt{5}$ b $3\sqrt{3}$ c $4\sqrt{2}$
d $3\sqrt{2}$ e $11\sqrt{5}$ f $\sqrt{3}$
g $6\sqrt{10}$ h $5\sqrt{2}$ i $-2\sqrt{21}$
j $-6\sqrt{11}$ k $-\sqrt{13}$ l $-7\sqrt{30}$

3 a $\sqrt{3} + 5\sqrt{2}$ b $3\sqrt{6} + 7\sqrt{11}$ c $4\sqrt{5} - 7\sqrt{2}$
d $-2\sqrt{2} + \sqrt{5}$ e $4\sqrt{3}$ f 0
g $-3\sqrt{2} - 3\sqrt{10}$ h $-2\sqrt{5} + 3\sqrt{15}$

4 a $\sqrt{2}$ b $5\sqrt{2}$ c $4\sqrt{3}$ d $\sqrt{5}$
e $7\sqrt{2}$ f $12\sqrt{3}$ g $8\sqrt{11}$ h $3\sqrt{2}$
i $5\sqrt{6}$ j $\sqrt{5}$ k $32\sqrt{2}$ l $20\sqrt{2}$

ISBN 978-1-108-77290-7 Cambridge University Press
 Updated June 2022

5 a $13\sqrt{2}$ b $9\sqrt{6}$
c $2\sqrt{5}-\sqrt{7}$ d $5\sqrt{5}+6\sqrt{7}$
e $\sqrt{6}-3\sqrt{2}$ f $2\sqrt{3}+11\sqrt{5}-5\sqrt{2}$
g $9\sqrt{3}+2\sqrt{2}$ h $11-9\sqrt{3}$
i $9+18\sqrt{2}$ j $-9\sqrt{2}+15\sqrt{5}$

6 a $\frac{5\sqrt{3}}{6}$ b $\frac{7\sqrt{5}}{12}$ c $\frac{\sqrt{2}}{30}$
d $\frac{\sqrt{7}}{6}$ e $\frac{-\sqrt{2}}{10}$ f $\frac{13\sqrt{3}}{14}$
g $\frac{13\sqrt{5}}{18}$ h $\frac{-7\sqrt{3}}{30}$ i $\frac{-11\sqrt{10}}{24}$

7 a $4\sqrt{3}+2\sqrt{5}$ cm b $14\sqrt{2}$ cm
c $\sqrt{10}+3\sqrt{2}$ cm d $2\sqrt{10}+4\sqrt{5}$ cm
e $4\sqrt{3}+\sqrt{30}$ m f $12\sqrt{3}$ cm

8 a $\sqrt{20}=2\sqrt{5}$
b $3\sqrt{72}=18\sqrt{2}$, $\sqrt{338}=13\sqrt{2}$

9 a $5\sqrt{3}-6\sqrt{3}+\sqrt{3}=0$
b $\sqrt{6}+2\sqrt{6}-3\sqrt{6}=0$
c $6\sqrt{2}-8\sqrt{2}+2\sqrt{2}=0$
d $2\sqrt{2}-3\sqrt{2}+\sqrt{2}=0$
e $4\sqrt{5}-7\sqrt{5}+3\sqrt{5}=0$
f $3\sqrt{2}-6\sqrt{3}-5\sqrt{2}+6\sqrt{3}+2\sqrt{2}=0$

10 a $6\sqrt{3}-3\sqrt{2}$, unlike surds
b $8\sqrt{2}+2\sqrt{5}$, unlike surds
c $5\sqrt{2}-6\sqrt{5}$, unlike surds
d $10\sqrt{10}+10\sqrt{3}$, unlike surds
e $20\sqrt{2}+30\sqrt{3}$, unlike surds
f $4\sqrt{5}-6\sqrt{6}$, unlike surds

11 a $\frac{7\sqrt{2}}{15}$ b $\frac{2\sqrt{3}}{3}$ c $\frac{\sqrt{5}}{12}$ d $\frac{-3\sqrt{2}}{4}$
e $\frac{\sqrt{3}}{2}$ f $\frac{-7\sqrt{7}}{15}$ g $-\sqrt{2}$ h $\frac{29\sqrt{6}}{28}$
i 0 j $8\sqrt{3}$ k $\frac{6\sqrt{6}}{35}$ l $\frac{29\sqrt{5}}{42}$

3C

Building understanding

1 a $\sqrt{\frac{15}{3}}=\sqrt{5}$ b $\sqrt{\frac{42}{7}}=\sqrt{6}$
c $\sqrt{6\times 5}=\sqrt{30}$ d $\sqrt{11\times 2}=\sqrt{22}$
2 a 6 b 7 c 5
3 a $2x+6$ b $10x-5$ c $30-24x$

Now you try

Example 6
a $\sqrt{15}$ b $24\sqrt{3}$ c 63

Example 7
a $-\sqrt{3}$ b $2\sqrt{2}$

Example 8
a $5\sqrt{6}-\sqrt{14}$
b $30\sqrt{2}-45$

Exercise 3C

1 a i $\sqrt{42}$ ii $\sqrt{55}$
b i $10\sqrt{21}$ ii $6\sqrt{35}$
c i 12 ii 45

2 a $\sqrt{15}$ b $\sqrt{21}$ c $\sqrt{26}$
d $\sqrt{35}$ e $-\sqrt{30}$ f $-\sqrt{30}$
g $\sqrt{66}$ h $\sqrt{6}$ i $\sqrt{70}$

3 a $\sqrt{10}$ b $\sqrt{6}$ c $-\sqrt{3}$
d $\sqrt{5}$ e $\sqrt{3}$ f $\sqrt{10}$
g $\sqrt{5}$ h $-\sqrt{13}$ i $-\sqrt{5}$

4 a 3 b 5 c 9
d $7\sqrt{2}$ e $2\sqrt{11}$ f $3\sqrt{6}$
g $5\sqrt{2}$ h $4\sqrt{6}$ i 10

5 a $10\sqrt{3}$ b $21\sqrt{2}$ c $12\sqrt{14}$
d $-50\sqrt{3}$ e $-18\sqrt{3}$ f $15\sqrt{5}$
g $42\sqrt{6}$ h $-60\sqrt{10}$ i $-20\sqrt{10}$
j $42\sqrt{2}$ k $24\sqrt{30}$ l $216\sqrt{7}$

6 a 11 b 13 c 12
d 125 e 147 f 162

7 a $2\sqrt{2}$ b $3\sqrt{6}$ c $\frac{\sqrt{5}}{2}$
d $\frac{-4}{\sqrt{13}}$ e $\frac{-1}{3\sqrt{7}}$ f $\frac{2\sqrt{5}}{3}$

8 a $\sqrt{6}+\sqrt{15}$ b $\sqrt{14}-\sqrt{10}$ c $-\sqrt{55}-\sqrt{65}$
d $-2\sqrt{15}-2\sqrt{21}$ e $6\sqrt{26}-3\sqrt{22}$ f $20-20\sqrt{2}$
g $30\sqrt{2}+15\sqrt{30}$ h $-12\sqrt{3}+12\sqrt{2}$ i $42+63\sqrt{2}$
j $90\sqrt{3}-24\sqrt{10}$ k $-16+24\sqrt{10}$ l $42\sqrt{2}+30$

9 a 28 b 18 c -75
d $\sqrt{2}-\sqrt{6}$ e $3\sqrt{3}+4$ f $-\sqrt{10}+\sqrt{5}$
g 2 h $8\sqrt{2}$ i $\sqrt{2}-6$

10 a $2\sqrt{6}$ b $\sqrt{30}$ c 6

11 a $\frac{3}{4}$ cm^2 b $2\sqrt{6}$ cm

12 a $\sqrt{6}\times\sqrt{6}=\sqrt{6\times 6}=\sqrt{36}=6$
b $-\sqrt{8}\times\sqrt{8}=-\sqrt{8\times 8}=-\sqrt{64}=-8$
c $-\sqrt{5}\times -\sqrt{5}=+\sqrt{5\times 5}=\sqrt{25}=5$

13 a Simplify each surd before multiplying.
b Allows for the multiplication of smaller surds, which is simpler.
c i $3\sqrt{2}\times 3\sqrt{3}=9\sqrt{6}$
ii $2\sqrt{6}\times 2\sqrt{5}=4\sqrt{30}$
iii $5\sqrt{2}\times 3\sqrt{5}=15\sqrt{10}$
iv $3\sqrt{6}\times 5\sqrt{3}=45\sqrt{2}$
v $6\sqrt{2}\times 4\sqrt{3}=24\sqrt{6}$
vi $6\sqrt{3}\times -10\sqrt{5}=-60\sqrt{15}$
vii $-12\sqrt{3}\times -2\sqrt{7}=24\sqrt{21}$
viii $7\sqrt{2}\times 10\sqrt{3}=70\sqrt{6}$
ix $12\sqrt{2}\times 12\sqrt{5}=144\sqrt{10}$

14 a 3 b 2 c -9
d $-\frac{1}{5}$ e $\frac{2}{5}$ f 3

15 a $54\sqrt{2}$ b $375\sqrt{3}$ c $162\sqrt{3}$
d 25 e 9 f $128\sqrt{2}$
g $-120\sqrt{5}$ h $-108\sqrt{2}$ i 720

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

j $14\sqrt{7}$　k $\frac{27\sqrt{2}}{2}$　l 81
m $100\sqrt{3}$　n 144　o $-96\sqrt{15}$
p $\frac{81\sqrt{3}}{25}$　q $\frac{5}{3\sqrt{3}}$　r $\frac{9\sqrt{6}}{2}$

16 a $19 - 2\sqrt{6}$　b 16
c $2\sqrt{15} - 85$　d $10\sqrt{3} - 37$
e $30 - 10\sqrt{2}$　f 0
g $4\sqrt{3} - 14$　h $47\sqrt{2} - 10\sqrt{30} + 11$

3D

Building understanding

1 a 1　b $\frac{1}{2}$　c -2　d 6
2 a $\sqrt{3}$　b 10　c $\sqrt{5}$　d $\sqrt{7}$
3 a 0.377…　b 2.886…　c 16.31…
All pairs of numbers are equal.

Now you try

Example 9
a $\frac{3\sqrt{2}}{2}$　b $\frac{4\sqrt{21}}{7}$　c $\frac{\sqrt{10}}{3}$　d $\frac{2\sqrt{7} - 7}{7}$

Exercise 3D

1 a i $\frac{3\sqrt{5}}{5}$　ii $\frac{7\sqrt{6}}{6}$
b i $\frac{2\sqrt{15}}{5}$　ii $\frac{3\sqrt{10}}{2}$
c i $\frac{\sqrt{21}}{5}$　ii $\frac{\sqrt{30}}{21}$
d i $\frac{\sqrt{2} - 2}{2}$　ii $\frac{4\sqrt{5} - 5}{5}$

2 a $\frac{\sqrt{2}}{2}$　b $\frac{\sqrt{7}}{7}$　c $\frac{3\sqrt{11}}{11}$　d $\frac{4\sqrt{5}}{5}$
e $\frac{5\sqrt{3}}{3}$　f $4\sqrt{2}$　g $\frac{\sqrt{15}}{3}$　h $\frac{\sqrt{14}}{7}$

3 a $\frac{\sqrt{6}}{3}$　b $\frac{\sqrt{35}}{7}$　c $\frac{\sqrt{66}}{11}$　d $\frac{\sqrt{10}}{5}$
e $\frac{\sqrt{21}}{3}$　f $\frac{\sqrt{42}}{7}$　g $\frac{\sqrt{30}}{3}$　h $\frac{\sqrt{34}}{2}$

4 a $\frac{4\sqrt{14}}{7}$　b $\frac{5\sqrt{6}}{3}$　c $\frac{3\sqrt{10}}{2}$
d $\frac{3\sqrt{42}}{7}$　e $\frac{7\sqrt{30}}{10}$　f $\frac{2\sqrt{105}}{15}$

5 a $\frac{4\sqrt{21}}{15}$　b $\frac{\sqrt{6}}{3}$　c $\frac{\sqrt{35}}{3}$　d $\frac{2\sqrt{2}}{5}$
e $\frac{2\sqrt{5}}{15}$　f $\frac{10}{9}$　g $\frac{9\sqrt{2}}{2}$　h $\frac{3\sqrt{7}}{2}$

6 a $\frac{\sqrt{3} + \sqrt{6}}{3}$　b $\frac{3\sqrt{7} + \sqrt{35}}{7}$　c $\frac{2\sqrt{5} - \sqrt{15}}{5}$
d $\frac{\sqrt{6} - \sqrt{10}}{2}$　e $\frac{\sqrt{35} + \sqrt{14}}{7}$　f $\frac{\sqrt{30} - \sqrt{21}}{3}$
g $\frac{2\sqrt{3} + \sqrt{42}}{6}$　h $\frac{5\sqrt{2} + 2\sqrt{5}}{10}$　i $\frac{\sqrt{30} - 5\sqrt{2}}{5}$
j $\frac{8\sqrt{3} - 15\sqrt{2}}{6}$　k $\frac{3\sqrt{2} + 2\sqrt{5}}{2}$　l $\frac{6\sqrt{5} + 5\sqrt{6}}{2}$

7 a $\frac{5\sqrt{3}}{3}$ cm^2　b $\frac{2}{3}$ m^2
c $\frac{\sqrt{10} + \sqrt{15}}{10}$ mm^2

8 a $\frac{2\sqrt{3} + 3\sqrt{2}}{6}$　b $\frac{6\sqrt{5} + 5\sqrt{2}}{10}$　c $\frac{9\sqrt{7} - 14\sqrt{3}}{21}$
d $\frac{5\sqrt{3} - 2\sqrt{2}}{6}$　e $\frac{2\sqrt{2} + 5\sqrt{3}}{12}$　f $\frac{9\sqrt{5} + 4\sqrt{3}}{30}$
g $\frac{-2\sqrt{14}}{15}$　h $\frac{6\sqrt{30} + 4\sqrt{6}}{9}$　i $\frac{3\sqrt{10} - 2\sqrt{42}}{9}$

9 As $\frac{\sqrt{x}}{\sqrt{x}}$ is equal to 1.

10 a $\frac{\sqrt{21} + \sqrt{7a}}{7}$　b $\frac{\sqrt{30} + \sqrt{5a}}{5}$
c $\frac{2\sqrt{3} + \sqrt{6a}}{6}$　d $1 - \sqrt{3a}$
e $1 - \sqrt{5a}$　f $1 - \sqrt{7a}$
g $\frac{4\sqrt{10a} + 5\sqrt{2}}{10}$　h $\frac{\sqrt{6a} + \sqrt{2}}{2}$
i $\frac{2\sqrt{14a} + 7\sqrt{2}}{14}$

11 a i 14　ii 2　iii 47
b Each question is a difference of perfect squares, and each answer is an integer.
c $\frac{4 + \sqrt{2}}{4 + \sqrt{2}}$
d i $\frac{12 + 3\sqrt{2}}{14}$　ii $\frac{-3\sqrt{3} - 3}{2}$
iii $2\sqrt{2} + \sqrt{6}$　iv $\frac{-(6 + 2\sqrt{30})}{7}$

12 a $\frac{5\sqrt{3} - 5}{2}$　b $2\sqrt{3} + 2$
c $3\sqrt{5} + 6$　d $-4 - 4\sqrt{2}$
e $\frac{-3 - 3\sqrt{3}}{2}$　f $\frac{42 + 7\sqrt{7}}{29}$
g $-12 - 4\sqrt{10}$　h $-14 - 7\sqrt{5}$
i $\frac{2\sqrt{11} + 2\sqrt{2}}{9}$　j $2\sqrt{5} - 2\sqrt{2}$
k $\sqrt{7} - \sqrt{3}$　l $\frac{\sqrt{14} - \sqrt{2}}{6}$
m $\frac{6 + \sqrt{6}}{5}$　n $\sqrt{14} + 2\sqrt{2}$
o $10 - 4\sqrt{5}$　p $\frac{b\sqrt{a} - b\sqrt{b}}{a - b}$
q $\frac{a\sqrt{a} + a\sqrt{b}}{a - b}$　r $\frac{a + b - 2\sqrt{ab}}{a - b}$
s $\frac{a - \sqrt{ab}}{a - b}$　t $\frac{a\sqrt{b} + b\sqrt{a}}{a - b}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

3E

Building understanding

1 a 3^4 b 7^6 c $6x^3$ d $8a^3b^2$

2

x	4	3	2	1	0
2^x	$2^4 = 16$	$2^3 = 8$	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$

3 a $2^2 \times 2^3 = 2 \times 2 \times 2 \times 2 \times 2$
$= 2^5$

b $\frac{x^5}{x^3} = \frac{x \times x \times x \times x \times x}{x \times x \times x}$
$= x^2$

c $(a^2)^3 = a \times a \times a \times a \times a \times a$
$= a^6$

d $(2x)^0 \times 2x^0 = 1 \times 2$
$= 2$

Now you try

Example 10

a x^7 b $14a^3b^5$

Example 11

a m^2 b $\frac{1}{2}xy^2$

Example 12

a a^6 b $27y^9$ c $\frac{4x^6}{49y^2z^4}$ d $4x^5y^4$

Example 13

a 2 b 6

Exercise 3E

1 a i x^7 ii x^8
b i $10a^3b^3$ ii $12a^4b^5$

2 a a^9 b x^5 c b^6
d $14m^5$ e $6s^7$ f $2t^{16}$
g $\frac{p^3}{5}$ h $\frac{c^7}{6}$ i $\frac{9}{25}s^2$
j $6x^3y^3$ k $15a^3b^6$ l $18v^9w^2$
m $150x^5y^6$ n $12r^7s^6$ o $20m^8n^{10}$

3 a x^3 b a c q^3
d b^4 e y^5 f d^5
g j h m^6 i $2xy^3$
j $3r^2s$ k $2p^2$ l $2m^4x$
m $5b^3$ n $4st$ o $\frac{1}{4}v^2$
p $\frac{1}{2}a$ q $-\frac{x}{3}$ r $-\frac{y^2}{2}$

4 a x^{10} b t^6 c $4a^6$ d $5y^{15}$
e $64t^6$ f $4u^4$ g $27r^9$ h $81p^{16}$
i $\frac{a^4}{b^6}$ j $\frac{x^9}{y^{12}}$ k $\frac{x^4y^6}{z^8}$ l $\frac{u^{16}w^8}{v^8}$
m $\frac{27f^6}{125g^3}$ n $\frac{9a^4b^2}{4p^2q^6}$ o $\frac{a^3t^9}{27g^{12}}$ p $\frac{256p^8q^{12}}{81r^4}$

5 a 8 b 3 c 1 d 1
e 5 f 3 g –5 h 3

6 a x^8 b x^2y^2 c x^6n^8 d xy^2
e m f r^4s^7 g $\frac{9x^8y^2}{2}$ h $2y^4$
i $2a^2b^2$ j $27m^7n^{14}$ k $-45a^8b^5$ l $\frac{16}{3}f^3$
m $2m^6n^3$ n $21y^3z^2$ o 1 p $-6m^2n^7$

7 a –27 b –27 c 81 d –81

8 a x^{12} b a^{105} c $\frac{a^{30}}{b^{15}}$

9 a 13 b 18 c 81 d 64
e 1 f 1 g 9 h 8

10 He has not included the minus sign inside the brackets, i.e. has only applied it afterwards. Need $(-2)^4$ not -2^4.

11 a 3 b 4 c 1
d 3 e 4 f 1

12 a 9 b 2 c 162 d –18

13 a ±2 b 5 c 2 d $\frac{7}{2}$

14 a $x = 2, y = 4$ or $x = 4, y = 2$ or $x = 16, y = 1$
b $x = 8, y = 2$ or $x = 4, y = 3$ or $x = 64, y = 1$, or $x = 2, y = 6$
c $x = 9, y = 2$, or $x = 3, y = 4$ or $x = 81, y = 1$
d $x = 1, y =$ any positive integer

3F

Building understanding

1 a $2^{-2}, 2^{-3}, 2^{-4}$ b x^{-1}, x^{-2}, x^{-3}

2 a $\frac{1}{3^2}$ b $\frac{1}{5^2}$ c $\frac{5}{4^2}$ or $\frac{5}{2^4}$ d $\frac{-2}{3^3}$

3 a $\frac{1}{a^b}$ b a^b

4 a $\frac{1}{25}$ b $\frac{1}{27}$ c $\frac{4}{49}$

Now you try

Example 14

a $\frac{1}{b^3}$ b $\frac{2y^3}{x^2}$ c $2x^4$

Example 15

a $\frac{5}{a^2b^4}$ b $\frac{n^8}{3m^7}$

Example 16

a $\frac{pq^3}{2}$ b $\frac{m^7}{16r^5}$

Exercise 3F

1 a i $\frac{1}{b^2}$ ii $\frac{1}{a^5}$
b i $\frac{4y^3}{x}$ ii $\frac{7y^2}{x^3}$
c i $2x^2$ ii $6x^7$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2 **a** $\frac{1}{x^5}$ **b** $\frac{1}{a^4}$ **c** $\frac{2}{m^4}$ **d** $\frac{3}{y^7}$
e $\frac{3a^2}{b^3}$ **f** $\frac{4m^3}{n^3}$ **g** $\frac{10y^5z}{x^2}$ **h** $\frac{3z^3}{x^4y^2}$
i $\frac{q^3r}{3p^2}$ **j** $\frac{d^2f^5}{5e^4}$ **k** $\frac{3u^2w^7}{8v^6}$ **l** $\frac{2b^3}{5c^5d^2}$

3 **a** x^2 **b** $2y^3$ **c** $4m^7$ **d** $3b^5$
e $2b^4d^3$ **f** $3m^2n^4$ **g** $\frac{4b^4a^3}{3}$ **h** $\frac{5h^3g^3}{2}$

4 **a** x **b** a^3 **c** $\frac{2}{b^4}$ **d** $\frac{3}{y^3}$
e $\frac{1}{xy}$ **f** $\frac{4y^2}{a^3}$ **g** $\frac{6}{a^5b^2}$ **h** $\frac{18b^4}{a^2}$
i $\frac{x}{2y}$ **j** $\frac{4m}{7n^3}$ **k** $\frac{q}{p^5}$ **l** $\frac{p}{q^2r}$
m $\frac{a^2}{b^2}$ **n** $\frac{m^2}{n}$ **o** $\frac{a}{b^3}$ **p** $\frac{4r^3}{3s^2}$

5 **a** $\frac{2x}{3}$ **b** $\frac{7d^2}{10}$ **c** $\frac{5}{3s^3}$ **d** $\frac{4}{3f^2}$
e $\frac{f^5}{g^5}$ **f** $\frac{1}{r^6s^2}$ **g** $\frac{wx^5}{2}$ **h** $\frac{5c^5d^4}{4}$

6 **a** $\frac{16}{x^4}$ **b** $\frac{1}{64m^6}$ **c** $\frac{2}{x^{21}}$ **d** $\frac{4}{d^6}$
e $\frac{9}{t^8}$ **f** $\frac{5}{x^4}$ **g** $\frac{81}{x^{20}}$ **h** $\frac{-8}{x^{15}}$
i $\frac{y^4}{16}$ **j** $\frac{h^{12}}{81}$ **k** $7j^8$ **l** $2t^6$

7 **a** a^7b^2 **b** $\frac{16p^4}{9q^2}$ **c** $54x^7y^{10}$
d $4a^8b^3$ **e** $\frac{324r^{11}}{s}$ **f** $\frac{2y^{14}}{x^3}$
g a^2b^{18} **h** $\frac{m^{14}}{n^8}$ **i** $\frac{27x}{2y}$

8 **a** $\frac{1}{25}$ **b** $\frac{1}{64}$ **c** $\frac{2}{49}$ **d** $\frac{-5}{81}$
e $\frac{1}{9}$ **f** 1 **g** 98 **h** -48
i $\frac{9}{4}$ **j** $\frac{-64}{125}$ **k** $\frac{1}{16}$ **l** 100

9 0.0041 cm

10 **a** **i** $\frac{3}{2}$ **ii** $\frac{7}{5}$ **iii** $\frac{y}{2x}$
b $\frac{b}{a}$

11 The negative index should only be applied to x not to 2:
$2x^{-2} = \frac{2}{x^2}$

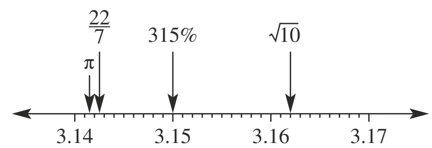
12 **a** $\frac{5}{6}$ **b** $\frac{5}{18}$ **c** $\frac{1}{3}$
d $-\frac{7}{12}$ **e** $\frac{71}{48}$ **f** $\frac{106}{9}$

13 Proof: $\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-1\times x} = 2^{-x}$

14 **a** -2 **b** -5 **c** -3 **d** -1
e -2 **f** -3 **g** -3 **h** -4
i 0 **j** 0 **k** 1 **l** 2
m -2 **n** 1 **o** -2 **p** 2

Progress quiz

1 **a** 3.16227766… irrational **b** $3.\overline{142857}$ rational
c 3.141592653… irrational **d** 3.15 rational



2 **a** $7\sqrt{2}$ **b** $10\sqrt{3}$ **c** $\frac{5\sqrt{2}}{2}$ **d** $\frac{5\sqrt{5}}{4}$

3 $\sqrt{192}$

4 **a** $3\sqrt{3}$ **b** $6\sqrt{2} + 2\sqrt{5}$
c $16\sqrt{3}$ **d** $17\sqrt{5} - 6\sqrt{3}$

5 **a** $-\sqrt{15}$ **b** $35\sqrt{6}$ **c** $\frac{2}{3\sqrt{5}}$ **d** 6 **e** 52

6 $66\sqrt{2}$

7 **a** $\frac{3\sqrt{7}}{7}$ **b** $\frac{2\sqrt{15}}{5}$ **c** $\frac{2\sqrt{3} - 3\sqrt{10}}{2}$

8 **a** a^5 **b** $12x^3y^4$ **c** h^4 **d** $\frac{1}{2}m^6n^3$
e a^6 **f** $9m^{10}$ **g** $\frac{4p^8q^6}{49r^2t^4}$ **h** 6

9 **a** $\frac{1}{x^3}$ **b** $\frac{2b^4}{a^2c^3}$ **c** $7m^2$ **d** $\frac{4}{5d^2}$
e $\frac{16}{k^8}$ **f** $\frac{a^6}{8}$ **g** $\frac{12m}{a^5}$ **h** $\frac{4d^5}{3c^2}$

10 **a** $\frac{a^4}{5b^5}$ **b** $\frac{9x^{10}d^9}{8c^{13}}$

3G

Building understanding

1 **a** 3 **b** 3 **c** 3 **d** 3
2 **a** 10^3 **b** 10^7 **c** 10^{-6} **d** 10^{-3}
3 **a** 4 **b** 5 **c** 3
d -4 **e** -3 **f** -5

Now you try

Example 17
a 20 480 **b** 0.000047

Example 18
a 7.94×10^6 **b** 2.71×10^{-4}

Exercise 3G

1 **a** **i** 3126 **ii** 5 040 000
b **i** 0.0028 **ii** 0.00000591

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2 a 3120 b 54 293
c 710 500 d 8 213 000
e 59 500 f −800 200
g −10 120 h 9 990 000
i 210 500 000 j −55 000
k 2 350 000 000 l 1 237 000 000 000

3 a 0.0045 b 0.0272 c 0.0003085
d 0.00783 e −0.000092 f 0.265
g 0.0001002 h −0.000006235
i 0.98 j −0.000000000545
k 0.000000000003285 l 0.000000875

4 a 6.24×10^3 b -5.73×10^5 c 3.02×10^4
d 4.24×10^5 e -1.01×10^4 f 3.50×10^7
g 7.25×10^4 h 3.56×10^5 i 1.10×10^8
j 9.09×10^5 k -4.56×10^6 l 9.83×10^9

5 a 2.42×10^{-3} b -1.88×10^{-2} c 1.25×10^{-4}
d 7.87×10^{-3} e 7.08×10^{-4} f 1.14×10^{-1}
g 6.40×10^{-6} h 7.89×10^{-5} i 1.30×10^{-4}
j 7.01×10^{-7} k 9.89×10^{-9} l -5.00×10^{-4}

6 a -2.4×10^4 b 5.71×10^6 c 7.0×10^8
d 4.88×10^3 e 1.9×10^{-3} f -7.05×10^{-4}
g 9.8×10^{-6} h -3.571×10^{-1} i 5.00×10^{-5}

7 a 7.7×10^6 km^2 b 2.5×10^6 c 7.4×10^9 km
d 1×10^{-2} cm e 1.675×10^{-27} kg f 9.5×10^{-13} g

8 a 2.85×10^{-3} b 1.55×10^{-3} c 4.41×10^{-8}
d 6.38×10^{-3} e 8.00×10^7 f 3.63×10^8
g 1.80×10^{-3} h 3.42×10^{15} i 8.31×10^{-2}

9 328 minutes

10 38 is larger than 10.

11 a 2.1×10^4 b 3.94×10^9 c 6.004×10^1
d 1.79×10^{-4} e 2×10^3 f 7×10^{-1}
g 1×10^7 h 6×10^6 i 4×10^{-3}
j 3.1×10^{-14} k 2.103×10^{-4} l 9.164×10^{-21}

12 a 9×10^4 b 8×10^9 c 6.4×10^9
d 1.44×10^{-8} e 4×10^4 f 6.25×10^{-12}
g 2.25×10^{-6} h 1.25×10^7 i 1×10^{-5}
j 1.275×10^{-4} k 1.8×10^{-1} l 2×10^2
m 8×10^{-1} n 4×10^{-14} o 2.5×10^4

13 $3 \times 10^{-4} = 3 \div 10\,000$

14 a i 9×10^{17} J ii 2.34×10^{21} J
iii 2.7×10^{15} J iv 9×10^{11} J
b i 1.11×10^8 kg ii 4.22×10^{-1} kg
iii 9.69×10^{-13} kg iv 1.89×10^{-19} kg
c 5.4×10^{41} J

3H

Building understanding

1 a 3, 2 b 5, 5 c 4, 4 d 5, 5

2 a 3 b 11 c 3 d 4
e 2 f 3 g 2 h 10

3 a 1.91, 1.91 b 1.58, 1.58 c 1.43, 1.43

Now you try

Example 19

a $11^{\frac{1}{2}}$ b $3^{\frac{1}{2}}x^{\frac{7}{2}}$ c $2x^{\frac{9}{4}}$ d $7^{\frac{3}{2}}$

Example 20

a $\sqrt[3]{5}$ b $(\sqrt[3]{11})^2$ or $\sqrt[3]{121}$

Example 21

a 5 b 3 c $\frac{1}{2}$

Exercise 3H

1 a i $13^{\frac{1}{2}}$ ii $19^{\frac{1}{2}}$
b i $6^{\frac{1}{2}}x^{\frac{3}{2}}$ ii $11^{\frac{1}{2}}x^{\frac{7}{2}}$
c i $4x^{\frac{5}{4}}$ ii $7x^{\frac{2}{3}}$
d i $6^{\frac{3}{2}}$ ii $13^{\frac{3}{2}}$

2 a $29^{\frac{1}{2}}$ b $35^{\frac{1}{3}}$ c $x^{\frac{2}{5}}$ d $b^{\frac{3}{4}}$
e $2^{\frac{1}{2}}a^{\frac{1}{2}}$ f $4^{\frac{1}{3}}t^{\frac{7}{3}}$ g $10^{\frac{1}{5}}t^{\frac{2}{5}}$ h $8^{\frac{1}{8}}m^{\frac{1}{2}}$

3 a $7x^{\frac{5}{2}}$ b $6n^{\frac{7}{3}}$ c $3y^3$
d $5p^{\frac{2}{3}}r^{\frac{1}{3}}$ e $2a^{\frac{4}{3}}b^{\frac{2}{3}}$ f $2g^{\frac{3}{4}}h^{\frac{5}{4}}$
g $5^{\frac{3}{2}}$ or $125^{\frac{1}{2}}$ h $7^{\frac{3}{2}}$ or $343^{\frac{1}{2}}$ i $4^{\frac{4}{3}}$ or $256^{\frac{1}{3}}$

4 a $\sqrt[5]{2}$ b $\sqrt[7]{8}$ c $\sqrt[3]{6}$ d $\sqrt[10]{11}$
e $\sqrt[3]{9}$ f $\sqrt[3]{49}$ g $\sqrt[5]{8}$ h $\sqrt[7]{81}$

5 a 6 b 3 c 4 d 7
e 2 f 5 g $\frac{1}{3}$ h $\frac{1}{2}$
i $\frac{1}{3}$ j $\frac{1}{10}$ k $\frac{1}{20}$ l $\frac{1}{10}$

6 a 4 b 8 c 216 d 32
e $\frac{1}{8}$ f $\frac{1}{9}$ g $\frac{1}{16}$ h $\frac{1}{125}$
i 125 j $\frac{1}{16}$ k $\frac{1}{81}$ l $\frac{1}{100}$

7 a a^2 b m^3 c x d $b^{\frac{1}{2}}$
e $s^{\frac{6}{7}}$ f $y^{\frac{1}{9}}$ g 1 h $\frac{a^{\frac{1}{2}}}{b}$

8 a $5s^2$ b $3t^2$ c $2t^2$ d $5t^4$
e x f b^4 g t^3 h m^2
i $4ab^4$ j $6m^2n$ k $2x^2y^3$ l $7r^3t^2$
m $\frac{5}{7}$ n $\frac{2x}{3}$ o $\frac{2}{x^2}$ p $10x$

9 a Method B
b i 32 ii 216 iii 128
iv 81 v 625 vi $\frac{1}{27}$
vii $\frac{32}{3125}$ viii $\frac{81}{10\,000}$

10 It equals 2 since $2^6 = 64$.

11 a i −3 ii −10
iii −2 iv −3
b i no ii yes
iii yes iv no
c y is a real number when n is odd, for $x < 0$.

3I

Building understanding

1 **a** i 4 ii 8 iii 16 iv 32
b i 3 ii 5 iii 6
2 **a** 16, 32, 64, 128, 256, 512
b 81, 243, 729, 2187, 6561
c 64, 256, 1024, 4096
d 125, 625, 3125
e 216, 1296
3 **a** 3^2 **b** 5^3 **c** 3^5
d 2^7 **e** 3^6

Now you try

Example 22

a $x = 3$ **b** $x = -3$ **c** $x = \frac{3}{2}$

Example 23
$x = 1$

Exercise 3I

1 **a** i $x = 2$ ii $x = 3$
b i $x = -2$ ii $x = -3$
c i $x = \frac{3}{2}$ ii $x = \frac{3}{2}$
2 **a** 3 **b** 3 **c** 2 **d** 2
e 3 **f** 3 **g** 4 **h** 3
i 4 **j** 5 **k** 4 **l** 3
3 **a** –2 **b** –2 **c** –2
d –4 **e** –5 **f** 3
g 2 **h** 6 **i** 3
4 **a** $\frac{3}{2}$ **b** $\frac{4}{3}$ **c** $\frac{3}{2}$ **d** $\frac{3}{2}$
e $\frac{1}{2}$ **f** $\frac{1}{3}$ **g** $\frac{1}{5}$ **h** $\frac{1}{4}$
i –2 **j** –4 **k** $-\frac{3}{2}$ **l** $-\frac{3}{2}$
5 **a** 1
b i 2 ii 32 iii 2^{60} iv 2^{1440}
c i 3 min ii 8 min iii 10 min
6 **a** $\frac{1}{2}$ **b** 1 **c** 3 **d** 1
e $\frac{3}{4}$ **f** 2 **g** 9 **h** $\frac{6}{7}$
i $\frac{15}{4}$ **j** $-\frac{11}{2}$ **k** 4 **l** $-\frac{3}{2}$
7 1 cent doubled every second for 30 seconds. Receive 2^{30} cents, which is more than 1 million dollars.
8 **a** i 1 ii 1 iii 1
b No solutions. If $a = 1$, then $a^x = 1$ for all values of x.
9 **a** 2 **b** 1 **c** $\frac{2}{3}$ **d** $\frac{3}{4}$
e $\frac{5}{4}$ **f** $\frac{1}{3}$ **g** $\frac{3}{10}$ **h** $-\frac{1}{2}$
10 **a** i 0.25 ii 0.125
iii 0.001 iv 0.0016
b i 5^{-2} ii 2^{-4}
iii 2^{-1} iv 5^{-4}
11 **a** –4 **b** –6 **c** –5
d $\frac{1}{2}$ **e** $-\frac{3}{2}$ **f** $-\frac{3}{4}$
12 **a** 1 **b** –1 **c** 8 **d** $-\frac{3}{2}$
e $-\frac{5}{2}$ **f** –2 **g** 3 **h** $\frac{2}{3}$
i $\frac{1}{5}$ **j** 2 **k** 0 **l** –2

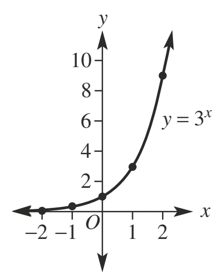
3J

Building understanding

1 **a**

x	–2	–1	0	1	2
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

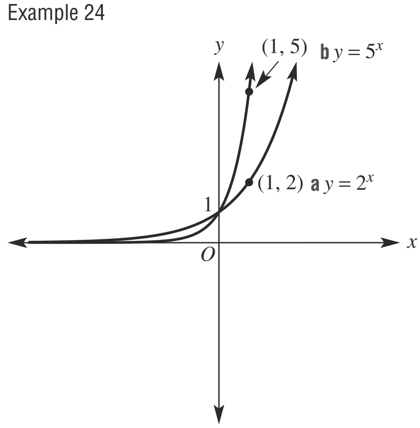
b



2 **a** asymptote **b** 1
c y-axis **d** x-axis
3 **a** $a^{-2} = \frac{1}{a^2} \neq -a^2$ **b** False since $3^{-2} = \frac{1}{3^2}$.
c $5^{-3}, 3^{-2}, 2^{-1}$ **d** $-9, -125, -\frac{1}{4}$

Now you try

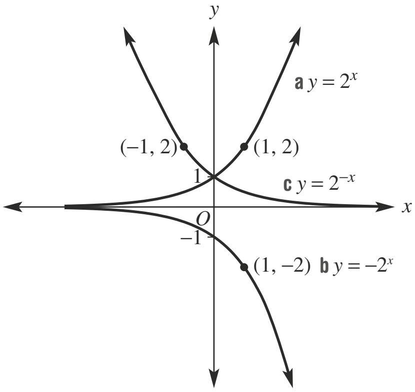
Example 24



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

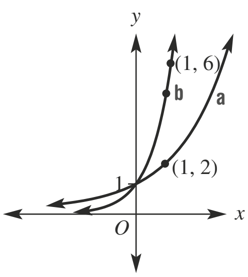
Example 25



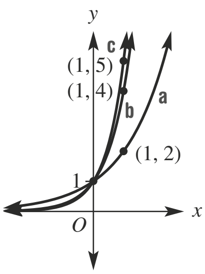
Example 26
(3, 27)

Exercise 3J

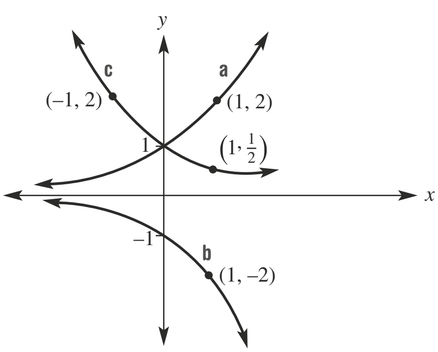
1



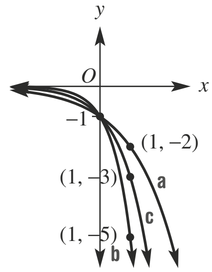
2



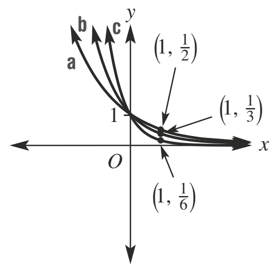
3



4



5



6 a i (0, 1) ii $\left(-1, \frac{1}{3}\right)$
iii (0, 1) iv (2, 9)
b i (4, –16) ii $\left(-1, -\frac{1}{2}\right)$
iii (0, –1) iv (2, –4)
c i $\left(1, \frac{1}{4}\right)$ ii (–3, 64)
iii (0, 1) iv $\left(1, \frac{1}{4}\right)$

7 a (2, 4) b (2, 9) c (1, –4) d (–3, 8)

8 a 1000
b i 2000 ii 8000
c i 2 years ii 4 years

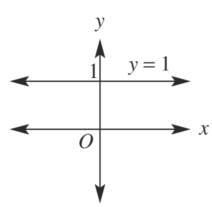
9 a $N = 2^t$ b $N = 2^{10} = 1024$ c 14 seconds

10 $x = 2.322$

11 a C b A c D
d E e F f B

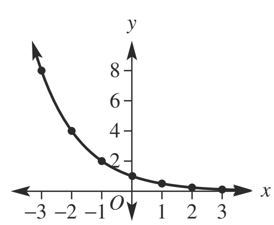
12 Substitute (2, 5) into the equation $y = 2^2 = 4 \neq 5$.

13 $y = 1$



14 It is the asymptote.

15 a



They are the same graph.

ISBN 978-1-108-77290-7

b i $y = \left(\frac{1}{3}\right)^x$ ii $\left(\frac{1}{5}\right)^x$ iii $y = \left(\frac{1}{10}\right)^x$

c i $y = 4^{-x}$ ii $y = 7^{-x}$ iii $y = 11^{-x}$

d $\frac{1}{a} = a^{-1}$, thus $\left(\frac{1}{a}\right)^x = (a^{-1})^x = a^{-x}$ as required (or similar)

3K

Building understanding

1 a $50 b $1050 c $52.50
d $55.13 e $1276.28

2 a 4.9 kg b $\frac{2}{100}$, 0.98 c 4.52 kg

3 a growth b decay c growth d decay

Now you try

Example 27
a $A = 50\,000(1.16)^n$ b $P = 10\,000(0.91)^n$

Example 28
a $V = 400\,000(1.07)^n$
b i $428 000 ii $490 017.20
c 3.3 years

Exercise 3K

1 a $V = 6000(1.12)^n$ b $P = 20\,000(0.92)^n$

2 a A = amount of money at any time, n = number of years of investment
$A = 200\,000 \times 1.17^n$
b A = house value at any time, n = number of years since initial valuation
$A = 530\,000 \times 0.95^n$
c A = car value at any time, n = number of years since purchase
$A = 14\,200 \times 0.97^n$
d A = size of oil spill at any time, n = number of minutes elapsed
$A = 2 \times 1.05^n$
e A = litres in tank at any time, n = number of hours elapsed
$A = 1200 \times 0.9^n$
f A = cell area at any time, n = number of minutes elapsed
$A = 0.01 \times 2^n$
g A = population at any time, n = number of years since initial census
$A = 172\,500 \times 1.15^n$
h A = mass of substance at any time, n = number of hours elapsed
$A = 30 \times 0.92^n$

3 a 1.1
b i $665 500 ii $1 296 871.23 iii $3 363 749.97
c 7.3 years

4 a 300 000
b i $216 750 ii $96 173.13 iii $42 672.53
c 3.1 years

5 a $V = 15\,000 \times 0.94^t$
b i 12 459 L ii 9727 L
c 769.53 L d 55.0 hours

6 a $V = 50\,000 \times 1.11^n$
b i $75 903.52 ii $403 115.58
c 6.64 years

7 a 3000
b i 7800 ii 20 280 iii 243 220
c 10 hours 11 minutes

8 a $D = 10 \times 0.875^t$, where t = number of 10 000 km travelled
b 90 000 c yes

9 a $T = 90 \times 0.92^t$
b i 79.4°C ii 76.2°C
c 3.22 minutes ≈ 3 minutes 13 seconds

10 a i $1610.51 ii $2143.59 iii $4177.25
b i $1645.31 ii $2218.18 iii $4453.92

11 a $2805.10 b $2835.25 c $2837.47

12 a i 90 g ii 72.9 g iii 53.1 g
b 66 years

13 a 60 L b 22.8 minutes

14 0.7%

3L

Building understanding

1 a $50 b $550 c $55 d $605 e $605

2

2	4200	210	4410
3	4410	220.50	4630.50
4	4630.50	231.53	4862.03
5	4862.03	243.10	5105.13

3 a $1102.50 b $1102.50
c $1157.63 d $1157.63

4 a $700(1.08)^2$ b $1000(1.15)^6$ c $850(1.06)^4$

Now you try

Example 29
$4502.19

Example 30
a $n = 60, r = \frac{5}{12}$ b $n = 12, r = 3.5$

Example 31
$8964.49

Exercise 3L

1 $6312.38

2 a $5105.13 b $11 946.33
c $13 652.22 d $9550.63

3 a $106 000 b $112 360 c $119 101.60
d $133 822.56 e $179 084.77 f $239 655.82

4 a 6, 3% b 60, 1% c 52, 0.173%
d 14, 2.625% e 32, 3.75% f 120, 0.8%

5 a $2254.32 b $87 960.39 c $1461.53
d $789.84 e $591.63 f $1407.76

ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

6 a $5075 b $5228.39 c $5386.42
7 $11 651.92
8 a i $3239.42 ii $3348.15 iii $3446.15
iv $3461.88 v $3465.96
b $226.54
9 a $P = 300, n = 12, r = 7\%, R = 14\%, t = 6$ years
b $P = 5000, n = 24, r = 2.5\%, R = 30\%, t = 2$ years
c $P = 1000, n = 65, r = 0.036\%, R = 0.936\%, t = 2.5$ years
d $P = 3500, n = 30, r = 0.0053\%, R = 1.9345\%, t = 30$ days
e $P = 10000, n = 10, r = 7.8\%, R = 7.8\%, t = 10$ years
f $P = 6000, n = 91, r = 0.22\%, R = 5.72\%, t = 3.5$ years
10 5.3% compounded bi-annually
11 a i approx. 6 years ii approx. 12 years
iii approx. 9 years iv approx. 5 years
v approx. 7 years vi approx. 4 years
b same answer as part **a** c yes

3M

Building understanding

1 B
2 $P = 750, r = 7.5, n = 5$
3 $I = 225, P = 300, r = 3, t = 25$
4 $240

Now you try

Example 32
a $4764.06 b $4720

Exercise 3M

1 a $8516.57 b $8400
2 a i $7146.10 ii $6955.64 iii $6858.57
iv $7260 v $7916.37
b $6000 at 5.7% p.a., for 5 years
3 a i $7080 ii $7080 iii $7428
iv $7200 v $7710
b 6000 at 5.7% p.a., for 5 years
4 a i I $240, $240 II $480, $494.40
III $1200, $1352.90 IV $2400, $3163.39
ii I $240, $243.60 II $480, $502.04
III $1200, $1375.67 IV $2400, $3224.44
iii I $240, $246.71 II $480, $508.64
III $1200, $1395.40 IV $2400, $3277.59
b compound interest c compound interest
5 a

Principal	Rate	Overall time	Interest	Amount
$7000	5%	5 years	$1750	$8750
$7000	10%	5 years	$3500	$10 500
$3300	10%	3 years	$990	$4290
$8000	10%	3 years	$2400	$10 400
$9000	8%	2 years	$1440	$10 440
$18 000	8%	2 years	$2880	$20 880

b i interest is doubled
ii no change
iii interest is doubled
6

Principal	Rate	Period	Overall time	Interest	Amount
$7000	4.56%	annually	5 years	$1750	$8750
$7000	8.45%	annually	5 years	$3500	$10 500
$9000	8%	fornightly	2 years	$1559.00	$10 559.00
$18 000	8%	fornightly	2 years	$3118.01	$21 118.01

7 a 8.45% b 8.19% c 8.12%
The more frequently the interest is calculated, the lower the required rate.
8 a i 4.2% ii 8.7%
b It increases by more than the factor.

Problems and challenges

1 3^n
2 a 5 b $\left(\frac{4}{9}\right)^x$
3 $\frac{1}{5}$
4 a -8 b 2^{2-a}
5 length $= 10\sqrt{2}$ cm, width $= 10$ cm
6 $\frac{-3 - \sqrt{2} + 7\sqrt{3}}{7}$
7 a $\frac{x - y}{xy}$ b $\sqrt{xy}(x - y)$
8 $12 + 8\sqrt{2}$
9 $x = 3.5$

Short-answer questions

1 a $2\sqrt{6}$ b $6\sqrt{2}$ c $30\sqrt{2}$ d $12\sqrt{6}$
e $\frac{2}{7}$ f $\frac{2\sqrt{2}}{3}$ g $5\sqrt{7}$ h $\frac{2\sqrt{5}}{5}$
2 a $4 + 7\sqrt{3}$ b $2\sqrt{5} + 2\sqrt{7}$ c $5\sqrt{2}$
d $4\sqrt{3} + 2\sqrt{2}$ e $2\sqrt{30}$ f $-12\sqrt{5}$
g $2\sqrt{5}$ h $\frac{\sqrt{7}}{3}$ i 0
3 a $2\sqrt{6} + 4\sqrt{2}$ b $12\sqrt{5} - 6$
c 11 d 48
4 a $\frac{\sqrt{6}}{6}$ b $5\sqrt{2}$ c $3\sqrt{6}$
d $2\sqrt{14}$ e $\frac{3\sqrt{2}}{4}$ f $\frac{5\sqrt{2}}{8}$
g $\frac{\sqrt{10} + 2\sqrt{2}}{2}$ h $\frac{4\sqrt{6} - 3}{3}$
5 a $25y^6$ b 6 c $20x^7y^{10}$
d $\frac{3x^2}{y^4}$ e $\frac{3y^4}{2x^3}$ f $\frac{27}{4b^8}$
6 a $21^{\frac{1}{2}}$ b $x^{\frac{1}{3}}$ c $m^{\frac{5}{3}}$ d $a^{\frac{2}{5}}$
e $10^{\frac{1}{2}}x^{\frac{3}{2}}$ f $2^{\frac{1}{3}}a^3b^{\frac{1}{3}}$ g $7^{\frac{3}{2}}$ h $4^{\frac{4}{3}}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

7 a 5 b 4 c $\frac{1}{2}$
d $\frac{1}{7}$ e $\frac{1}{10}$ f $\frac{1}{5}$

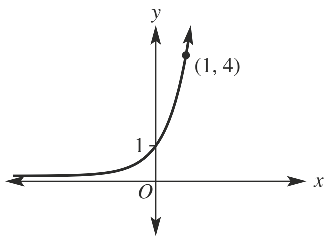
8 a i 3210 ii 4 024 000
iii 0.00759 iv 0.0000981
b i 3.08×10^{-4} ii 7.18×10^{-6}
iii 5.68×10^{6} iv 1.20×10^{8}

9 a $V = 800 \times 1.07^t$ b $V = 3000 \times 0.82^t$

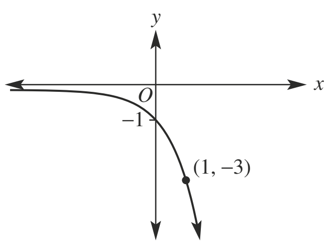
10 a \$1215.51 b \$3519.60 c \$5637.46

11 a 3 b 2 c 1 d 6
e −2 f −3 g $\frac{3}{2}$ h $\frac{2}{3}$
i 4 j 3 k −4 l 0

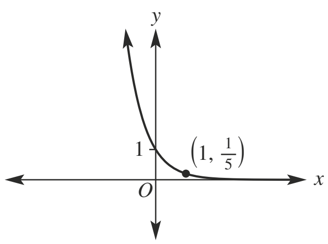
12 a



b



c



Multiple-choice questions

1 C	2 D	3 B	4 E	5 A
6 D	7 D	8 C	9 B	10 D
11 C	12 D	13 C	14 D	15 B

Extended-response questions

1 a $36\sqrt{15} + 3\sqrt{45} = 36\sqrt{15} + 9\sqrt{5}$ cm²
b $360\sqrt{3} + 144\sqrt{15} + 90 + 36\sqrt{5}$ cm³
c $4\sqrt{3} + 1$
d i 10000 cm² ii 1.6%

2 a $V = 10\,000 \times 1.065^n$
b i \$11 342.25 ii \$13 700.87
c 11.0 years
d i \$14 591 ii $V = 14\,591 \times 0.97^t$
iii \$12 917; profit of \$2917

Chapter 4

4A

Building understanding

1 a 0.799 b 0.951 c 1.192 d 0.931
2 a $\sin\theta$ b $\cos\theta$ c $\tan\theta$
3 a 1.80 b 2.94 c 3.42
d 2.38 e 22.33 f 12.47

Now you try

Example 1
a 2.74 b 25.03

Example 2
a 11.65 b 26.27

Exercise 4A

1 a 7.71 b 44.94
2 a 1.15 b 3.86 c 13.74 d 5.07
e 2.25 f 2.79 g 1.97 h 13.52
i 37.02 j 9.30 k 10.17 l 13.11
3 a 8.55 b 4.26 c 13.06 d 10.04
e 5.55 f 1.52 g 22.38 h 6.28
i 0.06 j 12.12 k 9.80 l 15.20
4 a $x = 2.5$ cm, $y = 4.33$ cm
b $x = 12.26$ cm, $y = 6.11$ cm
c $x = 0.20$ m, $y = 0.11$ m
5 a 125 m b 327 m
6 1.85 m
7 22.3 m
8 7.54 m
9 28.5 m
10 26.4 cm
11 a 4.5 cm b 8.5 mm
12 The student rounded tan 65° too early.
13 a 3.7 b 6.5 c 7.7
14 a i $a = c\sin\theta$ ii $b = c\cos\theta$
iii $\tan\theta = \frac{a}{b}$ iv $\tan\theta = \frac{c\sin\theta}{c\cos\theta} = \frac{\sin\theta}{\cos\theta}$
v Answers may vary
b i $a = c\sin\theta$
ii $b = c\cos\theta$
iii $c^2 = a^2 + b^2$
iv $c^2 = (c\sin\theta)^2 + (c\cos\theta)^2$
$c^2 = c^2(\sin\theta)^2 + c^2(\cos\theta)^2$
$\therefore 1 = (\sin\theta)^2 + (\cos\theta)^2$

4B

Building understanding

1 a 60° b $\frac{1}{2}$ c 0.75
2 a 23.58° b 60° c 11.31° d 5.74°
3 a tangent b cosine c sine

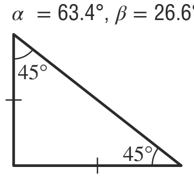
ISBN 978-1-108-77290-7 Cambridge University Press
 Updated June 2022

Now you try

Example 3
a 30° b 38.94°

Example 4
38.7°

Exercise 4B

1 a 30° b 54.31°
2 a 60° b 45° c 48.59°
d 30° e 52.12° f 32.74°
3 a $\alpha = 60°$, $\theta = 30°$ b $\alpha = 45°$, $\theta = 45°$
c $\alpha = 53.1°$, $\theta = 36.9°$ d $\alpha = 22.6°$, $\theta = 67.4°$
e $\alpha = 28.1°$, $\theta = 61.9°$ f $\alpha = 53.1°$, $\theta = 36.9°$
4 a 44.4°, 45.6° b 74.7°, 15.3°
c 58.3°, 31.7° d 23.9°, 66.1°
e 82.9°, 7.1° f 42.4°,47.6°
5 70.02°
6 31.1°
7 47.1°
8 a 66.4° b 114.1° c 32.0°
9 a i 45° ii 33.7° b 11.3°
10 a Once one angle is known, the other can be determined by subtracting the known angle from 90°.
b $\alpha = 63.4°$, $\beta = 26.6°$
11 a


b $\tan 45° = \frac{x}{x} = 1$
c $\sqrt{2}x$
d $\sin 45° = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}}$, $\cos 45°$ also equals $\frac{1}{\sqrt{2}}$.
12 a $\theta = 30°$ b $\alpha = 60°$ c $\sqrt{3}$
d i $\frac{1}{2}$ ii $\frac{1}{2}$ iii $\frac{\sqrt{3}}{2}$
iv $\frac{\sqrt{3}}{2}$ v $\frac{\sqrt{3}}{3}$ vi $\sqrt{3}$
e $AB = \frac{1}{2}x + \frac{\sqrt{3}}{2}x = \left(\frac{\sqrt{3}+1}{2}\right)x$

4C

Building understanding

1 $\angle ABC$ and $\angle DCB$
2 a B b C c A

Now you try

Example 5
70.02 m

Example 6
27.98°

Exercise 4C

1 1866.03 m
2 320 m
3 1509.53 m
4 32°
5 a 1.17 m b 1.50 m
6 8.69 cm
7 299 m
8 a 1.45° b 3.44° c 1.99°
9 yes
10 89.12 m
11 a i 8.7 cm ii 5 cm
b i 17.3 cm ii 20 cm
c Answers may vary.
12 321.1 km/h
13 a i 18° ii 72° iii 36° iv 54°
b i 0.77 m ii 2.38 m
iii 2.02 m iv 1.47 m
c 3.85 m d 4.05 m e Proof

4D

Building understanding

1 a 0°T b 45°T c 90°T d 135°T
e 180°T f 225°T g 270°T h 315°T
2 a 050°T b 060°T c 139°T
d 162°T e 227°T f 289°T
3 a 200°T b 082°T c 335°T d 164°T

Now you try

Example 7
a A is 060°T
B is 140°T
C is 250°T
D is 310°T
b i 240°T ii 130°T

Example 8
a 4.10 km b 19.28 km

Exercise 4D

1 a A is 050°T
B is 150°T
C is 215°T
D is 320°T
b i 230°T ii 140°T
2 a 220°T b 305°T c 075°T d 150°T
3 a 1.7 km b 3.6 km
4 a 121°T b 301°T
5 a 3.83 km b 6.21 km
6 a 14.77 cm b 2.6 cm
7 a 217°T b 37°T
8 a 1.414 km b 1.414 km c 2.914 km
9 a 1.62 km b 5.92 km c 2.16 km
10 10.032 km

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

11 a i 045°T ii 236.3°T iii 26.6°T iv 315°T
b i 296.6°T ii 116.6°T iii 101.3°T iv 246.8°T

12 a i 2.5 km ii 2.82 km iii 5.32 km
b i 4.33 km ii 1.03 km iii 5.36 km
c i 45.2°T ii 7.6 km

13 a 229.7°, 18.2 km b 55.1°,12.3 km

14 a 212.98 m
b i 99.32 m ii 69.20 m
c 30.11 m

15 a 38.30 km b 57.86 km c 33.50°

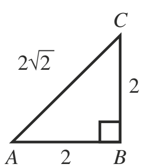
16 a 4.34 km b 2.07 km c 4.81 km

4E

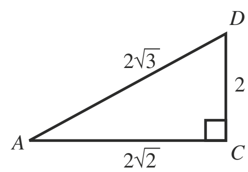
Building understanding

1 3

2 a



b



c 35.3° d 45°

Now You try

Example 9

a 31.896 m b 47.72°

Exercise 4E

1 a 37.609 m b 45.47°

2 a 1.888 m b 43.35°

3 a 57.409 m b 57.91°

4 a i 26.57° ii 11.18 cm
b 10.14°

5 a 7.31 m b 6.87 m

6 138.56 m

7 a i 2.25 m ii 2.59 m
b 49.03° c 3.43 m

8 a i 1.331 km ii 1.677 km
b 0.346 km

9 a camera C b 609.07 m

10 a 5.5 m b 34.5° c 34.7° d 0.2°

11 a 45° b 1.41 units c 35.26° d 1.73 units

12 a i 1.55 ii 1.27 iii 2.82
b 34.34°

13 22°

4F

Building understanding

1 $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

2 a 1.9 b 2.5

3 a 50.3° b 29.2°

Now you try

Example 10

7.3

Example 11

a 36.2° b 121.0°

Exercise 4F

1 3.8

2 a 7.9 b 16.5 c 19.1
d 9.2 e 8.4 f 22.7

3 a 38.0° b 51.5° c 28.8°
d 44.3° e 47.5° f 48.1°

4 a 1.367 km b 74° c 2.089 km

5 27.0°

6 131.0 m

7 a $\angle ABC = 80°$, $\angle ACB = 40°$ b 122 km

8 a $\angle ABC = 80°$ b 61.3 km c 53.9 km

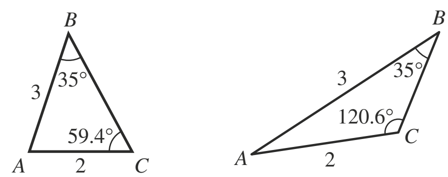
9 a 147.5° b 102.8° c 126.1°
d 100.5° e 123.9° f 137.7°

10 Impossible to find θ as such a triangle does not exist.

11 37.6° or 142.4°

12 a 59.4° or 120.6°

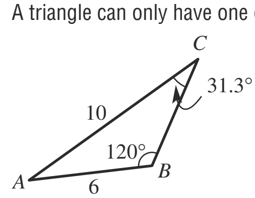
b



c 31.3°

d A triangle can only have one obtuse angle.

e



Progress quiz

1 a 12.58 b 38.14 c 15.44 d 6.59

2 a 39° b 58° c 52°

3 a i $\sqrt{80}$ or $4\sqrt{5}$ ii $\dfrac{8}{\sqrt{80}}$ or $\dfrac{2\sqrt{5}}{5}$
iii $\dfrac{4}{\sqrt{80}}$ or $\dfrac{\sqrt{5}}{5}$
b 26.6°

4 63°

5 23.84 m

6 *A* 060° *B* 150° *C* 288°

7 a 13.65 km b 048.4°

8 a 36.77 m b 61°

9 a $\sqrt{8}$ cm b 35°

10 a 8.7
b 66.7°

ISBN 978-1-108-77290-7

4G

Building understanding

1 a $c^2 = 3^2 + 4^2 - 2 \times 3 \times 4 \times \cos 105°$
b $7^2 = 5^2 + 9^2 - 2 \times 5 \times 9 \times \cos\theta$
2 a 9.6 b 1.5 c 100.3° d 36.2°

Now you try

Example 12
4.29 m

Example 13
26.53°

Exercise 4G

1 9.08 m
2 a 16.07 cm b 8.85 m c 14.78 cm
d 4.56 m e 2.86 km f 8.14 m
3 a 81.79° b 104.48° c 64.62°
d 61.20° e 92.20° f 46.83°
4 310 m
5 32.2°, 49.6°, 98.2°
6 a 145.9° b 208.2°
7 383 km
8 7.76 m
9 a cosine rule b sine rule c sine rule
d cosine rule e sine rule f cosine rule
10 a $\cos c = \dfrac{a^2 + b^2 - c^2}{2ab}$ b 121.9°
11 Obtuse, as cos of an obtuse angle gives a negative result.
12 a $AP = b - x$ b $a^2 = x^2 + h^2$
c $c^2 = h^2 + (b - x)^2$
d $c^2 = a^2 - x^2 + (b - x)^2 = a^2 + b^2 - 2bx$
e $\cos C = \dfrac{x}{a}$
f $x = a\cos C$ substitute into part **d**.

4H

Building understanding

1 a 3.7 b 48.0
2 a α b θ c β
3 a 56.44° or 123.56° b 45.58° or 134.42°

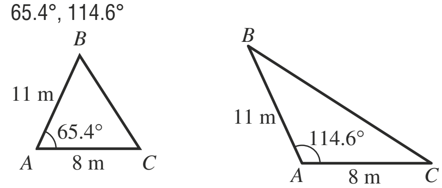
Now you try

Example 14
30.1 cm^2

Example 15
18.64

Exercise 4H

1 5.4 m^2
2 a 4.4 cm^2 b 26.4 m^2 c 0.9 km^2
d 13.7 m^2 e 318.4 m^2 f 76.2 cm^2
3 a 11.9 cm^2 b 105.6 m^2 c 1.6 km^2
4 a $x = 5.7$ b $x = 7.9$ c $x = 9.1$
d $x = 18.2$ e $x = 10.6$ f $x = 1.3$
5 a 59.09 cm^2 b 1.56 mm^2 c 361.25 km^2
6 a 35.03 cm^2 b 51.68 m^2 c 6.37 km^2
7 a 965.88 m^2 b 214.66 m^2 c 0.72 km^2
8 a 17.3 m^2 b 47.2 cm^2 c 151.4 km^2
9 a Area $= ab\sin\theta$
b Area $= \dfrac{1}{2}a^2\sin 60° = \dfrac{\sqrt{3}}{4}a^2$
c Area $= \dfrac{1}{2}a^2\sin(180° - 2\theta) = \dfrac{1}{2}a^2\sin 2\theta$
10 a i 129.9 cm^2 ii 129.9 cm^2
b They are equal because sin 60° and sin 120° are equal.
c Same side lengths with included angle 140°.
11 a 65.4°, 114.6°
b



12 a i 540° ii 108° iii 11.89 cm^2
iv 8.09 cm v 72°, 36° vi 19.24 cm^2
vii 43.0 cm^2
b 65.0 cm^2 c Answers may vary.

4I

Building understanding

1 a quadrant 1 b quadrant 3
c quadrant 4 d quadrant 2
2 a quadrants 1 and 2 b quadrants 2 and 4
c quadrants 2 and 3 d quadrants 1 and 4
e quadrants 1 and 3 f quadrants 3 and 4
3

θ	0°	90°	180°	270°	360°
$\sin\theta$	0	1	0	–1	0
$\cos\theta$	1	0	–1	0	1
$\tan\theta$	0	undefined	0	undefined	0

4 a 0.139 b –0.530 c –0.799
d –0.259 e 0.900 f –1.036

Now you try

Example 16
a 140° b 106° c 133°

Example 17
a quadrant 4, $\sin\theta$ is negative
$\cos\theta$ is positive
$\tan\theta$ is negative
b quadrant 3, $\sin\theta$ is negative
$\cos\theta$ is negative
$\tan\theta$ is positive

Example 18
a –sin 50° b –cos 54° c tan 80°

ISBN 978-1-108-77290-7

Cambridge University Press

Updated June 2022

Exercise 4I

1 a i 160° ii 105°
 b i 110° ii 138°
 c i 145° ii 101°

2 a 140° b 115° c 155°
 d 99° e 143° f 172°

3 a 30° b 86° c 70°
 d 9° e 21° f 37°

4 a quadrant 2, sin θ positive, cos θ negative, tan θ negative
 b quadrant 4, sin θ negative, cos θ positive, tan θ negative
 c quadrant 3, sin θ negative, cos θ negative, tan θ positive
 d quadrant 1, sin θ positive, cos θ positive, tan θ positive
 e quadrant 4, sin θ negative, cos θ positive, tan θ negative
 f quadrant 2, sin θ positive, cos θ negative, tan θ negative
 g quadrant 3, sin θ negative, cos θ negative, tan θ positive
 h quadrant 3, sin θ negative, cos θ negative, tan θ positive
 i quadrant 3, sin θ negative, cos θ negative, tan θ positive
 j quadrant 1, sin θ positive, cos θ positive, tan θ positive
 k quadrant 4, sin θ negative, cos θ positive, tan θ negative
 l quadrant 2, sin θ positive, cos θ negative, tan θ negative

5 a –sin 80° b cos 60° c tan 40° d sin 40°
 e –cos 55° f –tan 45° g –sin 15° h –cos 58°
 i tan 47° j sin 68° k cos 66° l –tan 57°

6 a 30° b 60° c 24°
 d 40° e 71° f 76°
 g 50° h 25° i 82°

7 a 42° b 47° c 34° d 9°
 e 33° f 62° g 14° h 58°

8 a $0 < \theta < 90°$ b $90° < \theta < 180°$
 c $270° < \theta < 360°$ d $180° < \theta < 270°$

9

θ_2	150°	315°	350°	195°	235°	140°	100°	35°	55°

10 a quadrant 4 b quadrant 1 c quadrant 2
 d quadrant 2 e quadrant 1 f quadrant 3

11 As $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and both sin θ and cos θ are negative over this range, tan θ is positive in the third quadrant.

12 As $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cos\theta = 0$ at 90° and 270°, the value of $\dfrac{\sin\theta}{\cos\theta}$ and, hence, tan θ is undefined at these values.

13 a true b true c false d true
 e true f false g true h false
 i false j true k true l false

14 a i 0.17 ii 0.17 iii 0.59 iv 0.59
 v 0.99 vi 0.99 vii 0.37 viii 0.37
 b $\sin a = \cos b$ when $a + b = 90°$.
 c i $90° - \theta$ ii $90° - \theta$
 d i 70° ii 5° iii 19° iv 38°
 e i $90° - \theta$ ii $\dfrac{b}{c}$ iii $\dfrac{b}{c}$
 f $\dfrac{2\sqrt{5}}{5}$

15 a i Proof
 ii True for these values.
 b i sin 60° = cos 30° = 0.866,
 sin 80° = cos 10° = 0.985,
 sin 110° = cos (–20°) = 0.940,
 sin 195° = cos (–105°) = –0.259
 ii Their values are the same.
 iii They add to 90°.
 iv $\sin\theta = \cos(90° - \theta)$
 v True for these values.
 c Answers may vary.

4J

Building understanding

1

Degrees	360°	180°	90°	60°	45°	30°	15°
Radians	$2\pi^c$	π^c	$\dfrac{\pi^c}{2}$	$\dfrac{\pi^c}{3}$	$\dfrac{\pi^c}{4}$	$\dfrac{\pi^c}{6}$	$\dfrac{\pi^c}{12}$

2 a $\dfrac{\pi}{180}$ b $\dfrac{180}{\pi}$

3 a $\dfrac{1}{\sqrt{2}}$ b $\dfrac{1}{\sqrt{2}}$ c 1 d $\dfrac{\sqrt{3}}{2}$ e $\dfrac{1}{2}$
 f $\dfrac{1}{\sqrt{3}}$ g $\sqrt{3}$ h $\dfrac{1}{2}$ i $\dfrac{\sqrt{3}}{2}$

4

$\theta°$	0	30	45	60	90	180	270	360
θ^c	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$	π	$\dfrac{3\pi}{2}$	2π
sin θ	0	$\dfrac{1}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{3}}{2}$	1	0	–1	0
cos θ	1	$\dfrac{\sqrt{3}}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{1}{2}$	0	–1	0	1
tan θ	0	$\dfrac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	0	undefined	0

Now you try

Example 19

a $\dfrac{3\pi^c}{4}$ b 150°

Example 20

a $\dfrac{\sqrt{3}}{2}$ b $\dfrac{\sqrt{2}}{2}$ c $-\dfrac{\sqrt{3}}{3}$

Exercise 4J

1 a i $\dfrac{\pi^c}{2}$ ii $\dfrac{5\pi^c}{4}$
 b i 135° ii 210°

2 a $\dfrac{\pi^c}{3}$ b $\dfrac{5\pi^c}{6}$ c $\dfrac{5\pi^c}{4}$ d $\dfrac{11\pi^c}{6}$
 e 135° f 210° g 300° h 165°

3 a $\dfrac{\sqrt{3}}{2}$ b $\dfrac{\sqrt{2}}{2}$ c $\sqrt{3}$ d $\dfrac{\sqrt{2}}{2}$
 e $-\dfrac{\sqrt{3}}{2}$ f $-\sqrt{3}$ g $\dfrac{\sqrt{2}}{2}$ h $-\dfrac{\sqrt{2}}{2}$
 i $\dfrac{\sqrt{3}}{2}$ j $-\dfrac{\sqrt{3}}{3}$ k $-\dfrac{1}{2}$ l $\dfrac{1}{2}$
 m –1 n 1 o 0 p undefined

ISBN 978-1-108-77290-7

Cambridge University Press

Updated June 2022

4 a 45°

b i $-\frac{\sqrt{2}}{2}$ ii $-\frac{\sqrt{2}}{2}$ iii 1

c 30°

d i $-\frac{1}{2}$ ii $\frac{\sqrt{3}}{2}$ iii $-\frac{\sqrt{3}}{3}$

e 60°

f i $\frac{\sqrt{3}}{2}$ ii $-\frac{1}{2}$ iii $-\sqrt{3}$

5 a $\frac{\sqrt{2}}{2}$ b 0 c $-\frac{\sqrt{3}}{2}$ d $-\frac{\sqrt{3}}{2}$

e -1 f $-\frac{\sqrt{3}}{2}$ g $-\frac{1}{2}$ h $-\sqrt{3}$

i $-\frac{\sqrt{2}}{2}$ j -1 k $-\frac{\sqrt{3}}{3}$ l $\frac{1}{2}$

m 0 n undefined o 1 p -1

6 a $\frac{\pi^c}{3}$ b $\frac{\pi^c}{4}$ c $\frac{\pi^c}{6}$ d $\frac{\pi^c}{3}$

e $\frac{\pi^c}{3}$ f $\frac{\pi^c}{6}$ g $\frac{\pi^c}{12}$ h $\frac{5\pi^c}{12}$

7 a $\frac{\sqrt{3}}{2}$ b $-\frac{\sqrt{2}}{2}$ c $-\frac{1}{2}$ d $-\frac{\sqrt{2}}{2}$

e $-\frac{1}{2}$ f $\frac{1}{2}$ g $-\frac{\sqrt{3}}{3}$ h $\sqrt{3}$

8 a $3\sqrt{2}$ b $3\sqrt{2}$ c $\frac{20\sqrt{3}}{3}$

d 14 e $5\sqrt{3}$ f 3

9 a $\frac{\pi^c}{4}$ b $\frac{\pi^c}{6}$ c $\frac{\pi^c}{3}$

10 a $\frac{\pi}{6}$ is the reference angle and $\frac{5\pi}{6}$ is in quadrant 2 with $\sin\theta$ positive.

b $\frac{\pi}{3}$ is the reference angle and $\frac{2\pi}{3}$ is in quadrant 2 with $\cos\theta$ negative.

c $\frac{\pi}{6}$ is the reference angle and $\frac{11\pi}{6}$ is in quadrant 4 with $\cos\theta$ positive.

d $\frac{\pi}{4}$ is the reference angle and $\frac{3\pi}{4}$ is in quadrant 2 with $\tan\theta$ negative.

e $\frac{\pi}{4}$ is the reference angle and $\frac{5\pi}{4}$ is in quadrant 3 with $\tan\theta$ positive

f $\frac{\pi}{3}$ is the reference angle and $\frac{4\pi}{3}$ is in quadrant 3 with $\sin\theta$ negative.

11 a 13

b i $\frac{5}{13}$ ii $\frac{12}{13}$ iii $\frac{5}{12}$

12 a i 60°, 120° ii 45°, 135° iii 60°, 300°

iv 150°, 210° v 45°, 225° vi 120°, 300°

b i $\frac{\pi^c}{3}, \frac{2\pi^c}{3}$ ii $\frac{5\pi^c}{4}, \frac{7\pi^c}{4}$ iii $\frac{3\pi^c}{4}, \frac{5\pi^c}{4}$

iv $\frac{\pi^c}{6}, \frac{11\pi^c}{6}$ v $\frac{\pi^c}{6}, \frac{7\pi^c}{6}$ vi $\frac{3\pi^c}{4}, \frac{7\pi^c}{4}$

4K

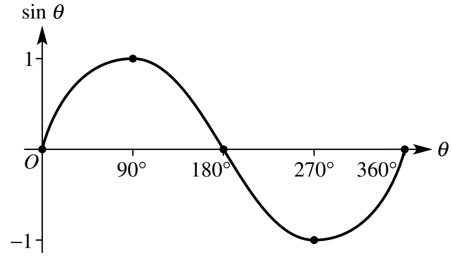
Building understanding

1 a

θ	0°	30°	60°	90°	120°	150°
$\sin\theta$	0	0.5	0.87	1	0.87	0.5

θ	180°	210°	240°	270°	300°	330°	360°
$\sin\theta$	0	–0.5	–0.87	–1	–0.87	–0.5	0

b

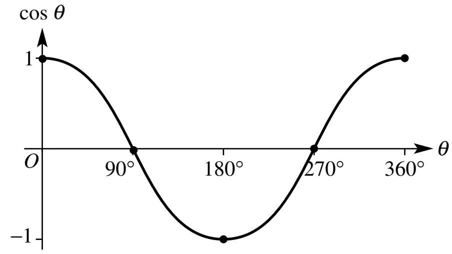


2 a

θ	0°	30°	60°	90°	120°	150°
$\cos\theta$	1	0.87	0.5	0	–0.5	–0.87

θ	180°	210°	240°	270°	300°	330°	360°
$\cos\theta$	–1	–0.87	–0.5	0	0.5	0.87	1

b



3 a i maximum = 1, minimum = –1

ii 0°, 180°, 360°

b i maximum = 1, minimum = –1 ii 90°, 270°

c i $90° < \theta < 270°$ ii $180° < \theta < 360°$

Now you try

Example 21

a ≈ –0.17 b ≈ 127° and 233°

Example 22

a false b true

Exercise 4K

1 a ≈ – 0.77

b ≈ 66° and 294°

2 a i 0.82 ii –0.98 iii 0.87 iv –0.77

v –0.17 vi 0.26 vii –0.42 viii 0.57

b i 37°, 323° ii 53°, 307° iii 73°, 287°

iv 84°, 276° v 114°, 246° vi 102°, 258°

vii 143°, 217° viii 127°,233°

Answers 4J

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

3 a i 0.42 ii 0.91 iii −0.64 iv −0.77
 v 0.34 vi −0.82 vii −0.64 viii 0.94
 b i 37°, 143° ii 12°, 168° iii 17°, 163°
 iv 64°, 116° v 204°, 336° vi 233°, 307°
 vii 224°, 316° viii 186°,354°

4 a true b false c false d true
 e false f true g true h true
 i true j false k true l true

5 a 110° b 60° c 350° d 260°
 e 27° f 326° g 233° h 357°

6 a 280° b 350° c 195° d 75°
 e 136° f 213° g 24° h 161°

7 a 30° b 60° c 15° d 70°
 e 55° f 80° g 55° h 25°
 i 36° j 72° k 63° l 14°

8 a 17.5°, 162.5° b 44.4°, 135.6°
 c 53.1°, 306.9° d 36.9°, 323.1°
 e 191.5°, 348.5° f 233.1°, 306.9°
 g 113.6°, 246.4° h 49.5°, 310.5°
 i 28.7°,151.3°

9 a 0, the maximum value of sin θ is 1.
 b 0, the minimum value of cos θ is −1.

10 a

θ	0°	30°	45°	60°	90°
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

 b i $\frac{1}{2}$ ii $-\frac{1}{2}$ iii $-\frac{\sqrt{2}}{2}$ iv 0
 v $\frac{1}{2}$ vi $-\frac{\sqrt{3}}{2}$ vii 0 viii $\frac{\sqrt{2}}{2}$
 ix $-\frac{\sqrt{3}}{2}$ x $-\frac{1}{2}$ xi $-\frac{\sqrt{2}}{2}$ xii $-\frac{1}{2}$
 xiii $-\frac{\sqrt{2}}{2}$ xiv $\frac{\sqrt{3}}{2}$ xv $-\frac{\sqrt{3}}{2}$ xvi $\frac{\sqrt{3}}{2}$

11 a 45°, 315° b 60°, 120° c 30°, 150°
 d 210°, 330° e 120°, 240° f 150°, 210°

12 a Graph is reflected in the x-axis.
 b Graph is reflected in the x-axis.
 c Graph is dilated and constricted from the x-axis.
 d Graph is dilated and constricted from the y-axis.
 e Graph is translated up and down from the x-axis.
 f Graph is translated left and right from the y-axis.

Problems and challenges

1 a 120°, 60° b 8.7 cm
2 225°
3 Use the cosine rule.
4 514 m
5 a 2 hours 9 minutes b 308°
6 17.93°
7 4.33 cm

Short-answer questions

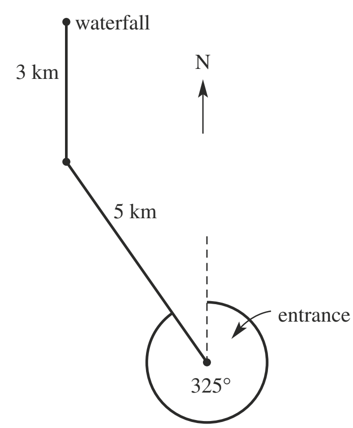
1 a 14.74 b 13.17 c $x = 11.55, y = 5.42$
2 a 45.6° b 64.8°
3 6.1 m
4 a $A = 115°, B = 315°, C = 250°, D = 030°$
 b i 295° ii 070°
5 a 98.3 km b 228.8 km c 336.8°
6 a 15.43 m b 52°
7 a i 15.5 cm ii 135.0 cm^2
 b i 14.9 cm ii 111.3 cm^2
8 28.1 m
9 a 52.6° b 105.4°
10 a 12.5 b 42.8°
11 a i sin 60° ii −cos 30°
 iii −tan 45° iv −sin 45°
 b i $\frac{\sqrt{3}}{2}$ ii $-\frac{\sqrt{3}}{2}$ iii −1 iv $-\frac{\sqrt{2}}{2}$
 c i negative ii positive
 iii negative iv positive
12 a $\frac{\pi^c}{3}$ b $\frac{5\pi^c}{4}$ c 150° d 300°
13 a 4 b $5\sqrt{3}$
14 a i 0.77 ii −0.97
 b i 53°, 127 ii 197°, 343° iii no value
 c i true ii true iii false

Multiple-choice questions

1 D 2 B 3 E 4 D 5 A
6 C 7 A 8 D 9 C 10 C

Extended-response questions

1 a



 b 2.9 km west
 c 7.7 km
 d i 21.9 m ii 38.0°
2 a 33.646° b 3177.54 m^2 c 41.00 m d 61.60 m
 e i 65.66°, 114.34°
 ii 80.1 m, 43.1 m

ISBN 978-1-108-77290-7

Chapter 5

5A

Building understanding

1 a $x^2 + 2x$ b $x^2 + 4x + 3$ c $x^2 + 8x + 16$
2 a $6x$ b $-20x$ c $-4x^2$ d $\frac{x}{2}$
e $-3x$ f $-18x$ g $5x$ h $-13x$

Now you try

Example 1
a $-2x + 8$ b $20x - 5x^2$ c $6x + \frac{3}{5}$ d $8x^2 - 3x$

Example 2
a $x^2 + 7x + 10$ b $x^2 - 4x + 4$ c $9x^2 - 4$

Example 3
a $6x^2 + 19x - 7$ b $3x^2 - 15x + 12$ c $11x - 9$

Exercise 5A

1 a i $-4x + 4$ ii $-2x + 12$
b i $6x - 3x^2$ ii $35x - 7x^2$
c i $12x + \frac{8}{5}$ ii $14x - \frac{7}{9}$
2 a $2x + 10$ b $3x - 12$ c $-5x - 15$
d $-4x + 8$ e $6x - 3$ f $12x + 4$
g $-10x + 6$ h $-20x - 15$ i $2x^2 + 5x$
j $3x^2 - x$ k $2x - 2x^2$ l $6x - 3x^2$
m $-6x^2 - 4x$ n $-18x^2 + 6x$ o $-10x + 10x^2$
p $-4x + 16x^2$ q $4x + \frac{8}{5}$ r $6x - \frac{15}{4}$
s $-2x - \frac{1}{3}$ t $-2x + \frac{3}{2}$ u $-9x + \frac{3}{8}$
v $-2x - \frac{14}{9}$ w $\frac{9}{4}x^2 + 6x$ x $\frac{14}{5}x - \frac{6}{5}x^2$
3 a $2x^2 + 3x$ b $6x^2 - 3x$ c $2x^2 + 7x$
d $8x^2 + 7x$ e $2x^2 - 2x$ f $25x - 12x^2$
4 a $x^2 + 10x + 16$ b $x^2 + 7x + 12$
c $x^2 + 12x + 35$ d $x^2 + 5x - 24$
e $x^2 + x - 30$ f $x^2 + x - 6$
g $x^2 - 4x - 21$ h $x^2 - 10x + 24$
i $x^2 - 13x + 40$
5 a $x^2 + 10x + 25$ b $x^2 + 14x + 49$
c $x^2 + 12x + 36$ d $x^2 - 6x + 9$
e $x^2 - 16x + 64$ f $x^2 - 20x + 100$
g $x^2 - 16$ h $x^2 - 81$
i $4x^2 - 9$ j $9x^2 - 16$
k $16x^2 - 25$ l $64x^2 - 49$
6 a $6x^2 + 13x + 5$ b $12x^2 + 23x + 10$
c $10x^2 + 41x + 21$ d $9x^2 - 9x - 10$
e $20x^2 + 2x - 6$ f $6x^2 + 5x - 25$
g $16x^2 - 25$ h $4x^2 - 81$
i $25x^2 - 49$ j $14x^2 - 34x + 12$
k $25x^2 - 45x + 18$ l $56x^2 - 30x + 4$
m $4x^2 + 20x + 25$ n $25x^2 + 60x + 36$
o $49x^2 - 14x + 1$
7 a 3 b 3 c 3 d 8 e 1 f 2
8 a $2x^2 + 14x + 24$ b $3x^2 + 27x + 42$
c $-2x^2 - 20x - 32$ d $-4x^2 - 44x - 72$
e $5x^2 + 5x - 60$ f $3x^2 + 6x - 45$
g $-3a^2 + 15a + 42$ h $-5a^2 + 30a + 80$
i $4a^2 - 36a + 72$ j $3y^2 - 27y + 60$
k $-2y^2 + 22y - 48$ l $-6y^2 + 42y - 72$
m $12x^2 + 48x + 45$ n $18x^2 + 12x - 48$
o $-6x^2 - 10x + 56$ p $2x^2 + 12x + 18$
q $4m^2 + 40m + 100$ r $2a^2 - 28a + 98$
s $-3y^2 + 30y - 75$ t $12b^2 - 12b + 3$
u $-12y^2 + 72y - 108$
9 a $2x^2 + 10x + 11$ b $2x^2 + 20x + 44$
c $2y^2 - 4y + 5$ d $2y^2 - y - 43$
e $-24a - 45$ f $b^2 + 54b + 5$
g $x^2 + 10x + 18$ h $x^2 - 14x + 40$
i $-4x^2 + 36x - 78$ j $-25x^2 - 30x + 5$
10 a $x^2 - 12x + 36\text{ cm}^2$ b $x^2 + 10x - 200\text{ cm}^2$
11 a $(a + b)(a - b) = a^2 - ab + ba - b^2 = a^2 - b^2$
b $(a + b)^2 = (a + b)(a + b) = a^2 + ab + ba + b^2$
$= a^2 + 2ab + b^2$
c $(a - b)^2 = (a - b)(a - b) = a^2 - ab - ba + b^2$
$= a^2 - 2ab + b^2$
d $(a + b)^2 - (a - b)^2 =$
$a^2 + ab + ba + b^2 - (a^2 - ab - ba + b^2) =$
$2ab + 2ab = 4ab$
12 a 618 b 220 c 567 d 1664
e 1386 f 891 g 3960 h 3480
13 a $-x^2 + 7x$ b $10a - 28$
c $4x^2 + 12x + 9$ d $4x + 8$
14 a $x^3 + 6x^2 + 11x + 6$ b $x^3 + 11x^2 + 38x + 40$
c $x^3 + 2x^2 - 15x - 36$ d $2x^3 - 13x^2 + 17x + 12$
e $2x^3 - x^2 - 63x + 90$ f $6x^3 - 35x^2 + 47x - 12$
15 a $2ab$ b $(a + b)^2 - c^2$
c $(a + b)^2 - c^2 = 2ab$
$c^2 = a^2 + 2ab + b^2 - 2ab$
$c^2 = a^2 + b^2$

5B

Building understanding

1 a 7 b -5 c $3a$ d $-3xy$
2 a If $x(x - 1) = x^2 - x$, then $x^2 - x = x(x - 1)$.
b If $2(1 - x) = 2 - 2x$, then $2 - 2x = 2(1 - x)$.
c If $(x + 2)(x - 2) = x^2 - 4$, then $x^2 - 4 = (x + 2)(x - 2)$.
d If $(3x - 7)(3x + 7) = 9x^2 - 49$, then
$9x^2 - 49 = (3x - 7)(3x + 7)$.

Now you try

Example 4
a $-2(x + 4)$ b $5a(3a + 4)$ c $(x + 2)(3 - a)$

Example 5
a $(x + 5)(x - 5)$ b $(4a + 3b)(4a - 3b)$
c $2(y + 7)(y - 7)$ d $(x + 8)(x - 4)$

Answers 5A

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 6

a $(x + \sqrt{7})(x - \sqrt{7})$ b $(x + 4\sqrt{2})(x - 4\sqrt{2})$
c $(x - 5 + \sqrt{2})(x - 5 - \sqrt{2})$

Example 7

$(x - 2)(x + a)$

Exercise 5B

1 a i $-4(x + 3)$ ii $-9(x + 4)$
b i $10a(a + 4)$ ii $17a(a + 2)$
c i $(x + 2)(4 - a)$ ii $(x + 5)(11 - a)$

2 a $3(x - 6)$ b $4(x + 5)$ c $7(a + b)$
d $3(3a - 5)$ e $-5(x + 6)$ f $-2(2y + 1)$
g $-3(4a + 1)$ h $-b(2a + c)$ i $x(4x + 1)$
j $x(5x - 2)$ k $6b(b - 3)$ l $7a(2a - 3)$
m $5a(2 - a)$ n $6x(2 - 5x)$ o $-x(2 + x)$
p $-4y(1 + 2y)$ q $ab(b - a)$ r $2xy(xz - 2)$
s $-12mn(m + n)$ t $3z^2(2xy - 1)$

3 a $(x - 1)(5 - a)$ b $(x + 2)(b + 3)$
c $(x + 5)(a - 4)$ d $(x + 2)(x + 5)$
e $(x - 4)(x - 2)$ f $(x + 1)(3 - x)$
g $(x + 3)(a + 1)$ h $(x - 2)(x - 1)$
i $(x - 6)(1 - x)$

4 a $(x + 3)(x - 3)$ b $(x + 5)(x - 5)$
c $(y + 7)(y - 7)$ d $(y + 1)(y - 1)$
e $(2x - 3)(2x + 3)$ f $(6a - 5)(6a + 5)$
g $(1 + 9y)(1 - 9y)$ h $(10 - 3x)(10 + 3x)$
i $(5x - 2y)(5x + 2y)$ j $(8x - 5y)(8x + 5y)$
k $(3a + 7b)(3a - 7b)$ l $(12a - 7b)(12a + 7b)$

5 a $2(x + 4)(x - 4)$ b $5(x + 3)(x - 3)$
c $6(y + 2)(y - 2)$ d $3(y + 4)(y - 4)$
e $3(x + 5y)(x - 5y)$ f $3(a + 10b)(a - 10b)$
g $3(2x + 3y)(2x - 3y)$ h $7(3a + 4b)(3a - 4b)$
i $(x + 9)(x + 1)$ j $(x - 7)(x - 1)$
k $(a + 5)(a - 11)$ l $(a - 8)(a - 6)$
m $(4x + 5)(2x + 5)$ n $(y + 7)(3y + 7)$
o $(3x + 11)(7x + 11)$ p $3x(3x - 10y)$

6 a $(x + \sqrt{7})(x - \sqrt{7})$ b $(x + \sqrt{5})(x - \sqrt{5})$
c $(x + \sqrt{19})(x - \sqrt{19})$ d $(x + \sqrt{21})(x - \sqrt{21})$
e $(x + \sqrt{14})(x - \sqrt{14})$ f $(x + \sqrt{30})(x - \sqrt{30})$
g $(x + \sqrt{15})(x - \sqrt{15})$ h $(x + \sqrt{11})(x - \sqrt{11})$
i $(x + 2\sqrt{2})(x - 2\sqrt{2})$ j $(x + 3\sqrt{2})(x - 3\sqrt{2})$
k $(x + 3\sqrt{5})(x - 3\sqrt{5})$ l $(x + 2\sqrt{5})(x - 2\sqrt{5})$
m $(x + 4\sqrt{2})(x - 4\sqrt{2})$ n $(x + 4\sqrt{3})(x - 4\sqrt{3})$
o $(x + 5\sqrt{2})(x - 5\sqrt{2})$ p $(x + 10\sqrt{2})(x - 10\sqrt{2})$
q $(x + 2 + \sqrt{6})(x + 2 - \sqrt{6})$
r $(x + 5 + \sqrt{10})(x + 5 - \sqrt{10})$
s $(x - 3 + \sqrt{11})(x - 3 - \sqrt{11})$
t $(x - 1 + \sqrt{7})(x - 1 - \sqrt{7})$
u $(x - 6 + \sqrt{15})(x - 6 - \sqrt{15})$
v $(x + 4 + \sqrt{21})(x + 4 - \sqrt{21})$
w $(x + 1 + \sqrt{19})(x + 1 - \sqrt{19})$
x $(x - 7 + \sqrt{26})(x - 7 - \sqrt{26})$

7 a $(x + 4)(x + a)$ b $(x + 7)(x + b)$
c $(x - 3)(x + a)$ d $(x + 2)(x - a)$
e $(x + 5)(x - b)$ f $(x + 3)(x - 4a)$
g $(x - a)(x - 4)$ h $(x - 2b)(x - 5)$
i $(x - 2a)(3x - 7)$

8 a $\left(x + \frac{\sqrt{2}}{3}\right)\left(x - \frac{\sqrt{2}}{3}\right)$ b $\left(x + \frac{\sqrt{3}}{2}\right)\left(x - \frac{\sqrt{3}}{2}\right)$
c $\left(x + \frac{\sqrt{7}}{4}\right)\left(x - \frac{\sqrt{7}}{4}\right)$ d $\left(x + \frac{\sqrt{5}}{6}\right)\left(x - \frac{\sqrt{5}}{6}\right)$
e $(x - 2 + 2\sqrt{5})(x - 2 - 2\sqrt{5})$
f $(x + 4 + 3\sqrt{3})(x + 4 - 3\sqrt{3})$
g $(x + 1 + 5\sqrt{3})(x + 1 - 5\sqrt{3})$
h $(x - 7 + 2\sqrt{10})(x - 7 - 2\sqrt{10})$
i $(\sqrt{3}x + 2)(\sqrt{3}x - 2)$
j $(\sqrt{5}x + 3)(\sqrt{5}x - 3)$
k $(\sqrt{7}x + \sqrt{5})(\sqrt{7}x - \sqrt{5})$
l $(\sqrt{6}x + \sqrt{11})(\sqrt{6}x - \sqrt{11})$
m $(\sqrt{2}x + 3)(\sqrt{2}x - 3)$
n $(\sqrt{5}x + 4)(\sqrt{5}x - 4)$
o $(\sqrt{3}x + \sqrt{10})(\sqrt{3}x - \sqrt{10})$
p $(\sqrt{13}x + \sqrt{7})(\sqrt{13}x - \sqrt{7})$

9 a $(x + 2)(y - 3)$ b $(a - 4)(x + 3)$
c $(a + 5)(x - 2)$ d $(y - 4)(x - 3)$
e $(a - 3)(2x - 1)$ f $(2a - 5)(x + 4)$

10 a $5(x + 2\sqrt{6})(x - 2\sqrt{6})$
b $3(x + 3\sqrt{6})(x - 3\sqrt{6})$
c $7(x + 3\sqrt{2})(x - 3\sqrt{2})$
d $2(x + 4\sqrt{3})(x - 4\sqrt{3})$
e $2(x + 3 + \sqrt{5})(x + 3 - \sqrt{5})$
f $3(x - 1 + \sqrt{7})(x - 1 - \sqrt{7})$
g $4(x - 4 + 2\sqrt{3})(x - 4 - 2\sqrt{3})$
h $5(x + 6 + 3\sqrt{2})(x + 6 - 3\sqrt{2})$

11 a 60 b 35 c 69 d 104
e 64 f 40 g 153 h 1260

12 a $4 - (x + 2)^2 = (2 - (x + 2))(2 + (x + 2)) = -x(x + 4)$
b i $-x(x + 6)$ ii $-x(x + 8)$
iii $x(10 - x)$ iv $(3 - x)(7 + x)$
v $(8 - x)(6 + x)$ vi $(6 - x)(14 + x)$

13 a $(x + a)^2 = x^2 + 2ax + a^2 \neq x^2 + a^2$
b If $x = 0$, then $(x + a)^2 = x^2 + a^2$. Or if $a = 0$, then $(x + a)^2 = x^2 + a^2$ is true for all real values of x.

14 $x^2 - \frac{4}{9} = \frac{1}{9}(9x^2 - 4) = \frac{1}{9}(3x + 2)(3x - 2)$

or: $x^2 - \frac{4}{9} = \left(x + \frac{2}{3}\right)\left(x - \frac{2}{3}\right)$
$= \frac{1}{3}(3x + 2)\frac{1}{3}(3x - 2)$
$= \frac{1}{9}(3x + 2)(3x - 2)$

15 a $-(2x + 5)$ b $-11(2y - 3)$
c $16(a - 1)$ d $20b$
e $-12s$ f $-28y$
g $(5w + 7x)(-w - x)$ h $(4d + 3e)(-2d + 7e)$
i $6f(2f + 6j)$ j 0

16 a $x^2 + 5y - y^2 + 5x$
$= x^2 - y^2 + 5x + 5y$
$= (x - y)(x + y) + 5(x + y)$
$= (x + y)(x - y + 5)$
b i $(x + y)(x - y + 7)$
ii $(x + y)(x - y - 2)$
iii $(2x + 3y)(2x - 3y + 2)$
iv $(5y + 2x)(5y - 2x + 3)$

ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

5C

Building understanding

1 a 9, 2 b 10, 2 c 5, –3 d 4, –3
 e –8, 3 f –10, 3 g –2, –5 h –12, –3

2 a Possible answer: $\frac{x-10}{x-10}=1$
 b Possible answer: $\frac{3(x-7)}{x-7}=3$
 c Possible answer: $\frac{-5(x+3)}{x+3}=-5$
 d Possible answer: $\frac{x+4}{3(x+4)}=\frac{1}{3}$

3 a $\frac{x}{2}$ b 3 c $\frac{1}{3}$ d $\frac{1}{4}$
 e $\frac{2}{3}$ f $x+1$ g $x-2$ h $\frac{x-3}{2}$

Now you try

Example 8

a $(x+3)(x+4)$ b $(x-6)(x-4)$
c $2(x-3)(x+2)$ d $(x-3)^2$

Example 9

a $x-4$ b $\frac{x+4}{2}$

Exercise 5C

1 a i $(x+1)(x+2)$ ii $(x+1)(x+5)$
 b i $(x-1)(x-3)$ ii $(x-6)(x-5)$
 c i $2(x-5)(x+1)$ ii $3(x-5)(x+2)$
 d i $(x-2)^2$ ii $(x-5)^2$

2 a $(x+6)(x+1)$ b $(x+3)(x+2)$
 c $(x+3)^2$ d $(x+5)(x+2)$
 e $(x+4)(x+3)$ f $(x+9)(x+2)$
 g $(x-1)(x+6)$ h $(x+3)(x-2)$
 i $(x+4)(x-2)$ j $(x-1)(x+4)$
 k $(x+10)(x-3)$ l $(x+11)(x-2)$
 m $(x-2)(x-5)$ n $(x-4)(x-2)$
 o $(x-4)(x-3)$ p $(x-1)^2$
 q $(x-6)(x-3)$ r $(x-2)(x-9)$
 s $(x-6)(x+2)$ t $(x-5)(x+4)$
 u $(x-7)(x+2)$ v $(x-4)(x+3)$
 w $(x+8)(x-4)$ x $(x-5)(x+2)$

3 a $2(x+5)(x+2)$ b $3(x+4)(x+3)$
 c $2(x+9)(x+2)$ d $5(x-2)(x+1)$
 e $4(x-5)(x+1)$ f $3(x-5)(x+2)$
 g $-2(x+4)(x+3)$ h $-3(x-2)(x-1)$
 i $-2(x-7)(x+2)$ j $-4(x-2)(x+1)$
 k $-5(x+3)(x+1)$ l $-7(x-6)(x-1)$

4 a $(x-2)^2$ b $(x+3)^2$
 c $(x+6)^2$ d $(x-7)^2$
 e $(x-9)^2$ f $(x-10)^2$
 g $2(x+11)^2$ h $3(x-4)^2$
 i $5(x-5)^2$ j $-3(x-6)^2$
 k $-2(x-7)^2$ l $-4(x+9)^2$

5 a $x+6$ b $x-3$ c $x-3$
 d $\frac{1}{x+7}$ e $\frac{1}{x-5}$ f $\frac{1}{x-6}$
 g $\frac{2}{x-8}$ h $\frac{x+4}{3}$ i $\frac{x-7}{5}$

6 a $\frac{5}{x+6}$ b $\frac{x-3}{3}$ c $\frac{2(x-1)}{x+5}$ d $\frac{4}{x+5}$
 e $\frac{4}{x+7}$ f $\frac{6}{x-2}$ g $\frac{x+2}{x-1}$ h $\frac{x-4}{x+6}$

7 a $x-\sqrt{7}$ b $x+\sqrt{10}$ c $x-2\sqrt{3}$
 d $\frac{1}{\sqrt{5}x-3}$ e $\frac{1}{\sqrt{3}x+4}$ f $\sqrt{7}x-\sqrt{5}$
 g $x+1-\sqrt{2}$ h $x-3+\sqrt{5}$ i $x-6-\sqrt{6}$

8 a $\frac{2(x+3)}{3(x-5)}$ b $\frac{x-3}{4}$ c $\frac{3}{x-3}$
 d $\frac{3}{2}$ e $\frac{x-2}{x+3}$ f $\frac{x+3}{x-1}$

9 $\frac{t^2-49}{5t-40}\times\frac{t^2-5t-24}{2t^2-8t-42}=$
$\frac{\cancel{(t-7)}(t+7)}{5\cancel{(t-8)}}\times\frac{\cancel{(t-8)(t+3)}}{2\cancel{(t-7)(t+3)}}=\frac{t+7}{10}$

10 a $x-3$ b $x+1$ c $x-8$
 d $\frac{6}{x-2}$ e $\frac{4}{x+5}$ f $\frac{x-7}{5}$

11 a $\frac{a^2+2ab+b^2}{a^2+ab}\times\frac{a^2-ab}{a^2-b^2}$
$=\frac{\cancel{(a+b)^2}}{a\cancel{(a+b)}}\times\frac{a\cancel{(a-b)}}{\cancel{(a+b)(a-b)}}$
$=1$
 b Answers will vary.

12 a $\frac{a-b}{a}$ b 1
 c $\frac{(a+b)^2}{(a-b)^2}$ d $\frac{(a+b)(a-b)}{a^2}$

13 a $\frac{3x-8}{(x+3)(x-4)}$ b $\frac{7x-36}{(x+2)(x-9)}$
 c $\frac{x-12}{(x+4)(x-4)}$ d $\frac{3x-23}{(x+3)(x-3)(x-5)}$
 e $\frac{x-14}{(x-3)(x+2)(x-6)}$ f $\frac{14x+9}{(x+3)(x+4)(x-8)}$
 g $\frac{9-3x}{(x+5)(x-5)(x-1)}$ h $\frac{4x+11}{(x-1)^2(x+4)}$

5D

Building understanding

1

ax^2+bx+c	$a\times c$	Two numbers which multiply to give $a\times c$ and add to give b
$6x^2+13x+6$	36	9 and 4
$8x^2+18x+4$	32	16 and 2
$12x^2+x-6$	–72	–8 and 9
$10x^2-11x-6$	–60	–15 and 4
$21x^2-20x+4$	84	–6 and –14
$15x^2-13x+2$	30	–3 and –10

ISBN 978-1-108-77290-7

2 a $(x+2)(x+5)$ b $(x-7)(x-2)$
c $(3x-4)(2x+1)$ d $(2x-1)(4x+3)$
e $(x+4)(5x-2)$ f $(2x-1)(6x-5)$

Now you try

Example 10
a $(2x+3)(3x+1)$ b $(4x-1)(2x+3)$

Example 11
1

Exercise 5D

1 a i $(4x+1)(2x+3)$ ii $(5x+2)(2x+3)$
b i $(2x+5)(3x-1)$ ii $(4x+3)(2x-1)$
2 a $(3x+1)(x+3)$ b $(2x+1)(x+1)$
c $(3x+2)(x+2)$ d $(3x-2)(x-1)$
e $(2x-1)(x-5)$ f $(5x-3)(x+1)$
g $(3x+1)(x-4)$ h $(3x+1)(x-1)$
i $(7x-5)(x+1)$ j $(2x-7)(x-1)$
k $(3x-4)(x+2)$ l $(2x-3)(x+4)$
m $(2x+1)(x-5)$ n $(13x+6)(x-1)$
o $(5x-2)(x-4)$ p $(4x-5)(2x-1)$
q $(3x-4)(2x+3)$ r $(5x-2)(2x+3)$
s $(3x+2)(2x+3)$ t $(4x-1)(x-1)$
u $(4x-5)(2x-1)$ v $(2x-5)(4x-3)$
w $(3x-2)(2x-3)$ x $(3x-2)(3x+5)$
3 a $(6x+5)(3x+2)$ b $(4x+3)(5x+6)$
c $(7x-2)(3x+4)$ d $(5x-2)(6x+5)$
e $(8x+3)(5x-2)$ f $(7x+2)(4x-3)$
g $(6x-5)(4x-3)$ h $(9x-2)(5x-4)$
i $(5x-2)(5x-8)$
4 a $2(3x+4)(x+5)$ b $3(2x+3)(x-4)$
c $3(8x+1)(2x-1)$ d $4(4x-5)(2x-3)$
e $8(2x-1)(x-1)$ f $10(3x-2)(3x+5)$
g $-5(5x+4)(2x+3)$ h $3(2x-3)^2$
i $5(4x-1)(x-1)$
5 a $2x-5$ b $4x-1$ c $3x-2$
d $\frac{2}{3x+2}$ e $\frac{2}{7x-2}$ f $\frac{4}{2x-3}$
g $\frac{x+4}{3x+1}$ h $\frac{3x-1}{2x+3}$ i $\frac{5x+4}{7x-2}$
j $\frac{3x-2}{5x-2}$ k $\frac{2x+3}{7x+1}$ l $\frac{2x-3}{4x-5}$
6 a $(3x-4)(x-5)$
b -10 m; the cable is 10 m below the water.
c $x=\frac{4}{3}$ or $x=5$
7 a $\frac{3x+4}{x-3}$ b $\frac{3x+2}{4}$ c $\frac{1-x}{3}$
d $\frac{4x-3}{5x+1}$ e 125 f $\frac{x+2}{5}$
g 1 h $\frac{(4x-5)^2}{(x-3)^2}$
8 $-12x^2-5x+3$
$=-(12x^2+5x-3)$
$=-(3x-1)(4x+3)$
$=(1-3x)(4x+3)$
a $(3-2x)(4x+5)$ b $(5-2x)(3x+2)$
c $(4-3x)(4x+1)$ d $(3-4x)(2x-3)$
e $(2-7x)(2x-5)$ f $(3-5x)(3x+2)$
9 Answers will vary.
10 a $\frac{9x+2}{(2x-3)(4x+1)}$ b $\frac{5x+15}{(3x-1)(2x+5)}$
c $\frac{16x^2+5x}{(2x-5)(4x+1)}$ d $\frac{7x-12x^2}{(3x-2)(4x-1)}$
e $\frac{8x-5}{(2x+1)(2x-1)(3x-2)}$ f $\frac{11-3x}{(3x+5)(3x-5)(3x-2)}$
g $\frac{2}{(2x-5)(3x-2)}$ h $\frac{12x+3}{(5x-2)(2x-3)(2x+7)}$

5E

Building understanding

1 a 9 b 1 c 4
d 16 e $\frac{25}{4}$ f $\frac{81}{4}$
2 a $(x+2)^2$ b $(x+4)^2$ c $(x+5)^2$
d $(x-6)^2$ e $(x-3)^2$ f $(x-9)^2$
3 a $(x+1+\sqrt{5})(x+1-\sqrt{5})$
b $(x+4+\sqrt{10})(x+4-\sqrt{10})$
c $(x-3+\sqrt{11})(x-3-\sqrt{11})$

Now you try

Example 12
a 36, $(x+6)^2$ b $\frac{81}{4}, \left(x-\frac{9}{2}\right)^2$

Example 13
a $(x+3+\sqrt{10})(x+3-\sqrt{10})$
b $(x-2)^2+3$ cannot be factorised.

Example 14
$\left(x+\frac{5+\sqrt{23}}{2}\right)\left(x+\frac{5-\sqrt{23}}{2}\right)$

Exercise 5E

1 a i 16, $(x+4)^2$ ii 49, $(x+7)^2$
b i $\frac{25}{4}, \left(x-\frac{5}{2}\right)^2$ ii $\frac{121}{4}, \left(x-\frac{11}{2}\right)$
2 a 9, $(x+3)^2$ b 36, $(x+6)^2$
c 4, $(x+2)^2$ d 16, $(x+4)^2$
e 25, $(x-5)^2$ f 1, $(x-1)^2$
g 16, $(x-4)^2$ h 36, $(x-6)^2$
i $\frac{25}{4}, \left(x+\frac{5}{2}\right)^2$ j $\frac{81}{4}, \left(x+\frac{9}{2}\right)^2$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

k $\frac{49}{4}, \left(x+\frac{7}{2}\right)^2$
l $\frac{121}{4}, \left(x+\frac{11}{2}\right)^2$
m $\frac{9}{4}, \left(x-\frac{3}{2}\right)^2$
n $\frac{49}{4}, \left(x-\frac{7}{2}\right)^2$
o $\frac{1}{4}, \left(x-\frac{1}{2}\right)^2$
p $\frac{81}{4}, \left(x-\frac{9}{2}\right)^2$

3 a $(x+2+\sqrt{3})(x+2-\sqrt{3})$
b $(x+3+\sqrt{7})(x+3-\sqrt{7})$
c $(x+1+\sqrt{5})(x+1-\sqrt{5})$
d $(x+5+\sqrt{29})(x+5-\sqrt{29})$
e $(x-4+\sqrt{3})(x-4-\sqrt{3})$
f $(x-6+\sqrt{26})(x-6-\sqrt{26})$
g $(x-2+\sqrt{7})(x-2-\sqrt{7})$
h $(x-4+\sqrt{21})(x-4-\sqrt{21})$
i $(x+7+\sqrt{43})(x+7-\sqrt{43})$

4 a not possible
b not possible
c $(x+4+\sqrt{15})(x+4-\sqrt{15})$
d $(x+2+\sqrt{2})(x+2-\sqrt{2})$
e $(x+5+\sqrt{22})(x+5-\sqrt{22})$
f $(x+2+\sqrt{10})(x+2-\sqrt{10})$
g not possible
h $(x-3+\sqrt{3})(x-3-\sqrt{3})$
i $(x-6+\sqrt{34})(x-6-\sqrt{34})$
j not possible
k $(x-4+\sqrt{17})(x-4-\sqrt{17})$
l not possible

5 a $\left(x+\frac{3+\sqrt{5}}{2}\right)\left(x+\frac{3-\sqrt{5}}{2}\right)$
b $\left(x+\frac{7+\sqrt{41}}{2}\right)\left(x+\frac{7-\sqrt{41}}{2}\right)$
c $\left(x+\frac{5+\sqrt{33}}{2}\right)\left(x+\frac{5-\sqrt{33}}{2}\right)$
d $\left(x+\frac{9+\sqrt{93}}{2}\right)\left(x+\frac{9-\sqrt{93}}{2}\right)$
e $\left(x-\frac{3+\sqrt{7}}{2}\right)\left(x-\frac{3-\sqrt{7}}{2}\right)$
f $\left(x-\frac{5+\sqrt{23}}{2}\right)\left(x-\frac{5-\sqrt{23}}{2}\right)$
g $\left(x-\frac{5+\sqrt{31}}{2}\right)\left(x-\frac{5-\sqrt{31}}{2}\right)$
h $\left(x-\frac{9+\sqrt{91}}{2}\right)\left(x-\frac{9-\sqrt{91}}{2}\right)$

6 a $2(x+3+\sqrt{5})(x+3-\sqrt{5})$
b $3(x+2+\sqrt{5})(x+2-\sqrt{5})$
c $4(x-1+\sqrt{5})(x-1-\sqrt{5})$
d $3(x-4+\sqrt{14})(x-4-\sqrt{14})$
e $-2(x+1+\sqrt{6})(x+1-\sqrt{6})$
f $-3(x+5+2\sqrt{6})(x+5-2\sqrt{6})$
g $-4(x+2+\sqrt{7})(x+2-\sqrt{7})$
h $-2(x-4+3\sqrt{2})(x-4-3\sqrt{2})$
i $-3(x-4+\sqrt{11})(x-4-\sqrt{11})$

7 a $3\left(x+\frac{3+\sqrt{5}}{2}\right)\left(x+\frac{3-\sqrt{5}}{2}\right)$
b $5\left(x+\frac{3+\sqrt{37}}{2}\right)\left(x+\frac{3-\sqrt{37}}{2}\right)$
c $2\left(x-\frac{5+\sqrt{17}}{2}\right)\left(x-\frac{5-\sqrt{17}}{2}\right)$
d $4\left(x-\frac{7+\sqrt{37}}{2}\right)\left(x-\frac{7-\sqrt{37}}{2}\right)$
e $-3\left(x+\frac{7+\sqrt{57}}{2}\right)\left(x+\frac{7-\sqrt{57}}{2}\right)$
f $-2\left(x+\frac{7+\sqrt{65}}{2}\right)\left(x+\frac{7-\sqrt{65}}{2}\right)$
g $-4\left(x-\frac{3+\sqrt{29}}{2}\right)\left(x-\frac{3-\sqrt{29}}{2}\right)$
h $-3\left(x-\frac{3+\sqrt{17}}{2}\right)\left(x-\frac{3-\sqrt{17}}{2}\right)$
i $-2\left(x-\frac{5+\sqrt{41}}{2}\right)\left(x-\frac{5-\sqrt{41}}{2}\right)$

8 a $x^2-2x-24$
$= x^2-2x+(-1)^2-(-1)^2-24$
$= (x-1)^2-25$
$= (x-1+5)(x-1-5)$
$= (x+4)(x-6)$
b Using a quadratic trinomial and finding two numbers that multiply to -24 and add to -2.

9 a If the difference of perfect squares is taken, it involves the square root of a negative number.
b i yes ii yes iii no iv no
v no vi yes vii yes viii no
c i $m \leqslant 4$ ii $m \leqslant 9$ iii $m \leqslant 25$

10 a $2(x+4)\left(x-\frac{3}{2}\right)$
b $3\left(x+\frac{2+\sqrt{13}}{3}\right)\left(x+\frac{2-\sqrt{13}}{3}\right)$
c $4\left(x-\frac{7+\sqrt{305}}{8}\right)\left(x-\frac{7-\sqrt{305}}{8}\right)$
d Unable to be factorised.
e $-2\left(x+\frac{3+\sqrt{41}}{4}\right)\left(x+\frac{3-\sqrt{41}}{4}\right)$
f $-3\left(x+\frac{7+\sqrt{13}}{6}\right)\left(x+\frac{7-\sqrt{13}}{6}\right)$
g Unable to be factorised.
h $-2\left(x-\frac{3+\sqrt{41}}{4}\right)\left(x-\frac{3-\sqrt{41}}{4}\right)$
i $2(x-1)\left(x+\frac{7}{2}\right)$
j $3\left(x+\frac{2+\sqrt{19}}{3}\right)\left(x+\frac{2-\sqrt{19}}{3}\right)$
k $-2\left(x+\frac{5}{2}\right)(x-1)$
l $-3\left(x+\frac{4}{3}\right)(x+1)$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Progress quiz

1 a $-8x^2 + \frac{10x}{3}$ b $4a^2 - 7a$
c $m^2 + 7m + 10$ d $k^2 - 6k + 9$
e $9m^2 - 4$ f $8h^2 - 6h - 35$
g $5x^2 - 35x + 60$ h $19p + 4$

2 a $4(a - 5)$ b $-6m(2m - 3)$
c $(x + 5)(4 - x)$ d $(a - 9)(a + 9)$
e $(4a - 11b)(4a + 11b)$ f $5(m - 5)(m + 5)$
g $(k - 5)(k + 9)$ h $(x - 3)(x + 1)$
i $(x - 2\sqrt{5})(x + 2\sqrt{5})$ j $(h + 3 - \sqrt{7})(h + 3 + \sqrt{7})$
k $(x + 5)(x + a)$ l $(x - 2m)(4x - 5)$

3 a $(x - 4)(x + 5)$ b $(a - 3)(a - 7)$
c $3(k - 9)(k + 2)$ d $(m - 6)^2$

4 a $x - 3$ b $\frac{x + 5}{2}$

5 a $(x + 4 + \sqrt{13})(x + 4 - \sqrt{13})$
b $(x - 6 + \sqrt{10})(x - 6 - \sqrt{10})$
c not possible
d $\left(x + \frac{5 + 3\sqrt{3}}{2}\right)\left(x + \frac{5 - 3\sqrt{3}}{2}\right)$

6 a $(3a + 2)(2a + 5)$ b $(2m - 3)(4m + 3)$
c $(3x - 2)(5x - 4)$ d $(2k - 7)(3k + 5)$

7 $\frac{2x + 5}{2x - 3}$

5F

Building understanding

1 a $0, -1$ b $0, 4$ c $3, -2$
d $\sqrt{3}, -\sqrt{3}$ e $\frac{1}{2}, -\frac{7}{3}$ f $-\frac{3}{8}, -\frac{3}{4}$

2 a $x^2 + 2x - 3 = 0$ b $x^2 - 5x + 6 = 0$
c $4x^2 + 4x - 3 = 0$ d $2x^2 - 6x - 5 = 0$
e $x^2 - 4x + 12 = 0$ f $3x^2 + 2x + 4 = 0$

3 a 2 b 1 c 2 d 2 e 1 f 1

Now you try

Example 15
a $x = 0, x = 3$ b $x = \pm\sqrt{11}$ c $x = \pm 3$

Example 16
a $x = -3$ or $x = 4$
b $x = -3$
c $x = -\frac{2}{3}$ or $x = \frac{1}{2}$

Example 17
a $x = -6$ or $x = 8$
b $x = -4$ or $x = 5$

Exercise 5F

1 a i $x = 0, 5$ ii $x = 0, 12$
b i $x = \pm\sqrt{13}$ ii $x = \pm\sqrt{19}$
c i $x = \pm 3$ ii $x = \pm 4$

2 a $x = 0, 4$ b $x = 0, 3$
c $x = 0, -2$ d $x = 0, 4$
e $x = 0, 5$ f $x = 0, -2$
g $x = \sqrt{7}, -\sqrt{7}$ h $x = \sqrt{11}, -\sqrt{11}$
i $x = \sqrt{5}, -\sqrt{5}$ j $x = 0, 2$
k $x = 0, -5$ l $x = 0, -\frac{1}{7}$
m $x = 2, -2$ n $x = 3, -3$
o $x = 6, -6$

3 a $x = -2, -1$ b $x = -3, -2$
c $x = 2, 4$ d $x = 5, 2$
e $x = -6, 2$ f $x = -5, 3$
g $x = 5, -4$ h $x = 8, -3$
i $x = 4, 8$ j $x = -2$
k $x = -5$ l $x = 4$
m $x = 7$ n $x = 12$
o $x = -9$

4 a $x = -\frac{3}{2}, -4$ b $x = -\frac{1}{2}, -\frac{7}{2}$ c $x = 5, \frac{7}{2}$
d $x = \frac{1}{2}, 11$ e $x = -\frac{5}{3}, 3$ f $x = -\frac{3}{5}, 2$
g $x = \frac{4}{3}, -\frac{5}{2}$ h $x = \frac{3}{7}, -4$ i $x = \frac{5}{4}, \frac{2}{5}$

5 a $x = -2, -6$ b $x = -1, 11$ c $x = 3$
d $x = 2$ e $x = \frac{3}{2}, -2$ f $x = \frac{2}{3}, \frac{5}{2}$

6 a $x = 6, -4$ b $x = 8, -4$ c $x = 3$
d $x = -2, -5$ e $x = 5, 3$ f $x = 3, -3$
g $x = 4, -4$ h $x = -1, -9$ i $x = 5, -1$
j $x = -5$ k $x = 8$ l $x = 8, -8$
m $x = 3, -1$ n $x = -\frac{2}{3}, -4$ o $x = -\frac{1}{4}, -\frac{3}{2}$

7 a $x = 12, -7$ b $x = -5, 14$ c $x = -9, 2$
d $x = \frac{5}{2}, -4$ e $x = -\frac{4}{5}, 2$ f $x = 2, -\frac{5}{6}$
g $x = -3, 1$ h $x = 1, \frac{1}{2}$ i $x = 3, -2$

8 a i $x = 1, -2$
ii $x = 1, -2$
b no difference
c $3x^2 - 15x - 18 = 3(x^2 - 5x - 6)$ and, as seen in part **a**, the coefficient of 3 makes no difference when solving.

9 This is a perfect square $(x + 8)^2$, which only has 1 solution; i.e. $x = -8$.

10 The student has applied the null factor law incorrectly; i.e. when the product does not equal zero. Correct solution is:
$x^2 - 2x - 8 = 7$
$x^2 - 2x - 15 = 0$
$(x - 5)(x + 3) = 0$
$x = 5$ or $x = -3$

11 a $x = -2, -1$ b $x = 1$ c $x = \frac{1}{2}, 5$
d $x = 8, -6$ e $x = -6, -2$ f $x = \frac{3}{2}, -4$
g $x = 8, -3$ h $x = 5, -3$ i $x = 2$
j $x = 4, -3$ k $x = 5, -2$ l $x = -5, 3$

ISBN 978-1-108-77290-7 Cambridge University Press
 Updated June 2022

5G

Building understanding

1 a Let x m be the width of the rectangle.
 b $x + 5$
 c $x(x + 5) = 24$
 d $x^2 + 5x - 24 = 0, x = -8, 3$
 e width = 3 m, length = 8 m
2 a width = 6 m, length = 10 m
 b width = 9 m, length = 7 m

Now you try

Example 18
width = 6 m, length = 8 m

Exercise 5G

1 width = 3 m, length = 4 m
2 width = 6 m, length = 9 m
3 height = 8 cm, base = 6 cm
4 height = 2 m, base = 7 m
5 8 and 9 or −9 and −8
6 12 and 14
7 15 m
8 a 6 b 13 c 14
9 1 m
10 father 64, son 8
11 5 cm
12 a 55
 b i 7 ii 13 iii 23
13 a 3.75 m
 b $t = 1$ second, 3 seconds
 c The ball will reach this height both on the way up and on the way down.
 d $t = 0$ seconds, 4 seconds
 e $t = 2$ seconds
 f The ball reaches a maximum height of 4 m.
 g No, 4 metres is the maximum height. When $h = 5$, there is no solution.
14 a $x = 0, 100$
 b The ball starts at the tee (i.e. at ground level) and hits the ground again 100 metres from the tee.
 c $x = 2$ m or 98 m
15 5 m × 45 m
16 150 m × 200 m

5H

Building understanding

1 a 1 b 100 c 4 d $\frac{25}{4}$
2 a $(x + \sqrt{3})(x - \sqrt{3}) = 0$
 b $(x + \sqrt{10})(x - \sqrt{10}) = 0$
 c $(x + 1 + \sqrt{5})(x + 1 - \sqrt{5}) = 0$
3 a $x = \sqrt{2}, -\sqrt{2}$
 b $x = \sqrt{7}, -\sqrt{7}$
 c $x = 3 - \sqrt{5}, 3 + \sqrt{5}$
 d $x = -5 - \sqrt{14}, -5 + \sqrt{14}$

Now you try

Example 19

a $x = 3 \pm \sqrt{7}$ b $x = -2 \pm 3\sqrt{2}$ c $x = \frac{5 \pm \sqrt{17}}{2}$

Exercise 5H

1 a i $x = 4 \pm \sqrt{13}$ ii $x = 6 \pm \sqrt{29}$
 b i $x = -2 \pm 2\sqrt{2}$ ii $x = -5 \pm 4\sqrt{2}$
2 a $x = -3 - \sqrt{6}, -3 + \sqrt{6}$
 b $x = -2 - \sqrt{2}, -2 + \sqrt{2}$
 c $x = -5 - \sqrt{10}, -5 + \sqrt{10}$
 d $x = -2 - \sqrt{6}, -2 + \sqrt{6}$
 e $x = -4 - \sqrt{19}, -4 + \sqrt{19}$
 f $x = -3 - \sqrt{14}, -3 + \sqrt{14}$
 g $x = 4 - \sqrt{17}, 4 + \sqrt{17}$
 h $x = 6 - \sqrt{39}, 6 + \sqrt{39}$
 i $x = 1 - \sqrt{17}, 1 + \sqrt{17}$
 j $x = 5 - \sqrt{7}, 5 + \sqrt{7}$
 k $x = 3 - \sqrt{5}, 3 + \sqrt{5}$
 l $x = 4 - \sqrt{7}, 4 + \sqrt{7}$
 m $x = -3 - \sqrt{13}, -3 + \sqrt{13}$
 n $x = -10 - \sqrt{87}, -10 + \sqrt{87}$
 o $x = 7 - \sqrt{55}, 7 + \sqrt{55}$
3 a $x = -4 - 2\sqrt{3}, -4 + 2\sqrt{3}$
 b $x = -3 - 2\sqrt{2}, -3 + 2\sqrt{2}$
 c $x = 5 - 2\sqrt{5}, 5 + 2\sqrt{5}$
 d $x = 2 - 3\sqrt{2}, 2 + 3\sqrt{2}$
 e $x = 5 - 2\sqrt{7}, 5 + 2\sqrt{7}$
 f $x = -4 - 2\sqrt{6}, -4 + 2\sqrt{6}$
 g $x = 1 - 4\sqrt{2}, 1 + 4\sqrt{2}$
 h $x = -6 - 3\sqrt{6}, -6 + 3\sqrt{6}$
 i $x = -3 - 5\sqrt{2}, -3 + 5\sqrt{2}$
4 a $x = \frac{-5 + \sqrt{17}}{2}, \frac{-5 - \sqrt{17}}{2}$
 b $x = \frac{-3 + \sqrt{5}}{2}, \frac{-3 - \sqrt{5}}{2}$
 c $x = \frac{-7 + \sqrt{29}}{2}, \frac{-7 - \sqrt{29}}{2}$
 d $x = \frac{3 + \sqrt{17}}{2}, \frac{3 - \sqrt{17}}{2}$
 e $x = \frac{1 + \sqrt{13}}{2}, \frac{1 - \sqrt{13}}{2}$
 f $x = \frac{-5 + \sqrt{33}}{2}, \frac{-5 - \sqrt{33}}{2}$
 g $x = \frac{7 + \sqrt{41}}{2}, \frac{7 - \sqrt{41}}{2}$
 h $x = \frac{9 + \sqrt{61}}{2}, \frac{9 - \sqrt{61}}{2}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

i $x = \frac{-1 + \sqrt{17}}{2}, \frac{-1 - \sqrt{17}}{2}$

j $x = \frac{-9 + 3\sqrt{5}}{2}, \frac{-9 - 3\sqrt{5}}{2}$

k $x = \frac{3}{2} + \sqrt{3}, \frac{3}{2} - \sqrt{3}$

l $x = -\frac{5}{2} + \sqrt{5}, -\frac{5}{2} - \sqrt{5}$

5 a 2 b 2 c 0 d 0
e 0 f 2 g 2 h 0
i 0 j 2 k 2 l 0

6 a No real solution. b $x = \frac{-5 \pm \sqrt{17}}{2}$
c $x = \frac{5 \pm \sqrt{17}}{2}$ d $x = \frac{-9 \pm \sqrt{69}}{2}$
e $x = \frac{-5 \pm \sqrt{21}}{2}$ f $x = 3 \pm \sqrt{5}$

7 a $x = \frac{-3 \pm \sqrt{29}}{2}$ b $x = \frac{-5 \pm \sqrt{61}}{2}$
c No real solutions. d $x = 4 \pm \sqrt{5}$
e $x = -5 \pm 2\sqrt{5}$ f No real solutions.

8 width $= \frac{-3 + \sqrt{89}}{2}$ cm, length $= \frac{3 + \sqrt{89}}{2}$ cm

9 a i 1.5 km ii 1.5 km
b i 0 km or 400 km iii 200 km
c $200 \pm 100\sqrt{2}$ km

10 a $x^2 + 4x + 5 = 0$
$(x + 2)^2 + 1 = 0$, no real solutions
b $\left(x - \frac{3}{2}\right)^2 + \frac{3}{4} = 0$, no real solutions

11 Factorise by quadratic trinomial; i.e. $(x + 6)(x - 5) = 0$, $6 \times (-5) = -30$, and $6 + (-5) = -1$.
Therefore, $x = -6, 5$.

12 a Use the dimensions of rectangle $BCDE$ and $ACDF$ and the corresponding side lengths in similar rectangles.
b $a = \frac{1 + \sqrt{5}}{2}$

13 a $x = -1 \pm \frac{\sqrt{6}}{2}$ b $x = -1 \pm \sqrt{5}$
c $x = 4 \pm \sqrt{11}$ d $x = \frac{3 \pm \sqrt{5}}{2}$
e $x = \frac{-5 \pm \sqrt{17}}{2}$ f $x = \frac{-1 \pm \sqrt{13}}{2}$

5I

Building understanding

1 a $a = 3, b = 2, c = 1$ b $a = 5, b = 3, c = -2$
c $a = 2, b = -1, c = -5$ d $a = -3, b = 4, c = -5$

2 a -8 b 49
c 41 d -44

3 a 1 b 0 c 2

Now you try

Example 20
a 2 solutions b no solutions c 1 solution

Example 21
a $x = \frac{-3 \pm \sqrt{5}}{2}$ b $x = \frac{1 \pm \sqrt{13}}{4}$

Exercise 5I

1 a 2 b 0 c 1

2 a 2 b 0 c 1 d 2
e 2 f 2 g 0 h 0
i 2 j 1 k 0 l 2

3 a $x = \frac{-3 \pm \sqrt{17}}{2}$ b $x = \frac{-7 \pm \sqrt{65}}{2}$
c $x = \frac{7 \pm \sqrt{29}}{2}$ d $x = 4$
e $x = -1, -4$ f $x = -1, -7$
g $x = \frac{-7 \pm \sqrt{65}}{8}$ h $x = \frac{-5 \pm \sqrt{37}}{6}$
i $x = \frac{2 \pm \sqrt{22}}{3}$ j $x = \frac{5 \pm \sqrt{65}}{4}$
k $x = -\frac{4}{3}, 1$ l $x = \frac{-3 \pm \sqrt{19}}{5}$

4 a $x = -2 \pm \sqrt{3}$ b $x = 3 \pm \sqrt{5}$
c $x = -3 \pm \sqrt{11}$ d $x = \frac{-3 \pm 3\sqrt{5}}{2}$
e $x = 2 \pm 2\sqrt{2}$ f $x = \frac{4 \pm \sqrt{10}}{3}$
g $x = \frac{1 \pm \sqrt{7}}{2}$ h $x = \frac{3 \pm 2\sqrt{3}}{3}$
i $x = \frac{4 \pm \sqrt{31}}{5}$

5 $\frac{-5 + \sqrt{105}}{2}$

6 a $x = \frac{3 \pm 2\sqrt{3}}{3}$ b $x = \frac{-2 \pm \sqrt{10}}{2}$
c $x = \frac{-5 \pm \sqrt{57}}{8}$ d $x = \frac{5 \pm \sqrt{17}}{4}$
e $x = \frac{-2 \pm \sqrt{13}}{3}$ f $x = 1 \pm \sqrt{6}$
g $x = \frac{1 \pm \sqrt{11}}{5}$ h $x = \frac{3 \pm \sqrt{41}}{4}$
i $x = \frac{5 \pm \sqrt{19}}{6}$

7 $\frac{3 + \sqrt{53}}{2}, \frac{-3 + \sqrt{53}}{2}$

8 $6\sqrt{2} + 10$ units

9 63 cm

10 When $b^2 - 4ac = 0$, the solution reduces to $x = \frac{-b}{2a}$; i.e. a single solution.

11 Answers will vary.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

12 $k = 6$ or -6

13 a i $k > 4$ ii $k = 4$ iii $k < 4$

b i $k > \frac{9}{8}$ ii $k = \frac{9}{8}$ iii $k < \frac{9}{8}$

c i $-2 < k < 2$

ii ± 2

iii $k > 2, k < -2$

d i no values

ii no values

iii all values of k

Problems and challenges

1 $b = -4, c = 1$

2 47

3 a $\pm 2, \pm 1$ b ± 3

4 a $x = 0, 1$ b $x = 1, -2$

5 144 cm^2

6 25 km/h

7 1.6 units

8 $x^2 - 2x + 2 = (x - 1)^2 + 1$, as $(x - 1)^2 \geqslant 0, (x - 1)^2 + 1 > 0$

9 Square area $-$ rectangle area $= \frac{(x - y)^2}{4} > 0$ for all x and y; hence, square area is greater than rectangle area.

10 $w : p = 1 : 3$; $t : q = 1 : 9$

Short-answer questions

1 a $-2x + 26$ b $3x^2 + 11x - 20$

c $25x^2 - 4$ d $x^2 - 12x + 36$

e $7x + 22$ f $12x^2 - 23x + 10$

2 a $x^2 + 4x + 4$

b $4x^2 + 18x$

c $x^2 + 3x + 21$

3 a $(x + 7)(x - 7)$ b $(3x + 4)(3x - 4)$

c $(2x + 1)(2x - 1)$ d $3(x + 5)(x - 5)$

e $2(x + 3)(x - 3)$ f $(x + \sqrt{11})(x - \sqrt{11})$

g $-2(x + 2\sqrt{5})(x - 2\sqrt{5})$ h $(x + 5)(x - 3)$

i $(x - 3 + \sqrt{10})(x - 3 - \sqrt{10})$

4 a $(x - 6)(x - 2)$ b $(x + 12)(x - 2)$

c $-3(x - 6)(x - 1)$

5 a $(3x + 2)(x + 5)$ b $(2x - 3)(2x + 5)$

c $(6x + 1)(2x - 3)$ d $(3x - 2)(4x - 5)$

6 a $\frac{2x}{x + 3}$ b $\frac{x - 4}{4}$

7 a $(x + 4 + \sqrt{6})(x + 4 - \sqrt{6})$

b $(x + 5 + \sqrt{29})(x + 5 - \sqrt{29})$

c $(x - 3 + 2\sqrt{3})(x - 3 - 2\sqrt{3})$

d $\left(x + \frac{3 + \sqrt{17}}{2}\right)\left(x + \frac{3 - \sqrt{17}}{2}\right)$

e $\left(x + \frac{5 + \sqrt{13}}{2}\right)\left(x + \frac{5 - \sqrt{13}}{2}\right)$

f $\left(x + \frac{7 + \sqrt{31}}{2}\right)\left(x + \frac{7 - \sqrt{31}}{2}\right)$

8 a $x = 0, -4$ b $x = 0, 3$

c $x = 5, -5$ d $x = 3, 7$

e $x = 4$ f $x = -9, 4$

g $x = -2, \frac{1}{2}$ h $x = \frac{2}{3}, -\frac{5}{2}$

i $x = \frac{1}{9}, -\frac{3}{2}$

9 a $x = 3, -3$ b $x = 5, -1$

c $x = 4, -7$ d $x = -3, 6$

10 length = 8 m, width = 6 m

11 a $x = -2 \pm \sqrt{7}$ b $x = 3 \pm 2\sqrt{2}$

c $x = \frac{3 \pm \sqrt{17}}{2}$ d $x = \frac{-5 \pm 3\sqrt{5}}{2}$

12 a 1 solution b 2 solutions

c 0 solutions d 2 solutions

13 a $x = \frac{-3 \pm \sqrt{33}}{2}$ b $x = 1 \pm \sqrt{5}$

c $x = \frac{2 \pm \sqrt{14}}{2}$ d $x = \frac{1 \pm \sqrt{37}}{6}$

Multiple-choice questions

1 D	2 B	3 C	4 A
5 B	6 D	7 C	8 C
9 E	10 C	11 A	12 B

Extended-response questions

1 a i $15 + 2x$ m

ii $12 + 2x$ m

b overall area $= 4x^2 + 54x + 180$ m^2

c trench area $= 4x^2 + 54x$ m^2

d Minimum width is 1 m.

2 a $S = 63\pi$ m^2

b 0.46 m

c i $420 = 3\pi r^2 + 12\pi r$

ii $3\pi r^2 + 12\pi r - 420 = 0$

iii $r = 4.97$ m; i.e. $\pi r^2 + 4\pi r - 140 = 0$.

Semester review 1

Linear relations

Short-answer questions

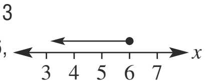
1 a $3 - 2x$ b 20

c $\frac{3a - 8}{4a}$ d $\frac{9x - 2}{(x + 2)(x - 3)}$

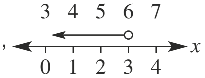
2 a i $x = -4$ ii $x = 2$

iii $x = 13$ iv $x = 2$

b i $x \leqslant 6$,



ii $x < 3$,

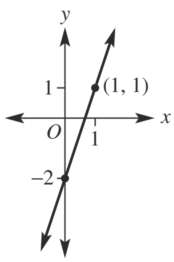


Answers 5I

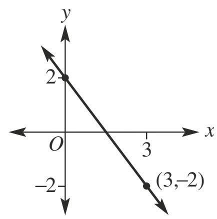
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

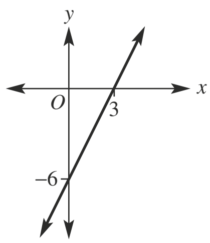
3 a i $m = 3, c = -2$



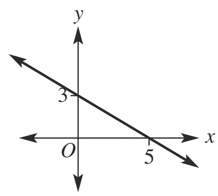
ii $m = -\frac{4}{3}, c = 2$



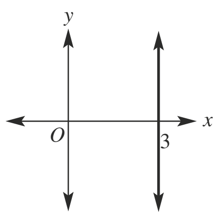
b i



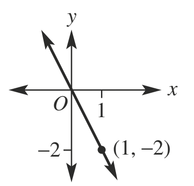
ii



iii



iv



4 a $y = -x + 3$ **b** $y = \frac{8}{5}x - \frac{9}{5}$

5 a $a = -3$ **b** $a = -4$

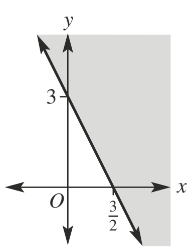
c $a = 1$ or $a = 7$ **d** $a = -4$

6 a $x = -3, y = -7$ **b** $x = -2, y = -4$

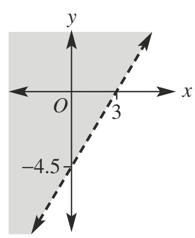
c $x = -1, y = 4$ **d** $x = 3, y = -5$

7 A hot dog costs \$3.50 a and soft drink \$2.

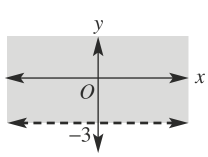
8 a



b



c



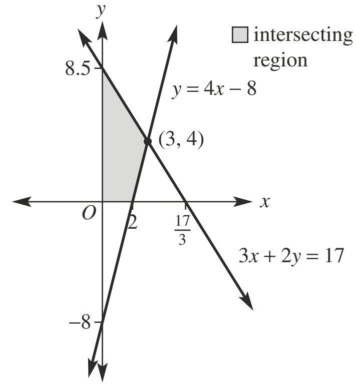
Multiple-choice questions

1 A **2** C **3** D
4 E **5** C

Extended-response question

a $x = 3, y = 4$

b, c



d 167 500 m^2

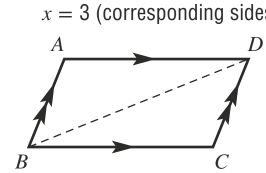
Geometry

Short-answer questions

1 a $AB = DE$ (given) $AC = DF$ (given)
$\angle BAC = 60° = \angle EDF$ (given).
$\therefore\ \triangle ABC \equiv \triangle DEF$ (SAS).
$a = 35$ (corresponding angles in congruent triangles)

b $BC = DC$ (given)
AC is common.
$\angle ABC = 90° = \angle ADC$ (given)
$\therefore\ \triangle ABC \equiv \triangle ADC$ (RHS).
$x = 3$ (corresponding sides in congruent triangles)

2



$\angle DBC = \angle BDA$ (alternate angles in parallel lines)
$\angle BDC = \angle DBA$ (alternate angles in parallel lines)
BD is common.
$\therefore\ \triangle BAD \equiv \triangle DCB$ (AAS).
Using congruence, $BC = AD$ and $AB = DC$, corresponding sides in congruent triangles.

3 a $x = 6.75$ **b** $x = 2$

4 a $x = 8$ **b** $x = 5$

c $a = 32, b = 65$ **d** $x = 40$

e $a = 55$ **f** $a = 90, b = 60, c = 70$

5 a $x = 20$ **b** $x = 8$

c $a = 63, b = 55$

6 a $x = \frac{47}{5}$ **b** $x = \frac{29}{3}$ **c** $x = \frac{39}{5}$

Multiple-choice questions

1 D **2** B **3** C
4 E **5** C

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

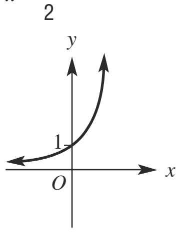
Extended-response question

a $CD = 6$ cm, chord theorem 2
b $OA = OD$ (radii of circle)
$OB = OC$ (radii of circle)
$AB = DC$ (chord theorem 2)
$\therefore \triangle OAB \equiv \triangle OCD$ (SSS).
c $OM = 4$ cm, area $= 12$ cm^2
d 30.6%
e $\angle BOD = 106.2°$

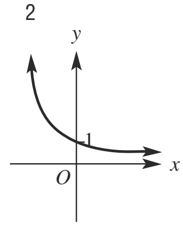
Indices and surds

Short-answer questions

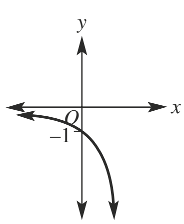
1 a $3\sqrt{6}$ b $20\sqrt{3}$ c $3\sqrt{6}$
d $\sqrt{10}$ e 21 f $48\sqrt{3}$
g $\sqrt{3}$ h $\frac{\sqrt{5}}{3}$ i $\frac{10\sqrt{2}}{7}$
2 a $7\sqrt{5} - \sqrt{7}$ b 0 c $-\sqrt{2} - 4$
3 a $2\sqrt{15} - 4\sqrt{3}$ b 45
4 a $\frac{3\sqrt{2}}{2}$ b $\frac{\sqrt{2}}{5}$ c $\frac{2\sqrt{5} - 5}{5}$
5 a $24x^{10}y^2$ b $3a^2b^2$ c $\frac{3b^2}{a^5}$ d $\frac{2x^2}{5y^3}$
6 a i 37 200 ii 0.0000049
b i 7.30×10^{-5} ii 4.73×10^9
7 a i $10^{\frac{1}{2}}$ ii $7^{\frac{1}{2}}x^3$, when $x > 0$
iii $4x^{\frac{3}{5}}$ iv $15^{\frac{3}{2}}$
b i $\sqrt{6}$ ii $\sqrt[5]{20}$
iii $\sqrt[4]{7^3}$ or $(\sqrt[4]{7})^3$
8 a $\frac{1}{5}$ b $\frac{1}{16}$ c 3 d $\frac{1}{2}$
9 a $x = 3$ b $x = 2$
c $x = \frac{3}{2}$ d $-\frac{1}{2}$
10 a



b



c



11 a \$2382.03 b \$7658.36

Multiple-choice questions

1 B 2 D 3 E 4 E 5 C

Extended-response question

a $V = 80\,000(1.08)^n$
b i \$86 400 ii \$108 839
c 11.91 years
d 6% per year

Trigonometry

Short-answer questions

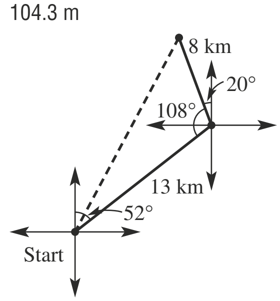
1 a $x = 19.5$ b $\theta = 43.8°, y = 9.4$
2 a i 150°T ii 330°T
b i 310°T ii 130°T
3 a 32.174 m b 52.2°
4 a $x = 9.8$ b $\theta = 125.3°$
5 95.1°
6 a $\frac{5\pi°}{18}$ b 135°
7 a $\tan\theta$
b i $\theta = 155°$ ii $\theta = 35°$ iii $\theta = 42°$
c i $\frac{1}{2}$ ii $\frac{\sqrt{2}}{2}$ iii $-\frac{\sqrt{3}}{3}$
8 a ≈ 0.34 b $\theta \approx 233°, 307°$ c yes

Multiple-choice questions

1 E 2 B 3 A
4 D 5 C

Extended-response question

a 104.3 m
b



c 17.242 km
d 206°T

Quadratic expressions and equations

Short-answer questions

1 a $9x^2 - 1$ b $4x^2 - 20x + 25$
c $-x^2 + 30x - 5$
2 a $(2x - y)(2x + y)$ b $(x + 2 + \sqrt{7})(x + 2 - \sqrt{7})$
c $3(x - 4)(x + 4)$ d $(x - 2)(x + 7)$
e $(x - 5)^2$ f $2(x - 6)(x - 2)$
3 a $(3x + 4)(x - 2)$
b $(3x - 1)(2x + 3)$
c $(5x - 4)(2x - 3)$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

4 a $x = 0, 3$ b $x = -4, \frac{1}{2}$ c $x = 0, -5$
d $x = 4, -4$ e $x = \sqrt{7}, -\sqrt{7}$ f $x = 2$
g $x = 8, -3$ h $x = -2, \frac{1}{3}$

5 a $x = -8, 5$ b $x = 3, 7$ c $x = -4, 5$

6 a i $(x - 3 + \sqrt{5})(x - 3 - \sqrt{5})$
ii $(x + 2)^2 + 3$, does not factorise further
iii $\left(x + \frac{3}{2} - \frac{\sqrt{5}}{2}\right)\left(x + \frac{3}{2} + \frac{\sqrt{5}}{2}\right)$
b i $x = 3 \pm \sqrt{5}$ ii no solutions
iii $x = \frac{-3 \pm \sqrt{5}}{2}$

7 a $x = \frac{-3 \pm \sqrt{57}}{4}$ b $x = 2 \pm \sqrt{10}$

Multiple-choice questions

1 C 2 B 3 D
4 B 5 D

Extended-response question

a $4x^2 + 40x$ b $44\ \text{m}^2$
c $x = 3$ d $x = 2.2$

Chapter 6

6A

Building understanding

1 a 40 mm b 9.6 cm c 1 m
d 8 m e 0.297 km f 510.2 cm

2 a $\frac{1}{4}$ b $\frac{1}{2}$ c $\frac{3}{4}$
d $\frac{1}{9}$ e $\frac{7}{24}$ f $\frac{11}{12}$

3 a 810 m b 9.4 km c 180 cm

Now you try

Example 1
a 20.2 m b 8.1 c $x = P - 13.2$

Example 2
a 21.99 cm b $r = \frac{C}{2\pi}$ c 1.91 cm

Example 3
a $10 + \frac{70\pi}{9}$ cm b 34.4 cm

Exercise 6A

1 a 36.6 cm b 5.1 cm c $x = P - 28.6$

2 a 21.8 m b 3.2 m c $x = P - 16.8$

3 a 43.98 cm b 7.54 m
c 89.22 mm d 3.46 km

4 a 75.40 m b $r = \frac{C}{2\pi}$ c 5.57 m

5 a i $8 + 4\pi$ m ii 20.6 m
b i $4 + \pi$ m ii 7.1 m
c i $2 + \frac{\pi}{3}$ km ii 3.0 km
d i $12 + 10\pi$ cm ii 43.4 cm
e i $10 + \frac{70\pi}{9}$ mm ii 34.4 mm
f i $6 + \frac{31\pi}{12}$ cm ii 14.1 cm

6 a 3 b 8.8 c 0.009
d 2.65 e 3.87 f 2.4

7 57.6 m

8 a 12.25 b 53.03 c 1.37
d 62.83 e 19.77 f 61.70

9 a 6π m b $10 + \frac{5\pi}{2}$ m c $\pi + 1$ km

10 a i 201 cm ii 1005 cm
b 4974
c $\frac{1\,000\,000}{\pi d}$

11 $r = \frac{2n}{\pi}$

12 $\pi\sqrt{2x}$

13 a $l = \frac{P - 2w}{2}$ or $\frac{1}{2}P - w$ b $l = 5 - w$
c $0 < w < 5$ d $0 < l < 5$

14 a 720° b 1080° c 540° d 1440°

6B

Building understanding

1 a $\sqrt{55}$ b $\sqrt{77}$
c $\sqrt{2}$ d $\sqrt{50} = 5\sqrt{2}$

2 a $x^2 + y^2 = z^2$ b $a^2 + d^2 = b^2$ c $2x^2 = c^2$

Now you try

Example 4
a 8.06 cm b 1.92 m

Example 5
a $\sqrt{61}$ b 10.49

Exercise 6B

1 a i 2.24 cm ii 10.77 m
b i 1.70 m ii 16.10 m

2 a 5 cm b 11.18 m c 16.55 km
d 1.81 mm e 0.43 km f 77.10 cm

3 a 4.58 m b 7.94 m c 0.63 m
d 1.11 cm e 14.60 cm f 0.09 cm

4 a i $\sqrt{34}$ ii 6.16
b i $\sqrt{80}$ (or $4\sqrt{5}$) ii 16.61
c i $\sqrt{10}$ ii 7.68
d i $\sqrt{89}$ ii 13.04

5 a no b yes c no
d no e yes f yes

ISBN 978-1-108-77290-7

6 a 2.86 m b 2.11 cm c 26.38 m
 d 4.59 cm e 0.58 km f 1.81 km
7 8.3 cm
8 a 13.19 mm b 13.62 m c 4.53 cm
 d 2.61 m e 12.27 km f 5.23 cm
9 a $2\sqrt{13}$ b $4\sqrt{2}$ c $\sqrt{181}$
10 a i 22.4 cm ii 24.5 cm
 b Investigation required.
11 $\frac{\sqrt{5}}{2}$ cm, using Pythagoras' theorem given that an angle in a semicircle is 90°.
12 a $4\sqrt{5}$ cm by $2\sqrt{5}$ cm b $3\sqrt{10}$ cm by $\sqrt{10}$ cm
 c $\sqrt{\frac{100}{101}}$ cm by $10\sqrt{\frac{100}{101}}$ cm = $\frac{10\sqrt{101}}{101}$ cm by $\frac{100\sqrt{101}}{101}$ cm
13 a i 5.41 m ii 4.61 m iii 5.70 m
 iv 8.70 m v 8.91 m vi 6.44 m
 b 7.91 m
14 Research required

6C

Building understanding

1 a πr^2 b $\frac{\theta}{360°} \times \pi r^2$ c l^2
 d $l \times w$
 e $\frac{1}{2}xy$, where x and y are the diagonals.
 f $\frac{1}{2}(b + l)h$ g $\frac{1}{2}bh$ h $\frac{1}{2}xy$
 i bh j $\frac{1}{2}\pi r^2$ k $\frac{1}{4}\pi r^2$
2 a i 10 ii 100
 b i 100 ii 10 000
 c i 1000 ii 1 000 000
 d 10 000

Now you try

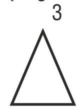
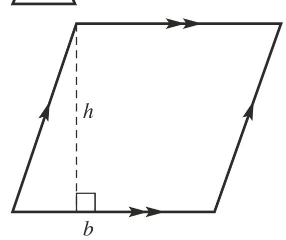
Example 6
a 35 000 cm^2 b 0.05 m^2

Example 7
a 12 m^2 b 1.43 cm^2 c 12.01 km^2

Example 8
a 2.7 b 2.16

Example 9
a $\frac{26\pi}{9} \approx 9.08$ cm^2 b $60 - \frac{9\pi}{2} \approx 45.86$ m^2

Exercise 6C

1 a i 150 mm^2 ii 50 000 cm^2 iii 200 000 m^2
 b i 0.0007 km^2 ii 0.45 m^2 iii 0.006 km^2
2 a 30 cm^2 b 2.98 m^2 c 0.205 km^2
 d 5000 cm^2 e 5 000 000 m^2 f 100 m^2
 g 230 cm^2 h 53 700 mm^2 i 2700 m^2
 j 10 000 000 mm^2 k 2 200 000 cm^2 l 0.000 145 km^2
3 a 25 cm^2 b 54.6 m^2 c 1.82 km^2
 d 0.025 mm^2 e 153.94 m^2 f 75 cm^2
 g 1472 m^2 h 0.05 mm^2 i 0.17 km^2
 j 2.36 km^2 k 1.1234 m^2 l 3.97 cm^2
4 a 2.88 b 14.35 c 1.44
 d 1.05 e 1.91 f 8.89
 g 1.26 h 0.52 i 5753.63
5 a 9π cm^2, 28.27 cm^2
 b $\frac{25}{2}\pi$ m^2, 39.27 m^2
 c $\frac{49}{3}\pi$ m^2, 51.31 m^2
 d $\frac{26}{9}\pi$ m^2, 9.08 m^2
 e 21π km^2, 65.97 km^2
 f $\frac{7}{8}\pi$ mm^2, 2.75 mm^2
6 43.2 m^2
7 a $\frac{25}{8}\pi + 25$ cm^2, 34.82 cm^2
 b 49 m^2
 c $\frac{289}{200}\pi + \frac{104}{25}$ m^2, 8.70 m^2
 d $\frac{(3969 - 441\pi)}{25}$ mm^2, 103.34 mm^2
 e $81\pi + 324$ km^2, 578.47 km^2
 f $\frac{49}{200}\pi - \frac{99}{400}$ m^2, 0.52 m^2
8 a 66 m^2 b 27 bags
9 a 100 ha b 200 000 m^2
 c 0.4 ha d 2.5 acres
10 a $a = \frac{2A}{h} - b$
 b i $3\frac{1}{3}$ ii 4.7 iii 0
 c  a triangle
11 a 

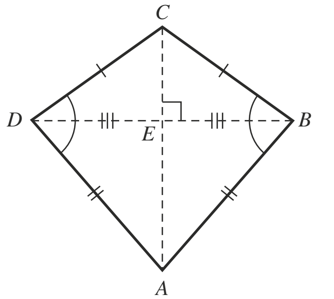
Let x be the base of each triangle.

$A = (b - x) \times h + \frac{1}{2}xh + \frac{1}{2}xh$
(i.e. rectangle and two triangles)
$A = bh - xh + xh$
$A = bh$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b



Let $x = AC$ and $y = BD$.

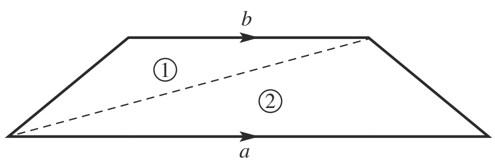
AC bisects BD, hence $DE = EB = \frac{1}{2}y$.

$A = \frac{1}{2} \times x \times \frac{1}{2}y + \frac{1}{2} \times x \times \frac{1}{2}y$
(i.e. area of $\triangle ACD$ plus area of $\triangle ABC$)

$A = \frac{1}{4}xy + \frac{1}{4}xy$

$A = \frac{1}{2}xy$

c Consider the following trapezium.



$A =$ Area ① + Area ②

$A = \frac{1}{2} \times a \times h + \frac{1}{2} \times a \times h$

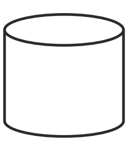
$A = \frac{1}{2}(a + b)h$

12 **a** 63.7% **b** 78.5% **c** 50% **d** 53.9%

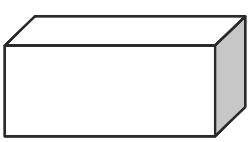
6D

Building understanding

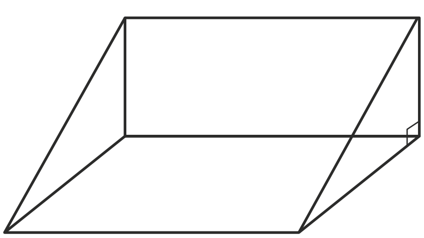
1 **a**



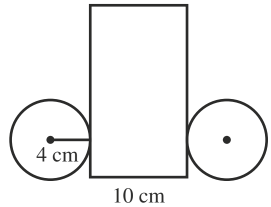
b



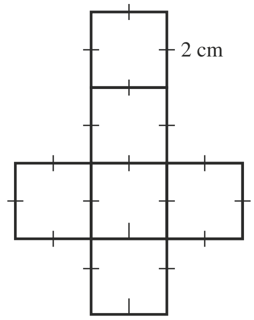
c



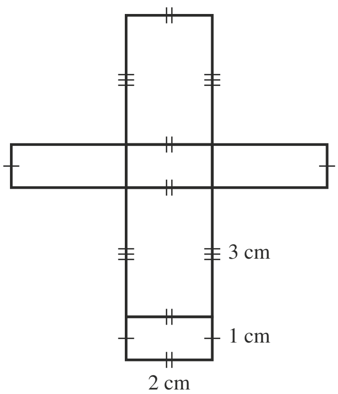
2 **a**



b



c



Now you try

Example 10
a 94 m^2 **b** 95.06 cm^2

Example 11
101.7 m^2

Exercise 6D

1 **a** 144 m^2 **b** 72.57 cm^2
2 **a** 90 cm^2 **b** 47.82 mm^2 **c** 111.3 cm^2
d 920 m^2 **e** 502.91 m^2 **f** 168.89 m^2
3 **a** 8.64 cm^2 **b** 96 mm^2 **c** 836.6 m^2
d 688 mm^2 **e** 4.74 cm^2 **f** 43.99 m^2
4 24.03 m^2
5 3880 cm^2
6 **a** 121.3 cm^2 **b** 10.2 m^2
c 236.5 m^2 **d** 2437.8 cm^2
7 **a** 66.2 **b** 17.9 **c** 243.1
d 207.3 **e** 2308.7 **f** 65.0
8 **a** 144.5 cm^2 **b** 851.3 m^2
c 1192.7 cm^2 **d** 4170.8 m^2
9 33.5 m^2

ISBN 978-1-108-77290-7

10 a $6x^2$ b $2(ab + ac + bc)$
c $\pi\left(\frac{1}{2}d\right)^2 + \frac{1}{2}\pi dh + dh$ d $\frac{1}{2}\pi r^2 + 2rh + \frac{1}{2}\pi rh$

11 a 6π b $\frac{11\pi}{2}$

12 a 0.79 m b 7.71 m

13 1 cm

14 a $4\pi r^2$ b $2x(x + 2y)$
c $2rh + \pi r(h + r)$ d $2rh + \frac{\theta}{180°}\pi r(h + r)$

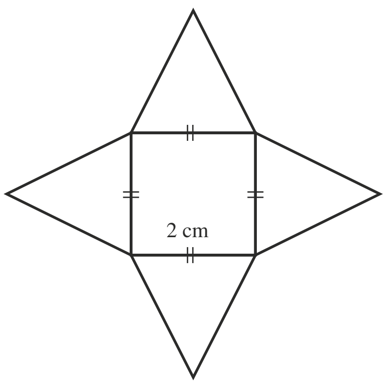
6E

Building understanding

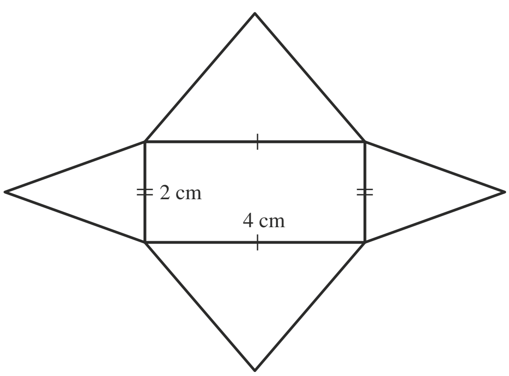
1 a $\frac{1}{2}bh$ b πr^2 c πrs

2 a $\sqrt{29}$ cm b $\sqrt{221}$ m c $\sqrt{109}$ cm

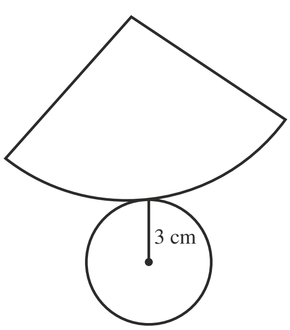
3 a



b



c



Now you try

Example 12
a 81.05 m^2 b 1160 mm^2

Example 13
a 12.7 cm b 12.6 cm

Exercise 6E

1 a 144.51 m^2 b 2100 mm^2

2 a 593.76 mm^2 b 0.82 m^2 c 435.90 km^2

3 a 64 m^2 b 105 cm^2 c 0.16 m^2

4 a 62.83 m^2 b 5.18 cm^2 c 1960.35 mm^2

5 a 10.44 cm b 126.7 cm^2

6 a 25.5 cm b 25.0 cm

7 a 18.9 cm b 17.8 cm

8 hat B

9 a 6.3 m b 66.6 m^2

10 a 105 cm^2 b 63 cm^2
c 163.3 cm^2 d 299.4 m^2
e 502.8 mm^2 f 76.6 m^2

11 Slant height, $s = \sqrt{r^2 + h^2}$,
so $\pi r(r + s) = \pi r(r + \sqrt{r^2 + h^2})$

12 Substitute $h = r$ into the equation given in Question **11**.
$$\begin{aligned}\pi r(r + \sqrt{r^2 + h^2}) &= \pi r(r + \sqrt{r^2 + r^2})\\ &= \pi r(r + \sqrt{2r^2})\\ &= \pi r(r + \sqrt{2}r)\\ &= \pi r^2(1 + \sqrt{2}) \text{ as required}\end{aligned}$$

13 182.3 cm^2

14 a $4\sqrt{26}$ cm b 306.57 cm^2 c $4\sqrt{2}$ cm
d 20.199 cm e 260.53 cm^2 f 85%

Progress quiz

1 a 36 cm b 26.85 cm c 30 cm

2 a i 5 ii 30 cm iii 30 cm^2
b i 41 ii 90 cm iii 180 cm^2
c i 6.40 ii 31.40 cm iii 50 cm^2

3 a $C = 25.13$ mm $A = 50.27$ mm^2
b $C = 55.29$ mm $A = 243.28$ mm^2

4 a 17.45 cm^2 b 29.32 cm^2

5 a 450 b 0.00045

6 a 3.86 cm^2 b 42.06 cm^2 c 28.54 cm^2

7 a 158 cm^2 b 2.12 m^2
c 434.29 cm^2 d 175.18 cm^2

8 a 58.90 cm^2 b 2.5 cm c 7.07 cm

9 13.75 cm

6F

Building understanding

1 a 80 cm^3 b 32 m^3 c 108 mm^3

2 a $V = abc$ b $V = y^2h$ c $V = \pi r^2h$

Now you try

Example 14
a 240 cm^3 b 50.27 m^3

Answers

6D

ISBN 978-1-108-77290-7

Example 15
67.5 cm^3
Example 16
393.1 cm^3

Exercise 6F

1 a 32 m^3 b 141.37 cm^3
2 a 2000 mm^3 b 200 000 cm^3 c 15 000 000 m^3
d 5.7 cm^3 e 0.0283 km^3 f 0.762 m^3
g 130 000 cm^3 h 1000 m^3 i 2094 mm^3
j 2700 mL k 0.342 ML l 0.035 kL
m 5720 kL n 74.25 L o 18 440 L
3 a 40 cm^3 b 10 500 m^3 c 259.7 mm^3
4 a 785.40 m^3 b 18.85 cm^3 c 1583.36 m^3
5 a 12 cm^3 b 1570.8 m^3 c 2.448 mm^3
6 a 30 km^3 b 196 cm^3 c 30 m^3
d 10 cm^3 e 0.002 m^3 f 4752.51 cm^3
g 0.157 m^3 h 1357.168 cm^3 i 24 m^3
7 1000
8 480 L
9 a 379.33 cm^3 b 223.17 m^3 c 6.808 m^3
d 716.46 mm^3 e 142.36 cm^3 f 42.85 cm^3
10 a 27 cm^3 b $3\sqrt{3}$ m^3
11 0.5 cm
12 He needs to use the perpendicular height of the oblique prism instead of 5.
13 $V = \frac{\theta}{360°}\pi r^2 h$
14 yes; 69.3 m^3
15 a $\frac{1}{\sqrt{2}}$ m b 5.8 m^3

6G

Building understanding

1 4 cm^3
2 15 m^3
3 a 10 m^3 b $\frac{8}{3}$ cm^3 c $58\frac{1}{3}$ mm^3

Now you try

Example 17
a 19.25 m^3 b 1119.19 mm^3

Exercise 6G

1 a 77 m^3 b 1588.86 mm^3
2 a 4 cm^3 b 585 m^3 c 50 km^3
d $\frac{8}{3}$ cm^3 e 8 cm^3 f 0.336 mm^3
3 a 0.82 m^3 b 9.38 mm^3 c 25 132.74 m^3
d 25.13 m^3 e 0.12 m^3 f 523.60 cm^3
4 47 mL
5 a 282.74 m^3 b 276 cm^3 c 48 m^3
d 56.88 mm^3 e 10.3488 m^3 f 70.79 m^3
6 4.76 cm
7 $\frac{2}{3}$
Wood wasted = volume of cylinder − volume of cone
Wood wasted = $\pi r^2 h - \frac{1}{3}\pi r^2 h$
Wood wasted = $\frac{2}{3}\pi r^2 h$
Wood wasted = $\frac{2}{3}$ of the volume of cylinder
8 a i $V = \frac{1}{3}x^2 h$ ii $V = \frac{1}{12}\pi x^2 h$
b $\frac{\pi}{4}$
9 a 3.7 cm
b i $h = \frac{3V}{\pi r^2}$ ii $r = \sqrt{\frac{3V}{\pi h}}$
10 a Similar triangles are formed so corresponding sides are in the same ratio.
b $\frac{1}{3}\pi(r_1^2 h_1 - r_2^2 h_2)$
c i 18.3 cm^3 ii 14.7 cm^3

6H

Building understanding

1 a 314.16 b 3.14
c 91.95 d 1436.76
2 $r = \sqrt{\frac{3}{\pi}}$
3 $r = \sqrt[3]{\frac{6}{\pi}}$
4 a $\frac{1}{2}$ b $\frac{1}{8}$ c $\frac{1}{4}$

Now you try

Example 18
TSA = 314.16 cm^2 V = 523.60 cm^3
Example 19
1.13 m
Example 20
a TSA = 128.33 cm^2
b V = 122.52 cm^3

Exercise 6H

1 TSA = 201.06 cm^2 V = 268.08 cm^3
2 a 50.27 cm^2, 33.51 cm^3
b 3.14 m^2, 0.52 m^3
c 18145.84 mm^2, 229 847.30 mm^3
d 1017.88 cm^2, 3053.63 cm^3
e 2.66 km^2, 0.41 km^3
f 5.81 m^2, 1.32 m^3
3 a 113.10 cm^2, 113.10 cm^3 b 201.06 m^2, 268.08 m^3
c 688.13 m^2, 1697.40 m^3 d 15.71 mm^2, 5.85 mm^3
e 21.99 m^2, 9.70 m^3 f 15.21 km^2, 5.58 km^3

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

4 a i 1.53 cm ii 3.50 cm iii 0.50 km
b i 0.89 m ii 3.09 cm iii 0.18 mm
5 a 113.10 cm^3 b 5654.9 cm^3 c 21 345.1 cm^3
6 11.5 cm
7 52%
8 a 32.72 cm^3 b 67.02 cm^3 c 0.52 m^3
9 1570.8 cm^2
10 a 4 m b 234.6 m^3
11 a 235.62 m^2 b 5.94 cm^2 c 138.23 mm^2
d 94.25 m^2 e 27.14 m^2 f 26.85 cm^2
12 a 5.24 m^3 b 942.48 m^3 c 10.09 cm^3
d 1273.39 cm^3 e 4.76 m^3 f 0.74 cm^3
13 a i 523.60 cm^3 ii 4188.79 cm^3
iii 14 137.17 cm^3
b 61.2 cm
14 a 5 cm b $5\sqrt{5}$ cm c 332.7 cm^2
15 a $r = \sqrt{\frac{S}{4\pi}}$ b $r = \sqrt[3]{\frac{3V}{4\pi}}$
16 a 4 times b 8 times
17 $V = \frac{4}{3} \times \pi r^3$

Substitute $\frac{d}{2}$ into r, giving:

$V = \frac{4}{3} \times \pi\left(\frac{d}{2}\right)^3$

$V = \frac{4}{3} \times \frac{\pi d^3}{8} = \frac{1}{3} \times \frac{\pi d^3}{2}$

$V = \frac{1}{6}\pi d^3$

18 $h = \frac{4}{3}r$
19 a i $\sqrt[3]{\frac{3}{4\pi}}$ ii $\sqrt[3]{36\pi}$ iii 1
iv 6 v 80.6%
b i $4\pi r^2$ ii $x = \sqrt[3]{\frac{4\pi}{3}}r$ iii $6\left(\frac{4\pi}{3}\right)^{\frac{2}{3}}r^2$
c Proof required. Example:

$$\frac{4\pi r^2}{6\left(\frac{4\pi}{3}\right)^{\frac{2}{3}}r^2} = \frac{2\pi}{3^{\frac{1}{3}}(4\pi)^{\frac{2}{3}}} = \frac{2\pi^{\frac{1}{3}}}{8^{\frac{1}{3}} \times 6^{\frac{1}{3}}} = \sqrt[3]{\frac{\pi}{6}}, \text{ as required.}$$

d They are the same.

6I

Building understanding

1 Some examples are 3.35, 3.37, 3.40 and 3.42.
2 a 347 cm b 3 m
3 6.65

Now you try

Example 21
a 44.5 cm to 45.5 cm
b 15.65 mm to 15.75 mm

Example 22
a 8.5 cm to 9.5 cm
b 34 cm to 38 cm
c 72.25 cm^2 to 90.25 cm^2

Exercise 6I

1 a i 54.5 cm to 55.5 cm
ii 31.5 cm to 32.5 cm
b i 32.25 mm to 32.35 mm
ii 108.35 mm to 108.45 mm
2 a i 1 cm ii 44.5 cm to 45.5 cm
b i 0.1 mm ii 6.75 mm to 6.85 mm
c i 1 m ii 11.5 m to 12.5 m
d i 0.1 kg ii 15.55 kg to 15.65 kg
e i 0.1 g ii 56.75 g to 56.85 g
f i 1 m ii 9.5 m to 10.5 m
g i 1 h ii 672.5 h to 673.5 h
h i 0.01 m ii 9.835 m to 9.845 m
i i 0.01 km ii 12.335 km to 12.345 km
j i 0.001 km ii 0.9865 km to 0.9875 km
k i 0.01 L ii 1.645 L to 1.655 L
l i 0.01 mL ii 9.025 mL to 9.035 mL
3 a 4.5 m to 5.5 m b 7.5 cm to 8.5 cm
c 77.5 mm to 78.5 mm d 4.5 mL to 5.5 mL
e 1.5 km to 2.5 km f 34.15 cm to 34.25 cm
g 3.85 kg to 3.95 kg h 19.35 kg to 19.45 kg
i 457.85 L to 457.95 L j 18.645 m to 18.655 m
k 7.875 km to 7.885 km l 5.045 s to 5.055 s
4 a \$4450 to \$4550
b \$4495 to \$4505
c \$4499.50 to \$4500.50
5 a 30 m b 15 g
c 4.6 km d 9.0 km
e 990 g f 990 g (nearest whole)
6 a 149.5 cm to 150.5 cm b 145 cm to 155 cm
c 149.95 cm to 150.05 cm
7 a 24.5 cm to 25.5 cm b 245 cm
c 255 cm
8 a 9.15 cm
b 9.25 cm
c 36.6 cm to 37 cm
d 83.7225 cm^2 to 85.5625 cm^2
9 a 9.195 cm
b 9.205 cm
c 36.78 cm to 36.82 cm
d 84.548025 cm^2 to 84.732025 cm^2
e Increasing the level of accuracy lowers the difference between the upper and lower limits of any subsequent working.
10 a Different rounding (level of accuracy being used)
b Cody used to the nearest kg, Jacinta used to the nearest 100 g and Luke used to the nearest 10 g.
c yes

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

11 a Distances on rural outback properties, distances between towns, length of wires and pipes along roadways
b building plans, measuring carpet and wood
c giving medicine at home to children, paint mixtures, chemical mixtures by students
d buying paint, filling a pool, recording water use

12 a $\pm 1.8\%$ b $\pm 5.6\%$ c $\pm 0.56\%$
d $\pm 0.056\%$ e $\pm 0.28\%$ f $\pm 0.056\%$
g $\pm 0.12\%$ h $\pm 0.071\%$

Problems and challenges

1 6
2 1.3 m
3 a As the sphere touches the top, bottom and curved surface, the height of the cylinder is $2r$, and the radius of the base is r. So the curved surface area $= 2 \times \pi \times r \times h$ and $h = 2r$, therefore this equals $4\pi r^2$, which is equal to the surface area of the sphere.
b 67%
4 $h = 4r$
5 $(4 - \pi)r^2$
6 $\sqrt{2} : 1$

Short-answer questions

1 a 23 cm b 2.7 cm^2 c 2 600 000 cm^3
d 8372 mL e 0.63825 m^2 f 3 000 000 cm^2
2 a 32 m b 28.6 m c 20.4 cm
3 a $\frac{7}{\pi}$ m b 15.60 m^2
4 a $\sqrt{65}$ b 8.31
5 a 16.12 m^2 b 216 m^2 c 38.5 m^2
d 78.54 cm^2 e 100.43 m^2 f 46.69 m^2
6 a 4.8 m b 25.48 m
7 a i 236 m^2 ii 240 m^3
b i 184 cm^2 ii 120 cm^3
c i 1407.43 cm^2 ii 4021.24 cm^3
d i 360 cm^2 ii 400 cm^3
e i 201.06 m^2 ii 268.08 m^3
f i 282.74 cm^2 ii 314.16 cm^3
8 a $\frac{175}{3\pi}$ cm b 17.6 cm
9 a 18 cm b $3\sqrt{61}$ cm c 2305.8 cm^2
10 12 m
11 a i 414.25 cm^2 ii 535.62 cm^3
b i 124 m^2 ii 88 m^3
c i 19.67 mm^2 ii 6.11 mm^3
12 a i 117.27 cm^2 ii 84.94 cm^3
b i 104 cm^2 ii 75 cm^3
c i 25.73 cm^2 ii 9.67 cm^3
13 a 4950π cm^3 b 1035π cm^2
14 a 7.5 m to 8.5 m
b 10.25 kg to 10.35 kg
c 4.745 L to 4.755 L

Multiple-choice questions

1 D 2 E 3 A 4 D 5 C 6 A
7 B 8 D 9 E 10 C 11 D 12 E

Extended-response questions

1 a 72 m^3 b $\sqrt{37}$ m
c 138.7 m^2 d 6 L, \$120
2 a 100 m b $50\sqrt{2}$ m
c 5000 m^2 d 36%
e Athlete A, 0.01 seconds

Chapter 7

7A

Building understanding

1 a maximum b $(-2, 4)$ c 2
d $-5, 1$ e $x = -2$
2 a minimum b $(1, -3)$ c -2
d $-1, 3$ e $x = 1$

Now you try

Example 1
a i minimum at $(-1, -3)$ ii $x = -1$
iii -3 and 1 iv -2
b i maximum at $(2, 0)$ ii $x = 2$
iii 2 iv -2

Example 2

	Formula	Maximum or minimum	Reflecfed in the x-axis (yes/no)	Turning point	y-value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = \frac{1}{2}x^2$	minimum	no	(0, 0)	$\frac{1}{2}$	wider
b	$y = (x - 2)^2$	minimum	no	(2, 0)	1	same
c	$y = -x^2 - 1$	maximum	yes	(0, −1)	−2	same

Exercise 7A

1 a i $(2, -5)$, min ii $x = 2$
iii $-1, 5$ iv -3
b i $(2, 0)$, max ii $x = 2$
iii 2 iv -1
c i $(2, 5)$, min ii $x = 2$
iii no x-intercept iv 7
d i $(-3, 0)$, min ii $x = -3$
iii -3 iv 4
e i $(2, -2)$, min ii $x = 2$
iii 1, 3 iv 6
f i $(0, 3)$, max ii $x = 0$
iii $-3, 3$ iv 3

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2

	Formula	Max or min	Reflected in the x-axis (yes/no)	Turning point	y-value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = 3x^2$	min	no	(0, 0)	$y = 3$	narrower
b	$y = \frac{1}{2}x^2$	min	no	(0, 0)	$y = \frac{1}{2}$	wider
c	$y = 2x^2$	min	no	(0, 0)	$y = 2$	narrower
d	$y = -4x^2$	max	yes	(0, 0)	$y = -4$	narrower
e	$y = -\frac{1}{3}x^2$	max	yes	(0, 0)	$y = -\frac{1}{3}$	wider
f	$y = -2x^2$	max	yes	(0, 0)	$y = -2$	narrower

3

	Formula	Turning point	Axis of symmetry	y-intercept	x-intercept
a	$y = (x + 3)^2$	(–3, 0)	$x = -3$	9	–3
b	$y = (x - 1)^2$	(1, 0)	$x = 1$	1	1
c	$y = (x - 2)^2$	(2, 0)	$x = 2$	4	2
d	$y = (x + 4)^2$	(–4, 0)	$x = -4$	16	–4

4

	Formula	Turning point	y-intercept	y-value when $x = 1$
a	$y = x^2 + 3$	(0, 3)	3	$y = 4$
b	$y = x^2 - 1$	(0, –1)	–1	$y = 0$
c	$y = x^2 + 2$	(0, 2)	2	$y = 3$
d	$y = x^2 - 4$	(0, –4)	–4	$y = -3$

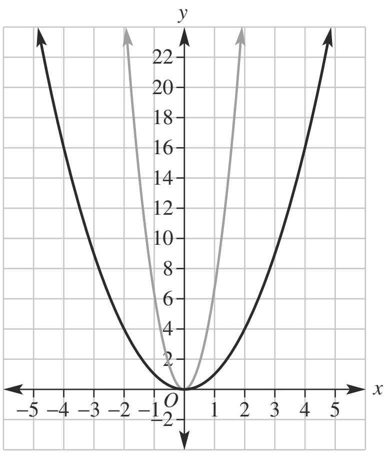
5 **a** $x = 0$ **b** $x = 0$ **c** $x = 0$
d $x = 0$ **e** $x = 0$ **f** $x = 2$
g $x = -1$ **h** $x = -3$ **i** $x = 0$
j $x = 0$ **k** $x = 0$ **l** $x = -4$

6 **a** (0, 0) **b** (0, 7) **c** (0, 0)
d (0, 0) **e** (0, –4) **f** (2, 0)
g (–1, 0) **h** (–3, 0) **i** (0, –3)
j (0, 2) **k** (0, –16) **l** (–4, 0)

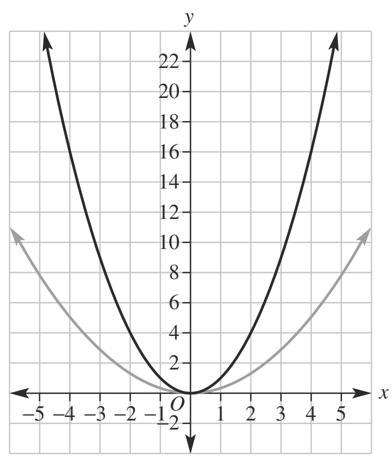
7 **a** 0 **b** 7 **c** 0
d 0 **e** –4 **f** 4
g 1 **h** –9 **i** –3
j 2 **k** –16 **l** –16

8 **a** H **b** C **c** G
d D **e** A **f** E
g B **h** F

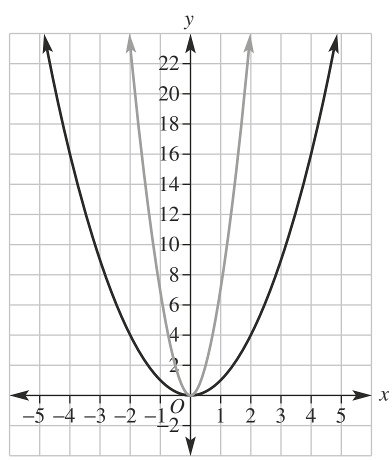
9 **a** **i**



ii



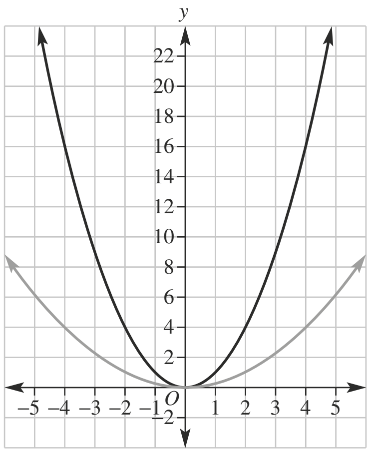
iii



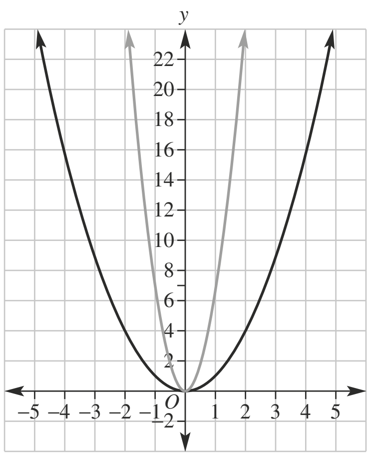
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

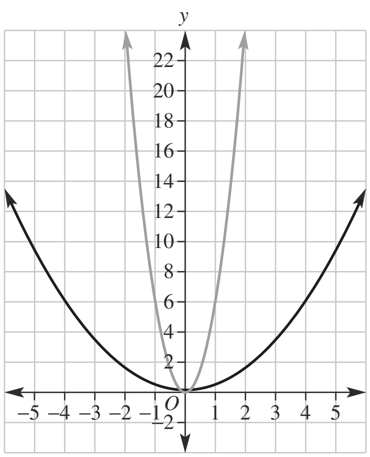
iv



v

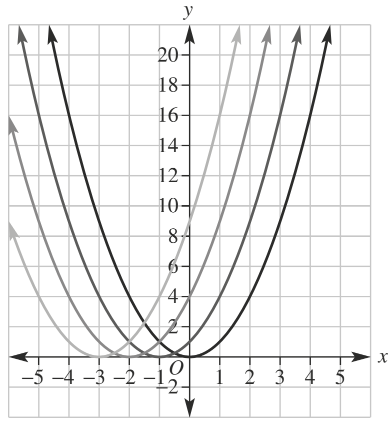


vi

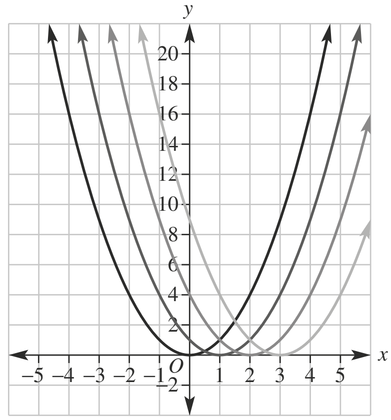


b The constant a determines the narrowness of the graph.

10 a i

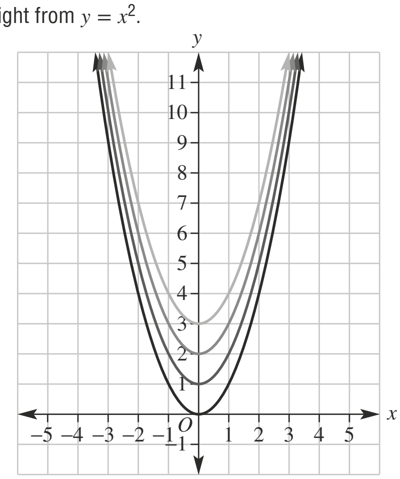


ii

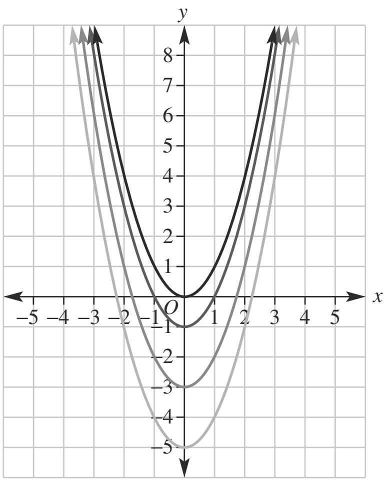


b The constant h determines whether the graph moves left or right from $y = x^2$.

11 a i



ii



b The constant k determines whether the graph moves up or down from $y = x^2$.

12 Answers could be:

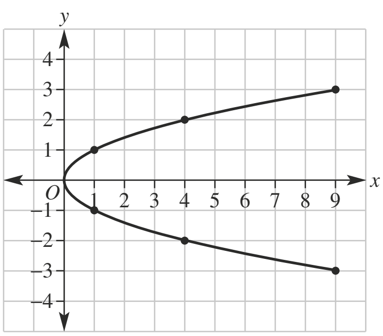
a $y = x^2 - 4$ **b** $y = (x - 5)^2$ **c** $y = x^2 + 3$

13 a $y = x^2 + 2$ **b** $y = -x^2 + 2$ **c** $y = (x + 1)^2$

d $y = (x - 2)^2$ **e** $y = 2x^2$ **f** $y = -3x^2$

g $y = (x + 1)^2 + 2$ **h** $y = \frac{1}{8}(x - 4)^2 - 2$

14 Parabola on its side.



7B

Building understanding

1 a (0, 0) **b** (0, 3) **c** (0, –4)

d (2, 0) **e** (–5, 0) **f** (0, 0)

2 a 5 **b** –3 **c** 4 **d** 2

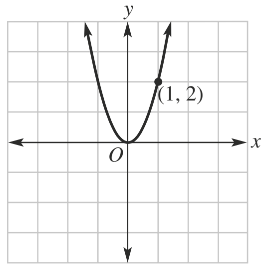
3 a up **b** right **c** left **d** down

e down **f** left **g** right **h** up

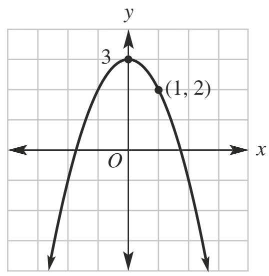
Now you try

Example 3

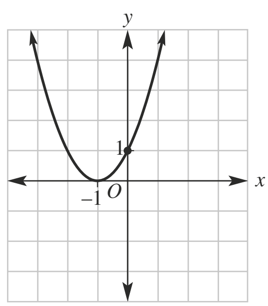
a



b

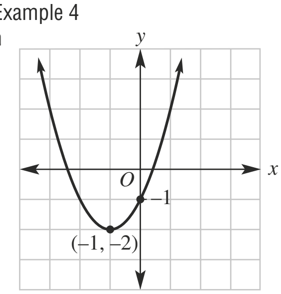


c

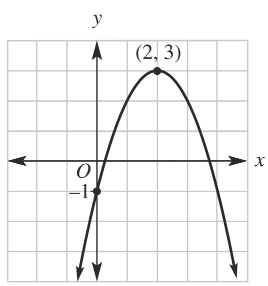


Example 4

a



b



Example 5

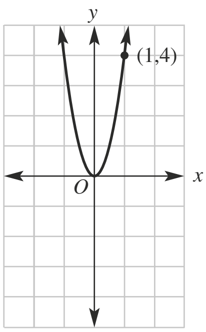
$y = 2x^2 - 1$

ISBN 978-1-108-77290-7

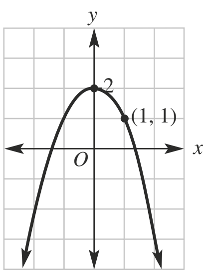
Cambridge University Press
Updated June 2022

Exercise 7B

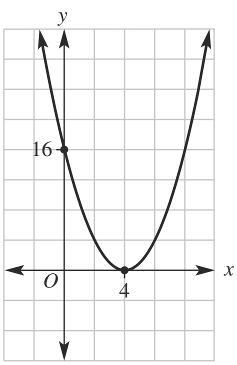
1 a



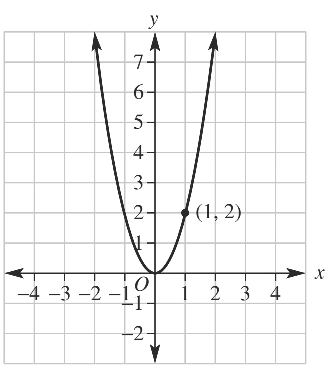
b



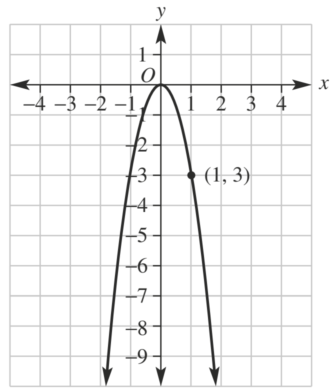
c



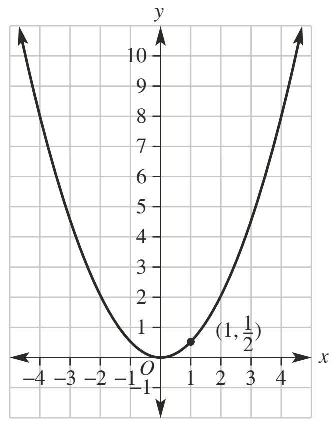
2 a



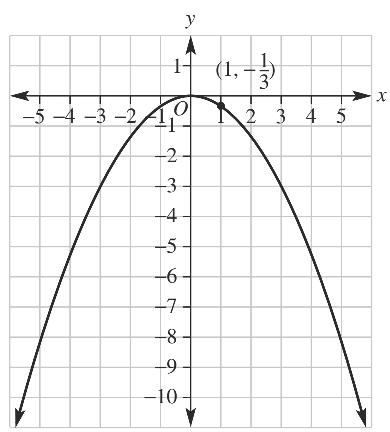
b



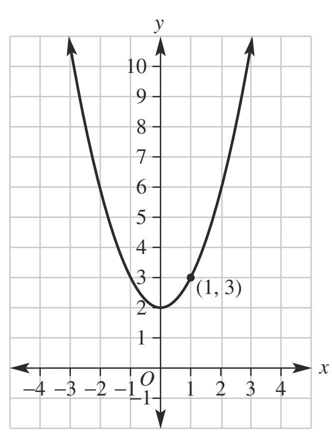
c



d



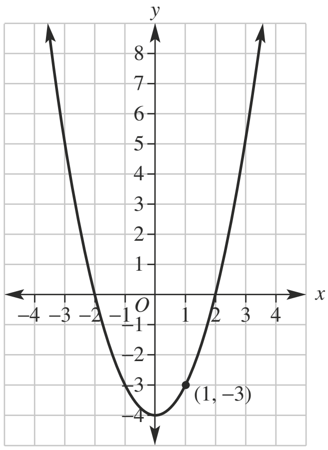
e



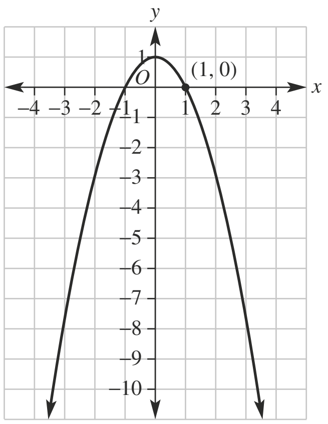
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

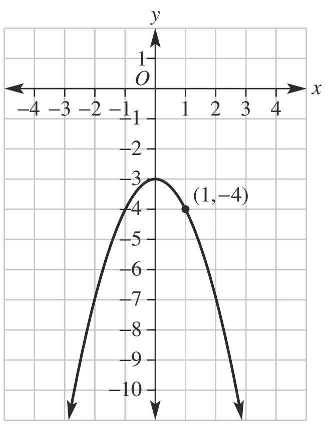
f



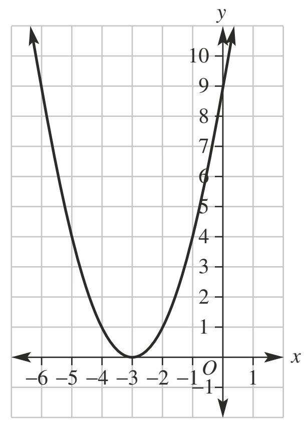
g



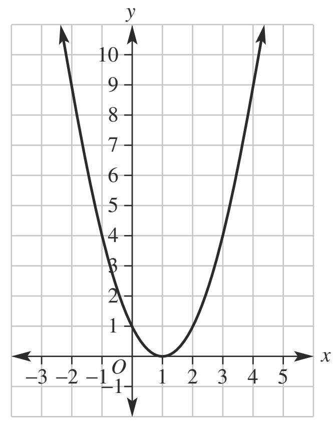
h



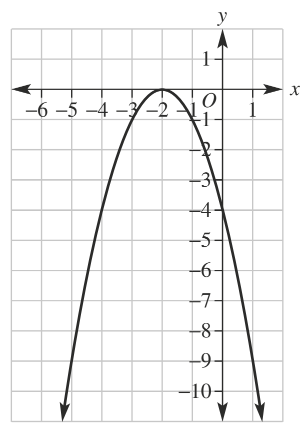
i



j



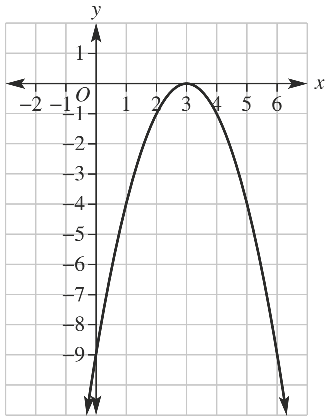
k



ISBN 978-1-108-77290-7

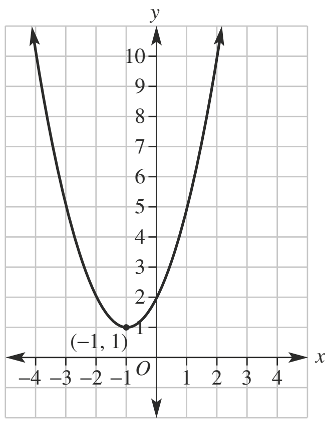
Cambridge University Press
Updated June 2022

l

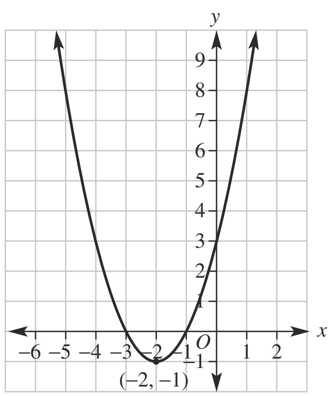


3 a (−3, 1) b (−2, −4) c (1, 3)
d (4, −2) e (3, −5) f (2, 2)
g (3, 3) h (2, 6) i (−1, 4)
j (2, −5) k (−1, −1) l (4, −10)

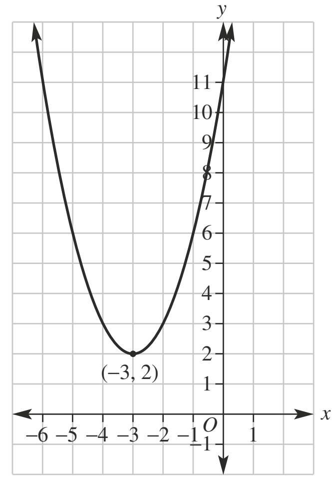
4 a



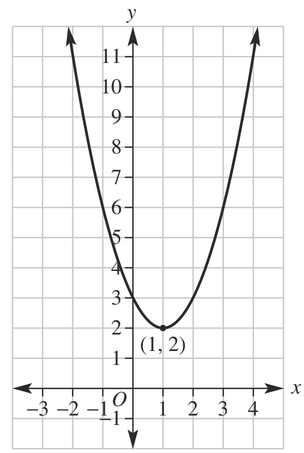
b



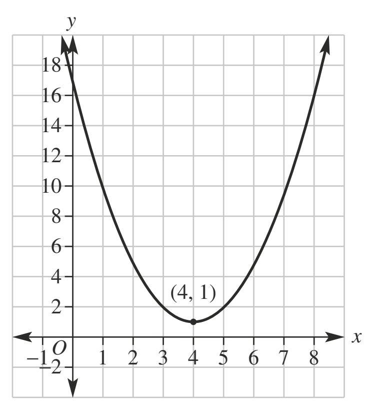
c



d

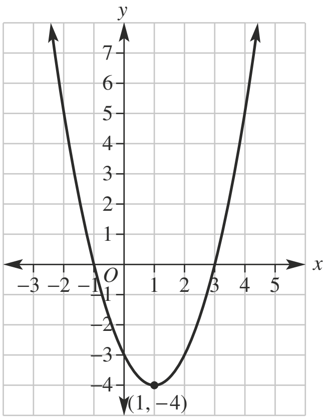


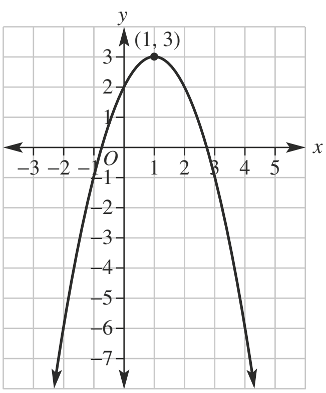
e

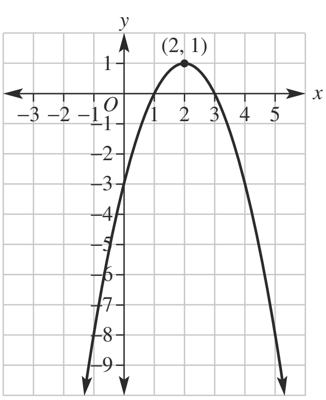


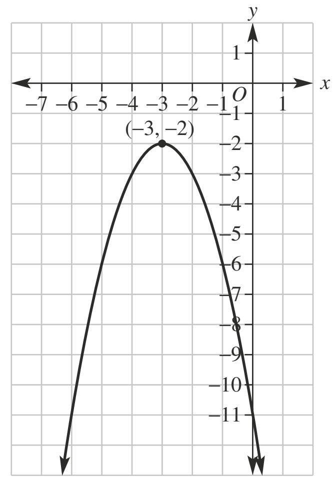
ISBN 978-1-108-77290-7

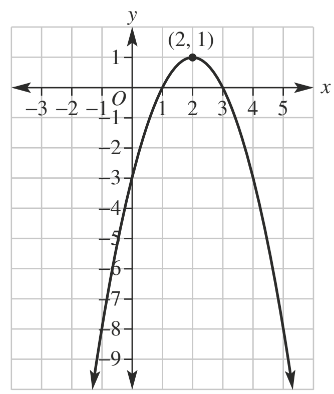
Cambridge University Press
Updated June 2022

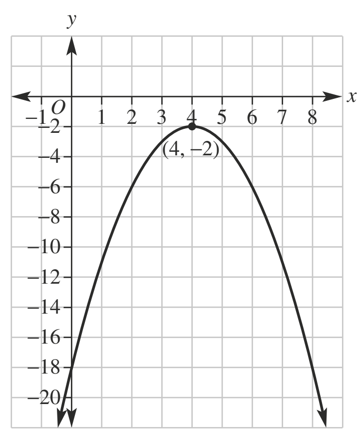
f


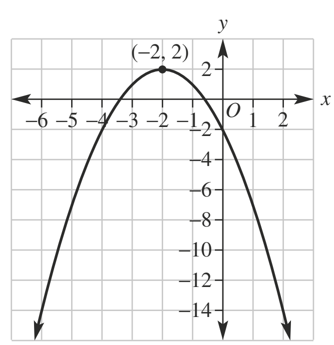
g


h


i


j


k


l


ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

5 **a** $y = -x^2$
b $y = (x + 2)^2$
c $y = x^2 - 5$
d $y = x^2 + 4$
e $y = (x - 1)^2$
f $y = -x^2 + 2$
g $y = -(x + 3)^2$
h $y = (x + 5)^2 - 3$
i $y = (x - 6)^2 + 1$

6 **a** $y = 6x^2$
b $y = x^2 + 4$
c $y = (x - 3)^2$
d $y = -(x + 2)^2$
e $y = \frac{1}{2}x^2$
f $y = -x^2 + 2$
g $y = x^2 - 1$
h $y = (x - 1)^2$
i $y = -7x^2$

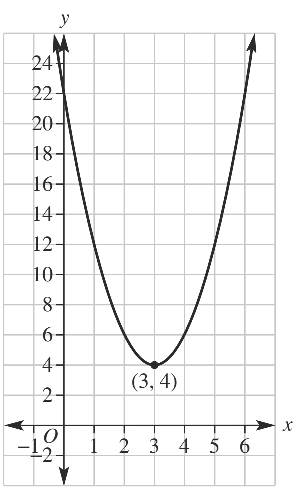
7 **a** maximum **b** (5, 25)
c 0 **d** 25 m
e **i** 21 m **ii** 21 m **iii** 0 m

8 **a** (1, 0) **b** (−2, 0) **c** (−3, 0)
d (0, −4) **e** (0, −2) **f** (0, 5)
g (−4, −1) **h** (−2, 3) **i** (5, 4)
j (−2, 3) **k** (−3, −5) **l** (3, −3)

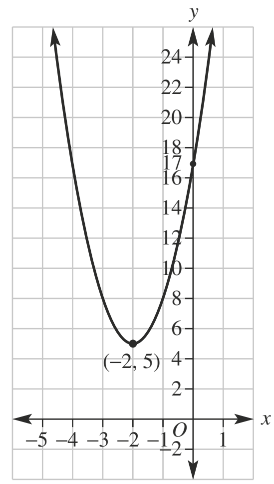
9 **a** translate 3 units right
b translate 2 units left
c translate 3 units down
d translate 7 units up
e reflect in x-axis
f translate 2 units left and 4 units down
g translate 5 units right and 8 units up
h reflect in x-axis, translate 3 units left
i reflect in x-axis, translate 6 units up

10 **a** (h, k) **b** $ah^2 + k$

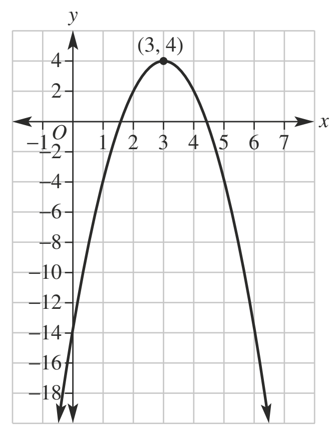
11 **a**



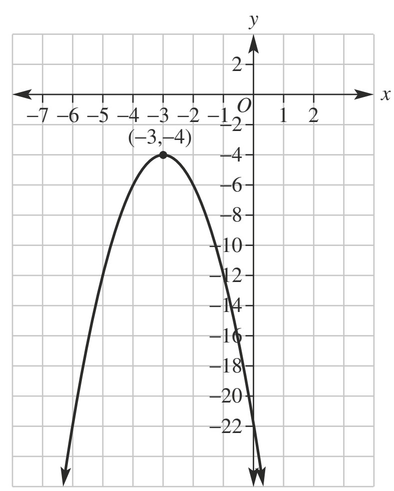
b



c



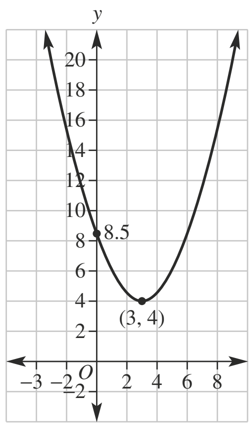
d



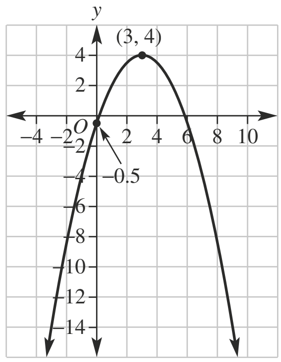
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

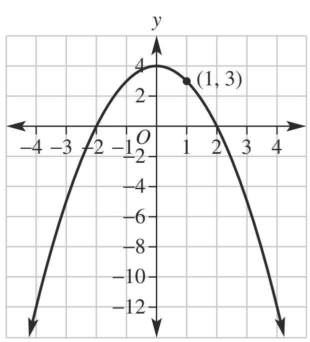
e



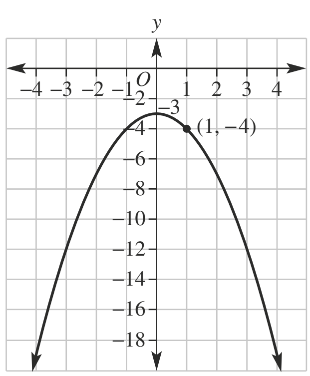
f



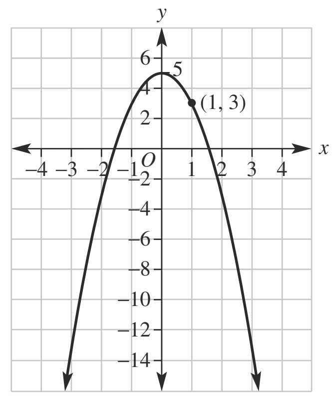
g



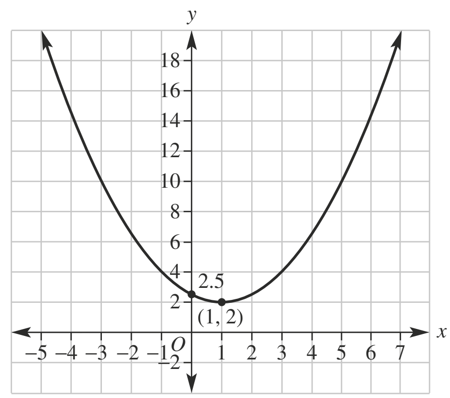
h



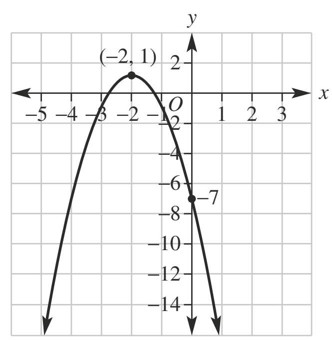
i



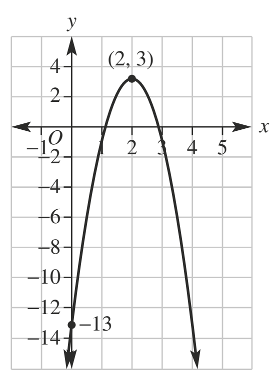
j



k



l



ISBN 978-1-108-77290-7

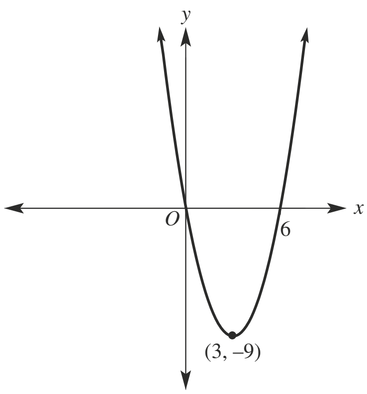
Cambridge University Press
Updated June 2022

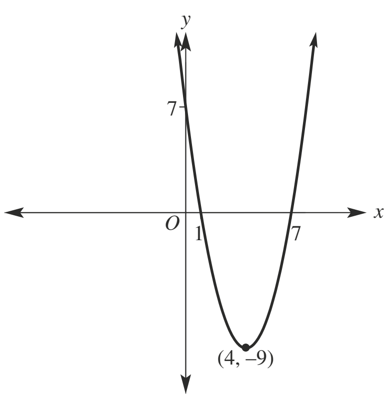
7C

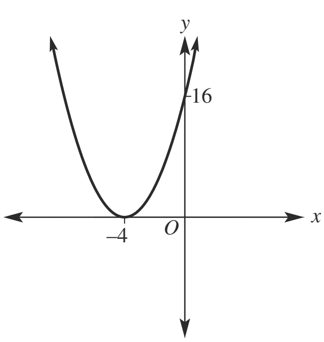
Building understanding

1 a $x = -1, x = 2$ b $x = 0, x = 3$
c $x = 0, x = -2$
2 a $x = 0, x = 4$ b $x = -4, x = 2$
c $x = 4$ d $x = \pm 5$
3 a 0
b −8
c 16
d −25
4 a (4, −4) b (−1, 9)

Now you try

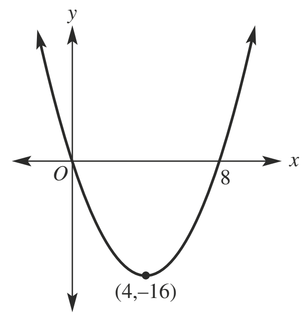
Example 6
a


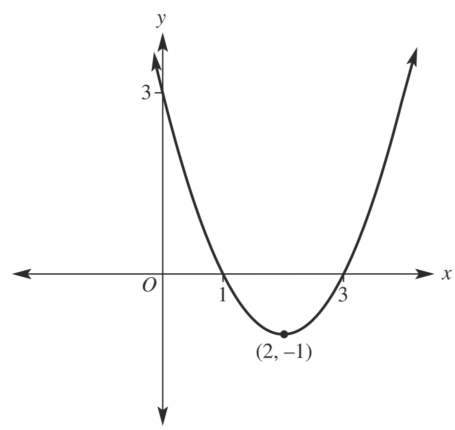
b


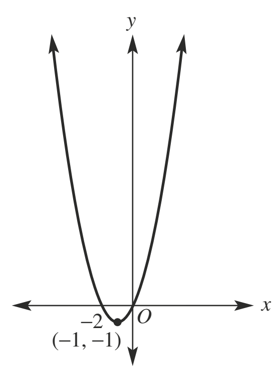
Example 7


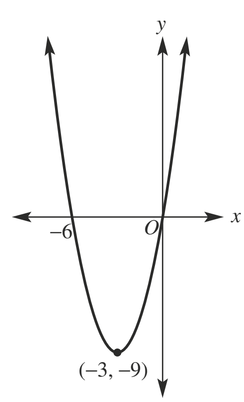
Example 8
$y = (x + 3)(x - 7)$
Turning point is (2, −25)

Exercise 7C

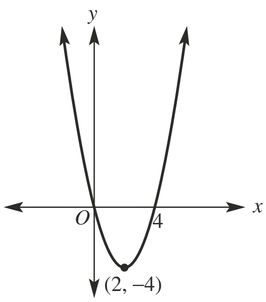
1 a


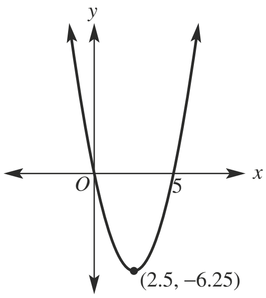
b


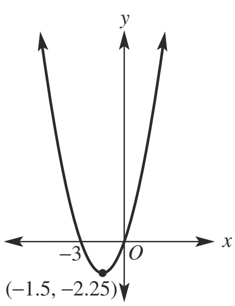
2 a


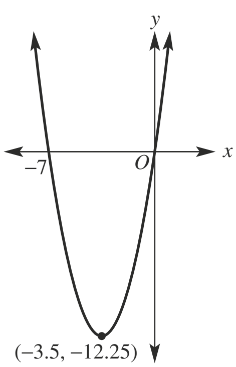
b


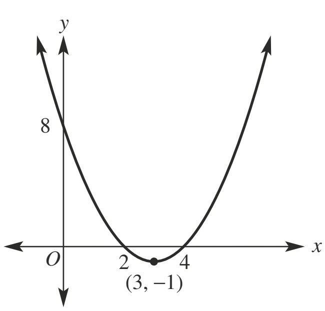
ISBN 978-1-108-77290-7

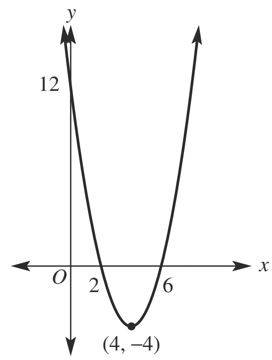
c


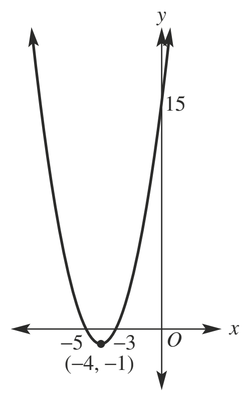
d


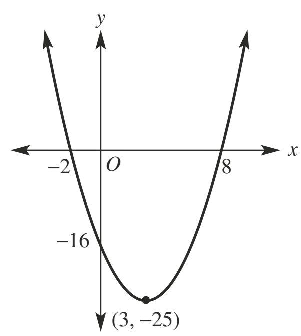
e


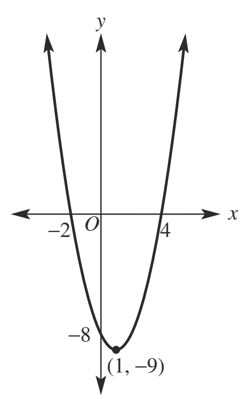
f


3 a


b


c


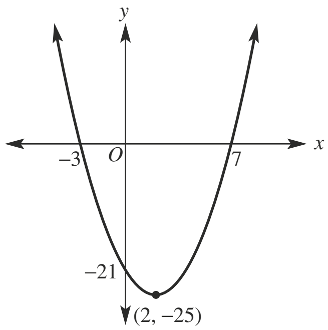
d


e


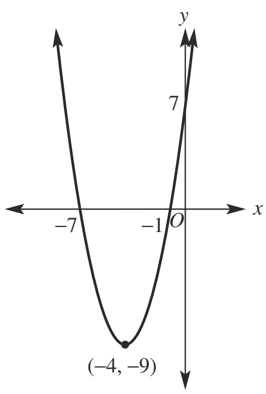
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

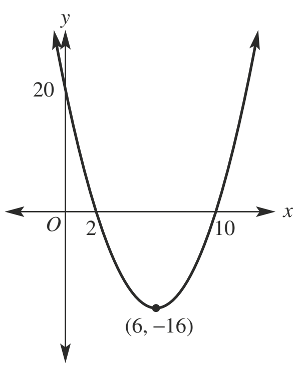
f



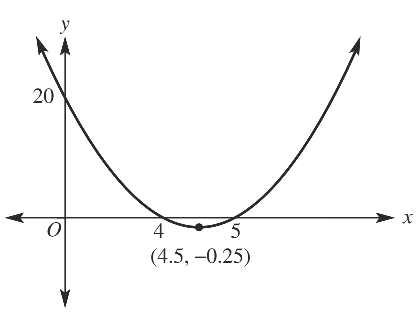
g



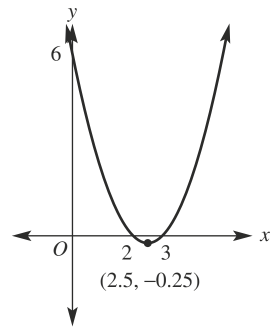
h



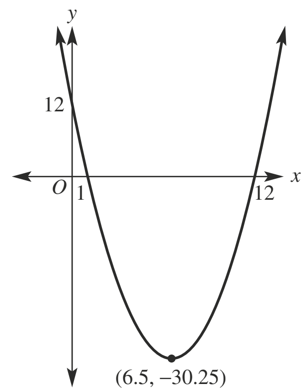
4 a



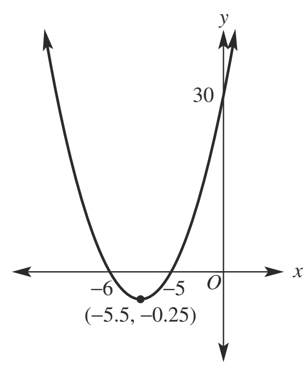
b



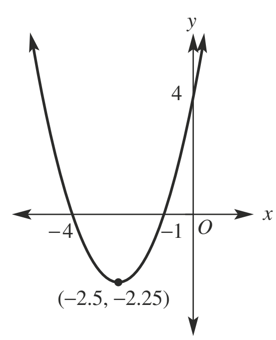
c



d



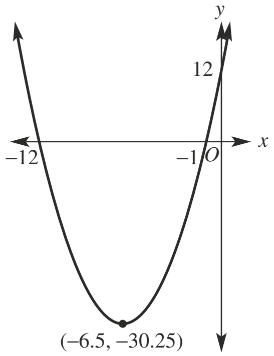
e



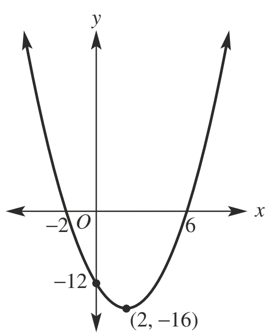
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

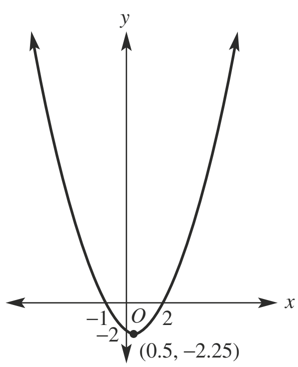
f



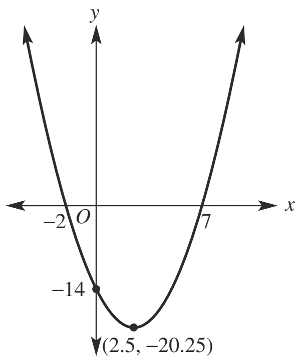
g



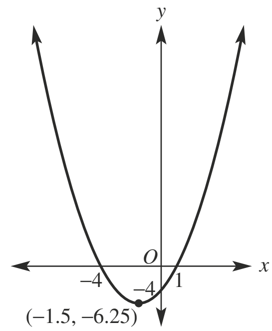
h



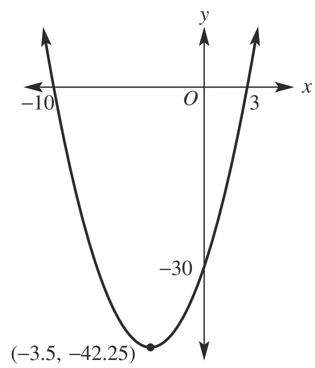
i



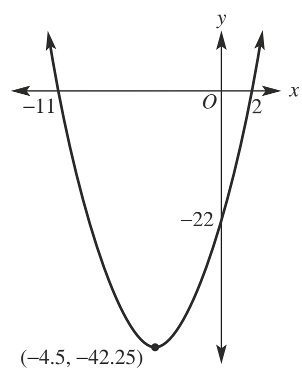
j



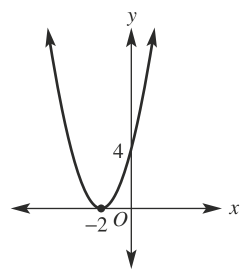
k



l

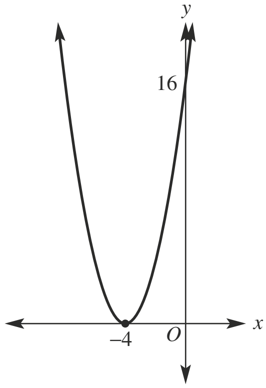


5 a

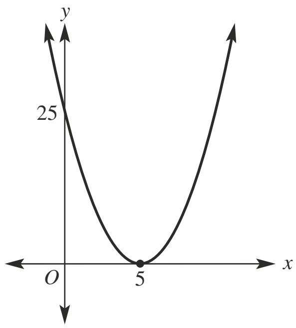


ISBN 978-1-108-77290-7

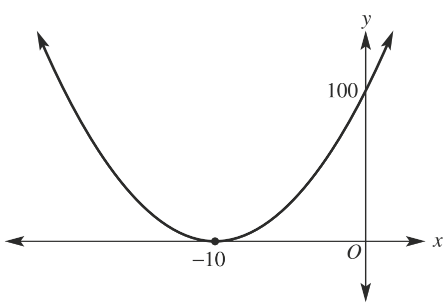
b



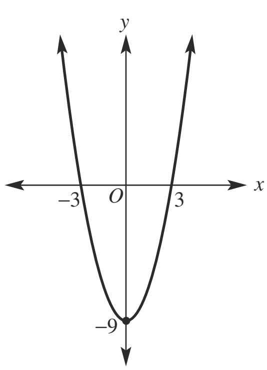
c



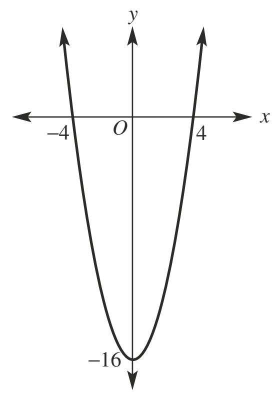
d



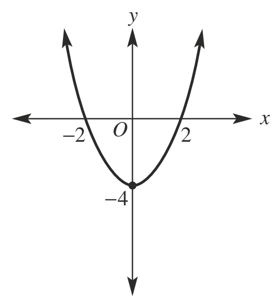
6 a



b



c

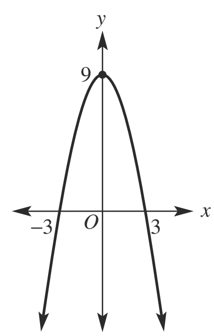


7 **a** (3.5, –4.5) **b** (3.5, –6.75) **c** (–3, –3)
d (–3, –4) **e** (0, –196) **f** (0, 196)
g (1, 0) **h** (1, 0) **i** (1, 8)

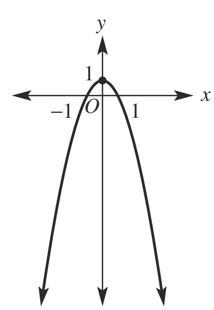
8 **a** $a = -1, b = -3$, TP (2, –1)
b $a = 5, b = -1$, TP (–2, –9)
c $a = 2, b = -6$, TP (2, –16)

9 **a** x-intercepts: $\sqrt{2}, -\sqrt{2}$; TP (0, –2)
b x-intercepts: $\sqrt{11}, -\sqrt{11}$; TP (0, –11)
c x-intercepts: $5\sqrt{2}, -5\sqrt{2}$; TP (0, –50)

10 a

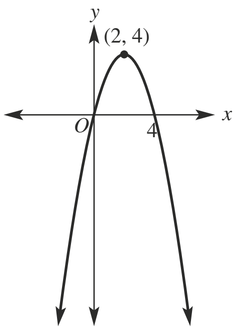


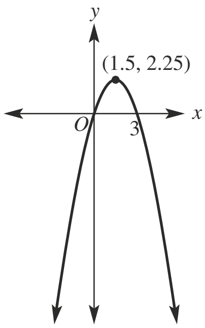
b

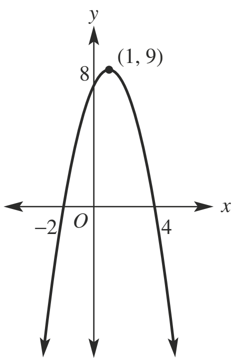


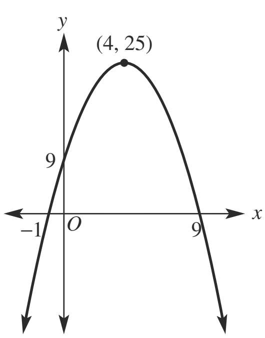
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

c


d


e


f


11 $a = -2$, TP $(1, 18)$

12 The coefficient does not change the x-intercepts.

13 **a** $y = x^2 - 2x + 1 = (x - 1)^2$
Only one x-intercept, which is the turning point.

b Graph has a minimum $(0, 2)$, therefore its lowest point is 2 units above the x-axis.

14 **a** $x = 4, x = -2$

b $(1, -9), (1, 9)$

c Same x-coordinate, y-coordinate is reflected in the x-axis.

15 **a** 0

b $0, -b$

c $\left(-\frac{b}{2}, -\frac{b^2}{4}\right)$

16 **a** $y = x(x - 4)$ **b** $y = x(x - 2)$

c $y = x(x + 6)$ **d** $y = (x + 3)(x - 3)$

e $y = (x + 2)(x - 2)$ **f** $y = (x + \sqrt{5})(x - \sqrt{5})$

g $y = (x + 4)(x - 2)$ **h** $y = (x - 1)(x - 5)$

i $y = (x + 1)(x - 3)$ **j** $y = -x(x - 4)$

k $y = -(x + 2)(x - 6)$ **l** $y = -(x - \sqrt{10})(x + \sqrt{10})$

7D

Building understanding

1 **a**
$$\begin{aligned} y &= x^2 + 2x - 5 \\ &= x^2 + 2x + \left(\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2 - 5 \\ &= (x + 1)^2 - 6 \end{aligned}$$
TP = $(-1, -6)$

b
$$\begin{aligned} y &= x^2 - 6x + 10 \\ &= x^2 - 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 10 \\ &= (x - 3)^2 + 1 \end{aligned}$$
TP = $(3, 1)$

2 **a** $x = \pm 3$ **b** $x = \pm\sqrt{3}$

c $x = 5, x = -3$ **d** $x = -4 \pm \sqrt{2}$

Now you try

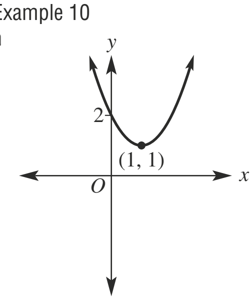
Example 9

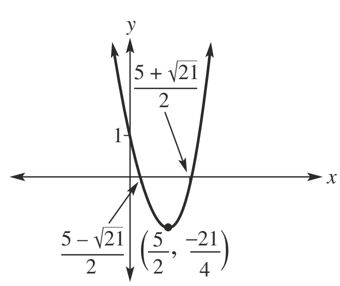
a maximum at $(-1,\ 18)$

b 16

c -4 and 2

Example 10

a


b


ISBN 978-1-108-77290-7 Cambridge University Press
 Updated June 2022

Exercise 7D

1 a maximum at (2, 3)
  b −9
  c 1 and 3

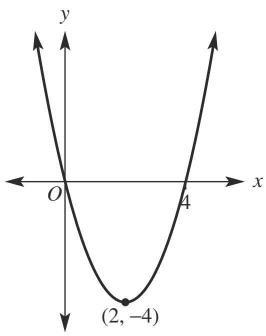
2 a min (3, 5) b max (1, 3)
  c max (−1, −2) d min (−2, −5)
  e min (−5, 10) f max (7, 2)
  g max (3, 8) h min (3, −7)

3 a 6 b −2 c 7 d 9
  e −16 f −55 g 3 h 1
  i −5 j −8 k 13 l −5

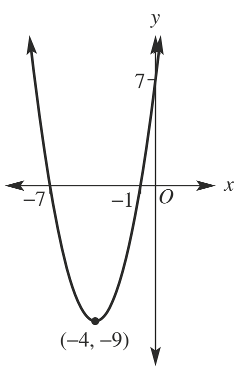
4 a $x = 5, x = 1$ b $x = -7, x = -1$
  c $x = 9, x = -3$ d $x = -2 \pm \sqrt{5}$
  e $x = 1 \pm \sqrt{10}$ f $x = 5 \pm \sqrt{3}$
  g $x = 4$ h $x = -6$
  i no x-intercept j no x-intercept
  k $x = 2 \pm \sqrt{5}$ l $x = 3 \pm \sqrt{10}$

5 a $x = -1, x = -5$ b $x = -3 \pm \sqrt{7}$
  c $x = -4 \pm \sqrt{21}$ d $x = -1 \pm \sqrt{7}$
  e no x-intercept f $x = 6 \pm \sqrt{41}$

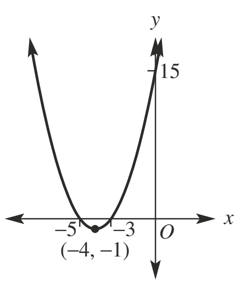
6 a



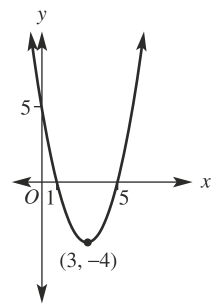
b



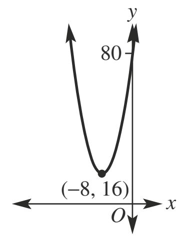
c



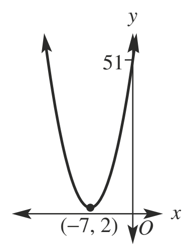
d



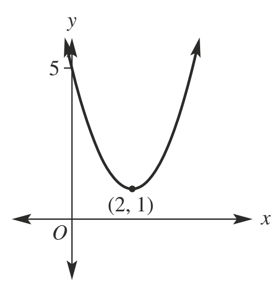
e



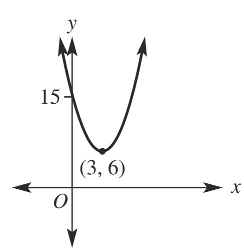
f



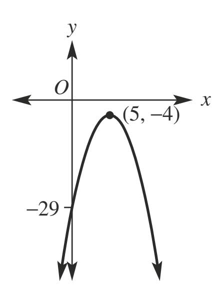
g



h



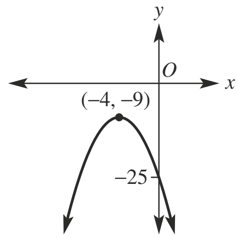
i



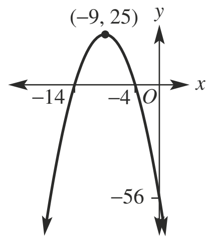
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

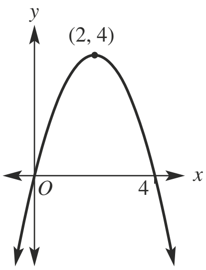
j



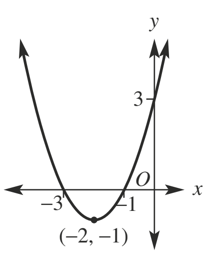
k



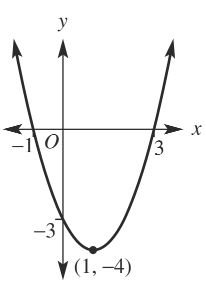
l



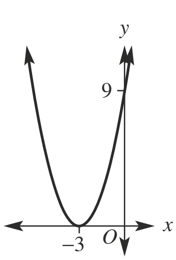
7 a



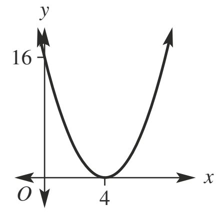
b



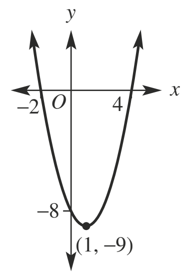
c



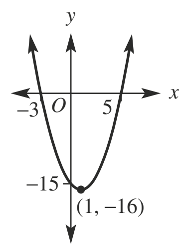
d



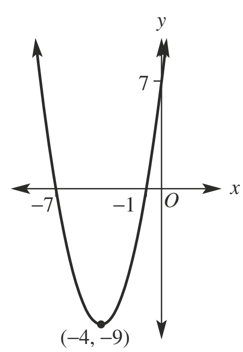
e



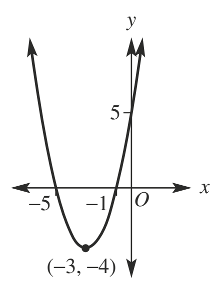
f



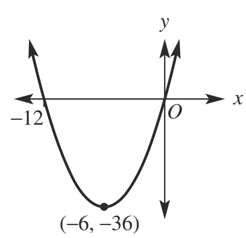
g



h

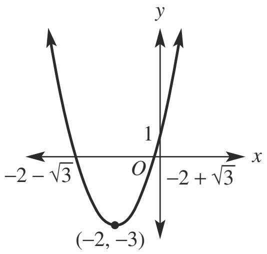


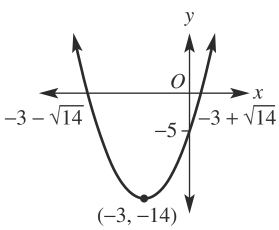
i

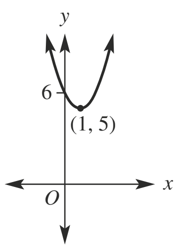


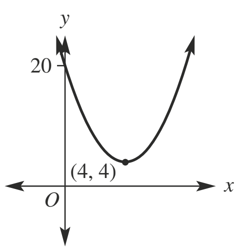
ISBN 978-1-108-77290-7

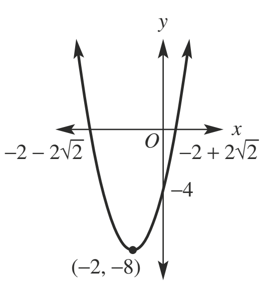
Cambridge University Press
Updated June 2022

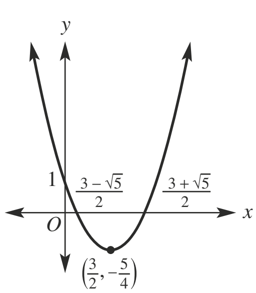
8 a


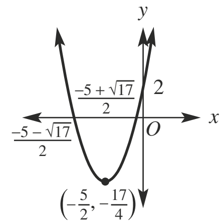
b


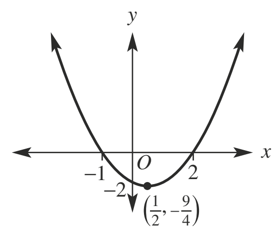
c


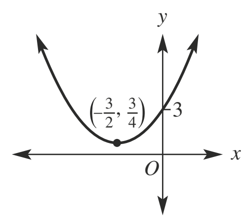
d


e


f


g


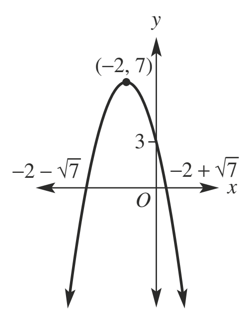
h


i


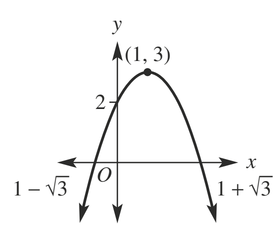
9 a 2 b 1 c 1
d 0 e 0 f 2

10 a $x = -1 \pm \sqrt{6}$ b $x = 3, x = 1$
c $x = 7, x = -1$ d $x = -2 \pm \sqrt{10}$
e $x = -2 \pm \sqrt{11}$ f $x = \dfrac{3 \pm \sqrt{5}}{2}$

11 a $y = -(x + 2)^2 + 7$



b $y = -(x - 1)^2 + 3$

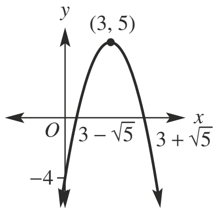


ISBN 978-1-108-77290-7

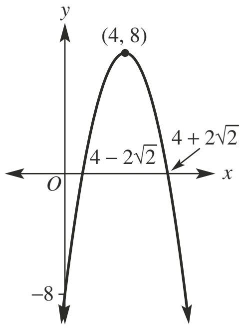
Cambridge University Press

Updated June 2022

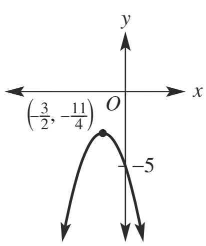
c $y = -(x - 3)^2 + 5$



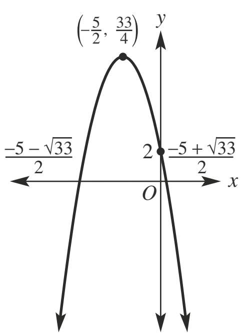
d $y = -(x - 4)^2 + 8$



e $y = -\left(x + \frac{3}{2}\right)^2 - \frac{11}{4}$



f $y = -\left(x + \frac{5}{2}\right)^2 + \frac{33}{4}$



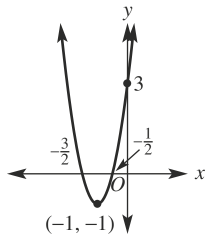
12 a $k > 0$
b $k = 0$
c $k < 0$

13 $x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$

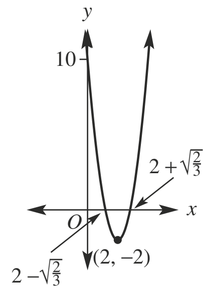
$$= \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + \frac{4c}{4}$$

$$= \left(x + \frac{b}{2}\right)^2 - \frac{b^2 - 4c}{4}$$

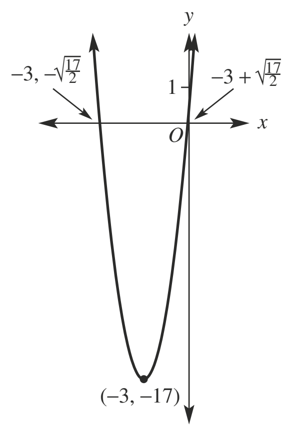
14 a $y = 4(x + 1)^2 - 1$



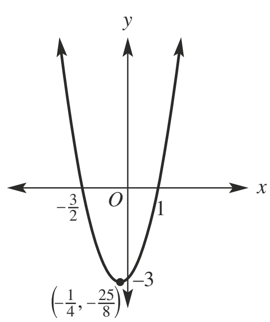
b $y = 3(x - 2)^2 - 2$



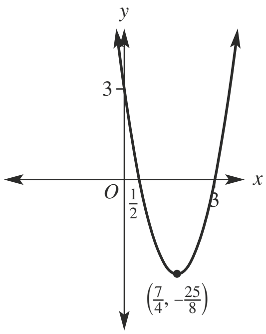
c $y = 2(x + 3)^2 - 17$



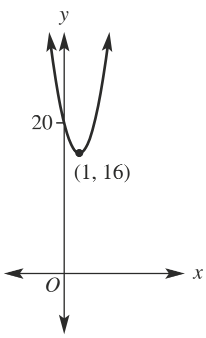
d $y = 2\left(x + \frac{1}{4}\right)^2 - \frac{25}{8}$



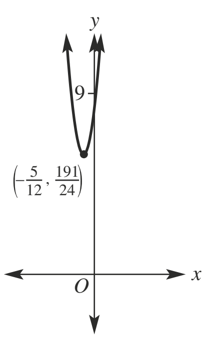
e $y = 2\left(x - \frac{7}{4}\right)^2 - \frac{25}{8}$



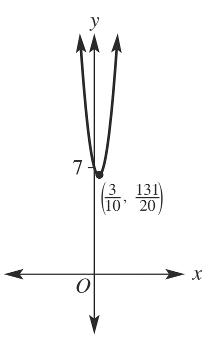
f $y = 4(x - 1)^2 + 16$



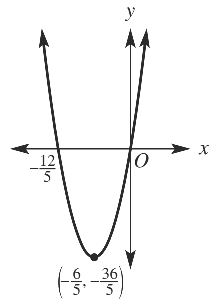
g $y = 6\left(x + \frac{5}{12}\right)^2 + \frac{191}{24}$



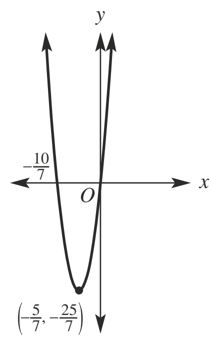
h $y = 5\left(x - \frac{3}{10}\right)^2 + \frac{131}{20}$



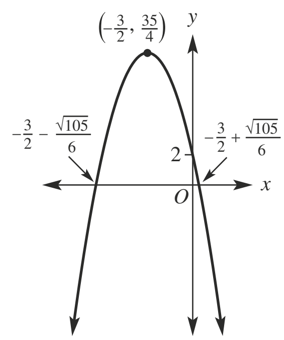
i $5\left(x + \frac{6}{5}\right)^2 - \frac{36}{5}$



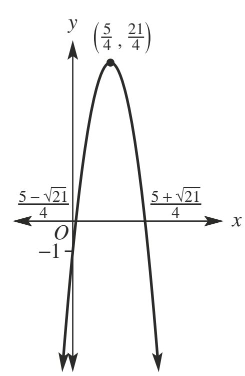
j $y = 7\left(x + \frac{5}{7}\right)^2 - \frac{25}{7}$



k $y = -3\left(x + \frac{3}{2}\right)^2 + \frac{35}{4}$



l $y = -4\left(x - \frac{5}{4}\right)^2 + \frac{21}{4}$



ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

7E

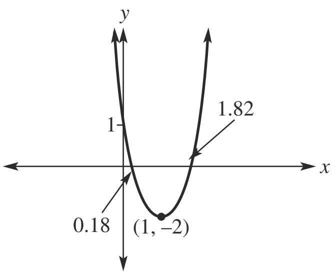
Building understanding

1 a 2 intercepts b 0 intercepts c 1 intercept
2 a $-1 \pm \sqrt{2}$ b 2.5, −1
c $\dfrac{-1 \pm \sqrt{17}}{4}$
3 a zero b positive c negative

Now you try

Example 11
a 2 b 1 c (1, −1)

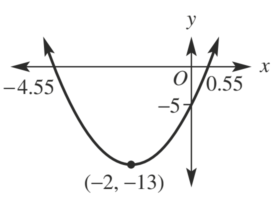
Example 12



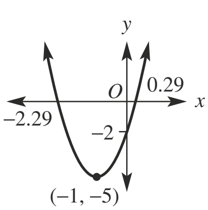
Exercise 7E

1 a 0 intercepts b 9 c (2, 1)
2 a 1 intercept b 0 intercepts c 2 intercepts
d 2 intercepts e 0 intercepts f 1 intercept
g 2 intercepts h 2 intercepts i 2 intercepts
j 0 intercepts k 2 intercepts l 2 intercepts
3 a 3 b 5 c −2
d −4 e 8 f −10
g 0 h 0 i −7
4 a (−1, 3) b (−2, −5) c (2, −1)
d (1, −5) e $\left(-\dfrac{3}{2}, 6\dfrac{1}{4}\right)$ f $\left(\dfrac{7}{2}, 5\dfrac{1}{4}\right)$
g $\left(-\dfrac{3}{4}, -5\dfrac{1}{8}\right)$ h $\left(\dfrac{3}{8}, -\dfrac{9}{16}\right)$ i (0, −9)
j $\left(\dfrac{1}{4}, -2\dfrac{3}{4}\right)$ k $\left(-\dfrac{1}{3}, \dfrac{1}{3}\right)$ l (0, 2)

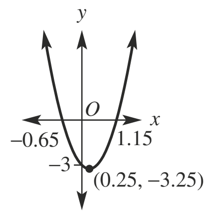
5 a



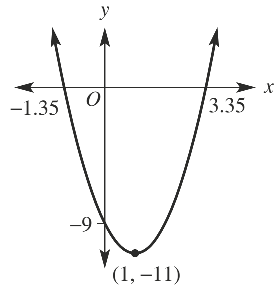
b



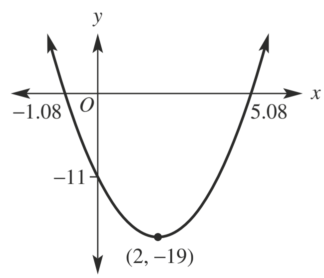
c



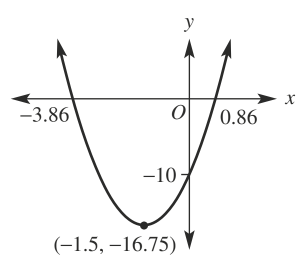
d



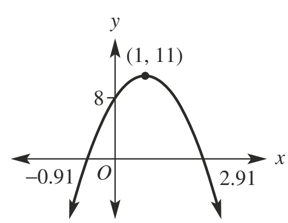
e



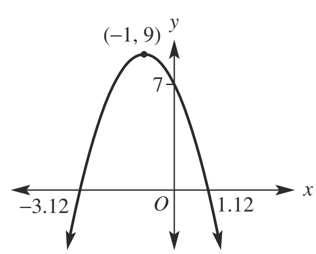
f



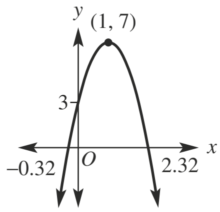
g



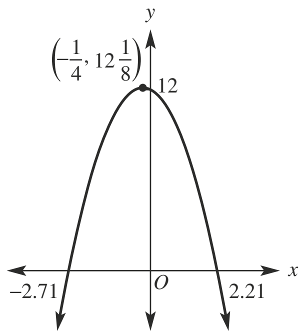
h



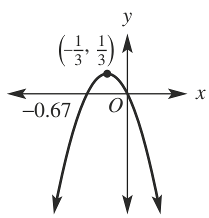
i



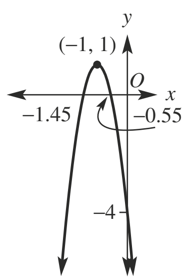
j



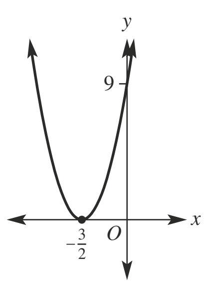
k



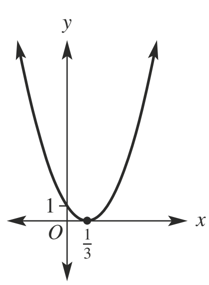
l



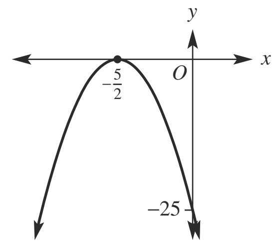
6 a



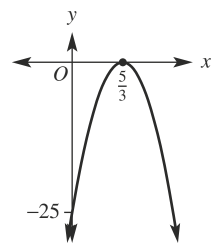
b



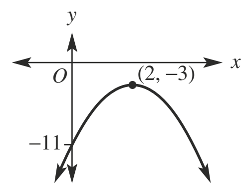
c



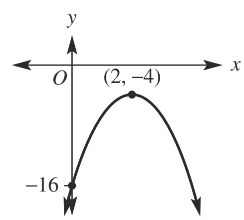
d



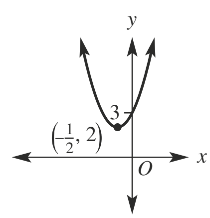
e



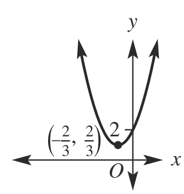
f



g



h



7 a $x = 1 \pm \frac{2\sqrt{3}}{3}$ **b** $x = -1 \pm \frac{\sqrt{10}}{2}$

c $x = 1 \pm \frac{\sqrt{10}}{2}$ **d** $x = \frac{-3 \pm \sqrt{15}}{2}$

e $x = 2 \pm \frac{\sqrt{6}}{2}$ **f** $x = 1 \pm \frac{\sqrt{30}}{5}$

8 $y = (x+1)^2 - 6 = x^2 + 2x - 5$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

9 a Anything with $b^2 - 4ac > 0$
b Anything with $b^2 - 4ac = 0$
c Anything with $b^2 - 4ac < 0$

10 Number under square root = 0, therefore $x = \frac{-b}{2a}$ (one solution)

11 $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$

12 $y = -\frac{b^2}{4a} + c$

13 $x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a} = 0$

$$x^2 + \left(\frac{b}{a}\right) + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

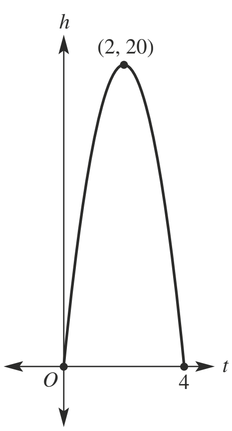
$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ as required}$$

7F

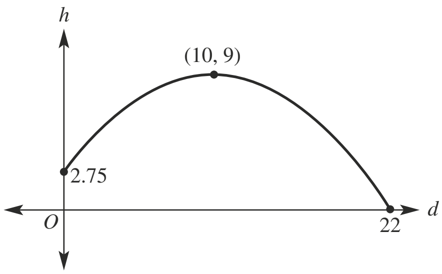
Building understanding

1 a



b 20 m
c 4 seconds

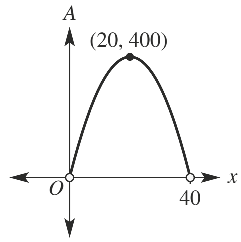
2 a



b 9 m
c 22 m

Now you try

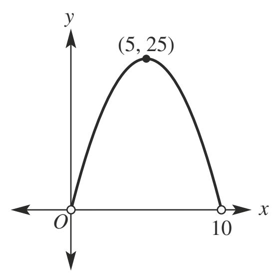
Example 13
a $40 - x$
b $A = x(40 - x)$
c $0 < x < 40$
d



e 400 cm^2
f square with side length 20 cm

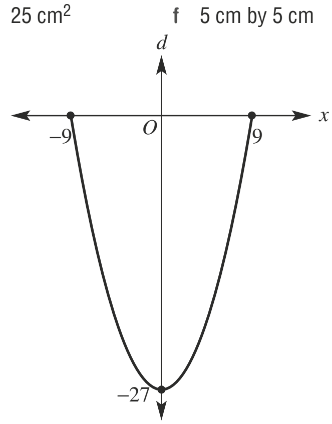
Exercise 7F

1 a $2 \times$ length $= 20 - 2x$
length $= 10 - x$
b $A = x(10 - x)$
c $0 < x < 10$
d



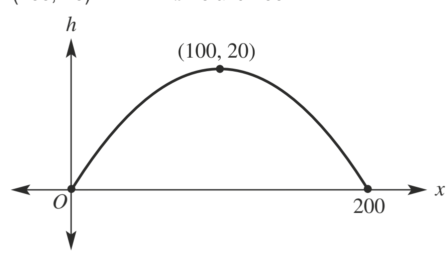
e 25 cm^2
f 5 cm by 5 cm

2 a



b 18 cm
c 27 cm

3 a (100, 20)
b 0 and 200
c



d 200 m
e 20 m

Answers

7E

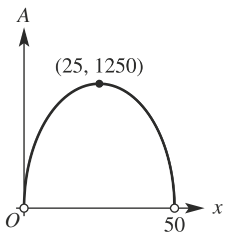
ISBN 978-1-108-77290-7

Cambridge University Press

Updated June 2022

4 **a** $100 - 2x$ **b** $A = x(100 - 2x)$ **c** $0 < x < 50$

d

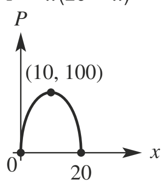


e 1250 m^2 **f** width = 25 m, length = 50 m

5 **a** $20 - x$

b $P = x(20 - x)$

c

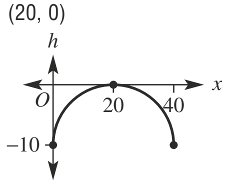


d **i** $x = 0$ or 20 **ii** $x = 10$

e 100

6 **a** (20, 0)

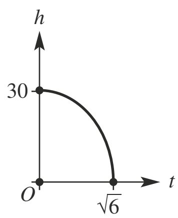
b



c 40 m **d** 10 m

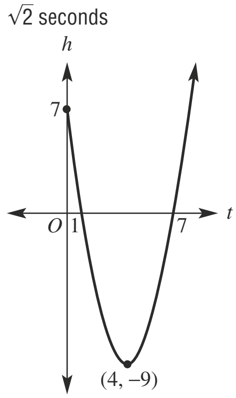
7 **a** $\sqrt{6}$ seconds

b



c $\sqrt{2}$ seconds

8 **a**



b **i** 1 second

ii 7 seconds

iii 4 seconds

c 9 m below sea level

d at 3 and 5 seconds

9 **a** 1 m

b No, 1 metre is the minimum height the kite falls to.

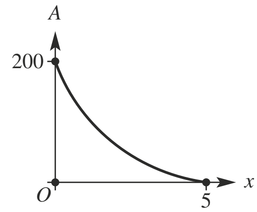
10 $P = x(64 - x)$ so maximum occurs at $x = 32$.

Maximum product $= 32(64 - 32) = 1024$

11 **a** $A = (20 - 2x)(10 - 2x)$

b min $x = 0$, max $x = 5$

c



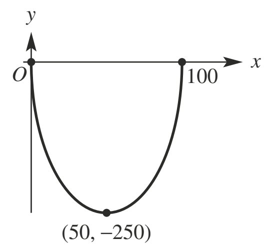
d Turning point occurs for an x-value greater than 5.

e 1 cm

12 **a** 6 m

b No, the maximum height reached is 4.5 m.

13 **a**



b **i** 2

ii none

c **i** (27.6, –200) and (72.4, –200)

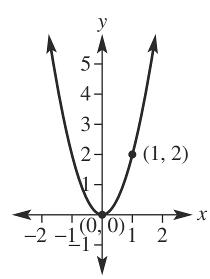
ii (1.0, –10) and (99.0, –10)

d The highway meets the edge of the river (50 metres along).

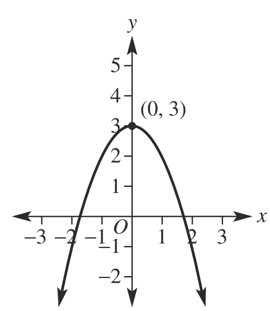
14 $5\frac{1}{24}$ m

Progress quiz

1 **a**



b

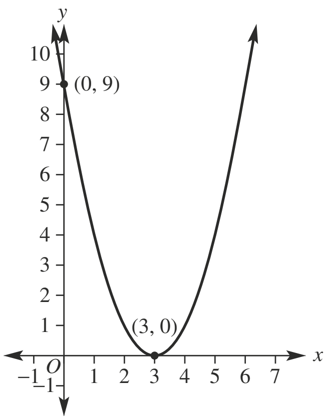


ISBN 978-1-108-77290-7

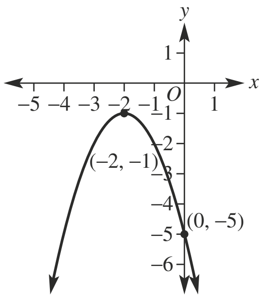
Cambridge University Press

Updated June 2022

c

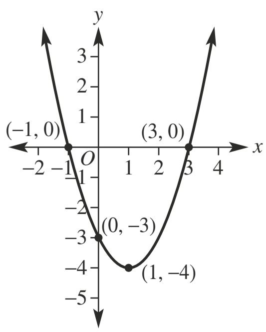


d

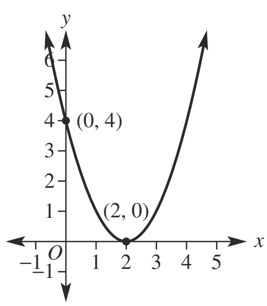


2 $y = 3x^2 + 2$

3 **a**



b



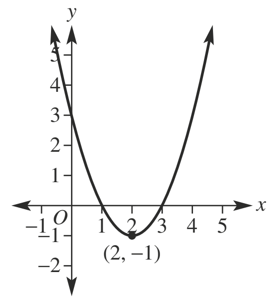
4 $a = 2, b = -4$; $y = (x + 2)(x - 4)$; Turning point is at $(1, -9)$.

5 **a** Turning point is a maximum at $(3, 8)$.

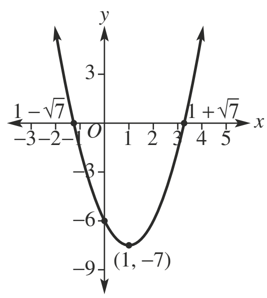
b y-intercept is at $(0, -10)$.

c x-intercepts at 5 and 1.

6 **a** $y = (x - 2)^2 - 1$



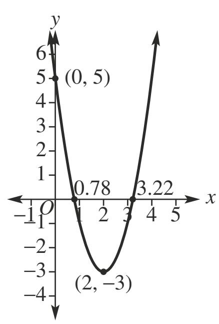
b $y = (x - 1)^2 - 7$



7

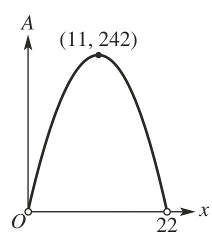
	i x-intercepts	ii y-intercepts	iii Turning point
a	$\triangle < 0$; no x-intercepts	(0, 5)	(2, 1)
b	$\triangle > 0$; two x-intercepts	(0, −7)	(−3, −16)
c	$\triangle = 0$; one x-intercept	(0, −16)	(−4, 0)

8



9 **a** $A = x(44 - 2x)$ or $A = 44x - 2x^2$

b



c 242 m^2; 11 m by 22 m

Answers

Ch7 Progress quiz

ISBN 978-1-108-77290-7

7G

Building understanding

1 a (2, 12) b (−1, −3)
2 a $x^2 + 3x + 6 = 0$ b $x^2 - 5x + 3 = 0$
c $x^2 + 3x - 12 = 0$
3 a $b^2 - 4ac < 0$ b $b^2 - 4ac > 0$ c $b^2 - 4ac = 0$

Now you try

Example 14
a (−1, 2) and (2, 2) b no intersection points

Example 15
a (0, 0) and (4, 16) b (1, 12)
c (−2, 3) and $\left(\frac{1}{2}, -\frac{3}{4}\right)$

Example 16
(−0.62, −1.85) and (1.62, 4. 85)

Example 17
a 2 solutions b 0 solutions

Exercise 7G

1 a (−2, 8) and (4, 8) b no intersection points
2 a (−3, 6) and (2, 6) b (−2, 12) and (6, 12)
c no solutions d (−3, −2) and $\left(-\frac{1}{2}, -2\right)$
e $\left(\frac{3}{2}, 0\right)$ f no solutions
3 a $x = 0, y = 0$ and $x = 3, y = 9$
b $x = 0, y = 0$ and $x = -2, y = 4$
c $x = -3, y = 9$ and $x = 6, y = 36$
d $x = 0, y = 5$ and $x = 3, y = 8$
e $x = -6, y = 34$ and $x = -2, y = 22$
f $x = -2, y = -3$ and $x = 3, y = 17$
g no solutions
h no solutions
i $x = -\frac{9}{2}, y = \frac{65}{2}$ and $x = -1, y = 8$
j $x = -\frac{5}{3}, y = -\frac{25}{3}$ and $x = 3, y = 1$
k $x = -3, y = 6$
l $x = -1, y = 2$
4 a $x = -4, y = 16$ and $x = 2, y = 4$
b $x = -1, y = 1$ and $x = 2, y = 4$
c $x = -1, y = 1$ and $x = \frac{1}{3}, y = \frac{1}{9}$
d $x = -2, y = 7$ and $x = -\frac{1}{2}, y = \frac{13}{4}$
e $x = -2, y = 0$ and $x = \frac{2}{3}, y = \frac{16}{9}$
f $x = -8, y = -55$ and $x = 2, y = 5$
5 a i no solutions
ii $x = -0.7, y = 1.5$ and $x = 2.7, y = 8.5$
iii $x = -1.4, y = -2.1$ and $x = 0.4, y = 3.1$
iv $x = -2.6, y = 8.2$ and $x = -0.4, y = 3.8$
b i $x = \frac{-1 \pm \sqrt{21}}{2}, y = \frac{-1 \pm \sqrt{21}}{2}$
ii $x = \frac{3 \pm \sqrt{13}}{2}, y = 3 \pm \sqrt{13}$
iii $x = \frac{-1 \pm \sqrt{13}}{2}, y = 1 \pm \sqrt{13}$
iv $x = \frac{-1 \pm \sqrt{17}}{2}, y = \pm\sqrt{17}$
6 a 2 b 0 c 2 d 0 e 1 f 2
7 Yes, the ball will hit the roof. This can be explained in a number of ways. Using the discriminant, we can see that the path of the ball intersects the equation of roof $y = 10.6$.
8 a $x = -1, y = -2$ and $x = -\frac{1}{2}, y = -\frac{7}{4}$
b $x = \frac{5}{2}, y = -\frac{15}{4}$ and $x = 2, y = -4$
c $x = 1, y = 8$ and $x = 2, y = 7$
d $x = -6, y = -14$ and $x = 2, y = 2$
9 a (−1, 4) and $\left(\frac{1}{2}, 5\frac{1}{2}\right)$ b 212 m
10 a (3, −4)
b i $c > -4$ ii $c = -4$ iii $c < -4$
11 a $1 + 4k$
b i $k > -\frac{1}{4}$ ii $k = -\frac{1}{4}$ iii $k < -\frac{1}{4}$
12 a Discriminant from resulting equation is less than 0.
b $k \geqslant 2$
13 a $m = 2$ or $m = -6$
b The tangents are on different sides of the parabola, where one has a positive gradient and the other has a negative gradient.
c $m > 2$ or $m < -6$

7H

Building understanding

1 a $f(x) = 8x$ b $f(x) = 9 - x^2$ c $f(x) = 2^x$
2 a true b true c false d false e true
3 a $y \geqslant 0$ b $y > 0$ c $y > 9$
d $0 \leqslant y \leqslant 1$ e $y \geqslant 0$

Now you try

Example 18
a 2 b 16 c $k^2 - 5k + 2$

Example 19
a function b function c not a function

Example 20
a Domain: all real x
Range: all real y
b Domain: all real x
Range: $y \leqslant 2$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 7H

1 a 3 b 11 c $c^2 - 2c + 3$

2 a 4 b 10 c 124
d 4 e 14 f $2a^2 - a + 4$

3 a function b function c not a function
d function e function f function

4 a function b function c function
d function e not a function f function
g not a function h function i not a function

5 a 0 b 2 c -4
d 230 e 0.176 f $2k^3 - k^2 + k$

6 a $f(0) = 0$, $f(2) = 8$, $f(-4) = -16$, $f(a) = 4a$, $f(a + 1) = 4a + 4$
b $f(0) = 1$, $f(2) = -3$, $f(-4) = -15$, $f(a) = 1 - a^2$, $f(a + 1) = -a^2 - 2a$
c $f(0) = -12$, $f(2) = 0$, $f(-4) = -12$, $f(a) = a^2 + 4a - 12$, $f(a + 1) = a^2 + 6a - 7$
d $f(0) = 9$, $f(2) = 25$, $f(-4) = 73$, $f(a) = 4a^2 - 9$, $f(a + 1) = 4a^2 + 8a + 13$

7 a all real x b all real x c all real x d all real x

8 a all real y b $y \geqslant 0$ c $y \geqslant 0$ d $y \leqslant 2$

9 a i 5 ii -2 iii 3
iv -15 v 5 vi -4
b $a = \frac{5}{3}$ represents the x-value of the point where the line graphs intersect.

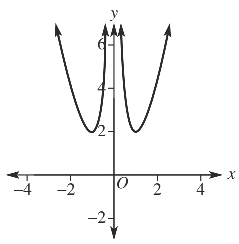
10 a i false ii false
b i false ii true
c i false ii false

11 $4x + 2h - 3$

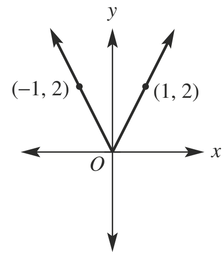
12 a They all pass the vertical line test, as each x-value has only one y-value.
b Vertical lines in the form $x = a$, since a single x-value has multiple y-values
c The y-value of the vertex is the maximum or minimum value of the parabola and therefore is essential when finding the range.
d i $y \geqslant -4$ ii $y \geqslant -12\frac{1}{4}$
iii $y \leqslant 1\frac{1}{8}$ iv $y \geqslant 1$

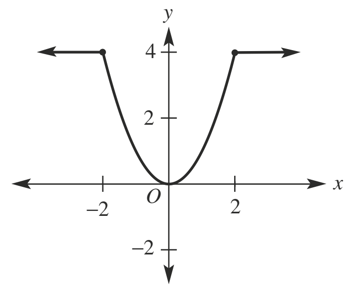
13 a $x \neq 1$ b $x \neq -\frac{1}{2}$ c $x \neq 1$

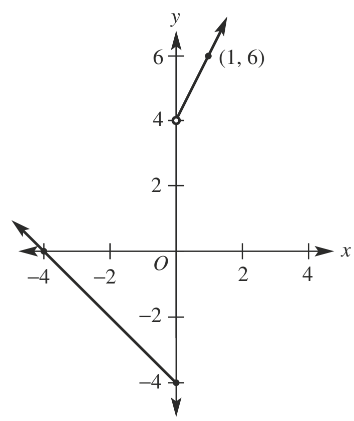
14 a $x \geqslant 0$ b $x \geqslant 2$ c $x \geqslant -2$ d $x \leqslant 2$

15 a $f(a) = f(-a) = a^2 + \frac{1}{a^2}$
b


c The y-axis is the axis of symmetry for the function.

16 a i


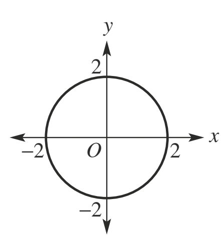
ii


iii


b iii
c i $y \geqslant 0$
ii $0 \leqslant y \leqslant 4$
iii $y \geqslant -4$
d i 8, 8, 2
ii 34, 18, -2

7I

Building understanding

1


2 a $x = \pm\sqrt{5}$ b $x = \pm 4$ c $y = \pm\sqrt{11}$

3 a (0, 0) b r

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

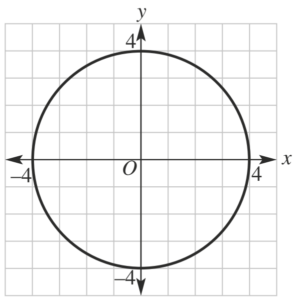
Now you try

Example 21

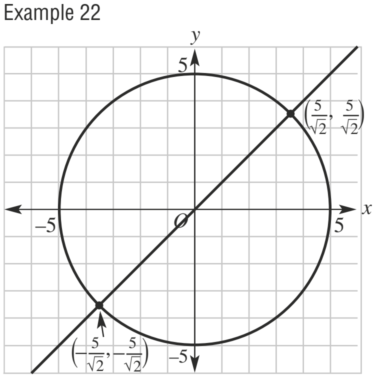
a (0, 0) **b** $r = 4$

c $y = \pm\sqrt{15}$ **d** $x = \pm\frac{\sqrt{63}}{2} = \pm\frac{3\sqrt{7}}{2}$

e



Example 22

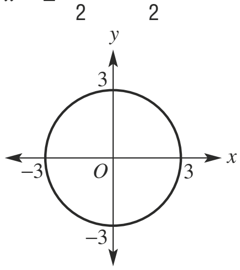


Exercise 7I

1 a (0, 0) **b** $r = 3$ **c** $y = \pm\sqrt{5}$

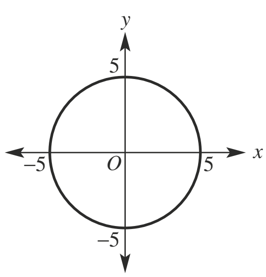
d $x = \pm\frac{\sqrt{27}}{2} = \frac{\pm 3\sqrt{3}}{2}$

e



2 a (0, 0) **b** $r = 5$ **c** $y = \pm\frac{\sqrt{19}}{2}$ **d** $x = \pm 3$

e



3 a $r = 6$ **b** $r = 9$
c $r = 12$ **d** $r = \sqrt{5}$
e $r = \sqrt{14}$ **f** $r = \sqrt{20} = 2\sqrt{5}$

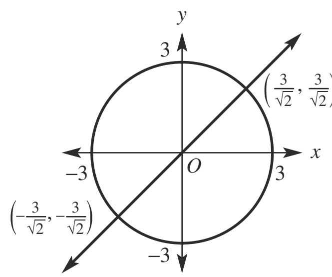
4 a $x^2 + y^2 = 4$ **b** $x^2 + y^2 = 49$
c $x^2 + y^2 = 10000$ **d** $x^2 + y^2 = 2601$
e $x^2 + y^2 = 6$ **f** $x^2 + y^2 = 10$
g $x^2 + y^2 = 121$ **h** $x^2 + y^2 = 0.25$

5 a $(1, \sqrt{3}), (1, -\sqrt{3})$
b $(-1, \sqrt{3}), (-1, -\sqrt{3})$
c $\left(\frac{1}{2}, \frac{\sqrt{15}}{2}\right), \left(\frac{1}{2}, -\frac{\sqrt{15}}{2}\right)$
d $\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right), \left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$
e (0, –2)
f (2, 0), (–2, 0)

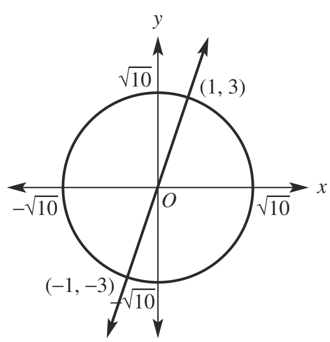
6 a x-intercepts: ± 1, y-intercepts: ± 1
b x-intercepts: ± 4, y-intercepts: ± 4
c x-intercepts: $\pm\sqrt{3}$, y-intercepts: $\pm\sqrt{3}$
d x-intercepts: $\pm\sqrt{11}$, y-intercepts: $\pm\sqrt{11}$

7 a $r = 2\sqrt{2}$ **b** $r = 2$ **c** $r = 3$
d $r = \sqrt{10}$ **e** $r = 2\sqrt{3}$ **f** $r = 2\sqrt{5}$

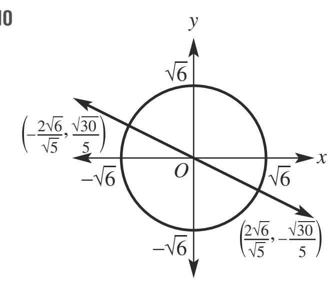
8



9



10

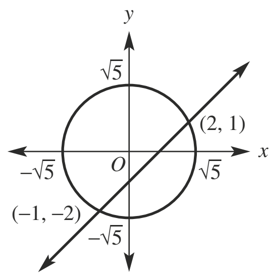


ISBN 978-1-108-77290-7

Cambridge University Press

Updated June 2022

11



Chord length $= 3\sqrt{2}$ units

12 a $m = \pm\sqrt{3}$

b $m > \sqrt{3}$ or $m < -\sqrt{3}$

c $-\sqrt{3} < m < \sqrt{3}$

13 a D b A c E d C e F f B

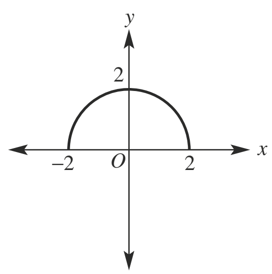
14 a $y = \pm\sqrt{16 - x^2} = \pm\sqrt{4^2 - x^2}$

b $x = \pm\sqrt{3 - y^2} = \pm\sqrt{(\sqrt{3})^2 - y^2}$

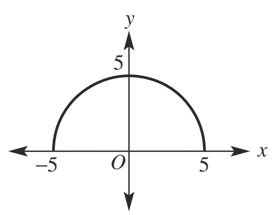
15 a Radius of graph is 2, so points are 2 units from (0, 0); i.e. < 2.

b Radius of graph is 1, so points are 1 unit from (0, 0); i.e. −1 is the leftmost point, which is not as far as −2.

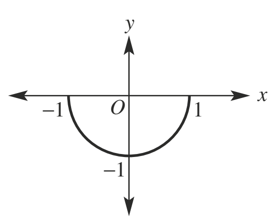
16 a



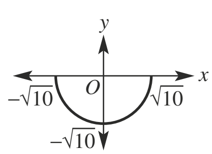
b



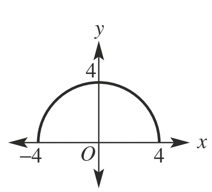
c



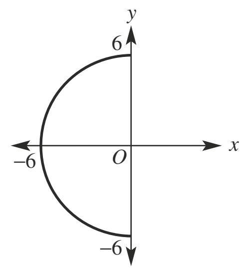
d



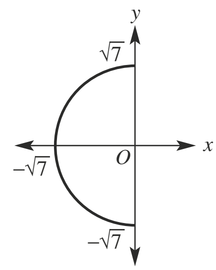
e



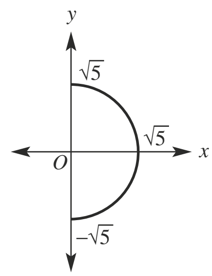
f



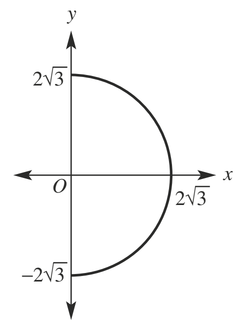
g



h



i



17 a $y = \sqrt{25 - x^2}$ b $y = -\sqrt{16 - x^2}$ c $x = \sqrt{4 - y^2}$

d $x = -\sqrt{1 - y^2}$ e $y = \sqrt{3 - x^2}$ f $y = -\sqrt{5 - x^2}$

g $x = \sqrt{10 - y^2}$ h $x = -\sqrt{8 - y^2}$ i $y = -\sqrt{18 - x^2}$

7J

Building understanding

1 a C b A c B

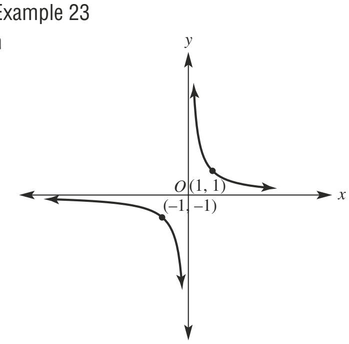
2 a $1 \div 0.1, 1 \div 0.01, 1 \div 0.001, 1 \div 0.00001$

b $x = \dfrac{1}{100}$ c 0.099 d 998

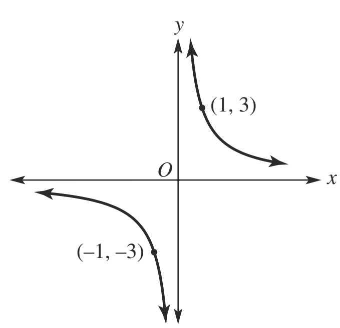
Now you try

Example 23

a



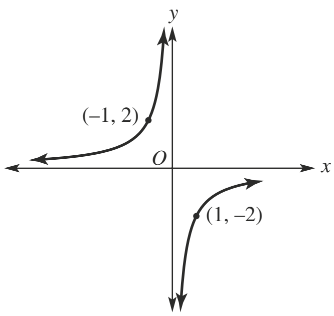
b



ISBN 978-1-108-77290-7

Updated June 2022

c



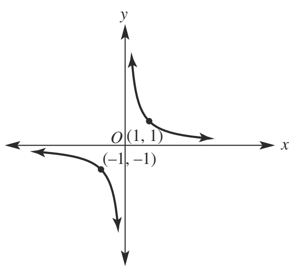
Example 24

a $\left(\frac{1}{4}, 4\right)$

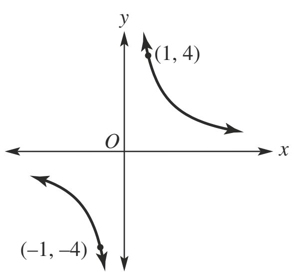
b $\left(-\frac{1}{\sqrt{3}}, -\sqrt{3}\right), \left(\frac{1}{\sqrt{3}}, \sqrt{3}\right)$

Exercise 7J

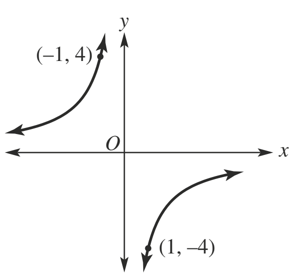
1 a



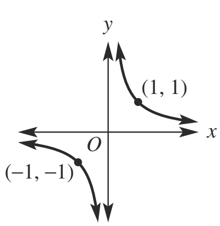
b



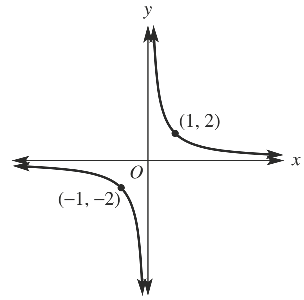
c



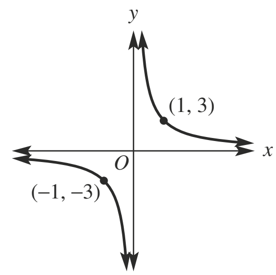
2 a



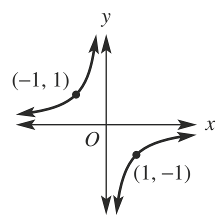
b



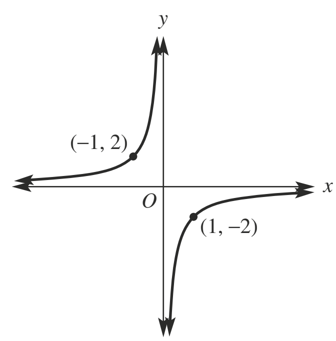
c



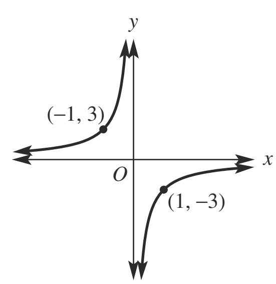
d



e



f



3 a (2, 1) b $\left(4, \frac{1}{2}\right)$ c (–1, –2) d $\left(-6, -\frac{1}{3}\right)$

4 a $\left(10, -\frac{1}{2}\right)$ b $\left(-4, \frac{5}{4}\right)$

c $\left(-7, \frac{5}{7}\right)$ d $\left(9, -\frac{5}{9}\right)$

ISBN 978-1-108-77290-7 Cambridge University Press
 Updated June 2022

5 a $(1, 3)$ b $(3, 1)$ c $\left(-\frac{3}{2}, -2\right)$ d $\left(-\frac{1}{2}, -6\right)$

6 a yes b yes c no d no

7 a $\left(\frac{1}{2}, 2\right)$ b $\left(\frac{1}{6}, 6\right)$
c $(-1, -1)$ d $\left(-\frac{1}{10}, -10\right)$
e $(1, 1), (-1, -1)$ f $\left(-\frac{1}{2}, -2\right), \left(\frac{1}{2}, 2\right)$
g $\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right), \left(-\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$
h $\left(\frac{1}{\sqrt{5}}, \sqrt{5}\right), \left(-\frac{1}{\sqrt{5}}, -\sqrt{5}\right)$

8 a $\left(\frac{2}{3}, -3\right)$ b $\left(-\frac{1}{2}, 4\right)$
c $\left(4, -\frac{1}{2}\right)$ d $\left(-6, \frac{1}{3}\right)$
e $(1, -2), (-1, 2)$ f $\left(\frac{1}{2}, -4\right), \left(-\frac{1}{2}, 4\right)$
g $(2, -1), (-2, 1)$ h $(\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$

9 a E b C c D d B e A f F

10 Yes, $x = 0$ or $y = 0$.

11 a zero b zero
c infinity d negative infinity

12 Greater the coefficient, the closer the graph is to the asymptote.

13 a i $x = \frac{1 \pm \sqrt{5}}{2}, y = \frac{-1 \pm \sqrt{5}}{2}$
ii $x = 1 \pm \sqrt{2}, y = -1 \pm \sqrt{2}$
iii $x = -1 \pm \sqrt{2}, y = 1 \pm \sqrt{2}$
b No intersection, $\triangle < 0$.
c $y = -x + 2, y = -x - 2$

7K

Building understanding

1 a direct proportion b inverse proportion
c inverse proportion d direct proportion
e neither f inverse proportion

2 a Straight line with y-intercept; neither direct nor inverse (indirect) proportion.
b Straight line starting at $(0, 0)$; direct proportion.
c Upward sloping curve so as x increases, y increases; neither direct nor inverse (indirect) proportion.
d Hyperbola shape so as x increases, y decreases; inverse (indirect) proportion.

3 a y is increasing at an increasing rate.
b y is increasing at a decreasing rate.
c y is decreasing at an increasing rate.
d y is decreasing at a decreasing rate.

Now you try

Example 25
a $a = \frac{3}{2}b$ b 60 c 8

Example 26
a $k = 60, n = \frac{60}{m}$ b 5 c 2

Example 27
Increasing distance from home
Constant then varying gradient
Constant then decreasing speed, decelerating

Exercise 7K

1 a i $m = 5h$ ii 30 iii $\frac{8}{5}$
b i $k = 24, y = \frac{24}{x}$ ii 3 iii 2

2 a i $p = 4q$ ii $p = 60$ iii $q = 25$
b i $p = 50q$ ii $p = 750$ iii $q = 4$

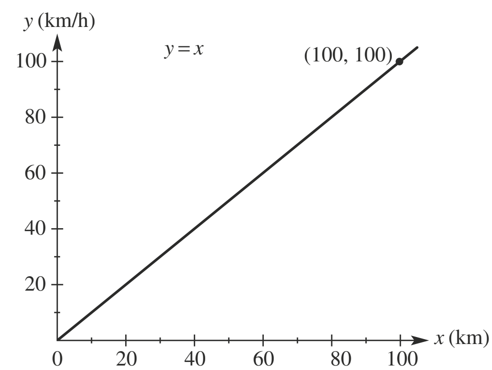
3 a i $k = 72, y = \frac{72}{x}$ ii $y = 2$ iii $x = 24$
b i $k = 50, y = \frac{50}{x}$ ii $y = 0.5$ iii $x = 0.5$

4 a Fixed distance from home, zero gradient, stationary.
b Decreasing distance from home, negative constant gradient, lower constant speed.
c Increasing distance from home, positive varying gradient, increasing speed, accelerating.
d Increasing distance from home, positive varying gradient, decreasing speed, decelerating.
e Decreasing distance from home, negative varying gradient, decreasing speed, decelerating.
f Decreasing distance from home, negative varying gradient, increasing speed, accelerating.

5 A Positive variable rate of change, increasing speed, accelerating.
B Positive constant rate of change, constant speed.
C Positive varying rate of change, decreasing speed, decelerating.
D Zero rate of change, stationary.
E Negative varying rate of change, increasing speed, accelerating.
F Negative constant rate of change, constant speed.
G Negative varying rate of change, decreasing speed, decelerating.

6 a $k = \$244$/tonne b $P = 244n$
c $\$33\,184$ d 1175 tonnes

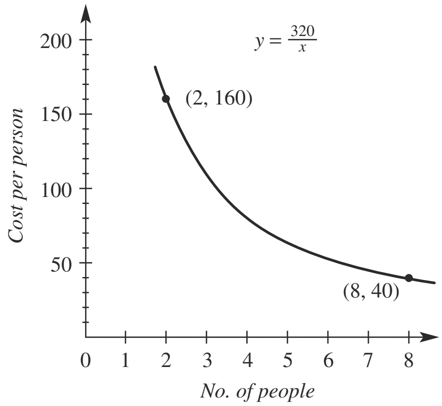
7 a $C = \frac{74}{s}$ b \$4.93 c \$2.47

8 a


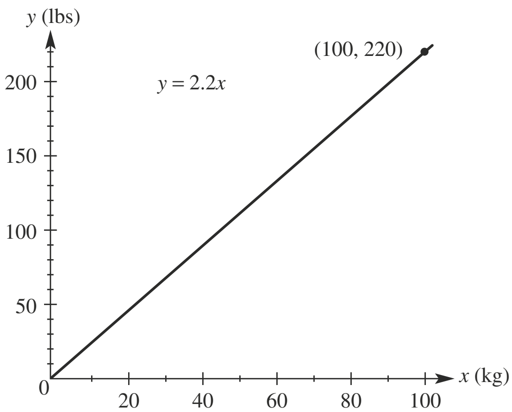
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

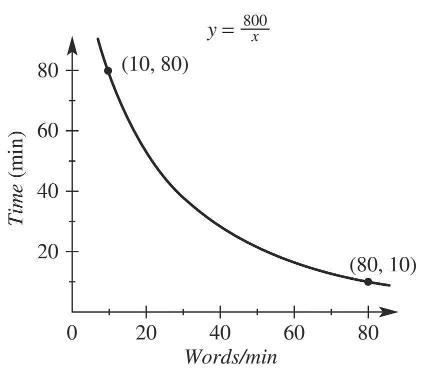
b



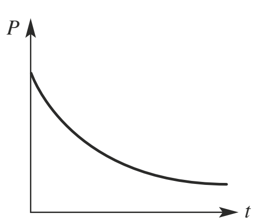
c



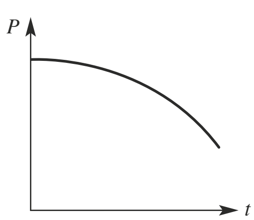
d



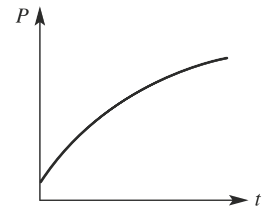
9 a



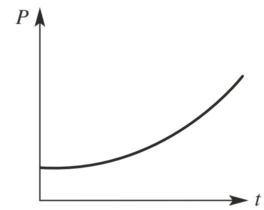
b



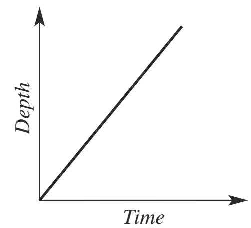
c



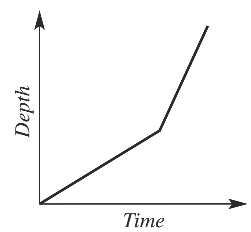
d



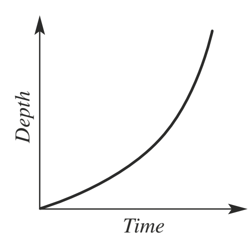
10 a



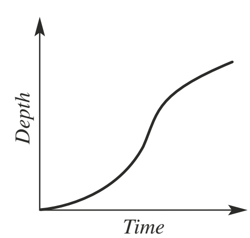
b



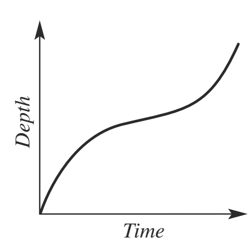
c



d



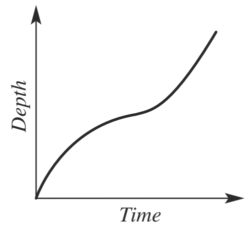
e



ISBN 978-1-108-77290-7

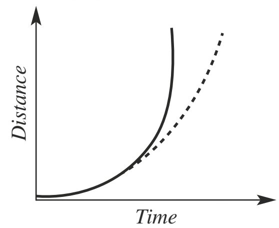
Cambridge University Press
Updated June 2022

f



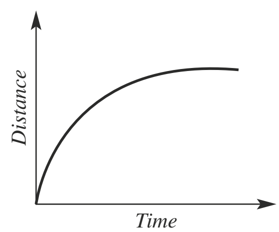
11 Corrected graphs are shown with a dashed line.

a



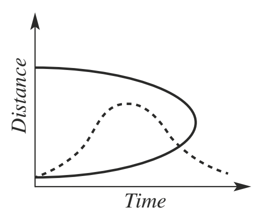
Vertical line incorrect. Can't change distance instantaneously.

b



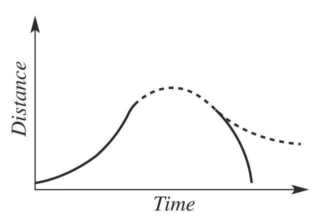
Graph correct.

c



Can't be in two places simultaneously. Curve must increase in gradient, turn, decrease in gradient.

d



Continuous motion means that no breaks in the curve are possible.
Final deceleration segment needs a curve becoming flatter, showing a decreasing gradient.

12 **A** & **d**: School bus; distance increases at an increasing rate (acceleration), then a constant rate (steady speed) and then a decreasing rate (deceleration) becoming a zero rate (stopped).
B & **a**: Soccer player; distance increases at a constant rate (steady speed), then a zero rate (stopped) and then at an increasing rate (acceleration).
C & **c**: Motor bike; distance increases at a constant rate (steady speed), then at an increasing rate (acceleration), and then at a decreasing rate (deceleration) becoming a zero rate (stopped).
D & **b**: Rocket booster; distance increases at an increasing rate (upward acceleration), then a decreasing rate (deceleration when detached) becoming zero (fleetingly stopped). Distance then decreases at an increasing rate (acceleration towards Earth) and finally distance decreases at a constant rate (steady fall to Earth with parachute).

13 Various solutions; check with your teacher.

7L

Building understanding

1 **a** down **b** right **c** left
d down **e** left **f** up

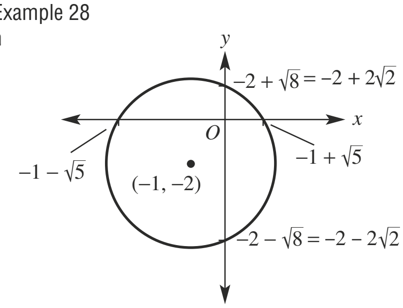
2 **a** $k = 3$ **b** $k = -1$ **c** $k = 4$

3 **a** $h = 0, k = 2$
b $h = 3, k = 0$
c $h = -2, k = -1$

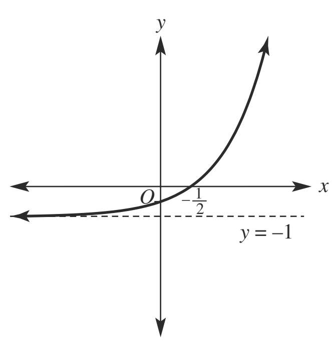
Now you try

Example 28

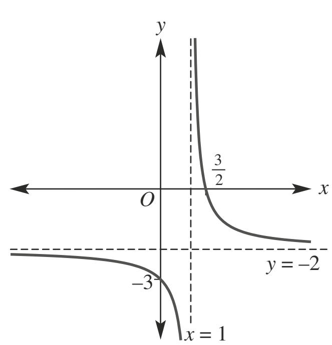
a



b



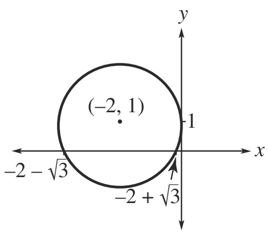
c



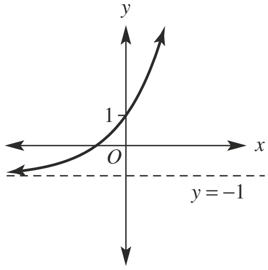
ISBN 978-1-108-77290-7 Cambridge University Press
 Updated June 2022

Exercise 7L

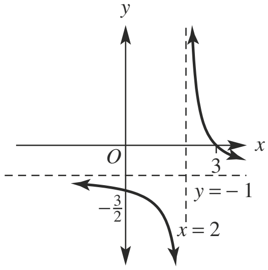
1 a



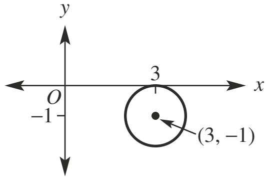
b



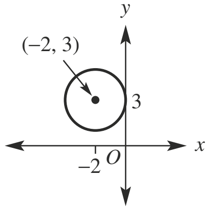
c



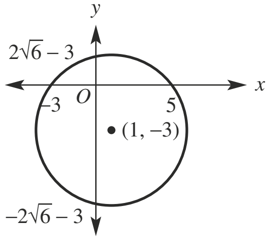
2 a



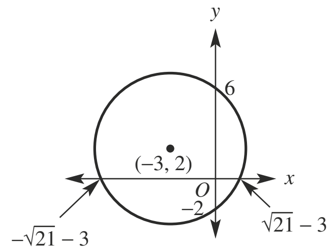
b



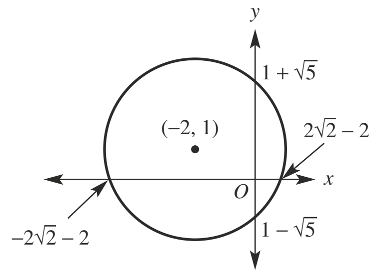
c



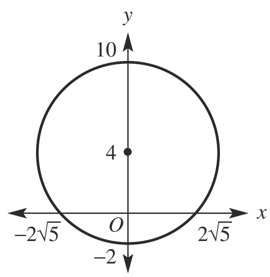
d



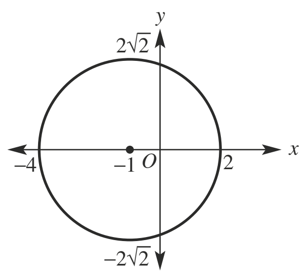
e



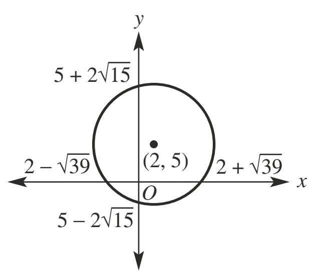
f



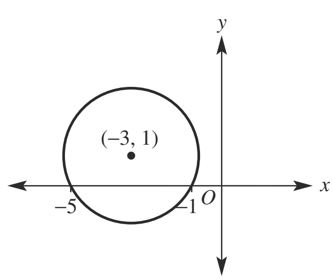
g



h



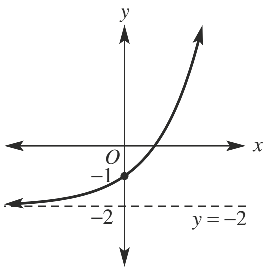
i



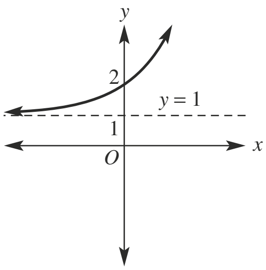
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

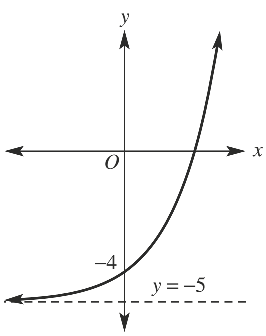
3 a



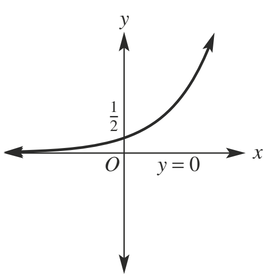
b



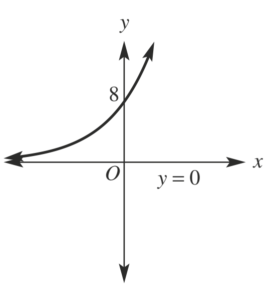
c



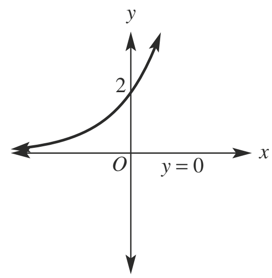
d



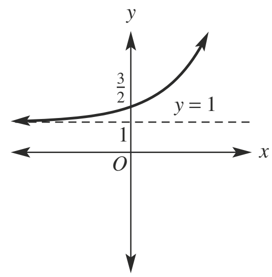
e



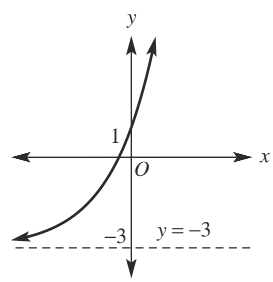
f



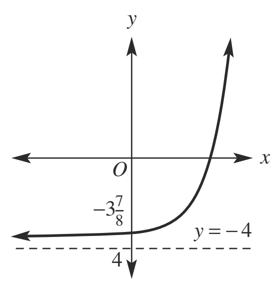
g



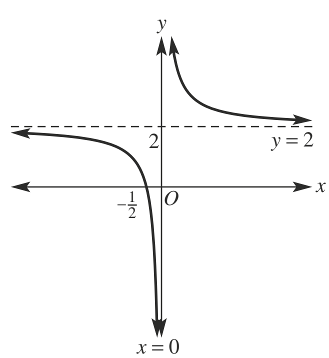
h



i



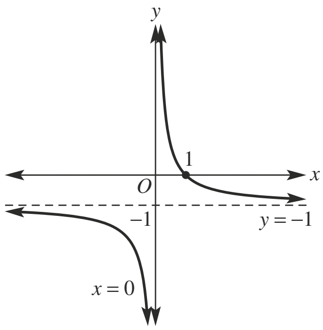
4 a



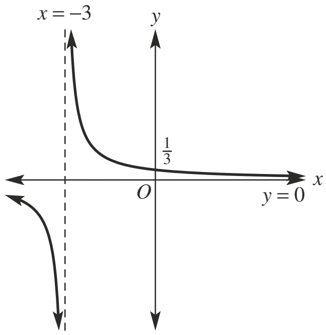
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

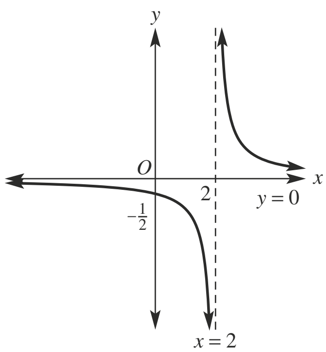
b



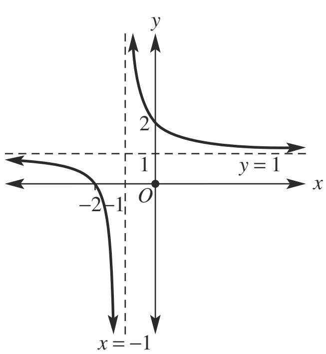
c



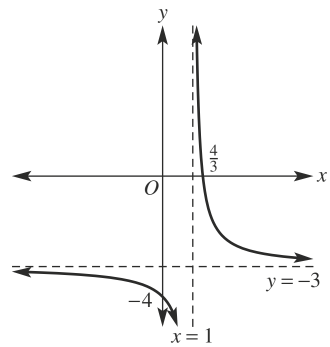
d



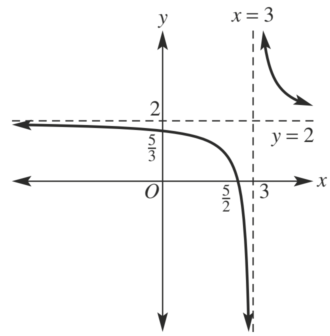
e



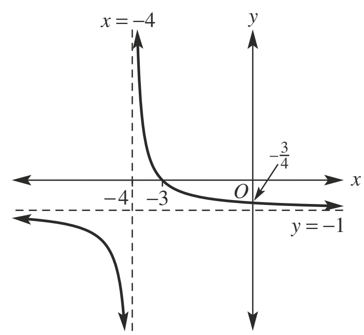
f



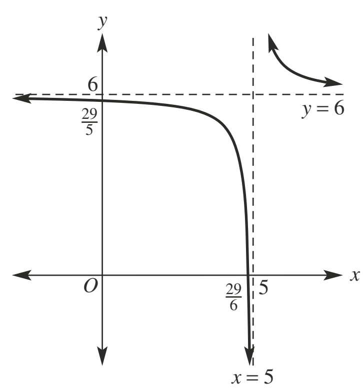
g



h



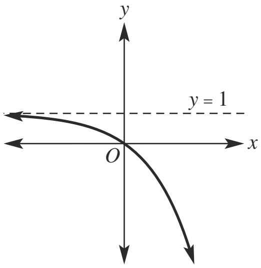
i

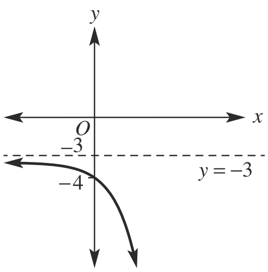


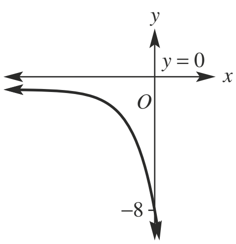
ISBN 978-1-108-77290-7

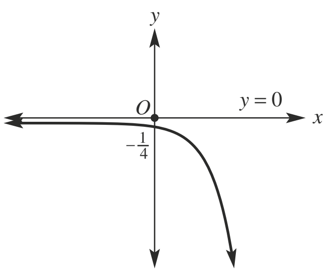
Cambridge University Press

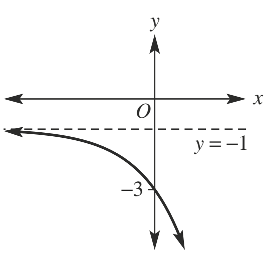
Updated June 2022

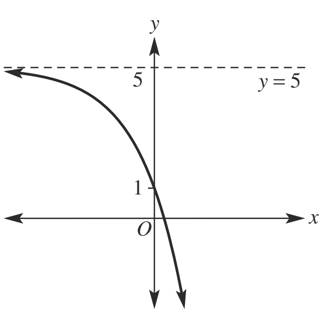
5 a 

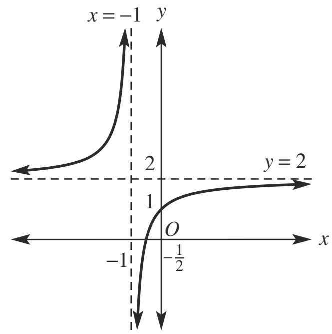
b 

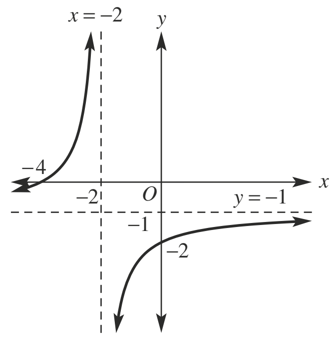
c 

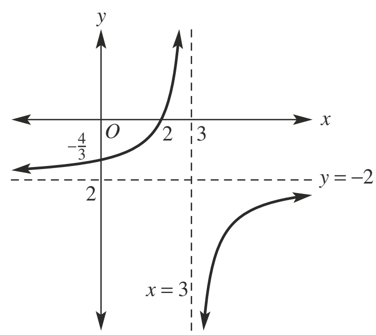
d 

e 

f 

6 a 

b 

c 

7 a $y = \dfrac{1}{x-2} - 1$ b $y = \dfrac{1}{x+1} + 3$

c $y = \dfrac{1}{x-1} + \dfrac{3}{2}$

8 a $\left(\dfrac{-3-\sqrt{5}}{2}, \dfrac{1-\sqrt{5}}{2}\right), \left(\dfrac{-3+\sqrt{5}}{2}, \dfrac{1+\sqrt{5}}{2}\right)$

b $(\sqrt{5}, 3+\sqrt{5}), (-\sqrt{5}, 3-\sqrt{5})$

c $\left(\dfrac{-1-\sqrt{11}}{2}, \sqrt{11}\right), \left(\dfrac{-1+\sqrt{11}}{2}, -\sqrt{11}\right)$

d $(1, 2), \left(-\dfrac{3}{5}, -\dfrac{6}{5}\right)$

e $(-6, 3), (-2, -1)$

f $(3, 0), (-3, -2)$

9 a max $x = 5$, min $x = 1$ b max $y = 0$, min $y = -4$

10 a $(x-2)^2 + (y-1)^2 = 8$ b $(x+2)^2 + y^2 = 25$

c $(x+5)^2 + (y+3)^2 = 18$ d $y = \dfrac{1}{x-1} + 1$

e $y = \dfrac{1}{x+2} - 1$ f $y = \dfrac{-1}{x+3}$

Answers

7L

11 a Solving $\frac{1}{x} = -x$ would require $x^2 = -1$, which is not possible.
b Circle has centre (1, –2) and radius 2, so maximum y-value on the circle is 0, which is less than 1.
c Exponential graph rises more quickly than the straight line and this line sits below the curve.
d Solving $\frac{2}{x+3} - 1 = \frac{1}{3x}$ gives a quadaratic with $\triangle < 0$, thus no points of intersection.

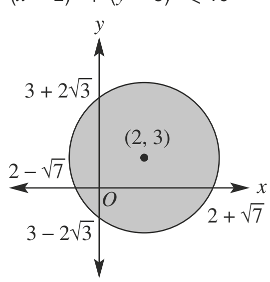
12 a $(x + 2)^2 + (y - 1)^2 = 4, C(-2, 1), r = 2$
b $(x + 4)^2 + (y + 5)^2 = 36, C(-4, -5), r = 6$
c $(x - 3)^2 + (y - 2)^2 = 16, C(3, 2), r = 4$
d $(x - 1)^2 + (y + 3)^2 = 15, C(1, -3), r = \sqrt{15}$
e $(x + 5)^2 + (y + 4)^2 = 24, C(-5, -4), r = 2\sqrt{6}$
f $(x + 3)^2 + (y + 3)^2 = 18, C(-3, -3), r = 3\sqrt{2}$
g $\left(x + \frac{3}{2}\right)^2 + (y - 3)^2 = \frac{29}{4}, C\left(-\frac{3}{2}, 3\right), r = \frac{\sqrt{29}}{2}$
h $\left(x + \frac{5}{2}\right)^2 + (y - 2)^2 = \frac{49}{4}, C\left(-\frac{5}{2}, 2\right), r = \frac{7}{2}$
i $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{3}{2}, C\left(\frac{1}{2}, -\frac{3}{2}\right), r = \sqrt{\frac{3}{2}}$
j $\left(x - \frac{3}{2}\right)^2 + \left(y - \frac{5}{2}\right)^2 = \frac{25}{2}, C\left(\frac{3}{2}, \frac{5}{2}\right), r = \frac{5}{\sqrt{2}}$

13 $(x + 2)^2 + (y - 3)^2 = -2$; radius can't be negative.

Problems and challenges

1 a $-\frac{2}{3} \leqslant x \leqslant \frac{1}{2}$
b $x < -\frac{3}{4}$ or $x > \frac{1}{3}$
c $\frac{7 - \sqrt{41}}{2} < x < \frac{7 + \sqrt{41}}{2}$

2 $(x - 2)^2 + (y - 3)^2 \leqslant 16$



3 a $b^2 - 4ac < 0$
b $b^2 - 4ac = 0$
c $b^2 - 4ac > 0$

4 $(x - 2)^2 + (y + 3)^2 = -15 + 9 + 4 = -2$, which is impossible.

5 a $k = \frac{1}{3}$ b $k < \frac{1}{3}$ c $k > \frac{1}{3}$

6 a $k = \pm\sqrt{20} = \pm 2\sqrt{5}$
b $k > 2\sqrt{5}$ or $k < -2\sqrt{5}$
c $-2\sqrt{5} < k < 2\sqrt{5}$

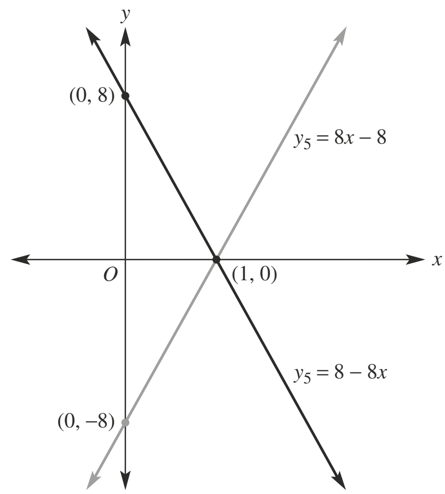
7 a $y = -(x + 1)(x - 3)$
b $y = \frac{3}{4}(x + 2)^2 - 3$
c $y = x^2 - 2x - 3$

8 $a = 2, b = -3, c = -8$; TP $\left(\frac{3}{4}, -\frac{73}{8}\right)$

9 20

10 $4\sqrt{3}$

11

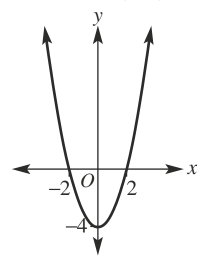


Short-answer questions

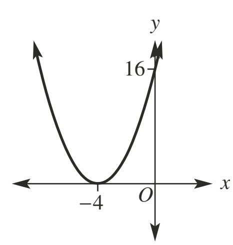
1 a minimum at (1, –4) b $x = 1$
c –1 and 3 d –3

2 a minimum at (2, 0)
b maximum at (0, 5)
c maximum at (–1, –2)
d minimum at (3, 4)

3 a



b

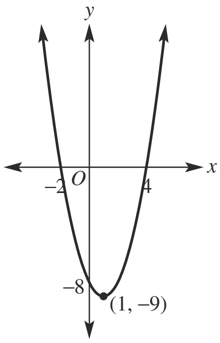


ISBN 978-1-108-77290-7

Cambridge University Press

Updated June 2022

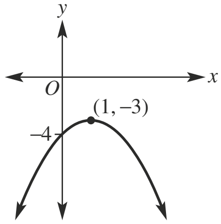
c



4 a i maximum at (1, –3) **ii** –4

iii no x-intercepts

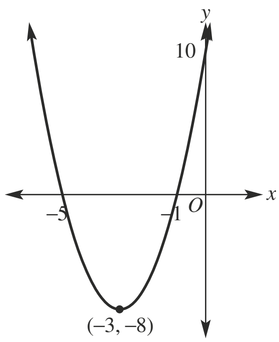
iv



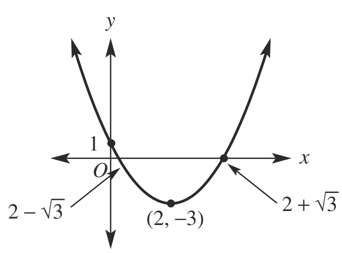
b i minimum at (–3, –8) **ii** 10

iii –1 and –5

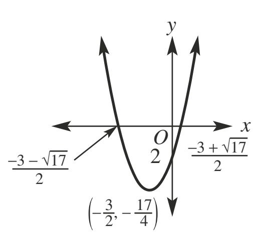
iv



5 a



b



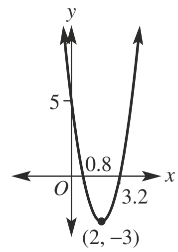
6 a 1 **b** 0 **c** 2 **d** 0

7 a i 5

ii (2, –3)

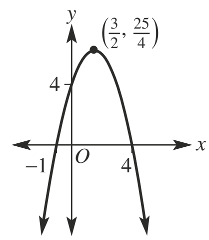
iii 0.8 and 3.2

iv



b i 4 **ii** $\left(\frac{3}{2}, \frac{25}{4}\right)$ **iii** –1 and 4

iv



8 a $x = 2, y = 10$ and $x = -6, y = 10$

b no solutions

c $x = \frac{1}{3}, y = \frac{10}{9}$ and $x = -1, y = 2$

9 Show $b^2 - 4ac = 0$.

10 a 9 **b** 24

c 3 **d** $2k^2 - k + 3$

11 a All real x, All real y.

b All real x, only $y = 4$.

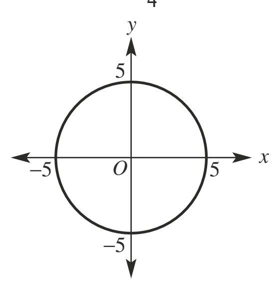
c Only $x = 1$, All real y.

d All real x except $x = 0$, All real y except $y = 0$.

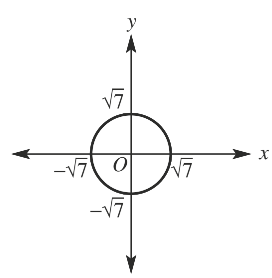
e All real x, $y \geqslant -3$.

f All real x, $y \geqslant -\frac{1}{4}$.

12 a

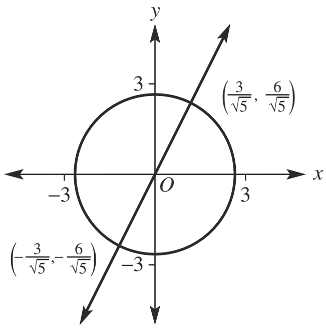


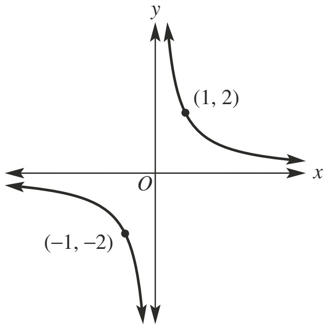
b

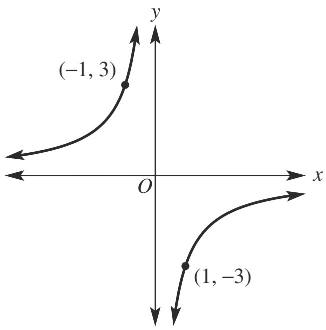


Answers

Ch7 Review

13 

14 a 

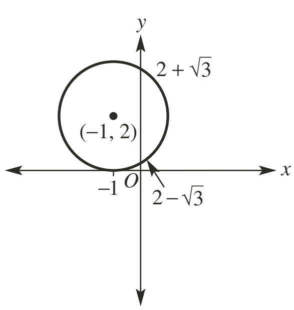
b 

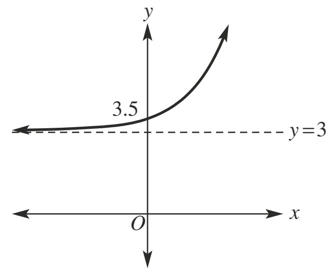
15 a $\left(\frac{4}{3}, 3\right)$

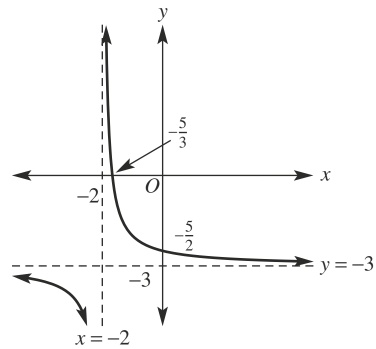
b $(\sqrt{2}, 2\sqrt{2})$ and $(-\sqrt{2}, -2\sqrt{2})$

16 a $y = 5x$

b i $k = 72$ ii 18 iii $\frac{720}{7}$

17 a 

b 

c 

Multiple-choice questions

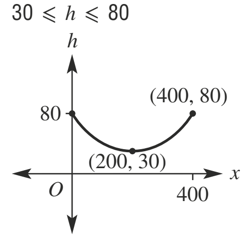
1 B	2 D	3 E	4 D	5 A
6 A	7 C	8 D	9 A	10 D
11 E	12 E	13 A	14 C	

Extended-response questions

1 a (200, 30)

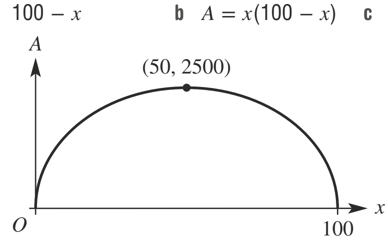
b $0 \leqslant x \leqslant 400$

c $30 \leqslant h \leqslant 80$

d 

e 400 m f 30 m g 80 m

2 a $100 - x$ b $A = x(100 - x)$ c $0 < x < 100$

d 

e 2500 m^2

f 50 m by 50 m

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Chapter 8

8A

Building understanding

1 a 2 b {H, T} c yes
d $\frac{1}{2}$ e $\frac{1}{2}$ f 1

2 a $\frac{1}{4}$ b $\frac{1}{6}$ c $\frac{3}{8}$ d 0

3 a 4 b 20 c 100

Now you try

Example 1

a $\frac{1}{11}$ b $\frac{2}{11}$ c $\frac{9}{11}$ d $\frac{3}{11}$

Example 2

a 0.1 b 0.09 c 0.87 d 0.9

Exercise 8A

1 a $\frac{1}{7}$ b $\frac{2}{7}$ c $\frac{5}{7}$ d $\frac{3}{7}$

2 a $\frac{3}{10}$ b $\frac{2}{5}$ c $\frac{3}{5}$ d $\frac{1}{2}$

3 a $\frac{1}{10}$ b $\frac{1}{2}$ c $\frac{1}{2}$ d $\frac{1}{2}$
e $\frac{2}{5}$ f $\frac{1}{5}$ g $\frac{3}{10}$ h $\frac{1}{10}$

4 a 0.09 b 0.43 c 0.47 d 0.91

5 a 0.62 b 0.03 c 0.97 d 0.38

6 a $\frac{1}{50}$ b $\frac{3}{10}$ c $\frac{49}{50}$

7 a $\frac{1}{2}$ b $\frac{3}{8}$ c $\frac{1}{4}$ d $\frac{5}{24}$ e 1 f 0

8 a $\frac{6}{25}$ b $\frac{1}{50}$ c $\frac{21}{25}$ d $\frac{2}{5}$ e $\frac{2}{25}$ f $\frac{4}{25}$

9 a i $\frac{7}{10}$ ii $\frac{1}{5}$ iii $\frac{1}{20}$ iv 0 v $\frac{1}{20}$
b $\frac{1}{10}$

10 a 59
b 4, as $\frac{41}{100}$ of 10 is closest to 4.
c 8, as $\frac{41}{100}$ of 20 is closest to 8.

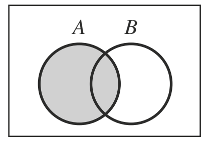
11 a $\frac{1}{4}$ b $\frac{1}{13}$ c $\frac{1}{52}$ d $\frac{1}{2}$
e $\frac{2}{13}$ f $\frac{4}{13}$ g $\frac{12}{13}$ h $\frac{9}{13}$

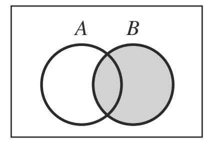
12 a $\frac{7}{15}$
b 15; any multiple of 15 is a possibility as 3 and 5 must be factors.

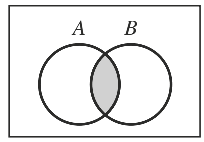
13 a 625π
b i 25π ii 200π iii 400π
c i $\frac{1}{25}$ ii $\frac{8}{25}$ iii $\frac{16}{25}$ iv $\frac{9}{25}$
v $\frac{24}{25}$ vi $\frac{17}{25}$ vii 1 viii $\frac{17}{25}$
d No, it doesn't.

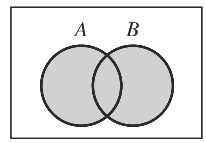
8B

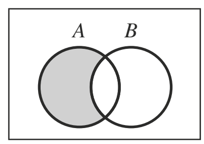
Building understanding

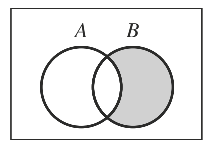
1 a


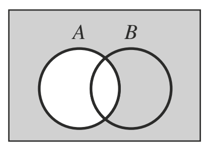
b


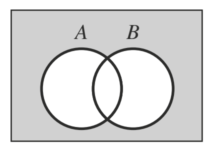
c


d


e


f


g


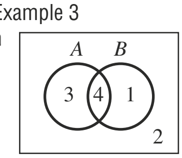
h


2 a $\varnothing$ b $\cap$ c $\cup$ d $\cup$, $\cup$

3 a no b yes c no

Now you try

Example 3

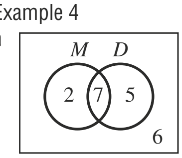
a


b i {2, 3, 5, 7}
ii {1, 2, 3, 4, 5, 6, 7, 8}

c i $\frac{7}{10}$ ii $\frac{2}{5}$ iii $\frac{4}{5}$

d No, $A \cap B \neq \varnothing$

Example 4

a


b i 2 ii 6

c i $\frac{9}{20}$ ii $\frac{1}{4}$ iii $\frac{7}{20}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Example 5

a

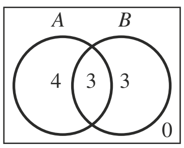
	A	A'	
B	2	3	5
B'	4	1	5
	6	4	10

b i 2 ii 3 iii 4 iv 1
v 6 vi 5 vii 9

c i $\frac{1}{5}$ ii $\frac{2}{5}$ iii $\frac{2}{5}$

Exercise 8B

1 a

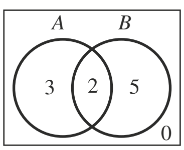


b i $A \cap B = \{2, 5, 8\}$
ii $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

c i $\frac{7}{10}$ ii $\frac{3}{10}$ iii 1

d No, since $A \cap B \neq \phi$

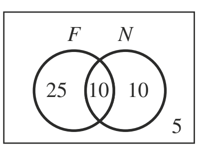
2 a



b i $A \cap B = \{2, 13\}$
ii $A \cup B = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$

c i $\frac{1}{2}$ ii $\frac{7}{10}$ iii $\frac{1}{5}$ iv 1

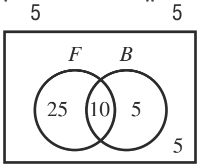
3 a



b i 25 ii 5

c i $\frac{2}{5}$ ii $\frac{1}{5}$ iii $\frac{1}{5}$

4 a



b i 25 ii 5

c i $\frac{7}{9}$ ii $\frac{2}{9}$ iii $\frac{8}{9}$ iv $\frac{2}{9}$ v $\frac{1}{9}$

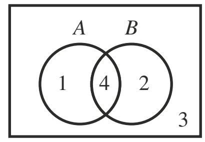
5 a

	A	A'	
B	2	6	8
B'	5	3	8
	7	9	16

b i 2 ii 6 iii 5 iv 3
v 7 vi 8 vii 13 viii 16

c i $\frac{1}{8}$ ii $\frac{9}{16}$ iii $\frac{5}{16}$

6 a



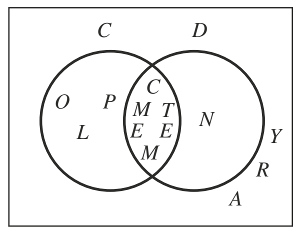
b

	A	A'	
B	4	2	6
B'	1	3	4
	5	5	10

c i 2 ii 3 iii $\frac{2}{5}$ iv $\frac{7}{10}$

7 a 4 **b** 10, 12 **c** a, c, e **d** nothing

8 a



b i $\frac{9}{13}$ ii $\frac{6}{13}$ iii $\frac{10}{13}$ iv $\frac{4}{13}$ v $\frac{3}{13}$

9 a

	A	A'	
B	3	3	6
B'	4	1	5
	7	4	11

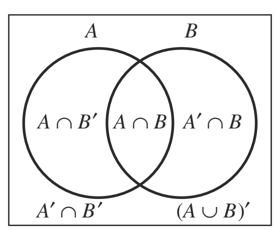
b

	A	A'	
B	2	7	9
B'	2	1	3
	4	8	12

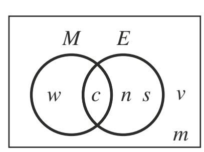
10 3

11 a $1 - a$ **b** $a + b$ **c** 0

12



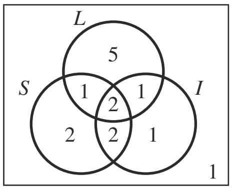
13 a



b i $\frac{1}{3}$ ii $\frac{2}{3}$ iii $\frac{1}{6}$ iv $\frac{2}{3}$ v $\frac{1}{3}$

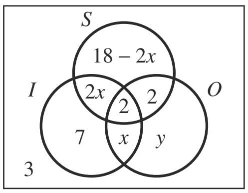
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

14 a 

b 1

c i $\frac{3}{5}$ ii $\frac{1}{3}$ iii $\frac{13}{15}$ iv $\frac{1}{15}$

15 a 

S = Own state
I = Interstate
O = Overseas

b i 4 ii 10

c i $\frac{5}{19}$ ii $\frac{1}{19}$ iii $\frac{7}{38}$ iv $\frac{35}{38}$ v $\frac{25}{38}$

8C

Building understanding

1 a i {4, 5, 6} ii {2, 4, 6}
iii {2, 4, 5, 6} iv {4, 6}

b No, $A \cap B \neq \phi$

c $\frac{2}{3}$

2 a 0.8 b 0.7

3 0.05

Now you try

Example 6

a i 13 ii 4 iii 1

b i $\frac{1}{4}$ ii $\frac{3}{4}$ iii $\frac{1}{52}$

c $\frac{4}{13}$

d $\frac{10}{13}$

Example 7

a 0.35 b 0.25

Exercise 8C

1 a i 13 ii 4 iii 1

b i $\frac{1}{4}$ ii $\frac{3}{4}$ iii $\frac{1}{52}$

c $\frac{4}{13}$

d $\frac{10}{13}$

2 a i {3, 6, 9, 12, 15, 18} ii {2, 3, 5, 7, 11, 13, 17, 19}

b i $\frac{1}{20}$ ii $\frac{13}{20}$

c $\frac{7}{20}$

3 a $\frac{1}{8}$ b $\frac{5}{24}$

4 a 0.1 b 0.2

5 a 0.3 b 0.1

6 a $\frac{3}{8}$ b $\frac{5}{32}$

7 a $\frac{4}{13}$ b $\frac{4}{13}$ c $\frac{7}{13}$

d $\frac{49}{52}$ e $\frac{10}{13}$ f $\frac{10}{13}$

8 a 0.4 b 0.45

9 Because $\Pr(A \cap B) = 0$ for mutually exclusive events

10 a $\Pr(A) < \Pr(A \cap B)$ b $\Pr(A) + \Pr(B) < \Pr(A \cup B)$

11 $\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C)$

12 a $\frac{3}{10}$ b $\frac{1}{4}$ c $\frac{3}{20}$

d $\frac{13}{20}$ e $\frac{9}{20}$ f $\frac{3}{5}$

13 a $\frac{1}{4}$ b $\frac{71}{500}$ c $\frac{33}{500}$

d $\frac{7}{100}$ e $\frac{1}{25}$ f $\frac{7}{500}$

8D

Building understanding

1 a $\frac{1}{3}$ b $\frac{1}{2}$

2 a i 2 ii 9

b $\frac{2}{9}$

3 a i 7 ii 10

b $\frac{7}{10}$ c $\frac{7}{12}$

Now you try

Example 8

a $\frac{7}{13}$ b $\frac{3}{13}$ c $\frac{3}{8}$ d $\frac{3}{7}$

Example 9

a

	A	A'	
B	9	6	15
B'	4	4	8
	13	10	23

b $\frac{4}{23}$ c $\frac{3}{5}$ d $\frac{9}{13}$

Exercise 8D

1 a $\frac{5}{12}$ b $\frac{1}{4}$ c $\frac{3}{8}$ d $\frac{3}{5}$

2 a i $\frac{9}{13}$ ii $\frac{3}{13}$ iii $\frac{3}{7}$ iv $\frac{1}{3}$

b i $\frac{14}{17}$ ii $\frac{4}{17}$ iii $\frac{4}{7}$ iv $\frac{2}{7}$

ISBN 978-1-108-77290-7 Cambridge University Press
 Updated June 2022

c i $\frac{3}{4}$ ii $\frac{5}{8}$ iii $\frac{5}{7}$ iv $\frac{5}{6}$

d i $\frac{7}{16}$ ii $\frac{1}{8}$ iii $\frac{1}{4}$ iv $\frac{2}{7}$

3 a i $\frac{7}{18}$ ii $\frac{1}{9}$ iii $\frac{1}{5}$ iv $\frac{2}{7}$

b i $\frac{4}{9}$ ii $\frac{1}{9}$ iii $\frac{1}{5}$ iv $\frac{1}{4}$

c i $\frac{8}{17}$ ii $\frac{7}{17}$ iii $\frac{7}{10}$ iv $\frac{7}{8}$

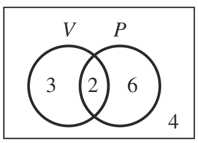
d i $\frac{3}{4}$ ii $\frac{1}{4}$ iii $\frac{2}{3}$ iv $\frac{1}{3}$

4 a

	A	A'	
B	9	6	15
B'	4	1	5
	13	7	20

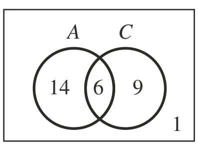
b $\frac{1}{5}$ c $\frac{3}{5}$ d $\frac{9}{13}$

5 a



b 4 c $\frac{2}{5}$ d $\frac{1}{4}$

6 a



	A	A'	
C	6	9	15
C'	14	1	15
	20	10	30

b i $\frac{3}{10}$ ii $\frac{7}{15}$

c $\frac{2}{5}$ d $\frac{3}{10}$

7 a

	A	A'	
B	2	2	4
B'	3	1	4
	5	3	8

i 1 ii $\frac{2}{5}$ iii $\frac{1}{2}$

b

	A	A'	
B	3	13	16
B'	5	6	11
	8	19	27

i 6 ii $\frac{3}{8}$ iii $\frac{3}{16}$

8 a $\frac{1}{13}$ b $\frac{1}{13}$ c $\frac{1}{4}$ d $\frac{1}{2}$

9 a $\frac{1}{3}$ b $\frac{1}{2}$

10 $\Pr(A|B) = \Pr(B|A) = 0$ as $\Pr(A \cap B) = 0$

11 a 1 b $\frac{1}{5}$

12 a $\Pr(A \cap B) = \Pr(A) \times \Pr(B|A)$ b 0.18

13 a 329 b $\frac{174}{329}$ c $\frac{81}{329}$

d $\frac{24}{155}$ e $\frac{31}{231}$ f $\frac{18}{31}$

Progress quiz

1 a $\frac{1}{10}$ b $\frac{1}{10}$ c $\frac{1}{20}$ d $\frac{2}{5}$ e $\frac{3}{5}$

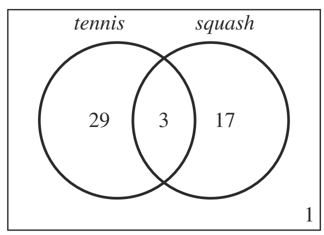
2 a 0.17 b 0.29 c 0.33 d 0.67

3 a $\frac{1}{13}$ b $\frac{1}{26}$ c $\frac{1}{52}$

d $\frac{1}{13}$ e $\frac{1}{13}$ f $\frac{12}{13}$

4 a

	Like tennis	Dislike tennis	Total
Like squash	3	17	20
Dislike squash	29	1	30
Total	32	18	50



b 29

c $\frac{1}{50}$

5 a 7 b 20 c 5

d $\frac{7}{33}$ e $\frac{13}{33}$ f $\frac{28}{33}$

6 a 0.83 b 0.17

7 a $\frac{1}{2}$ b $\frac{1}{6}$ c $\frac{1}{4}$ d $\frac{1}{3}$

8 a

	Like soft drink	Dislike soft drink	Total
Like water	15	5	20
Dislike water	20	0	20
Total	35	5	40

b $\frac{1}{4}$

ISBN 978-1-108-77290-7

Cambridge University Press

Updated June 2022

8E

Building understanding

1 a i 9 ii 6
b i $\frac{1}{3}$ ii $\frac{5}{9}$ iii $\frac{4}{9}$
c i 0 ii $\frac{2}{3}$ iii $\frac{1}{3}$
2 a 9 b 6

Now you try

Example 10
a $\frac{1}{36}$
b i $\frac{1}{18}$ ii $\frac{5}{18}$ iii $\frac{1}{12}$
c $\frac{1}{3}$

Example 11
a

		1st			
		R	R	B	W
2nd	R	X	(R, R)	(B, R)	(W, R)
	R	(R, R)	X	(B, R)	(W, R)
	B	(R, B)	(R, B)	X	(W, B)
	W	(R, W)	(R, W)	(B, W)	X

b i $\frac{1}{6}$ ii $\frac{1}{6}$ iii $\frac{1}{3}$ iv $\frac{1}{5}$

Exercise 8E

1 a

		1st roll			
		1	2	3	4
2nd roll	1	(1, 1)	(2, 1)	(3, 1)	(4, 1)
	2	(1, 2)	(2, 2)	(3, 2)	(4, 2)
	3	(1, 3)	(2, 3)	(3, 3)	(4, 3)
	4	(1, 4)	(2, 4)	(3, 4)	(4, 4)

b 16 c $\frac{1}{16}$
d i $\frac{1}{4}$ ii $\frac{5}{8}$ iii $\frac{13}{16}$

2 a

		1st toss	
		H	T
2nd toss	H	(H, H)	(T, H)
	T	(H, T)	(T, T)

b 4
c $\frac{1}{4}$
d i $\frac{1}{2}$ ii $\frac{3}{4}$
e 250

3 a

		1st		
		S	E	T
2nd	S	X	(E, S)	(T, S)
	E	(S, E)	X	(T, E)
	T	(S, T)	(E, T)	X

b i $\frac{1}{6}$ ii $\frac{2}{3}$ iii $\frac{2}{3}$ iv $\frac{1}{3}$ v 1

4 a

		1st				
		L	E	V	E	L
2nd	L	X	(E, L)	(V, L)	(E, L)	(L, L)
	E	(L, E)	X	(V, E)	(E, E)	(L, E)
	V	(L, V)	(E, V)	X	(E, V)	(L, V)
	E	(L, E)	(E, E)	(V, E)	X	(L, E)
	L	(L, L)	(E, L)	(V, L)	(E, L)	X

b 20
c i 8 ii 12 iii 12
d i $\frac{2}{5}$ ii $\frac{3}{5}$ iii $\frac{3}{5}$
e $\frac{1}{5}$

5 a

		Die 1					
		1	2	3	4	5	6
Die 2	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

b 36
c i 2 ii 6 iii 15
d i $\frac{1}{6}$ ii $\frac{1}{6}$ iii $\frac{35}{36}$ iv $\frac{1}{12}$
e $\frac{1}{6}$. Her guess is wrong.

6 a

		1st		
		O	L	D
2nd	C	(O, C)	(L, C)	(D, C)
	O	(O, O)	(L, O)	(D, O)
	L	(O, L)	(L, L)	(D, L)
	L	(O, L)	(L, L)	(D, L)
	E	(O, E)	(L, E)	(D, E)
	G	(O, G)	(L, G)	(D, G)
	E	(O, G)	(L, G)	(D, G)

b 21 c $\frac{1}{7}$

ISBN 978-1-108-77290-7

Updated June 2022

7 a i 100 ii 90
b i $\frac{1}{10}$ ii $\frac{1}{10}$ iii $\frac{4}{5}$
c $\frac{19}{100}$

8 a i $\frac{1}{4}$ ii $\frac{5}{8}$
b i $\frac{2}{5}$ ii $\frac{1}{10}$ iii $\frac{2}{3}$

9 a without b with c with d without

10 a 30
b i $\frac{1}{15}$ ii $\frac{1}{15}$ iii $\frac{2}{15}$ iv $\frac{4}{15}$
c $\frac{1}{18}$

11 a

		1st			
		2.5	5	10	20
2nd	2.5	5	7.5	12.5	22.5
	5	7.5	10	15	25
	10	12.5	15	20	30
	20	22.5	25	30	40

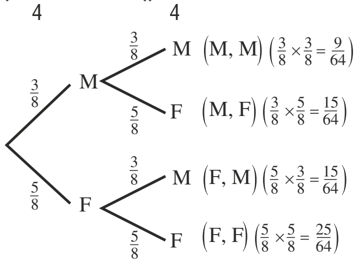
b 16
c i 1 ii 8 iii 8
d i $\frac{1}{16}$ ii $\frac{1}{8}$ iii $\frac{1}{4}$ iv $\frac{3}{16}$
e $\frac{7}{16}$

8F

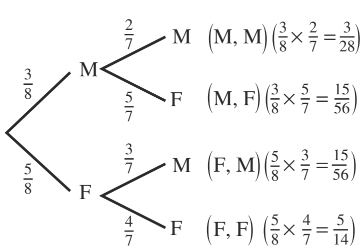
Building understanding

1 a i $\frac{2}{5}$ ii $\frac{3}{5}$ b i $\frac{2}{5}$ ii $\frac{3}{5}$
c i $\frac{1}{4}$ ii $\frac{3}{4}$

2 a



b

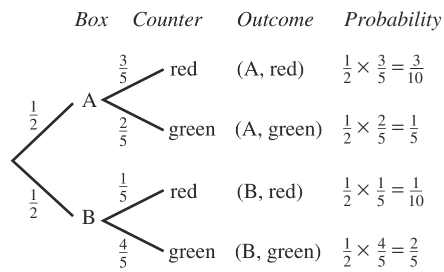


Now you try

Example 12

a $\frac{3}{5}$ b $\frac{1}{5}$

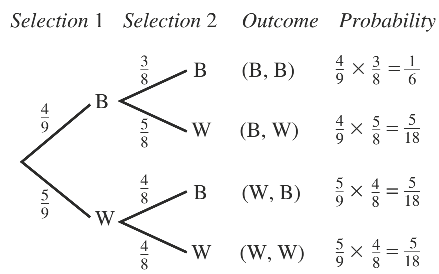
c



d $\frac{1}{10}$ e $\frac{2}{5}$

Example 13

a

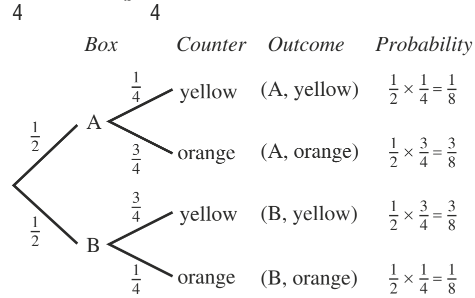


b i $\frac{5}{18}$ ii $\frac{1}{6}$ iii $\frac{5}{9}$
c i $\frac{20}{81}$ ii $\frac{16}{81}$ iii $\frac{40}{81}$

Exercise 8F

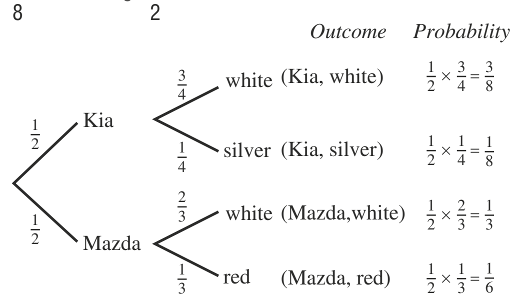
1 a $\frac{1}{4}$ b $\frac{3}{4}$

c



d $\frac{3}{8}$ e $\frac{1}{2}$

2 a



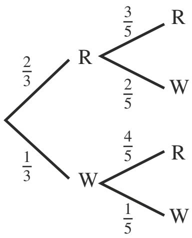
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b **i** $\frac{3}{8}$ **ii** $\frac{1}{6}$ **iii** $\frac{17}{24}$

iv $\frac{7}{24}$ **v** $\frac{5}{6}$ **vi** $\frac{1}{3}$

3 **a**

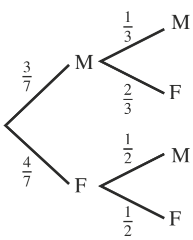


Outcome	*Probabilities*
(R, R)	$\frac{2}{3}\times\frac{3}{5}=\frac{2}{5}$
(R, W)	$\frac{2}{3}\times\frac{2}{5}=\frac{4}{15}$
(W, R)	$\frac{1}{3}\times\frac{4}{5}=\frac{4}{15}$
(W, W)	$\frac{1}{3}\times\frac{1}{5}=\frac{1}{15}$

b **i** $\frac{4}{15}$ **ii** $\frac{2}{5}$ **iii** $\frac{8}{15}$

c **i** $\frac{2}{9}$ **ii** $\frac{4}{9}$ **iii** $\frac{4}{9}$

4 **a**

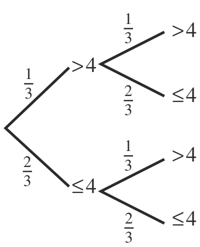


Outcome	*Probabilities*
(M, M)	$\frac{3}{7}\times\frac{1}{3}=\frac{1}{7}$
(M, F)	$\frac{3}{7}\times\frac{2}{3}=\frac{2}{7}$
(F, M)	$\frac{4}{7}\times\frac{1}{2}=\frac{2}{7}$
(F, F)	$\frac{4}{7}\times\frac{1}{2}=\frac{2}{7}$

i $\frac{1}{7}$ **ii** $\frac{2}{7}$ **iii** $\frac{4}{7}$ **iv** $\frac{3}{7}$

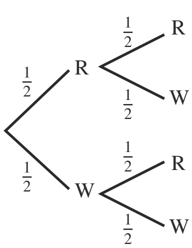
b **i** $\frac{9}{49}$ **ii** $\frac{16}{49}$ **iii** $\frac{24}{49}$ **iv** $\frac{25}{49}$

5 **a**



b **i** $\frac{1}{9}$ **ii** $\frac{4}{9}$ **iii** $\frac{5}{9}$ **iv** $\frac{4}{9}$

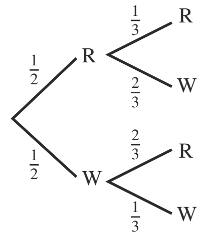
6 **a**



Outcome
(R, R)
(R, W)
(W, R)
(W, W)

i $\frac{1}{4}$ **ii** $\frac{1}{2}$ **iii** $\frac{3}{4}$ **iv** $\frac{3}{4}$

b

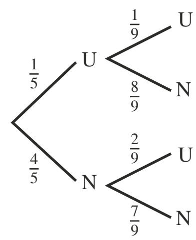


Outcome	*Probability*
(R, R)	$\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$
(R, W)	$\frac{1}{2}\times\frac{2}{3}=\frac{1}{3}$
(W, R)	$\frac{1}{2}\times\frac{2}{3}=\frac{1}{3}$
(W, W)	$\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$

i $\frac{1}{6}$ **ii** $\frac{2}{3}$ **iii** $\frac{5}{6}$ **iv** $\frac{5}{6}$

7 **a** **i** $\frac{1}{5}$ **ii** $\frac{4}{5}$

b



Outcome	*Probability*
(U, U)	$\frac{1}{5}\times\frac{1}{9}=\frac{1}{45}$
(U, N)	$\frac{1}{5}\times\frac{8}{9}=\frac{8}{45}$
(N, U)	$\frac{4}{5}\times\frac{2}{9}=\frac{8}{45}$
(N, N)	$\frac{4}{5}\times\frac{7}{9}=\frac{28}{45}$

i $\frac{1}{45}$ **ii** $\frac{16}{45}$ **iii** $\frac{44}{45}$

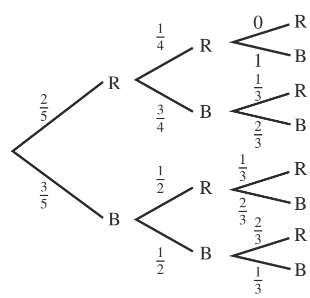
c 62.2%

8 **a** **i** 0.17 **ii** 0.11 **iii** 0.83

b **i** 0.1445 **ii** 0.0965 **iii** 0.8555

9 **a** $\frac{3}{7}$ **b** $\frac{4}{7}$

10 **a**

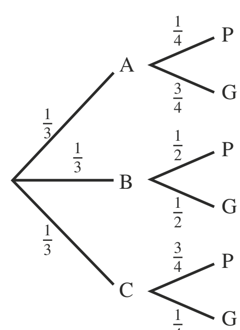


Outcome	*Probability*
(R, R, R)	0
(R, R, B)	$\frac{1}{10}$
(R, B, R)	$\frac{1}{10}$
(R, B, B)	$\frac{1}{5}$
(B, R, R)	$\frac{1}{10}$
(B, R, B)	$\frac{1}{5}$
(B, B, R)	$\frac{1}{5}$
(B, B, B)	$\frac{1}{10}$

i $\frac{1}{10}$ **ii** $\frac{3}{10}$ **iii** 0 **iv** $\frac{9}{10}$ **v** $\frac{9}{10}$

b **i** 1 **ii** $\frac{2}{5}$

11 **a**



Outcome	*Probability*
(A, P)	$\frac{1}{3}\times\frac{1}{4}=\frac{1}{12}$
(A, G)	$\frac{1}{3}\times\frac{3}{4}=\frac{1}{4}$
(B, P)	$\frac{1}{3}\times\frac{1}{2}=\frac{1}{6}$
(B, G)	$\frac{1}{3}\times\frac{1}{2}=\frac{1}{6}$
(C, P)	$\frac{1}{3}\times\frac{3}{4}=\frac{1}{4}$
(C, G)	$\frac{1}{3}\times\frac{1}{4}=\frac{1}{12}$

Answers

8F

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b 6

c **i** $\frac{1}{12}$ **ii** $\frac{1}{6}$ **iii** $\frac{1}{4}$

d $\frac{1}{2}$

12 **a** **i** $\frac{7}{8}$ **ii** $\frac{1}{8}$

b \$87.50 to player A, \$12.50 to player B

c **i** A \$68.75, B \$31.25 **ii** A \$50, B \$50

iii A \$81.25, B \$18.75 **iv** A \$34.38, B \$65.62

v Answers may vary.

8G

Building understanding

1 **a** **i** $\frac{1}{2}$ **ii** $\frac{1}{2}$

b yes

c $\frac{1}{2}$

2 **a** **i** $\frac{3}{10}$ **ii** $\frac{1}{3}$

b no

c no

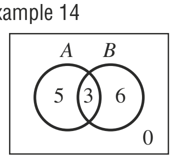
3 **a** with

b without

Now you try

Example 14

a

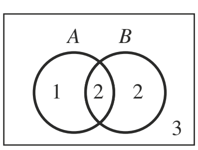


b **i** $\frac{4}{7}$ **ii** $\frac{1}{3}$

c No. $\Pr(A|B) \neq \Pr(A)$

Exercise 8G

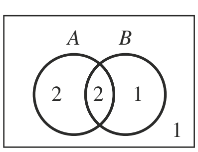
1 **a**



b **i** $\frac{3}{8}$ **ii** $\frac{1}{2}$

c not independent

2 **a**



b **i** $\frac{2}{3}$ **ii** $\frac{2}{3}$

c independent

3 **a** **i** $\frac{3}{4}, \frac{1}{2}$ **ii** not independent

b **i** $\frac{1}{4}, \frac{1}{4}$ **ii** independent

c **i** $\frac{1}{3}, \frac{1}{3}$ **ii** independent

d **i** $\frac{2}{7}, 0$ **ii** not independent

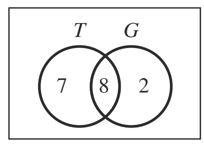
4 **a** $\Pr(A) = \frac{1}{2}$, $\Pr(A|B) = \frac{1}{2}$, independent

b $\Pr(A) = \frac{3}{10}$, $\Pr(A|B) = \frac{1}{4}$, not independent

c $\Pr(A) = \frac{5}{12}$, $\Pr(A|B) = \frac{3}{20}$, not independent

d $\Pr(A) = \frac{1}{9}$, $\Pr(A|B) = \frac{1}{9}$, independent

5 **a**



	T	T'	
G	8	2	10
G'	7	0	7
	15	2	17

i $\frac{15}{17}$ **ii** $\frac{7}{17}$ **iii** $\frac{4}{5}$

b no

6 **a** $\frac{1}{32}$ **b** $\frac{31}{32}$ **c** $\frac{31}{32}$

7 **a** $\frac{1}{216}$ **b** $\frac{1}{216}$ **c** $\frac{1}{72}$ **d** $\frac{1}{36}$

8 False; $\Pr(A|B) = 0$ but $\Pr(A) = \frac{2}{9}$.

9 **a** 6 **b** 22 **c** 2

10 **a** 0.24 **b** 0.76

11 $\frac{5}{6}$

Problems and challenges

1 **a** 0.16 **b** 0.192 **c** 0.144

2 0.593 75

3 **a** $\frac{7}{8}$ **b** 1 **c** $\frac{4}{7}$

4 **a** $\frac{1}{12}$ **b** $\frac{1}{2}$ **c** $\frac{3}{4}$ **d** $\frac{2}{3}$

5 $\frac{63}{64}$

6 $\frac{1}{13\,983\,816}$

7 $\frac{3}{5}$

8 $\frac{1}{12}$

9 true

10 $\frac{8}{9}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

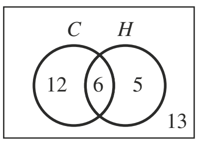
Short-answer questions

1 a $\frac{1}{8}$ b $\frac{1}{4}$ c $\frac{3}{8}$
d $\frac{5}{8}$ e $\frac{1}{2}$

2 a $\frac{5}{8}$ b $\frac{1}{2}$ c $\frac{5}{8}$

3 a i $\frac{2}{5}$ ii $\frac{1}{4}$ iii $\frac{1}{5}$
iv $\frac{1}{10}$ v $\frac{1}{20}$
b i $\frac{3}{5}$ ii $\frac{17}{20}$

4 a



b

	C	C'	
H	6	5	11
H'	12	13	25
	18	18	36

c 13
d i $\frac{1}{6}$ ii $\frac{5}{36}$ iii $\frac{1}{2}$

5 a 6 b $\frac{6}{13}$

6 a i 13 ii 4 iii 1
b i $\frac{3}{4}$ ii $\frac{1}{52}$
c $\frac{4}{13}$ d $\frac{10}{13}$

7 a 0.1 b 0.5

8 a $\frac{2}{5}$ b $\frac{1}{5}$

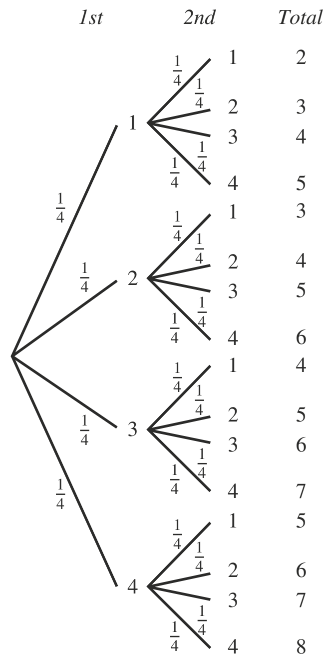
9 a i $\frac{4}{11}$ ii $\frac{5}{11}$ iii $\frac{1}{5}$
b No, $\Pr(A|B) \neq \Pr(A)$
c i $\frac{1}{2}$ ii $\frac{1}{4}$ iii $\frac{1}{2}$
d Yes, $\Pr(A|B) = \Pr(A)$

10 a

		1st				
		H	A	P	P	Y
2nd	H	(H, H)	(A, H)	(P, H)	(P, H)	(Y, H)
	E	(H, E)	(A, E)	(P, E)	(P, E)	(Y, E)
	Y	(H, Y)	(A, Y)	(P, Y)	(P, Y)	(Y, Y)

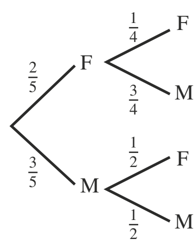
b 15
c i $\frac{1}{15}$ ii $\frac{2}{15}$ iii $\frac{13}{15}$

11 a



b i $\frac{1}{16}$ ii $\frac{1}{4}$ iii 0 iv 1

12



a $\frac{2}{5}$ b $\frac{3}{4}$ c $\frac{3}{10}$ d $\frac{3}{5}$ e $\frac{7}{10}$

13 a 0.12 b 0.58

Multiple-choice questions

1 A 2 B 3 D 4 C 5 B
6 A 7 C 8 A 9 E 10 E

Extended-response questions

1 a 3
b i $\frac{7}{15}$ ii $\frac{1}{15}$
c

	R	R'	
S	3	1	4
S'	3	8	11
	6	9	15

d i $\frac{1}{2}$ ii $\frac{3}{4}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2 a

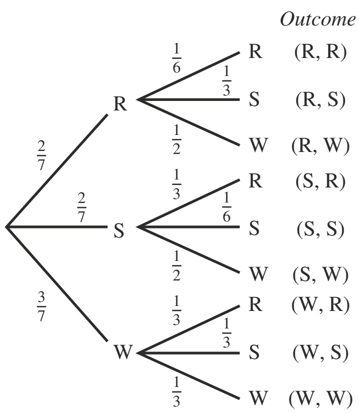
		1st		
		R	S	W
2nd	R	(R, R)	(S, R)	(W, R)
	S	(R, S)	(S, S)	(W, S)
	W	(R, W)	(S, W)	(W, W)

b **i** $\frac{1}{9}$ **ii** $\frac{1}{3}$ **iii** $\frac{5}{9}$ **iv** $\frac{4}{9}$

c 4

d $\frac{5}{9}$

e



i $\frac{1}{21}$ **ii** $\frac{10}{21}$ **iii** $\frac{6}{7}$ **iv** $\frac{16}{21}$

Chapter 9

9A

Building understanding

1 a C **b** D **c** A **d** B **e** E
2 a B **b** E **c** C
d D **e** F **f** A
3 a numerical **b** categorical **c** categorical
4 D

Now you try

Example 1
a numerical and continuous
b categorical and ordinal

Example 2
Students who study in the library may not reflect a typical group of Year 12 students.

Exercise 9A

1 a numerical and discrete
b numerical and continuous
c categorical and nominal
d categorical and ordinal

2 a numerical and discrete
b numerical and discrete
c categorical and nominal
d numerical and continuous
e categorical and ordinal

3 a No **b** No
c Yes **d** Yes

4 a D
b D is the most representative sample. A may pick out the keen students; B probably are good maths students who like maths; and C will have different-sized classes.

5 a For example, likely to be train passengers.
b For example, email will pick up computer users only.
c For example, electoral roll will list only people aged 18 years and over.

6 Check with your teacher.

7 a A small survey, misinterpreted their data.
b Survey more companies and make it Australia-wide.
c No, data suggest that profits had reduced, not necessarily that they were not making a profit. Also, sample size is too small.

8 a Graph A
b Graph B
c The scale on graph A starts at 23, whereas on graph B it starts at 5.
d Graph A because the scale expands the difference in column heights.

9 For example, showing only part of the scale, using different column widths, including erroneous data values.

10–12 Research required.

9B

Building understanding

1 a 10 **b** 1.4 **c** 1 **d** 1 **e** 90%

2

Class interval	Frequency	Percentage frequency
0–	2	20
10–	1	10
20–	5	50
30–40	2	20
Total	**10**	**100**

Now you try

Example 3

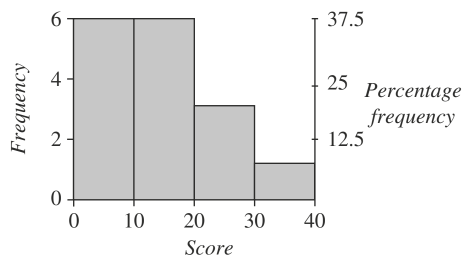
a

Class interval	Frequency	Percentage frequency
0–	6	37.5
10–	6	37.5
20–	3	18.75
30–40	1	6.25
	16	100

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b Number of phone texts



c

Stem	Leaf
0	2 5 5 7 7 9
1	0 1 2 4 4 8
2	2 4 9
3	2

2|4 means 24

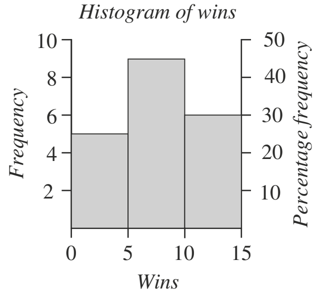
d Median = 11.5

Exercise 9B

1 a

Class interval	Frequency	Percentage frequency
0–	5	25
5–	9	45
10–15	6	30
Total	**20**	**100**

b Histogram of wins



c

Stem	Leaf
0	0 1 3 4 4 5 5 6 7 7 8 9 9 9
1	0 1 2 2 3 5

1|2 means 12

d 7.5

2 a

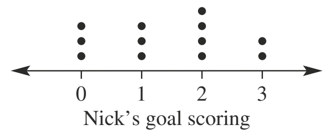
Type of transport	Frequency	Percentage frequency
Car	16	40
Train	6	15
Tram	8	20
Walking	5	12.5
Bicycle	2	5
Bus	3	7.5
Total	**40**	100

b i 6 ii car iii 40%
iv 17.5% v 42.5%

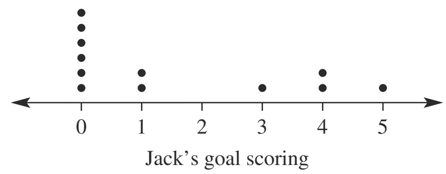
3 a symmetrical b negatively skewed
c positively skewed d symmetrical

4 a i 34.3 ii 38 iii 39
b i 19.4 ii 20 iii no mode

5 a



b



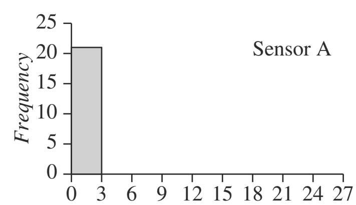
c Well spread performance.

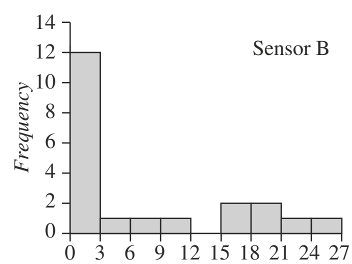
d Irregular performance, positively skewed.

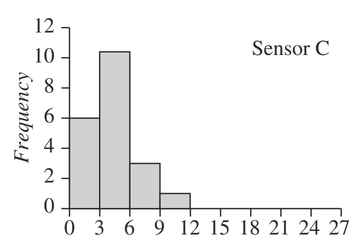
6 a

	Sensor A frequency	Sensor B frequency	Sensor C frequency
0–	21	12	6
3–	0	1	11
6–	0	1	3
9–	0	1	1
12–	0	0	0
15–	0	2	0
18–	0	2	0
21–	0	1	0
24–26	0	1	0
Total	**21**	**21**	**21**

b







c i low sensitivity ii very sensitive
iii moderately sensitive

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

7 **a**

Mass	Frequency	Percentage frequency
10–	3	6
15–	6	12
20–	16	32
25–	21	42
30–35	4	8
Total	**50**	**100**

b 50
c 32%
d At least 25 g but less than 30 g.
e 42%
f 94%

8 **a**

Section	Frequency	Percentage frequency
Strings	21	52.5
Woodwind	8	20
Brass	7	17.5
Percussion	4	10
Total	**40**	**100**

b 40 **c** 52.5% **d** 47.5%
e 9.3% **f** 65.6%

9 8 students scored between 20 and 30 and there are 32 students all together, so this class interval makes up 25% of the class.

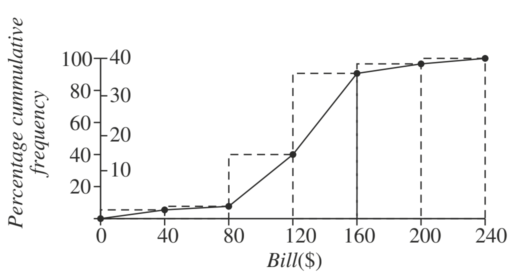
10 No discrete information, only intervals are given and not individual values.

11 $3 \leqslant a \leqslant 7, 0 \leqslant b \leqslant 4, c = 9$

12 **a**

Bill ($)	Frequency	Cumulative frequency	Percentage cumulative frequency
0–	2	2	5.4
40–	1	3	8.1
80–	12	15	40.5
120–	18	33	89.2
160–	3	36	97.3
200–240	1	37	100

b 15
c



d **i** \$130 **ii** \$100 **iii** \$150
e \$180
f approx. 20%

9C

Building understanding

1 **a** Min, lower quartile (Q_1), median (Q_2), upper quartile (Q_3), max
b Range is max – min; IQR is $Q_3 - Q_1$. Range is the spread of all the data, IQR is the spread of the middle 50% of data.
c An outlier is a data point (element) outside the vicinity of the rest of the data.
d If the data point is greater than $Q_3 + 1.5 \times \text{IQR}$ or less than $Q_1 - 1.5 \times \text{IQR}$.

2 **a** 2
b **i** 1 **ii** 3
c 2 **d** $-2, 6$ **e** yes; 8

3 **a** **i** 10.5 **ii** 7.5 **iii** 12
b 4.5 **c** 0.75, 18.75 **d** no

Now you try

Example 4
a Range = 9, IQR = 4.5
b Range = 0.9, IQR = 0.6

Example 5
a **i** 0 and 36 **ii** 24 **iii** 19 and 26
iv 7 **v** yes; 0
b Perhaps the bus was not taking passengers.

Exercise 9C

1 **a** min = 3, Q_1 = 4, median = 8, Q_3 = 10, max = 13; range = 10, IQR = 6
b min = 10, Q_1 = 10.5, median = 14, Q_3 = 15.5, max = 18; range = 8, IQR = 5
c min = 1.2, Q_1 = 1.85, median = 2.4, Q_3 = 3.05, max = 3.4; range = 2.2, IQR = 1.2
d min = 41, Q_1 = 53, median = 60.5, Q_3 = 65, max = 68; range = 27, IQR = 12

2 **a** min = 0, max = 17 **b** median = 13
c Q_1 = 10, Q_3 = 15 **d** IQR = 5
e 0
f Road may have been closed that day.

3 **a** **i** min = 4, max = 14 **ii** 7.5
iii Q_1 = 5, Q_3 = 9 **iv** IQR = 4
v no outliers
b **i** min = 16, max = 31 **ii** 25
iii Q_1 = 21, Q_3 = 27 **iv** IQR = 6
v no outliers

4 **a** **i** $5.\dot{3}$ **ii** 2.4
b **i** 2.5 **ii** 2
c **i** $2.9\dot{3}$ **ii** 0.5

5 **a** **i** min = 25, max = 128 **ii** 47
iii Q_1 = 38, Q_3 = 52.5 **iv** IQR = 14.5
v yes; 128 **vi** 51.25
b Median as it is not affected dramatically by the outlier.
c A more advanced calculator was used.

6 **a** no outliers **b** Outlier is 2.
c Outliers are 103, 182. **d** Outliers are 2, 8.

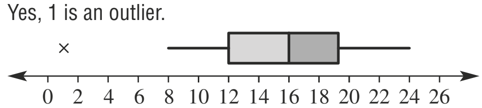
7 **a** IQR = 12 **b** No outliers.
c 24 **d** 22
8 1, 2, 3
9 **a** Increases by 5. **b** It is doubled.
c It is divided by 10.
10 **a** It stays the same. **b** It doubles.
c It is reduced by a scale factor of 10.
11 Answers may vary. Examples:
a 3, 4, 5, 6, 7 **b** 2, 4, 6, 6, 6 **c** 7, 7, 7, 10, 10
12 It is not greatly affected by outliers.
13 Answers will vary.

9D

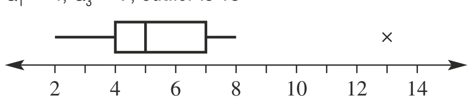
Building understanding

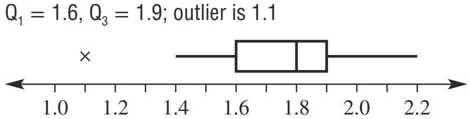
1 **a** 15 **b** 5 **c** 25 **d** 20
e 10 **f** 20 **g** 10
2 **a** 4 **b** 2 **c** 18 **d** 20
e It is.

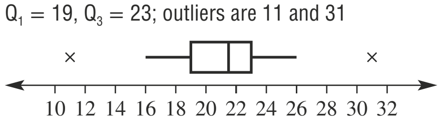
Now you try

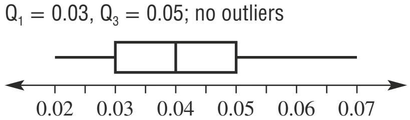
Example 6
a Yes, 1 is an outlier.
b


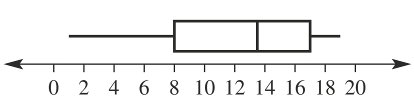
Exercise 9D

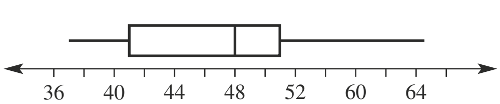
1 **a** **i** $Q_1 = 4$, $Q_3 = 7$; outlier is 13
ii


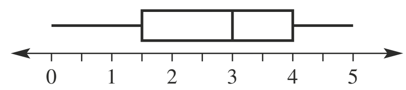
b **i** $Q_1 = 1.6$, $Q_3 = 1.9$; outlier is 1.1
ii


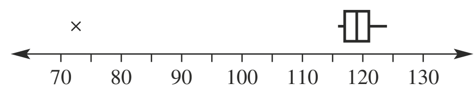
c **i** $Q_1 = 19$, $Q_3 = 23$; outliers are 11 and 31
ii


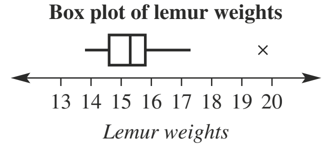
d **i** $Q_1 = 0.03$, $Q_3 = 0.05$; no outliers
ii


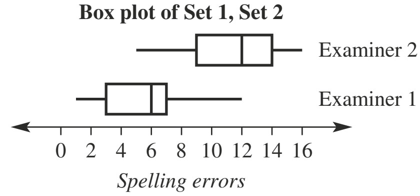
2 **a**


b


c


d


3 **a** Same minimum of 1.
b B
c **i** 5 **ii** 10
d Data points for B are more evenly spread than those for A.
4 **a** $Q_1 = 14.6$, $Q_2 = 15.3$, $Q_3 = 15.8$
b 19.7 kg
c **Box plot of lemur weights**


5 **a** They have the same median and upper quartile.
b B
c **i** 4 **ii** 5
d Set B is more spread out.
6 **a** A **b** B **c** B
7 **a** **Box plot of Set 1, Set 2**


b Yes, examiner 2 found more errors.
8 Answers may vary. Examples:
a **i, ii** Class results had a smaller spread in the top 25% and bottom 25% performed better.
iii State results have a larger IQR.
b The class did not have other results close to 0 but the school did.
9 Answers will vary.

9E

Building understanding

1 **a** larger **b** smaller
2 **a** B **b** A
3 A. The data values in A are spread farther from the mean than the data values in B.
4 **a** Gum Heights **b** Oak Valley

Now you try

Example 7
Mean = 2.8, $s = 1.6$

Example 8
a More data values are centred around the 10s and 20s for Year 12 compared to 20s and 30s for Year 7.
b The Year 7 data are more spread-out.
c Given their studies, Year 12s are more likely to watch less television.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 9E

1 mean = 3.8, $s = 2.4$

2 a mean = 6, $s = 2.2$ b mean = 3.6, $s = 2.6$
c mean = 8, $s = 3.8$ d mean = 32.5, $s = 3.6$

3 a mean = 2.7, $s = 0.9$ b mean = 14.5, $s = 6.6$

4 a The outer-suburb school has more data values in the higher range.
b There is less spread. Data values are closer to the mean.
c Students at outer-suburb schools may live some distance from the school. Answers will vary.

5 a false b true c true

6 a mean = 2, $s = 1.0$ b mean = 5.25, $s = 0.7$

7 a no b no c yes
d Yes, one of the deviations would be calculated using the outlier.

8 a No, standard deviation reflects the spread of the data values from the mean not the size of the data values.
b No. As for part **a**.

9 The IQRs would be the same, making the data more comparable. The standard deviation would be affected by the outlier.

10 a i 85.16 ii 53.16 iii 101.16
iv 37.16 v 117.16 vi 21.16
b i 66% ii 96% iii 100%
c i Research required
ii One SD from the mean = 68%
Two SDs from the mean = 95%
Three SDs from the mean = 99.7%
Close to answers found.

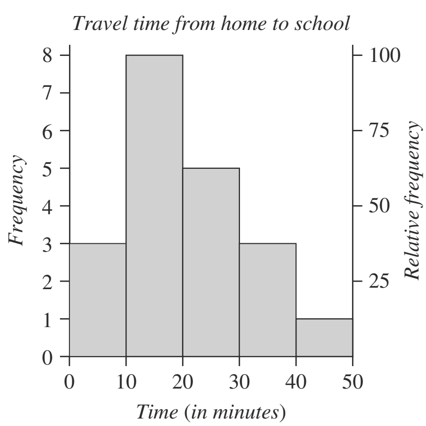
Progress quiz

1 a numerical and discrete
b categorical and nominal

2 a

Class interval	Frequency	Percentage frequency
0–	3	15
10–	8	40
20–	5	25
30–	3	15
40–50	1	5

b



c

Stem	Leaf
0	6 8 9
1	2 4 5 5 5 6 8 9
2	0 3 5 7 8
3	2 3 8
4	4

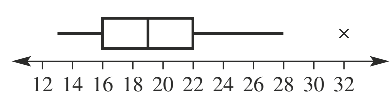
2|3 means 23

d Median = $\frac{18 + 19}{2} = 18.5$

3 a Range = 32 − 4 = 28; $Q_2 = 17$, $Q_1 = 12$, $Q_3 = 23$, IQR = 11
b Range = 6.6 − 4.2 = 2.4; $Q_2 = 5.2$, $Q_1 = 4.5$, $Q_3 = 6.1$, IQR = 1.6

4 a i min = 30, max = 69
ii median = 39
iii lower quartile = 34, upper quartile = 46
iv IQR = 12
v $Q_1 - 1.5 \times \text{IQR} = 16$; $Q_3 + 1.5 \times \text{IQR} = 64$
The outlier is 69.
b For example, school open day or grandparents day.

5 a $Q_1 = 16$, $Q_3 = 22$, IQR = 6
$Q_3 + 1.5 \times \text{IQR} = 31$
32 is an outlier.
b



6 2.7

9F

Building understanding

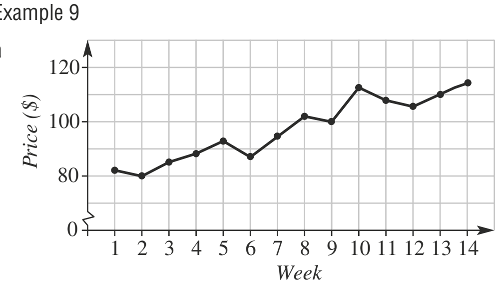
1 a linear b no trend
c non-linear d linear

2 a i 28°C ii 33°C
iii 33°C iv 35°C
b 36°C
c i 12 p.m. to 1 p.m. ii 3 p.m to 4 p.m.
d Temperature is increasing from 8 a.m. to 3 p.m. in a generally linear way. At 3 p.m. the temperature starts to drop.

Now you try

Example 9

a



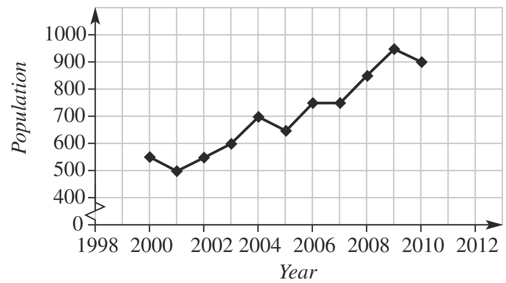
b General linear upward trend.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 9F

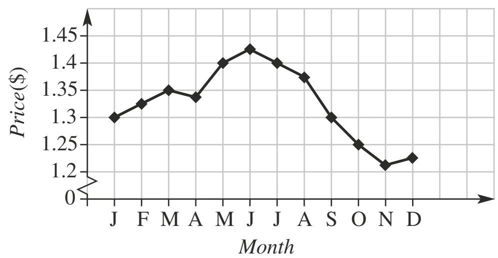
1 a



b Generally linear in a positive direction.

c **i** 500 **ii** 950

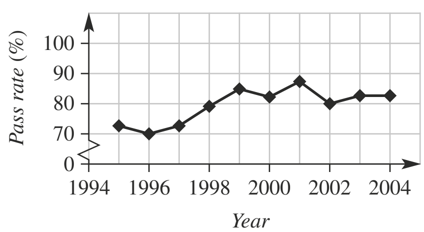
2 a



b The share price generally increased until it peaked in June and then continually decreased to a yearly low in November before trending upwards again in the final month.

c $0.21

3 a



b The pass rate for the examination has increased marginally over the 10 years, with a peak in 2001.

c 2001 **d** 11%

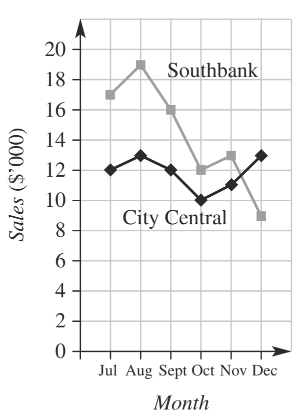
4 a linear

b **i** $650 000 **ii** $750 000

5 a **i** $6000 **ii** $4000

b 1

c



d **i** The sales trend for City Central for the 6 months is fairly constant.

ii Sales for Southbank peaked in August before taking a downturn.

e about $5000

6 a **i** 5.8 km

ii 1.7 km

b **i** Blue Crest slowly gets closer to the machine.

ii Green Tail starts near the machine and gets further from it.

c 8:30 p.m.

7 a The yearly temperature is cyclical and January is the next month after December and both are in the same season.

b no

c Northern hemisphere, as the seasons are opposite, June is summer.

8 a Increases continually, rising more rapidly as the years progress.

b Compound interest—exponential growth.

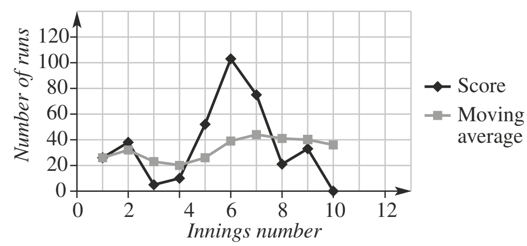
9 a Graphs may vary, but it should decrease from room temperature to the temperature of the fridge.

b No. Drink cannot cool to a temperature *lower* than that of the internal environment of the fridge.

10 a

Innings	1	2	3	4	5	6	7	8	9	10
Score	26	38	5	10	52	103	75	21	33	0
Moving average	26	32	23	20	26	39	44	41	40	36

b



c Innings number.

i The score fluctuates wildly.

ii The graph is fairly constant with small increases and decreases.

d The moving average graph follows the trend of the score graph but the fluctuations are much less significant.

9G

Building understanding

1 a unlikely **b** likely **c** unlikely

d likely **e** likely **f** likely

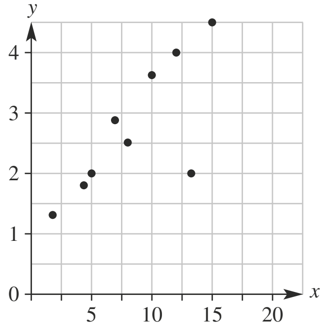
2 a y generally increases as x increases.

b y generally decreases as x increases.

ISBN 978-1-108-77290-7 Cambridge University Press
 Updated June 2022

Now you try

Example 10

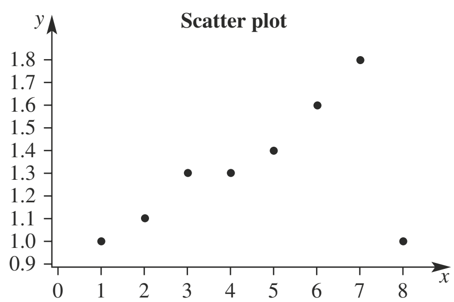
a


b positive

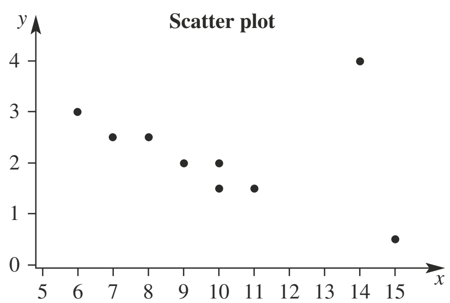
c strong

d (13, 2.0)

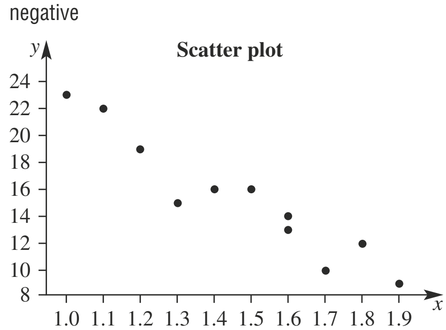
Exercise 9G

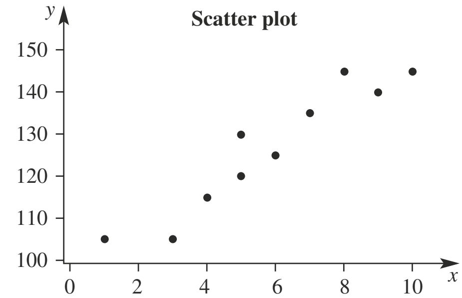
1 a


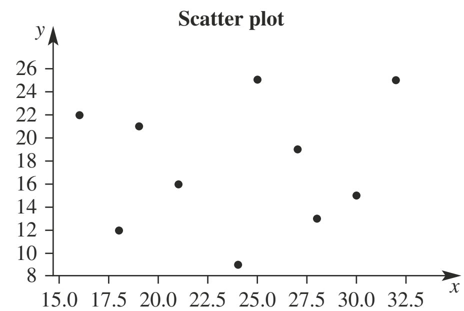
b positive c strong d (8, 1.0)

2 a


b negative c strong d (14, 4)

3 a negative


b positive


c none


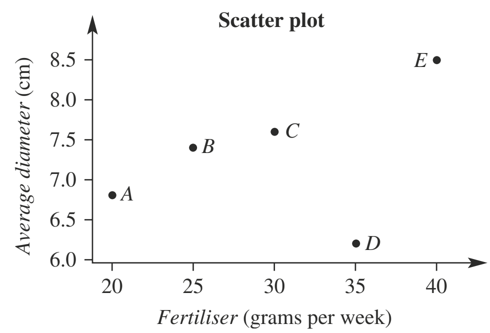
4 a none b weak negative

c positive d strong positive

5 a yes

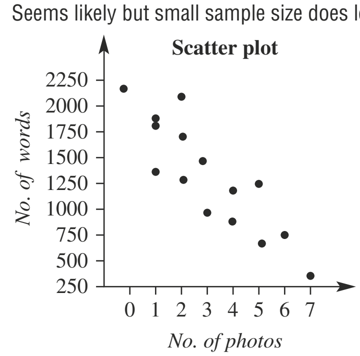
b decrease

c i yes ii car H

6 a


b D

c Seems likely but small sample size does lead to doubt.

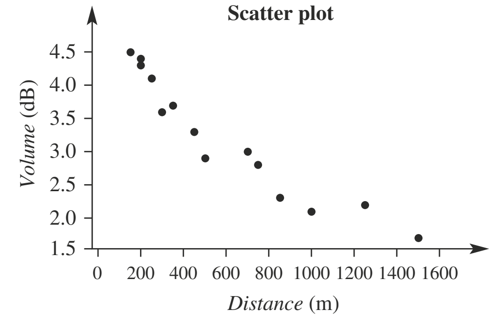
7 a


b Negative, weak correlation.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

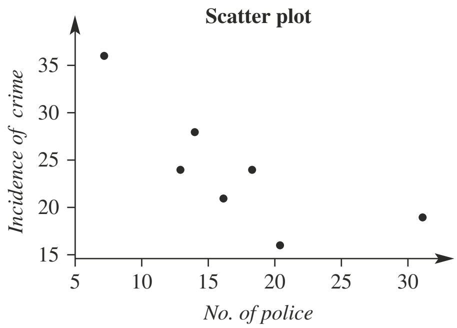
8 a



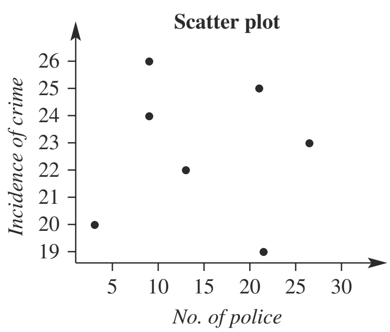
b negative

c As distance increases, volume decreases.

9 a i weak, negative correlation



ii no correlation



b Survey 1, as this shows an increase in the number of police has seen a decrease in the incidence of crime.

10 The positive correlation shows that as height increases it is predicted that ability to play tennis increases.

11 Each axis needs a better scale. All data are between 6 and 8 hours sleep and show only a minimal change in exam marks. Also, very small sample size.

12 a i students I, T   ii students G, S

b i students H, C   ii students B, N

c students C, G, H, S,

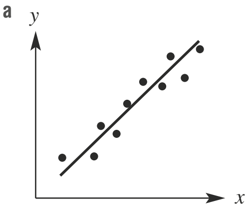
d students B, I, N, T

e no

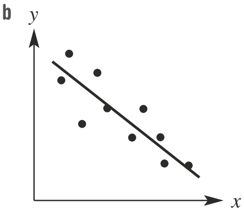
9H

Building understanding

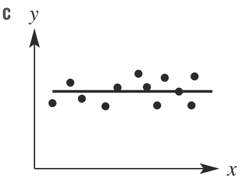
1 a



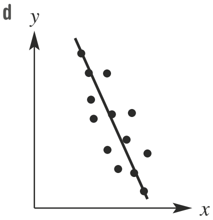
b



c



d



2 a $y = \frac{1}{2}x + \frac{7}{2}$ b $y = -\frac{2}{3}x + \frac{17}{3}$

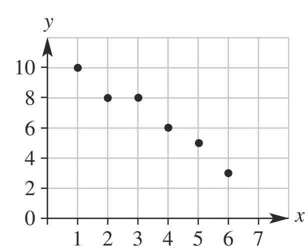
3 a i 17   ii $\frac{23}{4}$

b i $\frac{28}{5}$ ii $\frac{14}{5}$

Now you try

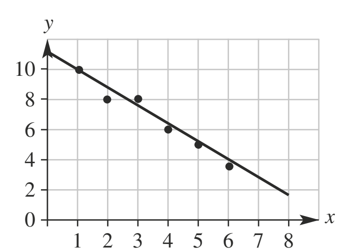
Example 11

a



b negative correlation

c



d i ≈ 7 ii ≈ 11

iii ≈ 8 iv ≈ 4.5

Example 12

a $y = \frac{5}{3}x + \frac{10}{3}$

b i 45 cm   ii 90 cm

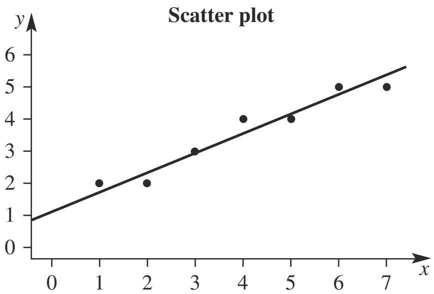
c i 34 kg   ii 46 kg

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

Exercise 9H

1 **a**

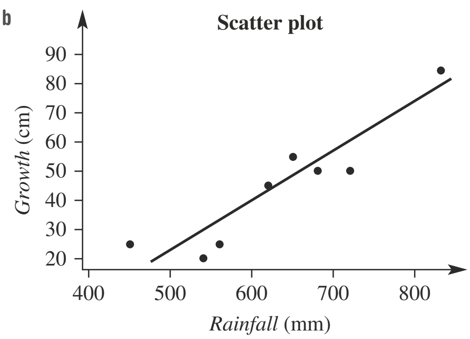


b positive correlation
c As above.
d All answers are approximate.
i 3.2 **ii** 0.9 **iii** 1.8 **iv** 7.4

2 **a** ≈ 4.5 **b** ≈ 6
c ≈ 0.5 **d** ≈ 50

3 **a** $y = \frac{3}{5}x + 18$
b **i** 42 **ii** 72
c **i** 30 **ii** 100

4 **a, b**



c **i** ≈ 25 cm **ii** ≈ 85 cm
d **i** ≈ 520 mm **ii** ≈ 720 mm

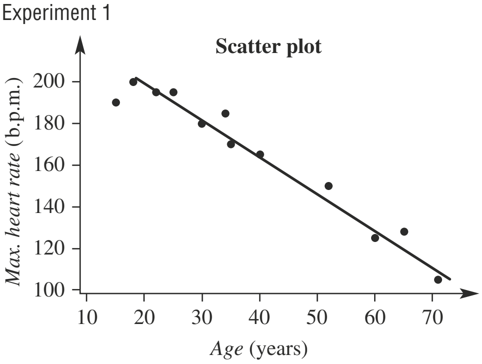
5 **a** $y = 5x - 5$
b 85 cm
c 21 kg

6 **a** Data do not appear to have any correlation.
b Too few data points to determine a correlation.

7 **a** Too few data points to determine a correlation.
b The data points suggest that the trend is not linear.

8 **a** **i** 50 **ii** 110
b It is possible to obtain scores of greater than 100%.

9 **a** Experiment 1



Experiment 2

b **i** ≈ 140 **ii** ≈ 125
c **i** ≈ 25 **ii** ≈ 22
d experiment 2
e Research required.

9I

Building understanding

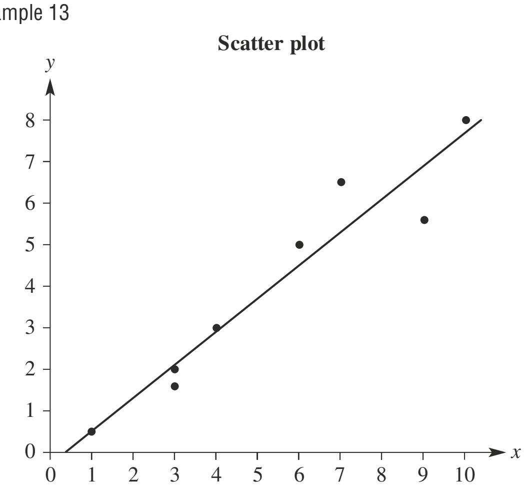
1 **a** **i** 12 **ii** 3.26
b **i** 7 **ii** 2

2 **a** There is no linear correlation.
b The correlation shown is not a linear shape.

Now you try

Example 13

a, c

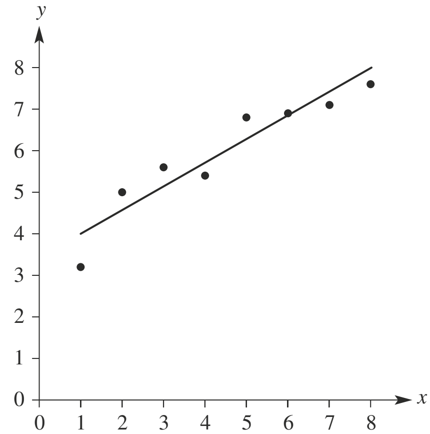


b $y = 0.80x - 0.29$
d **i** 3.32 **ii** 11.76

ISBN 978-1-108-77290-7

Exercise 9I

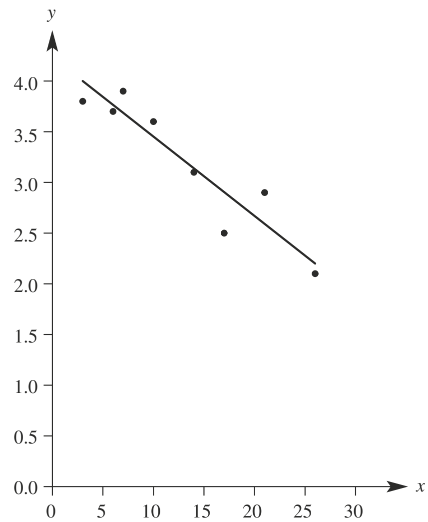
1 A a, c



b $y = 0.55x + 3.45$

d i 7.34 ii 10.11

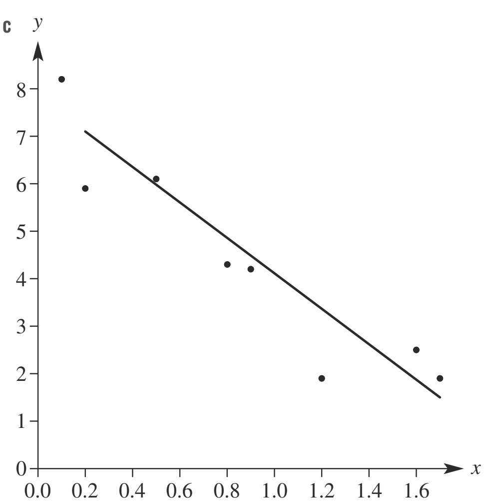
B a, c



b $y = -0.08x + 4.21$

d i 3.66 ii 3.28

C a, c



b $y = -3.45x + 7.42$

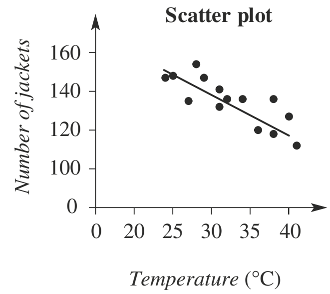
d i -16.74 ii -34.00

2 a $y = -3.55x + 43.04$

b $32 397

c 8 years

3 a, c



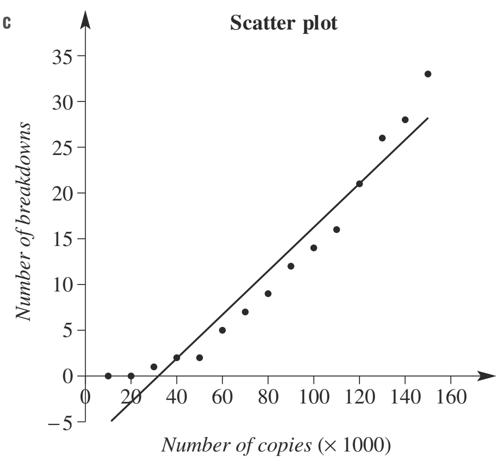
b $y = -1.72x + 190.57$

d i 139

ii 130

iii 113

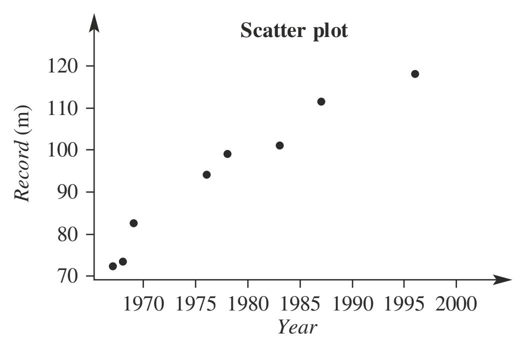
4 a, c



b $y = 0.24x - 7.27$ d 232 000 copies

e The regression line suggests that the photocopier will be considered for scrap because of the number of copies made, as it's likely to reach 200 000 copies before breaking down 50 times.

5 a



b $y = 1.60x - 3066.41$

c i 134 m ii 166 m

d No, records are not likely to continue to increase at this rate.

ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

6 All deviations are used in the calculation of the least squares regression.
7 A, as it has been affected by the outlier.
8 Research required.

Problems and challenges

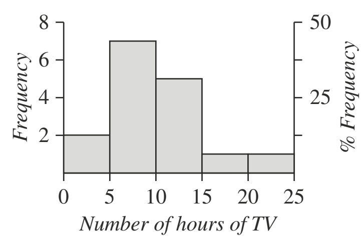
1 66 kg
2 88%
3 19
4 1.1
5 **a** larger by 3 **b** larger by 3 **c** no change
d no change **e** no change
6 $y = x^2 - 3x + 5$
7 $5.8 \leqslant a < 6.2$
8 Physics, Biology. The number of standard deviations from the mean shows the relative position of Emily's mark within the spread of all results from each class. This number gives a meaningful comparison of results. Physics 1.67, Maths 1.11, Biology 0.

Short-answer questions

1 **a**

Class interval	Frequency	Percentage frequency
0–	2	12.5
5–	7	43.75
10–	5	31.25
15–	1	6.25
20–25	1	6.25
Total	**16**	100

b



c It is positively skewed.

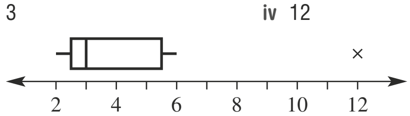
d

Stem	Leaf
0	1 3 5 6 6 7 8 8 9
1	0 1 2 3 4 6
2	4

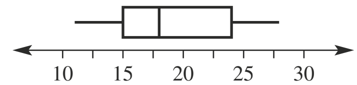
1|3 means 13

e 8.5 hours

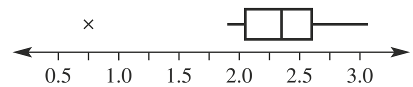
2 **a** **i** 10 **ii** $Q_1 = 2.5, Q_3 = 5.5$
iii 3 **iv** 12
v



b **i** 17 **ii** $Q_1 = 15, Q_3 = 24$
iii 9 **iv** none
v

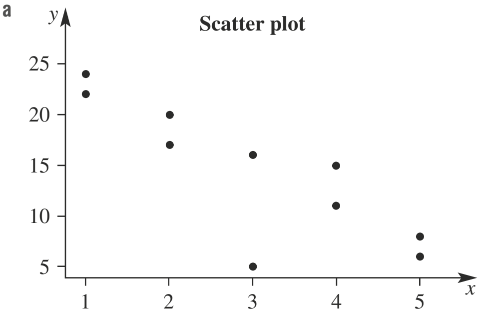


c **i** 2.4 **ii** $Q_1 = 2.1, Q_3 = 2.6$
iii 0.50 **iv** 0.7
v



3 **a** false **b** true **c** true **d** true

4 **a**



b negative **c** weak **d** (3, 5)

5 **a** $y = \frac{3}{5}x + \frac{7}{5}$
b **i** 3.8 **ii** 7.4
c **i** $2\frac{2}{3}$ **ii** $17\frac{2}{3}$

6 **a** mean = 7, $s = 2.5$
b mean = 4, $s = 3.0$

7 **a** The Cats **b** The Cats
c The Cats **d** The Cats

8 **a** non-linear
b linear

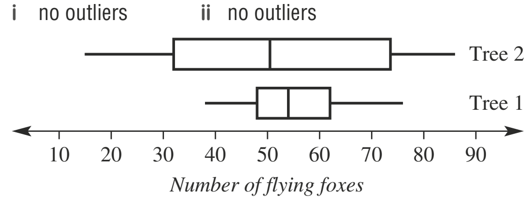
9 $y = -3.75x + 25.65$

Multiple-choice questions

1 D	2 B	3 D	4 C	5 A
6 C	7 C	8 D	9 E	10 B

Extended-response questions

1 **a** **i** 14 **ii** 41
b **i** no outliers **ii** no outliers
c

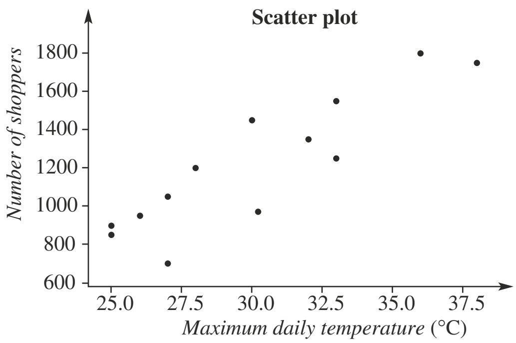


d More flying foxes regularly take refuge in tree 1 than in tree 2, for which the spread is much greater.

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

2 a



positive correlation

b $y = 74.56x - 1010.06$

c **i** 779

ii 33.7°C

Chapter 10

10A

Building understanding

1

x	0	1	2	3	4	5
2^x	1	2	4	8	16	32
3^x	1	3	9	27	81	243
4^x	1	4	16	64	256	1024
5^x	1	5	25	125	625	3125
6^x	1	10	100	1000	10 000	100 000

2 **a** 4 **b** 4 **c** 3 **d** 4

3 **a** $\frac{1}{10000}$ **b** $\frac{1}{2}$ **c** $\frac{1}{4}$ **d** $\frac{1}{27}$

Now you try

Example 1

a $10^2 = 100$ **b** $\log_3 81 = 4$

Example 2

a **i** 4 **ii** 5

b **i** −3 **ii** −2

c **i** 0.699 **ii** −0.347

Example 3

a 4 **b** 125

Exercise 10A

1 **a** **i** $10^4 = 10000$ **ii** $2^5 = 32$

b **i** $\log_2 8 = 3$ **ii** $\log_3 81 = 4$

2 **a** $2^4 = 16$ **b** $10^2 = 100$ **c** $3^3 = 27$

d $2^{-2} = \frac{1}{4}$ **e** $10^{-1} = 0.1$ **f** $3^{-2} = \frac{1}{9}$

3 **a** $\log_2 8 = 3$ **b** $\log_3 81 = 4$

c $\log_2 32 = 5$ **d** $\log_4 16 = 2$

e $\log_{10} \frac{1}{10} = -1$ **f** $\log_5 \frac{1}{125} = -3$

4 **a** 4 **b** 2 **c** 6 **d** 3

e 1 **f** 2 **g** 3 **h** 3

i 2 **j** 2 **k** 5 **l** 3

m 0 **n** 0 **o** 0 **p** undefined

5 **a** −3 **b** −2 **c** −2 **d** −3

e −2 **f** −4 **g** −4 **h** −1

i −1 **j** −3 **k** −5 **l** −1

m −3 **n** −1 **o** −2 **p** −2

6 **a** 0.699 **b** 1.672 **c** 2.210

d −0.097 **e** −0.770 **f** −1.431

7 **a** 3 **b** 5 **c** 6 **d** 4

e 3 **f** 2 **g** 16 **h** 81

i 1000 **j** $\frac{1}{9}$ **k** $\frac{1}{4}$ **l** $\frac{1}{343}$

m 3 **n** 2 **o** 4 **p** 8

q 3 **r** 10 **s** 2 **t** −1

8 **a**

Time (min)	0	1	2	3	4	5
Population	1	2	4	8	16	32

b $P = 2^t$ **c** 256 **d** 13 min

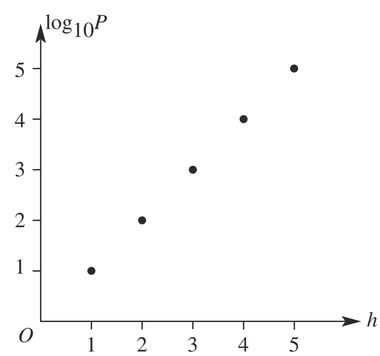
e $\log_2 10000$

9 **a** 16 **b** 26 **c** 6

10 **a**

h	0	1	2	3	4	5
P	1	10	100	1000	10 000	100 000
$\log_{10} P$	0	1	2	3	4	5

b



Graph is a straight line.

c $\log_{10} P = h$

11 **a** **i** 10 **ii** 100 **iii** 10 000

b **i** 10 **ii** 6 **iii** 3

12 Yes. $0 < b < 1 \Rightarrow \log_a b < 0$, when $a > 1$; e.g. $\log_2 \frac{1}{4} = -2$.

13 **a** $\frac{1}{4}$ **b** $\frac{1}{5}$ **c** $\frac{1}{2}$ **d** $\frac{1}{3}$

e $\frac{1}{2}$ **f** $\frac{1}{3}$ **g** $\frac{2}{3}$ **h** $\frac{4}{3}$

i $\frac{1}{2}$ **j** $\frac{1}{2}$ **k** $\frac{6}{5}$ **l** $\frac{4}{7}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

10B

Building understanding

1 a $\log_b xy = \log_b x + \log_b y$ b $\log_b \frac{x}{y} = \log_b x - \log_b y$
c $\log_a b^m = m \times \log_a b$ d $\log_a a = 1$
e $\log_c 1 = 0$ f $\log_a \frac{1}{b} = -\log_a b$

2 a 2 b 1 c 4
d 12 e 4 f −1

3 a 2 b 5 c 3
d −4 e 12 f 0

Now you try

Example 4
a $\log_a 24$ b $\log_a 2$ c $\log_a 16$

Example 5
a 0 b 1 c −2 d 1

Exercise 10B

1 a i $\log_a 14$ ii $\log_a 5$
b i $\log_a 2$ ii $\log_a 7$
c i $\log_a 9$ ii $\log_a 64$

2 a $\log_a 6$ b $\log_a 15$ c $\log_a 28$
d $\log_b 18$ e $\log_b 15$ f $\log_b 17$

3 a $\log_a 2$ b $\log_a 3$ c $\log_a 10$
d $\log_b 2$ e $\log_b\left(\frac{3}{2}\right)$ f $\log_b\left(\frac{7}{5}\right)$

4 a $\log_a 9$ b $\log_a 25$ c $\log_a 27$
d $\log_a 16$ e $\log_a 32$ f $\log_a 1000$

5 a 0 b 0 c 0 d 1
e 1 f 1 g 0 h 3
i $\frac{1}{3}$ j $\frac{2}{3}$ k 1 l $\frac{1}{2}$

6 a −2 b −3 c −3
d −1 e −2 f −5

7 a 1 b 1 c 3
d 2 e 2 f 2

8 a $\log_3 20$ b $\log_{10} 48$ c $\log_{10} 2$
d $\log_7 2$ e $\log_3 8$ f 0
g $\log_2\left(\frac{3}{4}\right)$ h $\log_5 6$

9 a $\frac{3}{2}$ b $\frac{5}{2}$ c $\frac{4}{3}$
d $\frac{3}{2}$ e $\frac{1}{3}$ f $\frac{4}{5}$

10 a $\log_a \frac{1}{x} = \log_a 1 - \log_a x = 0 - \log_a x = -\log_a x$ as required (using 2nd log law)
b $\log_a \frac{1}{x} = \log_a x^{-1} = -\log_a x$ as required (using 3rd log law)

11 $\log_a \sqrt[n]{x} = \log_a x^{\frac{1}{n}} = \frac{1}{n}\log_a x = \frac{\log_a x}{n}$ as required (using 3rd log law)

12 a Recall index law 1: $a^m \times a^n = a^{m+n}$
Now let $x = a^m$ and $y = a^n$ (1)
so $m = \log_a x$ and $n = \log_a y$ (2)
From (1), $xy = a^m \times a^n = a^{m+n}$
So $m + n = \log_a xy$
From (2), $m + n = \log_a x + \log_a y$
So $\log_a xy = \log_a x + \log_a y$, as required.
b Recall index law 2: $a^m \div a^n = a^{m-n}$
Now let $x = a^m$ and $y = a^n$ (1)
so $m = \log_a x$ and $n = \log_a y$ (2)
From (1), $x \div y = \frac{x}{y} = a^m \div a^n = a^{m-n}$
So $m - n = \log_a \frac{x}{y}$
From (2) $m - n = \log_a x - \log_a y$
So $\log_a \frac{x}{y} = \log_a x - \log_a y$, as required.
c Recall index law 3: $(am)^n = a^{mn}$
Let $x = a^m$
So $m = \log_a x$ (1)
$x^n = a^{mn}$ using index law 3
So $mn = \log_a x^n$
From (1): $n\log_a x = \log_a x^n$, as required.

10C

Building understanding

1 a $\log_2 8 = 3$ b $\log_4 2 = \frac{1}{2}$ c $\log_3 10 = x$

2 a 3 b 4 c 8 d 2

3 a 0.845 b −0.222
c −0.125 d 1.277

Now you try

Example 6
a 2.096 b 6.026

Example 7
a 3.459 b 8.530

Exercise 10C

1 a i 1.161 ii 2.335
b i 5.284 ii 4.140

2 a 1.465 b 3.459 c 1.594
d 6.871 e 1.177 f 2

3 a 1 b 1 c 3.969
d 1.727 e 6.579 f 1.528

4 a 2.585 b 1.893 c 1.209
d 1.129 e 1.559 f 6.579
g 3.322 h 1.262 i 0.356
j 3.969 k 3.106 l 1.137

5 a 2 days b 2.548 days c 3.322 days

6 a 14.21 years b 23.84 years c 47.19 years

7 a 10.48 years b 22.20 years c 91.17 years

8 a $A = 2000 \times 1.1^n$ b 7 years

9 a $F = 300\,000 \times 0.92^n$ b 8.3 years

10 a 69 years b 1386 years

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

11 a i $\frac{\log_{10} 7}{\log_{10} 2}$ ii $\frac{\log_{10} 16}{\log_{10} 3}$ iii $\frac{\log_{10} 1.3}{\log_{10} 5}$
b i $\frac{1}{\log_{10} 5}$ ii $\frac{3}{\log_{10} 2}$ iii $\frac{-1}{\log_{10} 3}$
c i 1.631 ii 1.167 iii −0.196

10D

Building understanding

1 a 5
b i 3 ii −2 iii 1 iv −1
c 2
2 a linear b quadratic c quartic
d quadratic e constant f cubic
3 a 3 b 4 c 6

Now you try

Example 8
a 14 b 8

Example 9
a yes b no

Exercise 10D

1 a −5 b 11 c 1 d −45
2 a 0 b 92 c −4 d 42
3 a 14 b 92 c 8 d 4
4 a, b, f are polynomials.
5 a −2 b 25 c −22
d −17 e 17 f −351
6 a $-\frac{1}{2}$ b −1 c $\frac{1}{2}$
7 a 0 b 4 c −108
8 a i 30 m ii 24 m iii 0 m
b Yes, when $5 < x < 7$.
9 a 8 b $n + 1$ c 1 d 1
10 a $-\frac{9}{8}$ b $-\frac{20}{27}$ c $\frac{5}{8}$ d $\frac{27}{64}$
e $\frac{16}{27}$ f $-\frac{216}{125}$ g $-\frac{1}{2}$ h $-\frac{9}{8}$
11 a $2k^3 - k^2 - 5k - 1$
b $2b^3 - b^2 - 5b - 1$
c $16a^3 - 4a^2 - 10a - 1$
d $-2a^3 - a^2 + 5a - 1$
e $-16a^3 - 4a^2 + 10a - 1$
f $-54k^3 - 9k^2 + 15k - 1$
g $2a^3b^3 - a^2b^2 - 5ab - 1$
h $-2a^3b^3 - a^2b^2 + 5ab - 1$
12 a i 10 ii 2 iii 1
iv −13 v −9 vi −18
b i 3 ii −11 iii −22
c $a = 2$ and $b = -1$

10E

Building understanding

1 a $x^2 + 2x$ b $x^2 + 6x - 55$
c $8x^2 - 26x + 15$
2 a $x^4 - 5x^3 + 4x^2 - 3$
b $-x^6 - 3x^4 + x^3 - x^2 + 13$
3 a, b, c are true.

Now you try

Example 10
a $3x^4 - x^3$ b $x^4 - 2x^3 + 4x^2 - 11x + 6$

Example 11
a $x^5 + x^4 - 5x^3 + 2x^2 + 7x - 12$
b $x^6 - 2x^4 + 6x^3 + x^2 - 6x + 9$

Exercise 10E

1 a i $x^4 - 2x^5$ ii $x^4 + 4x^3$
b i $x^5 + 3x^3 - 3x^2 + 2x - 3$
ii $x^5 - x^4 + x^3 - x^2 - x + 1$
2 a $x^3 - 3x^2$ b $x^4 - x^2$ c $2x^2 + 6x^3$
d $x^3 - x^4$ e $x^5 + 3x^4$ f $-3x^6 + 3x^3$
g $-2x^5 - 2x^4$ h $-x^7 + x^4$ i $-4x^7 + 8x^{10}$
3 a $x^5 + x^3 + 2x^2 + 2$
b $x^5 - x$
c $x^5 - x^4 - 3x^3 + 3x^2$
d $x^5 - x^3 - 2x^2 - 2x + 4$
e $x^5 + 2x^4 + 2x^3 - 2x^2 - 3x$
f $x^5 - 2x^4 + 5x^3 - 4x^2$
g $x^6 - x^5 + x^4 - 4x^3 + 2x^2 - x + 2$
h $x^6 - 5x^5 - x^4 + 8x^3 - 5x^2 - 2x + 2$
i $x^8 - x^6 + x^5 - 2x^4 - x^3 + 3x^2 + x - 3$
4 a $x^5 - 2x^4 + 2x^3 - 3x^2 + 3x - 1$
b $x^6 + 2x^4 - 2x^3 + x^2 - 2x + 1$
c $x^4 - 4x^3 + 6x^2 - 4x + 1$
5 a $x^5 + 3x^4 - x^3 - 9x^2 - 2x + 8$
b $x^4 + 2x^3 - 3x^2 - 4x + 4$
c $x^6 + 4x^5 + 2x^4 - 12x^3 - 15x^2 + 8x + 16$
6 a $x^3 + x^2 - 4x + 1$
b $x^3 - x^2 + 6x - 1$
c $2x^3 + 5x^2 - 23x + 5$
d $-x^5 + 5x^4 - 2x^3 + 5x^2 - x + 1$
e $-x^6 - 2x^4 - x^2 + 4$
f $-x^6 - x^4 - 10x^3 + 26x^2 - 10x + 1$
7 $(x^2 + x - 1)^4 = x^8 + 4x^7 + 2x^6 - 8x^5 - 5x^4 + 8x^3 + 2x^2 - 4x + 1$
8 $(x^2 - x - 1)^2 - (x^2 - x + 1)^2 = x^4 - 2x^3 - x^2 + 2x + 1 - (x^4 - 2x^3 + 3x^2 - 2x + 1) = 4x - 4x^2$ as required (or could use DOPS)
9 Yes. Multiplicative axiom $ab = ba$.
10 a 3 b 5 c 7 d 12
11 a m b m c $m + n$
d $2m$ e $2m$ f $3n$

Answers

10C

12 a $x^4 - x^3 + x^2 - x$
b $x^5 + 2x^4 - 3x^3$
c $x^3 + 4x^2 + x - 6$
d $6x^3 + 23x^2 - 5x - 4$
e $15x^3 - 11x^2 - 48x + 20$
f $x^5 + 3x^4 - x^3 - 3x^2 - 2x - 6$

Progress quiz

1 a $\log_2 32 = 5$ b $\log_{10} 1000 = 3$ c $\log_a a = 1$
2 a $10^2 = 100$ b $2^3 = 8$ c $7^0 = 1$
3 a $x = 4$ b $x = 4$ c $x = 0$
d $x = 625$ e $x = 5$ f $x = -6$
4 a 3 b 2 c 2
d -2 e $\frac{1}{2}$ f 1
5 a $x = 1.771$ b $x = 29.060$ c $x = 8.043$
6 The term involving $\sqrt{x}$ has a fractional index when written in index notation.
7 a 8 b 7
c $3k^4 - 2k^3 + k^2 + 7k + 8$
8 a 3 b 1 c 3 d $4x^3$
9 a $x^7 - 2x^5 + x^4$ b $x^4 + 2x^3 + 5x^2 - 2x - 6$
10 a $x^4 + x^3 + 3x + 2$
b $-x^4 + x^3 - x + 2$
c $x^8 + 4x^5 + 4x^2$
d $x^7 + x^5 + 4x^4 + 2x^2 + 4x$

10F

Building understanding

1 a 1 b 3 c 0
2 a If $182 \div 3 = 60$ remainder 2 then $182 = 3 \times 60 + 2$.
b If $2184 \div 5 = 436$ remainder 4 then $2184 = 5 \times 436 + 4$.
c If $617 \div 7 = 88$ remainder 1 then $617 = 7 \times 88 + 1$.

Now you try

Example 12
a $(x - 1)(x^2 + 2x - 2) + 1$ b $(x + 2)(3x^2 - 8x + 21) - 44$

Exercise 10F

1 a $(x - 1)(x^2 + 2x - 1) + 1$
b $(x + 2)(2x^2 - 5x + 14) - 30$
2 $P(x) = (x - 1)(x^2 + 2x) + 3$
3 $P(x) = (x + 1)(3x^2 - 4x + 5) - 3$
4 a $2x^3 - x^2 + 3x - 2 = (x - 2)(2x^2 + 3x + 9) + 16$
b $2x^3 + 2x^2 - x - 3 = (x + 2)(2x^2 - 2x + 3) - 9$
c $5x^3 - 2x^2 + 7x - 1 =$
$(x + 3)(5x^2 - 17x + 58) - 175$
d $-x^3 + x^2 - 10x + 4 =$
$(x-4)(-x^2 - 3x - 22) - 84$
e $-2x^3 - 2x^2 - 5x + 7 =$
$(x + 4)(-2x^2 + 6x - 29) + 123$
f $-5x^3 + 11x^2 - 2x - 20 =$
$(x - 3)(-5x^2 - 4x - 14) - 62$
5 a $6x^4 - x^3 + 2x^2 - x + 2$
$= (x - 3)(6x^3 + 17x^2 + 53x + 158) + 476$
b $8x^5 - 2x^4 + 3x^3 - x^2 - 4x - 6$
$= (x + 1)(8x^4 - 10x^3 + 13x^2 - 14x + 10) - 16$
6 a $x^2 - 2x + 3 - \frac{5}{x + 2}$
b $x^2 + 2x + 2 - \frac{1}{x - 1}$
c $x^3 - 3x^2 + 9x - 27 + \frac{79}{x + 3}$
d $x^3 + 4x^2 + 15x + 60 + \frac{240}{x - 4}$
7 $-1, 1, 2$
8
$$\begin{array}{r} 6x^2 - 7x - 3 \\ x - 5\overline{)6x^3 - 37x^2 + 32x + 15} \\ \underline{6x^3 - 30x^2} \\ -7x^2 + 32x \\ \underline{-7x^2 + 35x} \\ -3x + 15 \\ \underline{-3x + 15} \\ 0 \end{array}$$
Remainder of 0, as required.
9 a 4 b $\frac{13}{8}$
10 a -8 b $\frac{-253}{16}$ c $-\frac{41}{27}$
11 a $x^3 - x^2 + 3x + 2 = (x^2 - 1)(x - 1) + 4x + 1$
b $2x^3 + x^2 - 5x - 1 = (x^2 + 3)(2x + 1) - 11x - 4$
c $5x^4 - x^2 + 2 = 5x(x^3 - 2) - x^2 + 10x + 2$

10G

Building understanding

1 a -1 b 41 c -19 d -141
2 a 3 b -2
3 0

Now you try

Example 13
a 2 b -37

Example 14
a no b yes

Example 15
$k = -2$

Exercise 10G

1 a 5 b -7 c -55
2 a 3 b 11 c 27 d 57
e -127 f -33 g -13 h -69
3 a 3 b 20 c 36 d 5
4 **b**, **c** and **e** are factors of $P(x)$.
5 **b**, **d**, **f**, **g** are factors of $P(x)$.
6 a $x + 1$ b $x - 1$, $x + 1$ or $x + 2$
c $x + 2$ d $x - 2$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

7 **a** $x-2$, $x-1$ and $x+1$ **b** $x-3$, $x-1$ and $x+2$
c $x-3$, $x-2$ and $x+1$ **d** $x-5$, $x-1$ and $x+4$
8 **a** -4 **b** -2 **c** -14 **d** 96
9 -38
10 **a** 5 **b** 1 **c** 5 **d** -3
11 **a** -2 **b** 23
12 **a** $a=-1$ and $b=2$ **b** $a=3$ and $b=-4$

10H

Building understanding

1 $P(-1)=0$

2 **a** $x=-3$ or 1 **b** $x=-3$ or 4

Now you try

Example 16

a 2, -1, -6 **b** $\frac{1}{2}, -3, -\frac{2}{5}$

Example 17

$x=-1, -4, 3$

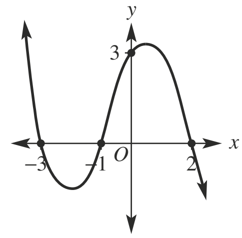
Exercise 10H

1 **a** **i** $-1, 4, 2$ **ii** $-5, 7, -3$
b **i** $\frac{5}{2}, -4, -\frac{2}{3}$ **ii** $-\frac{1}{5}, \frac{8}{3}, 2$
2 **a** $-3, 1, 2$ **b** $-7, -2, 1$ **c** $-4, 3, 4$
d $-\frac{1}{2}, -\frac{1}{3}, 3$ **e** $-\frac{2}{3}, -\frac{1}{2}, 3$ **f** $-\frac{2}{7}, \frac{1}{4}, \frac{2}{5}$
g $-\frac{12}{11}, -\frac{1}{2}, -\frac{11}{3}$ **h** $-\frac{3}{5}, -\frac{2}{19}, \frac{1}{2}$
3 **a** $(x-3)(x-2)(x+1)$; $-1, 2, 3$
b $(x+1)(x+2)(x+3)$; $-3, -2, -1$
c $(x-3)(x-2)(x-1)$; $1, 2, 3$
d $(x-4)(x-3)(x-1)$; $1, 3, 4$
e $(x-6)(x+1)(x+2)$; $-2, -1, 6$
f $(x-2)(x+3)(x+5)$; $-5, -3, 2$
4 **a** $x=1$ or $1+\sqrt{5}$ or $1-\sqrt{5}$ **b** $x=-2$
5 **a** $x=-1$, 3 or 5 **b** $x=-3$, -2 or 1
6 **a** $x=-4$, 1 or 3 **b** -2, -1 or 3
7 **a** 3 **b** 4 **c** n
8 **a** $x^2(x-1)$; 0, 1
b $x^2(x+1)$; -1, 0
c $x(x-4)(x+3)$; $-3, 0, 4$
d $2x^3(x+1)^2$; -1, 0
9 $0=x^4+x^2=x^2(x^2+1)$
No solution to $x^2+1=0$.
Thus, $x=0$ is the only solution.
10 The discriminant of the quadratic is negative, implying solutions from the quadratic factor are not real. $x=2$ is the only solution.
11 **a** $x=-4, -3, -2$ or 1
b $x=-2$ or 3
c $x=-3, -2, 1$ or 3
d $x=-2, \frac{1}{2}$, 1 or 2

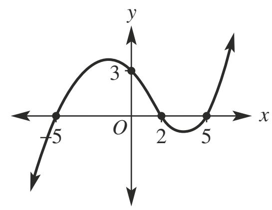
10I

Building understanding

1 **a**



b

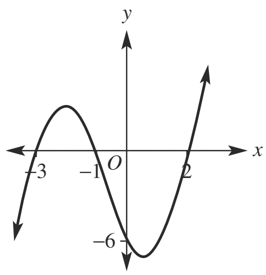


2 **a** y-intercept is 12.
x-intercepts are -1, 3, 4.
b y-intercept is 0.
x-intercepts are -7, 0, 5.

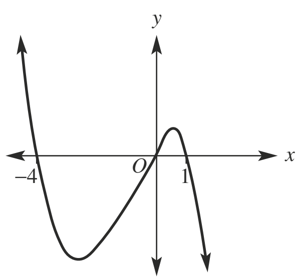
Now you try

Example 18

a

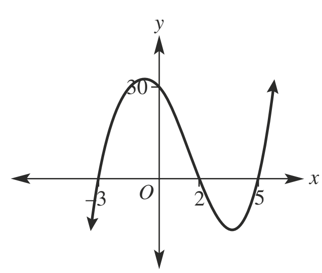


b



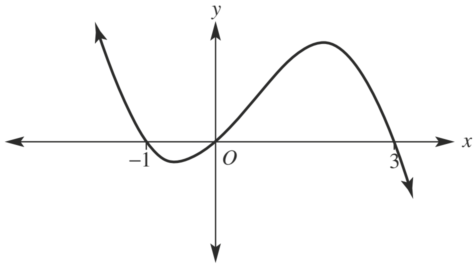
Exercise 10I

1 **a**

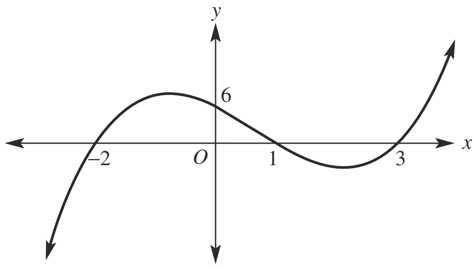


ISBN 978-1-108-77290-7 Cambridge University Press
Updated June 2022

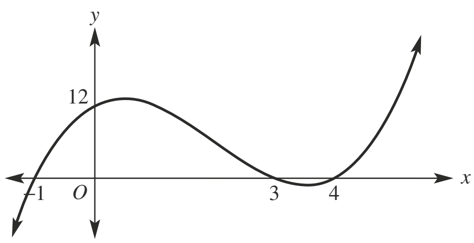
b



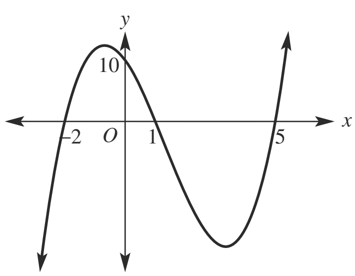
2 **a** y-intercept: 6
x-intercepts: –2, 1, 3



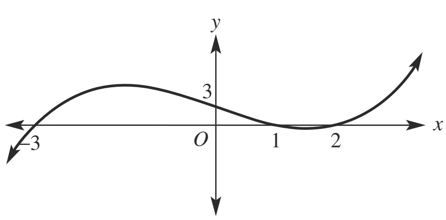
b y-intercept: 12
x-intercepts: –1, 3, 4



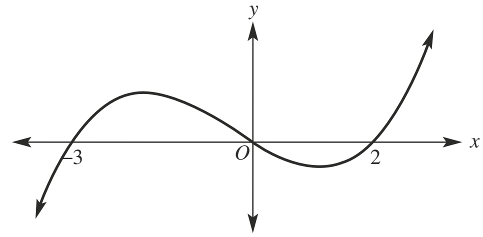
c y-intercept: 10
x-intercepts: –2, 1, 5



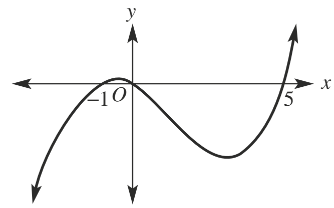
d y-intercept: 3
x-intercepts: –3, 1, 2



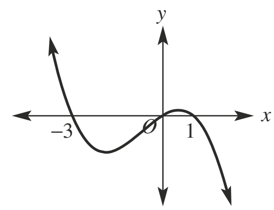
e y-intercept: 0
x-intercepts: –3, 0, 2



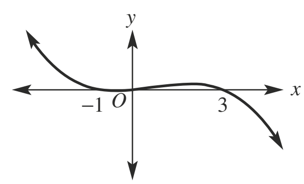
f y-intercept: 0
x-intercepts: –1, 0, 5



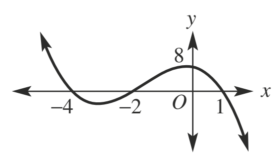
g y-intercept: 0
x-intercepts: –3, 0, 1



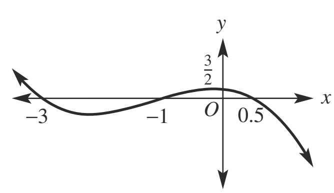
h y-intercept: 0
x-intercepts: –1, 0, 3



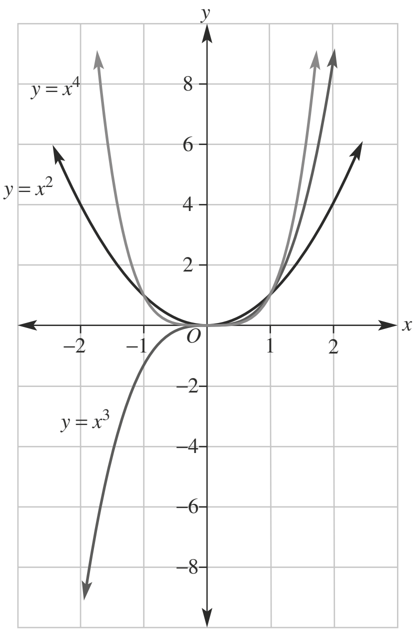
i y-intercept: 8
x-intercepts: –4, –2, 1



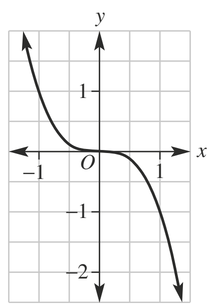
j y-intercept: $\frac{3}{2}$
x-intercepts: –3, –1, $\frac{1}{2}$



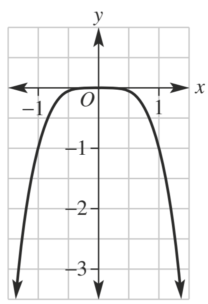
3



4 a



b



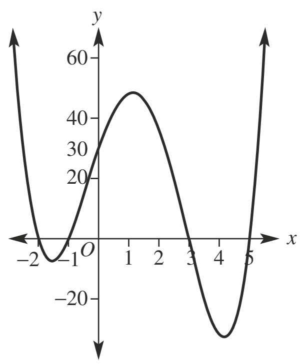
5 a $y = (x - 2)(x + 1)(x + 4)$

b $y = (x + 3)(x - 1)(x - 3)$

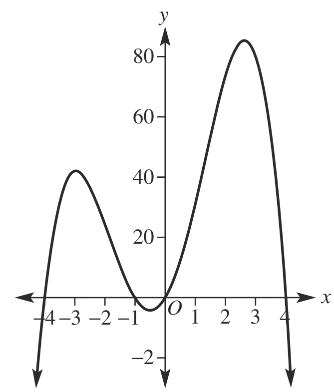
c $y = \frac{1}{2}x(x - 2)(x + 3)$

d $y = -\frac{1}{2}(x + 3)(x + 1)(x - 2)$

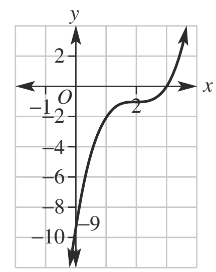
6 a



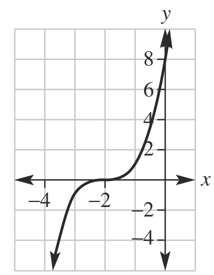
b



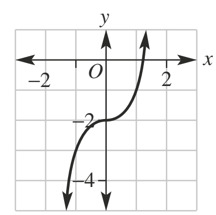
7 a



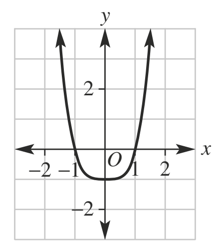
b



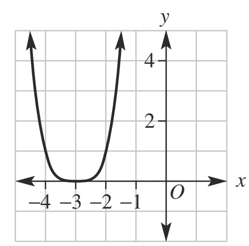
c



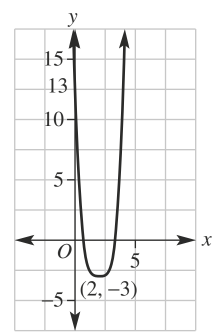
d



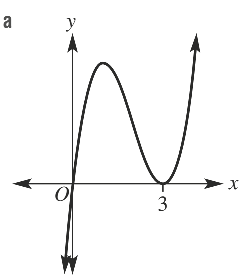
e



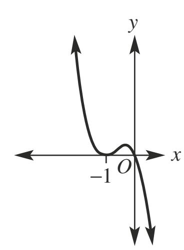
f



8 a



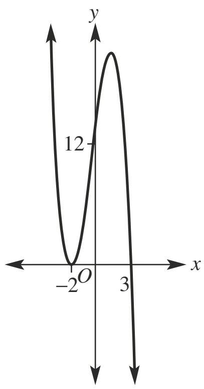
b



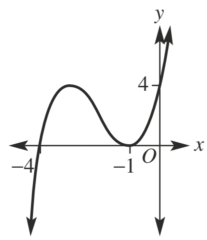
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

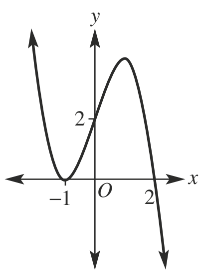
c



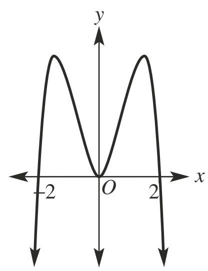
d



e



f

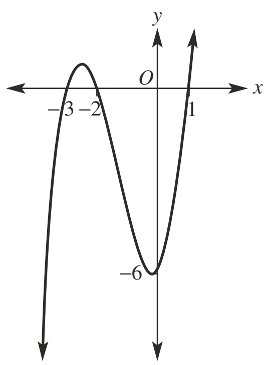


9 **a** **i** y-intercept $= (0, -6)$

ii $y = (x - 1)(x + 2)(x + 3)$

iii x-intercepts: $(-3, 0)$, $(-2, 0)$, $(-1, 0)$

iv

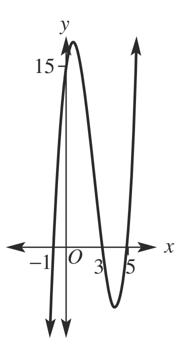


b **i** y-intercept $= (0, 15)$

ii $y = (x - 5)(x - 3)(x + 1)$

iii x-intercepts: $(-1, 0)$, $(3, 0)$, $(5, 0)$

iv

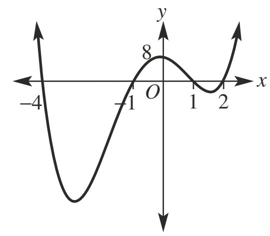


c **i** y-intercept $= (0, 8)$

ii $y = (x - 2)(x - 1)(x + 1)(x + 4)$

iii x-intercepts: $(-4, 0)$, $(-1, 0)$, $(1, 0)$, $(2, 0)$

iv

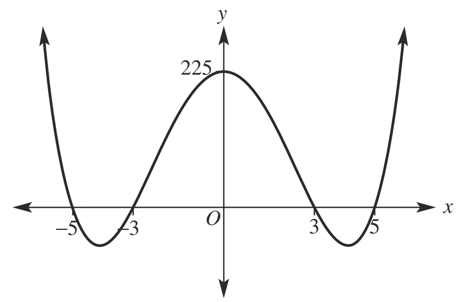


d **i** y-intercept $= (0, 225)$

ii $y = (x - 5)(x - 3)(x + 3)(x + 5)$

iii x-intercepts: $(-5, 0)$, $(-3, 0)$, $(3, 0)$, $(5, 0)$

iv



Problems and challenges

1 **a** 2 **b** 8 **c** $\frac{1}{2}$ **d** 3

2 **a** 1.43 **b** −1.43 **c** −2.71

3 $x = 6$

4 $2p - q + 2$

5 **a** $x > 2.10$ **b** $x \geqslant -2.81$

6 $a = 2 \times 3^{\frac{1}{4}}, b = \frac{1}{4}\log_2 3$

7 $\frac{\log_{10} 2}{\log_{10} 1.1} = \log_{1.1} 2$

8 −2

9 $a = 5, b = -2$

10 Proof using long division required.

a $(x^3 - a^3) \div (x - a) = x^2 + ax + a^2$

b $(x^3 + a^3) \div (x + a) = x^2 - ax + a^2$

11 **a** $2 \leqslant x \leqslant 5$ or $x \leqslant -1$

b $-4 < x < 1$ or $x > 4$

12 $y = \frac{1}{9}(x - 3)^2(x + 2)$

13 16

14 $y = -\frac{1}{10}x^2(x - 3)(x + 3)$

Short-answer questions

1 **a** $\log_2 16 = 4$ **b** $\log_{10} 1000 = 3$ **c** $\log_3 \frac{1}{9} = -2$

2 **a** $3^4 = 81$ **b** $4^{-2} = \frac{1}{16}$ **c** $10^{-1} = 0.1$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

3 a 3 b 4 c 4
d 0 e −3 f −3
g −1 h −4 i −2

4 a $\log_a 8$ b $\log_b 21$ c $\log_b 144$
d $\log_a 10$ e $\log_a 4$ f $\log_a 1000$
g 2 h 1 i $\frac{3}{2}$

5 a $x = \log_3 6$ b $x = \log_{1.2} 2$

6 a $\frac{\log_{10} 13}{\log_{10} 2}$ b $\frac{\log_{10} 2}{\log_{10} 0.8}$

7 a −1 b 1 c −2 d −34

8 a $x^4 + 3x^2 + 2$
b $x^5 - x^4 - 3x^3$
c $x^5 + x^4 - 3x^3 - x^2 - x + 3$
d $x^6 + 2x^4 - 4x^3 + x^2 - 4x + 3$

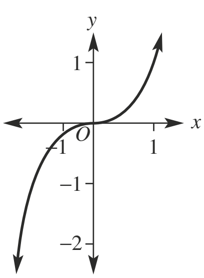
9 a $x^3 + x^2 + 2x + 3 = (x - 1)(x^2 + 2x + 4) + 7$
b $x^3 - 3x^2 - x + 1 = (x + 1)(x^2 - 4x + 3) - 2$
c $2x^3 - x^2 + 4x - 7 = (x + 2)(2x^2 - 5x + 14) - 35$
d $-2x^3 - x^2 - 3x - 4 = -(x - 3)(2x^2 + 7x + 24) - 76$

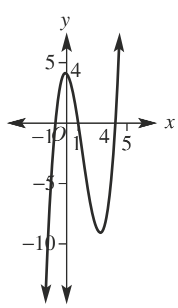
10 a −3 b −39 c −91 d 41

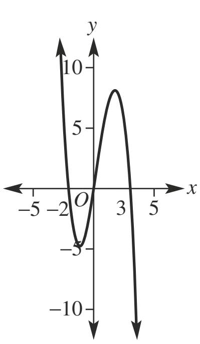
11 **b**, **c** and **d** are factors.

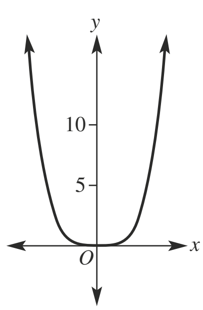
12 a $x = -2, 1$ or 3
b $x = -\frac{1}{3}, \frac{3}{2}$ or 5

13 a $(x - 1)(x + 2)(x + 3) = 0$ $x = -3, -2$ or 1
b $(x + 2)(x - 5)(x - 6) = 0$ $x = -2, 5$ or 6

14 a


b


c


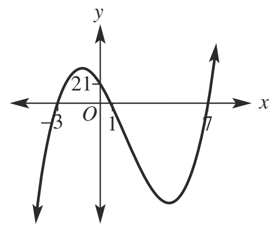
d


Multiple-choice questions

1 C 2 B 3 E 4 A 5 D
6 D 7 A 8 B 9 E 10 E

Extended-response questions

1 a B
b i \$121 000 ii \$115 369 iii \$272 034
c i 7.27 ii 6.17 iii 16.89

2 a i 32 ii 0
b There is no remainder; i.e. $P(1) = 0$.
c $x^2 - 4x - 21$
d $(x - 7)(x - 1)(x + 3)$
e $x = 7, 1$ or −3
f $P(0) = 21$
g


Semester review 2

Measurement

Short-answer questions

1 a 36 cm, 52 cm^2 b 1.3 m, 0.1 m^2
c 220 mm, 2100 mm^2

2 a 188.5 m^2, 197.9 m^3 b 50.3 cm^2, 23.7 cm^3
c 6.8 m^2, 1.3 m^3

3 a 1.8 cm b 58.8 cm^2

4 $\sqrt{\frac{27}{\pi}}$ cm

Multiple-choice questions

1 B 2 D 3 B 4 A 5 D

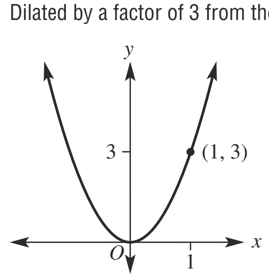
Extended-response question

a 753.98 cm^3 b 206.02 cm^3
c 17 cm d 1.79 cm

Parabolas and other graphs

Short-answer questions

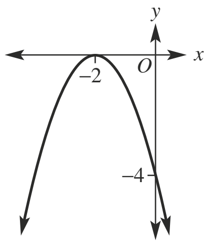
1 a Dilated by a factor of 3 from the x-axis.



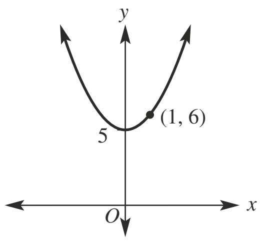
ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

b Reflected in x-axis and translated 2 units left.

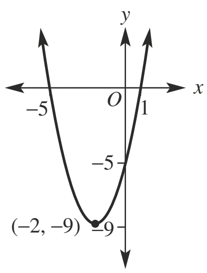


c Translated 5 units up.



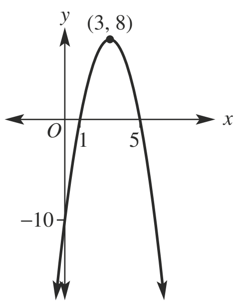
2 a −5 **b** −5, 1 **c** (−2, −9)

d

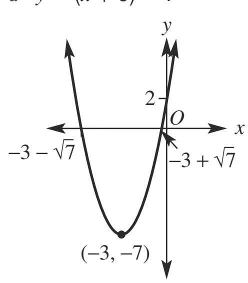


3 a Maximum at (3, 8). **b** −10 **c** 1, 5

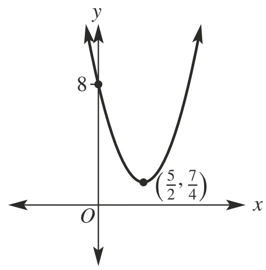
d



4 a $y = (x + 3)^2 - 7$

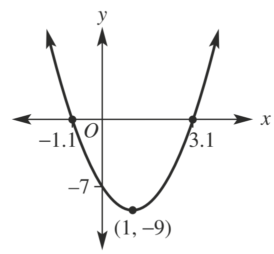


b $y = \left(x - \frac{5}{2}\right)^2 + \frac{7}{4}$



5 a Discriminant = 72, thus two x-intercepts.

b



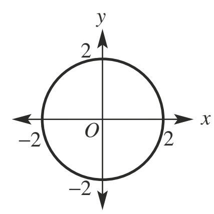
c (1, −9) and (−2, 9)

6 a Each x-value produces a unique y-value (any vertical line will cut the graph at most once).

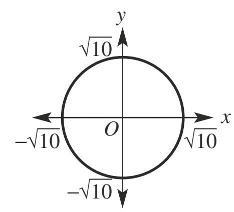
b **i** 3 **ii** 12 **iii** $(a - 2)^2 + 3$

c All real x, $y \geqslant 3$

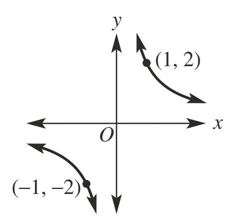
7 a



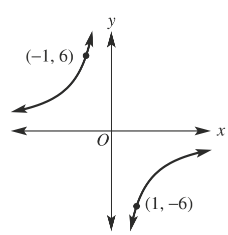
b



c



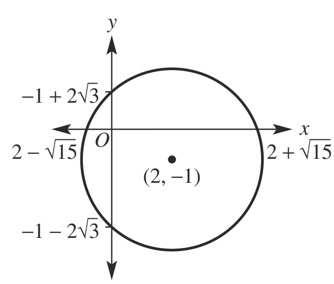
d



8 a $(\sqrt{3}, 2\sqrt{3}), (-\sqrt{3}, -2\sqrt{3})$

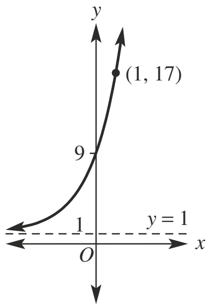
b $\left(\frac{1}{2}, 4\right), \left(-\frac{1}{2}, -4\right)$

9 a

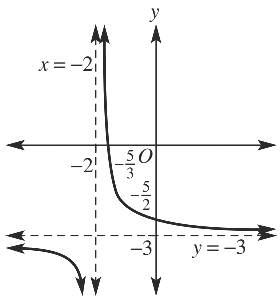


ISBN 978-1-108-77290-7 Cambridge University Press
 Updated June 2022

b



c

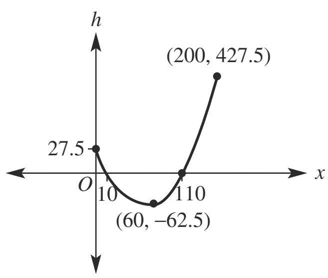


Multiple-choice questions

1 D **2** A **3** D **4** C **5** B

Extended-response question

a



b 27.5 m

c 10 m and 110 m from start

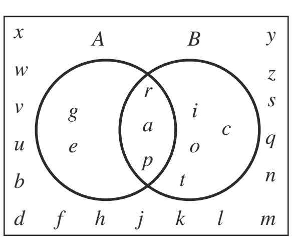
d 427.5 m

e 62.5 m

Probability

Short-answer questions

1 a



b **i** $\frac{5}{26}$ **ii** $\frac{3}{26}$ **iii** $\frac{9}{26}$ **iv** $\frac{19}{26}$

c No, $A \cap B \neq \varnothing$

2 a

	B	B'	
A	3	1	4
A'	4	4	8
	7	5	12

b **i** 3 **ii** 4 **iii** 5 **iv** 8

c **i** $\frac{1}{4}$ **ii** $\frac{1}{12}$ **iii** $\frac{7}{12}$ **iv** $\frac{3}{4}$

3 a 0.18 **b** 0.37

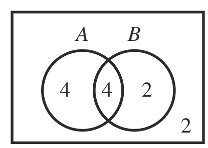
4 a

		Die 1			
		1	2	3	4
Die 2	1	2	3	4	5
	2	3	4	5	6
	3	4	5	6	7
	4	5	6	7	8

b 16

c **i** $\frac{3}{16}$ **ii** $\frac{5}{8}$ **iii** $\frac{1}{5}$

5 a



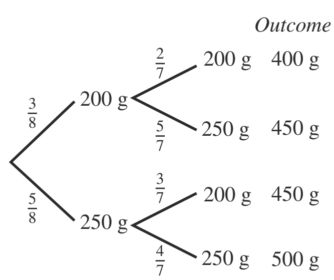
b **i** $\frac{2}{3}$ **ii** $\frac{2}{3}$

c Yes they are, since $\Pr(A|B) = \Pr(A)$

Multiple-choice questions

1 C **2** E **3** B
4 D **5** C

Extended-response question



a **i** $\frac{3}{28}$ **ii** $\frac{15}{28}$ **iii** $\frac{5}{14}$

b $\frac{9}{14}$ **c** $\frac{3}{5}$

ISBN 978-1-108-77290-7

Cambridge University Press
Updated June 2022

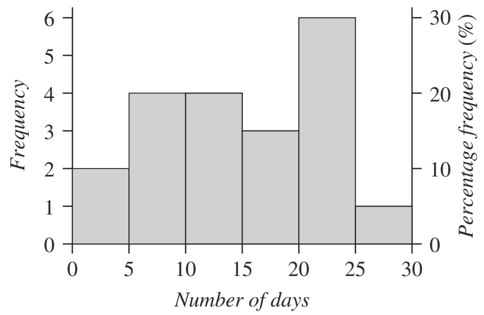
Statistics

Short-answer questions

1 **a**

Class interval	Frequency	Percentage frequency
0–	2	10%
5–	4	20%
10–	4	20%
15–	3	15%
20–	6	30%
25 – 30	1	5%
Total	**20**	**100%**

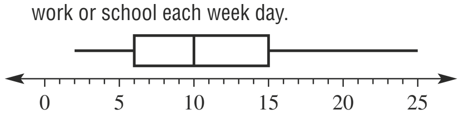
b



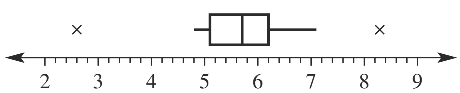
c **i** 14 **ii** 50%

iii 20–24 days, those that maybe catch public transport to work or school each week day.

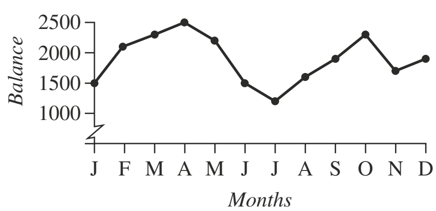
2 **a**



b



3 **a**

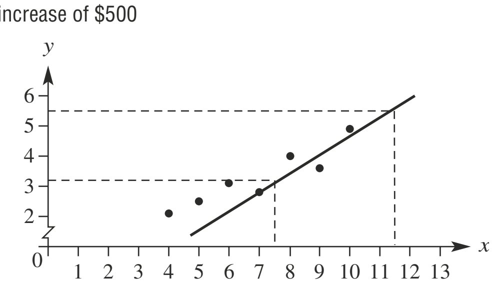


b Balance fluctuated throughout the year but ended up with more money after 12 months.

c May and June

d increase of $500

4 **a, c**



b positive

d **i** ≈ 3.2 **ii** ≈ 11.5

5 **a** **i** Under 40 **ii** Over 40

b Over 40: mean = 11, standard deviation = 7.3;
Under 40: mean = 24.1, standard deviation = 12.6

Multiple-choice questions

1 E **2** B **3** C
4 B **5** A

Extended-response question

a $y = 1.50x + 17.23$

b 41 cm

Logarithms and polynomials

Short-answer questions

1 **a** 3 **b** −2 **c** 3 **d** 0
e 2 **f** 2 **g** 1 **h** 3

2 **a** $x = 3$ **b** $x = 3$ **c** $x = 81$

3 **a** **i** $x = \log_3 30$ **ii** $x = \log_{2.4} 4$
b **i** $x = 2.460$ **ii** $x = 9.955$

4 **a** **i** 6 **ii** 0 **iii** −49 **iv** −5
b **i** $2x^6 + 6x^5 - 11x^4 - 25x^3 + 34x + 24$
ii $4x^6 - 12x^4 - 16x^3 + 9x^2 + 24x + 16$

5 $P(x) = (x - 3)(x^2 - x - 1) + 4$

6 **a** −24, not a factor **b** 0, a factor
c −40, not a factor

7 **a** $x = -1, 3, -6$ **b** $x = 0, \frac{5}{2}, -\frac{2}{3}$
c $x = -4, -2, 1$ **d** $x = -1, \frac{1}{2}, 2$

Multiple-choice questions

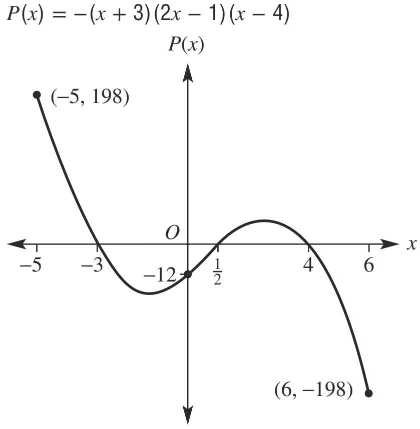
1 E **2** A **3** D
4 C **5** D

Extended-response question

a $P(-3) = 0$

b $P(x) = -(x + 3)(2x - 1)(x - 4)$

c



ISBN 978-1-108-77290-7