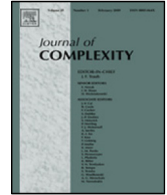




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The curse of dimensionality for numerical integration on general domains[☆]

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ABSTRACT

We prove the curse of dimensionality in the worst case setting for multivariate numerical integration for various classes of smooth functions. We prove the results when the domains are isotropic convex bodies with small diameter satisfying a universal ψ_2 -estimate. In particular, we obtain the result for the important class of volume-normalized ℓ_p^d -balls in the complete regime $2 \leq p \leq \infty$. This extends a result in a work of Hinrichs et al. (2014) to the whole range $2 \leq p \leq \infty$, and additionally provides a unified approach. The key ingredient in the proof is a deep result from the theory of Asymptotic Geometric Analysis, the thin-shell volume concentration estimate due to O. Guédon and E. Milman. The connection of Asymptotic Geometric Analysis and Information-based Complexity revealed in this work seems promising and is of independent interest.

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1. Introduction and main results

1.1. Introduction

In the last decade, understanding complex systems that depend on a huge amount of parameters and, more specifically, the study of high-dimensional geometric structures has become increasingly

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important. It has become apparent by now that the presence of high dimensions forces a certain regularity in the geometry of the space while, on the other hand, it unfolds various rather unexpected phenomena. Two independent and rather young mathematical disciplines that center around questions of this type from different perspectives are Information-based Complexity and Asymptotic Geometric Analysis. This work will bring together both areas for the first time to tackle one of the most frequent questions arising in and related to high-dimensional frameworks, namely the one for the curse of dimensionality.

A prominent example in this respect is multivariate numerical integration, which has received a lot of attention in previous years and is the central topic of this work. One is interested in approximating, up to some given error $\varepsilon > 0$, the value of a multivariate integral over a volume-normalized domain $K_d \subseteq \mathbb{R}^d$ (d considered large) of a function $f : K_d \rightarrow \mathbb{R}$ that belongs to some class \mathcal{F}_d of smooth functions. In this manuscript, we shall consider linear (deterministic) algorithms that use only function values to approximate such an integral. If the minimal number of function evaluations needed for this task in the worst case setting increases exponentially with the space dimension d , then we say that the function class \mathcal{F}_d suffers from the curse of dimensionality. For any unexplained notion or notation, we refer the reader to Section 2.

In [14], improving and extending previous results obtained in [13], Hinrichs, Novak, Ullrich and Woźniakowski proved the curse of dimensionality in the worst case setting for numerical integration for a number of classes of smooth d -variate functions. They considered different bounds on the Lipschitz constants for the directional or partial derivatives of $f \in \mathcal{C}^k(K_d)$ ($k \in \mathbb{N} \cup \{+\infty\}$), where $\mathcal{C}^k(K_d)$ denotes the space of k -times continuously differentiable functions on a domain $K_d \subseteq \mathbb{R}^d$ with $\text{vol}_d(K_d) = 1$. The assumption that the domain of integration has volume one guarantees that the integration problem is properly normalized. To be more specific, for $d \in \mathbb{N}$ and a volume-normalized domain $K_d \subseteq \mathbb{R}^d$, we define the class $\mathcal{C}_d^1(A_d, B_d, K_d)$ of smooth functions depending on two Lipschitz parameters, $A_d > 0$ and $B_d > 0$, as follows:

$$\mathcal{C}_d^1(A_d, B_d, K_d) := \left\{ f \in \mathcal{C}^1(K_d) : \|f\|_\infty \leq 1, \text{Lip}(f) \leq A_d \text{ and } \text{Lip}(D^\theta f) \leq B_d \text{ for all } \theta \in \mathbb{S}^{d-1} \right\},$$

where D^θ denotes the directional derivative in direction $\theta \in \mathbb{S}^{d-1}$ and

$$\text{Lip}(g) := \sup_{x, y \in K_d} \frac{|g(x) - g(y)|}{\|x - y\|_2}.$$

Classes of this type and variants thereof are classical objects in numerical analysis. Historically, the central question was the rate of error convergence for fixed dimension d . Roughly speaking, the classes $\mathcal{C}_d^1(A_d, B_d, K_d)$ are classes of \mathcal{C}^2 -functions, and the optimal rate of error convergence for integration of functions from these classes is $n^{-2/d}$, see [3] for $K_d = [0, 1]^d$. Only recently, motivated by modern applications, the emphasis was shifted to the study of the dependence of the error on the dimension. Specifically, in [13,14] the authors discussed necessary and sufficient conditions on the parameters A_d and B_d as well as on the domain K_d such that the curse of dimensionality holds for the class $\mathcal{C}_d^1(A_d, B_d, K_d)$. The results from their paper [14] are sharp and characterize the curse of dimensionality if the domain of integration is either the cube, i.e., $K_d = [0, 1]^d$, or a convex body with the property

$$\limsup_{d \rightarrow \infty} \frac{\text{rad}(K_d)}{\sqrt{d}} < \sqrt{\frac{2}{\pi e}}, \quad (1.1)$$

where $\text{rad}(K_d)$ denotes the radius of the set K_d (see [14, Theorem 4.1]). The latter are sets with rather small radius. For instance, the volume-normalized ℓ_p -balls \mathbb{D}_p^d satisfy (1.1) if and only if $2 \leq p < p_0$ where $p_0 \approx 170.5186$. However, this not only leaves a gap for the remaining class of ℓ_p -ball domains, but also the methods of proof are different for cubes and sets satisfying (1.1).

The main purpose of the present paper is to present a novel and unified approach to this problem that will, at the same time, allow us to complement and extend the results from [13,14] to a wider class of domains of integration. In particular, we study the case where these bodies satisfy a *uniform ψ_α -estimate*, $\alpha \in [1, 2]$ (see Section 2.2 and Proposition 4.1). This leads to a characterization of the

curse of dimensionality for ℓ_p -ball domains in the full regime $2 \leq p \leq \infty$, thereby closing the gap in [14].

The key observation is that one needs to put the domain of integration in the right perspective to unveil and understand those geometric aspects that are crucial in such a problem. This is where the theory of Asymptotic Geometric Analysis enters the stage and provides us with a suitable and natural framework to settle this question. To the best of our knowledge, this is the first work revealing such a connection, which is promising and of independent interest. The central idea is to exploit the fact that volume is highly concentrated in high-dimensional (isotropic) convex bodies. There are a number of deep results in this direction, among others, the pioneering works of Paouris [26] and Klartag [17]. The main ingredient in our proof is the famous thin-shell estimate due to Guédon and Milman [12], which shows that the Euclidean norm of an isotropic and log-concave random vector $X \in \mathbb{R}^d$ is highly concentrated around its expectation. The reason for using their estimate, despite the fact that it is the best known general bound, is the sensitivity of our estimates to the involved deviation parameters (see Remark 2.8). Under very natural geometric assumptions on the domain, reflecting a typical framework in Asymptotic Geometric Analysis, we are able to relax the radial condition (1.1) from [14]. As will become apparent later on, condition (1.1) uses, although in disguise, the well-known universal lower bound $1/\sqrt{2\pi e}$ for the isotropic constant. But as we shall demonstrate, the characterization of the curse of dimensionality does depend (to some extent) on the isotropic constant of the domain of integration (see also Remark 1.4), which is probably of independent interest again.

1.2. Main results

We shall now briefly specify the geometric framework we will be working in (including some additional explanations for readers less familiar with Asymptotic Geometric Analysis), followed by a presentation of the main results of this paper. For more information on the central (geometric) notions that appear here we refer the reader to the preliminaries.

A convex body $K_d \subseteq \mathbb{R}^d$ is a compact, convex set with non-empty interior and shall be called isotropic or in isotropic position if $\text{vol}_d(K_d) = 1$, the center of mass is at the origin and, for some $L_{K_d} \in (0, \infty)$,

$$\int_{K_d} \langle x, \theta \rangle^2 dx = L_{K_d}^2$$

for all $\theta \in \mathbb{S}^{d-1}$. We call L_{K_d} the isotropic constant of K_d . The maybe most prominent class of isotropic convex bodies are the volume normalized unit balls of the spaces ℓ_p^d ($1 \leq p \leq \infty$) and the value of their isotropic constants can be computed directly from Lemma 3.1 and Eq. (2.5). Let us remark that every centered convex body can be brought in isotropic position by a linear transformation and this position is uniquely determined up to orthogonal transformations. A convex body $K_d \subseteq \mathbb{R}^d$ is symmetric if $-x \in K_d$ whenever $x \in K_d$. The radius of such a symmetric convex body is defined as

$$\text{rad}(K_d) = \sup_{y \in K_d} \|y\|_2.$$

Let us now introduce an Orlicz norm $\|\cdot\|_{\psi_\alpha}$ that can be defined for any measurable function $f: \Omega \rightarrow \mathbb{R}$ on a probability space $(\Omega, \mathcal{F}, \mu)$ and $\alpha \geq 1$ via

$$\|f\|_{\psi_\alpha} := \inf \left\{ \lambda > 0 : \int_{\Omega} e^{|f(\omega)|/\lambda} d\mu(\omega) \leq 2 \right\}.$$

We say that an isotropic convex body K_d satisfies a ψ_α -estimate with constant $b_\alpha \in (0, \infty)$, for some $\alpha \in [1, 2]$, if

$$\|\langle \cdot, \theta \rangle\|_{\psi_\alpha} \leq b_\alpha \left(\int_{K_d} |\langle x, \theta \rangle|^2 dx \right)^{1/2}$$

for all $\theta \in \mathbb{S}^{d-1}$, where $\langle \cdot, \theta \rangle: K_d \rightarrow \mathbb{R}$. For short, we call an isotropic convex body with this property a ψ_α -body. As we shall see in Proposition 2.6, the unit balls of the spaces ℓ_p^d are ψ_2 -bodies with a

universal constant $C \in (0, \infty)$ for $p \geq 2$. For a better understanding, let us note that those Orlicz norms can be equivalently expressed in terms of L_q -norms (see, e.g., [9, Lemma 2.4.2]), i.e., there are absolute constants $c, C \in (0, \infty)$ such that

$$c \sup_{q \geq \alpha} \frac{\|f\|_{L_q(\mu)}}{q^{1/\alpha}} \leq \|f\|_{\psi_\alpha} \leq C \sup_{q \geq \alpha} \frac{\|f\|_{L_q(\mu)}}{q^{1/\alpha}}.$$

The family of ψ_α -norms plays a key rôle in the theory of Asymptotic Geometric Analysis. For more details, we refer the reader to Section 2.2 and to the monographs [2,9].

The first main result is the following.

Theorem 1.1. *Let $(K_d)_{d \in \mathbb{N}}$ be a sequence of symmetric, isotropic convex bodies that satisfy a uniform ψ_2 -estimate and*

$$\limsup_{d \rightarrow \infty} \frac{\text{rad}(K_d)}{\sqrt{d} L_{K_d}} < 2. \quad (1.2)$$

Then the curse of dimensionality holds for the class $\mathcal{C}_d^1(A_d, B_d, K_d)$ if

$$\limsup_{d \rightarrow \infty} \min \left\{ A_d \sqrt{d} L_{K_d}, B_d d L_{K_d}^2 \right\} > 0.$$

In addition to these lower bounds on the complexity of numerical integration, we will present lower bounds for the more general class of ψ_α -bodies with $\alpha \in [1, 2)$ (see Section 4). These bounds are, however, no longer exponential in d , but in $d^{\alpha/2}$. We shall present this result in Proposition 4.1.

Remark 1.2. We conjecture that the constant 2 in (1.2) could be replaced by an arbitrarily large constant. However, our proof heavily uses (a variant of) a famous result of Elekes [10] on the covering of convex hulls of n -element point sets in K_d with n balls of radius about $\text{rad}(K_d)/2$, see the proof of Proposition 4.1. For enlarging the 2, we would (at least) need a covering with balls of radius $c \cdot \text{rad}(K_d)$, $c < 1/2$, which has cardinality at most polynomial in d . We do not know results of this type.

Remark 1.3. In Asymptotic Geometric Analysis terms, our radial condition (1.2) simply means that such a body K_d is of “small diameter” (i.e., $\text{rad}(K_d) = a\sqrt{d} L_{K_d}$ for some $a \in [1, \infty)$) with a constant strictly smaller than 2. It follows directly from the isotropic condition that, for any isotropic convex body $K \subseteq \mathbb{R}^d$, $\sqrt{d} L_K \leq \text{rad}(K)$. Additionally, as shown by Kannan, Lovász and Simonovits in [15], we have the upper bound $\text{rad}(K) \leq (d+1)L_K$. We refer to the monograph [9] for more information.

Remark 1.4. If we compare the two conditions (1.1) and (1.2) on the radii, then we see that, contrary to condition (1.1) considered in [14], our condition (1.2) involves the isotropic constant L_{K_d} of the integral domain K_d . Using the well-known fact that among all isotropic convex bodies the (volume normalized) Euclidean ball, we write \mathbb{D}_2^d , minimizes the isotropic constant with $L_{\mathbb{D}_2^d} \geq 1/\sqrt{2\pi e}$ (see, e.g., [9, Proposition 3.3.1]), we obtain from our assumption (1.2) exactly the radius condition (1.1). The observation that this universal lower bound on the isotropic constant of the convex body under consideration, which appears in (1.1) only in disguise, can be substituted by the isotropic constant of the body itself, is the essential step that allows us to extend the results from [14].

Remark 1.5. As will become apparent, Theorem 1.1 is essentially Proposition 4.1 in the case $\alpha = 2$. As a matter of fact, in the statement of Theorem 1.1 the isotropic constants appearing in the two conditions involving the Lipschitz parameters $(A_d)_d$ and $(B_d)_d$ could have been omitted. The reason is that Bourgain proved in [8] that if K is a symmetric ψ_2 -body with constant $b \geq 1$, then its isotropic constant is bounded, even more precisely, $L_K \leq Cb \log(b+1)$ with an absolute constant $C \in (0, \infty)$. This bound has been improved by Klartag and Milman [18] to a linear dependence in b . However, we chose to include the constants $(L_{K_d})_d$ anyway to underline where the isotropicity enters the scene. Whether or not the isotropic constant is uniformly bounded above by an absolute constant in general is a famous open problem (see Remark 2.3).

We complement our lower bounds by the following concentration result, which easily implies an upper bound on the complexity of numerical integration in the space $\mathcal{C}_d^1(A_d, B_d, K_d)$. We stress that this result, as the last one, heavily relies on the thin-shell estimate of Guédon and Milman [12].

Theorem 1.6. *Let $(K_d)_{d \in \mathbb{N}}$ be a sequence of isotropic convex bodies. Assume that $(A_d)_{d \in \mathbb{N}}$ and $(B_d)_{d \in \mathbb{N}}$ satisfy*

$$\lim_{d \rightarrow \infty} \min \left\{ A_d \sqrt{d} L_{K_d}, B_d d L_{K_d}^2 \right\} = 0.$$

Then

$$\lim_{d \rightarrow \infty} \sup_{f \in \mathcal{C}_d^1(A_d, B_d, K_d)} \left| \int_{K_d} f(x) dx - f(0) \right| = 0.$$

In particular, the curse of dimensionality does not hold for $\mathcal{C}_d^1(A_d, B_d, K_d)$.

This theorem generalizes the corresponding result [14, Proposition 4.7] to arbitrary isotropic sets. Moreover, the necessary decay of the Lipschitz constants is the same as in [14, Theorem 4.1] if the hyperplane conjecture were shown to be true (see Remark 2.3).

The volume-normalized ℓ_p -balls, \mathbb{D}_p^d , are symmetric and isotropic ψ_2 -bodies whenever $2 \leq p \leq \infty$ (see [5, Proposition 10]). Moreover, we will show in Lemma 3.2 that those \mathbb{D}_p^d -balls satisfy (1.2). As a consequence, we obtain the curse of dimensionality for this important class of integral domains as a corollary to Theorem 1.1.

Corollary 1.7. *Let $2 \leq p \leq \infty$. Then the curse of dimensionality holds for the classes $\mathcal{C}_d^1(A_d, B_d, \mathbb{D}_p^d)$ if and only if*

$$\limsup_{d \rightarrow \infty} \min \left\{ \sqrt{d} A_d, d B_d \right\} > 0.$$

By Proposition 5.1, Proposition 5.2 and the fact that the isotropic constant is uniformly bounded for all ℓ_p -balls, the “only if”-part of Corollary 1.7 also holds for $1 \leq p < 2$. But our proof of the “if”-part is not applicable to this case, since the limit in (1.2) is infinity for $1 \leq p < 2$. Nevertheless, it is reasonable to conjecture that the complete result remains true also for $1 \leq p < 2$.

Remark 1.8. We note that there is a mistake in the formulation of the corresponding result from [14]. The assumptions on $L_{0,d}$ and $L_{1,d}$ in [14, Theorem 4.1] (which we call A_d and B_d) are not enough and should be replaced by an assumption as above.

The rest of the paper is organized as follows. In the next section, Section 2, we provide background material from both Asymptotic Geometric Analysis and Information-based Complexity. Having a broad readership in mind, we do this in slightly more detail than strictly necessary for the proof of our results. In Section 3, we present some auxiliary results in form of radial and volume estimates. The final Sections 4–5 are then devoted to the proofs of the main results mentioned above.

2. Preliminaries and notation

In this section we will present the notions and concepts from Information-based Complexity and Asymptotic Geometric Analysis needed throughout this work. Keeping a broad readership in mind, we try to keep this as detailed and self-contained as possible.

2.1. Background A—Information-based Complexity

Information-based Complexity (IBC) studies optimal algorithms and computational complexity for continuous problems like finding solutions of differential equations, integration and approximation, arising in different areas of application. The emphasis is on the dependence of the minimal error achievable within a certain class of algorithms using a given budget of n informations about, e.g., the

function to integrate or approximate. Typically, the functions under consideration depend on many variables, so also the dependence on the number d of variables, the dimension of the problem, is crucial. For recent monographs thoroughly treating many modern results in IBC, we refer the reader to [23–25].

In this subsection, we recall the necessary notions from IBC to precisely define our problem and explain notation already used in formulating the main results in the introduction.

2.1.1. The setting

Let \mathcal{F}_d be a class of continuous and integrable functions $f : K_d \rightarrow \mathbb{R}$, where $K_d \subseteq \mathbb{R}^d$ is measurable and of unit volume. For $f \in \mathcal{F}_d$, we want to approximate the integral

$$S_d(f) = \int_{K_d} f(x) dx$$

by algorithms

$$A_{n,d}(f) = \phi_{n,d}(f(x_1), f(x_2), \dots, f(x_n)),$$

where $\phi_{n,d} : \mathbb{R}^n \rightarrow \mathbb{R}$ is an arbitrary mapping and $x_j \in K_d$, $j \in \{1, \dots, n\}$ can be chosen adaptively. Adaptively means that the selection of x_j may depend on the already computed values $f(x_1), f(x_2), \dots, f(x_{j-1})$. The (worst case) error of approximation of the algorithm $A_{n,d}$ is defined as

$$e(A_{n,d}) := \sup_{f \in \mathcal{F}_d} |S_d(f) - A_{n,d}(f)|.$$

2.1.2. Complexity

For $\varepsilon > 0$, the information complexity, $n(\varepsilon, \mathcal{F}_d)$, is the minimal number of function values needed to guarantee that the error is at most ε , i.e.,

$$n(\varepsilon, \mathcal{F}_d) := \min \{ n \in \mathbb{N} : \exists A_{n,d} \text{ such that } e(A_{n,d}) \leq \varepsilon \}.$$

Hence, we minimize n over all possible choices of adaptive sample points $x_1, \dots, x_n \in K_d$ and mappings $\phi_{n,d} : \mathbb{R}^n \rightarrow \mathbb{R}$.

Remark 2.1. It is known by the result of Smolyak [30] on nonlinear algorithms and the result of Bahvalov [4] on adaption that, as long as the class \mathcal{F}_d is convex and symmetric, we may restrict the minimization of n by considering only *nonadaptive* choices of x_1, \dots, x_n and *linear* mappings $\phi_{n,d}$ (see also [23,31]). In this case, we have

$$n(\varepsilon, \mathcal{F}_d) = \min \left\{ n \in \mathbb{N} : \inf_{x_1, \dots, x_n \in K_d} \sup_{\substack{f \in \mathcal{F}_d, f(x_j)=0 \\ 1 \leq j \leq n}} |S_d(f)| \leq \varepsilon \right\}, \quad (2.1)$$

see, e.g., [23, Lemma 4.3].

In this paper, we always consider convex and symmetric \mathcal{F}_d so that we can use the previous formula for the information complexity $n(\varepsilon, \mathcal{F}_d)$. It is also well known that for convex and symmetric \mathcal{F}_d the total complexity, i.e., the minimal cost of computing an ε -approximation, insignificantly differs from the information complexity. For more details see, for instance, [23, Section 4.2.2].

2.1.3. The curse of dimensionality

By the curse of dimensionality we mean that the information complexity $n(\varepsilon, \mathcal{F}_d)$ is exponentially large in d . That is, there are positive numbers c , ε_0 and γ such that

$$n(\varepsilon, \mathcal{F}_d) \geq c(1 + \gamma)^d \quad \text{for all } \varepsilon \leq \varepsilon_0 \text{ and infinitely many } d \in \mathbb{N}. \quad (2.2)$$

There are many classes \mathcal{F}_d for which the curse of dimensionality has been proved for numerical integration and other multivariate problems (see [23–25] for such examples). From a computational point of view, the curse of dimensionality renders the problem intractable in high dimensions.

2.2. Background B—Asymptotic Geometric Analysis

We present here the background material from Asymptotic Geometric Analysis, which we organize by subtopic.

2.2.1. Convex bodies

We shall be working in \mathbb{R}^d equipped with the standard Euclidean structure $\langle \cdot, \cdot \rangle$ and use the notation $\text{vol}_d(A)$ to indicate the d -dimensional Lebesgue measure of a Borel subset A of \mathbb{R}^d . For $A \subseteq \mathbb{R}^d$, we define its radius to be

$$\text{rad}(A) := \inf_{x \in \mathbb{R}^d} \sup_{y \in A} \|y - x\|_2.$$

For symmetric convex sets $A \subseteq \mathbb{R}^d$, that is, sets that are convex and for which $-x \in A$ whenever $x \in A$, this simplifies to

$$\text{rad}(A) = \sup_{y \in A} \|y\|_2.$$

A convex body $K \subseteq \mathbb{R}^d$ is a compact and convex set with non-empty interior. We write $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$ for the Euclidean unit sphere in \mathbb{R}^d and $\sigma := \sigma_{d-1}$ for the uniform probability measure on \mathbb{S}^{d-1} . A convex body is said to be isotropic (or in isotropic position) if $\text{vol}_d(K) = 1$, its center of mass is at the origin, i.e.,

$$\int_K x \, dx = 0$$

and it satisfies the isotropic condition, i.e., there exists a constant $L_K \in (0, \infty)$ such that, for all $\theta \in \mathbb{S}^{d-1}$,

$$\int_K \langle x, \theta \rangle^2 \, dx = L_K^2.$$

We call L_K the isotropic constant of K .

Remark 2.2. There are several classical positions of convex bodies, e.g., John's position or the M -position, many of which arise as solutions to extremal problems. The isotropic position first arose from classical mechanics in the 19th century and has different research directions connected with it, one being the distribution of volume in convex bodies. In this work, we present yet another and new connection. For more information on the isotropic position we refer to [2, Chapter 2].

Remark 2.3. While the Euclidean ball minimizes the isotropic constant with $L_{\mathbb{B}_2^d} \geq 1/\sqrt{2\pi e}$, whether or not the isotropic constant is uniformly bounded above by an absolute constant in general is a famous open problem first posed by Bourgain in [7]. He obtained a general upper bound of order $\sqrt[4]{d} \log d$, the proof being based on the ψ_1 -behavior of linear functionals on convex bodies (see Section 2.2.3 for a definition). This was improved by Klartag to $\sqrt[4]{d}$ in [16]. Interestingly, while the problem remains open, there is no example of a convex body K with $L_K > 1$.

2.2.2. Isotropic log-concave probability measures

We say that a Borel probability measure μ on \mathbb{R}^d , which is absolutely continuous with respect to the Lebesgue measure, is centered if for all $\theta \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{R}^d} \langle x, \theta \rangle \, d\mu(x) = 0.$$

We say that μ is log-concave if, for all compact subsets A and B of \mathbb{R}^d and all $\lambda \in (0, 1)$,

$$\mu((1 - \lambda)A + \lambda B) \geq \mu(A)^{1-\lambda} \mu(B)^\lambda.$$

A function $f : \mathbb{R}^d \rightarrow [0, \infty)$ is said to be log-concave if it satisfies

$$f((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^\lambda$$

for all $x, y \in \mathbb{R}^d, \lambda \in (0, 1)$. We call it a log-concave density if additionally $\int_{\mathbb{R}^d} f(x) dx = 1$.

It was shown by Borell in [6] that any non-degenerate log-concave probability measure μ on \mathbb{R}^d (meaning it is not fully supported on any hyperplane) is absolutely continuous with respect to the Lebesgue measure and has a log-concave density f_μ , i.e., $d\mu(x) = f_\mu(x) dx$ (see also [9, Theorem 2.1.2]). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A random vector $X : \Omega \rightarrow \mathbb{R}^d$ will be called log-concave if its distribution $\mu(\cdot) = \mathbb{P}(X \in \cdot)$ is a log-concave probability measure on \mathbb{R}^d .

A Borel probability measure μ on \mathbb{R}^d , which is absolutely continuous with respect to the Lebesgue measure, is said to be isotropic if it is centered and satisfies the isotropic condition

$$\int_{\mathbb{R}^d} \langle x, \theta \rangle^2 d\mu(x) = 1,$$

for all $\theta \in \mathbb{S}^{d-1}$. A log-concave random vector X in \mathbb{R}^d is said to be isotropic if its distribution is isotropic, i.e., if $\mathbb{E}(X) = 0$ and $\mathbb{E}(X \otimes X) = \text{id}_d$, where id_d denotes the $d \times d$ identity matrix.

Remark 2.4. Notice that a convex body is isotropic if and only if the uniform probability measure on $\frac{K}{L_K}$ is isotropic. Examples of isotropic log-concave random vectors are standard Gaussian random vectors or random vectors uniformly distributed in $\frac{K}{L_K}$, where K is an isotropic convex body and L_K its isotropic constant.

2.2.3. ψ_α -estimates for linear functionals

Let $\alpha \in [1, 2]$ and let μ be a probability measure on \mathbb{R}^d . For a measurable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ define the Orlicz norm $\| \cdot \|_{\psi_\alpha(\mu)}$ by

$$\|f\|_{\psi_\alpha(\mu)} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^d} e^{|f(x)/\lambda|^\alpha} d\mu(x) \leq 2 \right\}.$$

Given some $\theta \in \mathbb{S}^{d-1}$ one says that θ defines a ψ_α -direction for μ with constant $C \in (0, \infty)$ if $f_\theta(x) = \langle x, \theta \rangle$ satisfies

$$\|f_\theta\|_{\psi_\alpha(\mu)} \leq C \left(\int_{\mathbb{R}^d} |f_\theta(x)|^2 d\mu(x) \right)^{1/2}.$$

Let $K \subseteq \mathbb{R}^d$ be a convex body with centroid at the origin. Such a body is said to be a ψ_α -body with constant $b_\alpha \in (0, \infty)$ if all directions $\theta \in \mathbb{S}^{d-1}$ are ψ_α with constant b_α , with respect to the uniform probability measure on K .

Remark 2.5. Every isotropic convex body K_d or, more generally, each log-concave probability measure on \mathbb{R}^d , satisfies a ψ_1 -estimate with some universal constant $C \in (0, \infty)$ that is independent of d and K_d (see, e.g., [9, Section 3.2.3]).

Regarding the unit balls of finite-dimensional ℓ_p spaces, the following result was obtained by Barthe, Guédon, Mendelson, and Naor [5, Proposition 10].

Proposition 2.6. *There exists $C \in (0, \infty)$ such that for every $d \in \mathbb{N}$ and every $p \geq 2$, \mathbb{B}_p^d is a ψ_2 -body with constant C .*

2.2.4. Thin-shell estimates

One of the driving forces of the theory of Asymptotic Geometric Analysis is the isotropic constant or hyperplane conjecture and with it the question of how volume is distributed in high-dimensional isotropic convex bodies. The last decade has seen major contributions in this direction. Among the most important ones are Paouris' result on the tail behavior of the Euclidean norm of an isotropic

log-concave random vector [26], Klartag's thin-shell estimate that resolved the central limit problem for log-concave measures [17], and the concentration results for the Euclidean norm of an isotropic log-concave random vector due to Guédon and Milman [12]. The latter result, more precisely [12, Theorem 1.1], plays a crucial rôle in our proofs. Denoting by $\|\cdot\|_{\text{HS}}$ the Hilbert–Schmidt norm and by $\|\cdot\|_{\text{op}}$ the operator norm, their result reads as follows.

Theorem 2.7. *Let X be an isotropic random vector in \mathbb{R}^d with log-concave density, which is in addition ψ_α , for some $\alpha \in [1, 2]$, with constant $b_\alpha \in (0, \infty)$. Assume that $A \in \mathbb{R}^{d \times d}$ satisfies $\|A\|_{\text{HS}}^2 = d$. Then, for all $t \geq 0$,*

$$\mathbb{P}\left(\left|\|AX\|_2 - \sqrt{d}\right| \geq t\sqrt{d}\right) \leq C \exp(-c\eta^{\frac{\alpha}{2}} \min\{t^{2+\alpha}, t\}),$$

where

$$\eta := \frac{d}{\|A\|_{\text{op}}^2 b_\alpha^2}.$$

This result combined with Paouris' theorem (see [27, Theorem 1.3]), gives the following deviation and small-ball estimate, respectively:

For all $t \geq 0$,

$$\mathbb{P}\left(\|AX\|_2 \geq (1+t)\sqrt{d}\right) \leq \exp(-c_1\eta^{\frac{\alpha}{2}} \min\{t^{2+\alpha}, t\}), \quad (2.3)$$

and, for all $t \in [0, 1]$,

$$\mathbb{P}\left(\|AX\|_2 \leq (1-t)\sqrt{d}\right) \leq C \exp\left(-c_2\eta^{\frac{\alpha}{2}} \max\{t^{2+\alpha}, \log \frac{c_3}{1-t}\}\right), \quad (2.4)$$

where $c_1, c_2, c_3, C \in (0, \infty)$ are absolute constants.

Remark 2.8. It will be essential later that (2.4) holds for any deviation parameter $t \in [0, 1]$. While Paouris' small-ball estimate from [27] is of the same flavor, it only holds for $t \geq C > 0$ for some absolute constant $C \in (0, \infty)$, while it is crucial in our proofs to take t arbitrarily small.

Remark 2.9. Let us note that recently Lee and Vempala [19,20] have made several improvements with regard to the thin-shell estimate, Poincaré constant and Lipschitz concentration constant. However, the Guédon–Milman concentration result is enough for our purposes.

2.2.5. Geometry of ℓ_p -balls

Let $d \in \mathbb{N}$ and consider the d -dimensional space \mathbb{R}^d . For any $1 \leq p \leq \infty$, the ℓ_p^d -norm, $\|\cdot\|_p$, of a vector $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ is given by

$$\|x\|_p := \begin{cases} \left(\sum_{i=1}^d |x_i|^p\right)^{1/p} & : p < \infty \\ \max\{|x_1|, \dots, |x_d|\} & : p = \infty. \end{cases}$$

We will denote by $\mathbb{B}_p^n := \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : \|x\|_p \leq 1\}$ the unit ball in ℓ_p^d and by \mathbb{D}_p^d the volume-normalized version, i.e.,

$$\mathbb{D}_p^d := \frac{\mathbb{B}_p^d}{\text{vol}_d(\mathbb{B}_p^d)^{1/d}}.$$

Recall that

$$\text{vol}_d(\mathbb{B}_p^d) = \frac{2^d \Gamma(1 + 1/p)^d}{\Gamma(1 + d/p)} \quad \text{and} \quad \text{rad}(\mathbb{B}_p^d) = d^{\max\{\frac{1}{2} - \frac{1}{p}, 0\}}.$$

Thus, if we define

$$\alpha_{d,p} := \text{vol}_d(\mathbb{B}_p^d)^{-1/d} = \frac{\Gamma(1 + d/p)^{1/d}}{2\Gamma(1 + 1/p)}, \quad (2.5)$$

then we can write

$$\mathbb{D}_p^d = \{x \in \mathbb{R}^d : \|x\|_p \leq \alpha_{d,p}\}.$$

The cone (probability) measure $\mathbf{m}_{\mathbb{B}_p^d}$ on \mathbb{B}_p^d is defined as

$$\mathbf{m}_{\mathbb{B}_p^d}(B) = \frac{\text{vol}_d(\{rx : x \in B, 0 \leq r \leq 1\})}{\text{vol}_d(\mathbb{B}_p^d)},$$

where $B \subseteq \mathbb{S}_p^{d-1}$ is a Borel subset. We remark that $\mathbf{m}_{\mathbb{B}_p^d}$ coincides with the normalized surface measure on \mathbb{S}_p^{d-1} if and only if $p = 1$, $p = 2$ or $p = \infty$. For a more detailed account to the relationship between the cone and the surface measure on ℓ_p -balls we refer the reader to [21,22].

We shall also use the following polar integration formula, stated here only for the case of ℓ_p^d -balls:

$$\int_{\mathbb{R}^d} f(x) \, dx = d \, \text{vol}_d(\mathbb{B}_p^d) \int_0^\infty \int_{\mathbb{S}_p^{d-1}} f(ry) r^{d-1} \, d\mathbf{m}_{\mathbb{B}_p^d}(y) \, dr, \quad (2.6)$$

where $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a non-negative measurable function (in fact, this may alternatively be used as a definition for the cone measure $\mathbf{m}_{\mathbb{B}_p^d}$ on \mathbb{B}_p^d).

Let us rephrase the following result of Schechtman and Zinn [29, Lemma 1] (independently obtained by Rachev and Rüschendorf in [28]) that provides a probabilistic representation of the cone measure $\mathbf{m}_{\mathbb{B}_p^d}$ of the unit ball of ℓ_p^d (see also [5] for an extension) and shall be used later with the previous polar integration formula.

Proposition 2.10. *Let $d \in \mathbb{N}$, $1 \leq p < \infty$, and g_1, \dots, g_d be independent real-valued random variables that are distributed according to the density*

$$f(t) = \frac{e^{-|t|^p}}{2\Gamma(1 + 1/p)}, \quad t \in \mathbb{R}.$$

Consider the random vector $G = (g_1, \dots, g_d) \in \mathbb{R}^d$ and put $Y := G/\|G\|_p$. Then Y is independent of $\|G\|_p$ and has distribution $\mathbf{m}_{\mathbb{B}_p^d}$.

3. Auxiliary computations

We present here some estimates that we need to prove the main results. In the first part, we show that the volume-normalized ℓ_p^d -balls satisfy the radial condition (1.2). In the second part, we estimate the volume of the intersection of a dilated Euclidean ball with a ψ_α -body.

3.1. Radial estimates for ℓ_p -balls

In order to show that the volume-normalized ℓ_p^d -balls satisfy the radial assumption (1.2), we need the following lemma.

Lemma 3.1. *Let $d \in \mathbb{N}$ and $1 \leq p \leq \infty$. Then*

$$L_{\mathbb{D}_p^d} = \alpha_{d,p} \cdot \gamma_{d,p}$$

with $\alpha_{d,p}$ from (2.5) and

$$\gamma_{d,p}^2 := \frac{1}{\text{vol}_d(\mathbb{B}_p^d)} \int_{\mathbb{B}_p^d} x_1^2 \, dx = \begin{cases} \frac{p \, \Gamma(1 + \frac{3}{p}) \, \Gamma(1 + \frac{d}{p})}{3(d+2) \, \Gamma(1 + \frac{1}{p}) \, \Gamma(\frac{d+2}{p})} & : 1 \leq p < \infty; \\ \frac{1}{3} & : p = \infty. \end{cases}$$

In particular, for $p \geq 2$,

$$\gamma_{d,p}^2 \geq \frac{d}{3(d+2)} \frac{\Gamma(1 + \frac{3}{p})}{\Gamma(1 + \frac{1}{p})} \left(\frac{p}{d}\right)^{2/p}.$$

Proof. From the definition of the isotropic constant (using it for the first standard unit vector, i.e., $\theta = e_1$) and a simple transformation, we obtain

$$L_{\mathbb{D}_p^d}^2 = \int_{\mathbb{D}_p^d} x_1^2 dx = \alpha_{d,p}^{d+2} \int_{\mathbb{B}_p^d} x_1^2 dx = \alpha_{d,p}^2 \cdot \gamma_{d,p}^2.$$

Now, for $p = \infty$, we have

$$\gamma_{d,\infty}^2 = \frac{1}{\text{vol}_d(\mathbb{B}_\infty^d)} \int_{\mathbb{B}_\infty^d} x_1^2 dx = \frac{1}{3}.$$

Let $1 \leq p < \infty$. Then, using the polar integration formula in (2.6) together with the probabilistic representation of the cone probability measure on \mathbb{B}_p^d (see Proposition 2.10), we obtain

$$\gamma_{d,p}^2 = \frac{1}{\text{vol}_d(\mathbb{B}_p^d)} \int_{\mathbb{B}_p^d} x_1^2 dx = \frac{p \Gamma(1 + \frac{3}{p}) \Gamma(1 + \frac{d}{p})}{3(d+2) \Gamma(1 + \frac{1}{p}) \Gamma(\frac{d+2}{p})}.$$

To prove the lower bound for $p \geq 2$, we use the inequality

$$\frac{\Gamma(x+1)}{\Gamma(x+\lambda)} \geq x^{1-\lambda} \quad \text{for } x > 0 \text{ and } \lambda \in (0, 1), \quad (3.1)$$

which is due to Gautschi [11]. Then, using (3.1) with the choice $x = d/p$ and $\lambda = 2/p$,

$$\begin{aligned} \gamma_{d,p}^2 &= \frac{p \Gamma(1 + \frac{3}{p}) \Gamma(1 + \frac{d}{p})}{3(d+2) \Gamma(1 + \frac{1}{p}) \Gamma(\frac{d}{p} + \frac{2}{p})} = \frac{d}{3(d+2)} \frac{p \Gamma(1 + \frac{3}{p}) \Gamma(1 + \frac{d}{p})}{d \Gamma(1 + \frac{1}{p}) \Gamma(\frac{d}{p} + \frac{2}{p})} \\ &\geq \frac{d}{3(d+2)} \frac{p \Gamma(1 + \frac{3}{p})}{d \Gamma(1 + \frac{1}{p})} \left(\frac{p}{d}\right)^{2/p-1} = \frac{d}{3(d+2)} \frac{\Gamma(1 + \frac{3}{p})}{\Gamma(1 + \frac{1}{p})} \left(\frac{p}{d}\right)^{2/p}. \quad \square \end{aligned}$$

We remark that \mathbb{D}_p^d are symmetric and isotropic ψ_2 -bodies whenever $2 \leq p \leq \infty$ (see Proposition 2.6). In view of Theorem 1.1, it is therefore left to prove that the sets \mathbb{D}_p^d satisfy assumption (1.2).

Lemma 3.2. For $2 \leq p \leq \infty$, we have

$$\limsup_{d \rightarrow \infty} \frac{\text{rad}(\mathbb{D}_p^d)}{\sqrt{d} L_{\mathbb{D}_p^d}} \leq \sqrt{3}.$$

Proof. We only consider the case $2 \leq p < \infty$ as the case $p = \infty$ can be proved in a similar way but is easier. From Lemma 3.1, we obtain that

$$\frac{\text{rad}(\mathbb{D}_p^d)}{\sqrt{d} L_{\mathbb{D}_p^d}} = \frac{\alpha_{d,p} d^{1/2-1/p}}{\sqrt{d} \alpha_{d,p} \gamma_{d,p}} = \frac{d^{-1/p}}{\gamma_{d,p}}$$

and

$$\limsup_{d \rightarrow \infty} \frac{\text{rad}(\mathbb{D}_p^d)}{\sqrt{d} L_{\mathbb{D}_p^d}} = \limsup_{d \rightarrow \infty} \frac{d^{-1/p}}{\gamma_{d,p}} \leq \sqrt{3} \sqrt{\frac{\Gamma(1 + \frac{1}{p})}{\Gamma(1 + \frac{3}{p})} p^{-2/p}}.$$

Since $\lim_{p \rightarrow \infty} \frac{\Gamma(1+\frac{1}{p})}{\Gamma(1+\frac{3}{p})} p^{-2/p} = 1$, it is enough to show that the function

$$p \mapsto \frac{\Gamma(1+\frac{1}{p})}{\Gamma(1+\frac{3}{p})} p^{-2/p}$$

is increasing in p for $p \geq 2$. Writing $x = \frac{1}{p}$ and taking natural logarithms, we have to show that

$$x \mapsto \ln \Gamma(1+3x) - \ln \Gamma(1+x) - 2x \ln x$$

is increasing in $x > 0$. The derivative of this function is

$$\begin{aligned} f(x) &:= 3\psi(1+3x) - \psi(1+x) - 2\ln x - 2 \\ &= -2\gamma - 2 - 2\ln x + 2x \sum_{k=1}^{\infty} \frac{4k+3x}{k(k+x)(k+3x)}. \end{aligned}$$

Here ψ is the Digamma function, which is the logarithmic derivative of the Gamma function, γ is the Euler–Mascheroni constant ($\gamma \approx 0.577$), and the last identity follows from the well-known series expansion

$$\psi(1+x) = -\gamma + \sum_{k=1}^{\infty} \frac{x}{k(k+x)},$$

which can be found in [1]. Using the integral test, we obtain for $x > 0$ that

$$2x \sum_{k=1}^{\infty} \frac{4k+3x}{k(k+x)(k+3x)} > 2x \int_1^{\infty} \frac{4k+3x}{k(k+x)(k+3x)} dk = 3\ln(1+3x) - \ln(1+x),$$

which shows that

$$f(x) > -2\gamma - 2 - 2\ln x + 3\ln(1+3x) - \ln(1+x).$$

To finish the proof, we finally show that $f(x) > 0$ for $x > 0$ by proving the estimate

$$g(x) := -2\ln x + 3\ln(1+3x) - \ln(1+x) > 2\gamma + 2.$$

A direct computation shows that

$$g'(x) = -\frac{2}{x} + \frac{9}{1+3x} - \frac{1}{1+x} = -\frac{2}{(1+3x)x(x+1)} < 0$$

for $x > 0$. This implies that g is decreasing and therefore,

$$g(x) > \lim_{t \rightarrow \infty} g(t) = \lim_{t \rightarrow \infty} \ln \frac{(1+3t)^3}{t^2(1+t)} = \ln 27 > 2\gamma + 2,$$

since $\ln 27 \approx 3.296$ and $2\gamma + 2 \approx 3.154$. \square

Remark 3.3. It is easy to see that the limit-superior in [Lemma 3.2](#) is actually a limit.

3.2. Volume estimates for intersections

We apply [Theorem 2.7](#) to obtain upper bounds on the volume of the intersection of ℓ_2 -balls with ψ_α -bodies. This will be crucial in the proofs of the main results.

Proposition 3.4. *Let $(K_d)_{d \in \mathbb{N}}$ be a sequence of isotropic convex bodies that satisfy a uniform ψ_α -estimate for some $\alpha \in [1, 2]$. Moreover, let $(r_d)_{d \in \mathbb{N}} \in [0, \infty)^{\mathbb{N}}$ satisfy*

$$\limsup_{d \rightarrow \infty} \frac{r_d}{L_{K_d}} < 1. \tag{3.2}$$

Then there exist absolute constants $c, C \in (0, \infty)$ such that

$$\text{vol}_d(K_d \cap r_d \sqrt{d} \mathbb{B}_2^d) \leq C \exp(-c d^{\alpha/2}).$$

Proof. Let X be a random vector chosen uniformly at random from $\frac{K}{L_K}$. The latter guarantees that the random vector is isotropic and log-concave (see Remark 2.4). We observe that, for any $\gamma \geq 0$,

$$\mathbb{P}\left(\|X\|_2 \leq \frac{r_d \sqrt{d}}{L_K}\right) = \text{vol}_d(K_d \cap r_d \sqrt{d} \mathbb{B}_2^d).$$

From the assumptions, we have that $r_d \leq a L_{K_d}$ for some constant $a \in (0, 1)$ if d is large enough. Therefore, using the small-ball estimate (2.4) with A chosen to be the identity matrix, we obtain that

$$\begin{aligned} \text{vol}_d(K_d \cap r_d \sqrt{d} \mathbb{B}_2^d) &\leq C \exp\left(-c_2 \left(\frac{d}{b_\alpha^2}\right)^{\alpha/2} \left(1 - \frac{r_d}{L_{K_d}}\right)^{2+\alpha}\right) \\ &\leq C \exp(-c d^{\alpha/2}), \end{aligned}$$

for large enough d , where $c_2, C \in (0, \infty)$ are the absolute constants from (2.4) and $c := c_2 a^{2+\alpha} / b_\alpha^\alpha$. \square

4. Proof of the main results

In this section we provide the proofs of our main results, Theorem 1.1 and Corollary 1.7. Both will be direct corollaries of the following more general result.

Proposition 4.1. Let $(K_d)_{d \in \mathbb{N}}$ be a sequence of symmetric and isotropic convex bodies that satisfy a uniform ψ_α -estimate for some $\alpha \in [1, 2]$ and

$$\limsup_{d \rightarrow \infty} \frac{\text{rad}(K_d)}{\sqrt{d} L_{K_d}} < 2. \quad (4.1)$$

Moreover, let $(A_d)_{d \in \mathbb{N}}$ and $(B_d)_{d \in \mathbb{N}}$ be such that

$$A_d \geq \frac{a}{\sqrt{d} L_{K_d}} \quad \text{and} \quad B_d \geq \frac{b}{d L_{K_d}^2} \quad \text{for some } a, b > 0.$$

Then there exist constants $c, \gamma, \varepsilon_0 \in (0, \infty)$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and $d \in \mathbb{N}$,

$$n(\varepsilon, \mathcal{F}_d) \geq c(1 + \gamma)^{d^{\alpha/2}}$$

with $\mathcal{F}_d := \mathcal{C}_d^1(A_d, B_d, K_d)$.

Proof of Proposition 4.1. The basic idea behind the proof is the same as in the proofs of the corresponding results from [14].

For given sample points $x_1, \dots, x_n \in K_d$, we construct a fooling function that is defined on the entire \mathbb{R}^d and satisfies the required bounds on the Lipschitz constants. This function will be zero at the points x_1, \dots, x_n and will have a large integral in K_d as long as n is not exponentially large in d . Moreover, this function will be zero on the entire convex hull of x_1, \dots, x_n .

To be precise, let \mathcal{P}_n be a set of n points in the symmetric ψ_α -body K_d and let $C_n := \text{conv}(\mathcal{P}_n)$ be their convex hull. Moreover, for a set $C \subseteq \mathbb{R}^d$ and $\delta \in (0, \infty)$ we define its $\delta\sqrt{d}$ -extension to be

$$C^{(\delta)} := C + \delta\sqrt{d} \mathbb{B}_2^d = \left\{x \in \mathbb{R}^d : \inf_{y \in C} \|x - y\|_2 \leq \delta\sqrt{d}\right\}.$$

By a result of Elekes [10], which was adapted to this setting in [14, Theorem 2.1], we know that, for arbitrary point sets $\mathcal{P}_n \subseteq K_d$ of cardinality n , the convex hull C_n is contained in the union of n balls

$$y_1 + \frac{\text{rad}(K_d)}{2} \cdot \mathbb{B}_2^d, \dots, y_n + \frac{\text{rad}(K_d)}{2} \cdot \mathbb{B}_2^d$$

for some $y_1, \dots, y_n \in K_d$. This implies that $C_n^{(\delta)}$ can be covered by n balls with radii $\frac{\text{rad}(K_d)}{2} + \delta\sqrt{d}$. Let $r_d := \frac{\text{rad}(K_d)}{2\sqrt{d}} + \delta$. We obtain

$$\text{vol}_d(C_n^{(\delta)} \cap K_d) \leq \sum_{i=1}^n \text{vol}_d(K_d \cap (y_i + r_d \sqrt{d} \mathbb{B}_2^d)).$$

Using the symmetry of K_d , which implies (by means of the dimension-free Brunn–Minkowski inequality) that the right hand side is maximized for $y_1 = \dots = y_n = 0$, we get

$$\text{vol}_d(C_n^{(\delta)} \cap K_d) \leq n \text{vol}_d(K_d \cap r_d \sqrt{d} \mathbb{B}_2^d).$$

Together with Proposition 3.4, this shows that

$$\text{vol}_d(C_n^{(\delta)} \cap K_d) \leq C n q^{d^{\alpha/2}} \quad (4.2)$$

for some $q \in (0, 1)$ if r_d satisfies (3.2) and d is large enough.

It is easy to check that r_d satisfies (3.2) whenever

$$\delta < L_{K_d} \left(1 - \frac{\text{rad}(K_d)}{2\sqrt{d} L_{K_d}} \right). \quad (4.3)$$

Clearly, such a $\delta \in (0, \infty)$ exists under the assumptions of Proposition 4.1 for large enough $d \in \mathbb{N}$ since, in general, $L_{K_d} \geq 1/\sqrt{2\pi e}$.

We now construct, for an arbitrary point set $\mathcal{P}_n \subseteq K_d$ of cardinality n , a function that fits our needs. For this we just use the distance function from the convex hull C_n and smooth it out using a suitable piecewise polynomial. That is, we consider the function $f : K_d \rightarrow \mathbb{R}$ defined as

$$f(x) := f_{\delta, \mathcal{P}_n}(x) = p_{\delta} \left(\inf_{y \in C_n} \|x - y\|_2 \right), \quad (4.4)$$

where

$$p_{\delta}(t) = \begin{cases} \frac{2}{\delta^2 d} t^2 & : t \leq \frac{\delta\sqrt{d}}{2}; \\ -\frac{2}{\delta^2 d} t^2 + \frac{4}{\delta\sqrt{d}} t - 1 & : t \in \left(\frac{\delta\sqrt{d}}{2}, \delta\sqrt{d} \right); \\ 1, & : t \geq \delta\sqrt{d}. \end{cases}$$

By direct computations, we obtain that

- $f \in \mathcal{C}^1(\mathbb{R}^d)$,
- $f(x) = 0$ for $x \in C_n$,
- $f(x) = 1$ for $x \notin C_n^{(\delta)}$,
- $\text{Lip}(f) \leq \frac{2}{\delta\sqrt{d}}$ and
- $\forall \theta \in \mathbb{S}^{d-1}: \text{Lip}(D^{\theta}f) \leq \frac{40}{\delta^2 d}$.

The precise computations can be found in [14, Section 4.1], but note that we use here a function that is zero only on C_n and not on $C_n^{(\delta)}$ (see [14, Remark 4.4]).

The properties above and (4.2) imply that

$$\int_{K_d} f(x) \, dx \geq \text{vol}_d(K_d \setminus C_n^{(\delta)}) = 1 - \text{vol}_d(K_d \cap C_n^{(\delta)}) \geq 1 - C n q^{d^{\alpha/2}}$$

with C, q as in (4.2) if K_d is a ψ_{α} -body with (1.2) and $\delta \in (0, \infty)$ with (4.3).

As this bound holds for arbitrary point sets \mathcal{P}_n with n elements, we see that the integral of the function f constructed above must be larger than $\varepsilon \in (0, 1)$ as long as $n < C^{-1} q^{-d^{\alpha/2}} (1 - \varepsilon)$. Taking into account the above properties of f this shows that the number of function evaluations that are

necessary to guarantee an error of at most ε for all functions in $\mathcal{F}_{d,\delta} := \mathcal{C}_d^1(A_d, B_d, K_d)$ with $A_d = \frac{2}{\delta\sqrt{d}}$ and $B_d = \frac{40}{\delta^2 d}$ satisfies

$$n(\varepsilon, \mathcal{F}_{d,\delta}) \geq c(1-\varepsilon)(1+\gamma)^{d^{\alpha/2}}$$

for some $c, \gamma \in (0, \infty)$ (see (2.1)) and d large enough. As this holds for all δ that satisfy (4.3), we obtain

$$n(\varepsilon, \mathcal{C}_d^1(A_d, B_d, K_d)) \geq c'(1+\gamma)^{d^{\alpha/2}}$$

for some $c', \gamma \in (0, \infty)$, large enough d and all $\varepsilon \in (0, 1/2)$, if $A_d \geq \frac{a_0}{\sqrt{d}L_{K_d}}$ and $B_d \geq \frac{b_0}{dL_{K_d}^2}$ for some $a_0, b_0 > 0$ that depend only on the precise value of the left hand side of (4.1).

The proof for arbitrary sequences (A_d) and (B_d) that satisfy the assumptions of the theorem follows from a scaling as in the proof of [14, Proposition 3.2]. For this, note that there exists a constant $M \in [1, \infty)$ such that $M \cdot A_d \geq \frac{a_0}{\sqrt{d}L_{K_d}}$ and $M \cdot B_d \geq \frac{b_0}{dL_{K_d}^2}$ with a_0, b_0 from above. Therefore,

$$n(\varepsilon, \mathcal{C}_d^1(M \cdot A_d, M \cdot B_d, K_d)) \geq c'(1+\gamma)^{d^{\alpha/2}}$$

for some $c', \gamma \in (0, \infty)$, large enough d and all $\varepsilon \in (0, 1/2)$. Using

$$\mathcal{C}_d^1(M \cdot A_d, M \cdot B_d, K_d) \subseteq M \cdot \mathcal{C}_d^1(A_d, B_d, K_d)$$

and $n(\varepsilon, M \cdot \mathcal{F}_d) = n(M \cdot \varepsilon, \mathcal{F}_d)$, this implies

$$n(\varepsilon, \mathcal{C}_d^1(A_d, B_d, K_d)) \geq c'(1+\gamma)^{d^{\alpha/2}}$$

for all $\varepsilon \in (0, 1/(2M))$ and large enough d . Clearly, we can now modify the constant c' such that this lower bound holds for all $d \in \mathbb{N}$. This proves the theorem. \square

Remark 4.2. In the case of volume-normalized ℓ_p -balls, the $\delta \in (0, \infty)$ that was used in the construction of the fooling function, see (4.4), can be chosen to be

$$\delta = \frac{1}{42} < \frac{1}{\sqrt{2\pi}e} \left(1 - \frac{\sqrt{3}}{2}\right) \approx 0.0324$$

for large enough d (see Lemma 3.2).

We finish this section with the proof of Theorem 1.1 and Corollary 1.7.

Proof of Theorem 1.1. Theorem 1.1 is just Proposition 4.1 in the case $\alpha = 2$. Note that the definition of the curse of dimensionality requires a lower bound as in Proposition 4.1 only for infinitely many $d \in \mathbb{N}$. \square

We now turn to the important class of ℓ_p -balls. Based upon this we will then be able to close the aforementioned gap left in [14] and prove the curse of dimensionality for numerical integration of smooth functions on ℓ_p -balls in the full regime $2 \leq p \leq \infty$.

Proof of Corollary 1.7. For the proof of the corollary it is enough to show that the sets \mathbb{D}_p^d with $2 \leq p \leq \infty$ satisfy the conditions of Theorem 1.1 and Theorem 1.6. This follows from the considerations from Section 2.2.5 and Lemma 3.2, see also Remark 4.2. \square

5. Proof of Theorem 1.6

The upper bounds on the complexity in the present setting (Theorem 1.6) can be proved in a very similar way as [14, Theorem 4.1]. However, as the emphasis in [14] was only on convex sets with small diameter, the authors did not find the precise dependence of the involved conditions on the isotropic constant. Since the proofs are quite short, we repeat them here.

For the statement of the following results, we define

$$I_q(K_d) := \int_{K_d} \|x\|_2^q \, dx$$

for measurable sets $K_d \subseteq \mathbb{R}^d$ and $q \geq 0$.

We start with the condition on the Lipschitz constant of the function. Afterwards we state the result on the Lipschitz constant of the derivatives.

Proposition 5.1. *Let $(K_d)_{d \in \mathbb{N}}$ be a sequence of measurable sets with $\text{vol}_d(K_d) = 1$ that contain the origin. Assume that*

$$\lim_{d \rightarrow \infty} A_d \cdot I_1(K_d) = 0.$$

Then

$$\lim_{d \rightarrow \infty} \sup_f \left| \int_{K_d} f(x) \, dx - f(0) \right| = 0,$$

where the supremum is taken over all f in the set $\{f \in \mathcal{C}(K_d) : \text{Lip}(f) \leq A_d\}$.

The next proposition deals with the optimality of the assumptions on the decay of $(B_d)_{d \in \mathbb{N}}$, i.e., the Lipschitz constants of the directional derivatives. However, in this case we have to assume more from the sets under consideration. Note that a set is called *star-shaped with respect to the origin* if, for every x from the set, the line segment from the origin to x is also contained in the set. Clearly, this is satisfied by every convex set that contains the origin.

Proposition 5.2. *Let $(K_d)_{d \in \mathbb{N}}$ be a sequence of measurable sets with $\text{vol}_d(K_d) = 1$, center of mass at the origin and star-shaped with respect to the origin. Assume that*

$$\lim_{d \rightarrow \infty} B_d \cdot I_2(K_d) = 0.$$

Then

$$\lim_{d \rightarrow \infty} \sup_f \left| \int_{K_d} f(x) \, dx - f(0) \right| = 0,$$

where the supremum is taken over all f in the set

$$\{f \in \mathcal{C}^1(K_d) : \text{Lip}(D^\theta f) \leq B_d \text{ for all } \theta \in \mathbb{S}^{d-1}\}.$$

Proof of Proposition 5.1. Using the estimate

$$|f(x) - f(0)| \leq A_d \|x\|_2,$$

valid for all $x \in K_d$ and $f \in \mathcal{C}(K_d)$ with $\text{Lip}(f) \leq A_d$, together with the triangle inequality, we obtain

$$\left| \int_{K_d} f(x) \, dx - f(0) \right| \leq A_d \int_{K_d} \|x\|_2 \, dx = A_d \cdot I_1(K_d). \quad \square \quad (5.1)$$

Proof of Proposition 5.2. Similar to [14, Proposition 4.7], we use that the mean value theorem implies

$$f(x) - f(0) = \langle \nabla f(y_x), x \rangle$$

for some y_x on the line segment between 0 and x , and $f \in \mathcal{C}^1(K_d)$. Moreover, since the center of mass of K_d is at the origin, we have $\int_{K_d} \langle \nabla f(0), x \rangle \, dx = 0$. Hence,

$$\begin{aligned} \int_{K_d} f(x) \, dx - f(0) &= \int_{K_d} (f(x) - f(0) - \langle \nabla f(0), x \rangle) \, dx \\ &= \int_{K_d} \langle \nabla f(y_x) - \nabla f(0), x \rangle \, dx. \end{aligned}$$

With $\theta_x := x/\|x\|_2$, we obtain $\langle \nabla f(y), x \rangle = D^{\theta_x} f(y) \|x\|_2$ and

$$\begin{aligned} |\langle \nabla f(y_x) - \nabla f(0), x \rangle| &= |D^{\theta_x} f(y_x) - D^{\theta_x} f(0)| \|x\|_2 \\ &\leq \text{Lip}(D^{\theta_x} f) \|y_x\|_2 \|x\|_2 \\ &\leq \text{Lip}(D^{\theta_x} f) \|x\|_2^2. \end{aligned}$$

If now $\text{Lip}(D^\theta f) \leq B_d$ for all $\theta \in \mathbb{S}^{d-1}$, we get

$$\left| \int_{K_d} f(x) dx - f(0) \right| \leq B_d \int_{K_d} \|x\|_2^2 dx = B_d \cdot I_2(K_d). \quad \square \quad (5.2)$$

Obviously, Propositions 5.1 and 5.2 hold also if the corresponding classes of functions are replaced by $\mathcal{C}_d^1(A_d, B_d, K_d)$. Moreover, it is clear from the definition that $I_2(K_d) = d L_{K_d}^2$ and $I_1(K_d) \leq \sqrt{I_2(K_d)} = \sqrt{d} L_{K_d}$ for isotropic K_d . Therefore, combining (5.1) and (5.2) proves the corresponding result in Theorem 1.6.

Finally, note that the above propositions imply that the error of the trivial algorithm $A_{1,d}(f) := f(0)$ goes to zero (as $d \rightarrow \infty$) for every $f \in \mathcal{C}_d^1(A_d, B_d, K_d)$, where A_d, B_d and K_d satisfy the above conditions.

Obviously, $A_{1,d}$ uses only one function evaluation. Therefore,

$$n(\varepsilon, \mathcal{C}_d^1(A_d, B_d, K_d)) = 1$$

for all $\varepsilon \in (0, 1)$ given that $d \geq d(\varepsilon)$ is large enough (see Section 2.1). Here we also used that $n(\varepsilon, \mathcal{C}_d^1(A_d, B_d, K_d)) > 0$ for $\varepsilon \in (0, 1)$, because the initial error equals 1. In particular, we do not have the curse of dimensionality.

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