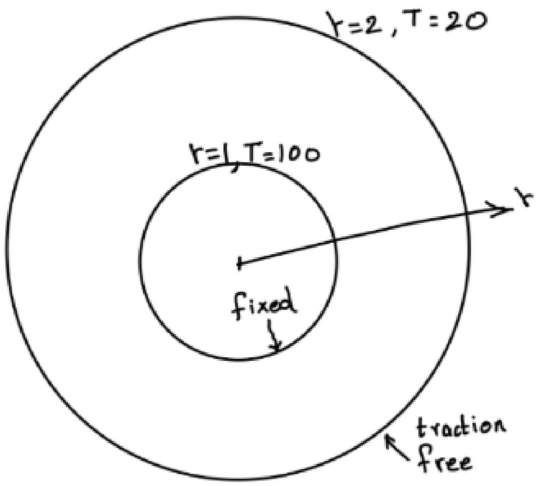
Assigment 2

Question 1

1. The steady-state heat conduction equation, in the radial direction r, of a long cylinder with inner diameter 1 unit and outer diameter 2 unit, is given by

$$\frac{\kappa}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + g = 0,$$



where κ is the thermal conductivity and g is the heat generated per unit area. Let $\kappa = 1$ and $g = 1$. Let the temperature on the inner surface of the cylinder be 100 units and that on the outer surface be 20 units.

(a) Using the Galerkin method, derive the elemental matrices that have to be coded for the finite element method. The residue has to be evaluated over the entire cylinder. (20)

(b) Write a finite element code to solve for the temperature distribution as a function of position r and compare with the analytical solution. (30)

Solution 1

- Code File: [Question 1 Mfile]

Deriving Governing Equation

Given differential Equation for temperature :

$\frac{k}{r}\frac{d}{dr}(r\frac{dT}{dr}) + g = 0$

Putting values of constants in the DE

$\frac{1}{r}\frac{d}{dr}(r\frac{dT}{dr}) + 1 = 0$

Solving the above equation for $r \in (1, 2)$

with Boundary conditions

$T(1) = 100$

$T(2) = 20$

$\dot{Q} = \frac{2\pi k(\Delta T)}{\ln(\frac{r_2}{r_1})} = \frac{2\pi(80)}{\ln(2)} = 725.117$

We get

$=> T(r) = \frac{(401-r^2)\log(2)+317\log(r)}{\log(16)}$

Finally we get our analytical solution any any x.

$$=> T(r) = \frac{(401-r^2)\log(2)+317\log(r)}{\log(16)}$$

Deriving Weak Form

$\int \delta T[\frac{1}{r}\frac{d}{dr}(r\frac{dT}{dr}) + 1]rdr = 0$

Note : r *is the Jacobian of cylindrical coordinate*

δT is residue that varies linearly like T since we are going to take linear approximation for **Galerkin method**

Evaluating further

$\int \delta T[\frac{d}{dr}(r\frac{dT}{dr}) + r]dr = 0$

Applying uv rule in first term

$\int [\frac{d}{dr}(\delta T r\frac{dT}{dr}) - \frac{d\delta T}{dr} r \frac{dT}{dr} + \delta T r]dr = 0$

$\delta T r\frac{dT}{dr}\Big|_1^2 = \int [r\frac{d\delta T}{dr}\frac{dT}{dr} - \delta T r]dr$

Heat flow per unite length at outer surface of cylinder

$\dot{Q} = 725.117$

According to our method our u varies linearly and it will be according to the following equation. Following are the elemental matrices to be integrated

$T = N_1T_1 + N_2T_2$

$N_1 = \frac{(r-r_2)}{(r_1-r_2)}$

$N_2 = \frac{(r-r_1)}{(r_2-r_1)}$

$\mathbf{N} = \begin{bmatrix} N_1 & N_2 \end{bmatrix}$

$\mathbf{B} = \mathbf{N}^{'} = \begin{bmatrix} N_1^{'} & N_2^{'} \end{bmatrix}$

$\delta T = \mathbf{N}\delta T$

$T = \mathbf{N}T$

$\frac{dT}{dr} = \mathbf{B}T$

Putting above matrices in our integral and simplifying

$\delta T r \frac{dT}{dr}\Big|_1^2 = \int [r \frac{d\delta T}{dr} \frac{dT}{dr} - \delta T r] dr$

$F^N \delta u^T = \delta T^T T(\int r \mathbf{B}^T \mathbf{B}) dr - \delta T^T \int r \mathbf{N}^{'} dr$

We can very conveniently compare above expression to

$F^N = KT - F^B$

$\mathbf{K} = \int r \mathbf{B}^T \mathbf{B} dr \; ; \; F^B = \int r \mathbf{N}^{'} dr$

For integrating above expression numerically we will be using **Gauss Quadrature** method and we will take **2** gauss points as 2 points will be sufficient to accurately give our answer to polynomial integration.

Analysis Procedure

- We will run [This code] on octave to get values of Temperature.
- We will also compare numerically calculated temperature and temperature with our analytical solution i.e $=> T(r) = \frac{(401 - r^2)\log(2) + 317\log(r)}{\log(16)}$
- We will plot graph of displacement for each node.

Observations

- Number of Elements : 10
- Root Mean Square Error (Displacements) = $4.0995 * 10^{-03}$

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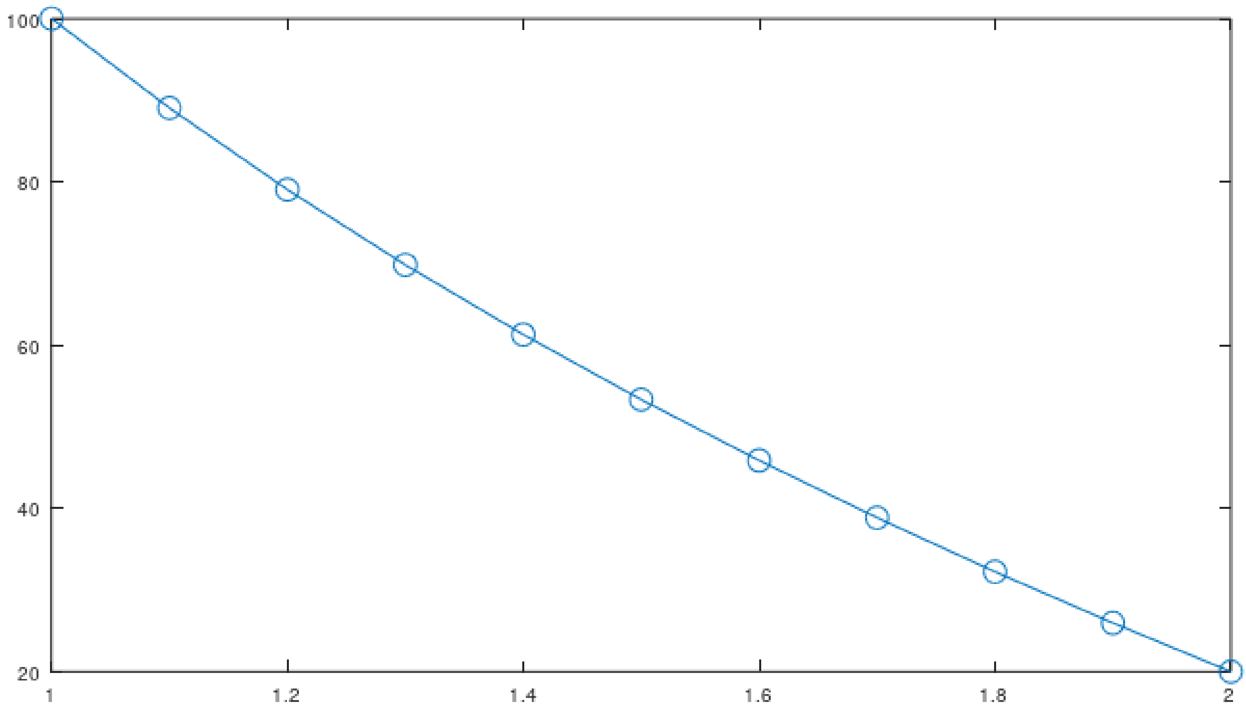
x	analytical Temperatures	numerical temperatures		
1	100	100		
1.1	89.05	89.054		
1.2	79.045	79.05		
1.3	69.83	69.836		
1.4	61.29	61.296		
1.5	53.329	53.335		
1.6	45.873	45.878		
1.7	38.859	38.863		
1.8	32.236	32.239		
1.9	25.962	25.963		
2	20	20		

- Graph of Temperature vs r



Question 2

2. Consider that the above cylinder is made of a material with Young's modulus E and thermal expansion coefficient α. Due to the temperature distribution in the cylinder, there will be a radial strain $\epsilon_{rr} = \frac{du}{dr}$ and circumferential strain $\epsilon_{\theta\theta} = \frac{u}{r}$, where u is the radial displacement. The corresponding thermal strains are $\epsilon_{rr}^{T} = \alpha T$ and $\epsilon_{\theta\theta}^{T} = \alpha T$. The linear momentum balance in the radial direction r in the is given by

$$\frac{1}{r}\frac{d}{dr}\left[r\sigma_{rr}\right] - \frac{1}{r}\sigma_{\theta\theta} = 0,$$

where the radial stress $\sigma_{rr} = E\left(\epsilon_{rr} - \epsilon_{rr}^{T}\right)$ and the circumferential stress $\sigma_{\theta\theta} = E\left(\epsilon_{\theta\theta} - \epsilon_{\theta\theta}^{T}\right)$.

(a) Using the Galerkin method, derive the elemental matrices that have to be coded for the finite element method to solve for the displacements. The residue has to be evaluated over the entire cylinder. (20)

(b) Write a finite element code to solve for the displacements u distribution as a function of position r. Assume that the internal surface of the cylinder is fixed and that the outer surface is traction free. The temperature T has to be taken from the solution of problem 1. Assume $\alpha = E = 1$. (30)

Solution 2

- Code File: Question 2 Mfile

Deriving Governing Equation

Given differential Equation for temperature :

$\frac{1}{r}\frac{d}{dr}(r\sigma_{rr}) - \frac{1}{r}\sigma_{\theta\theta} = 0$

$\sigma_{rr} = E(\frac{du}{dr} - \alpha T)$
$\sigma_{\theta\theta} = E(\frac{u}{r} - \alpha T)$

Putting values of constants in the DE

$\frac{1}{r}\frac{d}{dr}(r(\frac{du}{dr} - T)) - \frac{1}{r}(\frac{u}{r} - T) = 0$

Deriving Weak Form

$\int \delta u[\frac{1}{r}\frac{d}{dr}(r(\frac{du}{dr}-T))-\frac{1}{r}(\frac{u}{r}-T)]rdr=0$

Note : r *is the Jacobian of cylindrical coordinate*

δuu is residue that varies linearly like u since we are going to take linear approximation for **Galerkin method**

Evaluating further

$\int[\delta u\frac{d}{dr}(r(\frac{du}{dr}-T))-\delta u(\frac{u}{r}-T)]dr=0$

Applying uv rule in first term

$\int[\frac{d}{dr}\delta u(r(\frac{du}{dr}-T))-r\frac{d\delta u}{dr}(\frac{du}{dr}-T)-(\frac{u}{r}-T)]dr=0$

$\delta u(r(\frac{du}{dr}-T))\Big|_1^2=\int r\frac{d\delta u}{dr}(\frac{du}{dr}-T)dr+\int \delta u(\frac{u}{r}-T)dr=0$

Since $\sigma_{rr}=0$ at $r=2$ and $u=0$ at $r=1$

$0=\int r\frac{d\delta u}{dr}(\frac{du}{dr}-T)dr+\int \delta u(\frac{u}{r}-T)dr=0$

According to our method our u varies linearly and it will be according to the following equation. Following are the elemental matrices to be integrated

$u=N_1u_1+N_2u_2$

$N_1=\frac{(r-r_2)}{(r_1-r_2)}$

$N_2=\frac{(r-r_1)}{(r_2-r_1)}$

$\mathbf{N}=\begin{bmatrix}N_1 & N_2\end{bmatrix}$

$\mathbf{B}=\mathbf{N}^{'}=\begin{bmatrix}N_1^{'} & N_2^{'}\end{bmatrix}$

$\delta u=\mathbf{N}\delta u$

$T = \mathbf{N}T$

$u = \mathbf{N}u$

$\frac{du}{dr} = \mathbf{B}u$

Putting above matrices in our integral and simplifying

$0 = \int r\delta u^T \mathbf{B}^T(\mathbf{B}u - \mathbf{N}T)dr + \int \delta u \mathbf{N}^T(\frac{\mathbf{N}u}{r} - \mathbf{T})dr$

$\delta u^T u \int [r\mathbf{B}^T\mathbf{B} + \frac{\mathbf{N}^T\mathbf{N}}{r}]dr = T^T \delta u^T \int [r\mathbf{N}^T\mathbf{B} + \mathbf{N}^T\mathbf{N}]dr$

$Ku = TF^B$

We can very conveniently compare above expression to

$ KT = F^B$

$\mathbf{K} = \int [r\mathbf{B}^T\mathbf{B} + \frac{\mathbf{N}^T\mathbf{N}}{r}]\ ;\ F^B = \int [r\mathbf{N}^T\mathbf{B} + \mathbf{N}^T\mathbf{N}]dr$

For integrating above expression numerically we will be using **Gauss Quadrature** method and we will take **2** gauss points as 2 points will be sufficient to accurately give our answer to polynomial integration.

Analysis Procedure

- We will run [This code] on octave to get values of Temperature.
- We will plot graph of displacement for each node.

Observations

x	numerical displacement			
1	0			
1.1	11.9357			
1.2	21.9235			
1.3	30.3346			
1.4	37.4399			
1.5	43.4425			
1.6	48.4984			
1.7	52.7298			
1.8	56.2341			
1.9	59.0903			
2	61.3632			

Graph of Displacement vs r