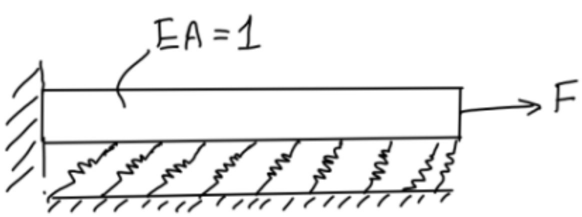
Github

FemAssignments

Assigment 1

Question 1

1. Consider a one-dimensional bar with a distributed spring of stiffness per unit length k as shown in the below figure.



   (a) Derive the governing differential equation and boundary conditions of the bar. (5)

   (b) Write the analytical solution of the above differential equation (no need to derive). (5)

   (c) Write a finite element code in octave to find the displacements and stresses. (10)

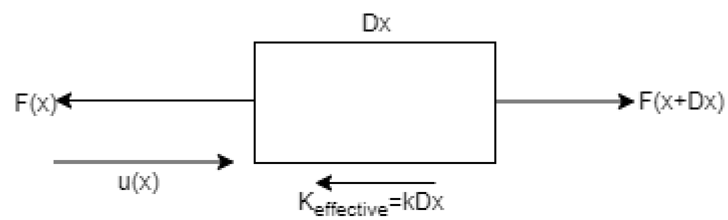
   (d) Compare the stresses and displacements from the finite element code with the analytical solution. (5)
   Assume $F = 1$ and $k = 1$.

Solution 1

- Code File: [Question 1 Mfile]

Deriving Governing Equation

Force Balance on an arbitrary element at distance x from bar



$$F(x + \Delta x) - F(x) - k\Delta x u(x) = 0$$

Dividing whole Equation by Δx

$\frac{F(x+\Delta x)-F(\Delta x)}{\Delta x} - ku(x)$ = 0

Take limit as $\Delta x \to 0$

$\frac{dF}{dx} - ku = 0$ {Note : Here $u = u(x)$}

$\sigma = \frac{F}{A}$ also $\sigma = E\frac{du}{dx}$

E = Elasticity modulus, **A**= Cross Sectional Area
Thus,

$$F = EA\frac{du}{dx}$$

Known Boundary Conditions

$$u(0) = 0$$

$EA \left.\frac{du}{dx}\right|_{x=l} = F$

and, Finally putting value of **F** in our governing equation

$\frac{d}{dx}[EA\frac{du}{dx}] - ku = 0$

Finally Putting values of constants in our equations, questions have asked us to take all constants as unity, so be it.

$\frac{d^2u}{dx^2} - u = 0$ for $x \in (0,1)$

$u(0) = 0$

$\frac{du}{dx} = 1$ at $x = 1$

Solving $\frac{d^2u}{dx^2} - u = 0$ for $x \in (0,1)$

with Boundary conditions

$u(0) = 0$

$\frac{du}{dx} = 1$ at $x = 1$

We get

$=> u(x) = \frac{e^x - e^{-x}}{e+e^{-1}}$

Finally we get our analytical solution any any x.

$$=> u(x) = \frac{e}{e^2+1}(e^x - e^{-x})$$

Deriving Weak Form

$$\int_1^2 \delta u(\frac{d^2u}{dx^2} - u)dx = 0$$

δu is weight function that varies linearly like u since we are going to take linear approximation for our calculations

Evaluating further

$$\int_1^2 (\frac{d}{dx}(\delta u\frac{du}{dx}) - \frac{d\delta u}{dx}\frac{du}{dx} - \delta uu)dx = 0$$

According to our method our u varies linearly and it will be according to the following equation.

$$u = N_1u_1 + N_2u_2$$

$$N_1 = \frac{(x - x_2)}{(x_1 - x_2)}$$

$$N_2 = \frac{(x - x_1)}{(x_2 - x_1)}$$

$$\mathbf{N} = \begin{bmatrix} N_1 & N_2 \end{bmatrix}$$

$$\mathbf{B} = \mathbf{N}^{'} = \begin{bmatrix} N_1^{'} & N_2^{'} \end{bmatrix}$$

$$\delta u = \mathbf{N}\delta u$$

$$u = \mathbf{N}u$$

$$\frac{du}{dx} = \mathbf{B}u$$

Putting above equations in our integral and simplifying

$$\delta u\frac{du}{dx}\Big|_1^2 = \int \delta u^T\mathbf{B}^T\mathbf{B}udx + \int \delta u^T\mathbf{N}^T\mathbf{N}udx$$

$$F^N \delta u^T = \delta u^T u(\int \mathbf{B}^T\mathbf{B} + \mathbf{N}^T\mathbf{N})dx$$

We can very conveniently compare above expression to $F = Kx$ as we write

$$\mathbf{K} = \int(\mathbf{B}^T\mathbf{B} + \mathbf{N}^T\mathbf{N})dx$$

there by leading to

$$F^N = \mathbf{K}u$$

For integrating above expression numerically we will be using **Gauss Quadrature** method and we will take **2** gauss points as 2 points will be sufficient to accurately give our answer to polynomial integration.

Analysis Procedure

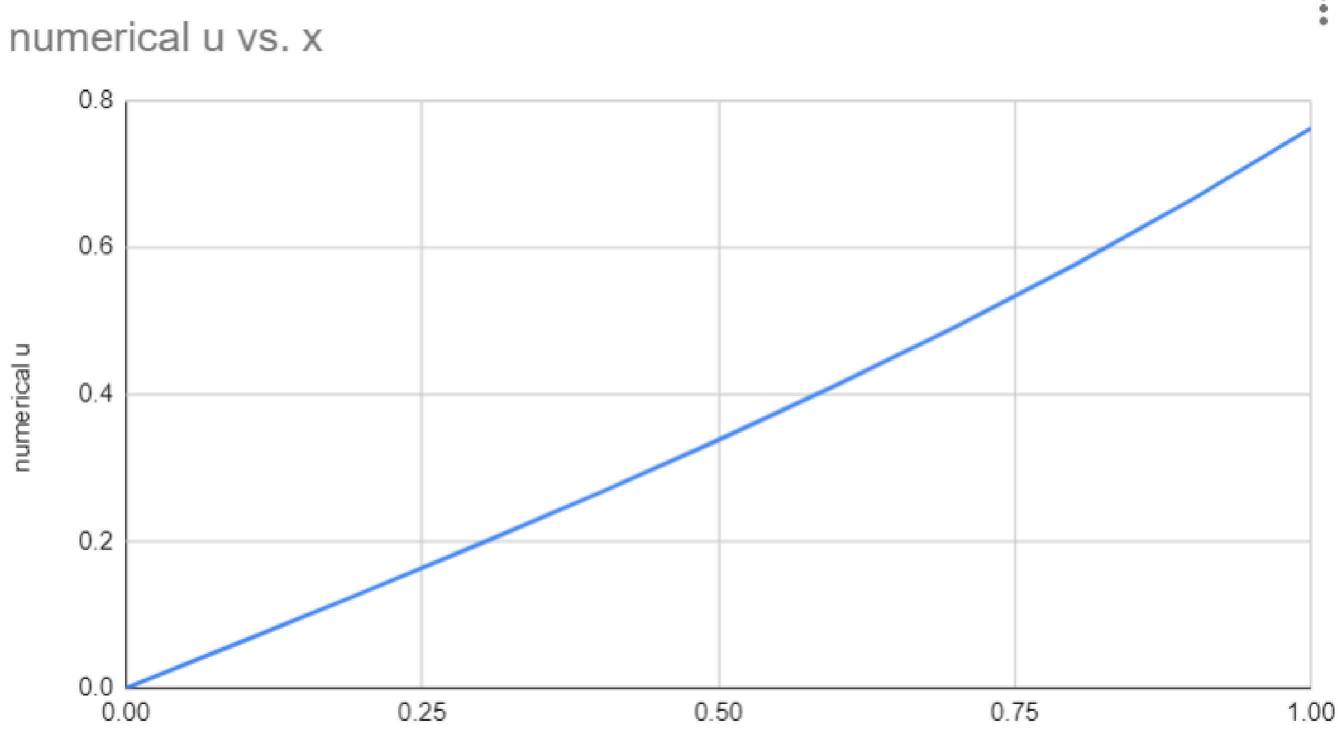
- We will run This code on octave to get values of displacements and stresses.
- We will also compare numerically calculated stresses and displacement with our analytical solution i.e $=> u(x) = \frac{e}{e^2+1}(e^x - e^{-x})$
- We will plot graph of displacement for each node.
- Additionally, we will also see how difference in values of displacements from analytical and numerical solution varies with changing number of elements.

Observations

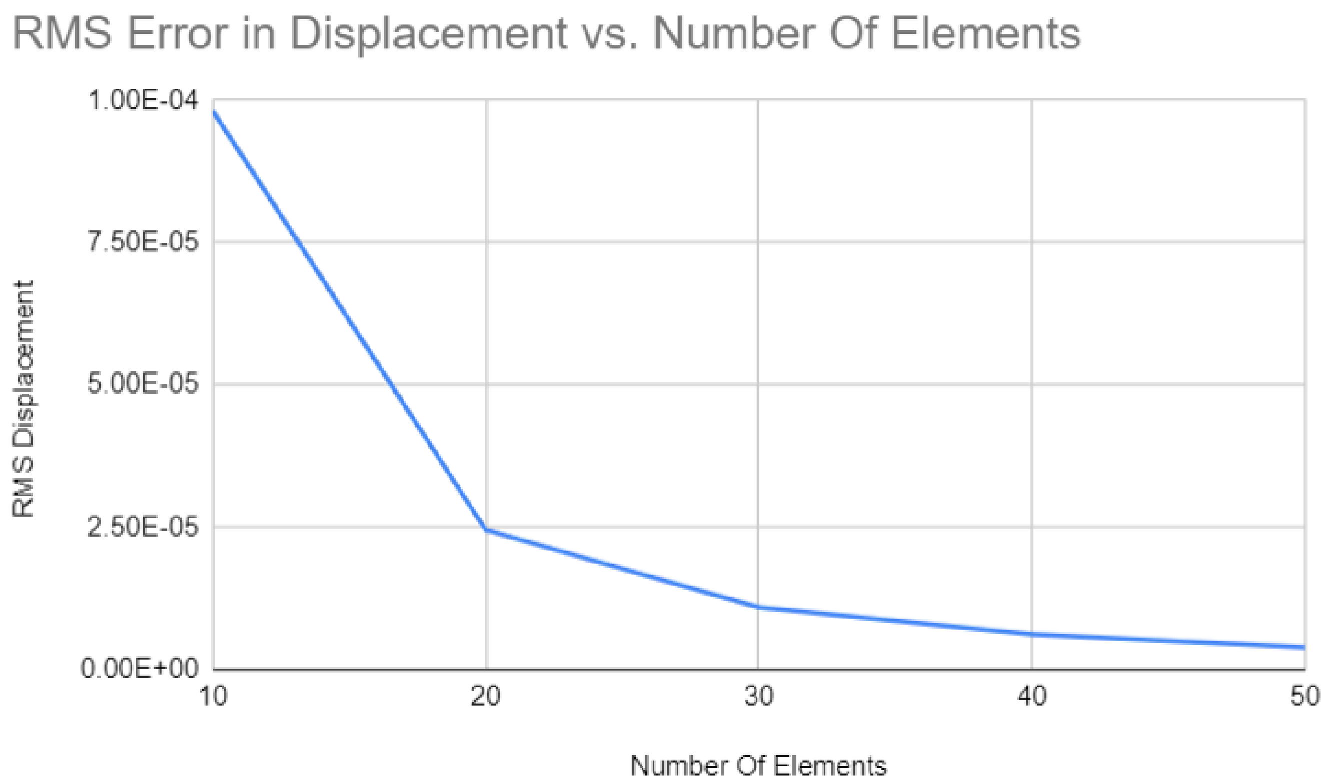
- Number of Elements : 10
- Root Mean Square Error (Displacements) = $9.8160 * 10^{-05}$

x	analytical displacement	numerical displacement	numerical stress	analytical stress
0	0	0	0.6489	0.6489
0.1	0.0649	0.0649	0.6554	0.6554
0.2	0.1305	0.1304	0.6685	0.6684
0.3	0.1973	0.1973	0.6883	0.6882
0.4	0.2662	0.2661	0.7149	0.7148
0.5	0.3377	0.3376	0.7487	0.7486
0.6	0.4126	0.4125	0.79	0.7898
0.7	0.4916	0.4915	0.8393	0.839
0.8	0.5755	0.5754	0.8969	0.8966
0.9	0.6652	0.6651	0.9635	0.9632
1	0.7616	0.7615	1	1

- Graph of Displacement vs x



Number Of Elements	RMS Error in Displacement
10	9.82E-05
20	2.45E-05
30	1.09E-05
40	6.13E-06
50	3.92E-06


RMS Error in Displacement vs. Number Of Elements
1.00E-04
7.50E-05
5.00E-05
2.50E-05
0.00E+00
RMS Displacement
10
20
30
40
50
Number Of Elements

Question 2

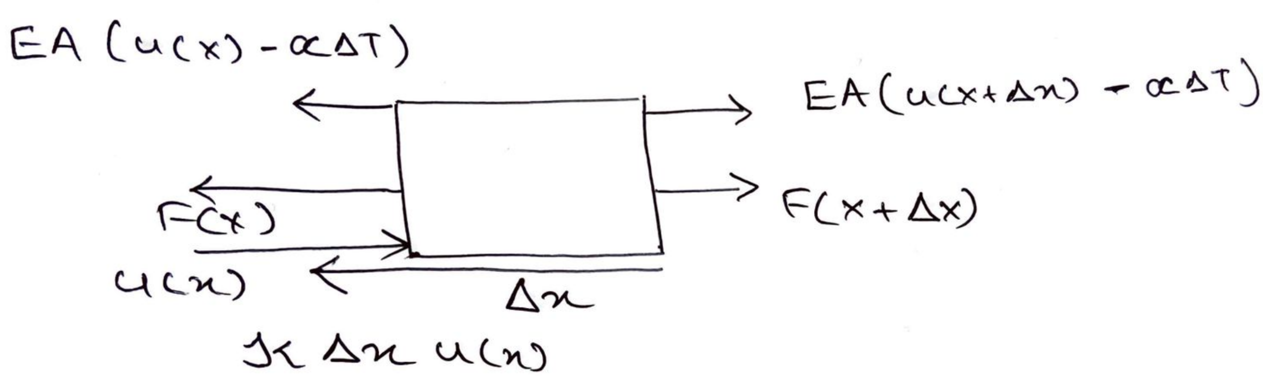
2. Solve the problem 1 when it is subjected to a temperature increase of $\Delta T = 1$. Recall that the stress is given by $\sigma = E(\epsilon - \alpha\Delta T)$, where $\alpha = 1$ is the coefficient of thermal expansion. Assume $E = 1$. (25)

Solution 2

- Code File: [Question 2 Mfile]

Deriving Governing Equation

Force Balance on an arbitrary element at distance x from bar



$u(x)$ is total displacement including temperature expansion.

Also, $\sigma = E(\epsilon - \alpha\Delta T)$

$$F(x+\Delta x) - F(x) - k\Delta x u(x) + EA\frac{[u(x+\Delta x) + \Delta x \alpha \Delta T]}{\Delta x} - EA\frac{[u(x) + \Delta x \alpha \Delta T]}{\Delta x} = 0$$

Dividing whole Equation by Δx

$$\frac{(F(x+\Delta x)-F(x))}{\Delta x} - ku(x) + EA\frac{[u(x+\Delta x)+\Delta x\alpha\Delta T]}{(\Delta x)^2} - EA\frac{[u(x)+\Delta x\alpha\Delta T]}{(\Delta x)^2} = 0$$

Take limit as $\Delta x \to 0$

Now;

$\frac{u(x)}{\Delta x} = \epsilon(x)$; $\frac{u(x+\Delta x)}{\Delta x} = \epsilon(x+\Delta x)$

$\frac{dF}{dx} - ku + \frac{d\epsilon}{dx} = 0$ {Note : Here $u = u(x)$}

$\sigma = \frac{F}{A}$ also $\sigma = E(\frac{du}{dx} - \alpha\Delta T)$

E = Elasticity modulus, **A**= Cross Sectional Area
Thus,

$$\frac{d\epsilon}{dx} = \frac{d^2u}{dx^2}$$

$$F = EA(\frac{du}{dx} - \alpha\Delta T)$$

Known Boundary Conditions

$$u(0) = 0$$

$$\frac{du}{dx}\Big|_{x=l} = \frac{F}{EA} + \alpha\Delta T$$

and, Finally putting value of **F** in our governing equation

$$\frac{d}{dx}[EA(\frac{du}{dx} - \alpha\Delta T)] - ku + \frac{d^2u}{dx^2} = 0$$

Finally Putting values of constants in our equations, questions have asked us to take all constants as unity, so be it.

$2\frac{d^2u}{dx^2} - u = 0$ for $x \in (0,1)$

$u(0) = 0$

$\frac{du}{dx} = 2$ at $x = 1$

$\{ \left.\frac{du}{dx}\right|_{x=1} = (\frac{F}{EA} = 1) + (\alpha \Delta T = 1) \}$

Solving $2\frac{d^2u}{dx^2} - u = 0$ for $x \in (0, 1)$

with Boundary conditions

$u(0) = 0$

$\frac{du}{dx} = 2$ at $x = 1$

We get

$=> u(x) = \frac{2\sqrt{2}e^{(1-x)/\sqrt{2}}(e^{\sqrt{2}x}-1)}{1+e^{\sqrt{2}}}$

Finally we get our analytical solution any any x.

$$=> u(x) = \frac{2\sqrt{2}e^{(1-x)/\sqrt{2}}(e^{\sqrt{2}x}-1)}{1+e^{\sqrt{2}}}$$

Deriving Weak Form

$\int_1^2 \delta u(2\frac{d^2u}{dx^2} - u)dx = 0$

δu is weight function that varies linearly like u since we are going to take linear approximation for our calculations

Evaluating further

$\int_1^2 (2\frac{d}{dx}(\delta u\frac{du}{dx}) - 2\frac{d\delta u}{dx}\frac{du}{dx} - \delta uu)dx = 0$

According to our method our u varies linearly and it will be according to the following equation.

$u = N_1 u_1 + N_2 u_2$

$N_1 = \frac{(x-x_2)}{(x_1-x_2)}$

$N_2 = \frac{(x-x_1)}{(x_2-x_1)}$

$\mathbf{N} = \begin{bmatrix} N_1 & N_2 \end{bmatrix}$

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$\delta u = \mathbf{N}\delta u$

$u = \mathbf{N}u$

$\frac{du}{dx} = \mathbf{B}u$

Putting above equations in our integral and simplifying

$= 2\ \delta u\frac{du}{dx}\Big|_1^2 = 2\int \delta u^T \mathbf{B}^T\mathbf{B}udx + \int \delta u^T\mathbf{N}^T\mathbf{N}udx$

$= F^N\delta u^T = \delta u^T u(\int \mathbf{B}^T\mathbf{B} + 0.5\mathbf{N}^T\mathbf{N})dx$

We can very conveniently compare above expression to $F = Kx$ as we write

$\mathbf{K} = \int(\mathbf{B}^T\mathbf{B} + 0.5\mathbf{N}^T\mathbf{N})dx$

there by leading to

$$F^N = \mathbf{K}u$$

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- We will plot graph of displacement for each node.

Observations

- Number of Elements : 10
- Root Mean Square Error (Displacements) = $6.6957 * 10^{-05}$

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x	analytical displacement	numerical displacement	stress	analytical stress
0	0	0	0.5877	0.5866
0.1	0.1588	0.1588	0.5957	0.5905
0.2	0.3184	0.3183	0.6116	0.6024
0.3	0.4795	0.4795	0.6356	0.6224
0.4	0.6431	0.6431	0.6678	0.6504
0.5	0.8099	0.8098	0.7083	0.6868
0.6	0.9807	0.9807	0.7574	0.7315

x	analytical displacement	numerical displacement	stress	analytical stress
0.7	1.1565	1.1564	0.8153	0.7849
0.8	1.338	1.3379	0.8822	0.8472
0.9	1.5263	1.5262	0.9586	0.9188
1	1.7221	1.722	1	1

- Graph of Displacement vs x