Problem 1

Let X and Y be i.i.d. Expo(1).

1. (10′) Find $P(\max(X,Y) \geq 1)$.
2. (10′) Find $P(\min(X,Y) \leq 1)$.

Problem 2

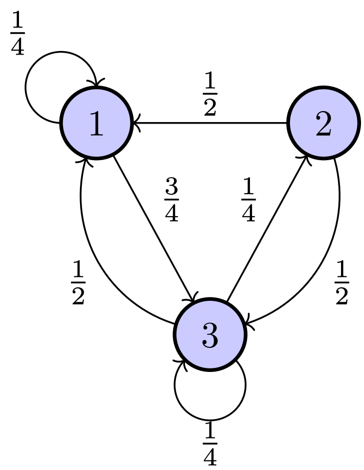
Let X and Y be two continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c, \text{if } 0 < x < 2, 0 < y < 2x \\ 0, \text{otherwise} \end{cases}$$

1. (5′) Find c.
2. (5′) Find the marginal PDF of X.
3. (5′) Find the marginal PDF of Y.

Problem 3

Given a Markov chain with state-transition diagram shown as follows:



1. (2′) Find $P(X_5 = 3 \mid X_4 = 2)$ and $P(X_4 = 1 \mid X_3 = 2)$.
2. (2′) Find the stationary distribution of the chain.
3. (2′) Find the period of state 2 and the period of the chain.
4. (2′) If $P(X_0 = 3) = \frac{1}{5}$, find $P(X_0 = 3, X_1 = 2, X_2 = 1)$.
5. (2′) Find $E(X_5 \mid X_3 = 1)$ and $\text{Var}(X_5 \mid X_3 = 1)$.

Problem 4

Given a coin with the probability p of landing heads. p is unknown and we need to estimate its value through data. In our data collection model, we have n independent tosses, result of each toss is either Head or Tail. Let X denote the number of heads in the total n tosses. Now we conduct experiments to collect data and find $X = k$. Then we need to find $\hat{p}$, the estimation of p.

1. (5′) Assume p is an unknown constant. Find $\hat{p}$ through the MLE (Maximum Likelihood Estimation) rule.
2. (5′) Assume p is a random variable with a prior distribution $p \sim \text{Beta}(1,1)$, where a and b are known constants. Find $\hat{p}$ through the MAP (Maximum a Posterior Probability) rule.
3. (5′) Assume p is a random variable with a prior distribution $p \sim \text{Unif}(0,1)$, where a and b are known constants. Find $\hat{p}$ through the MMSE (Minimal Mean Squared Error) rule.

Problem 5

($10'$) Let U_1 and U_2 be i.i.d. $\mathrm{Unif}(0,1)$. Let $V = \min(U_1, U_2)$, $Z = \max(U_1, U_2)$. Find the joint CDF of V and Z.

Problem 6

($5'$) Let $X_1, ..., X_n$ be i.i.d. $\mathrm{Unif}(0,1)$. Let $X_{(1)} < ... < X_{(n)}$ be the ordering of $X_1, ..., X_n$. Find the PDF of $X_{(n)} - X_{(1)}$.

Problem 7

($10'$) Find the expectation of the return value of the following program code.

```
float sum = 0.0;
while (sum <= 1.0) {
    sum += Unif(0, 1);
}
return sum;
```

Problem 8

Given a verse urn with N balls numbered $1, 2, ..., N$. We pick out all N balls without replacement. Let X_i be the number of the i-th ball, $i \in \{1, 2, ..., n\}$.

1. ($5'$) Find $\mathrm{Var}(X_i)$.
2. ($5'$) Find $\mathrm{Cov}(X_i, X_j)$, $i \neq j$ and $i, j \in \{1, 2, ..., n\}$.

Problem 9

($5'$) Let X and Y be bivariate normal and i.i.d. $\mathcal{N}(0,1)$. If $\mathrm{Corr}(X, Y) = \rho$, find $P(X > 0, Y > 0)$.