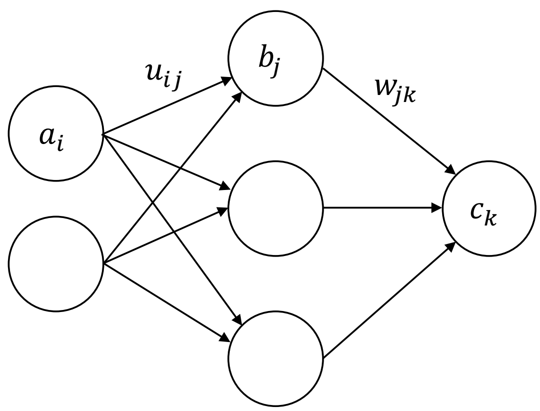
University of Illinois Spring 2018

CS 446: Machine Learning
Homework

Due on Tuesday, February 27, 2018, 11:59 a.m. Central Time

1. **[8 points]** Backpropagation

   Consider the deep net in the figure below consisting of an input layer, an output layer, and a hidden layer. The feed-forward computations performed by the deep net are as follows: every input a_i is multiplied by a set of fully-connected weights u_{ij} connecting the input layer to the hidden layer. The resulting weighted signals are then summed and combined with a bias e_j. This results in the activation signal $z_j = e_j + \sum_i a_i u_{ij}$. The hidden layer applies activation function g on z_j resulting in the signal b_j. In a similar fashion, the hidden layer activation signals b_j are multiplied by the weights connecting the hidden layer to the output layer w_{jk}, a bias f_k is added and the resulting signal h_k is transformed by the output activation function g to form the network output c_k. The loss between the desired target t_k and the output c_k is given by the MSE: $E = \frac{1}{2}\sum_k (c_k - t_k)^2$, where t_k denotes the ground truth signal corresponding to c_k. Training a neural network involves determining the set of parameters $\theta = \{U, W, e, f\}$ that minimize E. This problem can be solved using gradient descent, which requires determining $\frac{\partial E}{\partial \theta}$ for all θ in the model.



   (a) For $g(x) = \sigma(x) = \frac{1}{1+e^{-x}}$, compute the derivative $g'(x)$ of $g(x)$ as a function of $\sigma(x)$.

   Your answer:

$$\begin{aligned}
\frac{\partial g(x)}{\partial x} &= \frac{\partial \sigma(x)}{\partial x} \\
&= \frac{1}{(1+e^{-x})^2}(e^{-x}) \\
&= \frac{1}{(1+e^{-x})} \cdot \left(1 - \frac{1}{(1+e^{-x})}\right) \\
&= g(x) \cdot (1 - g(x)) \\
&= \sigma(x) \cdot (1 - \sigma(x))
\end{aligned}$$

   (b) We denote by $\delta_k = \frac{\partial E}{\partial h_k}$ the error signal of neuron k in the second linear layer of the network. Compute δ_k as a function of c_k, t_k, g' and h_k.

Your answer:

$$\begin{aligned}\delta_k &= \frac{\partial E}{\partial h_k}\\ &= \frac{\partial \frac{1}{2}\sum_k (c_k - t_k)^2}{\partial h_k}\\ &= (c_k - t_k) \cdot \frac{\partial c_k}{h_k}\\ &= (c_k - t_k) \cdot g'(h_k)\end{aligned}$$

(c) Compute $\frac{\partial E}{\partial w_{jk}}$. Use δ_k and b_j.

Your answer:

$$\begin{aligned}\frac{\partial E}{\partial w_{jk}} &= \frac{\partial E}{\partial h_k} \cdot \frac{\partial h_k}{\partial w_{jk}}\\ &= \delta_k \cdot \frac{\partial h_k}{\partial w_{jk}}\\ &= \delta_k \cdot b_j\end{aligned}$$

(d) Compute $\frac{\partial E}{\partial f_k}$. Use δ_k.

Your answer:

$$\begin{aligned}\frac{\partial E}{\partial f_k} &= \frac{\partial E}{\partial h_k} \cdot \frac{\partial h_k}{\partial f_k}\\ &= \delta_k \cdot \frac{\partial h_k}{\partial f_k}\\ &= \delta_k\end{aligned}$$

(e) We denote by $\psi_j = \frac{\partial E}{\partial z_j}$ the error signal of neuron j in the first linear layer of the network. Compute ψ_j as a function of δ_k, w_{jk}, g' and z_j.

Your answer:

$$\begin{aligned}\psi_j &= \frac{\partial E}{\partial z_j}\\ &= \frac{\partial E}{\partial h_k} \cdot \frac{\partial h_k}{\partial z_j}\\ &= \sum_k \delta_k \cdot \frac{\partial (f_k + w_{jk} \cdot g(z_j))}{\partial z_j}\\ &= \sum_k \delta_k \cdot w_{jk} \cdot g'(z_j)\end{aligned}$$

(f) Compute $\frac{\partial E}{\partial u_{ij}}$. Use ψ_j and a_i.

Your answer:

$$\begin{aligned}
\frac{\partial E}{\partial u_{ij}} &= \frac{\partial E}{\partial z_j} \cdot \frac{\partial z_j}{\partial u_{ij}} \\
&= \psi_j \cdot \frac{\partial (e_j + u_{ij} \cdot a_i)}{\partial u_{ij}} \\
&= \psi_j \cdot a_i
\end{aligned}$$

(g) Compute $\frac{\partial E}{\partial e_j}$. Use ψ_j.

Your answer:

$$\begin{aligned}
\frac{\partial E}{\partial e_j} &= \frac{\partial E}{\partial z_j} \cdot \frac{\partial z_j}{\partial e_j} \\
&= \psi_j \cdot \frac{\partial (e_j + u_{ij} \cdot a_i)}{\partial e_j} \\
&= \psi_j
\end{aligned}$$