Department of Electrical Engineering
Indian Institute of Technology Delhi

Analysis of Bicycle Dynamics and Stability Control

Project Report

ELL225-Control Engineering

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Contents

1 Abstract

The objective of this course project is to develop an autopilot for a rider-less bicycle to keep it in vertical upright position. Considering following quantities as: i) states: roll angle, roll rate, steer angle and steer rate, ii) input: steering torque and iii) output: roll angle, the dynamics of the bicycle can be expressed as a fourth order linear model. The linear model is velocity dependent (often referred to as linear parameter varying model or LPV), where one can obtain linear time-invariant state space models corresponding to the different fixed velocities. For this project, we have considered following three fixed velocities: $v_1 = 0$ m/s, $v_2 = 1$ m/s and $v_3 = 6$ m/s.

2 Introduction

Bicycles are statically unstable like the inverted pendulum, but can, under certain conditions, be stable in forward motion. A detailed model of a bicycle is complex because the system has many degrees of freedom and the geometry is intricate. We assume that the bicycle consists of four rigid parts, specifically, two wheels, a frame, and a front fork with handlebars. The influence of other moving parts, such as pedals, chain, and brakes, on the dynamics is disregarded. To include the rider in the analysis, the rider's upper body is modeled as a point mass that can move laterally with respect to the bicycle frame. The rider can also apply a torque to the handlebars. Since we do not consider extreme conditions and tight turns, we assume that the bicycle tire rolls without longitudinal or lateral slippage. Further, we often assume that the forward velocity is constant. To summarize, we simply assume that the bicycle moves on a horizontal plane and that the wheels always maintain contact with the ground.

2.1 Geometry

The parameters that describe the geometry of a bicycle are defined in Figure 1. The key parameters are wheelbase b, head angle λ, and trail c. The front fork is angled and shaped so that the contact point of the front wheel with the road is behind the extension of the steer axis. Trail is defined as the horizontal distance c between the contact point and the steer axis when the bicycle is upright with zero steer angle. A large trail improves stability but

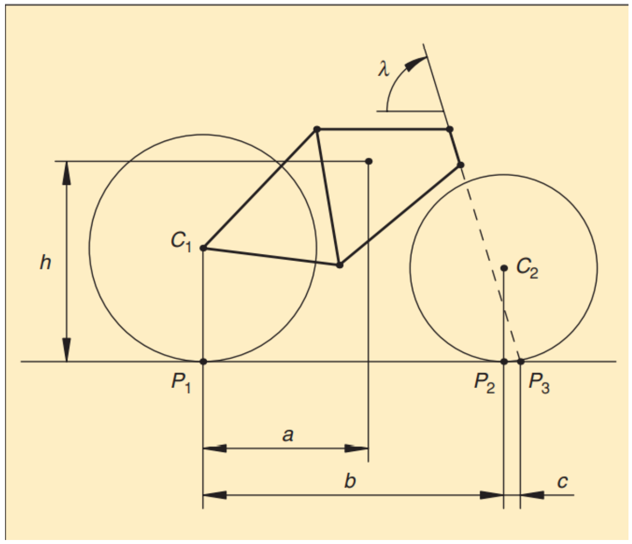


Figure 1: Parameters defining the bicycle geometry

makes steering less agile. Geometrically, it is convenient to view the bicycle as composed of two hinged planes, the frame plane and the front fork plane. The frame and the rear wheel lie in the frame plane, while the front wheel lies in the front fork plane. The planes are joined at the steer axis. The points P1 and P2 are the contact points of the wheels with the horizontal plane, and the point P3 is the intersection of the steer axis with the horizontal plane (Figure 1).

2.2 Coordinates

The coordinates used to analyze the system, which follow the ISO 8855 standard, are defined in Figure 2. There is an inertial system with axes ξηζ and origin O. The coordinate system xyz has its origin at the contact point P1 of the rear wheel and the horizontal plane. The x axis is aligned with the line of contact of the rear plane with the horizontal plane. The x axis also goes through the point P3 , which is the intersection between the steer axis and the horizontal plane. The orientation of the rear wheel plane is defined by the angle ψ, which is the angle between the ξ -axis and the x-axis. The z axis is vertical, and y is perpendicular to x and positive on the left side of the bicycle so that a right-hand system is obtained. The roll angle of the rear frame is positive when leaning to the right. The roll angle of the front fork plane is f . The steer angle δ is the angle of intersection between the rear and front planes, positive when steering left. The effective steer angle δf is the angle between the lines of intersection of the rear and front planes with the horizontal plane.

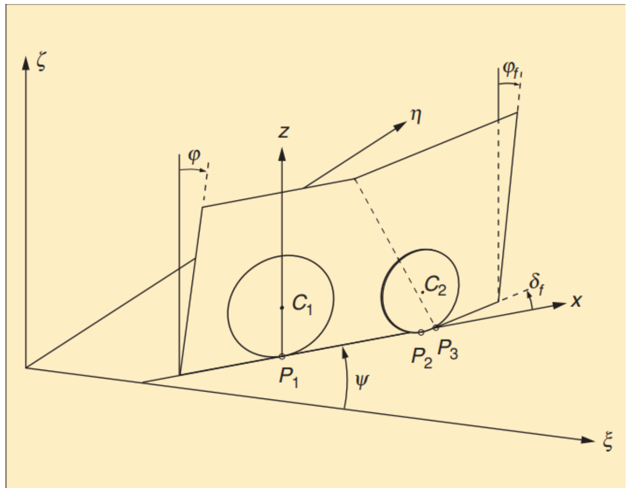


Figure 2: Coordinate Systems

3 Literature Review

Bicycles have intrigued scientists ever since they appeared in the middle of the 19$^{\text{th}}$ century. A thorough presentation of the history of the bicycle is given in the recent book [3]. Modelling bicycles became a popular topic for dissertations in the latter half of the last century [4]–[7]. The book [8] and its third edition [9] give a broad engineering perspective. Rankine, Klein, and Sommerfeld analyze bicycles in [10] and [11]. The first publications that used differential equations to describe the motion of an idealized bicycle appeared toward the end of the 19$^{\text{th}}$ century. One of the first computer simulations of a nonlinear bicycle model was presented in [12]. Nonlinear models are found in [13] and [14]–[16], in which the dynamic order ranges 2–20, depending on the assumptions made. Neˇımark and Fufaev [24] derived a comprehensive set of linearized models by approximating potential and kinetic energy by quadratic terms and applying Lagrange's equations to these expressions. The bicycle provides an ideal venue for illustrating modelling, dynamics, stabilization, and feedback. Many bicycle analyses aimed at understanding rider control are based on qualitative dynamics discussions that are too reduced to capture the ability of an uncontrolled moving bicycle to balance itself.

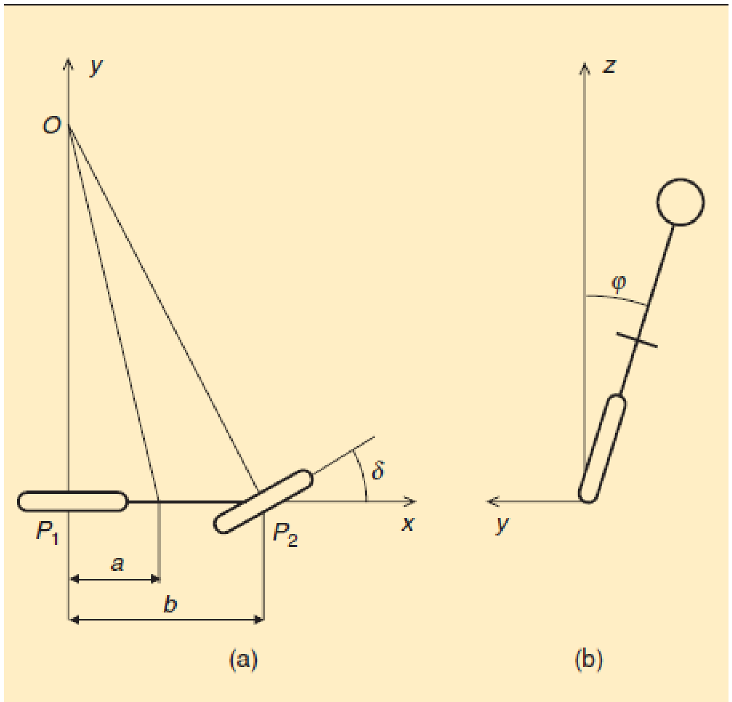


Figure 3: Schematic (a) top and (b) rear views of a naive (λ= 0) bicycle. The steer angle is δ, and the roll angle is φ. [1]

4 Theory

It is assumed that the bicycle rolls on the horizontal plane, the rider has fixed position and orientation relative to the frame, and that the forward velocity at the rear wheel v is constant. The head angle is 90° and that the trail c is zero. We also assume that the steer angle δ is the control variable. The rotational degree of freedom associated with the front fork then disappears, and the system is left with the roll angle φas the only degree of freedom. All angles are assumed to be small so that the equations can be linearized. An observer fixed to the coordinate system xyz experiences forces due to the acceleration of the coordinate system relative to inertial space. Let J denote the moment of inertia of this body with respect to the x-axis, and let $D = J_{xz}$ denote the inertia product with respect to the xz axes.

The angular momentum of the system with respect to the x axis is [17]:

$$L_{\mathrm{x}} = J\frac{d\varphi}{dt} - D\omega = J\frac{d\varphi}{dt} - \frac{VD}{b}\delta$$

$$J\frac{d^2\varphi}{dt^2} - mgh\varphi = \frac{VD}{b}\frac{d\delta}{dt} + \frac{mV^2h}{b}\delta \tag{1}$$

The term mgh is the torque generated by gravity. The terms on the right-hand side of (1) are the torques generated by steering, with the first term due to inertial forces and the second term due to centrifugal forces.
Approximating the moment of inertia as J $\approx$ mh^2 and D$\approx$ mgh, the model becomes

$$\frac{d^2\varphi}{dt^2} - \frac{g}{h}\varphi = \frac{aV}{bh}\frac{d\delta}{dt} + \frac{V^2}{bh}\delta$$

5 Methodology

Using Carvallo Whipple bicycle model,

$$M\ddot{q} + CV\dot{q} + (K_0 + K_2v^2)q = T$$

$$q = \begin{pmatrix}\phi\\ \delta\end{pmatrix}\ ; T = \begin{pmatrix}0\\ T_\delta\end{pmatrix} ; O = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} ; I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$q_1 = q; q_2 = \dot{q}; q_2 = \dot{q_1}$$

$$\dot{q}_2 = -M^{-1}CVq_2 - M^{-1}(K_0 + K_2V^2)q_1 + M^{-1}T$$

$$\begin{pmatrix}\dot{q}_1\\ \dot{q}_2\end{pmatrix} = \begin{pmatrix}O & I\\ -M^{-1}(K_0 + K_2V^2) & -M^{-1}CV\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix} + M^{-1}\begin{pmatrix}O\\ T\end{pmatrix}$$

5.1 Case 1: $\mathbf{v}_1 = 0$

5.1.1 State Space Representation

$$\begin{pmatrix} \dot{\phi} \\ \dot{\delta} \\ \ddot{\phi} \\ \ddot{\delta} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 15.08 & 0.10 & 0 & 0 \\ -2.56 & 17.23 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \delta \\ \dot{\phi} \\ \dot{\delta} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0.68 \\ 8.70 \end{pmatrix} T_\delta$$

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \delta \\ \dot{\phi} \\ \dot{\delta} \end{pmatrix}$$

5.1.2 Transfer Function Representation

$$G = \frac{0.6848s^2 - 10.9}{s^4 - 2.665x10^{-15}s^3 - 32.31s^2 + 4.263x10^{-14}s + 260.2}$$

5.1.3 Poles

$$p_1 = -4.13; p_2 = -3.90; p_3 = 3.90; p_4 = 4.13$$

5.1.4 Zeroes

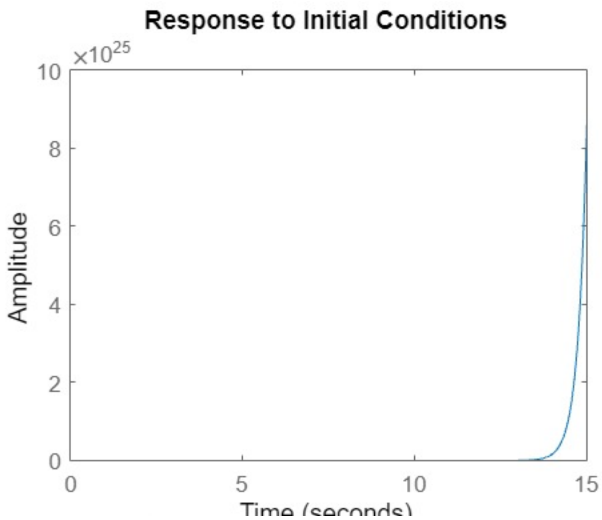
$$z_1 = -3.99; z_2 = 3.99$$

5.1.5 Eigenvalues of Matrix A

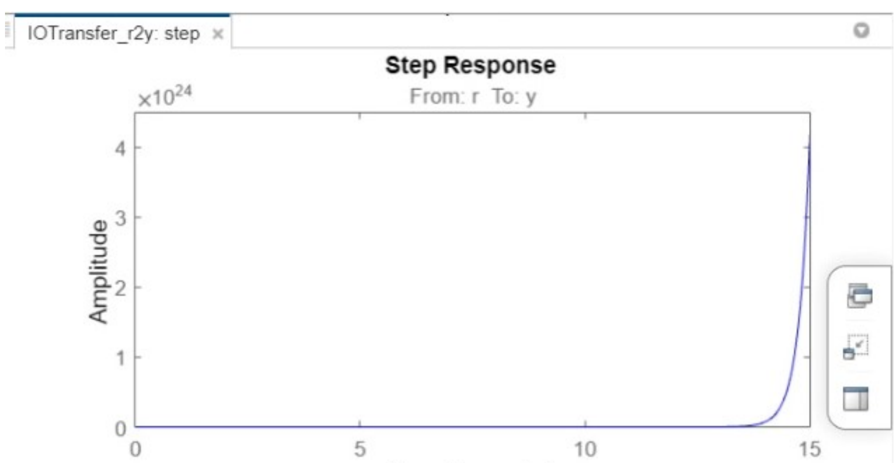
$$\text{Eigenvalues are } -4.13, -3.90, 3.90, 4.13$$

5.1.6 Zero Input Response

$$x_0 = \begin{pmatrix} 3 \\ 5 \\ 10 \\ 10 \end{pmatrix}$$



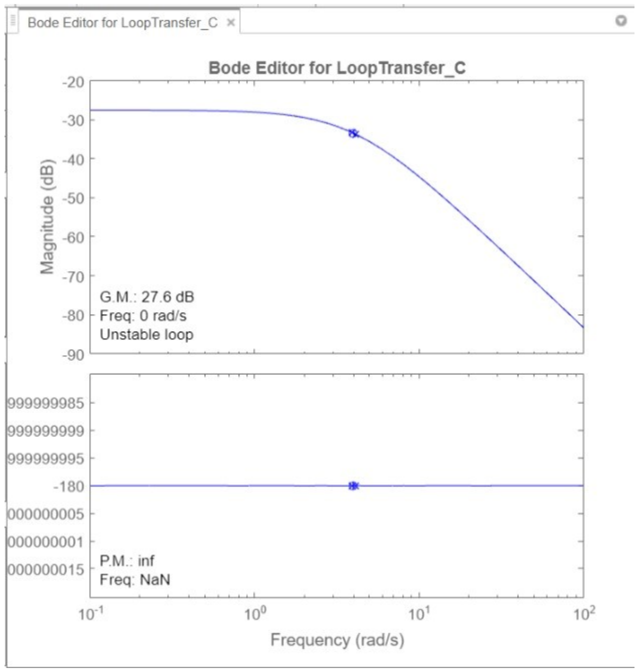
5.1.7 Unit Step Response



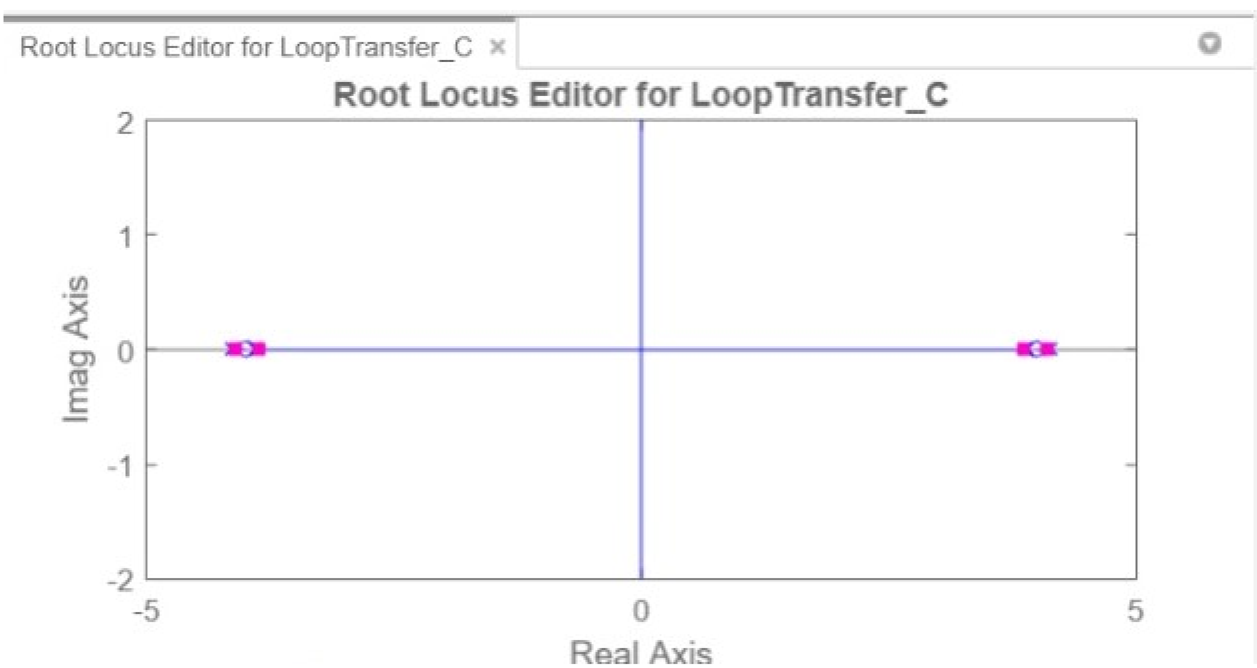
5.1.8 Stability

The system is unstable at this velocity because 2 of the poles lie in the right half plane.

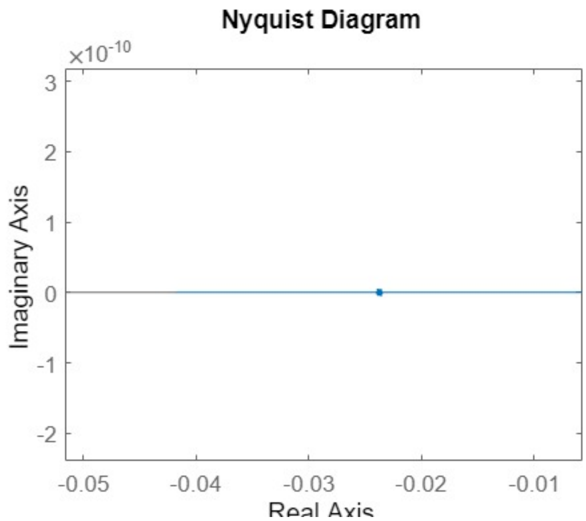
5.1.9 Bode Plot



5.1.10 Root Locus



5.1.11 Nyquist Plot



5.1.12 Gain Margin and Phase Margin

$$Gain\ Margin = 27.6dB;\ Phase\ Margin = \infty$$

5.2 Case 2: $\mathbf{v}_2 = 1$

5.2.1 State Space Representation

$$\begin{pmatrix} \dot{\phi} \\ \dot{\delta} \\ \ddot{\phi} \\ \ddot{\delta} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 15.08 & 1.62 & -0.29 & 0.83 \\ -2.56 & 16.38 & -3.79 & -1.79 \end{pmatrix} \begin{pmatrix} \phi \\ \delta \\ \dot{\phi} \\ \dot{\delta} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0.68 \\ 8.70 \end{pmatrix} T_\delta$$

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \delta \\ \dot{\phi} \\ \dot{\delta} \end{pmatrix}$$

5.2.2 Transfer Function Representation

$$G = \frac{0.6848s^2+8.473s+2.945}{s^4+2.097s^3-27.77s^2-23.71s+251.3}$$

5.2.3 Poles

$$p_1 = -4.44 + 0.476j; p_2 = -4.44 - 0.476j; p_3 = 3.39 + 1.03j; p_4 = 3.39 - 1.03j$$

5.2.4 Zeroes

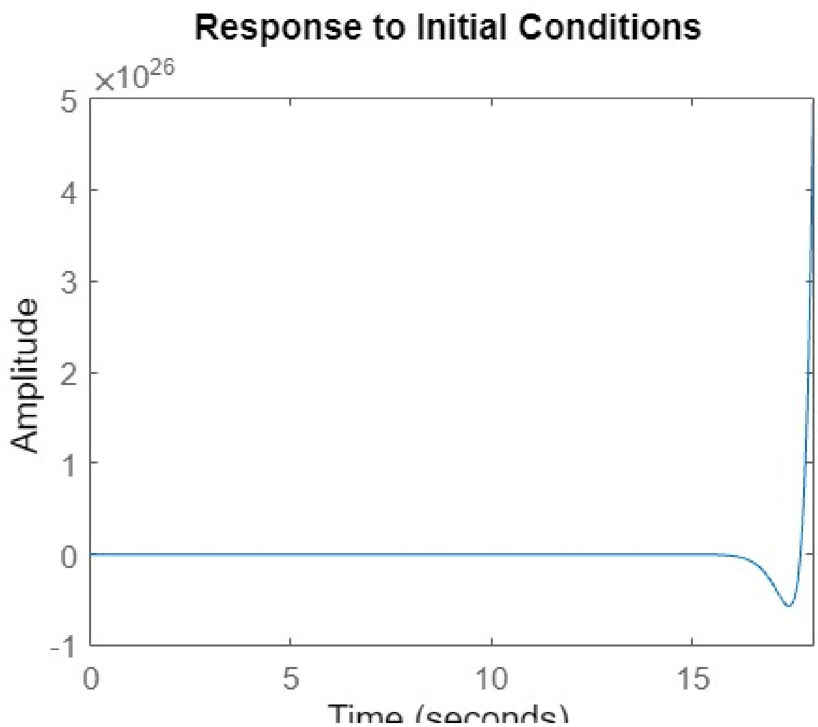
$$z_1 = -0.35; z_2 = -12.01$$

5.2.5 Eigenvalues of Matrix A

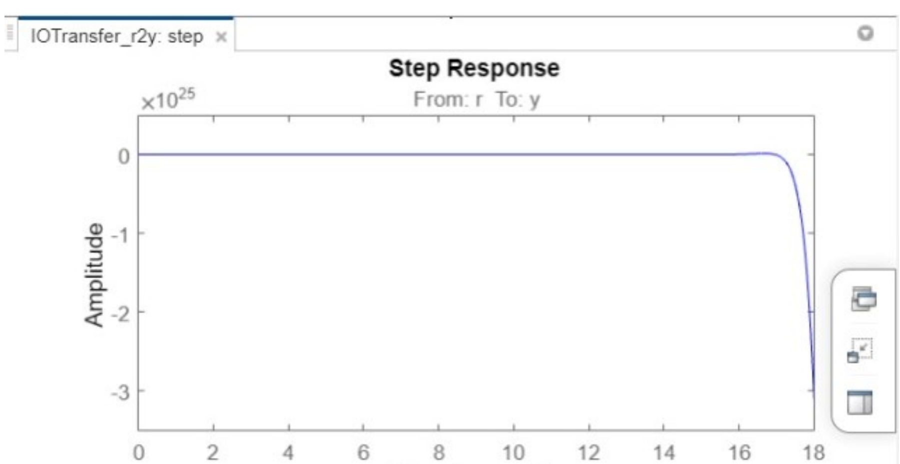
Eigenvalues are $-4.44 + 0.476j, -4.44 - 0.476j, 3.39 + 1.03j, 3.39 - 1.03j$

5.2.6 Zero Input Response

$$x_0 = \begin{pmatrix} 3 \\ 5 \\ 10 \\ 10 \end{pmatrix}$$



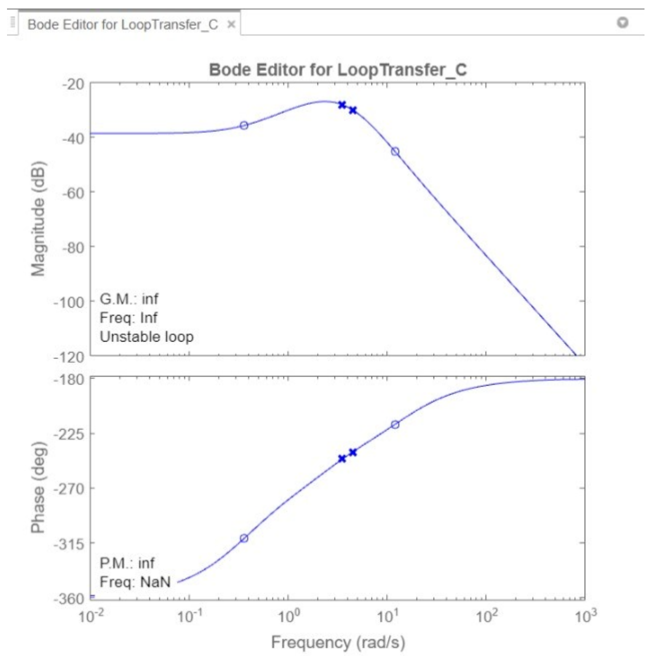
5.2.7 Unit Step Response



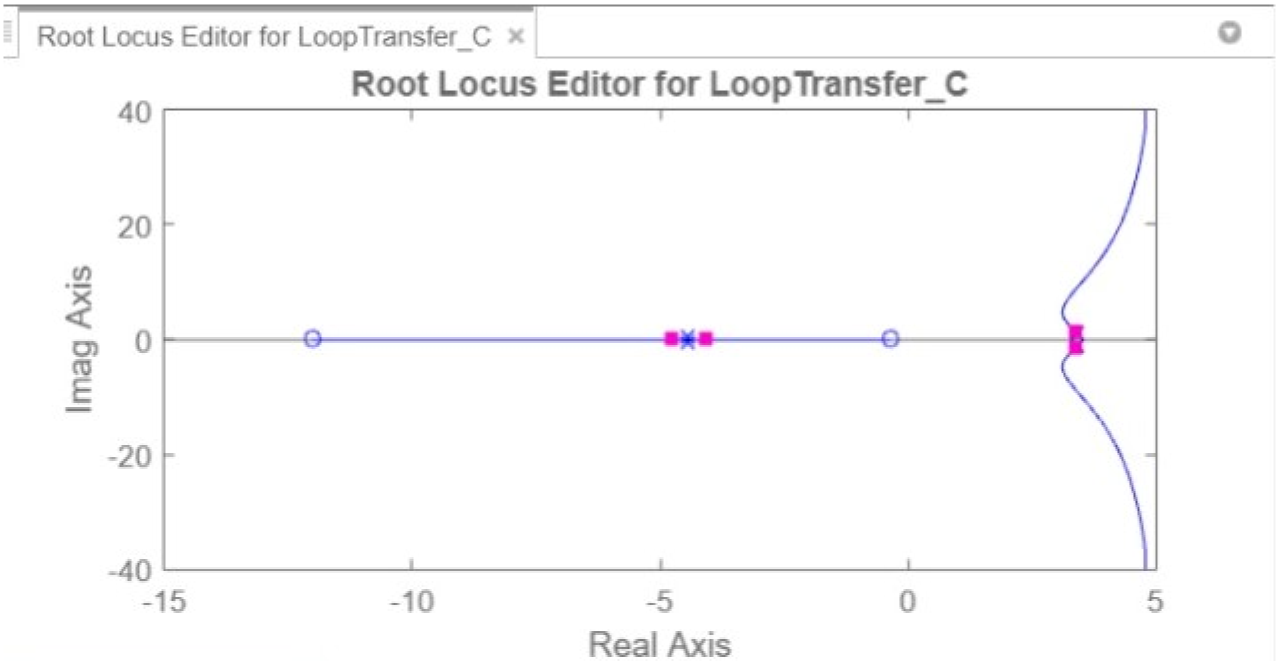
5.2.8 Stability

The system is unstable at this velocity because 2 of the poles lie in the right half plane.

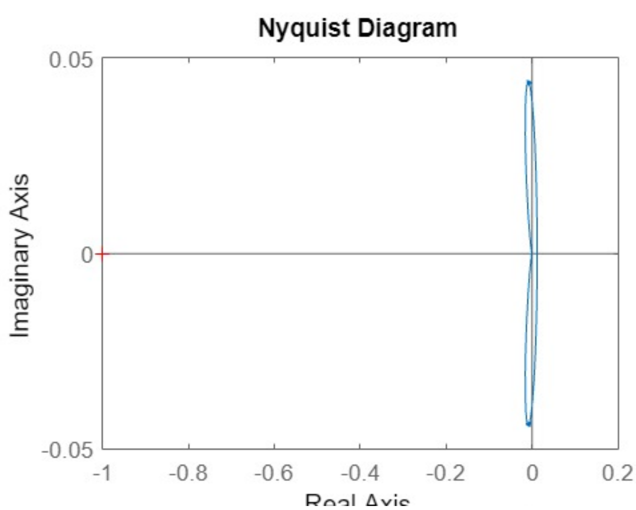
5.2.9 Bode Plot



5.2.10 Root Locus



5.2.11 Nyquist Plot



5.2.12 Gain Margin and Phase Margin

$$Gain\ Margin = \infty;\ Phase\ Margin = \infty$$

5.3 Case 3: $\mathbf{v}_3 = 6$

5.3.1 State Space Representation

$$\begin{pmatrix} \dot{\phi} \\ \dot{\delta} \\ \ddot{\phi} \\ \ddot{\delta} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 15.08 & 54.93 & -1.79 & 4.99 \\ -2.56 & -13.33 & -22.77 & -10.79 \end{pmatrix} \begin{pmatrix} \phi \\ \delta \\ \dot{\phi} \\ \dot{\delta} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0.68 \\ 8.70 \end{pmatrix} T_\delta$$

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \delta \\ \dot{\phi} \\ \dot{\delta} \end{pmatrix}$$

5.3.2 Transfer Function Representation

$$G = \frac{0.6848s^2+50.84s+487.4}{s^4+12.58s^3+131.2s^2+1125s-60.17}$$

5.3.3 Poles

$$p_1 = -1.11 + 10.36j; p_2 = -1.11 - 10.36j; p_3 = -10.41; p_4 = 0.05$$

5.3.4 Zeroes

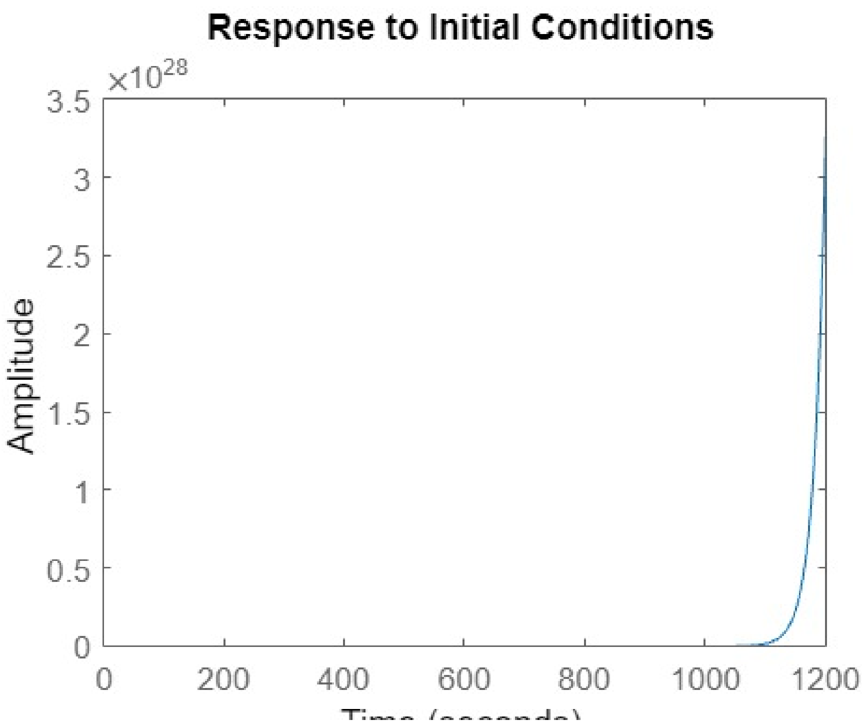
$$z_1 = -11.30; z_2 = -62.93$$

5.3.5 Eigenvalues of Matrix A

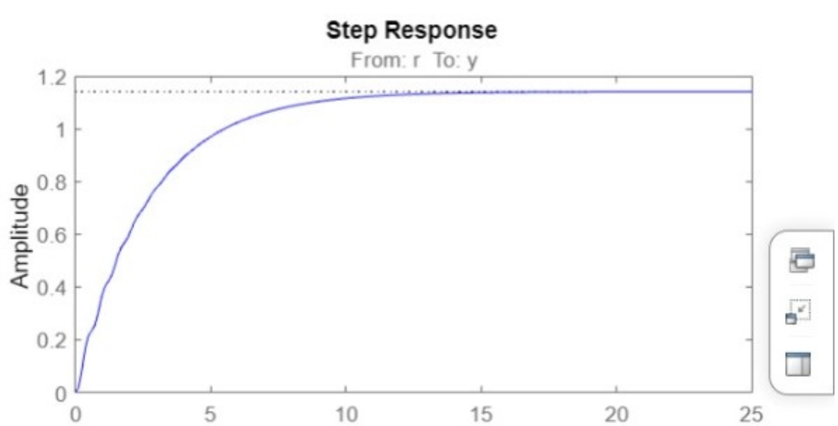
$$\text{Eigenvalues are } -1.11 + 10.36j, -1.11 - 10.36j, -10.41, 0.05$$

5.3.6 Zero Input Response

$$x_0 = \begin{pmatrix} 3 \\ 5 \\ 10 \\ 10 \end{pmatrix}$$



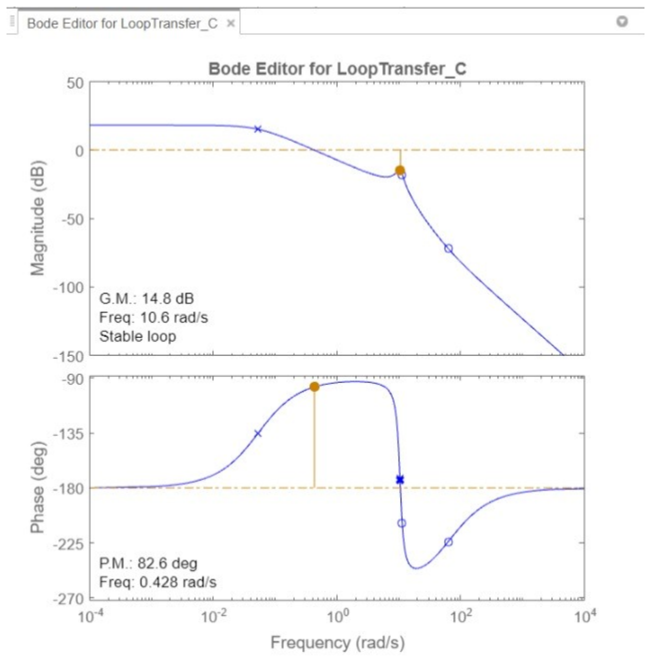
5.3.7 Unit Step Response



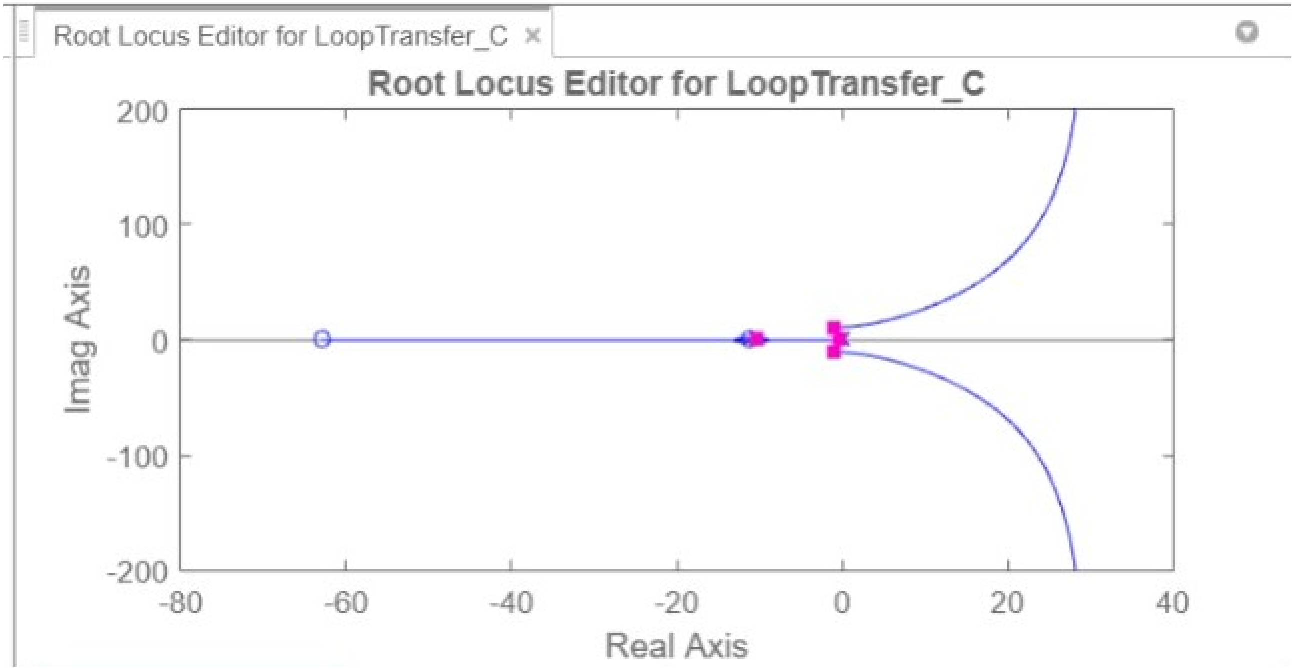
5.3.8 Stability

The system is unstable at this velocity because 1 of the poles lie in the right half plane.

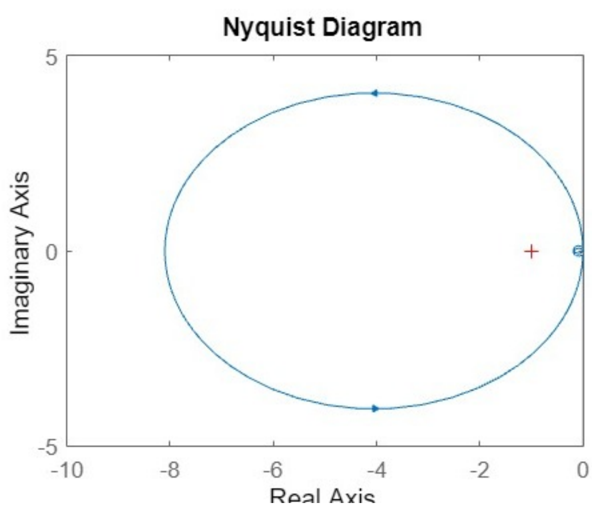
5.3.9 Bode Plot



5.3.10 Root Locus



5.3.11 Nyquist Plot



5.3.12 Gain Margin and Phase Margin

$$Gain\ Margin = 14.8dB; Phase\ Margin = 82.6deg$$

6 Controller Design

6.1 Case 1: $\mathbf{v}_1 = 0$

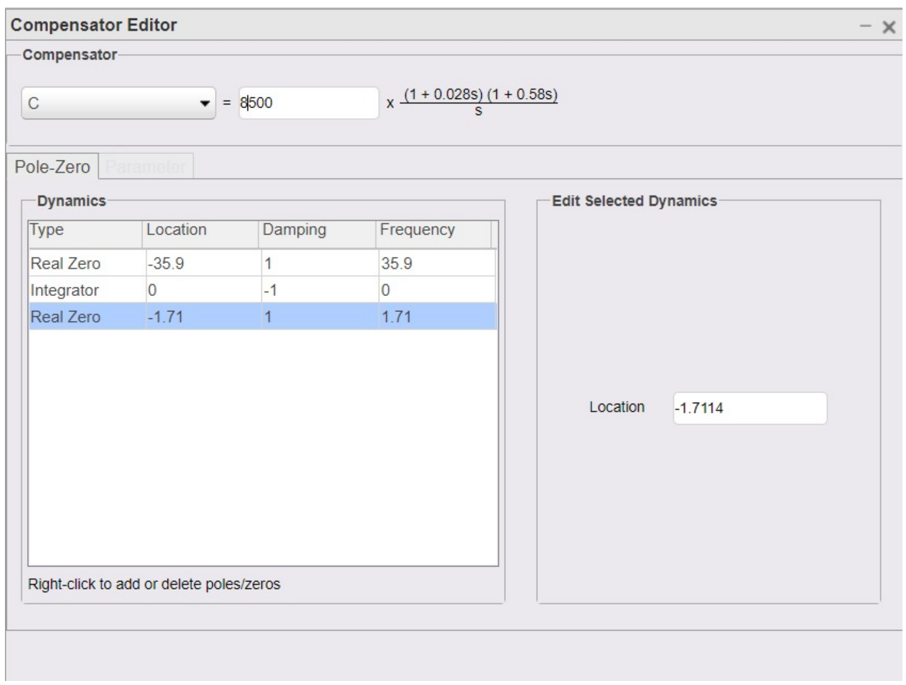
The only possible way to stabilize the system with zero forward velocity is through pole cancellations. However, this is a practically unreliable method. The issue is that if the extra zero does not completely cancel the pole (as is always the case in reality), a portion of the root locus gets trapped in the right-half plane. This results in an unstable closed-loop response. Thus, no feasible PID Controller or Lag-Lead Controller can be designed to minimize steady-state error and achieve the required transient conditions.

6.2 Case 2: $\mathbf{v}_2 = 1$

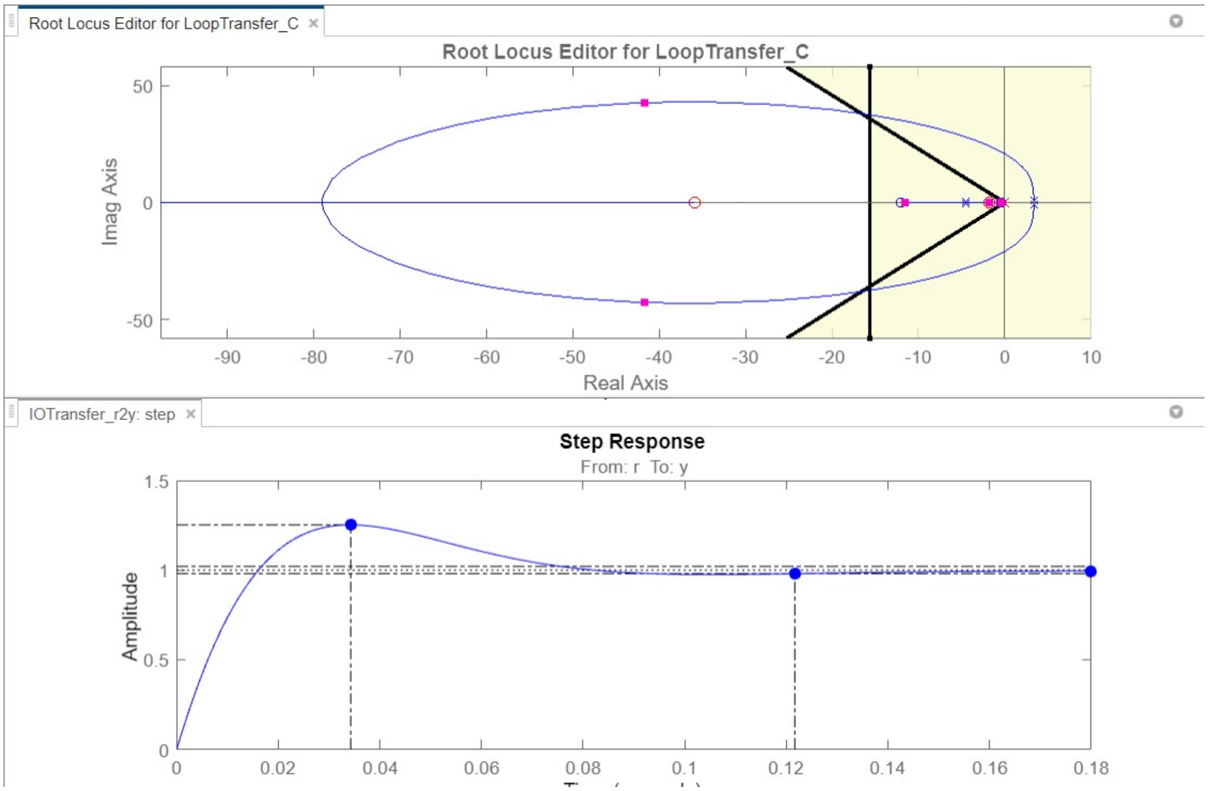
6.2.1 Design Specifications

$$Damping\ Ratio \geq 0.40$$
$$Settling\ Time \leq 0.25$$

6.2.2 Compensator Design



6.2.3 Updated Root Locus and Step Response



Damping Ratio = 0.401

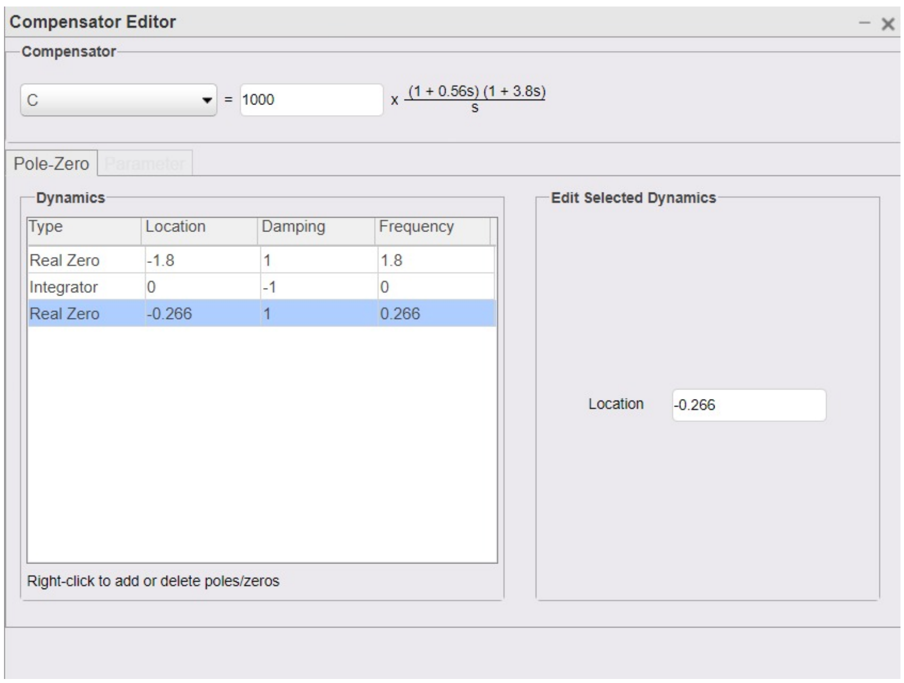
Settling Time= 0.122

6.3 Case 3: $\mathbf{v}_3 = 6$

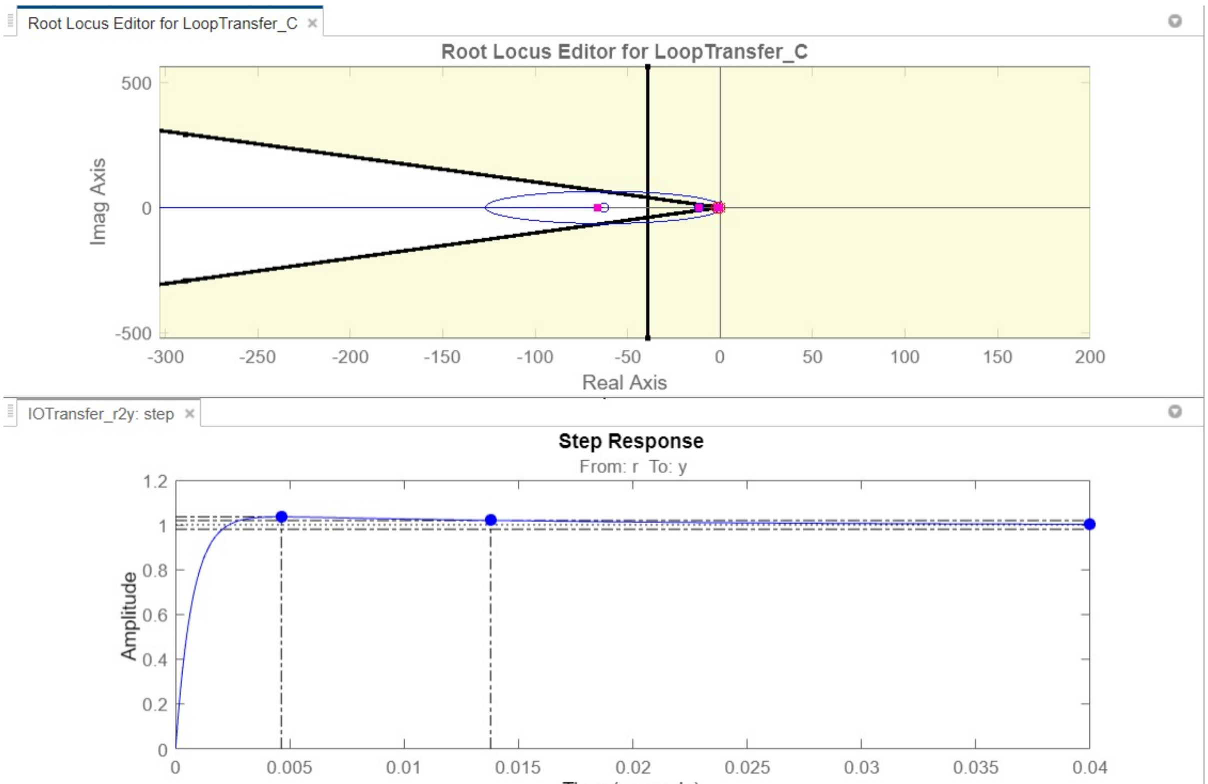
6.3.1 Design Specifications

$$Damping\,Ratio \geq 0.70$$
$$Settling\,Time \leq 0.10$$

6.3.2 Compensator Design



6.3.3 Updated Root Locus and Step Response



Damping Ratio = 0.72

Settling Time= 0.01

7 Conclusion

- The state space representation from the Carvallo Whipple bicycle model was obtained.
- Using MATLAB, the transfer function representation for velocities $v_1 = 0$ m/s, $v_2 = 1$ m/s and $v_3 = 6$ m/s were obtained, along with poles and zeroes for each case.
- Time response was computed for zero input with (any) non-zero initial states and unit step input.
- By observing the poles of the transfer function in each case, it can be concluded that the bicycle is unstable for all three cases.
- Using MATLAB, Nyquist plots, Bode plots and Root Locus were drawn for each case. Nyquist plots and Root Locus are for closed loop systems while Bode plots are for open loop systems. Such plots help in commenting on the stability of the system.
- The **sisotool** in MATLAB was used to design controllers for each case considering unity negative feedback. It was found that designing a controller for $v_1 = 0m/s$ case is not possible (excluding pole cancellation).

8 References

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9 Contribution

Name	**Contribution**
Divyansh Agarrwal	Theory, Methodology, Controller Design, Conclusio
Ekansh Singh	Documentation, Stability Analysis, Controller Design
Priyanshu Kumar	Introduction, Literature Review, Theory, Documentation
Animesh Jhawar	State Space Representation, Transfer Funtions, Zero Input and Step Response
Sanyam	Stability Analysis, Bode Plot, Root Locus, Nyquist Plot