Department of Electrical Engineering
Indian Institute of Technology Delhi

Analysis of Bicycle Dynamics and Stability Control

Project Report

ELL225 Control Engineering

Supervisor - Prof. Dr. Subashish Datta

Students :

Yajat Kapoor - 2021EE30471

Ayushmaan Pandey - 2021EE30709

Harshul Sagar - 2021EE10211

Contents

1 Abstract Idea

The aim of this course project is to develop an autopilot for a rider-less bicycle to keep it in vertical upright position. We are considering the following quantities as states: i) roll angle,ii) roll rate, iii)steer angle iv) steer rate and as input we have steering torque and our output is roll angle, the dynamics of the bicycle can be expressed as a fourth order linear model. For this project, we have considered following three fixed velocities: v1 = 0 m/s, v2=0.5m/s and v3 =5m/s.

2 Introduction

From a statistical viewpoint ,bicycles are unstable like the inverted pendulum. However,under translation they may be designed to be stable. A detailed model of a bicycle is complex because the system has many degrees of freedom and the mechanics is very intricate. We assume that the bicycle consists of four rigid portions, specifically, two wheels, a frame, and a front fork with handlebars. We have not taken the influence of other moving parts, such as pedals, chain, and brakes on the dynamics. We often assume that the forward velocity is constant. To summarize, we have assumed that the bicycle moves on a horizontal plane and that the wheels do not lose contact with the surface.

3 Methodology

Using Carvallo Whipple bicycle model, M$\ddot{\text{q}}$ + CV$\dot{q}$ + (K$_o + K_2 v_2)q = T$

$\dot{x}$(t)=A(v(t))$\cdot x(t) + B \cdot u(t)$, (1)

y(t)= C $\cdot x(t) + D \cdot u(t)$, (2)

where x(t)=$\begin{pmatrix} \phi \\ \delta \\ \dot{\phi} \\ \dot{\delta} \end{pmatrix}$ u(t)=T$_\delta(t)$ $y(t) = x(t)$.

$$A=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 13.67 & 0.225 - 1.319\, v^2(t) & -0.164\, v(t) & -0.552\, v(t) \\ 4.857 & 10.81 - 1.125\, v^2(t) & 3.621\, v(t) & -2.388\, v(t) \end{pmatrix} \qquad B=\begin{pmatrix} 0 \\ 0 \\ -0.339 \\ 7.457 \end{pmatrix}$$

$$C=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad D=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ Hence, G(s)=C(sI-A)}^{-1}B + D$$

In the given state space representation we examine the output for different velocities. Consider different cases v=0, v=0.5 and v=5 m/s. We shall be investigating each case individually by putting appropriate value of v in the state space representation and finding out G(s) and then finding the unit step response.

3.1 Case 1: v=0 m/s

3.1.1 State Space Model

$$\begin{pmatrix} \dot{\phi} \\ \dot{\delta} \\ \ddot{\phi} \\ \ddot{\delta} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 13.6700 & 0.2250 & 0 & 0 \\ 4.8570 & 10.8100 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \delta \\ \dot{\phi} \\ \dot{\delta} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -0.3390 \\ 7.4570 \end{pmatrix} \mathrm{T}$$

$$\mathrm{y} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi \\ \delta \\ \dot{\phi} \\ \dot{\delta} \end{pmatrix}$$

3.1.2 Transfer Function

The transfer function comes out to be $T(s) = \frac{-0.339s^2+5.342}{s^4+3.553e-^{15}s^3-24.48s^2-2.842e^{-14}s+146.7}$

3.1.3 Eigenvalues

Eigenvalues in this case are -3.7432, -3.2355, 3.7432, 3.2355

3.1.4 Poles

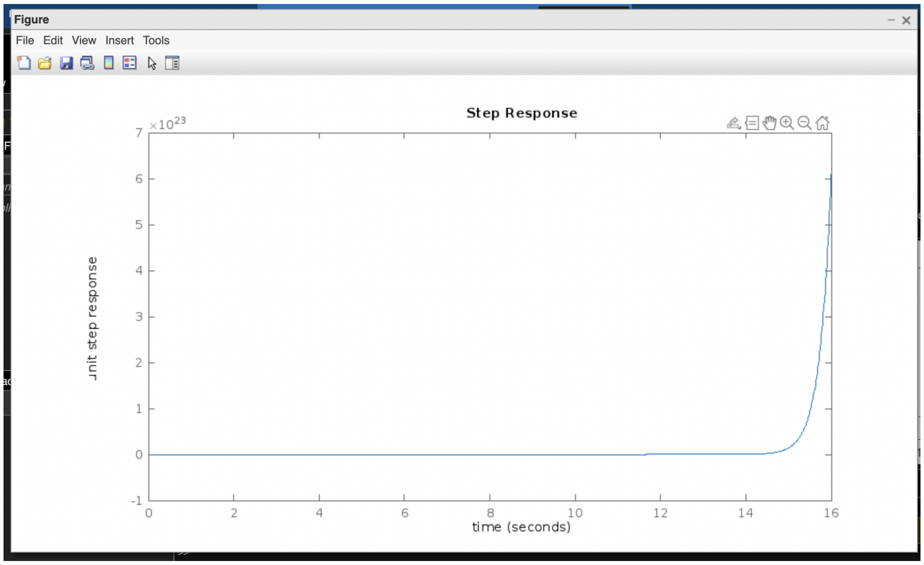
Poles of transfer function are -3.7432,-3.2355,3.7432,3.2355

3.1.5 Zeroes

Zeroes of transfer function are 3.9698, -3.9698

3.1.6 Unit Step Response

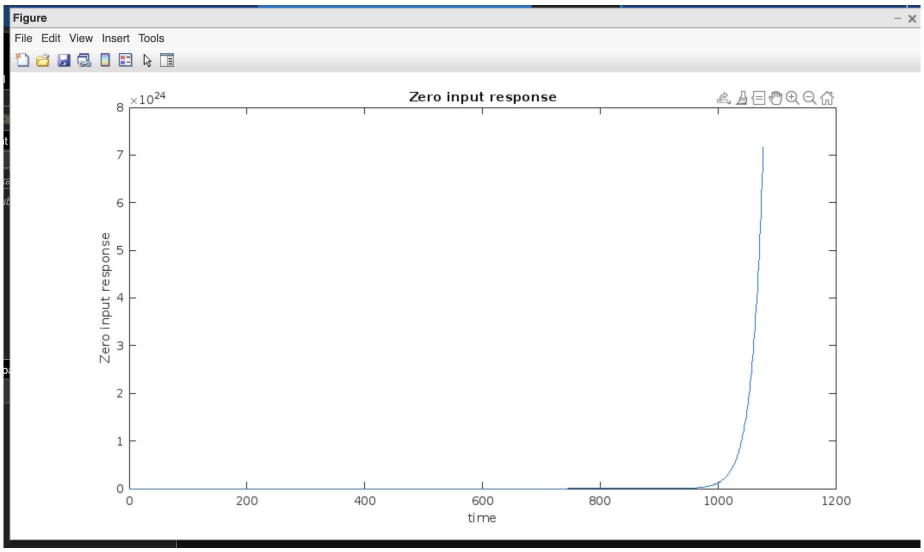
$y(t) = 0.005969e^{-3.743t} - 0.02418e^{3.236t} - 0.02418e^{-3.236t} + 0.005969e^{3.743t} + 0.03642$



3.1.7 Zero Input Response

$y(t) = 0.0113e^{-3.24t} + 0.0215e^{3.24t} + 0.354e^{-3.74t} + 0.613e^{3.74t}$

Zero input response is shown in the following plot



3.1.8 Stability

1. **Asymptotically Stable**
   As 2 eigenvalues are in Right Half of Complex Plane so natural response doesn't go to 0 at $t \to \infty$ so asymptotically unstable
2. **BIBO Stable**

   As 2 poles are in Right Half of Complex Plane so it is not BIBO stable as impulse response is not absolutely integrable.

3.1.9 Nyquist plot

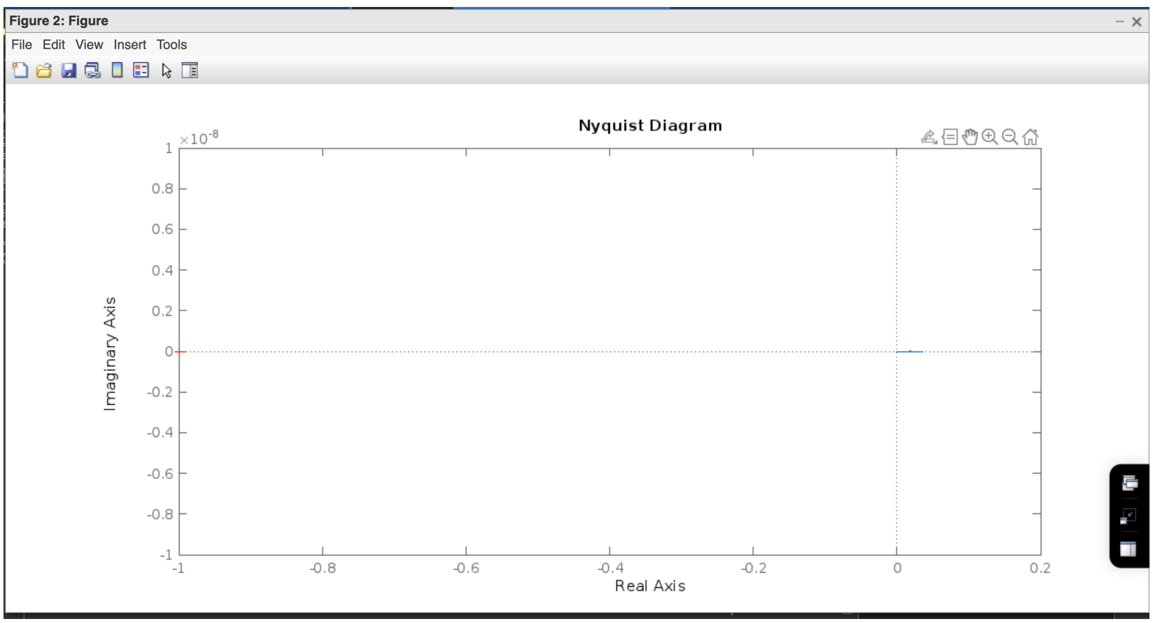


Figure 1: Nyquist Plot for v=0 m/s

By Cauchy's criteria we can observe that N = 0 , P = 2 hence Z = N + P = 0 + 2 = 2 . Hence the closed loop system is unstable.

3.1.10 Bode plot

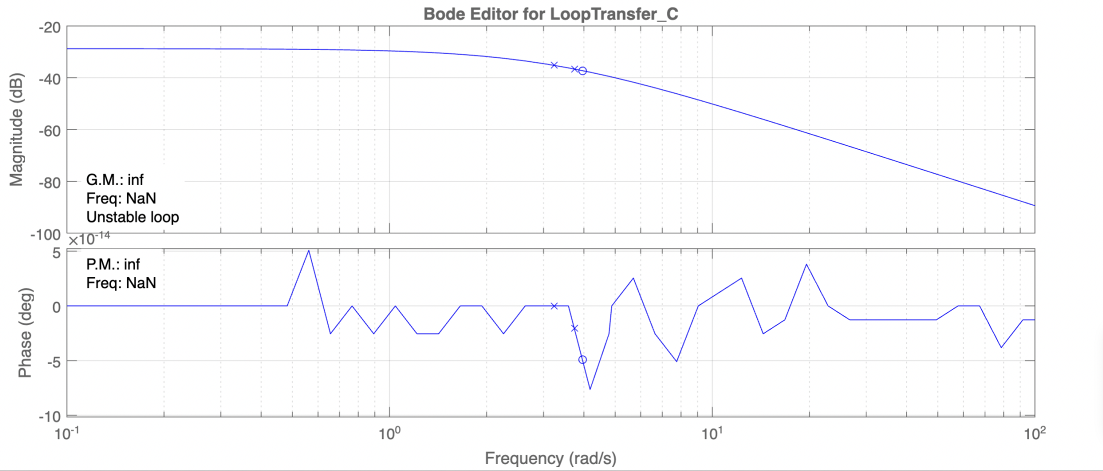


Figure 2: Bode Plot for v=0 m/s

Phase Margin=∞	Gain Margin=∞
Phase Cross over frequency=Not defined	Gain Cross over frequency=Not defined

3.1.11 Root Locus plot

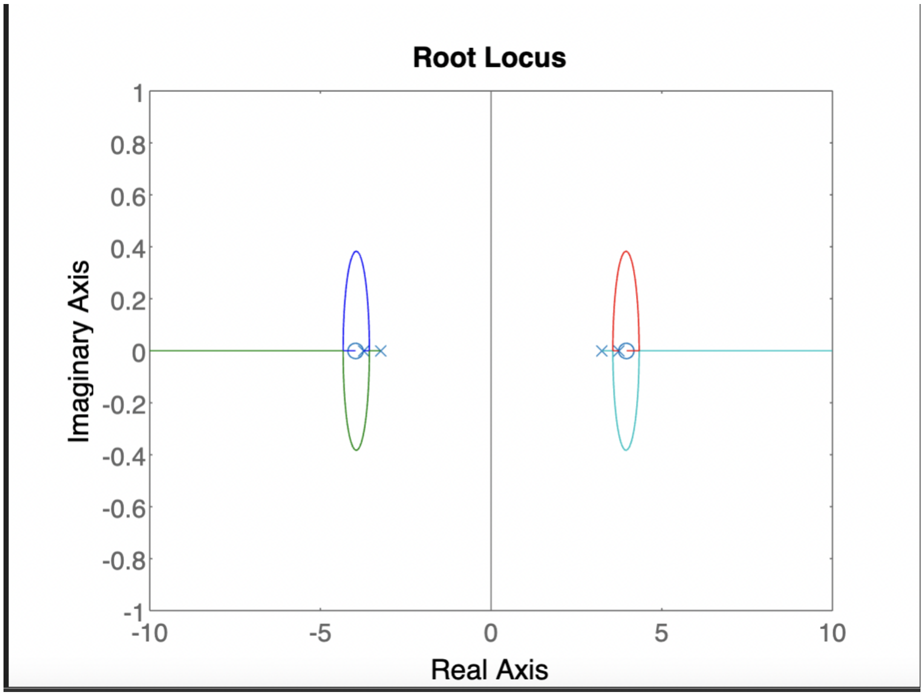


Figure 3: Root locus plot for v=0 m/s

3.2 Case 2: v=3.5 m/s

3.2.1 State Space Model

$$\begin{pmatrix} \dot{\phi} \\ \dot{\delta} \\ \ddot{\phi} \\ \ddot{\delta} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 13.6700 & -15.9328 & -0.5740 & -1.9320 \\ 4.8570 & -2.9712 & 12.6735 & -8.3580 \end{pmatrix} \begin{pmatrix} \phi \\ \delta \\ \dot{\phi} \\ \dot{\delta} \end{pmatrix} \quad + \quad \begin{pmatrix} 0 \\ 0 \\ -0.3390 \\ 7.4570 \end{pmatrix} \mathrm{T}$$

$$\mathrm{y} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi \\ \delta \\ \dot{\phi} \\ \dot{\delta} \end{pmatrix}$$

3.2.2 Transfer Function

$T(s) = \frac{-0.339s^2-17.24s-119.8}{s^4+8.932s^3+18.58s^2+98.76s+36.77}$

3.2.3 Eigenvalues

Eigenvalues in this case are -8.0753 + 0.0000i, -0.2301 + 3.3809i, -0.2301 - 3.3809i, -0.3965 + 0.0000i

3.2.4 Poles

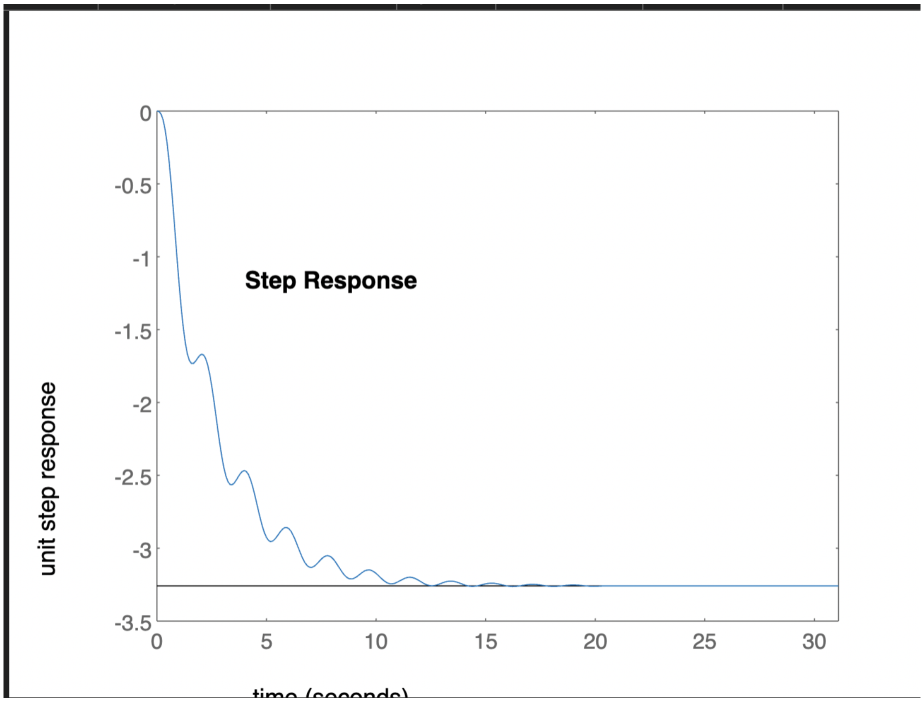
Poles of transfer function are -8.0753 + 0.0000i, -0.2301 + 3.3809i, -0.2301 - 3.3809i, -0.3965 + 0.0000i

3.2.5 Zeroes

Zeroes of transfer function are -42.5497, -8.3066;

3.2.6 Unit Step Response

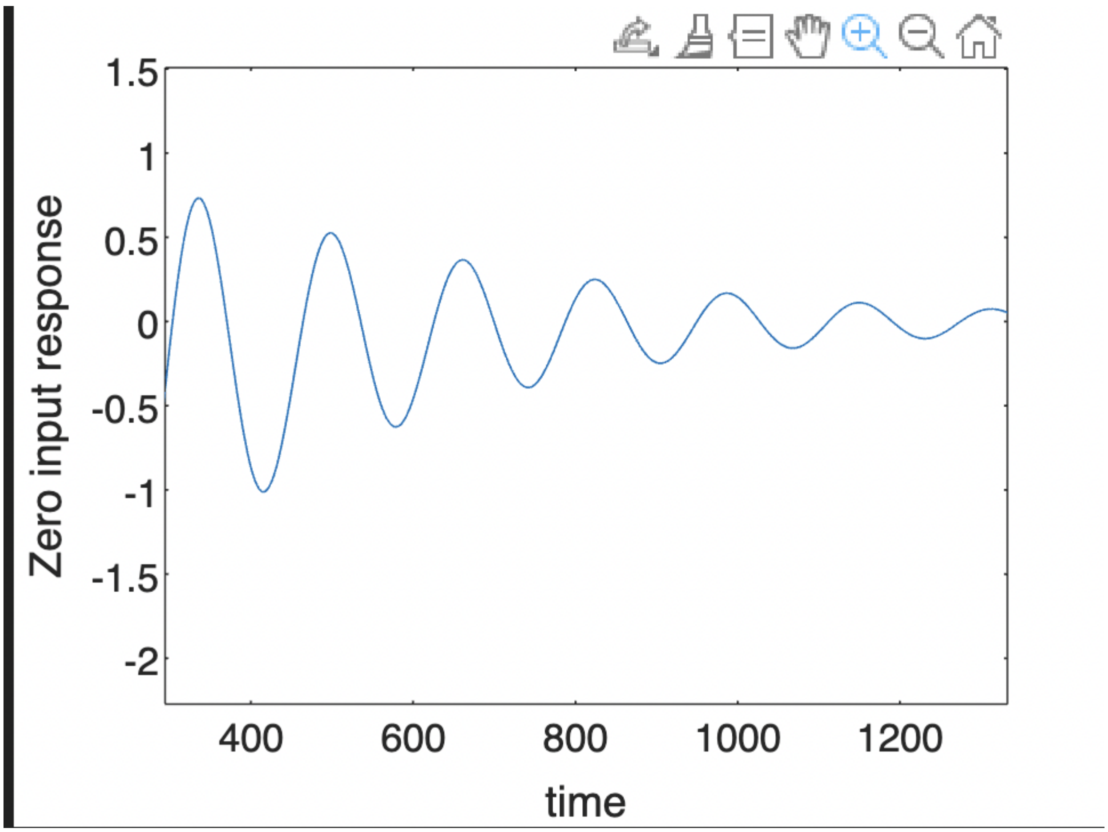
$y(t) = 3.24e^{-0.3965t} - 0.0005974e^{-8.075t} + e^{-0.2301t}cos(3.381t)(0.009598 + 0.1899i) + e^{-0.2301t}cos(3.381t)(0.009598 - 0.1899i) + e^{-0.2301t}sin(3.381t)(0.1899 - 0.009598i) + e^{-0.2301t}sin(3.381t)(0.1899 + 0.009598i) - 3.259$



3.2.7 Zero Input Response

$y(t) = e^{(t*(-0.23-3.4i))} * (1.1 + 0.44i) + e^{(t*(-0.23+3.4i))} * (1.1 - 0.44i) - e^{(-8.1*t)} * (4.0 * 10^{-3} + 2.7 * 10^{-19}i) - e^{(-0.4*t)} * (1.3 - 1.6 * 10^{-16}i) - (5.3 + 4.7 * 10{-}17i)$

Zero input response is shown in the following plot



3.2.8 Stability

1. **Asymtodically Stable** As all eigenvalues are in Left Half of Complex Plane so natural response goes to 0 at $t \to \infty$ so asymptodically stable
2. **BIBO Stable**
   As all poles are in Left Half of Complex Plane so it is BIBO stable as impulse response is absolutely integrable.

3.2.9 Nyquist plot

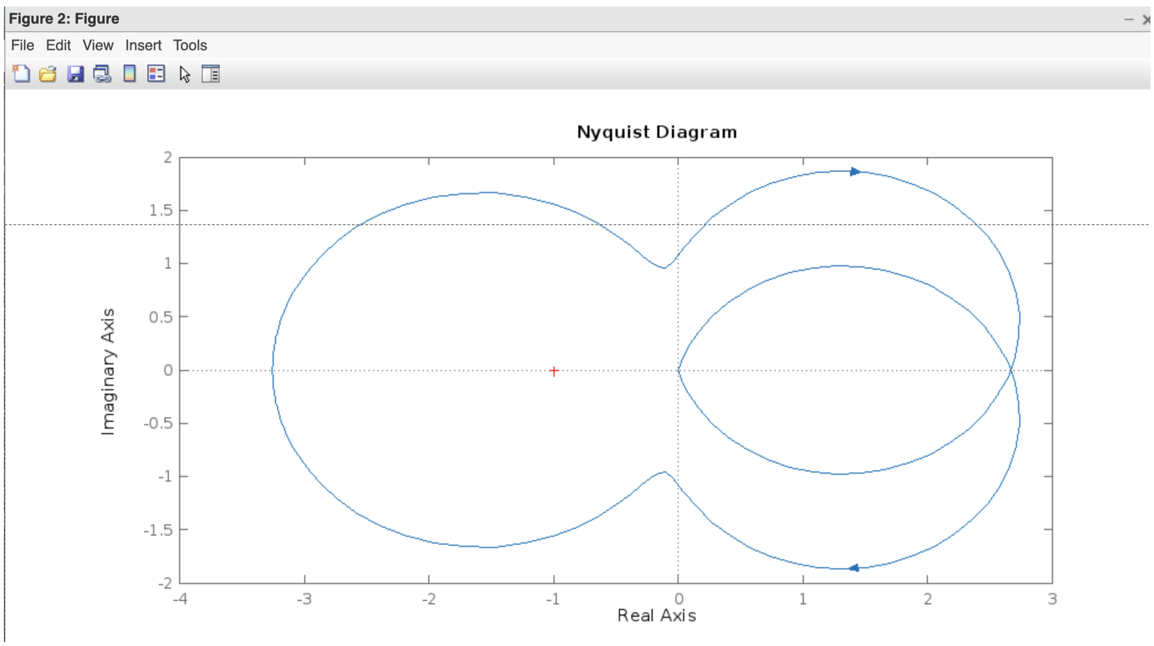


Figure 4: Nyquist Plot for v=3.5 m/s

By Cauchy's criteria we can observe that N = 1 , P = 0 hence Z = N + P = 0 + 1 = 1 . Hence the closed loop system is unstable.

3.2.10 Bode Plot

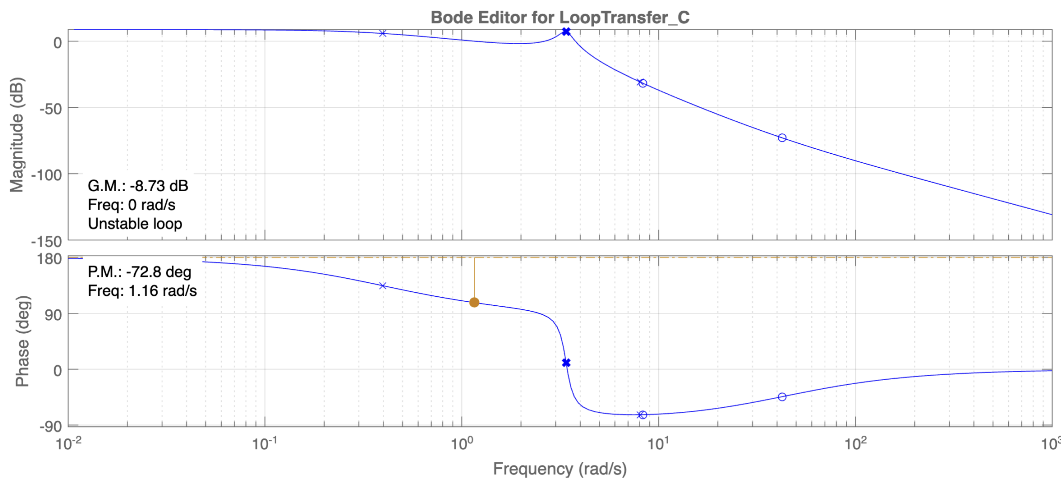


Figure 5: Bode Plot for v=3.5 m/s

Phase Margin=-72.8	Gain Margin=-8.73 dB
Phase Cross over frequency=0 rad/s	Gain Cross over frequency=1.16 rad/s

3.2.11 Root Locus

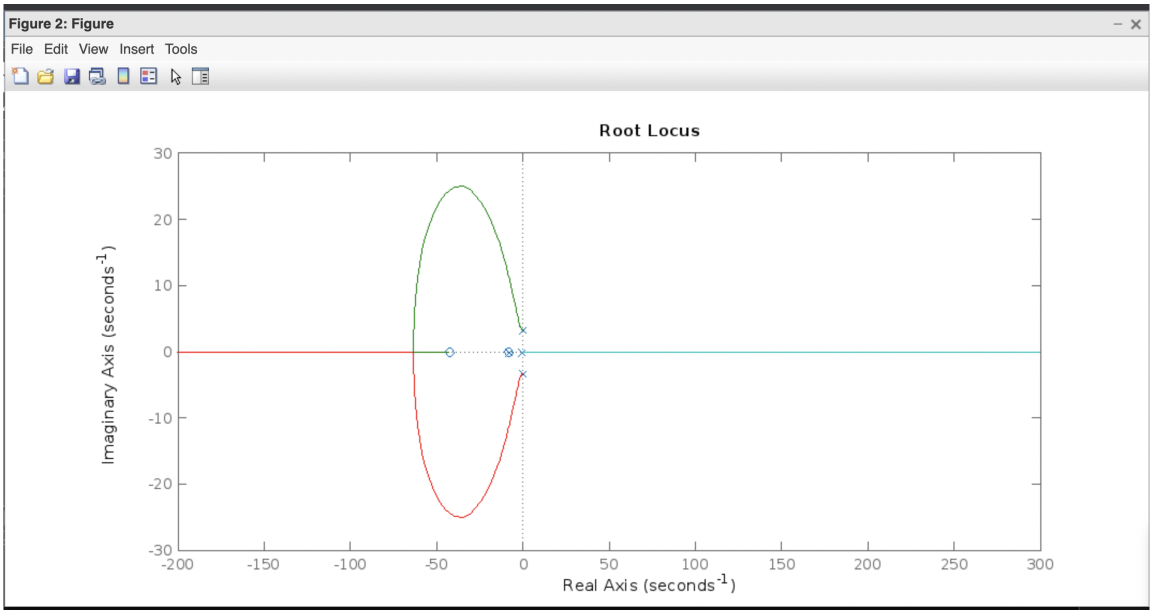


Figure 6: root locus Plot for v=3.5 m/s

3.3 Case 3: v=5 m/s

3.3.1 State Space Model

$$\begin{pmatrix} \dot{\phi} \\ \dot{\delta} \\ \ddot{\phi} \\ \ddot{\delta} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 13.6700 & -32.7500 & -0.8200 & -2.7600 \\ 4.8570 & -17.3150 & 18.1050 & -11.9400 \end{pmatrix} \begin{pmatrix} \phi \\ \delta \\ \dot{\phi} \\ \dot{\delta} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -0.339 \\ 7.457 \end{pmatrix} \mathrm{T}$$

$$\mathrm{y} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi \\ \delta \\ \dot{\phi} \\ \dot{\delta} \end{pmatrix}$$

3.3.2 Transfer Function

$T(s) = \frac{-0.339s^2-24.63s-250.1}{s^4+12.76s^3+63.41s^2+457.3s-77.63}$

3.3.3 Eigenvalues

Eigenvalues in this case are -10.8598 , -1.0330 + 6.4842i, -1.0330 - 6.4842i, 0.1658

3.3.4 Poles

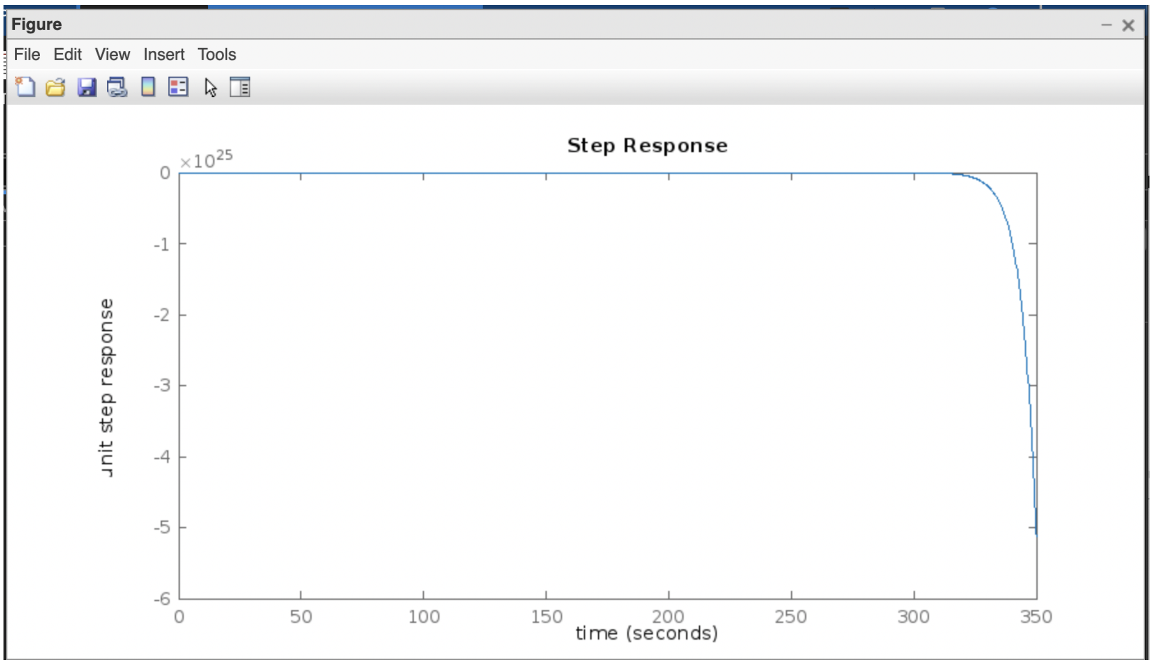
Eigenvalues in this case are -10.8598, -1.0330 + 6.4842i, -1.0330 - 6.4842i, 0.1658

3.3.5 Zeroes

Zeroes of transfer function are -60.4476, -12.2043

3.3.6 Unit Step Response

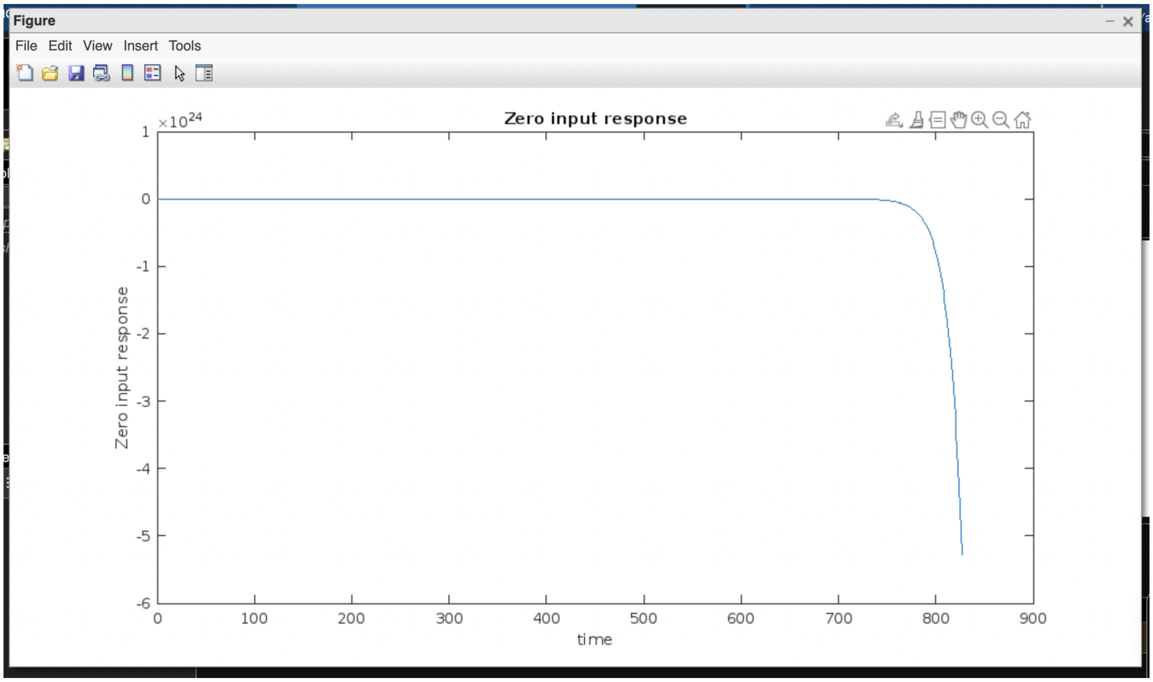
$y(t) = 3.222 - 0.001362e^{-10.86t} - e^{-1.033t}cos(6.484t)(0.0226) + e^{-1.033t}sin(6.484t)(0.07588) - 3.198e^{0.1658t}$



3.3.7 Zero Input Response

$y(t) = 0.538e^{0.166t} - 0.00617e^{-10.9t} + e^{-1.03t}cos(6.48t)(0.468) + e^{-1.03t}sin(6.48t)(0.204)$

Zero input response is shown in the following plot



3.3.8 Stability

1. **Asymptotically Stable** As 1 eigenvalue are in Right Half of Complex Plane so natural response doesn't go to 0 at $t \to \infty$ so asymptotically unstable
2. **BIBO Stable**

   As 1 pole of the transfer function is in Right Half of Complex Plane so it is not BIBO stable as impulse response is not absolutely integrable(so is BIBO unstable.

3.3.9 Nyquist Plot

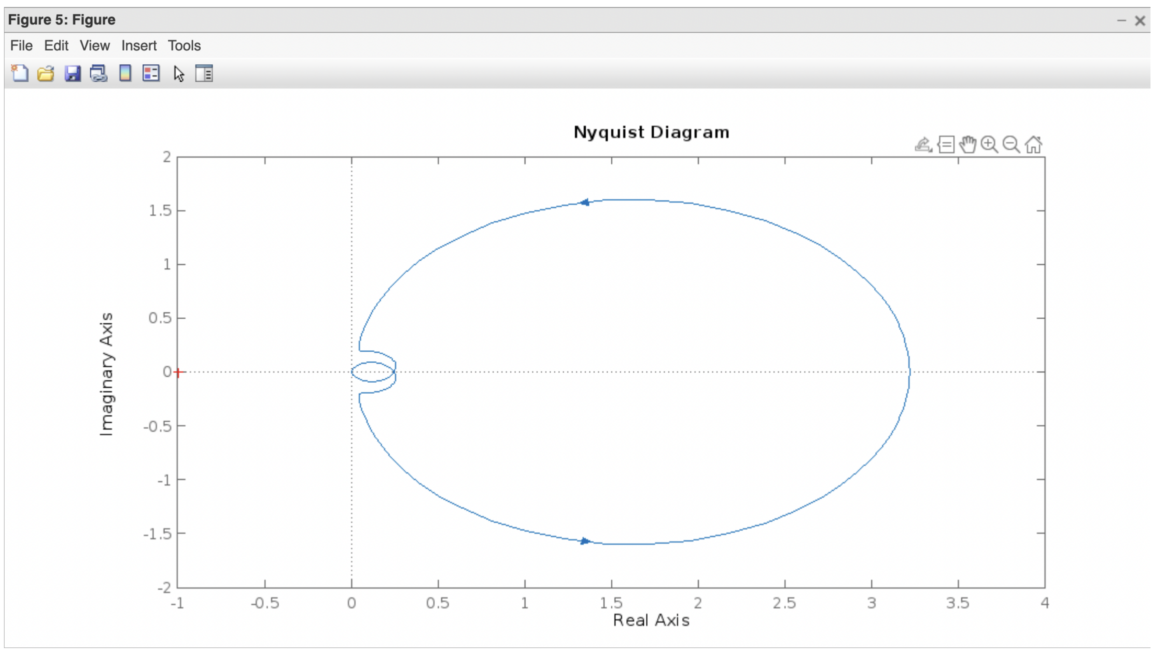


Figure 7: Nyquist Plot for v=5 m/s

By Cauchy's criteria we can observe that N = 0 , P = 1 hence Z = N + P = 0 + 1 = 1 . Hence the closed loop system is unstable.

3.3.10 Bode Plot

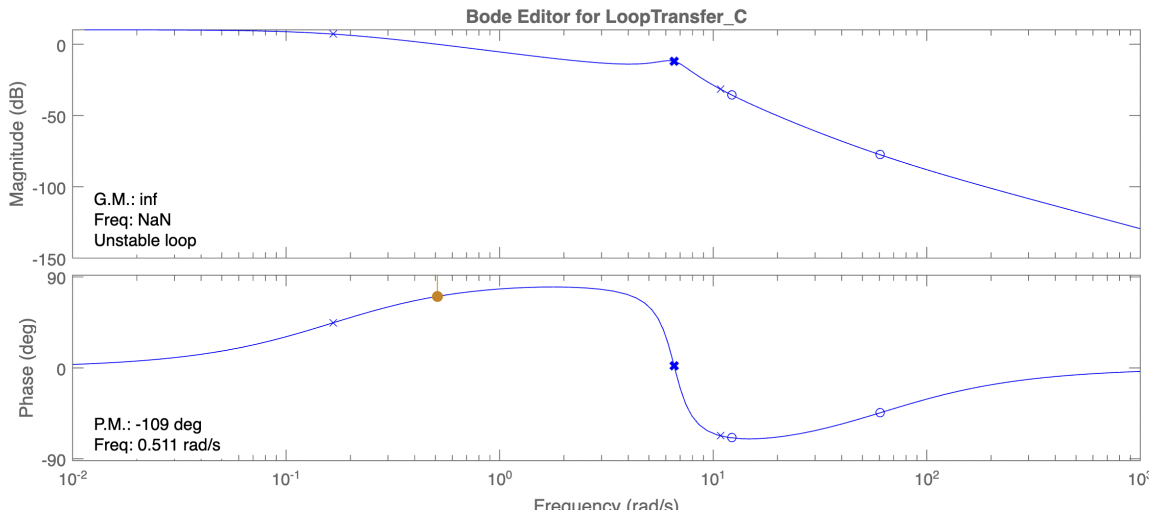


Figure 8: Bode Plot for v=5 m/s

Phase Margin=-109	Gain Margin=∞
Phase Cross over frequency=Not defined	Gain Cross over frequency=0.511 rad/s

3.3.11 Root Locus

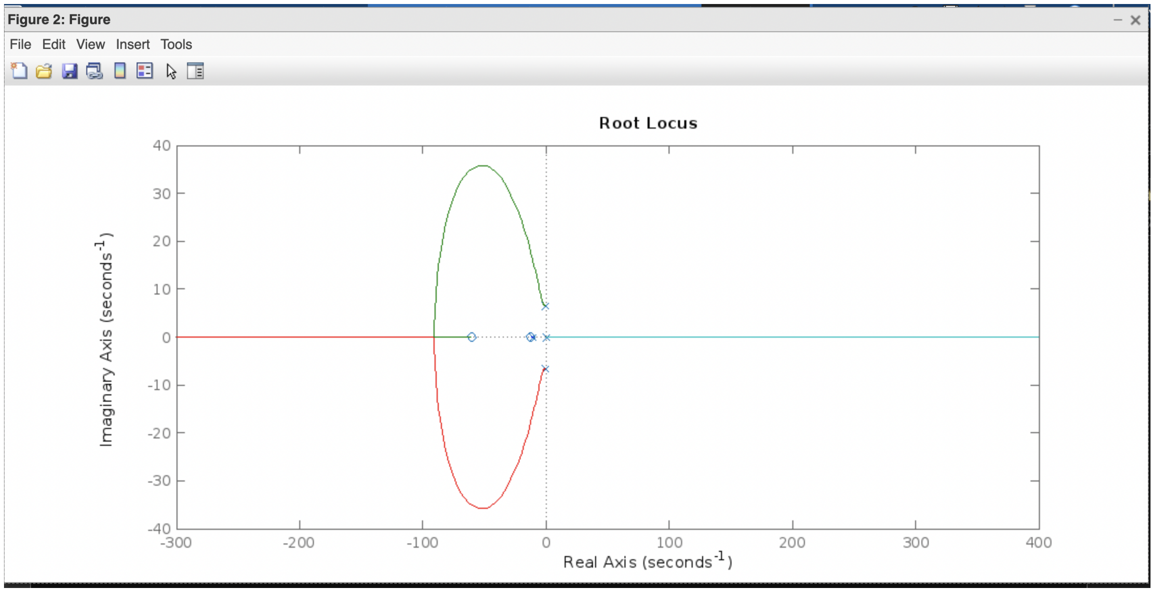


Figure 9: root locus Plot for v=5 m/s

4 Controller Design

4.1 v=0 m/s

For v=0 we tried to develop a PID controller, however if we consider the root locus plot of the compensated system some branch shall always lie in the right half plane leading to the system being unstable. One alternative to this is pole cancellation, i.e adding a zero at the same location so as to nullify it.While this is mathematically feasible but physically there would also be some error in realisation of such a compensator. Hence some branch of the root locus is bound to get trapped in the right half plane which will cause the system to be unstable and therefore such a design is not feasible.

4.2 v=3.5m/s

4.2.1 Design specification

Damping ratio
ζ=0.075

Settling time
$t_s < 15.5$s

4.2.2 Analysis

If we look at the root locus plot, then it can be inferred that for small values of K all the closed loop poles shall lie on the left half plane.Hence, a proportional controller shall suffice in this case. Note that, for a proportional controller once damping ratio is fixed we shall find the point of intersection of the damping ratio line with the root locus. From the location of the dominant pole, we can use its real part to find the settling time. Hence once damping ratio is fixed, settling time takes a unique value unlike in the case of a PID controller where damping ratio and settling time can be chosen independently. For the chosen value of damping ratio and settling time we got the value of the proportional controller K as 0.03721.One conclusion that we noticed was that our expected values of settling time slightly differed from the theoretical value.Since the real part of dominant pole is -0.254, we expect a settling time of 4/0.254 = 15.75. However the settling time upon examining the step response is around 11.6 seconds. This is because we have a pole very close to the dominant poles. Hence the second order approximation, under which we neglect the more negative poles will not work very effectively in this case. This is why we are obtaining slight difference in the value of settling time calculated theoretically using the dominant pole location and the value obtained from the step response. $C(s) = 0.03721$

4.2.3 Design specification

Damping ratio
ζ=0.075

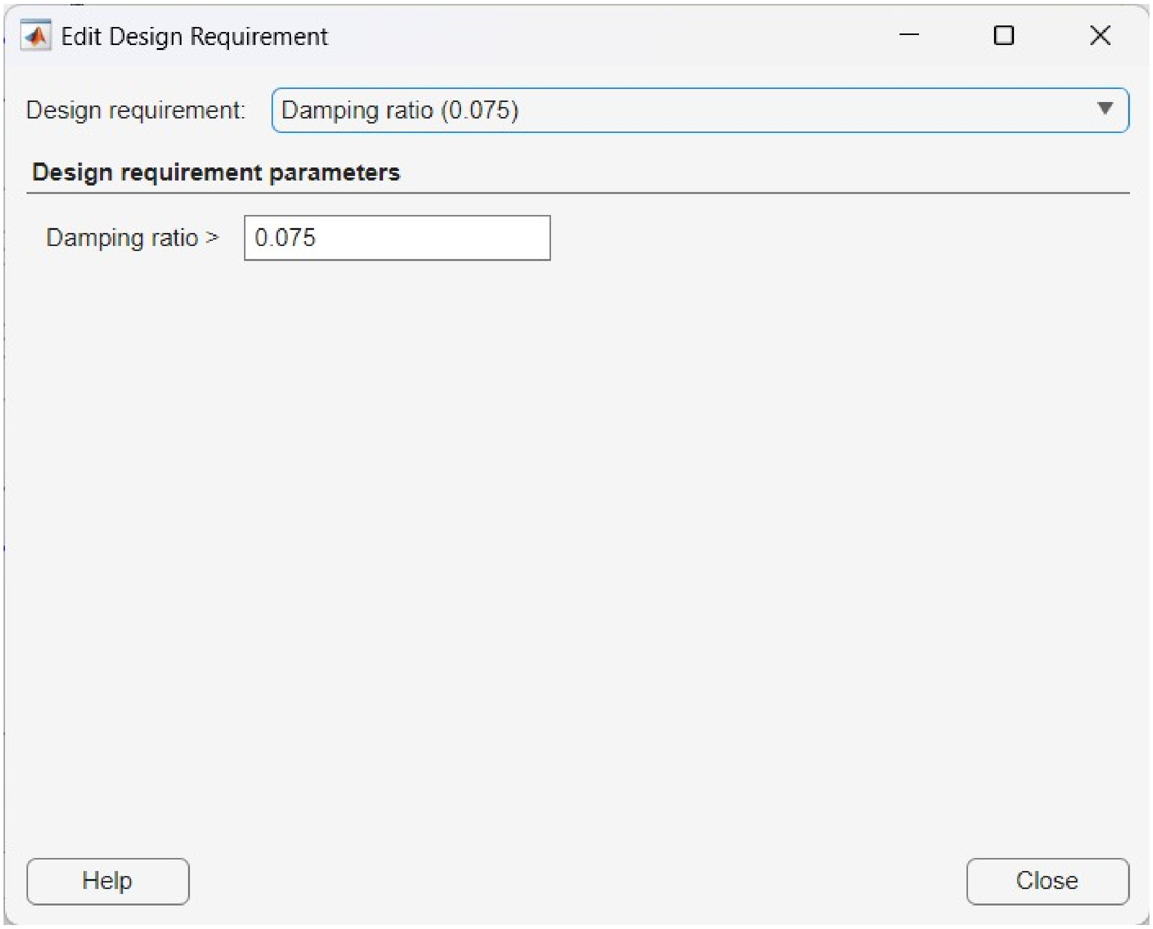


Figure 10: Damping ratio design requirement

Settling time
$t_s < 15.5$s

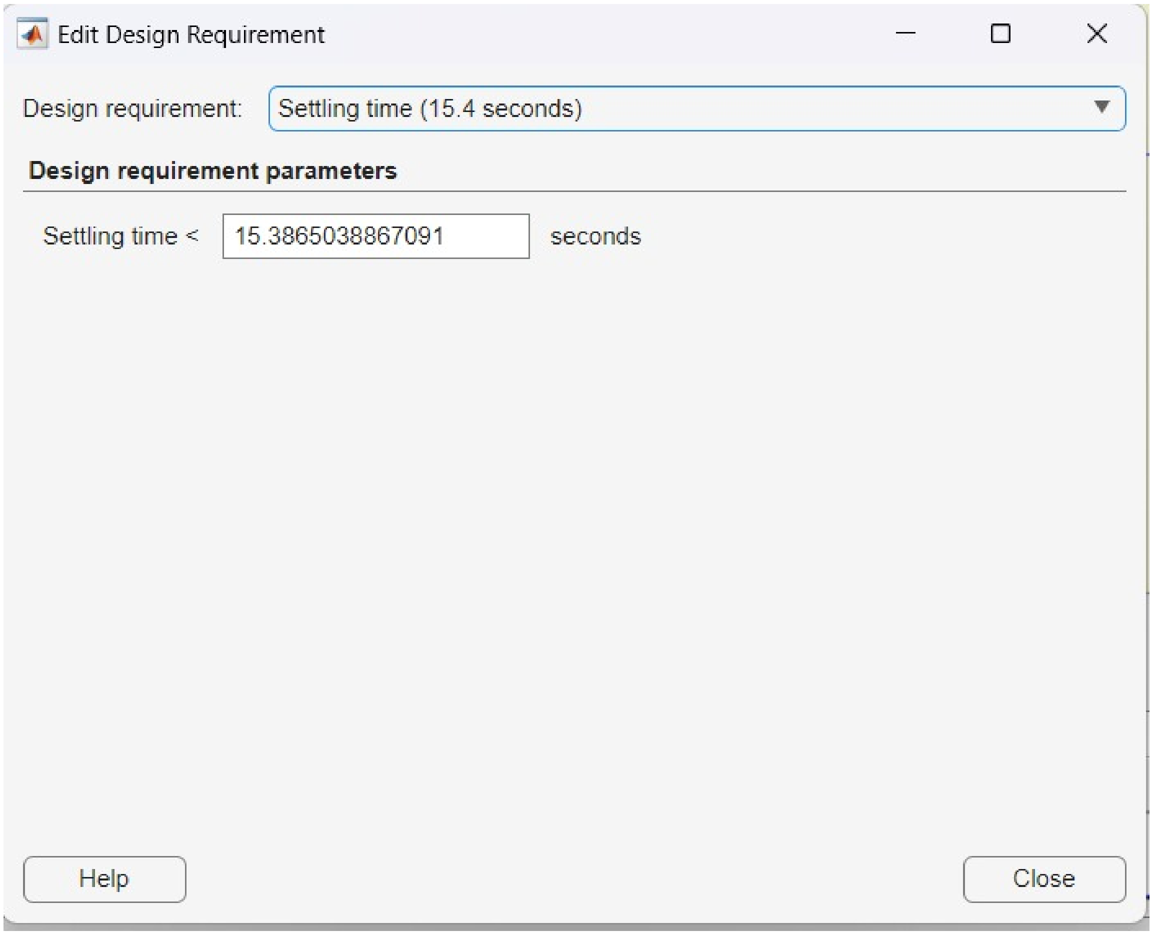
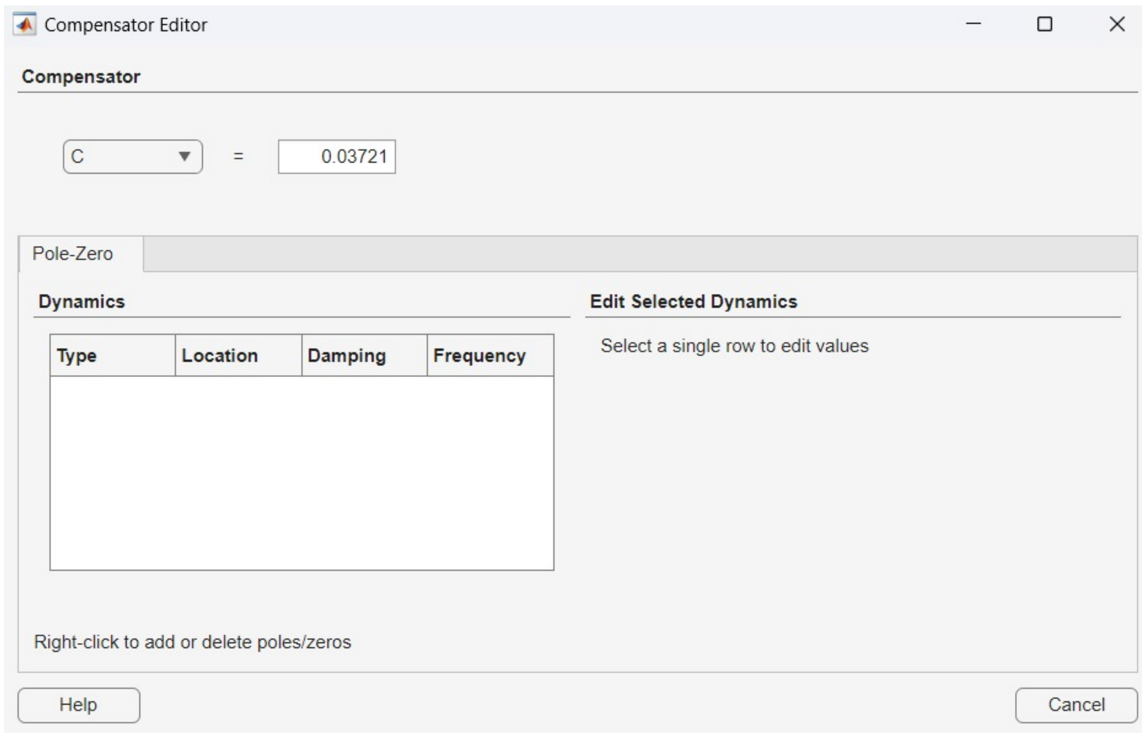
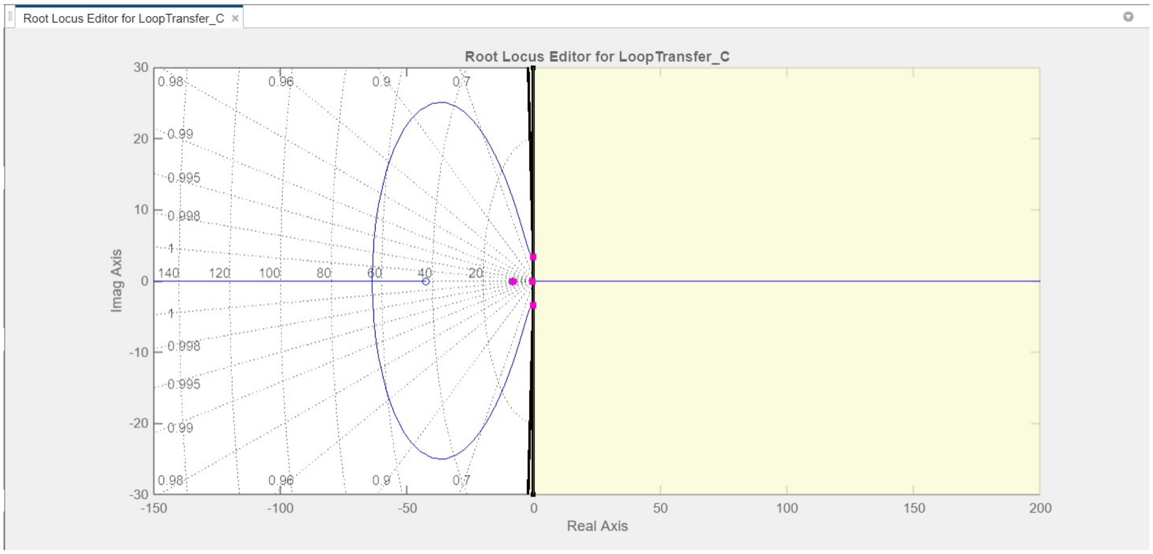


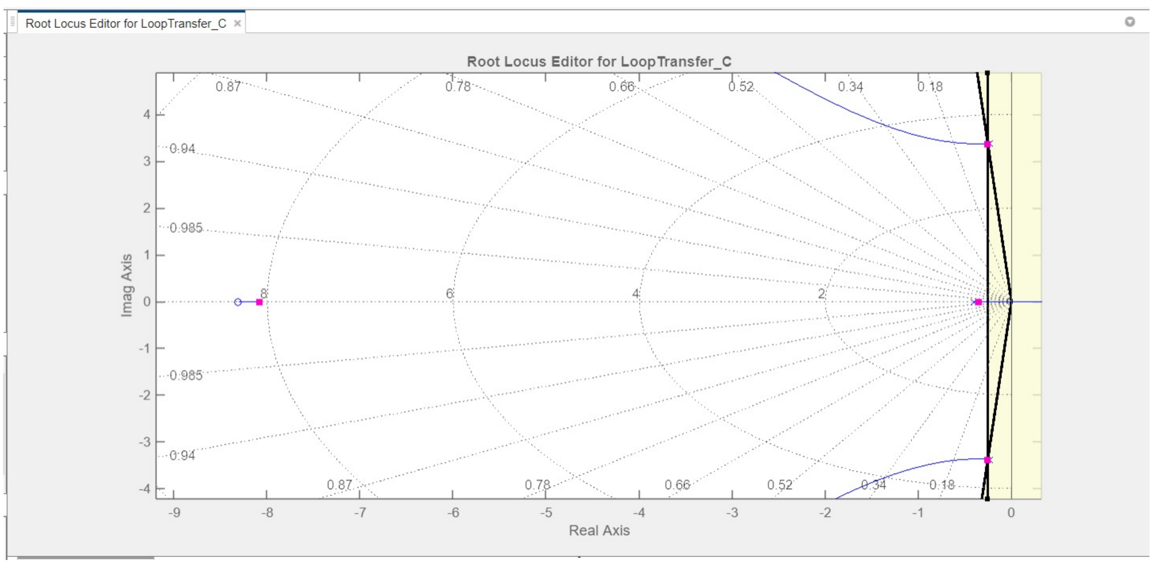
Figure 11: Settling time design requirement

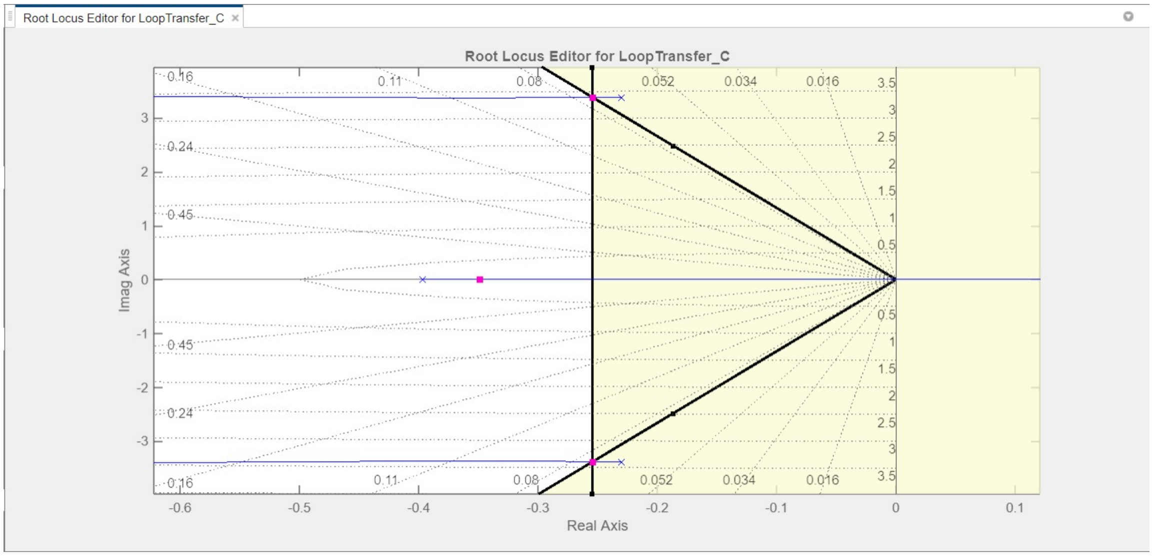
4.2.4 Compensator Design



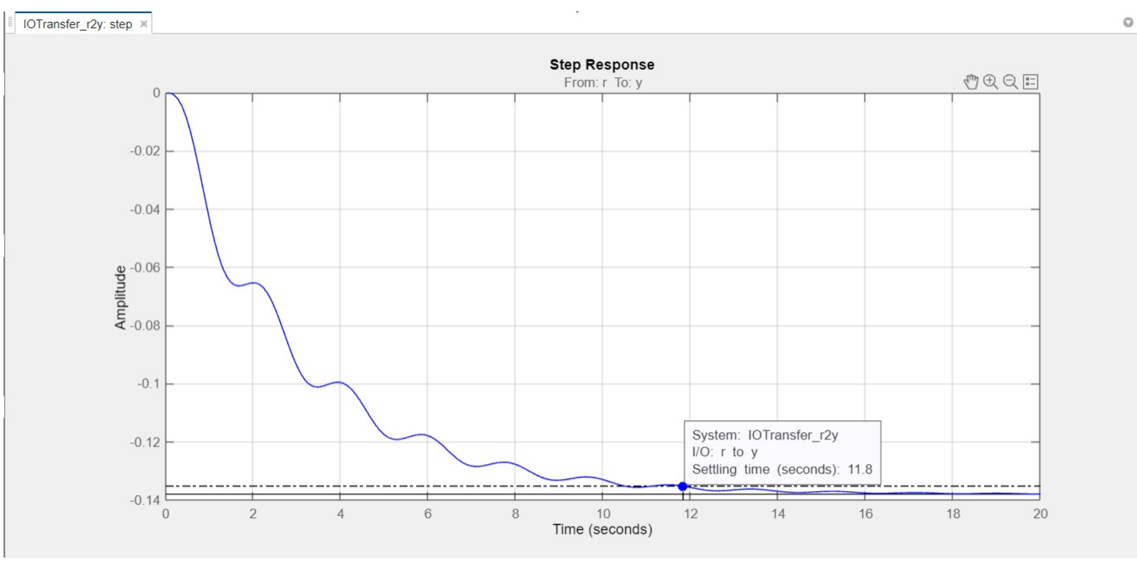
4.2.5 Root Locus plot of Compensated System

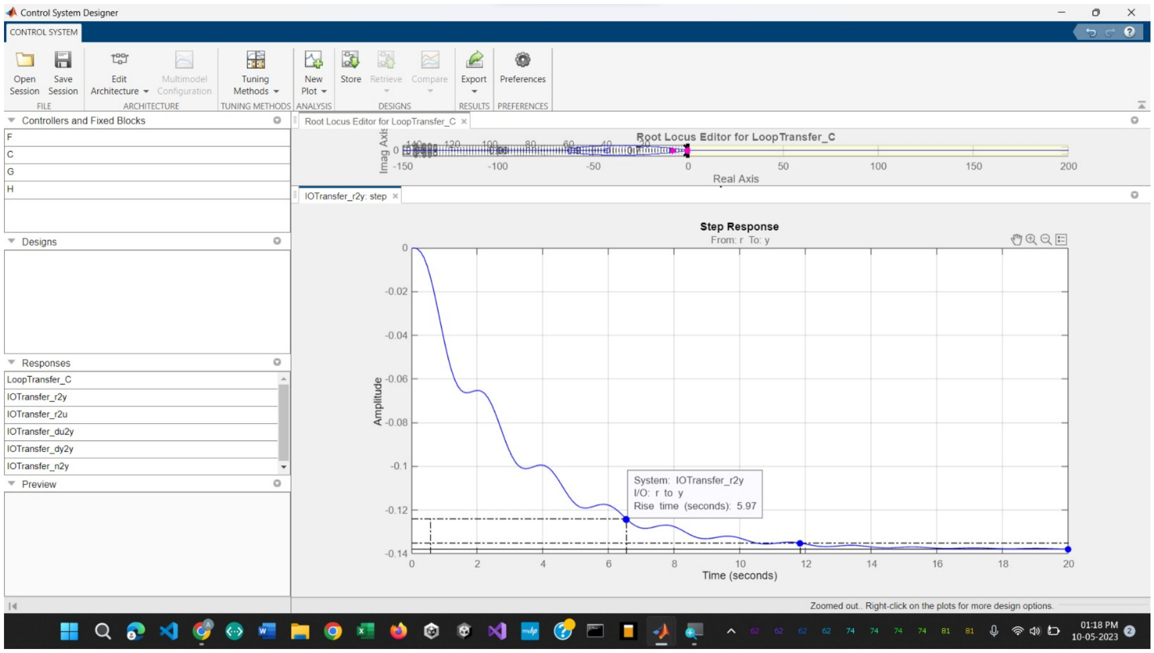


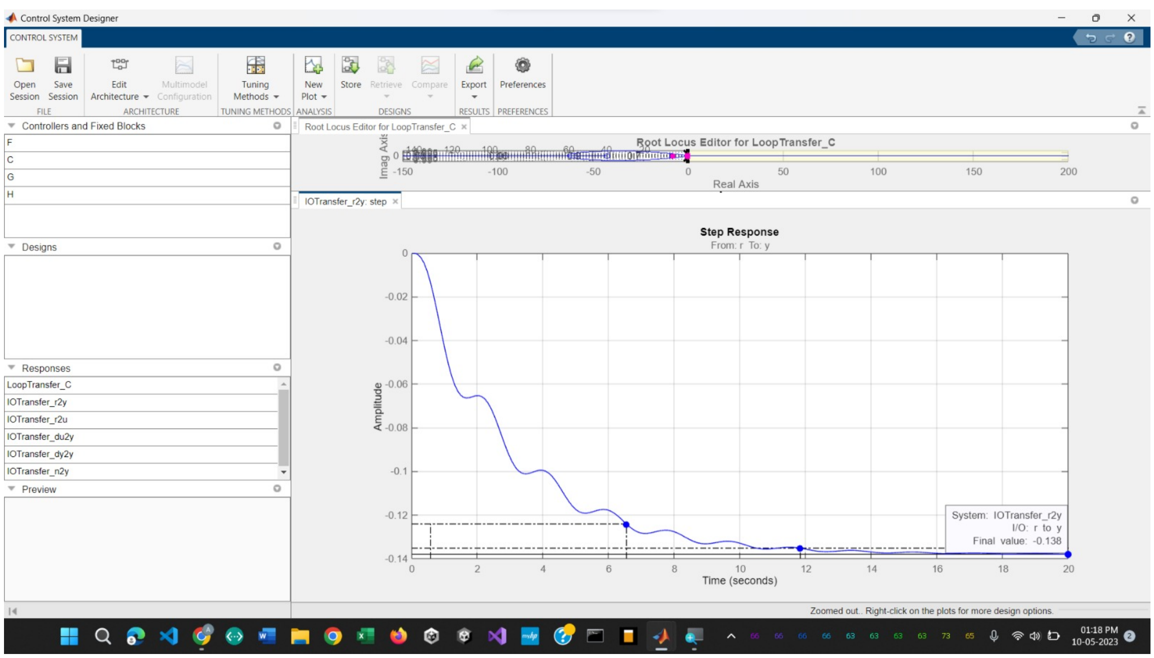




4.2.6 Step response plots of Compensated System




Control System Designer
Step Response
From: r To: y
Amplitude
Time (seconds)
System: IOTransfer_r2y
I/O: r to y
Rise time (seconds): 5.97
Root Locus Editor for LoopTransfer_C
Real Axis
Zoomed out.. Right-click on the plots for more design options.


Control System Designer
Step Response
From: r To: y
Amplitude
Time (seconds)
System: IOTransfer_r2y
I/O: r to y
Final value: -0.138
Root Locus Editor for LoopTransfer_C
Real Axis
Zoomed out.. Right-click on the plots for more design options.

4.3 v=5m/s

4.3.1 Design specification

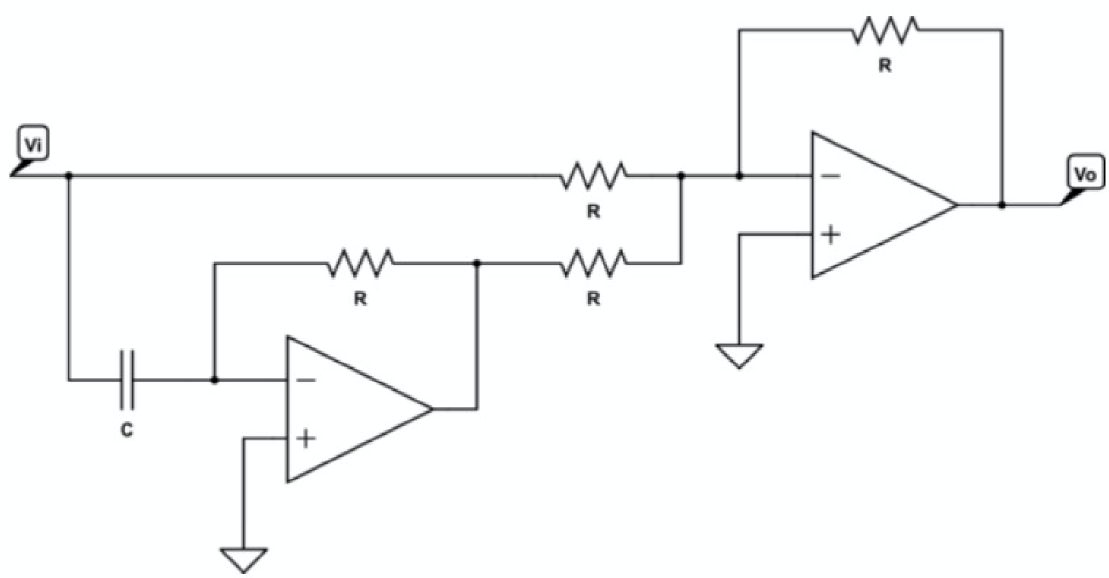
Damping ratio
ζ=0.045

Settling time
$t_s < 27$s

4.3.2 Analysis

Since a branch of the root locus lies completely in the right half plane, it is impossible to get a stable closed loop system by using only a proportional controller. Hence we shall be using a PID controller. First, we made the PD compensator for which we found the suitable value of zero that is to be added to get the suitable damping ratio and settling time. It turns out to be in the right half plane (approximately 0.9). A possible design scheme in order to realise such a compensator using Op amps is shown below. After doing so we design a PI compensator with a pole at origin and zero close to the origin.

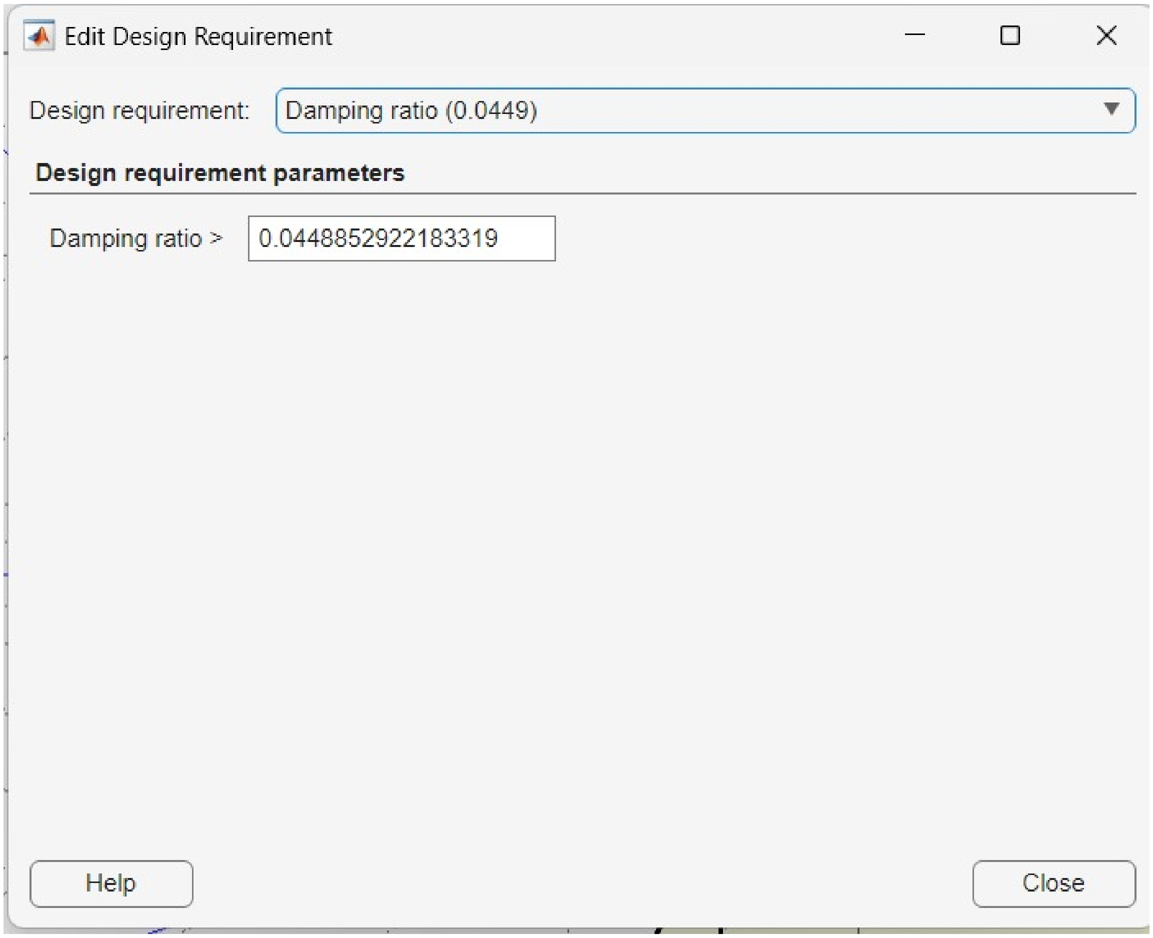
Hence $C(s) = \frac{(1-1.1s)(1+5s)}{s}$
The below circuit solves to $\frac{V0(s)}{V1(s)}$ = -(1-sRC) hence maybe used for synthesising compensator with pole in RHP.



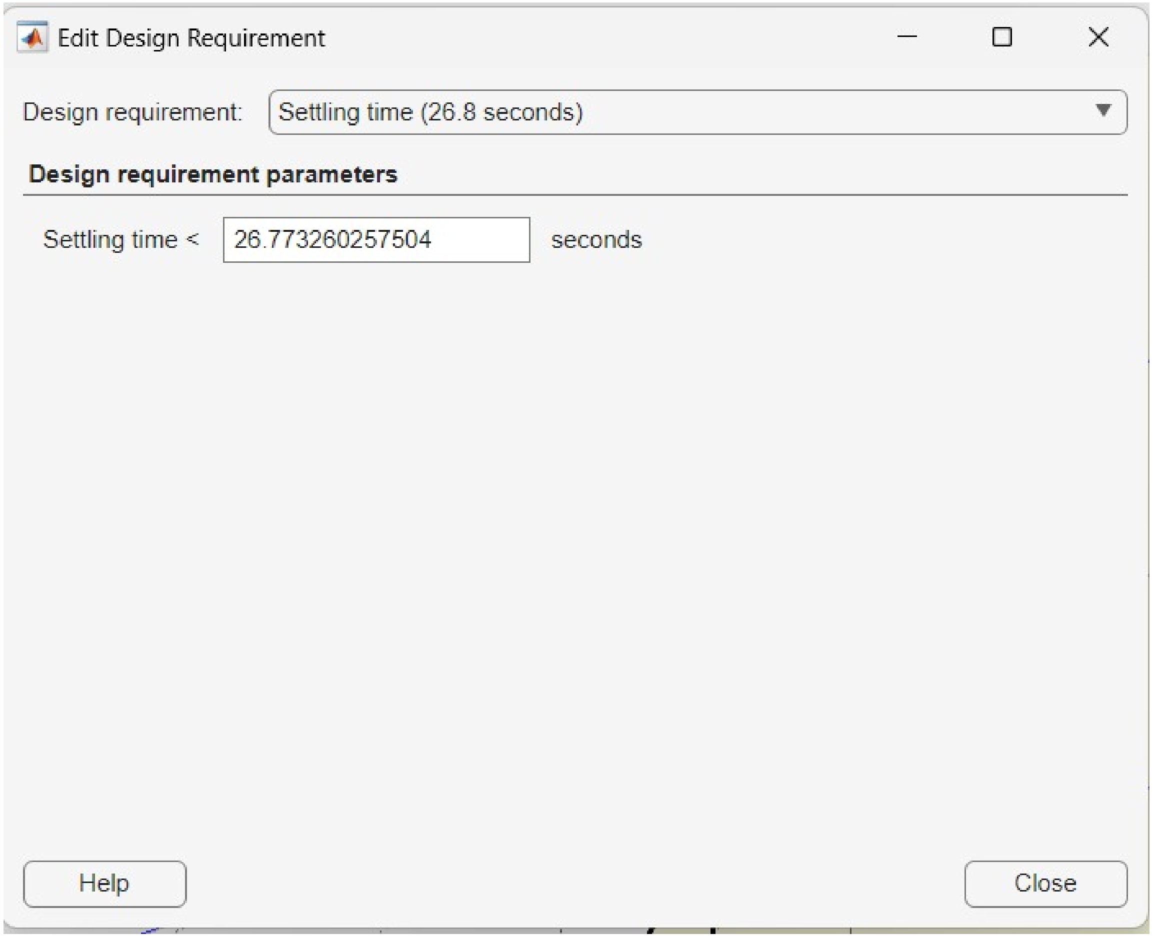
4.3.3 Design specification

Damping Ratio

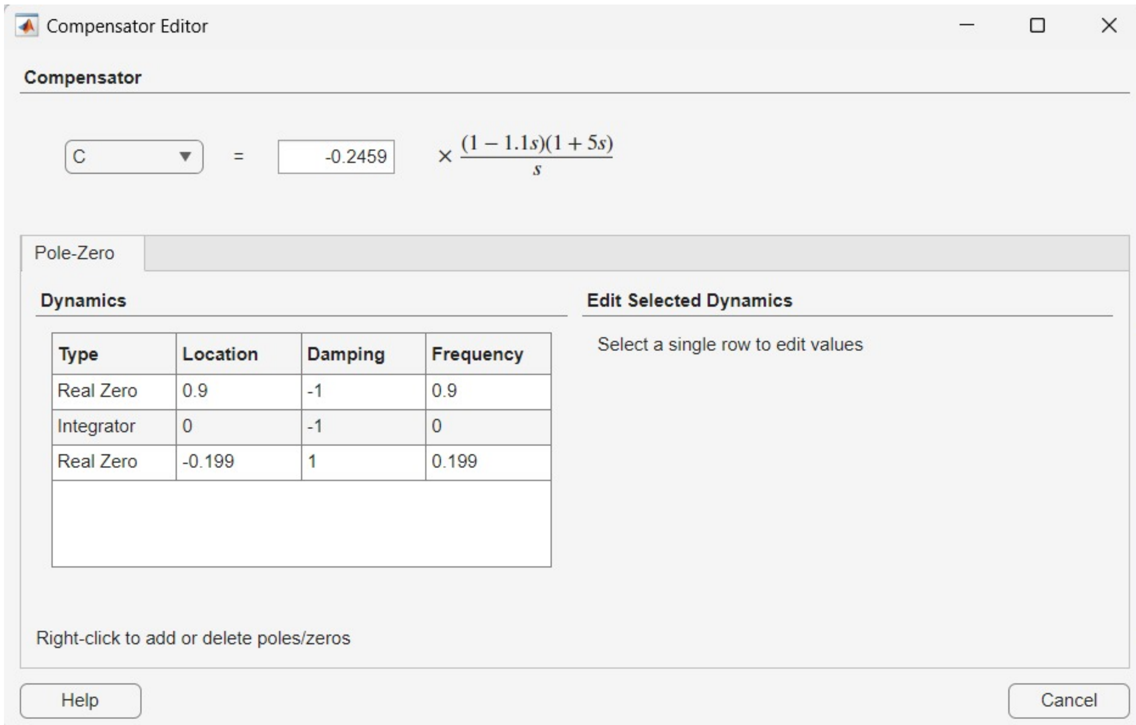
ζ=0.045



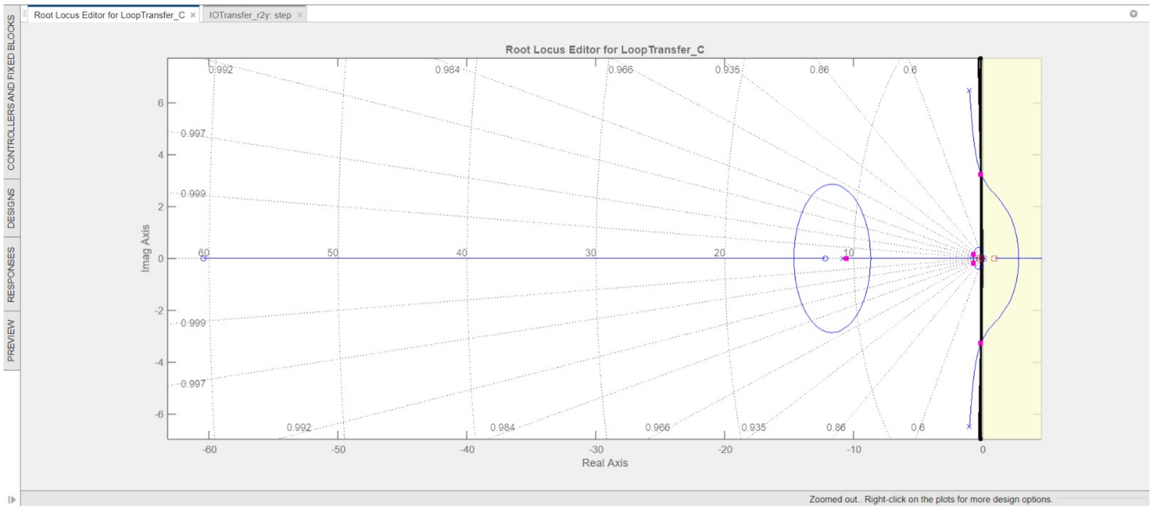
Settling Time<27 seconds

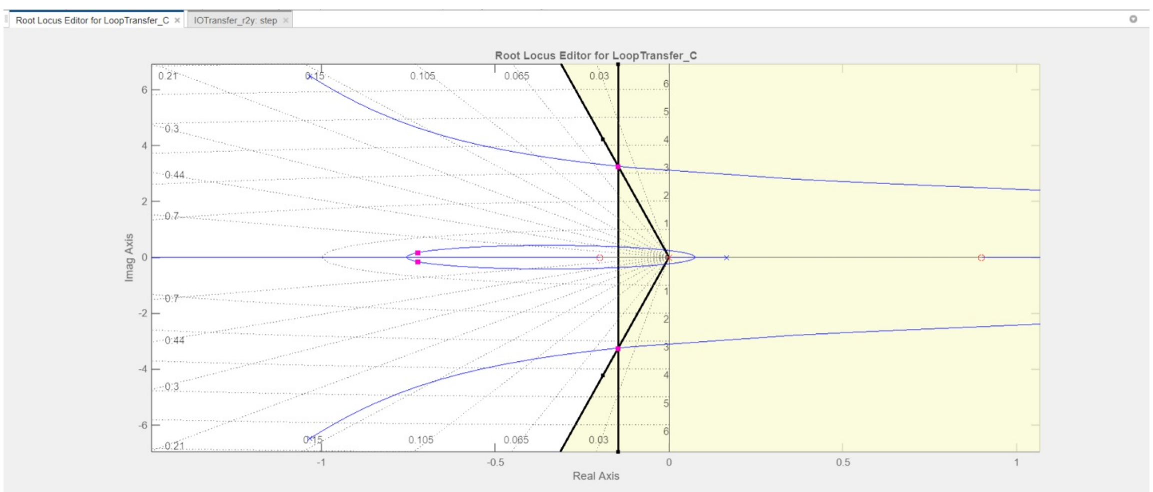


4.3.4 Compensator Design

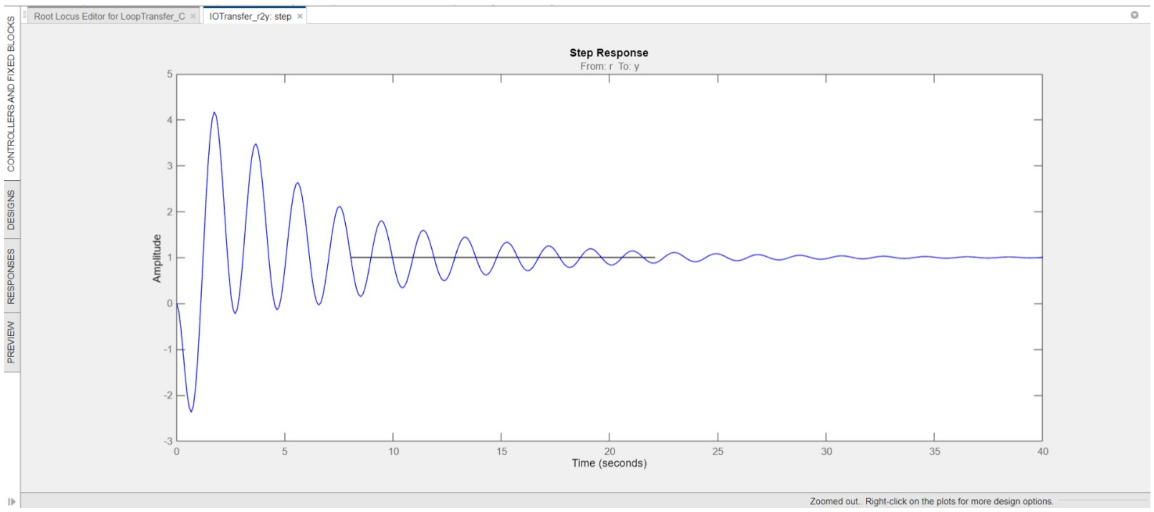


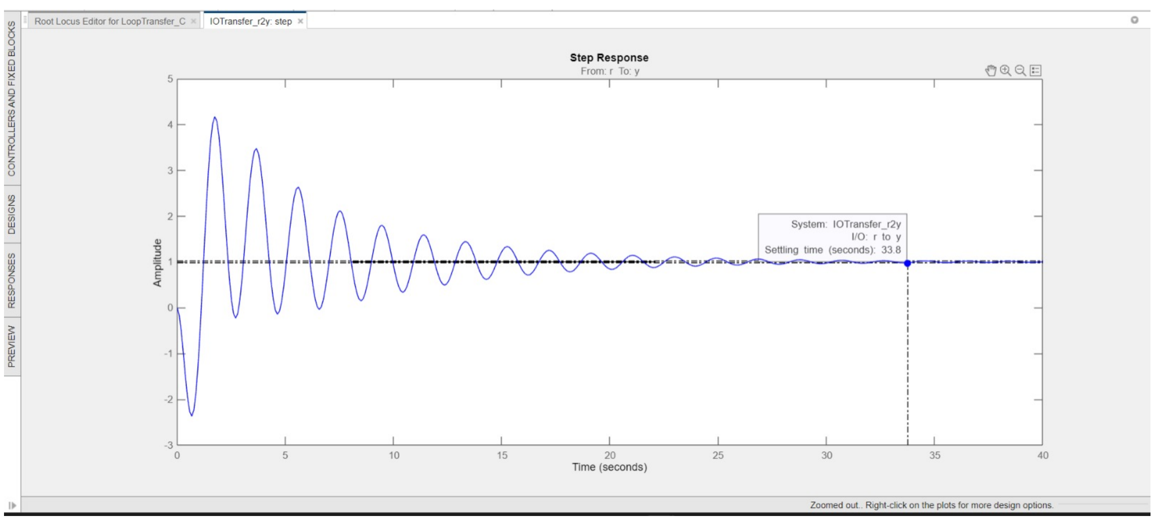
4.3.5 Root Locus plot of Compensated System

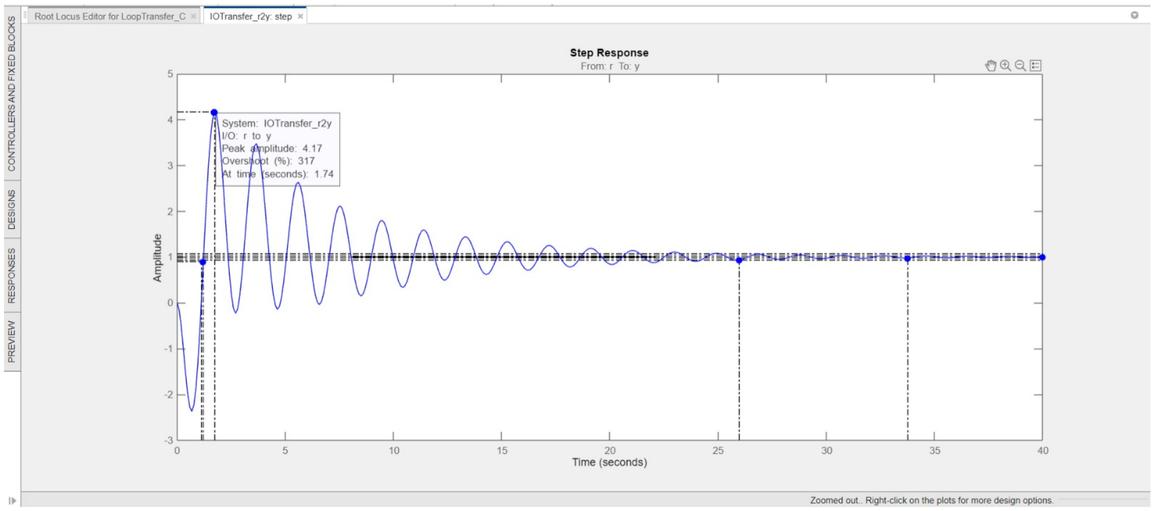


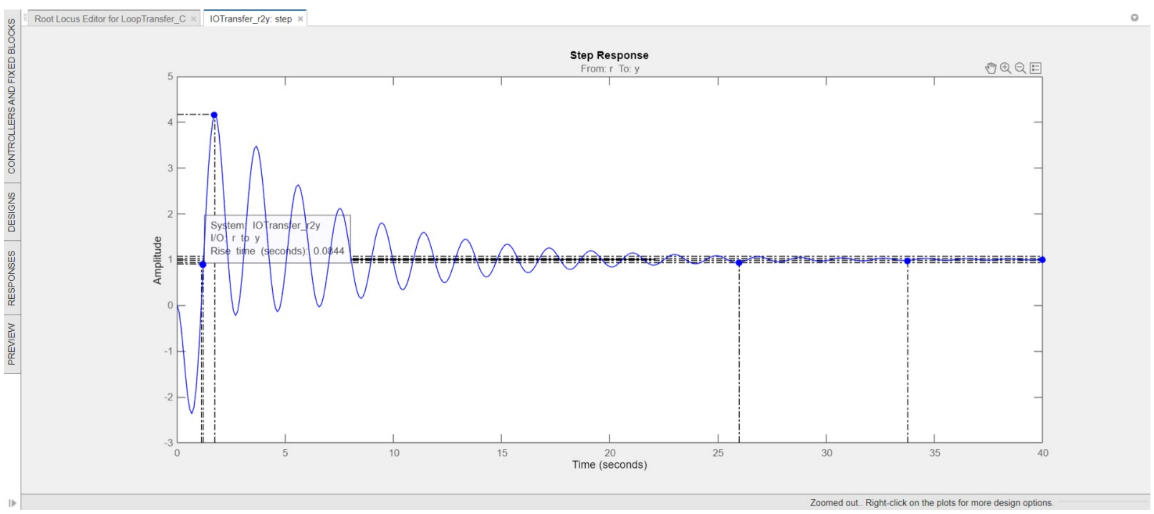


4.3.6 Step response plots of Compensated System

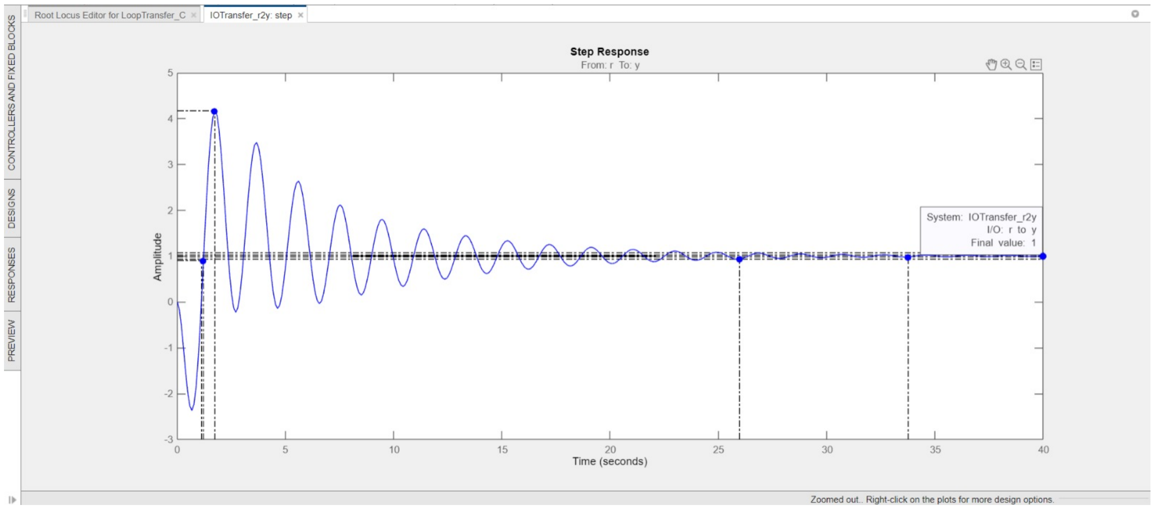








Steady State error is now made 0 using PI controller and shown in below image.



5 Conclusion

- State space representation of Carvallo Whipple bicycle model was obtained.
- The transfer functions, along with poles,zeroes and eigen values were obtained for $v_1 = 0$ m/s, $v_2 = 1$m/s and $v_3 = 6$ m/s.
- Computed time response for zero input and unit step input with the assumed initial state.
- Asymptomatic stability and BIBO stability of the system for the three velocities was discussed .
- Nyquist plots, Bode plots and Root Locus plots were drawn. The Root Locus depicts the locus of closed loop poles when we vary gain,while Bode plots and Nyquist plots are for open loop transfer function and with it we can predict the closed loop system stability.
- **sisotool** in Matlab was used to design the controllers, taking unity negative feedback. The controller for $v_1 = 0$ m/s could not be made, except using pole zero cancellation. The controllers for other two velocities were designed and implemented.