

计算方法 A 第 7 章习题答案

习题 7.2 方程 $x^3 - x^2 - 1 = 0$ 在 $x_0 = 1.5$ 附近有根, 将方程作三种同解变形, 可得三种迭代格式

$$(1) \quad x = 1 + \frac{1}{x^2}, \quad x_{k+1} = 1 + \frac{1}{x_k^2}$$

$$(2) \quad x^3 = 1 + x^2, \quad x_{k+1} = \sqrt[3]{1 + x_k^2}$$

$$(3) \quad x = \frac{1}{x-1}, \quad x_{k+1} = \frac{1}{\sqrt{x_k-1}}$$

判断各迭代格式在 $x_0 = 1.5$ 附近的收敛性, 选一种收敛最快的迭代格式, 计算在 $x_0 = 1.5$ 附近的根, 准确到 4 位小数.

解 取 $x_0 = 1.5$ 的邻域 $[1.4, 1.6]$ 作分析.

$$(1) \quad \varphi(x) = 1 + \frac{1}{x^2}, \quad |\varphi'(x)| = \left| -\frac{2}{x^3} \right| \leq \frac{2}{1.4^3} = L_1 = 0.7289 < 1, \text{ 所以迭代收敛.}$$

$$(2) \quad \varphi(x) = \sqrt[3]{1 + x^2}, \quad |\varphi'(x)| = \frac{2x}{3\sqrt[3]{(1+x^2)^2}} < \frac{2 \times 1.6}{3\sqrt[3]{(1+1.6^2)^2}} = L_2 = 0.5174 < 1, \text{ 所以迭代收敛.}$$

$$(3) \quad \varphi(x) = \frac{1}{\sqrt{x-1}}, \quad |\varphi'(x)| = \frac{1}{2\sqrt{(x-1)^3}} > \frac{1}{2 \times 0.6^{\frac{3}{2}}} = L_3 = 1.0758 > 1, \text{ 所以迭代发散.}$$

由于 $L_2 < L_1 < L_3$, 故迭代格式 (2) 收敛速度最快, 选取迭代格式 (2) 进行迭代. 要使结果具有 4 位小数, 只需要

$$|x^* - x_k| \leq \frac{L_2}{1 - L_2} |x_{k+1} - x_k| < \frac{1}{2} \times 10^{-4}$$

即 $|x_k - x_{k-1}| < \frac{1-L_2}{L_2} \times \frac{1}{2} \times 10^{-4} < \frac{1}{2} \times 10^{-4}$. 取 $x_0 = 1.5$, 计算结果如下:

$$x_1 = 1.48125, \quad x_2 = 1.47272, \quad x_3 = 1.46882, \quad x_4 = 1.46705, \quad x_5 = 1.46624,$$

$$x_6 = 1.46588, \quad x_7 = 1.46571, \quad x_8 = 1.46563, \quad x_9 = 1.46560.$$

由于 $|x_9 - x_8| < \frac{1}{2} \times 10^{-4}$, 取 $x^* \approx x_9$.

习题 7.3 设在区间 $[a, b]$ 上 $0 < m \leq f'(x) \leq M$, 证明对任意 $\lambda \in (0, \frac{2}{M})$, 取 $x_0 \in [a, b]$ 时, 由迭代格式

$$x_{k+1} = x_k - \lambda f(x_k), \quad k = 0, 1, 2, \dots$$

产生的迭代序列 x_k 收敛于方程 $f(x) = 0$ 的根 x^* .

证明 设 $\varphi(x) = x - \lambda f(x)$, 可知 $\varphi(x)$ 在 $(-\infty, +\infty)$ 上可导. 对于任意给定的 λ 值, 满足条件 $0 < m \leq f'(x) \leq M$ 时,

$$\varphi'(x) = 1 - \lambda f'(x) \implies 1 - \lambda M \leq \varphi'(x) \leq 1 - \lambda m < 1$$

又 $0 < \lambda < \frac{2}{M}$, 则 $0 < \lambda M < 2$ 时, $-1 < 1 - \lambda M < 1$, 所以 $-1 < 1 - \lambda M \leq \varphi'(x) \leq 1 - \lambda m < 1$. 若令 $L = \max\{|1 - \lambda M|, |1 - \lambda m|\}$, 则可知 $|\varphi'(x)| \leq L < 1$.

$$|x_k - x^*| \leq L|x_{k-1} - x^*| \leq \dots \leq L^k|x_0 - x^*| \rightarrow 0$$

即 $\lim_{k \rightarrow \infty} x_k = x^*$.

习题 7.8 怎样选取函数 $h(x)$, 使得当

$$\varphi(x) = x - \frac{f(x)}{f'(x)} + h(x) \left[\frac{f(x)}{f'(x)} \right]^2$$

时, 简单迭代格式至少三阶收敛.

解 令 $p(x) = -\frac{1}{f'(x)}$, $q(x) = \frac{h(x)}{f'^2(x)}$, 则

$$\varphi(x) = x + p(x)f(x) + q(x)f^2(x).$$

注意到 $f(x^*) = 0$, 因此 $\varphi(x^*) = x^*$, 即不动点存在. 若要简单迭代格式 $x_{k+1} = \varphi(x_k)$ 至少三阶收敛到 $f(x) = 0$ 的根 x^* . 根据收敛阶定理, 应有

$$\varphi'(x^*) = 0, \quad \varphi''(x^*) = 0.$$

于是

$$\begin{aligned}\varphi'(x^*) &= 1 + p(x^*)f'(x^*) + p'(x^*)f(x^*) + q'(x^*)f^2(x^*) + 2q(x^*)f(x^*)f'(x^*) \\ &= 1 + p(x^*)f'(x^*) = 1 - \frac{f'(x^*)}{f'(x^*)} = 1 - 1 = 0, \\ \varphi''(x^*) &= p''(x^*)f(x^*) + 2p'(x^*)f'(x^*) + p(x^*)f''(x^*) + q''(x^*)f^2(x^*) \\ &\quad + 4f(x^*)f'(x^*)q'(x^*) + 2q(x^*)(f(x^*)f''(x^*) + f'^2(x^*)) \\ &= 2p'(x^*)f'(x^*) + p(x^*)f''(x^*) + 2q(x^*)f'^2(x^*) = 0,\end{aligned}$$

即 $p(x), q(x)$ 满足关系

$$2p'(x^*)f'(x^*) + p(x^*)f''(x^*) + 2q(x^*)f'^2(x^*) = 0.$$

由于

$$p(x^*) = -\frac{1}{f'(x^*)}, \quad p'(x^*) = \frac{f''(x^*)}{f'^2(x^*)}, \quad q(x^*) = \frac{h(x^*)}{f'^2(x^*)},$$

代入得到

$$2\frac{f''(x^*)}{f'(x^*)} - \frac{f''(x^*)}{f'(x^*)} + 2h(x^*) = 0 \implies h(x^*) = -\frac{f''(x^*)}{2f'(x^*)}.$$

由 $f(x)$ 及 x^* 的任意性, 可知当 $h(x) = -\frac{1}{2} \frac{f''(x)}{f'(x)}$ 时, 上述简单迭代法至少三阶收敛.

习题 7.9 用松弛法和斯特森加速技术求解习题 7.2.

解 以习题 7.2 中的迭代格式 (1) 为例考虑加速技术. 松弛法:

$$\varphi(x) = 1 + \frac{1}{x^2}, \quad \varphi'(x) = -\frac{2}{x^3}, \quad \varphi'(x_0) = \varphi'(1.5) = -\frac{2}{1.5^3} = -\frac{16}{27}$$

迭代格式为

$$x_{k+1} = \frac{1 + \frac{1}{x_k^2} + \frac{16}{27}x_k}{1 + \frac{16}{27}}$$

计算如下:

$$x_1 = 1.46512, \quad x_2 = 1.46558, \quad x_3 = 1.46557,$$

故 $x^* = 1.4656$.

斯特森加速技术:

$$x_1 = \varphi(x_0) = 1.44444, \quad x_2 = \varphi(x_1) = 1.47929, \quad \bar{x} = x_2 - \frac{(x_2 - x_1)^2}{x_2 - 2x_1 + x_0} = 1.46586$$

$$x_0 = \bar{x} = 1.46586, \quad x_1 = \varphi(x_0) = 1.46539, \quad x_2 = \varphi(x_1) = 1.46569, \quad \bar{x} = x_2 - \frac{(x_2 - x_1)^2}{x_2 - 2x_1 + x_0} = 1.46557$$

$$x_0 = \bar{x} = 1.46557, \quad x_1 = \varphi(x_0) = 1.46557, \quad x_2 = \varphi(x_1) = 1.46557, \quad \bar{x} = x_2 - \frac{(x_2 - x_1)^2}{x_2 - 2x_1 + x_0} = 1.46557$$

由于 $|\bar{x} - x_2| < \frac{1}{2} \times 10^{-5}$, 故取 $x^* \approx \bar{x} = 1.4656$.

习题 7.13 证明 $f(x) = (x - x^*)^m g(x)$, 且 $g(x^*) \neq 0$, 当 $m > 1$ 时, 求解方程 $f(x) = 0$ 的牛顿法一阶收敛, 但令 $u = f(x)/f'(x)$ 时, 迭代法

$$x_{k+1} = x_k - m \frac{f(x_k)}{f'(x_k)} \quad \text{或者} \quad x_{k+1} = x_k - \frac{u(x_k)}{u'(x_k)}$$

二阶收敛.

证明 令 $\varphi(x) = x - \frac{f(x)}{f'(x)}$, 由于

$$\varphi(x^*) = 1 - \frac{1}{m} < 1,$$

所以一阶收敛.

令 $\varphi(x) = x - m \frac{f(x)}{f'(x)}$, 则

$$\varphi'(x^*) = 0, \quad |\varphi''(x^*)| < 1,$$

所以二阶收敛.

习题 7.17 给定 $x^{(0)} = (0.6, 0.8)^T$, 用布洛依登法求解如下方程组的近似解, 准确到两位小数

$$x_1^2 + x_2^2 - 1 = 0, \quad 2x_1 + x_2 - 1 = 0.$$

解 由题知

$$f(x) = \begin{pmatrix} x_1^2 + x_2^2 - 1 \\ 2x_1 + x_2 - 1 \end{pmatrix}, \quad f(x^{(0)}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$J_f(x) = \begin{pmatrix} 2x_1 & 2x_2 \\ 2 & 1 \end{pmatrix}, \quad A_0 = J_f(x^{(0)}) = \begin{pmatrix} 1.2 & 1.6 \\ 2 & 1 \end{pmatrix}, \quad A_0^{-1} = \begin{pmatrix} -0.5 & 0.8 \\ 1 & -0.6 \end{pmatrix}$$

$$x^{(1)} = x^{(0)} - A_0^{-1}f(x^{(0)}) = (-0.2, 1.4)^T, \quad f(x^{(1)}) = (0, 1)^T,$$

$$s^{(1)} = x^{(1)} - x^{(0)} = (-0.8, 0.6)^T, \quad y^{(1)} = f(x^{(1)}) - f(x^{(0)}) = (1, -1)^T,$$

$$A_1^{-1} = A_0^{-1} + \frac{(s^{(1)} - A_0^{-1}y^{(1)})s^{(1)T}A_0^{-1}}{s^{(1)T}A_0^{-1}y^{(1)}} = \begin{pmatrix} -0.25 & 0.55 \\ 0.5 & -0.1 \end{pmatrix}$$

$$x^{(2)} = x^{(1)} - A_1^{-1}f(x^{(1)}) = (0.05, 0.9)^T, \quad f(x^{(2)}) = (1.0, 0.0)^T,$$

$$s^{(2)} = x^{(2)} - x^{(1)} = (0.25, -0.5)^T, \quad y^{(2)} = f(x^{(2)}) - f(x^{(1)}) = (-1.1875, 0)^T,$$

$$A_2^{-1} = A_1^{-1} + \frac{(s^{(2)} - A_1^{-1}y^{(2)})s^{(2)T}A_1^{-1}}{s^{(2)T}A_1^{-1}y^{(2)}} = \begin{pmatrix} -0.2105 & 0.5263 \\ 0.4211 & -0.0526 \end{pmatrix}$$

$$x^{(3)} = x^{(2)} - A_2^{-1}f(x^{(2)}) = (0.0105, 0.9789)^T, \quad f(x^{(3)}) = (-0.1875, 0)^T,$$

$$s^{(3)} = x^{(3)} - x^{(2)} = (-0.0395, 0.0789)^T, \quad y^{(3)} = f(x^{(3)}) - f(x^{(2)}) = (0.1459, 0)^T,$$

$$A_3^{-1} = A_2^{-1} + \frac{(s^{(3)} - A_2^{-1}y^{(3)})s^{(3)T}A_2^{-1}}{s^{(3)T}A_2^{-1}y^{(3)}} = \begin{pmatrix} -0.2705 & 0.5623 \\ 0.5409 & -0.1246 \end{pmatrix}$$

$$x^{(4)} = x^{(3)} - A_3^{-1}f(x^{(3)}) = (-0.0007, 1.0014)^T, \quad f(x^{(4)}) = (-0.0416, 0)^T,$$

$$s^{(4)} = x^{(4)} - x^{(3)} = (-0.0124, 0.0248)^T, \quad y^{(4)} = f(x^{(4)}) - f(x^{(3)}) = (0.0444, 0)^T,$$

$$A_4^{-1} = A_3^{-1} + \frac{(s^{(4)} - A_3^{-1}y^{(4)})s^{(4)T}A_3^{-1}}{s^{(4)T}A_3^{-1}y^{(4)}} = \begin{pmatrix} -0.2531 & 0.5519 \\ 0.5062 & -0.1037 \end{pmatrix}$$

于是有

$$x^{(5)} \approx (9.48 \times 10^{-6}, 0.99998)^T, \quad \|x^{(5)} - x^{(4)}\|_{\infty} \leq \frac{1}{2} \times 10^{-2},$$

达到给定的精度, 因此取 $x^* \approx x^{(5)}$.