计算方法 A 第 6 章习题答案

习题 6.1 设函数 $f(x)$ 在区间 $[a,b]$ 上二阶导数连续, 试导出以下求积公式:

(1) $\int_a^b f(x)dx = (b-a)f(a) + \frac{(b-a)^2}{2}f'(\eta)$, $a < \eta < b$;

(2) $\int_a^b f(x)dx = (b-a)f(a) - \frac{(b-a)^2}{2}f'(\eta)$, $a < \eta < b$;

(3) $\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{24}f''(\eta)$, $a < \eta < b$.

解: (1) 左矩形公式: $f(x) \approx f(a)$, $\int_a^b f(x)dx \approx \int_a^b f(a)dx = f(a)(b-a)$, 因此

$$\int_a^b f(x)dx - f(a)(b-a) = \int_a^b f(x)dx - \int_a^b f(a)dx = \int_a^b (f(x)-f(a))dx$$

$$= \int_a^b f'(\xi)(x-a)dx = f'(\eta)\int_a^b (x-a)dx = \frac{1}{2}(b-a)^2 f'(\eta), \quad \xi, \eta \in (a,b)$$

(2) 右矩形公式: $f(x) \approx f(b)$, $\int_a^b f(x)dx \approx \int_a^b f(b)dx = f(b)(b-a)$, 因此

$$\int_a^b f(x)dx - f(b)(b-a) = \int_a^b f(x)dx - \int_a^b f(b)dx = \int_a^b (f(x)-f(b))dx$$

$$= \int_a^b f'(\xi)(x-b)dx = f'(\eta)\int_a^b (x-b)dx = -\frac{1}{2}(b-a)^2 f'(\eta), \quad \xi, \eta \in (a,b)$$

(3) 中矩形公式: $f(x) \approx f\left(\frac{a+b}{2}\right)$, 则

$$\int_a^b f(x)dx \approx \int_a^b f\left(\frac{a+b}{2}\right)dx = f\left(\frac{a+b}{2}\right)(b-a)$$

因此

$$\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)(b-a) = \int_a^b f(x)dx - \int_a^b f\left(\frac{a+b}{2}\right)dx = \int_a^b \left(f(x) - f\left(\frac{a+b}{2}\right)\right)dx$$

$$= \int_a^b \left(f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right) + \frac{1}{2}f''(\xi)\left(x - \frac{a+b}{2}\right)^2\right)dx$$

$$= f'\left(\frac{a+b}{2}\right)\int_a^b \left(x - \frac{a+b}{2}\right)dx + \frac{1}{2}f''(\eta)\int_a^b \left(x - \frac{a+b}{2}\right)^2 dx$$

$$= \frac{1}{24}f''(\eta)(b-a)^3, \quad \xi, \eta \in (a,b)$$

习题 6.7 导出下列求积公式及其截断误差估计式

(1) $\int_0^h f(x)dx \approx A_0 f(0) + B_0 f'(0) + A_1 f(h) + B_1 f'(h)$;

(2) $\int_0^{2h} f(x)dx \approx A_0 f(0) + A_1 f(h) + A_2 f(2h)$;

(3) $\int_{-1}^1 x^2 f(x)dx \approx A_0 f(x_0) + A_1 f(x_1)$;

(4) $\int_0^1 \sqrt{x} f(x) dx \approx A_0 f(x_0) + A_1 f(x_1)$;

(5) $\int_0^{+\infty} \mathrm{e}^{-x} f(x) dx \approx A_0 f(x_0) + A_1 f(x_1)$.

解: (1) 已知该式有四个待定系数, 故令 $f(x) = 1, x, x^2, x^3$, 代入求积公式使其精确成立, 得到

$$A_0 + A_1 = h, \quad B_0 + A_1 h + B_1 = \frac{1}{2} h^2, \quad A_1 h^2 + 2B_1 h = \frac{1}{3} h^3, \quad A_1 h^3 + 3B_1 h^2 = \frac{1}{4} h^4,$$

解得

$$A_0 = \frac{h}{2}, \quad A_1 = \frac{h}{2}, \quad B_0 = \frac{h^2}{12}, \quad B_1 = -\frac{h^2}{12}.$$

因此

$$\int_0^h f(x) dx \approx Q[f] = \frac{h}{2} f(0) + \frac{h^2}{12} f'(0) + \frac{h}{2} f(h) - \frac{h^2}{12} f'(h)$$

当 $f(x) = x^4$ 时,

$$I[f] = \int_0^h x^4 dx = \frac{1}{5} h^5, \quad Q[f] = \frac{1}{6} h^5 \neq I[f],$$

因此该公式具有3次代数精度. 根据广义佩亚诺定理, 取

$$e(x) = \frac{1}{4!} f^{(4)}(\xi) x^2 (x-h)^2,$$

此时有 $e(0) = e'(0) = e(h) = e'(h) = 0$,

$$\begin{aligned} R[f] = R[e(x)] &= \int_0^h e(x) dx - \frac{h}{12} [6e(0) + he'(0) + 6e(h) - he'(h)] \\ &= \int_0^h e(x) dx = \int_0^h \frac{1}{4!} f^{(4)}(\xi) x^2 (x-h)^2 dx \\ &= \frac{1}{4!} f^{(4)}(\eta) \int_0^h x^2 (x-h)^2 dx = \frac{h^5}{720} f^{(4)}(\eta) \end{aligned}$$

(2) 令 $f(x) = 1, x, x^2$, 代入求积公式使其精确成立, 得到

$$A_0 + A_1 + A_2 = 2h, \quad A_1 h + 2A_2 h = 2h^2, \quad A_1 h^2 + 4A_2 h^2 = \frac{8}{3} h^3,$$

解得

$$A_0 = \frac{h}{3}, \quad A_1 = \frac{4h}{3}, \quad A_2 = \frac{h}{3}.$$

因此

$$\int_0^{2h} f(x) dx \approx Q[f] = \frac{h}{3} f(0) + \frac{4h}{3} f(h) + \frac{h}{3} f(2h)$$

当 $f(x) = x^3$ 时,

$$I[f] = \int_0^{2h} x^3 dx = 4h^4, \quad Q[f] = \frac{h}{3} [4h^3 + 8h^3] = 4h^4 = I[f].$$

当 $f(x) = x^4$ 时,

$$I[f] = \int_0^{2h} x^4 dx = \frac{32}{5} h^5, \ \ Q[f] = \frac{h}{3}[4h^4 + 16h^4] = \frac{20}{3} h^5 \neq I[f].$$

因此该求积公式的代数精度为 $m = 3$. 根据广义佩亚诺定理, 取

$$e(x) = \frac{1}{4!} f^{(4)}(\xi) x(x-h)^2(x-2h),$$

此时 $e(0) = e(h) = e(2h) = 0$, 且

$$\begin{aligned} R[f] =& R[e(x)] = \int_0^{2h} e(x)dx - \frac{h}{3}[e(0) + 4e(h) + e(2h)] \\ =& \int_0^{2h} e(x)dx = \int_0^{2h} \frac{1}{4!} f^{(4)}(\xi) x(x-h)^2(x-2h)dx \\ =& \frac{1}{4!} f^{(4)}(\eta) \int_0^{2h} x(x-h)^2(x-2h)dx = \frac{h^5}{90} f^{(4)}(\eta). \end{aligned}$$

(3) 令 $f(x) = 1, x, x^2, x^3$, 代入求积公式使其精确成立, 有

$$A_0 - A_1 = \frac{2}{3}, \ \ -A_0 x_0 + A_1 x_1 = 0, \ \ A_0 x_0^2 + A_1 x_1^2 = \frac{2}{5}, \ \ A_0 x_0^3 + A_1 x_1^3 = 0,$$

解得

$$x_{0,1} = \pm \frac{\sqrt{15}}{5}, \ \ A_0 = A_1 = \frac{1}{3}.$$

因此求积公式为

$$\int_{-1}^1 x^2 f(x)dx = \frac{1}{3} f\left(\frac{\sqrt{15}}{5}\right) + \frac{1}{3} f\left(-\frac{\sqrt{15}}{5}\right).$$

又 $R[x^4] = \frac{4}{25} \neq 0$, 因此代数精度为3. 根据广义佩亚诺定理取

$$e(x) = \frac{1}{4!} f^{(4)}(\xi)(x + \frac{\sqrt{15}}{5})^2(x - \frac{\sqrt{15}}{5})^2,$$

此时 $e(-\frac{\sqrt{15}}{5}) = e(\frac{\sqrt{15}}{5}) = 0$, 且

$$R[f] = R[e] = \int_{-1}^1 x^2 e(x)dx = \int_{-1}^1 \frac{1}{4!} f^{(4)}(\xi) x^2 (x + \frac{\sqrt{15}}{5})^2 (x - \frac{\sqrt{15}}{5})^2 dx = \frac{1}{525} f^{(4)}(\eta)$$

(4) 截断误差 $R[f] = \int_0^1 \sqrt{x} f(x)dx - A_0 f(x_0) - A_1 f(x_1)$. 令

$$\begin{cases} R[1] = \dfrac{2}{3} - A_0 - A_1 = 0 \\ R[x] = \dfrac{2}{5} - A_0 x_0 - A_1 x_1 = 0 \\ R[x^2] = \dfrac{2}{7} - A_0 x_0^2 - A_1 x_1^2 = 0 \\ R[x^3] = \dfrac{2}{9} - A_0 x_0^3 - A_1 x_1^3 = 0 \end{cases}$$

解得

$$x_0 = \frac{5}{9} - \frac{\sqrt{280}}{63}, \ \ x_1 = \frac{5}{9} + \frac{\sqrt{280}}{63}, \ \ A_0 = \frac{1}{3} - \frac{\sqrt{280}}{300}, \ \ A_1 = \frac{1}{3} + \frac{\sqrt{280}}{300}.$$

因此求积公式为

$$\int_0^1 \sqrt{x} f(x) dx \approx \left(\frac{1}{3} - \frac{\sqrt{280}}{300}\right) f\left(\frac{5}{9} - \frac{\sqrt{280}}{63}\right) + \left(\frac{1}{3} + \frac{\sqrt{280}}{300}\right) f\left(\frac{5}{9} + \frac{\sqrt{280}}{63}\right)$$

且

$$R[x^4] = \frac{2}{11} - \left(\frac{1}{3} - \frac{\sqrt{280}}{300}\right)\left(\frac{5}{9} - \frac{\sqrt{280}}{63}\right)^4 - \left(\frac{1}{3} + \frac{\sqrt{280}}{300}\right)\left(\frac{5}{9} + \frac{\sqrt{280}}{63}\right)^4 = \frac{128}{43659} \neq 0$$

故其代数精度为3. 根据广义佩亚诺定理可取

$$e(x) = \frac{1}{4!} f^{(4)}(\xi)\big(x - \frac{5}{9} + \frac{\sqrt{280}}{63}\big)^2 \big(x - \frac{5}{9} - \frac{\sqrt{280}}{63}\big)^2,$$

此时有 $e(\frac{5}{9} - \frac{\sqrt{280}}{63}) = e(\frac{5}{9} + \frac{\sqrt{280}}{63}) = 0$, 则

$$R[f] = R[e(x)] = \int_0^1 \sqrt{x} e(x) dx - \left(\frac{1}{3} - \frac{\sqrt{280}}{300}\right) e\left(\frac{5}{9} - \frac{\sqrt{280}}{63}\right) - \left(\frac{1}{3} + \frac{\sqrt{280}}{300}\right) e\left(\frac{5}{9} + \frac{\sqrt{280}}{63}\right)$$

$$= \int_0^1 \sqrt{x} e(x) dx = \int_0^1 \frac{1}{4!} f^{(4)}(\xi) \sqrt{x}\big(x - \frac{5}{9} + \frac{\sqrt{280}}{63}\big)^2 \big(x - \frac{5}{9} - \frac{\sqrt{280}}{63}\big)^2 dx$$

$$= \frac{1}{24} f^{(4)}(\eta) \int_0^1 \sqrt{x}\big(x - \frac{5}{9} + \frac{\sqrt{280}}{63}\big)^2 \big(x - \frac{5}{9} - \frac{\sqrt{280}}{63}\big)^2 dx = \frac{16}{130977} f^{(4)}(\eta).$$

(5) 截断误差 $R[f] = \int_0^{+\infty} e^{-x} f(x) dx - A_0 f(x_0) - A_1 f(x_1)$. 令

$$\begin{cases} R[1] = 1 - A_0 - A_1 = 0 \\ R[x] = 1 - A_0 x_0 - A_1 x_1 = 0 \\ R[x^2] = 2 - A_0 x_0^2 - A_1 x_1^2 = 0 \\ R[x^3] = 6 - A_0 x_0^3 - A_1 x_1^3 = 0 \end{cases}$$

解得

$$x_0 = 2 - \sqrt{2}, \ x_1 = 2 + \sqrt{2}, \ A_0 = \frac{2 + \sqrt{2}}{4}, \ A_1 = \frac{2 - \sqrt{2}}{4}.$$

因此求积公式为

$$\int_0^{+\infty} e^{-x} f(x) dx \approx \frac{2 + \sqrt{2}}{4} f(2 - \sqrt{2}) + \frac{2 - \sqrt{2}}{4} f(2 + \sqrt{2}).$$

由于

$$R[x^4] = 24 - \frac{2 + \sqrt{2}}{4}(2 - \sqrt{2})^4 - \frac{2 - \sqrt{2}}{4}(2 + \sqrt{2})^4 = 4 \neq 0,$$

故其代数精度为3. 根据广义佩亚诺定理可取

$$e(x) = \frac{1}{4!} f^{(4)}(\xi)\big(x - 2 + \sqrt{2}\big)^2 \big(x - 2 - \sqrt{2}\big)^2,$$

此时有 $e(2-\sqrt{2}) = e(2+\sqrt{2}) = 0$, 则

$$
\begin{aligned}
R[f] =& R[e(x)] = \int_0^{+\infty} \mathrm{e}^{-x} e(x) dx \\
=& \int_0^{+\infty} \frac{1}{4!} f^{(4)}(\xi) \mathrm{e}^{-x} \big(x - 2 + \sqrt{2}\big)^2 \big(x - 2 - \sqrt{2}\big)^2 dx \\
=& \frac{1}{24} f^{(4)}(\eta) \int_0^{+\infty} \mathrm{e}^{-x} \big(x - 2 + \sqrt{2}\big)^2 \big(x - 2 - \sqrt{2}\big)^2 dx = \frac{1}{6} f^{(4)}(\eta).
\end{aligned}
$$

习题 6.11 确定下列数值微分公式的系数, 导出截断误差简单表达式

(1) $f'(0) \approx af(-h) + bf(0) + cf(h)$

(2) $f'(h) \approx af'(0) + b[f(2h) - f(h)]$

解: (1) 截断误差为
$$
R[f] = f'(0) - af(-h) - bf(0) - cf(h).
$$

分别取 $f = 1, x, x^2$, 令 $R[f] = 0$ 得到方程组
$$
\begin{cases}
a + b + c = 0 \\
(c - a)h = 1 \\
(a + c)h^2 = 0
\end{cases}
$$

解得
$$
a = -\frac{1}{2h}, \quad b = 0, \quad c = \frac{1}{2h}.
$$

故数值微分公式为
$$
f'(0) \approx -\frac{1}{2h} f(-h) + \frac{1}{2h} f(h).
$$

由于 $R[x^3] = h^2 \neq 0$, 所以代数精度 $m = 2$. 根据广义佩亚诺定理, 取
$$
e(x) = \frac{1}{6} f'''(\xi) x(x+h)(x-h),
$$

$$
R[f] = R[e] = e'(0) = -\frac{h^2}{6} f'''(\xi).
$$

(2) 截断误差为
$$
R[f] = f'(h) - af'(0) - b[f(2h) - f(h)].
$$

分别取 $f = 1, x, x^2$, 令 $R[f] = 0$ 得到方程组
$$
\begin{cases}
0 = 0 \\
a + bh = 1 \\
3bh^2 = 2h
\end{cases}
$$

解得 $a = \frac{1}{3}$, $b = \frac{2}{3h}$. 因此

$$f'(h) \approx \frac{1}{3} f'(0) + \frac{2}{3h} \big[f(2h) - f(h) \big].$$

由于 $R[x^3] = -\frac{5}{3} h^2 \neq 0$, 故代数精度 $m = 2$. 根据广义佩亚诺定理, 取

$$e(x) = \frac{1}{6} f'''(\xi) x^2 (x - 2h),$$

注意到 $e(0) = e'(0) = e(2h) = 0$, 于是

$$R[f] = R[e] = e'(h) - \frac{1}{3} e'(0) - \frac{2}{3h} \big[e(2h) - e(h) \big] = e'(h) + \frac{2}{3h} e(h) = -\frac{5}{18} h^2 f'''(\xi).$$