3.1 用雅可比迭代法与高斯一赛德尔迭代法求解方程组.

$$\begin{cases} -8x_1 + x_2 + x_3 = 1 \\ x_1 - 5x_2 + x_3 = 16 \\ x_1 + x_2 - x_3 = 7 \end{cases}$$

取初始向量 $X^{(0)} = (0,0,0)^T$，准确到两位数.

① 雅可比迭代法

雅可比格式为

$$\begin{cases} x_1^{(k+1)} = (x_2^{(k)} + x_3^{(k)} - 1)/8 \\ x_2^{(k+1)} = (x_1^{(k)} + x_3^{(k)} - 16)/5 \\ x_3^{(k+1)} = (x_1^{(k)} + x_2^{(k)} - 7)/4 \end{cases}$$

取初始向量 $X^{(0)} = (0,0,0)^T$ 代入以上格式得

$X^{(1)} = (-0.12, -3.20, -1.75)^T$

$X^{(2)} = (-0.74, -3.57\ -2.58)^T$

② 高斯一赛德尔迭代法

高斯一赛德尔迭代格式为

$$\begin{cases} x_1^{(k+1)} = (x_2^{(k)} + x_3^{(k)} - 1)/8 \\ x_2^{(k+1)} = (x_1^{(k+1)} + x_3^{(k)} - 16)/5 \\ x_3^{(k+1)} = (x_1^{(k+1)} + x_2^{(k+1)} - 7)/14 \end{cases}$$

将初始向量 $X^{(0)} = (0,0,0)^T$ 代入以上迭代得

$X^{(1)} = (-0.12, -3.22, -2.59)$

③ 逐次超松弛(SOR)迭代法 (取 $W=1.1$)

迭代格式为

$$\begin{cases} x_1^{(k+1)} = x_1^{(k)} + \frac{1.1}{-8}(8x_1^{(k)} - x_2^{(k)} - x_3^{(k)} + 1) \\ x_2^{(k+1)} = x_2^{(k)} + \frac{1.1}{-5}(-x_1^{(k+1)} + 5x_2^{(k)} - x_3^{(k)} + 16) \\ x_3^{(k+1)} = x_3^{(k)} + \frac{1.1}{-4}(-x_1^{(k+1)} - x_2^{(k+1)} + 4x_3^{(k)} + 7) \end{cases}$$

取 $X^{(0)}=(0,0,0)^T$

则 $X^{(1)}=(-0.14,\ -3.55,\ -2.94)^T$

3.4 欲用迭代法求解方程组

$$\begin{cases} x_1-4x_2-2x_4=11 \\ 4x_1+8x_3+3x_4=6 \\ 3x_2+x_3+5x_4=2 \\ 5x_1-x_2+x_3+2x_4=5 \end{cases}$$

写出收敛的高斯—赛德尔迭代法与 $w=0.8$ 的超松弛迭代法的迭代公式。给定初始向量 $X^{(0)}=(0,0,0,0)^T$，用高斯—赛德尔迭代法计算 $X^{(1)}$。

解：①高斯—赛德尔迭代法：

$$\begin{cases} x_1^{(k+1)}=4x_2^{(k)}+2x_4^{(k)}+11 \\ x_2^{(k+1)}=(-x_3^{(k)}-5x_4^{(k)}+2)/3 \\ x_3^{(k+1)}=(-4x_1^{(k+1)}-3x_4^{(k)})/8 \\ x_4^{(k+1)}=\cancel{(x_2-5x_1-x_3-2x_4)}\ (x_2^{(k+1)}-5x_1^{(k+1)}-x_3^{(k+1)})/2 \end{cases}$$

②$w=0.8$ 的超松弛迭代法

$$\begin{cases} x_1^{(k+1)}=x_1^{(k)}+0.8(4x_2^{(k)}+2x_4^{(k)}+11) \\ x_2^{(k+1)}=x_2^{(k)}+\frac{0.8}{3}(-x_3^{(k)}-5x_4^{(k)}+2)/3 \\ x_3^{(k+1)}=x_3^{(k)}+\frac{0.8}{8}(-4x_1^{(k+1)}-3x_4^{(k)}) \\ x_4^{(k+1)}=x_4^{(k)}+\frac{0.8}{2}(x_2^{(k+1)}-5x_1^{(k+1)}-x_3^{(k+1)}) \end{cases}$$

取 $X^{(0)}=(0,0,0)^T$

则 $X^{(1)}=(11,\ \frac{2}{3},\ -\frac{11}{2},\ -\frac{293}{12})$

3.8 (1) 求解线性方程组 $Ax=b$ (其中 A 为对称正定矩阵) 的迭代公式为

$$x^{(k+1)} = x^{(k)} - w(Ax^{(k)} - b) \qquad k=0,1,2,\cdots$$

证明当 $0<w<\frac{2}{\beta}$ 时. 上述迭代法收敛. (其中 $0<\alpha\le\lambda(A)\le\beta$)

proof: 将迭代公式改写为

$$x^{(k+1)} = (I-wA)x^{(k)} + wb \qquad k=0,1,2,\cdots$$

迭代矩阵 $B=I-wA$. 其特征值为 $\mu = 1-w\lambda(A)$

当 $0<w<\frac{2}{\beta}$ 时. 又 $0<\lambda(A)\le\beta$. 有

$$0 < w < \frac{2}{\lambda(A)}$$

$$\Rightarrow \quad |1-w\lambda(A)| < 1 \qquad \text{则 } |\mu|<1$$

这时 $\rho(B)<1$ 故迭代收敛.

3.10. 给定初始向量 $x^{(0)}=(0,0,0)^T$. 用共轭梯度法求解以下方程组

(1) $\begin{cases} 2x_1 - x_2 - x_3 = 2 \\ -x_1 + 2x_2 + x_3 = 0 \\ -x_1 + x_2 + 2x_3 = -2 \end{cases}$

(2) $\begin{cases} 4x_1 - x_2 + 2x_3 = 12 \\ -x_1 + 5x_2 + 3x_3 = 10 \\ 2x_1 + 3x_2 + 6x_3 = 18 \end{cases}$

解: (1)

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$

取 $d^{(0)} = r^{(0)} = b - Ax^{(0)} = b = (2,0,-2)^T$, 则

$$\|r^{(0)}\|_2^2 = 8 \qquad d^{(0)T}Ad^{(0)} = 24$$

故 $\alpha_0 = \frac{\|r^{(0)}\|_2^2}{d^{(0)T}Ad^{(0)}} = \frac{8}{24} = \frac{1}{3}$

$$x^{(1)} = x^{(0)} + \alpha_0 d^{(0)} = \frac{1}{3}(2,0,-2)^T$$

又 $r^{(1)} = b - Ax^{(1)} = \frac{1}{3}(0,4,0)^T$

$\|r^{(1)}\|_2^2 = \frac{1}{9} \times 16$

$\beta_0 = \frac{\|r^{(1)}\|_2^2}{\|r^{(0)}\|_2^2} = \frac{\frac{1}{9} \times 16}{8} = \frac{2}{9}$

$d^{(1)} = r^{(1)} + \beta_0 d^{(0)} = \frac{4}{9}(1, 3, -1)^T$

$d^{(1)T} A d^{(1)} = \frac{4^2 \times 12}{9^2}$

$\alpha_1 = \frac{\|r^{(1)}\|_2^2}{d^{(1)T} A d^{(1)}} = \frac{3}{4}$

$X^{(2)} = X^{(1)} + \alpha_1 d^{(1)} = (1, 1, -1)^T$

$r^{(2)} = b - AX^{(2)} = (0, 0, 0)^T$

迭代两次求得了方程组的解 $X^* = X^{(2)} = (1, 1, -1)^T$

(2) $\begin{cases} 4x_1 - x_2 + 2x_3 = 12 \\ -x_1 + 5x_2 + 3x_3 = 10 \\ 2x_1 + 3x_2 + 6x_3 = 18 \end{cases}$

$$\begin{pmatrix} 4 & -1 & 2 \\ -1 & 5 & 3 \\ 2 & 3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \\ 18 \end{pmatrix}$$

取 $d^{(0)} = r^{(0)} = b - AX^{(0)} = b = (12, 10, 18)^T$

$\|r^{(0)}\|_2^2 = 568 \qquad d^{(0)T} A d^{(0)} = 4724$

$\alpha_0 = \frac{\|r^{(0)}\|_2^2}{d^{(0)T} A d^{(0)}} = 0.1202$

$X^{(1)} = X^{(0)} + \alpha_0 d^{(0)} = (1.4428, 1.2024, 2.1643)^T$

$r^{(1)} = b - AX^{(1)} = (3.1025, -1.0618, -1.4784)^T$

$d^{(1)} = r^{(1)} + \beta d^{(0)} = (3.3758, -0.8340, -1.0684)^T$

$\|r^{(1)}\|_2^2 = 12.9384$

$d^{(1)T} A d^{(1)} = 52.4615$

$\alpha_1 = \frac{\|r^{(1)}\|_2^2}{d^{(1)T} A d^{(1)}} = 0.2466$

$$X^{(2)} = X^{(1)} + \alpha_1 d^{(1)} = (2.2754,\ 0.9967,\ 1.9008)^T$$

$$r^{(2)} = b - AX^{(2)} = (0.0935,\ 1.5897,\ -0.9455)^T$$

$$d^{(2)} = r^{(2)} + \beta_0 d^{(1)} = (-0.9884,\ -1.3686,\ -1.2287)$$

$$\|r^{(2)}\|_2^2 = 3.4298$$

$$d^{(2)T} A d^{(2)} = 4.6785$$

$$\alpha_2 = \frac{\|r^{(2)}\|_2^2}{d^{(2)T} A d^{(2)}} = 0.7331$$

$$X^{(3)} = X + \alpha_2 d^{(2)} = (3,\ 2,\ 1)^T$$

$$r^{(3)} = b - AX^{(3)} = (0,\ 0,\ 0)$$

故方程组的解 $X^* = X^{(3)} = (3,\ 2,\ 1)^T$

3.11 分别用循环阿诺尔迪算法和循环GMRES算法求解方程组 $Ax=b$，其中

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

取 $m=2$. 给定初始向量 $X^{(0)} = (0,0,0,0)^T$. 算出 $X^{(2)}$.

① 循环阿诺尔迪算法

$$r^{(0)} = b - AX^{(0)} = (1,1,1,1)^T$$

$$\|r^{(0)}\|_2^2 = 4 \qquad \|r^{(0)}\|_2 = 2$$

$$V_1 = \frac{r^{(0)}}{\|r^{(0)}\|_2} = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)^T$$

$$\tilde{V}_2 = AV_1 - (AV_1, V_1)V_1 = \left(-\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right)^T$$

$$V_2 = \frac{\sqrt{5}}{10}(-3,\ -1,\ 1,\ 3)^T$$

$$\tilde{V}_3 = AV_2 - (AV_2, V_1)V_1 - (AV_2, V_2)V_2$$
$$= \frac{\sqrt{5}}{10}(1,\ -1,\ -1,\ 1)^T$$

$$V_3 = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)^T$$

$$h_{11} = (AV_1, V_1) = \frac{5}{2} \qquad h_{12} = (AV_1, V_2) = -\frac{\sqrt{5}}{2}$$

$$h_{21} = (AV_2, V_1) = \frac{\sqrt{5}}{2} \qquad h_{22} = (AV_2, V_2) = \frac{1}{2}$$

$H_2=\begin{pmatrix}\frac{5}{2} & -\frac{\sqrt5}{2}\\ \frac{\sqrt5}{2} & \frac12\end{pmatrix}$

$\beta=\|r^{(0)}\|_2=2 \qquad e_1=(1,0)^T\in R^2$

$y=H_2^{-1}\beta e_1=\frac{2}{5}(1,-\sqrt5)^T$

$V_2=(v_1,v_2)$

$z=V_2y=(0.8.\ 0.4.\ 0,-0.4)^T$

$x^{(1)}=x^{(0)}+z=(0.8.\ 0.4,0,-0.4)^T$

$r^{(1)}=b-Ax^{(1)}=(0.2,-0.2,-0.2,0.2)^T$

$\|r^{(1)}\|_2^2=0.16 \qquad \|r^{(1)}\|_2=0.4$

$v_1=\frac{r^{(1)}}{\|r^{(1)}\|_2}=(\frac12,-\frac12,-\frac12,\frac12)$

$\tilde{v}_2=Av_1-(Av_1,v_1)v_1=(\frac14,\frac14,-\frac14,-\frac14)^T$

$v_2=\frac{\tilde{v}_2}{\|\tilde{v}_2\|}=(\frac12,\frac12,-\frac12,-\frac12)$

$h_{11}=(Av_1,v_1)=\frac12 \qquad h_{12}=(Av_1,v_2)=-\frac12$

$h_{21}=(Av_2,v_1)=\frac12 \qquad h_{22}=(Av_2,v_2)=\frac12$

$H_2=\begin{pmatrix}\frac12 & -\frac12\\ \frac12 & \frac12\end{pmatrix}$

$\beta=\|r^{(1)}\|_2=0.4 \qquad e_1=(1,0)^T$

$y=H_2^{-1}\beta e_1=(0.4,-0.4)^T$

$V=(v_1,v_2)$

$z=Vy=(0,-0.4,\ 0.\ 0.4)^T$

$x^{(2)}=x^{(1)}+z=(0.8,0.\ 0.\ 0)^T$

② 循环 GMRES 算法.

$r^{(0)}=b-Ax^{(0)}=(1,1,1,1)^T$

$\|r^{(0)}\|_2=2$

$v_1=\frac{r^{(0)}}{\|r^{(0)}\|_2}=(\frac12,\frac12,\frac12,\frac12)^T$

$\tilde{v}_2=Av_1-(Av_1,v_1)v_1=(-\frac34,-\frac14,\frac14,\frac34)^T$

$$V_2=\frac{\tilde{V}_2}{\|\tilde{V}_2\|}=\frac{\sqrt{5}}{10}(-3,-1,1,3)^T$$

$$\tilde{V}_3=AV_2-(AV_2,V_1)V_1-(AV_2,V_2)V_2$$
$$=\frac{\sqrt{5}}{10}(1,-1,-1,1)^T$$

$$V_3=\frac{\tilde{V}_3}{\|\tilde{V}_3\|}=(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})^T$$

$$h_{11}=(AV_1,V_1)=\frac{5}{2} \qquad h_{12}=(AV_1,V_2)=-\frac{\sqrt{5}}{2}$$

$$h_{21}=(AV_2,V_1)=\frac{\sqrt{5}}{2} \qquad h_{22}=(AV_2,V_2)=\frac{1}{2}$$

$$h_{32}=\|\tilde{V}_3\|=\frac{\sqrt{5}}{5} \qquad e_m=(0,1)^T$$

$$\bar{H}_m=\begin{pmatrix}H_m\\ h_{32}e_m^T\end{pmatrix}=\begin{pmatrix}\frac{5}{2} & -\frac{\sqrt{5}}{2}\\ \frac{\sqrt{5}}{2} & \frac{1}{2}\\ 0 & \frac{\sqrt{5}}{5}\end{pmatrix} \qquad V_2=(V_1,V_2)$$

$$y^{(m)}=\min_{y\in R^m}\|\beta e_1-\bar{H}_m y\|=(0.4516,\ -0.7213)^T$$

$$z^{(2)}=V_2\,y^{(m)}=(0.7097,\ 0.3871,\ 0.0645,\ -0.2581)^T$$

$$x^{(2)}=x^{(0)}+z^{(2)}=(0.7097,\ 0.3871,\ 0.0645,\ -0.2581)^T$$