CSE251
Basics of Computer Graphics
Module: Geometry

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Overview

Points and Frames

Rolling Wheel

Rotations

3D Rotations about an Axis

Arbitrary Axis, Point

Points and Frames in General

- Points go through changes in a common coordinate frame when a sequence of transformations is **viewed from right to left**
- Coordinate system goes through the same transformations when the sequence is **viewed from left to right**
- Composite transformations $P' = \mathbf{M_1 M_2 M_3}\ P$ relates the coordinates in successive coordinate frames as we go from left to right, starting with $\mathbf{X'Y'}$ coordinate frame to finally the $\mathbf{XY}$ frame.

Transforming the World Reference

- Consider $P_4 \;= \mathbf{M_4 M_3 M_2 M_1} \;\; P_0$
- Point P_0 undergoes 4 operations and get coordinates P_4
- Imagine the point having coordinates P_1, P_2, P_3 after operations $\mathbf{M_1}, \mathbf{M_2}, \mathbf{M_3}$
- We can also visualize coordinate frames $\mathbf{\Pi}_4, \mathbf{\Pi}_3, \mathbf{\Pi}_2, \mathbf{\Pi}_1, \mathbf{\Pi}_0$ in which a point has coordinates P_4 to P_0 respectively
- Operation $\mathbf{M}_i$ represents a change in coordinates from $\mathbf{\Pi}_i$ to $\mathbf{\Pi}_{i-1}$, resulting in new labels for the point.

Let us look at Ourselves

- Model IIIT Campus as a whole. Campus is our "world"
- Global coordinate frame $\mathbf{\Pi}_G$ for the campus: at the Gate
- Buildings: Himalaya, Vindhya, Bakul, Parul, ..., Palash. Each has a natural coordinate frame. $\mathbf{\Pi}_H$ is Himalaya's
- Himalaya has several rooms: B105, B204, B205, B304, etc., with own coordinate frames. $\mathbf{\Pi}_C$ is of B105 (our class)
- B105 has 55 desks, with coord frames $\mathbf{\Pi}_{Di}$ for desk i
- Desks are identical in geometry; the coord frame Π_{Di} places each in its location.

Consider a Desk

- Consider a corner point P of desk 37, with coordinates $(200, 30, 100)$ in $\mathbf{\Pi}_{D37}$. That is: $P_{D37} = (200, 30, 100)$
- Since our world is the campus, we have to ultimately describe everything in the coordinate frame $\mathbf{\Pi}_G$

$$P_G = \mathbf{M_{GH}}\ \mathbf{M_{HC}}\ \mathbf{M_{CD37}}\ P_{D37}$$

- $\mathbf{M_{GH}}$ aligns $\mathbf{\Pi}_G$ to $\mathbf{\Pi}_H$. $\mathbf{M_{HC}}$ aligns $\mathbf{\Pi}_H$ to $\mathbf{\Pi}_C$. $\mathbf{M_{CD37}}$ aligns $\mathbf{\Pi}_C$ to $\mathbf{\Pi}_{D37}$
- $P_G = \mathbf{M_{GH}} \overset{P_H}{\big|} \mathbf{M_{HC}} \overset{P_C}{\big|} \mathbf{M_{CD37}} \overset{P_{D37}}{\big|} P$ (for any point P on Desk37)
- We can place a given desk in any **building, room, place**!

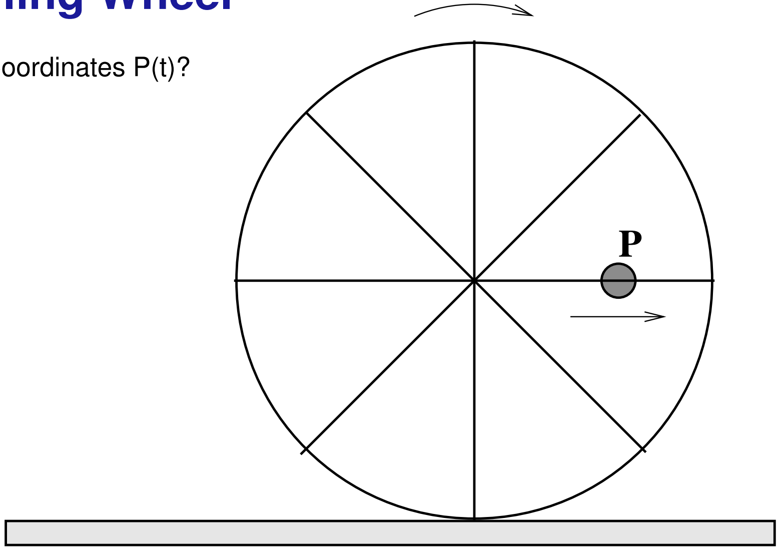
Walking on Stage

- Person walking horizontally on stage, with swinging arms
- How does the hand-tip move w.r.t each student? **How?**
- Student knows own position in room's reference frame
- Start at a student's eye. (That provides the viewpoint!)
- Align to room's reference frame using $\mathbf{M_1}$. Different matrix for each student, but everyone same now....
- Walk: pure translation. $\mathbf{M_2}$ aligns to person coord frame
- Arm swing: Simple harmonic motion with angle $\theta(t)$

Simpler viewpoints in newer coord frames.

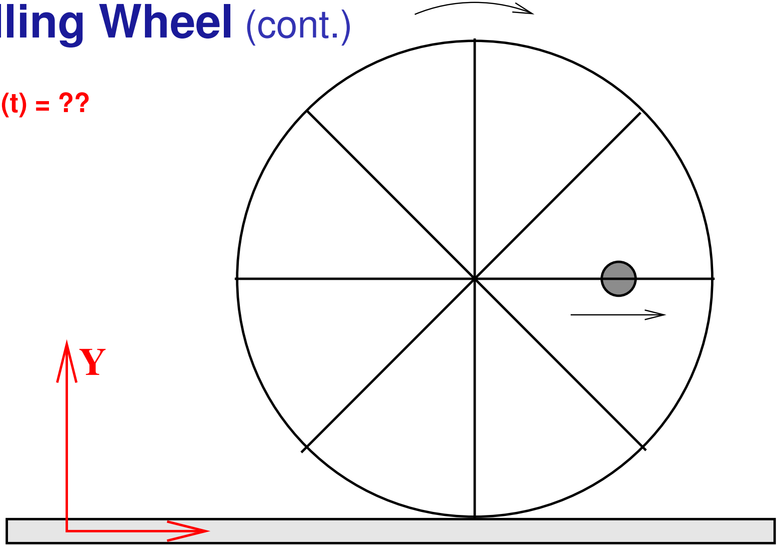
Rolling Wheel

Coordinates P(t)?



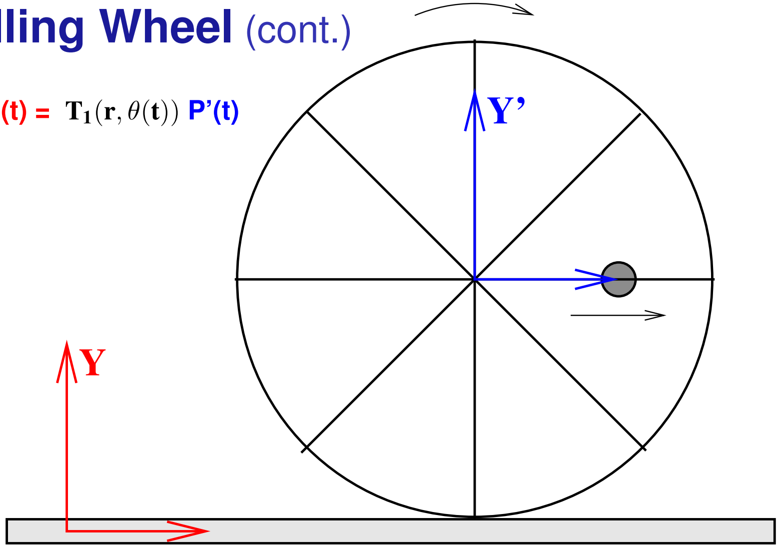
Rolling Wheel (cont.)

P(t) = ??



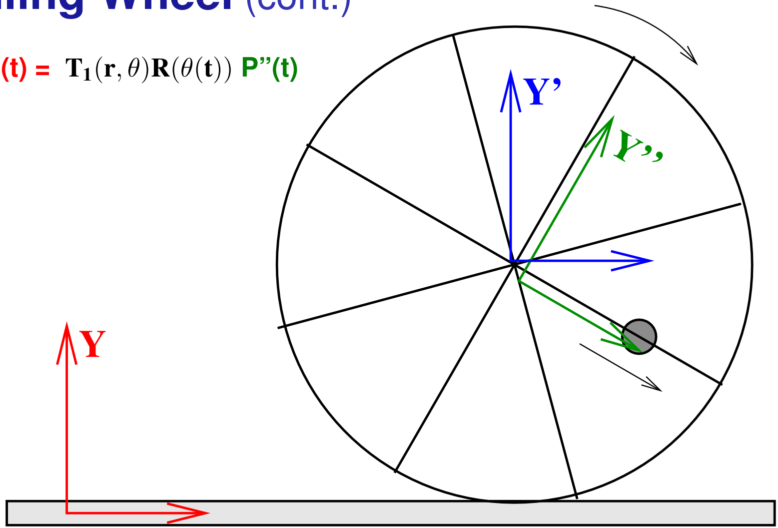
Rolling Wheel (cont.)

$\mathbf{P(t)} = \mathbf{T_1}(\mathbf{r}, \theta(\mathbf{t}))\ \mathbf{P'(t)}$



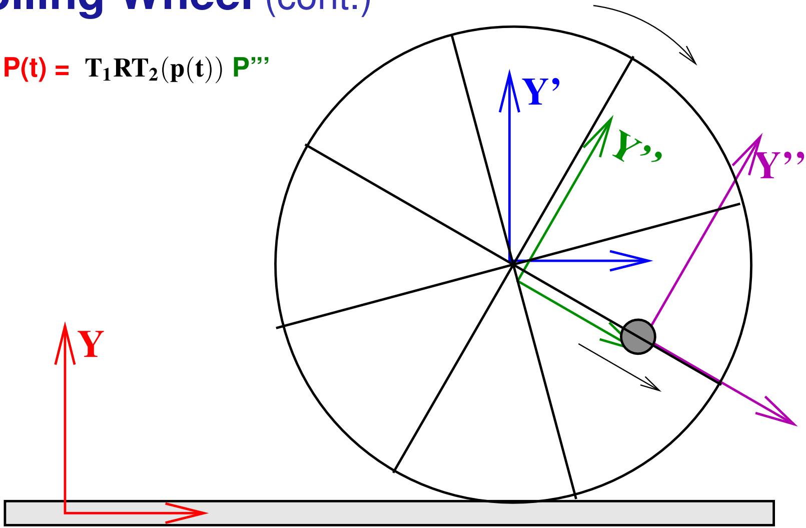
Rolling Wheel (cont.)

$\mathbf{P(t)} = \mathbf{T_1}(\mathbf{r}, \theta)\mathbf{R}(\theta(\mathbf{t}))\ \mathbf{P''(t)}$



Rolling Wheel (cont.)

P(t) = $T_1 R T_2(p(t))$ P'''

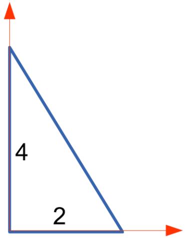


Final Transformation

- P(t) = $\mathbf{T_1(t)\ R(\theta(t))\ T_2(p(t))}$ P”’
- $\mathbf{T_1(t) = T(r\ \theta(t), r) = T(r\ \omega\ t, r)}$ (A translation matrix)
- $\mathbf{R(\theta(t)) = R_Z(\omega\ t)}$ (A normal rotation matrix)
- $\mathbf{T_2(t) = T(p(t), 0) = T(v\ t, 0)}$ (A translation matrix)
- **P”’** = $[0, 0, 1]^T$ (Origin of the bead)
- Lot simpler than thinking about it all together.
- What if we have a pendulum swinging freely on the bead?

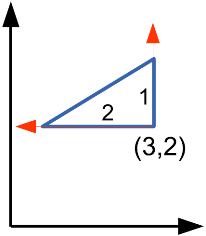
Given an object

- An object `traingleObj` is given. Can be drawn using `drawObject(triangleObj)`



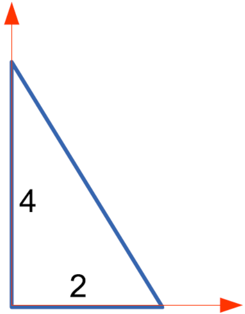
- `drawObject(triangleObj)` draws the object at (current) origin

Draw it in a different configuration

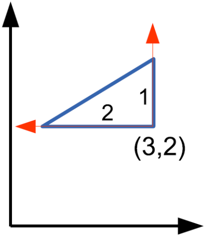


- Use `drawObject(triangleObj)`, with right transformations

Transformations



to

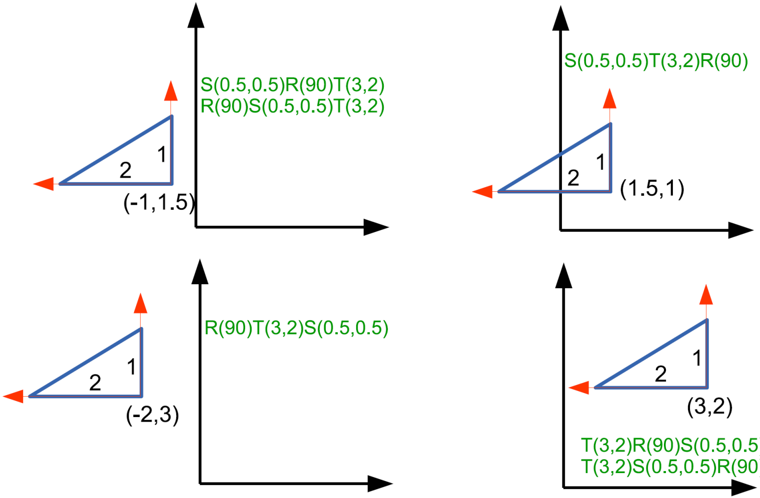


- What are the transformations? Combination of Translation, Rotation, Scaling!!
- Operations involved: $\mathbf{S(0.5, 0.5), T(3, 2), R(90)}$

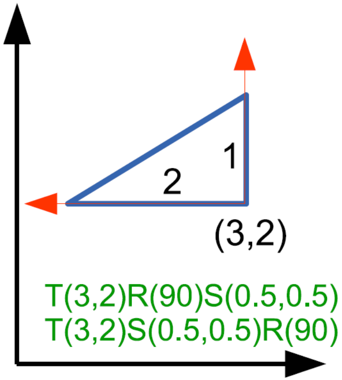
Which combination ?

1. $\mathbf{S(0.5, 0.5), R(90), T(3, 2)}$
2. $\mathbf{S(0.5, 0.5), T(3, 2), R(90)}$
3. $\mathbf{T(3, 2), R(90), S(0.5, 0.5)}$
4. $\mathbf{T(3, 2), S(0.5, 0.5), R(90)}$
5. $\mathbf{R(90), S(0.5, 0.5), T(3, 2)}$
6. $\mathbf{R(90), T(3, 2), S(0.5, 0.5)}$

Which combination ? (cont.)



Several Correct Situations

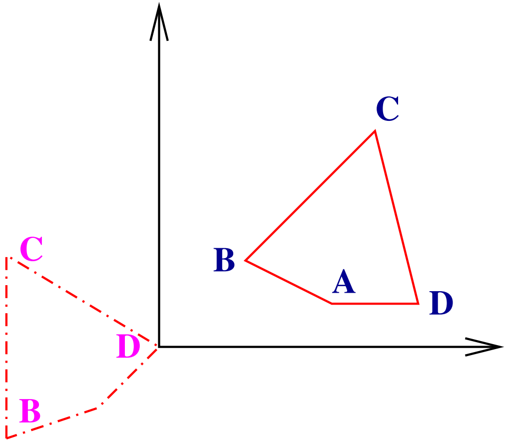


Another Similar Scenario

- A clock is hanging from a nail fixed to a flat plate. The plate is being translated with a velocity $\vec{\mathbf{v}}$ and acceleration $\vec{\mathbf{a}}$. The pendulum of the clock swings back and forth with a time period of 5 seconds and a max angle of $\pm\theta$. An ant travels from the bottom tip of the pendulum up to the centre.
- How do we compute the ant's position with respect to a fixed coordinate system coplanar with the plate?

  Please sketch the situation and work it out for yourself

A Transformation Problem

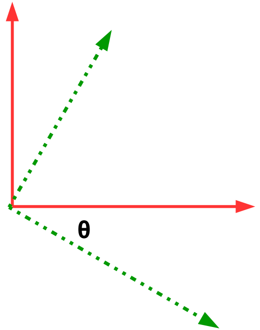


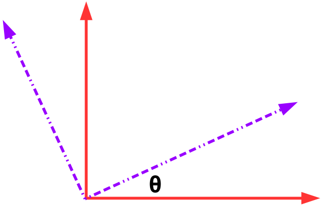
Bring **D** to origin and **BC** parallel to the Y axis as shown

2D Rotation: Observations

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- Orthonormal: $R^{-1} = R^T$
- Rows: vectors that **rotate to** coordinate axes
- Cols: vectors coordinate axes **rotate to**
- Invariants: distances, angles, parallelism.





Transformation Computation

- Step 1: Translate by $-\mathbf{D}$. What is the orientation of BC?
- Step 2: Rotate to have unit vector $\vec{\mathbf{u}} = [u_x \;\; u_y]^\mathsf{T}$ from **B** to **C** on the Y axis. That is the second row of **R** matrix
- The matrix for the total operation: $\mathbf{M} = \mathbf{T}(-\mathbf{D})\mathbf{R}$
- Two options for first row. $[u_y \;\; -u_x]^\mathsf{T}$ and $[-u_y \; u_x]^\mathsf{T}$
- **R** matrix: (a) $\begin{bmatrix} u_y & -u_x \\ u_x & u_y \end{bmatrix}$ or (b) $\begin{bmatrix} -u_y & u_x \\ u_x & u_y \end{bmatrix}$?
- Difference? The direction aligned to the X-axis!
- Option (a) is correct. **Why?** Draw Option (b)!

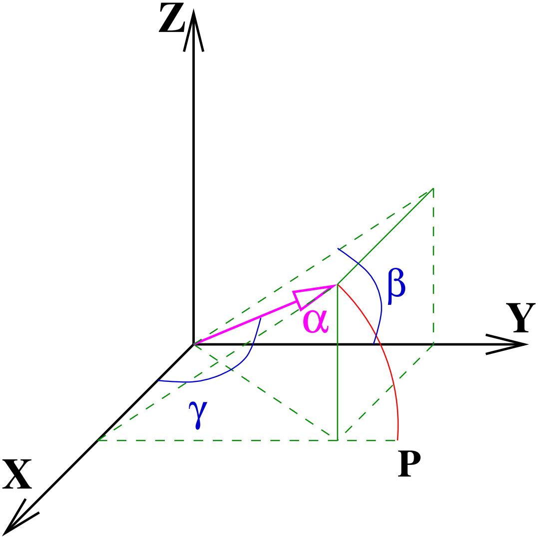
Rotation about an axis parallel to Z

- An axis parallel to Z axis, passing through point $(x, y, 0)$.
- Translate so that the axis passes through the origin: $\mathbf{T}(-x, -y, k)$ for any k!!
- Overall: $\mathbf{M} = \mathbf{T}(x, y, -k)\ \mathbf{R_Z}(\theta)\ \mathbf{T}(-x, -y, k)$
- Why shouldn't k matter? $\mathbf{R_Z}$ doesn't affect the z coordinate. So, whatever k is added first will be subtracted later

3D Rotation about an axis α

- What is $\mathbf{R}_{\alpha}(\theta)$?
- 3-step process:
  1. Apply $\mathbf{R}_{\alpha\mathbf{x}}$ to align α with the X axis.
  2. Rotate about X by angle θ.
  3. Undo the first rotation using $\mathbf{R}_{\alpha\mathbf{x}}^{-1}$
- Net result: $\mathbf{R}_{\alpha}(\theta) \quad = \quad \mathbf{R}_{\alpha\mathbf{x}}^{-1}\ \mathbf{R}_{\mathbf{x}}(\theta)\ \mathbf{R}_{\alpha\mathbf{x}}$
- Quite simple!? What is $\mathbf{R}_{\alpha\mathbf{x}}(\theta)$?
- **(We can align α with Y or Z axis also)**

3D Rotation about an axis α (cont.)



Computing $\mathbf{R}_\alpha$

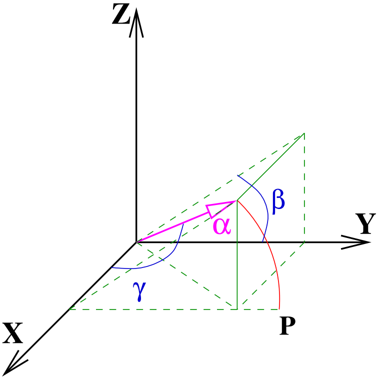
- First rotate by $-\beta$ about X axis. Vector α would lie in the XY plane, with tip at point **P**.
- $\beta = ?, \quad \tan\beta = ?$
- Next rotate by $-\gamma$ about Z axis. Vector α would coincide with the X axis.
- $\gamma = ?, \quad \tan\gamma = ?$

Computing $\mathbf{R}_\alpha$

- Rotate by $-\beta$ about X axis to bring α to XY plane
- $\tan\beta = \frac{\alpha_z}{\alpha_y}$
- Rotate by $-\gamma$ about Z axis to bring α to X axis
- $\tan\gamma = \frac{\sqrt{\alpha_y^2+\alpha_z^2}}{\alpha_x} = \frac{\sqrt{1-\alpha_x^2}}{\alpha_x}$ if $|\alpha| = 1$.
- $\mathbf{R}_{\alpha\mathbf{x}} = \mathbf{R}_\mathbf{z}(-\gamma)\mathbf{R}_\mathbf{x}(-\beta)$ and $\mathbf{R}_{\alpha\mathbf{x}}^{-1} = \mathbf{R}_\mathbf{x}(\beta)\mathbf{R}_\mathbf{z}(\gamma)$
- Alternative: Don't we know about **rotation matrices** and direction cosines that go **to/from coordinate axes**?

Final

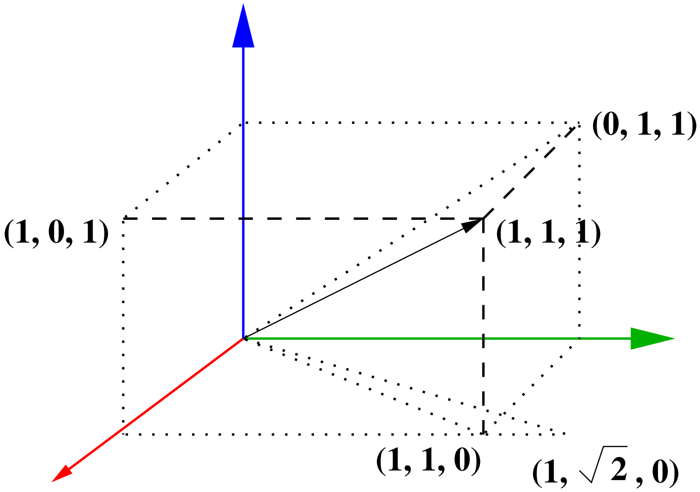
- $\mathbf{R}_\alpha(\theta) = \mathbf{R_x}(\beta)\mathbf{R_z}(\gamma) \quad \mathbf{R_x}(\theta) \quad \mathbf{R_z}(-\gamma)\mathbf{R_x}(-\beta)$



Alternate $\mathbf{R}_{\alpha\mathbf{x}}$

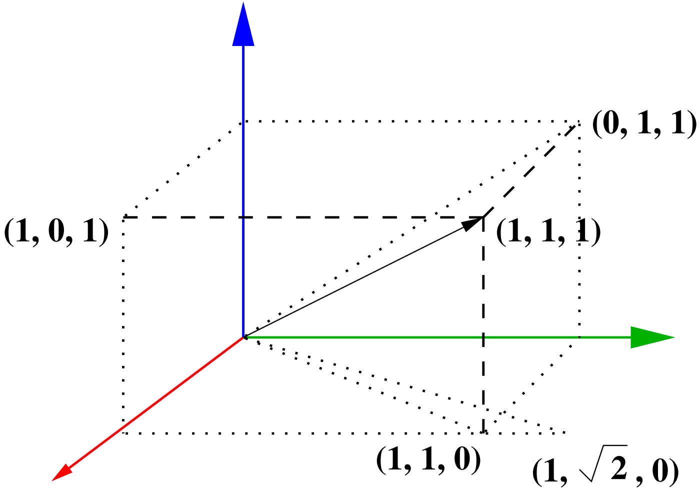
- After rotation, α will align with X-axis. Hence that is the first row $\mathbf{r_1}$ of the rotation matrix
- Find a direction orthogonal to α to be row $\mathbf{r_2}$. How?
- Take any vector $\mathbf{v}$ not parallel to α. $\mathbf{r_2} = \alpha \times \mathbf{v}$ will work!!
- Lastly, $\mathbf{r_3} = \mathbf{r_1} \times \mathbf{r_2}$ and $\mathbf{R}_{\alpha\mathbf{x}} = \begin{bmatrix} \alpha & 0 \\ \alpha \times \mathbf{v} & 0 \\ \mathbf{r_1} \times \mathbf{r_2} & 0 \\ \mathbf{0} & 1 \end{bmatrix}$
- Many possibilities, all with the same result (hopefully...)

Example: Rotation about $[1\ 1\ 1]^{T}$



$\beta = ?, \qquad \gamma = ?$

Example: Rotation about $[1\ 1\ 1]^T$



$$\tan\beta = 1, \qquad \tan\gamma = \sqrt{2}$$

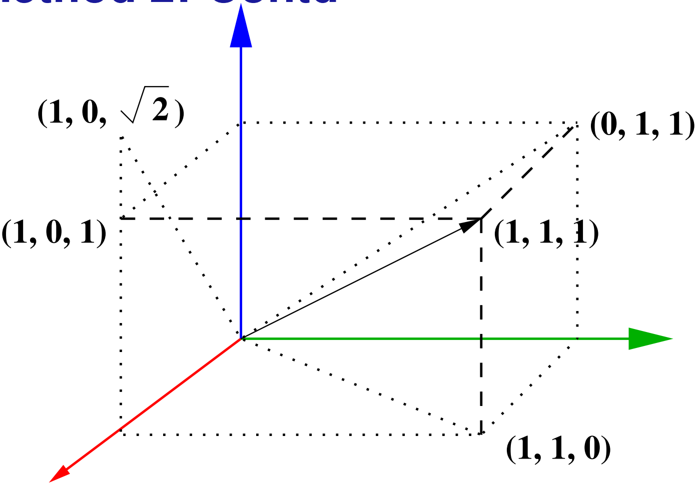
Computing $\mathbf{R}_{\alpha\mathbf{x}}$: Method 1

- Rotate by $-\pi/4$ about X. $\mathbf{R_X}(-\frac{\pi}{4}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$
- $\mathbf{R_Z}(-\arctan(\sqrt{\mathbf{2}})) = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $\mathbf{R}^{\mathbf{I}}_{\alpha\mathbf{x}} = \mathbf{R_Z}(-\tan^{-1}(\sqrt{\mathbf{2}}))\ \mathbf{R_X}(-\frac{\pi}{4}) = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Computing $\mathbf{R}_{\alpha\mathbf{x}}$: Method 2

- $[\mathbf{1}\ \mathbf{1}\ \mathbf{1}]^{\mathbf{T}}$ will lie on X-axis. First row $\mathbf{r_1} = [\frac{\mathbf{1}}{\sqrt{\mathbf{3}}}\ \ \frac{\mathbf{1}}{\sqrt{\mathbf{3}}}\ \ \frac{\mathbf{1}}{\sqrt{\mathbf{3}}}]^{\mathbf{T}}$.
- Second row: $\mathbf{r_2} \quad = \quad \alpha \times [\mathbf{1}\ \mathbf{0}\ \mathbf{0}]^{\mathbf{T}} \quad = \quad [\mathbf{0}\ \ \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\ \ \frac{-\mathbf{1}}{\sqrt{\mathbf{2}}}]^{\mathbf{T}}$
- Third row: $\mathbf{r_3} \quad = \quad \alpha \times [\mathbf{0}\ \ \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\ \ \frac{-\mathbf{1}}{\sqrt{\mathbf{2}}}]^{\mathbf{T}} \quad = \quad [\frac{\mathbf{2}}{\sqrt{\mathbf{6}}}\ \ \frac{-\mathbf{1}}{\sqrt{\mathbf{6}}}\ \ \frac{-\mathbf{1}}{\sqrt{\mathbf{6}}}]^{\mathbf{T}}$
- $$\mathbf{R}^{\mathbf{II}}_{\alpha\mathbf{x}} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{R_Y}(\tan^{-\mathbf{1}}(\sqrt{\mathbf{2}}))\ \mathbf{R_X}(\frac{\pi}{\mathbf{4}})$$

$\mathbf{R}_{\alpha\mathbf{x}}$ Method 2: Contd



Question: Which vector v yields the matrix of Method 1?