CHAPTER II

CONSERVATION LAWS

§6. Energy

DURING the motion of a mechanical system, the $2s$ quantities q_i and $\dot{q}_i$ ($i = 1, 2, \ldots, s$) which specify the state of the system vary with time. There exist, however, functions of these quantities whose values remain constant during the motion, and depend only on the initial conditions. Such functions are called *integrals of the motion*.

The number of independent integrals of the motion for a closed mechanical system with s degrees of freedom is $2s-1$. This is evident from the following simple arguments. The general solution of the equations of motion contains $2s$ arbitrary constants (see the discussion following equation (2.6)). Since the equations of motion for a closed system do not involve the time explicitly, the choice of the origin of time is entirely arbitrary, and one of the arbitrary constants in the solution of the equations can always be taken as an additive constant t_0 in the time. Eliminating $t+t_0$ from the $2s$ functions $q_i = q_i(t+t_0, C_1, C_2, \ldots, C_{2s-1})$, $\dot{q}_i = \dot{q}_i(t+t_0, C_1, C_2, \ldots, C_{2s-1})$, we can express the $2s-1$ arbitrary constants $C_1, C_2, \ldots, C_{2s-1}$ as functions of q and $\dot{q}$, and these functions will be integrals of the motion.

Not all integrals of the motion, however, are of equal importance in mechanics. There are some whose constancy is of profound significance, deriving from the fundamental homogeneity and isotropy of space and time. The quantities represented by such integrals of the motion are said to be *conserved*, and have an important common property of being additive: their values for a system composed of several parts whose interaction is negligible are equal to the sums of their values for the individual parts.

It is to this additivity that the quantities concerned owe their especial importance in mechanics. Let us suppose, for example, that two bodies interact during a certain interval of time. Since each of the additive integrals of the whole system is, both before and after the interaction, equal to the sum of its values for the two bodies separately, the conservation laws for these quantities immediately make possible various conclusions regarding the state of the bodies after the interaction, if their states before the interaction are known.

Let us consider first the conservation law resulting from the *homogeneity of time*. By virtue of this homogeneity, the Lagrangian of a closed system does not depend explicitly on time. The total time derivative of the Lagrangian can therefore be written

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i.$$

If L depended explicitly on time, a term $\partial L/\partial t$ would have to be added on the right-hand side. Replacing $\partial L/\partial q_i$, in accordance with Lagrange's equations, by $(\mathrm{d}/\mathrm{d}t)\, \partial L/\partial \dot{q}_i$, we obtain

$$\begin{aligned} \frac{\mathrm{d}L}{\mathrm{d}t} &= \sum_i \dot{q}_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \\ &= \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) \end{aligned}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right) = 0.$$

Hence we see that the quantity

$$E \equiv \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \tag{6.1}$$

remains constant during the motion of a closed system, i.e. it is an integral of the motion; it is called the *energy* of the system. The additivity of the energy follows immediately from that of the Lagrangian, since (6.1) shows that it is a linear function of the latter.

The law of conservation of energy is valid not only for closed systems, but also for those in a constant external field (i.e. one independent of time): the only property of the Lagrangian used in the above derivation, namely that it does not involve the time explicitly, is still valid. Mechanical systems whose energy is conserved are sometimes called *conservative* systems.

As we have seen in §5, the Lagrangian of a closed system (or one in a constant field) is of the form $L = T(q, \dot{q}) - U(q)$, where T is a quadratic function of the velocities. Using Euler's theorem on homogeneous functions, we have

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T.$$

Substituting this in (6.1) gives

$$E = T(q, \dot{q}) + U(q); \tag{6.2}$$

in Cartesian co-ordinates,

$$E = \sum_a \tfrac{1}{2} m_a v_a^2 + U(\mathbf{r}_1, \mathbf{r}_2, \ldots). \tag{6.3}$$

Thus the energy of the system can be written as the sum of two quite different terms: the kinetic energy, which depends on the velocities, and the potential energy, which depends only on the co-ordinates of the particles.

§7. Momentum

A second conservation law follows from the *homogeneity of space*. By virtue of this homogeneity, the mechanical properties of a closed system are unchanged by any parallel displacement of the entire system in space. Let us therefore consider an infinitesimal displacement $\boldsymbol{\epsilon}$, and obtain the condition for the Lagrangian to remain unchanged.

A parallel displacement is a transformation in which every particle in the system is moved by the same amount, the radius vector $\mathbf{r}$ becoming $\mathbf{r}+\boldsymbol{\epsilon}$. The change in L resulting from an infinitesimal change in the co-ordinates, the velocities of the particles remaining fixed, is

$$\delta L = \sum_a \frac{\partial L}{\partial \mathbf{r}_a}\cdot \delta \mathbf{r}_a = \boldsymbol{\epsilon}\cdot \sum_a \frac{\partial L}{\partial \mathbf{r}_a},$$

where the summation is over the particles in the system. Since $\boldsymbol{\epsilon}$ is arbitrary, the condition $\delta L = 0$ is equivalent to

$$\sum_a \partial L/\partial \mathbf{r}_a = 0. \tag{7.1}$$

From Lagrange's equations (5.2) we therefore have

$$\sum_a \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \mathbf{v}_a} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_a \frac{\partial L}{\partial \mathbf{v}_a} = 0.$$

Thus we conclude that, in a closed mechanical system, the vector

$$\mathbf{P} \equiv \sum_a \partial L/\partial \mathbf{v}_a \tag{7.2}$$

remains constant during the motion; it is called the *momentum* of the system. Differentiating the Lagrangian (5.1), we find that the momentum is given in terms of the velocities of the particles by

$$\mathbf{P} = \sum_a m_a \mathbf{v}_a. \tag{7.3}$$

The additivity of the momentum is evident. Moreover, unlike the energy, the momentum of the system is equal to the sum of its values $\mathbf{p}_a = m_a \mathbf{v}_a$ for the individual particles, whether or not the interaction between them can be neglected.

The three components of the momentum vector are all conserved only in the absence of an external field. The individual components may be conserved even in the presence of a field, however, if the potential energy in the field does not depend on all the Cartesian co-ordinates. The mechanical properties of the system are evidently unchanged by a displacement along the axis of a co-ordinate which does not appear in the potential energy, and so the corresponding component of the momentum is conserved. For example, in a uniform field in the z-direction, the x and y components of momentum are conserved.

The equation (7.1) has a simple physical meaning. The derivative $\partial L/\partial \mathbf{r}_a = -\partial U/\partial \mathbf{r}_a$ is the force $\mathbf{F}_a$ acting on the ath particle. Thus equation (7.1) signifies that the sum of the forces on all the particles in a closed system is zero:

$$\sum_a \mathbf{F}_a = 0. \tag{7.4}$$

In particular, for a system of only two particles, $\mathbf{F}_1 + \mathbf{F}_2 = 0$: the force exerted by the first particle on the second is equal in magnitude, and opposite in direction, to that exerted by the second particle on the first. This is the equality of action and reaction (*Newton's third law*).

If the motion is described by generalised co-ordinates q_i, the derivatives of the Lagrangian with respect to the generalised velocities

$$p_i = \partial L/\partial \dot{q}_i \tag{7.5}$$

are called *generalised momenta*, and its derivatives with respect to the generalised co-ordinates

$$F_i = \partial L/\partial q_i \tag{7.6}$$

are called *generalised forces*. In this notation, Lagrange's equations are

$$\dot{p}_i = F_i. \tag{7.7}$$

In Cartesian co-ordinates the generalised momenta are the components of the vectors $\mathbf{p}_a$. In general, however, the p_i are linear homogeneous functions of the generalised velocities $\dot{q}_i$, and do not reduce to products of mass and velocity.

PROBLEM

A particle of mass m moving with velocity $\mathbf{v}_1$ leaves a half-space in which its potential energy is a constant U_1 and enters another in which its potential energy is a different constant U_2. Determine the change in the direction of motion of the particle.

SOLUTION. The potential energy is independent of the co-ordinates whose axes are parallel to the plane separating the half-spaces. The component of momentum in that plane is therefore conserved. Denoting by θ_1 and θ_2 the angles between the normal to the plane and the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of the particle before and after passing the plane, we have $v_1 \sin\theta_1 = v_2 \sin\theta_2$. The relation between v_1 and v_2 is given by the law of conservation of energy, and the result is

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\left[1 + \frac{2}{mv_1^2}(U_1 - U_2)\right]}.$$

§8. Centre of mass

The momentum of a closed mechanical system has different values in different (inertial) frames of reference. If a frame K' moves with velocity $\mathbf{V}$

relative to another frame K, then the velocities $\mathbf{v}_a'$ and $\mathbf{v}_a$ of the particles relative to the two frames are such that $\mathbf{v}_a = \mathbf{v}_a' + \mathbf{V}$. The momenta $\mathbf{P}$ and $\mathbf{P}'$ in the two frames are therefore related by

$$\mathbf{P} = \sum_a m_a \mathbf{v}_a = \sum_a m_a \mathbf{v}_a' + \mathbf{V} \sum_a m_a,$$

or

$$\mathbf{P} = \mathbf{P}' + \mathbf{V} \sum_a m_a. \tag{8.1}$$

In particular, there is always a frame of reference K' in which the total momentum is zero. Putting $\mathbf{P}' = 0$ in (8.1), we find the velocity of this frame:

$$\mathbf{V} = \mathbf{P}/\sum m_a = \sum m_a \mathbf{v}_a / \sum m_a. \tag{8.2}$$

If the total momentum of a mechanical system in a given frame of reference is zero, it is said to be *at rest* relative to that frame. This is a natural generalisation of the term as applied to a particle. Similarly, the velocity $\mathbf{V}$ given by (8.2) is the velocity of the "motion as a whole" of a mechanical system whose momentum is not zero. Thus we see that the law of conservation of momentum makes possible a natural definition of rest and velocity, as applied to a mechanical system as a whole.

Formula (8.2) shows that the relation between the momentum $\mathbf{P}$ and the velocity $\mathbf{V}$ of the system is the same as that between the momentum and velocity of a single particle of mass $\mu = \Sigma m_a$, the sum of the masses of the particles in the system. This result can be regarded as expressing the *additivity of mass*.

The right-hand side of formula (8.2) can be written as the total time derivative of the expression

$$\mathbf{R} \equiv \sum m_a \mathbf{r}_a / \sum m_a. \tag{8.3}$$

We can say that the velocity of the system as a whole is the rate of motion in space of the point whose radius vector is (8.3). This point is called the *centre of mass* of the system.

The law of conservation of momentum for a closed system can be formulated as stating that the centre of mass of the system moves uniformly in a straight line. In this form it generalises the law of inertia derived in §3 for a single free particle, whose "centre of mass" coincides with the particle itself.

In considering the mechanical properties of a closed system it is natural to use a frame of reference in which the centre of mass is at rest. This eliminates a uniform rectilinear motion of the system as a whole, but such motion is of no interest.

The energy of a mechanical system which is at rest as a whole is usually called its *internal energy* E_i. This includes the kinetic energy of the relative motion of the particles in the system and the potential energy of their interaction. The total energy of a system moving as a whole with velocity V can be written

$$E = \tfrac{1}{2}\mu V^2 + E_i. \tag{8.4}$$

Although this formula is fairly obvious, we may give a direct proof of it. The energies E and E' of a mechanical system in two frames of reference K and K' are related by

$$\begin{aligned} E &= \tfrac{1}{2} \sum_a m_a v_a^2 + U \\ &= \tfrac{1}{2} \sum_a m_a(\mathbf{v}_a' + \mathbf{V})^2 + U \\ &= \tfrac{1}{2}\mu V^2 + \mathbf{V}\cdot \sum_a m_a \mathbf{v}_a' + \tfrac{1}{2} \sum_a m_a v_a'^2 + U \\ &= E' + \mathbf{V}\cdot \mathbf{P}' + \tfrac{1}{2}\mu V^2. \end{aligned} \tag{8.5}$$

This formula gives the law of transformation of energy from one frame to another, corresponding to formula (8.1) for momentum. If the centre of mass is at rest in K', then $\mathbf{P}' = 0$, $E' = E_i$, and we have (8.4).

PROBLEM

Find the law of transformation of the action S from one inertial frame to another.

Solution. The Lagrangian is equal to the difference of the kinetic and potential energies, and is evidently transformed in accordance with a formula analogous to (8.5):

$$L = L' + \mathbf{V}\cdot\mathbf{P}' + \tfrac{1}{2}\mu V^2.$$

Integrating this with respect to time, we obtain the required law of transformation of the action:

$$S = S' + \mu\mathbf{V}\cdot\mathbf{R}' + \tfrac{1}{2}\mu V^2 t,$$

where $\mathbf{R}'$ is the radius vector of the centre of mass in the frame K'.

§9. Angular momentum

Let us now derive the conservation law which follows from the *isotropy of space*. This isotropy means that the mechanical properties of a closed system do not vary when it is rotated as a whole in any manner in space. Let us therefore consider an infinitesimal rotation of the system, and obtain the condition for the Lagrangian to remain unchanged.

We shall use the vector $\delta\boldsymbol{\phi}$ of the infinitesimal rotation, whose magnitude is the angle of rotation $\delta\phi$, and whose direction is that of the axis of rotation (the direction of rotation being that of a right-handed screw driven along $\delta\boldsymbol{\phi}$).

Let us find, first of all, the resulting increment in the radius vector from an origin on the axis to any particle in the system undergoing rotation. The linear displacement of the end of the radius vector is related to the angle by $|\delta\mathbf{r}| = r\sin\theta\,\delta\phi$ (Fig. 5). The direction of $\delta\mathbf{r}$ is perpendicular to the plane of $\mathbf{r}$ and $\delta\boldsymbol{\phi}$. Hence it is clear that

$$\delta\mathbf{r} = \delta\boldsymbol{\phi}\times\mathbf{r}. \tag{9.1}$$

When the system is rotated, not only the radius vectors but also the velocities of the particles change direction, and all vectors are transformed in the same manner. The velocity increment relative to a fixed system of co-ordinates is

$$\delta \mathbf{v} = \delta \boldsymbol{\phi} \times \mathbf{v}. \tag{9.2}$$

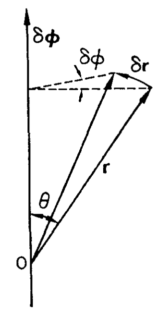

FIG. 5

If these expressions are substituted in the condition that the Lagrangian is unchanged by the rotation:

$$\delta L = \sum_a \left(\frac{\partial L}{\partial \mathbf{r}_a} \cdot \delta \mathbf{r}_a + \frac{\partial L}{\partial \mathbf{v}_a} \cdot \delta \mathbf{v}_a \right) = 0$$

and the derivative $\partial L/\partial \mathbf{v}_a$ replaced by $\mathbf{p}_a$, and $\partial L/\partial \mathbf{r}_a$ by $\dot{\mathbf{p}}_a$, the result is

$$\sum_a (\dot{\mathbf{p}}_a \cdot \delta \boldsymbol{\phi} \times \mathbf{r}_a + \mathbf{p}_a \cdot \delta \boldsymbol{\phi} \times \mathbf{v}_a) = 0$$

or, permuting the factors and taking $\delta \boldsymbol{\phi}$ outside the sum,

$$\delta \boldsymbol{\phi} \sum_a (\mathbf{r}_a \times \dot{\mathbf{p}}_a + \mathbf{v}_a \times \mathbf{p}_a) = \delta \boldsymbol{\phi} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \sum_a \mathbf{r}_a \times \mathbf{p}_a = 0.$$

Since $\delta \boldsymbol{\phi}$ is arbitrary, it follows that $(\mathrm{d}/\mathrm{d}t) \Sigma \mathbf{r}_a \times \mathbf{p}_a = 0$, and we conclude that the vector

$$\mathbf{M} \equiv \sum_a \mathbf{r}_a \times \mathbf{p}_a, \tag{9.3}$$

called the *angular momentum* or *moment of momentum* of the system, is conserved in the motion of a closed system. Like the linear momentum, it is additive, whether or not the particles in the system interact.

There are no other additive integrals of the motion. Thus every closed system has seven such integrals: energy, three components of momentum, and three components of angular momentum.

Since the definition of angular momentum involves the radius vectors of the particles, its value depends in general on the choice of origin. The radius vectors $\mathbf{r}_a$ and $\mathbf{r}_a'$ of a given point relative to origins at a distance $\mathbf{a}$ apart are related by $\mathbf{r}_a = \mathbf{r}_a' + \mathbf{a}$. Hence

$$\begin{aligned}\mathbf{M} &= \sum_a \mathbf{r}_a \times \mathbf{p}_a \\ &= \sum_a \mathbf{r}_a' \times \mathbf{p}_a + \mathbf{a} \times \sum_a \mathbf{p}_a \\ &= \mathbf{M}' + \mathbf{a} \times \mathbf{P}. \end{aligned} \tag{9.4}$$

It is seen from this formula that the angular momentum depends on the choice of origin except when the system is at rest as a whole (i.e. $\mathbf{P} = 0$). This indeterminacy, of course, does not affect the law of conservation of angular momentum, since momentum is also conserved in a closed system.

We may also derive a relation between the angular momenta in two inertial frames of reference K and K', of which the latter moves with velocity $\mathbf{V}$ relative to the former. We shall suppose that the origins in the frames K and K' coincide at a given instant. Then the radius vectors of the particles are the same in the two frames, while their velocities are related by $\mathbf{v}_a = \mathbf{v}_a' + \mathbf{V}$. Hence we have

$$\mathbf{M} = \sum_a m_a \mathbf{r}_a \times \mathbf{v}_a = \sum_a m_a \mathbf{r}_a \times \mathbf{v}_a' + \sum_a m_a \mathbf{r}_a \times \mathbf{V}.$$

The first sum on the right-hand side is the angular momentum $\mathbf{M}'$ in the frame K'; using in the second sum the radius vector of the centre of mass (8.3), we obtain

$$\mathbf{M} = \mathbf{M}' + \mu \mathbf{R} \times \mathbf{V}. \tag{9.5}$$

This formula gives the law of transformation of angular momentum from one frame to another, corresponding to formula (8.1) for momentum and (8.5) for energy.

If the frame K' is that in which the system considered is at rest as a whole, then $\mathbf{V}$ is the velocity of its centre of mass, $\mu\mathbf{V}$ its total momentum $\mathbf{P}$ relative to K, and

$$\mathbf{M} = \mathbf{M}' + \mathbf{R} \times \mathbf{P}. \tag{9.6}$$

In other words, the angular momentum $\mathbf{M}$ of a mechanical system consists of its "intrinsic angular momentum" in a frame in which it is at rest, and the angular momentum $\mathbf{R} \times \mathbf{P}$ due to its motion as a whole.

Although the law of conservation of all three components of angular momentum (relative to an arbitrary origin) is valid only for a closed system, the law of conservation may hold in a more restricted form even for a system in an external field. It is evident from the above derivation that the component of angular momentum along an axis about which the field is symmetrical is always conserved, for the mechanical properties of the system are unaltered by any rotation about that axis. Here the angular momentum must, of course, be defined relative to an origin lying on the axis.

The most important such case is that of a *centrally symmetric field* or *central field*, i.e. one in which the potential energy depends only on the distance from some particular point (the *centre*). It is evident that the component of angular momentum along any axis passing through the centre is conserved in motion in such a field. In other words, the angular momentum **M** is conserved provided that it is defined with respect to the centre of the field.

Another example is that of a homogeneous field in the z-direction; in such a field, the component M_z of the angular momentum is conserved, whichever point is taken as the origin.

The component of angular momentum along any axis (say the z-axis) can be found by differentiation of the Lagrangian:

$$M_z = \sum_a \frac{\partial L}{\partial \dot{\phi}_a}, \tag{9.7}$$

where the co-ordinate ϕ is the angle of rotation about the z-axis. This is evident from the above proof of the law of conservation of angular momentum, but can also be proved directly. In cylindrical co-ordinates r, ϕ, z we have (substituting $x_a = r_a \cos\phi_a$, $y_a = r_a \sin\phi_a$)

$$\begin{aligned} M_z &= \sum_a m_a(x_a\dot{y}_a - y_a\dot{x}_a) \\ &= \sum_a m_a r_a{}^2\dot{\phi}_a. \end{aligned} \tag{9.8}$$

The Lagrangian is, in terms of these co-ordinates,

$$L = \tfrac{1}{2}\sum_a m_a(\dot{r}_a{}^2 + r_a{}^2\dot{\phi}_a{}^2 + \dot{z}_a{}^2) - U,$$

and substitution of this in (9.7) gives (9.8).

PROBLEMS

PROBLEM 1. Obtain expressions for the Cartesian components and the magnitude of the angular momentum of a particle in cylindrical co-ordinates r, ϕ, z.

SOLUTION.
$$\begin{aligned} M_x &= m(r\dot{z} - z\dot{r})\sin\phi - mrz\dot{\phi}\cos\phi, \\ M_y &= -m(r\dot{z} - z\dot{r})\cos\phi - mrz\dot{\phi}\sin\phi, \\ M_z &= mr^2\dot{\phi}, \\ M^2 &= m^2r^2\dot{\phi}^2(r^2+z^2) + m^2(r\dot{z} - z\dot{r})^2. \end{aligned}$$

PROBLEM 2. The same as Problem 1, but in spherical co-ordinates r, θ, ϕ.

SOLUTION.
$$\begin{aligned} M_x &= -mr^2(\dot{\theta}\sin\phi + \dot{\phi}\sin\theta\cos\theta\cos\phi), \\ M_y &= mr^2(\dot{\theta}\cos\phi - \dot{\phi}\sin\theta\cos\theta\sin\phi), \\ M_z &= mr^2\dot{\phi}\sin^2\theta, \\ M^2 &= m^2r^4(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta). \end{aligned}$$

PROBLEM 3. Which components of momentum **P** and angular momentum **M** are conserved in motion in the following fields?

(a) the field of an infinite homogeneous plane, (b) that of an infinite homogeneous cylinder, (c) that of an infinite homogeneous prism, (d) that of two points, (e) that of an infinite homogeneous half-plane, (f) that of a homogeneous cone, (g) that of a homogeneous circular torus, (h) that of an infinite homogeneous cylindrical helix.

SOLUTION. (a) P_x, P_y, M_z (if the plane is the xy-plane), (b) M_z, P_z (if the axis of the cylinder is the z-axis), (c) P_z (if the edges of the prism are parallel to the z-axis), (d) M_z (if the line joining the points is the z-axis), (e) P_y (if the edge of the half-plane is the y-axis), (f) M_z (if the axis of the cone is the z-axis), (g) M_z (if the axis of the torus is the z-axis), (h) the Lagrangian is unchanged by a rotation through an angle $\delta\phi$ about the axis of the helix (let this be the z-axis) together with a translation through a distance $h\delta\phi/2\pi$ along the axis (h being the pitch of the helix). Hence $\delta L = \delta z\ \partial L/\partial z + \delta\phi\ \partial L/\partial\phi = \delta\phi(hP_z/2\pi + M_z) = 0$, so that $M_z + hP_z/2\pi = \text{constant}$.

§10. Mechanical similarity

Multiplication of the Lagrangian by any constant clearly does not affect the equations of motion. This fact (already mentioned in §2) makes possible, in a number of important cases, some useful inferences concerning the properties of the motion, without the necessity of actually integrating the equations.

Such cases include those where the potential energy is a homogeneous function of the co-ordinates, i.e. satisfies the condition

$$U(\alpha\mathbf{r}_1, \alpha\mathbf{r}_2, \ldots, \alpha\mathbf{r}_n) = \alpha^k U(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n), \tag{10.1}$$

where α is any constant and k the degree of homogeneity of the function.

Let us carry out a transformation in which the co-ordinates are changed by a factor α and the time by a factor β: $\mathbf{r}_a \to \alpha\mathbf{r}_a$, $t \to \beta t$. Then all the velocities $\mathbf{v}_a = \mathrm{d}\mathbf{r}_a/\mathrm{d}t$ are changed by a factor α/β, and the kinetic energy by a factor α^2/β^2. The potential energy is multiplied by α^k. If α and β are such that $\alpha^2/\beta^2 = \alpha^k$, i.e. $\beta = \alpha^{1-\frac{1}{2}k}$, then the result of the transformation is to multiply the Lagrangian by the constant factor α^k, i.e. to leave the equations of motion unaltered.

A change of all the co-ordinates of the particles by the same factor corresponds to the replacement of the paths of the particles by other paths, geometrically similar but differing in size. Thus we conclude that, if the potential energy of the system is a homogeneous function of degree k in the (Cartesian) co-ordinates, the equations of motion permit a series of geometrically similar paths, and the times of the motion between corresponding points are in the ratio

$$t'/t = (l'/l)^{1-\frac{1}{2}k}, \tag{10.2}$$

where l'/l is the ratio of linear dimensions of the two paths. Not only the times but also any mechanical quantities at corresponding points at corresponding times are in a ratio which is a power of l'/l. For example, the velocities, energies and angular momenta are such that

$$v'/v = (l'/l)^{\frac{1}{2}k}, \qquad E'/E = (l'/l)^k, \qquad M'/M = (l'/l)^{1+\frac{1}{2}k}. \tag{10.3}$$

The following are some examples of the foregoing.

As we shall see later, in *small oscillations* the potential energy is a quadratic function of the co-ordinates ($k = 2$). From (10.2) we find that the period of such oscillations is independent of their amplitude.

In a uniform field of force, the potential energy is a linear function of the co-ordinates (see (5.8)), i.e. $k = 1$. From (10.2) we have $t'/t = \sqrt{(l'/l)}$. Hence, for example, it follows that, in fall under gravity, the time of fall is as the square root of the initial altitude.

In the Newtonian attraction of two masses or the Coulomb interaction of two charges, the potential energy is inversely proportional to the distance apart, i.e. it is a homogeneous function of degree $k = -1$. Then $t'/t = (l'/l)^{3/2}$, and we can state, for instance, that the square of the time of revolution in the orbit is as the cube of the size of the orbit (*Kepler's third law*).

If the potential energy is a homogeneous function of the co-ordinates and the motion takes place in a finite region of space, there is a very simple relation between the time average values of the kinetic and potential energies, known as the *virial theorem*.

Since the kinetic energy T is a quadratic function of the velocities, we have by Euler's theorem on homogeneous functions $\Sigma \mathbf{v}_a \cdot \partial T/\partial \mathbf{v}_a = 2T$, or, putting $\partial T/\partial \mathbf{v}_a = \mathbf{p}_a$, the momentum,

$$2T = \sum_a \mathbf{p}_a \cdot \mathbf{v}_a = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_a \mathbf{p}_a \cdot \mathbf{r}_a\Big) - \sum_a \mathbf{r}_a \cdot \dot{\mathbf{p}}_a. \tag{10.4}$$

Let us average this equation with respect to time. The average value of any function of time $f(t)$ is defined as

$$\bar{f} = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau f(t)\,\mathrm{d}t.$$

It is easy to see that, if $f(t)$ is the time derivative $\mathrm{d}F(t)/\mathrm{d}t$ of a bounded function $F(t)$, its mean value is zero. For

$$\bar{f} = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau \frac{\mathrm{d}F}{\mathrm{d}t}\,\mathrm{d}t = \lim_{\tau\to\infty} \frac{F(\tau)-F(0)}{\tau} = 0.$$

Let us assume that the system executes a motion in a finite region of space and with finite velocities. Then $\Sigma \mathbf{p}_a \cdot \mathbf{r}_a$ is bounded, and the mean value of the first term on the right-hand side of (10.4) is zero. In the second term we replace $\dot{\mathbf{p}}_a$ by $-\partial U/\partial \mathbf{r}_a$ in accordance with Newton's equations (5.3), obtaining†

$$2\bar{T} = \overline{\sum_a \mathbf{r}_a \cdot \partial U/\partial \mathbf{r}_a}. \tag{10.5}$$

If the potential energy is a homogeneous function of degree k in the radius vectors $\mathbf{r}_a$, then by Euler's theorem equation (10.5) becomes the required relation:

$$2\bar{T} = k\bar{U}. \tag{10.6}$$

† The expression on the right of (10.5) is sometimes called the *virial* of the system.

Since $\bar{T}+\bar{U} = \bar{E} = E$, the relation (10.6) can also be expressed as

$$\bar{U} = 2E/(k+2), \qquad \bar{T} = kE/(k+2), \tag{10.7}$$

which express $\bar{U}$ and $\bar{T}$ in terms of the total energy of the system.

In particular, for small oscillations ($k = 2$) we have $\bar{T} = \bar{U}$, i.e. the mean values of the kinetic and potential energies are equal. For a Newtonian interaction ($k = -1$) $2\bar{T} = -\bar{U}$, and $E = -\bar{T}$, in accordance with the fact that, in such an interaction, the motion takes place in a finite region of space only if the total energy is negative (see §15).

PROBLEMS

PROBLEM 1. Find the ratio of the times in the same path for particles having different masses but the same potential energy.

SOLUTION. $t'/t = \sqrt{(m'/m)}$.

PROBLEM 2. Find the ratio of the times in the same path for particles having the same mass but potential energies differing by a constant factor.

SOLUTION. $t'/t = \sqrt{(U/U')}$.