Continuous Random variables

Random Variables

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A random variable where X can take on a range of values, not just particular ones.

<u>Examples</u>:

Heights

Distance a golfer hits the ball with their driver

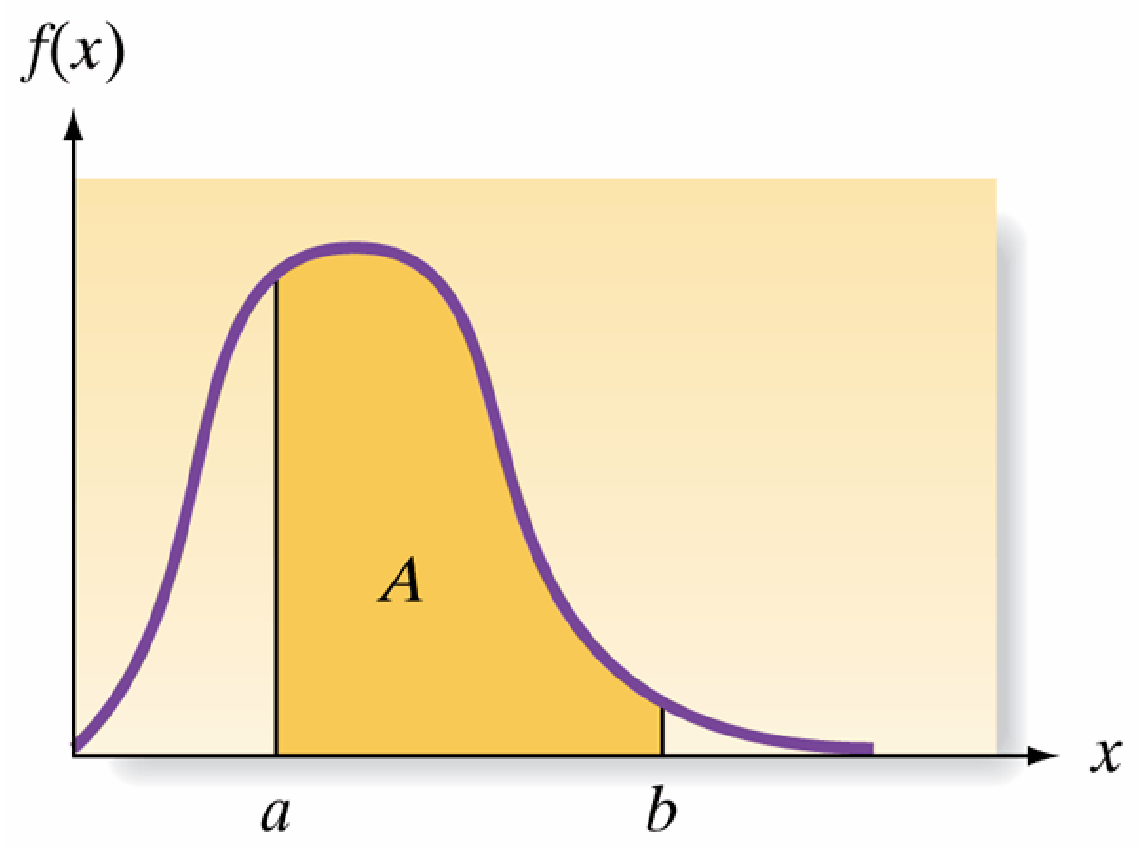
Time to run 100 meters

Electricity usage of a home.

For a discrete random variable, probabilities are given as a table of values, and the distribution can be graphed as a bar graph.

For a continuous random variable, probabilities are specified by a continuous function. The graph of the probability distribution function is a curve.

A probability $f(x)$ for a continuous random variable x



Definition

The **probability distribution for a continuous random variable, *x*,** can be represented by a smooth curve—a function of x, denoted $f(x)$. The curve is called a **density function** or **frequency function.** The probability that x falls between two values, a and b, i.e., $P(a < x < b)$, is the area under the curve between a and b.

We say that X is a *continuous* random variable if there

exists a nonnegative function f , defined for all real $x \in (-\infty, \infty)$

The function f is called the *probability density function* of the random variable X.

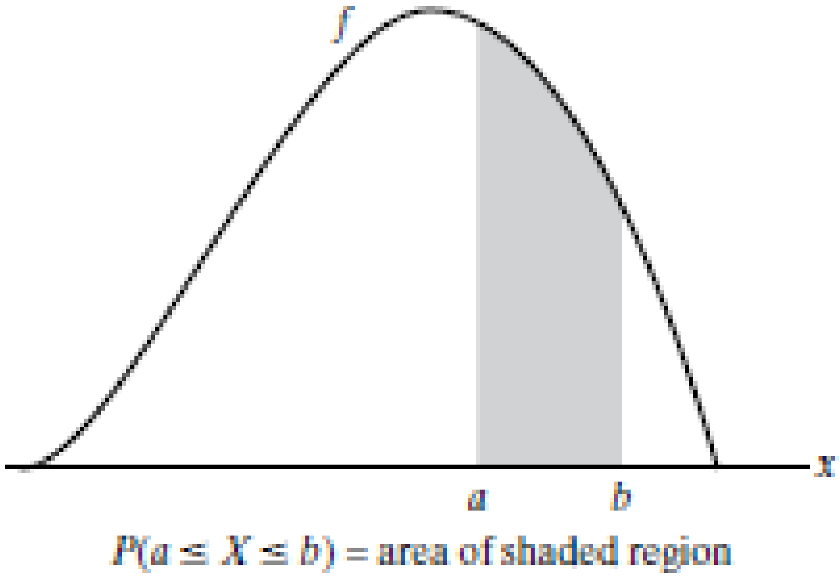


FIGURE 1: Probability density function f.

f must satisfy

$$1 = P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x)\, dx$$

$$P\{a \le X \le b\} = \int_a^b f(x)\, dx \qquad (1)$$

If we let $a = b$ in Equation (1), we get

$$P\{X = a\} = \int_a^a f(x)\,dx = 0$$

From another point of view, the probability at a single point is =0 as the sample space is Infinite, this appears in the denominator.

Suppose that X is a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} C(4x - 2x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of C?
(b) Find $P\{X > 1\}$.

Solution. **(a)** Since f is a probability density function, we must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$, implying that

$$C \int_0^2 (4x - 2x^2)\,dx = 1$$

$$C\left[2x^2 - \frac{2x^3}{3}\right]\Bigg|_{x=0}^{x=2} = 1$$

$$C = \frac{3}{8}$$

(b) $P\{X > 1\} = \int_1^{\infty} f(x)\,dx = \frac{3}{8}\int_1^2 (4x - 2x^2)\,dx = \frac{1}{2}$

The amount of time in hours that a computer functions before breaking down is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \lambda e^{-x/100} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

What is the probability that

(a) a computer will function between 50 and 150 hours before breaking down?
(b) it will function for fewer than 100 hours?

Solution. (a) Since

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \lambda \int_{0}^{\infty} e^{-x/100}\,dx$$

$$1 = -\lambda(100)e^{-x/100}\Big|_{0}^{\infty} = 100\lambda \quad \text{or} \quad \lambda = \frac{1}{100}$$

Hence, the probability that a computer will function between 50 and 150 hours before breaking down is given by

$$P\{50 < X < 150\} = \int_{50}^{150} \frac{1}{100} e^{-x/100}\,dx = -e^{-x/100}\Big|_{50}^{150}$$

$$= e^{-1/2} - e^{-3/2} \approx .384$$

$$P\{X < 100\} = \int_{0}^{100} \frac{1}{100} e^{-x/100}\,dx = -e^{-x/100}\Big|_{0}^{100} = 1 - e^{-1} \approx .633$$

In other words, approximately 63.3 percent of the time, a computer will fail before registering 100 hours of use. ■

The relationship between the cumulative distribution F and the probability density f is expressed by

$$F(a) = P\{X \in (-\infty, a]\} = \int_{-\infty}^{a} f(x)\,dx$$

Differentiating both sides of the preceding equation yields

$$\frac{d}{da}F(a) = f(a)$$

If X is a continuous random variable with distribution f, then

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

In either case, F is monotonic increasing, i.e.

$$F(a) \leq F(b) \quad \text{whenever} \quad a \leq b$$

and the limit of F to the left is 0 and to the right is 1:

$$\lim_{x \to -\infty} F(x) = 0 \qquad \text{and} \qquad \lim_{x \to \infty} F(x) = 1$$

Let X be a random variable with probability density function

$$f(x) = \begin{cases} c(1 - x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of c?

(b) What is the cumulative distribution function of X?

(a) $c\int_{-1}^{1}(1-x^2)dx = 1 \Rightarrow c = 3/4$

(b) $F(x) = \dfrac{3}{4}\int_{-1}^{x}(1-x^2)dx = \dfrac{3}{4}\left(x - \dfrac{x^3}{3} + \dfrac{2}{3}\right),\ -1 < x < 1$

A system consisting of one original unit plus a spare can function for a random amount of time X. If the density of X is given (in units of months) by

$$f(x) = \begin{cases} Cxe^{-x/2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

what is the probability that the system functions for at least 5 months?

$$\int xe^{-x/2}dx = -2xe^{-x/2} - 4e^{-x/2}$$. Hence,

$$c\int_0^\infty xe^{-x/2}dx = 1 \Rightarrow c = 1/4$$

0

$$P\{X > 5\} = \frac{1}{4}\int_{5}^{\infty} x e^{-x/2} dx = \frac{1}{4}[10e^{-5/2} + 4e^{-5/2}]$$

$$= \frac{14}{4} e^{-5/2}$$

the expected value of a discrete random variable X

$$E[X] = \sum_{x} xP\{X = x\}$$

If X is a continuous random variable having probability density function $f(x)$, then, because

$$f(x)\,dx \approx P\{x \le X \le x + dx\} \quad \text{for } dx \text{ small}$$

it is easy to see that the analogous definition is to define the expected value of X by

$$E[X] = \int_{-\infty}^{\infty} xf(x)\,dx$$

Find $E[X]$ when the density function of X is

$$f(x) = \begin{cases} 2x & \text{if } \quad 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} E[X] &= \int x f(x)\, dx \\ &= \int_0^1 2x^2\, dx \\ &= \frac{2}{3} \end{aligned}$$

If X is a continuous random variable with probability density function $f(x)$, then, for any real-valued function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx$$

Recall that :

Functions of Random Variables

If X is a random variable and $Y = g(X)$, then Y itself is a random variable. Thus, we can talk about its PMF, CDF, and expected value. First, note that the range of Y can be written as

$$R_Y = \{g(x) | x \in R_X\}.$$

If we already know the PMF of X, to find the PMF of $Y = g(X)$, we can write

$$\begin{aligned} P_Y(y) &= P(Y = y) \\ &= P(g(X) = y) \\ &= \sum_{x:g(x)=y} P_X(x) \end{aligned}$$

For some constant c, the random variable X has the probability density function

$$f(x) = \begin{cases} cx^4 & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find (a) $E[X]$ and (b) $\mathrm{Var}(X)$.

find c by using

$$1 = \int_0^2 cx^4 dx = 32c/5 \Rightarrow c = 5/32$$

(a) $E[X] = \frac{5}{32}\int_0^2 x^5 dx = \frac{5}{32}\frac{64}{6} = 5/3$

(b) $E[X^2] = \frac{5}{32}\int_0^2 x^6 dx = \frac{5}{32}\frac{128}{7} = 20/7 \Rightarrow \mathrm{Var}(X) = 20/7 - (5/3)^2 = 5/63$

The random variable X has the probability density function

$$f(x) = \begin{cases} ax + bx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E[X] = .6$, find (a) $P\{X < \frac{1}{2}\}$ and (b) $\mathrm{Var}(X)$.

$$1 = \int_0^1 (ax + bx^2)dx = a/2 + b/3$$

$$.6 = \int_0^1 (ax^2 + bx^3)dx = a/3 + b/4$$

$a = 3.6, b = -2.4.$

(a) $P\{X < 1/2\} = \int_0^{1/2}(3.6x - 2.4x^2)dx = (1.8x^2 - .8x^3)\Big|_0^{1/2} = .35$

(b) $E[X^2] = \int_0^1(3.6x^3 - 2.4x^4)dx = .42 \Rightarrow \mathrm{Var}(X) = .06$

The lifetime in hours of an electronic tube is a random variable having a probability density function given by

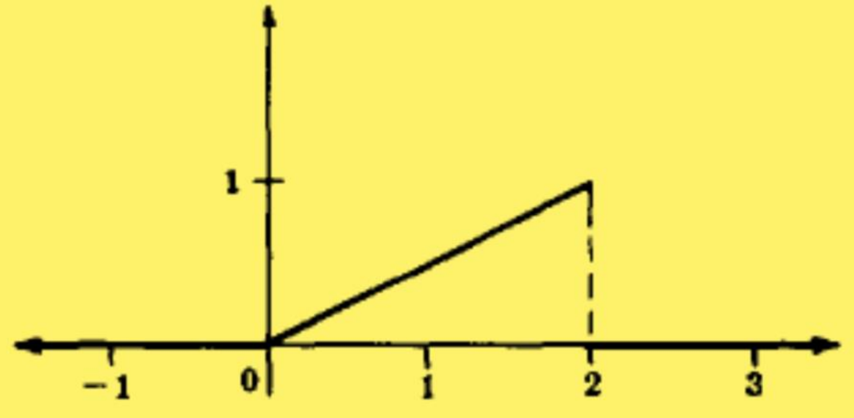
$$f(x) = xe^{-x} \qquad x \geq 0$$

Compute the expected lifetime of such a tube.

$$E[X] = \int_0^\infty x^2 e^{-x}\,dx \quad = 2$$

Let X be a continuous random variable with the following distribution:

$$f(x) = \begin{cases} \frac{1}{2}x & \text{if } 0 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$



Graph of f

Find the Distribution function

The cumulative distribution function F and its graph follows:

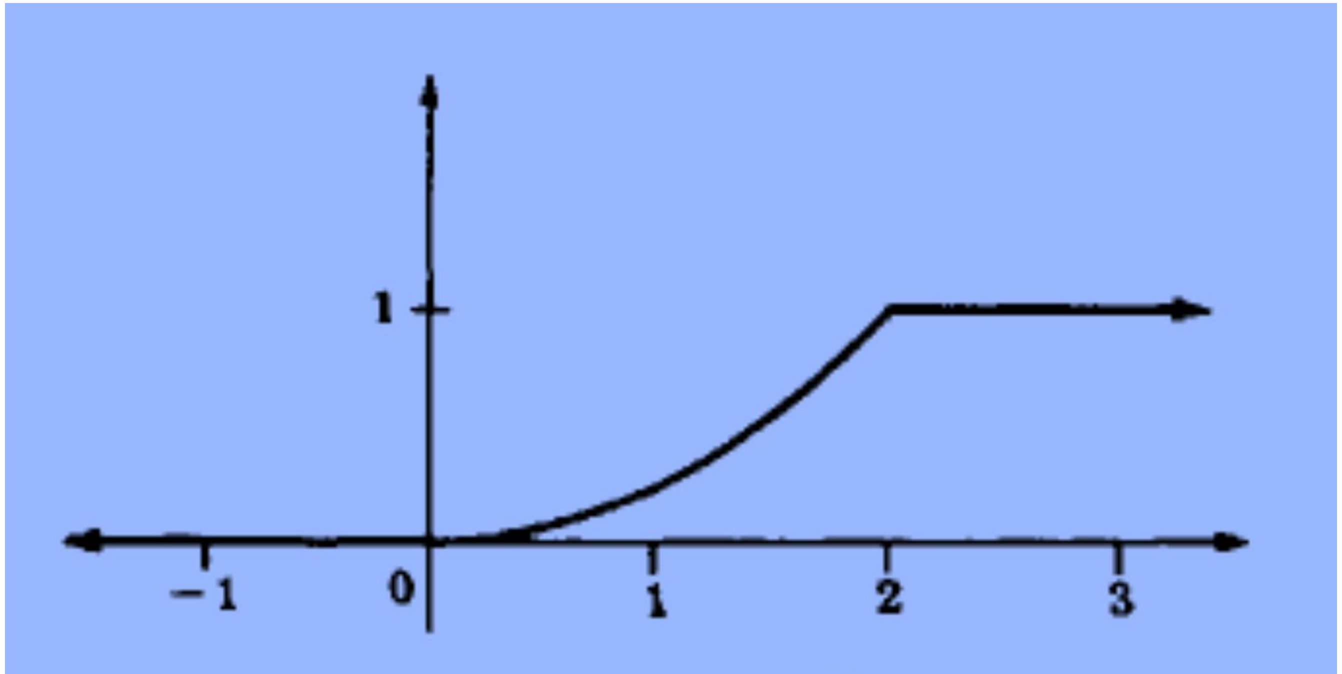
$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{4}x^2 & \text{if } 0 \leq x \leq 2 \\ 1 & \text{if } x > 2 \end{cases}$$

Here we use the fact that for $0 \leq x \leq 2$,

$$F(x) = \int_0^x \tfrac{1}{2}t\,dt = \tfrac{1}{4}x^2$$

The Cumulative probability at x=2 is 1, it has cumulated to 1 at 2

so after 2 , as the total probability cannot exceed 1, F(x)=1 for x>2



Graph of F

Find the probability density function given the distribution function

$$F(x) = \begin{cases} 0 & x \leq 0 \\ 10x^9 - 9x^{10} & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

$F'(x) = 90x^8 - 90x^9$ *for* $0 < x < 1$ *and* $F'(x) = 0$ *for* $-\infty < x \leq 0$ *and* $1 \leq x < \infty$, *which is just the pdf of* X.

The cumulative distribution function of random variable X is

$$F_X(x) = \begin{cases} 0 & x < -1, \\ (x+1)/2 & -1 \leq x < 1, \\ 1 & x \geq 1. \end{cases}$$

(a) What is $P[X > 1/2]$?

(b) What is $P[-1/2 < X \leq 3/4]$?

(c) What is $P[|X| \leq 1/2]$?

(d) What is the value of a such that $P[X \leq a] = 0.8$?

The CDF of X is

$$F_X(x) = \begin{cases} 0 & x < -1 \\ (x+1)/2 & -1 \le x < 1 \\ 1 & x \ge 1 \end{cases} \tag{1}$$

Each question can be answered by expressing the requested probability in terms of $F_X(x)$.

(a)

$$P[X > 1/2] = 1 - P[X \le 1/2] = 1 - F_X(1/2) = 1 - 3/4 = 1/4 \tag{2}$$

(b) This is a little trickier than it should be. Being careful, we can write

$$P[-1/2 \leq X < 3/4] = P[-1/2 < X \leq 3/4] + P[X = -1/2] - P[X = 3/4] \quad (3)$$

the probability that X takes on any specific value is zero. This implies $P[X = 3/4] = 0$ and $P[X = -1/2] = 0$.

Thus,

$$P[-1/2 \leq X < 3/4] = P[-1/2 < X \leq 3/4] = F_X(3/4) - F_X(-1/2) = 5/8 \quad (4)$$

(c)

$$P\left[|X| \leq 1/2\right] = P\left[-1/2 \leq X \leq 1/2\right] = P\left[X \leq 1/2\right] - P\left[X < -1/2\right] \tag{5}$$

Note that $P[X \leq 1/2] = F_X(1/2) = 3/4$. Since the probability that $P[X = -1/2] = 0$, $P[X < -1/2] = P[X \leq 1/2]$. Hence $P[X < -1/2] = F_X(-1/2) = 1/4$. This implies

$$P\left[|X| \leq 1/2\right] = P\left[X \leq 1/2\right] - P\left[X < -1/2\right] = 3/4 - 1/4 = 1/2 \tag{6}$$

(d) Since $F_X(1) = 1$, we must have $a \le 1$. For $a \le 1$, we need to satisfy

$$P[X \le a] = F_X(a) = \frac{a+1}{2} = 0.8 \tag{7}$$

Thus $a = 0.6$.

The cumulative distribution function of the continuous random variable V is

$$F_V(v) = \begin{cases} 0 & v < -5, \\ c(v+5)^2 & -5 \le v < 7, \\ 1 & v \ge 7. \end{cases}$$

(a) What is c?

(b) What is $P[V > 4]$?

(c) $P[-3 < V \le 0]$?

(d) What is the value of a such that $P[V > a] = 2/3$?

The CDF of V was given to be

$$F_V(v) = \begin{cases} 0 & v < -5 \\ c(v+5)^2 & -5 \leq v < 7 \\ 1 & v \geq 7 \end{cases} \tag{1}$$

(a) For V to be a continuous random variable, $F_V(v)$ must be a continuous function. This occurs if we choose c such that $F_V(v)$ doesn't have a discontinuity at $v = 7$. We meet this requirement if $c(7+5)^2 = 1$. This implies $c = 1/144$.

(b)

$$P[V > 4] = 1 - P[V \leq 4] = 1 - F_V(4) = 1 - 81/144 = 63/144 \tag{2}$$

(c)

$$P[-3 < V \leq 0] = F_V(0) - F_V(-3) = 25/144 - 4/144 = 21/144 \tag{3}$$

(d) Since $0 \leq F_V(v) \leq 1$ and since $F_V(v)$ is a nondecreasing function, it must be that $-5 \leq a \leq 7$. In this range,

$$P[V > a] = 1 - F_V(a) = 1 - (a+5)^2/144 = 2/3 \tag{4}$$

The unique solution in the range $-5 \leq a \leq 7$ is $a = 4\sqrt{3} - 5 = 1.928$.

For any random variable X,

(a) $E[X - \mu_X] = 0$,

(b) $E[aX + b] = aE[X] + b$,

(c) $\mathrm{Var}[X] = E[X^2] - \mu_X^2$,

(d) $\mathrm{Var}[aX + b] = a^2 \mathrm{Var}[X]$.

The method of calculating expected values depends on the type of random variable, discrete or continuous.

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx, \qquad \mathrm{Var}[X] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\, dx.$$

Our interpretation of expected values of discrete random variables carries over to continuous random variables. $E[X]$ represents a typical value of X, and the variance describes the dispersion of outcomes relative to the expected value. Furthermore, if we view the PDF $f_X(x)$ as the density of a mass distributed on a line, then $E[X]$ is the center of mass.

Random variable X has CDF

$$F_X(x) = \begin{cases} 0 & x < 0, \\ x/2 & 0 \leq x \leq 2, \\ 1 & x > 2. \end{cases}$$

(a) What is $E[X]$?

(b) What is $\mathrm{Var}[X]$?

(a) To find $E[X]$, we first find the PDF by differentiating the above CDF.

$$f_X(x) = \begin{cases} 1/2 & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

The expected value is then

$$E[X] = \int_0^2 \frac{x}{2}\, dx = 1$$

(b)

$$
\begin{aligned}
E\left[X^2\right] &= \int_0^2 \frac{x^2}{2}\,dx = 8/3 \\
\mathrm{Var}[X] &= E\left[X^2\right] - E\left[X\right]^2 = 8/3 - 1 = 5/3
\end{aligned}
$$

The random variable X has probability density function

$$f_X(x) = \begin{cases} cx & 0 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Use the PDF to find

(a) the constant c,

(b) $P[0 \le X \le 1]$,

(c) $P[-1/2 \le X \le 1/2]$,

(d) the CDF $F_X(x)$.

$$f_X(x) = \begin{cases} cx & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) From the above PDF we can determine the value of c by integrating the PDF and setting it equal to 1.

$$\int_0^2 cx\, dx = 2c = 1 \tag{2}$$

Therefore $c = 1/2$.

(b) $P[0 \leq X \leq 1] = \int_0^1 \frac{x}{2}\, dx = 1/4$

(c) $P[-1/2 \leq X \leq 1/2] = \int_0^{1/2} \frac{x}{2}\, dx = 1/16$

(d) The CDF of X is found by integrating the PDF from 0 to x.

$$F_X(x) = \int_0^x f_X(x')\, dx' = \begin{cases} 0 & x < 0 \\ x^2/4 & 0 \le x \le 2 \\ 1 & x > 2 \end{cases}$$