Science -I

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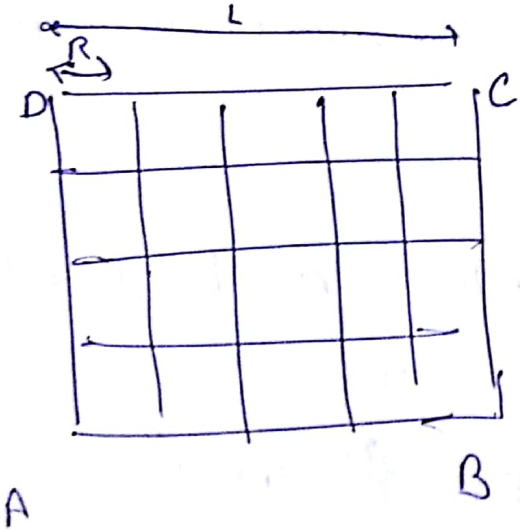
Project-I

PART-I (Grid Problem)

Given: A lattice of length L a
R is a grid size
N is the total no. of cells.

$N = L^2/R^2$



* Cat and Rat are on opposite sides.

Let N be the total number of steps.

Let $l = L/R$

Assumptions:

a) To simply the model, whenever both cat/rat are at the boundary/corner, they have a provision to wait / not move corresponding to the wall

b) i) probability of wait a boundary = 1/4

ii) probability of wait at corner = 1/2

(c) $N > \frac{L\sqrt{2}}{R}$. The number of steps should be greater than minimum required to meet.

Let us assume the cat and rat to be 2 molecules, which are doing a random walk.

Hence the probality of being at n_1, n_2 at the 2D-Space. out of N steps can be.

if it takes x steps in x-dir : then it takes $N-x$ steps in y direction.

$P(n_1)$ or Probality of being at n_1 after x steps.

$$= \frac{x!}{n_1!(x-n_1}$$

x_1 = steps taken in right, x_2 to steps taken in left.

$x_1 - x_2 = n_1$; $x_1 + x_2 = N$

$$P(n_1) = \frac{x!}{x_1!\, x_2!} (p_{right})^{x_1} (p_{left})^{x_2}.$$

similarly for y-axis.

y_R = steps up and y_2 = steps down.

$y_1 + y_2 = N-x$; $y_1 - y_2 = n_2$

$$P(n_2) = \frac{(N-x)!}{y_1!\, y_2!} (P_{up})^{y_1} (P_{down})^{y_2}$$

As the movement is mutually exclusive

$$P(n_1, n_2) = P(n_1) \cap P(n_2) = P(n_1) \times P(n_2)$$

$$P(n_1, n_2) = \frac{x!}{(x_1!)(x_2)} \times \left(\frac{1}{2}\right)^n \times \frac{(N-x_1)}{(y_1!)(y_2)!} \times \left(\frac{1}{2}\right)^n$$

Actual probability could be

$$P = \sum_{x=n_1}^{n_1-n_2} P(n_1, n_2) \Rightarrow \$$$

when cat and Rat meet

$$P(\text{meet}) = P_{rat}(n_1, n_2) \times P(cat(n_1, n_2))$$

$$P_{cat}(n_1, n_2) = P_{rat}(L-n_1, L-n_2)$$

$$L = L/R \; \star\star$$

$$P(\text{meet}) = P_{rat}(n_1, n_2) \times P_{rab}(L-n_1, L-n_2)$$

$$P(\text{meet, at all points}) = \sum_{n_1=0}^{L} \sum_{n_2=0}^{L} P_{rat}(n_1, n_2) + P_{rat}(L-n_1, L-n_2)$$

for coordinate x

$$\Rightarrow \sum \sum \left(\frac{1}{2}\right)^{4N} \frac{x!}{x_{rab}} \times \frac{(N-x_1^{rat})}{(y_1^{rat}!)(y_2^{rat})!} \times \frac{x_{cat}!}{x_1^{cat}! \, x_2^{cat}!} + \frac{N-x_1^{rab}}{(y_1^{rab}! \, y_2^{rab}!)}$$

$$\Rightarrow \sum \sum \left(\frac{1}{4}\right)^{4n}$$

Hence $P(\text{meet}) = \sum_{x=N}^{N} \sum_{n_1=0}^{L} \sum_{n_2=0}^{L} \left(\frac{1}{2}\right)^{4n} \frac{x_{rat}! \, (N-x_1)_{rab} \times x_{cat}! \times (N-x_2)!}{(x_1 rat)! \, (x_2 rat)! \, (y_1 rat)! \, (y_2 rat)! \, x_{cat}! (x_2 cat)! \times y_{1cat}! (y_2 cat)!}$

Summation for rat, cat

where $\Rightarrow$ ~~$n_1 + n_2 = N$~~

$x_1 rat + x_2 rat = x_1 cat + x_2 rab = x_{cat}$

$x_1 rat - x_2 rat = n_1 \qquad x_1 rab - x_2 cat = L - n_1$

$y_1 rat + y_2 rat = N - x_{rab} \qquad y_1 rab + y_2 rab = N - x_{cat}$

$y_1 rat - y_2 rat = n_2 \qquad y_1 cat - y_2 rab = L - n_2$

By substituting everything interms of n_1, n_2 and N followed by calculating the summation. We would get the value of P(meet).

$$P(meet) = \sum_{x_{rat}} \sum_{x_{cat}} \sum_{n_2} \sum_{n_1} \left(\frac{1}{4}\right)^{4N} (n_1+n_2)! \; (N$$

$$\Rightarrow \sum_{x_{cat}} \sum_{x_{rat}} \sum_{n_1} \sum_{n_2} \frac{x_{rat}! \; (N-x)!_{rat}}{\left(\frac{n_1+x_{rat}}{2}\right)! \left(\frac{L-n_1}{2}\right)!} \times \frac{1}{}$$

$$\Rightarrow \sum\sum\sum\sum \left(\frac{1}{2}\right)^{4N} \frac{(x_{rat}!)\,(N-x_{rat})!}{\left(\frac{n_1+x}{2}\right)! \left(\frac{x-n_1}{2}\right)! \left(\frac{n_2+N-x}{2}\right)! \left(\frac{n_2+N-x-n_2}{2}\right)!}$$

$$\times \frac{(x_{cat}!)\,(N-x_{cat})!}{\left(\frac{L-n_1+x_{rat}}{2}\right)! \left(\frac{x_{cat}+n_1-L}{2}\right)! \left(\frac{L-n_2+N-x_{cat}}{2}\right)! \left(\frac{N-x_{cat}-L+n_2}{2}\right)!}$$

We can calculate this using a PC and Hence

$$P(\text{Survival of Rat}) = 1 - P(meet)$$

$$\Rightarrow$$

Hence Probability of meet depends of N and L if $N < L\sqrt{2}$ the particles will never meet.

Case: 2

Assuming they start from center.

$$P(\text{meet}) = P(n_1, n_2) \times P(n_1 \times n_2)$$

$$= P(n_1 \times n_2)^2$$

$$= \sum_{n_1} \sum_{n_2} \left(\frac{1}{2}\right)^N \left(\frac{x!\,(N-x)!}{x_1!\; x_2!\; y_1!\; y_2!}\right)^2$$

x = Steps taken in x direction

x_1 = steps in right x_2 = steps in left

y_1 = step up y_2 = steps down

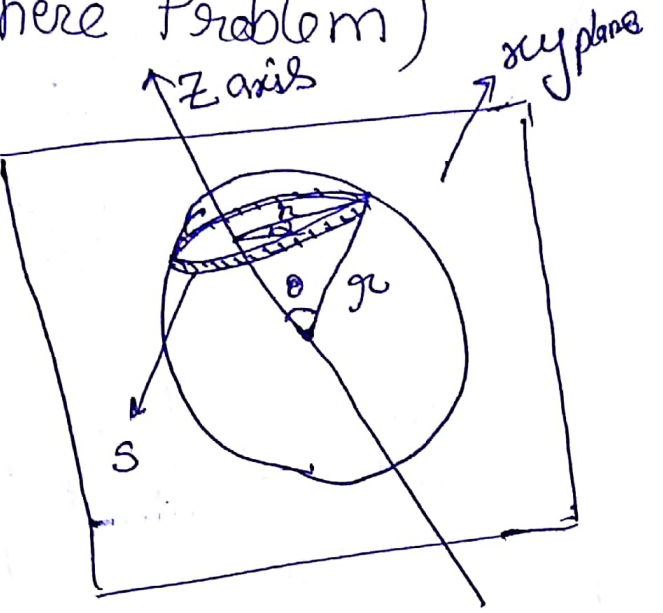
$x_1 + x_2 = x$ $y_1 + y_2 = N - x$

$x_1 - x_2 = n_1$ $y_1 - y_2 = n_2$

$P(\text{survival}) = 1 - \text{Probability (meet)}$

$$P(\text{survival}) = 1 - \left(\frac{1}{2}\right)^N \frac{x!\,(N-x)!}{(x_1!)(x_2!)\; y_1!\; (y_2)!}$$

PART-2 (Sphere Problem)



→ Z-axis is the Line cutting the sphere and emerging out of the sphere at another point.

→ Also, there is a plane xy, that has ~~Z~~ axis on it, which cuts the sphere into 2 hemisphere

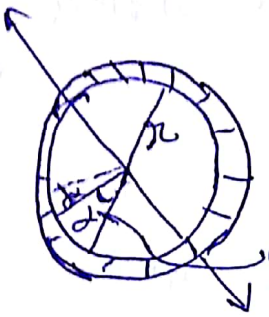
→ Blind cat and a blind rat are present on the sphere. We show their position using

i) θ : angle with the Z-axis

ii) α : angle with the line of intersection of plane with the plane.

Because: a set of points can have angle θ with Z-axis, as in the above example, all the points are on the circumference S.

The typical circles cut by the xy plane of the sphere will look like



Probability of being at a point which make angle α with the xy plane.

$$P(\alpha) = \frac{r\,d\alpha}{2\pi r} = \frac{d\alpha}{2\pi}$$

Probability of being at a point which makes an angle θ with the Z axis

$\Rightarrow$ $\frac{\text{Area occupied by the circular strip}}{\text{total surface area of the sphere.}}$

$$\Rightarrow \frac{(r\,d\theta)(2\pi r \sin\theta)}{4\pi r^2} = \frac{\sin\theta \cdot d\theta}{2}$$

Hence the probability of being at a point is.

$$P(\theta,\alpha) = \frac{\sin\theta\, d\theta\, d\alpha}{4\pi}$$

Analogy:

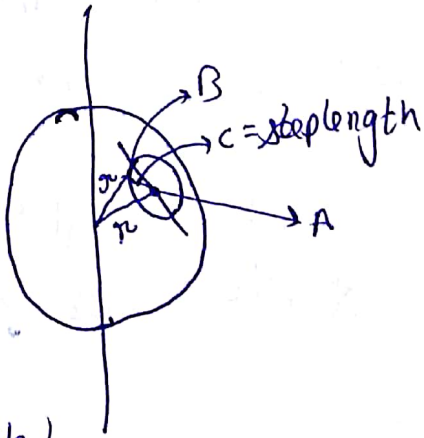
Displacement.

A: original position

B: new position

C = step length

angle of displacement = $\frac{\text{arc length (step length)}}{\text{radius}}$.

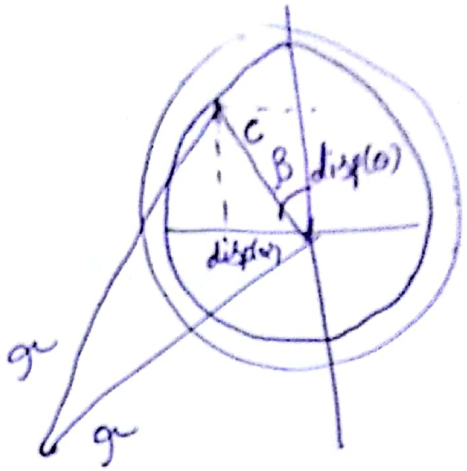


$$disp'\theta' = \frac{dis\theta}{r}$$

$$\theta_{new} = \theta - \frac{c\cos B}{r}$$

$$dis'\alpha' = \frac{disp}{r\sin\theta}$$

$$\alpha_{new} = \alpha + \frac{c\sin B}{r\sin\theta}$$



Assumption:

i) Basically assuming that B will be small enough so that the helix cab and rat travel individually happens to cover the total surface of the sphere.

Probality Densiby Funcbion, $\omega(\theta,\alpha)\, d\theta d\alpha = 1$

Probaliby at a point $= \frac{\sin\theta d\theta d\alpha}{4\pi}$

$$d\alpha \Rightarrow \frac{L\sin B}{r\sin\theta} \qquad d\theta = \frac{c\cos B}{r}$$

$$\omega(\theta,\alpha)\, d\theta d\alpha = \frac{\sin\theta}{4\pi} \times \frac{c\sin B\cos B}{r\sin\theta\, r}\, d\theta d\alpha$$

$$= \frac{c^2}{4\pi r^2}\sin B\cos B\, d\theta d\alpha$$

$$\text{Probaliby of meet} = \int_{\theta}\int_{\alpha} \omega(\theta,\alpha)^2\, d\theta d\alpha$$

$$\Rightarrow \int_{r=0}^{2\pi} \int_{\theta=0}^{2\pi} \frac{c^4}{16\pi^2 r^4} \sin^2\beta \cos^2\beta \, d\theta \, dr$$

$$\Rightarrow \frac{c^4}{16\pi^2 r^2} \sin^2\beta \cos^2\beta \iint_{r=0}^{2\pi} \int_{\theta=0}^{2\pi} d\theta \, dr .$$

$$\Rightarrow \frac{c^4 \sin^2\beta \cos^2\beta}{8 r^4}$$

Hence Probability of meet $\Rightarrow$ $P(\text{meet}) = \frac{c^4 \cos^2\beta \sin^2\beta}{r^4}$

where β is the angle of displacement.

Probability of Survival of the rat $\Rightarrow$

$1 - P(\text{meet}) \Rightarrow 1 - \frac{c^4 \cos^2\beta \sin^2\beta}{r^4}$.

Conclusion: Answer to Questions.

a) Survival/meeting is going to vary with time according to the above equation.

Since the probability of survival decreases with increase in time i.e. probability of survival $\propto \frac{1}{(\text{time})^c}$; c is a constant.

Because at large time, more number of steps have been taken.

b) i) it depends of L of the square. Hence as L => length of the grid increases; chance of survival increases.

ii) It depends on the radius of the sphere. meeting $\propto \frac{1}{r^4}$. Hence r increases, chance of survival increases.

c) i) If the number of steps is too large; probability for each cat and rat to be at a point = $1/L^2$

Hence probability of meet = $1/L^4$

Probability of survival = $1 - 1/L^4$

ii) Similarly for sphere portion. If a step is ~~this~~ very small probability of being at a point => $1/4\pi r^2$

Survival Probability = $1 - \left(\frac{1}{4\pi r^2}\right)^2 = 1 - \frac{1}{16\pi^2 r^4}$