14.2 Cauchy's Integral Theorem

We have just seen in Sec. 14.1 that a line integral of a function $f(z)$ generally depends not merely on the endpoints of the path, but also on the choice of the path itself. This dependence often complicates situations. Hence conditions under which this does ***not*** occur are of considerable importance. Namely, if $f(z)$ is analytic in a domain D and D is simply connected (see Sec. 14.1 and also below), then the integral will not depend on the choice of a path between given points. This result (Theorem 2) follows from Cauchy's integral theorem, along with other basic consequences that make ***Cauchy's integral theorem the most important theorem in this chapter*** and fundamental throughout complex analysis.

Let us begin by repeating and illustrating the definition of simple connectedness (Sec. 14.1) and adding some more details.

1. A **simple closed path** is a closed path (Sec. 14.1) that does not intersect or touch itself (Fig. 342). For example, a circle is simple, but a curve shaped like an 8 is not simple.

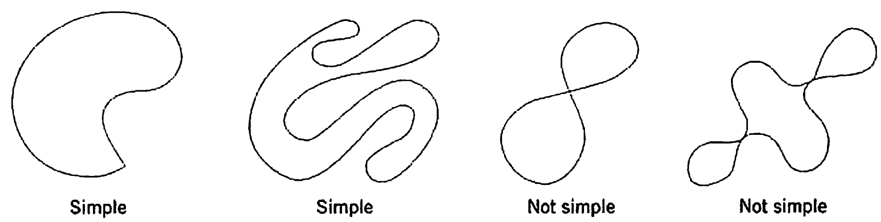


Fig. 342. Closed paths

2. A **simply connected domain** D in the complex plane is a domain (Sec. 13.3) such that every simple closed path in D encloses only points of D. *Examples:* The interior of a circle ("open disk"), ellipse, or any simple closed curve. A domain that is not simply connected is called **multiply connected.** *Examples:* An annulus (Sec. 13.3), a disk without the center, for example, $0 < |z| < 1$. See also Fig. 343.

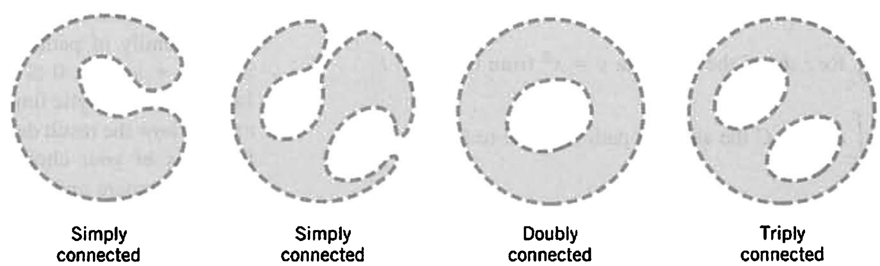


Fig. 343. Simply and multiply connected domains

More precisely, a **bounded domain** D (that is, a domain that lies entirely in some circle about the origin) is called **p-fold connected** if its boundary consists of p closed connected sets without common points. These sets can be curves, segments, or single points (such as $z = 0$ for $0 < |z| < 1$, for which $p = 2$). Thus, D has $p - 1$ **"holes"**, where "hole" may also mean a segment or even a single point. Hence an annulus is doubly connected ($p = 2$).

THEOREM 1

Cauchy's Integral Theorem

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

(1) $$\oint_C f(z)\,dz = 0.$$ See Fig. 344.

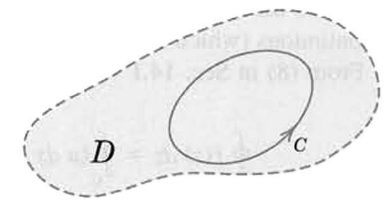


Fig. 344. Cauchy's integral theorem

Before we prove the theorem, let us consider some examples in order to really understand what is going on. A simple closed path is sometimes called a *contour* and an integral over such a path a **contour integral.** Thus, (1) and our examples involve contour integrals.

EXAMPLE 1 **No Singularities (Entire Functions)**

$$\oint_C e^z\,dz = 0, \qquad \oint_C \cos z\,dz = 0, \qquad \oint_C z^n\,dz = 0 \qquad (n = 0, 1, \cdots)$$

for any closed path, since these functions are entire (analytic for all z). ■

EXAMPLE 2 **Singularities Outside the Contour**

$$\oint_C \sec z\,dz = 0, \qquad \oint_C \frac{dz}{z^2 + 4} = 0$$

where C is the unit circle, $\sec z = 1/\cos z$ is not analytic at $z = \pm\pi/2, \pm 3\pi/2, \cdots$, but all these points lie outside C; none lies on C or inside C. Similarly for the second integral, whose integrand is not analytic at $z = \pm 2i$ outside C. ■

EXAMPLE 3 **Nonanalytic Function**

$$\oint_C \bar{z}\,dz = \int_0^{2\pi} e^{-it} ie^{it}\,dt = 2\pi i$$

where C: $z(t) = e^{it}$ is the unit circle. This does not contradict Cauchy's theorem because $f(z) = \bar{z}$ is not analytic. ■

EXAMPLE 4 **Analyticity Sufficient, Not Necessary**

$$\oint_C \frac{dz}{z^2} = 0$$

where C is the unit circle. This result does *not* follow from Cauchy's theorem, because $f(z) = 1/z^2$ is not analytic at $z = 0$. Hence *the condition that f be analytic in D is sufficient rather than necessary for* (1) *to be true.* ■

EXAMPLE 5 Simple Connectedness Essential

$$\oint_C \frac{dz}{z} = 2\pi i$$

for counterclockwise integration around the unit circle (see Sec. 14.1). C lies in the annulus $\frac{1}{2} < |z| < \frac{3}{2}$ where $1/z$ is analytic, but this domain is not simply connected, so that Cauchy's theorem cannot be applied. Hence ***the condition that the domain D be simply connected is essential.***

In other words, by Cauchy's theorem, if $f(z)$ is analytic on a simple closed path C and everywhere inside C, with no exception, not even a single point, then (1) holds. The point that causes trouble here is $z = 0$ where $1/z$ is not analytic. ■

PROOF Cauchy proved his integral theorem under the additional assumption that the derivative $f'(z)$ is continuous (which is true, but would need an extra proof). His proof proceeds as follows. From (8) in Sec. 14.1 we have

$$\oint_C f(z)\,dz = \oint_C (u\,dx - v\,dy) + i \oint_C (u\,dy + v\,dx).$$

Since $f(z)$ is analytic in D, its derivative $f'(z)$ exists in D. Since $f'(z)$ is assumed to be continuous, (4) and (5) in Sec. 13.4 imply that u and v have *continuous* partial derivatives in D. Hence Green's theorem (Sec. 10.4) (with u and $-v$ instead of F_1 and F_2) is applicable and gives

$$\oint_C (u\,dx - v\,dy) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

where R is the region bounded by C. The second Cauchy–Riemann equation (Sec. 13.4) shows that the integrand on the right is identically zero. Hence the integral on the left is zero. In the same fashion it follows by the use of the first Cauchy–Riemann equation that the last integral in the above formula is zero. This completes Cauchy's proof. ■

Goursat's proof *without the condition that $f'(z)$ is continuous*[1] is much more complicated. We leave it optional and include it in App. 4.

Independence of Path

We know from the preceding section that the value of a line integral of a given function $f(z)$ from a point z_1 to a point z_2 will in general depend on the path C over which we integrate, not merely on z_1 and z_2. It is important to characterize situations in which this difficulty of path dependence does not occur. This task suggests the following concept. We call an integral of $f(z)$ **independent of path in a domain *D*** if for every z_1, z_2 in D its value depends (besides on $f(z)$, of course) only on the initial point z_1 and the terminal point z_2, but not on the choice of the path C in D [so that every path in D from z_1 to z_2 gives the same value of the integral of $f(z)$].

[1]ÉDOUARD GOURSAT (1858–1936), French mathematician. Cauchy published the theorem in 1825. The removal of that condition by Goursat (see *Transactions Amer. Math. Soc.*, vol. 1, 1900) is quite important, for instance, in connection with the fact that derivatives of analytic functions are also analytic, as we shall prove soon. Goursat also made important contributions to PDEs.

THEOREM 2 **Independence of Path**

If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.

PROOF Let z_1 and z_2 be any points in D. Consider two paths C_1 and C_2 in D from z_1 to z_2 without further common points, as in Fig. 345. Denote by C_2^* the path C_2 with the orientation reserved (Fig. 346). Integrate from z_1 over C_1 to z_2 and over C_2^* back to z_1. This is a simple closed path, and Cauchy's theorem applies under our assumptions of the present theorem and gives zero:

$$(2') \qquad \int_{C_1} f\,dz + \int_{C_2^*} f\,dz = 0, \qquad \text{thus} \qquad \int_{C_1} f\,dz = -\int_{C_2^*} f\,dz.$$

But the minus sign on the right disappears if we integrate in the reverse direction, from z_1 to z_2, which shows that the integrals of $f(z)$ over C_1 and C_2 are equal,

$$(2) \qquad \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz \qquad \text{(Fig. 345).}$$

This proves the theorem for paths that have only the endpoints in common. For paths that have finitely many further common points, apply the present argument to each "loop" (portions of C_1 and C_2 between consecutive common points; four loops in Fig. 347). For paths with infinitely many common points we would need additional argumentation not to be presented here. ■

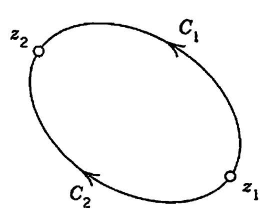


Fig. 345. Formula (2)

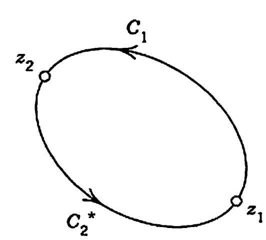


Fig. 346. Formula (2′)

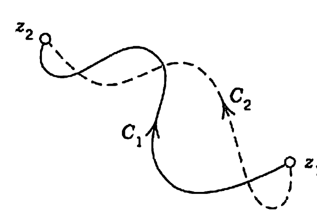


Fig. 347. Paths with more common points

Principle of Deformation of Path

This idea is related to path independence. We may imagine that the path C_2 in (2) was obtained from C_1 by continuously moving C_1 (with ends fixed!) until it coincides with C_2. Figure 348 shows two of the infinitely many intermediate paths for which the integral always retains its value (because of Theorem 2). Hence we may impose a continuous deformation of the path of an integral, keeping the ends fixed. As long as our deforming path always contains only points at which $f(z)$ is analytic, the integral retains the same value. This is called the **principle of deformation of path.**

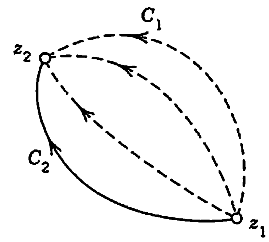


Fig. 348. Continuous deformation of path

EXAMPLE 6 A Basic Result: Integral of Integer Powers

From Example 6 in Sec. 14.1 and the principle of deformation of path it follows that

$$\oint (z - z_0)^m \, dz = \begin{cases} 2\pi i & (m = -1) \\ 0 & (m \neq -1 \text{ and integer}) \end{cases} \tag{3}$$

for counterclockwise integration around ***any simple closed path containing z_0 in its interior.***

Indeed, the circle $|z - z_0| = \rho$ in Example 6 of Sec. 14.1 can be continuously deformed in two steps into a path as just indicated, namely, by first deforming, say, one semicircle and then the other one. (Make a sketch). ■

Existence of Indefinite Integral

We shall now justify our indefinite integration method in the preceding section [formula (9) in Sec. 14.1]. The proof will need Cauchy's integral theorem.

THEOREM 3 Existence of Indefinite Integral

If $f(z)$ is analytic in a simply connected domain D, then there exists an indefinite integral $F(z)$ of $f(z)$ in D—thus, $F'(z) = f(z)$—which is analytic in D, and for all paths in D joining any two points z_0 and z_1 in D, the integral of $f(z)$ from z_0 to z_1 can be evaluated by formula (9) *in* Sec. 14.1.

PROOF The conditions of Cauchy's integral theorem are satisfied. Hence the line integral of $f(z)$ from any z_0 in D to any z in D is independent of path in D. We keep z_0 fixed. Then this integral becomes a function of z, call if $F(z)$,

$$F(z) = \int_{z_0}^{z} f(z^*) \, dz^* \tag{4}$$

which is uniquely determined. We show that this $F(z)$ is analytic in D and $F'(z) = f(z)$. The idea of doing this is as follows. Using (4) we form the difference quotient

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\left[\int_{z_0}^{z+\Delta z} f(z^*) \, dz^* - \int_{z_0}^{z} f(z^*) \, dz^*\right] = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z^*) \, dz^*. \tag{5}$$

We now subtract $f(z)$ from (5) and show that the resulting expression approaches zero as $\Delta z \to 0$. The details are as follows.

We keep z fixed. Then we choose $z + \Delta z$ in D so that the whole segment with endpoints z and $z + \Delta z$ is in D (Fig. 349). This can be done because D is a domain, hence it contains a neighborhood of z. We use this segment as the path of integration in (5). Now we subtract $f(z)$. This is a constant because z is kept fixed. Hence we can write

$$\int_z^{z+\Delta z} f(z)\, dz^* = f(z) \int_z^{z+\Delta z} dz^* = f(z)\, \Delta z. \qquad \text{Thus} \qquad f(z) = \frac{1}{\Delta z} \int_z^{z+\Delta z} f(z)\, dz^*.$$

By this trick and from (5) we get a single integral:

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z} \int_z^{z+\Delta z} [f(z^*) - f(z)]\, dz^*.$$

Since $f(z)$ is analytic, it is continuous. An $\epsilon > 0$ being given, we can thus find a $\delta > 0$ such that $|f(z^*) - f(z)| < \epsilon$ when $|z^* - z| < \delta$. Hence, letting $|\Delta z| < \delta$, we see that the *ML*-inequality (Sec. 14.1) yields

$$\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| = \frac{1}{|\Delta z|} \left| \int_z^{z+\Delta z} [f(z^*) - f(z)]\, dz^* \right| \leqq \frac{1}{|\Delta z|} \epsilon |\Delta z| = \epsilon.$$

By the definition of limit and derivative, this proves that

$$F'(z) = \lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z).$$

Since z is any point in D, this implies that $F(z)$ is analytic in D and is an indefinite integral or antiderivative of $f(z)$ in D, written

$$F(z) = \int f(z)\, dz.$$

Also, if $G'(z) = f(z)$, then $F'(z) - G'(z) \equiv 0$ in D; hence $F(z) - G(z)$ is constant in D (see Team Project 26 in Problem Set 13.4). That is, two indefinite integrals of $f(z)$ can differ only by a constant. The latter drops out in (9) of Sec. 14.1, so that we can use any indefinite integral of $f(z)$. This proves Theorem 3. ■

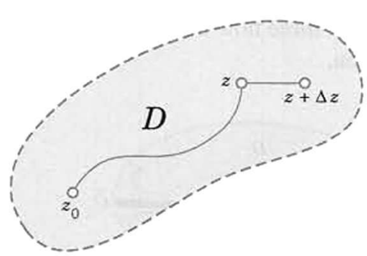


Fig. 349. Path of integration

Cauchy's Integral Theorem for Multiply Connected Domains

Cauchy's theorem applies to multiply connected domains. We first explain this for a **doubly connected domain** D with outer boundary curve C_1 and inner C_2 (Fig. 350). If a function $f(z)$ is analytic in any domain D^* that contains D and its boundary curves, we claim that

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz \qquad \text{(Fig. 350)} \tag{6}$$

both integrals being taken counterclockwise (or both clockwise, and regardless of whether or not the full interior of C_2 belongs to D^*).

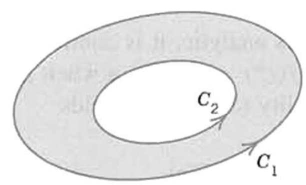


Fig. 350. Paths in (5)

PROOF By two cuts $\tilde{C}_1$ and $\tilde{C}_2$ (Fig. 351) we cut D into two simply connected domains D_1 and D_2 in which and on whose boundaries $f(z)$ is analytic. By Cauchy's integral theorem the integral over the entire boundary of D_1 (taken in the sense of the arrows in Fig. 351) is zero, and so is the integral over the boundary of D_2, and thus their sum. In this sum the integrals over the cuts $\tilde{C}_1$ and $\tilde{C}_2$ cancel because we integrate over them in both directions—this is the key—and we are left with the integrals over C_1 (counterclockwise) and C_2 (clockwise; see Fig. 351); hence by reversing the integration over C_2 (to counterclockwise) we have

$$\oint_{C_1} f\,dz - \oint_{C_2} f\,dz = 0$$

and (6) follows. ■

For domains of higher connectivity the idea remains the same. Thus, for a **triply connected domain** we use three cuts $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{C}_3$ (Fig. 352). Adding integrals as before, the integrals over the cuts cancel and the sum of the integrals over C_1 (counterclockwise) and C_2, C_3 (clockwise) is zero. Hence the integral over C_1 equals the sum of the integrals over C_2 and C_3, all three now taken counterclockwise. Similarly for quadruply connected domains, and so on.

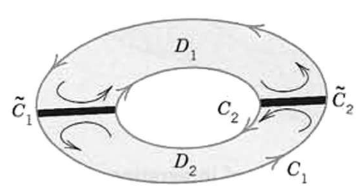


Fig. 351. Doubly connected domain

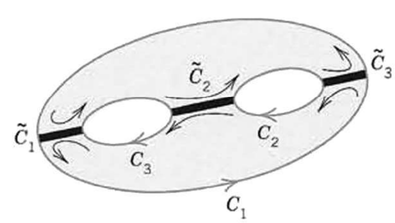


Fig. 352. Triply connected domain

PROBLEM SET 14.2

1–11 CAUCHY'S INTEGRAL THEOREM APPLICABLE?

Integrate $f(z)$ counterclockwise around the unit circle, indicating whether Cauchy's integral theorem applies. (Show the details of your work.)

1. $f(z) = \text{Re}\, z$
2. $f(z) = 1/(3z - \pi i)$
3. $f(z) = e^{z^2/2}$
4. $f(z) = 1/\bar{z}$
5. $f(z) = \tan z^2$
6. $f(z) = \sec(z/2)$
7. $f(z) = 1/(z^8 - 1.2)$
8. $f(z) = 1/(4z - 3)$
9. $f(z) = 1/(2|z|^3)$
10. $f(z) = \bar{z}^2$
11. $f(z) = z^2 \cot z$

12–17 COMMENTS ON TEXT AND EXAMPLES

12. **(Singularities)** Can we conclude in Example 2 that the integral of $1/(z^2 + 4)$ taken over (a) $|z - 2| = 2$, (b) $|z - 2| = 3$ is zero? Give reasons.
13. **(Cauchy's integral theorem)** Verify Theorem 1 for the integral of z^2 over the boundary of the square with vertices $1 + i$, $-1 + i$, $-1 - i$, and $1 - i$ (counterclockwise).
14. **(Cauchy's integral theorem)** For what contours C will it follow from Theorem 1 that

(a) $\displaystyle\oint_C \frac{dz}{z} = 0$, (b) $\displaystyle\oint_C \frac{\cos z}{z^6 - z^2}\, dz = 0$,

(c) $\displaystyle\oint_C \frac{e^{1/z}}{z^2 + 9}\, dz = 0$?

15. **(Deformation principle)** Can we conclude from Example 4 that the integral is also zero over the contour in Problem 13?
16. **(Deformation principle)** If the integral of a function $f(z)$ over the unit circle equals 3 and over the circle $|z| = 2$ equals 5, can we conclude that $f(z)$ is analytic everywhere in the annulus $1 < |z| < 2$?
17. **(Path independence)** Verify Theorem 2 for the integral of $\cos z$ from 0 to $(1 + i)\pi$ (a) over the shortest path, (b) over the x-axis to π and then straight up to $(1 + i)\pi$.

18. **TEAM PROJECT. Cauchy's Integral Theorem.**
(a) Main Aspects. Each of the problems in Examples 1–5 explains a basic fact in connection with Cauchy's theorem. Find five examples of your own, more complicated ones if possible, each illustrating one of those facts.
(b) Partial fractions. Write $f(z)$ in terms of partial fractions and integrate it counterclockwise over the unit circle, where

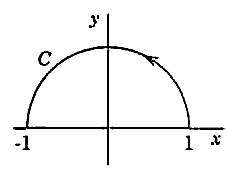
(i) $f(z) = \dfrac{2z + 3i}{z^2 + \frac{1}{4}}$ (ii) $f(z) = \dfrac{z + 1}{z^2 + 2z}$.

(c) Deformation of path. Review (c) and (d) of Team Project 34, Sec. 14.1, in the light of the principle of deformation of path. Then consider another family of paths with common endpoints, say, $z(t) = t + ia(t - t^2)$, $0 \leqq t \leqq 1$, and experiment with the integration of analytic and nonanalytic functions of your choice over these paths (e.g., z, $\text{Im}\, z$, z^2, $\text{Re}\, z^2$, $\text{Im}\, z^2$, etc).

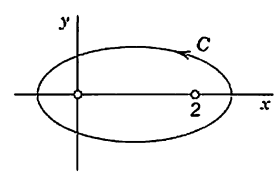
19–30 FURTHER CONTOUR INTEGRALS

Evaluate (showing the details and using partial fractions if necessary)

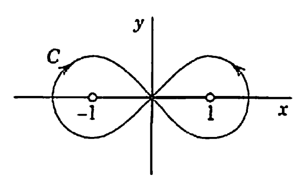
19. $\displaystyle\oint_C \frac{dz}{2z - i}$, C the circle $|z| = 3$ (counterclockwise)
20. $\displaystyle\oint_C \tanh z\, dz$, C the circle $|z - \frac{1}{4}\pi i| = \frac{1}{2}$ (clockwise)
21. $\displaystyle\oint_C \text{Re}\, 2z\, dz$, C as shown



22. $\displaystyle\oint_C \frac{7z - 6}{z^2 - 2z}\, dz$, C as shown



23. $\displaystyle\oint_C \frac{dz}{z^2 - 1}$, C as shown



24. $\displaystyle\oint_C \frac{e^{2z}}{4z}\, dz$, C consists of $|z| = 2$ (clockwise) and $|z| = \frac{1}{2}$ (counterclockwise)