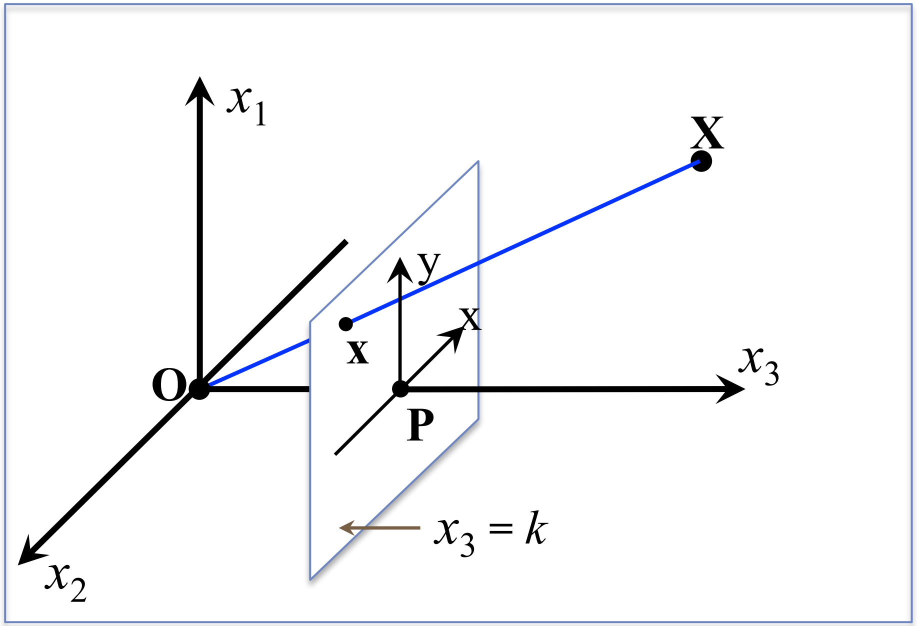
CSE578: Computer Vision

Spring'16

Projective Geometry a Review



Anoop M. Namboodiri

Center for Visual Information Technology

IIIT Hyderabad, INDIA

Points and Lines in $\mathcal{P}^2$

- Points represented by: $\mathbf{x} = [x \;\; y \;\; 1]^{\mathrm{T}}$.
- Consider the line equation: $ax + by + c = 0$.
- $[a\, b\, c][x\, y\, 1]^{\mathrm{T}} = \mathbf{l}.\mathbf{x} = \mathbf{l}^{\mathrm{T}}\, \mathbf{x} = 0$, where $\mathbf{l} = [a\, b\, c]^{\mathrm{T}}$.
- Lines are represented by 3-vectors, just like points.

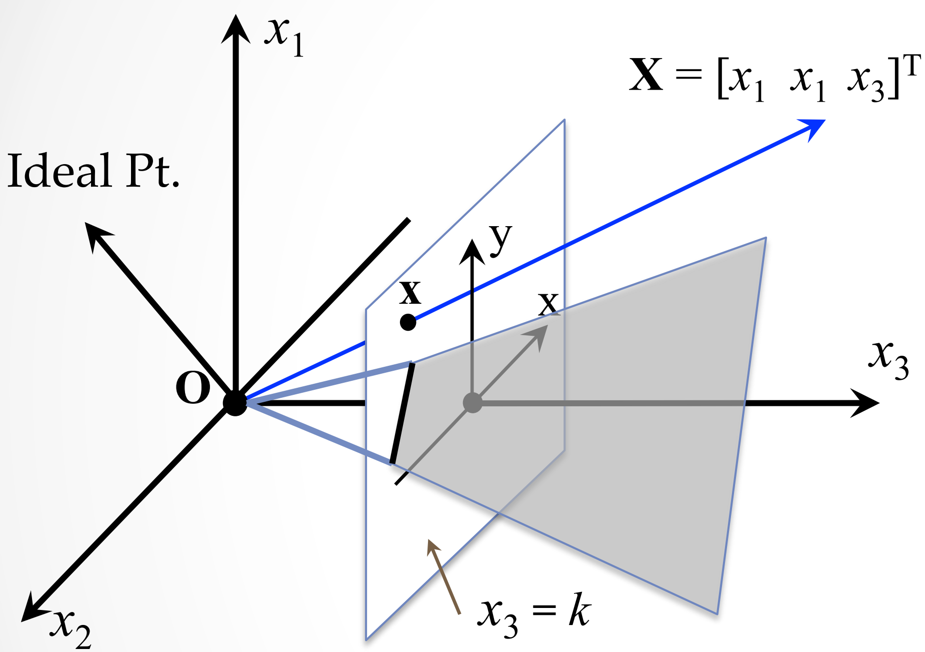
  Overall scale is unimportant.
- What does $\mathbf{l}^{\mathrm{T}}\, \mathbf{x} = 0$ describe?
  - All points $\mathbf{x}$ on a fixed line $\mathbf{l}$?
  - All lines $\mathbf{l}$ passing through a fixed point $\mathbf{x}$?

Points/Line at Infinity

- $\mathbf{x} = [x_1 \;\; x_2 \;\; x_3]^T$ represents $(x_1/x_3\,, x_2/x_3)$.
- What happens when $x_3 \rightarrow 0$?
- Becomes **point at infinity**, or **vanishing point** or ideal point in the direction (x_1, x_2).
- Points at infinity can be handled like any other point in projective geometry
- $[a \;\; b \;\; 0]^T$ are all points at infinity on the plane. They together form a line at infinity.
- What is the representation of $\mathbf{l}_\infty$?

$$\mathbf{l}_\infty = [0 \;\; 0 \;\; 1]^T$$

Visualizing Project Geometry of a Plane



- $\mathbf{x} = [x_1 \;\; x_2 \;\; x_3]^T$ represents rays from the origin in 3-space.
- The plane can be any cross section $\perp^r$ to x_3.
- Ideal points are rays on the $x_3 = 0$ plane.
- Lines are planes passing through the origin.

- Line at infinity, $\mathbf{l}_\infty$, corresponds to $x_3 = 0$.

Line joining 2 points

- Let x and y be points. We have: $\mathbf{l}^{\mathrm{T}}\mathbf{x} = \mathbf{l}^{\mathrm{T}}\mathbf{y} = 0$.
- Equation of $\mathbf{l}$? $y = y_1 + (x - x_1)\dfrac{y_2 - y_1}{x_2 - x_1}$
  - or: $(y_2 - y_1)\, x - (x_2 - x_1)\, y + (x_2 y_1 - x_1 y_2) = 0$.
  - Therefore, $\mathbf{l} = [(y_2 - y_1) \quad -(x_2 - x_1) \quad (x_2 y_1 - x_1 y_2)]^{\mathrm{T}}$.
- Considering them as vectors in 3-space, we want to find a vector L orthogonal to both P and Q.
- The cross-product $\mathbf{x} \times \mathbf{y}$ is a solution. Thus, $\mathbf{l} = \mathbf{x} \times \mathbf{y}$.
- $\mathbf{x} \times \mathbf{y} = [(y_2 - y_1) \quad - (x_2 - x_1) \quad (x_2 y_1 - x_1 y_2)]^{\mathrm{T}}$.

Example

- Line through (5,2) and (3,2): $\begin{bmatrix} i & j & k \\ 5 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$
- i.e., $y = 2$.
- Ideal point of the line $[0\ \ 1\ \ {-2}]^T$ is $[1\ \ 0\ \ 0]^T$. This is same as $[0\ \ 1\ \ k]^T$ for any k.
- Line joining $[3\ \ 4\ \ 0]^T$ and $[2\ \ 3\ \ 0]^T$ is $[0\ \ 0\ \ 1]^T$ or $\mathbf{l}_\infty$.

Point of Intersection of 2 lines

- Lines $\mathbf{l}$, $\mathbf{m}$ intersect at a point $\mathbf{x}$ with $\mathbf{l}^{\mathrm{T}}\mathbf{x} = \mathbf{m}^{\mathrm{T}}\mathbf{x} = 0$.
- $\mathbf{x} = \mathbf{l} \times \mathbf{m}$.
- $\mathbf{l}$: $a_1\, x + b_1\, y + c_1 = 0$; and $\mathbf{m}$: $a_2\, x + b_2\, y + c_2 = 0$.
- $x = (b_2c_1 - b_1c_2)/(a_2b_1 - a_1b_2)$.
- $y = (a_1c_2 - a_2c_1)/(a_2b_1 - a_1b_2)$.
- $\mathbf{x} = \;[(b_2c_1 - b_1c_2)\;\;(a_1c_2 - a_2c_1)\;\;(a_2b_1 - a_1b_2)]^{\mathrm{T}} \;=\; \mathbf{l} \times \mathbf{m}$.
- Duality at work: points and lines are interchangeable.

Example

- Intersection of x=1 and y=2: $\begin{bmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$
- Same as: $(1,2)$.
- Intersection of x=1 and x=2: $\begin{bmatrix} i & j & k \\ 1 & 0 & -1 \\ 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
- Ideal point of the line $\mathbf{l} = [a \;\; b \;\; c]^{\mathrm{T}}$ is $[\mathrm{b} \;\; -a \;\; 0]^{\mathrm{T}}$
- This is $\mathbf{l} \times \mathbf{l}_{\infty}$, the intersection of l with line at infinity!

Conics: 2nd order Entities

- General quadratic entity:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

- Rewrite using homogeneous coordinates as:

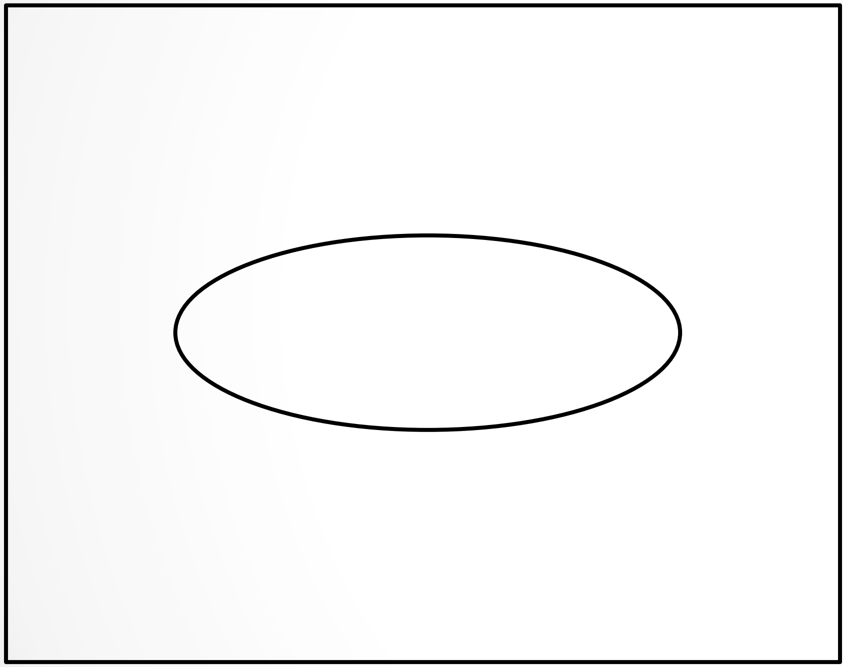
$$ax^2 + bxy + cy^2 + dxw + eyw + fw^2 = 0.$$

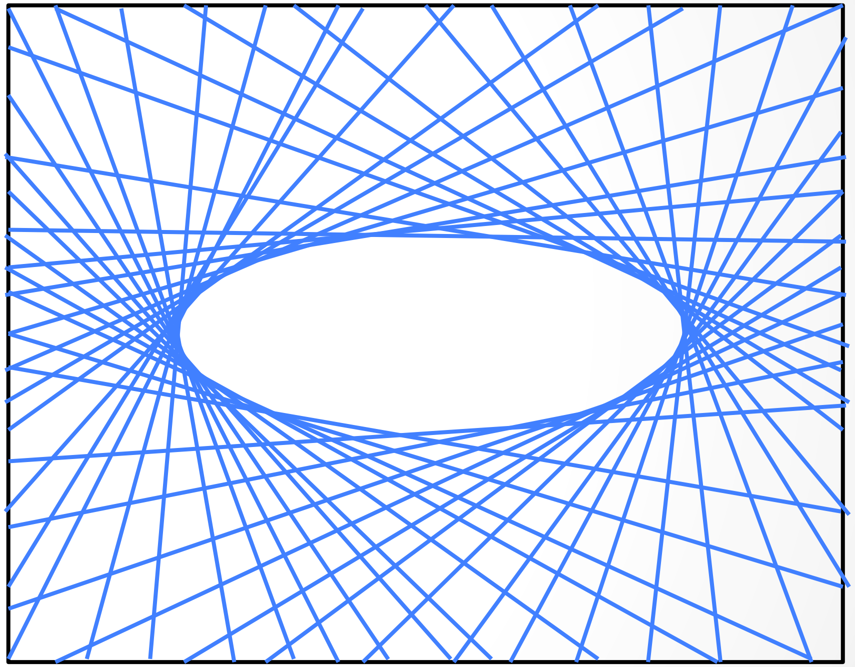
- Rewrite as:

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

- A symmetric C represents a conic: $\mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{x} = 0$. Covers circle, ellipse, parabola, hyperbola, etc.
- Degenerate conics include a line $(a = b = c = 0)$ and two lines when $\mathbf{C} = \mathbf{l}\mathbf{m}^{\mathrm{T}} + \mathbf{m}\mathbf{l}^{\mathrm{T}}$.

Point and Line Conics



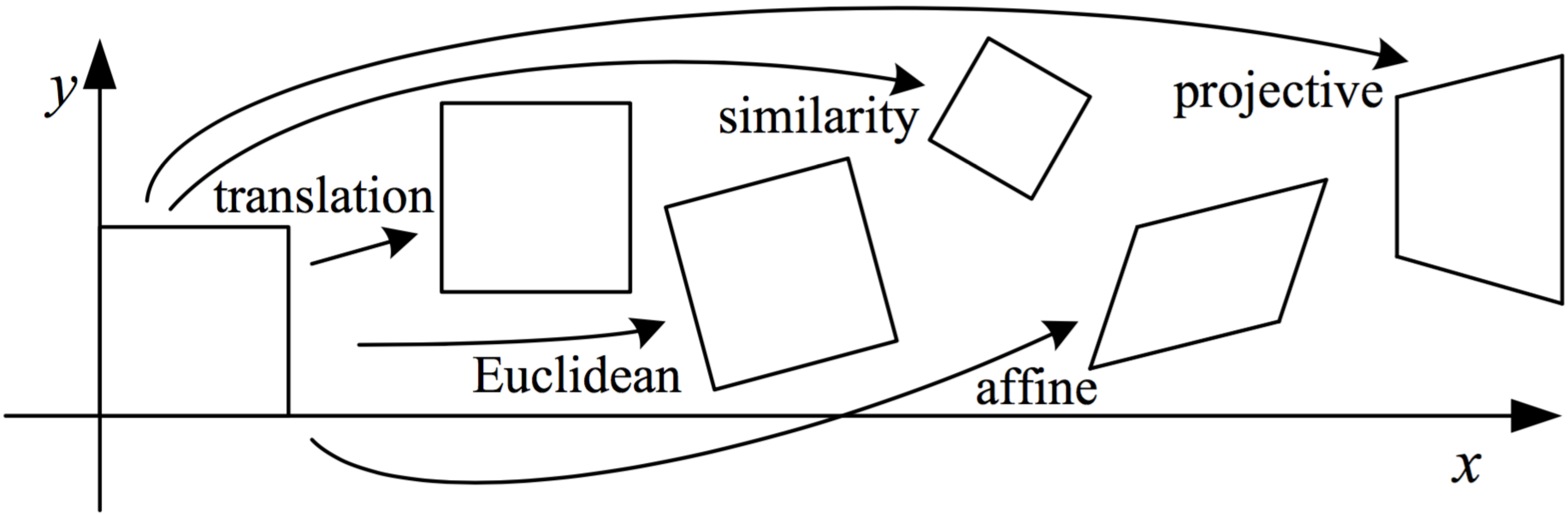


Properties of Conics

- $\mathbf{l} = \mathbf{Cx}$ gives the tangent line to the conic at $\mathbf{x}$.
  - A point $\mathbf{x}$ on the conic is on line $\mathbf{l} = \mathbf{Cx}$ as $\mathbf{x}^{\mathrm{T}}(\mathbf{Cx}) = \mathbf{0}$.
  - If $\mathbf{l}$ intersects the conic in another point $\mathbf{y}$;
    - $\mathbf{y}^{\mathrm{T}}\mathbf{Cy} = \mathbf{0}$ as $\mathbf{y}$ is on the conic; and
    - $(\mathbf{Cx})^{\mathrm{T}}\mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{Cy} = \mathbf{0}$ as $\mathbf{y}$ is on the line.
    - Thus, $\mathbf{Cy}$ is a line joining $\mathbf{x}$ and $\mathbf{y}$.
    - That is $\mathbf{Cy} = \mathbf{Cx}$ or $\mathbf{x} = \mathbf{y}$.
- Dual Conic: Conic defined by its tangent lines.
  - $\mathbf{l}^{\mathrm{T}}\mathbf{C}^{*}\mathbf{l} = \mathbf{0}$ or $\mathbf{l}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{l} = \mathbf{0}$ gives the set of lines tangential to $\mathbf{C}$.
- Point of tangency of $\mathbf{l}$ and $\mathbf{C}$ is given by: $\mathbf{C}^{-\mathrm{T}}\mathbf{l}$ or $\mathbf{C}^{-1}\mathbf{l}$.
  - Consider the point $\mathbf{x} = \mathbf{C}^{-1}\mathbf{l}$ on line $\mathbf{l}$.
  - It is also on the conic:

$$\mathbf{x}^{\mathrm{T}}\mathbf{Cx} = (\mathbf{C}^{-1}\mathbf{l})^{\mathrm{T}}\,\mathbf{C}\,(\mathbf{C}^{-1}\mathbf{l}) = \mathbf{l}^{\mathrm{T}}\mathbf{C}^{-\mathrm{T}}\,(\mathbf{CC}^{-1})\,\mathbf{l} = \mathbf{l}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{l} = \mathbf{0}$$

Hierarchy of Transformations



Hierarchy of Transformations

- Translation (2) $\rightarrow \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{bmatrix}$
- Euclidean (3) $\rightarrow \begin{bmatrix} c_\theta & -s_\theta & t_x \\ s_\theta & c_\theta & t_y \end{bmatrix}$
- Similarity (4) $\rightarrow \begin{bmatrix} 1+a & -b & t_x \\ b & 1+a & t_y \end{bmatrix}$
- Affine (6) $\rightarrow \begin{bmatrix} 1+a & b & t_x \\ c & 1+d & t_y \end{bmatrix}$
- Projective (8) $\rightarrow \begin{bmatrix} 1+h_{00} & h_{01} & h_{02} \\ h_{10} & 1+h_{11} & h_{12} \\ h_{20} & h_{21} & 1 \end{bmatrix}$

Projective Transformations

- A general non-singular 3×3 matrix H transforms points to other points. Overall scale of H is unimportant.
- $\mathbf{x}' = \mathbf{H}\,\mathbf{x}$ gives the transformed point.
- Linearity is preserved. $\mathbf{p}', \mathbf{q}', \mathbf{r}'$ collinear if $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are. In fact, that is the definition of the basic **projectivity** transformation.
- Such a transformation is called: **collineation, homography, projective transformation**.
- $\mathbf{l}' = \mathbf{H}^{-\mathbf{T}}\,\mathbf{l}$ gives the transformed line.
- $\mathbf{C}' = \mathbf{H}^{-\mathbf{T}}\,\mathbf{C}\,\mathbf{H}^{-1}$ is the transformed conic.
- $\mathbf{C}^{*\prime} = \mathbf{H}\,\mathbf{C}^{*}\,\mathbf{H}^{\mathbf{T}}$ is the transformed dual conic.

Isometric Transformation

- Transformations of the form, with $\delta = \pm 1$:

$$\begin{bmatrix} \delta \cos\theta & -\sin\theta & a \\ \delta \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix}$$

- Includes rotations, translations, reflections.
- Called *Euclidean* and *rigid* transformations.
- Preserves distance measurements, angles, parallelism, etc.

Similarity Transformations

- Transformations of the form for nonzero s:

$$\begin{bmatrix} s\cos\theta & -s\sin\theta & a \\ s\sin\theta & s\cos\theta & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

- Includes rotations, translations, uniform scaling
- Preserves angles, parallelism, ratio of distances, ratio of areas, circular points $\mathbf{I}$, $\mathbf{J}$
- 4 degrees of freedom; needs 2 point matches to estimate
- Geometric structure that is defined upto an unknown similarity transformation is called **metric structure**

Affine Transformations

- Transformations of the form: $\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$
- Includes rotations, translations, nonuniform scaling, shearing, etc.
- Preserves parallelism, ratio of lengths of parallel lines, ratio of areas, centroid, $\mathbf{l}_\infty$
- 6 degrees of freedom; needs 3 point matches
- Points at infinity map to other points at infinity

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix}$$

Projective Transformation

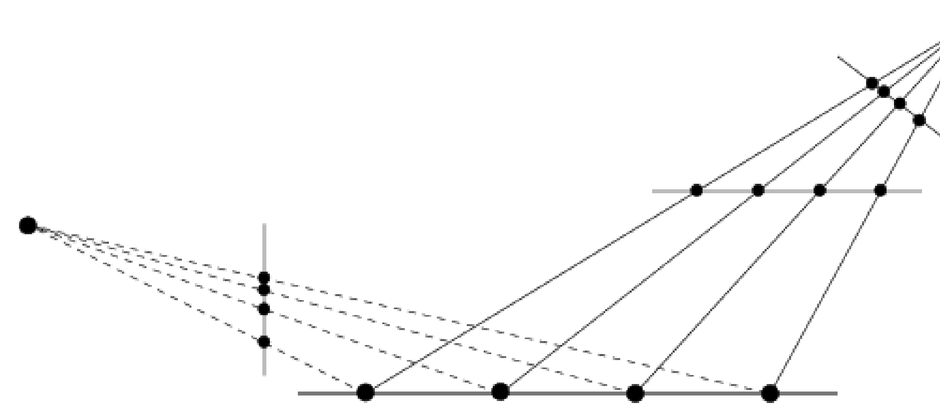
- Any general matrix $\mathbf{H}$, a general transformation.

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^{\mathbf{T}} & 1 \end{bmatrix} = \mathbf{H_P H_A H_S} = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{v}^{\mathbf{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

  where $\mathbf{K}$ is upper triangular with determinant 1
- Finite points can map to ideal points and vice versa.
- The impact of even slight projectivity is serious and non-intuitive. Yet, it models a pin-hole camera.
- Doesn't preserve parallelism, lengths, angles, or ratios of lengths. But, preserves cross-ratios.
- 8 degrees of freedom; needs 4 point matches

Cross-Ratios on a Line

Consider 4 points $\mathbf{X}_i,\ i = 1 \cdots 4$ on a line and its different projections $\mathbf{x}_i$



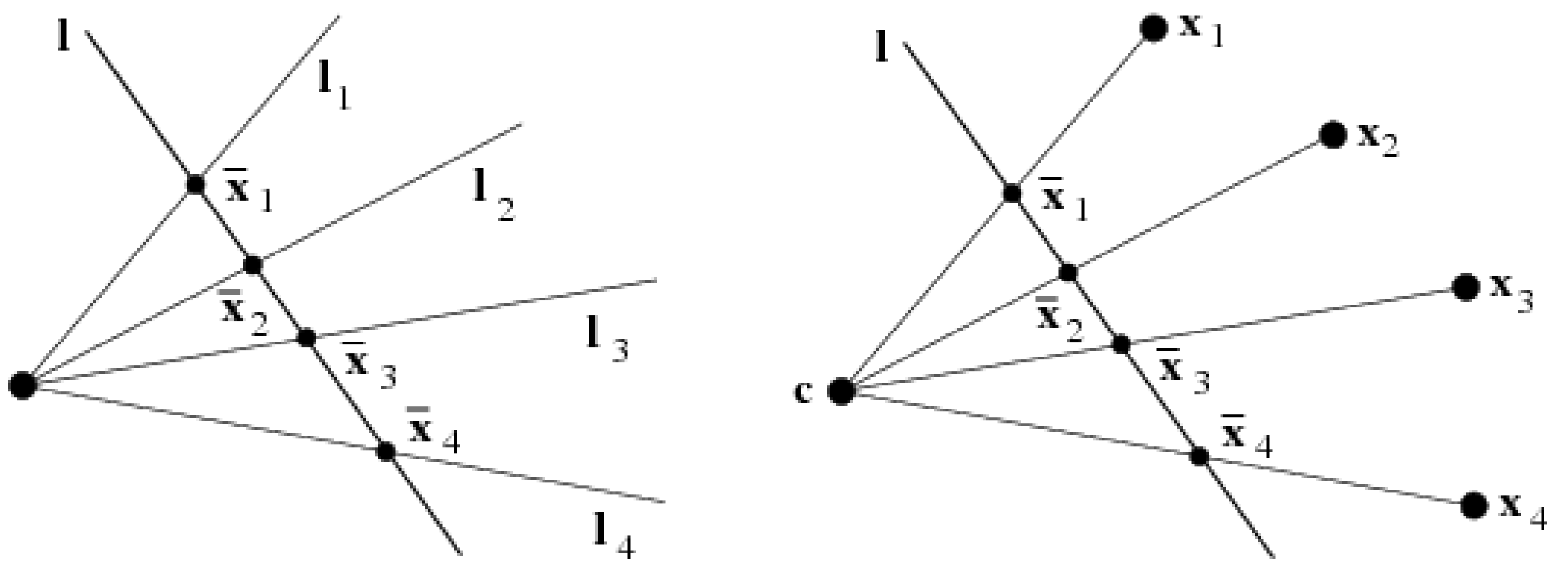
Cross ratio of 4 points defined as:

$$\text{Cross}(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}) = \frac{|\mathbf{x_1}\mathbf{x_2}||\mathbf{x_3}\mathbf{x_4}|}{|\mathbf{x_1}\mathbf{x_3}||\mathbf{x_2}\mathbf{x_4}|}, \quad \text{with } |\mathbf{x}\mathbf{y}| = \det \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

$|\mathbf{xy}|$ is the 1D signed distance along the line.

Homogeneous scale factors cancel each other. Hence cross-ratio is preserved under **any** projective transformation and can be measured in any projection.

Concurrent Lines



- For 4 concurrent co-planar lines, cross ratio of 4 points $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4$ measured on any line is constant.
- For 4 coplanar points and given a projection centre in the plane, Cross$(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4)$ on any line is constant.

Invariants for Different Types

Property	Euclidean	Similarity	Affine	Projective
Length				
Angle				
Length ratio				
Area ratio				
Parallelism				
Centroid				
Ratio of len ratio				
Collinearity				

Invariants for Different Types

Property	Euclidean	Similarity	Affine	Projective
Length	**Yes**	**No**	**No**	**No**
Angle	**Yes**	**Yes**	**No**	**No**
Length ratio	**Yes**	**Yes**	**No**	**No**
Area ratio	**Yes**	**Yes**	**Yes**	**No**
Parallelism	**Yes**	**Yes**	**Yes**	**No**
Centroid	**Yes**	**Yes**	**Yes**	**No**
Ratio of len ratio	**Yes**	**Yes**	**Yes**	**Yes**
Collinearity	**Yes**	**Yes**	**Yes**	**Yes**

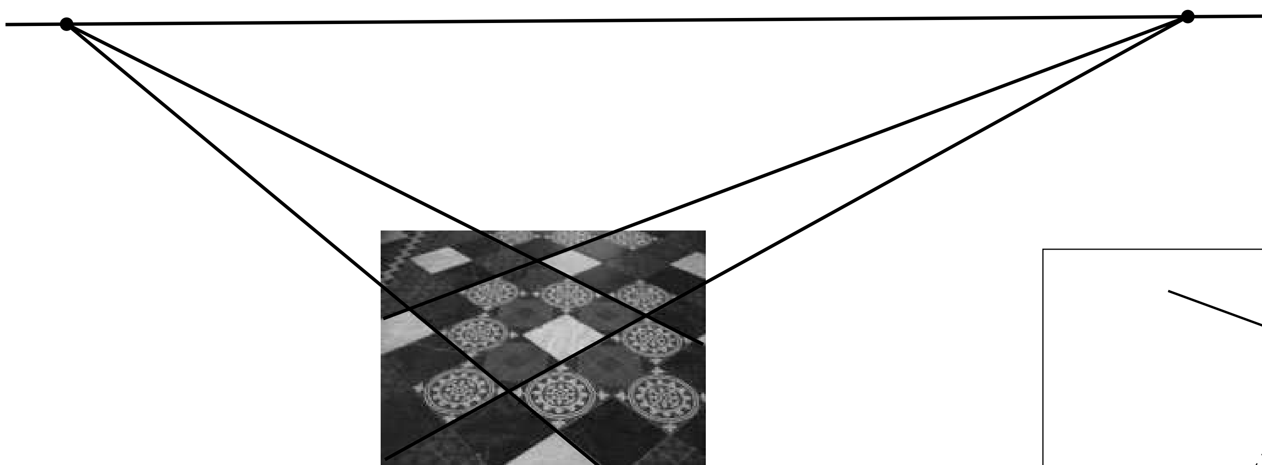
Circular Points

- Affinity maps $\mathbf{l}_\infty$ to itself. Conversely, any transformation that does it is an affine one
- General projectivity can map $\mathbf{l}_\infty$ to a finite line and vice versa
- A circle intersects $\mathbf{l}_\infty$ at **circular points**. Canonical (Euclidean) circle is: $x^2 + y^2 + dxw + eyw + fw^2 = 0$.
- Points on $\mathbf{l}_\infty$ have $w = 0$. Thus, $x^2 + y^2 = 0$.
- Circular points are given canonically by:

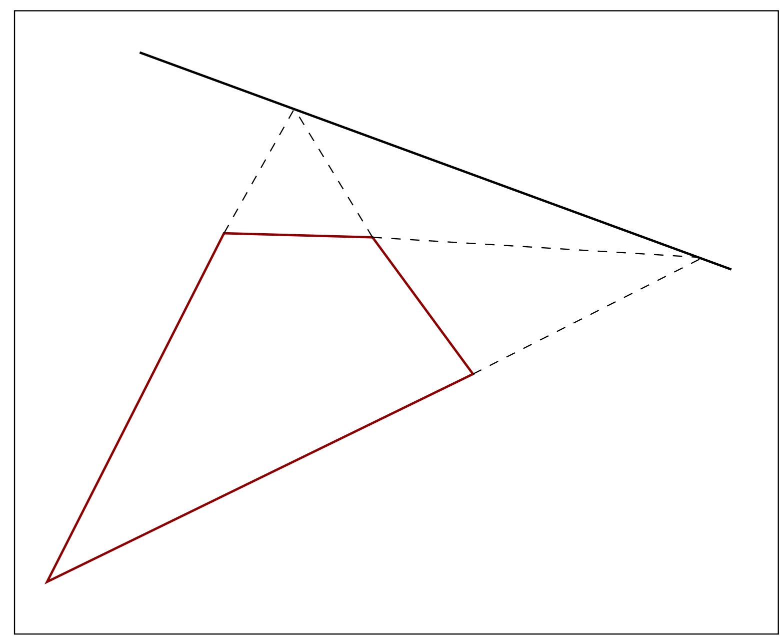
$$I = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

Finding Line at Infinity

Line at infinity can be found in the image from 2 sets of parallel lines.



Rectangle imaged as:

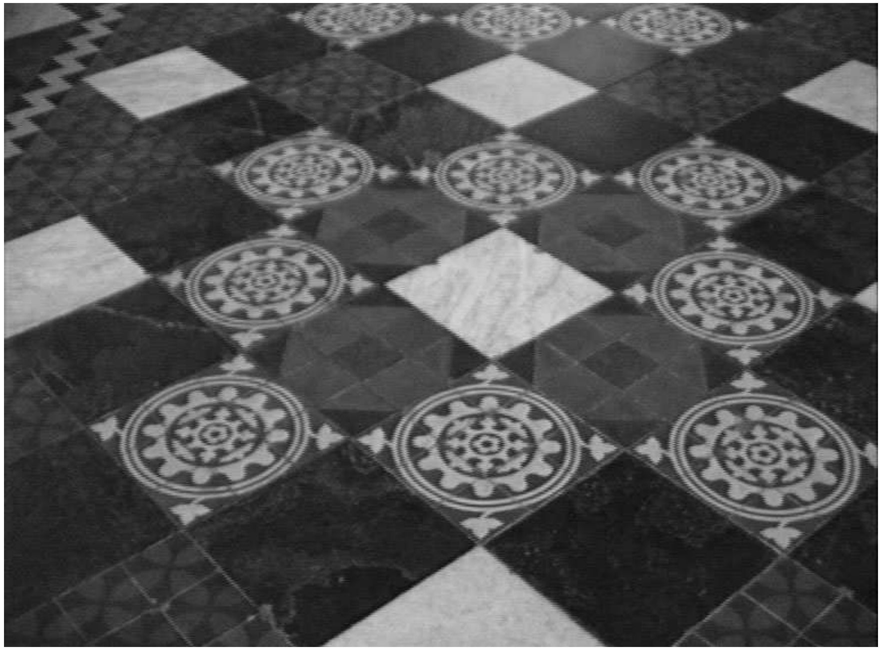


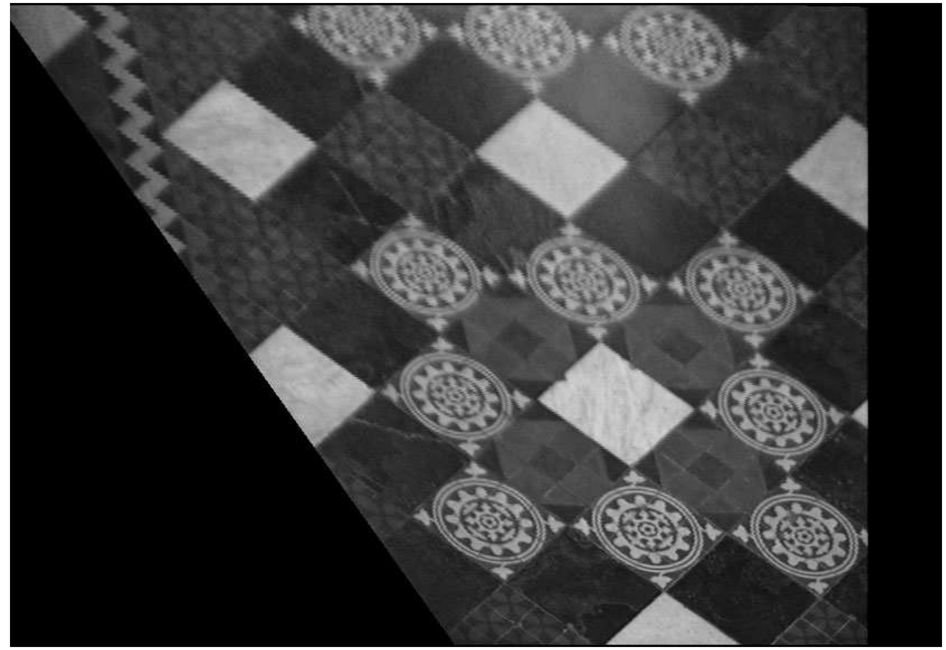
Affine Structure from Images

- Affine structure gives parallelism, ratio of areas, centroid, etc., and can be the basis of many decisions.
- Find $\mathbf{l}_\infty$ in image using parallel lines.
- Apply a transformation $\mathbf{H}$ that maps the line to $[0\ 0\ 1]^\mathbf{T}$
- Any $\mathbf{H} = \mathbf{H_A} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$ sends $\begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$ to $[0\ 0\ 1]^\mathbf{T}$ as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}^{-\mathbf{T}} \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \begin{bmatrix} l_3 & 0 & -l_1 \\ 0 & l_3 & -l_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Affine Rectification





Parallel lines are parallel, but right angles are not right angles.

Circular Points & Similarity

- Circular points are fixed under similarity

$$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} = s(\cos\theta + i\sin\theta) \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

- Conversely, any transformation that fixes circular points is a similarity.
- Thus, a transformation $\mathbf{H}$ that sends the circular points to their canonical form $\mathbf{I}$ and $\mathbf{J}$ leaves only a similarity transformation.

Dual Conic to Circular Points

- $\mathbf{C}^*_\infty = \mathbf{I}\mathbf{J}^\mathbf{T} + \mathbf{J}\mathbf{I}^\mathbf{T}$ is a dual conic defined by the circular points. It is fixed under similarity also.
- In canonical or Euclidean frame, $\mathbf{C}^*_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- We can see

$$\begin{bmatrix} s\mathbf{R}^\mathbf{T} & \mathbf{0} \\ \mathbf{t}^\mathbf{T} & 1 \end{bmatrix} \mathbf{C}^*_\infty \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} s^2 & 0 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv \mathbf{C}^*_\infty$$

Metric Structure from Images

- Identify circular points in the image.
- This can be done by finding a world circle in the image as a conic, finding the $\mathbf{l}_\infty$ in the image and finding their intersection
- Map one circular point to $\mathbf{I}$ and the other to $\mathbf{J}$. The transformation $\mathbf{H}$ that does it metric rectifies the image.
- $\mathbf{l}_\infty$ gives affine structure, the circle gives metric structure.
- Can be done using 2 non-parallel orthogonal line pairs instead of a circle or 5 orthogonal line pairs from projective!

Metric Rectification