Prove/Disprove: Let $L = \{\text{w} \in \{0,1\}^\star |$ w contains 1 in the third position from the end$\}$; for example $000100 \in L$ whereas $0011 \notin L$. Then any DFA recognizing L must have at least 8 states.

Proof. Answer is yes, there is no DFA with fewer than 8 states. We use proof by contradiction technique to prove the same. Assume that there is a DFA with 7 (or less than 7) states with q_0 as its start state. Now, consider all possible binary strings of length 3; we have exactly 8 different strings; say $w_1, w_2, \ldots, w_8$ where $w_i = x_{i1}x_{i2}x_{i3}$ and each $x_{ij} \in \{0,1\}$.

Let $\hat{\delta}(q_0, w_i) = r_i$ for each $i = 1, 2, \ldots, 8$. Since we have maximum of 7 different states, $r_i = r_j$ for some $i, j \in \{1, 2, \ldots, 8\}$ such that $i \neq j$. This implies, $\hat{\delta}(q_0, w_i) = \hat{\delta}(q_0, w_j)$. However, we know that w_i is different from w_j. Assume that k ($k \in \{1, 2, 3\}$) is the **first** position in which they differ. In each case (for $k = 1$ or 2 or 3), we produce two different stings (say u, v), of which, one's third symbol from the end is 1 and the other one's third symbol from the end is 0 such that both the strings u and v end up in the same state after reading them completely from the initial state. This leads to the contradiction to the fact that DFA must accept one string (the one whose third symbol from the end is 1) and reject the other string (whose third symbol from the end is 0).

1. Assume $k = 1$. We have only two possibilities and one of them must be true.

   (a) $w_i = 1x_{i2}x_{i3}$ and $w_j = 0x_{j2}x_{j3}$. Let $u = w_i$ and $v = w_j$. However, we have $\hat{\delta}(q_0, w_i) = \hat{\delta}(q_0, w_j)$. This implies, DFA either accepts or rejects both the stings $w_i(= u), w_j(= v)$. This is contradiction to the fact that, DFA must accept $w_i(= u)$ (since third symbol from the end is 1) and reject $w_j(= v)$ (since the third symbol from the end is 0).

   OR

   (b) $w_i = 0x_{i2}x_{i3}$ and $w_j = 1x_{j2}x_{j3}$. Let $u = w_i$ and $v = w_j$. The argument similar to the above one holds here too.

2. Assume $k = 2$. Again, we have only two possibilities and one of them must be true.

   (a) $w_i = x_{i1}1x_{i3}$ and $w_j = x_{i1}0x_{j3}$. Now, consider two strings $u = w_ix_{i1} = x_{i1}1x_{i3}x_{i1}$ and $v = w_jx_{i1} = x_{i1}0x_{j3}x_{i1}$. Then, we have $\hat{\delta}(q_0, u) = \hat{\delta}(\hat{\delta}(q_0, w_i), x_{i1}) = \hat{\delta}(\hat{\delta}(q_0, w_j), x_{i1}) = \hat{\delta}(q_0, v)$ (since $\hat{\delta}(q_0, w_i) = \hat{\delta}(q_0, w_j)$). Therefore, DFA either accepts or rejects both the stings u, v. This is contradiction to the fact that, DFA must accept u (since the third symbol from the end is 1) and reject v (since the third symbol from the end is 0).

   OR

   (b) $w_i = x_{i1}0x_{i3}$ and $w_i = x_{i1}1x_{j3}$. In this case too, the argument similar to the above one holds if we take $u = w_ix_{i1} = x_{i1}0x_{i3}x_{i1}$ and $v = w_jx_{i1} = x_{i1}1x_{j3}x_{i1}$.

3. Assume $k = 3$. Here also we have two possibilities:

   (a) $w_i = x_{i1}x_{i2}1$ and $w_j = x_{i1}x_{i2}0$. Now, consider two strings $u = w_i x_{i1}x_{i2} = x_{i1}x_{i2}1x_{i1}x_{i2}$ and $v = w_j x_{i1}x_{i2} = x_{i1}x_{i2}0x_{i1}x_{i2}$. Then, we have $\hat{\delta}(q_0, u) = \hat{\delta}(\hat{\delta}(q_0, w_i), x_{i1}x_{i2}) = \hat{\delta}(\hat{\delta}(q_0, w_j), x_{i1}x_{i2}) = \hat{\delta}(q_0, v)$ (since $\hat{\delta}(q_0, w_i) = \hat{\delta}(q_0, w_j)$). Therefore, `DFA` either accepts or rejects both the stings u, v. This is contradiction to the fact that, `DFA` must accept u (since the third symbol from the end is 1) and reject v (since the third symbol from the end is 0).

   OR

   (b) $w_i = x_{i1}x_{i2}0$ and $w_j = x_{i1}x_{i2}1$. In this case too, the argument similar to the above one holds if we take $u = w_i x_{i1}x_{i2} = x_{i1}x_{i2}0x_{i1}x_{i2}$ and $v = w_j x_{i1} = x_{i1}x_{i2}1x_{i1}x_{i2}$.

This completes the proof. $\square$