Introduction to Metaheuristics

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UCO Angers

2025-2026

based on lectures of Fabien Tricoire

Outline

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Objectives

Once this course is validated, you should:

- Know what is a metaheuristic
- Know why the world needs metaheuristics
- Be able to evaluate the quality of a metaheuristic
- Know how to design a metaheuristic for a given problem

During the course

- Several useful books, e.g.:
  - Handbook of Metaheuristics, Fred Glover & Gary A. Kochenberger, Kluwer's International Series, ISBN 1-4020-7263-5
  - Stochastic Local Search, Foundations and Applications, Holger H. Hoos & Thomas Stützle, Elsevier, ISBN 1-55860-872-9
  - Search Methodologies, Introductory Tutorials in Optimization and Decision Support Techniques, Edmund K. Burke & Graham Kendall, Springer, ISBN 0-387-23460-8
- Main source of information: these slides (you will get them)
- Interactive lectures
- Discussions
- Exercises

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Hard Optimisation Problems

Imagine a problem with. . .

- Some operational constraints
  - Example: visit all customers
- At least one optimal solution
  - Optimise = Minimise or Maximise an Objective function (a.k.a. Fitness)
  - Example: Minimise the sum of operational costs

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Such that finding this optimal solution is too hard...

- Even with the best programmer in the world
- Even with the best programming language
- Even with the best operating system
- Even with the fastest computer
- Even 20 years in the future

Such problems exist!

The Travelling Salesperson Problem (TSP)

A Travelling Salesperson wants to sell their goods to customers. . .

- Given data: distances using the road network
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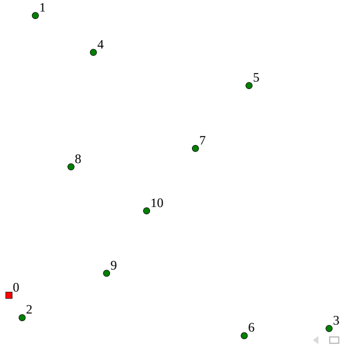

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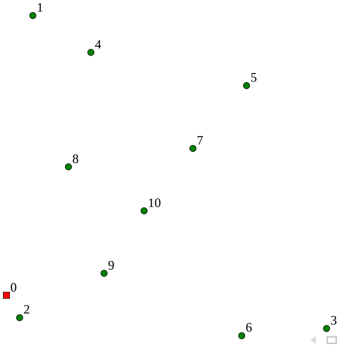

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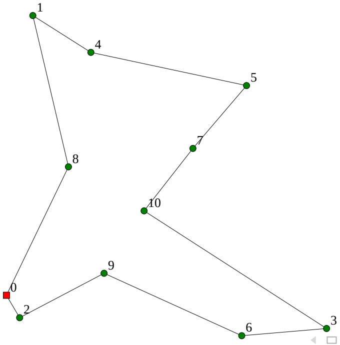

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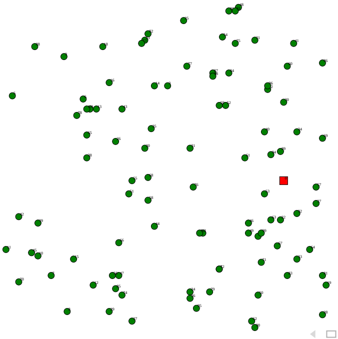

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Now suppose we have a computer a million times faster:

- 20 customers: 2 weeks
- 22 customers: 18 years
- 25 customers: 245,760 years

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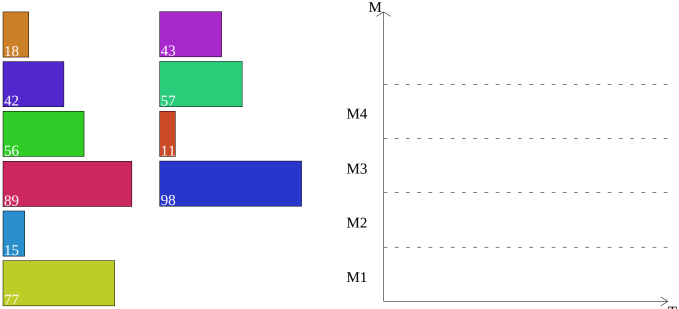

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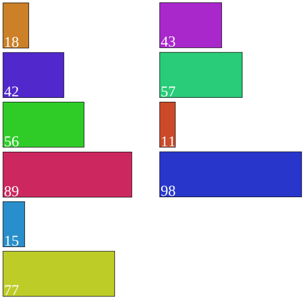

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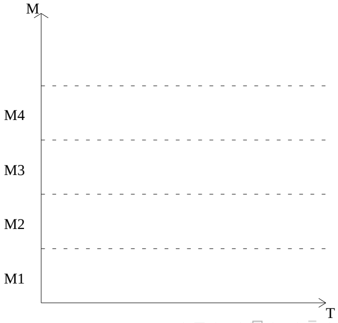

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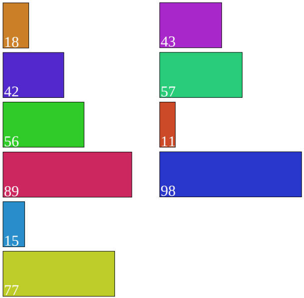

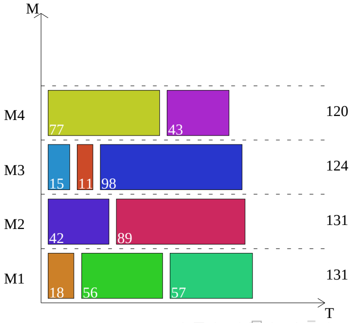

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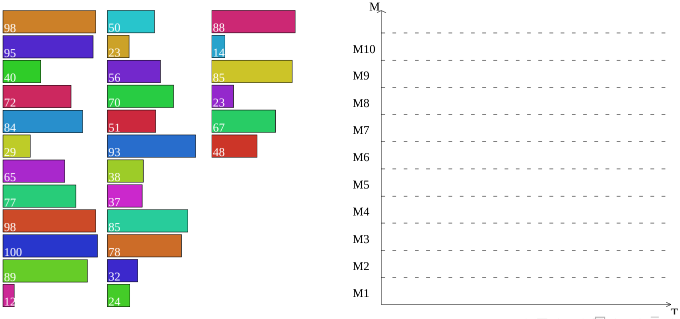

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Suppose our computer can process 500,000 solutions per second...

m	n	Time
4	10	2 s
5	15	17 h
6	20	232 years
8	25	$2.39 \cdot 10^9$ years
10	30	$6.34 \cdot 10^{16}$ years

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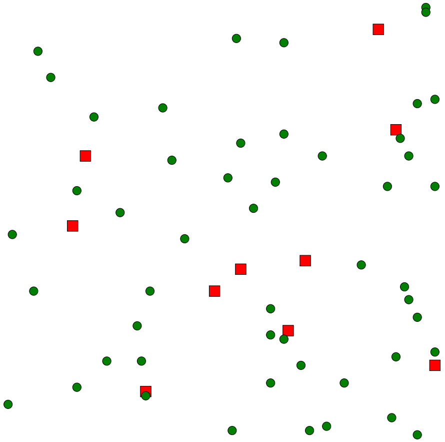
- Given data: cost induced by servicing customers through warehouses (e.g. distance)
- Constraint 1: No more than p warehouses may be open
- Constraint 2: Each customer must be serviced by a warehouse
- Decision 1: which warehouses should we open?
- Decision 2: which open warehouse should service which customer?
- Objective: minimise total service cost

Warehouse Location (a.k.a. p-median Problem)

An example! $(m = 10, p = 5)$



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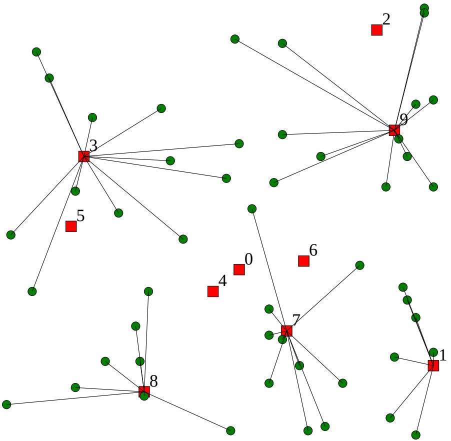

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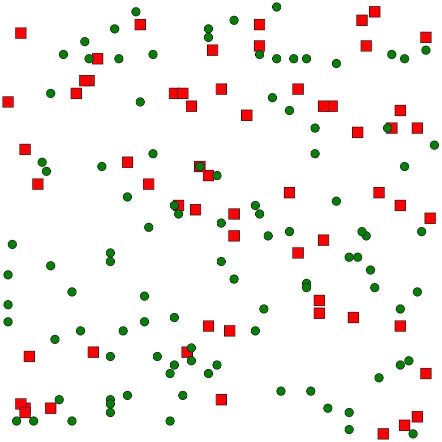

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- Which is equal to: $\sum_{k=1}^{p} \binom{m}{k}$

Warehouse Location

Suppose our recent computer can process 4,000 solutions per second...

m	p	Time
10	5	0.15 s
20	10	2.5 m
30	20	3 days
40	20	5 years
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- Question: Evaluation is quite time-consuming. Why?
- Answer: given a solution, finding the best open depot for each customer takes time.

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- Variables
  - Can be modified
  - Useful to store computation results
- Both kinds can be arranged into data structures
  - Example 1: Lists
  - Example 2: Arrays

Algorithms: basic operations

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Exercise: write an equivalent algorithm using repeat (...) until

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Exercise: write an equivalent algorithm using repeat (...) until

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$z \leftarrow 0$
repeat
    $z \leftarrow z + 2$
    $x \leftarrow x + 1$
until $x = 10$

"(..) allow a function to specify a return value to be passed back to the code that called the function" (Wikipedia)

Example function: $min(a, b)$

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Notation

- The Big $\mathcal{O}$ notation is used to represent complexity, and means "in the order of".
- For instance, our complete enumeration algorithm for the $P||C_{max}$ has time complexity $\mathcal{O}(m^n)$, since it evaluates m^n solutions.

(N.B.: the Big $\mathcal{O}$ notation is also used for other measures)

Introducing time complexity (2)

Exercise: what is the time complexity of the following algorithm?
(n and m are given constants, and drinking coffee is in $\mathcal{O}(1)$, a.k.a. constant time)

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    drink coffee
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Answer: $\mathcal{O}(m \cdot n)$

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Problems can be sorted into classes:

- P is the class of problems of polynomial time complexity
- NP is the class of problems that can be solved by a **N**on-deterministic **P**olynomial time algorithm (P $\subseteq$ NP)
- (this means it's actually hard to solve problems in NP $\setminus$ P)

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Time complexity of a problem: the time complexity of the fastest known algorithm to solve this problem.

Problems can be sorted into classes:

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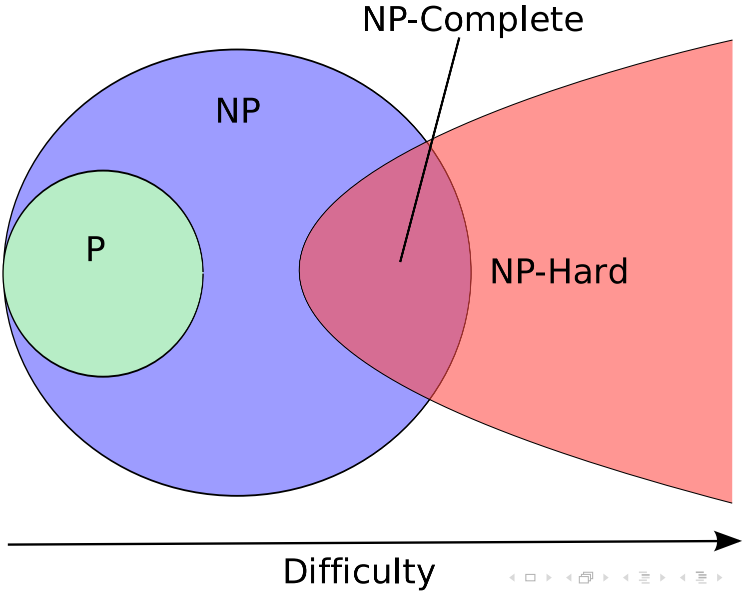
- NP-Complete problems are "the hardest problems in NP"; they are all equivalent!
- NP-hard problems are "at least as hard as NP-complete problems"

Most of the interesting combinatorial problems are NP-Hard, including TSP, $P||C_{max}$ and p-Median.

Complexity classes: an overview



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Bottom line

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  - May also be extremely long: a TSP with 15,112 nodes in 22.6 years on a computer from 2001
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- Heuristics
- Metaheuristics (general-purpose heuristic-based methods)

Outline

Heuristic approaches

A metaheuristic is a special kind of heuristic method. . .

(personal) Definition

Heuristic: a method based on common sense, empirical knowledge, and intuition.

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(personal) Definition

Heuristic: a method based on common sense, empirical knowledge, and intuition.

- This is neither a lecture on common sense nor on intuition.
- We will work on method and build knowledge.

Construction Vs. Improvement

Two critical questions

1. How do we construct a solution?
2. How do we improve a constructed solution?

Construction Vs. Improvement

Two critical questions

1. How do we construct a solution?
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Different questions call for different answers. . .

Two possible answers

1. With a construction heuristic
2. With an improvement heuristic

Construction heuristics

Idea: from a problem input, construct a solution to this problem.

Example: greedy algorithms

- Iteratively satisfy problem constraints
- Use a <u>greedy criterion</u> to achieve this
- Stop when all constraints are satisfied
- Remark: The greedy criterion should take the objective into account

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Exercise: design a greedy heuristic for the TSP

1. Which constraints do we iteratively satisfy?
2. What is the greedy criterion?

Greedy heuristic for the TSP

Nearest neighbour construction

	1	2	3	4	5	6	7	8	9	10
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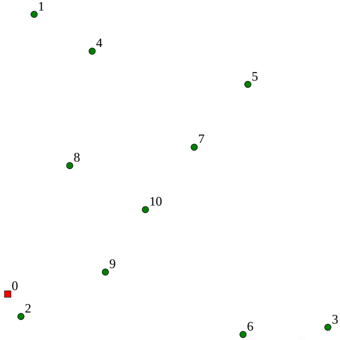

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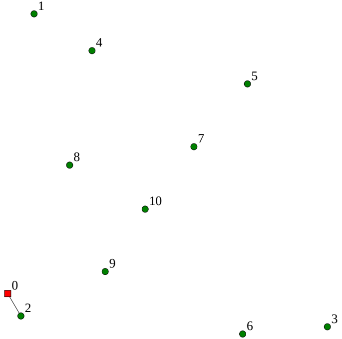

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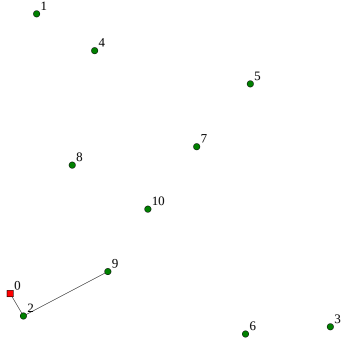

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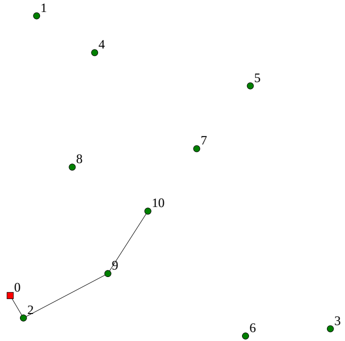

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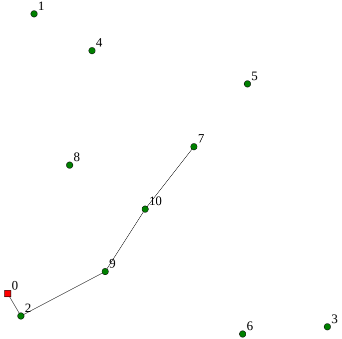

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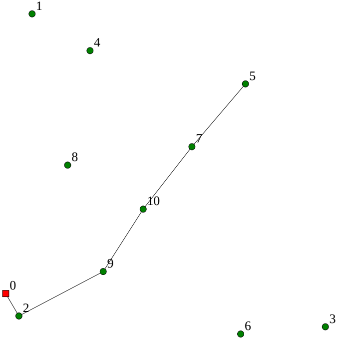

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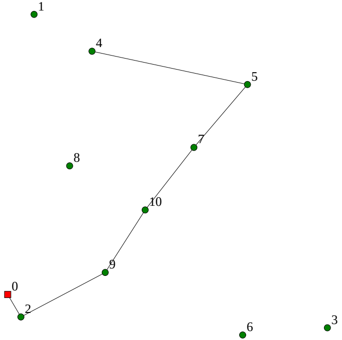

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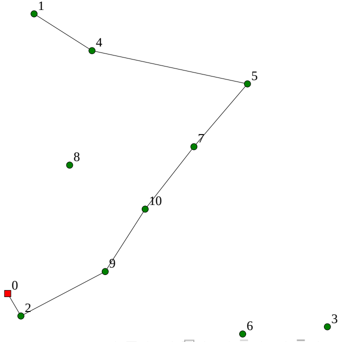

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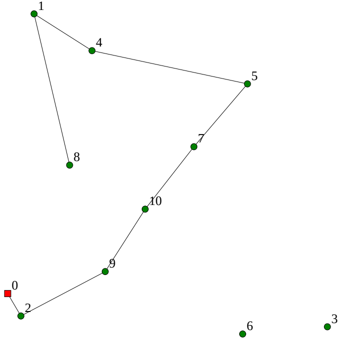

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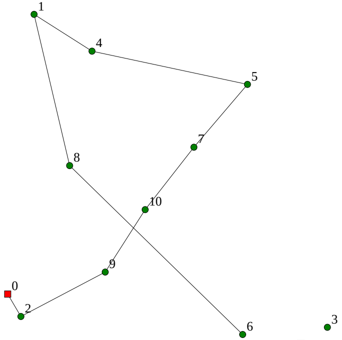

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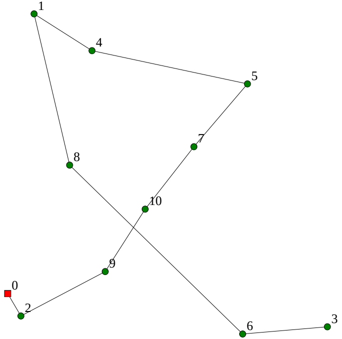

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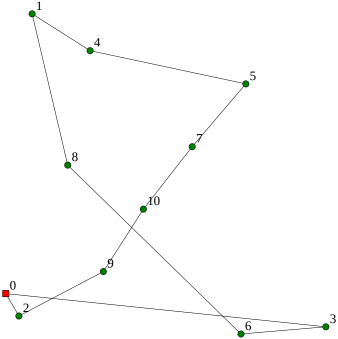

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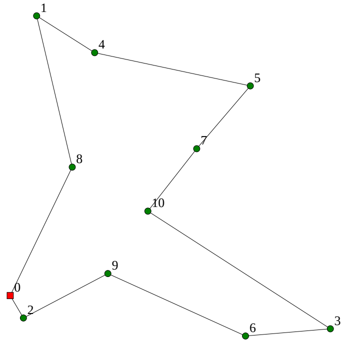
- Total cost = 335
- But there exists a better solution, with cost = 308 (9% better)



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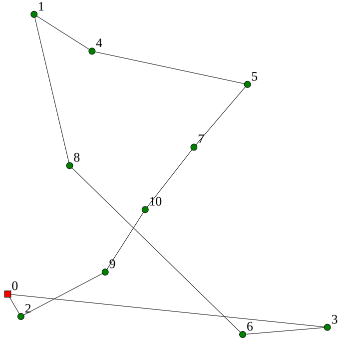
- Total cost = 335
- But there exists a better solution, with cost = 308 (9% better)
- So it would be nice to improve this tour...



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Neighbourhood: the set of solutions that can be obtained by applying a given operator to a given solution.

Improvement heuristics

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Solutions in the neighbourhood of x are its neighbours.

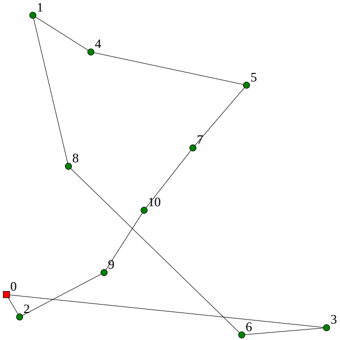
- Operator example for the TSP: move one node to another position in the tour.
- Exercise: find other neighbourhoods for the TSP.

Definition

Local Search: a heuristic aiming at improving a solution by performing neighbourhood exploration.

Improvement heuristics for the TSP

An example: the "node move" neighbourhood



Suppose that visiting order starts with node 2...
We can:

Improvement heuristics for the TSP

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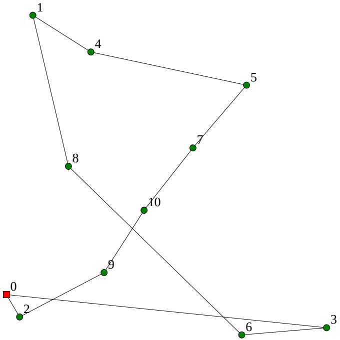

Suppose that visiting order starts with node 2...

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1. Move 2 at the end of the tour

Improvement heuristics for the TSP

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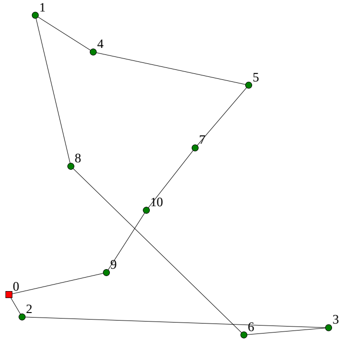

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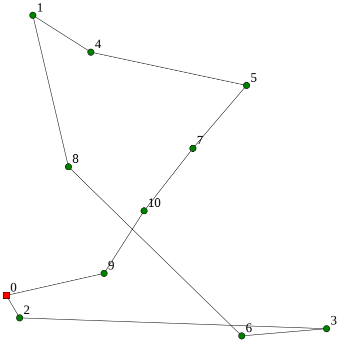

Suppose that visiting order starts with node 2...

We can:

1. Move 2 at the end of the tour
2. Move 6 after 3

Improvement heuristics for the TSP

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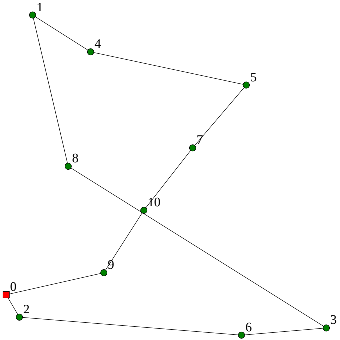

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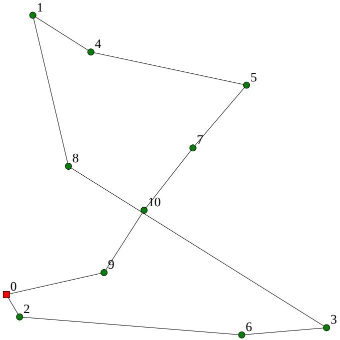

Suppose that visiting order starts with node 2. . .
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Improvement heuristics for the TSP

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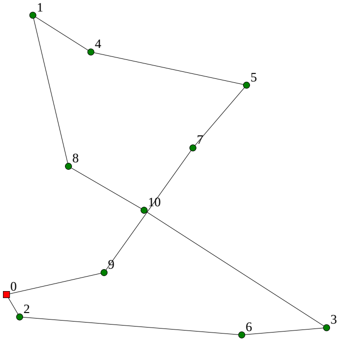

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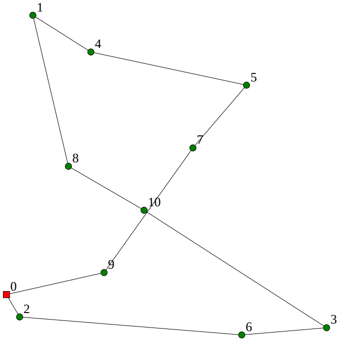
Improvement heuristics for the TSP

An example: the "node move" neighbourhood

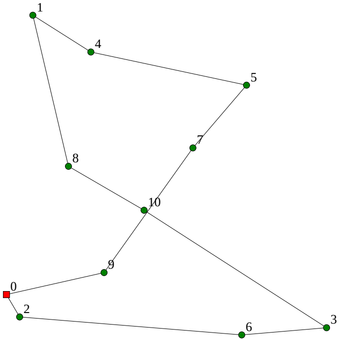


Suppose that visiting order starts with node 2...

We can:

1. Move 2 at the end of the tour
2. Move 6 after 3
3. Move 10 after 8

Then there is no more improvement!



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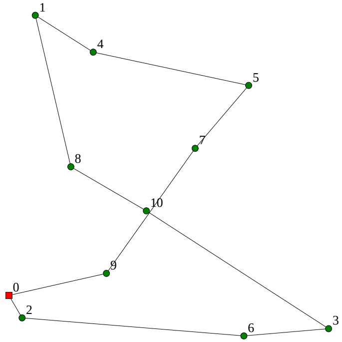

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We can:

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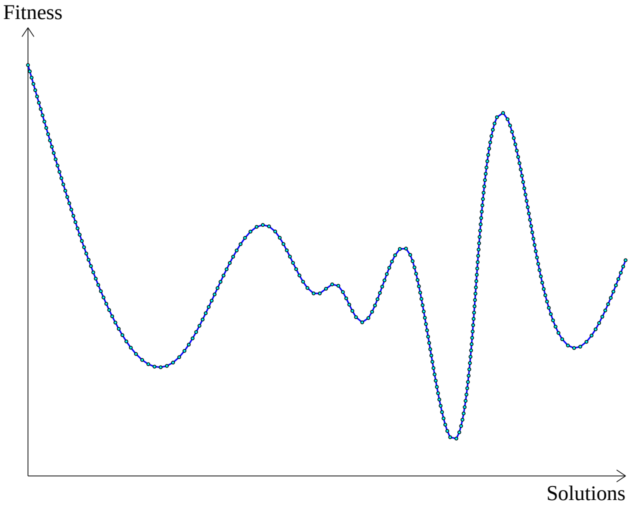
- We are still far from 308...

The local optimum issue

Assume neighbour solutions are adjacent points in this Solution Space. . .



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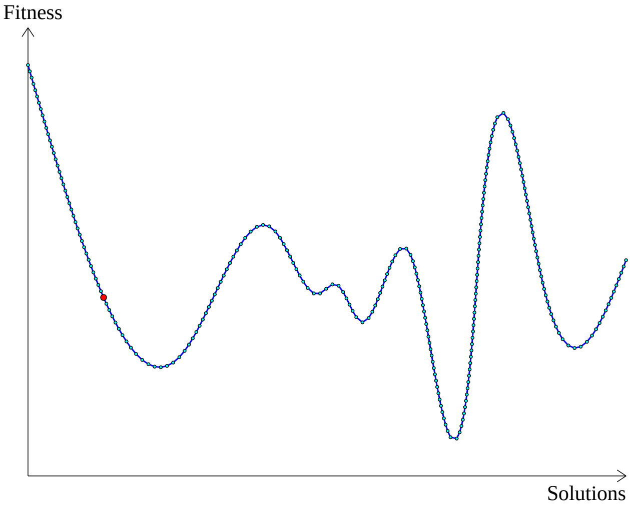

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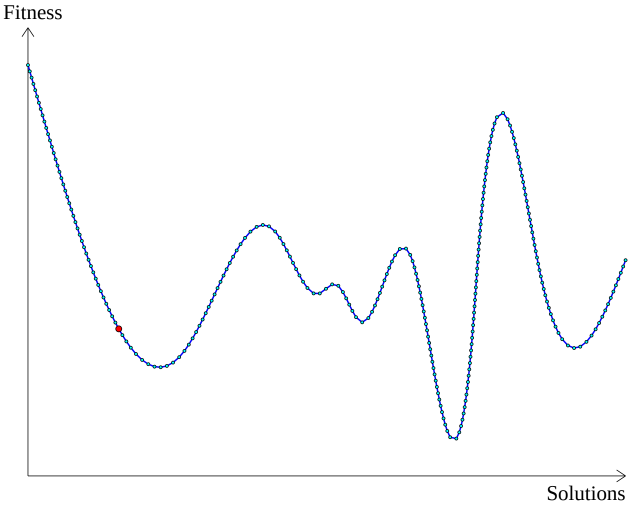

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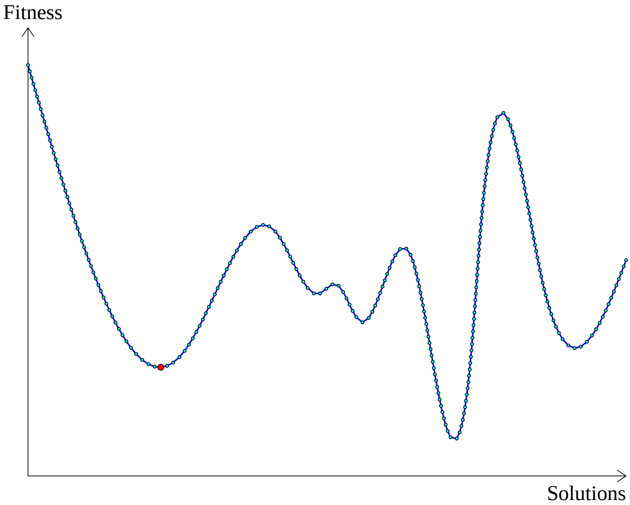

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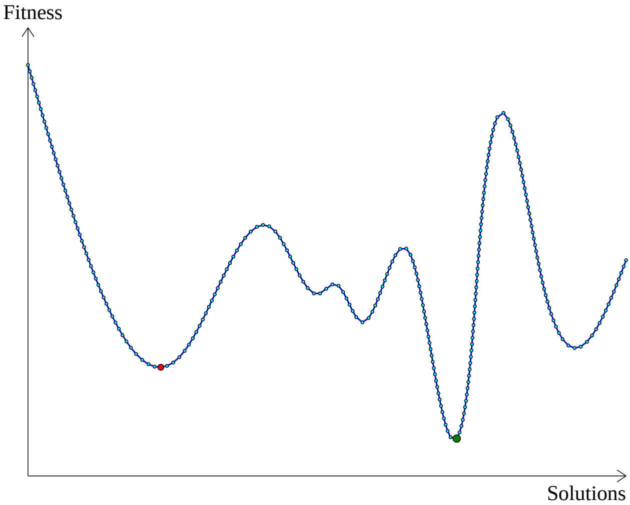

No more improvement in the neighbourhood!
Note that the global optimum is also a local optimum

Simple heuristics: limitations

A heuristic typically exploits problem structure

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Metaheuristics aim at dealing with these bad news!

Outline

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There is no commonly agreed definition!

(personal) Definition

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Other definitions:

- A top-level general strategy which guides other heuristics to search for feasible solutions in domains where the task is hard. (various sources on the web)
- A set of concepts that can be used to define heuristic methods that can be applied to a wide set of different problems. (Metaheuristics Network)

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Most metaheuristics include stochastic components (a.k.a. non-deterministic; e.g. randomness is involved in the process)

A few milestones

Term introduction

- 1986, Fred Glover: "Future Paths for Integer Programming and Links to Artificial Intelligence"
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A few dates

- 1965: first Evolution Strategy
- 1975: first Genetic Algorithms
- 1983: Simulated Annealing
- 1986: Tabu Search
- 1991: Ant Colony Optimisation
- 1997: Variable Neighbourhood Search

Evaluating Metaheuristics

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Relevant questions

1. Is the metaheuristic well-designed?
   $\rightarrow$ Qualitative analysis
2. Once applied to a given problem, does it behave well?
   $\rightarrow$ Quantitative analysis

Qualitative analysis

Again, no common agreement; personal suggestions:

Suggestion 1: Clarity, Modularity, Simplicity

A metaheuristic should be easy to apply to any given hard problem!

- Clarity: is it easy to understand what it does?
- Modularity: can it easily be applied to different problems?
- Simplicity: is it easy to implement?

Qualitative analysis (2)

Suggestion 2: Intensification

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Suggestion 3: Diversification

- Interesting solutions might reside in different "regions"
- A good metaheuristic should provide a broad exploration of the solution space!

Intensification and Diversification can be seen as opposite goals...

Quantitative analysis

Goal: once the metaheuristic is implemented, evaluate it.

Interesting criteria include:

- Ability to find good/optimal solutions
  - Comparison with optimal solutions (if available)
  - Comparison with Lower/Upper Bounds (LB/UB, if available)
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Keep this in mind!